Functional Analysis

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1 Introduction

Lecture 1: Introduction

Wed 12 Oct

Main reference is "Functional Analysis" by H.W. Alt.

1.1 Topological Spaces

Definition 1 (Topological space)
Let X be a set, a topology is a subset $\tau \subset P(X)$ is a topology if
- $\emptyset, X \in \tau$
- *any union of opens is open*
- *Finite intersections of opens are open.*

Definition 2 (Properties)
For $A \subset X$, $\overline{A}$ is the smallest closed set containing A and the interior A^o is the biggest open set contained in A.
Finally, the boundary is $\partial A = \overline{A} \setminus A^o$.
X is separable if $\exists$ a dense countable subset

Definition 3 (Sequences)
Let $x : \mathbb{N} \to X, \overline{x} \in X$, $\lim x_k = \overline{x} \iff$ any neighbourhood $U \in T$ of x eventually contains x_k

Definition 4 (Continuity)
A function $f : X \to Y$ is continuous if $\forall U \in \tau_Y, f^{-1}(U)$.
This is different from sequential continuity $x_n \to \overline{x} \implies f(x_n) \to f(\overline{x})$
.
f is continuous at $x \in X$ if $\forall V \in S$ st $f(x) \in V \implies f^{-1}(V) \in \tau_X$

Lecture 2: More recaps

Fri 14 Oct

1.2 Metric spaces

Definition 5 (Metric space)
X a set, $d : X \times X \to [0, \infty)$ is a matrix

Definition 6
X a set, d_1, d_2 metrics

1. *d_1 is topologically stronger than d_2 if τ_{d_1} is finer.*
2. *d_1 is uniformly stronger than d_2 if $\exists C > 0$ such that $d_2 \leq Cd_1$*
3. *d_1 is uniformly stronger than d_2 if $\exists C > 0$ such that $\frac{1}{C}d_1 \leq d_2 \leq Cd_1$*

Lemma 1
THe following are equivalent

1. *d_1 is topologically stronger than d_2*
2. *$\mathrm{Id} : (X, \tau_{d_1}) \to (X, \tau_{d_2})$ is continuous*
3. *If $x_n \to \overline{x}$ in d_1 then $x_n \to \overline{x}$ in d_2*
4. *$\forall x \in X \forall \epsilon > 0 \exists \delta_{\epsilon,x} > 0$ such that*
$$d(x, y) \leq \delta \implies d_2(x, y) < \epsilon$$

Definition 7
Let (X, d) be a metric space

1. *$A \subset X$ is bounded if $\exists \overline{x} \in X$ such that $\sup_{y \in A} d(x, y) < \infty$ or $A = \emptyset$*
2. *x_n is Cauchy if*
$$\lim_{n\to\infty} \sup_{i,j\geq n} d(x_i, x_j) = 0$$
3. *X complete if x Cauchy $\implies$ x convergent.*
4. *(Y, e) is a matric, $fX \to Y$ is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0$ such that $d(x, y) < \delta \implies e(f(x), f(y)) < \epsilon$.*

Define $X = \{x : \mathbb{N} \to \mathbb{R}$ such that $\exists N$ such that $x_i = 0$ eventually $\}$. This space, with p-norm is not complete, so we construct the completion.

Proposition 2
Let (X, d) a metric space and (Y, e) a complete metric space, $A \subset X, \phi : A \to Y$ uniformly continuous.
Then $\exists$ unique $\psi : \overline{A} \to Y$ such that ψ is uniformly continuous and $\phi = \psi|_A$.

Proof
If $x : \mathbb{N} \to A$ is Cauchy, then $\phi \circ x$ is also cauchy.
To prove this, let $\epsilon > 0$ and $\partial^\phi_\epsilon > 0$ be such that $d(x, y) < \delta \implies e(\phi(x), \phi(y)) < \epsilon$.

Let $N = N^x_\delta$ *be such that* $i,j \geq N \implies d(x_i,x_j) < \delta$ *, then* $e(\phi(x_i),\phi(x_j)) < \epsilon$

Now, let $a \in \overline{A}$ *, then* $\exists x_k$ *converging to* a*.*
x *is d-Cauchy and* $\phi \circ x$ *is e-cauchy.*
$\exists$ *a limit* $b^* = \lim \phi(x_k)$ *So we define* $\psi(a) = b^*$*.*
We now prove continuity/uniform continuity.
Let $a,b \in \overline{A}, x,y : \mathbb{N} \to A$ *and* $x_i \to b, y_j \to b$*.*
Then

$$e(\psi(a),\psi(b)) = \lim e(\phi(x_i),\phi(y_j))$$

Now, let $\epsilon > 0$*, then* $\exists \delta > 0$ *such that* $d(x,y) < \delta$*.*
Thus $e(\phi(x),\phi(y)) < \epsilon$
If $d(a,b) < \delta$ $\exists N$ *such that* $d(x_i,y_j) < \delta \forall i,j > N$

$$e(\phi(x_i),\phi(y_j)) < \epsilon \implies e(\psi(a),\psi(b) \leq \epsilon) \qquad \square$$

Theorem 3
If (X,d) *is a metric space, then there exists a complete metric space* (Y,e) *and an isometry* $\phi : X \to Y$ *such that* $Y = \overline{\phi(X)}$ *.*
Both are unique up to a bijective isometry.

Proof
Define $C_X \coloneqq \{x : \mathbb{N} \to X, x \text{ Cauchy } \}$ *and* $x\tilde{y}$ *if* $\lim_{j\to\infty} d(x_i,y_j) = 0$*.*
Write $Y = C_X/\sim$ *.*
For $x,y \in Y$*, define* $e(x,y) = \lim_{j\to\infty} d(x_i,x_j)$ *.*
Is this well defined ?
If $j,k \geq N$

$$|d(x_i,y_i) - d(x_k,y_k)| \leq d(x_i,x_k) + d(y_j,y_k)$$

And if $x\tilde{x}'$*, then*

$$\lim d(x_i,y_j) = \lim d(x'_j,y_j)$$

because

$$|d(x_i,y_j) - d(x'_j,y_j)| \leq d(x_j,x'_j) \to 0$$

To show that e *is a metric, most properties are obvious.*
We show that if $e(x,y) = 0$ *then* $\lim d(x_j,y_j) = 0 \implies x\tilde{y} \implies x = y$
Triangular equality holds because

$$e(x,y = \lim d(x_j,y_j \leq \limsup d(x_i,z_j) + d(z_j,y_j) = e(x,z) + e(z,y)$$

The isometry $\phi : X \to Y$ *simply sends* $x \mapsto [x]$*.*
We now show $[x] \in Y$*,* $\phi(x_k)$ *is a sequence in* Y*, we want to show that*

$\phi(x_k) \to [x]$.

$$\lim_{k\to\infty} e(\phi(x_k),[x]) = \lim_{k\to+\infty}\lim_{j\to\infty} d(x_k,x_j) = 0$$

Which shows $Y = \overline{\phi(X)}$ *Let* y^k *Cauchy* $\forall k \exists x_k \in X$ *such that* $e([y^k],\phi(x_k)) < 2^{-k}$.
We claim $[y^k] \to [x]$

$$d(x^k, x^h = e(\phi(x^k,\phi(x^h)))) \leq 2^{-k} + 2^{-h+e([y^k],[y^h])}$$

Thus $x \in C_X$ $[x] \in Y$

$$e([y^k],[x]) = \lim d(y^k_j, x_j) \leq \lim d(U^k_j, x_k) + d(x_k,x_j) \leq 2^{-k}$$

Finally, to show uniqueness, if (Y,e) *and* (Y',e') *are two completions.*
Let $\psi = \phi \circ (\phi')^{-1} : \phi'(X) \to Y$.
ψ *is an isometry so there is a unique extension* $\psi : Y' \to Y$ *and this is an isometry.* □

1.3 Norms, Banach Spaces

Throughout, $K = \mathbb{R}$ or $\mathbb{C}$

Definition 8 (Normed space)
$\|\cdot\| : X \to [0,\infty)$ *is a norm if*
- $\|x\| = 0 \iff x = 0$
- $\|\lambda x\| = |\lambda|\,\|X\|$
- $\|x + y\| \leq \|x\| + \|y\|$

Definition 9
c_0 *is the space* $c_0 = \{x : \mathbb{N} \to \mathbb{R}$ *s.t.* $\lim x_k = 0\}$ *together with* $\|x\|_{c_0} = \sup|x_k|$
For $p \in [1,\infty)$, $l_p = \left\{x : \mathbb{N} \to \mathbb{R} \text{ s.t. } \sum_{k\in\mathbb{N}} |x_k|^p < \infty\right\}$ *with* $\|x\|_{l_p} = (\sum |x_k|^p)^{\frac{1}{p}}$

Definition 10 (Banach Space)
A Banach space is a complete normed space.

Proposition 4
Any normed space has a completion which is Banach.

Proof

Let (Y, e) be the completion as above, define

$$[x] + [y] := [x + y] \text{ and } \lambda[x] := [\lambda x] \qquad \square$$

1.4 Basis of a normed space

Definition 11

Let $A \subset X$.

A is linearly independent if $\forall N \in \mathbb{N}, \forall a_i \in A \forall \lambda_i \in K, \sum_i \lambda_i a_i = 0 \implies \lambda_i = 0$.

We define

$$span(A) = \left\{ \sum(i) \lambda_i a_i, \lambda_i \text{ as above} \right\}$$

A is a Hamel basis if A is linearly independent and $X = span A$

Definition 12 (Schauder Basis)

$e : \mathbb{N} \to X$ is a Schauder basis if $\forall x \in X$ there is a unique $\lambda : \mathbb{N} \to K$ such that $x = \sum_{i=0}^{\infty} \lambda_i e_i \iff \lim \left\| x - \sum^N \lambda_i e_i \right\| = 0$

Lecture 3: Projections onto Hilbert spaces

Wed 19 Oct

Definition 13 (Equivalence of Norms)

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X

1. *$\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if the induced metrics are topologically stronger*
2. *$\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ if the induced metrics are equivalent*

Lemma 5

1. *If $\|\cdot\|_1$ is stronger than $\|\cdot\|_2 \implies \exists C > 0$ such that $\|x\|_2 \leq C \|x\|_1$*
2. *If $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2 \implies \exists C > 0$ such that $\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1$*

Proof

1. *If not, $\forall k \in \mathbb{N}, \exists v_k \in X$ such that $\|v_k\|_2 > k \|v_k\|_1$.*
   Let $w_k = \frac{v_k}{\|v_k\|_2}$ then $1 = \|w_k\|_2 > k \|w_k\|_1$.
   Thus $w_k \to 0 \in \|\cdot\|_1$, thus $w_k \to 0 \in \|\cdot\|_2$ which is a contradiction.
2. *Follows from 1.* $\square$

1.5 Scalar products and Hilbert spaces

Definition 14
Let H be a K-vector space.
A map $b : H \times H \to K$ is a scalar product if it satisfies

$$b(x, \lambda y + \mu z) = \lambda b(x, y) + \mu b(x, z)$$

$$b(\lambda x + \mu y, x) = \overline{\lambda} b(x, z) + \overline{\mu} b(y, z)$$

$b(x, y) = \overline{b(y, x)}$ *and* $b(x, x) > 0$.
(H, b) *is a pre-Hilbert space*

Example

1. K^d *with the usual scalar product*

2. $\ell^2(\mathbb{R})$

Proposition 7

1. $\|x\|_H = (x, x)^{\frac{1}{2}}$ *is a norm on* H
2. *Cauchy-Schwarz :* $|(x, y)| \leq \|x\| \|y\|$
3. $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Proof
To show Cauchy-Schwarz, note that $(x + ty, x + ty) \geq 0 \forall t \in K$, *thus*

$$(x, x) + t((x, y) + (y, x)) + t^2(y, y) \geq 0 \qquad \square$$

The middle term is $2t \operatorname{Re}(x, y)$, *if the scalar product isn't real, we may rotate* y *to make it real*

Proposition 8
Let $(X, \|\cdot\|)$ *be a normed space.*
If the parallelogram identity holds, then there is a scalar product b *such that* $\|x\| = b(x, x)^{\frac{1}{2}}$

Proof
Define $b(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right)$.
We want to check $b(x, \lambda y + \mu z) = \lambda b(x, y) + \mu b(x, z)$.
First, check $b(x, y + y') + b(x, y - y') = 2b(x, y)$

$$\frac{1}{4}\left[\|x + y + y'\|^2 - \|x - y - y'\|^2 + \|x + y - y'\|^2 - \|x - y - y'\|^2\right] = \frac{1}{2}(\|x + y\|^2 - \|x - y\|^2)$$

From the parallelogram identity, we get that the left hand side is

$$\frac{1}{4}\left[2\|x + y\|^2 + 2\|y'\|^2 - 2\|x - y\|^2 - 2\|y'\|^2\right]$$

and thus the equality above holds.
If $y' = y \implies b(x, 2y) = 2b(x, y)$ *and thus*

$$y' = ny \quad b(x, (n+1)y) = 2b(x, y) - b(x, y - ny)$$

and we conclude by induction that $b(x, ny) = nb(x, y)$.
Thus $b(x, qy) = qb(x, y) \forall q \in \mathbb{Q}$ *and by continuity, they agree on* $\mathbb{R}$.
Pick $v, w \in X$ *and* $y = \frac{v+w}{2}, y' = \frac{v-w}{2}$ *in the above equality, then*

$$b(x, v) + b(x, w) = 2b(x, \frac{v+w}{2}) \qquad \square$$

and we conclude from linearity.
For complex numbers, consider $s(x, y) = b(x, y) - ib(x, iy)$

Definition 15 (Hilbert Space)
(H, b) *is a Hilbert space if it is a complete pre-Hilbert space.*

Lemma 9
Every pre-Hilbert space has a completion, unique up to bijective isometry.

If $M \subset X$, then $p : X \to M$ is a projection if $p^2 = p$ and $p(X) = M$.
M is convex if $x, y \in M, t \in [0, 1]$, then $tx + (1-t)y \in M$

Theorem 10
Let H *be a Hilbert space,* $M \subset H$ *non-empty, closed, convex, then* $\exists$ *a unique map* $p : H \to M$ *such that*

$$\|x - px\| = d(x, M)$$

Proof
If $x \in M, px = x$.
If $x \notin M$, *let* $d = d(x, M) > 0$.
If $y, z \in M$ *are minimizers, then*

$$\frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x - z\|^2 = \left\|x - \frac{y+z}{2}\right\| + \left\|\frac{y-z}{2}\right\|^2$$

If $\|x - y\| = \|x - z\| = d$, *then* $\frac{y+z}{2} \in M \implies \|y - z\| \leq 0 \implies y = z \implies$ *uniqueness.*
To show existence, let $d = \inf_{y \in M} \|x - y\|$.
There is a sequence $y : \mathbb{N} \to M$ *such that* $\|x - y_k\| \to d$.

Thus

$$\frac{1}{2}\|x-y_h\|^2+\frac{1}{2}\|x-y_k\|^2=\left\|x-\frac{y_h+y_k}{2}\right\|+\frac{1}{4}\|y_h-y_k\|^2$$

The LHS goes to d^2 *and* $\left\|x-\frac{y_h+y_k}{2}\right\|\geq d^2$. □

Lemma 11
Let everything as above,, then $p: H\to M$ *is an orthogonal projection* $\iff \forall y\in M\forall x\in H, \mathrm{Re}(x-Px,y-px)\leq 0$

Proof
Let $f(t)=\|x-(ty+(1-t)px)\|^2$, *thus* $f(0)=\min f([0,1])$, *thus* $f'(0)\geq 0$

$$f(t)=\|x-Px\|^2-t\left[(x-px,y-px)+(y-px,x-px)\right]+t^2\|y-px\|^2$$

Thus $f'(0)=-Re(x-px,y-px)\geq 0$ □

Corollary 12
If $M\subset H$ *is a closed linear subspace, then*

$$M^\perp=\{x\in H:(x,m)=0\forall m\in M\}$$

is a closed linear subspace, $M\cap M^\perp=\{0\}, H=M\oplus M^\perp$ *and* $p:H\to M$ *satisfies* $x-px\in M^\perp$ *and* p *is linear*

Definition 16 (Orthonormal systemes)
Let (X,b) *be a pre-Hilbert space, the family* $(e_i)_{i\in I}$ *of vectors in* X *are orthonormal if* $b(e_i,e_j)=\delta_{ij}$

Lemma 13
If the $e_1,\ldots,e_n$ *are orthonormal, then* $\forall x$

$$\sum_i|(e_i,x)|^2\leq\|x\|^2$$

and if $e_1,\ldots,$ *are orthonormal, then* $\forall x$

$$\sum_i|(e_i,x)|^2\leq\|x\|^2$$

Proof

Let $y = \sum_i \lambda_i e_i \in span e_i$

$$g(\lambda) \left\| x - \sum_i \lambda_i e_i \right\|^2 = \|x\|^2 - \sum_i (\overline{\lambda_i}(e_i, x) + \lambda_i(x, e_i)) + \sum_i \lambda_i^2$$

Set $g(\lambda) \geq$ $\square$