ALGEBRAIC CURVES
EXERCISE SHEET 6

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let V, W be varieties and assume that W is affine.

(1) Show that there is a bijection $Hom_{Var}(V, W) \simeq Hom_{k-alg}(\mathcal{O}(W), \mathcal{O}(V))$.
(2) Show that there is a bijection $\mathcal{O}(V) \simeq Hom_{Var}(V, \mathbb{A}^1_k)$.
(3) Suppose $W = \mathbb{A}^1_k$. Let $k(T)$ be the field of rational functions of W. Show that there is a bijection $k(T) \simeq Hom_{Var}(\mathbb{A}^1_k, \mathbb{P}^1_k)$. (Hint: to build a morphism $\mathbb{A}^1_k \to \mathbb{P}^1_k$, take inspiration from exercise 4).

Exercise 2.

Let $n \geq 1$ and $f \in k[x_1, \ldots, x_n]$.

(1) Show that $\mathbb{A}^n_k - V(f)$ is affine. What is its ring of regular functions?
(2) Show that $\mathbb{A}^2_k - \{(0,0)\}$ is not affine. (Hint: compute the ring of regular functions).

Exercise 3.

Let $\varphi : V \to W$ be a morphism of affine varieties and $\varphi^\sharp : \Gamma(W) \to \Gamma(V)$ the corresponding morphism of coordinate rings. Let $P \in V$ and $Q = \varphi(P)$ and consider local rings $\mathcal{O}_P(V)$, $\mathcal{O}_Q(W)$ with maximal ideals $\mathfrak{m}_P, \mathfrak{m}_Q$. Show that $\varphi^\sharp$ extends uniquely to a ring homomorphism $\mathcal{O}_Q(W) \to \mathcal{O}_P(V)$ and that $\varphi^\sharp(\mathfrak{m}_Q) \subseteq \mathfrak{m}_P$.

Exercise 4.

Let $n \geq 1$ and V a variety. We use projective coordinates x_i, $1 \leq i \leq n+1$ on $\mathbb{P}^n_k$. Suppose there exist an open cover $(U_i)_{1 \leq i \leq n+1}$ of V and morphisms of varieties $\varphi_i : U_i \to \{x_i \neq 0\} \subseteq \mathbb{P}^n_k$, $1 \leq i \leq n+1$, such that $\forall i \neq j$, $(\varphi_i)_{\restriction U_i \cap U_j} = (\varphi_j)_{\restriction U_i \cap U_j}$. Show that there exists a unique morphism $\varphi : V \to \mathbb{P}^n_k$ such that $\varphi_{\restriction U_i} = \varphi_i$. We say that φ is obtained by *glueing* the φ_i, $1 \leq i \leq n+1$.

Exercise 5. *

Let $f \in k[x_1, x_2, x_3]$ an irreducible form of degree 2 and consider $V_P(f) \subseteq \mathbb{P}^2_k$.

(1) Show that, up to a linear change of coordinates, we can assume that $f = x_2^2 - x_1x_3$. (Hint: remember we classified similar subvarieties of $\mathbb{A}^2_k$).

(2) Show that the map:

$$\begin{array}{rcl} \mathbb{P}^1_k & \to & \mathbb{P}^2_k \\ (s:t) & \mapsto & (s^2 : st : t^2) \end{array}$$

induces an isomorphism $\mathbb{P}^1_k \simeq V_P(f)$. (Hint: take a look locally in the standard affine opens of projective space and use exercise 4).