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Math 261 – Discrete Optimization (Spring 2022)

Assignment 3

Problem 1

Let P be the polyhedron given by all $(x, y, z) \in \mathbb{R}^3$ such that

$$\begin{array}{rcrcrcl} -7x & + & y & + & 2z & \leq & 4 \\ 5x & - & 11y & + & 2z & \leq & 4 \\ -x & + & 7y & - & 10z & \leq & 4 \\ x & + & y & + & 2z & \leq & 4 \end{array}$$

Find all vertices of P.

Solution:

The vertices are all feasible intersection points of three of the planes given by the halfspaces. These are: $(-1, -1, -1), (0, 0, 2), (0, 2, 1), (3, 1, 0)$.

Problem 2

Given a vertex $\mathbf{v}$ of a polyhedron P, show that $\mathbf{v}$ must also be an extreme point of P.

Solution:

By definition, a point in P is an extreme point if it cannot be written as the convex combination of (at least 2) distinct points in P. So suppose for contradiction that a vertex $\mathbf{v}$ *can* be written as a convex combination of k points $\mathbf{x}_1, \ldots, \mathbf{x}_k \in P$, i.e. $\mathbf{v} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$ for some coefficients $\lambda_i \geq 0$ with $\sum_{i=1}^{k} \lambda_i = 1$. Since $\mathbf{v}$ is a vertex, there exists a vector $\mathbf{c}$ and a value β such that $\mathbf{c} \cdot \mathbf{v} = \beta$ *and* $\mathbf{c} \cdot \mathbf{x} < \beta$ for all $\mathbf{x} \in P$. Now,

$$\beta = \mathbf{c} \cdot \mathbf{v} = \mathbf{c} \cdot \left(\sum_{i=1}^{k} \lambda_i \mathbf{x}_i \right) = \sum_{i=1}^{k} \lambda_i \mathbf{c} \cdot \mathbf{x}_i < \sum_{i=1}^{k} \lambda_i \beta = \beta,$$

where the inequality is due to the fact that $\mathbf{c}\mathbf{x}_i < \beta$ holds for all $i = 1, \ldots, k$ (it is not active at any other point than $\mathbf{v}$). This yields the contradiction.

Problem 3

Given $\mathbf{x}^* = (0\ 1\ 1)^\intercal \in \mathbb{R}^3$ and the vector $\mathbf{d} = (1\ 1\ {-1})^\intercal \in \mathbb{R}^3$ decide if the ray $\{\mathbf{x}^* + \lambda \mathbf{d} : \lambda \in \mathbb{R}_{\geq 0}\}$ intersects the following hyperplanes while moving in the direction of $\mathbf{d}$. Give the order in which the trajectory passes the planes.

$$P_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : (1\ 2\ 3)\mathbf{x} = 0\right\} \qquad P_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : (3\ 2\ 1)\mathbf{x} = 4\right\}\}$$
$$P_3 = \left\{\mathbf{x} \in \mathbb{R}^3 : (1\ 1\ 1)\mathbf{x} = 2\right\} \qquad P_4 = \left\{\mathbf{x} \in \mathbb{R}^3 : (0\ 1\ 3)\mathbf{x} = -1\right\}$$

Solution:

The trajectory of $\mathbf{x}^*$ is given by the line $\{\mathbf{x}^* + \delta \mathbf{d} : \delta \geq 0\}$ where a point in the trajectory moves further away from $\mathbf{x}^*$ when δ becomes larger. To find the order in which $\mathbf{x}^*$ passes the planes we search the corresponding δ_i for which $\mathbf{x}^* + \delta_i \mathbf{d}$ is in the plane P_i or decide that such a δ does not exist.

P_1: $(1\ 2\ 3)((0\ 1\ 1)^\intercal + \delta(1\ 1\ \ {-1})^\intercal) = 5 + 0 \cdot \delta = 5 \neq 0$ for all δ, so $\mathbf{x}^*$ does not pass P_1 since it moves parallel to it.
P_2: $(3\ 2\ 1)((0\ 1\ 1)^\intercal + \delta(1\ 1\ \ {-1})^\intercal) = 3 + 4\delta = 4$ for $\delta = \frac{1}{4}$.
P_3: $(1\ 1\ 1)((0\ 1\ 1)^\intercal + \delta \cdot (1\ 1\ \ {-1})^\intercal) = 2 + \delta = 2$ for $\delta = 0$, so $\mathbf{x}^*$ is already on P_3.

P_4: $(0\ 1\ 3)((0\ 1\ 1)^\intercal + \delta\cdot(1\ 1\ \ -1)^\intercal) = 4 - 2\delta = -1$ for $\delta = \frac{5}{2}$.

The order in which $\mathbf{x}^*$ passes the planes is P_3, P_2, P_4. The plane P_1 will never be passed.

Problem 4

(a) Given an $m \times n$ matrix $\mathbf{A}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, assume that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that $\boldsymbol{\lambda}^\intercal \mathbf{A} = \mathbf{0}$ and $\boldsymbol{\lambda}\cdot\mathbf{b} > 0$. Show that the system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has no solution.

Solution:
Let $\boldsymbol{\lambda}, \mathbf{A}, \mathbf{b}$ have the properties listed above, and assume (for contradiction) that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{y} \in \mathbb{R}^n$. Then taking the dot product with $\boldsymbol{\lambda}$ on both sides, we have

$$\begin{aligned} \mathbf{A}\mathbf{y} &= \mathbf{b} \\ \boldsymbol{\lambda}^\intercal \mathbf{A}\mathbf{y} &= \boldsymbol{\lambda}\cdot\mathbf{b} \\ 0 &= \boldsymbol{\lambda}\cdot\mathbf{b} \end{aligned}$$

which becomes $0 > 0$ (a contradiction). Hence it must be that $A\mathbf{x} = \mathbf{b}$ did not have a solution.

(b) Prove that the system of linear equations

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

has no solution by finding a vector $\boldsymbol{\lambda}$ that satisfies part (a) of this problem.

Solution:
This question can seem overwhelming at first, but in the end, just think about what needs to be done — you need to show that a system of linear equations has no solution. And one way you know how to do this is by applying Gaussian elimination on the augmented matrix

$$\begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 5 & 4 & 1 & | & 2 \\ 7 & 5 & 1 & | & 4 \end{bmatrix}$$

The big difference here is that you will want to record the Gaussian elimination steps you made as a matrix multiplication. Doing this, you get

$$\begin{bmatrix} 1 & 0 & 0 \\ -5/2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 5 & 4 & 1 & | & 2 \\ 7 & 5 & 1 & | & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & | & 1 \\ 0 & 3/2 & 1 & | & -1/2 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}. \tag{1}$$

Notice the bottom row is the "Gaussian elimination way" of seeing that this system of linear equations has no solution. Now let's remember what we are trying to find: a vector $\boldsymbol{\lambda}$ which gives

$$\boldsymbol{\lambda}^\intercal \begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\lambda}^\intercal \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} > 0$$

Well the bottom row of (1) looks just like that. In particular, if we take $\boldsymbol{\lambda}^\intercal = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ then that gives

$$\boldsymbol{\lambda}^\intercal \begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\lambda}^\intercal \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 1$$

exactly as we wanted.

If anyone learned the LU-decomposition instead of Gaussian elimination (or prefers it because this is how computers tend to do this computation) then this would be similar: if you write $\mathbf{A} = \mathbf{LU}$ where $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular) then (1) is equivalent to having

$$\mathbf{L}^{-1}[\mathbf{A} \mid \mathbf{b}] = [\mathbf{U} \mid \mathbf{L}^{-1}\mathbf{b}].$$

(c) Show that if you can find a $\boldsymbol{\lambda}$ like the one in part (a) which *also* satisfies $\boldsymbol{\lambda} \geq \mathbf{0}$, then the polyhedron $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{Ax} \geq \mathbf{b}\}$ must be empty.

Solution:
It is pretty much the same proof as (a) — the one difference is that we need $\boldsymbol{\lambda} \geq 0$ in this case so that the inequalities do not reverse. Assume there exists a $\mathbf{y}$ which satisfies $A\mathbf{y} \geq \mathbf{b}$. Then taking the dot product with $\boldsymbol{\lambda}$ on both sides, we have

$$\begin{aligned} \mathbf{Ay} &\geq \mathbf{b} \\ \boldsymbol{\lambda}^\intercal \mathbf{Ay} &\geq \boldsymbol{\lambda} \cdot \mathbf{b} \\ 0 &\geq \boldsymbol{\lambda} \cdot \mathbf{b} \end{aligned}$$

and so

$$\boldsymbol{\lambda} \cdot \mathbf{b} \leq 0 \quad \text{and} \quad \boldsymbol{\lambda} \cdot \mathbf{b} > 0$$

which is a contradiction.

Problem 5

Consider the following linear program in inequality standard form:

$$\begin{array}{rlrcrcrcrr} \mathcal{P} = & \min & 5x & - & 3y & - & 2z & & & \\ & \text{s.t.} & 2x & - & 4y & + & 2z & \geq & -6, & (1) \\ & & -5x & + & 3y & - & 5z & \geq & -1, & (2) \\ & & -4x & + & 2y & - & z & \geq & -3, & (3) \\ & & x & & & & & \geq & 0, & (4) \\ & & & & y & & & \geq & 0, & (5) \\ & & & & & & z & \geq & -1 & (6) \end{array}$$

(a) Show that the point $\mathbf{v} = (0, 16/7, 11/7)^\intercal$ is a vertex of the feasible region of $\mathcal{P}$.

Solution:
The active constraints are the ones which are fulfilled with equality in $\mathbf{v}$. In our case, these are (1), (2) and (4). One can check that these are linearly independent, and since there are 3 of them (and the problem has 3 variables, this is a basic solution. Now we can check that $\mathbf{v}$ satisfies constraints (3), (5), (6) and that tells us it is a basic feasible solution. Since BFSs are the same as vertices, that makes $\mathbf{v}$ a vertex.

(b) Rewrite $\mathcal{P}$ in equality standard form.

Solution:
First we notice that $x \geq 0$ and $y \geq 0$, so they are already in the desired form. We could use a slack variable for z, or we could do something more clever and substitute z with $z' = z+1$, as this will get $z' \geq 0$ as we need. So we replace z with $z' - 1$ in each occurence of z (note that this also adds a constant term to the objective function.

$$\begin{array}{rlrcrcrcrr} \tilde{\mathcal{P}} = & 2 + \min & 5x & - & 3y & - & 2z' & & & \\ & \text{s.t.} & 2x & - & 4y & + & 2z' & \geq & -4 & (1) \\ & & -5x & + & 3y & - & 5z' & \geq & -6 & (2) \\ & & -4x & + & 2y & - & z' & \geq & -4 & (3) \\ & & x & & & & & \geq & 0 & (4) \\ & & & & y & & & \geq & 0 & (5) \\ & & & & & & z' & \geq & 0 & (6) \end{array}$$

It remains to get rid of the inequalities for the constraints (1), (2) and (3), which we do by adding the slack variables s_1, s_2 respectively s_3. This leads to the following linear program.

$$
\begin{array}{rrrrrrrrrrrl}
\mathcal{P}' = & 2 + \min & 5x & - & 3y & - & 2z' & & & & & \\
& \text{s.t.} & 2x & - & 4y & + & 2z' & - & s_1 & = & -4 & (1) \\
& & -5x & + & 3y & - & 5z' & - & s_2 & = & -6 & (2) \\
& & -4x & + & 2y & - & z' & - & s_3 & = & -4 & (3) \\
& & x & & & & & & & \geq & 0 & (4) \\
& & & & y & & & & & \geq & 0 & (5) \\
& & & & & & z' & & & \geq & 0 & (6) \\
& & & & & & & & s_1 & \geq & 0 & (7) \\
& & & & & & & & s_2 & \geq & 0 & (8) \\
& & & & & & & & s_3 & \geq & 0 & (9)
\end{array}
$$

(c) Let $\mathcal{P}'$ be the linear program you found in (b). Find a solution $\mathbf{v}'$ in $\mathcal{P}'$ that has the same cost value as $\mathbf{v}$ does in $\mathcal{P}$ (the $\mathbf{v}$ and $\mathcal{P}$ from part (a)). Determine which of the nonnegativity constraints in $\mathcal{P}'$ are active at $\mathbf{v}'$.

Solution:

Now we can get $\mathbf{v}'$ from $\mathbf{v}$ by doing the same transformations. Firstly $z = 11/7$ turns into $z' = 11/7 + 1 = 18/7$. Furthermore, since (1) and (2) were active in $\mathcal{P}$, the corresponding slack variables will be 0. Hence only s_3 needs a nonnegative slack variable, and we can find it by plugging in the variables for x, y, z' and solving for s_3, giving the solution

$$\mathbf{v}' = (0, 16/7, 18/7, 0, 0, 7)^\intercal$$

which has the same cost function that $\mathcal{P}$ does (by construction). A nonnegativity constraint is active precisely when it is set to 0, so these are (4), (7), (8) (corresponding to x, s_1, s_2.

(d) Try to prove that $\mathbf{v}$ is an optimal solution to $\mathcal{P}$.

Hint: We haven't discussed how to do this in class yet (we will), but the only thing missing at this point is a bit of cleverness (so try to do something clever). If you are stuck, try to get some inspiration from your solution to 4(c) and from the physical interpretation of Lagrange multipliers we talked about in class.

Solution:
The value of *primal* at $\mathbf{v}$ is -10. Hence to show that $\mathbf{v}$ is an optimal solution, we must somehow prove that every feasible solution to $\mathcal{P}$ satisfies $5x - 3y - 2z \geq -10$. To do this, we will take our inspiration from 4(c) and note that if we have a nonnegative combination of inequalities, that gives a new inequality, so we will try to construct the inequality we want from the constraints in $\mathcal{P}$.

On the other hand, the physical interpretation of Lagrange multipliers we saw tells us that at the optimal solution, the cost function needs to be a linear combination of the active constraints. So it makes sense to try to form $5x - 3y - 2z \geq -10$ as a combination of the active constraints, which we saw are (1), (2) and (4). In other words, we want to solve

$$\begin{bmatrix} 2 & -5 & 1 \\ -4 & 3 & 0 \\ 2 & -5 & 0 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -2 \end{bmatrix}$$

which is just a matrix equation with solution

$$s = 3/2 \quad t = 1 \quad u = 7.$$

So now if I take this linear combination of the constraints:

$$\begin{array}{rrcrcrlcl}
\frac{3}{2} \times (& 2x & - & 4y & + & 2z &) & \geq & \frac{3}{2} \times (-6) \\
+1 \times (& -5x & + & 3y & - & 5z &) & \geq & 1 \times (-1) \\
+7 \times (& x & & & & &) & \geq & 7 \times (\ 0\ ) \\
\hline
= & 5x & - & 3y & - & 2z & & \geq & -\ 10
\end{array}$$

and so since any feasible solution must satisfy (1), (2), (4) it must also satisfy $5x - 3y - 2z \geq -10$ (which means $\mathbf{v}$ gives the best possible value).