Serie 2

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $A, B \subset \mathbb{R}^d$. Show that if $A \subseteq B$ and $\mathrm{m}^*(B) = 0$, then $\mathrm{m}^*(A) = 0$.

Exercise 2. If A is a subset of $\mathbb{R}$, show that $\mathrm{m}^*(A) = \mathrm{m}^*(A \cap (0, \infty)) + \mathrm{m}^*(A \setminus (0, \infty))$.

The following exercise generalizes Exercise 2 to half-spaces in any dimension.

Exercise 3. If $A \subseteq \mathbb{R}^n$ and E is the half-plane $E := \{(x_1, \ldots, x_n) \in \mathbb{R}^d : x_n > 0\}$, show that $\mathrm{m}^*(A) = \mathrm{m}^*(A \cap E) + \mathrm{m}^*(A \setminus E)$.

We will see in the lecture that Exercises 2 and 3 prove that $(0, \infty)$ and half-spaces are Lebesgue measurable sets.

Exercise 4. We want to show that the notion of Lebesgue outer measure does not depend on whether we consider coverings by open, half-open or closed boxes. To this end, we introduce for a subset $\Omega \subseteq \mathbb{R}^n$

$$\overline{\mathrm{m}}^*(\Omega) = \inf \left\{ \sum_{j=1}^{\infty} \mathrm{vol}(B_j) : \quad \Omega \subseteq \bigcup_{j=1}^{\infty} B_j \text{ and for all } j \geq 1 \quad B_j = \prod_{i=1}^{n} [a_i^{(j)}, b_i^{(j)}] \right.$$
$$\left. \text{for some } -\infty < a_i^{(j)} < b_i^{(j)} < +\infty \right\}.$$

Show that for any subset $\Omega \subseteq \mathbb{R}^n$

$$\overline{\mathrm{m}}^*(\Omega) = \mathrm{m}^*(\Omega).$$

Exercise 5. Let $f \in C^1(\mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. Show that all the following statements are equivalent.

(i) f is convex.

(ii) For any $x, y \in \mathbb{R}^d$,

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle.$$

(iii) For any $x, y \in \mathbb{R}$,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0.$$

If, in addition, $f \in C^2(\mathbb{R}^d)$, then (i)-(iii) are equivalent to

(iv) For any $x, v \in \mathbb{R}^d$,

$$\langle \nabla^2 f(x) v, v \rangle \geq 0.$$

where $\nabla^2 f(x)$ denotes the Hessian matrix $\nabla^2 f(x) = \begin{pmatrix} \dfrac{\partial^2 f}{\partial^2 x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_d \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial^2 x_d} \end{pmatrix}$

Hint: For this last part, recall that for $g \in C^1$

$$\int_{t_0}^{t_1} g'(t)\, dt = g(t_1) - g(t_0).$$