Série 7 Exercice 8

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We show the double implication.
First, suppose there exists $s \in A$ such that $s^2 = b^2 - 4ac$.
Since $\gcd(a, b, c) = 1$, the polynomial $ax^2 + bx + c$ is primitive and we may apply Gauss's lemma which states that $ax^2 + bx + c$ is irreducible in $A[x]$ if and only if it is irreducible in $K[x]$.
Note that, in $K[x]$, we may write

$$\begin{aligned} a(x - \frac{-b+s}{2a})(x - \frac{-b-s}{2a}) &= a\left(x^2 - \frac{-b-s}{2a}x - \frac{-b+s}{2a}x + \frac{(-b+s)(-b-s)}{4a^2}\right) \\ &= ax^2 + bx + a\frac{b^2 - s^2}{4a^2} \\ &= ax^2 + bx + c \end{aligned}$$

Hence, $ax^2 + bx + c$ is not irreducible in $K[x]$ and thus also not in $A[x]$.

Now suppose $ax^2 + bx + c$ is not irreducible in $A[x]$, then it is also not irreducible in $K[x]$ by Gauss's lemma (as $ax^2 + bx + c$ is primitive by hypothesis) .
We now use the fact that a polynomial of degree two over a field is not irreducible if and only if it's zero set is non-empty (example 3.4.7.4 from the course notes).
Thus, rewrite (in $K[x]$)

$$\begin{aligned} ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a\left[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\ &= a\left[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}\right] \end{aligned}$$

Thus, if

$$a\left[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}\right] = 0$$

Then, there exists $\frac{s'}{d'} \in K[x], s', d' \in A[x]$ such that

$$(\frac{s'}{d'} + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

In particular, define $s = s'2a + bd'$ then we have that

$$\frac{s^2}{4a^2 d'^2} = \frac{b^2 - 4ac}{4a^2} \iff \frac{s^2}{d'^2} = b^2 - 4ac$$

Thus $\frac{s^2}{d^2} = \left(\frac{s}{d}\right)^2$ is an element of A satisfying the condition.

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a)

We view $x^2 + yx + 1$ as an element of $(\mathbb{C}[y])[x]$ and use the criteria established above.
Indeed, $x^2 + yx + 1$ is primitive as a polynomial over $\mathbb{C}[y]$ as $\gcd(1, y, 1) = 1$.
Furthermore, $y^2 - 4$ (the discrimant $b^2 - 4ac$ of the polynomial) may be rewritten as $(y-2)(y+2)$, and we claim that there does not exist a polynomial f such that $f^2 = (y-2)(y+2)$.
Indeed, this would mean that $\deg f = 1$, but then f is linear and thus has exactly one 0, however f^2 has two zero's, a contradiction.
Hence, the polynomial is irreducible.

b)

Simply write

$$y^2x^2 + yx^2 + yx + y^2 = y(yx^2 + x^2 + x + y)$$

Thus, the polynomial is not irreducible.

c)

We use the same trick as in a) and consider it as a polynomial over $\mathbb{C}[y]$.
Hence the discrimant is $y^2 - 4y^2 = -3y^2$ which is the square of $\sqrt{3}iy$.
Thus, we may write (this formula follows from our general computations in part 1)

$$(x - \frac{-y - \sqrt{3}iy}{2})(x - \frac{-y + \sqrt{3}iy}{2}) = x^2 + yx + y^2.$$