

Exercise 6

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Let

$$p(x) = x^k - \alpha_1 x^{k-1} - \dots - \alpha_k$$

be the characteristic polynomial of the linear recurrence.

By hypothesis q is a root with multiplicity m , we will use without proof that, for $0 \leq i < m$, q will be a root of $\frac{d^i}{dx^i} p(x)$.

First, notice that the case $i = 0$ is clear, indeed, we have

$$\begin{aligned} 0 &= q^k - \alpha_1 q^{k-1} - \dots - \alpha_k \\ q^k &= \alpha_1 q^{k-1} + \dots + \alpha_k \\ q^n &= \alpha_1 q^{n-1} + \dots + \alpha_k q^{n-k} \end{aligned}$$

Where, in the last step, we have simply multiplied by q^{n-k} .

Since this holds for all $n > k$, we have shown that q^n is a solution of the linear recurrence.

We will now prove the result for $i < m$.

Notice that, if $m \geq 1$, the result cited above implies in particular that

$$\begin{aligned} xq^{n-1} - \alpha_1(n-1)x^{n-2} - \dots - (n-k)x^{n-k-1} &= 0 \\ xq^n - \alpha_1(n-1)x^{n-1} - \dots - (n-k)x^{n-k} &= 0 \\ nq^n - \alpha_1(n-1)q^{n-1} - \dots - \alpha_k(n-k)q^{n-k} &= 0 \end{aligned}$$

Thus, nq^n also satisfies the linear recurrence relation.

In general, for $i < m$, repeating this process i times (ie. differentiating with respect to x and then multiplying by x) gives the equality

$$n^i q^n - \alpha_1(n-1)^i q^{n-1} - \dots - \alpha_k(n-k)^i q^{n-k} = 0$$

And thus, $n^i q^n$ is a solution to the linear recurrence.

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We will suppose that $q \neq 0$, indeed, if $q = 0$, the linear sequence would be the trivial sequence $\{0\}_{n=1}^{\infty}$. Suppose there exist factors $x_0, \dots, x_{m-1} \in \mathbb{R}$ satisfying

$$x_0 \{q^n\}_{n=1}^{\infty} + \dots + x_{m-1} \{n^{m-1}q^n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$$

Then, taking the $m-1$ first terms of each sequence, we get the linear system

$$\begin{cases} x_0q + x_1q + \dots x_{m-1}q = 0 \\ x_0q^2 + x_12q^2 + \dots + x_{m-1}2^{m-1}q^2 = 0 \\ \vdots \\ x_0q^{m-1} + x_1(m-1)q^{m-1} + \dots + x_{m-1}(m-1)^{m-1}q^{m-1} = 0 \end{cases}$$

Which simplifies to

$$\begin{cases} x_0 + x_1 + \dots x_{m-1} = 0 \\ x_0 + x_12 + \dots + x_{m-1}2^{m-1} = 0 \\ \vdots \\ x_0 + x_1(m-1) + \dots + x_{m-1}(m-1)^{m-1} = 0 \end{cases}$$

Putting the system into matrix form, we get

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & 2^{m-1} \\ 1 & 3 & \dots & 3^{m-1} \\ \vdots & & \ddots & \vdots \\ 1 & (m-1) & \dots & (m-1)^{m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We will now show that the determinant of this matrix is not equal to 0, an elementary result from linear algebra will thus show that $x_i = 0 \forall 0 \leq i \leq m-1$, which implies that the linear sequences are linearly independent.