Analysis I - notes (Fall 2019)

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Contents

1 Proofs

In this course we have to learn how to make mathematical proofs. Example:

Proposition 1.1. $\sqrt{2}$ *is not a rational number*

Proof. Assume it is, that is we may write

$$\sqrt{2} = \frac{a}{b}$$

for some integers a and $b \neq 0$. As $\sqrt{2} > 0$, a and b should have the same sign. If they are both negative, by multiplying both by -1 we may assume that they are positive. Furthermore, by dividing by their common factors we may assume that a and b are relatively prime. By multiplying the above equation by b we obtain

$$b\sqrt{2} = a.$$

Taking square of the equation yields

$$b^2 \cdot 2 = a^2$$

Hence a is even, so $a = 2r$. But then our equation becomes:

$$b^2 \cdot 2 = (2r)^2 = 4r^2$$

So, $b^2 = 2r^2$, from which we see that b is also even, contradicting the relatively prime assumption on a and b. We obtained a contradiction with our original assumption, hence $\sqrt{2}$ is not a rational number. $\square$

Remark 1.2. This proof is nice, but what is $\sqrt{2}$? Is it a real number? What is a real number anyways? We will get back to these questions.

But be careful, it is easy to write a wrong proof. Here is an example showing that 1 is the largest natural number:

Example 1.3. WRONG PROOF: We claim that 1 is the largest natural number. Indeed, let n be the largest natural number. Then $n \geq n^2$, so $0 \geq n^2 - n = n(n-1)$, so either n or $(n-1)$ is at most 0. So, $n \leq 0$ or $n \leq 1$.

Of course, this is all craziness after the absurd assumption in the first sentence. In fact, there does not need to be a largest element in a set of natural numbers.

Analysis mostly treats infinity. A great basic example is:

Proposition 1.4. $0.999999999\ldots = 1$

Proof. We give two proofs none of which are completely correct. Nevertheless we explain carefully the issues with them, and how these will be cleared up during this course.

(1) First an elementary proof:

$$9 * 0.99\ldots = (10-1) * 0.99\ldots = 10 * 0.99\ldots - 1 * 0.99\ldots = 9.99\ldots - 0.99\ldots = 9$$

So, $0.99\ldots = 9/9 = 1$.

This proof is reasonably OK. However, it assumes that we know what $0.99\ldots$ is. It also assumes that we can manipulate it as usually algebraically. None of these are that clear if you think about it deeper.

So, what is $0.99\ldots$?

What algebraic manipulations are allowed with it?

(2) Now, analysis defines $0.99\ldots$ precisely. $0.99\cdots := \sum_{i=1}^{\infty} \frac{9}{10^i}$.

But, what is $\sum_{i=1}^{\infty} \frac{9}{10^i}$?

By definition it is $\lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{9}{10^i}\right)$. Unfortunately we have not learned what lim is precisely, so we cannot quite continue in a precise way from here, nevertheless we continue the argument for completeness. If you are not comfortable with it now, it is completely OK, just skip to the next thing. However, before we proceed, we need to show an identity for the sum of elements in a geometric series:

$$a + a^2 + \cdots + a^n = \frac{a - a^{n+1}}{1-a} \tag{1.4.a}$$

To prove this equality, we just multiply the left side by $1 - a$ to obtain:

$$\begin{aligned}(a + a^2 + \cdots + a^n)(1-a) &= a - a \cdot a + a^2 - a^2 \cdot a + a^3 - \cdots - a^{n-1} \cdot a + a^n - a^n \cdot a \\ &= a - a^{n+1}\end{aligned}$$

This shows that (1.4.a) indeed holds.

And then we can proceed showing the statement:

$$\sum_{i=1}^{\infty}\frac{9}{10^i}=9\cdot\sum_{i=1}^{\infty}\frac{1}{10^i}=9\cdot\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{10^i}\right)=9\cdot\lim_{n\to\infty}\left(\frac{\frac{1}{10}-\frac{1}{10^{n+1}}}{1-\frac{1}{10}}\right)$$
$$=9\cdot\frac{\frac{1}{10}-\lim_{n\to\infty}\frac{1}{10^{n+1}}}{1-\frac{1}{10}}=9\cdot\frac{\frac{1}{10}}{1-\frac{1}{10}}=9\frac{1}{9}=1.$$

□

2 Basic notions

2.1 Sets

There are a couple of sets that we are going to work with:

(1) $\mathbb{N}$: set of natural numbers $\{0,1,2,\dots\}$. $\mathbb{N}$ is well ordered, that is, all its subsets contain a smallest element.

(2) $\mathbb{N}\subset\mathbb{Z}$: set of integer numbers $\{\dots,-1,0,1,\dots\}$ ($\subset$ means that $\mathbb{N}$ is a subset of $\mathbb{Z}$, that is, each element of $\mathbb{N}$ is contained in $\mathbb{Z}$).

(3) $\mathbb{Z}\subset\mathbb{Q}$: set of rational number of the form $\frac{a}{b}$, where $a\in\mathbb{Z}$ and $b\in\mathbb{Z}\setminus\{0\}$.

(4) $\mathbb{Q}\subset\mathbb{R}$: the set of real numbers. It is not easy to actually construct it. As you have seen above in case of $0.99\dots$, it is not clear how to make sense of it. In fact, in this course we do not construct it, although there are many equivalent constructions (you can just go to the wikipedia page "construction of the real numbers").

   So, instead of construction, in this course we list certain properties that uniquely defines $\mathbb{R}$ (we also do not prove this unicity, please believe it):

   (i) $\mathbb{R}$ is an *ordered field*, that is, there is division, multiplication, addition, subtraction and comparison in $\mathbb{R}$ (see page 2 of the book for a precise list of axioms).

   (ii) it satisfies the infimum axiom (to be discussed soon).

(5) generally if S is a set, one can always define $\{s\in S|\text{" condition"}\}$ meaning the set of those elements of S that satisfy the given condition. For example:

   (i) invertible elments: $\mathbb{R}^*:=\{x\in\mathbb{R}|x\neq 0\}$.

   (ii) half lines: $\mathbb{R}_+:=\{x\in\mathbb{R}|x\geq 0\}$, $\mathbb{R}_-:=\{x\in\mathbb{R}|x\leq 0\}$

   (iii) open bounded intervals: if $a<b\in\mathbb{R}$, then

   $$]a,b[:=\{x\in\mathbb{R}|a<x<b\}$$

   (iv) open ball centered around a with radius δ: if $a\in\mathbb{R},\delta\in\mathbb{R}_+$, then

   $$B(a,\delta):=]a-\delta,a+\delta[$$

   (v) closed bounded intervals: if $a<b\in\mathbb{R}$, then

   $$[a,b]:=\{x\in\mathbb{R}|a\leq x\leq b\}$$

   (vi) half closed half open intervals: $]a,b]$ and $[a,b[$, guess the definition or look it up in 1.1.5 in the book

   (vii) emptyset $\emptyset$: it has no elements

$\longleftarrow$ End of 1. class, on 18.09.2019.

2.2 Bounds

For this section let $S \neq \emptyset$ denote a subset of $\mathbb{R}$.

Definition 2.1. $a \in \mathbb{R}$ is an *upper* (resp. *lower*) bound of S if $x \leq a$ (resp. $x \geq a$) holds for all $x \in S$.

If S has an upper/lower bound then it is called bounded from above/from below.

If S is bounded both from above and below, then it is called bounded.

The upper and lower bounds are not unique.

Example 2.2. (1) $\{\sin(n^2) | n \in \mathbb{Z}\}$ is bounded; examples of lower bounds are -5 and -13, example of upper bounds are 1 and 27;

(2) $\{n^2 | n \in \mathbb{Z}\}$ is not bounded, but it is bounded from below

(3) $\{n^3 | n \in \mathbb{Z}\}$ is not bonded from above/below

Definition 2.3. The maximum (resp. minimum) of S denoted by $\max S$ (resp. $\min S$) is an upper (resp. lower) bound of S which is contained in S

Example 2.4. (1) $]1, 2[$ does not have a minimum or a maximum

(2) $[1, 2]$ has both

(3) $S := \left\{ \frac{n-1}{n} \middle| n \in \mathbb{Z}^*_+ \right\}$ has a minimum, as the elements are larger as n increases, so $\frac{1-1}{1} = 0$ is the minimum. However it does not have a maximum, because:

   (i) if $a < 1$, then $1 - \frac{1}{n} = \frac{n-1}{n} > a$ whenever $n > \frac{1}{1-a}$, so a cannot even be an upper bound

   (ii) however, if $a \geq 1$, then $a \notin S$.

The last example hints that we might need a new notion, as 1 is almost like the maximum of S, just it is not in S. This is given by the next definition:

Definition 2.5. The *supremum* $\operatorname{Sup} S$ (resp. infimum $\operatorname{Inf} S$) is the smallest upper bound (resp. largest lower bound), if it exists at all.

Example 2.6. (1) $\operatorname{Sup} \left\{ \frac{n-1}{n} \middle| n \in \mathbb{Z}^*_+ \right\} = 1$.

(2) $\operatorname{Sup}]a, b[= b$.

(3) $\operatorname{Inf}]a, b[= a$.

(4) $\operatorname{Inf}\{n^3 | n \in \mathbb{Z}\}$ and $\operatorname{Sup}\{n^3 | n \in \mathbb{Z}\}$ do not exist.

Axiom 2.7. Infimum axiom Each non-empty subset of $\mathbb{R}^*_+$ admits an infimum.

Corollary 2.8. *Each non-empty bounded from above (resp. below) subset $S \subset \mathbb{R}$ admits a supremum (resp. infimum).*

Proof. Idea: shift/reflect S across the origin to turn it into a subset of $\mathbb{R}^*_+$. Then apply the infimum axiom. □

Proposition 2.9. *If $S \subset \mathbb{N}$, then* $\operatorname{Inf} S = \operatorname{Min} S$.

Proof. Set $d := \operatorname{Inf} S$, which exists by Corollary 2.8. We have to show that $d \in S$. So, assume $d \notin S$. Then, as $\operatorname{Inf} S$ is the largest lower bound of S, for each $\varepsilon > 0$, $d + \varepsilon$ is not a lower bound. Hence:

$$\text{For each } \varepsilon > 0, \text{ there is } s_\varepsilon \in S, \text{ such that } s_\varepsilon < d + \varepsilon. \tag{2.9.a}$$

Apply (2.9.a) for $\varepsilon' := \frac{1}{2}$. This yields $s_{\varepsilon'}$ such that

$$d < s_{\varepsilon'} < d + \varepsilon'$$

Apply then again the above property of S, but now for $\varepsilon'' := s_{\varepsilon'} - d$. We obtain $s_{\varepsilon''} \in S$ such that

$$d < s_{\varepsilon''} < d + \varepsilon'' = s_{\varepsilon'} < d + \varepsilon' = d + \frac{1}{2}.$$

In particular, $0 < s_{\varepsilon'} - s_{\varepsilon''} < d + \frac{1}{2} - d = \frac{1}{2}$. This is a contradiction, because $s_{\varepsilon'}, s_{\varepsilon''} \in \mathbb{N}$ and the difference of two natural numbers is an integer. Hence, our assumption was false, or with other words $d \in S$. ☐

2.3 $\mathbb{Q}$ vs $\mathbb{R}$

2.3.1 Integer part

Let x be a positive real number, for example $\pi^2 + \pi$. According to Corollary 2.8 and Proposition 2.9, the set $S := \{n \in \mathbb{N} | n > x\}$ has a minimum, say N. Then $N - 1$ is not in S. We call it the *integer part* of x and we denote it by $[x]$. We call $\{x\} = x - [x]$ the *fractional part* of x.

For example $[\pi^2 + \pi] = 13$, and $\{\pi^2 + \pi\}$ is what it is, not a number that a human can write down with decimals. For rational numbers, things are a bit easier. For example, $\left[\frac{3}{2}\right] = 1$ and $\left\{\frac{3}{2}\right\} = \frac{1}{2}$.

If x is negative and it is not in $\mathbb{Z}$, we take $[x] = -[-x] - 1$; if x is in $\mathbb{Z}$ we take $[x] = x$. Then, the fractional part is defined as for positive numbers. For example $[-7.5] = -8$ and $\{-7.5\} = 0.5$.

2.3.2 $\sqrt{2}$ is a real number

We have seen in Proposition 1.1 that $\sqrt{2}$ is not a rational number. But why is it a real number? The reason is that $\sqrt{2}$ is $\operatorname{Sup}\{x \in \mathbb{R} | x^2 \leq 2\}$ (note this is obviously non-empty as 1 is in it, also it is bounded above by 2, so Sup exists according to Corollary 2.8; moreover, the Sup has to be bigger than 1, so it is positive). We do not prove here that the above Sup is indeed equal to $\sqrt{2}$, as it is a quite particular computation (see page 9 of the book).

2.3.3 Rational numbers are everywhere

The question is if one can find rational numbers between arbitrary two real numbers. So, for example, is there a rational number c, such that $0 < c < \pi$? The left inequality, that is, that $0 < c$, is easy to guarantee. One just has to choose a positive number. How does one guarantee the inequality on right inequality? Well, as soon as c is positive, $c < \pi$ is equivalent to $\frac{1}{c} > \frac{1}{\pi}$. So, if one chooses $\frac{1}{c}$ to be any integer that is bigger than we are fine. For example one can choose

$$\frac{1}{c} = \left[\frac{1}{\pi}\right] + 1.$$

That is, if we call this integer number n, then we would have $c := \frac{1}{n}$, which is clearly rational.

Proposition 2.10. *If $a < b$ are real numbers, then there is a rational number c, such that $a < c < b$.*

Proof. *Easy case, we assume $a = 0$:*

We have $\left[\frac{1}{b}\right] + 1 > \frac{1}{b}$, moreover $\left[\frac{1}{b}\right] + 1$ is a positive integer. We conclude that

$$0 < \frac{1}{\left[\frac{1}{b}\right] + 1} < b\,,$$

so we can take $\frac{1}{\left[\frac{1}{b}\right]+1}$ as c.

← End of 2. class, on 23.09.2019.

General case:

$$n := \left[\frac{1}{b-a}\right] + 1 \Rightarrow n > \frac{1}{b-a} \Rightarrow \frac{1}{n} < b - a$$

$$a = \frac{an}{n} < \frac{[an]+1}{n} \le \frac{an+1}{n} = a + \frac{1}{n} < a + b - a = b$$

Furthermore, $\frac{[an]+1}{n}$ is a rational number. We conclude that we can take $c = \frac{[an]+1}{n}$ (this is not the unique rational number between a and b, it is just one example of a rational number between a and b). □

2.3.4 Irrational numbers are everywhere

Proposition 2.11. *If $a < b$ are real numbers, then there is an irrational number c, such that $a < c < b$.*

Proof. Apply Proposition 2.10 to $\frac{a}{\sqrt{2}} < \frac{b}{\sqrt{2}}$. This yields a rational number d such that $\frac{a}{\sqrt{2}} < d < \frac{b}{\sqrt{2}}$. Additionally we can assume that $d \neq 0$, because if it is we can replace it by the rational number given by applying Proposition 2.10 to 0 and $\frac{b}{\sqrt{2}}$. In particular, $a < \sqrt{2}d < b$, where $d \neq 0$. Furthermore, $\sqrt{2}d$ is irrational, as if it was rational, then $\sqrt{2} = \frac{\sqrt{2}d}{d}$ would also be rational. □

2.4 *Absolute value*

Definition 2.12. If $x \in \mathbb{R}$, then the *absolute value* $|x|$ of x is defined as follows:

$$|x| = \begin{cases} \;\;x & \text{if } x \ge 0 \\ -x & \text{if } x \le 0. \end{cases}$$

For example $|3| = 3$, $|-5| = 5$, $|-\pi| = \pi$, $|0| = 0$ and $|5| = 5$. It is useful to draw the graph. Another way to introduce the absolute value is the distance between x and 0 on the real line.

The absolute value behaves well with respect to the multiplication; for example, by using Definition 2.12 we obtain:

$$|5 \cdot (-3)| = |-15| = 15 = 5 \cdot 3 = |5| \cdot |-3|.$$

Similarly,

$$\left|(-\sqrt{2}) \cdot (-4)\right| = \left|4\sqrt{2}\right| = 4\sqrt{2} = \sqrt{2} \cdot 4 = \left|-\sqrt{2}\right| \cdot |-4|.$$

This is true for any pair of real numbers, which one can prove by looking at all the possibilities of the signs of the numbers involved ("positive"·"positive"; "positive"· "negative"; "negative"·"negative"). With symbols: $|xy| = |x||y|$.

Analogously, one has $\left|\frac{5}{-4}\right| = \frac{|5|}{|-4|}$, and with symbols $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.

The absolute value is also needed to relate powers and roots. For example $\sqrt{(-3)^2} = \sqrt{9} = 3 = |-3|$ and $\sqrt{(7)^2} = \sqrt{49} = 7 = |7|$. With symbols: $\sqrt{x^2} = |x|$.

Last but not least, let us remark that $-\sqrt{3}$ is between $-\left|-\sqrt{3}\right|$ and $|-\sqrt{3}|$. This is true for any number, and with symbols one writes that $-|x| \leq x \leq |x|$.

A deep property of the absolute value is the triangle inequality. Can you draw a triangle with sides of length 1, 4, and 600 ? I do not think so. But you can draw a triangle of sides 3,4 and 6 (give it a try, you might need a compass).

The reason is that for every triangle, the sum of the length of two edges is always bigger then the length of the third edge. This implies a triangle inequality for the absolute value, we will understand better the relation with triangles when dealing with complex numbers, let us give a couple of examples now:

$$|3+(-7)| \leq |3|+|-7|$$

and

$$|(-5)+(-4)| \leq |-5|+|-4|$$

In general one has the following result

Proposition 2.13 (Triangle inequality)**.** *For any pairs of real numbers x and y one has*

$$|x+y| \leq |x|+|y|$$

Proof. Recall that $x \leq |x|$ and $y \leq |y|$. So, if $x+y \geq 0$, then $|x+y| = x+y \leq |x|+|y|$.

Similarly, $x \geq -|x|$ and $y \geq -|y|$. So, if $x+y \leq 0$, then $|x+y| = -x-y \leq |x|+|y|$. $\square$

One has also the reverse triangle inequality

$$|x-y| \geq ||x|-|y||$$

you can look it up on the book at page 10.

2.5 Extended real number line

The extended real line is the set

$$\overline{\mathbb{R}} := \{-\infty, +\infty\} \cup \mathbb{R}$$

The symbols $+\infty$ and $-\infty$ are called plus infinity and minus infinity, *they are not numbers*, just symbols, so be very carefull to do no treat them as numbers.

Later in the course we will use extensively these symbols. For the time being, we just want to use them to define the following subsets of $\mathbb{R}$. For any real number a, the set $[a, +\infty[$ is the set of all real number x greater or equal than a. Similarly

$$]a, +\infty[= \{x \text{ such that } x > a\}$$

$$]-\infty, a] = \{x \text{ such that } x \leq a\}$$

$$]-\infty, a[= \{x \text{ such that } x < a\}$$

and finally $]-\infty, +\infty[$ is the full set of real numbers $\mathbb{R}$. These sets are also called interval, or extended intervals.

3 COMPLEX NUMBERS

To define the complex number we have to introduce a new number called i, the imaginary unit. This number is the square root of -1, so it has the property that $i^2 = -1$. The introduction of this new number can somehow be compared to the introduction of 0, or of the negative numbers. Let us give a more formal definition.

Definition 3.1. A *complex number* is an expression of the form $x + yi$, where x and y are real numbers, and i is the imaginary unit. The set of complex numbers is denoted by $\mathbb{C}$. Often elements of $\mathbb{C}$ are denoted with the letter z, so we will often write $z = x + yi$.

Taking $y = 0$, one sees that $\mathbb{R} \subset \mathbb{C}$, so for example 0, 3, and $-\pi$ are complex numbers. Other examples of complex numbers are $5 - i$, $3i$, $-2i$ and $\frac{1}{2} + \sqrt{2}i$. Complex numbers are not ordered, it makes no sense to ask if a complex number is bigger than another; in particular, it does not make sense to ask if a complex number is positive or negative

It is remarkable that the equation $x^2 = -1$ has no solution in the set of real numbers, but two distinct solutions in the set of complex numbers, namely i and $-i$.

We can add and multiply complex numbers using the standard formal properties of addition and multiplication, always remembering that $i^2 = -1$.

Example 3.2.

- $(5 + 3i) + (2 - i) = (2 + 5) + (3 - 1)i = 7 + 2i$
- $(1 - 2i)(3 + 4i) = 3 - 6i + 4i - 8i^2 = 3 - 6i + 4i + 8 = 11 - 2i$

It is very important to identify the complex number with the plane, which is called the complex plane. We have to use x and y as Cartesian co-ordinates. The real line can be identified with the lines $\{y = 0\}$. Complex numbers become vectors, and the sum of complex numbers is equal to the sum of vectors. Multiplication of a complex number by a positive real number corresponds to scaling the length of the vector.

A complex number z has a real part and an imaginary part, denoted by $\mathrm{Re}(z)$ and $\Im(z)$, these are the x and y co-ordinate in the plane.

Example 3.3.

- $\mathrm{Re}(5 + 3i) = 5$, $\Im(5 + 3i) = 3$
- $\mathrm{Re}(-3i) = 0$, $\Im(-3i) = -3$

The modulus $|z|$ of a complex number z is its distance from the origin in the complex plane, it can be computed using Pythagorean Theorem.

Example 3.4.

- $|3 - 4i| = \sqrt{3^2 + 2^2} = \sqrt{25} = 5$,
- $|-3i| = 3$

The formula is $|x + yi| = \sqrt{x^2 + y^2}$. Of course we have a triangle inequality

$$|z + w| \leq |z| + |w|$$

The conjugate of a complex number z is denoted by $\overline{z}$ is obtained by changing the sign of the imaginary part, so $\overline{x + iy} = x - iy$. It is important to understand that geometrically in the complex plane this corresponds to reflection across the real line.

Example 3.5.

- $\overline{3 - 4i} = 3 + 4i$,
- $\overline{3i} = -3i$,
- $\overline{1} = 1$.

Remark that conjugation is compatible with all operations by explicit computation (so one writes all numbers involved as $x+iy$ and expands all the obtained parentheses). Similarly, we have

$$z\overline{z} = (x+iy)\overline{(x+iy)} = (x+iy)(x-iy) = x^2 + ixy - ixy - i^2y^2 = x^2 + y^2 = |z|^2$$

We also have the following relation between conjugation, real part and imaginary part

$$\mathrm{Re}(z) = \frac{1}{2}(z+\overline{z}) \quad \text{and} \quad \Im(z) = \frac{1}{2}(z-\overline{z})$$

We can associate to every non-zero complex number an angle, called argument or phase, in the following way. In the complex plane we we have the half line $\mathbb{R}_+$ of the positive real numbers. Given a non-zero complex number z, we can take the half line L_z starting from the origin and passing trough z. The argument or phase of z is the angle between $\mathbb{R}_+$ and L_z, moving in the anti-clockwise direction. The argument of z is denoted by $\arg(z)$.

Example 3.6. The argument of 3 is zero. The argument of i is $\frac{\pi}{2}$, the argument of $\frac{\sqrt{2}}{2}(1+i)$ is $\frac{\pi}{4}$.

More generally, for any positive real number R and any angle α, we have that the argument of $R(\cos(\alpha)+\sin(\alpha)i)$ is α.

Take now a non-zero complex number z, we have seen that its distance from the origin is $|z|$, and let α be its argument. The number $\frac{z}{|z|}$ has distance 1 from the origin, so it lies on the trigonometric circle, hence its x and y co-ordinates are just $\cos(\alpha)$ and $\sin(\alpha)$. We conclude that

$$z = |z|(\cos(\alpha)+\sin(\alpha)i)$$

This is called the *polar form* of z, and it is very important. (Conversely, when we write a complex number as $x+iy$, we can say that we are using the Cartesian form, or Cartesian representation)

Example 3.7. The polar form of $1+i$ is $\sqrt{2}\left(\cos(\frac{\pi}{4})+\sin(\frac{\pi}{4})i\right)$

The multiplication of complex numbers become particularly easy if we use the polar form, from this point of view the multiplication is equivalent to add the angles and multiply the absolute values.

Example 3.8. Let α and β be two numbers. Then

$$(5(\cos(\alpha)+\sin(\alpha)i))(3(\cos(\beta)+\sin(\beta)i)) =$$

$$= 15(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) + (\cos(\alpha)\sin(\beta)+\sin(\alpha)\cos(\beta))i =$$

$$= 15(\cos(\alpha+\beta)+\sin(\alpha+\beta)i)$$

← End of 3. class, on 25.09.2019.

Example 3.9.

- The modulus of $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is 1, and the argument is $\frac{\pi}{3}$, so

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{2017} = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

  because $1^{2017}=1$, so the absolute values does not change; then $2017 = 336\cdot 6+1$, so $2017\cdot\frac{\pi}{3} = 336\cdot 2\pi + \frac{\pi}{3}$, so also the argument does not change.

The above example shows that the polar form is really useful!!

Division of two complex numbers. We would like to take two complex numbers z and w, with $w\neq 0$, and write $\frac{z}{w}$ in the form $x+yi$. A key idea is to make the denominator a real number by multiplying with the conjugate.

Example 3.10.

$$\frac{2-3i}{5+i} = \frac{(2-3i)(5-i)}{(5+i)(5-i)} = \frac{7-17i}{26} = \frac{7}{26} - \frac{17}{26}i$$

In fact, we may write down a general formula using $w\overline{w} = |w|^2$, which we have used already in the above example. So, we have

$$\frac{z}{w} = \frac{z\overline{w}}{\overline{w} \cdot w} = \frac{z\overline{w}}{|w|^2}.$$

Example 3.11.

$$\frac{1}{3-\sqrt{3}i} = \frac{3+\sqrt{3}i}{12} = \frac{1}{4} + \frac{\sqrt{3}}{4}i,$$

or

$$\frac{i}{1-i} = \frac{i(1-i)}{2} = \frac{1}{2} + \frac{1}{2}i$$

We can also use the polar form to divide complex numbers. As with multiplication the moduli (plural of the modulus) multiplied and the arguments added up, with division, we have to do the inverse. That is, moduli are divided and arguments are subtracted:

$$|z|(\cos(\alpha) + \sin(\alpha)i)/|w|(\cos(\beta) + \sin(\alpha)i) = \frac{|z|}{|w|}(\cos(\alpha - \beta) + \sin(\alpha - \beta)i)$$

Note that because of the presence of cos and sin, one can add any multiple of 2π to the argument on the right hand side.

Example 3.12. The inverse of

$$3\left(\cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right)\right)$$

is

$$\frac{1}{3}\left(\cos\left(-\frac{2\pi}{7}\right) + i\sin\left(-\frac{2\pi}{7}\right)\right)$$

3.1 Euler formula

The following is a formal definition, called the Euler formula. It is very important that you do not try to understand it as the powers of something, as we have not defined i-th powers. So, just think about it as a shortcut for the argument part of the polar form.

Definition 3.13. Let α be a real number, then

$$e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$$

We can now write

$$|z|(\cos(\alpha) + \sin(\alpha)i) = |z|e^{i\alpha}$$

Example 3.14.

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

and now we can write out the multiplication as

$$zw = \left(|z|e^{i\alpha}\right)\left(|w|e^{i\beta}\right) = |z||w|e^{i(\alpha+\beta)}$$

3.2 Finding solutions of equations among complex numbers

The main importance of complex numbers is that any polynomial equation has a solution among the complex numbers. This is called the *fundamental theorem of algebra.* We are not going to formally state it and prove it, but we will see a few examples.

We have already learned that the equation $z^2 = -1$ has no solutions in the real numbers, but two solutions in the complex numbers, i and $-i$. We are going to learn how to solve slightly more general equations, thanks to the polar form.

Suppose that we want to solve the equation

$$z^n = Re^{i\alpha}$$

where z is the unknown, n is a positive integer, R is a positive real number, and α is an angle. Then the solutions of this equation are always of the form

$$z = \sqrt[n]{R}e^{i\beta}$$

where β is an angle such that $n\beta = \alpha$, as an angle. In particular, β has to be equal to $\frac{\alpha}{n} + \frac{2k\pi}{n}$, where k is an integer between 0 and $n-1$, so that the above equation has always exactly n distinct solutions.

Example 3.15. The equation

$$z^2 = 3e^{i\frac{\pi}{5}}$$

has two solutions: $\sqrt{3}e^{i\frac{\pi}{10}}$ and $\sqrt{3}e^{i\left(\frac{\pi}{10}+\pi\right)}$

The equation

$$z^3 = 27e^{i\frac{\pi}{7}}$$

has three solutions: $3e^{i\frac{\pi}{21}}$, $3e^{i\left(\frac{\pi}{21}+\frac{2\pi}{3}\right)}$ and $3e^{i\left(\frac{\pi}{21}+\frac{4\pi}{3}\right)}$

One can also use the usual quadratic formula (the proof is the same as usually, complete the square, etc.).

Example 3.16. For example the solutions of

$$z^2 + 2z + 3 = 0$$

are

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 3}}{2} = \frac{-2 \pm \sqrt{-8}}{2} = -1 \pm \sqrt{-2} = -1 \pm i\sqrt{2}$$

In any case, the message is that everything works as usually, you just always have solutions, contrary to the real case, and for some equations (say $z^n = a$, where $a \neq 0$) it is even guaranteed that there are distinct solutions.

4 Sequences

Definition 4.1. A sequence is a function $x : \mathbb{N} \to \mathbb{R}$, where we (by tradition) denote the value of n by x_n, and the whole sequence by (x_n).

The mother of all sequences is the following:

Example 4.2. *Arithmetic progression*:

$$x_0 = a, x_1 = a + b, x_2 = a + 2b, \ldots, x_n = a + nb, \ldots,$$

for some real numbers a and b.

For example, the arithmetic progression given by $a = 1$ and $b = 2$ is $x_0 = 1, x_1 = 3, x_2 = 5, \ldots$, that is, the sequence takes up as values all the positive odd numbers.

Definition 4.3. For a sequence (x_n) we have:

(x_n) is $\begin{Bmatrix} \textit{bounded from below} \\ \textit{bounded from above} \\ \textit{bounded} \end{Bmatrix}$, if the set $\{x_n | n \in \mathbb{N}\}$ of its values is $\begin{Bmatrix} \text{bounded from below} \\ \text{bounded from above} \\ \text{bounded} \end{Bmatrix}$.

Definition 4.4. A sequence (x_n) is $\begin{Bmatrix} \textit{constant} \\ \textit{increasing} \\ \textit{strictly increasing} \\ \textit{decreasing} \\ \textit{strictly decreasing} \end{Bmatrix}$, if for each $n \in \mathbb{N}$, $\begin{Bmatrix} x_n = x_{n+1} \\ x_n \leq x_{n+1} \\ x_n < x_{n+1} \\ x_n \geq x_{n+1} \\ x_n > x_{n+1} \end{Bmatrix}$.

Definition 4.5. A sequence is $\begin{Bmatrix} \textit{monotone} \\ \textit{strictly monotone} \end{Bmatrix}$ if it $\begin{Bmatrix} \text{increases or decreases} \\ \text{strictly increases or strictly decreases} \end{Bmatrix}$.

Example 4.6. An arithmetic progression $x_n = a + nb$ is

(1) bounded from above $\Leftrightarrow b \leq 0 \Leftrightarrow$ decreasing

(2) bounded from below $\Leftrightarrow b \geq 0 \Leftrightarrow$ increasing

(3) bounded $\Leftrightarrow b = 0 \Leftrightarrow$ constant

(4) strictly increasing $\Leftrightarrow b > 0$

(5) strictly decreasing $\Leftrightarrow b < 0$

Example 4.7. A *geometric progression* is of the form $x_n = aq^n$ for some real numbers a and q.

For example, if $a = 1$ and

(1) $q = \frac{1}{2}$, then $x_n = \frac{1}{2^n}$.

(2) $q = -\frac{1}{2}$, then $x_n = \frac{1}{(-2)^n}$

(3) $q = -1$, then $x_n = (-1)^n$

(4) $q = 2$, then $x_n = 2^n$

(5) $q = -2$, then $x_n = (-2)^n$

We will analyze for what values of a and q is x_n bounded in Example 4.17.

4.1 Recursive sequences

We call a sequence *recursive* if x_n is given in terms of $x_{n-1}, \ldots, x_{n-j}$ for some fixed integer $j > 0$ (so j does not depend on n).

Example 4.8. Fibonacci sequence: $x_{n+2} = x_{n+1} + x_n$, $x_0 = 1$, $x_1 = 1$.
Then we have $x_2 = 2$, $x_3 = 3$, $x_4 = 5$, $x_5 = 8$, ...

Example 4.9. $x_n = \sqrt{4 + x_{n-1}}$, $x_0 = 1$. Is this sequence bounded?

The answer to the above question can be given using *induction*. This is a method of proving something for all natural numbers n, by proving first that it holds for $n = 0$, and then that it holds for n assuming that it holds for $n - 1$. The latter step is called the induction step, and the assumption that the statement holds for $n - 1$ is the induction hypothesis. Below is the example:

Proposition 4.10. *The sequence* $x_n = \sqrt{4 + x_{n-1}}$, $x_0 = 1$ *is bounded.*

Proof. As a warm-up we do first the easy case, that is, x_n is bounded from below. We show that $0 \le x_n$ for each n. Indeed, for $x_0 = 1 \ge 0$, so we only have to prove the induction step. We assume that $0 \le x_{n-1}$, and we need to show that $0 \le x_n$. However, this is straightforward:

$$x_n = \sqrt{4 + x_{n-1}} \ge \sqrt{4+0} = 2 \ge 0,$$

where at the first inequality we used our induction hypothesis.

Now, comes the real deal, showing that x_n is bounded from above. In fact, here the hardest is probably finding one correct upper bound. For example 2 is obviously not an upper bound, as $x_2 = \sqrt{5} > 2$. We claim that 3 is an upper bound, and we show it by induction. Indeed, $x_0 = 1 < 3$, so we are only left to show the induction step. That is, assume that $x_{n-1} < 3$. Then:

$$x_{n+1} = \sqrt{4 + x_{n-1}} < \sqrt{4+3} = \sqrt{7} < 3,$$

where at the first inequality we used our induction hypothesis.

□

Example 4.11. $x_0 = 0$, $x_n = x_{n-1} + (-1)^n n^2$. Equivalently, $x_n = \sum_{i=1}^n (-1)^i i^2$.

Is x_n bounded (in any direction)?

Proposition 4.12. *For the above sequence, $x_{2m} = (2m+1)m$ for every $m \in \mathbb{N}$.*

Proof. We prove by induction.

$x_0 = 0 = (2 \cdot 0 + 1) \cdot 0$, so this is OK.

We need to show then the induction step, so we assume that $x_{2(m-1)} = (2m-1)(m-1)$. Then

$$x_{2m} = x_{2(m-1)} - (2m-1)^2 + (2m)^2 = \underbrace{x_{2(m-1)} - (2m)^2 + 4m - 1 + (2m)^2}_{\text{foiling out the middle square}}$$

$$= \underbrace{x_{2(m-1)} + 4m - 1}_{\text{cancelling the two } (2m)^2 \text{ terms}} = \underbrace{(2m-1)(m-1) + 4m - 1}_{\text{using the induction hypothesis}} = \underbrace{2m^2 - 3m + 1 + 4m - 1}_{\text{foiling out the paretheses}}$$

$$= 2m^2 + m = (2m+1)m$$

□

Example 4.13. Getting back to our example, we see by the previous proposition that $x_{2m} = (2m+1)m$, so x_n is not bounded from above, because for any real number b,

$$(2m+1)m \ge m^2 > b,$$

where

$$m^2 > b \Leftrightarrow m > \sqrt{b}.$$

So, b cannot be an upper bound, as for $m > \sqrt{b}$, $x_{2m} > b$.

To see the boundedness from below, we compute also

$$x_{2m+1} = x_{2m} - (2m+1)^2 = (2m+1)m - (2m+1)^2 = -(2m+1)(m+1),$$

and we see that it is also not bounded from below in a similar fashion.

⟵ End of 4. class, on 30.09.2019.

Definition 4.14. If $0 \le k \le n$ are integers, then $\binom{n}{k}$ is the number of possible ways one can choose a subset of unordered k elements from a set of n elements. With formulas:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 1}.$$

One can show using induction also:

Proposition 4.15. *For any $x, y \in \mathbb{R}$, we have $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.*

Proof. See pages , 253 and 254 from the book. □

Corollary 4.16. Bernoulli inequality *If $q > 1$ is a real number, then $q^n \geq 1 + n(q-1)$*

Proof. Simply apply the binomial formula

$$(1+(q-1))^n = \sum_{i=0}^{n} \binom{n}{i} (q-1)^i \geq \binom{n}{0}(q-1)^0 + \binom{n}{1}(q-1)^1 = 1 + n(q-1).$$

□

Example 4.17. Let $x_n = aq^n$ be a geometric progression for some real numbers a and q.

We see that (x_n) is bounded if either $a = 0$ or $|q| \leq 1$, because for $a = 0$ we have $x_n = 0$, and in the other case by induction we can show that $|x_n| \leq a$. Indeed, for x_0 it is true by the definition of x_n that $|x_0| = a$. Futhermore, the induction step is also OK, because if $|x_{n-1}| \leq a$, then

$$|x_n| = |aq^n| = |aq^{n-1}||q| = |x_{n-1}||q| \leq a \cdot 1 = a.$$

Using the Bernoulli inequality we can prove that (x_n) is not bounded in any other case. So, assume that $|q| > 1$ and $a \neq 0$. Then

$$|aq^n| = |a||q|^n \geq \underbrace{|a|(1 + n(|q|-1))}_{\text{Bernoulli inequality}}$$

The latter expression is not bounded because we have

$$|a|(1 + n(|q|-1)) \leq b \Leftrightarrow n \leq \frac{\frac{b}{|a|} - 1}{|q| - 1},$$

which does not hold for $n \geq \left[\frac{\frac{b}{|a|} - 1}{|q|-1} \right] + 1$. So, no b can be an upper bound for $|x_n|$.

One can show similarly:

(1) x_n is bounded $\Leftrightarrow |q| \leq 1$ or $a = 0$,

(2) x_n is increasing $\Leftrightarrow \begin{cases} q \geq 1 \text{ and } a \geq 0, \text{ or} \\ 0 \leq q \leq 1 \text{ and } a \leq 0. \end{cases}$

(3) x_n is strictly increasing $\Leftrightarrow \begin{cases} q > 1 \text{ and } a > 0, \text{ or} \\ 0 < q < 1 \text{ and } a < 0\ . \end{cases}$

(4) x_n is decreasing $\Leftrightarrow \begin{cases} 0 \leq q \leq 1 \text{ and } a \geq 0, \text{ or} \\ q \geq 1 \text{ and } a \leq 0. \end{cases}$

(5) x_n is strictly decreasing $\Leftrightarrow \begin{cases} 0 < q < 1 \text{ and } a > 0, \text{ or} \\ q > 1 \text{ and } a < 0\ . \end{cases}$

(6) x_n is constant $\Leftrightarrow q = 1$ or $a = 0$

(7) x_n is bounded $\Leftrightarrow q \leq 1$ or $a = 0$

(8) x_n is bounded from above $\Leftrightarrow$ bounded or $q > 1$ and $a \leq 0$

(9) x_n is bounded from below $\Leftrightarrow$ bounded or $q > 1$ and $a \geq 0$

There are more examples of proof by induction on page 252 of the book. Also, there are further examples of sequences on page 16 of the book, which I suggest you take a look at.

4.2 Limit of a sequence

Definition 4.18. Let (x_n) be a sequence. We say that x_n is *convergent* to some $x \in \mathbb{R}$, if for each $0 < \varepsilon \in \mathbb{R}$, there exists $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$, $|x_n - x| \leq \varepsilon$. If the x in the definition exists then it is called the limit of x_n, and we denote it by $\lim_{n\to\infty} x_n$.

If (x_n) is not convergent, it is called *divergent*.

Proposition 4.19. *If a sequence (x_n) converges, then its limit is unique.*

Proof. Assume that $x \neq y \in \mathbb{R}$ are two limits. Then, for each $0 < \varepsilon \in \mathbb{R}$ there are $n_\varepsilon^x, n_\varepsilon^y \in \mathbb{N}$ such that for all $n \geq n_\varepsilon^x$ we have:

$$|x - x_n| \leq \varepsilon$$

and for all $n \geq n_\varepsilon^y$ we have

$$|y - x_n| \leq \varepsilon.$$

So, if we set $n_\varepsilon := \max\{n_\varepsilon^x, n_\varepsilon^y\}$, then both of the above inequalities hold for all integers $n \geq n_\varepsilon$. In particular, for such n, we have

$$|y - x| \leq \underbrace{|y - x_n| + |x_n - x|}_{\text{triangle inequality}} \leq \varepsilon + \varepsilon = 2\varepsilon$$

Since, this holds for all $0 < \varepsilon \in \mathbb{R}$, we obtain that $y = x$. $\square$

Example 4.20. A convergent sequence is $x_n := 1 - \frac{1}{\sqrt{n}}$.

Indeed, $\lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n}}\right) = 1$, because for any $0 < \varepsilon \in \mathbb{R}$:

$$\left|1 - \frac{1}{\sqrt{n}} - 1\right| = \left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} \leq \varepsilon,$$

if $\sqrt{n} > \frac{1}{\varepsilon} \Leftrightarrow n > \frac{1}{\varepsilon^2}$. So, we may set $n_\varepsilon := \left[\frac{1}{\varepsilon^2}\right] + 1$

Example 4.21. A divergent sequence is $x_n := (-1)^n$. Indeed, if x_n was convergent with limit x, then for $\varepsilon := \frac{1}{2}$ there would exist $n_{\frac{1}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{1}{2}}$, we would have $|x_n - x| \leq \frac{1}{2}$. In particular, if $n' \geq n_{\frac{1}{2}}$ is any other integer, then we would have:

$$|x_n - x_{n'}| \leq \underbrace{|x_n - x| + |x - x_{n'}|}_{\text{triangle inequality}} \leq \frac{1}{2} + \frac{1}{2} = 1$$

However, in our sequence $|x_n - x_{n+1}| = 2 > 1$. This is a contradiction.

Remark 4.22. In fact, the above argument shows that if (x_n) is a convergent sequence, then for all $0 < \varepsilon \in \mathbb{R}$ there is an $n_\varepsilon \in \mathbb{N}$ such that for all $n, n' \geq n_\varepsilon$, $|x_n - x_{n'}| < \varepsilon$. We will call this the Cauchy criterion for convergence and we will learn it later more in detail.

Also, with similar arguments as above we may show that:

Proposition 4.23. *If (x_n) is convergent, then it is bounded.*

Proof. Set $x := \lim_{n\to\infty} x_n$. By definition of the limit, for $\varepsilon := 1$ we have an $n_1 \in \mathbb{N}$, such that for all integers $n \geq n_1$, $|x_n - x| \leq 1$. Set

$$R := \max\{|x_0|, |x_1|, \ldots, |x_{n_1-1}|, |x+1|, |x-1|\}.$$

Then, R is an upper and $-R$ is a lower bound. Indeed, they are bounds for $x_0, \ldots, x_{n_1-1}$ just because R is at least as big as the absolute values of all these elements of the sequence, by definition of R. Furthermore, R and $-R$ are also bounds for the other elements of the sequence, because these elements are lying in the interval $[x-1, x+1]$, which is again bounded by $-R$ and R, by definition of R. $\square$

Example 4.24. $x_n = n^2$ cannot be convergent as it is not bounded.

Example 4.25. The backwards direction of the above proposition is not true. That is, if a sequence (x_n) is bounded, then it is not necessary convergent. An example is $x_n := (-1)^n$. On the other hand, a little later we will see that a monotone, bounded sequence is convergent.

4.2.1 Limits and algebra

Proposition 4.26. *Let (x_n) and (y_n) be two convergent sequences. Set $x := \lim_{n\to\infty} x_n$ and $y := \lim_{n\to\infty} y_n$. Then:*

(1) $(x_n + y_n)$ is also convergent, and $\lim_{n\to\infty} (x_n + y_n) = x + y$,

(2) $(x_n \cdot y_n)$ is also convergent, and $\lim_{n\to\infty} (x_n \cdot y_n) = x \cdot y$,

(3) if $y \neq 0$, then $\left(\frac{x_n}{y_n}\right)$ is also convergent, and $\lim_{n\to\infty} \left(\frac{x_n}{y_n}\right) = \frac{x}{y}$ (note that because $y \neq 0$, there is an n_0 such that $y_n \neq 0$ for $n \geq n_0$, so dividing by y_n makes sense at least for $n \geq n_0$), and

(4) if there is an $n_0 \in \mathbb{N}$, such that $x_n \leq y_n$ for each integer $n \geq n_0$, then $x \leq y$.

Proof. We prove only the first one and we refer to (2.3.3 and 2.3.6 in the book for the proofs of the others).

So, fix $0 < \varepsilon \in \mathbb{R}$. We want to prove that for big n, $|(x_n + y_n) - (x + y)|$ is smaller than ε. However, all we know that $|x_n - x|$ and $|y_n - y|$ are small for big n. Luckily,

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq \underbrace{|x_n - x| + |y_n - y|}_{\text{triangle inequality}}. \tag{4.26.a}$$

So, to make $|(x_n + y_n) - (x + y)|$ be smaller than ε, we have to make the sum of $|x_n - x|$ and $|y_n - y|$ smaller than ε. This we can attain for example if we make both $|x_n - x|$ and $|y_n - y|$ smaller than $\frac{\varepsilon}{2}$ (however, this is an arbitrary division, the proof would work with any two positive numbers that add up to ε, for example with $\frac{\varepsilon}{3}$ and $\frac{2\varepsilon}{3}$).

So, after this initial discussion we can make a formal proof: there are integers $n^x_{\frac{\varepsilon}{2}}$ and $n^y_{\frac{\varepsilon}{2}}$, such that

(1) whenever $n \geq n^x_{\frac{\varepsilon}{2}}$, then $|x - x_n| \leq \frac{\varepsilon}{2}$, and

(2) whenever $n \geq n^y_{\frac{\varepsilon}{2}}$, then $|y - y_n| \leq \frac{\varepsilon}{2}$.

Set $n_\varepsilon := \max\left\{n^x_{\frac{\varepsilon}{2}}, n^y_{\frac{\varepsilon}{2}}\right\}$. Then, (4.26.a) tells us that for every $n \geq n_\varepsilon$ we have

$$|(x_n + y_n) - (x + y)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This shows that $(x_n + y_n)$ converges and the limit is $x + y$. $\square$

Example 4.27. With the above machinery we can already compute the limits of fractions of polynomials, called rational functions.

(1) $x_n := \frac{n^2+2n+3}{4n^2+5n+6}$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n^2+2n+3}{4n^2+5n+6} = \underbrace{\lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{3}{n^2}}{4+\frac{5}{n}+\frac{6}{n^2}}}_{\text{dividing both the numerator and the denominator by } n} = \underbrace{\frac{\lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(4+\frac{5}{n}+\frac{6}{n^2}\right)}}_{\text{using the above division rule for limits}}$$

$$= \underbrace{\frac{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{2}{n} + \lim_{n\to\infty}\frac{3}{n^2}}{\lim_{n\to\infty} 4 + \lim_{n\to\infty}\frac{5}{n} + \lim_{n\to\infty}\frac{6}{n^2}}}_{\text{the above addition rule of limits}} = \underbrace{\frac{1+\lim_{n\to\infty}\frac{2}{n}+3\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}{4+\lim_{n\to\infty}\frac{5}{n}+6\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)^2}}_{\text{the above product rule of limits, and limits of constant sequences}} = \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

Comments:

(i) dividing the numerator and the denominator by n is an operation that one cannot perform for $n=0$. So, after the second equality sign the sequence does not make sense for $n=0$. But this is fine, as the 0-th term (and in fact, even any finitely many first terms) of the sequence does not matter for the limit computation, so you can think about the 0-th term being anything (for example 0) after the second equality sign.
The same issue shows up many times later in our computations when we are computing limits of sequences of the form $\frac{2}{n}$, or $\frac{3}{n^2}$

(ii) for any number $c \in \mathbb{R}$: $\lim_{n\to\infty}\frac{c}{n} = 0$, as for $0 < \varepsilon \in \mathbb{R}$ we may choose $n_\varepsilon := \left[\frac{c}{\varepsilon}\right]+1$, and for this choice we have for each integer $n \geq n_\varepsilon$:

$$\left|\frac{c}{n}\right| < \frac{c}{\frac{c}{\varepsilon}} = \varepsilon$$

(iii) in the step where we use that limits behave well with respect to fractions, we should check first that the limit of the denominator is not 0. However, following our argument, we see that this limit is 4, so we are fine.

(2) $x_n = \frac{n+2}{3n^2+4n+5}$. Here we will not give the above explanations again (as they are the same):

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{n+2}{3n^2+4n+5} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{2}{n^2}}{3+\frac{4}{n}+\frac{5}{n^2}} = \frac{0+0}{3+0+0} = 0$$

(3) $x_n = \frac{n^2+2n+3}{4n+5}$. For $n \geq 1$, we have $0 \leq \frac{3}{n}$ and $1 \geq \frac{5}{n}$. Hence, for $n \geq 1$:

$$x_n = \frac{n^2+2n+3}{4n+5} = \frac{n+2+\frac{3}{n}}{4+\frac{5}{n}} \geq \frac{n+2}{5}$$

This shows that (x_n) is not bounded and hence cannot be convergent by Proposition 4.23.

Using the method of the above exercise one can show (see page 22 of the book for a precise proof, although there is an unnecessary assumption in the book about $y_n \neq 0$ for all $n \in \mathbb{N}$, in fact it is enough if $y_n \neq 0$ for some $n \in \mathbb{N}$):

Proposition 4.28. *If (x_n) and (y_n) are sequences given by formulas*

$$x_n = a_0 + a_1 n + \cdots + a_p n^p, \text{ with } a_p \neq 0$$

and

$$y_n = b_0 + b_1 n + \cdots + b_q n^q, \text{ with } b_q \neq 0,$$

then

(1) if $p \leq q$, then $\left(\frac{x_n}{y_n}\right)$ is convergent, and

(i) if $p = q$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a_p}{b_q}$,

(ii) if $p < q$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = 0$,

(2) if $p > q$, then $\left(\frac{x_n}{y_n}\right)$ is divergent.

$\longleftarrow$ End of 5. class, on 02.10.2019.

Theorem 4.29. Squeeze Theorem *Let x_n, y_n and z_n be three sequences satisfying:*

(1) (x_n) and (z_n) are convergent, with the same limit, say a,

(2) there is an $n_0 \in \mathbb{N}$, such that for all integers $n \geq n_0$, $x_n \leq y_n \leq z_n$

then (y_n) is also convergent, with limit a.

Proof. For each $\varepsilon > 0$, there are natural numbers n_ε^x and n_ε^z, such that for each integer $n \geq n_\varepsilon^x$, we have $a - \varepsilon < x_n$ and for each integer $n \geq n_\varepsilon^z$ we have $a + \varepsilon > z_n$.

Set $n_\varepsilon := \max\{n_\varepsilon^x, n_\varepsilon^z, n_0\}$. Then, for each integer $n \geq n_\varepsilon$ we have:

$$a - \varepsilon < x_n \leq y_n \leq z_n < a + \varepsilon$$

which in particular implies that $|y_n - a| < \varepsilon$.

$\square$

Example 4.30. $\lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{\sqrt{n}}\right) = 0$, because we may squeeze $\frac{1}{n} + \frac{1}{\sqrt{n}}$ with 0 from below and with $\frac{2}{\sqrt{n}}$ from above, and both of the latter sequences converge to 0. Indeed:

(1) $0 \leq \frac{1}{n} + \frac{1}{\sqrt{n}}$ holds for every integer $n \geq 1$.

(2) For every integer $n \geq 1$ we also have:

$$\frac{1}{n} + \frac{1}{\sqrt{n}} \leq \underbrace{\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}}_{n \leq n^2 \Rightarrow \sqrt{n} \leq n} = \frac{2}{\sqrt{n}}.$$

(3) $\lim_{n\to\infty} 0 = 0$ obviously.

(4) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ by the computation of Example 4.20.

Example 4.31. In general a geometric sequence $x_n = aq^n$ is convergent if and only if $a = 0$ or $-1 < q \leq 1$. Indeed:

The $a = 0$, the $q = 1$, the $q = 0$ and the $q = -1$ cases are clear, so we may assume that $a \neq 0$, and $|q| \neq 0, 1$. Then:

(1) In the $|q| > 1$ case, we already showed in Example 4.17 that x_n is not bounded. Hence, according to Proposition 4.23 it is also not convergent.

(2) In the $|q| < 1$ case, we show that $\lim_{n\to\infty} aq^n = 0$. For that we should understand when $|aq|^n < \varepsilon$ for any $0 < \varepsilon \in \mathbb{N}$:

$$|aq|^n \leq \varepsilon \Leftrightarrow \frac{|a|}{\varepsilon} \leq \left(\frac{1}{|q|}\right)^n \tag{4.31.b}$$

According to the Bernoulli inequality,

$$\left(\frac{1}{|q|}\right)^n \geq 1+n\left(\frac{1}{|q|}-1\right) > n\left(\frac{1}{|q|}-1\right) \tag{4.31.c}$$

putting (4.31.b) and (4.31.c) together we see that $|aq|^n \leq \varepsilon$ holds as soon as we guarantee that $\frac{|a|}{\varepsilon} < n\left(\frac{1}{|q|}-1\right)$. Lastly the latter inequality is equivalent to $n \geq n_\varepsilon$, where

$$n_\varepsilon = \left[\frac{\frac{|a|}{\varepsilon}}{\left(\frac{1}{|q|}-1\right)}\right]+1.$$

Example 4.32. Now, we are ready to state our next squeeze example.

We claim that $\lim\limits_{n\to\infty} \frac{2^n}{n!} = 0$. Indeed, we have for all integers $n \geq 3$:

$$0 \leq \frac{2^n}{n!} \leq \frac{2^n}{2\cdot 3^{n-2}} = \frac{9}{2}\cdot\left(\frac{2}{3}\right)^n$$

Furthermore $\lim\limits_{n\to\infty} 0 = 0$ and $\lim\limits_{n\to\infty} \frac{9}{2}\cdot\left(\frac{2}{3}\right)^n = \frac{9}{2}\cdot\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{9}{2}\cdot 0 = 0$ by Example 4.31. So, squeeze theorem concludes our claim.

Example 4.33. $x_n = \sqrt[n]{n}$ (there is a different proof in the book, on page 24, check it out too):

We squeeze x_n with $1 \leq x_n \leq y_n := 1+\frac{1}{\sqrt{n}}$. As the limit of both sides is 1, and x_n is not smaller than 1, it is enough to prove the second inequality, for high enough values of n. For that consider the following equivalence of inequalities:

$$\sqrt[n]{n} \leq 1+\frac{1}{\sqrt{n}} \quad \Leftrightarrow \quad n \leq \left(1+\frac{1}{\sqrt{n}}\right)^n = \sum_{i=0}^{n}\binom{n}{i}\frac{1}{(\sqrt{n})^i}$$

Note that the sum on the right hand side for $i=4$ is

$$\frac{n(n-1)(n-2)(n-3)}{24}\frac{1}{(\sqrt{n})^4} = \frac{n(n-1)(n-2)(n-3)}{24n^2}.$$

So, we know the desired inequality $\left(\text{i.e., that } \sqrt[n]{n} \leq 1+\frac{1}{\sqrt{n}}\right)$ as soon as $n \geq 4$ and $n \leq \frac{n(n-1)(n-2)(n-3)}{24n^2}$. The latter is equivalent to

$$\frac{24n^2}{(n-1)(n-2)(n-3)} \leq 1.$$

However, we have just learned that

$$\lim_{n\to\infty}\frac{24n^2}{(n-1)(n-2)(n-3)} = 0,$$

so there is an integer n_1, such that for each $n \geq n_1$,

$$\left|\frac{24n^2}{(n-1)(n-2)(n-3)}\right| \leq 1.$$

Corollary 4.34. *If* $\lim\limits_{n\to\infty} x_n = 0$ *and* (y_n) *is bounded, then* $\lim\limits_{n\to\infty} x_n y_n = 0$.

Proof. Note that showing $\lim x_n y_n = 0$ is equivalent to showing $\lim |x_n y_n| = 0$ (by the definition, as $|x_n y_n - 0| = ||x_n y_n| - 0|$). Similary, from the assumption $\lim_{n\to\infty} x_n = 0$ we obtain that also that $\lim_{n\to\infty} |x_n| = 0$

As y_n is bounded, there is an integer $M > 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Hence, we may squeeze $|x_n y_n|$:

$$0 \leq |x_n y_n| \leq |x_n| M,$$

where both sides converge to 0. This shows that so does $|x_n y_n|$. $\square$

Example 4.35. $\lim_{n\to\infty} \frac{1}{n^2} \sin(n) = 0$. Note that here $\sin(n)$ does not converge in itself. So, we may not apply the previous multiplication rule of limits. However, we may apply the previous corollary, as $\lim_{n\to\infty} \frac{1}{n^2} = 0$, and $\sin(n)$ is bounded (by -1 and 1).

Example 4.36. Define the recursive sequence $x_{n+1} = \frac{\sin(x_n)}{2}$, $x_0 = 1$. Then we have:

$$\frac{|x_{n+1}|}{|x_n|} = \frac{\frac{|\sin(x_n)|}{2}}{|x_n|} \leq \frac{1}{2},$$

as $\frac{|\sin(x)|}{|x|} \leq 1$ for all $x \in \mathbb{R}$ ($|x|$ measures the length of the circle segment of angle x, where we count multiple revolutions too, and $|\sin(x)|$ gives the absolute value of the y-coordinate of the endpoint of the circle segment).

In particular, we have

$$|x_n| = \frac{|x_n|}{|x_{n-1}|} |x_{n-1}| \leq \frac{1}{2} |x_{n-1}|.$$

Iterating this we obtain

$$|x_n| \leq \frac{1}{2} |x_{n-1}| \leq \frac{1}{2^2} |x_{n-2}| \leq \cdots \leq \frac{1}{2^{n-1}} |x_1| \leq \frac{1}{2^n}$$

So, we may use the Squeeze Theorem (Theorem 4.29) to show that $\lim_{n\to\infty} x_n = 0$, squeezing with:

$$0 \leq x_n \leq \frac{1}{2^n}.$$

Corollary 4.37. QUOTIENT CRITERION *Let (x_n) be a sequence such that*

$$q = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$$

exists. If $q < 1$, then (x_n) converges to 0, and if $q > 1$, then (x_n) diverges.

Proof. We show here only the $q < 1$ case, the other one is similar, and is left as a homework. Note that as q is the limit of non-negative numbers, it is automatically non-negative. So, we have $0 \leq q < 1$.

Set $\varepsilon := \frac{1-q}{2}$. Then, there is a $n_\varepsilon \in \mathbb{N}$, such that for all integers $n \geq n_\varepsilon$:

$$q - \varepsilon < \frac{|x_{n+1}|}{|x_n|} < q + \varepsilon = q + \frac{1-q}{2} = \frac{q+1}{2} =: \overline{q} < 1.$$

In particular, then $|x_{n_\varepsilon + i}| \leq |x_{n_\varepsilon}| \overline{q}^i$ (for all $i \in \mathbb{N}$), so we may squeeze $|x_n|$ (for every integer $n \geq n_\varepsilon$):

$$0 \leq |x_n| \leq |x_{n_\varepsilon}| \overline{q}^{n-n_\varepsilon},$$

because

$$\lim_{n\to\infty} |x_{n_\varepsilon}| \overline{q}^{n-n_\varepsilon} = \underbrace{\frac{|x_{n_\varepsilon}|}{\overline{q}^{n_\varepsilon}} \lim_{n\to\infty} \overline{q}^n = \frac{|x_{n_\varepsilon}|}{\overline{q}^{n_\varepsilon}} \cdot 0}_{|\overline{q}|<1} = 0.$$

$\square$

Example 4.38. Some examples showing that we are not able to say anything in the $q = 1$ case:

(1) If $x_n := n$, then (x_n) is divergent and $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{n+1}{n} = 1$

(2) If $x_n := \frac{n+1}{n}$, then (x_n) is convergent to 1 (see the computation one line above), and $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = \lim_{n\to\infty} \frac{\frac{n+2}{n+1}}{\frac{n+1}{n}} = \lim_{n\to\infty} \frac{(n+2)n}{(n+1)^2} = 1.$

4.2.2 Limits of recursive sequences

Example 4.39. The Fibonacci sequence is $x_0 = x_1 = 1$ and $x_{n+1} = x_n + x_{n-1}$. In particular then if we define $y_n := \frac{x_{n+1}}{x_n}$ we obtain $y_{n+1} = 1 + \frac{1}{y_n}$, and $y_0 = 1$. We call this the sequence of Fibonacci quotients.

Proposition 4.40. *If (y_n) is the sequence of Fibonacci quotients, then for each integer $n > 0$ we have $1 \leq y_n \leq 2$.*

Proof. We prove the above statements by induction on n.

The $n = 0$ case: by definition we have $2 \geq y_0 = 1$.

So, assume we know that statement for n and then we prove it for $n + 1$ below:

$$y_{n+1} = 1 + \frac{1}{y_n} \geq 1 + \frac{1}{2} \geq 1,$$

and

$$y_{n+1} = 1 + \frac{1}{y_n} \leq 1 + \frac{1}{1} = 2,$$

□

Example 4.41. Let us continue with our Fibonacci quotient example.

Let us assume now that (y_n) is convergent, and let y be the limit. Then, as $y_n \geq 1$, it follows that $y \geq 1$. Furthermore, we have

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{n+1} = \underbrace{\lim_{n\to\infty}\left(1 + \frac{1}{y_n}\right)}_{\text{definition of } y_{n+1}} = \underbrace{1 + \frac{1}{\lim_{n\to\infty} y_n}}_{\text{algebraic rules of limit}} = 1 + \frac{1}{y}$$

This yields an equation which we are able to solve:

$$y = 1 + \frac{1}{y} \quad \Leftrightarrow \quad y^2 = y + 1 \quad \Leftrightarrow \quad y = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

As we have seen that $y \geq 1$, the only possibility that can happen is $y = \frac{1+\sqrt{5}}{2}$.

←
End of 6. class, on 07.10.2019.

However, we do not know at this point that (y_n) converges. As we have figured out that if it converges the only possible limit is $\frac{1+\sqrt{5}}{2}$, we may show that (y_n) converges by showing that $z_n := \left| y_n - \frac{1+\sqrt{5}}{2} \right|$ converges to 0.

$$z_{n+1} = \left| y_{n+1} - \frac{1+\sqrt{5}}{2} \right| = \underbrace{\left| 1 + \frac{1}{y_n} - 1 - \frac{1}{\frac{1+\sqrt{5}}{2}} \right|}_{\substack{\text{we apply the definition of the sequence to } y_{n+1}\text{, and} \\ \text{then as we found } \frac{1+\sqrt{5}}{2} \text{ as the solution of } y = 1 + \frac{1}{y} \\ \text{we may replace } \frac{1+\sqrt{5}}{2} \text{ by } 1 + \frac{1}{\frac{1+\sqrt{5}}{2}}}} = \left| \frac{1}{y_n} - \frac{1}{\frac{1+\sqrt{5}}{2}} \right| = \frac{\left| y_n - \frac{1+\sqrt{5}}{2} \right|}{y_n \frac{1+\sqrt{5}}{2}} \leq \frac{|z_n|}{\frac{1+\sqrt{5}}{2}}$$

So, we obtain that

$$0 \leq z_n \leq \frac{|z_0|}{\left(\frac{1+\sqrt{5}}{2}\right)^n},$$

where

$$\lim_{n\to\infty} \frac{|z_0|}{\left(\frac{1+\sqrt{5}}{2}\right)^n} = 0.$$

So, the Squeeze Theorem (Theorem 4.29) shows that $\lim_{n\to\infty} z_n = 0$. This in turn implies, by the definition of z_n that $\lim_{n\to\infty} y_n = \frac{1+\sqrt{5}}{2}$.

Summarizing, we showed that for the Fibonacci sequence (x_n) that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1+\sqrt{5}}{2}$$

This number is also called the Golden ratio.

The general approach of finding the limit of a recursive sequence x_n is the one we just performed for y_n. That is, the steps are:

(1) assuming that there exists a limit, computing what it can be using the algebraic rules of limits

(2) showing some kind of upper or/and lower bounds that exclude all but one of the possibilities for the limit (obtained in the previous point),

(3) showing that $\lim_{n\to\infty} |x_n - x| = 0$, where x is the only possibility left for the limit by the above points.

Example 4.42. This method of finding the limit does not always work. For example:

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1})$$

gives $x = \frac{1}{2}(x + x)$.

Example 4.43. On the other hand, if the above limit finding step has no solution, we automatically know that (y_n) is divergent. For example if we change one sign in the Fibonacci quotient sequence, we obtain:

$$y_n = 1 - \frac{1}{y_{n-1}}.$$

For simplicity (so that we do not have to worry about when $y_{n-1} = 0$), let us consider the related recursive sequence:

$$y_n^2 = y_{n-1} - 1.$$

Let us assume y_n is convergent to y, and take limits. Then we obtain

$$y^2 - y + 1 \Leftrightarrow y = \frac{1 \pm \sqrt{(-1)^2 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2}.$$

Since there is no real solution, the limit cannot exist.

4.2.3 Approaching infinities

Definition 4.44. We say that a sequence (x_n) approaches $+\infty$ (resp. $-\infty$) if for all real numbers $A \in \mathbb{R}$ there is an $n_A \in \mathbb{N}$ such that for all integers $n \geq n_A$, $x_n \geq A$ (resp. $x_n \leq A$).

In the first case we write $\lim_{n\to\infty} x_n = +\infty$, and $\lim_{n\to\infty} x_n = -\infty$ in the second case. In both cases the sequence is DIVERGENT.

Example 4.45. Using the Bernoulli inequality argument of Example 4.17, we can show that $\lim x_n = +\infty$ for every geometric progression $x_n = aq^n$ with $a > 0$ and $q > 1$. An example is $x_n = 3 \cdot 2^n$.

Similarly, $\lim x_n = -\infty$ for every geometric progression $x_n = aq^n$ with $a < 0$ and $q > 1$. An example is $x_n = -3 \cdot 2^n$.

On the other hand, neither is true for $x_n = aq^n$ with $a \neq 0$ and $q < 0$. For example $x_n = (-2)^n$ approaches neither $+\infty$ nor $-\infty$.

The infinite limits satisfy some algebraic rules, and do not satisfy others. Check out page 29 and 30 of the book for full list. Here we note just a few examples:

Example 4.46. If $\lim_{n\to\infty} x_n = +\infty$ and (y_n) is bounded from below, then $\lim_{n\to\infty} x_n + y_n = +\infty$. For example:

$$\lim_{n\to\infty} (2^n + \sin(n)) = +\infty,$$

because $\lim_{n\to\infty} 2^n = +\infty$ and $\sin(n) \geq -1$.

Similarly we have:

(1) $\lim_{n\to\infty} x_n = -\infty$ and (y_n) is bounded from above $\quad\Rightarrow\quad \lim_{n\to\infty} (x_n + y_n) = -\infty$,

(2) $\lim_{n\to\infty} x_n = +\infty$ and there is a natural number n_0 and a real number $A > 0$ such that for every integer $n \geq n_0$, $y_n \geq A \quad\Rightarrow\quad \lim_{n\to\infty} (x_n \cdot y_n) = +\infty$,

(3) $\lim_{n\to\infty} |x_n| = +\infty$ and (y_n) is bounded $\quad\Rightarrow\quad \lim_{n\to\infty} \frac{y_n}{x_n} = 0$

(4) etc.

Again, these are only some examples, page 29 and 30 of the book for full list.

It is particularly important that one cannot drop the boundedness assumptions on y_n. That is, in those cases "anything can happen", as shown for addition by the next example:

Example 4.47. Here we show examples of sequences (x_n) and (y_n), for which $\lim_{n\to\infty} x_n = +\infty$, $\lim_{n\to\infty} y_n = -\infty$, but the $\lim_{n\to\infty} (x_n + y_n)$ all different behaviors:

(1) $\lim_{n\to\infty} (\underbrace{n}_{x_n} + \underbrace{(-n)}_{y_n}) = 0$, so $x_n + y_n$ can be convergent,

(2) $\lim_{n\to\infty} (\underbrace{2n}_{x_n} + \underbrace{(-n)}_{y_n}) = n$, so $\lim_{n\to\infty} (x_n + y_n) = +\infty$ can happen,

(3) $\lim_{n\to\infty} (\underbrace{n}_{x_n} + \underbrace{(-2n)}_{y_n}) = -n$, so $\lim_{n\to\infty} (x_n + y_n) = -\infty$ can happen,

(4) $\lim_{n\to\infty} (\underbrace{(2n + (-1)^n n)}_{x_n} + \underbrace{(-2n)}_{y_n}) = (-1)^n n$, so $x_n + y_n$ unbounded and hence divergent without approaching $+\infty$ or $-\infty$,

(5) etc.

It is a homework to cook up similar examples for multiplication and division. For example, a famous case where "anything can happen" for multiplication is $\lim_{n\to\infty} x_n = +\infty$ and $\lim_{n\to\infty} y_n = 0$.

Similarly to the argument for finite limits, we can prove squeeze theorems for infinite limits:

Theorem 4.48. Squeeze Theorem for approaching infinities *Let (x_n) and (y_n) be two sequences for which there is an integer n_0 such that for every integer $n \geq n_0$, $x_n \leq y_n$.*

(1) If $\lim_{n\to\infty} x_n = +\infty$, then $\lim_{n\to\infty} y_n = +\infty$.

(2) If $\lim_{n\to\infty} y_n = -\infty$, then $\lim_{n\to\infty} x_n = -\infty$.

Example 4.49. (1) We compute $\lim_{n\to\infty} \frac{n!}{2^n}$. We have $\frac{n!}{2^n} \geq \frac{2\cdot 3\cdot 3\cdots 3\cdot 3}{2^n} = \frac{1}{2}\left(\frac{3}{2}\right)^{n-2}$, and $\lim_{n\to\infty} \frac{1}{2}\left(\frac{3}{2}\right)^{n-2} = +\infty$ according to Example 4.45. Hence, Theorem 4.48 yields $\lim_{n\to\infty} \frac{n!}{2^n} = +\infty$.

(2) Similarly, but using the other point of Theorem 4.48 we obtain $\lim_{n\to\infty} -\frac{n!}{2^n} = -\infty$.

There is also a quotient rule, for which we also refer to the book.

4.3 Monotone sequences

We call a sequence (x_n) monotone if it is increasing or decreasing.

Theorem 4.50. *If x_n is bounded and increasing (resp. decreasing), then (x_n) is convergent and*

$$\lim_{n\to\infty} x_n = \mathrm{Sup}\{x_n | n \in \mathbb{N}\} \qquad (\textit{resp. } \lim_{n\to\infty} x_n = \mathrm{Inf}\{x_n | n \in \mathbb{N}\}).$$

Proof. We prove only the increasing case. We leave as a homework to change the words in it to obtain a proof for the decreasing case.

Set $S := \mathrm{Sup}\{x_n | n \in \mathbb{N}\}$ and let $0 < \varepsilon \in \mathbb{R}$ be arbitrary. By definition, S is the smallest upper bound, so $S - \varepsilon$ is not an upper bound. Hence, there is an $n_\varepsilon \in \mathbb{N}$ such that $S - \varepsilon < x_{n_\varepsilon}$. In particular, for any integer $n \geq n_\varepsilon$:

$$S - \varepsilon < \underbrace{x_{n_\varepsilon} \leq}_{\text{prev. line}} \underbrace{x_n \leq}_{(x_n) \text{ is montone}} \underbrace{S}_{S \text{ is an upper bound}} < S + \varepsilon.$$

□

Example 4.51. Introduction of e.

Let $n \in \mathbb{Z}_+$. We claim that $\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}$. Indeed:

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &= \sum_{i=0}^{n} \binom{n}{i} \frac{1}{n^i} = \sum_{i=0}^{n} \frac{1}{i!} \frac{n(n-1)\ldots(n-i+1)}{n^i} \\ &= \sum_{i=0}^{n} \frac{1}{i!} \frac{n(n-1)\ldots(n-i+1)}{n^i} = \sum_{i=0}^{n} \frac{1}{i!}\left(1 - \frac{1}{n}\right)\ldots\left(1 - \frac{i-1}{n}\right) \qquad (4.51.a)\end{aligned}$$

Similarly,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} = \sum_{i=0}^{n+1} \frac{1}{i!}\left(1 - \frac{1}{n+1}\right)\ldots\left(1 - \frac{i-1}{n+1}\right) > \sum_{i=0}^{n} \frac{1}{i!}\left(1 - \frac{1}{n+1}\right)\ldots\left(1 - \frac{i-1}{n+1}\right) \qquad (4.51.b)$$

So, to prove our claim it is enough to show that each term on the right side of (4.51.a) is at most as big as the corresponding term on the right side of (4.51.b). However, that is clear as $\frac{j}{n} > \frac{j}{n+1}$ for any $j \in \mathbb{Z}$.

Having proved our claim, we see that $x_n := \left(1+\frac{1}{n}\right)^n$ (set $x_0 = 1$ as for $n = 0$ the expression does not have a meaning) is a monotone increasing sequence, with $x_0 = 1$, $x_1 = 2$. Is it bounded? Well yes, because:

$$\left(1+\frac{1}{n}\right)^n = \sum_{i=0}^{n} \frac{1}{i!}\left(1-\frac{1}{n}\right)\dots\left(1-\frac{i-1}{n}\right) \leq \sum_{i=0}^{n}\frac{1}{i!} \leq 1+\sum_{i=1}^{n}\frac{1}{2^{i-1}} = 1+\frac{1-\frac{1}{2^n}}{\frac{1}{2}} = 3-\frac{1}{2^n} \leq 3,$$

where, for evaluating the sum, we used the formula that we have already shown earlier (for $a = \frac{1}{2}$):

$$(1+\dots+a^{n-1}) = \frac{1-a^n}{1-a}.$$

So, indeed, (x_n) is not only increasing, but also bounded by above by 3. So, there exists $\lim\limits_{n\to\infty} x_n$ according to Theorem 4.50.

Definition 4.52. We define $e := \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

$\longleftarrow$ End of 7. class, on 09.10.2019.

Theorem 4.50 aslo gives another method for showing the existence of limits for recursive sequences:

Example 4.53. Consider the recursive sequence $x_0 = 2$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$.

First we claim that $x_n > 0$ for all integers $n \in \mathbb{N}$. This is certainly true for $n = 0$, and if we assume it for $n-1$, then the recursive formula gives it to us also for n. Hence, indeed, it is true for all $n \in \mathbb{N}$. In particular, the division in the definition does make sense.

Next, we claim that $x_n \geq 1$ for all integers $n \geq 1$. Indeed, a similar induction shows this. That is, for $n = 0$, we have $x_0 = 2 \geq 1$. Furthermore,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \geq 1 \Leftrightarrow x_n + \frac{1}{x_n} \geq 2 \Leftrightarrow x_n^2 + 1 \geq 2x_n \Leftrightarrow (x_n-1)^2 \geq 0, \tag{4.53.c}$$

where we used that we already know that $x_n > 0$, when we multiplied by x_n. So, by (4.53.c), the induction step works too. That is, assuming $x_n \geq 1$, we obtain that $x_{n+1} \geq 1$ holds as well.

Next, we claim that x_n is decreasing. Indeed,

$$x_n - x_{n+1} = x_n - \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) = \frac{1}{2}\left(x_n - \frac{1}{x_n}\right) \geq 0,$$

where we obtained the last inequality using that $x_n \geq 1 \geq \frac{1}{x_n}$.

So, (x_n) is decreasing (hence bounded from above) and also bounded from below by 1. In particular, x_n is convergent, and $\lim\limits_{n\to\infty} x_n \geq 1$. Hence to find the actual limit we may just apply limit to the recursive equation to obtain that if y is the limit, then

$$y = \frac{1}{2}\left(y + \frac{1}{y}\right) \Leftrightarrow \frac{y}{2} = \frac{1}{2y} \Leftrightarrow y^2 = 1$$

As we also know that $y \geq 1$, $y = 1$ has to hold. So, $\lim\limits_{n\to\infty} x_n = 1$.

4.4 Liminf, limsup

Definition 4.54. Let (x_n) be a bounded sequence. We define a new sequence

$$(y_n) := \mathrm{Sup}\{x_k | n \leq k \in \mathbb{N}\} \text{ (resp. } \mathrm{Inf}\{x_k | n \leq k \in \mathbb{N}\})$$

This is a decreasing, (resp. increasing) sequence, as Sup (resp. Inf) is taken over smaller and smaller sets as n increases. Furthermore y_n is bounded by the same bound as x_n. Hence, according to Theorem 4.50, y_n is convergent, and we call its limit the limsup (resp. liminf) of x_n. We denote it by $\lim\limits_{n\to\infty}\sup x_n$ (resp. $\lim\limits_{n\to\infty}\inf x_n$).

Example 4.55. $x_n = (-1)^n$. Then,

$$\lim_{n\to\infty} \operatorname{Sup}\{x_k | n \le k \in \mathbb{N}\} = \lim_{n\to\infty} \operatorname{Sup}\{-1, 1\} = \lim_{n\to\infty} 1 = 1,$$

and

$$\lim_{n\to\infty} \operatorname{Inf}\{x_k | n \le k \in \mathbb{N}\} = \lim_{n\to\infty} \operatorname{Sup}\{-1, 1\} = \lim_{n\to\infty} -1 = -1.$$

Hence,

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} y_n = 1,$$

and

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} z_n = -1.$$

4.5 Subsequences

Definition 4.56. If (x_n) is a sequence, then a subsequence is a sequence of the form (x_{n_k}) with variable being k and with $k \mapsto n_k$ being a strictly increasing function of k.

Example 4.57. $x_n := (-1)^n$, then both the constant 1 sequence and the constant -1 sequences are subsequences.

Example 4.58. $x_n = n^2$, then $y_k := x_{n_k} = k^6$ is a subsequence by setting $n_k = k^3$.

Example 4.59. $x_n = \left(1 + \frac{2}{n}\right)^n$ and $n = 2k$, then we get

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \left(1 + \frac{2}{2k}\right)^{2k} = \lim_{k\to\infty} \left(\left(1 + \frac{1}{k}\right)^k\right)^2 = \left(\lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k\right)^2 = e^2.$$

But is the same true for x_n? This is something you have seen already in the exercise session, but we will indicate also another argument in Example 4.64, which you will work then out in the exercise session.

The next proposition answers the connection between the convergence of a sequence and a subsequence.

Proposition 4.60. *If $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then all subsequences (x_{n_k}) converge also to a.*

The proof of Proposition 4.60 is just about invoking the definition, so we do not spell out the details here.

Example 4.61. Similarly, if $x_n = (-1)^n \left(1 + \frac{1}{n}\right)^n$, then

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{Sup}\{x_k | n \le k \in \mathbb{N}\} &= \lim_{n\to\infty} \operatorname{Sup}\left\{ (-1)^k \left(1 + \frac{1}{k}\right)^k \middle| k \ge n \right\} \\
&= \underbrace{\lim_{n\to\infty} \operatorname{Sup}\left\{ \left(1 + \frac{1}{k}\right)^k \middle| k \ge n, k \text{ is even} \right\}}_{\text{we can throw away the negative terms for Sup}} = \underbrace{\lim_{n\to\infty} e}_{\substack{\text{First, Sup is the same as lim according to Theorem 4.50. Second, lim is the same as the lim} \\ \text{for all } k \in \mathbb{N} \text{ according to Proposition 4.60}}} = e,
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{Inf}\{x_k | n \le k \in \mathbb{N}\} &= \lim_{n\to\infty} \operatorname{Inf}\left\{ (-1)^k \left(1 + \frac{1}{k}\right)^k \middle| k \ge n \right\} \\
&= \underbrace{\lim_{n\to\infty} \operatorname{Inf}\left\{ -\left(1 + \frac{1}{k}\right)^k \middle| k \ge n, k \text{ is odd} \right\}}_{\text{we can throw away the positive terms for Inf}} = \underbrace{\lim_{n\to\infty} -e}_{\substack{\text{First, Inf is the same as lim according to Theorem 4.50. Second, lim is the same as the lim} \\ \text{for all } k \in \mathbb{N} \text{ according to Proposition 4.60}}} = -e,
\end{aligned}$$

Theorem 4.62. BOLZANO-WEIERSTRASS *Every bounded sequence contains a convergent subsequence.*

Proof. We define n_k with induction on k. We set $n_0 = 0$. So, assume n_{k-1} is defined. Set then $s_k := \operatorname{Sup}\{x_n | n > n_{k-1}\}$. Then there is a integer $n_k > n_{k-1}$ such that

$$x_{n_k} > s_k - \frac{1}{k}.$$

We claim that (x_{n_k}) is convergent. Indeed, this follows from the squeeze principle, as we have:

$$s_k - \frac{1}{k} < x_{n_k} < s_k,$$

where $\lim\limits_{k\to\infty} s_k = \lim\limits_{n\to\infty} \sup x_n$ as it is a subsequence of $\operatorname{Sup}\{x_{n'} | n' > n\}$, and

$$\lim_{k\to\infty} s_k - \frac{1}{k} = \lim_{k\to\infty} s_k - \lim_{k\to\infty} \frac{1}{k} = \lim_{n\to\infty} \sup x_n - 0 = \lim_{n\to\infty} \sup x_n.$$

□

Example 4.63. Sometimes, it is possible to write down explicitly the convergent subsequences, for example for $x_n = \sin\left(\frac{n\pi}{4}\right)\left(1 + \frac{1}{n}\right)^n$, if we set

(1) $n_k = 8k + 1$, then $\lim\limits_{k\to\infty} x_{n_k} = \frac{1}{\sqrt{2}} e$,

(2) $n_k = 8k + 2$, then $\lim\limits_{k\to\infty} x_{n_k} = e$,

(3) $n_k = 8k + 5$, then $\lim\limits_{k\to\infty} x_{n_k} = -\frac{1}{\sqrt{2}} e$,

(4) etc.

However, sometimes this is not quite possible, and we just know that the convergent subsequence exists, for example for $x_n = \sin(n)\left(1 + \frac{1}{n}\right)^n$.

Example 4.64. Let $a > 0$ be an integer. Then for $x_n = \left(1 + \frac{a}{n}\right)^n$, we may consider the subsequence $n_k = ak$ to obtain:

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} \left(1 + \frac{a}{ak}\right)^{ak} = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^{ak} = \lim_{k\to\infty} \left(\left(1 + \frac{1}{k}\right)^k\right)^a = e^a.$$

There is an exercise on the exercise sheet that x_n is increasing and bounded for $a = 2$ (where the method should be similar than for $a = 1$ using the binomial expansion). In particular, x_n is convergent. However, if it is convergent we may find the limit as the limit of any of its subsequence. So, $\lim\limits_{n\to\infty} x_n = e^a$. ⟵ End of 8. class, on 14.10.2019.

4.6 Cauchy convergence

Definition 4.65. A sequence (x_n) is *Cauchy convergent* (or equivalently it is a *Cauchy sequence*), if for every $0 < \varepsilon \in \mathbb{R}$ there is an $n_\varepsilon \in \mathbb{N}$ such that for every integer $n, m \geq n_\varepsilon$, $|x_n - x_m| \leq \varepsilon$.

Example 4.66. $x_n := 1 - \frac{1}{n}$, then

$$|x_n - x_m| = \left|1 - \frac{1}{n} - 1 + \frac{1}{m}\right| = \left|\frac{1}{m} - \frac{1}{n}\right| \leq \frac{1}{m} + \frac{1}{n} < \varepsilon$$

if $n, m \geq \frac{2}{\varepsilon}$, as then

$$\frac{1}{m} + \frac{1}{n} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So, x_n is Cauchy convergent.

Theorem 4.67. *(x_n) is convergent if and only if it is Cauchy convergent.*

Proof. (1) First we assume that (x_n) is convergent, and then we show that it is Cauchy convergent. Let $x := \lim_{n \to \infty} x_n$ and $0 < \varepsilon \in \mathbb{R}$ arbitrary. Then there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n \geq n_{\frac{\varepsilon}{2}}$, we have $|x_n - x| \leq \frac{\varepsilon}{2}$. Then, for any integers $n, m \geq n_{\frac{\varepsilon}{2}}$ we have

$$|x_n - x_m| = |(x_n - x) + (x - x_m)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

(2) For the other direction, assume that (x_n) is Cauchy convergent. We first claim that then (x_n) is bounded. Indeed, there is an $n_1 \in \mathbb{N}$ such that for all integers $n \geq n_1$, $|x_n - x_m| \leq 1$. Then, an upper bound for $|x_n|$ is

$$\max\{|x_0|, \ldots, |x_{n_1 - 1}|, |x_{n_1}| + 1\}.$$

So, (x_n) is bounded. Hence it contains a convergent subsequence x_{n_k} converging to $x \in \mathbb{R}$. Fix then a $0 < \varepsilon \in \mathbb{R}$. As (x_n) is Cauchy, there is an $n_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all integers $n, m \geq n_{\frac{\varepsilon}{2}}$,

$$|x_n - x_m| < \frac{\varepsilon}{2}.$$

Now, there is a k such that $n_k \geq n_{\frac{\varepsilon}{2}}$ and $|x_{n_k} - x| \leq \frac{\varepsilon}{2}$. For this value of k and any integer $n \geq n_{\frac{\varepsilon}{2}}$ we have:

$$|x_n - x| \leq |(x_n - x_{n_k}) + (x_{n_k} - x)| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

5 Series

Definition 5.1. A series is a sequence (S_n) associated to another sequence (x_n) using the formula

$$S_n = \sum_{k=0}^{n} x_k$$

Remark 5.2. Definition 5.1 also allows the sum to start at $k = 1, 2, \ldots$, by assigning (secretly) the value 0 to x_k for lower values of k. With other words, $S_n = \sum_{k=1}^{n} x_k$, $S_n = \sum_{k=2}^{n} x_k$, ..., are perfectly fine series.

Example 5.3. The following are a few examples of series, and sequences S_n associated to them. Note that in the case of the first example one has an explicit expression for S_n without involving sums. However, in the other cases, we are not able to provide such formulas. So, one just has to take it as it is, so as a sequence obtained by adding the first n elements of the given other sequence.

(1) $S_n = \sum_{k=0}^{n} \frac{1}{2^k} = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^{n+1}}\right).$

(2) $S_n = \sum_{k=0}^{n} \frac{1}{k!}$

(3) $S_n = \sum_{k=1}^{n} \frac{1}{k}$

(4) $S_n = \sum_{k=1}^{n} (-1)^k \frac{1}{k}$

(5) $S_n = \sum_{k=1}^{n} \frac{1}{k^2}$

(6) $S_n = \sum_{k=1}^{n} \frac{1}{k^s}$

Definition 5.4. A series $S_n = \sum_{k=0}^{n} x_k$ is

$$\left.\begin{array}{r} \text{convergent} \\ \text{divergent} \\ \text{absolute convergent} \end{array}\right\}, \text{ if } \left\{\begin{array}{l} (S_n) \text{ is convergent} \\ (S_n) \text{ is divergent} \\ (S'_n) \text{ is convergent, where } S'_n = \sum_{k=0}^{n} |x_k|. \end{array}\right.$$

If it is convergent, then we use the notation $\sum_{k=0}^{\infty} x_k := \lim_{n\to\infty} S_n$.

Example 5.5.

$$\sum_{k=0}^{n} \frac{1}{2^k} = 2\left(1 - \frac{1}{2^{n+1}}\right) \Rightarrow \sum_{k=0}^{\infty} \frac{1}{2^k} = \lim_{k\to\infty} 2\left(1 - \frac{1}{2^{n+1}}\right) = 2.$$

By Cauchy's convergence criterion we have:

Proposition 5.6. *$\sum_{k=0}^{\infty} x_k$ is convergent if and only if for every $0 < \varepsilon \in \mathbb{R}$, there is an $n_\varepsilon \in \mathbb{N}$ such that for every integer $m > n \geq n_\varepsilon$,*

$$\left| \sum_{k=n+1}^{m} x_k \right| < \varepsilon.$$

Example 5.7. *We show that $\sum_{k=1}^{\infty} \frac{1}{k}$ is not convergent.*

Indeed, assume that for $\varepsilon = \frac{1}{4}$, Cauchy's criterion is satisfied with $n_1 \in \mathbb{N}$. Then, for $n := n_1$ and $m = 2n$,

$$\frac{1}{4} = \varepsilon > \left| \sum_{k=n+1}^{2n} x_k \right| = \left| \sum_{k=n+1}^{2n} \frac{1}{k} \right| \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = \frac{1}{2}.$$

This is a contradiction.

An immediate consequence of the Cauchy convergence criterion (Proposition 5.6) is the following:

Proposition 5.8. *If $\sum_{k=0}^{\infty} x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.*

Proof. Indeed, by Proposition 5.6, for every $0 < \varepsilon \in \mathbb{R}$, there is an $n_\varepsilon \in \mathbb{N}$ such that for all integers $m > n \geq n_\varepsilon$,

$$\left| \sum_{k=n+1}^{m} x_k \right| \leq \varepsilon.$$

In particular, if we choose $m = n+1$, then we obtain that

$$\varepsilon \geq \left| \sum_{k=n+1}^{n+1} x_k \right| = |x_{n+1}|.$$

This implies that $\lim_{n\to\infty} x_n = 0$. $\square$

Example 5.9. *We show that* $\sum_{k=0}^{\infty} \cos(n)$ *is not convergent.*

Indeed, according to Proposition 5.8, for this it is enough to see that $x_n := \cos(n)$ does not converge to 0. Assume it does. Then, so do all its subsequences. However, consider the subsequence given by $n_k := \lfloor 2k\pi \rfloor$. We have

$$x_{n_k} = \underbrace{\cos(\lfloor 2k\pi \rfloor) \geq \cos(2k\pi - 1)}_{\substack{\cos(x) \text{ is an increasing function in the interval} \\ 2k\pi - \frac{\pi}{2} \leq x \leq 2k\pi, \text{ and furthermore, as } \frac{\pi}{2} > \\ 1,\ 2k\pi - 1 \text{ is in this interval}}} = \cos(-1) > 0.$$

This contradicts the earlier assumption that all subsequences of x_n converge to 0.

One can use the Cauchy criterion (Proposition 5.6) to show:

Proposition 5.10. Squeeze theorem *Assume there is an $n_0 \in \mathbb{N}$ such that $0 \leq x_n \leq y_n$ for every integer $n \geq n_0$. Then:*

(1) If $\sum_{k=0}^{\infty} y_k$ *is convergent, then* $\sum_{k=0}^{\infty} x_k$ *is also convergent.*

(2) If $\sum_{k=0}^{\infty} x_k$ *is divergent, then* $\sum_{k=0}^{\infty} y_k$ *is also divergent.*

Proof. For every $n, m \geq n_0$ we have

$$0 \leq \left| \sum_{k=n+1}^{m} x_k \right| \leq \left| \sum_{k=n+1}^{m} y_k \right|.$$

So, if the Cauchy type criterion of Proposition 5.6 is verified for y_k then it also holds for x_k and if it does not hold for x_k it must also fail for y_k. $\square$

Definition 5.11. If $0 < s$ is a rational number, say $s = \frac{a}{b}$ then one defines $n^s := \sqrt[b]{n^a}$ for all $n \in \mathbb{N}$. This does not depend on the representation of s as $\frac{a}{b}$. That is, if we replace $\frac{a}{b}$ by $\frac{ca}{cb}$ (where $c \in \mathbb{N}$), then:

$$\sqrt[cb]{n^{ca}} = \sqrt[b]{\sqrt[c]{n^{ca}}} = \sqrt[b]{n^a}.$$

Also, we have $\sqrt[b]{n^a} = \left(\sqrt[b]{n}\right)^a$.

Example 5.12. $2^{\frac{2}{3}} = \sqrt[3]{4}$

Example 5.13. *If $0 < s = \frac{a}{b} < 1$ is a rational number, then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is divergent*, because Proposition 5.10 applies. That is, for each $n \geq 1$ we have:

$$0 \leq \frac{1}{n^s} = \frac{1}{\left(\sqrt[b]{n}\right)^a} \leq \underbrace{\frac{1}{\left(\sqrt[b]{n}\right)^b}}_{b>a} = \frac{1}{n}.$$

Example 5.14. *On the other hand, we show that if $s > 1$ be a rational number, then $\sum_{k=1}^{\infty} \frac{1}{k^s}$ is convergent.*

Indeed, in this situation:

$$S_n := \sum_{k=1}^{n} \frac{1}{k^s} \leq \sum_{k=1}^{2n+1} \frac{1}{k^s} = 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \frac{1}{(2k+1)^s}$$
$$\leq 1 + \sum_{k=1}^{n} \frac{1}{(2k)^s} + \sum_{k=1}^{n} \frac{1}{(2k)^s} = 1 + \frac{2}{2^s} S_n = 1 + \frac{1}{2^{s-1}} S_n$$

By taking the two ends of this stream of inequalities we obtain:

$$S_n \leq 1 + 2^{1-s} S_n \quad \Leftrightarrow \quad S_n \leq \frac{1}{1 - 2^{1-s}}$$

Hence, S_n is bounded from above. As it is also increasing, it is convergent by Theorem 4.50.

$\longleftarrow$ End of 9. class, on 16.10.2019.

In the above example we have seen that:

Proposition 5.15. *If $x_n \geq 0$, then*

$$\sum_{k=0}^{\infty} x_k = \begin{cases} \textit{convergent} & \textit{if } S_n = \sum_{k=0}^{n} x_k \textit{ is bounded} \\ +\infty \ (\textit{divergent}) & \textit{if } S_n \textit{ is not bounded.} \end{cases}$$

Proof. If $x_n \geq 0$, then S_n is increasing, so Theorem 4.50 yields us the first case. The second case is just a direct consequence of the definition of approaching $+\infty$ (Definition 4.44). □

Example 5.16. According to Example 5.7, $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ is not absolute convergent.

However, it is convergent by the following argument. First let us examine the odd terms of S_n, that is, the sequence S_{2m+1}:

$$S_{2m+1} := -1 + \sum_{k=2}^{2m+1} (-1)^k \frac{1}{k} = -1 + \sum_{k=1}^{m} \left((-1)^{2k} \frac{1}{2k} + (-1)^{2k+1} \frac{1}{2k+1} \right)$$
$$= -1 + \sum_{k=1}^{m} \left(\frac{1}{2k} - \frac{1}{2k+1} \right) = -1 + \sum_{k=1}^{m} \frac{1}{2k(2k+1)}$$
$$\leq -1 + \sum_{k=1}^{m} \frac{1}{2k \cdot 2k} = -1 + \frac{1}{4} \sum_{k=1}^{m} \frac{1}{k^2} \leq \underbrace{-1 + \frac{1}{1 - 2^{1-2}}}_{\text{by Example 5.14}} = 1$$

So, S_{2m+1} is an increasing bounded sequence, hence it is convergent according to Theorem 4.50. Set $S := \lim_{m \to \infty} S_{2m+1}$. The question, is what happens when we put in the even terms. By the definition of our series we have:

$$S_{2m} = S_{2m-1} + \frac{1}{2m}.$$

In particular, we may write the sequence S_n as the sum $S_n = x_n + y_n$ of two other sequence defined by

$$x_n := \begin{cases} S_n & \text{if } n \text{ is odd} \\ S_{n-1} & \text{if } n \text{ is even} \end{cases}, \text{ and } \quad y_n := \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}.$$

Now, x_n converges to S by the following argument: x_n is increasing and its values are the same as those of S_{2m+1}. Hence, any upper bound for S_{2m+1} is automatically an upper bound for x_n too. In particular, x_n is not only increasing but also bounded. Therefore, x_n is convergent by Theorem 4.50 and then it must converge to the same limit as its subsequence S_{2m+1} by Proposition 4.60. This concludes the above claim that $\lim_{n\to\infty} x_n = S$.

By the definition of y_n we also have $\lim_{n\to\infty} y_n = 0$ (we leave the ϵ-details to the reader). Hence, by the algebraic rules of the limit (Proposition 4.26), S_n is convergent too, and

$$\lim_{n\to\infty} S_n = \underbrace{\lim_{n\to\infty}(x_n + y_n)}_{S_n = x_n + y_n} = \underbrace{\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n}_{\text{Proposition 4.26}} = S + 0 = S.$$

Example 5.16 generalizes to:

Proposition 5.17. Leibniz criterion: *if (x_k) is decreasing, and $\lim_{n\to\infty} x_k = 0$, then $\sum_{k=0}^{\infty}(-1)^k x_k$ is convergent.*

Proof. We refer to the book for the proof, but many of the main ideas are already presented in Example 5.16. □

Example 5.18. $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{1}{\sqrt{k}}$ is convergent by Proposition 5.17, because $\lim_{k\to\infty}\frac{1}{\sqrt{k}} = 0$.

So, we have seen in Example 5.16 that $\displaystyle\sum_{k=1}^{\infty}(-1)^k\frac{1}{k}$ is convergent but not absolute convergent. The inverse of this is not possible, that is:

Proposition 5.19. *If $\displaystyle\sum_{n=0}^{\infty} x_n$ is absolute convergent, then it is convergent.*

Proof. If we apply the Cauchy criterion for convergence in both cases, then in the absolute convergence case we have to show that

$$\sum_{k=n+1}^{m} |x_k| \le \varepsilon \tag{5.19.a}$$

and in the case of "usual" convergence

$$\left|\sum_{k=n+1}^{m} x_k\right| \le \varepsilon. \tag{5.19.b}$$

As we have by the triangle equality

$$\left|\sum_{k=n+1}^{m} x_k\right| \le \sum_{k=n+1}^{m} |x_k|,$$

we see that (5.19.a) implies (5.19.b). □

Proposition 5.20. Bernoulli inequality (negative case).
If $-1 < x < 0$, then $(1+x)^n \ge 1 + nx$

Proof. We prove the statement by induction. For $n = 1$, we have $1 + x$ on both sides. So, the initial step is OK. To prove the induction step, assume that we know the statement for n. Then, it is also true for $n + 1$ because of the following computation:

$$(1+x)^{n+1} = (1+x)(1+x)^n \ge (1+x)(1+nx) = (1+(n+1)x + nx^2) \ge 1 + (n+1)x.$$

□

Example 5.21. In the last example, we compute $\sum_{k=0}^{\infty} \frac{1}{k!}$. First, $S_n := \sum_{k=0}^{n} \frac{1}{k!}$ is definitely an increasing sequence. Furthermore, it is bounded, because

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n} \frac{1}{2^k} \leq 1 + \sum_{k=0}^{\infty} \frac{1}{2^k} = 3.$$

So, $\sum_{k=0}^{\infty} \frac{1}{k!}$ is convergent.

This is one of the rare occasions when we are actually able to compute an infinite sum. We will use that we have showed earlier in Example 4.51 that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdot \dots \cdot \left(1 - \frac{k-1}{n}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e, \tag{5.21.c}$$

where the above notation is a bit loose for the $k = 0$ and $k = 1$ terms, as there we mean that the terms with the big parentheses do not exist at all. So the $k = 0$ and $k = 1$ terms are both 1.

Having recalled (5.21.c), we may use it to find the limit of our present sum too:

$$\begin{aligned}
\sum_{k=0}^{n} \frac{1}{k!} \geq \sum_{k=0}^{n} \frac{1}{k!} \left(1 - \frac{1}{n}\right) \cdot \dots \cdot \left(1 - \frac{k-1}{n}\right) &\geq 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(1 - \frac{k-1}{n}\right)^{k-1} \\
\underbrace{\geq 2 + \sum_{k=2}^{n} \frac{1}{k!} \left(1 - \frac{(k-1)^2}{n}\right)}_{\text{Proposition 5.20}} &= \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{n} \sum_{k=2}^{n} \frac{(k-1)^2}{k!} \\
&\geq \sum_{k=0}^{n} \frac{1}{k!} - \frac{1}{n} \sum_{k=2}^{n} \frac{1}{(k-2)!} \geq \sum_{k=0}^{n} \frac{1}{k!} - \frac{3}{n}
\end{aligned}$$

So, we see that $\sum_{k=0}^{\infty} \frac{1}{k!} \geq e$ and that $e \geq \sum_{k=0}^{\infty} \frac{1}{k!} - \lim_{n\to\infty} \frac{3}{n} = \sum_{k=0}^{\infty} \frac{1}{k!}$. Then, it follows that $\sum_{k=0}^{\infty} \frac{1}{k!} = e$

Last there are two theorems in the book that you should know, although for most examples that they are used, squeeze is usually easier to use.

$\longleftarrow$ End of 10. class, on 21.10.2019.

Proposition 5.22. CAUCHY CONVERGENCE/DIVERGENCE CRITERION

(1) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\limsup_{n\to\infty} \sqrt[n]{|x_n|} < 1$, then $\sum_{k=0}^{\infty} x_n$ is absolutely convergent.

(2) If $\left(\sqrt[n]{|x_n|}\right)$ is bounded and $\limsup_{n\to\infty} \sqrt[n]{|x_n|} > 1$ or if $\left(\sqrt[n]{|x_n|}\right)$ is not bounded, then $\sum_{k=0}^{\infty} x_n$ is divergent.

Proposition 5.23. ALEMBERT'S CRITERION

Let (x_n) be a sequence such that $\left(\frac{|x_{n+1}|}{|x_n|}\right)$ is convergent and let the limit be ρ.

(1) if $\rho < 1$, then $\sum_{k=0}^{\infty} x_n$ is absolutely convergent.

(2) if $\rho > 1$, then $\sum_{k=0}^{\infty} x_n$ is divergent.

The ideas behind the proofs. (1) For the convergence statements, for both sequences one tries to do the squeeze given by $0 \le |x_n| \le q$ for some $q < 1$. The only question is what q should one choose. In the case of Cauchy criterion we obtain that if ρ is the limsup, then $\sqrt[n]{|x_n|} \le \rho + \varepsilon$ after finitely many steps, so, then $|x_n| \le (\rho+\varepsilon)^n$. So, we may set $q = \rho + \varepsilon$ in this case. The case of Alembert is similar. There $\frac{x_{n+1}}{x_n} \le \rho + \varepsilon$ after finitely many steps, say after $n \ge n_\varepsilon$. So, we have $|x_n| \le q^{n-n_\varepsilon} |x_{n_\varepsilon}|$ if we set $q = \rho + \varepsilon$ here too.

(2) For the divergence statements one just shows that $|x_n|$ does not converge to 0. In the case of Cauchy there are infinitely many elements with $\sqrt[n]{|x_n|} \ge 1$ which is equivalent to $|x_n| \ge 1$. In the case of Alembert, $\frac{|x_{n+1}|}{|x_n|} > 1$ after finitely many steps, that is, $|x_{n+1}| > |x_n|$.

Example 5.24. It can be proved that $\sum_{k=0}^{\infty} \frac{k}{2^k}$ is convergent using both Proposition 5.22 and Proposition 5.23, as

$$\lim_{k\to\infty} \sqrt[k]{|x_k|} = \lim_{k\to\infty} \sqrt[k]{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{\sqrt[k]{k}}{2} = \frac{\lim_{k\to\infty} \sqrt[k]{k}}{2} = \frac{1}{2},$$

and

$$\lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|} = \lim_{k\to\infty} \frac{\frac{k+1}{2^{k+1}}}{\frac{k}{2^k}} = \lim_{k\to\infty} \frac{k+1}{2k} = \frac{1}{2}.$$

□

6 Real functions of 1-variable

We are going to consider functions $f : E \to \mathbb{R}$ where E is a subset of $\mathbb{R}$. Being a function here means that f assigns a single element $f(x)$ to each $x \in E$. We call E the *domain* of f. The set of values of f:

$$R(f) := \{f(x) \in \mathbb{R} | x \in E\}$$

is called the *range* of f.

Whenever $E' \subset E$ is a smaller set, we can restrict f to E', regard it as a function $f|_{E'} : E' \to \mathbb{R}$. The latter is called the restriction of f over E'.

6.1 Basic properties of functions

If $f, g : E \to \mathbb{R}$ are functions with the same domain, then we say that $f < g$ or $f \le g$ if for all points of E the value of f is smaller/smaller or equal than the value of g. (With formula: for all $x \in E$, we have $f(x) < g(x)$ (resp. $f(x) \le g(x)$)).

If $f : E \to \mathbb{R}$ is a function then the absolute value function $|f|$ is defined by $|f|(x) := |f(x)|$. Geometrically we reflect the negative part across the x-axis.

Example 6.1. Draw the absolute value function of $f(x) = x^3$.

An *even* function is the one the graph of which can be reflected across the y-axis, so $f(x) = f(-x)$.

An *odd* function is the one the graph of which can be reflected across the origin, so $f(x) = -f(-x)$.

Example 6.2. $\cos(x)$ is an even function, and $\sin(x)$ is an odd function. x^3 is an odd function, $|x^3|$ is an even function, but $|x^3 + 1|$ neither even nor odd.

A *periodic* function is one such that $f(x) = f(x+P)$ for some real number $P > 0$ (which we call the period).

Example 6.3. $\cos(x)$ and $\sin(x)$ are periodic with period 2π, and $\{x\}$ is periodic with period 1

The inverse function is the function f^{-1} such that $f^{-1}(x) = y \Leftrightarrow f(y) = x$. Most functions DO NOT have an inverse!!

Example 6.4. $\sin(x)$ does not have an inverse, however $\sin(x)|_{[\frac{-\pi}{2},\frac{\pi}{2}]}$ does have one, which is denoted by $\arcsin(x)$.

A function $f : E \to \mathbb{R}$ is *injective* if for every $x, y \in E$, we have $f(x) \neq f(y)$. With other words every horizontal line intersects the graph of f in at most 1 point. As a function can take only one value at each point, we obtain that f has an inverse if and only if f is injective. Furthermore, $\mathrm{Dom}(f^{-1}) = R(f)$ and $R(f^{-1}) = \mathrm{Dom}(f)$ will hold.

Remark 6.5. To simplify things compared to the book, here in class we always take the target of our functions to be $\mathbb{R}$. The reason is simple: if you take a function $f : E \to F$ where both E and F are subsets of $\mathbb{R}$, then you can always replace the target by $\mathbb{R}$ and obtain a function $f' : E \to \mathbb{R}$ which takes the same value at each point, and hence it is basically the same function. It is not the same function if one reads the definitions of mathematics word by word, but for all "practical" and mathematical purposes f' is the same as f, as you can recover from f' the function f by restricting back the target to F.

In particular, you might find imprecise the above statement about injectivity being equivalent to invertibility, because you might insist that invertibility is equivalent to bijectivity, as stated in the book. If that is the case, then note that there is not much difference practically between injectivity and surjectivity if you are allowed to restrict the target. Indeed, for a function $f : E \to \mathbb{R}$ denote by $\tilde{f} : E \to R(f)$ the function obtained by restricting the range of f to $R(f)$. Then: $\tilde{f}$ is bijective if and only if f is injective.

A number y_0 is an upper (resp. lower) bound for $f : E \to \mathbb{R}$ if y_0 is an upper (resp. lower) bound for the range $R(f)$ of f. This then defines when a function is bounded (resp. bounded from above, from below)

A number y_0 is a supremum (resp. infimum) for $f : E \to \mathbb{R}$ if y_0 is the supremum (resp. infimum) of the range $R(f)$ of f. We write

$$\mathrm{Sup}_{x\in E} f(x), \text{ and } \mathrm{Inf}_{x\in E} f(x)$$

Example 6.6. $\frac{3}{2}$ is an upper bound of $\sin(x)$ and $\cos(x)$, but

$$\mathrm{Sup}_{x\in\mathbb{R}} \cos(x) = 1, \text{ and } \mathrm{Sup}_{x\in\mathbb{R}} \sin(x) = 1.$$

Also,

$$\mathrm{Sup}_{x\in\mathbb{R}_+} 1 - \frac{1}{x} = 1$$

Definition 6.7. A function $f : \mathbb{R} \supseteq E \to \mathbb{R}$ has a *local maximum* (resp., local minimum) at $x_0 \in D$ if there is a real number $\delta > 0$ such that for every $x \in E$ if $|x - x_0| \leq \delta$ then $f(x) \leq f(x_0)$ (resp., $f(x) \geq f(x_0)$).

Example 6.8. (1) $\cos(x)$ has a local maximum at $2k\pi$ and a local minimum at $(2k+1)\pi$ for any $k \in \mathbb{Z}$.

(2) We have not learned yet how to compute, but we will learn that $x(x-1)(x+1) = x^3 - x$ has a local maximum at $x = -\frac{1}{\sqrt{3}}$

(3) We have also not learned yet how to compute, but we will learn that $\sin(x) + \frac{1}{\sqrt{2}}x$ has a local maximum at $2k\pi + \frac{3\pi}{2}$ and a local minimum at $2k\pi + \frac{5\pi}{2}$

A function $f : E \to \mathbb{R}$ is

$$\left.\begin{array}{l}\text{increasing}\\ \text{decreasing}\end{array}\right\} \text{if} \left\{\begin{array}{l}\forall x, y \in E : x \leq y \Rightarrow f(x) \leq f(y)\\ \forall x, y \in E : x \leq y \Rightarrow f(x) \geq f(y).\end{array}\right.$$

Example 6.9. $f(x) = x^3$ is increasing, $f(x) = -x^3$ is decreasing and $f(x) = x^2$ is neither. ⟵ End of 11. class, on 23.10.2019.

6.2 Limits of functions and continuity

The starting point of our investigation is what happens with the function $f(x) := \frac{\sin(x)}{x}$ in $x = 0$. For the first sight, $f(0)$ is not defined so there is nothing to be discussed. However, if we approach $x = 0$ with the sequence $\frac{1}{n}$, then we obtain a limit. Indeed,

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \underbrace{\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right)}_{\text{Exercise 11.b of Sheet 3}} = 1.$$

So, if we set $f(0) = 1$ then maybe $f(x)$ becomes a nice "continuous" function. We will make sense of this below. We will define precisely why $f(0) = 1$ makes $f(x)$ "continuous". But first, we should start with a condition that guarantees that for a function $f : E \to \mathbb{R}$ and for a point $x_0 \in E$ there are enough "test sequences" approaching $x = x_0$, that is, sequences as $\frac{1}{n}$ was above for $x = 0$:

Definition 6.10. A function $f : E \to \mathbb{R}$ is defined on *a pointed neighborhood* of $x_0 \in \mathbb{R}$ if there is an interval of the form $]x_0 - \alpha, x_0 + \alpha[$ contained in $E \cup \{x_0\}$.

Example 6.11. $\frac{\sin(x)}{x}$ is defined on any pointed neighborhood of 0. Indeed, it is defined on $E := \mathbb{R} \setminus \{0\}$, and hence

$$]-\alpha, +\alpha[\subseteq E \cup \{0\} = \mathbb{R}.$$

Definition 6.12. Assume $f : E \to \mathbb{R}$ is defined on a pointed neighborhood of $x_0 \in \mathbb{R}$. Then, $\lim\limits_{x\to x_0} f(x) = l$ if one of the following two equivalent definitions holds:

(1) For every $0 < \varepsilon \in \mathbb{R}$ there is a $0 < \delta \in \mathbb{R}$ such that:

$$\forall x \in E : 0 < |x - x_0| \leq \delta \Rightarrow |f(x) - l| \leq \varepsilon.$$

(2) For every sequence $(x_n) \subset E \setminus \{x_0\}$ for which $\lim\limits_{n\to\infty} x_n = x_0$, we have $\lim\limits_{n\to\infty} f(x_n) = l$.

With everyday language the two definitions mean the following:

(1) whenever x is close to x_0, $f(x)$ is also close to l (More precisely: for every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is closer to x_0 than δ then $f(x)$ is closer to l than ε.)

(2) whenever (x_n) converges to x_0, then $(f(x_n))$ converges to l.

We comment a bit about why the above two definitions are equivalent.

- (1) $\Rightarrow$ (2): Take a sequence (x_n) for which $\lim\limits_{n\to\infty} x_n = x_0$. We have to show that $\lim\limits_{n\to\infty} f(x_n) = l$. So, fix $\varepsilon > 0$. Then, this yields a $\delta > 0$ as in definition (i). For this δ, there is an n_δ such that $|x_0 - x_n| \leq \delta$ for $n \geq n_\delta$, and hence for all such n, $|l - f(x_n)| < \varepsilon$.

- NOT (1) $\Rightarrow$ NOT (2): The negation of (i) is that there is an $\varepsilon > 0$ such that for each $\delta > 0$ there is an $y_\delta \in [x_0 - \delta, x_0 + \delta]$ such that $|f(y_\delta) - l| > \varepsilon$. So, $x_n := y_{\frac{1}{n}}$ converges to x_0, but all $f(x_n)$ have distance at least ε from l, so $(f(x_n))$ does not converge to l.

In general, we try to use definition (2) more as it is simpler. Luckily, it is almost always enough for proving that a limit does not exist. We usually use definition (1) only when (2) does not work.

Example 6.13. We show that $\lim_{x\to 2} x^2 = 4$ using point (1) of Definition 6.12. For this we have to do the following: for any $\varepsilon > 0$ we have to give $\delta > 0$ such that

$$|x - 2| \le \delta \Rightarrow |x^2 - 4| \le \varepsilon.$$

For that note that

$$(x - 2)(x + 2) = x^2 - 4.$$

Furthermore, if $|x - 2| \le 1$, then $5 \ge x + 2 \ge 3$, and so

$$|x^2 - 4| = |x - 2||x + 2| \le 5|x - 2|. \tag{6.13.a}$$

In particular, $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$ works, because if

$$|x - 2| \le 1, \text{ and } |x - 2| \le \frac{\varepsilon}{5},$$

then we have

$$\underbrace{|x^2 - 4| \le 5|x - 2|}_{(6.13.a)} \le 5\frac{\varepsilon}{5} = \varepsilon.$$

Example 6.14. The above example is not hard to show also with point(2) of Definition 6.12 (which usually does not work for trickier limits such as $\frac{\sin(x)}{x}$ at $x = 0$). We verify this here too: assume that (x_n) is a sequence such that $\lim_{n\to\infty} x_n = 2$. Then, by the algebraic properties of the limit, that is, by Proposition 4.26, we know that $\lim_{n\to\infty} x_n^2 = 2^2 = 4$.

Example 6.15. Similar argument as above shows that for any $x_0 \in \mathbb{R}$, $\lim_{x\to x_0} x^n = x_0^n$ for any $n \in \mathbb{N}$.

Definition 6.16. $f : E \to \mathbb{R}$ is *continuous* at $x_0 \in E$, if $\lim_{x\to x_0} f(x) = f(x_0)$ (including that the limit of $f(x)$ at x_0 exists).

Note that the main difference between having a limit and being continuous is that in the latter case $f(x_0)$ agrees with the limit. In particular, if we want to test continuity with conditions as in Definition 6.12, then we may drop the restrictions there made about not checking things at x_0. More precisely, we have the following (compare with Definition 6.12):

Proposition 6.17. *Assume $f : E \to \mathbb{R}$ is defined on a ball around $x_0 \in \mathbb{R}$. Then, f is continuous at x_0 if and only if one of the following two equivalent definitions hold:*

(1) For every $0 < \varepsilon \in \mathbb{R}$ there is a $0 < \delta \in \mathbb{R}$ such that:

$$\forall x \in E : |x - x_0| \le \delta \Rightarrow |f(x) - f(x_0)| \le \varepsilon.$$

(2) For every sequence $(x_n) \subset E$ for which $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Example 6.15 yields:

Corollary 6.18. *If $n \in \mathbb{N}$, then $f(x) = x^n$ is continuous at every $x_0 \in \mathbb{R}$. Or, using the next definition, we can say $f : \mathbb{R} \to \mathbb{R}$ is continuous.*

Definition 6.19. If $E \to \mathbb{R}$ is such that for each $x_0 \in E$, E contains a ball around x_0 (for example E is an open interval), then we say that f *is continuous* if it is continuous at every $x_0 \in E$.

Example 6.20. Set
$$f(x) := \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$
Then we have $\lim_{x\to 0} f(x) = 0$, as in the definition we assumed $0 < |x - x_0| \leq \delta$, so the function value 1 for $x_0 = 0$ does not cause any problem.

Example 6.21. We claim that $\lim_{x\to 0} \cos(x) = 1$. Indeed, let (x_n) be a sequence converging to 0. Then,
$$0 \leq |\cos(x_n) - 1| = \left|2\sin^2\left(\frac{x_n}{2}\right)\right| \leq 2\frac{x_n^2}{4} = \frac{x_n^2}{2},$$
using the inequality $|\sin(x)| \leq |x|$. So, squeeze theorem tells us that $\lim_{n\to\infty} |\cos(x_n) - 1| = 0$.

Example 6.22. (1) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$
and let (x_n) and (y_n) be two sequences defined by
$$x_n := \frac{1}{n}, \text{ and } y_n := \frac{\sqrt{2}}{n}.$$
Then:
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} f(x_n) = 1, \text{ but } \lim_{n\to\infty} f(y_n) = -1$$
So, we exhibited two sequences converging to 0, for which the associated function value sequences have different limits. In particular, the limit of f at $x_0 = 0$ does not exist.

(2) In fact, one can show that the example of the previous point is not continuous at any $x_0 \in \mathbb{R}$. The main idea is that one can find using Proposition 2.10 and Proposition 2.11 sequences $(x_n) \subseteq \mathbb{Q}$ and $(y_n) \subseteq \mathbb{R} \setminus \mathbb{Q}$ that converge to x_0. Then, $f(x_n) = 1$ and $f(y_n) = -1$, so the limits are also 1 and -1 respectively which contradicts point (2) of Proposition 6.17.

Example 6.23. *We show that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.* Indeed, consider the sequences $x_n := \frac{1}{\pi\left(2n+\frac{1}{2}\right)}$ and $x_n' := \frac{1}{\pi\left(2n+\frac{3}{2}\right)}$. Then, first $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} x_n' = 0$. However,
$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{1}{2}\right)}}\right) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) = 1,$$
but
$$\lim_{n\to\infty} \sin\left(\frac{1}{x_n'}\right) = \sin\left(\frac{1}{\frac{1}{\pi\left(2n+\frac{3}{2}\right)}}\right) = \sin\left(\pi\left(2n + \frac{3}{2}\right)\right) = -1.$$
So, point (2) of Definition 6.12 is not satisfied, and hence the limit does not exist

6.2.1 Limit and algebra

Point (2) of Definition 6.12 creates the possibility of moving all the statements about limits of sequences to limits of functions. Indeed, say we are working at x_0, and we want to prove that if l and k are the limits of $f(x)$ and $g(x)$ (at x_0), then $l+k$ is the limit of $(f+g)(x)$. So take a sequence (x_n) converging to x_0. We know that $\lim_{n\to\infty} f(x_n) = l$ and $\lim_{n\to\infty} g(x_n) = k$. But then we have learned about sequences that $\lim_{n\to\infty} f(x_n) + f'(x_n) = l + k$, which exactly shows that $\lim_{n\to\infty}(f+f')(x_n) = l+k$.

We collect all the statements one can show along the same arguments:

Proposition 6.24. *Let f and g be two functions such that a pointed neighborhood of x_0 is in the domain of both f and g. Assume that the limits of f and g at x_0 exist and they are l and k, respectively. Then,*

(1) the limit of $f+g$ exists at x_0 and $\lim_{x\to x_0}(f+g)(x) = l+k$

(2) the limit of $f\cdot g$ exists at x_0 and $\lim_{x\to x_0}(f\cdot g)(x) = l\cdot k$

(3) if $k \neq 0$, then the limit of $\frac{f}{g}$ exists at x_0 and $\lim_{x\to x_0}\left(\frac{f}{g}\right)(x) = \frac{l}{k}$

(4) if $f(x) \leq g(x)$ for any x in a pointed neighborhood of x_0, then $l \leq k$.

(5) SQUEEZE: if there is a third function $h(x)$ such that there is also a pointed neighborhood of x_0 in the domain of h, and:

(i) on some pointed neighborhood of x_0 we have $f(x) \leq h(x) \leq g(x)$, and

(ii) $l = k$,

then $\lim_{x\to x_0} h(x) = l$.

Example 6.25. The main example for using point (5) of Proposition 6.24 is that $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$. Indeed, by geometric reasons we have

$$\frac{\sin(x)}{x} \leq 1, \text{ and } \operatorname{tg}(x) \geq x \Rightarrow \frac{\sin(x)}{x} \geq \cos(x).$$

So, squeeze tells us the limit, as

$$\cos(x) \leq \frac{\sin(x)}{x} \leq 1.$$

Example 6.26. $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$. Indeed, we can squeeze it with

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

The above proposition has all the nice consequences about continuity:

Proposition 6.27. *If $f, g : E \to \mathbb{R}$ are continuous functions at $x_0 \in E$, then so are*

(1) $\alpha f + \beta g$ for any $\alpha, \beta \in \mathbb{R}$,

(2) $f \cdot g$, and

(3) if $g|_E$ is nowhere zero (meaning that for all $x \in E : g(x) \neq 0$), then $\frac{f}{g}$ too.

Example 6.28. ◦ if $p(x)$ is a polynomial of 1 variable, so something of the form $a_0 + a_1x + a_2x^2 + \cdots + a_rx^r$, then $p(x)$ is continuous.

◦ $f(x) := \frac{1}{x}$ is continuous on $\mathbb{R} \setminus 0$.

◦ $\frac{x}{x^2-3x+1}$ is continuous on $\mathbb{R} \setminus \left\{ \frac{3\pm\sqrt{5}}{2} \right\}$,

◦ In general, if $p(x)$ and $q(x)$ are two polynomials, then $\frac{p(x)}{q(x)}$ is continuous on $\{x \in \mathbb{R} | q(x) \neq 0\}$ (which is the whole real line minus finitely many points). ⟵ End of 12. class, on 28.10.2019.

6.2.2 Limit and composition

Let us recall that:

Definition 6.29. If $f : E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ are functions such that $R(f) \subseteq G$ then we may define the *composition* $g \circ f$ (order matters!!) of f with g by

$$(g \circ f)(x) = g(f(x)).$$

Example 6.30. Let us take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$, and $g : \mathbb{R} \to \mathbb{R}$ given by $g(y) = y^3 + y + 1$. Then we have

$$(g \circ f)(x) = (x^2 + 1)^3 + (x^2 + 1) + 1 = x^6 + 3x^4 + 4x^2 + 3. \tag{6.30.b}$$

Let us look at an example about whether composition of continuous functions is continuous or not.

Example 6.31. Consider the functions defined in Example 6.30. *We want to show that $g \circ f$ is continuous at $x = 0$.* As $(g \circ f)(0) = 3$ by (6.30.b), for this we have to show that $\lim_{x\to 0} g \circ f(x) = 3$. However, this limit statement is immediate from equation (6.30.b). So, indeed $g \circ f$ is continuous at $x = 0$.

In general the situation is just as nice as in Example 6.31:

Proposition 6.32. *If $f : E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be two functions, such that*

(1) $f(E) \subseteq G$,

(2) f is continuous at x_0,

(3) g is continuous at $y_0 := f(x_0)$,

then $g \circ f$ is continuous at x_0.

Proof. We verify condition (2) of Proposition 6.17. Let $(x_n) \subseteq E$ be a sequence such that

$$\lim_{n\to\infty} x_n = x_0. \tag{6.32.c}$$

According to (6.32.c) and our assumption (2), we have:

$$\lim_{n\to\infty} f(x_n) = y_0. \tag{6.32.d}$$

Hence

$$\lim_{n\to\infty} (g \circ f)(x_n) = \underbrace{\lim_{n\to\infty} g(f(x_n)) =}_{\text{Definition 6.29}} \underbrace{g(y_0)}_{\text{(6.32.d) and condition (3)}}.$$

□

Let us examine a bit further Example 6.31. In the proof of Proposition 6.32 we showed that if

$$\lim_{x \to x_0} f(x) = y_0 \qquad \text{and} \qquad \lim_{y \to y_0} g(y) = l, \tag{6.32.e}$$

then $\lim_{x \to x_0} g \circ f(x) = l$ holds under the assumption that f and g are continuous. We would be tempted to say that the same holds if we only assume the limit conditions of (6.32.e). However, as we will see it in Example 6.34, this does not work. The reason is that in Definition 6.12, contrary to Proposition 6.17, there is nothing said about the behavior at x_0 and y_0. So, we have to assume that $f(x)$ avoids y_0 in a pointed neighborhood of x_0. The precise statement about composition of functions is as follows:

Proposition 6.33. *Let $f : E \to \mathbb{R}$ and $g : G \to \mathbb{R}$ be functions and let $x_0 \in E$ be a point such that*

(1) $f(E) \subset G$,

(2) $\lim_{x \to x_0} f(x_0) = y_0$,

(3) $\lim_{y \to y_0} g(y_0) = l$

(4) there is a pointed neighborhood $B(x_0, r) \setminus \{x_0\} \subset E$ such that for every x in this neighborhood, $f(x) \neq y_0$.

Then: $\lim_{x \to x_0} (g \circ f)(x) = l$

Proof. We verify point (2) of Definition 6.12. So, take a sequence $(x_n) \subset E \setminus \{x_0\}$ such that

$$\lim_{n \to \infty} x_n = x_0. \tag{6.33.f}$$

In particular, by throwing away finitely many elements of the sequence, we may assume that

$$(x_n) \subset B(x_0, r) \setminus \{x_0\} \tag{6.33.g}$$

(note that $B(x_0, r) \setminus \{x_0\} \subset E \setminus \{x_0\}$ by our assumption (4)). By our assumption (2) and by (6.33.f), we have

$$\lim_{n \to \infty} f(x_n) = y_0. \tag{6.33.h}$$

Lastly, by our assumption (4) and (6.33.g) we have

$$(f(x_n)) \subset G \setminus \{y_0\}. \tag{6.33.i}$$

Hence, by our assumption (3) and by (6.33.i), we have

$$\lim_{n \to \infty} (g \circ f)(x_n) = \lim_{n \to \infty} g(f(x_n)) = l.$$

□

The following is the example that condition (4) of Proposition 6.33 is necessary. That is, if we did not have condition (4), the statement of Proposition 6.33 would not hold.

Example 6.34. Consider:

$$g(x) = \begin{cases} 0 & \text{, for } x \neq 0 \\ 1 & \text{, for } x = 0 \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{, for } x \neq 0 \\ 0 & \text{, for } x = 0 \end{cases}$$

Then, $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$. However, $\lim_{x\to 0}(g\circ f)(x) \neq 0$, because the following two sequences induce function value sequences with different limits:

$$x_n := \frac{1}{\pi n} \qquad \text{and} \qquad y_n := \frac{1}{\pi n + \frac{\pi}{2}},$$

as

$$\lim_{n\to\infty} (g\circ f)(x_n) = \lim_{n\to\infty} 1 = 1 \qquad \text{and} \qquad \lim_{n\to\infty} (g\circ f)(y_n) = \lim_{n\to\infty} 0 = 0.$$

Also, note that condition (4) of Proposition 6.33 below is not satisfied in this example, as $f(x) = 0$ for $x = \frac{1}{\pi n}$, so there is no pointed neighborhood of 0 such that the function value avoids 0.

Example 6.35. A positive example for applying Proposition 6.33 is during the argument of showing that $\lim_{x\to 0} \dfrac{\sin(x^2)}{x^2} = 1$. Indeed, if we set $g(x) := \frac{\sin(x)}{x}$, and $f(x) = x^2$, then condition (4) of Proposition 6.33 is also satisfied, as $f(x) \neq 0$ for $x \neq 0$.

$\longleftarrow$ End of 13. class, on 30.10.2019.

6.2.3 Infinite limits

Definition 6.36. A *(pointed) neighborhood of* $+\infty$ (resp. $-\infty$) is an interval of the form $]a, +\infty[$ (resp. $]-\infty, a[$).

Definition 6.37. Let x_0 and l be either a real number, or $+\infty$, or $-\infty$, and let $f : E \to \mathbb{R}$ be a function such that a pointed neighborhood of x_0 is contained in E. We say that the limit of $f(x)$ at x_0 is l, which we denote by $\lim_{x\to x_0} f(x) = l$, if for each sequence $(x_n) \subset E \setminus \{x_0\}$, whenever $\lim_{n\to\infty} x_n = x_0$ we have $\lim_{n\to\infty} f(x_n) = l$.

Example 6.38. *We claim that* $\lim_{x\to 0} \frac{1}{x^2} = +\infty$. Indeed, if $\lim_{n\to\infty} x_n = 0$, then $\lim_{n\to\infty} \frac{1}{x_n^2} = +\infty$ according to the algebraic properties of limits of sequences.

Example 6.39. *We claim that* $\lim_{x\to 0} \frac{1}{x}$ *does not exist*, as for $x_n = \frac{1}{n}$ we have $\lim_{n\to\infty} \frac{1}{x_n} = \lim_{n\to\infty} n = +\infty$, and for $y_n = \frac{-1}{n}$ we have $\lim_{n\to\infty} \frac{1}{y_n} = \lim_{n\to\infty} -n = -\infty$.

Proposition 6.40. *Let x_0 be either a real number, or $\pm\infty$, and let $f, g : E \to \mathbb{R}$ be functions.*

(1) Addition rule. *If*

- $\lim_{x\to x_0} f(x) = +\infty$ *(resp. $-\infty$), and*
- $g(x)$ *is bounded from below (resp. above)*

then $\lim_{x\to x_0} (f+g)(x) = +\infty$ *(resp. $-\infty$).*

(2) Product rule. *If*

- $\lim_{x\to x_0} |f(x)| = +\infty$,
- $|g(x)|$ *is bounded from below by a positive number (that is, there is a $\delta > 0$ such that $|g(x)| \geq \delta$ for all $x \in E$), and*
- $f(x)g(x) > 0$ *(resp. < 0) for all $x \in E$,*

then $\lim_{x\to x_0} f(x)g(x) = +\infty$ *(resp. $-\infty$).*

(3) First division rule. *If*

- $f(x)$ *is bounded,*
- $g(x)$ *is nowhere zero, and*
- $\lim_{x\to x_0} |g(x)| = +\infty$.

Then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

(4) Second division rule. *If*

- $\lim_{x\to x_0} g(x) = 0$,
- $|f(x)|$ *is bounded from below by a positive number (that is, there is a* $\delta > 0$ *such that* $|f(x)| \geq \delta$ *for all* $x \in E$*), and*
- $f(x)/g(x) > 0$ *(resp.* < 0*) for all* $x \in E$,

then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = +\infty$ *(resp.* $-\infty$*).*

(5) Squeeze. *If* $f(x) \leq g(x)$*, and*

- *if* $\lim_{x\to x_0} f(x) = +\infty$*, then* $\lim_{x\to x_0} g(x) = +\infty$
- *if* $\lim_{x\to x_0} g(x) = -\infty$*, then* $\lim_{x\to x_0} f(x) = -\infty$

Example 6.41. Point (1) of Proposition 6.40 can be used to show:

- $\lim_{x\to 0} \underbrace{\frac{1}{x^2}}_{\to\infty} + \underbrace{\cos(x)}_{\text{bounded}} = +\infty$
- $\lim_{x\to +\infty} \underbrace{\cos(x)}_{\text{bounded}} \underbrace{-x}_{\to -\infty} = -\infty$.

Example 6.42. The assumptions in point (1) of Proposition 6.40 are important, as otherwise we can have all different kinds of limits. We give examples of this using the functions $f(x) = x^3$, $g(x) = x^2$ and $h(x) = x^3 + 1$. We have

- $\lim_{x\to +\infty} \pm f(x) = \lim_{x\to +\infty} \pm x^3 = \pm\infty$,
- $\lim_{x\to +\infty} \pm h(x) = \lim_{x\to +\infty} \pm (x^3 + 1) = \pm\infty$, and
- $\lim_{x\to +\infty} \pm g(x) = \lim_{x\to +\infty} \pm x^2 = \pm\infty$.

On the other hand:

- $\lim_{x\to +\infty} f(x) - g(x) = \lim_{x\to +\infty} x^3 - x^2 = \lim_{x\to +\infty} x^2(x-1) = +\infty$,
- similarly $\lim_{x\to +\infty} g(x) - f(x) = -\infty$, and
- $\lim_{x\to +\infty} f(x) - h(x) = -1$.

In particular, never use addition law for limits of the type $(+\infty) + (-\infty)$.

Example 6.43. The above assumptions for point (2) of Proposition 6.40 are also important. We give examples of this using the functions $f(x) = x$, $g(x) = \frac{\cos(x)}{x}$ and $h(x) = (-1)^{[x]}$. We have

- $\lim_{x \to +\infty} |f(x)| = +\infty$,
- $|g(x)|$ is not bounded from below, and
- $|h(x)|$ is bounded from below, but $f(x)h(x) \not\to 0$.

Then:

- $\lim_{x \to +\infty} f(x)g(x) = \lim_{x \to +\infty} \cos(x)$ does not exist, and
- $\lim_{x \to +\infty} f(x)h(x) = \lim_{x \to +\infty} x(-1)^{[x]}$ does not exist, on the other hand
- the product law applies to $(f(x)h(x))(h(x))$ and yields $\lim_{x \to +\infty} (f(x)h(x))(h(x)) = +\infty$

Never try to use product rule to limits of the type $0 \cdot \infty$.

Example 6.44. The assumptions of the first division rule are also important. One can can show that in the $\frac{\pm\infty}{\pm\infty}$ case anything can happen for example using $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$,$(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}$ with limit at 0:

(1) $\lim_{x \to 0} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x \to 0} x = 0$,

(2) $\lim_{x \to 0} \frac{(-1)^{\left[\frac{1}{x}\right]}\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to 0} (-1)^{\left[\frac{1}{x}\right]}$ does not exist and bounded,

(3) $\lim_{x \to 0} \frac{\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x \to 0} \frac{1}{x}$ does not exist and unbounded, and

(4) $\lim_{x \to 0} \frac{\frac{1}{x^3}}{\frac{1}{x}} = \lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Similar examples show that the assumptions are important for the second division rule. Never try to use division rules to limits of the form $\frac{\pm\infty}{\pm\infty}$ and $\frac{0}{0}$.

6.2.4 One sided limits

The main question is how to make sense of limits such as at 0 of $\sqrt{x}$, as here the domain does not contain a pointed neighborhood of 0. The solution for this is the introduction of the notions of left and right limits.

Definition 6.45. A function $f : E \to \mathbb{R}$ is defined on the left (resp. right) of $x_0 \in E$, if E contains an interval of the form $]x_0 - \delta, x_0[$ (resp. $]x_0, x_0 + \delta[$).

Definition 6.46. Let $f : E \to \mathbb{R}$ be a function, such that $x_0 \in E$, and f is defined on the left (resp. right) of x_0. Let l be either a real number or $\pm\infty$. Then, $\lim_{x \to x_0^-} f(x) = l$ (resp. $\lim_{x \to x_0^+} f(x) = l$) if for all sequences $(x_n) \subset \{x \in E | x < x_0\}$ (resp. $(x_n) \subset \{x \in E | x > x_0\}$) we have:

$$\lim_{n \to \infty} x_n = x_0 \quad \Rightarrow \quad \lim_{n \to \infty} f(x_n) = l.$$

Example 6.47. Consider $f(x) := \sqrt{x} : \mathbb{R}_+ \to \mathbb{R}$. *We claim that* $\lim_{x \to 0^+} \sqrt{x} = 0$.

Indeed, fix a sequence $(x_n) \subset \mathbb{R}_+^*$ such that $\lim_{n \to \infty} x_n = 0$. We have to show that then $\lim_{n \to \infty} \sqrt{x_n} = 0$ too. So, we need to show that for each $\varepsilon > 0$, there is an n_0 such that for every integer $n \geq n_0$, $\sqrt{x_n} \leq \varepsilon$. However, we know that $\lim_{n \to \infty} x_n = 0$. So, we know that there is an n_0 such that $|x_n| < \varepsilon^2$ for all $n \geq n_0$. But then, for any such n we also have $\sqrt{x_n} < \varepsilon$.

Proposition 6.48. *Let $f : E \to \mathbb{R}$ be a function such that there is a pointed neighborhood of x_0 contained in E, and both*

$$l_1 := \lim_{x \to x_0^-} f(x) \qquad \text{and} \qquad l_2 := \lim_{x \to x_0^+} f(x)$$

exists. Then

$$\lim_{x \to x_0} f(x) \text{ exists } \Leftrightarrow l_1 = l_2.$$

Furthermore, if $l_1 = l_2$, then $\lim\limits_{x \to x_0} f(x)$ agrees with this common value.

Example 6.49. Consider the function $f(x) = \{x\}$. Both left and right limits exist at all points, and furthermore:

$$\lim_{x \to x_0^-} \{x\} = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z} \\ 1 & \text{if } x \in \mathbb{Z} \end{cases} \qquad \text{and} \qquad \lim_{x \to x_0^+} \{x\} = \begin{cases} \{x\} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

Hence, according to Proposition 6.48,

$$\lim_{x \to x_0} \{x\} \text{ exists } \Leftrightarrow x \notin \mathbb{Z}.$$

6.2.5 Monotone functions

For monotone functions seemingly left and right limits always exist:

Example 6.50. (1) Let

$$f(x) := \operatorname{sgn}(x) = \begin{cases} 1 & \text{, if } x > 0 \\ 0 & \text{, if } x = 0 \\ -1 & \text{, if } x < 0 \end{cases}$$

Then

$$\lim_{x \to x_0^-} \operatorname{sgn}(x) = -1 \qquad \text{and} \qquad \lim_{x \to x_0^+} \operatorname{sgn}(x) = 1$$

Note that these limits exist and neither of them agree with $f(0) = 0$.

(2) Let $f(x) := [x]$. Then:

$$\lim_{x \to x_0^-} [x] = \begin{cases} [x] & \text{if } x \notin \mathbb{Z} \\ x - 1 & \text{if } x \in \mathbb{Z} \end{cases} \qquad \text{and} \qquad \lim_{x \to x_0^+} [x] = \begin{cases} [x] & \text{if } x \notin \mathbb{Z} \\ x & \text{if } x \in \mathbb{Z} \end{cases}$$

So, the left and right limits exist, despite having different values whenever $x \in \mathbb{Z}$.

Proposition 6.51. *If $f : E \to \mathbb{R}$ is monotone, then at each point $x_0 \in E$:*

(1) if f is defined on the left of x_0, $\lim\limits_{x \to x_0^-} f(x)$ exists,

(2) if f is defined on the right of x_0, $\lim\limits_{x \to x_0^+} f(x)$ exists, and

(3) if f is defined in a neighborhood of $\pm\infty$, then $\lim\limits_{x \to \pm\infty} f(x)$ exists.

Proof. We treat only the increasing case, as the decreasing one follows from that by regarding $-f$ instead of f. Also, we treat only the first case as the others are similar. Set:

$$l := \operatorname{Sup}\{f(x) | x \in E, x < x_0\}. \tag{6.51.j}$$

Let

$$(x_n) \subset \{x \in E | x < x_0\} \quad \text{such that} \quad \lim_{n \to \infty} x_n = x_0. \tag{6.51.k}$$

We have to show that $\lim_{n\to\infty} f(x_n) = l$. Fix a $\varepsilon > 0$. Then, by the definition of l, there is an $x' \in \{x \in E | x < x_0\}$, such that

$$f(x') > l - \varepsilon. \tag{6.51.l}$$

According to (6.51.k), there is an $n_0 \in \mathbb{N}$ such that for all integers $n \geq n_0$ we have

$$x' < x_n < x_0. \tag{6.51.m}$$

However, then for all integers $n \geq n_0$ we have:

$$l \geq \underbrace{f(x_n)}_{(6.51.j) \text{ and } (6.51.m)} \geq \underbrace{f(x')}_{f \text{ is increasing and } (6.51.m)} \geq \underbrace{l - \varepsilon.}_{(6.51.l)}$$

This shows that $\lim_{n\to\infty} f(x_n) = l$ indeed.

□

6.2.6 More on continuity

First, we note that there are more algebraic rules of continuity (we already discussed addition, multiplication and division in Proposition 6.27):

Proposition 6.52. *If $f, g : E \to \mathbb{R}$ are functions that are continuous at $x_0 \in E$, then so are:*

(1) $|f|$,

(2) $\max\{f, g\}$, *where*

$$\max\{f, g\}(x) := \max\{f(x), g(x)\}$$

(3) $\min\{f, g\}$ *(defined similarly),*

(4) $f^+ := \max\{f, 0\}$,

(5) $f^- := \min\{f, 0\}$.

Example 6.53. We can use for example the continuity of the absolute value for squeezing. For example, let

$$g(x) := \begin{cases} 1 & \text{, for } x \in \mathbb{Q} \\ x & \text{, for } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

We claim that $g(x)$ is continuous at $x_0 = 1$. The main idea is that u we can squeeze $f(x) - 1$:

$$-|x-1| \leq f(x) - 1 \leq |x-1|.$$

According to point (1) of Proposition 6.52, so by the continuity of the absolute value, we have

$$\lim_{x\to 1} -|x-1| = \lim_{x\to 1} |x-1| = 0,$$

by squeeze (point (5) of Proposition 6.24) we obtain that $\lim_{x\to 1} f(x) - 1 = 0$, and hence $\lim_{x\to 1} f(x) = 1 = f(1)$. Hence, f *is continuous at* $x_0 = 1$. ⟵ End of 14. class, on 04.11.2019.

Second, we introduce a stronger version of continuity:

Definition 6.54. A function $f : E \to \mathbb{R}$ is *uniformly* continuous if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in E$ we have:

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon.$$

Proposition 6.55. *If $f : E \to \mathbb{R}$ is uniformly continuous then it is continuous.*

Example 6.56. *The function $f(x) := x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous (but it is continuous* as it is a polynomial). Indeed, we have:

$$|x^2 - y^2| = |x + y| \cdot |x - y|.$$

So, for any $\varepsilon > 0$ and $\delta > 0$ we may find x and y such that $|x+y| > \frac{2\varepsilon}{\delta}$, and $|x-y| = \frac{\delta}{2}$ (that is, chose two numbers that are very big but very close to each other). However, then $|x - y| \leq \delta$, but

$$|x^2 - y^2| > \frac{2\varepsilon}{\delta}\frac{\delta}{2} = \varepsilon.$$

Example 6.57. *We show that $\cos(x) : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and hence continuous.* Indeed,

$$|\cos(x) - \cos(y)| = 2\left|\sin\left(\frac{x+y}{2}\right)\right|\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2\left|\sin\left(\frac{x-y}{2}\right)\right| \leq 2|x-y|.$$

So, if we set $\delta = \frac{\varepsilon}{2}$, then we have

$$|x-y| \leq \delta \Rightarrow |\cos(x) - \cos(y)| \leq 2|x-y| \leq 2\delta = 2\frac{\varepsilon}{2} = \varepsilon$$

Example 6.57 together with Proposition 6.32 implies that functions such as $\cos(x^2)$, $\cos^2(x)$, etc. are continuous.

Lastly, we introduce left and right continuity, and we use this to define continuity on a closed interval.

Definition 6.58. Let $f : E \to \mathbb{R}$ be a function, and $x_0 \in E$.

(1) f *is left continuous at* x_0, if $\lim\limits_{x \to x_0^-} f(x) = f(x_0)$.

(2) f *is right continuous at* x_0, if $\lim\limits_{x \to x_0^+} f(x) = f(x_0)$.

In Definition 6.19 we defined what it means to be continuous on an open interval. For functions the domains of which are closed intervals the definition has to use left and right limits as well at the two endpoints:

Definition 6.59. A function $f : [a, b] \to \mathbb{R}$ is *continuous* if it is continuous at c for all $a < c < b$, it is left continuous at b and right continuous at a.

Example 6.60. $\sqrt{1-x^2} : [-1, 1]$ is continuous. Indeed this is true by the following (where we use that $g(y) = \sqrt{y}$ is continuous on $\mathbb{R}_+^*$, which will be a consequence of our general theorem about the continuity of the inverse. Indeed, by applying the statement of Theorem 6.73 to $f(x) = x^2$ one obtains that $f^{-1} = g$ is continuous on $\mathbb{R}_+^*$):

(1) if $-1 < c < 1$, then $\sqrt{1-x^2}$ at c is continuous because $\sqrt{1-x^2}$ is the composition of $\sqrt{y}$ and $1 - x^2$, and the latter is continuous at c and the former is continuous at $1 - c^2$ (as $1 - c^2 > 0$).

(2) $\sqrt{1-x^2}$ is left continuous at 1, because for all (x_n) converging to 1 from the left we have $\lim\limits_{n\to\infty} \sqrt{1-x_n^2} = 0$, as $\lim\limits_{n\to\infty} 1 - x_n^2 = 0$, and $\lim\limits_{x\to 0^+} \sqrt{y} = 0$ according to Example 6.47.

(3) $\sqrt{1-x^2}$ is right continuous at -1 by almost verbatim the same argument as the previous point, one only needs to take $\lim\limits_{n\to\infty} x_n = -1$ instead of 1.

6.2.7 Consequences of Bolzano-Weierstrass

Continuous functions on bounded closed intervals behave nicely. Theorem 6.61 states that they attain their minima and maxima, and Theorem 6.63 states that they are uniformly continuous.

Theorem 6.61. *If $f : [a,b] \to \mathbb{R}$ is continuous for some $a, b \in \mathbb{R}$, then there are $c, d \in [a,b]$ such that*

$$M := \mathrm{Sup}_{x\in[a,b]} f(x) = f(c) \qquad m := \mathrm{Inf}_{x\in[a,b]} f(x) = f(d).$$

In particular the above Sup *and* Inf *is* Max *and* Min*, respectively.*

Proof. We only prove the statement for Sup, as the proof for Inf is verbatim the same, only some of the signs has to be reversed.

First we prove that f is bounded from above, and hence Sup does make sense. Assume that f is not bounded from above. That is, for each integer $n > 0$ there is $x_n \in [a,b]$ such that $f(x_n) \geq n$. According to Bolzano-Weierstrass (Theorem 4.62), there is a convergent subsequence $(x_{n_k}) \subset (x_n)$. Set $c := \lim\limits_{k\to\infty} x_{n_k}$. Then $c \in [a,b]$, and the following stream of equalities yields a contradiction:

$$\mathbb{R} \ni f(c) = \underbrace{\lim_{k\to\infty} f(x_{n_k})}_{f\,:\,[a,b]\to\mathbb{R} \text{ is continuous}} = \underbrace{+\infty}_{f(x_{n_k})\geq n_k \geq k} .$$

This concludes the statement that f is bounded from above.

Having proved that f is bounded from above, $\mathrm{Sup}_{x\in[a,b]} f(x)$ makes sense. In what follows we prove that $\mathrm{Sup}_{x\in[a,b]} f(x) = \mathrm{Max}_{x\in[a,b]} f(x)$: there is a sequence $(x_n) \subset [a,b]$ such that $f(x_n) \geq M - \frac{1}{n}$. In particular, $\lim\limits_{n\to\infty} f(x_n) = M$. According to Bolzano-Weierstrass (Theorem 4.62), there is a convergent subsequence $(x_{n_k}) \subset (x_n)$. Set $c := \lim\limits_{k\to\infty} x_{n_k}$. Then $c \in [a,b]$, and

$$f(c) = \underbrace{\lim_{k\to\infty} f(x_{n_k})}_{f\,:\,[a,b]\to\mathbb{R} \text{ is continuous}} = \underbrace{\lim_{n\to\infty} f(x_n)}_{\text{Proposition 4.60}} = M.$$

□

Example 6.62. The above theorem is not true, if the domain is not $[a,b]$ for $a, b \in \mathbb{R}$. For example, take $f(x) = \pm\frac{1}{x^2+1} : \mathbb{R} \to \mathbb{R}$, then f does not attain its minimum/maximum as the function is nowhere 0, but it converges to 0 as x goes to $\pm\infty$.

Theorem 6.63. *If $f : [a,b] \to \mathbb{R}$ is continuous for some $a, b \in \mathbb{R}$, then f is uniformly continuous.*

Proof. Assume that f is not uniformly continuous. Then there is a $\varepsilon > 0$ such that for every $\frac{1}{n}$ there are x_n and $y_n \in [a,b]$ such that $|x_n - y_n| \leq \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon$. By Bolzano-Weierstrass (Theorem 4.62) we may assume that $\lim\limits_{n\to\infty} x_n = x_0 \in [a,b]$. However, then the condition $|x_n - y_n| \leq \frac{1}{n}$ yields that we have also $\lim\limits_{n\to\infty} y_n = x_0$. Using again $|x_n - y_n| \leq \frac{1}{n}$ together with the continuity of f we obtain that $|f(x_0) - f(x_0)| \geq \varepsilon$. This is a contradiction. □

Example 6.64. Again, it is important that the domain is $[a,b]$ for $a, b \in \mathbb{R}$. We have seen that $x^2 : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous on $\mathbb{R}$.

Theorem 6.65. Intermediate value theorem *If $f : [a,b] \to \mathbb{R}$ is continuous then it takes each value between $M := \max\limits_{x\in[a,b]} f(x)$ and $m := \min\limits_{x\in[a,b]} f(x)$ at least once. That is, for each $c \in [m, M]$ there is a $d \in [a,b]$ such that $f(d) = c$.*

Idea. We give only the idea and we refer to the precise proof to page 81-82 of the book.

We know by the above theorem that there are $a', b' \in [a, b]$ such that $m = f(a')$ and $M = f(b')$. Hence, by replacing a with a' and b with b' (and some algebraic manipulation in the case when $b' < a'$), we may assume that $f(a) = m$, $f(b) = M$ and $m < c < M$. Then, the idea is to consider

$$S := \{x \in [a, b] | f(x) < c\}$$

Set $d := \operatorname{Sup} S$. By the definition of Sup, there is a sequence $(x_n) \subset S$ converging to d from the left and let y_n be any sequence converging to d from the right. Applying continuity to the first sequence shows that $f(d) \leq c$, and by applying it to the second one shows that $f(d) \geq c$. So, $f(d) = c$. ⟵ End of 15. class, on 06.11.2019.

□

Example 6.66. With other words, the above theorem says that $f([a, b]) = [m, M]$. So, the image of $[1, 2]$ for example via a continuous function cannot be $[1, 2] \cup [3, 4]$.

Example 6.67. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, such that $f(0) = 1$, $f(1) = 3$, $f(2) = -1$, then f takes the value 2 at least two times. Indeed, our assumptions say that the maximum of $f|_{[0,1]}$ is at least 3 and the minimum of $f|_{[0,1]}$ is at most 1. Hence, Theorem 6.65 applied to $f|_{[0,1]}$ yields that there is at least one $c \in [0, 1]$ such that $f(c) = 2$. Similarly, Theorem 6.65 applied to $f|_{[0,1]}$ yields that there is at least one $d \in [1, 2]$ such that $f(d) = 2$. Furthermore, $c \neq d$, because $c = d$ can only happen if $c = d = 1$. However, $f(c) = 3 \neq 2$. Hence, c and d are two distinct real numbers at which f takes the value 2.

We will apply the above theoretical result to find solutions of equations of the form $f(x) = x$. For example one can ask, if there is a solution of $\cos(x) = x$ for some $x \in \left[0, \frac{\pi}{2}\right]$. Corollary 6.68 lets us answer this question.

Corollary 6.68. Banach fixed point theorem for closed intervals

If $f : [a, b] \to [a, b]$ is a continuous function (where $a, b \in \mathbb{R}$), then it has a fixed point. That is, there is an $x \in [a, b]$ such that $f(x) = x$.

Proof. Set $g(x) := f(x) - x$. Then $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$. So, by the intermediate value theorem, there is a real number $c \in [a, b]$ such that $0 = g(c)$, which means that $f(c) = c$. □

Example 6.69. $\cos(x) : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ can be regarded as $\cos(x) : \left[0, \frac{\pi}{2}\right] \to \left[0, \frac{\pi}{2}\right]$. Then the above theorem says that there is a fixed point x for which $\cos(x) = x$.

Definition 6.70. $f : E \to \mathbb{R}$ is *strictly increasing* (resp. decreasing) if $f(x) < f(y)$ (resp. $f(y) > f(x)$) for every $x < y$ in E.

$f : E \to \mathbb{R}$ is *strictly monotone*, if it is *strictly increasing* or *strictly decreasing*.

Corollary 6.71. *If $f :]a, b[\to \mathbb{R}$ is strictly monotone and continuous (where $a, b \in \mathbb{R}$), then the range $R(f)$ is an open interval.*

Proof. Set $S := \operatorname{Sup}\{ f(x) \mid x \in]a, b[\}$ and $I := \operatorname{Inf}\{ f(x) \mid x \in]a, b[\}$.

First, we show that $S, I \notin R(f)$. In fact, we show only the statement about S as a proof for the statement about I can be proven by just reversing the signs in the proof for the statement about S. So, let us assume the contrary, that is, that $S = f(c)$ for some $c \in]a, b[$. Choose a $c < d \in]a, b[$ (here we are using that the interval is open!). Then, as f is strictly increasing $f(d) > f(c) = S$, which is a contradiction with the definition of S.

Second, we show that $R(f) =]I, S[$, which will finish our proof. So, take $M \in]I, S[$. By the definition of S and I, there are $c, d \in]a, b[$ such that $f(c) < M < f(d)$. However, then the intermediate value theorem (Theorem 6.65) tells us that $M \in R(f)$.

□

Theorem 6.72. *Let $f : E \to F$ be a continuous function on an interval. Then, f is strictly monotone if and only if it is injective.*

Proof. We do not prove this in class, read the proof from page 84-85 of the book. $\square$

Theorem 6.73. *If $f : E \to F$ is continuous, strictly monotone and surjective function between intervals, then f^{-1} (which exists by Theorem 6.72) is also continuous.*

Proof. We only show the case when E is an open interval. In this case, F is also an open interval according to Corollary 6.71. Fix a $0 < \varepsilon \in \mathbb{R}$ and a $y_0 \in F$. Set $x_0 := f^{-1}(y_0)$. According to Corollary 6.71, there are $c, d \in \mathbb{R}$ such that

$$f(]x_0 - \varepsilon, x_0 + \varepsilon[) =]c, d[\tag{6.73.n}$$

In particular, there is a $\delta > 0$ such that for every $y \in F$ we have the implication

$$|y - y_0| \le \delta \Rightarrow y \in]c, d[. \tag{6.73.o}$$

For example, $\delta := \frac{\min\{|c-y_0|,|d-y_0|\}}{2}$ is a good choice of such a δ.

We show that with the above choice of δ the definition of the continuity of f^{-1} at y_0 is satisfied. That is, for every $y \in F$,

$$|y - y_0| \le \delta \Rightarrow \underbrace{y \in]c, d[}_{(6.73.o)} \Rightarrow \underbrace{|f^{-1}(y) - x_0| \le \varepsilon.}_{(6.73.n)}$$

$\square$

Example 6.74. Neither of the functions $\sin(x)$, $\cos(x)$, $\operatorname{tg}(x)$ and $\operatorname{cotg}(x)$ are invertible if considered as functions $\mathbb{R} \to \mathbb{R}$, as they are not injective. However, if we restrict their domains adequately they become strictly montones, and then according to Theorem 6.73 their inverses are continuous too. To ome of these inverses we assign special names and notations:

(1) $\operatorname{Arcsin}(x)$ is the inverse of $\sin(x)|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$. For example, $\operatorname{Arcsin}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$, and $\operatorname{Arcsin}\left(-\frac{1}{2}\right) \neq \frac{7\pi}{6}$, despite having $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$ too.

(2) $\operatorname{Arccos}(x)$ is the inverse of $\cos(x)|_{[0,\pi]}$.

(3) $\operatorname{Arctg}(x)$ is the inverse of $\operatorname{tg}(x)|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$.

(4) $\operatorname{Arccotg}(x)$ is the inverse of $\operatorname{cotg}(x)|_{[0,\pi]}$.

WARNING: according to the book's notation which is used also in the course and in the exam, $\operatorname{Arctg}(x)$ is the inverse of the adequate restriction of $\operatorname{tg}(x)$ and not of $\operatorname{cotg}(x)$!

7 Differentiation

Let $f : E \to \mathbb{R}$ be a real valued one variable function. We would like to approximate it with a linear one. That is, we would like to write

$$f(x) = f(x_0) + a(x - x_0) + r(x), \tag{7.0.a}$$

where a is a real number, and $r(x)$ is small in a neighborhood of x_0. The question is how small we would like it to be that we can also attain. Definitely, we would like it to be smaller than linear. The precise mathematical wording of this is that

$$\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0. \tag{7.0.b}$$

Considering the graph of the function, $f(x_0) + a(x - x_0)$ would be a tangent line to the graph at $(x_0, f(x_0))$.

Now, the question is if the above a and $r(x)$ exist, which again visually means whether there is a tangent line to the graph of the function at $(x_0, f(x_0))$. Equation (7.0.a) implies that

$$\frac{r(x)}{x - x_0} + a = \frac{f(x) - f(x_0)}{x - x_0} \tag{7.0.c}$$

By applying limit to both sides of this equation, using (7.0.b), we obtain that

$$a = \underbrace{\lim_{x \to x_0} \frac{r(x)}{x - x_0} + a}_{\text{by (7.0.b)}} = \underbrace{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}_{\text{by (7.0.c)}}.$$

So, the existence of the real number a, which we call the derivative of f at x_0, and of the behavior of the error term described in (7.0.b) is equivalent to the existence of the latter limit. In particular, instead of using equations (7.0.a) and (7.0.b), we may equivalently define the derivative also by: $\longleftarrow$ End of 16. class, on 11.11.2019.

Definition 7.1. A function $f : E \to \mathbb{R}$ is differentiable at $x_0 \in E$, if the function (of x)

$$\frac{f(x) - f(x_0)}{x - x_0} : E \setminus \{x_0\} \to \mathbb{R}$$

admits a limit in x_0. In this case we call the limit the derivative of f at x_0 and we denote it by $f'(x_0)$. The function $x_0 \mapsto f(x_0)'$ is called the derivative function of f (where the domain is the set of all points of E where the above limit exists).

We say that f is differentiable if it is differentiable at all $x_0 \in E$.

Remark 7.2. By the above introduction, $f'(x_0)$ can be also defined to be the unique number that satisfies

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + r(x), \tag{7.2.d}$$

such that $\lim_{x \to x_0} \frac{r(x)}{x - x_0} = 0$.

Example 7.3. *We show that* $(x^2)' = 2x$.

For this we need to find at each x_0 the limit

$$\lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = \lim_{x \to x_0} x + x_0 = 2x_0.$$

Example 7.4. *Similarly, if* $a \in \mathbb{Z}_+$, *then* $(x^a)' = ax^{a-1}$.

Indeed,

$$\lim_{x \to x_0} \frac{x^a - x_0^a}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^{a-1} + x^{a-2}x_0 + x^{a-3}x_0^2 + \cdots + x^1 x_0^{a-2} + x_0^{a-1})}{x - x_0}$$
$$= \lim_{x \to x_0} x^{a-1} + x^{a-2}x_0 + x^{a-3}x_0^2 + \cdots + x x_0^{a-2} + x_0^{a-1} = a x_0^{a-1}.$$

Example 7.5. *We show that* $\sin(x)' = \cos(x)$*:*

$$\lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{2\cos\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x - x_0}$$
$$= \lim_{x \to x_0} \cos\left(\frac{x + x_0}{2}\right) \cdot \lim_{x \to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = \cos(x_0)$$

Example 7.6. *We show that* $\cos(x)' = -\sin(x)$*:*

$$\lim_{x\to x_0} \frac{\cos(x)-\cos(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{-2\sin\left(\frac{x+x_0}{2}\right)\sin\left(\frac{x-x_0}{2}\right)}{x-x_0}$$
$$= \lim_{x\to x_0} -\sin\left(\frac{x+x_0}{2}\right) \cdot \lim_{x\to x_0} \frac{\sin\left(\frac{x-x_0}{2}\right)}{\frac{x-x_0}{2}} = -\sin(x_0)$$

Proposition 7.7. *If* $f : E \to \mathbb{R}$ *is differentiable at* x_0*, then it is continuous at* x_0*.*

Proof. This is a consequence of the following computation:

$$\lim_{x\to x_0} f(x) = \underbrace{\lim_{x\to x_0} f(x_0) + (x-x_0)f(x_0)' + r(x)}_{\text{by (7.2.d)}} = f(x_0) + \lim_{x\to x_0} r(x).$$
$$= f(x_0) + \lim_{x\to x_0} \frac{r(x)}{x-x_0} \cdot \lim_{x\to x_0} (x-x_0) = f(x_0).$$

□

Example 7.8. The above implication is not true backwards, that is, if f is continuous at x_0, it does not have to be differentiable. For example, consider $f(x) := |x|$. This is continuous at $x_0 = 0$ (Proposition 6.52), but it is not differentiable at 0, because that would mean that $\lim_{x\to 0} \frac{|x|}{x}$ exists. However, Proposition 6.48 together with the following computation implies the opposite.

$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \neq 1 = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} \frac{|x|}{x}.$$

7.1 Algebraic properties of derivation

7.1.1 Addition

Proposition 7.9. *If* $f, g : E \to \mathbb{R}$ *are differentiable at* x_0*, then so is* $\alpha f + \beta g$ *for any* $\alpha, \beta \in \mathbb{R}$*, and furthermore*

$$(\alpha f + \beta g)'(x_0) = \alpha f'(x_0) + \beta g'(x_0).$$

Proof.

$$(\alpha f + \beta g)'(x_0) = \lim_{x\to x_0} \frac{(\alpha f + \beta g)(x) - (\alpha f + \beta g)(x_0)}{x-x_0}$$
$$= \alpha \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} + \beta \lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0} = \alpha f'(x_0) + \beta g'(x_0)$$

□

Example 7.10. $\left(5x^3 + 6x^2\right)' = \left(5x^3\right)' + \left(6x^2\right)' = 15x^2 + 12x$

7.1.2 Multiplication

Proposition 7.11. *If* $f, g : E \to \mathbb{R}$ *are differentiable at* x_0*, then so is* $f \cdot g$*, and furthermore*

$$(f \cdot g)'(x_0) = (fg' + f'g)$$

Proof.

$$\begin{aligned}
(f\cdot g)'(x_0) &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}\\
&= \lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)+f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\\
&= \lim_{x\to x_0}\frac{f(x)(g(x)-g(x_0))+(f(x)-f(x_0))g(x_0)}{x-x_0}\\
&= \left(\lim_{x\to x_0} f(x)\right)\cdot\left(\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\right)+g(x_0)\left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)\\
&= f(x_0)g'(x_0)+g(x_0)f'(x_0)
\end{aligned}$$

□

Example 7.12.

$$\left(x^2\cos(x)\right)' = \left(x^2\right)'\cos(x)+x^2\left(\cos(x)\right)' = 2x\cos(x)+x^2(-\sin(x)) = x(2\cos(x)-x\sin(x))$$

7.1.3 Division

Proposition 7.13. *If $f,g:E\to\mathbb{R}$ are differentiable at x_0, and $g(x_0)\neq 0$, then $\frac{f}{g}$ is also differentiable at x_0, and furthermore*

$$\left(\frac{f}{g}\right)'(x_0) = \left(\frac{gf'-fg'}{g^2}\right)(x_0)$$

In particular,

$$\left(\frac{1}{g}\right)'(x_0) = \left(\frac{-g'}{g^2}\right)(x_0)$$

Proof.

$$\begin{aligned}
\lim_{x\to x_0}\frac{\frac{f}{g}(x)-\frac{f}{g}(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x)}{g(x)g(x_0)(x-x_0)}\\
&= \lim_{x\to x_0}\frac{f(x)g(x_0)-f(x_0)g(x_0)+f(x_0)g(x_0)-f(x_0)g(x)}{g(x)g(x_0)(x-x_0)}\\
&= \frac{g(x_0)}{g(x_0)\cdot\lim_{x\to x_0} g(x)}\left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)-\frac{f(x_0)}{g(x_0)\cdot\lim_{x\to x_0} g(x)}\left(\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}\right)\\
&= \left(\frac{gf'-fg'}{g^2}\right)(x_0)
\end{aligned}$$

□

Example 7.14. If $b>0$ is an integer and $x\neq 0$, then:

$$\left(\frac{1}{x^b}\right)' = -\frac{\left(x^b\right)'}{x^{2b}} = -\frac{bx^{b-1}}{x^{2b}} = \frac{-b}{x^{b+1}}.$$

That is, by setting $a=-b$ we obtain $(x^a)'=ax^{a-1}$. In particular, this shows that: *the formula $(x^a)'=ax^{a-1}$ holds for all integer a* (not only the non-negative ones).

Example 7.15. If $x\neq k\pi+\frac{\pi}{2}$ for some $k\in\mathbb{Z}$ (or equivalently, if $\cos(x)\neq 0$), then

$$\begin{aligned}
\operatorname{tg}(x)' = \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{\cos(x)(\sin(x))'-\sin(x)(\cos(x))'}{\cos(x)^2}\\
&= \frac{\cos(x)\cos(x)-\sin(x)(-\sin(x))}{\cos(x)^2} = \frac{1}{\cos^2(x)}
\end{aligned}$$

← End of 17. class, on 13.11.2019.

7.1.4 Inversion of functions

Proposition 7.16. *Let $f : I =]a, b[\to F$ be a bijective continuous function (so f is strictly monotone, and f^{-1} exists and is continuous by Theorem 6.73), and let $x_0 \in I$ be such that $f'(x_0) \neq 0$.*

Then f^{-1} is differentiable at $y_0 := f(x_0)$, and we have

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

Proof. The idea behind the proof of the proposition is that if we set $y = f(x)$ and $y_0 = f(x_0$, we have

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\frac{y-y_0}{f^{-1}(y)-f^{-1}(y_0)}} = \frac{1}{\frac{f(f^{-1}(y))-f(f^{-1}(y_0))}{f^{-1}(y)-f^{-1}(y_0)}}.$$

Check page 109 for the precise proof. □

Example 7.17. If $f(x) = x^b$ for some integer $b \geq 1$, then $f^{-1}(y) = \sqrt[b]{y} = y^{\frac{1}{b}}$. So, $f'(x) = bx^{b-1}$, and

$$\left(y^{\frac{1}{b}}\right)' = \left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{b\left(y^{\frac{1}{b}}\right)^{b-1}} = \frac{1}{b} y^{-\frac{b-1}{b}} = \frac{1}{b} y^{\frac{1}{b}-1}.$$

So, for $c = \frac{1}{b}$ (where $b \in \mathbb{Z}_+^*$), the formula for $(y^c)' = cy^{c-1}$. That is the formula is the same as in the case of c being an integer.

Example 7.18. If $f(x) = \sin(x)|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$, then $f^{-1}(y) = \operatorname{Arcsin}(y)$. Then, $f'(x) = \cos(x)|_{[-\frac{\pi}{2},\frac{\pi}{2}]}$, and so

$$\operatorname{Arcsin}'(y) = \frac{1}{\cos(\operatorname{Arcsin}(y))} = \frac{1}{\sqrt{1-\sin^2(\operatorname{Arcsin}(y))}} = \frac{1}{\sqrt{1-y^2}}.$$

Proposition 7.19. *If $f : E \to F$ is differentiable at $x_0 \in E$, $g : G \to H$ is differentiable at $f(x_0)$, and $f(E) \subset G$, then $g \circ f : E \to H$ is differentiable at x_0, and*

$$(g \circ f)'(x_0) = ((g' \circ f) \cdot f')(x_0)$$

Idea of the proof.

$$\begin{aligned}
\lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{x - x_0} &= \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \lim_{x\to x_0} \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \cdot \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = g'(f(x_0)) \cdot f'(x_0)
\end{aligned}$$

□

Example 7.20. Let $f(x) = x^2$ and $g(y) = \cos(y)$. Then $f'(x) = 2x$ and $g'(y) = -\sin(y)$. In particular,

$$\cos(x^2)' = (g \circ f)'(x) = (g' \circ f)(x) \cdot f'(x) = -\sin(x^2)2x.$$

Example 7.21. Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}^+$, $f(x) = x^a$ and $g(y) = y^{\frac{1}{b}}$. Then according to Example 7.14 and Example 7.17, $f(x)' = ax^{a-1}$ and $g(y)' = \frac{1}{b} y^{\frac{1}{b}-1}$. Hence,

$$\left(x^{\frac{a}{b}}\right)' = \left((x^a)^{\frac{1}{b}}\right)' = (g \circ f)'(x) = (g' \circ f)(x) \cdot f'(x) = \frac{1}{b}\left(x^a\right)^{\frac{1}{b}-1} ax^{a-1} = \frac{a}{b} x^{\frac{a}{b}-a+a-1} = \frac{a}{b} x^{\frac{a}{b}-1}$$

So, *the formula $(x^r)' = rx^{r-1}$ holds also when r is any rational number* (as it did for $r \in \mathbb{Z}$ in Example 7.14).

Our last example is the inverse of the exponential function. However, for this we have to define first the exponential function itself:

Definition 7.22. For $x \in \mathbb{R}$, we define

$$e^x := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

Remark 7.23. Applying Definition 7.22, to $x = 0$ yields $e^0 = 1$. Furthermore, according to Example 5.21, $e^1 = e$.

Proposition 7.24. *For any $x, y \in \mathbb{R}$, $e^{x+y} = e^x e^y$.*

Proof. See pages 151-153 of the book. □

Corollary 7.25. *For any $x \in \mathbb{R}$, $e^{-x} = \frac{1}{e^x}$.*

Proof.

$$e^x \cdot e^{-x} = \underbrace{e^{x+(-x)}}_{\text{Proposition 7.24}} = e^0 = \underbrace{1}_{\text{Remark 7.23}} .$$

Dividing by e^x yields the statement (e^x cannot be 0, since then $e^x \cdot e^{-x} = 1$ could not hold). □

Corollary 7.26. *For every $x \in \mathbb{R}$, $e^x > 0$.*

Proof. For $x \geq 0$, then all the terms in the infinite sum in Definition 7.22 is at least zero, and the first term is 1. This implies the statement for $x \geq 0$.

So, we may assume from now that $x < 0$. We have $e^x = \frac{1}{e^{-x}}$ by Corollary 7.25. However, as now $-x > 0$ holds, the previous paragraph tells us that $e^{-x} > 0$, and hence also $\frac{1}{e^{-x}} > 0$. □

Proposition 7.27. $(e^x)' = e^x$

Proof. We need to show that

$$\lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} = e^{x_0}.$$

This is equivalent to show that

$$0 = \lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} - e^{x_0} = \left(\lim_{x \to x_0} \frac{e^{x - x_0} - 1}{x - x_0} - 1 \right) e^{x_0}$$

By setting $y = x - x_0$, we see that it is enough to show that

$$\lim_{y \to 0} \frac{e^y - 1}{y} - 1 = 0. \tag{7.27.a}$$

However, for $0 < |y| \leq 1$:

$$0 \leq \left| \frac{e^y - 1}{y} - 1 \right| = \left| \frac{\sum_{k=0}^{\infty} \frac{y^k}{k!} - 1}{y} - 1 \right| = \left| \sum_{k=2}^{\infty} \frac{y^{k-1}}{k!} \right| \leq \sum_{k=2}^{\infty} \frac{|y|^{k-1}}{k!} \leq |y| \sum_{k=2}^{\infty} \frac{|y|^{k-2}}{(k-2)!}$$

$$= |y| \sum_{k=0}^{\infty} \frac{|y|^k}{k!} \leq |y| e$$

Hence, squeeze (point (5) of Proposition 6.24) tells us that (7.27.a) holds indeed. □

Proposition 7.28. *We have* $\lim_{x \to +\infty} e^x = +\infty$*, and* $\lim_{x \to -\infty} e^x = 0$*.*

Proof. According to Definition 7.22, for all $x > 0$, $e^x \geq 1 + x$. As $\lim_{x \to +\infty} 1 + x = +\infty$, squeeze (point (5) of Proposition 6.40) shows that $\lim_{x \to +\infty} e^x = +\infty$. Then Corollary 7.25, Corollary 7.26 and point (3) of Proposition 6.40 show that $\lim_{x \to -\infty} e^x = 0$. □

Proposition 7.29. *The function* $e^x : \mathbb{R} \to \mathbb{R}$ *is strictly increasing.*

Proof. Choose $y > x \in \mathbb{R}$. We have to show that $e^y > e^x$. This is shown by the following computation:

$$e^y - e^x = \underbrace{(e^{y-x} - 1)}_{\substack{> 1 \text{ by Definition 7.22, using} \\ y > x}} \cdot \underbrace{e^x}_{\substack{> 0 \text{ by Corol-} \\ \text{lary 7.26}}} > 0$$

□

Corollary 7.30. *The range* $R(e^x)$ *of* $e^x : \mathbb{R} \to \mathbb{R}$ *is* $]0, +\infty[= \mathbb{R}_+^*$*.*

Proof. Follows immediately from Proposition 7.28 and Proposition 7.29. □

Definition 7.31. We call $\mathrm{Log}(x) : \mathbb{R}_+^* \to \mathbb{R}$ the inverse of e^x.

Example 7.32. If $f(x) = e^x$, then $f^{-1}(x) = \mathrm{Log}(x)$ and $f'(x) = e^x$ (Proposition 7.27). Hence:

$$(\mathrm{Log}(x))' = \frac{1}{e^{\mathrm{Log}(x)}} = \frac{1}{x}.$$

7.2 One sided derivatives

Definition 7.33. If $f : E \to \mathbb{R}$ is a function and $x_0 \in E$, then we say that the left (resp. right) derivative of f exists at x_0 if the function

$$\frac{f(x) - f(x_0)}{x - x_0} : E \setminus \{x_0\} \to \mathbb{R}$$

admits a left (resp. right limit). The value of this limit is then the left (resp. right) derivative.

Example 7.34. For $f(x) = |x|$ at $x = 0$ the left derivative is -1 and the right derivative is 1.

Proposition 7.35. *Let* $f : E \to \mathbb{R}$ *be a function and* $x_0 \in E$ *a real number. Then* f *is differentiable at a point* x_0 *if and only if both its left and right derivatives exist and they agree. Furthermore, then the value of the derivative is the same as the common value of the left and the right derivatives.*

Proof. This is an immediate consequence of Proposition 6.48, Definition 7.1 and Definition 7.33. □

7.3 Higher derivatives

We may iterate derivation, obtaining second, third, etc. derivatives.

Example 7.36. For example, the second derivative of $\mathrm{Arcsin}(x)$ is:

$$\mathrm{Arcsin}\left(\frac{1}{x}\right)'' = \mathrm{Arcsin}\left(x^{-1}\right)'' = \left(\left(-x^{-2}\right) \frac{1}{\sqrt{1 - x^{-2}}}\right)'$$
$$= -\left((x^4 - x^2)^{-\frac{1}{2}}\right)' = (4x^3 - 2x)\frac{1}{2}(x^4 - x^2)^{-\frac{3}{2}}.$$

$\longleftarrow$
End of 18. class, on 18.11.2019.

Example 7.37. Here we show an example of a function $f(x)$ such that $f(x)$ is differentiable two times, but not three times. That is, $f'(x)$ and $f''(x)$ exist for every $x \in \mathbb{R}$, but $f'''(0)$ does not exist.

Set $f(x) := |x^3|$. *First, we claim that $f'(x)$ exists for all $x \in \mathbb{R}$, and:*

$$f'(x) = \begin{cases} 3x^2 & \text{for } x \geq 0 \\ -3x^2 & \text{for } x < 0 \end{cases}$$

This is immediate at $x \neq 0$ from the formula

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 0 \\ -x^3 & \text{for } x \leq 0 \end{cases}$$

To conclude the above first claim we just have to compute the left and the right derivatives of $f(x)$ at $x = 0$, and show that both are 0. Indeed:

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^3 - 0}{x} = \lim_{x \to 0^+} x^2 = 0,$$

and

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x^3 - 0}{x} = \lim_{x \to 0^-} -x^2 = 0.$$

This concludes our first claim.

Similarly, one can prove that $f''(x)$ exists for all $x \in \mathbb{R}$ and

$$f''(x) = \begin{cases} 6x & \text{for } x \geq 0 \\ -6x & \text{for } x < 0 \end{cases}$$

With other words, $f''(x) = 6|x|$. However, as $|x|$ is not differentiable at $x = 0$, we obtain that $f'''(0)$ does not exist.

Definition 7.38. $f : E \to F$ is called a C^n function if its n-th derivatives exists at all $x_0 \in E$ and they are all continuous. The n-th derivative of f at $x \in E$ is denoted by $f^{(n)}(x)$.

Example 7.39. (1) According to Example 7.4, x, x^2, etc. are C^n for all n, and for $a \in \mathbb{N}$:

$$(x^a)^{(n)} = \begin{cases} 0 & \text{if } n > a \\ a \cdot (a-1) \cdot \dots \cdot (a-n+1)x^{a-n} & \text{for } n \leq a \end{cases}$$

(2) $|x| : \mathbb{R} \to \mathbb{R}$ is not C^1, according to Example 7.34.

(3) $|x^3| : \mathbb{R} \to \mathbb{R}$ is C^2 but not C^3, according to Example 7.37.

7.4 Local extrema

In the next proposition f being differentiable at x_0 means that $f'(x_0)$ exists (Definition 7.1), and local extremum means either a local minimum or a local maximum.

Proposition 7.40. *If $f : E \to \mathbb{R}$ is differentiable at x_0, and f admits a local extremum at x_0, then $f'(x_0) = 0$.*

Proof. We present the local maximum case, as one just need to reverse a few signs, to obtain from it the case of local minimum. In this case, by Definition 6.7, there is a real number $\delta > 0$ such that

$$|x - x_0| \leq \delta \Rightarrow f(x) \leq f(x_0). \tag{7.40.a}$$

However, then

$$\lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{(x-x_0)} \leq \underbrace{\lim_{x\to 0^+} \frac{0}{(x-x_0)}}_{x>x_0,\text{ and (7.40.a)}} = 0, \tag{7.40.b}$$

and

$$\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{(x-x_0)} \geq \underbrace{\lim_{x\to 0^+} \frac{0}{(x-x_0)}}_{x<x_0,\text{ and (7.40.a)}} = 0. \tag{7.40.c}$$

As $f(x)$ is differentiable, at x_0 the two above limits agree (Proposition 7.35). Hence the following stream of inequalities have to be all equalities, which conclude our proof:

$$0 \leq \underbrace{\lim_{x\to x_0^-} \frac{f(x)-f(x_0)}{(x-x_0)}}_{(7.40.c)} = f'(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0)}{(x-x_0)} \underbrace{\leq 0.}_{(7.40.c)}$$

□

Example 7.41. (1) For $f(x) = x^2$, we have $f'(x) = 2x$. Hence, $f'(x) = 0$ if and only if $x = 0$. So, $x = 0$ is the only option for a local extrema, and it is indeed a local minimum.

(2) For $f(x) = x^3$, we have $f'(x) = 3x^3$. So $f'(x) = 0$ if and only if $x = 0$ (as in the previous case). However, $f(0) = 0$ is not a local extremum. This underlines that Proposition 7.40 yields only a necessary, but not a sufficient condition for having a local extremum.

Definition 7.42. If $f : E \to \mathbb{R}$ is a function such that $f'(x_0) = 0$ for some $x_0 \in E$, then we call x_0 a stationary point.

So, if $f'(x_0) = 0$, we cannot be sure that there is a local extrema there. However, if the domain of f is a closed, bounded interval $[a, b]$, then we know that there is also a global maximum and minimum (Theorem 6.61). Therefore, using Proposition 7.40, we know that the local extremuma can be only either at x such that $f'(x) = 0$, or at $x = a$, or at $x = b$. This gives an algorithmic way of finding global extrema, an example of which is shown below:

Example 7.43. We compute the minimum and the maximum of $f(x) = \frac{4}{3}x^3 + \frac{3}{2}x^2 - x + 2$ on $\left[-2, \frac{1}{2}\right]$. By the discussion in the paragraph before the example we have to compute:

$$f'(x) = 4x^2 + 3x - 1,$$

and then find the solutions of the equation $f'(x) = 0$. These are:

$$x = \frac{-3 \pm \sqrt{25}}{8} = \frac{-3 \pm 5}{8} = -1, \text{ and } x = \frac{1}{4}.$$

Then, we have to compute the function values at these two points, and at the endpoints of our interval. The point, where the function value is the maximal yields the maximum and where the function value is minimal yields the minimum of $f(x)$ on $\left[-2, \frac{1}{2}\right]$:

x	$f(x)$
-2	$-\frac{4}{3}8 + \frac{3}{2}4 + 2 + 2 = \frac{36-64}{6} + 4 = 4 - \frac{28}{6} = -\frac{4}{6}$
-1	$-\frac{4}{3} + \frac{3}{2} + 1 + 2 = \frac{9-8}{6} + 3 = 3 + \frac{1}{6}$
$\frac{1}{4}$	$\frac{4}{3\cdot 64} + \frac{3}{2\cdot 16} - \frac{1}{4} + 2 = \frac{4+18-48}{192} + 2 = \frac{-26}{192} + 2 = 2 - \frac{13}{96}$
$\frac{1}{2}$	$\frac{4}{3\cdot 8} + \frac{3}{2\cdot 4} - \frac{1}{2} + 2 = \frac{4+9-12}{24} + 2 = 2 + \frac{1}{24}$

So, $f(x)$ on $\left[-2, \frac{1}{2}\right]$ takes its minimum at $x = -2$ and its maximum at $x = -1$.

⟵ End of 19. class, on 20.11.2019.

7.5 Rolle's theorem and consequences

Most of the rest of what we learn about differentiation are consequences of Rolle's theorem (Theorem 7.44) and of the Mean value theorem (Theorem 7.46), where the latter itself is a consequence of the former. We cover first these two theorems.

Theorem 7.44. Rolle's theorem *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, and $f(a) = f(b)$, then there is a $c \in]a, b[$ for which $f'(c) = 0$.*

Proof. If f is constant, then $f'(x) = 0$, and so we are ready.

If f is not constant, then according to Theorem 6.61 f has both a maximum and a minimum on $[a, b]$. However, as f is non-constant, one of these values have to be not equal to $f(a) = f(b)$. Formally, this means that there is a $c \in [a, b]$, such that $f(c) \neq f(a) = f(b)$. In particular, we must have $a < c < b$. Then, f is differentiable at c, and as f has a (local) extremum at c, we have $f'(c) = 0$ according to Proposition 7.40. □

Example 7.45. Differentiable is needed here. For example if one takes $f(x) = |x|$ on $[-1, 1]$, then $f(-1) = f(1)$, but there is no point, where the derivative is 0.

Theorem 7.46. Mean value theorem *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, then there is a $c \in]a, b[$ for which $f'(c)(b - a) = f(b) - f(a)$.*

Proof. Apply the previous theorem to $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x - a)$ □

Example 7.47. Let f be a C^1 function (see Definition 7.38) on $[-1, 1]$ such that $f(1) = 2$, $f(0) = 4$ and $f(1) = 3$. We show using Theorem 7.46 that there is a $c \in] - 1, 1[$ such that $f'(c) = 1$:

(1) Applying Theorem 7.46 to $f|_{[-1,0]}$ we obtain that there is an $a \in] - 1, 0[$ such that $f(a) = \frac{4-2}{0-(-1)} = 2$.

(2) Applying Theorem 7.46 to $f|_{[0,1]}$ we obtain that there is a $b \in]0, 1[$ such that $f(b) = \frac{3-4}{1-0} = -1$.

(3) As f is C^1, f' is continuous. Hence, Theorem 6.65 implies that there is a $c \in]a, b[\subset] -1, 1[$ such that $f'(c) = 1$.

Corollary 7.48. *If $f, g : [a, b] \to \mathbb{R}$ are continuous functions (where a and b are real numbers) that are differentiable over $]a, b[$ such that $f'(x) = g'(x)$ for each $x \in]a, b[$, then there is a real number c such that $f(x) = g(x) + c$.*

Proof. By regarding $f - g$, it is enough to show the other statement that $f'(x) = 0$ (for all $x \in]a, b[$) implies that f is a constant function. Assume it is not. Then, there it has two different function values, say at c and $d \in [a, b]$. By replacing a and b with c and d we may assume that this is at the endpoints. However, then the mean value theorem for the derivative tells us that then there has to be a $c \in]a, b[$, such that $f'(c) = \frac{f(b)-f(a)}{b-a} \neq 0$. This is a contradiction. □

7.5.1 Monotone functions and differentials

Corollary 7.49. *If $f : [a, b] \to \mathbb{R}$ is a continuous function (where a and b are real numbers), such that $f|_{]a,b[}$ is differentiable, then*

(1) f is increasing (resp. decreasing) if and only if $f'(x) \geq 0$ (resp. ≤ 0)

(2) if $f'(x) > 0$ (resp. < 0), then f is strictly increasing (resp. strictly decreasing)

Proof. We only prove the increasing case of (1), as the others are similar.

- First we assume that f is increasing. Then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

- Second, let us assume that $f'(x) \geq 0$ everywhere, and we also assume that f is not increasing. Hence, there are $a \leq c < d \leq b$, such that $f(c) > f(d)$. However, then the mean value theorem (Theorem 7.46) tell us that then there has to be a $c < e < d$ such that $f'(e) = \frac{f(d)-f(c)}{d-c} < 0$.

□

Example 7.50. Strictly increasing does not imply that $f'(x) > 0$! For example, $f(x) = x^3$ is strictly increasing, but $f'(0) = 3 \cdot 0^2 = 0$.

Example 7.51. Here we find the real numbers a, for which $f(x) := \sin(x) + ax$ monotone. As $f(x)$ is differentiable on $\mathbb{R}$ this happens if and only if $f'(x)$ is everywhere non-negative or everywhere non-positive. So, let us compute $f'(x)$:

$$f'(x) = \cos(x) + a$$

So:

- f is increasing if and only if $a \geq 1$
- f is decreasing if and only if $a \leq -1$

We define a few more functions that we check that are increasing or decreasing using Corollary 7.49. Recall we only defined X^a for $a \in \mathbb{Q}$ (so not for any real number a), and we defined a^x only for $a = e$. Below we extend the above functions for all real numbers a, as well as we introduce the hyperbolic trigonometric functions:

Definition 7.52. (1) The *hyperbolic trigonometric* functions are defined below, and they are called hyperbolic sine/cosine/tangent/cotangent:

$$\begin{aligned} \mathrm{sh}(x) &:= \tfrac{e^x - e^{-x}}{2} \\ \mathrm{ch}(x) &:= \tfrac{e^x + e^{-x}}{2} \\ \mathrm{th}(X) &:= \tfrac{\mathrm{sh}(x)}{\mathrm{ch}(x)} \\ \coth(X) &:= \tfrac{\mathrm{ch}(x)}{\mathrm{sh}(x)} \end{aligned}$$

(2) For any $a \in \mathbb{R}_+^*$, the a-based exponential functions is:

$$a^x := e^{x \cdot \mathrm{Log}(a)}.$$

The domains of all the above functions is $\mathbb{R}$. And the domain of the following functions is $\mathbb{R}_+^*$:

(3) For any $a \in \mathbb{R}$, the a-th power functions is:

$$x^a := e^{a \cdot \mathrm{Log}(x)}.$$

(4) $\mathrm{Log}_a := \frac{\mathrm{Log}(x)}{\mathrm{Log}(a)} = (a^x)^{-1}$.

Remark 7.53. In the special cases where the functions of Definition 7.52 have been already defined (so x^a when $a \in \mathbb{Q}$, and a^x when $a = e$), they agree with the previously defined functions. This will be an exercise on the exercise sheet.

Proposition 7.54. *We have:*

(1) $\mathrm{sh}(x)' = \mathrm{ch}(x)$

(2) $\mathrm{ch}(x)' = \mathrm{sh}(x)$

(3) $\mathrm{th}(x)' = \frac{1}{\mathrm{ch}(x)^2}$

(4) $\mathrm{coth}(x)' = \frac{-1}{\mathrm{sh}(x)^2}$

(5) $(x^a)' = ax^{a-1}$

(6) $(a^x)' = \mathrm{Log}(a) \cdot a^x$

(7) $\mathrm{Log}_a(x)' = \frac{1}{\mathrm{Log}(a) \cdot x}$

Proof. These will be exercises on the exercise sheet. □

Example 7.55. Using Corollary 7.49 and Proposition 7.54 we obtain that the all the functions of Definition 7.52 are either monotone, or become monotone when restricted to $\mathbb{R}_+^*$ or to $\mathbb{R}_-^*$.

For example: $\mathrm{sh}(x)' = \mathrm{ch}(x) = \frac{e^x + e^{-x}}{2} > 0$. We leave the rest as an exercise.

7.5.2 L'Hôpital's rule

L'Hôpital rule gives a method to determine limits of fractions, where both the denominator and numerator approach 0, $-\infty$ or $+\infty$ (meaning, both approach the same out of these three possibilities). For example, one can ask what is $\lim\limits_{x \to +\infty} \dfrac{e^x}{x}$. This we cannot answer using the algebraic rules of Proposition 6.24, but luckily Theorem 7.56 yields a method:

Theorem 7.56. *[*L'Hôpital rule*] Assume we are in the following situation:*

(1) $f, g : I :=]a, b[\to \mathbb{R}$ *are differentiable functions, and we are in one of the following situation:*

(i) $x_0 \in I$,

(ii) $x_0 = a \in \mathbb{R}$,

(iii) $x_0 = b \in \mathbb{R}$,

(iv) $a = -\infty$ *and* $x_0 = -\infty$ *or*

(v) $b = +\infty$ *and* $x_0 = +\infty$,

(2) $g(x) \neq 0$ *and* $g'(x) \neq 0$ *for all* $x \in I \setminus \{x_0\}$ *(to be able to divide with them),*

(3) $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \alpha$ *for* $\alpha = 0$ *or* $\pm\infty$.

Then, in the respective cases we have the following implications for any $\mu \in \overline{\mathbb{R}}$:

Cases (1|i), (1|iv) and (1|v)	$\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = \mu$
Case (1|ii)	$\lim\limits_{x\to x_0^+} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim\limits_{x\to x_0^+} \frac{f(x)}{g(x)} = \mu$
Case (1|iii)	$\lim\limits_{x\to x_0^-} \frac{f'(x)}{g'(x)} = \mu \quad \Rightarrow \quad \lim\limits_{x\to x_0^-} \frac{f(x)}{g(x)} = \mu$

Proof. We prove only the $\alpha = 0$ and $x_0 \in]a, b[$ case, and we refer to page 121-122 of the book for the rest.

As f and g are differentiable at x_0 they are also continuous there, and hence

$$f(x_0) = \lim_{x\to x_0} f(x) = \alpha = 0 \qquad \text{and} \qquad g(x_0) = \lim_{x\to x_0} g(x) = \alpha = 0. \tag{7.56.a}$$

So, by the mean value theorem for derivatives, there is a real number $c(x)$ between x and x_0 such that

$$f'(c(x)) = \frac{f(x) - f(x_0)}{x - x_0}. \tag{7.56.b}$$

In particular, $c(x) : I \setminus x_0 \to I \setminus x_0$ is a function such that $\lim\limits_{x\to x_0} c(x) = x_0$. Then:

$$\mu = \underbrace{\lim_{x\to x_0} \frac{f'(x)}{g'(x)}}_{\text{definition of } \mu} = \underbrace{\lim_{x\to x_0} \frac{f'(c(x))}{g'(c(x))}}_{\text{Proposition 6.33}} = \underbrace{\lim_{x\to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}}_{(7.56.b)} = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \underbrace{\lim_{x\to x_0} \frac{f(x)}{g(x)}}_{(7.56.a)}$$

□

Example 7.57.

$$\lim_{x\to 0} \frac{\operatorname{Arcsin}(x)}{\sin(x)} = \lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\cos(x)} = 1$$

Example 7.58.

$$\lim_{x\to +\infty} \frac{e^x}{x^2} = \lim_{x\to +\infty} \frac{e^x}{2x} = \lim_{x\to +\infty} \frac{e^x}{2} = +\infty$$

So, e^x goes to $+\infty$ as x goes to $+\infty$ quicker than x^2 (in fact a similar argument shows that it goes faster than any polynomial).

Example 7.59.

$$\lim_{x\to 0^+} \frac{\operatorname{Log}(x)}{\frac{-1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to 0^+} x = 0$$

and

$$\lim_{x\to +\infty} \frac{\operatorname{Log}(x)}{x} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{1} = 0$$

So, Log goes to $-\infty$ as x goes to 0 and to $+\infty$ as x goes to $+\infty$ slower than $\frac{1}{x}$ and x, respectively.

7.5.3 Taylor expansion

Definition 7.60. Let $f : E \to \mathbb{R}$ be a function such that there is a neighborhood of $a \in E$ which is contained in the domain (so in E). The n-th order expansion of f around a is an equality

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n\epsilon(x)$$

where a_i are real number, and $\epsilon(x)$ is a function $E \to \mathbb{R}$ such that $\lim_{x\to a} \epsilon(x) = 0$.

Proposition 7.61. *In the above situation, if the n-th order expansion around a exists, then a_i are uniquely determined.*

Proof. Let a_i and $\epsilon(x)$ be giving one and let a_i' and $\epsilon'(x)$ be giving another expansion. We show by induction on i that $a_i = a_i'$. For $i = 0$ this is given by passing to the limit as x goes to a of the two expansion:

$$\begin{aligned} a_0 = \lim_{x\to a} a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n\epsilon(x) = \lim_{x\to a} f(x) \\ = \lim_{x\to a} a_0' + a_1'(x-a) + a_2'(x-a)^2 + \cdots + a_n'(x-a)^n + (x-a)^n\epsilon'(x) = a_0' \end{aligned}$$

Then, we have to prove the induction step. So, let us assume that we know that $a_j = a_j'$ for $j = 0, \ldots, i-1$. Then we have:

$$\begin{aligned} a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + (x-a)^n\epsilon(x) = f(x) \\ = a_0' + a_1'(x-a) + a_2'(x-a)^2 + \cdots + a_n'(x-a)^n + (x-a)^n\epsilon'(x) \\ = a_0 + a_1(x-a) + \cdots + a_{i-1}(x-a)^{i-1} + a_i'(x-a)^i + \cdots + a_n'(x-a)^n + (x-a)^n\epsilon'(x) \end{aligned}$$

So, take the two endpoints of this stream of equalities, and subtract from both $a_0 + a_1(x-a) + \cdots + a_{i-1}(x-a)^{i-1}$ and then divide both by $(x-a)^i$. This yields that

$$a_i + a_{i+1}(x-a) + \cdots + a_n(x-a)^n + (x-a)^n\epsilon(x) = a_i' + a_{i+1}'(x-a) + \cdots + a_n'(x-a)^n + (x-a)^n\epsilon'(x).$$

Taking limit of this equation as x goes to a yields that $a_i = a_i'$, which concludes the induction step.

So, we know that $a_i = a_i'$ for each i. However, then it follows also that $\epsilon(x) = \epsilon'(x)$ for each $x \in E$. □

$\longleftarrow$ End of 20. class, on 25.11.2019.

Theorem 7.62. *Let $n \geq 0$ be an integer. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is $n+1$ times differentiable on I, and let $a \in I$ be an arbitrary real number. Then, for each $x \in I$ there is an x' (strictly) between a and x such that*

$$f(x) = \left(\sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i \right) + f^{(n+1)}(x')\frac{(x-a)^{n+1}}{(n+1)!}.$$

Proof. To understand the proof, note that the statement for $n = 0$ is just the Mean value theorem (Theorem 7.46). Indeed, that says that there is an x' between a and x such that $f'(x') = \frac{f(x)-f(a)}{x-a}$. If we multiply by $x-a$, we obtain

$$f(x) = f(a) + f'(a + \theta_{x,a}(x-a))(x-a).$$

Recall that the proof of Theorem 7.46 was an application of Rolle's theorem to the function $g(y) := f(y) - f(a) - \frac{f(x)-f(a)}{x-a}(y-a)$. Furthermore, this was working, because $g(a) = g(x)$, and because $g'(y) = f'(y) - \frac{f(x)-f(a)}{x-a}$. So, $g'(y)$ being 0 yielded exactly the above equation.

Define

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

And consider

$$g(y) = f(y) - P_n(y) + \frac{P_n(x) - f(x)}{(x-a)^{n+1}}(y-a)^{n+1}$$

Then, we have $0 = g(x) = g(a) = g'(a) = \cdots = g^{(n)}(a)$. This means that there is a y_1 between a and x such that $g'(y_1) = 0$ by Rolle's theorem. But then applying Rolle's theorem again we obtain a y_2 between a and y_1 such that $g^{(2)}(y_2) = 0$. Iterating this process we obtain y_{n+1} between a and x such that $g^{(n+1)}(y_{n+1}) = 0$. In particular, by setting $x' := y_{n+1}$, we obtain

$$0 = g^{(n+1)}(x') = f(x') + \frac{P_n(x) - f(x)}{(x-a)^{n+1}}(n+1)!.$$

Reorganizing the latter equation yields exactly the statement of the theorem. $\square$

Corollary 7.63. *Let $n \geq 0$ be a real number. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times continuously differentiable on I, and let $a \in I$ be an arbitrary real number. Then, the n-th order expansion of f around a exists and is*

$$\sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j.$$

The idea behind the proof of the corollary is that by the previous theorem the error term is $f^{(n+1)}(x') - f^{(n+1)}(x)$, which converges to zero as x goes to a as x' is between a and x, and $f^{(n)}$ is continuous. For the precise proof we refer to page 126 of the book.

Example 7.64. Applying Corollary 7.63 to $f(x) = \frac{1}{1-x}$ and $a = 0$ yields

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + x^n \epsilon(x)$$

as we have

$$\left(\frac{1}{1-x}\right)^{(n)} = n!(1-x)^{n+1} \qquad \Rightarrow \qquad f^{(n)}(0) = n! \qquad \Rightarrow \qquad \frac{f^{(n)}(0)}{n!} = 1.$$

Example 7.65. Applying Corollary 7.63 to $f(x) = e^x$ and $a = 0$ yields

$$e^x = \sum_{j=0}^{n} \frac{x^n}{n!} + x^n \epsilon(x),$$

as

$$(e^x)^{(n)} = e^x \qquad \Rightarrow \qquad f^{(n)}(0) = 1 \qquad \Rightarrow \qquad \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}.$$

Example 7.66. Similarly, the $2n+1$-th order expansion of $\cos(x)$ is

$$\cos(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j}}{(2j)!} + x^{2n+1}\epsilon(x)$$

and the $2n+2$-th order expansion of $\sin(x)$ is

$$\sin(x) = \sum_{j=0}^{n} (-1)^n \frac{x^{2j+1}}{(2j+1)!} + x^{2n+2}\epsilon(x)$$

Note that

$$\begin{aligned}
e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \dots \\
&= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \dots \\
&= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) = \cos(x) + i\sin(x)
\end{aligned}$$

which gives a way to figure out the expansions for $\cos(x)$ and $\sin(x)$.

Look up $\frac{1}{\sqrt{1-x}}$ from the book (page 127).

Example 7.67. One can also figure out expansions of products, sums, compositions, etc. For example the 3-rd order expansion of $\sin(\cos(x))$ is as follows:

$$\begin{aligned}
\cos(\sin(x)) &= \cos\left(x - \frac{x^3}{6} + x^3\epsilon_1(x)\right) \\
&= 1 - \frac{\left(x - \frac{x^3}{6} + x^3\epsilon_1(x)\right)^2}{2} + \left(x - \frac{x^3}{6} + x^3\epsilon_{\sin}(x)\right)^3 \epsilon_2(\sin(x)) = 1 - \frac{x^2}{2} + x^3\epsilon(x),
\end{aligned}$$

where $x^3\epsilon(x)$ is the sum of all terms of the form x^3 times something going to 0 as x goes to 0. In particular, $\lim_{x\to 0}\epsilon(x) = 0$ and hence the above is indeed the 3-rd order expansion. One warning should be given though here: the base-point of the expansion of the outside function should be the value of the inside function at the base-point of the inside expansion. So, for example, $\sin(\cos(x))$ at 0 is not easy to compute this way, because one would need the expansion of sin around $\cos(0) = 1$, for which there is no nice formula.

Another example is the composition of $\frac{1}{1-y}$ and $e^x - 1$:

$$\begin{aligned}
\frac{1}{1-(e^x-1)} &= \frac{1}{1 - \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_1(x)\right)} \\
&= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_1(x)\right) + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_1(x)\right)^2 \\
&\quad + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_1(x)\right)^3 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_1(x)\right)^3 \epsilon_2(\sin(x)) \\
&= 1 + x + \left(\frac{1}{2} + 1\right)x^2 + \left(\frac{1}{6} + 2\cdot 1\cdot\frac{1}{2} + 1\right)x^3 + x^3\epsilon(x) = 1 + x + \frac{3}{2}x^2 + \frac{13}{6}x^3 + x^3\epsilon(x).
\end{aligned}$$

Similarly, one can write the order 3 expansion of $\frac{1}{1-x}\cdot e^x$ around 0:

$$\frac{1}{1-x}\cdot e^x = (1+x+x^2+x^3+x^3\epsilon_1(x))\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^3\epsilon_2(x)\right) = 1+2x+\frac{5}{2}x^2+\frac{8}{3}x^2+x^3\epsilon(x)$$

See pages 127-131 for more examples.

Example 7.68. One can use expansions also to replace L'Hospital arguments. For example, if one would like to determine $\lim_{x\to 0}\frac{(e^x-1-x)+x\sin(x)}{\cos(x)-1}$, then can compute the 2-nd order expansions first:

$$\begin{aligned}
(e^x - 1 - x) + x\sin(x) &= \left(1 + x + \frac{x^2}{2} + x^2\epsilon_1(x) - 1 - x\right) + x(x + x^2\epsilon_2(x)) \\
&= \frac{x^2}{2} + x^2 + x^2\underbrace{(\epsilon_1(x) + \epsilon_2(x))}_{=:\epsilon_3(x)} = \frac{3}{2}x^2 + x^2\epsilon_3(x)
\end{aligned}$$

$$\cos(x) - 1 = 1 - \frac{x^2}{2} + x^2\epsilon_4(x) - 1 = -\frac{x^2}{2} + x^2\epsilon_4(x)$$

Then we have:

$$\lim_{x\to 0}\frac{(e^x - 1 - x) + x\sin(x)}{\cos(x) - 1} = \lim_{x\to 0}\frac{\frac{3}{2}x^2 + x^2\epsilon_3(x)}{-\frac{x^2}{2} + x^2\epsilon_4(x)} = \lim_{x\to 0}\frac{\frac{3}{2} + \epsilon_3(x)}{-\frac{1}{2} + \epsilon_4(x)} = -3.$$

7.5.4 Application of Taylor expansion to local extrema and inflection points

We have seen that if f has a local extremum at a, then $f'(a) = 0$ (Proposition 7.40). However, we have also seen that there is no backwards implication (Example 7.41). Nevertheless is there some sufficient conditions for local extrema?

Let us assume that we have a function f whose first many derivatives are 0 at a and the n-th one is the first that is not-zero, for some even number n. Let us assume that $f^{(n)}(a) > 0$. Then, we have an n-th order expansion:

$$f(x) = f(a) + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\epsilon(x).$$

So, there will be a small neighborhood of a where $|\varepsilon(x)| < \frac{1}{2}\cdot\frac{f^{(n)}(a)}{n!}$ holds. In particular, on this neighborhood we have:

$$f(a) + \frac{1}{2}\cdot\frac{f^{(n)}(a)}{n!}(x)(x-a)^n \le f(x) \le f(a) + \frac{3}{2}\cdot\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

This shows that a is a local minimum. There is a similar argument for local maximum too yielding the following:

Theorem 7.69. *Let $n \ge 2$ be an even integer. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times differentiable on I, and let $a \in I$ be an arbitrary real number. If $f'(a) = \dots = f^{(n-1)}(a) = 0$ and*

(1) if $f^{(n)} > 0$, then f has a local minimum, and

(2) if $f^{(n)} < 0$, then f has a local maximum.

Example 7.70. Consider the function $f(x) = \sin(x) + \frac{1}{2}x|_{[0,2\pi]}$.

We have $f'(x) = 0$ if and only if $\cos(x) = -\frac{1}{2}$ if and only if $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$. Whether or not we have a maximum or minimum at these points is decided by the sign of $f''(x) = -\sin(x)$.

- At $x = \frac{2\pi}{3}$, $f(x)'' < 0$, so $f(x)$ has a local maximum, and
- At $x = \frac{4\pi}{3}$, $f(x)'' > 0$, so $f(x)$ has a local minimum.

$\longleftarrow$ End of 21. class, on 27.11.2019.

Question 7.71. What if we have n odd in the above statement instead of even?

Then, our function locally looks like $(x-a)^3$, $(x-a)^5$, or $(x-a)^7$, etc. This looks opposite to what a local maximum is. That is, on one side we have exactly bigger than 0 and on the other side smaller than that. Such behavior is called an inflection point. That is, for an inflection point we require that on one side of a the function is above the tangent line, and on the other side it is below it:

Definition 7.72. If $f : E \to \mathbb{R}$ is a function such that f is differentiable at $a \in E$, then we say that f has an *inflection point* at a if there is an $\delta > 0$ such that either

(1) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x-a) > 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x-a) < 0$, or

(2) $\{x \in E | a < x < a + \delta\} \Rightarrow f(x) - f(a) - f'(a)(x - a) < 0$, and
$\{x \in E | a - \delta < x < a\} \Rightarrow f(x) - f(a) - f'(a)(x - a) > 0$.

Theorem 7.73. *Let $n \geq 3$ be an odd integer. Let $f : I \to \mathbb{R}$ be a function on an open interval I, which is n times differentiable on I, and let $a \in I$ be an arbitrary real number. If $f''(a) = \cdots = f^{(n-1)}(a) = 0$ and $f^{(n)}(a) \neq 0$, then f has an inflection point at a.*

The reason why this is true is the same as for the above theorem on local maximum/minimum.

Example 7.74. Consider $f(x) = \sin(x) - x$. Then $f'(x) = \cos(x) - 1$, $f''(x) = -\sin(x)$ and $f'''(x) = -\cos(x)$. Hence, $f'(0) = f''(0) = 0$, and $f'''(0) \neq 0$. Hence $f(x)$ has an inflection point at $x = 0$ according to Theorem 7.73.

7.5.5 Convex and concave functions

Definition 7.75. A function $f : I \to \mathbb{R}$ on an open interval is called convex (resp. concave) if for every $a, b \in I$ and every $\lambda \in [0, 1]$ we have:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

(resp. $f(\lambda a + (1 - \lambda)b) \geq \lambda f(a) + (1 - \lambda)f(b)$).

Geometrically, the above definition means the following. We may assume that $a < b$. Then, $x := \lambda a + (1 - \lambda)b$ is a point between a and b. There are two possibilities:

(1) either $f(x)$ is below the line segment connecting $(a, f(a))$ and $(b, f(b))$ (this is characterized by the fact that the slope of f between a and b is at least as big as the slope of f between a and x and at most as big as the slope of f between x and b), or

(2) $f(x)$ is above the line segment connecting $(a, f(a))$ and $(b, f(b))$ (this is characterized by the fact that the slope of f between a and b is at most as big as the slope of f between a and x and at least as big as the slope of f between x and b).

Now, f is convex, if and only if, the first case holds for all choices of $x = \lambda a + (1 - \lambda)b$ (where $\lambda \in [0, 1]$). Furthermore, f is concave if and only if the second case holds for all such x.

Theorem 7.76. *Let $f : I \to \mathbb{R}$ be a differentiable function on an open interval. Then f is convex (resp. concave) if and only if $f' : I \to \mathbb{R}$ is an increasing (resp. decreasing) function.*

Proof. We prove only the statements about convexity, as f is convex if and only if $-f$ is concave.

(1) First, let us assume that f is convex. Let $a < b$ be points of I. We want to prove that $f'(a) \leq f'(b)$. By the above characterization of convexity we have

$$\frac{f(b) - f(\lambda a + (1 - \lambda)b)}{b - (\lambda a + (1 - \lambda)b)} \geq \frac{f(b) - f(a)}{b - a}, \text{ and } \frac{f(\lambda a + (1 - \lambda)b) - f(a)}{(\lambda a + (1 - \lambda)b) - a} \leq \frac{f(b) - f(a)}{b - a}.$$

Now, as λ goes to 0, the left side of the first inequality converges to $f'(b)$, and as λ goes to 1 the left side of the second inequality converges to $f'(a)$. This yields:

$$f'(b) \geq \frac{f(b) - f(a)}{b - a} \geq f'(a)$$

(2) For, the other direction let us assume that f' is increasing. Fix $a < b \in I$. Set $x := \lambda a + (1 - \lambda)b$ for any $\lambda \in]0, 1[$ (for $\lambda = 0$ and 1 the convexity inequality is automatic). Then, the mean value theorem tells us that there are $a < x_1 < x < x_2 < b$ such that $\frac{f(x)-f(a)}{x-a} = f'(x_1)$ and $\frac{f(b)-f(x)}{b-x} = f'(x_2)$. In particular, by our assumption that the derivative is increasing it follows that $\frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(x)}{b-x}$. But this shows that f is convex by the above characterization of convexity in terms of slopes.

□

If f' is differentiable (or equivalently, f is differentiable twice), then f' being increasing is equivalent to f'' being at least 0.

Corollary 7.77. *Let $f : I \to \mathbb{R}$ be a two times differentiable function on an open interval. Then f is convex (resp. concave) if and only if $f''(x) \geq 0$ (resp. $f''(x) \leq 0$) for all $x \in I$.*

Example 7.78. $(e^x)'' = e^x$ so $e^x : \mathbb{R} \to \mathbb{R}$ is convex

Example 7.79. $\mathrm{Log}(x)'' = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$, so $\mathrm{Log} : \mathbb{R}^*_+ \to \mathbb{R}$ is concave.

Example 7.80. Here we explain why a differentiable convex function f is "above" the tangent line at $(a, f(a))$. More precisely this means that $f(x) \geq f(a) + f'(a)(x - a)$.

We explain this for $x > a$, leaving the $x < a$ case to the reader. So, assuming that $x > a$ we want to show that $f(x) \geq f(a) + f'(a)(x - a)$, or equivalently that

$$\frac{f(x) - f(a)}{x - a} \geq f'(a). \tag{7.80.c}$$

Indeed, by the Mean value theorem (Theorem 7.46) there is an $a < x' < x$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(x'). \tag{7.80.d}$$

As f is convex, f' is increasing (Theorem 7.76), and hence $f'(x') \geq f(a)$. Putting this together with (7.80.d) we obtain (7.80.c).

In particular, we obtain that e^x is "above" $1+x$ (the tangent at $0, 1$). Or, using the concave version, $\mathrm{Log}(x)$ is "below" $x - 1$.

7.5.6 Lipschitz continuity and Banach fixed point theorem

Example 7.81. The guiding question that we are going to answer is the following: we have seen in Example 6.69, by using Corollary 6.68, that the equation $\cos(x) = x$ has a solution x_0 between 0 and $\frac{\pi}{2}$. In fact, as $\cos(x)$ is bounded above by 1, $x_0 \leq 1$, or equivalently $x_0 \in [0, 1]$. As $\cos(x)' = -\sin(x)$, if we define $k := \sin(1) < 1$, then $|\cos(x)'| \leq k$ for every $x \in]0, 1[$.

In particular, *we claim that for any $x \neq y \in [0, 1]$*, we have

$$\left| \frac{\cos(x) - \cos(y)}{x - y} \right| \leq k.$$

Indeed, suppose the contrary. That is, there are $x, y \in [0, 1]$ such that

$$\left| \frac{\cos(x) - \cos(y)}{x - y} \right| > k.$$

Then by the mean value theorem (Theorem 7.46) applied to $\cos|_{[x,y]}$, there is a z strictly between x and y such that $|\cos(z)'| > k$. This contradicts the definition of k, and hence concludes our claim.

The property of $\cos|_{[0,1]}$ claimed in the above claim is called being k-Lipschitz (Definition 7.82), and it is the main property needed to state Theorem 7.84.

Definition 7.82. Let $k > 0$ be a real number. A function $f : E \to \mathbb{R}$ is *k-Lipschitz* if for every $x, y \in E$, $|f(x) - f(y)| \leq k|x - y|$. If $k < 1$, then we call k-Lipschitz functions *k-contractions.*

Proposition 7.83. *$f : I \to \mathbb{R}$ a function on an interval is k-Lipschitz (for any $k > 0$), then it is uniformly continuous on I (and hence also continuous on I).*

Proof. One can take $\varepsilon := k\delta$ in Definition 6.54. $\square$

Theorem 7.84. Banach fixed point theorem - contraction version *Let I be either $\mathbb{R}$ or a closed interval, and let $0 < k < 1$ be a real number. If $f : I \to I$ is a k-contraction, then there is a unique $c \in I$ such that $f(c) = c$ (so f has a fixed point). Furthermore, $c = \lim_{n\to\infty} x_n$, where x_n is a recursive sequence defined by $x_{n+1} = f(x_n)$ with any choice of $x_1 \in I$.*

$\longleftarrow$ End of 22. class, on 02.12.2019.

Idea of the proof. By Definition 7.82, we have $|x_{n+1} - x_n| \leq k^{n-1}|x_2 - x_1|$. Hence, by the triangle inequality for any $m > n$ we have

$$|x_m - x_n| \leq \sum_{i=n}^{m-1} |x_{i+1} - x_i| \leq \sum_{i=n}^{m-1} k^{i-1}|x_2 - x_1| = \frac{k^{n-1} - k^{m-1}}{1-k}|x_2 - x_1| \leq \frac{k^{n-1}}{1-k}|x_2 - x_1|.$$

As $k < 1$, the above inequality shows that (x_n) is a Cauchy sequence. Hence, $c := \lim_{n\to\infty} x_n$ exists and $c \in I$. The following stream of equalities conclude that there does exist a fixed point:

$$f(c) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = c.$$

Furthermore, the fixed point is unique, because if $c \neq d$ are two different fixed points, then:

$$|c - d| = |f(c) - f(d)| \leq k|c - d|,$$

which is a contradiction. $\square$

Example 7.85. Continuing on Example 7.81, we can approximate the fixed point $x_0 \in [0, 1]$

of $\cos(x)$ by the iteration method of Theorem 7.84 starting with $x_1 = 1$ or with $x_1 = 0.4$:

n	x_n	$x_{n+1} = \cos(x_n)$
1	1	0.54030230586814
2	0.54030230586814	0.857553215846393
3	0.857553215846393	0.654289790497779
4	0.654289790497779	0.793480358742566
5	0.793480358742566	0.701368773622757
6	0.701368773622757	0.763959682900654
7	0.763959682900654	0.722102425026708
8	0.722102425026708	0.75041776176376
9	0.75041776176376	0.73140404242251
10	0.73140404242251	0.744237354900557
11	0.744237354900557	0.735604740436347
12	0.735604740436347	0.741425086610109
13	0.741425086610109	0.737506890513243
14	0.737506890513243	0.740147335567876
15	0.740147335567876	0.738369204122323
16	0.738369204122323	0.739567202212256
17	0.739567202212256	0.738760319874211
18	0.738760319874211	0.739303892396906
19	0.739303892396906	0.738937756715344
20	0.738937756715344	0.739184399771494
21	0.739184399771494	0.739018262427412
22	0.739018262427412	0.739130176529671
23	0.739130176529671	0.739054790746917
24	0.739054790746917	0.739105571926536
25	0.739105571926536	0.739071365298945
26	0.739071365298945	0.739094407379091
27	0.739094407379091	0.739078885994992
28	0.739078885994992	0.739089341403393
29	0.739089341403393	0.739082298522402
30	0.739082298522402	0.739087042695332

n	x_n	$x_{n+1} = \cos(x_n)$
1	0.4	0.921060994002885
2	0.921060994002885	0.604975687265943
3	0.604975687265943	0.822515925550386
4	0.822515925550386	0.680379541565672
5	0.680379541565672	0.777334009662464
6	0.777334009662464	0.712785945518353
7	0.712785945518353	0.756542961958452
8	0.756542961958452	0.727213302346576
9	0.727213302346576	0.747029870594107
10	0.747029870594107	0.733710193866485
11	0.733710193866485	0.742695063534227
12	0.742695063534227	0.736648630284655
13	0.736648630284655	0.740724195468833
14	0.740724195468833	0.737980048873721
15	0.737980048873721	0.739829079879756
16	0.739829079879756	0.738583797313267
17	0.738583797313267	0.73942274621522
18	0.73942274621522	0.738857670920033
19	0.738857670920033	0.739238335432225
20	0.739238335432225	0.738981925685749
21	0.738981925685749	0.73915465111206
22	0.73915465111206	0.739038303337701
23	0.739038303337701	0.73911667757351
24	0.73911667757351	0.739063884188215
25	0.739063884188215	0.739099446648462
26	0.739099446648462	0.7390754914386
27	0.7390754914386	0.73909162799748
28	0.73909162799748	0.739080758236075
29	0.739080758236075	0.739088080246627
30	0.739088080246627	0.739083148056105

So, we see that Theorem 7.84 yields a very efficient method of approximating fixed points of contractions numerically. Indeed, the last three entries in both cases had already the same first 5 digits. One can show that at least the first 4 digits are correct here, so the truncation at 4 digits of the fixed point is 0.7390.

7.6 Leftover from study of functions

7.6.1 Asymptotes

There are three kinds of asymptotes:

(1) If for some $c \in \mathbb{R}$, $\lim\limits_{x \to c^-} f(x) = \pm\infty$ or $\lim\limits_{x \to c^+} f(x) = \pm\infty$, then f has a vertical asymptote.

(2) If for some $c \in \mathbb{R}$, $\lim\limits_{x \to +\infty} f(x) = c$ or $\lim\limits_{x \to -\infty} f(x) = c$, then f has a horizontal asymptote.

(3) If for some $a \neq 0, b \in \mathbb{R}$, $\lim\limits_{x \to +\infty} f(x) - ax = b$ or $\lim\limits_{x \to -\infty} f(x) - ax = b$, then we say that f has a slant asymptote.

Example 7.86. (1) vertical asymptote: $f(x) = \frac{1}{1-x}$ at $x = 1$.

(2) horizontal asymptote: $f(x) = 2 - e^{-x}$ at $y = 2$,

(3) $f(x) = 2 + 3x + \frac{1}{x^2}$ for $a = 3$ and $b = 2$.

8 Integration

8.1 Definition

The idea behind integration is that the integral $\int_a^b f(x)dx$ of a bounded function f on a closed interval $[a, b]$ should be the area under the graph of f. However, it is not that easy to say what this area means and when it is computable at all. If it is computable, we say that the function is integrable (Definition 8.11), and the value of this area is then called the integral $\int_a^b f(x)dx$ of f.

Now, the idea of trying to define the area under the graph of f is simple. We start with the only area that we can compute trustably, that is of rectangles, and then we try to approximate the area under the graph of f from above and from below using rectangles. These approximations are called upper and lower Darboux sums (Definition 8.3). We say that the area under the graph of f is computable, which as above means that the function is integrable, if these two approximations meet in the limit. This is spelled out in precise mathematical terms below.

Definition 8.1. A *partition* $\sigma = (x_i)$ of a bounded interval $[a, b]$ is an ordered collection $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of points of $[a, b]$.

The *norm* of σ is

$$\max\{x_i - x_{i-1} | 1 \le i \le n\}.$$

A *refinement* $\sigma' = (x'_i)$ of σ is a partition such that each value of x_i shows up amongst x'_i.

The *regular partition* of length n is $x_i := a + i\frac{b-a}{n}$.

Proposition 8.2. *Each two partitions have a common refinement, and each partition can be refined to another one with arbitrarily small norm.*

Definition 8.3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\sigma = (x_i)$ a partition of $[a, b]$. Then, the *upper Darboux sum* of f with respect to σ is

$$\overline{S}_\sigma = \sum_{i=1}^{n} M_i(x_i - x_{i-1}),$$

where $M_i := \operatorname{Sup}_{x \in [x_{i-1}, x_i]} f(x)$. The *lower Darboux sum* of f with respect to σ is

$$\underline{S}_\sigma = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

where $m_i := \operatorname{Inf}_{x \in [x_{i-1}, x_i]} f(x)$.

Example 8.4. Let us consider a constant function $f(x) = c$. Then for any partition σ,

$$\underline{S}_\sigma = \overline{S}_\sigma = \sum_{i=1}^{n} c(x_i - x_{i-1}) = \underbrace{cx_n - cx_0}_{\text{telescopic sum}} = c(b - a).$$

Example 8.5. Consider $f := x|_{[a,b]}$, and $\sigma_n = \left(a + \frac{i(b-a)}{n}\right)$ the regular partition. Then

$$\overline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)\frac{b-a}{n} = a(b-a) + \frac{n(n+1)}{2}\frac{(b-a)^2}{n^2}$$

and

$$\underline{S}_{\sigma_n} = \sum_{i=1}^{n} \left(a + (i-1)\frac{b-a}{n} \right) \frac{b-a}{n} = a(b-a) + \frac{(n-1)n}{2} \frac{(b-a)^2}{n^2},$$

where in both cases we used the following statement, which can be proved using induction for example

$$\sum_{i=1}^{n} i = \frac{(n+1)n}{2}.$$

Note that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n} = a(b-a) + \frac{(b-a)^2}{2} = \frac{b^2}{2} - \frac{a^2}{2}$.

Proposition 8.6. *Let M and m be upper and lower bounds for $f : [a,b] \to \mathbb{R}$. Then, for any partition σ of $[a,b]$, $m(b-a) \leq \overline{S}_\sigma, \underline{S}_\sigma \leq M(b-a)$. In particular, the sets*

$$\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

and

$$\{\underline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

are bounded.

Proof. Immediate from the definition □

Definition 8.7. Let $f : [a,b] \to \mathbb{R}$ be a bounded function, then the upper Darboux integral of f (on $[a,b]$) is

$$\overline{S} := \operatorname{Inf}\{\overline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

and the lower Darboux integral of f (on $[a,b]$) is

$$\underline{S} := \operatorname{Sup}\{\underline{S}_\sigma | \sigma \text{ is a partition of } [a,b]\}$$

Example 8.8. Using the above computation for the constant function Example 8.4, we see that if f is the constant function on $[a,b]$, then $\overline{S} = \underline{S} = (b-a)c$.

Proposition 8.9. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function.*

(1) If σ is a partition of $[a,b]$ and σ' is a refinement of σ, then:

$$\underline{S}_\sigma \leq \underline{S}_{\sigma'}, \text{ and } \overline{S}_\sigma \geq \overline{S}_{\sigma'}.$$

(2) If σ is a partition of $[a,b]$, then:

$$\underline{S}_\sigma \leq \overline{S}_\sigma.$$

Corollary 8.10. *If $f : [a,b] \to \mathbb{R}$ is a bounded function, then $\underline{S} \leq \overline{S}$.*

Proof. It is enough to prove that $\underline{S}_{\sigma_1} \leq \overline{S}_{\sigma_2}$ for any partitions σ_1 and σ_2 of $[a,b]$. However, this follows straight from Proposition 8.9. Indeed, if σ is a common refinement of σ_1 and σ_2, then Proposition 8.9 yields that

$$\underline{S}_{\sigma_1} \leq \underbrace{\underline{S}_\sigma}_{\text{Proposition 8.9.(1)}} \leq \underbrace{\overline{S}_\sigma}_{\text{Proposition 8.9.(2)}} \leq \underbrace{\overline{S}_{\sigma_2}}_{\text{Proposition 8.9.(1)}}.$$

□

Definition 8.11. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We say that f is *integrable*, if $\overline{S} = \underline{S}$, in which case this common value is called the *integral of f between a and b*, and it is denoted by

$$\int_a^b f(x)dx.$$

Remark 8.12. Using Corollary 8.10, f is integrable if one shows a sequence σ_n of partitions such that $\lim_{n\to\infty} \overline{S}_{\sigma_n} = \lim_{n\to\infty} \underline{S}_{\sigma_n}$. Indeed, this follows immediately as,

$$\lim_{n\to\infty} \underline{S}_{\sigma_n} \leq \underline{S} \leq \overline{S} \leq \lim_{n\to\infty} \overline{S}_{\sigma_n}. \tag{8.12.a}$$

Example 8.13. Using Example 8.4, the constant functions are integrable on $[a, b]$, and

$$\int_a^b c\, dx = (b-a)c$$

Example 8.14. Using Remark 8.12 and the computation of Example 8.5 for $f(x) := x|_{[a,b]}$, we see that $x|_{[a,b]}$ is integrable, and

$$\int_a^b x dx = \frac{b^2}{2} - \frac{a^2}{2}$$

Example 8.15. Consider the function $[0, 2] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 3 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, for all partition σ, $\overline{S}_\sigma = 6$, and $\underline{S}_\sigma = 0$. So, $\overline{S} = 6$, $\underline{S} = 0$, and hence f is not integrable.

Proposition 8.16. *If $f : [a, b] \to \mathbb{R}$ is continuous then it is integrable. More precisely, we know that the above assumptions imply uniform continuity of f. So, fix $\varepsilon > 0$. Let $\delta > 0$ be the constant in the definition of uniform continuity associated to $\frac{\varepsilon}{b-a}$ (that is, $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \frac{\varepsilon}{b-a}$), and let σ be a partition of $[a, b]$ with norm at most δ. Then $\overline{S}_\sigma - \underline{S}_\sigma \leq \varepsilon$.*

Proof. The statement is the direct consequence of the definition:

$$\overline{S}_\sigma - \underline{S}_\sigma = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=1}^{n} \frac{\varepsilon}{b-a}(x_i - x_{i-1}) = \underbrace{\frac{\varepsilon}{b-a}(b-a)}_{\text{telescopic sum}} = \varepsilon$$

□

8.2 Basic properties

Proposition 8.17. *If $f, g : [a, b] \to \mathbb{R}$ are integrable and $\alpha, \beta \in \mathbb{R}$, then*

(1) If f extends over $[b, c]$ for some $b < c \in \mathbb{R}$ and it is also integrable over $[b, c]$, then it is integrable over $[a, c]$, and

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

(2) $\alpha f + \beta g$ is then also integrable.

$$\int_a^b (\alpha f + \beta g)(x)dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$$

(3) If $f \leq g$, then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

(4) $|f|$ is then also integrable, and

$$\int_a^b |f(x)|dx \geq \left| \int_a^b f(x)dx \right|$$

Proof. All these statements are the same. One writes up the inequalities for lower and for upper Darboux sums for fixed partitions. Then these remain valid when taking Sup or Inf. This gives inequalities in both direction, which then implies equalities.

For example, let us look at how this goes in the case of point (2) (we leave the rest to the reader). Let σ, and τ be partitions for $[a, b]$ and $[b, c]$ respectively. Then these induce a partition ρ for $[a, c]$ and by definition we have

$$\overline{S}_\rho^{[a,c]} = \overline{S}_\sigma^{[a,b]} + \overline{S}_\tau^{[b,c]} \qquad \text{and} \qquad \underline{S}_\rho^{[a,c]} = \underline{S}_\sigma^{[a,b]} + \underline{S}_\tau^{[b,c]}.$$

As this is true for all σ and τ, we obtain by taking Inf and Sup that

$$\overline{S}^{[a,c]} \leq \overline{S}^{[a,b]} + \overline{S}^{[b,c]}, \text{ and } \underline{S}^{[a,c]} \geq \underline{S}^{[a,b]} + \underline{S}^{[b,c]}. \tag{8.17.a}$$

However, as f is integrable both on $[a, b]$ and on $[b, c]$, we have $\int_a^b f(x)dx = \overline{S}^{[a,b]} = \underline{S}^{[a,b]}$ and $\int_b^c f(x)dx = \overline{S}^{[b,c]} = \underline{S}^{[b,c]}$. So, (8.17.a) yields that

$$\int_a^b f(x)dx + \int_b^c f(x)dx \leq \underline{S}^{[a,c]} \leq \overline{S}^{[a,c]} \leq \int_a^b f(x)dx + \int_b^c f(x)dx.$$

As the two ends of (8.2) are the same, we have everywhere equalities. This concludes both the integrability of f over $[a, c]$ as well as the statement of (1). □

Example 8.18.

$$\int_a^b (1+x)dx = \underbrace{\int_a^b 1dx + \int_a^b xdx}_{\text{point (2) of Proposition 8.17}} = \underbrace{(b-a)}_{\text{Example 8.13}} + \underbrace{\frac{b^2-a^2}{2}}_{\text{Example 8.14}}.$$

8.3 Fundamental theorem of calculus

Here we learn how to compute integrals using the anti-derivative in Theorem 8.24. The latter is defined below:

Definition 8.19. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. A function $G : [a, b] \to \mathbb{R}$ is called an *anti-derivative* of f if G is continuous on $[a, b]$, it is differentiable on $]a, b[$, and $G'(x) = f(x)$ for all $x \in]a, b[$.

Remark 8.20. According to Corollary 7.48 an anti-derivative, if exists, is determined up to adding a constant. For that reason, many times the anti-derivative of f is denoted by $\int f(x)dx + c$. Also, sometimes it is also called the indefinite integral, and what we defined as the integral sometimes is called the definite integral. We use the integral/anti-derivative naming in this course.

Example 8.21.

function	e^x	$\cos(x)$	$\sin(x)$	$\frac{1}{x}$	x	...
anti-derivative	e^x	$\sin(x)$	$-\cos(x)$	$\operatorname{Log}\lvert x\rvert$	$\frac{x^2}{2}$	...

The next statement is not too interesting in itself, however it is needed in the proof of Theorem 8.24. ⟵ End of 23. class, on 04.12.2019.

Theorem 8.22. $\boxed{\text{MEAN VALUE THEOREM FOR INTEGRALS}}$

If $f : [a, b] \to \mathbb{R}$ is continuous, then there is a $c \in [a, b]$, such that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Proof. As $[a, b]$ is closed and f is continuous, by Theorem 6.61, f takes its maximum and minimum. Set $M := \max_{x\in[a,b]} f(x)$ and $m := \min_{x\in[a,b]} f(x)$. By Theorem 6.65, f takes all values in $[m, M]$. However, by Proposition 8.6, we have

$$m \leq \frac{\int_a^b f(x)dx}{b - a} \leq M,$$

so there is a $c \in [a, b]$ such that $f(c)$ equals the above fraction, which is exactly the statement of the theorem. $\square$

So far we defined $\int_a^b f(x)dx$ only for $a < b$. If $a = b$, then we define it to be 0, and if $a > b$, then we define $\int_a^b f(x) := -\int_b^a f(x)dx$. With these notations our previously proven rules give that if $f : [a, b] \to \mathbb{R}$ is continous, and $c, d \in [a, b]$ are any points, then

$$\int_a^c f(x)dx + \int_c^d f(x)dx = \int_a^d f(x)dx.$$

We use this notation in the following proof, which is an essential step towards our final goal, Theorem 8.24.

Theorem 8.23. $\boxed{\text{FUNDAMENTAL THEOREM OF CALCULUS I}}$

Let $f : [a, b] \to \mathbb{R}$ be continuous. Then,

$$F(x) := \int_a^x f(t)dt$$

is an anti-derivative of f.

Proof. Fix $x_0 \in]a, b[$. Then, for any $x_0 \neq x \in]a, b[$:

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0}\int_{x_0}^x f(t)dt = \underbrace{f(c(x))}_{\text{Theorem 8.22}},$$

for a real number $c(x)$ between x and x_0. Hence:

$$\lim_{x\to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x\to x_0} f(c(x)) = \underbrace{\lim_{x\to x_0} f(x)}_{\lim_{x\to x_0} c(x)=x_0} = \underbrace{f(x_0)}_{f \text{ is continuous}}.$$

$\square$

Theorem 8.24. $\boxed{\text{FUNDAMENTAL THEOREM OF CALCULUS II}}$

Let $f : [a, b] \to \mathbb{R}$ be continuous and let G be an anti-derivative of f. Then,

$$\int_a^b f(x)dx = G(b) - G(a).$$

Proof. We have already shown in Theorem 8.23 that $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f. As both F and G are anti-derivatives, they differ in a constant, say c. So, $G + c = F$. Then:

$$G(b) - G(a) = (G(b) + c) - (G(a) + c) = F(b) - F(a) = \int_a^b f(x)dx - \int_a^a f(x)dx = \int_a^b f(x)dx.$$

□

Notation 8.25. The $G(b) - G(a)$ in Theorem 8.24 is usually denoted by

$$G(x)|_a^b \qquad \text{or} \qquad G(x)|_{x=a}^{x=b}.$$

Example 8.26.

$$\int_{-5}^{-1} \frac{1}{x} = \big(\operatorname{Log}|x|\big)\big|_{x=-5}^{x=-1} = \operatorname{Log} 1 - \operatorname{Log} 5 = -\operatorname{Log} 5$$

8.4 Substitution

Theorem 8.27. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let $\phi : [\alpha, \beta] \to [a, b]$ be a continuously differentiable function. Then:*

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t)dt. \tag{8.27.a}$$

Proof. Define $G(x) := \int_a^x f(u)du$. By Theorem 8.23, G is an anti-derivative of f, and hence Theorem 8.24 tells us that the left side of the equation equals $G(\phi(\beta)) - G(\phi(\alpha))$. So, it is enough to show that the right side equals the same quantity.

For that, note that by the chain rule $G(\phi(t))' = G'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$. However, then Theorem 8.24 applied to the integral on the right side of the equation we obtain that this integral equals $G(\phi(\beta)) - G(\phi(\alpha))$. □

Example 8.28. First an example, where we go from the right side of (8.27.a) to the left side.

$$\int_0^1 \sqrt{e^x} e^x dx = \underbrace{\int_1^e \sqrt{u} du}_{u=e^x \quad (e^x)'=e^x} = \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_{u=1}^{u=e} = \frac{2}{3}\left(e^{\frac{3}{2}} - 1\right)$$

Example 8.29. The integral below computes the area of a quarter of a circle of radius 1, so the result should be $\frac{\pi}{4}$. Indeed, the above computation shows that are train of thought is correct. Note that, opposite to the previous example, in this argument at our first substitution we go from the left side of (8.27.a) to the right side.

$$\int_0^1 \sqrt{1 - x^2} dx = \underbrace{\int_0^{\frac{\pi}{2}} \sqrt{1 - (\sin(t))^2} \cos(t)dt}_{x=\sin(t) \quad \sin(t)'=\cos(t)} = \int_0^{\frac{\pi}{2}} \sqrt{\cos(t)^2} \cos(t)dt$$

$$= \int_0^{\frac{\pi}{2}} |\cos(t)| \cos(t)dt = \underbrace{\int_0^{\frac{\pi}{2}} \cos(t)\cos(t)dt}_{t\in\left[0,\frac{\pi}{2}\right] \Rightarrow \cos(t)\geq 0 \Rightarrow |\cos(t)|=\cos(t)} = \int_0^{\frac{\pi}{2}} \frac{\cos(2t) + 1}{2} dt$$

$$= \underbrace{\int_0^{\pi} \frac{\cos(u) + 1}{2} \frac{1}{2} du}_{t=\frac{u}{2}} = \frac{1}{4} \int_0^{\pi} \cos(u) + 1du = \frac{1}{4} (\sin(u) + u)|_{u=0}^{u=\pi}$$

$$= \frac{1}{4}(\sin(\pi) + \pi - \sin(0) - 0) = \frac{\pi}{4}$$

Example 8.30. Recall that $\mathrm{sh}(x) : \mathbb{R} \to \mathbb{R}$ is an odd function and it is strictly increasing (indeed, $\mathrm{sh}(x)' = \mathrm{ch}(x) > 0$). In particular, it has an inverse, which we denote by $\mathrm{Argsh}(x) : \mathbb{R} \to \mathbb{R}$. With this we may compute similarly :

$$\int_0^1 \sqrt{1+x^2}dx = \underbrace{\int_0^{\mathrm{Argsh}(1)} \sqrt{1+(\mathrm{sh}(t))^2}\,\mathrm{ch}(t)dt}_{x=\mathrm{sh}(t) \qquad \mathrm{sh}(t)'=\mathrm{ch}(t)} = \int_0^{\mathrm{Argsh}(1)} \sqrt{\mathrm{ch}(t)^2}\,\mathrm{ch}(t)dt$$

$$= \int_0^{\mathrm{Argsh}(1)} |\,\mathrm{ch}(t)^2|\,\mathrm{ch}(t)dt = \underbrace{\int_0^{\mathrm{Argsh}(1)} \mathrm{ch}(t)\,\mathrm{ch}(t)dt}_{\mathrm{ch}(t)>0 \ \Rightarrow\ |\,\mathrm{ch}(t)|=\mathrm{ch}(t)} = \int_0^{\mathrm{Argsh}(1)} \frac{\mathrm{ch}(2t)+1}{2}dt$$

$$= \underbrace{\int_0^{2\,\mathrm{Argsh}(1)} \frac{\mathrm{ch}(u)+1}{2}\frac{1}{2}du}_{t=\frac{u}{2}} = \frac{1}{4}\int_0^{2\,\mathrm{Argsh}(1)} \mathrm{ch}(u)+1du = \frac{1}{4}\left(\mathrm{sh}(u)+u\right)|_{u=0}^{u=2\,\mathrm{Argsh}(1)}$$

$$= \frac{\mathrm{sh}(2\,\mathrm{Argsh}(1)) + 2\,\mathrm{Argsh}(1)}{4} = \frac{2\,\mathrm{sh}(\mathrm{Argsh}(1))\,\mathrm{ch}(\mathrm{Argsh}(1)) + 2\,\mathrm{Argsh}(1)}{4}$$

$$= \frac{2\,\mathrm{sh}(\mathrm{Argsh}(1))\sqrt{1+\mathrm{sh}(\mathrm{Argsh}(1))^2} + 2\,\mathrm{Argsh}(1)}{4}$$

$$= \frac{2\cdot 1\cdot\sqrt{1+1^2} + 2\,\mathrm{Argsh}(1)}{4} = \frac{2\sqrt{2} + 2\,\mathrm{Argsh}(1)}{4}$$

Example 8.31. Substitution can be used the generally integrate $\cos(x)^n$ and $\sin(x)^n$ for $n > 0$ an integer. The simplest case is when n odd. Here is an example of that:

$$\int_0^{\frac{\pi}{2}} \cos(x)^5 dx = \int_0^{\frac{\pi}{2}} \cos(x)(1-\sin(x)^2)^2 dx = \underbrace{\int_0^1 (1-u^2)^2 du}_{u(x)=\sin(x) \qquad u(x)'=\cos(x)}$$

$$= \int_0^1 1 - 2u^2 + u^4 du = u - \frac{2u^3}{3} + \frac{u^5}{5}\bigg|_{u=0}^{u=1} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{6}{15} = \frac{2}{5}$$

On the other hand when the power is odd half angle formulas, helps to get it down to odd power:

$$\int_0^{\frac{\pi}{2}} \sin^4(x)dx = \int_0^{\frac{\pi}{2}} \left(\frac{1-\cos(2x)}{2}\right)^2 dx = \int_0^{\frac{\pi}{2}} \frac{1}{4} - \frac{\cos(2x)}{2} + \frac{\cos(2x)^2}{4}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}dx - \int_0^{\frac{\pi}{2}} \frac{\cos(2x)}{2}dx + \int_0^{\frac{\pi}{2}} \frac{\cos(2x)^2}{4}dx = \frac{\pi}{8} - \frac{\sin(2x)}{4}\bigg|_{x=0}^{x=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \frac{\cos(4x)+1}{8}dx$$
$$= \frac{\pi}{8} + \frac{\pi}{16} + \frac{\sin(4x)}{32}\bigg|_{x=0}^{x=\frac{\pi}{2}} = \frac{\pi}{8} + \frac{\pi}{16} = \frac{5\pi}{16} \qquad (8.31.b)$$

8.5 Integration by parts

Theorem 8.32. *If $f, g : I \to \mathbb{R}$ are two continuously differentiable functions on an open interval, and $a < b$ elements of I, then*

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx,$$

Proof. Equivalently, it is enough to show that

$$\int_a^b (f(x)g'(x) + f'(x)g(x))dx = f(x)g(x)|_a^b.$$

However, this follows immediately from Theorem 8.24 as

$$(f(x)g(x))' = f(x)g'(x) + f'(x)g(x),$$

by the product rule. $\square$

Generally integration by parts are useful for products. The main question is how you distribute f and g. There is a rule which works in most cases (but not always!). The idea that on this list you find the first type of function that you have in your product (of two different type of functions), and you assign g' to be that:

(1) E(xponential)

(2) T(rigonometric)

(3) A(lgebraic, that is, polynomial)

(4) L(ogarithm)

(5) I(nverse trigonometric).

Here are some examples (omitting the limits, so put $(\_)|_{x=a}^b$ around every function not between an integral and dx, and put limits a and b on every integral sign). Or alternatively you can treat these as the rules for anti-derivatives.

Example 8.33.

$$\int xe^x dx = \underbrace{xe^x - \int e^x dx}_{g'(x)=e^x \quad g(x)=e^x \quad f(x)=x \quad f'(x)=1} = xe^x - e^x = (x-1)e^x$$

$\longleftarrow$ End of 24. class, on 09.12.2019.

Example 8.34.

$$\int \sin(x)e^x dx = \underbrace{\sin(x)e^x - \int \cos(x)e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\sin(x), f'(x)=\cos(x)} = \underbrace{\sin(x)e^x - \cos(x)e^x + \int (-\sin(x))e^x dx}_{g'(x)=e^x, g(x)=e^x, f(x)=\cos(x), f'(x)=-\sin(x)}$$

So, comparing the two endpoints we have the equality:

$$\int \sin(x)e^x dx = \frac{e^x(\sin(x) - \cos(x))}{2}.$$

Example 8.35.

$$\int \operatorname{Log}(x)dx = \underbrace{x\operatorname{Log}(x) - \int 1dx}_{f(x)=\operatorname{Log}(x), f'(x)=\frac{1}{x}, g'(x)=1, g(x)=x} = x\operatorname{Log}(x) - x$$

Example 8.36.

$$\int \operatorname{Arctg}(x) = \underbrace{x\operatorname{Arctg}(x) - \int \frac{x}{1+x^2}dx}_{f(x)=\operatorname{Arctg}(x), f'(x)=\frac{1}{1+x^2}, g'(x)=1, g(x)=x} = \underbrace{x\operatorname{Arctg}(x) - \frac{1}{2}\int \frac{1}{u}du}_{u(x)=1+x^2, u(x)'=2x}$$

$$= x\operatorname{Arctg}(x) - \frac{1}{2}\operatorname{Log}|u| = x\operatorname{Arctg}(x) - \frac{1}{2}\operatorname{Log}|1+x^2|$$

8.6 Integrating rational functions

A *rational function*, is a function of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials with real coefficients.

By a theorem called the FUNDAMENTAL THEOREM OF ALGEBRA (which we do not prove as it is really-really hard), $Q(x)$ can be written as

$$Q(x) = (x - a_1)^{k_1} \dots (x - a_n)^{k_n} (x^2 + 2b_1 x + c_1)^{l_1} \dots (x^2 + 2b_m x + c_m)^{l_m}, \tag{8.36.a}$$

where a_i, b_i and c_i are real numbers, $k_i, l_i > 0$ are integers, and there is no real number x_0 such that $x_0^2 + 2b_i x_0 + c_i = 0$ (for any $1 \leq i \leq m$).

Remark 8.37. The fundamental theorem of algebra is originally for polynomials $R(x)$ with complex coefficients, and states that they can factored into linear terms with complex coefficients. That is, there are complex numbers $d - i$ such that

$$R(x) = (x - d_1)^{s_1} \dots \left(x - d_n'\right)^{s_n}. \tag{8.37.b}$$

This does work also for $R(x) := Q(x)$, however it possibly yields d_i that are complex and not real. Then, the expression cannot be used for integration because we did not learn integration of complex valued functions.

So, the idea is to collect the d_i that are real numbers. These become the a_i in (8.36.a). Furthermore, because $Q(x)$ has real coefficients, (8.37.b) for $R(x) = Q(x)$ is invariant under conjugation. Hence, whenever d_i is not real, then its conjugate has to also show up with the same power. That is, we have a factor of the right hand side of (8.37.b) of the form:

$$(x - d_i)^{s_i} \left(x - \overline{d_i}\right)^{s_i} = \left((x - d_i)\left(x - \overline{d_i}\right)\right)^{s_i} = \left(x - 2\left(d_i + \overline{d_i}\right) + d_i \overline{d_i}\right)^{s_i} = \left(x - 2\,\mathrm{Re}(d_i) + |d_i|^2\right)^{s_i}$$

Then we set $b_j = -\mathrm{Re}(d_i)$, $l_j = s_i$ and $c_j = |b_i|^2$, and we obtain one of the terms of the form $(x^2 + 2b_j x + c_j)^{l_j}$ in (8.36.a).

Example 8.38. Take $Q(x) = x^3 + x^2 - 2$, and consider the factorization as in (8.36.a). As the degree of Q is three, there must be a linear term (the product of the non-linear terms has even degree). This correspond to a real root of $Q(x)$, so let us search for it.

(1) *Finding the real root.*

1st try: $Q(0) = -2$. As $\lim_{x \to +\infty} Q(x) = +\infty$, according to the Intermediate value theorem (Theorem 6.65) Q has a root greater than 0.

2nd try: $Q(1) = 0$. We found the root, great.

(Unfortunately, for high degrees there is no algorithm of finding the roots, one just has to do trial and error using the intermediate theorem, hoping that the makers of the exercise set up a nice root.)

(2) *Factoring out the linear term.*

$$\begin{array}{ccccccc} x^3 & + & x^2 & & & - & 2 \quad : \quad x - 1 \quad = \quad x^2 + 2x + 2 \\ x^3 & - & x^2 & & & & \\ \hline & & 2x^2 & & & - & 2 \\ & & 2x^2 & - & 2x & & \\ \hline & & & & 2x & - & 2 \\ & & & & 2x & - & 2 \\ \hline & & & & & & 0 \end{array}$$

Hence, we have

$$(x^2 + 2x + 2)(x - 1) = x^3 + x^2 - 2$$

A particular consequence of (8.36.a) is the following:

Proposition 8.39. *Any rational function* $\frac{P(x)}{Q(x)}$ *can be written as*

$$\frac{P(x)}{Q(x)} = \alpha_1 R_1(x) + \cdots + \alpha_t R_t(x),$$

where the α_i *are real numbers, and* $R_i(x)$ *are of the form:*

(1) polynomial, or

(2) $\frac{1}{(x-r)^p}$*, or*

(3) $\frac{x+c}{(x^2+2rx+s)^p}$.

Instead of giving a proof, we explain the idea behind Proposition 8.39 in the following example:

Example 8.40.

$$\frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2} = \frac{Ax+B}{(x^2+x+1)^2} + \frac{Cx+D}{x^2+x+1} = \frac{Ax+B+(Cx+D)(x^2+x+1)}{(x^2+x+1)^2}$$
$$= \frac{Cx^3+(C+D)x^2+(A+C+D)x+(B+D)}{(x^2+x+1)^2}$$

So, we have

$$\begin{array}{cl} C & = 4 \\ C+D & = 9 \\ A+C+D & = 11 \\ B+D & = 8 \end{array}$$

This yields

$$C = 4 \Rightarrow 4+D = 9 \Rightarrow D = 5 \Rightarrow A+4+5 = 11; B+5 = 8 \Rightarrow A = 2; B = 3$$

That is,

$$\frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2} = \frac{2x+3}{(x^2+x+1)^2} + \frac{4x+5}{x^2+x+1}.$$

Having the decomposition stated in Proposition 8.39, the question is how we integrate these terms separately:

Example 8.41. ∘ $p > 1$, then

$$\int \frac{1}{(x-r)^p}dx = \frac{(x-r)^{1-p}}{1-p}$$

∘ $p = 1$, then

$$\int \frac{1}{(x-r)}dx = \operatorname{Log}|x-r|$$

Example 8.42.

$$\int \frac{x+c}{(x^2+2rx+s)^p} = \frac{1}{2}\int \frac{2(x+r)}{(x^2+2rx+s)^p}dx + \int \frac{c-r}{(x^2+2rx+s)^p}dx,$$

where

$$\int \frac{2(x+r)}{(x^2+2rx+s)^p} = \begin{cases} \operatorname{Log}|x^2+2rx+s| & \text{if } p = 1 \\ \frac{(x^2+2rx+s)^{1-p}}{1-p} & \text{if } p > 1 \end{cases}$$

Hence, according to the last two examples, we know how to integrate all the terms in Proposition 8.39, except we have not figured out yet for $p > 0$ the integral

$$\int \frac{1}{(x^2+2rx+s)^p}dx = \int \frac{1}{((x+r)^2+(s-r^2))^p}dx = \underbrace{\frac{1}{(s-r^2)^p}\int \frac{1}{\left(\left(\frac{x+r}{\sqrt{s-r^2}}\right)^2+1\right)^p}dx}_{s-r^2>0,\text{ as } x^2+2rx+s \text{ has no real roots}}$$

$$= \underbrace{\frac{1}{(s-r^2)^{p-\frac{1}{2}}}\int \frac{1}{(u^2+1)^p}du}_{u=\frac{x+r}{\sqrt{s-r^2}}}$$

Set

$$I_p := \int \frac{1}{(u^2+1)^p}du.$$

Luckily, we have $I_1 := \operatorname{Arctg}(u)$. Furthermore, if $p \geq 1$, then we can obtain a recursive formula as follows:

$$I_p = \int \frac{1}{(u^2+1)^p}du = \underbrace{\frac{u}{(u^2+1)^p} - \int \frac{-pu\cdot 2u}{(u^2+1)^{p+1}}du}_{\text{Integration by parts with } f(u)=\frac{1}{(u^2+1)^p},\, g'(u)=1}$$

$$= \frac{u}{(u^2+1)^p} + 2p\int \frac{u^2+1-1}{(u^2+1)^{p+1}}du = \frac{u}{(u^2+1)^p} + 2pI_p - 2pI_{p+1}$$

So, by looking at the two ends of the equation, we obtain the recursive equality:

$$I_{p+1} = \frac{\frac{u}{(u^2+1)^p} + (2p-1)I_p}{2p}.$$

Be careful, there is an error on page 201 of the book about this, where intead of $2p-1$, $2(p-1)$ is written!!

Example 8.43. Let us compute I_2 for example:

$$I_2 = \frac{\frac{u}{u^2+1} + I_1}{2} = \frac{1}{2}\left(\frac{u}{u^2+1} + \operatorname{Arctg}(u)\right)$$

Example 8.44. Let us get back to Example 8.40:

$$\int \frac{4x^3+9x^2+11x+8}{(x^2+x+1)^2}dx = \underbrace{\int \frac{2x+3}{(x^2+x+1)^2}dx + \int \frac{4x+5}{x^2+x+1}dx}_{\text{We apply the result of Example 8.40}}$$

$$= \underbrace{\int \frac{2x+3}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx + \int \frac{4x+5}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx}_{\text{We complete the square in the denominators}}$$

$$= \underbrace{\int \frac{(2x+1)+2}{\left(\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\right)^2}dx + \int \frac{(4x+2)+3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx}_{\text{We express also the numertor in terms of a multiple of } x+\frac{1}{2}}$$

$$= \underbrace{\int \left(\frac{2}{\sqrt{3}}\right)^3 \frac{\frac{2}{\sqrt{3}}(2x+1)+\frac{4}{\sqrt{3}}}{\left(\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right)^2}dx + \int \frac{2}{\sqrt{3}} \frac{\frac{2}{\sqrt{3}}(4x+2)+\frac{6}{\sqrt{3}}}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}dx}_{\substack{\text{We multiply the numerators and the denominators by adequate multiples of } \frac{2}{\sqrt{3}}, \\ \text{to make them of the form } u^2+1 \text{ or } (u^2+1)^2}}$$

$$= \underbrace{\frac{4}{3}\int \frac{2u+\frac{4}{\sqrt{3}}}{(u^2+1)^2}du + \int \frac{4u+\frac{6}{\sqrt{3}}}{u^2+1}du}_{u=\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right) \quad \Rightarrow \quad x=\frac{\sqrt{3}}{2}u-\frac{1}{2} \quad \Rightarrow \quad x(u)'=\frac{\sqrt{3}}{2}}$$

$$= \frac{4}{3}\int \frac{2u}{(u^2+1)^2}du + \frac{16}{3\sqrt{3}}\int \frac{1}{(u^2+1)^2}du + 2\int \frac{2u}{u^2+1}dx + \frac{6}{\sqrt{3}}\int \frac{1}{u^2+1}du$$

$$= \frac{4}{3}\frac{-1}{u^2+1} + \frac{8}{3\sqrt{3}}\left(\frac{u}{u^2+1}+\operatorname{Arctg}(u)\right) + 2\operatorname{Log}|u^2+1| + \frac{6}{\sqrt{3}}\operatorname{Arctg}(u)$$

$$= \frac{4}{3}\frac{-1}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1} + \frac{8}{3\sqrt{3}}\left(\frac{\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)}{\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1}+\operatorname{Arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right)$$

$$+ 2\operatorname{Log}\left|\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)^2+1\right| + \frac{6}{\sqrt{3}}\operatorname{Arctg}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)$$

⟵ End of 25. class, on 11.12.2019.

8.6.1 Rational functions in exponentials

There is a method of integrating functions obtained by plugging in e^x into a rational function. We explain it via the next example:

Example 8.45.

$$\int \frac{1}{e^x+1}dx = \int \frac{1}{(e^x+1)e^x}e^x dx = \underbrace{\int \frac{1}{(t+1)t}dt}_{t(x)=e^x \quad t(x)'=e^x} = \int \frac{1}{t} - \frac{1}{t+1}dt$$

$$= \operatorname{Log}|t| - \operatorname{Log}|t+1| = \operatorname{Log}|e^x| - \operatorname{Log}|e^x+1| = x - \operatorname{Log}|e^x+1|$$

8.6.2 Rational functions in roots

There is a method of integrating functions obtained by plugging in $\sqrt{x}$ into a rational function. We explain it via the next example:

Example 8.46.

$$\int \frac{1}{\sqrt{x}+1}dx = \int \left(\frac{1}{\sqrt{x}+1}2\sqrt{x}\right)\frac{1}{2}\frac{1}{\sqrt{x}}dx = \underbrace{\int \frac{1}{t+1}2tdt}_{t(x)=\sqrt{x} \quad t(x)'=\frac{1}{2}\frac{1}{\sqrt{x}}}$$

$$= \int 2 - \int \frac{2}{t+1}dt = 2t - 2\operatorname{Log}|t+1| = 2\sqrt{x} - 2\operatorname{Log}|\sqrt{x}+1|$$

There is a similar substitution if we have a rational function of $\sqrt[n]{x}$ for any n (check out the book!).

8.7 Improper integrals

The question is how to make sense of integrals of the form $\int_1^{+\infty} \frac{1}{x^2}dx$. Or more generally, we have a real valued function f that is continuous on an interval I of the form $[a,b[$, $]a,b]$ or $]a,b[$, where $a,b \in \overline{\mathbb{R}}$, but either f does not extend continuously to $[a,b]$, or (in the case when a or b are $\pm\infty$) if $[a,b]$ does not exist at all.

Definition 8.47. In the above case we define the *improper integral* of f on I as

(1) if $I=[a,b[$, then

$$\int_a^{b-} f(t)dt := \lim_{x \to b-} \left(\int_a^x f(t)dt\right)$$

(2) if $I=]a,b]$, then

$$\int_{a^+}^{b} f(t)dt := \lim_{x \to a+} \left(\int_x^b f(t)dt\right)$$

(3) if $I=]a,b[$,

$$\int_{a^+}^{b^-} f(t)dt := \int_{a^+}^{c} f(t)dt + \int_c^{b-} f(t)dt$$

for any $c \in I$ (it is an easy exercise that the sum does not depend on c),

assuming that the above limits exist. Furthermore, if the limits exist we say that the integrals converge.

If the above limits diverge, we say that the corresponding improper integral is divergent.

Remark 8.48. By abuse of notation many times the $+$ and the $-$ is forgotten from the lower and upper limits.

Example 8.49.

$$\int_{0^+}^{1} \frac{1}{\sqrt{t}} dt = \lim_{x\to 0^+} \left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}} \Bigg|_{t=x}^{t=1} \right) = \lim_{x\to 0^+} 2 - 2\sqrt{x} = 2$$

Example 8.50.

$$\int_{0^+}^{1} \frac{1}{t} dt = \lim_{x\to 0^+} \left(\mathrm{Log}(t)|_{t=x}^{t=1} \right) = \lim_{x\to 0^+} -\mathrm{Log}(x) = +\infty$$

So, $\int_{0^+}^{1} \frac{1}{t} dt$ is divergent.

Example 8.51.

$$\int_{0^+}^{1} \mathrm{Log}(t) dt = \lim_{x\to 0^+} \left(\mathrm{Log}(t)t - t \right)|_{t=x}^{t=1} = -1 - \lim_{x\to 0^+} (\mathrm{Log}(x)x - x) = -1 - \lim_{x\to 0^+} (\mathrm{Log}(x)x)$$

Here, we may compute $\lim_{x\to 0^+} (\mathrm{Log}(x)x)$ using L'Hospital's rule:

$$\lim_{x\to 0^+} (\mathrm{Log}(x)x) = \lim_{x\to 0^+} \frac{\mathrm{Log}(x)}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0.$$

Hence, $\int_{0^+}^{1} \mathrm{Log}(t) dt = -1$.

Definition 8.52. In the situations of Definition 8.47, we say that the integral is absolute convergent if the integrals with f replaced by $|f|$ are also convergent.

The following is an immediate consequence of Cauchy's convergence criterion:

Proposition 8.53. *If an improper integeral is absolute convergent, then it is also convergent.*

Example 8.54. The backwards implication of Proposition 8.53 does not hold, as shown by the next example.

$$\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt = \underbrace{\lim_{x\to+\infty} \frac{-\cos(x)}{x} - \frac{-\cos\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} - \int_{\frac{\pi}{4}}^{+\infty} \frac{-\cos(t)}{-t^2} dt}_{g'=\sin(t) \quad f=\frac{1}{t} \quad g=-\cos(t) \quad f'=\frac{1}{-t^2}} = \frac{\sqrt{2}}{\pi} - \int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2} dt. \tag{8.54.a}$$

So, $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$ is convergent if so is $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2} dt$. However, the latter is convergent because it is absolute convergent:

$$\int_{\frac{\pi}{4}}^{+\infty} \left| \frac{\cos(t)}{t^2} \right| dt \leq \int_{\frac{\pi}{4}}^{+\infty} \frac{1}{t^2} dt = \lim_{x\to+\infty} \left(\frac{-1}{t} \Bigg|_{\frac{\pi}{4}}^{x} \right) = \frac{4}{\pi} + \lim_{x\to+\infty} \frac{1}{x} = \frac{4}{\pi}.$$

This yields that $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$ is convergent.

However, be careful, the fact that $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\cos(t)}{t^2} dt$ is absolute convergent, does not mean that so is $\displaystyle\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$. That is, equation (8.54.a) does not work for $\frac{\sin(t)}{t}$ replaced by $\left|\frac{\sin(t)}{t}\right|$. And, in fact, $\int_{\frac{\pi}{4}}^{+\infty} \frac{\sin(t)}{t} dt$ is not absolute convergent, because

$$\int_{\frac{\pi}{4}}^{n\pi} \left|\frac{\sin(t)}{t}\right| dt \geq \sum_{k=1}^{n} \int_{k\pi-\frac{3\pi}{4}}^{k\pi-\frac{\pi}{4}} \left|\frac{\sin(t)}{t}\right| dt \geq \sum_{k=1}^{n} \frac{\pi}{2} \left(\min_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} \frac{|\sin(t)|}{t} \right)$$

$$\geq \sum_{k=1}^{n} \frac{\pi}{2} \left(\frac{\min_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} |\sin(t)|}{\max_{t\in\left[k\pi-\frac{3\pi}{4},k\pi-\frac{\pi}{4}\right]} t} \right) = \sum_{k=1}^{n} \frac{\pi}{2} \frac{\frac{1}{\sqrt{2}}}{k-\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} \sum_{k=1}^{n} \frac{1}{k}.$$

As $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent, $\lim_{n\to\infty} \int_{\frac{\pi}{4}}^{n\pi} \left|\frac{\sin(t)}{t}\right| dt$ does not exist. Therefore, $\int_{\frac{\pi}{4}}^{\infty} \left|\frac{\sin(t)}{t}\right| dt$ is divergent.

Example 8.55. A typical application of improper integral is to give an upper bound on infinite sums. For example

$$\sum_{k=10}^{\infty} \frac{1}{k^2} \leq \int_{9}^{+\infty} \frac{1}{x^2} dx = \left.\frac{-1}{x}\right|_{x=9}^{x\to+\infty} = \left(\lim_{n\to\infty x\to+\infty} \frac{-1}{x} \right) - \frac{-1}{9} = \frac{1}{9}$$

9 Power series

Definition 9.1. A *power series* centered at $x_0 \in \mathbb{R}$ is an expression of the form

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

for some sequence a_k of real numbers.

The *domain of convergence* of $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ is

$$D := \left\{ x \in \mathbb{R} \middle| \sum_{k=0}^{\infty} a_k (x - x_0)^k \text{ is convergent} \right\}$$

Note that $x_0 \in D$ always, as the only non-zero term in $\sum_{k=0}^{\infty} a_k (x_0 - x_0)^k$ is a_0.

Theorem 9.2. *There is a real number $R \geq 0$ (called* radius of convergence*) such that*

$$if \begin{Bmatrix} |x - x_0| < R \\ \\ |x - x_0| > R \end{Bmatrix}, then \begin{cases} \sum_{k=0}^{\infty} a_k (x - x_0)^k \textit{ is convergent} \\ \sum_{k=0}^{\infty} a_k (x - x_0)^k \textit{ is divergent} \end{cases}$$

Proof. Let D be the domain of convergence. We have to show that if $x' \in D$, then for all x with $0 < |x - x_0| < |x' - x_0|$, $x \in D$.

So, we assume that $\sum_{k=0}^{\infty} a_k (x' - x_0)^k$ is convergent. In particular, $\lim_{k\to\infty} a_k (x' - x_0)^k \to 0$, and hence the sequence $b_k := a_k (x' - x_0)^k$ is bounded. So, there is a real number B, such that

$|b_k| \leq B$. However, then if we set $y := \frac{x-x_0}{x'-x_0}$, which is smaller than 1 by our assumption, we get that

$$0 \leq \left|a_k(x-x_0)^k\right| = \left|b_k y^k\right| \leq B|y|^k$$

Since $\sum_{k=0}^{\infty} B|y|^k$ is a geometric series, it is convergent. In particular by the squeeze theorem for series (Proposition 5.10), $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is absolutely convergent, and hence also convergent.

□

Theorem 9.3. *Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series, with radius of convergence R.*

(1) If $l := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists for some $l \in \overline{\mathbb{R}}$, then $R = \frac{1}{l}$ (for $l = 0$ and $l = +\infty$, then $R = +\infty$ and $R = 0$, respectively).

(2) If $L := \lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists for some $L \in \overline{\mathbb{R}}$, then $R = \frac{1}{L}$.

Proof. Let $x \neq x_0$. We want to decide when $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent

(1) We use the quotient criterion:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = \lim_{n\to\infty} \left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = |x-x_0| l$$

So, by the quotient criterion, that is, Proposition 5.22, $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent if $|x-x_0| < \frac{1}{l}$ and it is divergent if $|x-x_0| > \frac{1}{l}$.

(2) We use the Alembert's criterion:

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x-x_0)^n|} = \lim_{n\to\infty} |\sqrt[n]{a_n}|\, |x-x_0| = L|x-x_0|$$

So, by Alembert's criterion, that is, Proposition 5.23, $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ is convergent if $|x-x_0| < \frac{1}{L}$ and it is divergent if $|x-x_0| > \frac{1}{L}$.

□ ⟵ End of 26. class, on 16.12.2019.

Example 9.4. We have seen earlier that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is convergent for all $x \in \mathbb{R}$, which we can verify now easier:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

So, indeed, the radius of convergence of the above series is ∞.

Example 9.5. According to Theorem 9.3, the radius of convergence of

$$\sum_{n=0}^{\infty} nx^n$$

is $\frac{1}{l}$, where

$$l = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

So, the radius of convergence is 1.

Example 9.6. According to Theorem 9.3, the radius of convergence of

$$\sum_{n=0}^{\infty} e^{n+1} x^n$$

is $\frac{1}{l}$, where

$$l = \lim_{n\to\infty} \frac{e^{n+1}}{e^n} = \lim_{n\to\infty} e = e.$$

So, the radius of convergence is $\frac{1}{e}$

9.1 Taylor series

Definition 9.7. If $f : I \to \mathbb{R}$ is a function on an open interval such that it is differentiable n-times for all integer $n > 0$ at $x_0 \in I$ (we say that f is infinitely many times differentiable at x_0), then the Taylor series of f is

$$T_f(x) := \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

Remark 9.8. Note that the terms of the Taylor series of f are the same as those of the Taylor expansions of f (Section 7.5.3, in particular Theorem 7.62). Hence, the Taylor series equals f if and only if the error term computed in Theorem 7.62 converges to 0 as n goes to ∞. This yields the following proposition.

Proposition 9.9. *In the situation of Definition 9.7, if f is infinitely many times differentiable on I, then $f(x) = T_f(x)$ for an $x \in I$, whenever*

$$\lim_{n\to\infty} \left(\operatorname{Sup}_{y\in I} \left| f^{(n)}(y) \right| \right) \frac{|x - x_0|^n}{n!} = 0$$

Example 9.10. We have seen that the Taylor series of $\operatorname{Log}(1 + x)$ is

$$T(x) := \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} x^k.$$

Let us determine the radius of convergence:

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{n+1}}{\frac{(-1)^{n-1}}{n}} \right| = 1,$$

so the radius is 1. In particular, according to Theorem 9.3, $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ converges whenever $|x| < 1$ and diverges whenever $|x| > 1$.

The next question whether for $x \in]-1,1[$ we have

$$\operatorname{Log}(1+x) = T(x). \tag{9.10.a}$$

We answer (9.10.a) using Proposition 9.9. However, although (9.10.a) holds on $]-1,1[$, unfortunately, the methods of Proposition 9.9 are sufficient to prove it only over $\left]-\frac{1}{2},1\right[$. And in fact, as the argument is half as long on $I := \left]-\frac{1}{2},\frac{1}{2}\right[$, we present it only in the latter case.

To apply Proposition 9.9, first we have to compute the n-th derivative of $\operatorname{Log}(1+x)$. This is

$$\operatorname{Log}(1+x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(1+x)^{n-1}}. \tag{9.10.b}$$

Then, for $x \in I = \left]-\frac{1}{2},\frac{1}{2}\right[$ we have

$$\begin{aligned} 0 \leq \lim_{n\to\infty} \left(\operatorname{Sup}_{y\in I} \left|\operatorname{Log}(1+y)^{(n)}\right| \right) \frac{|x|^n}{n!} &= \lim_{n\to\infty} \left(\operatorname{Sup}_{y\in I} \frac{(n-1)!}{|1+y|^{n-1}} \right) \frac{|x|^n}{n!} \\ &= \lim_{n\to\infty} \frac{(n-1)!}{\left(\frac{1}{2}\right)^{n-1}} \frac{|x|^n}{n!} \leq \lim_{n\to\infty} \frac{(n-1)!}{\left(\frac{1}{2}\right)^{n-1}} \frac{\left(\frac{1}{2}\right)^n}{n!} = \lim_{n\to\infty} \frac{1}{2n} = 0 \end{aligned}$$

So, we obtained by Proposition 9.9 that for every $x \in \left]-\frac{1}{2},\frac{1}{2}\right[$,

$$\operatorname{Log}(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{k} x^k.$$

We note once more that the statement holds also over $x \in]-1,1[$, for which we refer to page 506-507 of the book.

Remark 9.11. We note that the second part of Example 9.10, sometimes completely fails, that is, the Taylor series equals the original function sometimes only in the center point x_0. The famous example is

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & x = 0 \end{cases}$$

We claim that f is infinitely many times differentiable in 0*, and $f^{(n)}(0) = 0$ for every integer $n \geq 0$.* We only give the idea of how to prove this claim: one proves by induction that outside 0, $f^{(n)}$ is a function which is a sum of terms of the form $\frac{e^{-\frac{1}{x^2}}}{x^j}$, for some $j \geq 0$, and furthermore $f^{(n)}(0) = 0$

Having the above claim, we obtain that the Taylor series of f around 0 is the constant 0 function. So, the radius of convergence is $+\infty$, but apart from 0 there is no point where the Taylor series equals f.

Example 9.12. Similar example as above is the Taylor series for $\frac{1}{x+1}$. Using the Taylor expansion, computed in Example 7.64, the Taylor series is

$$\sum_{k=0}^{\infty} (-1)^k x^k$$

As in Example 9.10, using Theorem 9.3, the radius of convergence is 1. Furthermore, using Proposition 9.9 as in Example 9.10, for $\left]-\frac{1}{2},\frac{1}{2}\right[$,

$$\frac{1}{x+1} = \sum_{k=0}^{\infty} (-1)^k x^k. \tag{9.12.c}$$

Note that here one can show quickly by using ad-hoc methods instead of Proposition 9.9 that (9.12.c) holds over $]-1,1[$ too. Indeed, by the summation formula for the geometric series, for every $|x|<1$ we have

$$\sum_{k=0}^{\infty}(-1)^k x^k = \lim_{n\to\infty}\sum_{k=0}^{n}(-1)^k x^k = \underbrace{\lim_{n\to\infty}\frac{1+(-1)^n x^{n+1}}{1+x} = \frac{1}{1+x}}_{|x|<1 \quad \Rightarrow \quad \lim\limits_{n\to\infty}(-1)^n x^{n+1}=0}.$$

We finish with the following theorem, which we do not prove. See pages 505 and 506 of the book for the proof.

Theorem 9.13. *Let $\sum_{k=0}^{\infty} a_k(x-x_0)^k$ be a power series, let R be its radius of convergence and set $D := \{x\in\mathbb{R}||x-x_0|<R\}$. Then, we may define the function $f(x): D\to\mathbb{R}$ as $f(x) := \sum_{k=0}^{\infty} a_k(x-x_0)^k$.*

Then, for every x for which $|x-x_0|<R$ we have

(1)

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-x_0)^{k-1},$$

(2)

$$\int_{x_0}^{x} f(t)dt = \sum_{k=0}^{\infty}\frac{a_k}{k+1}(x-x_0)^{k+1}.$$

Example 9.14. If we want to take the derivative and integral of $\operatorname{Log}(x)$, then we can take it over $]0,2[$ also term by term as a power series. So, for $x\in]0,2[$:

$$\operatorname{Log}(x)' = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}k}{k}(x-1)^{k-1} = \sum_{k=1}^{\infty}(-1)^{k-1}(x-1)^{k-1} = \sum_{k=0}^{\infty}(-1)^k(x-1)^k = \frac{1}{1+(x-1)} = \frac{1}{x}.$$

and

$$\int_1^x \operatorname{Log}(t)dt = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k(k+1)}(x-1)^{k+1} = \sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{1}{k}-\frac{1}{k+1}\right)(x-1)^{k+1}$$

$$= (x-1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k + \sum_{k=1}^{\infty}(-1)^{k+2}\frac{1}{k+1}(x-1)^{k+1}$$

$$= (x-1)\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k + \sum_{k=2}^{\infty}(-1)^{k+1}\frac{1}{k}(x-1)^k$$

$$= (x-1)\operatorname{Log}(x) + (\operatorname{Log}(x)-(x-1)) = x\operatorname{Log}(x) - x + 1$$

where the result of the second computation is exactly what we obtained in Example 8.35

$\longleftarrow$ End of 27. class, on 18.12.2019.