Exercise 10

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We will write V for the set of vertices of G.
Let $e \in E(G) \setminus E(T)$, we can write e as $e = \{a, b\}$, $a, b \in V$.
Since $E(T)$ is connected, there exists a path in T of the form $(a, v_1, \dots, v_n, b), v_i \in G$.
We define $K = T + e - \{v_n, b\}$.
Clearly, K is still spanning since it still contains v_n, a and b[1] .
We now show that K is a tree.
First, we show that K still is connected.
Indeed, consider two vertices $x, y \in V$.
Consider the path of T which would connect x to y : $x, u_0, \dots, u_n, y$.
If there exists $0 \leq i \leq n$ such that $\{u_i, u_{i+1}\} = e$, replace u_i, u_{i+1} in the path with the path connecting a and b, we are left with a path contained in K.
Hence, K is connected.
We now show that K contains no cycle.
For the sake of contradiction, suppose K contains a cycle of the form $c_0, \dots, c_k, c_0$.
If $\forall 0 \leq j \leq k, \{c_j, c_{j+1}\} \neq e$, then the cycle is contained in T which is a contradiction since T is a tree.
Hence, suppose there exists a j such that $\{c_j, c_{j+1}\} = e$, without loss of generality, suppose that $c_j = a$ and $c_{j+1} = b$.
If that were the case, we could again create a new cycle of the form $c_0, \dots, c_{j-1}, a, v_1, \dots, v_n, b, c_{j+2}, \dots$

1. v_n is still contained in K because $\{v_{n-1}, v_n\}$ is contained in K