École Polytechnique Fédérale de Lausanne
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Math 261 – Discrete Optimization (Spring 2021)

Practice Problems (for the final)

These are practice problems for the exam. The final question is not a "question" but a set of True/False questions. This is just for practice — there will not be a set of True/False questions on the final.

Problem 1

Let $V = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^n$ and let $\mathbf{x}$ be a point in the convex hull of V.
Prove or give a counterexample: there exist real numbers $\lambda_1, \ldots, \lambda_m$ with the following properties

1. $\lambda_i \geq 0$ for all i
2. $\sum_i \lambda_i = 1$
3. $\sum_i \lambda_i \mathbf{v}_i = \mathbf{x}$
4. At most n of the λ_i are non-zero.

Problem 2

A refinery can produce gas of two types of qualities Q_1 and Q_2, composed of three kinds of crude oil C_1, C_2, C_3. To produce one unit of gas Q_1 or Q_2, one can combine fractional quantities of the three kinds of crude oils such that they sum up to 1. The following restrictions apply:

- For C_1: at most 30% can be used for Q_1 and at most 50% can be used for Q_2
- For C_2: at least 40% must be used for Q_1 and at least 10% must be used for Q_2
- For C_3: at most 50% can be used for Q_1

Furthermore, there are caps on the amount of each crude oil one can process, costs per unit for each crude oil, and a price per unit for each gas as follows:

	max quantity	cost per unit
C_1	3000	3
C_2	2000	6
C_3	4000	4

	price per unit
Q_1	5.5
Q_2	4.5

Formulate the problem of maximizing revenue (income by selling the gas minus cost of raw material) as a linear program. **You do not need to transform your LP into standard form.**

Problem 3

Let P and Q be polyhedra in $\mathbb{R}^n$ and define the set

$$R = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in P, \mathbf{y} \in Q\}$$

Show that for any extreme point $\mathbf{z}_* \in R$, there exist extreme points $\mathbf{x}_* \in P$ and $\mathbf{y}_* \in Q$ such that $\mathbf{z}_* = \mathbf{x}_* + \mathbf{y}_*$. **You may assume (without proof) that R is a polyhedron.**

Problem 4

Let $\mathbf{A}$ be matrix with independent columns (and no 0 rows) and let $P = \{\mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ be a polyhedron. Prove or find a counterexample for the following:

1. If two different bases β_1 and β_2 lead to the same basic solution $\mathbf{x}$, then $\mathbf{x}$ is a degenerate solution.
2. If $\mathbf{x}$ is a degenerate basic solution, then there must be two different bases β_1 and β_2 which lead to $\mathbf{x}$.
3. If P contains a degenerate basic feasible solution $\mathbf{y}$, then there is a row from $\mathbf{A}$ (and $\mathbf{b}$) that can be removed that does not change the polytope P.

Problem 5

Let $P \subseteq \mathbb{R}^n$ be a polytope for which $\mathbf{x} \geq \mathbf{0}$ for all $\mathbf{x} \in P$. Prove or find counterexamples for the following statements:

1. There exists a matrix $\mathbf{A}$ and vector $\mathbf{b}$ for which
$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$$
2. There exists a matrix $\mathbf{B}$ and vector $\mathbf{d}$ for which
$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{B}\mathbf{x} = \mathbf{d}, \mathbf{x} \geq 0\}$$

Problem 6

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that the following are equivalent (if and only if):

- The system $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ has a feasible solution.
- Every vector $\boldsymbol{\lambda} \in \mathbb{R}^m$ with $\boldsymbol{\lambda}^\intercal \mathbf{A} \geq \mathbf{0}$ satisfies $\boldsymbol{\lambda} \cdot \mathbf{b} \geq 0$.

Note: you cannot simply say "Farkas' Lemma" — you need to prove any version of Farkas' Lemma that you wish to use.

Problem 7

Consider the set of linear programs

$$\begin{aligned} \mathcal{P}_t = \min\ & x + y + z \\ \text{s.t. } & x + 2y + 3z = 3 \\ & -x + 2y + 6z = t \\ & x, \quad y, \quad z \geq 0 \end{aligned}$$

where t is a parameter (not a variable). Write down a linear program $\mathcal{P}'_t$ with the following properties:

1. If the optimal value of $\mathcal{P}'_t$ is 0, then $\mathcal{P}_t$ is feasible and every optimal solution to $\mathcal{P}'_t$ is a feasible solution for $\mathcal{P}_t$
2. If the optimal value of $\mathcal{P}'_t$ is not zero, then $\mathcal{P}_t$ is infeasible.

Note: you do not need to solve the linear program, but you should explain why it has the required properties.

Problem 8

Consider the following transportation problem: $G_1, \ldots, G_i$ are goods, $W_1, \ldots, W_j$ are warehouses and $C_1, \ldots, C_k$ are clients. Let $s_{i,j} \geq 0$ be the amount of good G_i that is available in warehouse W_j and let $d_{i,k}$ be the amount of good G_i that has been ordered by client C_k. Finally, let $p_{i,j,k}$ be the cost to send one unit of good G_i from warehouse W_j to client C_k.
Formulate an integer program that determines the cheapest way to transport the goods to the clients such that:

- The demand of each client is satisfied, and
- the supplies of the warehouses are not exceeded.

Be sure to explain the meaning of any variables you create.

Problem 9

Consider the integer program

$$\begin{aligned} \mathcal{P} = \min \quad & x + y + z \\ \text{s.t.} \quad & 2x + y + z \geq 9 \\ & x + 2y + 2z \geq 11 \\ & x + y \geq 3 \\ & x, \ y, \ z \geq 0 \\ & x, \ y, \ z \in \mathbb{Z} \end{aligned}$$

(a) Find (with proof) the optimal value of $\mathcal{P}$.

(b) Find a stronger formulation of $\mathcal{P}$. That is, find an integer program $\mathcal{P}'$ for which $\text{feas}(\mathcal{P}) = \text{feas}(\mathcal{P}')$ and $\text{feas}(\mathcal{P}'_{\mathbb{R}}) \subsetneq \text{feas}(\mathcal{P}_{\mathbb{R}})$ (where $X_{\mathbb{R}}$ denotes the LP relaxation of X). Be sure to prove that both conditions hold!

Problem 10

1. Let x be a variable. For each of the two constraints

$$(a)\ |x| \leq 1 \qquad \text{and} \qquad (b)\ |x| \geq 2$$

   determine whether it is possible to implement the constraint as part of a linear program. If it is possible, show how. If it is not possible, explain.

2. Show (with explanation!) how to implement the constraint $1 \leq |x| \leq 2$ in a mixed integer program. *Hint: consider the quantity $x + az$ where a is a well-chosen integer and z is a binary variable.*

Problem 11

For directed graph $D = (V, E)$ with truncated incidence matrix $\tilde{\mathbf{A}}$, let

$$\begin{aligned} \mathcal{P} = \min \quad & \mathbf{c} \cdot \mathbf{f} \\ \text{s.t.} \quad & \tilde{\mathbf{A}}\mathbf{f} = \tilde{\mathbf{b}} \\ & \mathbf{f} \geq \mathbf{0} \end{aligned}$$

be an uncapacitated network flow problem. If β is a basis for $\tilde{\mathbf{A}}$, what can be said about $\det\left[\tilde{\mathbf{A}}_\beta\right]$? Explain why this is true (you should try to give the main ideas that one would use in a proof but you do not need to show any of the details).

Problem 12

Let $\mathcal{P} = \max\{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be a linear program in equality standard form which has a bounded feasible region. Let $\mathbf{y}$ be a non-degenerate basic feasible solution for P with column basis matrix $\mathbf{B}$. For a variable x_j not in the column basis, adding x_j (and only x_j) to the basis will require $\mathbf{y}$ to move to $\mathbf{y} + \theta\mathbf{d}^j$ for some real number θ and some vector $\mathbf{d}^j$.

(a) Find (with proof)[1] a formula for the direction vector $\mathbf{d}^j$.

[1]Note: "with proof" here means you need to explain why everything you write is correct. Answers that use results from class without explaining why they are true will not receive full credit.

(b) Given a vector $\mathbf{d}^j$ from part (a), show (with proof) how one would find the values of $\theta \in \mathbb{R}$ for which $\mathbf{y} + \theta \mathbf{d}^j$ is feasible.

(c) Now assume you are given *arbitrary* vector $\mathbf{z}$. Show (with proof) how one would find the values of $\theta \in \mathbb{R}$ for which $\mathbf{y} + \theta \mathbf{z}$ is feasible.

Problem 13

Let $P \subseteq \mathbb{R}^n$ be a bounded polyhedron, $\mathbf{c} \in \mathbb{R}^n$ a vector and $t \in \mathbb{R}$ a scalar. Consider the polyhedron

$$Q = \{\mathbf{x} \in P : \mathbf{c} \cdot \mathbf{x} = t\}$$

Show that each vertex of Q is either a vertex of P or a convex combination of two adjacent vertices of P.

Problem 14

Consider an exercise session with n teachers and m students. Each student wants to find out an answer to one question. Teacher i has ability to answer at most q_i questions during the session. Additionally, not every teacher can answer every question: Teacher i can only answer questions of students from the set $S_i \subseteq \{1, \ldots, m\}$. The goal is to find the maximum number of *distinct* questions that can be answered under these constraints (that is, two teachers answering the same student's question only counts for one question answered).

(a) Write an explicit integer program which would solve this problem (and explain why it does).

(b) Describe an algorithm that will solve this problem in polynomial time (and prove it is correct).

Problem 15

Let $D = (V, E)$ be a directed graph with edge-cost function $c : E \to \mathbb{R}$ for which there are no negative cost cycles. Design an algorithm which finds a minimum cost cycle in time at most $O(|V||E|^2)$ and prove that it is correct.

Note: The term "cycle" in this problem refers to a "directed cycle" (all edges point the same way).

Problem 16

Let $\mathcal{P} = \min\{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ be a linear program in equality standard form which has a finite optimal value. Define the function

$$L_P(\mathbf{y}) = \min_{\mathbf{x} \geq 0} \left\{\mathbf{c}^\intercal \mathbf{x} - \mathbf{y}^\intercal (\mathbf{A}\mathbf{x} - \mathbf{b})\right\}.$$

(a) Show that if $\mathbf{x}_*$ is an optimal solution to P and $\boldsymbol{\lambda}_*$ is an optimal solution to its dual, then

$$L_P(\boldsymbol{\lambda}_*) \leq \mathbf{c} \cdot \mathbf{x}_*.$$

(b) Show that if $\mathbf{x}_*$ is an optimal solution to P and $\boldsymbol{\lambda}_*$ is an optimal solution to its dual, then

$$L_P(\boldsymbol{\lambda}_*) \geq \mathbf{c} \cdot \mathbf{x}_*.$$

Problem 17

True or False:

(a) If the linear program

$$\begin{aligned} \min \ & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t. } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

is feasible and bounded, then there exists a basis β which gives an optimal solution.

(b) If the linear programs

$$\begin{aligned} \mathcal{P} = \min \ & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t. } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{D} = \max \ & \mathbf{b} \cdot \mathbf{y} \\ \text{s.t. } & \mathbf{A}^\intercal \mathbf{y} \leq \mathbf{c} \end{aligned}$$

are both feasible, then they must have the same optimal value.

(c) A set C is convex if and only if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$ for all $\mathbf{x}, \mathbf{y} \in C$ and all $\lambda \in \mathbb{R}$.

(d) If a polytope P is defined by a finite number of constraints, then it can only have a finite number of feasible solutions.

(e) If $\mathbf{x}_*$ and $\mathbf{y}_*$ are feasible solutions to a linear program, then

$$\frac{\mathbf{x}_* + \mathbf{y}_*}{2}$$

is a feasible solution as well.

(f) If $\mathbf{x}_*$ and $\mathbf{y}_*$ are optimal solutions to a linear program, then

$$\frac{\mathbf{x}_* + \mathbf{y}_*}{2}$$

is an optimal solution as well.

(g) If $\mathbf{x}_*$ is a feasible solution to a linear program $\mathcal{P}$ with cost function $\mathbf{c}$ and $\boldsymbol{\lambda}_*$ is a feasible solution to the dual linear program $\mathcal{D}$ with cost function $\mathbf{b}$ and

$$\mathbf{x}_* \cdot \mathbf{c} = \boldsymbol{\lambda}_* \cdot \mathbf{b}$$

then $\mathbf{x}_*$ and $\boldsymbol{\lambda}_*$ are optimal solutions.

(h) If β is a basis for a linear program and $\boldsymbol{\lambda}$ is the dual solution associated to β, then $\boldsymbol{\lambda}$ is feasible if and only if the reduced costs in the dual are nonnegative.

(i) Given a matrix $\mathbf{A}$ and vector $\mathbf{b}$, there exists vectors $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{y}$ for which

$$\mathbf{y}^\intercal \mathbf{A} \geq \mathbf{0}^\intercal \quad \text{and} \quad \mathbf{y} \cdot \mathbf{b} < 0 \quad \text{and} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$

(j) If a linear program is in equality standard form, then its dual will be in equality standard form as well.

(k) For every integer program, there exists a linear program with the same set of feasible solutions.