

Topology I

Course by Viktoryia Ozornova

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1 Homology Theories

Lecture 1: Introduction

Mon 10 Oct

Aim : Study further algebraic invariants of topological spaces.

We want to assign to pairs of topological spaces abelian groups.

$$h_n : T \rightarrow \text{Ab} \quad \forall n \in \mathbb{Z}$$

and to pairs continuous maps, we want to assign a map $h_n(f) : h_n(X) \rightarrow h_n(Y)$ which is functorial. Here T is the category of pairs of topological spaces $A \subset X$ with morphisms $f : (X, A) \rightarrow (Y, B)$ such that $f(A) \subset B$.

To relate h_n for different $n \in \mathbb{N}$, we will construct connecting morphisms $\partial_n : h_n(X, A) \rightarrow h_{n-1}(A, \emptyset)$.

Axiom 1 (Eilenberg-Steenrod Axiom)

A (generalised) homology theory consists of functors $h_n : T \rightarrow \text{Ab}$ and natural connecting homomorphisms $\partial_n : h_n(X, A) \rightarrow h_{n-1}(A, \emptyset)$ ¹ satisfying

— *Homotopy invariance :*

If $f, g : (X, A) \rightarrow (Y, B)$ are homotopic continuous maps of pairs then the induced maps $h_n(f) = h_n(g)$. Here homotopy of pairs means that there exists $H : X \times [0, 1] \rightarrow Y$ such that $H(A \times [0, 1]) \subset B$

— *Long exact sequence of a pair (LES) :*

Given a pair of topological spaces (X, A) there is a long exact sequence of abelian groups.

Denote $i : (A, \emptyset) \rightarrow (X, \emptyset)$ and $j : (X, \emptyset) \rightarrow (X, A)$, then

$$h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A, \emptyset)$$

— *Excision*

Given $B \subset A \subset X$ subspaces such that $\overline{B} \subset A^\circ$, the inclusion induces a group isomorphism

$$h_n(X \setminus B, A \setminus B) \rightarrow h_n(X, A)$$

We add another axiom to "make things easier"

— *Additivity :*

Given a family of pairs of spaces $(X_i, A_i)_{i \in I}$, the inclusions induce an isomorphism

$$\bigoplus h_n(X_i, A_i) \rightarrow h_n(\coprod X_i, \coprod A_i)$$

This is the end of the axioms for a generalised homology theory, the homology theory is called an ordinary homology theory if the Dimension Axiom holds, namely

$$h_n(pt) = 0 \forall n \neq 0$$

1. From now on, we write $h_n(A) := h_n(A, \emptyset)$

The abelian group $h_0(pt)$ is called the coefficient group of (h_n, ∂_n)

Lemma 2

If $f : X \rightarrow Y$ is a homotopy equivalence, then $\forall n \in \mathbb{Z}$ we obtain $h_n(f) : h_n(X) \rightarrow h_n(Y)$ to be an isomorphism for any homology theory (h_n, ∂_n)

Proof

Choose $g : Y \rightarrow X$ such that $g \circ f \simeq \text{Id}_X$ and $f \circ g \simeq \text{Id}_Y$, then by functoriality and homotopy invariance $\text{Id}_{h_n(X)} = h_n(\text{Id}_X) = h_n(g) \circ h_n(f)$, by symmetry, $h_n(f)$ and $h_n(g)$ are inverses. \square

Similarly, if $f : (X, A) \rightarrow (Y, B)$ is a homotopy equivalence of pairs, then the same result holds.

Example

For any such homology theory

$$h_n(\mathbb{R}^k) \simeq h_n(pt) \simeq h_n(D^k)$$

Lecture 2: Homology Theories

Wed 12 Oct

Recall that the natural homomorphisms ∂_n are natural, in the sense that the compositions

$$h_{n-1}(f) \circ \partial_n : h_n(X, A) \rightarrow h_{n-1}(A) \rightarrow h_{n-1}(B)$$

and

$$\partial_n \circ h_n(f) : h_n(X, A) \rightarrow h_n(Y, B) \rightarrow h_{n-1}(B)$$

coincide.

Today, we compute the homology groups $h_*(S^k)$ for $k \geq 0$ for a given ordinary homology theory h_* . Here, the k -sphere is defined as a subspace of \mathbb{R}^{k+1} .

Recall from the exercises that $h_*(pt \amalg pt) = h_*(pt) \oplus h_*(pt)$ for ordinary homology theories concentrated in degree 0.

There are two maps $\pm : pt \rightarrow S^0$ and one natural map $S^0 \rightarrow pt$ called the "fold" map.

By functoriality, the composition $h_*(pt) \rightarrow h_*(S^0) \rightarrow h_*pt$ is the identity.

To compute $h_*(S^k)$, we use two LES

$$\dots \xrightarrow{\partial_{n+1}} h_n(S^k) \xrightarrow{h_*\iota} h_n(D^{k+1}) = 0 \xrightarrow{h_*\iota} h_n(D^{k+1}, S^k) \rightarrow h_{n-1}(S^k) \rightarrow h_{n-1}(D^{k+1}) = 0 \dots$$

As $h_n(D^{k+1}) = 0$ for $n \neq 0$, we have an isomorphism $\partial_n : h_n(D^{k+1}, S^k) \rightarrow h_{n-1}(S^k)$.

The inclusion $D^k \subset S^k$ (as the upper hemisphere) gives rise to another LES

$$0 = h_n D^k \xrightarrow{h_*\iota} h_n S^k \xrightarrow{h_*\iota} h_n(S^k, D^k) \xrightarrow{\partial_n} h_{n-1} D^k = 0 \rightarrow h_{n-1} S^k \dots$$

And thus we also get an isomorphism $h_n \iota : h_n S^k \rightarrow h_{n-1} D^k$. The inclusion of the north pole $pt \subset D^k \subset S^k$ induces, using excision, the isomorphism $h_n(S^k \setminus pt, D^k \setminus pt) \simeq h_n(S^k, D^k)$ of the following diagram

$$\begin{array}{ccccc} h_n(D^k, S^{k-1}) & \xleftarrow{\simeq} & h_n(S^k \setminus pt, D^k \setminus pt) & \xrightarrow{\simeq} & h_n(S^k, D^k) \\ \simeq \partial_n \downarrow & & \partial_n \downarrow & & \downarrow \partial_n \\ h_{n-1}(S^{k-1}) & \xrightarrow{h_{*}\iota} & h_{n-1}(D^k \setminus pt) & \longrightarrow & h_{n-1}(D^k) \end{array}$$

We know that the bottom row of this diagram is an ES.

In particular $h_n(D^k, S^{k-1}) \simeq h_n(S^k, D^k)$.

The isomorphism $\partial_n : h_n(D^k, S^{k-1}) \rightarrow h_{n-1}(S^{k-1})$ now almost allows us to use induction to find the homology groups.

We now consider the case $n \in \{0, 1\}$ (This part of the proof is not complete yet)

$$h_1(D^k) = 0 \rightarrow h_1 S^k \rightarrow h_1(S^k, D^k) \xrightarrow{\partial_1} h_0 D^k \rightarrow h_0 S^k \rightarrow h_0(S^k, D^k) \rightarrow h_{-1} D^k = 0$$

The case $n \in \{0, 1\}$ gives a split short exact sequence

$$0 \rightarrow h_0 D^k \rightarrow h_0 S^k \rightarrow h_0(S^k, D^k) \simeq h_0(D^k, S^{k-1}) \rightarrow 0$$

The homotopy equivalence $pt \rightarrow D^k$ gives a split of this exact sequence $h_0 S^k \rightarrow h_0 pt \rightarrow h_0 D^k$.

The boundary homomorphism $h_1(S^k, D^k) \rightarrow h_0 D^k$ being 0 using results from the exercise sheet.

Now by induction, $h_n S^k = 0$ for all $n < 0$ and $h_0 S^k = h_0(pt)$ for all $k > 0$.

We also have that $h_n S^1 \simeq h_{n-1} S^0$ for $n \notin \{0, 1\}$.

What about $h_1 S^1$?

$$h_1(D^1, S^0) \rightarrow h_1(S^1, D^1) \rightarrow h_0(D^1)$$

and

$$h_1(D^1, S^0) \rightarrow h_0 S^0 \rightarrow h_0(D^1)$$

Where the last morphism is induced by the fold map, namely $h_0 S^0 = h_0 pt \oplus h_0 pt \rightarrow h_0(pt)$ and $(x, y) \mapsto x + y$.

We have

$$h_1 D^1 \rightarrow h_1(D^1, S^0) \rightarrow h_0 S^0 = h_0 pt \oplus h_0 pt \rightarrow h_0 D^1$$

We were able to show isomorphisms $h_n S^k \simeq h_{n-1} S^{k-1}$ for $n \notin \{0, 1\}$, $h_0 S^k \simeq h_0 pt$ for $k > 0$ and $h_1 S^1 \simeq h_0 pt$.

What about $h_1 S^k$ for $k > 1$?

We have isomorphisms

$$h_1 S^k \rightarrow h_1(S^k, D^k) \xrightarrow{\partial} h_0 D^k \simeq h_0 S^k$$

and

$$h_1(D^k, S^{k-1}) \simeq h_1(S^k, D^k) \rightarrow h_0 S^{k-1} \simeq h_0 D^k$$

and thus $h_1 S^k = 0$ for $k > 1$.

Proposition 4

For any ordinary homology theory (h_*, ∂_*) , the following holds

$$h_n S^k = \begin{cases} h_0 pt \oplus h_0 pt & \text{if } k = 0 = n \\ 0, & k > 0, n \notin \{0, k\} \\ h_0 pt & \text{if } k > 0 \text{ and } n \in \{0, k\} \\ 0, & \text{else} \end{cases}$$

We add one additional assumption, that there exists an ordinary homology theory with coefficient group $h_0 pt \simeq \mathbb{Z}$

Corollary 5

S^k and S^l are not homotopy equivalent for $k \neq l$

Proof

$$h_k S^k \simeq h_0 pt \neq h_k S^l = 0$$

□

Corollary 6 (Brouwer fixed point theorem)

Any continuous map $f : D^n \rightarrow D^n$ has a fixed point.

Proof

Assume $f : D^n \rightarrow D^n$ is a map without a fixed point.

Consider $g : D^n \rightarrow S^{n-1}$ sending $x \mapsto \frac{x-f(x)}{\|x-f(x)\|}$, by assumption, this is continuous.

Next, we claim that $g|_{S^{n-1}}$ is homotopic to $\text{Id}_{S^{n-1}}$ via the map

$$H(x, t) := \frac{x - tf(x)}{\|x - tf(x)\|}$$

If $t = 1$, the denominator is $\neq 0$, if $t < 1$

$$\|tf(x)\| = t\|f(x)\| < \|f(x)\| \leq 1$$

Hence, $\|x - tf(x)\| \neq 0$ and H is a well defined continuous map.

Now, consider

$$h_{n-1} S^{n-1} \xrightarrow{\text{ind}} h_{n-1} D^n \xrightarrow{h_{n-1}(g)} h_{n-1} S^{n-1}$$

By homotopy equivalence $h_{n-1}(g) \circ ind$ is the identity.

For $n > 1$, this implies that the identity factors through 0, which is a contradiction.

The special case $n = 1$ gives

$$h_0 S^0 \rightarrow h_0 D^1 \rightarrow h_0 S^0$$

If the coefficient group is \mathbb{Z} , this is a contradiction. □

2 Constructing singular homology

We want to construct a (ordinary) homology theory.

The idea is to study X by mapping topological simplices into X , here the topological n simplex is defined as

$$\Delta^n = \left\{ (t_0, \dots, t_n) \mid t_i \geq 0 \forall i, \sum_i t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

We define

$$Sing_n(X) = \{ f : \Delta^n \rightarrow X \text{ continuous} \}$$

in general, this set is huge.

Lecture 3: Singular homology

Mon 17 Oct

Goal : Find a way to organise the information in $Sing_n(X)$!

1. Relate $Sing_n(X)$ for different n to each other
2. Linearize!

We'll call $Sing_n(X)$ the n -th component of the singular set.

We think of the edges of the simplices as being ordered.

There are maps $\Delta^1 \rightarrow \Delta^n$ which are inclusions into the edges.

In fact, for every subset $S \subset \{0, \dots, n\}$, there is a continuous injective map $\Delta^k \rightarrow \Delta^n$, where $k = |S|$.

Now, for any $k < n$, we have restriction maps $Sing_n(X) \rightarrow Sing_k(X)$.

Define the category Δ_{inj} , whose objects are $[n]$ for every $n \in \mathbb{N}$ and whose morphisms $[k] \rightarrow [n]$ are order preserving injective maps.

The composition is just the composition of maps.

For X a fixed topological space, we get a contravariant functor $Sing.(X) : \Delta_{inj} \rightarrow \text{Set}$.

Given $\alpha : [k] \rightarrow [n]$ an injective order preserving map, we get

$$Sing_n(X) \rightarrow Sing_k(X)$$

with precomposition by α .

Lemma 7

Δ_{inj} can also be described as the category with objects $[n]$ and generated by maps $d^i : [n] \rightarrow [n+1]$ subject to the relations

$$d^j d^i = d^i d^{j-1}$$

for $0 \leq i < j \leq n$

Proof (Sketch)

This relation is indeed satisfied in Δ_{inj}

$$\{0 < \dots < n-2\} \xrightarrow{d^i} \{0 < \dots < n-1\} \xrightarrow{d^j} \{0 < \dots < n\}$$

Here

$$k \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k \geq i \end{cases} \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k+1 \leq j \\ k+2, k+2 \geq j+1 \end{cases}$$

One can compute that the composition $d^i d^{j-1}$ gives the same map.

What remains to show is that, subject to these relations, any order preserving injective map can be written as a composition of maps d^i .

If α is missing $i_1 < i_2 < \dots < i_{n-k}$, then α can be written as

$$\alpha = d^{i_{n-k}} d^{i_{n-k-1}} \dots d^{i_1}$$

□

We'll call d^i the i -th coface map.

A contravariant functor $\Delta_{inj} \rightarrow \text{Set}$ is called a semi-simplicial set.

Definition 1 (Singular Chain Complex)

A (non-negatively graded) singular chain complex of a space X has as chain groups

$$S_n X = \mathbb{Z} \langle \text{Sing}_n(X) \rangle$$

and differentials $\partial_n : S_n(X) \rightarrow S_{n-1}(X)$ defined on generators as

$$\partial_n(\sigma : \Delta^n \rightarrow X) \mapsto \sum_{i=0}^n (-1)^i \sigma \circ d^i$$

Lemma 8

The singular chain complex of a space is a chain complex.

Proof

By linearity, it is enough to check this on generators $\sigma : \Delta^n \rightarrow X$.

$$\begin{aligned}
\delta_{n-1}\delta_n\sigma &= \delta_{n-1}\left(\sum_{i=0}^n(-1)^i\sigma\circ d^i\right) \\
&= \sum_{i=0}^n(-1)^i\sum_{j=0}^{n-1}(-1)^j\sigma\circ d^i\circ d^j \\
&= \sum_{i=0}^n\sum_{j=0}^{n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&\quad + \sum_{0\leq i\leq j\leq n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&\quad + \sum_{0\leq i<j'\leq n-1}(-1)^{i+j'-1}\sigma\circ d^{j'}\circ d^i \\
&= 0
\end{aligned}$$

□

Lemma 9

We get a functor from chain complexes with chain maps to graded abelian groups, which is just taking homology.

Definition 2 (Singular Homology)

The singular homology $H_\bullet X$ (with integer coefficients) on a space X is the homology of the singular chain complex.

Lecture 4: Homology Theories

Wed 19 Oct

Lemma 10

Homology defines a functor $Ch \rightarrow gr\ Ab$

Proof (Sketch)

Let $f : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$, then $H_n(f) = f_*$ sending $x \in \ker(d_n)/\text{Im}(d_{n+1})$ to $[f(x)]$ □

Example

Let's compute the singular homology of the point.

Clearly $S_* = \mathbb{Z}$ and the maps induced by restriction are the identity.

Hence, the boundary maps will be

$$\dots \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus $\forall n > 0$, we get $H_n(pt) = 0$ and $H_0(pt) = \mathbb{Z}$.

Now we want to define homology for pairs.

Let $A \subset X$ be a pair of spaces.

We want to associate a singular chain complex $(S_\bullet(X, A), \delta_\bullet)$.

More generally, any continuous map $f : X \rightarrow Y$ induces $Sing_n(X) \rightarrow Sing_n(Y)$ by postcomposition.

Thus we get a functor $Sing_n(-) : \mathcal{T} \rightarrow \text{Set}$.

This in turn defines a chain map by extending $S_n f$ linearly to $S_n X$.

This defines a chain map $C_n X \rightarrow C_n Y$ since

$$\sigma \in S_n X \rightarrow f \circ \sigma \rightarrow \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ d_i$$

and

$$\sigma \in S_n X \rightarrow \sum_{i=0}^n (-1)^i \sigma \circ d^i \rightarrow \sum_{i=0}^n (-1)^i (f \circ \sigma \circ d_i)$$

coincide.

For an inclusion of subspaces $A \subset X$, we get an induced map $S_\bullet(i) : (S_\bullet A, \delta_\bullet) \rightarrow (S_\bullet X, \delta_\bullet)$ which is levelwise injective.

Definition 3 (Singular chain complex of a pair)

The singular chain complex of a pair is defined to be the quotient chain complex $S_\bullet X / S_\bullet A$.

Then the singular homology of the pair (X, A) is the homology of this chain complex.

For any pair (X, A) there is a short exact sequence of chain complexes

$$0 \rightarrow (S_\bullet A, \delta_\bullet) \rightarrow (S_\bullet X, \delta_\bullet) \rightarrow (S_\bullet(X, A), \delta_\bullet) \rightarrow 0$$

(ie. levelwise short exact)

What about coefficient groups $\neq \mathbb{Z}$.

Definition 4

Given a pair of spaces (X, A) and G an abelian group G , define the singular chain complex of (X, A) with coefficient in G as follows

$$S_n(X, A; G) = S_n(X, A) \otimes_{\mathbb{Z}} G$$

with the natural induced differentials. The singular homology of (X, A) with coefficients in G is the homology of this new chain complex.

Proposition 12

For any short exact sequence of chain complexes $0 \rightarrow C_{\bullet} \rightarrow D_{\bullet} \rightarrow E_{\bullet} \rightarrow 0$, we get a long exact sequence of homology groups

$$\dots \rightarrow H_n C_{\bullet} \rightarrow H_n D_{\bullet} \rightarrow H_n E_{\bullet} \rightarrow H_{n-1} C_{\bullet} \rightarrow \dots$$

which is natural in short exact sequences of chain complexes; w

Proof

The definition of the map $\partial_n : H_n E \rightarrow H_{n-1} C$ is a standard diagram chase.

We then prove that :

1. γ is in the kernel of $d_{n-1}^C : C_{n-1} \rightarrow C_{n-2}$

$$f_{n-2} d_{n-1}^C \gamma = d_{n-1}^D f_{n-1} \gamma = 0$$

as f_{n-2} is injective, $d_{n-1}^C \gamma = 0$

2. The choice of β is independent on the choice of γ .

Suppose β' is also such that $g_n \beta = g_n \beta'$.

We want to show that $\gamma - \gamma'$ is in the image of d_n^C .

As $g_n(\beta - \beta') = 0 \exists \tilde{\gamma} : f_n \tilde{\gamma} = \beta - \beta'$

$$f_{n-1} d_n^C \tilde{\gamma} = d_n^D f_n \tilde{\gamma} = d_n^D \beta - d_n^D \beta' = f_{n-1}(\gamma - \gamma').$$

Thus $d_n^D \tilde{\gamma} = \gamma - \gamma'$

3. Independence of the choice of representative α .

We want to show that if $\alpha = d_n^E \tilde{\alpha}$, then $\gamma = 0$.

This again is a standard diagram chase. So we conclude that $\partial_n : H_n E \rightarrow H_{n-1} C$ is a well defined map, it is easy to check that it is linear.

It remains to show that the long sequence above is exact, which is part of the homework. \square

We want to show that the connecting homomorphisms are natural, namely, for two short exact sequences

$$0 \rightarrow C_{\bullet} \rightarrow D_{\bullet} \rightarrow E_{\bullet} \rightarrow 0$$

$$0 \rightarrow C'_{\bullet} \rightarrow D'_{\bullet} \rightarrow E'_{\bullet} \rightarrow 0$$

with $\phi : C_\bullet \rightarrow C'_\bullet, \psi, \eta$ etc which make the diagram commute, we get, for every n a commutative diagram

$$H_n E \xrightarrow{\partial_n} H_{n-1} C_\bullet \rightarrow H_{n-1} C'_\bullet = H_n E \xrightarrow{H_n \eta} H_n E'_\bullet \xrightarrow{\partial'_{n-1}} H_n C'_\bullet$$

Lecture 5: General results about singular homology

Mon 24 Oct

Proposition 13

Relative singular homology with coefficients in an abelian group G defines a functor $H_*(-, -; G) : \text{Top}^{(2)} \rightarrow \text{gr Ab}$ and connecting homomorphisms $\partial_n : H_n(X, A, G) \rightarrow H_{n-1}(A; G)$ such that the LES for homology theories is satisfied.

Proof

$$H_n(X, A; G) = H_n(S_n X / S_n A \otimes G, \bar{\delta}_n).$$

Let $f : (X, A) \rightarrow (Y, B)$ be a map of pairs of spaces.

We have already shown that taking homology is functorial.

We still need to show that $\mathbb{T}^{(2)} \rightarrow \text{Ch}$ mapping $(X, A) \mapsto (S_*(X)/S_*(A), \delta_*)$ is a functor.

Recall that $- \otimes G$ is a functor from chain complexes to chain complexes.

We get a map of short exact sequences

$$0 \rightarrow S_* A \rightarrow S_* X \rightarrow S_* X / S_* A \rightarrow 0$$

to

$$0 \rightarrow S_* B \rightarrow S_* Y \rightarrow S_* Y / S_* B \rightarrow 0 \quad \square$$

induced by f , where the map from $S_* X / S_* A \rightarrow S_* Y / S_* B$ is induced by the universal property.

This map of chain complexes is clearly functorial, by definition of the $f_* : S_* X \rightarrow S_* Y$.

Remark

In general, tensoring with G does not preserve exact sequences but it does preserve split short exact sequences.

Thus, we want to show that, for every n , $0 \rightarrow S_n A \rightarrow S_n X \rightarrow S_n X / S_n A \rightarrow 0$ is split short exact.

Notice that $\text{Sing}_n(X) = \{\sigma : \Delta^n \rightarrow A \rightarrow X\} \coprod \{\sigma : \Delta^n \rightarrow X \mid \text{Im } \sigma \not\subset A\}$.

Thus it is clear that the short exact sequence above is split.

Hence, after tensoring with G , we still obtain a map of short exact sequences as above, thus we have a map of long exact sequences in homology of the form

$$\dots \rightarrow H_n A \rightarrow H_n X \rightarrow H_n(X, A) \xrightarrow{\partial_n^{(X, A)}} H_{n-1} A \rightarrow H_{n-1} X \rightarrow \dots$$

$$\dots \rightarrow H_n B \rightarrow H_n Y \rightarrow H_n(Y, B) \xrightarrow{\partial_n^{(Y, B)}} H_{n-1} B \rightarrow H_{n-1} Y \rightarrow \dots$$

Where the vertical maps are all induced by f_* , (here H_n is homology with coefficients in G)

We now want to show that singular homology is homotopy invariant, namely, if $f, g : (X, A) \rightarrow (Y, B)$ are homotopic maps of pairs, ie. \exists a continuous map $H : X \otimes [0, 1] \rightarrow Y$ such that the restriction to $A \times [0, 1]$ is contained in B which restricts to f (resp. g) on $X \times 0$ (resp. $X \times 1$).

Then, we want to show that $H_n f = H_n g : H_n(X, A; G) \rightarrow H_n(Y, B; G)$ coincide.

Definition 5 (Homotopic chain maps)

Two chain maps $\phi, \psi : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$ are chain homotopic if there exists a family of linear maps $h_n : C_n \rightarrow C'_{n+1} \forall n \geq 0$ such that

$$\phi_n - \psi_n = d'_{n-1} h_n + h_{n-1} d'_n : C_n \rightarrow C'_n$$

The family h_n is then called a chain homotopy.

Proposition 15

Given two homotopic chain maps $\phi, \psi : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$, the induced maps on homology coincide.

Proof

Pick $[x] \in H_n C$, we want to compare $[\phi(x)]$ and $[\psi(x)]$, we need to show that $\phi(x) - \psi(x) \in \text{Im } d'_{n+1} = [d' h(x) + h d(x)]$.

As $x \in \ker d$, $h d(x) = 0$ and $d' h(x) \in \text{Im } d'_{n+1}$ □

So now we want to show that homotopic maps of pairs induce homotopic maps of chain complexes.

The key idea will be to notice that, in general $\Delta^n \times [0, 1]$ is not a simplex in general but it can be decomposed as a union of $n + 1$ $(n + 1)$ simplices.

We can consider $\Delta^n \times [0, 1] \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$.

Notice that any convex hull of $n + 2$ linearly independent vectors in \mathbb{R}^{n+2} is homeomorphic to the $n + 1$ simplex which is compatible with the ordering of the vertices.

For $i \in \{0, n\}$, consider $\tau_i = \text{conv}((e_0, 0), (e_1, 0), \dots, (e_i, 0), (e_{i+1}, 1), \dots, (e_n, 0))$.

We want to build a map $S_n(X) \rightarrow S_{n+1}(Y)$, we do this by noticing that a map $\sigma : \Delta^n \rightarrow X$ induces $\Delta^n \times [0, 1] \xrightarrow{\sigma \times \text{Id}} X \times [0, 1] \xrightarrow{H} Y$.

And thus $\tau_i \subset \Delta^n \times [0, 1] \rightarrow X \times [0, 1] \rightarrow Y$ gives an $n + 1$ simplex in Y .

The claim we will prove next time is that this induces a chain homotopy $S_* f, S_* g$

Lecture 6: Homotopy invariance

Wed 26 Oct

If $f, g : X \rightarrow Y$ are homotopic maps, then we construct a map $h_n S_n X \rightarrow S_{n+1} Y$ by defining it

$$h_n(\sigma) = \sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \circ \tau_i$$

We want to show this is a chain homotopy, ie $S_n g - S_n f = h d + d h$.

Notice that $\tau_i \circ d^{i+1} = \tau_{i+1} \circ d^{i+1}$, $\tau_0 \circ d^0 = \text{inclusion of } \Delta^n \times \{1\}$ and $\tau_n \circ d^n = \text{inclusion of } \Delta^n \times \{0\}$.

We need to analyze

$$\begin{aligned} h_{n-1} \delta_n(\sigma) + \delta_{n+1} h_n(\sigma) &= h_{n+1} \left(\sum_{j=0}^n (-1)^j \sigma \circ d^j \right) + \delta_{n+1} \left(\sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \circ \tau_i \right) \\ &= \sum_{j=0}^n (-1)^j h_{n+1}(\sigma \circ d^j) + \sum_{k=0}^{n+1} (-1)^k \sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \tau_i \circ d^k \\ &= \sum_{j=0}^n (-1)^j h_{n+1}(\sigma \circ d^j) + \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^n (-1)^{i+k} H(\sigma \times \text{Id}) \tau_i \circ d^k + g(\sigma) - f(\sigma) \end{aligned}$$

Notice that $h_{n-1}(\sigma \circ d^k) = \sum_{i=0}^{k-1} (-1)^i H(\sigma \times \text{Id}) (\tau_i \circ d^{k+1}) + \sum_{i=k+1}^n (-1)^{i+1} (\sigma \times \text{Id}) (\tau_i \circ d^k)$.

Putting everything together yields

$$\begin{aligned} &= \sum_{j=0}^n (-1)^j \left(\sum_{i=0}^{j-1} \right) (-1)^i H(\sigma \times \text{Id}) (\tau_i \circ d^{j+1}) \\ &+ \sum_{i=j-1}^n (-1)^{j+1} H(\sigma \times \text{Id}) (\tau_i \circ d^j) + \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^n (-1)^{i+k} H(\sigma \times \text{Id}) \tau_i \circ d^k \\ &+ g(\sigma) - f(\sigma) \end{aligned}$$

The rest of the proof is cursed and I won't write it down, Dieck's Book seems to have a complete proof.

We still have to show that H_n respects homotopy of pairs

Proposition 16

The functor $H_*(-, -; G) \rightarrow \text{Ab}$ is homotopy invariant

Proof

Let $f, g : (X, A) \rightarrow (Y, B)$ be given homotopic maps via $H : X \times [0, 1] \rightarrow Y$.

This gives us chain homotopies $(f|_A)_*$ to $(g|_A)_*$.

f_*, g_* induce maps $S_* X / S_* A \rightarrow S_* Y / S_* B$ and the chain homotopies between

f_* and g_* induce the same chain homotopies between these chain maps.
 Moreover, tensoring with G maps chain homotopies to chain homotopies. \square

2.1 Excision

Proposition 17

$H_*(-, -; G)$ satisfies the excision axiom.

Proof

Consider the cover $X = (X \setminus B)^\circ \cup A^\circ$ and write $U = \{X \setminus B, A\}$

Let $S_n^U(X) = \mathbb{Z} \langle \sigma : \Delta^n \rightarrow X \mid \text{Im } \sigma \subset X \setminus B \text{ or } A \rangle \subset S_n(X)$ \square

Lecture 7: Excision

Mon 31 Oct

Proof

We want to show that $H_n(S_\bullet^U(X), \delta_\bullet) \rightarrow H_n(X)$ is an isomorphism.

A chain map inducing isomorphisms on all homology groups is also called a quasi-isomorphism.

We'll use barycentric subdivision to make our simplices smaller.

First, let's recall the lebesgue lemma

Lemma 18

Let K be a compact metric space and $(U_i)_{i \in I}$ an open cover of K .

Then, there is an $\epsilon > 0$ s.t. any ϵ -ball around any point in K is contained in a single open set U_i .

To prove this, for every $x \in K$, we choose $\delta(x) > 0$ such that the $\delta(x)$ ball around x is contained in a U_i of the cover.

Now, we look at $\left\{ B_{\frac{\delta(x)}{2}}(x) \right\}_{x \in K}$, this is an open cover of K , so there is a finite subcover $B_{\frac{\delta(x_i)}{2}}(x_i)$.

Set $\epsilon = \min_j \frac{\delta(x_j)}{2}$.

Given any $x \in K$, we want to show that $B_\epsilon(x)$ is completely contained in some U_i .

We can find x_l such that $d(x, x_l) < \frac{\delta(x_l)}{2}$, let $y \in B_\epsilon(x)$ then $\delta(x_l, y) \leq \frac{\delta(x_l)}{2} + \epsilon \leq \delta(x_l)$

We can apply the Lebesgue lemma to the open cover $\{\sigma^{-1}(X \setminus \overline{B}), \sigma^{-1}(A^\circ)\}$ is an open cover of compact metric spaces Δ^n .

Thus, there exists $\epsilon > 0$ such that any open ϵ -ball in Δ^n is mapped by σ either to $X \setminus B$ or to A .

We'll now prove the following proposition

Proposition 19

There is a chain map (called the Barycentric subdivision) $Sd : S_\bullet X \rightarrow S_\bullet X$ which is

- Natural in X
- Chain homotopic to the identity
- If $X = \Delta^n$, then any summand τ of $Sd(\sigma)$ for $\sigma : \Delta^k \rightarrow \Delta^n$ has the following property

$$\text{diam}(\text{Im } \tau) \leq \frac{k}{k+1} \text{diam}(\text{Im } \sigma)$$

Let $v_0, \dots, v_n \in \mathbb{R}^N$, then their barycenter is $\frac{v_0 + \dots + v_n}{n+1}$.

We will consider the following auxiliary "cone" map.

For any $\tau : \Delta^k \rightarrow \Delta^n$ and b the barycenter of Δ^n , define $\rho_b(\tau) : \Delta^{k+1} \rightarrow \Delta^n$ by

$$\rho_b(t_0, \dots, t_{k+1}) \mapsto \begin{cases} t_0 b + \tau(\frac{1}{1-t_0}(t_1, \dots, t_{k+1}))(1-t_0) \\ b \text{ if } t_0 = 1 \end{cases}$$

This is indeed continuous.

What is the relation of the cone construction and the boundary of a simplex?

$$\begin{aligned} \delta_{k+1} \rho_b(\tau) &= \sum_{j=0}^{k+1} (-1)^j \rho_b(\tau) \circ d^j \\ &= \tau + \sum_{j'=0}^k (-1)^{j'+1} \rho_b(\tau \circ d^j) \\ &= \tau - \rho_b(\delta_k \tau) \end{aligned}$$

So we obtain a linear map $\rho_b : S_k(\Delta^n) \mapsto S_{k+1}(\Delta^n)$ with the property

$$\delta_{k+1} \rho = \text{Id} - \rho_b \circ \delta_k$$

We define a map $Sd : S_n(X) \rightarrow S_n X$.

For $n = 0$, $Sd_0 = \text{Id}_{S_0 X}$.

Given some $n > 0$ and $\sigma : \Delta^n \rightarrow X$, define

$$Sd(\sigma) = \sigma_*(\rho_b Sd(\delta_n i_n))$$

where i_n is the identity of the Δ^n simplex, considered as an element of $\text{Sing}_n(\Delta^n)$

We claim that Sd is a chain map.

Given $\sigma : \Delta^n \rightarrow X$, we want to compute

$$\delta_n Sd \sigma = \delta_n \sigma_*(\rho_b Sd(\delta_n i_n))$$

We can switch δ_n and σ_* since σ_* is post composition and δ_n is precomposition.

$$\begin{aligned} &= \sigma_*(\delta_n \rho_b Sd(\delta_n i_n)) \\ &= \sigma_*(Sd(\delta_n i_n) - \rho_b(\delta_{n-1} Sd(\delta_n i_n))) \end{aligned}$$

Notice that

$$\delta_{n-1} Sd(\delta_n i_n) = Sd(\delta_{n-1} \delta_n i_n) = 0$$

Thus

$$\delta_n Sd(\sigma) = \sigma_* Sd(\delta_n i_n) = Sd(\delta_n \sigma) \quad \square$$

For naturality, if $f : X \rightarrow Y$ is a map of topological spaces, one can explicitly check that $Sd \circ f_* = f_* \circ Sd$

Lecture 8: still excision

Wed 02 Nov

I missed the first half of the lecture because I overslept, so there is a part missing here.

Lemma 20

The map of chain complexes $S^U(X) \rightarrow S(X)$ is a quasi-isomorphism

Proof

Let $\sum_k a_k \sigma_k \in S_N X$ be in the kernel of δ_n , ie. $[\sum_k a_k \sigma_k]$ represents an element in $H_n(X)$.

To see surjectivity, notice that

$$Sd(\sum_k a_k \sigma_k) = \sum_k a_k \sigma_k - h(\delta(\sum_k a_k \sigma_k)) - \delta h(\sum_k a_k \sigma_k)$$

The middle term is 0 by our hypothesis, and we see that $Sd(\sum_k a_k \sigma_k)$ is representing the same element in homology.

We can apply Sd arbitrarily many times and hence $Sd^l(\sum_k a_k \sigma_k)$ is too.

Now, for every k , $\sigma_k^{-1} A^\circ, \sigma_k^{-1}(X \setminus \overline{B})$ forms an open covering of Δ_n .

There exists a Lebesgue Number $\epsilon_k > 0$ for this open cover.

There is an $l_k \in \mathbb{N}$ such that $(\frac{n}{n+1})^{l_k} < \frac{\epsilon_k}{\sqrt{2}}$.

Now, for any simplex τ in the barycentric subdivision of Δ_n $\text{diam} \tau \leq \epsilon$.

Now $(\sigma_k)_*(\tau) \subset X \setminus \overline{B}$ or $\subset A^\circ$.

Setting $l = \max_k l_k$, we see that $Sd^l(\sum_k a_k \sigma_k)$ is a preimage for $\sum_k a_k \sigma_k$.

For injectivity, let $\sum_k a_k \sigma_k$ be an element of $S_\bullet^U X$ in $\ker \delta$ which is in the image of δ .

Let $\sum_j b_j \tau_j$ be such that $\delta(\sum_j b_j \tau_j) = \sum_k a_k \sigma_k$.

There exists an m such that $Sd^m(\sum b_j \tau_j) \in S^U X$, now $\delta Sd^m(\sum b_j \tau_j) = Sd^m(\sum a_k \sigma_k)$.

Thus, $\sum a_k \sigma_k$ is a boundary.

This concludes the proof. \square

We can now use that a quasi-isomorphism of complexes of free abelian groups has a homotopy inverse.

Lecture 9: Conclusion of Excision

Mon 07 Nov

We proved last week that $S_*^{\{X \setminus B, A\}X \rightarrow S_* X}$ is in fact a quasi-isomorphism. In homework, we proved that any quasi-isomorphism between chain complexes of levelwise free abelian groups is a chain homotopy. With this, we can now prove excision

Proof

There are maps between the two following SES

$$0 \rightarrow S_*(A \setminus B) \rightarrow S_*(X \setminus B) \rightarrow S_*(X \setminus B, A \setminus B) \rightarrow 0$$

$$0 \rightarrow S_*(B) \rightarrow S_*(X) \rightarrow S_*(X, B) \rightarrow 0$$

which are induced by inclusion.

There are maps $S_* A \rightarrow S_*^U X$ and $S_*(X \setminus B) \rightarrow S_*^U(X)$ and a quotient map $S_*^U X \rightarrow S_*(X \setminus B, A \setminus B)$.

As the map of chain complexes is a quasi-isomorphism, they are all chain homotopic and thus tensoring with G preserves the quasi-isomorphism.

Applying the LES in homology, we get maps between the two following short exact sequences

$$H_n(A; G) \rightarrow H_n(S_*^{\{X \setminus B, A\}}(X) \otimes G) \rightarrow H_n(X \setminus B, A \setminus B; G) \xrightarrow{\partial} H_{n-1}(A; G) \rightarrow H_{n-1}(S_*^{\{X \setminus B, A\}}(X) \otimes G)$$

$$H_n(A; G) \rightarrow H_n(X; G) \rightarrow H_n(X, A; G) \xrightarrow{\partial} H_{n-1}(A; G) \rightarrow H_{n-1}(X; G)$$

Now, applying the five lemma, we see that $H_n(X \setminus B, A \setminus B; G) \simeq H_n(X, A; G)$ \square

It remains to show additivity.

Let $\{(X_i, A_i)\}_{i \in I}$ be a family, we need to show that the inclusions induce a map

$$\bigoplus_{i \in I} H_n(X_i, A_i) \rightarrow H_n\left(\coprod_{i \in I} X_i, \coprod_{i \in I} A_i\right)$$

Notice that the homology functor $H_n : Ch_{\geq 0} \rightarrow gr \text{ Ab}$ commutes with arbitrary direct sums.

Furthermore, from the definition, it is clear that $S_n(\coprod_i X_i, \coprod_i A_i; G) \simeq \bigoplus_i S_n(X_i, A_i; G)$.

It is easy to check that $H_n(pt; \mathbb{Z}) = 0$ for all $n \neq 0$, tensoring the chain complex

(of singular chains of a point)

$$\dots \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \dots$$

with G , it is clear that $H_n(pt; G) = 0 \forall n \geq 1$ and $H_0(pt, G) = G$

Lecture 10: computations

Wed 09 Nov

$$h_n(D^k, S^{k-1})$$

Example

We know that

$$h_n(D^k, S^{k-1}) = \begin{cases} h_0(pt), n = k \\ 0 \text{ else} \end{cases}$$

for any ordinary homology theory.

This is almost $h_n S^k$ but S^k is the collapse D^k/S^{k-1} .

Example

Consider $D^k \setminus 0 \rightarrow D^k$.

Using homotopy equivalence, we get

$$h_n(D^k) \rightarrow h_n(D^k, D^k \setminus \{0\}) \rightarrow h_{n-1}(D^k \setminus 0) \rightarrow h_{n-1}(D^k)$$

Thus

$$h_n(D^k, D^k \setminus 0) = \begin{cases} h_0(pt) & \text{if } n = k \\ 0 & \end{cases}$$

Compare this to the quotient $D^k/D^k \setminus 0$ which has two points, with exactly one of them being open.

This is called the Sierpinski space \mathcal{S} , call it's points a and b with a open.

Notice that there is a bijection between continuous maps $X \rightarrow \mathcal{S} \leftrightarrow \{U \subset X \text{ open}\}$.

We claim that \mathcal{S} is contractible, there is a unique map $\mathcal{S} \rightarrow pt$ and we define a map $pt \rightarrow a$.

We define the map $\mathcal{S} \times [0, 1] \rightarrow \mathcal{S}$ to be the one induced by the open set $\{a\} \times [0, 1] \cup \{a, b\} \times [0, 1]$.

Thus, in this case relative homology and homology of the quotient don't agree.

Our next goal will be to make the difference between these pairs of spaces precise.

Definition 6

A subspace $A \subset X$ is called neighborhood deformation retract (NDR) if $A \subset X$ is closed and there is a neighborhood $A \subset V \subset X$ such that A is a deformation retract of V .

Ie. there exists a continuous map $r : V \rightarrow A$ such that $r|_A = \text{Id}_A$ and there exists a continuous map $H : V \times [0, 1] \rightarrow V$ such that $H_0 = \text{Id}_V$, $H_1 = \iota \circ r$ and A is fixed by H .

Example

$S^{k-1} \rightarrow D^k$ is a NDR, choosing $r : D^k \setminus 0 \rightarrow S^{k-1}$ sending $x \mapsto \frac{x}{\|x\|}$ with the obvious homotopy.

Example

$D^k \setminus 0 \rightarrow D^k$ is not closed so in particular not a NDR.

A slightly more elaborate example is $A = \{\frac{1}{n} | n \in \mathbb{N}\} \cup 0 \subset \mathbb{R}$.

This is a closed subset but not a deformation retract as $V \xrightarrow{r} A$ can not be both continuous and the identity on A for any neighborhood V of A in $X = \mathbb{R}$.

Theorem 25

Let $A \subset X$ be a NDR and let (h_*, ∂_*) be any ordinary homology theory.

Then the quotient map $q : X \rightarrow X/A$ induces isomorphisms

$$h_n(X, A) \rightarrow h_n(X/A, A/A) \forall n$$

2.2 Pushouts of topological spaces

Proposition 26

Given $C \leftarrow A \rightarrow B$, it can always be completed to a pushout.

Lecture 11: pushouts

Mon 14 Nov

Lemma 27

Let $X \leftarrow A \xrightarrow{f} B$ be a pushout diagram with pushout $B \xrightarrow{j} Y \xleftarrow{g} X$.

Then if i is an embedding, open embedding or closed embedding, then j is too.

Definition 7 (Locally compact)

A topological space is called locally compact if for all $z \in Z$ and any neighborhood $U \subset Z$, there exists a compact neighborhood $z \in C \subset U$

Example

Any compact Hausdorff space Z is locally compact.

Proposition 29

Let $X \xrightarrow{p} Y$ be a quotient map and Z a locally compact topological space, then

$X \times Z \xrightarrow{p \times \text{Id}} Y \times Z$ is a quotient.

Lecture 12: how much of a homology theory is determined by the axioms?

Wed 16 Nov

Corollary 30

Let D be the pushout of $C \leftarrow A \rightarrow B$ and Z a locally compact space, then $D \times Z$ is the pushout of $C \times Z \leftarrow A \times Z \rightarrow B \times Z$.

Proposition 31

Pushouts of NDR are NDR.

Proof

Let (X, A) be a NDR and let Y be the pushout of $A \leftarrow A \rightarrow B$.

Let V be a neighborhood of A and $r : V \rightarrow A$ be a retraction and $H : V \times [0, 1] \rightarrow V$ with $H_0 = \text{Id}$, $H_1 = \iota \circ r$ and which fixes A .

Take the pushout W of $V \leftarrow A \rightarrow B$, we get an induced map to Y which is an embedding and W is a neighborhood of B in Y .

Universal properties give a retract $s : W \rightarrow B$.

To check this is a NDR, we use the above proposition to get an induced map $W \times [0, 1] \rightarrow W$, this turns out to be the desired homotopy. \square

Remark

W can be chosen as a pushout of $V \leftarrow A \rightarrow B$.

Now, we want to show that for $A \rightarrow X$ a NDR and h_* an ordinary homology theory, the map $X \rightarrow X/A$ induces isomorphisms on homology.

Proof

Let V be the neighborhood of A , there are maps induced by inclusion

$$h_n(X, A) \rightarrow h_n(X, V) \xleftarrow{\cong} h_n(X \setminus A, V \setminus A)$$

The quotient map induces maps to

$$h_n(X/A, A/A) \rightarrow h_n(X/A, V/A) \leftarrow h_n((X \setminus A)/A, (V \setminus A)/A)$$

We show $h_n(X, A) \rightarrow h_n(X, V)$ is an isomorphism.

Compare the LES of

$$h_n(A) \rightarrow h_n X \rightarrow h_n(X, A) \rightarrow h_{n-1} A \rightarrow h_{n-1} X$$

and

$$h_n(V) \rightarrow h_n X \rightarrow h_n(X, V) \rightarrow h_{n-1} V \rightarrow h_{n-1} X$$

the inclusions induces isomorphisms on all groups except $h_n(X, A) \rightarrow h_n(X, V)$ and we conclude by the five lemma. We can also show that the maps $h_n(X/A, A/A) \rightarrow h_n(X/A, V/A)$ are isomorphisms by notice that $A/A \rightarrow V/A$ is a NDR. \square

We now want to identify nice spaces on which h_*X is already determined by its coefficient group.

Definition 8 (CW complexes)

Let (X, A) be a pair of spaces.

The pair (X, A) together with a filtration $X_{-1} = A \subset X_0 \subset X_1 \subset \dots \subset X_n \subset \dots \subset X$ is a relative CW-complex if

1. There exist pushout diagrams $\forall n \geq 0$ such that X_n is the pushout of

$$\coprod_{i \in I_n} D_n \xleftarrow{\Phi_i} \coprod_{i \in I_n} S^{n-1} \xrightarrow{\phi_i} X_{n-1}$$

The ϕ_i are called attaching maps and Φ_i are the "characteristic maps".

2. $X \simeq \varprojlim_{n \in \mathbb{N}} X_n$ where X carries the weak topology.
If $A = \emptyset$, X together with a filtration is called a CW-complex.

Lecture 13: stuff

Mon 21 Nov

Remark

If $X = X_n$ for some $n \in \mathbb{N}_0$, the CW-complex is finite dimensional.

For a fixed filtration of a space, the images $\Phi_i(D^{\circ, n})$ are well defined and identified with path-connected components of $X_n \setminus X_{n-1}$ and they are called n -cells.

Definition 9 (Cellular map)

A map of CW-complexes $f : X \rightarrow Y$ is called cellular if $f(X_n) \subset Y_n \forall n$.

Lecture 14: dimension of cw-complexes

Wed 23 Nov

If X is a CW-complex, then the dimension of n is the largest n such that X_j is constant $\forall j \geq n$.

It turns out that $\dim X \geq n \iff \exists$ an embedding $D^n \rightarrow X$.

Example

$$S^\infty = \varprojlim_i S^i$$

3 Homology of CW-complexes

For a relative CW-complex (X, A) , we want to define $H_*^{cell}(X, A)$, the cellular homology of (X, A) .

Definition 10 (Cellular chain complex)

The cellular chain complex of a relative CW-complex (X, A) with respect to an ordinary homology theory (h_*, ∂_*) is defined as $C_n(X, A) = h_n(X_n, X_{n-1}) \forall n \geq 0$.

The maps of the complex are given by $\text{incl}_* \circ \partial$, one can check that this is in fact a chain complex.

We want to give a different description of the cellular chain complex which lends itself to computations.

Notice that $h_n(X_n, X_{n-1}) = h_n(X_n/X_{n-1}) \simeq h_n(\bigvee_{i \in I_n} S^n, pt)$.

To give a better characterisation of the differentials, choose pushouts $\coprod_i D^n \rightarrow \coprod_i S^{n-1} \rightarrow X_{n-1}$ for every n , then the differentials $\bigoplus_{i \in I_n} G \rightarrow \bigoplus_{j \in I_{n-1}} G$ are induced by the maps $G_{i_0} \rightarrow G_{j_0}$ which in turn are induced on the $n-1$ st homology by the continuous maps $S^{n-1} \xrightarrow{\phi_{i_0}} X_{n-1} \rightarrow X_{n-1}/X_{n-2} \rightarrow \bigvee S^{n-1} \rightarrow S^{n-1}$ where the last map collapses the j_0 component.

Lecture 15: calculation of homology on CW complexes

Mon 28 Nov

Choose pushouts $\coprod_{i \in I_n} D^n \rightarrow \coprod_{i \in I_n} S^{n-1} \rightarrow X_{n-1}$ for every X_n , an isomorphism $h_k S^k \rightarrow G$ and collapsing map isomorphisms $k_n : D^n/S^{n-1} \rightarrow S^n$.

The differential maps $h_n(X_n, X_{n-1})$ can be computed componentwise via

$$h_n(D^n, S^{n-1}) \xrightarrow{\text{inclusion into } i_0} \bigoplus_{i \in I_n} h_n(D^n, S^{n-1}) \rightarrow h_n(\coprod D^n, \coprod S^{n-1}) \rightarrow h_n(X_n, X_{n-1}) \rightarrow h_{n-1}(X_{n-1}, X_{n-2})$$

These maps are called $d(i_0, j_0)$ and are induced on homology via the maps $S^{n-1} \xrightarrow{m(i_0, j_0)} S^{n-1}$ defined in such a way to commute with the inclusions to the X_n .

Lecture 16: Cellular homology coincides with its ordinary homology theory

Wed 30 Nov

Theorem 35

Given an ordinary homology theory (h_*, ∂_*) and a relative CW-complex (X, A) such that either $h_* = H_*^{\text{sing}}$ or (X, A) is finite dimensional then

$$H_k^{\text{cell}}(X, A) \simeq h_k(X, A)$$

Lemma 36

1. $h_k(X_{n+1}, X_n) = 0$ if $k \neq n+1$
2. $h_k(X_n, A) = 0$ if $k > 0$

3. If $k \leq l \leq n$, $h_k(X_n, X_l) \rightarrow h_k(X_{n+1}, X_l)$ is an isomorphism.

Proof

1. As $X_n \subset X_{n+1}$ is a NDR, the map $h_k(X_{n+1}, X_n) \rightarrow h_k(X_{n+1}/X_n, X_n/X_n) \simeq h_k(\bigvee_{i \in I_{n+1}} S^{n+1}, pt) = 0$
2. By induction, for $n = -1$, $h_k(A, A) = 0$, for $n = 0$, $h_k(X_0, A) = h_k(A \coprod_{i \in I_0} pt, A) = 0$.

For the induction step, consider the LES of the triple

$$h_k(X_{n-1}, A) \rightarrow h_k(X_n, A) \rightarrow h_k(X_n, X_{n-1})$$

The leftmost and rightmost terms are 0

3. By induction, the case $n = l$

$$0 = h_k(X_l, X_l) \rightarrow h_k(X_{l+1}, X_l) = 0$$

For the induction step, $h_k(X_n, X_l) \xrightarrow{\sim} h_k(X_{n+1}, X_l) \rightarrow h_k(X_{n+2}, X_l)$

The LES for the triple (X_{n+2}, X_n, X_l) looks like

$$h_{k+1}(X_{n+2}, X_n) \rightarrow h_k(X_n, X_l) \rightarrow h_k(X_{n+2}, X_l) \rightarrow h_k(X_{n+2}, X_n)$$

The LES of the triple (X_{n+2}, X_{n+1}, X_n)

$$0 = h_k(X_{n+1}, X_n) \rightarrow h_k(X_{n+2}, X_n) \rightarrow h_k(X_{n+2}, X_{n+1}) = 0$$

Hence $h_k(X_{n+2}, X_n) = 0$.

Thus, the rightmost and leftmost terms in the sequence

$$h_{k+1}(X_{n+2}, X_n) \rightarrow h_k(X_n, X_l) \rightarrow h_k(X_{n+2}, X_l) \rightarrow h_k(X_{n+2}, X_n)$$

vanish and we get the desired isomorphism. \square

We can now prove that homology and cellular homology coincide on nice enough CW-complexes.

Proof

Recall that the cellular homology was defined as the homology of

$$h_{k+1}(X_{k+1}, X_k) \xrightarrow{\partial} h_k(X_k, X_{k-1}) \xrightarrow{\partial} h_{k-1}(X_{k-1}, X_{k-2}) \quad \square$$

Lecture 17: stuff

Mon 05 Dec

Theorem 37

Assume (h_*, ∂_*) , (h'_*, ∂'_*) are two ordinary homology theories and $\rho : (h_*, \partial_*) \rightarrow (h'_*, \partial'_*)$ is a natural transformation of homology theories such that

— $\rho_{pt} : h_* pt \rightarrow h'_* pt$ is an isomorphism

Then ρ is an isomorphism for all (f.d.) CW-complexes.

Definition 11 (Natural transformation of homology theories)

A natural transformation $\rho : (h_*, \partial_*) \rightarrow (h'_*, \partial'_*)$ is a family of natural transformations $\rho : h_* \rightarrow h'_*$ compatible with the boundary morphisms.

4 Homotopy groups

Given a based topological space (X, x_0) ,

$$\pi_1(X, x_0) \simeq \{ (S^1, 1) \mapsto (X, x_0) \text{ continuous map} \} / \text{homotopy}$$

Lecture 18: homotopy theory

Mon 12 Dec

Lecture 19: Higher Homotopy groups

Mon 12 Dec