Series 2 Exercise 7

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19 mars 2022

1 $\nu(1) = 0$ and $\nu(-1) = 0$

Indeed, note that

$$\nu(1 \cdot 1) = \nu(1) + \nu(1) = \nu(1) \iff 2\nu(1) = \nu(1) \iff \nu(1) = 0$$

Now for the second part, notice that since $-1 \cdot -1 = 1$ we get

$$\nu(-1 \cdot -1) = \nu(-1) + \nu(-1) = \nu(1) = 0 \iff 2\nu(-1) = 0 \iff \nu(-1) = 0$$

2 R_ν is a subring of K

To show R_ν is a subring, we have to show that $1, 0 \in R_\nu$ and that R_ν is closed under addition and multiplication.

Using the first part of the exercise, we immediatly get that $1 \in R_\nu$ since $\nu(1) \geq 0$ and by definition $0 \in R_\nu$.

R_ν is closed under multiplication

Let $x, y \in R_\nu \setminus \{0\}$, we get that $\nu(x \cdot y) = \nu(x) + \nu(y) \geq 0$ since $\nu(x), \nu(y) \geq 0$ by hypothesis.
If either x or y is equal to 0, then clearly $x \cdot y = 0 \in R_\nu$.
Hence R_ν is closed under multiplication.

R_ν is closed under addition

Indeed, let $x, y \in R_\nu$, now $\nu(x + y) \geq \min(\nu(x), \nu(y)) \geq 0$ hence $x + y \in R_\nu$.

This shows that R_ν is a subring of K.

3 K is the fraction field of R_ν

We explicit an isomorphism between K and $\operatorname{Frac} R_\nu$.
Let $j : R_\nu \to K$ be the inclusion of R_ν in K, this is obviously a ring homomorphism.
Now applying the universal property of the fraction field to j, we get a unique ring homomorphism $\phi : \operatorname{Frac} R_\nu \to K$ making the following diagramm commute :

$$\begin{tikzcd}
R_\nu \arrow[r, "j"] \arrow[d, "\iota"'] & K \\
\operatorname{Frac} R_\nu \arrow[ur, "\exists!\phi"'] &
\end{tikzcd}$$

Recall from the proof of the universal property of the fraction field that ϕ is defined by $\phi(\frac{a}{b}) = \iota(a) \cdot \iota(b)^{-1}$.
We now show ϕ is injective.
Indeed, suppose $\phi(\frac{a}{b}) = \phi(\frac{c}{d})$, then $j(a)j(b)^{-1} = j(c)j(d)^{-1} \implies j(a)j(d) = j(c)j(b) \iff j(ad) = j(cb)$, ie. that $ad = cb$, which in turn implies $\frac{a}{b} = \frac{c}{d}$ in $\operatorname{Frac} R_\nu$. Here we used the fact that j is injective.
Thus, we only need to show that ϕ is surjective.
Let $a \in K$, if $\nu(a) \geq 0$, then clearly $\phi(\frac{a}{1}) = j(a) \cdot 1 = a$.
If $\nu(a) < 0$, then notice that $\frac{1}{\frac{1}{a}} \in \operatorname{Frac} R_\nu$ since $\nu(1) = \nu(\frac{a}{a}) = \nu(a) + \nu(\frac{1}{a}) \iff \nu(a) = -\nu(\frac{1}{a})$, hence if a has a negative valuation, $\frac{1}{a} \in R_\nu$
Finally the following calculation shows that ϕ is surjective.

$$\phi\left(\frac{1}{\frac{1}{a}}\right) = j(1) \cdot j\left(\frac{1}{a}\right)^{-1} = a$$

Hence $\operatorname{Frac} R_\nu \simeq K$ which concludes the proof.

4 For every $x \in \mathbb{Z}, \nu(x) \geq 0$

Obviously $\nu(0)$ is undefined so I guess this is a typo and I show the result for $\mathbb{Z} \setminus \{0\}$.

Indeed, since $\nu(1) = \nu(-1) = 0$, we get $\forall x \in \mathbb{Z}, x > 0$

$$\nu(x) = \nu(\underbrace{1 + \ldots + 1}_{x \text{ times}}) \underbrace{\geq}_{\text{since } \nu \text{ is a valuation}} 0$$

Similarly, if $x < 0$, we may write $\nu(x) = \nu(\underbrace{-1 \ldots - 1}_{x \text{ times}}) \geq 0$ by the same argument as above.

5 If $\nu(p) = 0$ for all primes p, then ν is trivial.

First, note that, since we may write any integer as product of primes, we get that for all $x \in \mathbb{Z} \setminus \{0\}$, $\nu(x) = \nu\left(\prod_{i=1}^{n} p_i\right) = \sum_{i=1}^{n} \nu(p_i) = 0$, where $\prod_{i=1}^{n} p_i$ is the decomposition of x into prime factors.
For the general case, first notice that $\forall x \in \mathbb{Q}$, we have that $\nu(1) = \nu(\frac{x}{x}) = \nu(x) + \nu(\frac{1}{x}) = 0$, hence $\nu(x^{-1}) = -\nu(x)$.
Hence, for $\frac{a}{b} \in \mathbb{Q}, a, b \in \mathbb{Z}$, we get $\nu(\frac{a}{b}) = \nu(a) - \nu(b) = 0$, thus implying ν is trivial.

6 $\nu(p) \neq 0$ happens for at most one prime

Let p, q be primes in $\mathbb{Z}$ and suppose $\nu(p), \nu(q) \neq 0$, then by part 4, we know that $\nu(p), \nu(q) > 0$.
By Bezout's equality, there exist $a, b \in \mathbb{Z}$ such that $ap + bq = 1$.
Applying ν to the above equality, we get

$$\nu(ap + bq) = \nu(1) = 0 \geq \min(\nu(ap), \nu(bq))$$

Hence, either $\nu(ap) \leq 0$ or $\nu(bq) \leq 0$. Without loss of generality, suppose $\nu(ap) \leq 0$, then $\nu(a) + \nu(p) \leq 0$ which means that $\nu(a) < 0$ (since by hypothesis, $\nu(p) > 0$), however, this contradicts part 3.

7 p-adic valuation

Suppose $\nu(p) = c$, then clearly, $\nu(p^i) = i \cdot c$.
Furthermore, if $a, b \in \mathbb{Z}$ are coprime to p, then

$$\nu(\frac{a}{b}) = \nu(p_{a,1} \dots p_{a,n}) - \nu(p_{b,1} \dots p_{b,m}) \underbrace{=}_{\text{all } p_{k,j} \text{ are prime to } p.} 0$$

, where $p_{a,1} \dots p_{a,n}$ (resp. $p_{b,1} \dots p_{b,m}$) is the prime decomposition of a (resp. b) and the last equality follows from part 6.
Combining both observations above, we get that $\forall \frac{c}{d} \in \mathbb{Q}$, we may write

$$\nu(\frac{c}{d}) = \nu(p^i \frac{c'}{d'}) = \nu(p^i) + \nu(\frac{c'}{d'}) = i \cdot c$$

where in the first equality, we have simply isolated all factors from c and d which are powers of p, hence implying that c' and d' are both coprime to p [1].
We now show that ν_p is indeed a discrete valuation on $\mathbb{Q}$ when $c = 1$.
Let $p^i \frac{a}{b}, p^j \frac{c}{d} \in \mathbb{Q}$ be fractions of the form stated in the instruction.

1. One can always do this, simply consider the prime decomposition of c and d and isolate all powers of p.

We then have $\nu\left(p^i\frac{a}{b}p^j\frac{c}{d}\right) = \nu(p^{i+j}\frac{ac}{bd})$, since both ac and bd are prime to p, we get $\nu\left(p^{i+j}\frac{ac}{bd}\right) = (i+j)\cdot c = i+j$, showing the first property of a discrete valuation.
Now suppose without loss of generality, that $i < j$, then we may write

$$\nu(p^i\frac{a}{b} + p^j\frac{c}{d}) = \nu\left(p^i\left(\frac{a}{b} + p^{j-i}\frac{c}{d}\right)\right)$$

Furthermore, note that $\frac{a}{b} + p^{j-i}\frac{c}{d} = \frac{ad+p^{j-i}cb}{bd}$, notice that bd is clearly prime to p, furthermore, we may write

$$ad + p^{j-i}cb = p^k l \text{ for some integers } k \text{ and } l.$$

From this, we deduce that

$$\nu(p^i\frac{a}{b} + p^j\frac{c}{d}) = \nu(p^i) + \nu(p^k l) \geq \nu(p^i) \geq \min(i,j)$$

Where the last inequality followed from our assumption that $i < j$.

8 Valuation ring of ν_p is not $\mathbb{Z}$

Indeed, to show this we simply have to find an element of R_{ν_p} which is not in $\mathbb{Z}$, to see this take any integer $a \in \mathbb{Z}$ prime to p and note that

$$\nu(\frac{p}{a}) = 1 \implies \frac{p}{a} \in R_{\nu_p}$$

But obviously $\frac{p}{a} \notin \mathbb{Z}$.