The Steenrod Algebra and Its Dual

David Wiedemann

October 19, 2023

These are notes for the seminar "Advanced Topics in Homotopy Theory" given by Prof. Stefan Schwede and Dr. Jack Davies in Bonn during the WS2023/24. Our goal is to present the main results of Milnor's paper "The Steenrod Algebra and its Dual" [Mil58].

Contents

1 Hopf Algebras

1.1 Bi-Algebras

We start by studying Hopf algebras independently. Throughout, let k be a field.

Definition 1 (Algebra) *An **Algebra** is a triple* $(\mathcal{A}, \mu, \eta)$ *with* $\mathcal{A}$ *a* k*-vector space together with two maps* $\mu \colon A \otimes A \to A$ *(multiplication) ,* $\eta \colon k \to A$ *(unit) making the following diagrams commute*

$$\begin{tikzcd}
\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mathrm{Id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{Id}"'] & A \otimes \mathcal{A} \arrow[d, "\mu"] \\
\mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"'] & \mathcal{A}
\end{tikzcd}$$

$$\begin{tikzcd} k \otimes \mathcal{A} \arrow[r, "i\otimes\eta"] \arrow[dr] & \mathcal{A} \otimes \mathcal{A} \arrow[d, dashed, "\mu"] & \mathcal{A} \otimes k \arrow[l, "\eta\otimes i"'] \arrow[dl] \\ & \mathcal{A} & \end{tikzcd}$$

Dualizing these definitions, we unsurprisingly obtain

Definition 2 (Coalgebra) *A **coalgebra** is a triple (C, Δ, ϵ) where C is a k-vector space togethere with two maps $\Delta\colon C \to C \otimes C$ (comultiplication) and $\epsilon\colon C \to k$ (augmentation) making the following diagrams commute*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{Id}\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes\mathrm{Id}"'] & C \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd} & C \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\ k \otimes C & C \otimes C \arrow[l, "\epsilon\otimes\mathrm{Id}"] \arrow[r, "\mathrm{Id}\otimes\epsilon"'] & C \otimes k \end{tikzcd}$$

Since taking duals commutes with tensor products, notice that the dual $C^\vee$ naturally gets an algebra structure.
We define (co-)algebra morphisms in the obvious way.

Definition 3 (Bialgebra) *A **bialgebra** is a tuple $(\mathcal{A}, \mu, \eta, \Delta, \epsilon)$ such that $(\mathcal{A}, \mu, \epsilon)$ is an algebra, $(\mathcal{A}, \Delta, \epsilon)$ is a coalgebra and such that Δ and ϵ are algebra morphisms*

Equivalently, one can also require μ and ϵ to be coalgebra morphisms.
If $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ is a graded algebra, we define the **dual algebra** by

$$\mathcal{A}^* := A_n^*, \text{ with } A_n^* = \hom(A_{-n}, k)$$

We call a graded algebra $\mathcal{A}$ **graded commutative** if for all homogeneous elements $\alpha, \beta \in \mathcal{A}$, we have $\alpha\beta = (-1)^{\dim\alpha\dim\beta}\beta\alpha$. (omitting μ for sanity reasons) The graded algebra $\mathcal{A}$ is **connected** if $\mathcal{A}_0$ is generated by 1, equivalently $\eta\colon k \to \mathcal{A}_0$ is an isomorphism.
We can similarly define the notion of a graded coalgebra and of a connected coalgebra.

1.2 Antipode maps

Let C be a bi-algebra as above and let $f, g\colon C \to C$ be linear maps, we define the convolution $f * g$ of f with g as the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f\otimes g} C \otimes C \xrightarrow{\mu} C.$$

Definition 4 (Antipode) *An antipode* $S\colon C \to C$ *is an endomorphism such that*

$$S * \mathrm{Id} = \mathrm{Id} * S = \eta \circ \epsilon.$$

Definition 5 (Hopf Algebra) *A Hopf Algebra is a bi-algebra with an antipode*

For specific classes of bialgebras, there is a way of constructing an antipode map.

Theorem 1 *Let* $\mathcal{A}$ *be a graded bialgebra such that* $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i a_i \otimes b_i$ *with* $\dim a_i, \dim b_i > 0$*, then* $\mathcal{A}$ *admits an antipode map.*

Proof Let $x \in \mathcal{A}$, to define S, we proceed inductively on the degree of x. If $\dim x = 0$, we define $S(x) = x$.
Inductively, suppose we've defined S for all x of degree $< n$ and write $\Delta(x) = x \otimes 1 + 1 \otimes x + \sum_i a_i \otimes b_i$ as above. Since Δ respects the grading, we may suppose that $\dim b_i < n$, we let

$$S(x) \coloneqq -x - \sum_i a_i S(b_i)$$

One now easily checks that S is an antipode. $\square$

2 The Steenrod Algebra

Let p be a prime.

Definition 6 (Stable Cohomology operation) *A* ***stable*** $\bmod p$ ***cohomology operation*** θ *of type* $r \in \mathbb{Z}$ *is a family of natural transformations* $(\theta_n)_{n \in \mathbb{N}}$

$$\theta_n\colon H^n(-, \mathbb{F}_p) \to H^{n+r}(-, \mathbb{F}_p)$$

such that the following diagram commutes for every space X

$$\begin{array}{ccc} H^n(X, \mathbb{F}_p) & \xrightarrow{\theta_n} & H^{n+r}(X, \mathbb{F}_p) \\ \downarrow & & \downarrow \\ H^{n+1}(\Sigma X, \mathbb{F}_p) & \xrightarrow[\theta_{n+1}]{} & H^{n+r+1}(\Sigma X, \mathbb{F}_p) \end{array}$$

We can trivially compose two cohomology operations θ, θ' of type r (resp. r') to obtain a cohomology operation of type $r + r'$, this motivates the following definition.

Definition 7 (Steenrod Algebra) *The* $\bmod p$ ***Steenrod Algebra*** $\mathcal{A}_p$ *is the ring freely generated by the stable cohomology operations. This ring comes with a natural grading coming from the type of the cohomology operation.*

For those familiar with (maps of) spectra, the most natural way to define the Steenrod algebra is by the formula $\mathcal{A}_p = H\mathbb{F}_p^*(H\mathbb{F}) = \bigoplus_n H\mathbb{F}_p^n(H\mathbb{F}_p)$.

Remark 2 *Notice that if* θ *and* θ' *are two cohomology operations of different types, their sum* $\theta + \theta'$ *in* $\mathcal{A}_p$ *does* ***not*** *define a cohomology operation in any natural way.*
Despite this, $\mathcal{A}_p$ *still naturally acts on the* ***full*** *cohomology* $H^*(X)$ *of a space, when viewed as an abelian group.*

As we will establish in the next section, $\mathcal{A}_p$ carries a Hopf algebra structure which makes $H^*(X)$ into a (Hopf-)module. Before showing this, we present structural results about the Steenrod algebra.

2.1 Steenrod Powers

From now on, $H^*(-)$ will always denote mod p cohomology for a fixed prime p.

Definition 8 (Steenrod Powers) *Suppose* $p > 2$, *the* ***Steenrod powers*** *are the stable cohomology operations*

$$P^i\colon H^q(-, \mathbb{F}_p) \to H^{q+2i(p-1)}(-, \mathbb{F}_p)$$

uniquely determined by the following properties

1. $P^0 = \mathrm{Id}$
2. *if* $x \in H^{2n}(X, A, \mathbb{F}_p)$, *then* $P^n x = x^p$
3. *if* $x \in H^n(X, A)$, *then* $P^i x = 0$ *for all* $2i > n$
4. $\delta P^i = P^i \delta$ *where* δ *is the boundary homomorphism*
5. $P^i(xy) = \sum_{j+k=i} P^j x P^k y$

Definition 9 (Steenrod Squares) *The* ***Steenrod squares*** *are the unique stable* mod 2 *cohomology operations*

$$Sq^i\colon H^q(-, \mathbb{F}_2) \to H^{q+i}(-, \mathbb{F}_2)$$

uniquely determined by

1. $P^0 = \mathrm{Id}$
2. *if* $x \in H^n(X, A, \mathbb{F}_2)$, *then* $Sq^n(x) = x^2$
3. *if* $x \in H^n(X, A, \mathbb{F}_2)$, *then* $Sq^i x = 0$ *for all* $i > n$
4. $Sq^n(xy) = \sum_{i+j=n} Sq^i x Sq^j y$
5. $\delta Sq^i = Sq^i \delta$

The natural transformation $\beta\colon H^n(-) \to H^{n+1}(-)$ induced by the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ is also stable, we call it the **Bockstein morphism**.
For $p = 2$, the Bockstein coincides with Sq^1. It is a famed result of Steenrod that these operations generate the Steenrod algebra.

Theorem 3 (Structure of the Steenrod Algebra) *[SE62, Ch. VI, Sec. 2] Let p be an odd prime. Call a sequence $I = (\epsilon_0, s_1, \epsilon_1, s_2, \dots)$* ***admissible*** *if it is finite, $s_i \geq 1, \epsilon = 0, 1$ and $s_i \geq ps_{i+1} + \epsilon_i$. The set*

$$P^I := \beta^{\epsilon_0} P^{s_1} \beta^{\epsilon_1} P^{s_2}, \quad I \text{ admissible}$$

is a basis for the Steenrod algebra.

There is a similar result for $p = 2$, which we do not make explicit.

3 The Diagonal Morphism

From now on, p is a prime different from 2 and $\mathcal{A} := \mathcal{A}_p$.

The main goal of this talk is to present a proof that $\mathcal{A}_p$ has the structure of a Hopf algebra and to make its structure more explicit.
Throughout, let X be a space. We start by constructing the diagonal morphism $\psi^* \colon \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$.

Proposition 4 *There is a unique diagonal morphism $\psi^* \colon \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ such that*

1. *For all $\theta \in \mathcal{A}^*$, $\psi^*(\theta) = \sum_i \theta_i' \otimes \theta_i$" and $\alpha, \beta \in H^*(X)$ we have*

$$\theta(\alpha \smile \beta) = \sum (-1)^{\dim \theta_i'' \dim \alpha} \theta_i'(\alpha) \smile \theta_i''(\beta)$$

2. *The morphism ψ^* is a ring morphism.*

Proof Let $\mathcal{A}^* \otimes \mathcal{A}^*$ act on $H^*(X) \otimes H^*(X)$ by

$$(\theta' \otimes \theta'')(\alpha \otimes \beta) = (-1)^{\dim \theta'' \dim \alpha} \theta'(\alpha) \otimes \theta''(\beta)$$

and we let $c : H^*(X) \otimes H^*(X) \to H^*(X)$ denote the cup product.
ψ^* exists
Let $R \subset \mathcal{A}^*$ be the set of all θ such that

$$\theta(\alpha \smile \beta) = c\rho(\alpha \otimes \beta)$$

for some $\rho \in \mathcal{A}^* \otimes \mathcal{A}^*$. We want to show that $R = \mathcal{A}^*$.
Notice that R is closed under multiplication and addition. If $\theta_1, \theta_2 \in R$, then

$$\theta_1\theta_2(\alpha \smile \beta) = c\rho_1\rho_2(\alpha \otimes \beta) \text{ and } (\theta_1 + \theta_2)(\alpha \smile \beta) = c((\rho_1 + \rho_2)(\alpha \otimes \beta))$$

Hence, it suffices to show that R contains the Bockstein and the Steenrod powers which follows from the formulas

$$\delta(\alpha \smile \beta) = \delta\alpha \smile \beta + (-1)^{\dim \alpha} \alpha \smile \delta(\beta)$$
$$P^n(\alpha \smile \beta) = \sum_{i+j=n} P^i(\alpha) \smile P^j(\beta)$$

ψ^* **is unique**
Let $K \coloneqq K(\mathbb{F}_p, n+1)$ and $\gamma \in H^{n+1}(K)$ correspond to the identity map, the map

$$\begin{aligned} \mathrm{ev}_\gamma \colon \mathcal{A}_i^* &\to H^{n+1+i}(K) \\ \theta &\mapsto \theta\gamma \end{aligned}$$

is an isomorphism for all $i \leq n$, it follows that

$$\begin{aligned} j\colon\ (\mathcal{A}^* \otimes \mathcal{A}^*)_i &\to H^{2n+2+i}(K \times K) \\ \theta \otimes \theta' &\mapsto (-1)^{\dim\theta' \dim\gamma}\theta(\gamma) \otimes \theta'(\gamma) \end{aligned}$$

is too.
Let $\theta \in \mathcal{A}_i^*$, suppose ρ, ρ' both satisfy the required equality, then

$$j(\rho) = c\rho\left((\gamma \otimes 1) \otimes (1 \otimes \gamma)\right) = c\rho'\left((\gamma \otimes 1) \otimes (1 \otimes \gamma)\right) = j(\rho')$$

The unicity of ψ^* implies that it is a ring morphism. □

Remark 5 *From this proof, we can in particular single out the action of ψ^* on generators, namely, it follows that*

$$\begin{aligned} \psi^*(\delta) &= \delta \otimes 1 + 1 \otimes \delta \\ \psi^*(P^n) &= \sum_{i+j=n} P^i \otimes P^j. \end{aligned}$$

Theorem 6 (The Steenrod Algebra is a Hopf Algebra) *The maps*

$$\mathcal{A} \xrightarrow{\psi^*} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\phi^*} \mathcal{A}$$

Give $\mathcal{A}$ the structure of a Hopf algebra. Furthermore ϕ^* is associative and ψ^* is associative and commutative.

Proof It suffices to show that ψ^* is associative and commutative.

Associativity

It suffices to check the identity

$$(\psi^* \otimes 1)\psi^* = (1 \otimes \psi^*)\psi^*$$

This identity clearly holds on generators, namely

$$\begin{aligned} (\psi^* \otimes 1)\,(\delta \otimes 1 + 1 \otimes \delta) &= \delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta \\ &= (1 \otimes \psi^*)\,(\delta \otimes 1 + 1 \otimes \delta) \end{aligned}$$

and

$$(\psi^* \otimes 1)\left(\sum_{i+j=n} P^i \otimes P^j\right) = \sum_{i+j=n}\left(\sum_{i'+j'=i} P^{i'} \otimes P^{j'}\right) \otimes P^j$$
$$= \sum_{i+j+k=n} P^i \otimes P^j \otimes P^k$$
$$= (1 \otimes \psi^*)\left(\sum_{i+j=n} P^i \otimes P^j\right).$$

(Graded) Commutativity

Let

$$\begin{aligned} T\colon \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \otimes \mathcal{A} \\ \theta \otimes \theta' &\mapsto (-1)^{\dim\theta \dim\theta'} \theta' \otimes \theta. \end{aligned}$$

We have to check that $\psi^* = T\psi^*$, which one can check again on generators:

$$T(1 \otimes \delta + \delta \otimes 1) = 1 \otimes \delta + \delta \otimes 1$$

and

$$T(\sum_{i+j=n} P^i \otimes P^j) = \sum_{i+j=n} (-1)^{4ij(p-1)^2} P^j \otimes P^i \qquad \square$$

4 The dual Steenrod Algebra

For the rest of this talk, we focus on the dual Steenrod algebra $\mathcal{A}_* := \mathcal{A}^\vee$, whose multiplication is induced by ψ^*. Our goal is to fully determine the structure of $\mathcal{A}_*$.
To single out an appropriate set of generators for $\mathcal{A}_*$, we analyze how $\mathcal{A}_*$ (co-)acts on the cohomology ring of a specific space. We start by describing this co-action formally and then introduce the relevant space.

4.1 The coaction of $\mathcal{A}_*$

Given that we are working over a vector space, cohomology and homology are dual. Hence, given $\theta \in \mathcal{A}$ and $\mu \in H_*$, the rule

$$\theta \cdot \mu(\alpha) := \mu(\theta(\alpha)) \text{ for all } \alpha \in H^*$$

gives a well defined action

$$\lambda_*\colon \mathcal{A} \otimes H_* \to H_*$$

We denote the dual of this action by $\lambda^*\colon H^* \to \mathcal{A}_* \otimes H^*$. The restriction of λ_*

$$\lambda_i\colon \mathcal{A} \otimes H^{n+i} \to H^n$$

also gives rise to dual morphisms $\lambda^i\colon H^n \to \mathcal{A}_* \otimes H^{n+i}$ which satisfy

$$\lambda^* = \lambda^1 + \lambda^2 + \dots .^{1}$$

We can also understand the action of $\mathcal{A}$ better in terms of λ^*.

Lemma 7 *Let* $\lambda^*(\alpha) = \sum_i \alpha_i \otimes \omega_i$ *and* $\theta \in \mathcal{A}$, *then*

$$\theta\alpha = \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \theta, \omega_i \rangle \alpha_i$$

Proof By definition of the action, we have

$$\begin{aligned}
\langle \mu, \theta\alpha \rangle &= \langle \mu\theta, \alpha \rangle \\
&= \langle \mu \otimes \theta, \lambda^* \alpha \rangle \\
&= \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \mu, \alpha_i \rangle \langle \theta, \omega_i \rangle
\end{aligned} \qquad \square$$

And the general equality follows.

4.2 Generators for $\mathcal{A}_*$

Fix some large integer N and let $X = S^{2N+1}/\mathbb{Z}_p = sk_{2N+1}K(\mathbb{F}_p, 1)$. The (mod p) cohomology ring of X has the following properties

$$H^1(X) = \langle \alpha \rangle, H^2(X) = \langle \beta \rangle, H^{2i}(X) = \langle \beta^i \rangle, H^{2i+1}(X) = \langle \alpha\beta^i \rangle,$$

where $\beta = \delta\alpha$ and $i \leq N$

Notation 8 *We define*

$$M^k \coloneqq P^{p^{k-1}} \cdots P^p P^1$$

Lemma 9 *For all* $\theta \in \mathcal{A}$

$$\theta\beta = \begin{cases} \beta^{p^k} & \textit{if } \theta = M_k \\ 0 & \textit{else.} \end{cases}$$

Proof Let $\mathcal{P} = 1 + P^1 + P^2 + \dots$, from the properties of the Steenrod powers, we notice that

$$\mathcal{P}\beta = \beta + \beta^p \text{ thus } \mathcal{P}\left(\beta^{p^r}\right) = \beta^{p^r} + \beta^{p^{r+1}}.$$

Hence $P^{p^r}(\beta^{p^r}) = \beta^{p^{r+1}}$ and $P^j(\beta^{p^r})$ for $j \neq p^r$ and $j > 0$. From this, we deduce the statement. $\square$

[1] Elements in H^* are always finite sums, so this sum should be understood as $\bigoplus_i \lambda^i$

We will now explicitly determine a basis for $\mathcal{A}_*$.

Lemma 10 *There exist elements* $\tau_i, \in \mathcal{A}_*^{2p^k-1}$ *such that*

$$\lambda^*\alpha = \alpha \otimes 1 + \beta \otimes \tau_0 + \ldots + \beta^{p^r} \otimes \tau_r.$$

Similarly, there exist elements $\xi_i \in \mathcal{A}_*^{2p^i-2}$ *with* $\xi_0 = 1$ *such that*

$$\lambda^*\beta = \beta \otimes \xi_0 + \beta^p \otimes \xi_1 + \ldots + \beta^{p^r} \otimes \xi_r$$

Proof From the above, it follows that

$$\lambda^*\beta = \lambda^0\beta + \lambda^{2p-2}\beta + \ldots + \lambda^{2p^k-2}\beta.$$

As the cohomology of X is one-dimensional in all degrees, we deduce that $\lambda^{2p^k-2}(\beta) = \beta^{p^k} \otimes \xi^k$. The exact same argument works for $\lambda^*\alpha$. $\square$

We now study the evaluation pairing $\mathcal{A}_* \times \mathcal{A} \to \mathbb{F}_p$. We easily establish the following lemma

Lemma 11 *We have* $\xi_k(M_k) = 1$ *but* $\xi_k(\theta) = 0$ *for any other monomial. Furthermore*

$$\langle M_k\delta, \tau_k\rangle = 1$$

and $\langle \theta, \tau_k\rangle$ *for any other monomial.*

Proof We know that

$$M_k\beta = \beta^{p^k} = \sum_i (-1)^{2p^i \dim \xi^i} \langle M_k, \xi_i\rangle \beta^{p^i}$$

Proving the equality. The second equality follows from the same argument applied to α and $M_k\delta$. $\square$

We are ready to prove the main structure theorem for the dual Hopf algebra.

Theorem 12 *There is a graded isomorphism*

$$\mathcal{A}_* \simeq \Lambda[\tau_0, \tau_1, \ldots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots], \quad \text{where } \dim \tau_i = 2p^i - 1, \dim \xi_i = 2p^i - 2.$$

Here $\Lambda[\tau_0, \ldots]$ *denotes the exterior algebra and* $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$ *is the polynomial algebra. This isomorphism is graded*

Proof Let $\mathcal{I}$ be the set of finite sequences $(\epsilon_0, r_1, \epsilon_1, \ldots)$ with $\epsilon_i = 0, 1$ and $r_i \in \mathbb{N}$. Given $I \in \mathcal{I}$, we define

$$\omega(I) \coloneqq \tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\xi_2^{r_2}\cdots.$$

We claim it is sufficient to show that the set of $\omega(I)$ form a basis for $\mathcal{A}_*$. Indeed, the τ_i, ξ_j then don't observe any additional identities and the graded commutativity gives

the desired isomorphism.
We may order the set $\mathcal{I}$ colexicographically, ie. $(a_1, \epsilon_1, a_2, \cdots) < (b_1, \epsilon_1', b_2, \cdots)$ if $a_i < b_i$ for the largest i such that a_i and b_i differ (remember that the sequences are finite).

We also associated to a $J = (\epsilon_0, r_1, \epsilon_1, \ldots) \in \mathcal{I}$ an element of $\mathcal{A}$.

$$\theta(J) = \delta^{\epsilon_0} P^{s_1} \delta^{\epsilon_1} P^{s_2} \cdots ,$$

where $s_j = \sum_{i=k}^{\infty} (\epsilon_i + r_i) p^{i-k}$.

One can check that the $\theta(J)$ are the basic monomials of the Cartan basis for $\mathcal{A}$.

To show the isomorphism, we show that the basic monomials in $\mathcal{A}$ form an "almost dual" basis to the set of $\omega(I)$.
More precisely, we will show the following lemma.

$$\text{Let } I < J \in \mathcal{I}, \text{ then } \langle \theta(J), \omega(I) \rangle = 0 \text{ if } I < J, \text{ furthermore } \langle \theta(I), \omega(I) \rangle = \pm 1. \qquad (\star)$$

The proof of $(\star)$ will constitute the main part of the proof, let us see how to conclude given $(\star)$.
Let $\mathcal{I}_n \subset \mathcal{I}$ be the set of sequences such that $\dim \omega(I) = \dim \theta(I) = n$. The matrix $(\langle \theta(J), \omega(I) \rangle)_{I,J \in \mathcal{I}_n}$ is upper-triangular with ± 1 on the diagonal, hence, the pairing is non-degenerate and the $\omega(I)$ generate the n-th graded part of $\mathcal{A}_*$. $\square$

References

[Mil58] John Milnor. "The Steenrod Algebra and its Dual". **in**(1958).

[SE62] Norman Earl Steenrod **and** David Bernard Alper Epstein. "Cohomology Operations". **in***Ann. of Math. Stud.*: (1962).