Série 1

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1 Problem Set 2 Exercise 3

We first note that we may rewrite

$$\begin{aligned} P = \quad & \min \; 2x + 3|y-10| \\ \text{s.t.} \quad & |x-2| + y \leq 5 \end{aligned}$$

as

$$\begin{aligned} P' = \quad & \min \; 2x + 3w \\ \text{s.t.} \quad & |x-2| + y \leq 5 \\ & w \geq y - 10 \\ & w \geq 10 - y \end{aligned}$$

Applying the same trick for $|x-2|$ yields

$$\begin{aligned} P'' = \quad & \min \; 2x + 3w \\ \text{s.t.} \quad & z + y \leq 5 \\ & w \geq y - 10 \\ & w \geq 10 - y \\ & z \geq x - 2 \\ & z \geq 2 - x \end{aligned}$$

Note that positivity conditions on w and z are not needed as at least one $y-10$ or $10-y$ (respectively $x-2$ or $2-x$) will be positive.

We now justify why P and P' have the same objective value.
First of, suppose P has an optimal solution $v = (x_0, y_0)$, we then pretend $v_* = (x_0, y_0, |y_0 - 10|)$ (where the last coordinate is the value of w) also is an optimal solution for P' which takes the same objective value as P.

Indeed, the fact that it takes the same objective value as P is clear.
Now suppose P' has a smaller optimal solution than P, then, as the constraints on x are left unchanged, this implies the value of w is smaller than the value of $|y - 10|$, but this is impossible since we force $w \geq y - 10$ and $w \geq 10 - y$.
The very same argument applied to go from P' to P'' works to show P'' has the same optimal objective value as P'.
Hence P'' is a linear program with the same optimal objective value as P

2 Problem Set 3 Exercise 2

Let v be a vertex of a polyhedron P, by definition this implies there exists a vector c such that $c \cdot v > c \cdot y \forall y \in P \setminus \{v\}$.
Now suppose v is not an extreme point, this means there exists $x, y \in P \setminus \{v\}$ such that $v = \lambda x + (1 - \lambda)y$.
Now first note that $c \cdot x \neq c \cdot y$, indeed, if $c \cdot x = c \cdot y$, then

$$c \cdot v = c \cdot (\lambda x + (1 - \lambda)y) = \lambda c \cdot x + (1 - \lambda)c \cdot x = c \cdot x$$

Which is impossible since v is the unique optimal solution.
Hence, we may suppose without loss of generality that $c \cdot x > c \cdot y$, but then

$$c \cdot x = \lambda c \cdot x + (1 - \lambda)c \cdot x > \lambda c \cdot x + (1 - \lambda)c \cdot y = c \cdot v$$

Which contradicts v being a vertex of P.

3 Problem Set 3 Exercise 3

We set v_i to be the vectors corresponding to the different planes $P_1, \dots, P_4$ and k_i the corresponding constant terms. To determine in which order the ray passes through the P_i, we have to solve the equations

$$v_i \cdot (x^* + \lambda d) = k_i$$

for λ.
We first solve the equation in general and then plug in the values, ie. we solve

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right) = k$$

$$\begin{aligned} a\lambda + b + b\lambda + c - c\lambda &= k \\ (a + b - c)\lambda &= k - b - c \\ \lambda &= \frac{k - b - c}{a + b - c} \end{aligned}$$

Now pluging in the different values for a, b, c and k , we get

$$\lambda_1 = \text{“} -\frac{5''}{0} \text{ is undefined , since}$$
$$(123)x^* \neq 0$$

We conclude the ray never passes through P_1

$$\lambda_2 = \frac{1}{4}$$
$$\lambda_3 = 0$$
$$\lambda_4 = \frac{5}{2}$$

From this we conclude that the ray

— Never passes through P_1

— Then passes through

1. P_3
2. P_2
3. P_4