Functional Analysis

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1 Introduction

Lecture 1: Introduction

Wed 12 Oct

Main reference is "Functional Analysis" by H.W. Alt.

1.1 Topological Spaces

Definition 1 (Topological space)
Let X be a set, a topology is a subset $\tau \subset P(X)$ is a topology if
- $\emptyset, X \in \tau$
- *any union of opens is open*
- *Finite intersections of opens are open.*

Definition 2 (Properties)
For $A \subset X$, $\overline{A}$ is the smallest closed set containing A and the interior A^o is the biggest open set contained in A.
Finally, the boundary is $\partial A = \overline{A} \setminus A^o$.
X is separable if $\exists$ a dense countable subset

Definition 3 (Sequences)
Let $x : \mathbb{N} \to X, \overline{x} \in X$, $\lim x_k = \overline{x} \iff$ any neighbourhood $U \in T$ of x eventually contains x_k

Definition 4 (Continuity)
A function $f : X \to Y$ is continuous if $\forall U \in \tau_Y, f^{-1}(U)$.
This is different from sequential continuity $x_n \to \overline{x} \implies f(x_n) \to f(\overline{x})$
.
f is continuous at $x \in X$ if $\forall V \in S$ st $f(x) \in V \implies f^{-1}(V) \in \tau_X$

Lecture 2: More recaps

Fri 14 Oct

1.2 Metric spaces

Definition 5 (Metric space)
X a set, $d : X \times X \to [0, \infty)$ is a matrix

Definition 6
X a set, d_1, d_2 metrics

1. d_1 *is topologically stronger than* d_2 *if* τ_{d_1} *is finer.*
2. d_1 *is uniformly stronger than* d_2 *if* $\exists C > 0$ *such that* $d_2 \leq Cd_1$
3. d_1 *is uniformly stronger than* d_2 *if* $\exists C > 0$ *such that* $\frac{1}{C}d_1 \leq d_2 \leq Cd_1$

Lemma 1
THe following are equivalent

1. d_1 *is topologically stronger than* d_2
2. $\mathrm{Id} : (X, \tau_{d_1}) \to (X, \tau_{d_2})$ *is continuous*
3. *If* $x_n \to \overline{x}$ *in* d_1 *then* $x_n \to \overline{x}$ *in* d_2
4. $\forall x \in X \forall \epsilon > 0 \exists \delta_{\epsilon,x} > 0$ *such that*

$$d(x, y) \leq \delta \implies d_2(x, y) < \epsilon$$

Definition 7
Let (X, d) be a metric space

1. $A \subset X$ *is bounded if* $\exists \overline{x} \in X$ *such that* $\sup_{y \in A} d(x, y) < \infty$ *or* $A = \emptyset$
2. x_n *is Cauchy if*
$$\lim_{n\to\infty} \sup_{i,j \geq n} d(x_i, x_j) = 0$$
3. X *complete if x Cauchy* $\implies$ *x convergent.*
4. (Y, e) *is a matric,* $f X \to Y$ *is uniformly continuous if* $\forall \epsilon > 0 \exists \delta > 0$ *such that* $d(x, y) < \delta \implies e(f(x), f(y)) < \epsilon$.

Define $X = \{x : \mathbb{N} \to \mathbb{R}$ such that $\exists N$ such that $x_i = 0$ eventually $\}$. This space, with p-norm is not complete, so we construct the completion.

Proposition 2
Let (X, d) a metric space and (Y, e) a complete metric space, $A \subset X, \phi : A \to Y$ uniformly continuous.
Then $\exists$ unique $\psi : \overline{A} \to Y$ such that ψ is uniformly continuous and $\phi = \psi|_A$.

Proof
If $x : \mathbb{N} \to A$ is Cauchy, then $\phi \circ x$ is also cauchy.
To prove this, let $\epsilon > 0$ and $\partial^{\phi}_{\epsilon} > 0$ be such that $d(x, y) < \delta \implies e(\phi(x), \phi(y)) < \epsilon$.

Let $N = N_\delta^x$ *be such that* $i, j \geq N \implies d(x_i, x_j) < \delta$ *, then* $e(\phi(x_i), \phi(x_j)) < \epsilon$

Now, let $a \in \overline{A}$ *, then* $\exists x_k$ *converging to* a*.*
x *is d-Cauchy and* $\phi \circ x$ *is e-cauchy.*
$\exists$ *a limit* $b^* = \lim \phi(x_k)$ *So we define* $\psi(a) = b^*$*.*
We now prove continuity/uniform continuity.
Let $a, b \in \overline{A}, x, y : \mathbb{N} \to A$ *and* $x_i \to b, y_j \to b$*.*
Then

$$e(\psi(a), \psi(b)) = \lim e(\phi(x_i), \phi(y_j))$$

Now, let $\epsilon > 0$*, then* $\exists \delta > 0$ *such that* $d(x, y) < \delta$*.*
Thus $e(\phi(x), \phi(y)) < \epsilon$
If $d(a, b) < \delta$ $\exists N$ *such that* $d(x_i, y_j) < \delta \forall i, j > N$

$$e(\phi(x_i), \phi(y_j)) < \epsilon \implies e(\psi(a), \psi(b) \leq \epsilon)$$

□

Theorem 3
If (X, d) *is a metric space, then there exists a complete metric space* (Y, e) *and an isometry* $\phi : X \to Y$ *such that* $Y = \overline{\phi(X)}$ *.*
Both are unique up to a bijective isometry.

Proof
Define $C_X := \{x : \mathbb{N} \to X, x \text{ Cauchy } \}$ *and* $x\tilde{y}$ *if* $\lim_{j\to\infty} d(x_i, y_j) = 0$*.*
Write $Y = C_X / \sim$ *.*
For $x, y \in Y$*, define* $e(x, y) = \lim_{j\to\infty} d(x_i, x_j)$ *.*
Is this well defined ?
If $j, k \geq N$

$$|d(x_i, y_i) - d(x_k, y_k)| \leq d(x_i, x_k) + d(y_j, y_k)$$

And if $x\tilde{x}'$*, then*

$$\lim d(x_i, y_j) = \lim d(x'_j, y_j)$$

because

$$|d(x_i, y_j) - d(x'_j, y_j)| \leq d(x_j, x'_j) \to 0$$

To show that e *is a metric, most properties are obvious.*
We show that if $e(x, y) = 0$ *then* $\lim d(x_j, y_j) = 0 \implies x\tilde{y} \implies x = y$
Triangular equality holds because

$$e(x, y = \lim d(x_j, y_j \leq \limsup d(x_i, z_j) + d(z_j, y_j) = e(x, z) + e(z, y)$$

The isometry $\phi : X \to Y$ *simply sends* $x \mapsto [x]$*.*
We now show $[x] \in Y$*,* $\phi(x_k)$ *is a sequence in* Y*, we want to show that*

$\phi(x_k) \to [x]$.

$$\lim_{k\to\infty} e(\phi(x_k),[x]) = \lim_{k\to+\infty}\lim_{j\to\infty} d(x_k,x_j) = 0$$

Which shows $Y = \overline{\phi(X)}$ *Let* y^k *Cauchy* $\forall k \exists x_k \in X$ *such that* $e([y^k],\phi(x_k)) < 2^{-k}$.
We claim $[y^k] \to [x]$

$$d(x^k,x^h = e(\phi(x^k,\phi(x^h)))) \leq 2^{-k} + 2^{-h+e([y^k],[y^h])}$$

Thus $x \in C_X$ $[x] \in Y$

$$e([y^k],[x]) = \lim d(y_j^k,x_j) \leq \lim d(U_j^k,x_k) + d(x_k,x_j) \leq 2^{-k}$$

Finally, to show uniqueness, if (Y,e) *and* (Y',e') *are two completions.*
Let $\psi = \phi \circ (\phi')^{-1} : \phi'(X) \to Y$.
ψ *is an isometry so there is a unique extension* $\psi : Y' \to Y$ *and this is an isometry.* □

1.3 Norms, Banach Spaces

Throughout, $K = \mathbb{R}$ or $\mathbb{C}$

Definition 8 (Normed space)
$\|\cdot\| : X \to [0,\infty)$ *is a norm if*
- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda| \, \|X\|$
- $\|x+y\| \leq \|x\| + \|y\|$

Definition 9
c_0 *is the space*$c_0 = \{x : \mathbb{N} \to \mathbb{R}$ *s.t.* $\lim x_k = 0\}$ *together with* $\|x\|_{c_0} = \sup |x_k|$
For $p \in [1,\infty)$, $l_p = \left\{x : \mathbb{N} \to \mathbb{R} \text{ s.t. } \sum_{k\in\mathbb{N}} |x_k|^p < \infty\right\}$ *with* $\|x\|_{l_p} = (\sum |x_k|^p)^{\frac{1}{p}}$

Definition 10 (Banach Space)
A Banach space is a complete normed space.

Proposition 4
Any normed space has a completion which is Banach.

Proof
Let (Y, e) be the completion as above, define

$$[x] + [y] \coloneqq [x + y] \text{ and } \lambda[x] \coloneqq [\lambda x]$$ □

1.4 Basis of a normed space

Definition 11
Let $A \subset X$.
A is linearly independent if $\forall N \in \mathbb{N}, \forall a_i \in A \forall \lambda_i \in K, \sum_i \lambda_i a_i = 0 \implies \lambda_i = 0$.
We define

$$span(A) = \left\{ \sum(i) \lambda_i a_i, \lambda_i \text{ as above} \right\}$$

A is a Hamel basis if A is linearly independent and $X = spanA$

Definition 12 (Schauder Basis)
$e : \mathbb{N} \to X$ is a Schauder basis if $\forall x \in X$ there is a unique $\lambda : \mathbb{N} \to K$ such that $x = \sum_{i=0}^{\infty} \lambda_i e_i \iff \lim \left\| x - \sum^N \lambda_i e_i \right\| = 0$