TIL

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Lecture 1: Tensor Products preserve Flatness

Wed 31 Aug

Let S be flat over R and $\phi : R \to T$ a morphism of rings, then $S \otimes_R T$ is a flat T-module.
Recall that a module S is flat iff $\mathrm{Tor}_n^R(S, M) = 0$ for any R module M.
Let $0 \to A \to B \to C \to 0$ be an SES of $T-$modules, then tensoring with $S \otimes_R T$ gives

$$0 \to A \otimes_T (S \otimes_R T) \to B \otimes_T (S \otimes_R T) \to C \otimes_T (S \otimes_R T) \to 0$$

But recall that $A \otimes_T (S \otimes_R T) = A \otimes_R S$ where A is given the structure of an $R-$module through ϕ, from this it is immediate that $S \otimes_R T$ is flat.

This is useful for a version of going down for flat extensions, if $\phi : R \to S$ is a morphism which makes S into a flat $R-$module, if $P' \subset P$ are two prime ideals of R and Q is an ideal of S whose contraction is P then there exists a prime ideal $Q' \subset Q$ whose contraction is P'.

To show this (10.11 in Eisenbud), we first reduce to the case $P' = 0$ by modding out by P', now R is a domain and we can replace S with $S/P'S$
Now we use the above to show that $S/P'S = S \otimes R/P'$ is flat over R/P'.
Now every nonzero divisor of R (which is just $R \setminus 0$ since R is a domain) is a nonzerodivisor on S by some lemma.
We now choose Q' a minimal prime of S, hence, it's elements consist of zerodivisors (since $AssS$ contains all minimal primes and is equal to 0 and the set of zerodivisors) and thus $Q' \cap R = 0$

$\dim R[x] = 1 + \dim R$

The inequality $\dim R[x] \leq 1 + \dim R$ is obvious, since for a chain of ideals P_i in R, we can form $P_1 R[x] \subset \ldots \subset P_n R[x] \subset P_n R[x] + (x)$, for the other inequality, Eisenbud claims that the inequality holds if we assume the result that :
For P prime in R, an ideal Q maximal wrt to the property of contracting to P $\dim R[x]_Q = 1 + \dim R_P$.

Indeed, once we have this result, take Q to be maximal in $R[x]$ and $P = R \cap Q$. The other inequality then follows from

$$1 + \dim R \geq 1 + \dim R_p = \dim R[x]_Q$$

Taking Q to be a maximal ideal in a maximal chain for $R[x]$, we get $\dim R[x]_Q = \dim R[x]$

Lecture 2: Crossed Homomorphisms

Mon 05 Sep

For a finite group G together with a G-module A, we can nicely describe the first cohomology group $H^1(G, A)$.
Indeed, letting $d_n : C_n(G, A) \to C_{n+1}(G, A)$ be the boundary maps, we get that $d_1(f)(g, h) = gf(h) - f(gh) + f(g)$ so an element of the kernel d_1 is precisely an element satisfying $f(gh) = gf(h) - f(g)$. We can also describe coboundaries as elements of the form $f(g) = ga - a$ which are called principal crossed homomorphisms.

From this we can deduce Hilbert's theorem 90 which says that for a finite galois extension K/L with galois group G, we have $H^1(G, K^\times) = 0$, here $K^\times$ is made into a G-module in the obvious way.
To prove this, we use that the Galois automorphisms are linearly independent in $End(K, K)$ thus for a cocycle $\{a_\sigma\}$ (ie a map $G \to A$ satisfying $f(\sigma\tau) = \sigma f(\tau) f(\sigma)^{-1} = \sigma a_\tau a_\sigma^{-1}$) hence there is some $\gamma \in K$ such that

$$\beta = \sum_{\sigma \in G} a_\sigma \sigma(\gamma)$$

is nonzero. We can now easily check that

$$\sigma(\beta) = a_\sigma^{-1}(\beta)\beta \implies a_\sigma = \frac{\beta}{\sigma(\beta)}$$

Which in turn means that 1-cocycles satisfy the 1-coboundary condition and thus $H^1(G, K^\times) = 0$

Lecture 3: Exterior Derivatives

Tue 06 Sep

We start with a smooth manifold M, on this, we have differential k-forms, these are maps which, to each $p \in M$ associate an alternating k-form ω_p.
We think of 1-forms as projections, ie. differential forms act on vector fields in the obvious way, by defining $(\omega(X))_p = \omega_p(X_p)$, as such, at least locally, differential forms can be thought of as projections of a vector field onto a subvector. Somehow, k-forms then represent higher dimensional projections (as they act on k vector fields at the same time and will somehow locally describe a k-dimensional volume) To describe smoothness of these k-forms, we turn to vector bundles, first, we can define the cotangent bundle (usually written T^*M), this is just the disjoint union $\bigcup_{p \in M} T_p^*M$.
We topologise this in a way analoguous to the tangent bundle.
For a coordinate chart $(U, y_1, \ldots, y_n) \ni p$, we have a canonical choice of basis for T_p^*M which is just the projections onto the coordinates (equivalently, the dual basis to $\frac{\partial}{\partial y_i}$).
Now we have trivialization maps $\phi : \bigcup_{p \in U} T_p^*M \to U \times \mathbb{R}^n$ and we can endow T^*M with the topology induced from the trivialization maps.

For k-forms, we have to refine the construction a bit.
First of, we recall that an alternating k-form on a v.s. V is a multilinear map $V^k \to \mathbb{R}$ which is alternating (respects sign of signature when we permute the coefficients).
These alternating forms form a vector space called the kth exterior power of V which we denote by $\bigwedge^k(V^*)$, we can now, again, form a vector bundle over M, written $\bigwedge^k(T^*M)$ and called the kth exterior power and defined, as a set, to be $\bigcup_{p\in M}\bigwedge^k(T_p^*M)$ and topologized as above(with corresponding basis explained below)
Indeed, a natural choice for a basis of $\bigwedge^k(T_p^*M)$ is the set $dx^I = dx_{i_1}\wedge\ldots\wedge dx_{i_k}$ where I runs over all growing multiindices.

We can now get to the first important property of k-forms, namely a pullback of a k-form.
Recall that every smooth map $F : M \to N$ induces a pushforward on tangent spaces by sending, for $f \in C^\infty_{F(p)}(N)$, X a tangent vecotr at p (ie. a derivation $C^\infty_p(M) \to \mathbb{R}$)

$$(F_{*,p}X)(f) = X(f \circ F)$$

One important thing to note is that this pushforward is not (in general) defined globally as a point $p' \in N$ may have non-trivial fibers, while the differentials at the different points of the fiber may disagree.
This means we can NOT pushforward vector fields.

In this respect differential forms are better behaved because we can pull them back With the same setup as above, let ω be a differential k-form on N, we can define the pullback of ω, written $F^*\omega$.
This pullback is just precomposition with $F_*^{\times n}$, namely, for $v_1, \ldots, v_k \in T_pM$, we have

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(F_{*,p}v_1, \ldots)$$

This pullback is linear and in fact, pullback of C^∞ forms are smooth, though this is thougher to prove. The definition of wedge product of differential forms is simply their wedge product evaluated pointwise and pullbacks commute with wedge products.

We now finally get to the exterior derivative, we define an antiderivation on a grader algebra $A = \bigoplus A^k$ as an $\mathbb{R}$-linear endomorphism of A which satisfies an equivalent of the chain rule, namely an antiderivation D needs to satisfy for two homogeneous elements ω and τ of degree k, l respectively

$$D(\omega\tau) = D(\omega)\cdot\tau + (-1)^k\omega\cdot D\tau$$

We say that D is of degree n if for a homogeneous piece ω of degree k, the degree of $D\omega$ is $k+n$.

We now define the exterior algebra on M as a direct sum $\Omega^*(M) = \bigoplus \Omega^k(M)$, here $\Omega^k(M)$ represents the set of all smooth sections of $\bigwedge^k(T^*M) \to M$.
We define the exterior derivative to be the unique antiderivation satisfying

— D is of degree 1

— $D \circ D = 0$

— For a function f (which is just an element of $\Omega^0(M)$) and for a vector field X, we have $Df(X) = Xf$

From these properties we can see how D can be expressed in local charts, if $(U, y_1, \ldots, y_n)$ is such a chart on M, and if $\omega = \sum a_I dx^I$ is k-form on U, we have

$$\begin{aligned} D\omega &= \sum D(a_I dx^I) \\ &= \sum da_I \wedge dx^I + \sum a_I D(dx^I) \\ &= \sum da_I \wedge dx^I \\ &= \sum_I \sum_{j \in [n]} \frac{\partial a_I}{\partial y_j} dy_j \wedge dx^I \end{aligned}$$

Using the fact that D is a local operator and partitions of unity, we can show that D is uniquely determined and independent of the choice of coordinate charts.
Using this, we can define the unique exterior derivative of a manifold M, we denote it by d.

Lecture 4: distribution of ideals in number rings

Thu 15 Sep

The basic result we want to show is that ideals are approximatively evenly distributed among class group, more precisely : Let R be a number ring corresponding to $K/\mathbb{Q}$ (an extension of degree n) and let C be a class in it's ideal class group, let $i_C(t)$ denote the number of ideals I in C satisfying $\|I\| \leq t$, then

$$i_C(t) = \kappa t + \epsilon(t), \quad \epsilon(t) \text{ is } O(t^{1-\frac{1}{n}})$$

To show this, we resort to a few tricks.
We first start off by noticing a bijection between the set of ideals we want to count and another "nice" set, namely, there is a one-to-one correspondence

$$A := \{ \text{ Ideals } I \text{ in } C \text{ such that } \|I\| \leq t\} \leftrightarrow \{ \text{ Principal ideals } (\alpha) \subset J \text{ such that } \|\alpha\| \leq t\,\|J\|\} =: B$$

Where J is a fixed element of the class C^{-1}.
The correspondence is given as follows, to an ideal $I \in C$, we associate IJ, by

multiplicativity of the norm, it is clear that IJ is in the right hand set.
As an inverse, for a principal ideal $(\alpha) \subset J$, let I be an ideal such that $IJ = (\alpha)$ (exists by some previous theorem), then I is the corresponding ideal in the left set.
If the group of units of R (call it U) is finite, and we are able to count the elements whose norm is $\leq t\,\|J\|$ (call it n) , then we would have $i_C(t)|U| = n$.

To count elements of B, we now resort to the second main trick in the proof : we construct a subset of R which has a representative for every coset of U.
Then intersecting this subset with the set of all elements of norm smaller than t, we will be able to count elements.
Recall from the previous chapter that $U = W \times V$ with V free abelian and W cyclic.
To construct this subset, we recall the log maps defined a chapter earlier, namely, we consider the compositions

$$V \subset U \subset R - 0 \to \wedge_R - 0 \to \mathbb{R}^{r+s}$$

Here, as usual, r is the number of real embeddings of K and $2s$ is the number of complex ones, recall also that $\wedge_R$ is a $r + 2s$ dimensional lattice.
We recall that units get mapped onto the hyperplane $H : \sum_i y_i = 0$, because for u a unit, it's norm is 1. We also remember that the kernel of this composition is W and thus the composition $V \to \mathbb{R}^{r+s}$ is injective. We denote it's image by $\wedge_U$.
We replace $\wedge_R \subset \mathbb{R}^n \simeq \mathbb{R}^r \times \mathbb{C}^s$ and extend the log mapping by defining

$$\log(x_1, \ldots, x_r, z_1, \ldots, z_s) \coloneqq (\log|x_1|, \ldots, 2\log|z_1|, \ldots)$$

It is less an extension and more so a redefinition made to agree with the usual log mapping.
Note that with this definition, counting elements of R satisfying $\|\alpha\| \leq t\,\|J\|$ is equivalent to counting elements of $\mathbb{R}^r \times \mathbb{C}^s$ (in the image of the inclusion) satisfying $N(x) \leq t\,\|J\|$

We get onto the third trick in this proof, namely, instead of finding representatives of cosets directly in R, we will search for them in $\mathbb{R}^{r+s}$ and then pull them back to representatives in R. To this end we use the following proposition :

Proposition 1
If $f : G \to G'$ is a morphism of abelian groups and $S \subset G$ gets mapped isomorphically onto it's image in G', then a set of representatives of cosets of $f(S) \subset G'$ pull back to representatives of cosets in G

Preuve

Let D' be a set of coset representatives of $f(S)$, let aS be a coset in G, let d be a representative for $f(aS)$, then any element of $f^{-1}(d)$ is a representative for aS. $\square$

We apply this to $V \subset R$, and so we have to find a set of coset representatives in $\mathbb{R}^{r+s}$ for V.

Here's a general procedure to find such a set, start of with a fundamental parallelotope F for $\wedge_U$ and let $D' = F \oplus \mathbb{R}v$, where we choose $v = (1, \ldots, 2)$ for technical reasons (though any v not in H would work.), then $D = \log^{-1}(D')$ is a set of coset representatives in $\mathbb{R}^r \times \mathbb{C}^s$.

We can check that this indeed is a set of coset representatives by using the proposition above. Indeed, let $x \in \mathbb{R}^{r+s}$, let x_v be it's projection onto v, then $x - x_v$ is an element of the subvector space spanned by the image of V and thus has a representative in F. (this is because the group of units has rank $r+s-1$).

Now stuff gets technical, define $D_a = \{x : N(x) \leq a\}$, then $D_a = a^{\frac{1}{n}} D_1$, which means we want to estimate the number of points $(t \|J\|)^{\frac{1}{n}} D_1$ and this is not something im about to latex.