Algebraic Geometry I

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List of Theorems

Lecture 1: Intro

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Quick Motivation

We study schemes.
These are objects that "look locally" like (Spec A, A).
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— *$\forall U \subset X$ open, an object in C $\mathcal{F}(U)$*

— *$\forall V \subset U \subset X$ open, a morphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$*

such that

— *$\forall U$ open, $\rho_{U,U}$ is the identity on $\mathcal{F}(U)$*

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*

— *$\mathcal{F}(U)$ is called the space of sections of $\mathcal{F}$ over U*

— *Elements of $\mathcal{F}(X)$ are called global sections.*

— *There are alternative notations for $\mathcal{F}(U)$: $\Gamma(U, F)$ or $H_0(F)$*

— *The ρ_{UV} are called restriction maps, for $s \in \mathcal{F}(U)$, we write $s|_V :=$ $\rho_{UV}(s)$ and is called restriction of s to V.*

Example

— *For any object A in C, we define the constant presheaf $\underline{A}'$ defined by $\underline{A}'(U) = A$ and with restriction maps the identity.*

- *The presheaf of continuous functions :* C^0*.*
  We define $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*
- *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

  This example is universal in a certain sense

Remark
Define the category Ouv_X *with*
- *objects* $U \subset X$ *open subsets*
- *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{tikzcd}
\mathcal{F}_1(U) \arrow[r, "\rho(U)"] \arrow[d] & \mathcal{F}_2(U) \arrow[d] \\
\mathcal{F}_1(V) \arrow[r, "\rho(V)"'] & \mathcal{F}_2(V)
\end{tikzcd}$$

Example
- *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$
- *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
- *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark
There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$ *.*
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*
— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$

Example

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

— *$PSh_C(X)$ is the category of presheaves of C on X*
— *$Sh_C(X)$ is the category of sheaves of C on X*
If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then $\operatorname{Spec} A$ is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :

— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*