Exercise 7. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. *(From Exercise sheet 2)*
Show that for $n \in \mathbb{Z}_{\geq 1}$ we have

$$n = \sum_{d|n} \phi(n).$$

Exercise 2. Show that $\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$.

Exercise 3. Compute the values for $\mu(10!), \phi(10!)$, and $\mu(2021)$.

Exercise 4. How many necklaces can a necklace designer think of if the necklaces have the following constraints?

(1) There are to be n beads in the necklace and each bead can be one of r possible colours.
(2) If there are n distinct beads to choose from and all of them have to be used in the necklace.
(3) If there are n distinct beads to choose from and all of them have to be used in the necklace, but the necklaces are only for display.

To clarify, a worn necklace cannot be flipped over (because the neck obstructs!). A necklace that is only on display can be flipped over.

Exercise 5. Let $\Lambda(n)$ be a function defined for $n \in \mathbb{Z}_{\geq 1}$ by the formula

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime number } p \text{ and } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Show that

$$\sum_{d|n} \Lambda(d) = \log n.$$

Exercise 6. Show that for $n \geq 1$

$$\sum_{d|n} |\mu(d)| = 2^{w(n)},$$

where $w(n) =$ number of distinct prime divisors of n.

Exercise 7. (*Optional, attempt only if you know Burnside's lemma*)
How many necklaces can a necklace designer think of if there are to be n beads in the necklace and each bead can be one of r possible colours, but the necklaces are only on display.