Serie 6

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega := (0,1)\times(0,1)$. Investigate the existence and equality of $\int_\Omega f\, d(x,y)$, $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ for

(i) $f(x,y) := \frac{x^2-y^2}{(x^2+y^2)^2}$.

(ii) $f(x,y) := (1-xy)^{-a}$ for $a > 0$.

Compare your result with Fubini's Theorem.

Solution:

(i) We have

$$\int_0^1\int_0^1 f(x,y)\,dy\,dx = \int_0^1\int_0^1 \partial_y\left(\frac{y}{x^2+y^2}\right)dy\,dx = \int_0^1 \frac{1}{x^2+1}\,dx = [\arctan(x)]_0^1 = \frac{\pi}{4}$$

and

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1\int_0^1 \partial_x\left(\frac{-x}{x^2+y^2}\right)dx\,dy = \int_0^1 \frac{-1}{y^2+1}\,dy = [-\arctan(y)]_0^1 = -\frac{\pi}{4}$$

This computation does not contradict Fubini as $f \notin L^1(\Omega)$. Indeed, we compute by Tonelli:

$$\begin{aligned}\int_0^1\int_0^1 |f(x,y)|\,dy\,dx &= \int_0^1\int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx + \int_0^1\int_x^1 \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\,dx \\ &= \int_0^1\int_0^x \partial_y\left(\frac{y}{x^2+y^2}\right)dy\,dx + \int_0^1\int_x^1 \partial_y\left(\frac{-y}{x^2+y^2}\right)dy\,dx \\ &= \int_0^1 \left[\frac{y}{x^2+y^2}\right]_0^x dx + \int_0^1 \left[\frac{-y}{x^2+y^2}\right]_x^1 dx \\ &= \int_0^1 \frac{x}{2x^2} - \frac{1}{1+x^2} + \frac{x}{2x^2}\,dx = \int_0^1 \frac{1}{x} - \frac{1}{1+x^2}\,dx = \infty.\end{aligned}$$

(ii) Let $f_a(x,y) = (1-xy)^{-a}$. Note that $f_a(x,y)$ is positive on Ω so $\int_\Omega f_a(x,y)d(x,y) = \int_0^1\int_0^1 f_a(x,y)\,dy\,dx = \int_0^1\int_0^1 f_a(x,y)\,dx\,dy$ by Tonelli's theorem. We show that $\int_\Omega f_a(x,y)d(x,y)$ is finite if and only if $0 < a < 2$. Let's consider two cases: $a = 1$ and $a \neq 1$.

- Case $a = 1$: In this case the integral is finite and we can actually compute it.

$$\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{dx}{1-xy}\right) dy &= \int_0^1 \left(-\frac{1}{y}\int_1^{1-y} \frac{ds}{s}\right) dy \quad \text{by setting } s = 1-xy, ds = -y\,dx \\
&= \int_0^1 \frac{1}{y}[\log(s)]_{1-y}^1\,dy = \int_0^1 \frac{-\log(1-y)}{y}\,dy \\
&= \int_0^1 \sum_{k=1}^\infty \frac{y^{k-1}}{k}\,dy \quad \text{by expanding } \log(1-y) \text{ into its Maclaurin series} \\
&= \sum_{k=1}^\infty \int_0^1 \frac{y^{k-1}}{k}\,dy = \sum_{k=1}^\infty \left[\frac{1}{k^2}y^k\right]_0^1 dy = \sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6},
\end{aligned}$$

  where we interchanged the serie and the integral by either monotone convergence (the serie has positive terms) or by uniform convergence of the Maclaurin serie.

- Case $a \neq 1$: We have

$$\begin{aligned}
\int_0^1 \left(\int_0^1 \frac{dx}{(1-xy)^a}\right) dy &= \int_0^1 -\frac{1}{y}\left(\int_1^{1-y}\frac{ds}{s^a}\right) dy \\
&= \int_0^1 -\frac{1}{y}\left[\frac{s^{1-a}}{1-a}\right]_1^{1-y} dy \\
&= \int_0^1 -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}\,dy.
\end{aligned}$$

  Observe that the integrand $y \mapsto -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}$ is continuous and bounded (by $\frac{1}{1-a}$) on $(0,1)$ if $0 < a < 1$, and thus has a finite integral.

  So assume $a > 1$. In this case, observe that

$$(1-y)^{1-a} \leq -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a} \leq \frac{1}{a-1}(1-y)^{1-a}, \qquad \forall y \in (0,1).$$

  Note that the integral $\int_0^1 (1-y)^{1-a}dy$ is finite if and only if $a-1 < 1$, i.e., $a < 2$. Hence, we conclude that the integral $\int_0^1 -\frac{1}{y}\cdot\frac{(1-y)^{1-a}-1}{1-a}\,dy$ is finite if and only if $a < 2$.

We conclude that f_a is absolutely integrable if and only if $0 < a < 2$. Since f is absolutely integrable, we can apply Fubini's theorem, which is consistent with our computations.

Exercise 2. The *Dirichelet integral* is the improper integral defined by

$$\int_0^\infty \frac{\sin(x)}{x}\,dx.$$

It is a very simple example of an integral that exists in the Riemann sense but **not** in the Lebesgue sense. Explain why and then compute the value of this integral (understood as a Riemann-integral) by restricting the the domain of integration to $[0,t]$ and then taking the limit as $t \to \infty$.

Solution:

You might have already encountered this integral in other analysis courses and you were maybe asked to show that the function $x \mapsto \frac{\sin(x)}{x}$ is *Riemann-integrable* but not *absolutely integrable.* Let's recall the proofs of theses two properties :

- The function $x \mapsto \frac{\sin(x)}{x}$ is Riemann-integrable on $[0, \infty)$ because it can be extended by continuity at 0, and in particular it is Riemann-integrable on $[0, \pi]$. Also, $\forall t \geq \pi$

$$\int_\pi^t \frac{\sin(x)}{x}\, dx \underset{\text{IBP}}{=} \left[\frac{-\cos(x)}{x}\right]_\pi^t - \int_\pi^t \frac{\cos(x)}{x^2}\, dx = \frac{-\cos(t)}{t} - \frac{1}{\pi} - \int_\pi^t \frac{\cos(x)}{x^2}\, dx.$$

  We know that the integral $\int_\pi^\infty \frac{\cos(x)}{x^2}\, dx$ exists and equals $\lim_{t\to\infty} \int_\pi^t \frac{\cos(x)}{x^2}\, dx$ because $\left|\frac{\cos(x)}{x^2}\right|$ is dominated by $\frac{1}{x^2}$. We conclude that $\int_\pi^t \frac{\sin(x)}{x}\, dx$ converges as $t \to \infty$.

- The function is not not absolutely integrable because

$$\begin{aligned}\int_0^\infty \left|\frac{\sin(x)}{x}\right| dx &= \lim_{n\to\infty} \sum_{k=0}^n \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin(x)}{x}\right| dx \\ &\geq \lim_{n\to\infty} \sum_{k=0}^n \frac{1}{(k+1)\pi} \underbrace{\int_{k\pi}^{(k+1)\pi} |\sin(x)|\, dx}_{=2} \\ &= \frac{2}{\pi} \sum_{k=1}^\infty \frac{1}{k} = \infty.\end{aligned}$$

  For the first equality, we implicitly used the monotone convergence theorem.

To compute the value of the integral, we use Fubini's theorem, which can only be applied to absolutely integrable functions. We therefore restrict the domain of integration to a bounded one to understand the integral as a Lebesgue integral:

$$\int_0^t \frac{\sin x}{x}\, dx = \int_0^t \sin x \int_0^\infty e^{-ax}\, da\, dx \underset{\text{Fubini}}{=} \int_0^\infty \underbrace{\int_0^t \sin x e^{-ax}\, dx}_{=:I_t(a)}\, da$$

Now the inner integral $I_t(a)$ is taken on a bounded domain so we can again re-interpret it as a Riemann-integral and use integration by parts:

$$I_t(a) = \left[-\cos(x)e^{-ax} - a\sin(x)e^{-ax}\right]_0^t - a^2 I_t(a) \Longrightarrow I_t(a) = \frac{1 - e^{-at}(\cos(t) + a\sin(t))}{1+a^2}$$

Notice that the function I_t is dominated by $a \mapsto \frac{2}{1+a^2}$, which is Lebesgue integable, and that $\lim_{t\to\infty} I_t(a) = \frac{1}{1+a^2}$. We can therefore use the dominated convergence theorem to conclude:

$$\lim_{t\to\infty} \int_0^t \frac{\sin(x)}{x}\, dx = \lim_{t\to\infty} \int_0^\infty I_t(a) = \int_0^\infty \frac{da}{1+a^2} = \frac{\pi}{2}.$$

Exercise 3. We show that translations are continuous on $L^p(\mathbb{R}^n)$. In other words, let $f \in L^p(\mathbb{R}^n)$ for

$1 \le p < \infty$ and prove that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p \, dx = 0.$$

Hint: Begin by showing the result for $f \in C_c^\infty(\mathbb{R}^n)$ and then approximate any function in $L^p(\mathbb{R}^n)$ by functions in $C_c^\infty(\mathbb{R}^n)$ in order to conclude.

Solution: We split the proof in two steps.

Step 1: We prove the result in the case of $f \in C_c^\infty(\mathbb{R}^n)$.

If $f \in C_c^\infty(\mathbb{R}^n)$, there exists $R > 0$ such that $\operatorname{supp} f \subseteq B(0,R)$. For any $\varepsilon \in \mathbb{R}^d$ such that $|\varepsilon| < 1$, we define the function $G_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by $G_\varepsilon(x) = |f(x+\varepsilon) - f(x)|^p$. Note that $|G_\varepsilon(x)| \le 2^p \|f\|^p_{C^0(\mathbb{R}^n)}$ if $x \in B(0, R+1)$ and $G_\varepsilon(x) = 0$ if $x \notin B(0,R+1)$; in particular G_ε is bounded and has compact support, hence $G_\varepsilon \in L^1(\mathbb{R}^n)$. Moreover, $G_\varepsilon(x) \to 0$ pointwise as $|\varepsilon| \to 0$. Thus, by the dominated convergence theorem, we have

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p \, dx = \lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} G_\varepsilon(x) \, dx = 0.$$

Step 2: We conclude the result for a general $f \in L^p(\mathbb{R}^n)$ by approximation.

Let $\tau > 0$. By density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, for any $\tau > 0$ there exists $g \in C_c^\infty(\mathbb{R}^n)$, such that

$$||g - f||^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |g(x) - f(x)|^p \, dx \le \frac{\tau}{3^p},$$

which directly implies that for every ε we also have

$$\int_{\mathbb{R}^n} |g(x+\varepsilon) - f(x+\varepsilon)|^p \, dx \le \frac{\tau}{3^p}.$$

Since $g \in C_c^\infty(\mathbb{R}^n)$ we know from Step 1 that

$$\lim_{|\varepsilon|\to 0} \int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p \, dx = 0,$$

and therefore for all $\varepsilon = \varepsilon(\tau)$ small enough, we have

$$\int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p \, dx \le \frac{\tau}{3^p}.$$

We conclude by the triangular inequality that

$$\begin{aligned}
&\int_{\mathbb{R}^n} |f(x+\varepsilon) - f(x)|^p \, dx = \int_{\mathbb{R}^n} |f(x+\varepsilon) - g(x+\varepsilon) + g(x+\varepsilon) - g(x) + g(x) - f(x)|^p \, dx \\
&\le \int_{\mathbb{R}^n} 3^{p-1} \left\{ |f(x+\varepsilon) - g(x+\varepsilon)|^p + |g(x+\varepsilon) - g(x)|^p + |g(x) - f(x)|^p \right\} \, dx \\
&= 3^{p-1} \left\{ \int_{\mathbb{R}^n} |f(x+\varepsilon) - g(x+\varepsilon)|^p \, dx + \int_{\mathbb{R}^n} |g(x+\varepsilon) - g(x)|^p \, dx + \int_{\mathbb{R}^n} |g(x) - f(x)|^p \, dx \right\} \le \tau.
\end{aligned}$$

The previous inequality being true for any τ, we deduce the result letting $\tau \to 0$.

Exercise 4. We prove a kind of continuity of the Lebesgue measure under translations.

(i) Let A be a measurable set and $\mathrm{m}(A) < \infty$. Show that

$$\lim_{|\varepsilon|\to 0} \mathrm{m}\big((A+\varepsilon)\setminus A\big) = 0.$$

(ii) Show that the result in (i) is false if $\mathrm{m}(A) = \infty$.

(iii) Show that the result in (i) is false if A is not measurable. (Replace m by m^*.)

Solution:

(i) Let $f = \chi_A$. Since $f \in L^1(\mathbb{R}^n)$, using exercise 3, we get

$$\lim_{|\varepsilon|\to 0} \int |\chi_A(x-\varepsilon) - \chi_A(x)|dx = 0.$$

Since

$$|\chi_A(x-\varepsilon) - \chi_A(x)| = \chi_{[(A+\varepsilon)\setminus A]\cup[A\setminus(A+\varepsilon)]}(x),$$

it follows that

$$\int |\chi_A(x-\varepsilon) - \chi_A(x)|dx = \mathrm{m}\big([(A+\varepsilon)\setminus A]\cup[A\setminus(A+\varepsilon)]\big).$$

Finally, using

$$\mathrm{m}((A+\varepsilon)\setminus A) \le \mathrm{m}([(A+\varepsilon)\setminus A]\cup[A\setminus(A+\varepsilon)])$$

we deduce the result.

(ii) Define

$$A := \bigcup_{n=1}^{\infty}(n, n+1/2).$$

Then for all $0 < \varepsilon < 1/2$,

$$(A+\varepsilon)\setminus A = \bigcup_{n=1}^{\infty}[n+1/2, n+1/2+\varepsilon),$$

and by additivity,

$$\mathrm{m}\big((A+\varepsilon)\setminus A\big) = \sum_{n=1}^{\infty}\varepsilon = \infty.$$

(iii) Let $V \subset [0,1]$ be the non-measurable Vitali. Recall from the construction that V has the following property:

$$(V+\varepsilon)\cap V = \emptyset \quad \forall \varepsilon \in \mathbb{Q}\cap[-1,1]\setminus\{0\}.$$

It follows from translation invariance of the outer measure that for any $\varepsilon \in \mathbb{Q}\cap[-1,1]\setminus\{0\}$, it holds

$$\mathrm{m}^*\big((V+\varepsilon)\setminus V\big) = \mathrm{m}^*(V+\varepsilon) = \mathrm{m}^*(V).$$

Notice that necessarily $\mathrm{m}^*(V) > 0$ (otherwise V would be measurable), hence

$$\lim_{\substack{|\varepsilon| \to 0 \\ \varepsilon \in \mathbb{Q} \cap [-1,1] \setminus \{0\}}} \mathrm{m}^*((V + \varepsilon) \setminus V) = \mathrm{m}^*(V) > 0\,.$$

Exercise 5. For $f \in L^1(\mathbb{R}^n)$, we define for $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx\,. \tag{1}$$

We will call $\hat{f}$ the Fourier transform of f . Prove that

(i) $\hat{f}$ is well-defined, i.e. that the integral on the right-hand side of (1) converges for every $\xi \in \mathbb{R}^n$,

(ii) $\hat{f}$ is a bounded function on $\mathbb{R}^n$,

(iii) $\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$.

Hint: Write (and justify it) that for any $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \left\{ f(x) - f(x-\xi') \right\} e^{-2\pi i x\cdot\xi}\, dx \text{ with } \xi' = \frac{\xi}{2|\xi|^2},$$

and use a previous exercise.

Solution: As for the well-definition (i), we use the monotonicity of the integral together with the fact $|e^{-2\pi i x\cdot\xi}| = 1$ for every $x, \xi \in \mathbb{R}^n$ to estimate

$$|\hat{f}(\xi)| = \left| \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \right| \le ||f||_{L^1(\mathbb{R}^n)}$$

for every $\xi \in \mathbb{R}^n$. Hence the right-hand side of (1) converges for every $\xi \in \mathbb{R}^n$ and $\hat{f}$ is well-defined. The very same computation shows that

$$\sup_{\xi\in\mathbb{R}^n} |\hat{f}(\xi)| \le ||f||_{L^1(\mathbb{R}^n)}\,,$$

i.e. $\hat{f}$ is a bounded function of ξ . Regarding (iii), we note that

$$\begin{aligned}\hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx \\ &= \int_{\mathbb{R}^n} f(z-\xi') e^{-2\pi i (z-\xi')\cdot\xi}\, dz \\ &= -\int_{\mathbb{R}^n} f(z-\xi') e^{-2\pi i z\cdot\xi}\, dz \text{ with } \xi' = \frac{\xi}{2|\xi|^2},\end{aligned}$$

hence

$$\hat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}^n} \left\{ f(x) - f(x-\xi') \right\} e^{-2\pi i x\cdot\xi}\, dx.$$

Then by monotonicity of the integral

$$|\hat{f}(\xi)| \le \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x-\xi')||e^{-2\pi i x\cdot\xi}|\, dx = \frac{1}{2}\int_{\mathbb{R}^n} |f(x) - f(x-\xi')|\, dx.$$

Finally, since $|\xi'| \to 0$, as $|\xi| \to \infty$, $\lim_{|\xi|\to\infty} |\hat{f}(\xi)| = 0$ due to Exercise 3.

Exercise 6. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two measurable functions.

(i) Assuming the result of Exercise 10, prove that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2n}$ (as a function of $(x,y) \in \mathbb{R}^{2n}$).

(ii) Show that if f and g are integrable on $\mathbb{R}^n$, then $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2n}$ (as a function of $(x,y) \in \mathbb{R}^{2n}$).

(iii) We define the convolution of two integrable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\, dy.$$

Show that $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$ (that is, $y \mapsto f(x-y)g(y)$ is an integrable function on $\mathbb{R}^n$ for a.e. $x \in \mathbb{R}^n$ fixed).

(iv) Show that $f * g$ is integrable whenever f and g are integrable, and that

$$\|f * g\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},$$

with equality if f and g are non-negative.

(v) Recall that the Fourier transform $\hat{f}$ of an integrable function $f \in L^1(\mathbb{R}^n)$ defined in (1). Check first that $\hat{f}$ is bounded and continuous function of ξ. Then prove that for $f, g \in L^1(\mathbb{R}^n)$ integrable, one has

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \qquad \forall \xi \in \mathbb{R}^n\,.$$

Solution:

(i) Using the result of Exercise 8, we know that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x,y) = f(x-y)$ (as a function of $(x,y) \in \mathbb{R}^{2n}$) is measurable. Now we prove that the function $G : \mathbb{R}^{2n} \to \mathbb{R}$ given by $G(x,y) = g(y)$ (as function of $(x,y) \in \mathbb{R}^{2n}$) is measurable. Indeed, notice

$$\{(x,y) \in \mathbb{R}^{2n} : G(x,y) > \alpha\} = \{(x,y) \in \mathbb{R}^{2n} : g(y) > \alpha\} = \mathbb{R}^n \times \{y \in \mathbb{R}^n : g(y) > \alpha\}$$

which is measurable since it is the product of two measurable sets. Finally, the fact that $f(x-y)g(y)$ is a measurable function of $(x,y) \in \mathbb{R}^{2n}$ follows from the fact $f(x-y)g(y) = F(x,y)G(x,y)$ and that the product of two measurable functions is measurable.

(ii) Since the function $f(x-y)g(y)$ is measurable, using Tonelli's theorem

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} |f(x-y)g(y)|\, d(x,y) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)||g(y)|\, dx\, dy \\
&= \int_{\mathbb{R}^n} |g(y)| \int_{\mathbb{R}^n} |f(x-y)|\, dx\, dy \\
&= \|f\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |g(y)|\, dy \\
&= \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)} < \infty.
\end{aligned}$$

(iii) By (ii), $f(x-y)g(y)$ is integrable on $\mathbb{R}^{2n}$ and thus, using Fubini's theorem, we have

$$\int_{\mathbb{R}^n} |f(x-y)g(y)|dy < \infty \qquad \text{for a.e } x \in \mathbb{R}^n$$

and hence $(f * g)(x)$ is well-defined for a.e. $x \in \mathbb{R}^n$.

(iv) We have (similar to the computation in (ii))

$$\begin{aligned}\int_{\mathbb{R}^n} |(f * g)(x)|\, dx = \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\, dy \right| dx &\le \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x-y)g(y)|\, dy\, dx \\ &\le \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)},\end{aligned}$$

and thus

$$\|f * g\|_{L^1(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

If f and g are non-negative, then, again by Fubini,

$$\begin{aligned}\|f * g\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} f(x-y)g(y)\, dy \right| dx &&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y)g(y)\, dy\, dx \\ &= \int_{\mathbb{R}^n} g(y) \left(\int_{\mathbb{R}^n} f(x-y)\, dx \right) dy &&= \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}\, ,\end{aligned}$$

where the last equality uses again the non-negativity of f and g together with a change of variables.

(v) Let $f \in L^1(\mathbb{R}^n)$. We have for $\xi \in \mathbb{R}^n$

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}^n} |f(x) e^{-2\pi i x \cdot \xi}|\, dx \le \|f\|_{L^1} \|\phi_\xi\|_{L^\infty} = \|f\|_{L^1},$$

where, $\phi_\xi(x) = e^{-2\pi i x \cdot \xi}$, hence $\|\hat{f}\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$, which shows that $\hat{f}$ is a bounded function. Now for the continuity, let $\xi_n \to \xi$. Then

- $f(x)e^{-2\pi i x \cdot \xi_n} \to f(x) e^{-2\pi i x \cdot \xi}$ for every $x \in \mathbb{R}^n$,
- $|f(x)e^{-2\pi i x \cdot \xi_n}| \le |f(x)|$ for every $x \in \mathbb{R}^n$,
- $|f| \in L^1(\mathbb{R}^n)$ by assumption.

We can thus apply the dominated convergence theorem to deduce

$$\lim_{n\to\infty} \hat{f}(\xi_n) = \lim_{n\to\infty} \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi_n}\, dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi}\, dx = \hat{f}(\xi),$$

which proves continuity. Hence $\hat{f}$ is a bounded continuous function. Since f, g are integrable $f * g \in L^1(\mathbb{R}^n)$ by (iv) and hence, by the above discussion, the Fourier transforms of f, g and $f * g$ is well-defined and a continuous and bounded function. In particular, they are defined for every $\xi \in \mathbb{R}^n$ and not only almost everywhere. Finally, by Fubini

$$\begin{aligned}
\widehat{(f * g)}(\xi) &= \int_{\mathbb{R}^n} (f * g)(x) e^{-2\pi i x \cdot \xi} \, dx \\
&= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} f(x-y) g(y) \, dy \right\} e^{-2\pi i x \cdot \xi} \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) g(y) e^{-2\pi i x \cdot \xi} \, dy \, dx \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i (x-y) \cdot \xi} g(y) e^{-2\pi i y \cdot \xi} \, dx \, dy \\
&= \int_{\mathbb{R}^n} g(y) e^{-2\pi i y \cdot \xi} \int_{\mathbb{R}^n} f(x-y) e^{-2\pi i (x-y) \cdot \xi} \, dx \, dy \\
&= \int_{\mathbb{R}^n} g(y) e^{-2\pi i y \cdot \xi} \hat{f}(\xi) \, dy = \hat{f}(\xi) \hat{g}(\xi).
\end{aligned}$$

Exercise 7. Show that there does not exist a function $I \in L^1(\mathbb{R}^n)$ such that

$$f * I = f \text{ for all } f \in L^1(\mathbb{R}^n).$$

Hint: Use the Fourier transform and Exercise 6.

Solution: Suppose for a contradiction that such an I exists. Recall that that for $f, g \in L^1(\mathbb{R}^n)$, we have $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. Thus $\hat{f}(\xi)\hat{I}(\xi) = \hat{f}(\xi)$, and therefore $\hat{I}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$. However, since $I \in L^1(\mathbb{R}^n)$, we have $\hat{I}(\xi) \to 0$ as $|\xi| \to \infty$ (see Exercise 3), giving the desired contradiction.

Exercise 8. From every L^p-convergent sequence $\{f_\nu\}_{\nu>0}$ ($1 \le p < \infty$), we can always extract a subsequence that converges pointwise almost everywhere. Yet, it might happen that the full sequence $f\nu$ converges pointwise nowhere. Here we discuss such an example. For all $\nu > 0$, we define

$$h(\nu) := \max\{k \in \mathbb{N} : 2^k \le \nu\}, \quad I_\nu := \left\{ \frac{\nu - 2^{h(\nu)}}{2^{h(\nu)}} \right\} + [0, 1/2^{h(\nu)}], \quad f_\nu := \chi_{I_\nu}.$$

More precisely, $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}$, $f_5 = \chi_{[1/4,2/4]}$, $f_6 = \chi_{[2/4,3/4]}$, $f_7 = \chi_{[3/4,1]}$, $f_8 = \chi_{[0,1/8]}, \cdots$.

(i) Show that f_ν converges in $L^p(0,1)$ for $1 \le p < \infty$.

(ii) Show that f_ν converges pointwise nowhere on $[0,1]$.

(iii) Find a subsequence of f_ν which converges pointwise a.e. on $[0,1]$.

Solution:

(i) With $1 \le p < \infty$ we have

$$\|f_\nu\|_{L^p} = \mathrm{m}(I_\nu)^{1/p} = \frac{1}{2^{h(\nu)/p}} \to 0 \text{ as } \nu \to \infty$$

and thus $f_\nu \to 0$ in $L^p(0,1)$ as $\nu \to \infty$.

(ii) For all $n \in \mathbb{N}$, the family $\{I_{2^n}, I_{2^n+1}, \ldots, I_{2^{n+1}-1}\}$ covers $[0,1]$. It follows that for all $x \in [0,1]$ and $k \in \mathbb{N}$, there is $\nu_k^0, \nu_k^1 > k$ such that

$$x \notin I_{\nu_k^0} \quad \text{and} \quad x \in I_{\nu_k^1}.$$

Thus, for all $x \in [0,1]$, there are two subsequences (depending on x) of $\{f_\nu\}$, given by $\{f_k^0\}_{k=1}^\infty$ and $\{f_k^1\}_{k=1}^\infty$, such that

$$\lim_{k\to\infty} f_k^0(x) = 0 \quad \text{and} \quad \lim_{k\to\infty} f_k^1(x) = 1\,.$$

We deduce that $\lim_{\nu\to\infty} f_\nu(x)$ does not exist.

(iii) Consider the subsequence $\tilde{f}_k := f_{2^k} = \chi_{[0,2^{-k}]}$ for $k \ge 0$. Then, for all $x > 0$, there is $k \ge 1$ such that $x > 2^{-k}$, hence $\lim_{k\to\infty} \tilde{f}_k(x) = 0$. However, if $x = 0$, we have $\lim_{k\to\infty} \tilde{f}_k(0) = 1$. We conclude that $\tilde{f}_k \to 0$ a.e.

Exercise 9. Throughout this exercise, we assume $\Omega \subseteq \mathbb{R}^n$ is measurable.

(i) Show that for any $f, g \in L^\infty(\Omega)$, we have $\|f+g\|_{L^\infty(\Omega)} \le \|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\Omega)}$.

(ii) Show that $L^p(\Omega)$ is a real vector space $1 \le p \le \infty$.

(iii) Show that if $\Omega \subset \mathbb{R}^n$ is bounded and $1 \le p < q \le \infty$, then $L^q(\Omega) \subseteq L^p(\Omega)$. More precisely, find a constant $K = K(\Omega, p, q)$ such that

$$\|f\|_{L^p(\Omega)} \le K \|f\|_{L^q(\Omega)}.$$

(iv) Show that if $\Omega \subset \mathbb{R}^n$ is bounded and if $f \in L^\infty(\Omega)$, then

$$\lim_{p\to\infty} \|f\|_{L^p(\Omega)} = \|f\|_{L^\infty(\Omega)}\,.$$

Hint: For (iv), use (iii) to show the following inequalities

$$\limsup_{p\to\infty} \|f\|_{L^p(\Omega)} \le \|f\|_{L^\infty(\Omega)},$$
$$\liminf_{p\to\infty} \|f\|_{L^p(\Omega)} \ge \|f\|_{L^\infty(\Omega)} - \varepsilon \quad \forall\, \varepsilon > 0\,.$$

For the second inequality, study the set $A_\varepsilon := \{x \in \Omega : \ |f(x)| \ge \|f\|_{L^\infty} - \varepsilon\}$.

Solution:

(i) Note that if $|f(x)|+|g(x)|>\alpha+\beta$, then $|f(x)|>\alpha$ or $|g(x)|>\beta$, hence

$$\begin{aligned}\{x\in\Omega:|f(x)+g(x)|>\|f\|_{L^\infty}+\|g\|_{L^\infty}\}&\subseteq\{x\in\Omega:|f(x)|+|g(x)|>\|f\|_{L^\infty}+\|g\|_{L^\infty}\}\\&\subseteq\{x\in\Omega:|f(x)|>\|f\|_{L^\infty}\}\cup\{x\in\Omega:|g(x)|>\|g\|_{L^\infty}\}.\end{aligned}$$

Therefore, since

$$\mathrm{m}(\{x\in\Omega:|f(x)|>\|f\|_{L^\infty}\})=\mathrm{m}(\{x\in\Omega:|g(x)|>\|g\|_{L^\infty}\})=0$$

by the definition of the essential supremum, we conclude

$$\mathrm{m}(\{x\in\Omega:|f(x)+g(x)|>\|f\|_{L^\infty}+\|g\|_{L^\infty}\})=0\,.$$

(ii) Let $f,g\in L^p(\Omega)$ and $\lambda,\mu\in\mathbb{R}$. It is clear that $\lambda f+\mu g$ is measurable. If $p=\infty$, it follows from (i), that $\lambda f+\mu g\in L^\infty(\Omega)$. Now, assume $1\le p<\infty$. By convexity of the function $x\to|x|^p$, it holds for any $x,y\in\mathbb{R}$

$$|x+y|^p\le 2^{p-1}(|x|^p+|y|^p).$$

Thus,

$$||\lambda f+\mu g||^p_{L^p(\Omega)}=\int_\Omega|\lambda f(x)+\mu g(x)|^p\,dx\le 2^{p-1}\left(|\lambda|^p||f||^p_{L^p(\Omega)}+|\mu|^p||g|^p_{L^p(\Omega)}\right)<+\infty\,,$$

which proves $\lambda f+\mu g\in L^p(\Omega)$ and hence $L^p(\Omega)$ is a real vector space.

(iii) Let Ω bounded and $1\le p<q\le\infty$. Fix $f\in L^q(\Omega)$. First assume that $1<q<\infty$. Using Hölder's inequality with exponents q/p and $q/(q-p)$ (note that this is an admissible choice only if $1\le p<q$!)

$$\int_\Omega|f|^p\,dx=\int_\Omega|f|^p|1|\,dx\le\left(\int_\Omega(|f|^p)^{q/p}\,dx\right)^{p/q}\left(\int_\Omega|1|\,dx\right)^{1-p/q}=(\|f\|_{L^q})^p|\Omega|^{1-p/q},$$

where $|\Omega|$ is the area of Ω, i.e. $|\Omega|=\int_\Omega 1\,dx=\mathrm{m}(\Omega)$. Taking pth-roots, we obtain

$$\|f\|_{L^p}\le\|f\|_{L^q}|\Omega|^{\frac{1}{p}-\frac{1}{q}}.$$

Now, for $q=\infty$,

$$\|f\|_{L^p}=\left(\int_\Omega|f|^p\,dx\right)^{1/p}\le\left(\int_\Omega\|f\|^p_{L^\infty}\,dx\right)^{1/p}=|\Omega|^{1/p}\|f\|_{L^\infty}.\qquad(2)$$

(iv) Let $f\in L^\infty(\Omega)$. Using (2), we get, since $|\Omega|^{1/p}\to 1$ as $p\to\infty$, that

$$\limsup_{p\to\infty}\|f\|_{L^p}\le\|f\|_{L^\infty}.\qquad(3)$$

In order to show the reverse inequality, fix $0 < \varepsilon < \|f\|_{L^\infty}$ and consider the set

$$A_\varepsilon := \{x \in \Omega : \ |f(x)| \geq \|f\|_{L^\infty} - \varepsilon\}.$$

By the definition of the essential supremum, it is clear that $\mathrm{m}(A_\varepsilon) > 0$. Thus,

$$\int_\Omega |f|^p \, dx \geq \int_{A_\varepsilon} |f|^p \, dx \geq \mathrm{m}(A_\varepsilon)(\|f\|_{L^\infty} - \varepsilon)^p > 0,$$

and therefore, taking the pth root,

$$\|f\|_{L^p} \geq \mathrm{m}(A_\varepsilon)^{1/p}(\|f\|_{L^\infty} - \varepsilon).$$

Since $\mathrm{m}(A_\varepsilon) > 0$ we have $\mathrm{m}(A_\varepsilon)^{1/p} \to 1$ as $p \to \infty$, hence taking the limit $p \to \infty$, we obtain

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_\infty - \varepsilon.$$

Since ε was arbitrary, we deduce

$$\liminf_{p\to\infty} \|f\|_{L^p} \geq \|f\|_{L^\infty}. \tag{4}$$

Exercise 10 ($\star$)**.** Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Prove that the function $F : \mathbb{R}^{2n} \to \mathbb{R}$ defined by $F(x, y) = f(x - y)$, $(x \in \mathbb{R}^n,\ y \in \mathbb{R}^n)$ is measurable.

Solution: For any $\alpha \in \mathbb{R}$ we will show that

$$\widehat{E}_\alpha = \{(x, y) \in \mathbb{R}^{2n} : F(x, y) < \alpha\}$$

is measurable. Then for any set $\mathcal{A} \subset \mathbb{R}^n$, we define the set $\widetilde{\mathcal{A}}$ by

$$\widetilde{\mathcal{A}} := \{(x, y) \in \mathbb{R}^{2n} : x - y \in \mathcal{A}\}$$

Define

$$E_\alpha = \{z \in \mathbb{R}^n : f(z) < \alpha\}$$

and note that $\widehat{E}_\alpha = \widetilde{E}_\alpha = \{(x, y) : x - y \in E_\alpha\}$, so that is suffices to show that $\widetilde{E}_\alpha$ is measurable.
Step 1: As a preliminary result, we show that any measurable set can be written as the union of a countable union and intersection of closed sets and a measure 0 set.
Let E be a measurable set and consider E^c. We first want to show that there exists $\mathcal{O}$, a countable union and intersection of open sets, such that

$$\mathrm{m}(\mathcal{O} \setminus E^c) = 0\,. \tag{5}$$

To show (5), assume first that $\mathrm{m}(E^c) = \mathrm{m}^*(E^c) < +\infty$. By definition of the outer measure, there exists for any $k \in \mathbb{N}$ an open set O_k such that $E^c \subset O_k$ and $\mathrm{m}^*(O_k \setminus E^c) \leq 1/k$. Clearly,

$E^c \subset \bigcap_{k=1}^\infty O_k$ and

$$\mathrm{m}^*\left(\bigcap_{k=1}^\infty O_k \setminus E^c\right) \leq \mathrm{m}^*(O_j \setminus E^c) \leq \frac{1}{j}, \quad \forall j \in \mathbb{N}.$$

We conclude $\mathrm{m}(\bigcap_{k=1}^\infty O_k \setminus E^c) = 0$ and the desired set is therefore given by $\mathcal{O} := \bigcap_{k=1}^\infty O_k$.

If now $\mathrm{m}(E^c) = \infty$, then we consider $F_n := E^c \cap B(0,n)$. Since F_n is bounded, $\mathrm{m}(F_n) < +\infty$ and by the above procedure, we find for every $n \in \mathbb{N}$ a collection of open sets $\{O_{n,k}\}_{k=1}^\infty$ such that

$$\mathrm{m}\left(\bigcap_{k=1}^\infty O_{n,k} \setminus F_n\right) = 0\,.$$

We use that by construction $E^c = \bigcup_{n=1}^\infty F_n$ and we conclude by subadditivity that

$$\mathrm{m}\left(\left[\bigcup_{n=1}^\infty \bigcap_{k=1}^\infty O_{n,k}\right] \setminus E^c\right) \leq \sum_{n=1}^\infty \mathrm{m}\left(\bigcap_{k=1}^\infty O_{n,k} \setminus F_n\right) = 0\,.$$

The desired set is therefore given by $\mathcal{O} := \bigcup_{n=1}^\infty \bigcap_{k=1}^\infty O_{n,k}$. This finishes the proof of (5).

We now define $Z := \mathcal{O} \setminus E^c$, where $\mathcal{O}$ is given by (5). By construction, we have $E^c = \mathcal{O} \setminus Z$ and by taking complements on both sides, we obtain

$$E = \mathcal{O}^c \cup Z\,,$$

where $\mathcal{O}^c = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty O_{n,k}^c$ with $O_{n,k}^c$ closed (as complements of open sets).

Step 2: We show that for any open set $\mathcal{O} \subset \mathbb{R}^n$, the set $\widetilde{\mathcal{O}}$ is open and for every closed set $\mathcal{C} \subset \mathbb{R}^n$ the set $\widetilde{\mathcal{C}}$ is closed.

Indeed, let $\mathcal{O}$ open and $z \in \widetilde{\mathcal{O}}$, then z writes $z = (x,y)$, where $x - y \in \mathcal{O}$. Since $\mathcal{O}$ is open, let $r > 0$ be such that $B(x-y,r) \subset \mathcal{O}$. All the norms being equivalent in finite dimension, we define the norm on $\mathbb{R}^{2n}$ as $\|(u,v)\|_{\mathbb{R}^{2n}} = \max(\|u\|_{\mathbb{R}^n}, \|v\|_{\mathbb{R}^n})$. Then, for any $h = (a,b) \in B\left(z, \frac{r}{2}\right)$, by the triangular inequality,

$$\|(a-b)-(x-y)\|_{\mathbb{R}^n} \leq \|a-x\|_{\mathbb{R}^n} + \|b-y\|_{\mathbb{R}^n} \leq 2\|(x,y)-(a,b)\|_{\mathbb{R}^{2n}} \leq r\,,$$

therefore $a-b$ is in $\mathcal{O}$ and h is in $\widetilde{\mathcal{O}}$. This proves that $\widetilde{\mathcal{O}}$ is open. The claim about closed sets follows by writing a closed set $\mathcal{C}$ as $\mathcal{C} = \mathbb{R}^n \setminus \mathcal{O}$ for some open set $\mathcal{O}$.

Step 3: We prove that if $\mathcal{A}$ can be written as a countable union and intersection of closed sets, then the same holds for $\widetilde{\mathcal{A}}$.

Indeed, assume that there is a countable family $\{\mathcal{C}_{n,k}\}_{n,k\in\mathbb{N}}$ of closed sets such that

$$\mathcal{A} = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \mathcal{C}_{n,k}.$$

Then clearly,

$$\widetilde{\mathcal{A}} = \bigcap_{n=1}^\infty \bigcup_{k=1}^\infty \widetilde{\mathcal{C}}_{n,k},$$

and due to the previous step every $\widetilde{\mathcal{C}}_{n,k} \subset \mathbb{R}^{2n}$ is a closed set.

Step 4: We prove that $\widetilde{E}_\alpha$ is measurable.

Since E_α is measurable, there is a set $\mathcal{A}$ that can be written as a countable union and intersection of closed sets $\mathcal{A}$ and measure 0 set $\mathcal{Z}$ such that $E_\alpha = \mathcal{A} \cup \mathcal{Z}$ and therefore $\widetilde{E}_\alpha = \widetilde{\mathcal{A}} \cup \widetilde{\mathcal{Z}}$. Due to the previous step combined with the fact that any countable union and intersection of measurable sets is measurable, it suffices to prove that $\widetilde{\mathcal{Z}}$ is measurable. Actually, we will prove that $\mathrm{m}^*(\widetilde{\mathcal{Z}}) = 0$ and therefore it is measurable. For any $n \in \mathbb{N}$, let $\mathcal{O}^n$ be an open set such that $\mathcal{Z} \subset \mathcal{O}^n$ such that $\mathrm{m}(\mathcal{O}^n) < 1/n$. Then define,

$$B_k = \{(x,y) \in \mathbb{R}^{2n} : |y| \leq k\} = \mathbb{R}^n \times B(y,k),$$

and put $\widetilde{\mathcal{Z}}_k := \widetilde{\mathcal{Z}} \cap B_k$ and $\widehat{\mathcal{O}}^n_k = \widetilde{\mathcal{O}}^n \cap B_k$. Note that $\widetilde{\mathcal{Z}}_k \subset \widehat{\mathcal{O}}^n_k$. Now we will prove that

$$\mathrm{m}(\widehat{\mathcal{O}}^n_k) = \mathrm{m}(\mathcal{O}^n)\,\mathrm{m}(B_k).$$

First note that $\chi_{\widehat{\mathcal{O}}^n_k}(x,y) = \chi_{\widetilde{\mathcal{O}^n} \cap B_k}(x,y) = \chi_{\mathcal{O}^n}(x-y)\chi_{B_k}(y)$, and therefore

$$\mathrm{m}(\widehat{\mathcal{O}}^n_k) = \int_{\mathbb{R}^{2n}} \chi_{\mathcal{O}}(x-y)\chi_{B_k}(y)d(x,y) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \chi_{\mathcal{O}^n}(x-y)\,dx \right) \chi_{B_k}(y)\,dy = \mathrm{m}(\mathcal{O}^n)\,\mathrm{m}(B_k),$$

where the second equality is justified by Tonelli's theorem. Thus since $\mathrm{m}(\mathcal{O}^n) \to 0$ as $n \to \infty$, for any fixed $k \in \mathbb{N}$, $\mathrm{m}(\widehat{\mathcal{O}}^n_k) \to 0$, as $n \to \infty$. Then, since $\widetilde{\mathcal{Z}}_k \subset \widehat{\mathcal{O}}^n_k$, for all n we get

$$m^*(\widetilde{\mathcal{Z}}_k) = 0.$$

Then finally, since $\widetilde{\mathcal{Z}} = \bigcup_{k=1}^\infty \widetilde{\mathcal{Z}}_k$,

$$0 \leq \mathrm{m}^*(\widetilde{\mathcal{Z}}) \leq \sum_{k=1}^\infty \mathrm{m}^*(\widetilde{\mathcal{Z}}_k) = 0.$$

This proves that $\widetilde{\mathcal{Z}}$ is measurable and we deduce the result.