Discrete Mathematics

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List of Theorems

Lecture 1: Introduction

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1 Counting

1.1 Finite sets

Let A be a finite set. We denote by $|A|$ the cardinality of A.

Definition 1 (First Numbers)
We denote by $[n]$ the set of n first natural numbers.

1.2 Bijections

Theorème 1
If there exists a bijection between finite sets A and B then $|A| = |B|$.

1.3 Operations with finite sets

— union
— intersection
— product
— exponentiation
— quotient

Definition 2 (Cartesion product)

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Theorème 2

$$|A \times B| = |A||B|$$

Definition 3 (Disjoint union)
Define

$$A \sqcup B = A \times \{0\} \cup B \times \{1\}$$

Theorème 3

$$|A \sqcup B| = |A| + |B|$$

Definition 4 (Exponential object)

$$A^B = \{f | f \text{ is a function from } A \text{ to } B \}$$

Theorème 4

$$|A^B| = |A|^{|B|}$$

Definition 5 (Binomial coefficient)

A binomial coefficient $\binom{n}{k}$ is the number of ways in which one can choose k objects out of n distinct objects assuming order doesn't matter.

Proposition 5

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Proposition 6

The following identities hold :

1. $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

2. *$\binom{n}{k}$ is the k-th element in the n-th line of Pascal's triangle.*

Preuve

Each subset of $[n+1]$ either contains $n+1$ or not.
Number of $(k+1)$-element subsets containing $n+1$ is $\binom{n}{k}$
Number of $(k+1)$-element subsets not containing $n+1$ is $\binom{n}{k+1}$ $\square$

Proposition 7

The number of subsets of an n-element set is 2^n, since we have

$$2^n = \sum \binom{n}{i}$$

Proposition 8

The number of subsets of even cardinality is the same as even cardinality : 2^{n-1}

Preuve

Consider

$$\phi : 2^{[n]} \to 2^{[n]}$$

defined by

$$\phi(A) = A\Delta\{1\} = \begin{cases} A\setminus\{1\}\,, & \textit{if } 1\in A \\ A\cup\{1\}\,, & \textit{otherwise} \end{cases}$$

□

The cardinality of subsets A and $\phi(A)$ always have different parity.
Since $\phi\circ\phi = \mathrm{Id}$ we deduce that ϕ is a bijection between the set of odd and even subsets is the same.

Theorème 9

$$(1+x)^n = \sum \binom{n}{i} x^i$$

Preuve

In lecture notes. □

Proposition 10

Assume we have k identical objects and n different persons. Then ne number of ways in which one can distribute this k objects among the n persons equals

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Equivalently, it is the number of solutions of the equation $x_1 + ... + x_n = k$

Preuve

Let $\mathcal{A}$ be the set of all solutions of the equation. Let $\mathcal{B}$ be the set of all subsets of cardinality $n-1$ in $k+n-1$.
we construct a bijection $\psi : \mathcal{A} \to \mathcal{B}$ in the following way

$$A = (x_1, \ldots, x_n) \mapsto B = \{x_1 + 1, x_1 + x_2 + 2 \ldots\}$$

It suffices to show that ψ is invertible. Let $B \in \mathcal{B}$. Suppose that $b_1 \ldots, b_{n-1}$ are the elements of B, ordered. Then the preimage is an n-tuple of integers $(x_1, \ldots)$ defined by

$$x_1 = b_1 - 1$$
$$x_i = b_i - b_{i-1}$$
$$x_n = k + n - 1 - b_{n-1}$$

□

It is easy to see from these equations that the $x_i are non-negative and their sums yield k$.