Exercise 8

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We prove the result by induction on n.
The case $n = 1$ is clear, indeed, by definition, a vertex of degree 1 is a leaf, so the whole graph contains at least one leaf.
Let $G = (V, E)$ be the tree we consider.
Suppose the result shown for $n > 1$ vertices, we now show it for $n+1$ vertices.
Let $\mathcal{V}$ be the vertex of degree $n + 1$.
Let $\{e_1 = \{\mathcal{V}, v_1\}, \ldots, e_{n+1} = \{\mathcal{V}, v_{n+1}\}\}$ be all the edges of G which are connected to $\mathcal{V}$.
Now consider the set of all paths starting at v_{n+1} and which do not pass through $\mathcal{V}$, we denote this set by P.
We can now consider the set of all edges which are contained in a path of P, ie. we consider

$$\mathcal{E} = \{\{a, b\} \in E | \exists \{c_1, \ldots, c_k\} \in P \quad \exists i : c_i = a, c_{i+1} = b\}$$

and we let

$$V' = \{a \in V | \exists e \in \mathcal{E} \text{ such that } a \in e\}.$$

We can now consider the graph $G' = (V \setminus V', E \setminus (\mathcal{E} \cup \{\mathcal{V}, v_{n+1}\}))$.
Note that $\mathcal{V}$ now is of degree n, so by induction hypothesis, G' has at least n leaves.
However, we can now consider the path of maximum length in P, the last element of this path is a leaf, since if it wasn't a leaf, we could extend the path.
This leaf, however, is not contained in G', which means that G has at least $n + 1$ leaves. This concludes the proof by induction.