Topology I

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1 Homology Theories

Lecture 1: Introduction

Mon 10 Oct

Aim : Study further algebraic invariants of topological spaces.
We want to assign to pairs of topological spaces abelian groups.

$$h_n : T \to \text{Ab} \quad \forall n \in \mathbb{Z}$$

and to pairs continuous maps, we want to assign a map $h_n(f) : h_n(X) \to h_n(Y)$ which is functorial. Here T is the category of pairs of topological spaces $A \subset X$ with morphisms $f : (X, A) \to (Y, B)$ such that $f(A) \subset B$.
To relate h_n for different $n \in \mathbb{N}$, we will construct connecting morphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$.

Axiom 1 (Eilenberg-Steenrod Axiom)
A (generalised) homology theory consists of functors $h_n : T \to \text{Ab}$ and natural connecting homomorphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$ [1] *satisfying*

- *Homotopy invariance :*
  If $f, g : (X, A) \to (Y, B)$ are homotopic continous maps of pairs then the induced maps $h_n(f) = h_n(g)$. Here homotopy of pairs means that there exists $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subset B$
- *Long exact sequence of a pair (LES) :*
  Given a pair of topological spaces (X, A) there is a long exact sequence of abelian groups.
  Denote $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$, then

$$h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A, \emptyset)$$

- *Excision*
  Given $B \subset A \subset X$ subspaces such that $\overline{B} \subset A^o$, the inclusion induces a group isomorphism

$$h_n(X \setminus B, A \setminus B) \to h_n(X, A)$$

  We add another axiom to "make things easier"
- *Additivity :*
  Given a family of pairs of spaces $(X_i, A_i)_{i \in I}$, the inclusions induce an isomorphism

$$\bigoplus h_n(X_i, A_i) \to h_n(\coprod X_i, \coprod A_i)$$

  This is the end of the axioms for a generalised homology theory, the homology theory is called an ordinary homology theory if the <u>Dimension Axiom</u> holds, namely

$$h_n(pt) = 0 \forall n \neq 0$$

1. From now on, we write $h_n(A) \coloneqq h_n(A, \emptyset)$

The abelian group $h_0(\text{ } pt \text{ })$ is the called the coefficient group of (h_n, ∂_n)

Lemma 2
If $f : X \to Y$ is a homotopy equivalence, then $\forall n \in \mathbb{Z}$ we obtain $h_n(f) : h_n(X) \to h_N(Y)$ to be an isomorphism for any homology theory (h_n, ∂_n)

Preuve
Choose $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$, then by functoriality and homotopy invariance $\mathrm{Id}_{h_n(X)} = h_n(\mathrm{Id}_X) = h_n(g) \circ h_n(f)$, by symmetry, $h_n(f)$ and $h_n(g)$ are inverses. $\square$

Similarly, if $f : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs, then the same result holds.

Example
For any such homology theory

$$h_n(\mathbb{R}^k) \simeq h_n(\text{ } pt \text{ }) \simeq h_n(D^k)$$