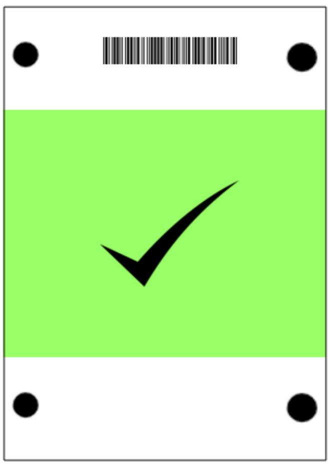
Matériel à utiliser / *Writing equipment*

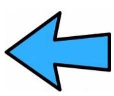
OBLIGATOIRE ***MANDATORY***	 + 	Un **stylo bille** ou **feutre fin** à encre **noire** ou **bleu foncé** pour marquer vos réponses. *A **ball-pen** or **felt-tip** pen with **black** or **dark blue** ink to mark your responses.*
	 + 	Un crayon, une gomme, etc. pour le reste (brouillon, calculs, etc.) *A pencil, eraser, etc. for the rest of the writing (calculations, scratch, etc.)*

Où écrire / *Where to write or mark*



Ne pas écrire dans l'entête

Do not write in the header



Utiliser l'espace central

Use the central space

Ne pas écrire dans le pied de page

Do not write in the footer

Instructions pour les examens QCM
Respecter ces instructions

Instructions for multiple-choice exams
Make sure to follow these instructions

Comment faire / *How to do*

Choisir une réponse / *Select an answer*

« ☒ » ou/*or* « ■ » ou/*or* « ☑ »	Faire une **croix propre et nette** dans la case correspondante (ou **noircir** ou **cocher** la case). *Make a **clear and neat cross** in the corresponding box (or **darken** completely or **check** the box).*

Ne PAS choisir une réponse / *NOT Select an answer*

« ☐ »	Laisser la case entièrement vide. *Leave the box entirely empty.*

Corriger une réponse / *Correct an answer*

« ☐ » ou/*or* « »	**Gommer** proprement toute trace de crayon dans la case ou recouvrir **toute** la case marquée au correcteur blanc, de manière propre et uniforme. ***Erase** completely any pencil marks in the box or cleanly paint over **whole** box using correction fluid, erasing the whole box.*

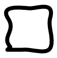
EPFL

Instructions pour les examens QCM
Respecter ces instructions

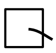
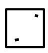
Instructions for multiple-choice exams
Make sure to follow these instructions

Ce qu'il ne faut PAS faire / *What should NOT be done*

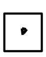
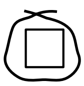
Ne pas corriger une réponse ainsi / *Do not correct an answer like this*

«  »	Ne pas **redessiner** une case effacée, cela pourrait être lu comme une réponse sélectionnée. *Do not **redraw** the box if deleted, it could be read as a selected answer.*
«  »	Ne pas **remplir** ou **tracer** la case pour annuler une croix. Effacer le contenu et éventuellement la case. La sélection sera sinon validée. *Do not **darken** or **draw on** the box to cancel a mark. Erase content or the whole box otherwise it will be read as a selected answer*

Ne pas laisser de traces / *Do not leave traces*

«  » ou/*or* «  » ou/*or* «  »	Ne pas laisser de traces dans les cases. Ceci pourrait être mal interprété lors de la lecture. *Do not leave traces in the boxes. It could be misinterpreted when reading.*

Ne pas remplir ainsi / *Do not fill like this*

«  » ou/*or* «  » ou/*or* «  » ou/*or* «  »	Ne pas faire de points, et ne pas utiliser de stylos ou feutres d'une autre couleur que noir ou bleu foncé. Le choix risque d'être illisible et mal interprété. *Do not make dots, and do not use pens or felts with other ink than black or dark blue. The marks could be unreadable and misinterpreted.*
«  »	Ne pas entourer la case pour sélectionner une réponse. Ceci serait lu comme une non réponse. *Do not circled the box to select an answer. This would be read as no answer.*

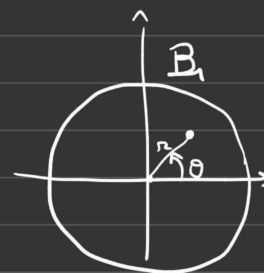
Tout manquement au suivi de ces règles peut entrainer des pertes de points.

Failure to adequately follow these instructions may mean that you will forfeit marks.

En cas de **doute**, signalez-le à votre enseignant-e en fin d'examen.

If in **doubt**, let your professor know at the end of the exam.

Exam: • instructions
• same morning, seating by email.



2021-6-2
W14

Recall: Laplace eq in a disk (week 7.3)

$$\boxed{\partial_{xx} u + \partial_{yy} u = 0}$$

In a disc, we rewrite it in polar coordinates $(w(r,\theta) = u(r\cos\theta, r\sin\theta))$
with Dirichlet boundary cond.

$$\boxed{\begin{cases} r^2 \partial_{rr} w + r \partial_r w + \partial_{\theta\theta} w = 0 \\ w(1,\theta) = f(\theta) \end{cases}}$$

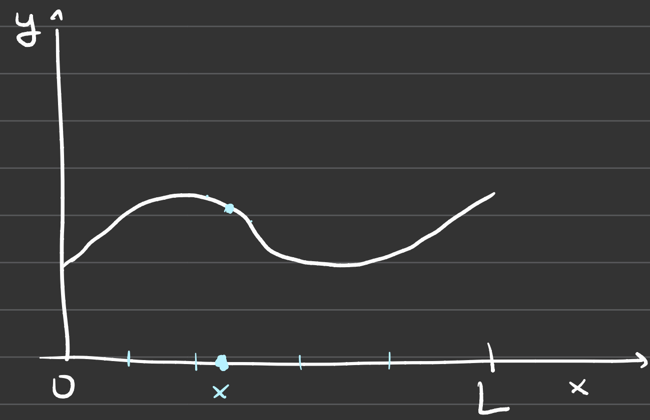
(LEpol)

Exercise
• Formal solution is $\underbrace{\frac{a_0}{2} + \sum r^n (a_n \cos n\theta + b_n \sin n\theta)}_{v(r,\theta)}$
who are they?
• Show that if $f \in L^1(0,2\pi) \Rightarrow v$ this function is $C^\infty((0,1) \times (0,2\pi))$.
What happens in original variables?
• $f \in C^3 \Rightarrow \lim_{r \to 1} v(r,\theta) = f(\theta)$
(hint: $0 \leq 1 - r^n \leq 1 - e^{-\log(\frac{1}{2}) n} \leq \log(\frac{1}{2}) n \ldots$)

The wave equation

Ex 1 (derivation of wave eq)

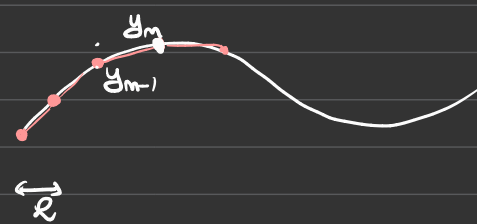
Consider a vibrating rope $y = u(x,t)$



Model the rope as N masses
with x coordinate $x_m = \frac{L}{N} m$.

y-coord $y_m(t)$ to be det.

Mass above $x_m = \rho \underbrace{\frac{L}{N}}_{h}$



Each mass satisfies • Newton law
• forces are generated by neighb.
and equal $\frac{1}{h}(y_m - y_{m-1})$.

① Eq satisfied by y_m

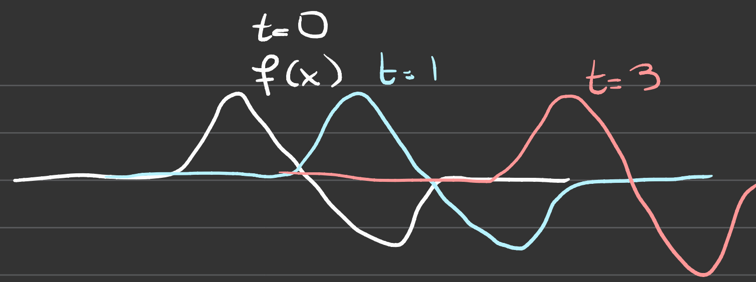
② Let $N \to \infty$ and find a PDE.

① $$\rho \cancel{h}\, y_m''(t) = \underbrace{\frac{1}{h^2}\left[-y_m(t) + y_{m-1}(t) + y_{m+1}(t) - y_m(t)\right]}_{\text{discretization of second deriv!}}$$

② Fix x as $N \to \infty$

$$\boxed{\rho\, \partial_{tt} u(x,t) = \partial_{xx} u(x,t)}$$

THE WAVE EQUATION



Ex 2 why wave equation?

$$f(x \pm ct) = u(x,t)$$

velocity of wave going to the right with −

Does u solve the W.E $\boxed{\partial_{tt} u = c^2 \partial_{xx} u}$? Y

$$\partial_{tt} u = (\pm c)^2 f''(x-ct)$$

$$\partial_{xx} u = f''(x-ct)$$

Rmk also in $\mathbb{R}^d$ with $c \in \mathbb{R}^d$ given and $\partial_{xx} \rightsquigarrow \Delta$

Ex 3 (change of units) Let $a,b>0$, u sol, U s.t.

$$U(\underbrace{ax}_{=X}, \underbrace{bt}_{=T}) = u(x,t)$$

Then U solves

$$\partial_{tt} U = \frac{a^2}{b^2}\frac{\cancel{b^2} c^2}{\cancel{a^2}} \partial_{xx} U \ ?$$

Indeed $U(X,T) = u\left(\frac{X}{a}, \frac{T}{b}\right)$

$$\partial_{TT} U = \partial_{tt} u(\ ) \frac{1}{b^2}$$

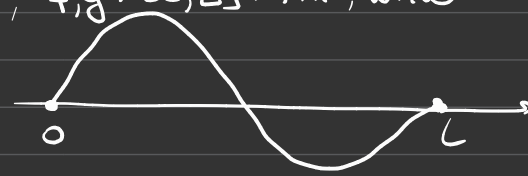
$$\partial_{XX} U = \frac{1}{a^2} \partial_{xx} u(\dots)$$

Then

$$\partial_{tt} U = \frac{1}{b^2} \partial_{tt} u(\dots) = \frac{1}{b^2} c^2 \partial_{xx} u(\dots) = \frac{a^2 c^2}{b^2} \partial_{xx} U.$$

Choosing $a = \frac{\pi}{L}$, $b = c\frac{\pi}{L}$, we reduce to $c=1$, $L=\pi$.

Ex 4 (WE in a bdd interval) Let $L, c>0$, $f, g : [0,L] \to \mathbb{R}$, with $f(0)=f(L)=g(0)=g(L)=0$



$$\begin{cases} \partial_{tt} u = c^2 \partial_{xx} u & x \in (0,L),\ t \in (0,\infty) \\ u(0,t) = 0 = u(L,t) & \\ u(x,0) = f(x) & \\ \partial_t u(x,0) = g(x) & \forall x \in (0,L) \end{cases}$$

① Find $u : (0,L) \times (0,\infty) \to \mathbb{R}$ s.t. $u(x,t) = v(x)w(t)$, ignoring initial cond

② write the formal sol.

① WE $\iff$ $\cancel{v(x)}\,\dfrac{w''(t)}{w(t)} = c^2\,\dfrac{v''(x)}{v(x)}\,\cancel{w(t)}$

$$\iff (*) \begin{cases} \dfrac{v''}{v} = -\lambda \\ v(0) = v(L) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{w''(t)}{c^2 w(t)} = -\lambda \end{cases}$$

Gen sol of $(*)$; $v'' + \lambda v = 0$

$\lambda > 0$ $\quad \alpha \sin\sqrt{\lambda}x + \cancel{\beta \cos\sqrt{\lambda}x}$

$v(L) = 0 \Rightarrow \sin\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = \pi n \quad n \in \mathbb{N}$

$\Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2 \quad n \in \mathbb{N}$

Overall for $(*)$

$\lambda = \left(\frac{n\pi}{L}\right)^2 \qquad v_n(x) = \sin\left(\frac{n\pi}{L}x\right).$

For the second one $w'' + \lambda c^2 w = 0$

$$w_n(t) = \alpha_n \sin\left(\frac{n\pi}{L}ct\right) + \beta_n \cos\left(\frac{n\pi}{L}ct\right).$$

A formal sol is

$$u(x,t) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right)\left(\alpha_m \sin\left(\frac{m\pi}{L}ct\right) + \beta_m \cos\left(\frac{m\pi}{L}ct\right)\right).$$

Now I determine α_m and β_m depending on initial cond

- $f(x) = u(x,0) = \sum \sin\left(\frac{m\pi}{L}x\right)\beta_m$

So β_m must be equal to the F.c. of f in sines only

$$\beta_m := \frac{2}{L}\int_0^L f(x) \sin\left(\frac{m\pi}{L}x\right) dx$$

- $g(x) = \partial_t u(x,0) = \sum_{m=1}^{\infty} \sin\left(\frac{m\pi}{L}x\right) \underbrace{\alpha_m \frac{m\pi}{L} c}\, \cancel{\cos \frac{m\pi}{L}ct}$ $t=0$ makes 1

So α_m must be

$$\alpha_m \cancel{\frac{m\pi c}{L}} = \frac{2}{cm\pi \cancel{L}}\int_0^L g(x) \sin\left(\frac{m\pi x}{L}\right) dx$$

Ex 4 Assume $f \in C^4$, $g \in C^3$. Prove that $u \in C^2((0,L) \times (0,\infty))$ and

$$f(x) = \lim_{t \to 0} u(x,t)$$

$$g(x) = \lim_{t \to 0} \partial_t u(x,t) \qquad \text{unif in } x.$$

f.c. of g times $\frac{c}{m}$

Reg $f, g \Rightarrow |\beta_m| \le \frac{C}{m^4}$ $\qquad |\alpha_m| \le \frac{C}{m^4}$.

Formally

$$\partial_{xx} u(x,t) = \sum_{m=1}^{\infty} \left(\frac{m\pi}{L}\right)^2 \left[\alpha_m \cos\left(\frac{m\pi c}{L} t\right) + \beta_m \sin\left(\frac{m\pi c}{L} t\right)\right] \overset{-\sin}{\cos} \frac{m\pi}{L} x \quad (*)$$

The serie in RHS converges absolutely

$$| \quad | \le \sum m^2 C \frac{1}{m^4} \le C \sum \frac{1}{m^{2}} \qquad (**)$$

converges as soon as this number is >1

Rigorously, define

$$u_N = \sum_{m=1}^{N} \dots$$ same as in u

same lemma from 1st to 2nd derivative

$$\partial_x u_N = \sum_{m=1}^{N} \text{ same terms as in } (*)$$

$$|\partial_x u_N - \text{RHS of } (*)| \le \sum_{m=N+1}^{\infty} \frac{m\pi}{L} \left[\frac{C}{m^4} + \cancel{\frac{C}{m^4}}\right]$$

this goes to 0 as $N \to \infty$

Hence $\partial_x u_N \longrightarrow$ RHS of $(*)$ uniformly

By $\boxed{\text{Prop W13}}$ $\quad (*)$ holds!

Same also for derivatives in t.

Question is $u \in C^\infty$? Not in general!!!

A sol of WE is $f(x-ct)$. It is not (for some time) gaining regularity wrt the initial datum!

$$f \in C^4 \Rightarrow f(x) = \sum \beta_m \sin\left(\frac{m\pi}{L}x\right) \qquad \forall x$$

$$|u(x,t) - f(x)| = \left| \sum \underbrace{\sin\left(\frac{m\pi}{L}x\right)}_{\leq 1} \left(\alpha_m \sin\left(\frac{m\pi}{L}ct\right) - \beta_m \left[\cos\left(\frac{m\pi}{L}ct\right) - 1\right]\right)\right|$$

$$\leq C \sum_{m=1}^{m_0} \frac{1}{m^4} \Big[\underbrace{\sin\left(\frac{m\pi}{L}ct\right)}_{\to 0 \text{ as } t \to 0} + \underbrace{\cos\left(\frac{m\pi}{L}ct\right) - 1}_{\to 0 \text{ as } t \to 0}\Big]$$

$$C \sum_{m=m_0+1}^{\infty} \frac{1}{m^4}$$

Fix $\varepsilon > 0$. The second term is smaller then $\frac{\varepsilon}{2}$ for m_0 large enough, that I fix (m_0) now.
The first term converges to 0 as $t \to \infty$ and it's a finite sum $\Rightarrow$ provided t is suff small, it is smaller than $\varepsilon/2$.

$\square$

Ex 5 (D'Alembert formula) Assume $c=1$, $L=\pi$

$$\begin{cases} \partial_{tt} u = \partial_{xx} u & x\in(0,\pi),\ t\in(0,\infty) \\ u(0,t) = 0 = u(L,t) & \oplus \\ \left.\begin{array}{l} u(x,0) = f(x) \\ \partial_t u(x,0) = g(x) \end{array}\right\} & \forall x \in (0,\pi) \end{cases}$$

Observe that $u(x,t) = F(x+t) + G(x-t)$ solves $\partial_{tt}u = \partial_{xx}u$

Extend f and g oddly on $(-\pi,\pi)$ and then periodically on $\mathbb{R}$.

Choose F and G to satisfy the initial conditions

$$\begin{cases} F(x) + G(x) = f(x) \quad (\star) \Rightarrow F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = g(x) \end{cases}$$

$$\Rightarrow \begin{cases} 2F' = f' + g \\ 2G' = f' - g \end{cases} \Rightarrow \begin{cases} F = \frac{1}{2}\left[f(x) + \int_0^x g(y)\,dy\right] + C_1 \\ G = \frac{1}{2}\left[f(x) - \int_0^x g(y)\,dy\right] + C_2 \end{cases}$$

From $(\star)$, $C_1 + C_2 = 0$

So

$$\boxed{u(t,x) = \frac{f(x+t) + f(x-t)}{2} + \int_{x-t}^{x+t} g(y)\,dy}.$$

$\oplus$ is automatically satisfied

$$u(0,t) = \underbrace{\frac{f(t) + f(-t)}{2}}_{=0} + \underbrace{\int_{-t}^{t} g(y)\,dy}_{=0}.$$

oddly ext

Same in π... $\square$

A POSTERIORI OVERVIEW OF THE COURSE - THE CORNERSTONES

⚠ disclaimer: all the material discussed in class can be useful at the exam

1. Lebesgue measure

- Existence of the Lebesgue measure theorem (construction with outer measure, measurable sets and functions)
- The Cantor set

2. Lebesgue integration, convergence theorems, L^p spaces

- Lebesgue integral (from simple to positive, to signed functions)
- $\boxed{\text{3 Convergence theorems}}$ (monotone, Fatou, dominated)
- $\boxed{L^p \text{ spaces}}$, L^p is a norm, Holder and Minkowski ineq, density of smooth functions, convolutions)

3. Fourier analysis

- Approximation by trigonometric polynomials
- $\boxed{\text{Fourier and Plancherel theorems}}$
- Pointwise and uniform convergence of Fourier series
- Trigonometric Fourier series (also in sin only...)

4. Fourier transform

- Inversion formula and Plancherel identity
- Good kernels and transform of convolutions

5. Applications to PDEs

- Solutions of Heat, Laplace, wave equations

$\boxed{\text{Methods}}$ are important, even more than single results! For example, separation of variables, sol via Fourier series / transf, showing regularity and finding how is the b.c. taken (unif, in L^2...)

What's next?

- ODE $\qquad \dot{x}(t) = f(t, x(t))$.

- PDEs
19th Hilbert problem: $\min_{\substack{u: B_1 \to \mathbb{R} \\ u|_{\partial B_1} \text{ given}}} \int F(\nabla u)$ is the min C^∞?

When $F(x) = |x|^2$ minimizers solve $\Delta u = 0$.

- Calculus of variations $\qquad$ look at min problems

- Harmonic analysis $\qquad \underset{\substack{\uparrow \\ \text{given}}}{\Delta u = f} \qquad f \in L^p \Rightarrow D^2 u \in L^p$?

- Optimal transport

Functional analysis...