Algebraic Geometry I

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List of Theorems

Lecture 1: Intro

Mon 10 Oct

Quick Motivation

We study schemes.
These are objects that "look locally" like $(\operatorname{Spec} A, A)$.
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— *$\forall U \subset X$ open, an object in C $\mathcal{F}(U)$*

— *$\forall V \subset U \subset X$ open, a morphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$*

such that

— *$\forall U$ open, $\rho_{U,U}$ is the identity on $\mathcal{F}(U)$*

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*

— *$\mathcal{F}(U)$ is called the space of sections of $\mathcal{F}$ over U*

— *Elements of $\mathcal{F}(X)$ are called global sections.*

— *There are alternative notations for $\mathcal{F}(U)$: $\Gamma(U, F)$ or $H_0(F)$*

— *The ρ_{UV} are called restriction maps, for $s \in \mathcal{F}(U)$, we write $s|_V := \rho_{UV}(s)$ and is called restriction of s to V.*

Example

— *For any object A in C, we define the constant presheaf $\underline{A}'$ defined by $\underline{A}'(U) = A$ and with restriction maps the identity.*

— *The presheaf of continuous functions :* C^0.
We define $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*

— *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

This example is universal in a certain sense

Remark
Define the category Ouv_X *with*

— *objects* $U \subset X$ *open subsets*

— *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho(U)} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow[\rho(V)]{} & \mathcal{F}_2(V) \end{array}$$

Example

— *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$

— *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$

— *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark
There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \ \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$.
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*
— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$

Example

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

— *$PSh_C(X)$ is the category of presheaves of C on X*
— *$Sh_C(X)$ is the category of sheaves of C on X*
If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then $\operatorname{Spec} A$ is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \textit{ satisfies } i \textit{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6
Let $(I, \leq)$ be a pair where I is a set and $\leq$ is a binary relation.
$(I, \leq)$ is called a preorder if ll is reflexive and transitive.
$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric
$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example

1. *Let $I = \{U \subset X | U \textit{ open }\}$ and $U \leq V \iff V \subset U$.*
   Then I is a directed poset.
2. *For $x \in X$, let*
$$I_x = \{U \subset X | U \textit{ open } x \in U\}$$
   This is a directed poset.

Definition 7
Let $(I, \leq)$ be a directed set and C a category.
A direct system in C indexed by I is a pair $\left(\{A_i\}, \{\rho_{ij}\}_{i,j \in I, i \leq j}\right)$.
Where the A_i are objects in C, the $\rho_{ij} : A_i \to A_j$ are morphisms in C such that

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example
If $\mathcal{F}$ is a presheaf of C on X and I_X as in the second example above, then

$$\left(\{\mathcal{F}(U_i)_{U_i \in I_X}\}, \{\rho_{U_i, U_j}\}\right)$$

is a direct system.

Definition 8 (direct limit)
Let $(I, \leq)$ be a directed set, C a category.
Let $\left(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I}\right)$ be a directed system, then the direct limit is a pair $(\lim_{i \in I} A_i, \{\rho_i\}_{i \in I})$ where $\lim A_i$ is in C and $\rho_i : A_i \to \lim A_i$ such that

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects A in C and morphisms $f_i : A_i \to A$ such that*

   $$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

   $\exists ! f : \lim_{i \in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

Remark
The direct limit is unique up to unique isomorphism.

Example
Write $(*) = (\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.
Let $*$ be a direct systement in Set.
Let $\lim_{i \in I} A_i := A_i / \sim$ where $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ such that $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
This is the direct limit of $*$.
If $*$ is a system in Ab *, let $\lim A_i := \bigoplus A_i / N$ with $N = \langle a_i - \rho_{ij}(a_i) \rangle$.*
The natural map $\bigcup A_i / \sim \to \bigoplus A_i / N$ is a bijection

Remark
Taking the direct limits in (Ab) *is exact in the following sense :*
$\forall$ directed sets I, $\forall$ direct systems $\{M_i\}, \{N_i\}, \{P_i\}$ indexed by I and for all

collections of commutative diagrams, we get

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

Definition 9
Let C be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of C on X.
Then the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x.
For $s \in \mathcal{F}(U)$, we write s_x for the image of s in $\mathcal{F}_x$ and call it the germ of s at x.

Remark
A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$

Remark
Let $x \in X, \mathcal{F}$ a presheaf of Set, Ab

1. $\forall U \subset X$ *open*, $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

Definition 10 (Sheafification)
Let $\mathcal{F}$ be a presheaf of sets (...) on X.
The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

Remark

1. $\mathcal{F}^+$ *is a sheaf*
2. *Sheafification is functorial.*
   For $\rho : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, the collection $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ sending $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. $\exists$ *a natural morphism* $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ *defined by* $\iota_F(U)(s) : x \to s_x$
4. $\forall s \in \mathcal{F}^+(U)$ *there is an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

Proposition 20

$\forall$ *morphisms* $\phi : \mathcal{F} \to \mathcal{G}$ *such that* $\mathcal{G}$ *is a sheaf, there exists a unique morphism* $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ *such that* $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ *open, let* $s \in \mathcal{F}^+(U)$ $\exists$ *an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.

Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, *we have to set*

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since $\mathcal{G}$ *is a sheaf and*

$$\phi(U_i)(s_i)|_{U_i \cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i \cap U_j}) = \phi(U_j)(s_i)|_{U_i \cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ *with* $t|_{U_i} = \phi(U_i)(s_i)$.

For ϕ^+ *to be a morphism, we have to set* $\phi^+(U)(s) = t$.

We still have to check that ϕ^+ *is compatible with restriction maps.* $\square$

Remark

The proposition above shows that $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ *naturally in the presheaf and the sheaf* $\mathcal{G}$.

Hence $(-)^+$ *is left-adjoint to the forgetful functor* $Sh(X) \to Psh(X)$

Proposition 22

$X = \operatorname{Spec} A$ $\forall a \in A$ *there exist isomorphisms* $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *such that* $\forall b \in A$ *with* $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow[\sim]{} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *by sending* $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.

Clearly, these make the diagram commute.

This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.

Let $I = Ann(s) = \{r \in A | rs = 0\}$.

Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, *we have* $I \not\subset p$, *hence* $V(I) \subset V(a) \implies a \in \sqrt{I}$.

Thus there exists $m \geq 1$ *such that* $a^m s = 0$, *hece* $\frac{s}{a^n} = 0$.

To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, *by definition of* $\mathcal{O}_X$ *and because* $D(h_i)$ *form a basis, we find* $a_i, g_i, h_i \in A$ *such that*

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

and $s(q) = \frac{a_i}{g_i}$ *for all* $q \in D(h_i)$.

1. *Claim 1 : Can choose* $g_i = h_i$
2. *Claim 2 : Can choose* I *finite*
3. *Claim 3 : Can choose* a_i, h_i *such that* $h_j a_i = h_i a_j$.

Using these claims, since $D(a) = \bigcup D(h_i)$ *, we find* $n > 0, b_j \in A$ *such that* $a^n = \sum b_j h_j$.
Write $c = \sum a_i b_i$.
Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.
Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.
We now prove the claims

1. *We have* $D(h_i) \subset D(g_i)$ *thus* $V(g_i) \subset V(h_i)$ *and thus* $h_i \in \sqrt{(g_i)}$.
   So there exists $c_i \in A$ *and* $n > 1$ *such that* $h_i^n = c_i g_i$.
   Now, we replace h_i *by* h_i^n *and* a_i *by* $a_i c_i$. *Then*
   $$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$
2. *We have* $D(a) \subset \cup D(h_i) \iff V(\sum h_i) = \cap V(h_i) \subset V(a)$.
   This is equivalent to saying that $a \in \sqrt{\sum (h_i)}$.
   Thus there exists $n \geq 1$ *such that* $a^n \in \sum_i (h_i)$.
   So there exist finitely many $b_i \in A$ *such that* $a^n = \sum b_j h_j$
3. *On* $D(h_i) \cap D(h_j) = D(h_i h_j)$, *we have*
   $$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$
   Thus
   $$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$
   Thus, there exists $N_j \geq 1$ *such that* $(h_i h_j)^{N_j}(h_j a_i - h_i a_j) = 0$.
   From claim 2, I *is finite, so we can choose* N *big enough such that* N *works for all* $D(h_i)$.
   Now, we replace h_i *by* h_i^{N+1} *and* a_i *by* $h_i^N a_i$ *and we get* $h_j a_i - h_i a_j = 0 \in A$. $\square$

Corollary 23
Take $X = \operatorname{Spec} A$, *then* $\forall p \in \operatorname{Spec} A \exists$ *isomorphisms* $\phi_p : A_p \to \mathcal{O}_{X,p}$ *such that the appropriate diagram commutes.*

Proof

1. *Observe* $\lim_{a \in A \setminus p} = A_a \simeq A_p$ *(check universal property)*
2. *Observe that* $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$

Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

Definition 11 (Subsheaf)
Let $\mathcal{F}$ be a (pre)sheaf on X.
Then a sub(pre)sheaf of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by $\mathcal{F}$.

Definition 12 (Kernel, cokernel of presheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *The presheaf kernel of ϕ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$*
2. *The presheaf image is defined as $\mathrm{Im}^{pre}(\phi)(U) = \mathrm{Im}(\phi(U))$*
3. *The presheaf cokernel is $\mathrm{coker}^{pre}(\phi)(U) = \mathrm{coker}(\phi(U))$.*

In each case, the restriction maps are induced by those in of $\mathcal{F}$ or $\mathcal{G}$.

Lemma 24
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf kernel is a sheaf.

Proof
Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
Since $\mathcal{F}$ is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.
Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.
Thus

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

Since $\mathcal{G}$ is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$. □

Example
By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are.

Definition 13 (Cokernel/image of morphisms of sheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *sheaf kernel : $\ker^{pre}(\phi)$*

2. *sheaf image* $(\mathrm{Im}^{pre}(\phi))^+$
3. *sheaf cokernel* $(\mathrm{coker}^{pre}(\phi))^+$

Lemma 26 (cokernels are cokernels)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. $\ker \phi \to \mathcal{F}$ *is a categorical kernel in* $Sh(X)$
2. $\mathcal{G} \to \operatorname{coker} \phi$ *is a categorical cokernel in* $Sh(X)$.

Proof

1. *This means that for each commutative diagram with solid arrows, the dotted arrow is unique*
   "Insert cokernel/kernel diagram here"
   This holds for every open U and so the kernel is a sheaf.
2. *The appropriate diagram commutes and we use the universal property of sheafification.* □

Proposition 27
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. *ϕ is a monomorphism in $Sh(X)$*
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

Proof
Recall ϕ is a monomorphism if for every $\psi : \mathcal{F}' \to \mathcal{F}, \phi \circ \psi = 0 \implies \psi = 0$.
The implication $1 \implies 2$ *follows by applying the monomorphism property to* $\ker \phi \to \mathcal{F}$ $2 \implies 1$ *If $\phi \circ \psi = 0$, then ψ factors through the kernel* $\ker \phi \to \mathcal{F}$ *and so $\psi = 0$*
$2 \iff 3$ *Since* $\ker(\phi)(U) = \ker(\phi(U))$
$3 \implies 4$ *Follows because taking direct limits is exact.*
$4 \implies 3$ *Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.*
So $s_x = 0 \forall x \in U$ and so $s = 0$ □

Proposition 28
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. *ϕ is an epimorphism in $Sh(X)$*
2. $\mathrm{coker}(\phi) = 0$
3. $\mathrm{coker}(\phi_x) = 0$

Proof

Recall that ϕ is an epimorphism if for every $\psi : \mathcal{G} \to \mathcal{G}', \psi \circ \phi = 0 \implies \psi = 0$

$1 \implies 2$ *Apply epimorphism property to $\mathcal{G} \to \operatorname{coker}(\phi)$*

$2 \implies 3$ *We have*

$$\begin{aligned} 0 &= (\operatorname{coker} \phi)_x \\ &= (\operatorname{coker}^{pre} \phi)_x = \operatorname{coker}(\phi_x) \end{aligned}$$

□

$3 \implies 1$

Let $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\psi \circ \phi = 0$, this implies that $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$.

Since ϕ_x is an epimorphism of abelian groups, we get $\psi_x = 0$.

As the hom sheaf is a sheaf, we get that $\psi = 0$

Remark

If $\operatorname{coker}(\phi(U)) = 0 \forall U \subset X \implies \operatorname{coker}(\phi) = 0$ but the converse is not true.

Corollary 30

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent

1. *ϕ is an isomorphism*
2. *$\phi(U)$ is an isomorphism $\forall U \subset X$ open*
3. *ϕ_x is an isomorphism $\forall x \in X$*

Proof

$1 \implies 2$ *since taking sections is a functor*

$2 \implies 3$ *since taking limits is functorial*

$2 \implies 1$ *because $(\phi(U))^{-1}$ defines a morphism of sheaves*

$3 \implies 2$ *Need to show surjectivity of $\phi(U)$.*

Let $t \in \mathcal{G}(U)$, since ϕ_x is an isomorphism $\forall x \in U$, we find $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$.

There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ such that $(s_{V_x})_x = s_x$

Since

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

we can choose $V + x$ sufficiently small such that $\phi(V_x)(s_{V_x}) = t|_{V_x}$.

Since $\phi(V_x \cap V_y)$ is injective and $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$, we have $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$.

Thus there exists $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ and $\phi(U)(s)|_{V_x} = t|_{V_x}$ and thus $\phi(U)(s) = t$. □

Definition 14 (Exact Sequence of sheaves)

A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ is called exact if $\ker \phi_2 = \operatorname{Im} \phi_1$

Corollary 31
A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.

Lecture 4: locally ringed spaces, (affine) Schemes (!)

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Corollary 32
A sequence of sheaves is exact if and only if it is exact on all stalks.

Proof
If $\ker(\phi_{2,x}) = \operatorname{Im}(\phi_{1,x}) \forall x \in X$, *thus* $(\phi_{2,x} \circ \phi_{1,x}) = (\phi_2 \circ \phi_1)_x$.
Thus $\phi_2 \circ \phi_1 = 0$ *because the hom sheaf is a sheaf.*
Thus ϕ_1 *factors as* $\mathcal{F}_1 \to \operatorname{Im}\phi_1 \to \ker\phi_2 \to \mathcal{F}_2$ *as* ψ_x *is an isomorphism,* ψ *is an isomorphism.* □

Corollary 33
Let $\phi : \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves, then* $\operatorname{Im}\phi = \ker(\mathcal{G} to \operatorname{coker}\phi)$

Corollary 34
$Sh(X)$ *is an abelian category.*

1.5 Direct and inverse image, ringed spaces

Definition 15
Let $f : X \to Y$ *be a continuous map.*
We define the direct image of $\mathcal{F}$ *by* f *on* Y *defined by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

We can check that $f_*\mathcal{F}$ is a sheaf with restriction maps induced by $\mathcal{F}$.
If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, then the $(f_*\phi)(X) = \phi(f^{-1}(V))\mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}V)$ define a morphism of sheaves.
Thus we get a functor $f_* : Sh(X) \to Sh(Y)$.

Definition 16 (inverse image)
Let $f : X \to Y$ *be a continuous map and let* $\mathcal{G}$ *be a sheaf on* Y.

The inverse image of $\mathcal{G}$ along f is the sheafification of the presheaf

$$f^{-1,pre}(\mathcal{G})$$

defined by

$$f^{-1,pre}(\mathcal{G})(U) = \varinjlim_{f(U)\subset V} \mathcal{G}(V)$$

We can again check that the if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on Y, we define $f^{-1}\phi : \varinjlim \mathcal{F}(V) \to \varinjlim \mathcal{G}(V)$ using the maps induced by ϕ.
Thus we get a functor $Sh(Y) \to Sh(X)$.

Lemma 35
Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

1. *$\forall y \in Y$ there is a natural isomorphism*

$$(f_*\mathcal{F})_y \simeq \varinjlim_{y\in V\subset Y} \mathcal{F}(f^{-1}(V))$$

In particular $forall x \in X$ there is a natural map $(f_*\mathcal{F})_{f(x)\to\mathcal{F}_x}$

2. *$\forall x \in X$ there is a natural isomorphism $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$*

Proof
The isomorphisms are immediate from the definition.
The morphism $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ is given by

$$(f_*\mathcal{F})_{f(x)} = \varinjlim \mathcal{F}(f^{-1}(V)) = \varinjlim_{x\in f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \to \varinjlim_{x\in U} \mathcal{F}(U) = \mathcal{F}_x$$ $\square$

Proposition 36
If $f : X \to Y$ is a continuous map, then $f_* : Sh(X) \to Sh(Y)$ is right-adjoint to $f^{-1} : Sh(Y) \to Sh(X)$

Corollary 37
$f^{-1} : Sh(Y) \to Sh(X)$ is exact

Proof
Let $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ be exact in $Sh(Y)$.
Thus $\forall y \in Y, 0 \to \mathcal{G}_{1,y} \to \mathcal{G}_{2,y} \to \mathcal{G}_{3,y} \to 0$ is exact.
In particular it is exact at $f(x)\forall x \in X$ and thus the associated inverse image

sequence is exact. $\square$

Corollary 38
$f_* : Sh(X) \to Sh(Y)$ *is left-exact.*

Proof
Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *be exact in* $Sh(X)$.
Recall that the section functor is left-exact, thus $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ *is exact* $\forall U \subset X$.
Thus $0 \to (f_*\mathcal{F}_1)_y \to (f_*\mathcal{F}_2)_y \to (f_*\mathcal{F}_3)_y$ *is exact* $\forall y \in Y$ *and thus* $0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ *is exact.* $\square$

Example
f_* *is usually not right-exact.*
Eg, if $f : X \to \{*\}$ *and* $\mathcal{F}$ *is a sheaf on* X*, then* $(f_*\mathcal{F})(\emptyset) = 0$ *and* $(f_*\mathcal{F})(\{*\}) = \mathcal{F}(X)$ *and taking sections is not exact.*

Definition 17 (Ringed space)
A ringed space is a pair $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X.
A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a pair* $(f, f^\sharp)$ *where* $f : X \to Y$ *is a continous map and* $f^\sharp$ *is a morphism* $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark
Ringed spaces form a category, if $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ *define their composition to be* $(g \circ f, g_*(f^\sharp \circ g^\sharp))$

Example

1. *For every ring* A, $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a ringed space.*
2. *For any field* K *and any topological space* X*, define a sheaf* $Fun_{X,K}(U) = \{s : U \to K\}$.
   There is a functor $\top \to$ (*Ringed spaces*) *sending* $X \mapsto (X, Fun_{X,K})$ *where for* $f : X \to Y$ $f^\sharp$ *is the pullback (precomposition).*
3. (X, C^0_X) *is a ringed space*

Observe that for many of these examples of ringed spaces, the stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 18 (Morphism of local rings)
A morphism of local rings $\phi : A \to B$ *with maximal ideals* m_A *and* m_B *is called local if* $m_A = \phi^{-1}(m_B)$

Example

1. *For all ring homomorphism $\phi : A \to B$ and all $q \in \operatorname{Spec} B$ the induced map $A_{\phi^{-1}(q)} \to B_q$ is local.*
2. *A ring homomorphism $\phi : A \to K$ from a local ring A to a field iff $m_A = \ker \phi$*

Definition 19 (Locally ringed space)
A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,x}$ is local $\forall x \in X$.
A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \xrightarrow{f_x^\sharp} (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$$

is local.

Remark
The category of locally ringed spaces is a subcategory of the category of ringed spaces

Definition 20 (Affine Scheme)
An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that $X = \operatorname{Spec} A$ and $\mathcal{O}_X$ is the structure sheaf.

Definition 21 (Scheme)
A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $X = \bigcup_{i \in I} U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.
A morphism of schemes is a morphism of the underlying ringed spaces.

Example

1. *If $(X, \mathcal{O}_X)$ is a scheme and $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is not necessarily a scheme (even if X is affine).*
2. *If $(X, \mathcal{O}_X)$ is a scheme and $X = \{*\}$, then X is affine.*
   Then $\operatorname{Spec} A = \{*\}$ iff every $a \in A$ is either a unit or nilpotent.

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Remark
By abuse of notation, we write X is a scheme with $\mathcal{O}_X$ implicit.

Lemma 46
Let X be a topological space with basiis for the topology $\{v_i\}_{i\in I}$.
Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X.
For any collection of morphisms $\phi_i : \mathcal{F}(V_i) \to \mathcal{G}(V_i)$ such that $\rho_{ij} \circ \phi_i = \phi_j$, then $\exists! \phi : \mathcal{F} \to \mathcal{G}$ which restricts to ϕ_i on the V_i.

Proposition 47
Let $(X, \mathcal{O}_X)$ be a locally ringed space and A a ring, then the map $\hom((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \to \hom(A, \mathcal{O}_X(X))$ which maps $(f, f^\sharp) \to f^\sharp(\operatorname{Spec} A)$ is a natural bijection.

In particular, for all locally ringed spaces $(X, \mathcal{O}_X)$, there is a natural affinization morphism $aff_X : X \to \operatorname{Spec} \mathcal{O}_X(X)$

Corollary 48
Every morphism of locally ringed spaces $(X, \mathcal{O}_X) \to \operatorname{Spec} A$ factors uniquely through aff_X.

Corollary 49
A locally ringed space is an affine scheme iff the affinization is an isomorphism.

Corollary 50
The functor
$$(affSch) \to (Ring)^{op}$$
mapping $(X, \mathcal{O}_X) \to \mathcal{O}_X(X)$ is an equivalence of categories.

Proof
Fully faithful is the proposition above.
Essential surjectiveness is immediate as for any ring, we can look at $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ as $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$. $\square$

We now prove the statement

Proof
We use that there exists a natural isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(a)) \simeq A_a$.
Naturality follows from functoriality of $f^\sharp(-)$.
We have to construct an inverse, let $\phi : A \to \mathcal{O}_X(X)$ be a ring homomorphism, we need to define $(f, f^\sharp) : (X, \mathcal{O}_X) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.
We map $x \mapsto \ker(A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_{X,x}/m_x)$.
We claim that f is continuous.

It suffices to show that $X_{\phi(a)} = f^{-1}(D(a)) = \{x \in X | \phi(a)_x \notin m_x\} \subset X$ *is open.*
Take $x \in X_{\phi(a)}$*, then* $\phi(a)_x \notin m_x \implies \phi(a)_x \in \mathcal{O}_{X,x}^{\times}$.
Thus $\exists x \in V \subset X$ *and* $b \in \mathcal{O}_X(V)$ *such that* $\phi(a)|_V b = 1 \in \mathcal{O}_X(V)$.
Thus $\phi(a)_y b_y = 1 \forall y \in V \implies \phi(A)_y \notin m_y \implies V \subset X_{\phi(a)} \implies X_{\phi(a)}$ *is open.*
To define $f^\sharp$*, observe that* $\forall a \in A, \phi(a)|_{X_{\phi(a)}} \in \mathcal{O}_X(X_{\phi(a)})$ *is a unit in every stalk, hence a unit.*
Thus there is a unique morphism such that $A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{\phi(a)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))} \mathcal{O}_X(X_{\phi(a)})$ *so we get a morphism* $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_X$.
We still have to show that this map is a morphism of locally ringed spaces.
We claim that $\forall x \in X$*, the map* $f_x^\sharp : A_{f(x)} \to \mathcal{O}_{X,x}$ *is a local homomorphism.*
The diagram induces a commutative diagram

$$A \xrightarrow{\phi}\!\!\to \mathcal{O}_X(X) \xrightarrow{\pi_2} \mathcal{O}_{X,x} = A \xrightarrow{\pi_1} A_{f(x)} \xrightarrow{f_x^\sharp} \mathcal{O}_{X,x}$$

Note that $p_1^{-1}(f_x^{\sharp,-1}(m_x)) = \pi_1^{-1} \circ \pi_2^{-1}(m_x) = f(x)$ *by definition.*
Thus $f_x^{\sharp,-1}(m_x) = f(x) A_{f(x)}$.
Now, we need to show that this construction is in fact an inverse.
By construction, if $(f, f^\sharp)$ *comes from* ϕ*, then* $\phi = f^\sharp(\operatorname{Spec} A)$.
Conversely, let $(f, f^\sharp) : X \to \operatorname{Spec} A$ *be a morphism and let* $(f', f'^\sharp) : X \to \operatorname{Spec} A$ *be associated to* $f^\sharp(\operatorname{Spec} A)$.
We need to show that $(f, f^\sharp) = (f', f'^\sharp)$.
$\forall x \in X, \exists$ *a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_{X,x} = A \to A_{f(x)} \to O_{X,x}$$

As $f_x^\sharp$ *and* $f_x'^\sharp$ *are local,* $f(x) = f'(x)$*. Now,* $\forall a \in A$*, there is a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))}\!= f'^\sharp(D(a)) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A),a})$$

$\square$

Example
For every locally ringed space $(X, \mathcal{O}_X)$*, there exists a unique morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$ *because* $\exists! \mathbb{Z} \to \mathcal{O}_X(X)$.
If $(X, \mathcal{O}_X)$ *is a locally ringed space such that each* $\mathcal{O}_X(U)$ *has characteristic* $p > 0$*, then* $\exists!$ *morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{F}_p, \mathcal{O}_{\operatorname{Spec} \mathbb{F}_p})$.

Definition 22 (Scheme over another scheme)
Let S be a sscheme. The category of schemes over S, Sch/S is the category whose objects are morphisms $X \to S$ and morphisms are commutative triangles.

Example
Let K be a field.
The affine n-space over k is denoted $\mathbb{A}^n_k$ is $\operatorname{Spec} k[x_1, \dots, x_n]$.
If k is algebraically closed, then

$$k^n \simeq \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq \mathbb{A}^n_k \simeq \hom_{k-alg}(k[x_1, \dots, x_n], k)$$

If $\phi : A \to B$ is a surjective ring homomorphism, then the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism onto $V(I)$ where $I = \ker \phi$.
In particular, if $I \subset K[x_1, \dots, x_n]$, $k = \overline{k}$ is an ideal, then $V(I) = \{(a_1, \dots, a_n) \in k^n | f(a_1, \dots, a_n) = 0 \forall f \in I\}$ is the image of $\operatorname{Spec}_{max} k[x_1, \dots, x_n] / I \to \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq k^n$.

Example (glueing two schemes)
If X_1, X_2 are two schemes and $U_i \subset X_i$ are open subsets,

$$(\phi, \phi^\sharp) : (U_1, \mathcal{O}_X|_{U_i}) \simeq (U_2, \mathcal{O}_X|_{U_2})$$

is an isomorphism.
We define the scheme $(X, \mathcal{O}_X)$ by glueing X_1 and X_2 over U_1 as follows.
As a set, $X = X_1 \coprod X_2 / \sim$ where $x_1 \sim \phi(x_1)$.
Note, there are natural maps $\pi_i : X_i \to X$.
We say that a subset $U \subset X$ is open $\iff$ $\pi_i^{-1}(U) \subset X_i$ open for $i = 1, 2$.
We define the structure sheaf as $\mathcal{O}_X(U) = \ker(\mathcal{O}_{X_1}(\pi_1^{-1}(U)) \oplus \mathcal{O}_{X_2}(\pi_2^{-1}(U)) \to \mathcal{O}_{X_1}(\pi_1^{-1}(U) \cap U_1))$.
Then X is a scheme.

Example (Explicit example of glueing)
Take $X_1 = X_2 = \mathbb{A}^1_K$ and $U_1 = U_2 = \mathbb{A}^1_K \setminus 0$.
Notice that $U \simeq \operatorname{Spec} k[x, x^{-1}]$.

1. *Taking the glueing map $\phi = \mathrm{Id}$, we get a line with two origins.*
2. *Taking $\phi^\sharp(U_2) : x \mapsto \frac{1}{x}$, we get the projective line $\mathbb{P}^1_k$.*
   The k-rational points of this scheme are in correspondence with lines in k^2, namely
   $$P^1_k(k) \simeq k^2 \setminus \{0\} / k^\times.$$

2 Properties of schemes

2.1 Topological properties

Definition 23
A scheme $(X, \mathcal{O}_X)$ is called

1. *connected if X is*
2. *irreducible if $\forall U_1, U_2$ open non empty their intersection is non-empty.*
3. *quasi-compact if X is.* [a]
4. *quasi-separated if X is, ie. $\forall U_1, U_2$ open and quasi-compact, their intersection is quasi-compact.*

a. All affine schemes are quasi-compact, but $\mathbb{A}_k^\infty \setminus 0$ is not quasi-compact

Lecture 6: Topological properties

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Remark
$\operatorname{Spec} R \times S = \operatorname{Spec} R \coprod \operatorname{Spec} S$ *but* $\operatorname{Spec} \prod_i R_i \not\simeq \coprod_i \operatorname{Spec} R_i$ *for infinite products*

Lemma 56
Affine schemes are quasi-compact and quasi-separated.

Proof
Let $X = \operatorname{Spec} A$ be an affine scheme.
Quasi-compactness has already been proven.
If $U \subset X$ is open and qc., then $U = \bigcup_{i \in I_U} D(a_i), a_i \in A$ and I_U finite.
For $U_1, U_2 \subset X$ qc. open, then

$$U_1 \cap U_2 = \bigcup_{i \in I_{U_1}, j \in I_{U_2}} D(a_1) \cap D(a_2) = \bigcup D(a_1 a_2)$$

Check that a finite union of qc spaces is qc □

Remark
Let X be a topological space, then $\forall$ subsets $V \subset X$ and $U \subset X$, then

$$U \cap V \neq \emptyset \iff U \cap \overline{V} \neq \emptyset$$

Thus V is irreducible iff it's closure is.

If X is irreducible, then every non-empty open is dense.

2.2 Scheme-Theoretic Properties

Definition 24 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ *is a pair* $(U, \mathcal{O}_U)$ *with* U *open in* X *and* $\mathcal{O}_U := \mathcal{O}_X|_U$

If P is a property of rings, when do we say that $(X, \mathcal{O}_X)$ satisfies P?

1. $\forall U \subset X, \mathcal{O}_X(U)$ satisfies P (usually too strong)
2. $\forall U \subset X$ open and affine, $\mathcal{O}_X(U)$ satisfies P
3. $\exists$ an open affine cover $U = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P
4. $\forall x \in X \exists x \in U$ open affine such that $\mathcal{O}_X(U)$ satisfies P.
5. $\forall x \in X, \mathcal{O}_{X,x}$ satisfies P.

Observe that $1 \implies 2 \implies 3 \iff 4$.

Lemma 58
For $P =$ *"reduced ring", then all 5 are equivalent.*

Proof
From commutative algebra, we know that a ring A *is reduced* $\iff A_p$ *is reduced* $\forall p \in \operatorname{Spec} A$.
This implies that $2 \iff 3 \iff 4 \iff 5$.
Let's show $2 \implies 1$.
Let $U \subset X$ *open and* $s \in \mathcal{O}_X(U)$ *such that* $s^n = 0$, *then* $s^n|_V = 0 \forall V \subset U$ *affine.*
Thus, $s|_V = 0 \forall V \subset U$ *open affine and as* $\mathcal{O}_X$ *is a sheaf* $s = 0$. ☐

Definition 25 (Reduced Scheme)
A scheme $(X, \mathcal{O}_X)$ *is called reduced if* $\mathcal{O}_X(U)$ *is reduced* $\forall U \subset X$ *open.*

Definition 26
Let P *be a property of rings or of open affines* $\operatorname{Spec} A \hookrightarrow X$ *of a scheme* X

- P *is called affine-local if* $\forall a_1, \ldots, a_n \in A$ *such* $(a_1, \ldots, a_n) = A$. A *satisfies* P *every* A_{a_i} *satisfies* P
- P *is called stalk-local if* A *satisfies* $P \iff A_p$ *satisfies* $P \forall p \in \operatorname{Spec} A$.

Remark
Being stalk-local is stronger than being affine local.
This is becauses $A \to A_a$ *induces* $(A_a)_{pA_a} \simeq A_p \forall p \in D(a)$

Example

1. *Reduced is stalk-local*
2. *Normal*
3. *regular*
4. *Cohen-Macaulay*

Example

1. *Integrality is not affine-local (consider* $A = k \times k$ *)*
2. *Factorial is not affine-local*
3. *Noetherian is not stalk-local (consider* $A = \prod_i \mathbb{F}_2$ *)*

Lemma 62
Being Noetherian is affine-local.

Why do we care ?

For affine-local properties, 2 and 4 of our list are equivalent.

Proof
If A *is noetherian, then any quotient and any localization is.*
Assume $(a_1, \dots, a_n) = A$ *and* A_{a_i} *are Noetherian.*
Let $\phi_i : A \to A_{a_i}$ *be the localization maps.*
Claim : $\forall$ *ideals* $I \subset A$, $I = \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$.
One inclusion is clear.
Let $b \in \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$, *thus there exists* $N > 0$ *and* $b_i \in I$ *such that* $b = \frac{b_i}{a_i^N} \in A_{a_i}$.
Thus there exists an $M > 0$ *such that* $a_i^M(a_i^N b - b_i) = 0$ *in* A.
Set $k = M + N$, *note that* $1 = (a_1^k, \dots, a_n^k)$.
We can write $1 = \sum_{i=1}^n c_i a_i^k$ *for some* $c_i \in A$.
Thus $b = \sum c_i a_i^k b = \sum c_i a_i^M b_i \in I$.
Let $I_1 \subset \ldots \subset I_n \subset$ *be an ascending chain of ideals in* A, *then we get an ascending chain of ideals* $\phi_1(I_1)A_{a_i} \subset \ldots \subset \phi_i(I_n)A_{a_i}$.
This becomes constant because A_{a_i} *is noetherian and* $\exists N > 0$ *such that* $\phi_i(I_k)A_{a_i} = \phi_i(I_N)A_{a_i} \forall kggN$ $\square$

Lemma 63
Let P *be an affine-local property of rings. Let* $(X, \mathcal{O}_X)$ *be a scheme, then the following are equivalent.*

1. *Every open affine* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P
2. $\exists$ *an open affine cover* $X = \cup \operatorname{Spec} A_i$ *such that each* $\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P.

Proof

$1 \implies 2$ *is clear.*

$2 \implies 1$.

Let $\operatorname{Spec} A \hookrightarrow X$ *open and affine.*

Write $\operatorname{Spec} A = \cup \operatorname{Spec} A_{a_i}$ *with* $a_i \in A$ *such that* $A_{a_i} \simeq (A_i)_{b_i}$ *for some* $b_i \in A_i$.

$\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P, *implies* $(\operatorname{Spec}(A_i)_{b_i}) \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A_{a_i} \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P $\square$

Lemma 64

Let $\operatorname{Spec} A, \operatorname{Spec} B \subset X$ *be open affines, then for every point* $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$ *there exist* $a \in A$ *and* $b \in B$ *such that* $A_a \simeq B_b$ *such that* $x \in D(a) \subset \operatorname{Spec} A$ *and* $x \in D(b) \subset \operatorname{Spec} B$ *and the isomorphism* $\operatorname{Spec} A_a \simeq \operatorname{Spec} B_b$ *commutes with the inclusions to* X.

Proof

$\operatorname{Spec} A \cap \operatorname{Spec} B \subset \operatorname{Spec} A$ *is open.*

Thus, there exists $a \in A$ *with* $x \in D(a) \subset \operatorname{Spec} A \cap \operatorname{Spec} B$.

We can assume wlog that $\operatorname{Spec} A \to X$ *factors throguh* $\operatorname{Spec} B$.

Write $\phi : B \to A$ *for the induced map of rings.*

Since $\operatorname{Spec} A \subset \operatorname{Spec} B$ *is open* $\exists b \in B$ *and* $B \to A \to B_b$ *is just localization of* B *at* b.

THen $A \to B_b$ *satisfies the universal property of* $A \to A_{\phi(b)}$.

So we get a commutative square $B \to A \to A_{\phi(b)}$ *and* $B \to B_b \to A_{\phi(b)}$ *and we get an isomorphism* $B_b \simeq A_{\phi(b)}$. $\square$

Definition 27

Let P *be an affine-local property of rings.*

A scheme $(X, \mathcal{O}_X)$ *is called locally* P *if* $\mathcal{O}_X(U)$ *satisfies* $P \forall U \subset X$ *open affine.*

Definition 28 (Noetherian scheme)

A scheme $(X, \mathcal{O}_X)$ *is called Noetherian if it is locally Noetherian and qc.*

Definition 29 (Integral scheme)
A scheme $(X, \mathcal{O}_X)$ is called integral if $\mathcal{O}_X(U)$ is an integral domain $\forall U \subset X$ open and non-empty.

Lemma 65
For a scheme $(X, \mathcal{O}_X)$, the following are equivalent.

1. *X is integral*
2. *X is reduced and irreducible.*
3. *$\forall U \subset X$ open affine, $\mathcal{O}_X(U)$ is integral.*

Proof
$1 \implies 3$ *is clear.*
$3 \implies 2.$
Reduced is clear.
Let $U_1, U_2 \subset X$ open with $U_1 \cap U_2 = \emptyset$.
Wlog, the U_i are affine.
Then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$.
Thus $\mathcal{O}_X(U_1) = 0$ or $\mathcal{O}_X(U_2) = 0$ which implies U_1 or $U_2 = \emptyset$.
$2 \implies 1$
Let $U \subset X$ be open.
Assume $\exists a, b \in \mathcal{O}_X(U)$ such that $ab = 0$.
Let $U_a = \{x \in U | a_x \notin m_x\}$ and similarly U_b.
Note that $U_a \cap U_b = \emptyset$ since $\forall x \in U_a \cap U_b, a_x$ and b_x are units.
Thus $U_a = \emptyset$ or $U_b = \emptyset$.
If $U_a = \emptyset \forall \operatorname{Spec} A \subset U \forall p \in \operatorname{Spec} A$

$$(a|_{\operatorname{Spec} A})_p \in pA_p$$

thus $a|_{\operatorname{Spec} A} \in p \forall p \in \operatorname{Spec} A$.
Thus $a|_{\operatorname{Spec} A}$ is nilpotent.
But since X is reduced, $a|_{\operatorname{Spec} A} = 0$.
Covering U by affines, $a = 0$ (as A was arbitrary). □

Lecture 7: Open and Closed Subschemes

Mon 31 Oct

3 Open and closed subschemes and immersions

Definition 30 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ is a pair $(U, \mathcal{O}_U)$, with $U \subset X$ open and $\mathcal{O}_U = \mathcal{O}_X|_U$.

Lemma 66
If A is a ring and $a \in A$, then there is an isomorphism of locally ringed spaces $(\operatorname{Spec} A_a, \mathcal{O}_{\operatorname{Spec} A_a}) \simeq (D(a), \mathcal{O}_{\operatorname{Spec} A}|_{D(a)})$.
In particular, open subschemes of schemes are schemes.

Proof
From commutative algebra, localization $A \to A_a$ induces a homeomorphism $\operatorname{Spec} A_a \to D(a) \subset \operatorname{Spec} A$.
On sheaves, we want to give morphisms $\mathcal{O}_{\operatorname{Spec} A}|_{D(a)}(U) \to \mathcal{O}_{\operatorname{Spec} A_a}(f^{-1}(U))$.
If $s : U \to \coprod_{p \in U} A_p \to (f^{-1}(U) \to U \xrightarrow{s} \coprod_{p \in U} A_p \to \coprod_{p \in U} (A_a)_{pA_a})$, using $A_p \simeq (A_a)_{pA_a}$. $\square$

Note that, if $i : U \to X$ is the inclusion of an open, then $(i, i^\sharp) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ with

$$i^\sharp(V) : \mathcal{O}_X(V) \xrightarrow{\rho_{V, V \cap U}} \mathcal{O}_X(V \cap U) = i_* \mathcal{O}_U(V)$$

is a morphism of schemes.

Remark
If $i : U \to X$ is an inclusion of an open, then there are in general many sheaves of rings $\mathcal{F}$ on U such that $\exists i^\sharp$ such that $(i, i^\sharp) : (U, \mathcal{F}) \to (X, \mathcal{O}_X)$ is a morphism of schemes.

FOr example, if $X = \operatorname{Spec} k$, $U = \operatorname{Spec} {}^{k[x]}\!/\!_{x^2}$then $k \subset {}^{k[x]}\!/\!_{x^2}$ induces a morphism $(f, f^\sharp) : U \to X$ such that $f = \mathrm{Id}_X$.

Definition 31 (Open immersion)
An open immersion (or open embedding) is a morphism of schemes $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that f is a homeomorphism onto an open subset $U \subset Y$ $\mathcal{O}_Y|U \simeq (f_* \mathcal{O}_X)|_U$.

Example
Let k be a fieldand let $\iota : \operatorname{Spec} k \to X = \mathbb{A}^n$ be the closed point corresponding to

0.
Then

$$(\mathcal{O}_X)|_{\operatorname{Spec} k}(\operatorname{Spec} k) = (i^{-1}\mathcal{O}_X)(\operatorname{Spec} k)$$
$$= \varprojlim_{0 \in U \subset \mathbb{A}^n} \mathcal{O}_X(U) = \mathcal{O}_{X,0} = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$$

But $\operatorname{Spec} k[x_i]_{(x_1,\ldots,x_n)}$ *has more than one point.*
Thus, $(\operatorname{Spec} k, (\mathcal{O}_X)|_{\operatorname{Spec} k})$ *is not a scheme.*

Observe : If $Z \subset \operatorname{Spec} A$ is a closed subset, then $Z = V(I)$ for some ideal I. Then the map $\operatorname{Spec} {}^{A}\!/\!_{I} \to \operatorname{Spec} A$ induced by the quotient map is a homeomorphism onto $V(I)$ and this gives a scheme structure on Z (which depends on I !).

Definition 32 (Ideal sheaves)
Let $(X, \mathcal{O}_X)$ *be a scheme, then*

1. *An ideal sheaf on* $(X, \mathcal{O}_X)$ *is a subsheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that* $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ *is an ideal for all* $U \subset X$ *is open.*
2. *For an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$*, the quotient sheaf* ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}}$ *is the cokernel sheaf of the inclusion, namely, the sheafification of the sheaf* $U \mapsto {}^{\mathcal{O}_X(U)}\!/\!_{\mathcal{I}(U)}$.

Definition 33 (Closed Subsceme)
Let $(X, \mathcal{O}_X)$ *be a scheme, then a closed subscheme of* $(X, \mathcal{O}_X)$ *consists of a subset* $Z \subset X$ *and an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that*

1. $Z = \left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\}$
2. $(Z, ({}^{\mathcal{O}_X}\!/\!_{I})|_Z)$ *is a scheme*

Remark
By 1, Z *is closed, indeed, for* $1 \in ({}^{\mathcal{O}_X}\!/\!_{(X)})$*, we have*

$$\left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\} = \operatorname{Supp} 1$$

Remark
The morphism ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} \to i_*(({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})|_Z)$ *is an isomorphism.*

If $Z \subset X$ is a closed subscheme determined by $\mathcal{I}$, then $(i, i^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ where $i : Z \to X$ is the inclusion and $i^\sharp : \mathcal{O}_X \to {}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} = i_*\mathcal{O}_Z$ is a morphism of schemes.

Example
Condition 2 in the definition of closed subscheme is not automatic, even if X is affine.

Definition 34 (Closed immersion)
A closed immersion (or closed embedding) is a morphism of schemes $(f, f^\sharp) : X \to Y$ such that f is a homeomorphism onto a closed subset and $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is surjective on stalks.

Remark
The morphism $(i, i^\sharp)$ of the inclusion of closed subscheme is a closed immersion.

Example
If A is a ring and $I \subset A$ is an ideal, then the morphism $\operatorname{Spec} {}^{A}\!/_{I} \to \operatorname{Spec} A$ is a closed immersion.

Indeed, by CA, this is a homeomorphism onto $V(I)$.
The map $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A/I}$ is surjective because $f^{\sharp_p} : A_p \to ({}^{A}\!/_{I})_{pfaktorAI}$, which is the localization of a surjective map.

From now on, $V(I) \subset \operatorname{Spec} A$ for the closed subscheme determined by I.

Proposition 74
If $X = \operatorname{Spec} A$ is affine, then the map $I \to V(I)$ is a bijection between ideals of A and closed subschemes.

Proof
Let $Z \subset X$ be a closed subscheme determined by $\mathcal{I}$.
Let $I_Z = \ker(\mathcal{O}_X(X) \to \mathcal{O}_Z(Z))$.
Note, if $Z = V(I)$ for some ideal, then $I = I_Z$.
So we have to show that $Z = V(I_Z)$.
The morphism $\phi : \mathcal{O}_X(X) \to \mathcal{O}_Z(Z)$ factors through ${}^{\mathcal{O}_X(X)}\!/_{I_Z} = {}^{A}\!/_{I_Z}$ so $\iota : Z \to X$ factors through $V(I_Z)$.
Replace A by ${}^{A}\!/_{I_Z}$, we may assume ϕ is injective and we have to show that ϕ is an isomorphism.

Claim 1

$\forall U \subset Z$ open affine and $s \in A$, we have $D(s) \cap U = D(\phi(s)|_U) \subset U$ and $V(S) \cap U = V(\phi(s)|_U) \subset U$.
It suffices to prove the first equality.
$\forall p \in Z$, the following diagram commutes

$$A \xrightarrow{\phi} \mathcal{O}_Z(Z) \to \mathcal{O}_{Z,p}$$

and $A \to A_p \xrightarrow{\iota_p^\sharp} \mathcal{O}_{Z,p}$ *and* $i_p^\sharp$ *is local.*
Now

$$D(\phi(s)|_U) = \{p \in U | \phi(s)_p \notin m_p \subset \mathcal{O}_{Z,p}\} = \left\{p \in U | i_p^\sharp(s_p) \notin m_p \subset \mathcal{O}_{Z,p}\right\}$$
$$= \{p \in U | s_p \notin m_p \subset A_p\} = D(s) \cap U \square$$

Claim 2

We show $Z \to X$ *is surjective. Since* Z *is closed in* X*, it suffices to show that* $\forall s \in A$ *such that* $Z \subset V(s)$*, we have* $V(s) = X$*.*
Choose such an $s \in A$*.*
As closed subspaces of qc. spaces are qc.
We can cover Z *by finitely many open affines* U_i*.*
By claim 1, $U_i \subset V(\phi(s)|_{U_i}) \subset U_i$*.*
Thus $\phi(s)|_{U_i} \in p \forall p \in \operatorname{Spec} \mathcal{O}_Z(U_i)$*, thus* $\phi(s)|_{U_i}$ *is nilpotent.*
Thus, there exists $n_i > 1$ *such that* $(\phi(s)|_{U_i})^{n_i} = 0$*.*
And there exists N *such that* $\phi(s)^N = \phi(s^N) = 0$ *in* $\mathcal{O}_Z(Z)$*.*
Thus $s^N = 0$ *as* ϕ *is injective by hypothesis, thus* $V(s) = X$*.*

Claim 3.

$i^\sharp$ *is an isomorphism.*
Since $Z \to X$ *is a closed subscheme,* $i^\sharp$ *is surjective on stalks and thus surjective.*
To show injectivity, it suffices to show that $\forall a \in A$ *such that* $i_p^\sharp(\frac{a}{1}) = 0 \implies \frac{a}{1} = 0 \in A_p$*.*
Since $i_p^\sharp(\frac{a}{1}) = 0$*,* $\exists p \in U \subset Z$ *open affine such that* $\phi(a)|_U = 0$*.*
Choose a finite open affine cover, $Z = U \cup \bigcup_{i=1}^n U_i$*.*
Choose $s \in A$ *such that* $p \in D(s) \subset U \subset X$*.*
Then $\phi(sa)|_U = 0$*, thus* $\phi(sa)|_{D(s) \cap U_i} = 0 = \phi(sa)|_{D(\phi(s)|_{U_i})}$*.*
Thus, there exists $N > 0$ *such that* $\phi(sa)|_{U_i} \cdot (\phi(s)|_{U_i})^N = 0 \in \mathcal{O}_Z(U_i)$*.*
Thus $\phi(s^{N+1}a)|_{U_i} = 0$*, thus* $\phi(s^{N+1}a) = 0 \implies s^{N+1}a = 0 \in A$ *which implies* $\frac{a}{1} = 0$ *in* A_p*.*

Lecture 8: Fiber Products

Fri 04 Nov

Corollary 75
Closed subschemes of affine schemes are affine.
Moreover, if $\phi : A \to B$ *is a morphism of rings, then* ϕ *is surjective iff* $\operatorname{Spec} \phi$ *is a closed immersion.*

Remark
For all closed subsets $Z \subset X$ of a scheme X, there is an ideal sheaf $\mathcal{I}$ on X making Z into a closed subscheme.
To prove this, if $U \subset X$ is affine, then $Z \cap U$ is closed in U hence $Z \cap U = V(I)$ for some ideal in $\mathcal{O}_X(U)$.
Then you take radicals and glue them together on a cover of Z.
This structure is called the reduced induced scheme structure on Z.

3.1 Fiber Products

Definition 35 (Fiber product)
Let C be a category, given two morphism $\pi_X : X \to S$ and $\pi_Y : Y \to S$, the fiber product $X \times_S Y$ of X and Y over S is an object together with morphisms p_x to X and p_y to Y which is universal.

Remark
Alternatives names sometimes are fibre product, fibered produc or pullback.
Fiber products are unique up to unique isomorphism.
If S is terminal in C, then the fiber product is just the product.

Remark
If a square is a fiber product, we call the diagram cartesian.

Lemma 79
Assume all fiber products exist.
Let "commutative thingy".
Then

$$(X_1 \times_{S_1} Y_1) \times_{X_0 \times_{S_0} Y_0} (X_2 \times_{S_2} Y_2) = (X_1 \times_{X_0} X_2) \times_{S_1 \times_{S_0} S_2} (Y_1 \times_{Y_0} Y_2)$$

Corollary 80
- *If C admits fiber products, then $X \times_S Y = Y \times_S X$*
- *A composition of two pullback squares is a pullback*
- *For a zigzag $X \to S, Y \to S, Y \to T, Y \to Z$,*

$$(X \times_S Y) \times_T Z = X \times_S (Y \times_T Z)$$

- *For maps $X \to S \to T$ and $Y \to S$*

$$X \times_S Y \to X \times_T Y \to S \times_T S$$

  and

$$X \times_S Y \to S \to S \times_T S$$

is a pullback.

Example

1. *If* $\pi_X : X \to S, \pi_Y : Y \to S$ *are in* (Set), *then* $X \times_S Y = \{(x, y) | \pi_X(x) = \pi_Y(y)\} \subset X \times Y$ *together with the two projections.*
2. *If* X *and* Y *are groups (or rings) and* π_X, π_Y *are homomorphisms as above, then* $X \times_S Y$ *is, as a set, the fiber product of the underlying sets, with the obvious groups (resp. ring) structures.*

Goal for today

Theorem 82 (Fiber products of schemes exist)
Fiber products exist in (Sch) *and also in* (Sch/S)

Why do we care ?

Allows us to talk about fibers, graphs, diagonals...
Recall that every point $y \in Y$ of a scheme Y has a natrual scheme structure given by the residue field $\mathcal{O}_{Y,y} / \mathfrak{m}_y = k(y)$

Definition 36 (Fibers)
Let $f : X \to Y$ *be a morphism of schemes over* S.

1. *For any* $y \in Y$, *let* $k(y)$ *be the residue field, then the fiber of* f *over* y
$$f^{-1}(y) = X_y = X \times_Y \operatorname{Spec} k(y)$$
2. *The geometric fiber of* f *over* Y *is*
$$X_{\overline{y}} = X \times_Y \operatorname{Spec} \overline{k(y)}$$
3. *a closed fiber is a fiber over a closed point*
4. *For all integral schemes* Y, *there is a unique point* $\eta \in Y$ *such that* $\overline{\{\eta\}} = Y/$
This is called the generic point of Y.
The fiber over the generic point is called the generic fiber of f.
5. *The morphism*
$$\Gamma_f := (\mathrm{Id}, f) : X \to X \times_S Y$$
is called the graph of f.

6. *The morphism*

$$\Delta_{X/Y} = \Gamma_{\mathrm{Id}_X} : X \to X \times_Y X$$

is called the diagonal of X over Y.

Proposition 83

If $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, S = \operatorname{Spec} C$ and $\pi_X : X \to S, \pi_Y : Y \to S$ are morphisms of schemes then $X \times_S Y$ exists in (Sch) and is given by $\operatorname{Spec}(A \otimes_C B)$ together with the maps induced by the natural maps $A \to A \otimes_C B, B \to A \otimes_C B$

Proposition 84

We use the universal property of $A \otimes_C B$ and the equivalence of categories to show that it is a pullback in the category of affine schemes.
For Z a scheme, there is a map from the affinization of Z to $\operatorname{Spec} B$ and $\operatorname{Spec} A$ which then induce a map $aff Z \to \operatorname{Spec} A \otimes_C B$.

Example

1. *If $X = Y = \mathbb{A}^1_k$, the fiber product over $\operatorname{Spec} k$, then $X \times_{\operatorname{Spec} K} Y = \mathbb{A}^2_k$.*
2. *If $X - Y = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$, then*

$$X \times_S Y = \operatorname{Spec} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Spec} {\mathbb{C}[x]}\big/{(x^2+1)} = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$$

   Note that $X \times_S Y$ has two points but X, Y, S each have only one.
3. *Take $X = \operatorname{Spec} k[x,y,z]/(z^2 - x), Y = \operatorname{Spec} k[z]$, let $f : X \to Y$ induced by mapping $z \to z$.*
   Let $\lambda \in Y$ be the point corresponding to $(z - \lambda)$, then

$$f^{-1}(\lambda) = \operatorname{Spec} k[x,y,z]/(z^2 - xy) \otimes_{k[z]} k[z]/(z - \lambda) = \operatorname{Spec} k[x,y]/(\lambda^2 - xy)$$

$$= \begin{cases} \mathbb{A}^1 \setminus 0 \text{ if } x = 0 \\ V(xy) \subset \mathbb{A}^2 \text{ if } \lambda = 0 \end{cases}$$

4. *If we take a non-rational point in the above example, say $\lambda = (z^2 + 1)$, then*

$$f^{-1}(y) = \operatorname{Spec} \mathbb{R}[x,y,z]/(-1 - xy, z^2 + 1) = \operatorname{Spec} \mathbb{C}[x,y]/(-1 - xy)$$

We now start the proof that fiber products exist.

Proof

Claim 1

If $X \times_S Y$ exists and $U \subset X$ open, then the open subscheme $p_x^{-1}(U) \subset X \times_S Y$ is a fiber product of U and Y over S.
For Z a scheme and two commuting maps $Z \to X \times_S Y$ and $Z \to U$, there

is a map on the topological level $Z \to p_X^{-1}(U)$ and it also exists on the level of schemes.

Claim 2

If $X = \bigcup U_i$ is an open cover such that $U_i \times_S Y$ exists $\forall i \in I$, then $X \times_S Y$ exists.
We postpone the proof of this until monday.

Now, if S is affine, consider the open affine covers $X = \bigcup_i U_i, Y = \bigcup V_i$, then $U_i \times_S V_j$ exists $\forall i, j$, thus $U_i \times_S Y$ exists $\forall i$ and thus $X \times_S Y$ exists.

If S is not affine, let $S = \bigcup_i W_i$ be an open affine cover.
Set $U_i = \pi_X^{-1}(W_i), V_i = \pi_Y^{-1}(W_i)$.
Now, $U_i \times_{W_i} V_i$ exists and now $U_i \times_{W_i} V_i = U_i \times_S Y$ by one of the identities. □

Lecture 9: fiber products exist

Mon 07 Nov

We finish the proof by showing that if $X = \bigcup_{i \in I} U_i$ is an open cover such that each $U_i \times_S Y$ exists, then $X \times_S Y$ exists.

Proof
We know that $p_{U_i}^{-1}(U_i \cap U_j) \simeq (U_i \cap U_j) \times_S Y \simeq p_{U_j}^{-1}(U_i \cap U_j)$ via unique isomorphisms compatible with the projections.
There is a unique scheme T with maps to Y and X such that $p_X^{-1}(U_i) \simeq U_i \times_S Y$.
We claim that T is $X \times_S Y$.
Let Z be a scheme with morphisms $Z \xrightarrow{f_X} X$ and $Z \xrightarrow{f_y} Y$ which commutes with projections to S.
Let $V_i = f_X^{-1}(U_i)$, we get unique morphisms $V_i \to U_i \times_S Y \to T$ which is unique if $p_X \circ f_i = p_Y \circ f_i$.
By claim 1, f_i and f_j coincide on $U_i \cap V_j$ thus they glue to a unique morphism $f : Z \to T$ □

Corollary 86
Let $\pi_X : X \to S, \pi_Y : Y \to S$ be a diagram of schemes, let $S = \bigcup_i W_i, U_i = \pi_X^{-1}(W_i), V_i = \pi_y^{-1}(W_i)$ and $U_i = \bigcup_j U_{ij}, V_i = \bigcup_j V_{ij}$ be open covers.
Then $X \times_S Y = \bigcup_{i \in I} \bigcup_{j,k} U_{ij} \times_{W_i} V_{ik}$ is an open cover.

Proposition 87
Let $f : X \to Y$ be a morphism of schemes.
Then for every $y \in Y$, the map $g : f^{-1}(y) \to X$ is a homeomorphism onto the

set-theoretic fiber $f_{set}^{-1}(y)$*.*

Proof

Without loss of generality, Y *is affine.*
We can also assume that X *is affine, because if* $X = \bigcup_i U_i$ *is an open cover and each* $g|_{g^{-1}(U_i)} : g^{-1}(U_i) \to U_i$ *is a homeomorphism onto* $f_{set}^{-1}(y) \cap U_i$*, then* g *is a homeomorphism onto* $f_{set}^{-1}(y)$*.*

So let $X = \operatorname{Spec} B, Y = \operatorname{Spec} B, y = p \in \operatorname{Spec} A$*, then we claim that* $B \otimes_A k(y) \simeq {S^{-1}B}\big/{pS^{-1}B}$*.*
Furthermore, the isomorphism is compatible with the maps from B *and* $k(y) = {A_p}\big/{pA_p}$ *where* $S = \operatorname{Im}(A \setminus p \to B)$*.*
To prove this, we check that ${S^{-1}B}\big/{pS^{-1}B}$ *satisfies the universal property of* $B \otimes_A k(y)$*.*
Let C *be a ring with morphisms* ${A_p}\big/{pA_p} \xrightarrow{f_A} C$ *and* $B \xrightarrow{f_B} C$ *compatible with the morphisms from* A π_A, π_B*.*
Notice that $\pi_A(A \setminus p) \subset ({A_p}\big/{pA_p})^\times$ *and sends* $\pi_A(p) = 0$*.*
Thus $f_B(S) \subset C^\times, f_B(pB) = 0$*.*
Thus there exists a unique $f : {S^{-1}B}\big/{pS^{-1}B} \to C$ *such that* $f \circ p_B = f_B$*.*
Thus

$$f \circ \pi_A = f \circ p_B \circ \pi_B = f_A \circ \pi_A$$

As π_A *is an epimorphism,* $f \circ p_A = f_A$*.*

We now have to check $\operatorname{Spec} {S^{-1}B}\big/{pS^{-1}B} \to \operatorname{Spec} B$ *is a homeomorphism onto* $f_{set}^{-1}(y)$*.*

We know it's a homeomorphism onto it's image by general commutative algebra.
THe image is $\left\{q \in \operatorname{Spec} B | S \cap q = \emptyset, pS^{-1}B \subset qS^{-1}B\right\}$*.*
But this is just the set-theoretic fiber. $\square$

4 Properties of Morphisms

4.1 Properties of properties of morphisms

Remark

If $f : X \to Y$ *and* $g : Y; \to Y$ *are morphisms of schemes, let* $X_{Y'} = X \times_Y Y'$*. Then we call* $f_{y'} : X_{Y'} \to Y'$ *the base change of* f *along* g*.*

Remark

In the following, whenever we say P *is a property of morphisms of schemes, we assume that* P *is satisfied by isomorphisms.*

Definition 37
Let P be a property of morphisms of schemes, we say that P satisfies

1. *(COMP) : P is stable under composition*
2. *(CANC) : if $g \circ f$ satisfies P, then f does*
3. *(BC) : if it is stable under base change that is $\forall f : X \to Y, g : Y' \to Y$ such that f satisfies P, also $f_{Y'}$ satisfies P.*
4. *(LOCT) : If it local on the target, ie. if $\forall f : X \to Y$ and $\forall$ open covers $Y = \bigcup V_I$ f satisfies $P \iff f_{V_i}$ satisfies $P \forall i \in I$*
5. *(LOCS) : If it is local on the source ie. if $\forall f : X \to Y$ and $\forall$ open covers $X = \bigcup_i U_i$ f satisfies $P \iff f|_{U_i}$ satisfies P.*

Definition 38
Let P be a property of morphisms of schemes, then a morphism $f : X \to Y$ is called universally P if $\forall Y' \to Y, f_{Y'}$ satisfies P.

Lemma 90
Let $f : X \to Y$ be a morphism of schemes over S, then the diagram $X \xrightarrow{\Gamma_f} X \times_S Y$ over $Y \to Y \times_S Y$ is cartesian.

Proof
This is a special case of the "magic square" with the isomorphism $X \to X \times_Y Y$ □

Definition 39
Let P be a property of morphisms of schemes, then we say that $f : X \to Y$ satisfies Δ_P if $\Delta_{X/Y}$ satisfies P.

Lemma 91
The following hold

1. *If P satisfies (BC), then Δ_p satisfies (BC)*
2. *If P satisfies (BC) and (COMP), then Δ_P satisfies (COMP).*
3. *If P satisfies (LOCT), then Δ_P satisfies (LOCT)*
4. *If P satisfies (BC) ,(COMP) and $f, g : X \to Z$ satisfy P as morphisms of schemes over S and $X \to S, Z \to S$ satisfy Δ_p, then $(f, g) : X \to Y \times_S Z$ satisfy P*

Lemma 92
Let P be a property of morphisms of schemes.
Assume that P satisfies stability under base change and composition.
Let $f : X \to Y, g : X' \to Y'$ be morphisms of schemes over S satisfying P, then the product $f \times_S g$ satisfies P.

Proof
There are maps

$$X \times_S X' \xrightarrow{\mathrm{Id} \times_S g} X \times_S Y' \xrightarrow{f \times_S \mathrm{Id}} Y \times_S Y'$$

which compose to $f \times_S g$.
But both maps are base changes and thus $f \times_S g$ satisfy P. $\square$

Theorem 93 (Cancellation Theorem)
Let P be a property of morphisms of schemes and P satisfies stability under composition and base change.
Let $f : X \to Y, g : Y \to Z$ two morphisms such that $g \circ f$ satisfies P and $\Delta_{Y/Z}$ satisfies P then f satisfies P.

Proof
Write f as the composition

$$X \xrightarrow{(\mathrm{Id}, f)} X \times_Z Y \to Y$$ $\square$

But (Id, f) is a base change of the diagonal and p_y is a base change of $g \circ f$.

4.2 Topological properties

Definition 40
A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes

1. *is injective, surjective, bijective if f is.*
2. *is open, resp. closed if f is.*
3. *quasi-compact if $f^{-1}(V)$ is quasi-compact for all open quasi-compact $V \subset Y$.*
4. *quasi-separated if $\Delta_{X/Y}$ is quasi-compact.*
5. *has finite fibers if $f^{-1}(y)$ is finite as a set.*

Lecture 10: geometric meaning of separated and proper morphisms

Mon 14 Nov

5 Valuative Criteria

Definition 41 (Specializations)
Let X be a topological space

1. *$x, x' \in X$. If $x' \in \overline{\{x\}}$ we say that x specializes to x' (or x' is a specialization) or x' generalizes to x.*
2. *A subset $V \subset X$ is called closed under specialization if it contains all the specializations of all it's points.*

Remark
Closed subsets are closed under specialization (the converse is not true in general)

Definition 42 (Relative specialization)
Let $f : X \to Y$ be a continuous map.
We say that specializations lift along f if $\forall x \in X$ and any specialization y of $f(x)$ in Y, there exists $x' \in X$ mapping to the specialization such that x' specializes to x.

Remark
Both concepts are stable under composition and local on the target.

Example
Let K be an alg. closed field, let X be the affine line with two origins over K.

1. *Look at the morphism $f : \mathbb{A}^1 \to X$ mapping to the upper origin.*
   The generic point corresponding to the upper point of X lifts along f, but the other generic point does not.
2. *The map $g : X \to \mathbb{A}^1$ lifts the generic point of $\mathbb{A}^1$ non-uniquely.*

Lemma 97
Let $f : X \to Y$ be a quasi-compact morphism. Then f is closed if and only if specializations lift along f.

Proof
If f is closed, let $x \in X$ and $f(x) \rightsquigarrow y'$.
Then $y' \in \overline{\{f(x)\}} = \overline{\left\{f(\overline{\{x\}})\right\}} = f(\{\overline{x}\})$.
So $\exists x' \in \overline{\{x\}}$ such that $f(x') = y'$.

Conversely, assume specializations lift.
Let $Z \subset X$ be closed, equipped with the reduced induced scheme structure.
As $\operatorname{Im}(X \to Y) = \operatorname{Im}(Z \to X \to Y)$, we may assume wlog that $Z = X$.
Then $f(X)$ is stable under specialization.
We have to show $f(X)$ is closed.
Closed, qc, liftability of specializations are LOCT so wlog Y is affine.
Since f is qc. $\exists$ a surjection $\coprod \operatorname{Spec} A_i \to X$ where the disjoint union is affine.
But $\coprod \operatorname{Spec} A_i \simeq \operatorname{Spec}(\prod A_i)$, so we can assume that X is affine.
We have reduced to a commutative algebra claim :
Let $\phi : A \to B$ be a morphism of rings. If the image T of $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed under specializations, then it is closed.
Wlog, ϕ is injective.
Recall $p \in \operatorname{Spec} A$ is called minimal if it is minimal wrt inclusion.
Every $p' \in \operatorname{Spec} A$ is a specialization of a minimal prime.
So it suffices to show that T contains all minimal primes of A.
Let $p \in \operatorname{Spec} A$ be a minimal prime, consider the fiber product $\operatorname{Spec} B_p = \operatorname{Spec} A_p \times_{\operatorname{Spec} A} \operatorname{Spec} B$.
Since localiztion is exact, if $A \to B$ is an inclusion, $A_p \to B_p$ is too, so p has a preimage in B.
Hence $T = \operatorname{Spec} A$ $\square$

Definition 43 (Valuation Rings)
Let A be a local integral domain with maximal ideal m_A contained in a field K

1. *If $B \subset K$ is a local ring with maximal ideal m_B, we say that B dominates A if $A \subset B$ and $m_B \cap A = m_A$*
2. *The ring A is a valuation ring if it is maximal w.r.t. domination*
3. *The ring A is called discrete valuation ring if it is a Noetherian valuation ring.*

Lemma 98
Let $A \subset K$ be a local subring, then A is dominated by a valuation ring with fraction field K.

Proof
Apply Zorn's lemma to the appropriate set $M = \{A_k \subset K, A_i \text{ local }, A_i \text{ dominated by } A\}$.
Let $\{A_i\}_{i \in I}$ be a totally ordered subset of M.
Let $B = \bigcup_i A_i, m_B = \bigcup_i m_{A_i}$.

B is a ring, $m_B \subset B$ is an ideal with $m_B \cap A_i = m_{A_i}$.
$b \in B$ is a unit $\iff \exists c \in B$ such that $bc = 1 \iff \exists i \in I, bc = 1 \in A_i \iff \exists i$ such that $b \in A_i \setminus m_{A_i} \iff b \notin m_B$.
$B \in M$ and dominates all A_i.
So M has a maximal element. $\square$

Lemma 99
Let X be a scheme and let $x, x' \in X$ with $x \to x'$.
Then there exists a valuation ring A and a morphism $f : \operatorname{Spec} A \to X$ such that $f(\eta_A), f(m_A) = x'$.
More precisely, given a field extension $g : k(x) \to K$, we may assume A has fraction field K.

Proof
Since $x' \in \overline{\{x\}}$, we have $\mathcal{O}_{X,x'} \to k(x) \xrightarrow{g} K$.
By the lemma above, there is a valuation ring A with fraction field K dominating $\phi(\mathcal{O}_{X,x'})$.
This map induces $\operatorname{Spec} A \to X$ with the desired properties. $\square$

Definition 44
Let $f : X \to Y$ be a morphism of schemes.
We say that f satisfies the existence part of the valuative criterion if for every commutative diagram of solid arrows $\operatorname{Spec} A \to Y \leftarrow X$ commuting with $A \leftarrow \operatorname{Spec} K \to X$ where A is a valuation ring with field of fractions K, there is a lift $\operatorname{Spec} A \to X$ making the diagram commute.
f satisfies the uniquenes part of the valuative criterion if there is at most one such map.

Theorem 100 (Valuative criterion)
Lt $f : X \to Y$ be a morphism of schemes, TFAE

1. *Specializations lift along any base change*
2. *f satisfies the existence part of the valuative criterion.*

Proof
$1 \implies 2$
Wlog $Y = \operatorname{Spec} A$.
Let $x \in X$ be the image of g.
As specializations lift, $\exists x' \in X$ such that $x \to x'$ and $f(x') = m_A$.
We get a diagram of rings $A \to K \xleftarrow{g^\sharp_{x'}} \mathcal{O}_{X,x'} \xleftarrow{f^\sharp_{x'}} A$ commuting with $A \to K$.
Since $f^\sharp_{x'}$ is local, $g^\sharp_{x'}(\mathcal{O}_{X,x'})$ dominates A, hence $g^\sharp_{x'}(\mathcal{O}_{X,x'}) = A$.

We get a dotted arrow $\mathcal{O}_{X,x'} \to A$.
$2 \implies 1$
Let $Y' \to Y$ be any morphism, then there is a lift $\operatorname{Spec} A \to X_{Y'}$ *by universal property of the fiber product.*
So it suffcies to show that specializations lift along f.
Let $x \in X$ and $f(x) \to y'$.
Let $K = k(x)$ and consider $g : k(f(x)) \to k(x)$ induced by f.
By the lemma, we get a lift $\operatorname{Spec} A \to X$ *and the image of m_A precisely is the preimage of the specialization.* □

Now

1. If $f : X \to Y$ is qc.
   f universally closed $\iff$ f satisfies existence in valuative criterion
2. $f : X \to Y$ is qs
   f is separated $\iff$ f satisfies uniqueness in the valuative criterion
3. $f : X \to Y$ is qs and of finite type
   f is proper $\iff$ f satisfes existence and uniqueness in valuative criterion.

Lecture 11: quasi-coherent sheaves

Fri 18 Nov

6 Quasi-coherent sheaves

Recall that affine schemes are equivalent to rings, what about modules?

Definition 45
Let $(X, \mathcal{O}_X)$ be a ringed space

1. *A presheaf of $\mathcal{O}_X$ modules is a presheaf $\mathcal{F}$ on X such that*
   (a) *$\forall U \subset X$ open, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module*
   (b) *$\forall V \subset U \subset X$ open, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is an $\mathcal{O}_X(U)$ module homeomorphism, where $\mathcal{F}(V)$ is considered a $\mathcal{O}_X(U)$ module via the restriction map.*
2. *A morphism of presheaves of $\mathcal{O}_X$-modules is a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ such that $\forall U \subset X$ open, $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.*
3. *For two sheaves of $\mathcal{O}_X$-modules, the presheaf* $\hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ *is an $\mathcal{O}_X$-module, called the hom-sheaf.*
4. *The category of $\mathcal{O}_X$-modules is denoted by $Mod(X, \mathcal{O}_X)$.*

Remark
There is a forgetful functor $Mod(X, \mathcal{O}_X) \to Sh(X)$.

The sheafification of a presheaf of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$ module.
$Mod(X, \mathcal{O}_X)$ is an abelian category

Definition 46
Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

If $A \to B$ is a ring homomorphism, there is a restriction of scalars functor ${}_A- : Mod(B) \to Mod(A)$ and extension of scalars $- \otimes_A B : ModA \to Mod(B)$, we can do the same for morphisms of sheaves of rings.

Definition 47
Let $f : X \to Y$ be a morphism of ringed spaces

1. *The direct image functor $f_* : Mod(X, \mathcal{O}_X) \to Mod(Y, \mathcal{O}_Y)$ is the composition $Mod(X, \mathcal{O}_X) \xrightarrow{f_*} Mod(Y, f_*\mathcal{O}_X) \xrightarrow{\mathcal{O}_Y -} Mod(Y, \mathcal{O}_Y)$.*
2. *The inverse image functor*

$$f^* : Mod(Y, \mathcal{O}_Y) \to Mod(X, \mathcal{O}_X)$$

is the composition $Mod(Y, \mathcal{O}_Y) \xrightarrow{f^{-1}} Mod(X, f^{-1}\mathcal{O}_Y) \xrightarrow{- \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X} Mod(X, \mathcal{O}_X)$.

Definition 48
Let $(X, \mathcal{O}_X)$ be a ringed space.
A sheaf of $\mathcal{O}_X$-modules is

1. *free if $\mathcal{F} = \mathcal{O}_X^{\oplus I}$ for some set I*
2. *locally free if $\exists$ an open cover $\bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ is free.*
3. *locally free of finite rank if $\exists$ an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r_i}$ for all $i \in I$ and some $r_i \in \mathbb{N}$*
4. *invertible (or line bundle) if its locally free of rank 1.*

Definition 49
Let $(X, \mathcal{O}_X)$ be a ringed space.
A sheaf of $\mathcal{O}_X$-modules is called

1. *quasi-coherent if there exists an open cover $X = \bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ appears in an exact sequence*

$$\mathcal{O}_{U_i}^{\oplus I_i} \to \mathcal{O}_{U_i}^{\oplus J_i} \to \mathcal{F}|_{U_i} \to 0$$

2. *coherent if*
   — *there exists an open cover* $X = \bigcup U_i$ *and* $n_i \in \mathbb{N}$ *and a surjection*
   $$\mathcal{O}|_{U_i}^{\oplus n_i} \to \mathcal{F}|_{U_i} \to 0$$
   — $\forall U \subset X$ *open,* $n \in \mathbb{N}$ *and all surjections*
   $$\phi : \mathcal{O}_U^{\oplus n} \to \mathcal{F}|_U$$
   $\ker \phi$ *satisfies a.*

We get categories $Coh(X, \mathcal{O}_X)$ and $QCoh(X, \mathcal{O}_X)$ of coherent and quasi-coherent sheaves.
In general, $QCoh$ is not abelian, though for schemes it is.

Definition 50
Let $X = \operatorname{Spec} A$ *be an affine scheme.*
Let M *be an* A*-module.*
The sheaf of $\mathcal{O}_X$*-modules associated to* M *is* $\tilde{M} : U \mapsto \{s : U \to \coprod M_p | s$ *satisfies i) and ii)* $\}$

1. $\forall p \in U, s(p) \in M_p$
2. $\forall p \in U \exists m \in M, a \in A$ *and* $V \subset U$ *open such that*
$$p \in V \subset D(a) \text{ such that } s(q) = \frac{m}{a} \in M_q \forall q \in V$$

As in the affine case, we get that

Proposition 102
Let $X = \operatorname{Spec} A$ *and* M *an* A*-module*

1. $\forall a \in A$*, there exist natural isomorphisms*
$$\phi_A : M_a \to \tilde{M}(D(a))$$
   and these isomorphisms commute with the localization maps $M_a \to M_b$*.*
2. $\forall p \in \operatorname{Spec} A$*, there exist natural isomorphisms* $\phi_p : M_p \to \tilde{M}_p$ *which are also natural wrt the maps* $M_a \to M_p \; \forall a \in A \setminus p$

We get a functor $A - mod \to Mod(X, \mathcal{O}_X)$ if $X = \operatorname{Spec} A$ where the maps are induced by postcomposition with the obvious map $\coprod M_p \to \coprod N_p$.

Lemma 103
Let $X = \operatorname{Spec} A \forall A$*-modules* M, N*, the maps* $\hom(M, N) \leftrightarrow \hom(\tilde{M}, \tilde{N})$ *sending* $\phi : M \to N$ *to* $\tilde{\phi}$ *and* $f : \tilde{M} \to \tilde{N}$ *to* $f(X)$ *are mutually inverse.*

The proof of this will follow from an exercise.

Proof

Let $X = \operatorname{Spec} A$ be an affine scheme

1. *A sequence of modules $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact iff the associated sequence of sheaves is exact*
2. *If $\phi : M \to N$ is a morphism of A-modules, then $\widetilde{\ker \phi} = \ker(\tilde{\phi})$ and similarly for the cokernel.*
3. *If $\{M_i\}$ is a family of A-modules, then $\bigoplus \tilde{M}_i = \widetilde{\bigoplus M_i}$.*
4. *If $\{M_i\}, \{\phi_{ij}\}$ is a directed system, then direct limits commute with $\tilde{\cdot}$*

$\square$

Theorem 104 (Criterion for quasi-coherence)
Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$ module, then the following are equivalent

1. *$\forall \operatorname{Spec} A \subset X$ open affine, $\mathcal{F}|_{\operatorname{Spec} A} \simeq \tilde{M}$ for some A-module M.*
2. *The same for some cover*
3. *$\mathcal{F}$ is quasi-coherent*
4. *$\forall \operatorname{Spec} A = U \subset X$ open affine and every $a \in A$, the natural map $F(U)_a \to \mathcal{F}(D(a))$ is an isomorphism.*

Lecture 12: criterion for quasi-coherence and graded rings

Mon 21 Nov

Proof

$1 \implies 2$ *is clear.*
$2 \implies 3$ *Choose a presentation $A_i^{\oplus I_i} \to A_i^{\oplus J_i} \to M_i \to 0$.*
Taking $\tilde{\cdot}$, this is the same as $\mathcal{O}_{U_i}^{\oplus I_i} \to \mathcal{O}_{U_i}^{\oplus J_i} \to \widetilde{M_i} \to 0$.
This is exact as $\tilde{\cdot}$ preserves exact sequences.
$4 \implies 1$ *Write $U = \operatorname{Spec} A$ and $M = \mathcal{F}(U)$.*
We have compatible isomorphisms $M_a \simeq \mathcal{F}(D(a)) \forall a \in A$.
By some lemma, these glue to an isomorphism $\widetilde{M} \simeq \mathcal{F}|_U$.
$3 \implies 4$ *Wlog $X = \operatorname{Spec} A$, so X is qc.*
Choose finitely many b_i such that $X = \bigcup_{i=1}^n D(b_i)$ and exact sequences

$$\widetilde{A_{b_i}}^{\oplus I_i} \to \widetilde{A_{b_i}}^{\oplus J_i} \to \mathcal{F}|_{D(b_i)}$$

call the first map $\tilde{\phi}$.
There exists $\phi : A_{b_i}^{\oplus I_i} \to A_{b_i}^{\oplus J_i}$ inducing $\tilde{\phi}$.
Thus $\mathcal{F}|_{D(b_i)} = \operatorname{coker}(\tilde{\phi}) = \widetilde{\operatorname{coker} \phi}$.
Let $a \in A$.

Since $\mathcal{F}$ is a sheaf, we have a map of exact sequences from

$$0 \to \mathcal{F}(X) \to \prod_{i=1}^{n} \mathcal{F}(D(b_i)) \to \prod_{i,j} \mathcal{F}(D(b_i b_j))$$

to

$$0 \to \mathcal{F}(X)_a \to \prod_{i=1}^{n} \mathcal{F}(D(b_i))_a \to \prod_{i,j} \mathcal{F}(D(b_i b_j))$$

to

$$0 \to \mathcal{F}(D(a)) \to \prod_{i} \mathcal{F}(D(ab_i)) \to \prod_{i,j} \mathcal{F}(D(ab_i b_j))$$

Since $\mathcal{F}|_{D(b_i)}$ is $\widetilde{M}$ for some A_{b_i}-module, M, the map $\prod \mathcal{F}(D(b_i))_a \to \prod_i \mathcal{F}(D(ab_i))$ is an isomorphism.
Similarly, the map $\prod_{i,j} \mathcal{F}(D(b_i b_j))_a \to \prod_{i,j} \mathcal{F}(D(ab_i b_j))$ is an isomorphism.
By the five-lemma, the map $\mathcal{F}(X)_a \to \mathcal{F}(D(a))$ is an isomorphism. □

Corollary 105
Let $X = \operatorname{Spec} A$.
Then $\widetilde{-} : A - mod \to Mod(X, \mathcal{O}_X)$ induces an equivalence between $A - mod$ and $QCoh(X, \mathcal{O}_X)$

Remark
Let $X = \operatorname{Spec} A$

1. *If $\widetilde{M} \in Coh(X, \mathcal{O}_X)$, then M is finitely generated.*

2. *If A is Noetherian and M is a f.g. M-module, then $\widetilde{M} \in Coh(X, \mathcal{O}_X)$.*

Moreover, if X is a locally Noetherian schemen and $\mathcal{F} \in Mod(X, \mathcal{O}_X)$, then the following are equivalent

$\mathcal{F} \in Coh(X, \mathcal{O}_X) \iff \forall U \subset X$ open affine there is a f.g. A-module M such that $\widetilde{M} \simeq \mathcal{F}|_U$

Corollary 107
If $(X, \mathcal{O}_X)$ is a scheme, then $QCoh(X, \mathcal{O}_X)$ is abelian

Proof (Sketch)
If X is affine, this is clear using the equivalence of categories.
If not, we need to check that kernels,cokernels, finite direct sums of qcoh sheaves are qcoh.
As all these conditions are local, wee conclude. □

Proposition 108
Let $\phi : A \to B$ be a morphism of rings and let $f : X = \operatorname{Spec} B \to \operatorname{Spec} A = Y$ be the induced map on spectra.

Let $\mathcal{F} = \widetilde{M} \in QCoh(X, \mathcal{O}_X), \mathcal{G} = \widetilde{N} \in QCoh(Y, \mathcal{O}_Y)$, then there is a natural isomorphism

$$f_*\widetilde{M} = \widetilde{({}_A M)}$$

and

$$f^*\widetilde{N} = \widetilde{(N \otimes_A B)}$$

In particular, $f_*\mathcal{F}$ is qcoh and $f^*\mathcal{G}$ is too.

Proof

Take $a \in A$, we have

$$\begin{aligned}(f_*\tilde{M})(D(a)) &= \widetilde{M}(D(\phi(a))) \\ &= M_{\phi(A)} \\ &= ({}_A M)_a \\ &= \widetilde{({}_A M)}(D(a))\end{aligned}$$

These morphisms glue to a natural isomorphism $f_*\widetilde{M} \simeq \widetilde{({}_A M)}$.

We have a chain of natural isomorphisms $\hom_{\mathcal{O}_X}(f^*\widetilde{N}, \mathcal{F}) \simeq \hom_{\mathcal{O}_Y}(\widetilde{N}, f_*\mathcal{F}) \simeq \hom_{\mathcal{O}_Y Y}(\widetilde{N}, \widetilde{({}_A M)}) \simeq \hom_A(N, {}_A M) \simeq \hom_B(N \otimes_A B, M) \simeq \hom_{\mathcal{O}_X}(\widetilde{N \otimes_A B}, \mathcal{F})$.

By the Yoneda lemma, this implies that $f^*\widetilde{N} \simeq \widetilde{N \otimes_A B}$. $\square$

Proposition 109

Let $f : X \to Y$ be a morphism of schemes.

Then,

1. $\forall \mathcal{G} \in QCoh(Y, \mathcal{O}_Y), f^*\mathcal{G} \in QCoh(X, \mathcal{O}_X)$
2. If X is locally noetherian, then $\forall \mathcal{G} \in Coh(Y, \mathcal{O}_Y), f^*\mathcal{G} \in Coh(X, \mathcal{O}_X)$
3. If f is qc. and qs. $\forall \mathcal{F} \in QCoh(X, \mathcal{O}_X), f_*\mathcal{F} \in QCoh(Y, \mathcal{O}_Y)$
4. If f is proper and Y is locally Noetherian, then $\forall F \in Coh(X, \mathcal{O}_X), f_*\mathcal{F} \in Coh(Y, \mathcal{O}_Y)$

Proof

Wlog. $Y = \operatorname{Spec} A$ affine since quasi-coherence is local.

For 1 and 2, cover X by open affines and assume wlog that X is affine.

Then $f^*\widetilde{M} = \widetilde{(M \otimes_A B)}$.

Since f is qc., $X = \bigcup_{i=1}^n U_i$ with $U_i = \operatorname{Spec} B_i$.

Since f is qs, we can write $U_i \cap U_j = \bigcup_k U_{ijk}$ with $U_{ijk} = \operatorname{Spec} B_{ijk}$.

Since $\mathcal{F}$ is a sheaf, we have an exact sequence

$$0 \to f_*\mathcal{F} \to \prod_{i=1}^n (f|_{U_i})_*(\mathcal{F}|_{U_i}) \to \prod_{ij}\prod_k (f|_{U_{ijk}})_*(\mathcal{F}|_{U_{ijk}})$$

This is exact as for every $V \subset Y$ open, giving $s \in f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ which is the same as giving $s_i \in \mathcal{F}(f^{-1}(V) \cap U_i) = \mathcal{F}(f|_{U_i}^{-1}(V)) =$

$((f|_{U_i})_*(\mathcal{F}|_{U_i}))(V)$ *such that the* s_i *agree on intersections.* $\square$

Proposition 110
Let $(X, \mathcal{O}_X)$ *be a scheme and* $\mathcal{I} \subset \mathcal{O}_X$ *a sheaf of ideals.*
Let $Z = \operatorname{Supp}(\mathcal{O}_X/\mathcal{I}) = \{x \in X | (\mathcal{O}_X/\mathcal{I})_x \neq 0\}$ *and* $\mathcal{O}_Z = (\mathcal{O}_X/I)|_Z$.
Then $(Z, \mathcal{O}_Z)$ *is a scheme iff* $\mathcal{I}$ *is quasi-coherent.*
Z *is ascheme and* $\mathcal{I}$ *is quasi-coherent are local on* X*, so wlog* $X = \operatorname{Spec} A$.
If $\mathcal{I}$ *is qcoh., then* $\mathcal{I} = \widetilde{I}$ *for some ideal* $I \subset A$.

Lecture 13: Proj construction

Fri 25 Nov

7 The proj construction

7.1 Algebraic preliminaries

Definition 51 (Graded rings)

1. *A* $(\mathbb{N})-$ *graded ring is a ring* A *together with a decomposition* $A = \bigoplus_{d=0}^{\infty} A_d$ *of it's underlying additive group such that* $A_d \cdot A_{d'} \subset A_{d+d'}$.
2. *A graded morphism of graded rings is a ring morphism* $\phi : A \to B$ *such that* $\phi(A_d) \subset B_d \forall d \geq 0$
3. *A graded module over a graded ring is an* A*-module* M *together with a decomposition* $M = \bigoplus_{i=0}^{\infty} M_d$ *which is compatible with the grading from* A.
4. *A graded morphism of graded modules is what you'd expect*
5. *The irrelevant ideal of a graded ring is* $A_+ = \bigoplus_{d=1}^{\infty} A_d$.
6. *An element* $a \in A$ *is called homogeneous if* $a \in A_d$ *for some* $d \geq 0$
7. *An ideal* $I \subset A$ *is homogeneous if* $I = \bigoplus_d (I \cap A_d)$
8. *The homogenization of an ideal* $I \subset A$

Example
For any ring $A_0, A = A_0[x_0, \ldots, x_n]$ *becomes a graded ring if we set* A_d *to be the subset of polynomials of degree* d.

Lemma 112
Let A *be a graded ring, then*

1. A_0 *is a subring*
2. *If* $\mathfrak{p} \subset A$ *is a prime, then so is* $\mathfrak{p}^h \subset A$
3. *An ideal is homogeneous iff it can be generated by homogeneous*

elements

4. *A homogeneous ideal $I \subset A$ is prime iff $\forall a, b \in A$ homogeneous such that $ab \in I$, either $a \in I$ or $b \in I$*
5. *Being homogeneous is stable under taking radicals*

Definition 52 (Localizations of graded rings)
Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a graded ring and M a graded A-module

1. *Let $S \subset A$ be a multiplicatively closed subset consisting of homogeneous elements then the homogeneous localization of M at S is $(S^{-1}M)_0 = \left\{ \frac{a}{b} \in S^{-1}M \middle| \deg a = \deg b \right\}$*
2. *If $a \in A$ is homogeneous, then $S = \left\{ a^i \right\}$ consists of homogeneous elements and we let $M_{(a)} = (S^{-1}M)_0$ is the homogeneous localization of M at a.*
3. *If $p \subset A$ is a homogeneous prime ideal, we set $S = \{a \in A \setminus p | a \text{ homogeneous } \}$ and let $M_{(p)} = (S^{-1}M)_0$ be the homogeneous localization of M at p.*

7.2 The projective spectrum

Definition 53
Let A be a graded ring.

1. *The projective spectrum of A is $Proj A = \{p \in \operatorname{Spec} A | p \text{ homogeneous}, A_+ \not\subset p\}$*
2. *The Zariski topology on $Proj A$ is defined by defining the closed subsets $V_+(I) = \{p \in Proj A | I \subset p\}$ for subsets $I \subset A$.*
3. *The principal open subsets associated to a homogeneous element $a \in A_+$ is $D_+(a) = Proj(A) \setminus V_+(\{a\}) = \{p \in Proj A | a \notin p\}$.*

Lemma 113
The following identities hold

1. *For any subset $I \subset A$, we have $V_+(I) = V_+((I)) = V_+((I)^h)$*
2. *For any two subsets $I, J \subset A$,*

$$V_+(I) \cup V_+(J) = V_+(I \cap J)$$

3. *For abritrarily many J_i*

$$\cap_i V_+(J_i) = V_+(\sum_i J_i)$$

4.

$$V_+(A_+) = V_+(A) = \emptyset$$

5.

$$V_+((0)) = Proj A$$

6. *If $a = \sum a_i \in A$ is an element and the a_i are homogeneous of degree i, then $D(a) \cap Proj(A) = (D(a_0) \cap Proj A) \cup (\bigcup D_+(a_i))$.*
7. *If $a = a_0 \in A$, then $D(a_0) \cap Proj A = \bigcup_{b \in A_d, d \geq 1} D_+(a_0 b)$*
8. *In particular, the Zariski topology on $Proj A$ is a topology by the above properties*

Lemma 114
Let I, J be homogeneous ideals in a graded ring, then $V_+(I) \subset V_+(J) \iff J \cap A_+ \subset \sqrt{I}$.
In particular, if $a, b \in A_+$ are homogeneous, then $D_+(b) \subset D_+(a) \iff D(b) \subset D(a)$.

Proof
If $J \cap A_+ \subset \sqrt{I}$ and let $\mathfrak{p} \in V_+(I) \implies I \subset \mathfrak{p} \implies \sqrt{I} \subset p \implies JA_+ \subset J \cap A_+ \subset \mathfrak{p} \implies p \in V_+(J)$
Assume $V_+(I) \subset V_+(J)$.
Let $\mathfrak{p} \in \operatorname{Spec} A$ with $I \subset \mathfrak{p}$, then $I \subset \mathfrak{p}^h$.
If $A_+ \not\subset p^h$, then $p^h \in V_+(I) \subset V_+(J) \implies J \subset p^h \implies J \subset p$.
If $A_+ \subset p^h$, then $J \cap A_+ \subset p$.
Finally, $D_+(b) \subset D_+(a) \iff V_+(a) \subset V_+(b) \iff (b) \cap A_+ \subset \sqrt{(a)} \iff (b) \subset \sqrt{(a)} \iff D(b) \subset D(a)$ □

Corollary 115
Let A be a graded ring, then $Proj A = \emptyset$ iff $A_+ \subset \sqrt{0}$.

7.3 The structure sheaf on $Proj A$ and quasi-coherent sheaves

If M is a graded A-module and $D_+(b) \subset D_+(a)(\iff D(b) \subset D(a))$, then there is a map $M_{(a)} \to M_{(b)}$ which commutes with the maps $M_{(a)} \to M_a$

Lemma 116

If $A = \bigoplus_{d=0}^{\infty} A_d$ a graded ring, $M = \bigoplus_{d=0}^{\infty} M_d$ is a graded A-module and $a, b \in A_+$ are homogeneous with $D_+(b) \subset D_+(a)$.

There is a map $M_{(a) \to M_a \to M_b}$ which factors through an $A_{(a)}$ linear map $M_{(a)} \to M_{(b)}$ via the map $(M_{(a)})_{b^{\deg a}/a^{\deg b}} \simeq M_{(b)}$.

Proof

$D(b) \subset D(a) \implies \exists n > 0, c \in A$ such that $b^n = ac$.

Since b^n is homogeneous and a is homogeneous, we can choose c homogeneous.

Take $\frac{e}{a^m} \in M_{(a)}$ with $\deg e = m \deg a$.

Then, in M_b, we have

$$\frac{e}{a^m} = \frac{c^m e}{c^m a^m} = \frac{c^m e}{b^{nm}} \in M_{(b)}$$ □

In particular, we get homeomorphisms $\phi_a : D_+(a) \to \operatorname{Spec} A_{(a)}$

Proof

Note that $\phi_a(p) = pA_a \cap A_{(a)}$.

Define $\psi_a : \operatorname{Spec} A_{(a)} \to D_+(a)$ by sending q to $\bigoplus_d \left\{ b \in A_d | \frac{b^{\deg a}}{a^d} \in q \right\}$ □

Definition 54

Let A be a graded ring, M a graded A-module.

Let $X = ProjA$.

1. *The structure sheaf $\mathcal{O}_X$ on X is the unique sheaf of rings on X wich agrees with $\widetilde{A_{(a)}}$ on $D_+(a)$*

Lecture 14: Twisting sheaves

Mon 28 Nov

Definition 55

Let A be a graded ring and M a graded A-module.

Let $X = ProjA$

1. *The structure sheaf $\mathcal{O}_X$ on X is the unique sheaf of rings on $ProjA$ such that $\mathcal{O}_X|_{D_+(a)} = \widetilde{A_{(a)}}$*
2. *The sheaf of $\mathcal{O}_X$-modules associated to M is the unique sheaf $\widetilde{M}$ such that $\widetilde{M}|_{D_+(a)} = \widetilde{M_{(a)}}$*
3. *The n-th (Serre) twisting sheaf on $ProjA$ is $\widetilde{A(n)}$ where $A(n) = \bigoplus_{d \in \mathbb{Z}} A(n)_d$, where $A(n)_d = A_{n+d}$*
4. *The n-th twist $\widetilde{M(n)}$ of $\widetilde{M}$ is defined as $\widetilde{M(n)}$ where we shift the*

grading as above.

$Proj A$ is is a scheme and $\widetilde{M} \in QCoh(Proj A)$

Corollary 117

Let A be a graded ring, M a graded A-module, $X = Proj A$.
For every $p \in D_+(b) \subset D_+(a) \subset Proj A$.
We find that $\widetilde{M}_p \simeq M_{(p)}$

Proof

By definition of $\tilde{M}$ we have $\widetilde{M}|_{D_+(a)} = \widetilde{M_{(a)}}$.
We need to check that $M(p) = \lim_{p \in D_+(a)} M_{(a)}$ $\square$

Remark

Thinking of $\mathcal{O}_{Proj A}$ and $\widetilde{M}$ as their own sheafification, we obtain descriptions $\mathcal{O}(U)$ and $\widetilde{M}(U)$ similar to the description of the structure sheaf.

Definition 56

Let A_0 be a ring, $n \geq 1$, $A = A_0[x_0, \ldots, x_n]$.
Equip A with the standard grading.
The scheme $Proj A$ is called Projective n-space over A and is denoted $\mathbb{P}^n_{A_0}$

Remark

For any ring A_0, $\mathbb{P}^n_{A_0} \simeq \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A_0$.
If X is any scheme, we define $\mathbb{P}^n_X = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X$

Remark

Consider $\mathbb{P}^n_{A_0} = Proj A_0[x_0, \ldots, x_n]$.

1. $D_+(x_i) \simeq \operatorname{Spec} A_{(x_i)} = \operatorname{Spec} A_0[\frac{x_0}{x_i}, \ldots] \simeq \mathbb{A}^n_{A_0}$
2. *If $A_0 = K$ is a field, then $V_+(x_i) \subset \mathbb{P}^n_{A_0}$ with its reduced induced scheme structure is isomorphic to $\mathbb{P}^{n-1}_K$*

Definition 57

Let A be a graded ring, then the affine cone over $Proj A$ is $\operatorname{Spec} A$.

Definition 58

A graed ring A is finitely generated in degree 1 if it is generated by finitely many elements of A_1 as an A_0 algebra.

Lemma 121
Assume A is f.g. in degree in degree 1, then there exists $n > 0$ and a closed immersion $Proj A \to \mathbb{P}^n_{A_0}$ over $\operatorname{Spec} A_0$.

Proof
There exists a graded surjection $A_0[x_1, \ldots, x_n] \to A$ and then apply an exercise. □

Remark
If A is f.g. over A_0, then $A^{(n)} = \bigoplus A^{(n)}_d$ with $A^{(n)}_d = A_{nd}$ is f.g. in degree 1 for a suitable n.
By an exercise on the next sheet $Proj A \simeq Proj A^{(n)}$.

Theorem 123
Let A be a graded ring, f.g. in degree 1 then $\pi_A : Proj A \to \operatorname{Spec} A_0$ is proper.

Proposition 124
We know that $\mathbb{P}^n_{A_0} = \simeq \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A_0$ and that there is a closed immersion $Proj A \to \mathbb{P}^n_{A_0}$.
Since proper is stable stable under base change and (COMP) and closed immersions are proper, it suffices to show $\pi : \mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is proper.
We want to show that π is separated, of finite type and satisfies the existence in the valuative criterion.

π is separated

Suffices to show that $D_+(x_i) \cap D_+(x_j) \to D_+(x_i) \times_{\operatorname{Spec} \mathbb{Z}} D_+(x_j)$ is a closed immersion.
We have to show that $\mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \otimes \mathbb{Z}[\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j}] \to \mathbb{Z}[\frac{x_0^2}{x_i x_j}, \frac{x_0 x_1}{x_i x_j} \ldots]$ is surjective, which is obvious.

π is of finte type

$\mathbb{P}^n_{\mathbb{Z}} = \bigcup D_+(x_i) = \bigcup \operatorname{Spec} \mathbb{Z}[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$ *and these specs are of finite type.*

Lecture 15: main theorem of elimination theory

Fri 02 Dec

Remark
We proved that $\pi : \operatorname{Proj} A_0[x_0, \ldots, x_n] \to \operatorname{Spec} A_0$ is closed $\forall$ rings A_0.
In particular, given homogeneous polynomials $f_1, \ldots, f_n \in A_0[x_0, \ldots, x_n]$.
In particular, there exists an ideal $I \subset A_0$ such that $\phi : A_0 \to K$ where K is a

field with induced map $\phi' : A_0[x_0, \dots, x_n] \to K[x_0, \dots, x_n]$, then $\phi'(f_1), \dots, \phi'(f_m)$ have a common solution in $\overline{K}$ (ie., the fiber $V_+(f_1, \dots, f_n) \to \operatorname{Spec} A_0$ over the K-rational point corresponding to ϕ is non-empty.) if and only if $\phi(I) = 0$ (ie. the K-rational point corresponding to ϕ factors through $V(I)$)

8 Quasi-Coherent sheaves on $\operatorname{Proj} A$

Let A be a graded ring, $X = \operatorname{Proj} A, \phi : M \to N$ is a morphism of graded A-modules.
$\forall a \in A$ homogeneous, we get compatible maps $M_{(a)} \to N_{(a)}$ which glue to a morphism $\tilde{\phi} : \widetilde{M} \to \widetilde{N}$.
Note that $\tilde{\cdot}$ is exact but not an equivalence of categories

Example
Let A be a graded ring, $M = \bigoplus M_d$, $N = \bigoplus N_d$ such that $\exists n > 0$ such that $M_d = N_d \forall d > n$, the resulting qcoh. sheaves are isomorphic.

However

Proposition 127
Let $A = A_0[x_0, \dots, x_n]$ be the standard graded polynomial ring.
Then $\exists$ a natural isomorphism $A_d \simeq \Gamma(\operatorname{Proj} A, \mathcal{O}_{\operatorname{Proj} A}(d))$

Definition 59
Let A be a graded ring, $X = \operatorname{Proj} A$, $\mathcal{F}$ a qcoh. sheaf, then

1. *The n-th twist of $\mathcal{F}$ is $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$*
2. *The graded module associated to the sheaf $\mathcal{F}$ is*

$$\Gamma_*(\mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F}(d))$$

We still have to check how $\Gamma_*(\mathcal{F})$ become an A-module?
What is the relation between

$$\widetilde{M}(n) \text{ and } \widetilde{M(n)}$$

Definition 60
Let M, N be graded A-modules, then $M \otimes_A N$ becomes a graded module by setting

$$(M \otimes_A N)_d = \bigoplus_{d_1 + d_2 = d} (M_{d_1} \otimes_{\mathbb{Z}} N_{d_2}) / (ma \oplus n - m \oplus an | a + v + w = d, m)$$

Remark

This satisfies the universal property of the tensor product in the category of graded rings.

Example

$$(M \otimes_A A(n))_d = M_{d+n}$$

Lemma 130

Let A be a graded ring and M, N garded A-modules, $X = \operatorname{Proj} A$. Then, there is a natural morphism

$$\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_A N}$$

Lecture 16: saturated graded modules

Mon 05 Dec

Lemma 131

For every $d > 0, d|n$ and $a \in A_d$, we have $\mathcal{O}_X(n)|_{D_+(a)} \simeq \mathcal{O}_X|_{D_+(a)}$ and the morphisms

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(m_n); \mathcal{O}_X(n) \otimes \mathcal{F}(m) \to F(m+n); \widetilde{M}(n) \to \widetilde{M(n)}$$

restrict to isomorphisms.

Proof

Multiplication by $a^{\frac{n}{d}}$ defines a graded morphism $M \to M(n)$ which induces an isomorphism $M_{(a)} \to M(n)_{(a)}$

The composition

$$\widetilde{M}|_{D_+(a)} \to \widetilde{M(n)}|_{D_+(a)} \to (\widetilde{M(n)} \otimes \widetilde{A(m)})|_{D_+(a)} \to \widetilde{M(n+m)}$$

is just multiplication by $a^{m+n/d}$. $\square$

Corollary 132

Let A be a graded ring generated in degree 1, let $X = \operatorname{Proj} A$, then

1. *$\mathcal{O}_X(n)$ is an invertible sheaf $\forall n \in \mathbb{Z}$*
2. *There is a natural isomorphism $\mathcal{O}_X(n_1) \otimes \mathcal{O}_X(n_2) \to \mathcal{O}_X(n_1 + n_2) \forall n_1, n_2 \in \mathbb{Z}$*
3. *There exists a natural isomorphism $\widetilde{M(n)} \xrightarrow{\sim} \widetilde{M}(n)$*

Recall that there is a natural map $M_d \to \Gamma(\operatorname{Proj} A \widetilde{M}(d))$ and we obtain a

natural map $M \to \Gamma_*(\widetilde{M})$.
If this map is an isomorphism, we call the module M saturated.

Example

$A = A_0[x_0, \dots, x_n]$ is saturated as a graded module over itself.

Theorem 134

Let A be a graded ring, f.g. in degree 1.
Let $X = \operatorname{Proj} A$, the the functors

$$\widetilde{()} : \{ \text{ graded saturated } A\text{-modules } \} \leftrightarrow QCoh(X, \mathcal{O}_X) : \Gamma_*(\cdot)$$

are essential inverses.

Proof

By some lemma, there is a closed immersion $X \to \mathbb{P}^n_{A_0}$ and hence X is qcqs.
For any $\mathcal{F} \in QCoh(X, \mathcal{O}_X)$, $a \in A_d, d \geq 1$. Define $\Gamma_*(\mathcal{F})_{(a)} = (\bigoplus \Gamma(X, \mathcal{F}(n)))_{(a)} \to \Gamma(D_+(a), \mathcal{F})$ by sending $\frac{m}{a^n} \to m|_{D_+(a)} \cdot a^{-n}$.
These are compatible with restriction and we get a map $\beta : \widetilde{\Gamma_*(F)} \to \mathcal{F}$.
We claim that β is an isomorphism.
It suffices to check $\beta(D_+(a))$ is bijective $\forall a \in A_1$.
Set $s = \alpha(a)$.
Then

$$X_s = \{x \in X | s_x \notin \mathfrak{m}_x \mathcal{O}_X(1)_x\} = \bigcup_{b \in A_1} \left\{ x \in D_+(b) | (\frac{d}{b})_x \notin \mathfrak{m}_x \right\} = \bigcup (D_+(a) \cap D_+(b)) = D_+(a)$$

To see surjectivity, let $m' \in \Gamma(D_+(a), \mathcal{F})$, by exercise 47, $\exists n > 0$ and $m \in \Gamma(X, \mathcal{F}(n))$ such that

$$m|_{D_+(a)} = m' \otimes s|^{\otimes n}_{D_+(a)} = a^n m'$$

Hence $\beta(D_+(a))(m) = m'$.
To see injectivity, assume $a^{-n} m|_{D_+(a)} = 0 \in \Gamma(D_+(a), \mathcal{F})$.
Hence $m|_{D_+(a)} = 0 \in \Gamma(D_+(a), \mathcal{F}(n))$.
Now, using an exercise, there exists an $n' > 0$ such that $\otimes s^{\otimes n'} = 0 \in \Gamma(X, \mathcal{F}(n, n'))$.
Hence $\frac{m}{a^n} = 0 \in \Gamma_*(\mathcal{F})_{(a)}$.

By definition, we want to show that $\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathcal{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F})$ is the identity.
Thus $\Gamma_*(\mathcal{F})$ is the identity. □

Corollary 135
Let A be a graded ring, f.g. in degree 1

1. *For every closed subscheme $Z \subset \operatorname{Proj} A$ given by an ideal sheaf $\mathcal{I}_Z$, there exists a homogeneous ideal $I_Z \subset A$ such that $\widetilde{I_Z} = \mathcal{I}_Z$. In particular $Z \to \operatorname{Proj} A$ is given by the quotient map*
2. *If A is saturated as a module over itself, then I_Z can be chosen to be saturated and then I_Z be the unique saturated homogeneous ideal $I_Z \subset A$ such that $\mathcal{I}_Z = \widetilde{I_Z}$.*

9 Morphisms to projective space via invertible sheaves

Recall $\hom(X, \operatorname{Spec} A) \simeq \hom(A, \mathcal{O}_X(X))$ (this is affinization).
In particular, we understand morphisms of schemes to $\mathbb{A}^n_{\mathbb{Z}}$ which are equivalent to choosing n global sections and if we fix $X \to \operatorname{Spec} A$, then

$$\hom_{Sch/\operatorname{Spec} A}(X, \mathbb{A}^n_A) \simeq \mathcal{O}_X(X)^n$$

What about $\hom_{Sch/\operatorname{Spec} A}(X, \mathbb{P}^n_A)$?
The answer is similar but we use invertible sheaves.
Recall, if $f : X \to Y$ is a morphism of schemes and $\mathcal{F} \in QCoh(Y, \mathcal{O}_Y)$, then there is a map $\mathcal{F} \to f_* f^* \mathcal{F}$.
Taking $\Gamma(Y, -)$, we get a map $f_* : \Gamma(Y, \mathcal{F}) \to \Gamma(X, f^*\mathcal{F})$ called the pullback oof sections.
Explicitly, $f^*(s)$ is the image of $s \otimes 1$ under

$$\mathcal{F}(Y) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X(X) \to (f^{-1}\mathcal{F})(X) \otimes_{f^{-1}\mathcal{O}_Y(X)} \mathcal{O}_X(X) \to f^*\mathcal{F}(X)$$

Proposition 136
Let A be a ring and let $f : X \to \mathbb{P}^n_A$ a morphism of schemes over $\operatorname{Spec} A$, then

1. *The sheaf $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is invertible*
2. *For $s_i = f^*x_i \in \Gamma(X, \mathcal{L})$, the germs $(s_i)_x$ generated $\mathcal{L}_x$ as an $\mathcal{O}_{X,x}$-module $\forall x \in X$.*

Proof
Follows from the general fact that the inverse image of an invertible sheaf is invertible (locally $A \otimes_A B \simeq B$).

$$\mathcal{L}_x \simeq \mathcal{O}_{\mathbb{P}^n_A}(1)_{f(x)} \otimes \mathcal{O}_{X,x}$$

One checks that $(s_i)_x \mapsto (x_j)_{f(x)} \otimes 1$. It suffices to show that the $(x_i)_{f(x)}$

generated $\mathcal{O}_{\mathbb{P}^n}(1)_{f(x)}$.
But this is clear because multiplication by x_i *is an isomorphism on* $\mathcal{O}_{\mathbb{P}^n}|_{D_+(x_i)} \to \mathcal{O}_{\mathbb{P}^n}(1)|_{D_+(x_i)}$ $\square$

Definition 61 (Globally Generated)
Let $\mathcal{L}$ *be an invertible sheaf*

1. *A collection* $\{s_i\}_{i\in I}$ *of global sections* $s_i \in \Gamma(X, \mathcal{L})$ *is said to generated* $\mathcal{L}$ *if the induced map* $\mathcal{O}_X^{\oplus I} \to \mathcal{F}$ *is surjective.*
2. $\mathcal{L}$ *is called (finitely) globally generated if it admits (finitely) many global sections that generated it.*

Lecture 17: criterion for morphisms to projective space being closed immersions

Fri 09 Dec

We say that $(\mathcal{L}, s_0, \ldots, s_n)$ and $(\mathcal{M}, t_0, \ldots, t_n)$ are isomorphic if there exists an isomorphism $\phi : \mathcal{L} \to \mathcal{M}$ such that $\phi(x)(s_i) = t_i \forall i$

Theorem 137
Let X *be a scheme over* $\operatorname{Spec} A$*, then the association*

$$\theta : (f : X \to \mathbb{P}^n_A) \mapsto (f^*\mathcal{O}(1), f^*x_0, \ldots, f^*x_n)$$

is a natural bijection between $\hom(X, \mathbb{P}^n_A)$ *and the set of isomorphism classes of* $(n+2)$ *tuples* $(\mathcal{L}, s_0, \ldots, s_n)$ *where* $\mathcal{L}$ *is invertible and the* $s_i \in \Gamma(X, \mathcal{L})$ *generate* $\mathcal{L}$.

Proof
Let $f : X \to \mathbb{P}^n_A$ *be any morphism, then let* $X_i = \{x \in X | (f^*(x_i)) \notin \mathfrak{m}_x f^*\mathcal{O}_{\mathbb{P}^n}(1)_x\} = \left\{x \in X | (x_i)_{f(x)} \notin \mathfrak{m}_{f(x)}\mathcal{O}_{\mathbb{P}^n}(1)_{f(x)}\right\} = f^{-1}\{D_+(x_i)\}$.
The induced map $f_i : X_i \to D_+(x_i)$ *is determined by a morphism of rings*

$$A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \mapsto \Gamma(X_i, \mathcal{O}_{X_i})$$

sending $\frac{x_j}{x_i} \mapsto s_{ji}$.
By naturality of the pullback of sections

$$A[x_0, \ldots, x_n]_{(x_i)} \xrightarrow{\sim} \Gamma(D_+(x_i), \mathcal{O}_{\mathbb{P}^n}) \to \Gamma(X_i, \mathcal{O}_{X_i})$$

the trivialisations of the first twist commute with the maps

$$A[x_0, \ldots, x_n](1)_{(x_i)} \xrightarrow{\sim} \Gamma(D_+(x_i), \mathcal{O}_{\mathbb{P}^n}(1)) \xrightarrow{f^*} \Gamma(X_i, f^*\mathcal{O}_{\mathbb{P}^n}(1))$$

So we can write $s_{ji} = (f^*x_j)|_{D_+(x_i)}(f^*x_i)|^{-1}_{D_+(x_i)}$.
So the s_{ji} *is uniquely determined by the isomorphism class of*

$(f^*\mathcal{O}_{\mathbb{P}^n}(1), f^*x_0, \ldots)$.
Let's show this association is surjective.
Let $(\mathcal{L}, s_0, \ldots, s_n)$ *be as in the statement, set*

$$X_{s_i} = \{x \in X | (s_i)_x \notin \mathfrak{m}_x \mathcal{L}_x\}$$

and define $f_i : X_{s_i} \to \mathbb{P}^n_A$ *to be the map associated to the map of rings*

$$A[\frac{x_0}{x_i}, \ldots] \to \Gamma(X_{s_i}, \mathcal{O}_{X_{s_i}}); \frac{x_j}{x_i} \mapsto s_j|_{X_{s_i}} \cdot (s_i)|_{X_{s_i}}^{-1}$$

The f_i *coincide on* $X_{s_i} \cap X_{s_j}$ *and we get a map* $f : X \to \mathbb{P}^n_A$.
Using an exercise, one can show that there is an isomorphism ϕ : $f^*\mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{L}$ *sending* $f^*x_i \mapsto s_i$ $\square$

Remark
By the Yoneda lemma, this theorem fully characterises $\mathbb{P}^n_A$

Remark
If $X = \operatorname{Spec} k$ *in this theorem, we get that*

$$\mathbb{P}^n_A(k) = \left\{ \text{1-dimensional quotients of } k^{n+1} / k^\times \right\}$$

Remark
The morphism $f : X \to \mathbb{P}^n$ *associated to* $(\mathcal{L}, s_0, \ldots, s_n)$ *factors through* $V_+(I) \subset \mathbb{P}^n_A$ *(I saturated homogeneous).*
This factors iff $g(s_0, \ldots, s_n) = 0 \forall g \in I$.
In particular, $f(X)$ *is contained in a hyperplane, ie. in* $V_+(g)$ *for some homogeneous polynomial* $g \in A[x_0, \ldots, x_n](1)$ *iff the* s_i *are A-linearly dependent.*

Remark
If the s_i *do not globally generate* $\mathcal{L}$*, we still get a morphism* $X \supsetneq \bigcup X_{s_i} \to \mathbb{P}^n_A$.

Remark
If $A = k$ *is a field and we replace the* s_i *by* t_i *with* $\langle s_0, \ldots, s_n \rangle = \langle t_0, \ldots, t_n \rangle$ *then the induced morphisms differ by an automorphism of* $\mathbb{P}^n$.

Remark
If $A = k$ *is a field, then the morphism* $X \to \mathbb{P}^n_k$ *induced by* $s_0, \ldots, s_n$ *is given on k-rational points by* $x \mapsto [s_0(x) : \ldots : s_n(x)]$ *where* $s_i(x)$ *is the image of* s_i *in* $\mathcal{L}_x / \mathfrak{m}_x \mathcal{L}_x$

Proposition 144
The morphism $f : X \to \mathbb{P}^n_A$ *induced* $(\mathcal{L}, s_0, \ldots, s_n)$ *is a closed immersion iff*

1. X_{s_i} *is affine*
2. *The maps* $A[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}] \to \Gamma(X_{s_j}, \mathcal{O}_{X_{s_j}})$ *sending* $\frac{x_j}{x_i} \mapsto s_j|_{X_{s_i}}$ *is surjective*

Proof
Recall that $X_{s_i} = f^{-1}(D_+(x_i))$.
Being a closed immersion is (LOCT).
f *is a closed immersion* $\iff$ $f_{D_+(x_i)}$ *is a closed immersion* $\forall i$ $\iff$ *both conditions hold.* □

There is a better criterion under stronger assumptions.

Proposition 145
Let X *be a proper scheme over an algebraically closed field* k *and let* $f : X \to \mathbb{P}^n_k$ *be induced by* $(\mathcal{L}, s_0, \ldots, s_n)$.
Suppose that $f_*\mathcal{O}_X$ *is coherent.*
Let $V = \langle s_0, \ldots, s_n \rangle \subset \Gamma(X, \mathcal{L})$.
Then f *is a closed immersion iff*

1. *Elements of* V *separate points, ie. for any two distinct closed points* $x, y \in X$ $\exists s \in V$ *such that* $s_x \in \mathfrak{m}_x\mathcal{L}_x$ *and* $s_y \notin \mathfrak{m}_y\mathcal{L}_y$ *or vice-versa.*
2. *Elements of* V *separate tangent vectors, ie. for each closed point* $x \in X$ *the set* $\{s \in V | s_x \in \mathfrak{m}_x\mathcal{L}_x\}$ *spans the* k*-vector space* ${}^{\mathfrak{m}_x\mathcal{L}_x}\!/\!_{\mathfrak{m}_x^2\mathcal{L}_x}$

Lemma 146
Let X *be a scheme locally of finite type over a field* k.
Let $x \in X$, *the following are equivalent*

1. $x \in X$ *is closed*
2. $k \to k(x)$ *is finite*
3. $k \to k(x)$ *is algebraic.*

In particular, if $k = \overline{k}$, *then the set of closed points is the set of rational points.*

Proof
If $x \in X$ *is closed, we find* $\operatorname{Spec} A \subset X$ *open affine such that* $x = \mathfrak{m} \in \operatorname{Spec} A$.
Using the NSS, we get that $k(x) = A/m$ *is a finite extension.*
The implicaiton finite $\to$ *algebraic is clear.*
Now, suppose $x \in X$ *such that* $k \subset k(x)$ *is algebraic.*
Choose $x \in \operatorname{Spec} A \subset X$.
So $x = p$ *for some prime ideal.*
Consider

$$k \to A \to A/p \to\to \frac{(}{A}/p) = k(x)$$

$k(x)|k$ *is algebraic implies that* A/p *is integral over* k *and thus* A/p *is a field.* □

Lemma 147

Let $f : X \to Y$ be a closed morphism of schemes locally of f.t. over a field, then f is injective iff it is injective on closed points.

Proof

One direction is obvious.
Let $x, y \in X$ such that $f(x) = f(y)$.
Now

$$\overline{f(x)} = \overline{f(y)} = f(\overline{x}) = f(\overline{y})$$

We claim that the set of closed points $\overline{\{x\}}_{cl} = \overline{\{y\}}_{cl}$ are equal.
Take $z \in \overline{\{x\}}_{cl}$, then $f(z) \subset f(\overline{\{x\}})$, then $f(z) \in f(\overline{x}) = f(\overline{y})$.
Thus there is a $v \in \overline{\{y\}}$ such that $f(v) = f(z)$.
Because $f(z)$ is closed, $\overline{\{v\}} \subset f^{-1}(f(z))$.
Thus, there exists $w \in \overline{\{v\}}_{cl} \subset \overline{\{y\}}_{cl}$ such that $f(w) = f(v) = f(z)$ and $w = z$.
hence $\overline{\{x\}}_{cl} = \overline{\{y\}}_{cl}$.
Now, since $\overline{\{x\}}_{cl} = \overline{\{y\}}_{cl}$ there is an open affine $\operatorname{Spec} A \subset X$ such that $x = \mathfrak{p} \in \operatorname{Spec} A, y = \mathfrak{q} \in \operatorname{Spec} A$.
Now to show the equality, we notice that $p = \bigcap_{p \subset m} m = \bigcap_{q \subset m} m = q$ because A is Jacobson. □

Lemma 148

Let $\phi : A \to B$ be a local homomorphism of Noetherian local rings, assume that

1. *The induced map $A/m_A \to B/m_B$ is an isomorphism*
2. *The induced map $m_A \to m_B/m_{B^2}$ is surjective*
3. *B is a finitely generated A-module*

Then ϕ is surjective.

Proof

Let $I = m_A B \subset B$.
Since ϕ is local, $I \subset m_B$.
By the assumption above, $I \to m_B/m_B^2$ is surjective.
Thus $m_B = I + m_B^2$ and thus $I = m_B$.
We know that $1 \in B$ generates $B/m_A B = B/m_B \simeq A/m_A$ as an A-module.
$1 \in B$ generates B as an A-module and thus ϕ is surjective. □

Lecture 18: very ample invertible sheaves

Mon 12 Dec

9.1 (quasi)-projective morphisms and (very) ample invertible sheaves

Definition 62

Let $f : X \to Y$ be a morphism of schemes, then f is called

1. *projective : if $\exists n \in \mathbb{N}$ and a closed immersion $\iota : X \to \mathbb{P}^n$ over Y.*
2. *quasi-projective : if $f = g \circ h$ with h an open immersion and g injective.*
3. *locally-projective : if $\exists$ an open cover $Y = \bigcup V_i$ such that f_{V_i} is projective*
4. *An invertible sheaf $\mathcal{L}$ on X is called (f-relatively) very ample if there exists an immersion $X \to \mathbb{P}^n_Y$ over Y such that $\mathcal{L} = \iota^*\mathcal{O}(1)$*

Remark

In particular, f is quasi-projective $\iff$ $\exists f$-relatively very ample invertible sheaf on X and f is projective iff f is proper and quasi-projective.

Definition 63

If X is a noetherian scheme, then an invertible sheaf $\mathcal{L}$ on X is called ample if $\forall \mathcal{F} \in Coh(X, \mathcal{O}_X) \exists n_0 \geq 0$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n} \forall n \geq n_0$ is globally generated.

Theorem 150 (Serre)

Let $f : X \to \operatorname{Spec} A$ be a proper morphism and A is Noetherian.
Assuma A is noetherian and $\mathcal{L}$ is an f-relatively very ample invertible sheaf, then $\mathcal{L}$ is ample.

Proof

Choose an immersion $X \to \mathbb{P}^n_A$ such that $\iota^*\mathcal{O}(1) \simeq \mathcal{L}$.
Since f is proper, ι is a closed immersion.
Let $\mathcal{F} \in Coh(X, \mathcal{O}_X)$, then $\iota_*\mathcal{F} \in Coh(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A})$.
From the projection formula, there is a natural isomorphis $\Gamma(\mathbb{P}^n_A, \iota_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_A}(m)) = \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m})$.
So $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated $\iff$ $\iota_*\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_A}(m)$ is globally generated.
Wlog, $X = \mathbb{P}^n_A$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_A}(1)$.
Set $\mathcal{F}_i = \mathcal{F}|_{D_+(x_i)}$, then $\mathcal{F}_i = \widetilde{M_i}$ for some f.g. $B_i = A[\frac{x_0}{x_i}, \dots, x_{\frac{n}{x_i}}]$-module M_i.
Choose generators s_{ij} of M_i.

By qcqs, we can choose $m_0 \geq 0$ such that $x_i^{m_0} \cdot s_{ij} = t_{ij}|_{D_+(x_i)}$ for some $t_{ij} \in \Gamma(\mathbb{P}^n_A, \mathcal{F}(m_0))$.
Then, for every $m \geq m_i$, the sections $x_i^{m-m_0} \cdot t_{ij} \in \Gamma(\mathbb{P}^n_A, \mathcal{F}(m))$ generate $\mathcal{F}(m)$. □

Remark
The proof shows that $\mathcal{F}(m)$ is generated by finitely many global sections, this can be used to show that $f_*\mathcal{F}$ is coherent if $\mathcal{F}$ is a very ample invertible sheaf.

Lemma 152
Let $\mathcal{L}$ be an invertible sheaf, then the following are equivalent

1. *$\mathcal{L}$ is ample*
2. *$\mathcal{L}^{\otimes n}$ is ample $\forall n > 0$*
3. *$\mathcal{L}^{\otimes n}$ is ample for some $n > 0$*

Proof
$1 \implies 2 \implies 3$ is clear.
$3 \implies 1$.
Let $\mathcal{F} \in Coh(X, \mathcal{O}_X)$, set $\mathcal{F}' = \bigoplus_{i=0}^{n-1}(\mathcal{F} \otimes \mathcal{L}^{\otimes i})$, this is a coherent sheaf.
So $\exists m_0 \geq 0$ such that $\forall m \geq m_0$.
So $\exists m_0 \geq 0$ such that $\forall m \geq m_0$

$$(\mathcal{F}' \otimes \mathcal{L}^{\otimes m}) = \bigoplus_{i=0}^{n-1}(\mathcal{F} \otimes \mathcal{L}^{\otimes mn+i})$$

is globally generated.
Thus $\mathcal{F} \otimes \mathcal{L}^{\otimes(nm+i)}$ is globally generated $\forall m \geq m_0 i = 0, \ldots, n-1$ and thus $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ is globally generated $\forall k \geq k_0 = nm_0$. □

Theorem 153
Let $f : X \to \operatorname{Spec} A$ be a morphissm of finite type with A noetherian, $\mathcal{L}$ invertible, then $\mathcal{L}$ is ample iff $\exists n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample.

Lecture 19: picard group

Fri 16 Dec

9.2 Picard groups and divisors

We want to study the connection between invertible sheaves on X and closed subschemes of codimension 1.

Definition 64 (Picard Group)
Let X be a scheme, the picard group of X is the group $Pic(X)$ of isomorphism classes of invertible sheaves on X.

The group law is given by the tensor product.
The inverse is given by $\mathcal{L}^{-1} = \hom_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.
The neutral element is $\mathcal{O}_X$.
Note that there is a natural map $s \otimes t^* \mapsto t^*(s)$ from $\mathcal{L} \times \mathcal{L}^{-1} \to \mathcal{O}_X$.

Definition 65 (Dimension)
Let X be a topological space, A a ring, then

- *The dimension of X is $\dim X = \sup\{n \in \mathbb{N} | \exists \emptyset \subsetneq X_0 \ldots \subsetneq X_n\}$ where each X_i is closed and irreducible.*
- *For $Z \subset X$ closed and irreducible, the codimension of Z in X is $codim(Z, X) = \sup\{n \in \mathbb{N} | \exists Z = Z_0 \subsetneq \ldots \subsetneq Z_n \subset X\}$ with each Z_i closed irreducibe.*
- *For $X = \operatorname{Spec} A$, we get the usual notions of krull dimension and height.*

Note that $ht(p) = codim(V(p), \operatorname{Spec} A)$ and $\dim \operatorname{Spec} A = \sup ht(p)$.

Definition 66 (Regular ring)
Let A be a Noetherian local ring with maximal ideal and residue field k. Then A is regular if $\dim_k m/m^2 = \dim A$

Proposition 154
If A is a Noetherian local ring of dimension 1, then

1. *A is regular*
2. *A is normal*
3. *A is a dvr*

Definition 67 (Divisor)
Let X be a noetherian scheme

- *A prime divisor on X is an integral closed subscheme of codim. 1*
- *A Weil divisor on X is a finite formal sum of prime divisors.*
- *The group of Weil divisors on X is denoted by $Div(X)$*
- *A weil divissor $D = \sum n_Z Z$ is called effective if $n_Z \geq 0$ for all prime divisors Z.*
- *If D, D' are two weil divisors with $D - D'$ is effective, we write $D \geq D'$.*

— *The support of a Weil divisor* $D = \sum n_Z Z$ *is* $\operatorname{Supp} D = \bigcup_{n_Z \neq 0} Z$
— *If* $U \subset X$ *is open, the restriction of a Weil divisor* D *on* X *to* U *is* $D|_U = \sum n_Z(Z \cap U)$.

Next, we want to associate a Weil divisor to every pair $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf and s is a rational section (a section of $\mathcal{L}$ over some non-empty open).

Definition 68
Let X *be a Noetherian scheme, we call* X *regular in codimension 1 if for every* $x \in X$ *such that* $\dim \mathcal{O}_{X,x} = 1$*, the local ring is regular.*

In what follows, we fix X a noetherian integral and (R1) scheme.
For $(\mathcal{L}, s)$ as above and $Z \subset X$ prime, we can define the valuation $v_Z(s)$ of s along z as follows.
Let $\eta_Z \in Z$ be the generic point.
Choose $\eta_Z \in V \subset X$ open and an isom $\phi : \mathcal{L}|_V \simeq \mathcal{O}_V$.
Let s' be the image of s under $\mathcal{L}(U) \to \mathcal{L}(V \cap U) \to \mathcal{O}_V(V \cap U) \to k(X)$.
Then we set $v_Z(s) = v(s')$ where v is the valuation associated to $\mathcal{O}_{X,\eta_Z} \subset k(X)$

Definition 69
For $(\mathcal{L}, s)$ *as above, we say thhat* s *has a root along* Z *if* $v_Z(s) > 0$ *and has a pole along* Z *if* $v_Z(s) < 0$

Note $v_Z(s) = v_Z(s|V) \forall \emptyset \neq V \subset U$ and that if there exists and isomorphism $\phi : \mathcal{L} \to \mathcal{M}$ and $\phi(s) = t$ then $v_Z(s) = v_Z(t)$.
We write $\sim$ for the equivalence relation on the set of pairs $(\mathcal{L}, s)$ induced by isomos and restrictions.
Note that $\{(\mathcal{L}, s)/\sim\}$ is a group wrt $\otimes$ and there exists a surjective group homomorphism

$$\{(\mathcal{L}, s)\} / \sim \to Pic(X)$$

with kernel $\{\mathcal{O}_X, s\} / \sim = k(X)^\times$.
We would like to define

$$div : \{(\mathcal{L}, s)\} / \sim \to Div(X)$$

sending $(\mathcal{L}, s) \mapsto \sum v_Z(s) Z$.

Lemma 155
For $(\mathcal{L}, s)$ *as above, there are only finitely many prime divisors* $Z \subset X$ *such that* $v_Z(s) \neq 0$.

Proof
Let $V \subset X$ open affine, non-empty.
Then $\dim(X \backslash V) < \dim X$ and $X \backslash V$ has only finitely many irreducible components.
Write $s = \frac{a}{b}$ for $a, b \in A$ and then $v_Z(s) = v_Z(a) - v_Z(b)$ so wlog $s \in A$.
Let $Z = V(p)$ with $p \in \operatorname{Spec} A$ of ht 1.
Since $s \in A$, $v_Z(s) \geq 0$ and $v_Z(s) > 0 \iff s \in p$ and we conclude with Krull PIT. $\square$

So we get a map

$$div : \{(\mathcal{L}, s)\} / \sim \to Div(X)$$

is well defined.

Lecture 20: principal divisors, class group

Mon 19 Dec

Throughout, X is integral, Noetherian and $R1$.

Definition 70 (Principal divisors)

1. *A weil divisor D on X is called principal if $D = div(f)$ for some $f \in K(X)^\times$.*
2. *Two Weil divisors D, D' on X are called linearly equivalent if $D - D'$ is principal.*
3. *The (Weil divisor) class group $Cl(X)$ of X is the quotient of $Div(X)$ by the subgroup of principal divisors.*

Lemma 156
Assume X is noetherian, integral, normal and let s be a rational section of an invertible sheaf.
If $v_Z(s) \geq 0 \forall Z \subset X$ prime, then s extends uniquely to a global section of $\mathcal{L}$.

Proposition 157
Let X be an integral, normal, Noetherian scheme, then div is injective.

Proof
Assume $div(\mathcal{L}, s) = 0$. In particular $v_Z(s) = 0 \forall Z \subset X$ prime.
By the above, wlog, s is a global section.
Consider $s \times - : \mathcal{O}_X \to \mathcal{L}$.

If this is an iso, then $(\mathcal{O}_X, 1) \simeq (\mathcal{L}, s)$.
To check that $s\cdot$ *is an iso, we can work locally, ie., we may assume* $X = \operatorname{Spec} A$ *and* $\mathcal{L} = \mathcal{O}_X$ *is trivial,* $s \in A$.
Since $v_Z(s) = 0 \forall Z \subset X$ *prime, we have* $v_Z(\frac{1}{s}) = 0$ ☐

Definition 71
Let D *be a Weil divisor on* X*, the sheaf asscoiated to* D *is*

$$\mathcal{O}_X(D) : U \to \left\{ f \in K(X)^\times | div|_U f + D|_U \geq 0 \right\} \cup \{0\}$$

where $div|_U f$ *is the principal divisor asscoiated to* f *considered as a rational function on* U*. Note that* $1 \in K(X)$ *is a rational section of* $\mathcal{O}_X(D)$.

This is a sheaf.

Lemma 158
$\mathcal{O}_X(D)$ *is quasi-coherent.*

Proof
By a theorem, it suffices to show that for all open affines $\operatorname{Spec} A = U \subset X$ *open and* $a \in A \setminus \{0\}$*, the map*

$$\rho : \mathcal{O}_X(D)(U)_a \to \mathcal{O}_X(D)(D(a))$$

is an isomorphism. ☐

Proposition 159
Let X *be a normal integral and Noetherian scheme.*
Let $(\mathcal{L}, s)$ *be a pair of an invertible sheaf and a rational section* s.
Then, there exists an isomorphism $\mathcal{O}_X(div(\mathcal{L}, s)) \to \mathcal{L}$ *that sends* 1 *to* s.

Proof
Let $\emptyset \neq U \subset X$ *such that* $s \in \mathcal{L}(U)$.
Let $V \subset X$ *be any open, let* $t \in \mathcal{O}_X(div(\mathcal{L}, s))(V) \subset K(X)$.
Then $t|_{V \cap U} \in \mathcal{O}_X(div(\mathcal{L}, s))(V) \subset K(X)$ *and*

$$divt + div(s)|_{U \cap V} \geq 0$$

Then $t|_{U \cap V}$ *has no poles, so* $t \in \mathcal{O}_X(V \cap U)$.
So $st \in \mathcal{L}(V \cap U)$ *which we can extend to* $\mathcal{L}(V)$.
So we get a morphism $\mathcal{O}_X(div(s)) \to \mathcal{L}$. ☐

Definition 72

A weil divisord D on X is called locally principal if $\forall x \in X \exists x \in U \subset X$ such that $D|_U$ is principal.

Corollary 160

Let D be a locally principal Weil divisor on a normal, integral, Noetherian scheme.
Then $\mathcal{O}_X(D)$ is invertible.

Proof

We have $\mathcal{O}_U(D|_U) = \mathcal{O}_X(D)|_U$.
If $D|_U$ is principal, then $\mathcal{O}_U(D|_U) \simeq \mathcal{O}_U$ □

Proposition 161

Let X be a normal, integral, Noetherian.
Let D be a Weil divisor on X such that $\mathcal{O}_X(D)$ is invertible, then $D = div(\mathcal{O}_X(D), 1)$, in particular D is locally principal.

Proposition 162

If A is a Noetherian integral domain, tfae
A is factorial $\iff$ every height 1 prime in A is principal.

Proposition 163

Factorial domains are normal.

Theorem 164

All Noetherian regular local rings are factorial.

Definition 73 (Factorial scheme)

A scheme X is called factorial if $\mathcal{O}_{X,x}$ is factorial $\forall x \in X$.

Proposition 165

Let X be a factorial, integral, Noetherian scheme, then, every Weil divisor on X is locally principal.
In particular div is an isomorophism and $Pic(X) \simeq Cl(X)$

Proof

Wlog, $X = \operatorname{Spec} A$, $D \subset X$ prime with ideal $p \subset A$.
Note that $D|_{X \setminus D} = 0$.
Let $x \in D \subset X$, then $ht(pA_x) = 1$ and A_x is factorial and $pA_x = (f_x)$ for some $f_x \in A_x$.

In particular, $v_D(f_x) = 1$, so $div(\mathcal{O}_X, f_x)|_U = D|_U$ for some open neighborhood $x \in U \subset X$ □

Corollary 166
If A is factorial, then $Pic(\operatorname{Spec} A) = Cl(\operatorname{Spec} A) = 0$

Example

1. *If A is a Dedekind domain, then $Cl(A)$ are fractional ideals mod principal fractional ideals and this is isomorphic to $Pic(\operatorname{Spec} A)$*
2. *If k is a field, then*
$$Pic(\mathbb{A}^n_k) = Cl(\mathbb{A}^n_K) = 0$$

Proposition 168
Let X be a normal, integral, Noetherian scheme and $Z \subset X$ is integral and closed, let $U = X \setminus Z$, then

1. *The restriction map $Cl(X) \to Cl(U)$ is surjective*
2. *If $codim(Z, X) \geq 2$, then $Cl(X) \simeq Cl(U)$*
3. *If $codim(Z, X) = 1$, then there is an exact sequence*
$$\mathbb{Z} \to Cl(X) \to Cl(U) \to 0$$

Lecture 21: cartier divisors

Mon 09 Jan

Corollary 169
If k is a field, then
$$Aut_{(Sch/k)}(\mathbb{P}^n_k) \simeq PGL_{n+1}(k)$$

9.3 Cartier divisors and linear systems

Definition 74
Let X be a scheme.
An effective cartier divisor on X is a closed subscheme $D \subset X$ whose ideal sheaf is invertible.
The invertible sheaf associated to an effective cartier divisor is
$$\mathcal{O}_X(D) = \mathcal{I}_D^{-1}$$

Given $D \subset X$, we have a SES
$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

where "$\mathcal{O}_D = \iota_* \mathcal{O}_D$ " where ι is the immersion.
Tensoring with $\mathcal{O}_X(D)$, we get

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D)$$

Definition 75
Let X be a scheme, $\mathcal{L}$ an invertible sheaf on X and $s \in \Gamma(X, \mathcal{L})$

1. *s is called regular if the associated morphism $\mathcal{O}_X \to \mathcal{L}$ is injective*
2. *The Zero scheme $Z(s) \subset X$ associated to s is the closed subscheme of X with ideal sheaf the image of the map $\mathcal{L}^{-1} \to \mathcal{O}_X$*

Lemma 170
Let X be a scheme, $Z \subset X$ a closed subscheme TFAE

1. *Z is an effective cartier divisor*
2. *the ideal sheaf is locally generated by a non-zero divisor*
3. *$Z = Z(s)$ for a regular section s of some invertible sheaf*

Proof
I_Z is invertible iff it is locally genertaed by a non-zero divisor so $1 \iff 2$.
To show $1 \implies 3$, simply note that $Z(1_D) = D$
$3 \implies 1$ if $\mathcal{O}_X \to \mathcal{L}$ injective corresponds to a global section s, then $\mathcal{L}^{-1} \to \mathcal{O}_X$ is injective and $\mathcal{L}^{-1}$ is isomorphic to it's image. $\square$

If D_1, D_2 are effective Cartier divisors, then $\mathcal{I}_{D_1} \otimes \mathcal{I}_{D_2} \simeq \mathcal{I}_{D_1} \times \mathcal{I}_{D_2}$

Definition 76
Two effective Cartier divisors $D, D' \subset X$ are called linearly equivalent, denoted $D \sim D'$ if $\mathcal{O}_X(D) \simeq \mathcal{O}_X(D')$ as invertible sheaaves.

Definition 77
Let X be an integral scheme over a field k

1. *If $D \subset X$ is an effective Cartier iv, the complete linear system $|D|$ associated to D is the set of all effective Cartier divisors D' with $D \sim D'$*
2. *A linear system is the set V of effective Cartier divisors corresponding to a finite dimensional k-subspace $W \subset \Gamma(X, \mathcal{O}_X(D))$ for some effective Cartier Divisor. The dimension of V is $\dim W - 1$*
3. *The base locus of a lineat system V is the intersection of all elements of V*

4. *A lineat system is called base-point free if it's base locus is empty.*

Remark

A linear system V of dimension n determines an equivalenc class of morphisms $f : X \supset U \to \mathbb{P}^n_k$, we call V very ample if it is an immersion.

Definition 78

Let P be a property of schemes (or morphisms of schemes).
Let $f : X \to \operatorname{Spec} k$ be a scheme over a field k, then X (or f) is called geometrically P if X_K or f_K is P for all field extensions $K|k$.

Proposition 172

Let X be a proper scheme over a field k.
Assume X is geometrically reduced and connected, then $\Gamma(X, \mathcal{O}_X) = k$ and thus $\Gamma(X, \mathcal{O}_X^\times) = k^\times$

Lecture 22: Differentials

Fri 13 Jan

Lemma 173

Let $f : X \to Y$ be a surjective morphism over a scheme S.
Assume $\pi_X : X \to S$ is proper and $\pi_Y : Y \to S$ is separated, then π_Y is proper.

Proof

Need to show π_Y is universally closed.
Let $T \to S$ be any scheme and $Z \subset Y_T$ closed.
By some exercise, $f_T : X_T \to Y_T$ is surjective.
So $\pi_{Y,T}(Z) = \pi_{X,T}(f_T^{-1}(Z))$ and the lhs is closed. □

Proposition 174

Let X be a proper scheme over a field k.
Assume X is geometrically raduced and geometrically connected.
Then $\Gamma(X, \mathcal{O}_X) = k$ and $\Gamma(X, \mathcal{O}_X^\times) = k^\times$

Proof

By the above, $\Gamma(X_{\overline{K}}, \mathcal{O}_{X_{\overline{K}}}) = \Gamma(X, \mathcal{O}_X) \otimes_K \overline{K}$.
So wlog, K algebraic closed.
Consider any morphism $f : X \to \mathbb{A}^1_K$.
Since $X \to \operatorname{Spec} K$ is proper and $\mathbb{A}^1_K \to \operatorname{Spec} k$ is separated so $f(X)$ is closed.
We know $f(X)$ is not $\mathbb{A}^1_K$ since $\mathbb{A}^1_K \to \operatorname{Spec} K$ is not proper.
So $f(X)$ is closed, connected, reduced ie. a closed point.

In particular, $f(X)$ is a rational point.
This means that $\Gamma(X, \mathcal{O}_X) \simeq \hom_{(Sch/k)}(X, \mathbb{A}^1_K) = \hom(\operatorname{Spec} k, \mathbb{A}^1_K) = k$ $\square$

Corollary 175
Let X be a geometrically integral scheme, proper over a field k.
Then, for any effective cartier divisor D on X, there is a bijection

$$|D| \leftrightarrow \frac{\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\}}{k^\times}$$

10 Differentials

Recall that a local ring A is regular if $\dim_{A/m} \frac{m}{m^2} = \dim A$.
We want to introduce a sheaf Ω such that $\Omega_X \otimes_{\mathcal{O}_{Y,x}} k(x) \simeq m/m^2$

Definition 79 (Derivations)
Let A be a ring, B an algebra over A and M a B-module.
An A-linear derivation from B to M is a map $d : B \to M$ such that
- $d(b + b') = db + db' \forall b, b' \in B$
- $d(bb') = db \cdot b' + b' \cdot db$
- $da = 0 \forall a \in A$

Definition 80
Let A be a ring and B an A-algebra, then the module of relative differentials forms (or Kaehler differentials) of B over A is the universal B-module $\Omega_{B/A}$ together with an A-linear derivation $d : B \to \Omega_{B/A}$ through which all derivations $d : B \to M$ factor.

Example
Let A be a ring and $B = A[x_1, \ldots, x_n]$ then $\Omega_{B/A}$ is the free B-module generated by the dx_i

Proposition 177
Let A be a ring, B, C A-algebras, $S \subset B$ mult. closed, $S' \subset A$ mult. closed mapping to units in B, then there are natural isoms.

$$\Omega_{B \otimes_A C/A} \simeq (\Omega_{B/A} \otimes_A C) \oplus (\Omega_{C/A} \otimes_A B)$$
$$\Omega_{S^{-1}B/A} \simeq S^{-1}\Omega_{B/A}$$
$$\Omega_{B/A} \simeq \Omega_{B/S'^{-1}A}$$

Now, if $f : X \to Y$ is a morphism of schemes, $\operatorname{Spec} A = V \subset Y$ open affine and $U = f^{-1}(V) = \bigcup \operatorname{Spec} B_i$, then the $\widetilde{\Omega_{B_i/A}}$ glue to a sheaf.

Definition 81
Let $f : X \to Y$ be a morphism of schemes, the qcoh. sheaf $\Omega_{X/Y}$ is called the sheaf of relative differentials or cotangent sheaf.

Proposition 178
Let $f : X \to Z$ be a morphism of schemes

1. *Let $g : Y \to Z$ be another morphism of schemes, then there is a natural isom*
$$\Omega_{X \times_Z Y/Z} \simeq pr_1^* \Omega_{X/Z} \oplus pr_2^* \Omega_{Y/Z}$$
2. *Let $U \subset X$ open, then there is a natural isom $\Omega_{X/Y/}|_U \simeq \Omega_{U/Y}$*
3. *Let $V \subset Y$ be an open subscheme, such that f factors through V then there is a natural isom $\Omega_{X/V} \simeq \Omega_{X/Y}$*
4. *Let $g : Y \to Z$ be a morphism of schemes, then there is a natural exact sequence of $\mathcal{O}_X$-modules*
$$f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$
5. *Let $Z \subset X$ be a closed subscheme with ideal sheaf $\mathcal{I}$, then there is a natural right exact sequence*
$$\mathcal{I}/\mathcal{I}^2 \to \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \to \Omega_{Z/Y} \to 0$$
6. *Let $g : Y' \to Y$ be a morphism of schemes, then there is a natural isom.*
$$\Omega_{X_{Y'}/Y'} \simeq g'^* \Omega_{X/Y}$$
where $g' : X_{Y'} \to X$ is the natural projection map.

Corollary 179
Let B be a localization of a f.g. A-algebra, then $\Omega_{B/A}$ is a f.g. B-module. In particular, if $f : X \to Y$ is of finite type and X is Noetherian, then $\Omega_{X/Y}$ is coherent.

When is the conormal sequence left-exact ?

Proposition 180
Let $f : X \to Y$ be a morphism of schemes and $\iota : Z \to X$ be a closed subscheme with ideal sheaf $\mathcal{I}$.
Assume there is a left-inverse $X \to Z$, then the conormal sequence is exact.

Lecture 23: smoothness and jacobian criterion

Mon 16 Jan

10.1 Reminder of field theory

Definition 82

1. *$K \subset L$ is said to be finite separable if the extension is finite and if $\forall \alpha \in L$, the minimal polynomial m_α has distinct roots.*
2. *K is perfect if every finite extension is separable.*
3. *$K \subset L$ is f.g. separable if $\exists$ a finite transcendence basis $x_1, \dots, x_n$ such that L is finite separable over $k(x_1, \dots, x_n)$.*

Lemma 181

Assume

- *$\mathcal{O}_{X,x}$ Noetherian*
- *$k \subset k(x)$ is f.g. separable*

then the conormal sequence is left-exact and split.

Proof

It is enough to show that there is a $\beta : k(x) \to \mathcal{O}_{X,x}$.
If $k(x)/k$ is purely transcendental, we define $\beta : k(x) \to \mathcal{O}_{X,x}$ mapping x_i to some lift.
If $k(x) = k(x_1, \dots, x_n, y)$ where y has min. pol $P(T)$.
We claim we can replace $\mathcal{O}_{X,x}$ by $\mathcal{O}_{X,x}/m_x^2$ because any derivation of $\mathcal{O}_{X,x}$ factors through $\mathcal{O}_{X,x}/m_x^2$.
We map $k(x) \to \mathcal{O}_{X,x}/m_x^2$ by sending x_i to some lift and we send $y \to y + b$ where we choose b such that $\beta(P(y)) = 0$. □

Lemma 182

Let k be a field, then $\dim k(x)\Omega_{k(x)/k} = trdegk(x)/k$.

Corollary 183 (Affine Jacobian Criterion)

Let $A = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$, denote by $\frac{\partial f_i}{\partial x_j}(x)$ the image of $\frac{\partial f_i}{\partial x_j}$ under $\mathcal{O}_{X,x} \to \frac{\mathcal{O}_{X,x}}{m_x}$.
Define the jacobian matrix at a point $x \in X$ as what you'd expect.
The cokernel of the Jacobian (seen as a map $k(x)^{\oplus m} \to k(x)^{\oplus n}$) then is

$$\operatorname{coker} J_A(x) \simeq \Omega_{A/k} \otimes_A k(x)$$

10.2 Regularity vs. smoothness

Definition 83

— *A Noetherian scheme X is said to be regular at $x \in X$ if $\mathcal{O}_{X,x}$ is regular.*
  It is said to be regular if it is regular everywhere.

— *A scheme X over k is smooth at x if $\dim \Omega_{X/k} \otimes k(x) = \dim \mathcal{O}_{X,x}$*

Corollary 184
If X is of f.t. over a perfect field K, then X is regular $\iff$ X is smooth.

Lecture 24: Canonical sheaf

Fri 20 Jan

Proposition 185
If X is of finite type over a perfect field k, then tfae

1. *X/k is smooth*
2. *X/k is smooth at every closed $x \in X$*
3. *On every connected component X_i of X, $\Omega_{X/k}$ is locally free of rank $\dim X - i$.*

Corollary 186 (Generically smooth)
If X is an integral scheme of finite type over a perfect field k, then there is a dense open set $U \subset X$ such that U is regular.

Theorem 187
If X is integral smooth of f.t. over a perfect field k.
Let $Z \subset X$ be a irreducible closed subscheme.
Z is smooth iff

1. *$\Omega_{Z/k}$ is locally free.*
2. *The conormal sequence is exact.*

Theorem 188 (Euler Sequence)
Let A be a ring, $Y = \operatorname{Spec} A$, then there is a SES

$$0 \to \Omega_{X/Y} \to \mathcal{O}_X(-1)^{\oplus n+1} \to \mathcal{O}_X \to 0$$

where $X = \mathbb{P}^n_A$.

Definition 84 (Canonical sheaf)
Let X be a smooth scheme of f.t. over a perfect field.
Then the canonical bundle/sheaf is

$$\omega_X = \Lambda^{\dim X} \Omega_{X/k}$$

Lecture 25: hypersurfaces

Mon 23 Jan

Corollary 189
Let k be a field, then $\omega_{\mathbb{P}^n/k} \simeq \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$

Proposition 190
Let D be a smooth effective Cartier divisor on a smooth scheme of f.t. over a perfect field k.
Then $\omega_D = (\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D))|_D$

11 Curves and function fields

Definition 85 (Varieties)
Let k be a field.

- *A variety over k is a geometrically integral separated scheme of f.t. over k.*
- *A variety X is called complete if X is proper.*
- *A curve is a variety of dimension 1*

Lemma 191
A curve X over k is regular iff it is normal.
If X is normal, then $\mathcal{O}_{X,x}$ is a DVR $\forall x \in X$ closed.

Theorem 192
There is an equivalence of categories

$$\{ \text{ regular complete curves } /k\} \leftrightarrow \{ \text{ f.g. field extensions } K/k \text{ of trdeg} K/k = 1 \ \}$$

where morphisms on the left are dominant morphisms.

Lemma 193
Let A be an integral domain, f.g. over a field k.
Let $K = FracA$ and $K \subset L$ a finite extension.
Then, the integral closure B of A in L is f.g. as A-module, f.g. as k-algebra, normal and has $FracB = L$.

Lemma 194
Let $f : X \to Y$ be a finite morphism of Noetherian schemes and $\mathcal{L}$ is an ample invertible sheaf on Y,, then $f^*\mathcal{L}$ is ample.

Corollary 195
Every regular complete curve X over a field k is projective.

Lecture 26: degrees on curves and riemann roch

Fri 27 Jan

Lemma 196
Let $f : X \to Y$ be a finite morphism of Noetherian schemes, let $\mathcal{L}$ be an ample invertible sheaf on Y, then $f^*\mathcal{L}$ is ample.

11.1 Degrees on curves

Lemma 197
Let $f : X \to Y$ be a dominant morphism between complete curves over k, then f is finite.

Definition 86 (Degree)
Let $f : X \to Y$ be a dominant morphism of curves over k

1. *The degree of f is $\deg f = [K(X) : K(Y)]$*
2. *f is separable if $K(Y) \subset K(X)$ is separable.*

Proposition 198
Let $f : X \to Y$ be a finite morphism of curves over k.
Assume Y is regular, then

$$f_*\mathcal{O}_X$$

is locally free of rank $\deg f$.

A uniformizer of a DVR is a generator of the maximal ideal.
Let $f : X \to Y$ be a finite morphism of regular curves over a field k.

Let $y \in Y$ be a closed point and $t \in \mathcal{O}_{Y,y}$, then

$$\deg f = \sum_{f(x)=y} v_X(f^\sharp(\varpi_y))[k(x):k(y)]$$

Definition 87
Let $f : X \to Y$ be a finite morphismof regular curves over a field

1. *The ramification index of f at a closed point $x \in X$ is defined as $e_f(x) = v_x(f_x^\sharp t)$ where t is a uniformizer ov $\mathcal{O}_{Y,f(x)}$*
2. *A closed point $x \in X$ is called a ramification point if $e_f(x) > 1$*
3. *The inertia degree of f at a closed point $x \in X$ is $[k(x) : k(y)]$.*

It is clear how to pull back $(\mathcal{L}, s)$ along $f : X \to Y$, namely, set $f^*(\mathcal{L}, s) = (f^*\mathcal{L}, f^{*s})$

Definition 88
Let $f : X \to Y$ be a finite morphism of regular curves over k, let $D = \sum_{p \in Y} a_p p$ be a Weil divisor on Y.
Then the pullback of D along f is $f^*D = \sum_{p\in Y} \sum_{f(x)=p} a_p e_f(q) f_f(q) q$

Definition 89
Let $D = \sum_{p\in X} a_p p$ be a Weil divisor on a regular curve x.
The degree of D is $\deg D = \sum_{p\in x} a_p[k(p) : k]$

Proposition 199
Let $f : X \to Y$ be a finite morphism of regular curves, let D be a Weil divisor, then deg$(f^*D) = \deg f \deg D$

Proposition 200
Let D be a principal divisor on a regular complete curve X.
Then $\deg D = 0$

Lemma 201
Let X be a regular complete curve over a field k and $\mathcal{L}$ is an invertible sheaf.
If $\deg(\mathcal{L}) < 0$, then $H^0(X, \mathcal{L}) = 0$.

In the following, let $h^0(X, \mathcal{L}) = \dim_K H^0(X, \mathcal{L})$, then

Theorem 202 (Riemann-Roch)
Let X be a smooth, complete, geometrically integral curve over a perfect field k.
Let $\mathcal{L} \in Pic(X)$, then $h^0(X, \mathcal{L}) < \infty$ and

$$h^0(X, \mathcal{L}) - h^0(X, \omega_X \otimes \mathcal{L}^{-1}) = \deg \mathcal{L} - h^0(X, \omega_X) + 1$$

Definition 90
Let X be as before, then the geometric genus of X is $g(X) = h^0(X, \omega_X)$