Serie 6

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega := (0,1)\times(0,1)$. Investigate the existence and equality of $\int_\Omega f\, d(x,y)$, $\int_0^1 \int_0^1 f(x,y)\, dx\, dy$ and $\int_0^1 \int_0^1 f(x,y)\, dy\, dx$ for

(i) $f(x,y) := \frac{x^2-y^2}{(x^2+y^2)^2}$.

(ii) $f(x,y) := (1-xy)^{-a}$ for $a > 0$.

Compare your result with Fubini's Theorem.

Exercise 2. The *Dirichelet integral* is the improper integral defined by

$$\int_0^\infty \frac{\sin(x)}{x}\, dx.$$

It is a very simple example of an integral that exists in the Riemann sense but **not** in the Lebesgue sense. Explain why and then compute the value of this integral (understood as a Riemann-integral) by restricting the the domain of integration to $[0,t]$ and then taking the limit as $t \to \infty$.

Exercise 3. We show that translations are continuous on $L^p(\mathbb{R}^n)$. In other words, let $f \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ and prove that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p\, dx = 0.$$

Hint: Begin by showing the result for $f \in C_c^\infty(\mathbb{R}^n)$ and then approximate any function in $L^p(\mathbb{R}^n)$ by functions in $C_c^\infty(\mathbb{R}^n)$ in order to conclude.

Exercise 4. We prove a kind of continuity of the Lebesgue measure under translations.

(i) Let A be a measurable set and $\mathrm{m}(A) < \infty$. Show that

$$\lim_{|\varepsilon|\to 0} \mathrm{m}\big((A+\varepsilon) \setminus A\big) = 0.$$

(ii) Show that the result in (i) is false if $\mathrm{m}(A) = \infty$.

(iii) Show that the result in (i) is false if A is not measurable. (Replace m by m^*.)

Exercise 5. For $f \in L^1(\mathbb{R}^n)$, we define for $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx\,. \tag{1}$$

We will call $\hat{f}$ the Fourier transform of f. Prove that

(i) $\hat{f}$ is well-defined, i.e. that the integral on the right-hand side of (1) converges for every $\xi \in \mathbb{R}^n$,

(ii) $\hat{f}$ is a bounded function on $\mathbb{R}^n$,

(iii) $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0$.

Hint: Write (and justify it) that for any $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} \{f(x) - f(x - \xi')\}\, e^{-2\pi i x \cdot \xi}\, dx \text{ with } \xi' = \frac{\xi}{2|\xi|^2},$$

and use a previous exercise.

Exercise 6. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions.

(i) Assuming the result of Exercise 10, prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2n}$ (as a function of $(x, y) \in \mathbb{R}^{2n}$).

(ii) Show that if f and g are integrable on $\mathbb{R}^n$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2n}$ (as a function of $(x, y) \in \mathbb{R}^{2n}$).

(iii) We define the convolution of two integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) g(y)\, dy.$$

Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$ (that is, $y \mapsto f(x-y)g(y)$ is an integrable function on $\mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$ fixed).

(iv) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

with equality if f and g are non-negative.

(v) Recall that the Fourier transform $\hat{f}$ of an integrable function $f \in L^1(\mathbb{R}^n)$ defined in (1). Check first that $\hat{f}$ is bounded and continuous function of ξ. Then prove that for $f, g \in L^1(\mathbb{R}^n)$ integrable, one has

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \qquad \forall \xi \in \mathbb{R}^n\,.$$

Exercise 7. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \text{ for all } f \in L^1(\mathbb{R}^n).$$

Hint: Use the Fourier transform and Exercise 6.

Exercise 8. From every L^p-convergent sequence $\{f_\nu\}_{\nu>0}$ $(1 \le p < \infty)$, we can always extract a subsequence that converges pointwise almost everywhere. Yet, it might happen that the full sequence $f\nu$ converges pointwise nowhere. Here we discuss such an example. For all $\nu > 0$, we define

$$h(\nu) := \max\{k \in \mathbb{N} : 2^k \le \nu\}, \quad I_\nu := \left\{\frac{\nu - 2^{h(\nu)}}{2^{h(\nu)}}\right\} + [0, 1/2^{h(\nu)}], \quad f_\nu := \chi_{I_\nu}.$$

More precisely, $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,2/4]}$, $f_6 = \chi_{[2/4,3/4]}$, $f_7 = \chi_{[3/4,1]}$, $f_8 = \chi_{[0,1/8]}, \cdots$.

(i) Show that f_ν converges in $L^p(0,1)$ for $1 \le p < \infty$.

(ii) Show that f_ν converges pointwise nowhere on $[0,1]$.

(iii) Find a subsequence of f_ν which converges pointwise a.e. on $[0,1]$.

Exercise 9. Throughout this exercise, we assume $\Omega \subseteq \mathbb{R}^n$ is measurable.

(i) Show that for any $f, g \in L^\infty(\Omega)$, we have $\|f+g\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}$.

(ii) Show that $L^p(\Omega)$ is a real vector space $1 \le p \le \infty$.

(iii) Show that if $\Omega \subset \mathbb{R}^n$ is bounded and $1 \le p < q \le \infty$, then $L^q(\Omega) \subseteq L^p(\Omega)$. More precisely, find a constant $K = K(\Omega, p, q)$ such that

$$\|f\|_{L^p(\Omega)} \le K\|f\|_{L^q(\Omega)}.$$

(iv) Show that if $\Omega \subset \mathbb{R}^n$ is bounded and if $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)} .$$

Hint: For (iv), use (iii) to show the following inequalities

$$\limsup_{p\to\infty} \|f\|_{L^p(\Omega)} \le \|f\|_{L^\infty(\Omega)},$$
$$\liminf_{p\to\infty} \|f\|_{L^p(\Omega)} \ge \|f\|_{L^\infty(\Omega)} - \varepsilon \quad \forall\, \varepsilon > 0 .$$

For the second inequality, study the set $A_\varepsilon := \{x \in \Omega : \ |f(x)| \ge \|f\|_{L^\infty} - \varepsilon\}$.

Exercise 10 ($\star$)**.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Prove that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x,y) = f(x-y)$, $(x \in \mathbb{R}^n,\ y \in \mathbb{R}^n)$ is measurable.