Exercise 12

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1

We prove the double implication.
$\Rightarrow$
Let us denote by A the adjacency matrix.
First suppose that G contains a cycle of length three, without loss of generality, we can suppose that the three vertices of the cycle are numbered by $1, 2$ and 3.
Hence, $(1, 2) \in E(G)$, and we deduce that the $(1, 2)$ entry of the adjacency matrix is different from 0.
Now consider the $(1, 2)$ entry of A^2, applying the formula for matrix multiplication yields

$$\left(A^2\right)_{1,2} = \sum_{i=1}^{n} A_{1,i} A_{i,2}$$

Note that if $i = 3$, by defintion $A_{1,3} A_{3,1} = 1$, and, since all other terms of the sum are nonnegative, $(A^2)_{1,2} \geq 1$.
$\Leftarrow$
Now suppose that G is a graph such that $A_{i,j} \neq 0$ and $(A^2)_{i,j} \neq 0$.
This implies that the vertices i and j are adjacent.
Furthermore, this implies that

$$\sum_{k=1}^{n} A_{i,k} A_{k,j} \neq 0.$$

This implies that there exists $l \in [n]$ such that $A_{i,l} = A_{l,j} = 1$, and hence $(i, j), (i, l), (l, j) \in E(G)$, which means G contains a triangle.

2

We will proceed by induction on the number n of vertices of T which are not leafs.
If $n = 1$, let v be the vertice of degree different to 1 and $l_1, \ldots, l_k$ the set of all leafs.

Since $f(l_i) = g(l_i) \forall i \in [k]$ and since f and g are bijections on the set of vertices, we immediatly deduce that $f(v) = g(v)$.
Suppose the result shown for all natural numbers smaller than n, we will now show it for $n+1$.
Let $l_1, \ldots, l_k$ again be the set of all the leafs.
We first prove that all vertices v which are adjacent to a leaf satisfy $f(v) = g(v)$.
Indeed, let l be a leaf adjacent to v , then $(v,l) \in E(T)$ implies that $f(v), f(l)$ are adjacent and that $g(v), g(l)$ are adjacent.
However, since by hypothesis $g(l) = f(l)$, and since a graph isomorphism preserves the degrees of all vertices, we deduce that $f(v) = g(v)$.
We now consider a new tree $T' = T - \{l_1, \ldots, l_k\}$.
Note that all leafs of $l \in T'$ satisfy that $f(l) = g(l)$.
Also note that, since a tree always contains a leaf, the number of vertices which are not leafs of T' is smaller than the number of vertices which are not leafs of T.
Thus, we can apply our induction hypothesis to see that $\forall v \in V(T')$, $g(v) = f(v)$ and this concludes the proof.