The Steenrod Algebra and Its Dual

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These are notes for the seminar "Advanced Topics in Homotopy Theory" given by Prof. Stefan Schwede and Dr. Jack Davies in Bonn during the WS2023/24. Our goal is to present the main results of Milnor's paper "The Steenrod Algebra and its Dual".

1 Hopf Algebras

1.1 Bi-Algebras

We start by studying Hopf algebras independently. Throughout, let k be a field.

Definition 1 (Algebra) *An **Algebra** is a triple $(\mathcal{A}, \mu, \eta)$ with $\mathcal{A}$ a k-vector space together with two maps $\mu \colon A \otimes A \to A$ (multiplication) , $\eta \colon k \to A$ (unit) making the following diagrams commute*

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mathrm{Id} \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{Id}"'] & A \otimes \mathcal{A} \arrow[d, "\mu"] \\ \mathcal{A} \otimes \mathcal{A} \arrow[r, "\mu"'] & \mathcal{A} \end{tikzcd}$$

$$\begin{tikzcd} k \otimes \mathcal{A} \arrow[r, "i \otimes \eta"] \arrow[dr] & \mathcal{A} \otimes \mathcal{A} \arrow[d, dashed, "\mu"] & \mathcal{A} \otimes k \arrow[l, "\eta \otimes i"'] \arrow[dl] \\ & \mathcal{A} & \end{tikzcd}$$

Dualizing these definitions, we unsurprisingly obtain

Definition 2 (Coalgebra) *A **coalgebra** is a triple (C, Δ, ϵ) where C is a k-vector space togethere with two maps $\Delta \colon C \to C \otimes C$ (comultiplication) and $\epsilon \colon C \to k$ (augmentation) making the following diagrams commute*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\mathrm{Id} \otimes \Delta"] \\ C \otimes C \arrow[r, "\Delta \otimes \mathrm{Id}"'] & C \otimes C \otimes C \end{tikzcd}$$

$$\begin{tikzcd}
 & C \arrow[dl] \arrow[d, "\Delta"] \arrow[dr] & \\
k \otimes C & C \otimes C \arrow[l, "\epsilon \otimes \mathrm{Id}"'] \arrow[r, "\mathrm{Id} \otimes \epsilon"'] & C \otimes k
\end{tikzcd}$$

Since taking duals commutes with tensor products, notice that the dual $C^\vee$ naturally gets an algebra structure.
We define (co-)algebra morphisms in the obvious way.

Definition 3 (Bialgebra) *A **bialgebra** is a tuple $(\mathcal{A}, \mu, \eta, \Delta, \epsilon)$ such that $(\mathcal{A}, \mu, \epsilon)$ is an algebra, $(\mathcal{A}, \Delta, \epsilon)$ is a coalgebra and such that Δ and ϵ are algebra morphisms*

Equivalently, one can also require μ and ϵ to be coalgebra morphisms.
If $\mathcal{A} = \bigoplus_{n \in \mathbb{N}} \mathcal{A}_n$ is a graded algebra, we define the **dual algebra** by

$$\mathcal{A}^* := A_n^*, \text{ with } A_n^* = \hom(A_{-n}, k)$$

We call a graded algebra $\mathcal{A}$ **graded commutative** if for all homogeneous elements $\alpha, \beta \in \mathcal{A}$, we have $\alpha\beta = (-1)^{\dim \alpha \dim \beta} \beta\alpha$. (omitting μ for sanity reasons)

1.2 Antipode maps

Let C be a bi-algebra as above and let $f, g \colon C \to C$ be linear maps, we define the convolution $f * g$ of f with g as the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} C \otimes C \xrightarrow{\mu} C.$$

Definition 4 (Antipode) *An antipode $S \colon C \to C$ is an endomorphism such that*

$$S * \mathrm{Id} = \mathrm{Id} * S = \eta \circ \epsilon.$$

Definition 5 (Hopf Algebra) *A Hopf Algebra is a bi-algebra with an antipode*

It can be shown that $*$ makes the set of endomorphisms of C into a group whose identity is $\eta \circ \epsilon$.

2 The Steenrod Algebra

Let p be a prime and let $H\mathbb{F}_p$ be the spectrum representing $\bmod p$ cohomology.

Definition 6 (Steenrod Algebra) *The* $\bmod p$ *Steenrod Algebra is the cohomology ring $\mathcal{A}_p := H\mathbb{F}_p^*(H\mathbb{F}_p) := \bigoplus_n \varprojlim_k H^{n+k}(K(\mathbb{F}_p, n), \mathbb{F}_p)$.*
This ring comes with a natural grading, we denote its i-th graded component $\mathcal{A}_p^i$.

Recall that an element $\theta \in \mathcal{A}_i^*$ defines a stable cohomology operation $\theta \colon H^n(-, \mathbb{F}_p) \to H^{n+i}(-, \mathbb{F}_p)$. Thus, $\mathcal{A}^*$ is the ring of all stable $\bmod p$ cohomology operations with composition.

2.1 Steenrod powers

From now, $H^*(-)$ will always denote mod p cohomology for a fixed prime p.

Definition 7 (Steenrod Powers) *The* ***Steenrod powers*** *are the cohomology operations*

$$P^i \colon H^q(-, \mathbb{F}_p) \to H^{q+2i(p-1)}(-, \mathbb{F}_p)$$

uniquely determined by the following properties

1. $P^0 = \mathrm{Id}$
2. *if* $x \in H^{2n}(X, A, \mathbb{F}_p)$, *then* $P^n x = x^p$
3. *if* $x \in H^n(X, A)$, *then* $P^i x = 0$ *for all* $2i > n$
4. $\delta P^i = P^i \delta$ *where* δ *is the boundary homomorphism*
5. $P^i(xy) = \sum_{j+k=i} P^j x P^k y$

The natural transformation $\beta \colon H^n(-) \to H^{n+1}(-)$ induced by the short exact sequence $0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0$ is also stable.
It is a famed result of Steenrod that these operations generate the Steenrod algebra.

2.2 The Diagonal Morphism

The main goal of this talk is to present a proof that $\mathcal{A}_p$ has the structure of a Hopf algebra and to make its structure more explicit.
Throughout, let X be a space. We start by constructing the diagonal morphism $\psi^* \colon \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$.

Proposition 1 *There is a unique diagonal morphism* $\psi^* \colon \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ *such that*

1. *For all* $\theta \in \mathcal{A}^*$, $\psi^*(\theta) = \sum_i \theta_i' \otimes \theta_i$" *and* $\alpha, \beta \in H^*(X)$ *we have*
$$\theta(\alpha \cup \beta) = \sum (-1)^{\dim \theta_i'' \dim \alpha} \theta_i'(\alpha) \cup \theta_i''(\beta)$$
2. *The morphism* ψ^* *is a ring morphism.*

Proof Let $\mathcal{A}^* \otimes \mathcal{A}^*$ act on $H^*(X) \otimes H^*(X)$ by

$$(\theta' \otimes \theta'')(\alpha \otimes \beta) = (-1)^{\dim \theta'' \dim \alpha} \theta'(\alpha) \otimes \theta''(\beta)$$

and we let $c : H^*(X) \otimes H^*(X) \to H^*(X)$ denote the coproduct.
ψ^* **exists**
Let $R \subset \mathcal{A}^*$ be the set of all θ such that

$$\theta(\alpha \cup \beta) = c\rho(\alpha \otimes \beta)$$

for some $\rho \in \mathcal{A}^* \otimes \mathcal{A}^*$. We want to show that $R = \mathcal{A}^*$.
Notice that R is closed under multiplication and addition. If $\theta_1, \theta_2 \in R$, then

$$\theta_1\theta_2(\alpha \cup \beta) = c\rho_1\rho_2(\alpha \otimes \beta) \text{ and } (\theta_1 + \theta_2)(\alpha \cup \beta) = c((\rho_1 + \rho_2)(\alpha \otimes \beta))$$

Hence, it suffices to show that R contains the Bockstein and the Steenrod powers which follows from the formulas

$$\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^{\dim \alpha}\alpha \cup \delta(\beta)$$
$$P^n(\alpha \cup \beta) = \sum_{i+j=n} P^i(\alpha) \cup P^j(\beta)$$

ψ^* is unique
Let $K := K(\mathbb{F}_p, n+1)$ and $\gamma \in H^{n+1}(K)$ correspond to the identity map, the map

$$\begin{aligned} \underset{\gamma}{\mathrm{ev}}\colon \mathcal{A}^*_i &\to H^{n+1+i}(K) \\ \theta &\mapsto \theta\gamma \end{aligned}$$

is an isomorphism for all $i \leq n$, it follows that

$$\begin{aligned} j\colon\ (\mathcal{A}^* \otimes \mathcal{A}^*)_i &\to H^{2n+2+i}(K \times K) \\ \theta \otimes \theta' &\mapsto (-1)^{\dim \theta' \dim \gamma}\theta(\gamma) \otimes \theta'(\gamma) \end{aligned}$$

is too.
Let $\theta \in \mathcal{A}^*_i$, suppose ρ, ρ' both satisfy the required equality, then

$$j(\rho) = c\rho\left((\gamma \otimes 1) \otimes (1 \otimes \gamma)\right) = c\rho'\left((\gamma \otimes 1) \otimes (1 \otimes \gamma)\right) = j(\rho')$$

The unicity of ψ^* implies that it is a ring morphism.

Remark 2 *From this proof, we can in particular single out the action of ψ^* on generators, namely, it follows that*

$$\psi^*(\delta) = \delta \otimes 1 + 1 \otimes \delta$$
$$\psi^*(P^n) = \sum_{i+j=n} P^i \otimes P^j.$$

Theorem 3 (The Steenrod Algebra is a Hopf Algebra) *The maps*

$$\mathcal{A} \xrightarrow{\psi^*} \mathcal{A} \otimes \mathcal{A} \xrightarrow{\phi^*} \mathcal{A}$$

Give $\mathcal{A}$ the structurre of a Hopf algebra. Furthermore ϕ^* is associative and ψ^* is associative and commutative.

Proof It suffices to show that ψ^* is associative and commutative.

Associativity

It suffices to check the identity

$$(\psi^* \otimes 1)\psi^* = (1 \otimes \psi^*)\psi^*$$

This identity clearly holds on generators, namely

$$\begin{aligned}(\psi^* \otimes 1)\,(\delta \otimes 1 + 1 \otimes \delta) &= \delta \otimes 1 \otimes 1 + 1 \otimes \delta \otimes 1 + 1 \otimes 1 \otimes \delta \\ &= (1 \otimes \psi^*)\,(\delta \otimes 1 + 1 \otimes \delta)\end{aligned}$$

and

$$\begin{aligned}(\psi^* \otimes 1)\left(\sum_{i+j=n} P^i \otimes P^j\right) &= \sum_{i+j=n}\left(\sum_{i'+j'=i} P^{i'} \otimes P^{j'}\right) \otimes P^j \\ &= \sum_{i+j+k=n} P^i \otimes P^j \otimes P^k \\ &= (1 \otimes \psi^*)\left(\sum_{i+j=n} P^i \otimes P^j\right).\end{aligned}$$

Commutativity

Let

$$\begin{aligned}T\colon \mathcal{A} \otimes \mathcal{A} &\to \mathcal{A} \otimes \mathcal{A} \\ \theta \otimes \theta' &\mapsto (-1)^{\dim\theta \dim\theta'}\theta' \otimes \theta.\end{aligned}$$

We have to check that $\psi^* = T\psi^*$, which one can check again on generators:

$$T(1 \otimes \delta + \delta \otimes 1) = 1 \otimes \delta + \delta \otimes 1$$

and

$$T(\sum_{i+j=n} P^i \otimes P^j) = \sum_{i+j=n} (-1)^{4ij(p-1)^2} P^j \otimes P^i$$

3 The dual Steenrod Algebra

For the rest of this talk, we focus on the dual Steenrod algebra $\mathcal{A}_* := \mathcal{A}^\vee$, whose multiplication is induced by ψ^*. Our goal is to fully determine the structure of $\mathcal{A}_*$.
To single out an appropriate set of generators for $\mathcal{A}_*$, we analyze how $\mathcal{A}_*$ (co-)acts on the cohomology ring of a specific space. We start by describing this co-action formally and then introduce the relevant space.

3.1 The coaction of $\mathcal{A}_*$

Given that we are working over a vector space, cohomology and homology are dual. Hence, given $\theta \in \mathcal{A}$ and $\mu \in H_*$, the rule

$$\theta \cdot \mu(\alpha) := \mu(\theta(\alpha)) \text{ for all } \alpha \in H^*$$

gives a well defined action

$$\lambda_* \colon \mathcal{A} \otimes H_* \to H_*$$

We denote the dual of this action by $\lambda^* \colon H^* \to \mathcal{A}_* \otimes H^*$. The restriction of λ_*

$$\lambda_i \colon \mathcal{A} \otimes H^{n+i} \to H^n$$

also gives rise to dual morphisms $\lambda^i \colon H^n \to \mathcal{A}_* \otimes H^{n+i}$ which satisfy

$$\lambda^* = \lambda^1 + \lambda^2 + \dots .^1$$

We can also understand the action of $\mathcal{A}$ better in terms of λ^*.

Lemma 4 *Let $\lambda^*(\alpha) = \sum_i \alpha_i \otimes \omega_i$ and $\theta \in \mathcal{A}$, then*

$$\theta\alpha = \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \theta, \omega_i \rangle \alpha_i$$

Proof By definition of the action, we have

$$\begin{aligned} \langle \mu, \theta\alpha \rangle &= \langle \mu\theta, \alpha \rangle \\ &= \langle \mu \otimes \theta, \lambda^* \alpha \rangle \\ &= \sum_i (-1)^{\dim \alpha_i \dim \omega_i} \langle \mu, \alpha_i \rangle \langle \theta, \omega_i \rangle \end{aligned}$$

And the general equality follows.

3.2 Generators for $\mathcal{A}_*$

Fix some large integer N and let $X = S^{2N+1}/\mathbb{Z}_p = sk_{2N+1}K(\mathbb{F}_p, 1)$. The (mod p) cohomology ring of X has the following properties

$$H^1(X) = \langle \alpha \rangle, H^2(X) = \langle \beta \rangle, H^{2i}(X) = \langle \beta^i \rangle, H^{2i+1}(X) = \langle \alpha\beta^i \rangle,$$

where $\beta = \delta\alpha$ and $i \leq N$

Notation 5 *We define*

$$M^k := P^{p^{k-1}} \cdots P^p P^1$$

[1] Elements in H^* are always finite sums, so this sum should be understood as $\bigoplus_i \lambda^i$

Lemma 6 *For all $\theta \in \mathcal{A}$*

$$\theta\beta = \begin{cases} \beta^{p^k} & \text{if } \theta = M_k \\ 0 & \text{else.} \end{cases}$$

Proof Let $\mathcal{P} = 1 + P^1 + P^2 + \ldots$, from the properties of the Steenrod powers, we notice that

$$\mathcal{P}\beta = \beta + \beta^p \text{ thus } \mathcal{P}\left(\beta^{p^r}\right) = \beta^{p^r} + \beta^{p^{r+1}}.$$

Hence $P^{p^r}(\beta^{p^r}) = \beta^{p^{r+1}}$ and $P^j(\beta^{p^r})$ for $j \neq p^r$ and $j > 0$. From this, we deduce the statement.

We will now explicitly determine a basis for $\mathcal{A}_*$.

Lemma 7 *There exist elements $\tau_i, \in \mathcal{A}_*^{2p^k-1}$ such that*

$$\lambda^*\alpha = \alpha \otimes 1 + \beta \otimes \tau_0 + \ldots + \beta^{p^r} \otimes \tau_r.$$

Similarly, there exist elements $\xi_i \in \mathcal{A}_*^{2p^i-2}$ with $\xi_0 = 1$ such that

$$\lambda^*\beta = \beta \otimes \xi_0 + \beta^p \otimes \xi_1 + \ldots + \beta^{p^r} \otimes \xi_r$$

Proof From the above, it follows that

$$\lambda^*\beta = \lambda^0\beta + \lambda^{2p-2}\beta + \ldots + \lambda^{2p^k-2}\beta.$$

As the cohomology of X is one-dimensional in all degrees, we deduce that $\lambda^{2p^k-2}(\beta) = \beta^{p^k} \otimes \xi^k$. The exact same argument works for $\lambda^*\alpha$.

We now study the evaluation pairing $\mathcal{A}_* \times \mathcal{A} \to \mathbb{F}_p$. We easily establish the following lemma

Lemma 8 *We have $\xi_k(M_k) = 1$ but $\xi_k(\theta)$ for any other monomial.*
Furthermore

$$\langle M_k\delta, \tau_k\rangle = 1$$

and $\langle\theta, \tau_k\rangle$ for any other monomial.

Proof We know that

$$M_k\beta = \beta^{p^k} = \sum_i (-1)^{2p^i \dim \xi^i} \langle M_k, \xi_i\rangle \beta^{p^i}$$

Proving the equality. The second equality follows from the same argument applied to α and $M_k\delta$.

We are ready to prove the main structure theorem for the dual Hopf algebra.

Theorem 9 *There is an isomorphism*

$$\mathcal{A}_* \simeq \Lambda[\tau_0, \tau_1, \ldots] \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots].$$

where $\Lambda[\tau_0, \ldots]$ denotes the exterior algebra and $\mathbb{F}_p[\xi_1, \xi_2, \ldots]$ is the polynomial algebra. This isomorphism respects the grading. define weights on generators

Proof Let $\mathcal{I}$ be the set of finite sequences $(\epsilon_0, r_1, \epsilon_1, \ldots)$ with $\epsilon_i = 0, 1$ and $r_i \in \mathbb{N}$. Given $I \in \mathcal{I}$, we define

$$\omega(I) := \tau_0^{\epsilon_0} \xi_1^{r_1} \tau_1^{\epsilon_1} \xi_2^{r_2} \cdots .$$

We claim it is sufficient to show that the set of $\omega(I)$ form a basis for $\mathcal{A}_*$. Indeed, the τ_i, ξ_j then don't observe any additional identities and the graded commutativity gives the desired isomorphism.
We may order the set $\mathcal{I}$ colexicographically, ie. $(a_1, \epsilon_1, a_2, \cdots) < (b_1, \epsilon_1', b_2, \cdots)$ if $a_i < b_i$ for the largest i such that a_i and b_i differ (remember that the sequences are finite).

We also associated to a $J = (\epsilon_0, r_1, \epsilon_1, \ldots) \in \mathcal{I}$ an element of $\mathcal{A}$.

$$\theta(J) = \delta^{\epsilon_0} P^{s_1} \delta^{\epsilon_1} P^{s_2} \cdots ,$$

where $s_j = \sum_{i=k}^{\infty} (\epsilon_i + r_i) p^{i-k}$.

One can check that the $\theta(J)$ are the basic monomials of the Cartan basis for $\mathcal{A}$.

To show the isomorphism, we show that the basic monomials in $\mathcal{A}$ form an "almost dual" basis to the set of $\omega(I)$.
More precisely, we will show the following lemma.

$$\text{Let } I < J \in \mathcal{I}, \text{ then } \langle \theta(J), \omega(I) \rangle = 0 \text{ if } I < J, \text{ furthermore } \langle \theta(I), \omega(I) \rangle = \pm 1. \qquad (\star)$$

The proof of $(\star)$ will constitute the main part of the proof, let us see how to conclude given $(\star)$.
Let $\mathcal{I}_n \subset \mathcal{I}$ be the set of sequences such that $\dim \omega(I) = \dim \theta(I) = n$. The matrix $(\langle \theta(J), \omega(I) \rangle)_{I,J \in \mathcal{I}_n}$ is upper-triangular with ± 1 on the diagonal, hence, the pairing is non-degenerate and the $\omega(I)$ generate the n-th graded part of $\mathcal{A}_*$.