Exercise 2. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. Prove the following.

$$\frac{2^n}{n+1} \leq \binom{n}{\lfloor n/2 \rfloor} \leq \frac{2^n}{2}$$

Compare this with the result obtained in the class using Stirling's formula.

Exercise 2. Consider the following claim.

Statement: Any function $f : [n] \to [m]$ is a constant function.
Proof: We will prove this by induction over k that for any $A \in \binom{[n]}{k}$, we have $|f(A)| = 1$. Clearly this is true when $k = 1$. Assume the hypothesis for k and consider a set $A \in \binom{[n]}{k+1}$ and elements $x, y \in A$. Then $|A \setminus \{x\}| = k$ and $y \in A \setminus \{x\}|$ so by induction $f(A \setminus \{x\}) = \{t\}$ for some $t \in [m]$ and hence $f(y) = t$, since $t \in A \setminus \{x\}$. Similarly $f(A \setminus \{y\}) = t'$ and $f(x) = t'$ for some $t' \in [m]$ by induction. But t must be equal to t' because for any $z \in A \setminus \{x, y\}$ we get $f(z) = t = t'$. Therefore, it must be the case that $f(x) = t$ and hence $f(A) = \{t\}$. Hence, $|f(A)| = 1$ and when $k = n$, we get our claim. ■

Do you agree with this proof? If not, explain why.

Exercise 3. Show that in a group of 20 people and friendship is mutual, show that there exist two people who have the same number of friends?

Exercise 4. Determine the number of permutations of the set $[n]$

(1) with exactly one fixed point.
(2) with exactly k fixed points.

Exercise 5.

(1) How many positive integers are there that divide 10^{40} or 20^{30} ?
(2) How many positive integers less than or equal to 385 are there such that they are not divisible by neither of the following numbers: $5, 7, 11$?

Exercise 6. Prove the following.

(1) $\varphi(n)$ is even for $n \geq 3$.
(2) For every natural number n, we get

$$\sum_{d|n} \varphi(d) = n,$$

where the sum is taken over all divisors d that divide n.