Assignment 3

David Wiedemann

9 mai 2022

1 Problem Set 7, Exercise 3

To each edge $e \in E$, we associate a number $a(e) \in \{0, 1\}$ We will say that e is in M if $a(e) = 1$ and that e is not in M if $a(e) = 0$.
Now, for each vertex $v \in V$, we denote by $E(v) \subset E$ the set of all edges containing v.
The condition for M to be a matching then reads as $\sum_{e \in E(v)} a(e) \leq 1 \forall v \in V$.
Thus, the problem of finding a matching with a maximal number of edges reads as

$$\begin{aligned} \max \sum_{e \in E} a(e) \\ \text{s.t.} \sum_{e \in E(v)} a(e) \leq 1 \forall v \in V \\ a(e) \in \{0, 1\} \end{aligned}$$

2 Every tree has at least one leaf vertex

Recall that a tree is a connected graph that contains no cycles and that a leaf is a vertex of degree 1.
Let $T = (V, E)$ be a (finite) tree.
Suppose by contradiction that T does not contain any leaf, we construct a path as follows. We pick some vertex $v_1 \in V$, as T is connected, there is some vertex v_2 connected to v_1.
As v_2 is not a leaf it is at least of degree 2, so there is a vertex v_3 different from v_1 which is connected to v_2.
Proceeding inductively in this way, we get a path $v_1, v_2, \ldots,$ which always satisfy $v_{n+1} \neq v_{n-1}$ (ie. we aren't going back and forth).
As the set V is finite, this path must cross itself again at some point, ie. there is some index j and some $i < j - 1$ satisfying $v_i = v_j$.
We may pick j minimal among all indexes with this property, then the walk $v_i, v_{i+1}, \ldots, v_{j-1}, v_j = v_i$ is in fact a cycle (as it doesn't intersect itself by our minimality assumption on j).
But this contradicts the fact that T is a tree.

3 Problem Set 9, Exercise 2.a)

Let x be a basic solution for $\mathcal{P}$, then there exists a basis β of n elements such that

$$A_\beta x_\beta = b$$

and such that for every index $i \notin \beta$, $x_i = 0$.
As the set of collumn vectors of A_β is linearly independent, we know that A_β is invertible and thus by hypothesis $\det A_\beta = \pm 1$.
Cramer's rule tells us that for all $j \in \beta$, the value of x_j is given by

$$x_j = \frac{\det A_\beta^j}{\det A_\beta}$$

where A_β^j denotes the matrix A_β where we've replaced the j-th collumn with the vector b.
But then $\det A_\beta^j$ is an integer and as $\det A_\beta = \pm 1$, we conclude that in fact x is a vector with integer entries.