

Algebraic Curves

David Wiedemann

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Lecture 1: Introduction

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Let K be a field, given a set of polynomials $S = \{f_1, \dots\}$, we can consider $V(S) = \{(x_1, \dots) \in K^n \mid f_i(x_1, \dots) = 0 \forall i\}$.

Notice that if $a_1, \dots \in K[x_1, \dots]$ then also $\sum_i a_i(x)f_i(x) = 0$ only depends on the ideal generated by S .

If $I(S)$ happens to be prime, we call V an algebraic variety.

1 Affine algebraic sets

1.1 Recollection on commutative algebra

All rings are commutative and with unit.

Let R be a ring.

- R is an integral domain, or just domain if there are no zero divisors, ie, $\forall a, b \in R$ s.t.

$$a \cdot b = 0 \implies a = 0 \text{ or } b = 0$$

- Any domain can be embedded into its quotient ring.

- A proper ideal I is maximal if it's not contained in any other proper ideal
- A proper ideal I is prime if

$$\forall a, b \in R, ab \in I \implies a \in I \text{ or } b \in I$$

- A proper ideal I is radical if

$$a^n \in I \implies a \in I$$

- For any ideal $I \subset R$, the radical \sqrt{I} is the smallest radical ideal containing I

Lemme 1

$I \subset R$ is maximal $\iff R/I$ is a field

Lemme 2

$I \subset R$ is prime $\iff R/I$ is a domain

Lemme 3

radical $\iff R/I$ has no nilpotent elements.

Given a subset $S \subset R$ we can consider the ideal generated by S

$$I(S) = \left\{ \sum_i a_i s_i \right\}$$

I is finitely generated if $I = I(S)$ with S finite.

- We say that R is Noetherian \iff \nexists a chain of strictly increasing ideals. Equivalently, every ideal is finitely generated.

Theorème 4

- *In fact, hilbert's basis theorem says that, if R is Noetherian, then $R[x]$ is noetherian.*

In particular $K[x_1, \dots, x_n]$ is Noetherian

- I is in principal if it is generated by one element.
- A domain is called a principal ideal domain (PID) if every ideal is principal.
- $a \in R$ is irreducible if a is not a unit, nor zero and if

$$a = b.c$$

then either b or c are units.

- A pid $(a) \subset R$ is prime $\iff a$ is irreducible.
- R is a UFD if R is a domain and elements in R can be factored uniquely up to units and reordering into irreducible elements.

Theorème 5

R is a UFD $\implies R[x]$ is a UFD

And, if R is a PID, then R is a UFD

Theorème 6 (Gauss Lemmma)

- *R is a UFD and $a \in R[X]$ irreducible, then also $a \in Q(R)[X]$ is irreducible.*

- Localization

Let R be a domain, if $S \subset R$ is a multiplicative subset, then the localization of R at S is defined as

$$S^{-1}R = \left\{ x \in Q(R) \mid x = \frac{a}{b}, b \in S \right\}$$

If M is an R -module, we have similarly

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \left\{ \frac{m}{s} = \frac{m'}{s'} \iff ms' = sm' \right\}$$

If $p \subset R$ is a prime ideal, then it's complement is a multiplicative subset and we define

$$R_p = (R \setminus p)^{-1}R$$

- There is a 1-1 correspondence between $p \subset R$ prime and ideals of R_p , furthermore R_p is a local ring
- Localization is exact, in particular, given $I \subset p$ the short exact sequence

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$$

gets sent to

$$0 \rightarrow I_p \rightarrow R_p \rightarrow (R/I)_p \rightarrow 0$$

ie. localization commutes with taking quotients.

1.2 Polynomial rings

For $a \in \mathbb{N}^n$, we set

$$X^a = X_1^{a_1} \dots \in k[X_1, \dots]$$

Thus for any $F \in k[X_1, \dots, X_n]$, we can write it as

$$F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$$

F is homogeneous or a form of degree d if the coefficients $\lambda_a = 0$ unless $a_1 + \dots + a_n = d$.

Any F can be written uniquely as $F = F_0 + \dots + F_d$ where F_i is a form of degree i .

The derivative of $F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$ with respect to X_i is $F_{X_i} = \frac{\partial F}{\partial X_i}$.

If F is a form of degree d we have

Theorème 7 (Euler's theorem)

$$\sum_{i=1}^n \frac{\partial F}{\partial X_i} X_i = dF$$