Questions

David Wiedemann

I have decided to first present the proof of the Kuznetsov conjectures for Pfaffian cubics following Huybrecht's book as it would best tie in with Saverio's talk. Here are a few questions relating to the proofs of these.

1. The geometric input we need for this proof is lemma 6.2.20, Saverio told me he sketched a proof of this, but do you think it makes sense to sketch a proof of this again? Especially the fact that the fibers are $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$?

2. In the proof of lemma 3.10, I am clear on how the proof works when the fiber is $\mathbb{P}^1 \times \mathbb{P}^1$, but I would like to at least have some idea of how the $\mathbb{F}_2$ case works, potentially using it as an exercise. (or is this bad practice?) In fact, I am not even sure what $\mathcal{O}_X(1)$ restricts to on a fiber isomorphic to $\mathbb{F}_2$.

3. Turning to lemma 3.11, there are certain claims made in the proof that I am not confident about. Why is $\mathcal{I}_{\Sigma_p}$ simple? This is simply stated and I tried to use the LES in ext to show this but didn't get far.

4. I have the same question for $\mathrm{Hom}(\mathcal{I}_{\Sigma_{p_1}}, \mathcal{I}_{\Sigma_{p_2}}) = 0$.

5. Is $\mathcal{I}$ the ideal sheaf associated to the reduced induced scheme structure on Σ

6. Finally, do you think it makes sense to present the computation of $\chi(\mathcal{I}_{\Sigma_{p_1}}, \mathcal{I}_{\Sigma_{p_2}})$? Of course, the proof that $\chi(\mathcal{O}_{\Sigma_p}, \mathcal{O}_{\Sigma_p}) = 10$ is not within the scope of the talk, but I can put that into one "geometric input" lemma.

I believe the introduction together with the proof of this result will occupy a bit more than half of the talk and I have been debating what to present in the second part. As my main goal should be to motivate the Kuznetsov conjectures in general, two things come to mind.
First off, I am thinking of defining what the twisted derived category of sheaves on a variety is using Azumaya algebras and mentioning proposition 3.3 as well as its corollary 3.5. This would imply presenting the main geometry of the situation (in particular,

proposition 6.1.10). Unfortunately I would not go into the details of the proof of any lemma as it is very similar to the first proof I gave and I don't have the time.

Another option I have been considering is to follow Kuznetsov's expository paper https://arxiv.org/abs/1509.09115 to present a more theoretical reason that motivates the formulation of the Kuznetsov conjectures. In this paper, he defines the Griffith's component of a k-linear triangulated category and shows that, if it is well-defined, it is a birational invariant of $D^b(X)$. He then uses this Griffith's component and another conjecture to argue that, if $\mathcal{A}_X \not\simeq S$, then the geometric dimension of $\mathcal{A}_X$ is not 2 and hence X could not be rational. The main argument is presented on page 21 of the paper while a few definitions and results are defined before. I believe this second option has a few distinct advantages which I would like your feedback on.

- It has a very different flavour than the proof of proposition 6.3.9, while the proof of 6.3.3 is very similar. For time reasons, I don't think I could do the proof of 6.3.3 justice.
- It doesn't rely on extra geometric input that has not been covered in the seminar up until now
- The whole argument is reasonably easy to follow and manages to omit the use of Hochschild homology which is out of the scope of my talk.