

Algebraic Geometry I

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Quick Motivation

We study schemes.

These are objects that "look locally" like $(\text{Spec } A, A)$.

Examples include

- A itself
- Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)

Let C be a category. A presheaf \mathcal{F} of C on X consists of

- $\forall U \subset X$ open, an object in C $\mathcal{F}(U)$
- $\forall V \subset U \subset X$ open, a morphism $\rho_{U,V} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$

such that

- $\forall U$ open, $\rho_{U,U}$ is the identity on $\mathcal{F}(U)$
- Restriction maps are compatible

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark

Usually, $C = \text{Set}, \text{Ab}, \text{Ring}, \text{etc.}$

In particular, we usually assume the objects in C have elements.

Remark

- Elements of $\mathcal{F}(U)$ are called sections of \mathcal{F} over U .
- $\mathcal{F}(U)$ is called the space of sections of \mathcal{F} over U
- Elements of $\mathcal{F}(X)$ are called global sections.
- There are alternative notations for $\mathcal{F}(U) : \Gamma(U, \mathcal{F})$ or $H_0(U, \mathcal{F})$
- The $\rho_{U,V}$ are called restriction maps, for $s \in \mathcal{F}(U)$, we write $s|_V := \rho_{U,V}(s)$ and is called restriction of s to V .

Example

- For any object A in C , we define the constant presheaf \underline{A} defined by $\underline{A}(U) = A$ and with restriction maps the identity.

- The presheaf of continuous functions : C^0 .
We define $C^0(U) := \{f : U \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ and the restriction maps are the natural restrictions.
- More generally, if $\pi : Y \rightarrow X$ is continuous, we can look at the presheaf of continuous sections of π , here

$$\mathcal{F}_\pi(U) := \{s : U \rightarrow Y \mid s \text{ continuous } \pi \circ s = \text{Id}\}$$

This example is universal in a certain sense

Remark

Define the category Ouv_X with

- objects $U \subset X$ open subsets
- morphisms $U \rightarrow V$ are either empty or the inclusion $U \rightarrow V$ if $U \subset V$

Then a presheaf of C on X is just a contravariant functor $\text{Ouv}_X^{\text{op}} \rightarrow C$

Definition 2 (Morphism of presheaves)

A morphism $\phi : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ of presheaves on X consists of a collection of morphisms $\rho(U) : \mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U)$ which are natural.

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho(U)} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow{\rho(V)} & \mathcal{F}_2(V) \end{array}$$

Example

- Every morphism of objects $A \rightarrow B$ in C yields a morphism $\underline{A} \rightarrow \underline{B}$
- If $X = \mathbb{R}^n$, let C^∞ be the presheaf of smooth functions, then for every open U , there is an inclusion $C^\infty(U) \rightarrow C^0(U)$ and these inclusions induce a morphism of sheaves $C^\infty \rightarrow C^0$
- If $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ are continuous, we get $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \rightarrow \mathcal{F}_{\pi_1}$ by mapping a section $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \rightarrow \pi_2 \circ s$

Remark

There is an equivalence of categories

$$\text{Presheaves of } C \text{ on } X \simeq \text{Fun}(\text{Ouv}_X^{\text{op}}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)

Let $C = \text{Set}, \text{Ab}, \text{Ring}$.

A sheaf \mathcal{F} of C on X is a presheaf such that $\forall U \subset X$ open and all open covers $U = \bigcup_{i \in I} U_i$

- $\forall s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i} \forall i \in I$ then $s = t$
- $\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

- The section s of the gluing condition is unique by the locality condition.
- If C has products, then a presheaf is called a sheaf if

$$\mathcal{F}(U) \rightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i, j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \rightarrow \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$

Example

1. If \mathcal{F} is a sheaf, let $U \cap \emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element
2. C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open
 - Two continuous functions $f, g : U \rightarrow \mathbb{R}$ that coincide on an open cover are equal
 - Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \rightarrow \mathbb{R}$, the function $f : U \rightarrow \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.

Definition 4 (Morphisms of sheaves)

A morphism of sheaves $\rho : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

- $PSh_C(X)$ is the category of presheaves of C on X
 - $Sh_C(X)$ is the category of sheaves of C on X
- If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \rightarrow PSh_C(X)$.

By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then $\text{Spec } A$ is the set of prime ideals of A .

The closed subsets of the Zariski topology on $\text{Spec } A$ are of the form $V(M) = \{p \in \text{Spec } A \mid M \subset p\}$.

A basis of this topology is given by $D(a) = \{p \in \text{Spec } A \mid a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)

Let A be a ring and $X = \text{Spec } A$, then the structure sheaf \mathcal{O}_X on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \rightarrow \prod_{p \in \text{Spec } A} A_p \mid s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ and $V \subset U$ open with $p \in V \subset D(b)$ with $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark

\mathcal{O}_X is a sheaf of rings :

- $\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addition

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6

Let (I, \leq) be a pair where I is a set and \leq is a binary relation.

(I, \leq) is called a preorder if \leq is reflexive and transitive.

(I, \leq) is called a poset if it is preordered and \leq is antisymmetric

(I, \leq) is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example

1. Let $I = \{U \subset X \mid U \text{ open}\}$ and $U \leq V \iff V \subset U$.

Then I is a directed poset.

2. For $x \in X$, let

$$I_x = \{U \subset X \mid U \text{ open } x \in U\}$$

This is a directed poset.

Definition 7

Let (I, \leq) be a directed set and C a category.

A direct system in C indexed by I is a pair $(\{A_i\}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.

Where the A_i are objects in C , the $\rho_{ij} : A_i \rightarrow A_j$ are morphisms in C such that

1. $\rho_{ii} = \text{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example

If \mathcal{F} is a presheaf of C on X and I_X as in the second example above, then

$$(\{\mathcal{F}(U_i)_{U_i \in I_X}\}, \{\rho_{U_i, U_j}\})$$

is a direct system.

Definition 8 (direct limit)

Let (I, \leq) be a directed set, C a category.

Let $(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I})$ be a directed system, then the direct limit is a pair $(\lim_{i \in I} A_i, \{\rho_i\}_{i \in I})$ where $\lim A_i$ is in C and $\rho_i : A_i \rightarrow \lim A_i$ such that

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. For all objects A in C and morphisms $f_i : A_i \rightarrow A$ such that

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

$$\exists! f : \lim_{i \in I} A_i \rightarrow A \text{ such that } f \circ \rho_i = f_i$$

Remark

The direct limit is unique up to unique isomorphism.

Example

Write $(*) = (\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.

Let $*$ be a direct system in Set .

Let $\lim_{i \in I} A_i := A_i / \sim$ where $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ such that $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.

This is the direct limit of $*$.

If $*$ is a system in Ab , let $\lim A_i := \bigoplus A_i / N$ with $N = \langle a_i - \rho_{ij}(a_i) \rangle$.

The natural map $\bigcup A_i / \sim \rightarrow \bigoplus A_i / N$ is a bijection

Remark

Taking the direct limits in (Ab) is exact in the following sense :

\forall directed sets I , \forall direct systems $\{M_i\}, \{N_i\}, \{P_i\}$ indexed by I and for all

collections of commutative diagrams, we get

$$0 \rightarrow \lim M_i \rightarrow \lim N_i \rightarrow \lim P_i \rightarrow 0$$

Definition 9

Let C be a category with direct limits. Let $x \in X$ be a point, \mathcal{F} a presheaf of C on X .

Then the stalk of \mathcal{F} at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x .

For $s \in \mathcal{F}(U)$, we write s_x for the image of s in \mathcal{F}_x and call it the germ of s at x .

Remark

A morphism of sheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \rightarrow \mathcal{G}_x \forall x \in X$

Remark

Let $x \in X, \mathcal{F}$ a presheaf of Set, Ab

1. $\forall U \subset X$ open, $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ open and $t \in \mathcal{F}(U)$ such that $t_x = s$.

Definition 10 (Sheafification)

Let \mathcal{F} be a presheaf of sets (\dots) on X .

The sheafification of \mathcal{F} is the sheaf \mathcal{F}^+ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \rightarrow \prod_{x \in U} \mathcal{F}_x \mid s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U, s(x) \in \mathcal{F}_x$
2. $\forall x \in U, \exists V \subset U$ open and $t \in \mathcal{F}(V)$ such that $t_x = s(x) \forall x \in V$

Remark

1. \mathcal{F}^+ is a sheaf
2. Sheafification is functorial.
For $\rho : \mathcal{F} \rightarrow \mathcal{G}$ a morphism of presheaves, the collection $\rho^+(U) : \mathcal{F}^+(U) \rightarrow \mathcal{G}^+(U)$ sending $s \rightarrow (\prod_{x \in U} \rho_x) \circ s$
3. \exists a natural morphism $\iota_{\mathcal{F}} : \mathcal{F} \rightarrow \mathcal{F}^+$ defined by $\iota_{\mathcal{F}}(U)(s) : x \rightarrow s_x$
4. $\forall s \in \mathcal{F}^+(U)$ there is an open cover $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, the map $\iota_{\mathcal{F},x} : \mathcal{F}_x \rightarrow \mathcal{F}_x^+$ is an isomorphism.

Proposition 20

\forall morphisms $\phi : \mathcal{F} \rightarrow \mathcal{G}$ such that \mathcal{G} is a sheaf, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ open, let $s \in \mathcal{F}^+(U) \ni$ an open cover $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.

Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, we have to set

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since \mathcal{G} is a sheaf and

$$\phi(U_i)(s_i)|_{U_i \cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i \cap U_j}) = \phi(U_j)(s_j)|_{U_i \cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ with $t|_{U_i} = \phi(U_i)(s_i)$.

For ϕ^+ to be a morphism, we have to set $\phi^+(U)(s) = t$.

We still have to check that ϕ^+ is compatible with restriction maps. \square

Remark

The proposition above shows that $\text{hom}_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \text{hom}_{Psh(X)}(\mathcal{F}, \mathcal{G})$ naturally in the presheaf and the sheaf \mathcal{G} .

Hence $(-)^+$ is left-adjoint to the forgetful functor $Sh(X) \rightarrow Psh(X)$

Proposition 22

$X = \text{Spec } A \forall a \in A$ there exist isomorphisms $\phi_a : A_a \rightarrow \mathcal{O}_X(D(a))$ such that $\forall b \in A$ with $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow{\sim} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \rightarrow \mathcal{O}_X(D(a))$ by sending $\frac{s}{a^n} \mapsto (p \rightarrow \frac{s}{a^n} \in A_p)$.

Clearly, these make the diagram commute.

This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.

Let $I = \text{Ann}(s) = \{r \in A | rs = 0\}$.

Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, we have $I \not\subset p$, hence $V(I) \subset V(a) \implies a \in \sqrt{I}$.

Thus there exists $m \geq 1$ such that $a^m s = 0$, hence $\frac{s}{a^n} = 0$.

To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, by definition of \mathcal{O}_X and because $D(h_i)$ form a basis, we find $a_i, g_i, h_i \in A$ such that

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

and $s(q) = \frac{a_i}{g_i}$ for all $q \in D(h_i)$.

1. Claim 1 : Can choose $g_i = h_i$
2. Claim 2 : Can choose I finite
3. Claim 3 : Can choose a_i, h_i such that $h_j a_i = h_i a_j$.

Using these claims, since $D(a) = \bigcup D(h_i)$, we find $n > 0, b_j \in A$ such that $a^n = \sum b_j h_j$.

Write $c = \sum a_i b_i$.

Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.

Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.

We now prove the claims

1. We have $D(h_i) \subset D(g_i)$ thus $V(g_i) \subset V(h_i)$ and thus $h_i \in \sqrt{(g_i)}$.
So there exists $c_i \in A$ and $n > 1$ such that $h_i^n = c_i g_i$.
Now, we replace h_i by h_i^n and a_i by $a_i c_i$. Then

$$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$

2. We have $D(a) \subset \bigcup D(h_i) \iff V(\sum h_i) = \bigcap V(h_i) \subset V(a)$.
This is equivalent to saying that $a \in \sqrt{\sum (h_i)}$.
Thus there exists $n \geq 1$ such that $a^n \in \sum_i (h_i)$.
So there exist finitely many $b_i \in A$ such that $a^n = \sum b_j h_j$
3. On $D(h_i) \cap D(h_j) = D(h_i h_j)$, we have

$$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$

Thus

$$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$

Thus, there exists $N_j \geq 1$ such that $(h_i h_j)^{N_j} (h_j a_i - h_i a_j) = 0$.

From claim 2, I is finite, so we can choose N big enough such that N works for all $D(h_i)$.

Now, we replace h_i by h_i^{N+1} and a_i by $h_i^N a_i$ and we get $h_j a_i - h_i a_j = 0 \in A$. \square

Corollary 23

Take $X = \text{Spec } A$, then $\forall p \in \text{Spec } A \exists$ isomorphisms $\phi_p : A_p \rightarrow \mathcal{O}_{X,p}$ such that the appropriate diagram commutes.

Proof

1. Observe $\lim_{a \in A \setminus p} A_a \simeq A_p$ (check universal property)
2. Observe that $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$

Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

Definition 11 (Subsheaf)

Let \mathcal{F} be a (pre)sheaf on X .

Then a sub(pre)sheaf of \mathcal{F} is a (pre)sheaf \mathcal{G} such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by \mathcal{F} .

Definition 12 (Kernel, cokernel of presheaves)

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves

1. The presheaf kernel of ϕ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$
2. The presheaf image is defined as $\text{Im}^{pre}(\phi)(U) = \text{Im}(\phi(U))$
3. The presheaf cokernel is $\text{coker}^{pre}(\phi)(U) = \text{coker}(\phi(U))$.

In each case, the restriction maps are induced by those in \mathcal{F} or \mathcal{G} .

Lemma 24

If \mathcal{F} and \mathcal{G} are sheaves, then the presheaf kernel is a sheaf.

Proof

Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.

Since \mathcal{F} is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.

Thus

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

Since \mathcal{G} is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$. \square

Example

By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if \mathcal{F} and \mathcal{G} are.

Definition 13 (Cokernel/image of morphisms of sheaves)

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves

1. sheaf kernel : $\ker^{pre}(\phi)$

2. sheaf image $(\text{Im}^{pre}(\phi))^+$
3. sheaf cokernel $(\text{coker}^{pre}(\phi))^+$

Lemma 26 (cokernels are cokernels)

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves

1. $\ker \phi \rightarrow \mathcal{F}$ is a categorical kernel in $Sh(X)$
2. $\mathcal{G} \rightarrow \text{coker } \phi$ is a categorical cokernel in $Sh(X)$.

Proof

1. This means that for each commutative diagram with solid arrows, the dotted arrow is unique
"Insert cokernel/kernel diagram here"
 This holds for every open U and so the kernel is a sheaf.
2. The appropriate diagram commutes and we use the universal property of sheafification. \square

Proposition 27

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ is a monomorphism in $Sh(X)$
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

Proof

Recall ϕ is a monomorphism if for every $\psi : \mathcal{F}' \rightarrow \mathcal{F}$, $\phi \circ \psi = 0 \implies \psi = 0$.
 The implication $1 \implies 2$ follows by applying the monomorphism property to $\ker \phi \rightarrow \mathcal{F}$.
 $2 \implies 1$ If $\phi \circ \psi = 0$, then ψ factors through the kernel $\ker \phi \rightarrow \mathcal{F}$ and so $\psi = 0$.
 $2 \iff 3$ Since $\ker(\phi)(U) = \ker(\phi(U))$
 $3 \implies 4$ Follows because taking direct limits is exact.
 $4 \implies 3$ Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.
 So $s_x = 0 \forall x \in U$ and so $s = 0$ \square

Proposition 28

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ is an epimorphism in $Sh(X)$
2. $\text{coker}(\phi) = 0$
3. $\text{coker}(\phi_x) = 0$

Proof

Recall that ϕ is an epimorphism if for every $\psi : \mathcal{G} \rightarrow \mathcal{G}'$, $\psi \circ \phi = 0 \implies \psi = 0$

1 \implies 2 Apply epimorphism property to $\mathcal{G} \rightarrow \text{coker}(\phi)$

2 \implies 3 We have

$$\begin{aligned} 0 &= (\text{coker } \phi)_x \\ &= (\text{coker}^{pre} \phi)_x = \text{coker}(\phi_x) \end{aligned} \quad \square$$

3 \implies 1

Let $\psi : \mathcal{G} \rightarrow \mathcal{G}'$ such that $\psi \circ \phi = 0$, this implies that $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$.

Since ϕ_x is an epimorphism of abelian groups, we get $\psi_x = 0$.

As the hom sheaf is a sheaf, we get that $\psi = 0$

Remark

If $\text{coker}(\phi(U)) = 0 \forall U \subset X \implies \text{coker}(\phi) = 0$ but the converse is not true.

Corollary 30

If $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves, then the following are equivalent

1. ϕ is an isomorphism
2. $\phi(U)$ is an isomorphism $\forall U \subset X$ open
3. ϕ_x is an isomorphism $\forall x \in X$

Proof

1 \implies 2 since taking sections is a functor

2 \implies 3 since taking limits is functorial

2 \implies 1 because $(\phi(U))^{-1}$ defines a morphism of sheaves

3 \implies 2 Need to show surjectivity of $\phi(U)$.

Let $t \in \mathcal{G}(U)$, since ϕ_x is an isomorphism $\forall x \in U$, we find $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$.

There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ such that $(s_{V_x})_x = s_x$
Since

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

we can choose $V + x$ sufficiently small such that $\phi(V_x)(s_{V_x}) = t|_{V_x}$.

Since $\phi(V_x \cap V_y)$ is injective and $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$, we have $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$.

Thus there exists $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ and $\phi(U)(s)|_{V_x} = t|_{V_x}$ and thus $\phi(U)(s) = t$. \square

Definition 14 (Exact Sequence of sheaves)

A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ is called exact if $\ker \phi_2 = \text{Im } \phi_1$

Corollary 31

A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.

Lecture 4: locally ringed spaces, (affine) Schemes (!)

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Corollary 32

A sequence of sheaves is exact if and only if it is exact on all stalks.

Proof

If $\ker(\phi_{2,x}) = \text{Im}(\phi_{1,x}) \forall x \in X$, thus $(\phi_{2,x} \circ \phi_{1,x}) = (\phi_2 \circ \phi_1)_x$.

Thus $\phi_2 \circ \phi_1 = 0$ because the hom sheaf is a sheaf.

Thus ϕ_1 factors as $\mathcal{F}_1 \rightarrow \text{Im } \phi_1 \rightarrow \ker \phi_2 \rightarrow \mathcal{F}_2$ as ψ_x is an isomorphism, ψ is an isomorphism. \square

Corollary 33

Let $\phi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves, then $\text{Im } \phi = \ker(\mathcal{G} \text{ to coker } \phi)$

Corollary 34

$Sh(X)$ is an abelian category.

1.5 Direct and inverse image, ringed spaces**Definition 15**

Let $f : X \rightarrow Y$ be a continuous map.

We define the direct image of \mathcal{F} by f on Y defined by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

We can check that $f_\mathcal{F}$ is a sheaf with restriction maps induced by \mathcal{F} .*

If $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on X , then the $(f_\phi)(X) = \phi(f^{-1}(V))\mathcal{F}(f^{-1}(V)) \rightarrow \mathcal{G}(f^{-1}V)$ define a morphism of sheaves.*

Thus we get a functor $f_ : Sh(X) \rightarrow Sh(Y)$.*

Definition 16 (inverse image)

Let $f : X \rightarrow Y$ be a continuous map and let \mathcal{G} be a sheaf on Y .

The inverse image of \mathcal{G} along f is the sheafification of the presheaf

$$f^{-1,pre}(\mathcal{G})$$

defined by

$$f^{-1,pre}(\mathcal{G})(U) = \varprojlim_{f(U) \subset V} \mathcal{G}(V)$$

We can again check that if $\phi : \mathcal{F} \rightarrow \mathcal{G}$ is a morphism of sheaves on Y , we define $f^{-1}\phi : \varprojlim \mathcal{F}(V) \rightarrow \varprojlim \mathcal{G}(V)$ using the maps induced by ϕ . Thus we get a functor $Sh(Y) \rightarrow Sh(X)$.

Lemma 35

Let $f : X \rightarrow Y$ be a continuous map, \mathcal{F} a sheaf on X and \mathcal{G} a sheaf on Y .

1. $\forall y \in Y$ there is a natural isomorphism

$$(f_*\mathcal{F})_y \simeq \varprojlim_{y \in V \subset Y} \mathcal{F}(f^{-1}(V))$$

In particular for all $x \in X$ there is a natural map $(f_*\mathcal{F})_{f(x)} \rightarrow \mathcal{F}_x$

2. $\forall x \in X$ there is a natural isomorphism $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$

Proof

The isomorphisms are immediate from the definition.

The morphism $(f_*\mathcal{F})_{f(x)} \rightarrow \mathcal{F}_x$ is given by

$$(f_*\mathcal{F})_{f(x)} = \varprojlim \mathcal{F}(f^{-1}(V)) = \varprojlim_{x \in f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \rightarrow \varprojlim_{x \in U} \mathcal{F}(U) = \mathcal{F}_x \quad \square$$

Proposition 36

If $f : X \rightarrow Y$ is a continuous map, then $f_* : Sh(X) \rightarrow Sh(Y)$ is right-adjoint to $f^{-1} : Sh(Y) \rightarrow Sh(X)$

Corollary 37

$f^{-1} : Sh(Y) \rightarrow Sh(X)$ is exact

Proof

Let $0 \rightarrow \mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_3 \rightarrow 0$ be exact in $Sh(Y)$.

Thus $\forall y \in Y, 0 \rightarrow \mathcal{G}_{1,y} \rightarrow \mathcal{G}_{2,y} \rightarrow \mathcal{G}_{3,y} \rightarrow 0$ is exact.

In particular it is exact at $f(x) \forall x \in X$ and thus the associated inverse image

| sequence is exact. □

Corollary 38

$f_* : Sh(X) \rightarrow Sh(Y)$ is left-exact.

Proof

Let $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$ be exact in $Sh(X)$.

Recall that the section functor is left-exact, thus $0 \rightarrow \mathcal{F}_1(U) \rightarrow \mathcal{F}_2(U) \rightarrow \mathcal{F}_3(U)$ is exact $\forall U \subset X$.

Thus $0 \rightarrow (f_*\mathcal{F}_1)_y \rightarrow (f_*\mathcal{F}_2)_y \rightarrow (f_*\mathcal{F}_3)_y$ is exact $\forall y \in Y$ and thus $0 \rightarrow f_*\mathcal{F}_1 \rightarrow f_*\mathcal{F}_2 \rightarrow f_*\mathcal{F}_3$ is exact. □

Example

f_* is usually not right-exact.

Eg, if $f : X \rightarrow \{*\}$ and \mathcal{F} is a sheaf on X , then $(f_*\mathcal{F})(\emptyset) = 0$ and $(f_*\mathcal{F})(\{*\}) = \mathcal{F}(X)$ and taking sections is not exact.

Definition 17 (Ringed space)

A ringed space is a pair (X, \mathcal{O}_X) where X is a topological space and \mathcal{O}_X is a sheaf of rings on X .

A morphism of ringed spaces $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f : X \rightarrow Y$ is a continuous map and $f^\#$ is a morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$.

Remark

Ringed spaces form a category, if $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, $(g, g^\#) : (Y, \mathcal{O}_Y) \rightarrow (Z, \mathcal{O}_Z)$ define their composition to be $(g \circ f, g_* (f^\# \circ g^\#))$

Example

1. For every ring A , $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ is a ringed space.
2. For any field K and any topological space X , define a sheaf $\text{Fun}_{X,K}(U) = \{s : U \rightarrow K\}$.
There is a functor $\top \rightarrow (\text{Ringed spaces})$ sending $X \mapsto (X, \text{Fun}_{X,K})$ where for $f : X \rightarrow Y$ $f^\#$ is the pullback (precomposition).
3. (X, C_X^0) is a ringed space

Observe that for many of these examples of ringed spaces, the stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 18 (Morphism of local rings)

A morphism of local rings $\phi : A \rightarrow B$ with maximal ideals m_A and m_B is called local if $m_A = \phi^{-1}(m_B)$

Example

1. For all ring homomorphism $\phi : A \rightarrow B$ and all $q \in \text{Spec } B$ the induced map $A_{\phi^{-1}(q)} \rightarrow B_q$ is local.
2. A ring homomorphism $\phi : A \rightarrow K$ from a local ring A to a field iff $m_A = \ker \phi$

Definition 19 (Locally ringed space)

A locally ringed space is a ringed space (X, \mathcal{O}_X) such that $\mathcal{O}_{X,x}$ is local $\forall x \in X$.

A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that

$$f_x^\# : \mathcal{O}_{Y,f(x)} \xrightarrow{f_x^\#} (f_* \mathcal{O}_X)_{f(x)} \rightarrow \mathcal{O}_{X,x}$$

is local.

Remark

The category of locally ringed spaces is a subcategory of the category of ringed spaces

Definition 20 (Affine Scheme)

An affine scheme is a locally ringed space (X, \mathcal{O}_X) such that $X = \text{Spec } A$ and \mathcal{O}_X is the structure sheaf.

Definition 21 (Scheme)

A scheme is a locally ringed space (X, \mathcal{O}_X) such that there exists an open cover $X = \bigcup_{i \in I} U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.

A morphism of schemes is a morphism of the underlying ringed spaces.

Example

1. If (X, \mathcal{O}_X) is a scheme and $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is not necessarily a scheme (even if X is affine).
2. If (X, \mathcal{O}_X) is a scheme and $X = \{*\}$, then X is affine.
Then $\text{Spec } A = \{*\}$ iff every $a \in A$ is either a unit or nilpotent.

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Remark

By abuse of notation, we write X is a scheme with \mathcal{O}_X implicit.

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Lemma 46

Let X be a topological space with basis for the topology $\{v_i\}_{i \in I}$.

Let \mathcal{F} and \mathcal{G} be sheaves on X .

For any collection of morphisms $\phi_i : \mathcal{F}(V_i) \rightarrow \mathcal{G}(V_i)$ such that $\rho_{ij} \circ \phi_i = \phi_j$, then $\exists! \phi : \mathcal{F} \rightarrow \mathcal{G}$ which restricts to ϕ_i on the V_i .

Proposition 47

Let (X, \mathcal{O}_X) be a locally ringed space and A a ring, then the map $\text{hom}((X, \mathcal{O}_X), (\text{Spec } A, \mathcal{O}_{\text{Spec } A})) \rightarrow \text{hom}(A, \mathcal{O}_X(X))$ which maps $(f, f^\#) \rightarrow f^\#(\text{Spec } A)$ is a natural bijection.

In particular, for all locally ringed spaces (X, \mathcal{O}_X) , there is a natural affinization morphism $\text{aff}_X : X \rightarrow \text{Spec } \mathcal{O}_X(X)$

Corollary 48

Every morphism of locally ringed spaces $(X, \mathcal{O}_X) \rightarrow \text{Spec } A$ factors uniquely through aff_X .

Corollary 49

A locally ringed space is an affine scheme iff the affinization is an isomorphism.

Corollary 50

The functor

$$(\text{affSch}) \rightarrow (\text{Ring})^{\text{op}}$$

mapping $(X, \mathcal{O}_X) \rightarrow \mathcal{O}_X(X)$ is an equivalence of categories.

Proof

Fully faithful is the proposition above.

Essential surjectiveness is immediate as for any ring, we can look at $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ as $\mathcal{O}_{\text{Spec } A}(\text{Spec } A) = A$. \square

We now prove the statement

Proof

We use that there exists a natural isomorphism $\mathcal{O}_{\text{Spec } A}(D(a)) \simeq A_a$.

Naturality follows from functoriality of $f^\#(-)$.

We have to construct an inverse, let $\phi : A \rightarrow \mathcal{O}_X(X)$ be a ring homomorphism, we need to define $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (\text{Spec } A, \mathcal{O}_{\text{Spec } A})$.

We map $x \mapsto \ker(A \xrightarrow{\phi} \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x}/m_x)$.

We claim that f is continuous.

It suffices to show that $X_{\phi(a)} = f^{-1}(D(a)) = \{x \in X \mid \phi(a)_x \notin m_x\} \subset X$ is open.

Take $x \in X_{\phi(a)}$, then $\phi(a)_x \notin m_x \implies \phi(a)_x \in \mathcal{O}_{X,x}^\times$.

Thus $\exists x \in V \subset X$ and $b \in \mathcal{O}_X(V)$ such that $\phi(a)|_V b = 1 \in \mathcal{O}_X(V)$.

Thus $\phi(a)_y b_y = 1 \forall y \in V \implies \phi(a)_y \notin m_y \implies V \subset X_{\phi(a)} \implies X_{\phi(a)}$ is open.

To define f^\sharp , observe that $\forall a \in A, \phi(a)|_{X_{\phi(a)}} \in \mathcal{O}_X(X_{\phi(a)})$ is a unit in every stalk, hence a unit.

Thus there is a unique morphism such that $A \xrightarrow{\phi} \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X_{\phi(a)}) = A \rightarrow A_a \xrightarrow{\exists! f^\sharp(D(a))} \mathcal{O}_X(X_{\phi(a)})$ so we get a morphism $f^\sharp : \mathcal{O}_{\text{Spec } A} \rightarrow f_* \mathcal{O}_X$.

We still have to show that this map is a morphism of locally ringed spaces.

We claim that $\forall x \in X$, the map $f_x^\sharp : A_{f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local homomorphism.

The diagram induces a commutative diagram

$$A \xrightarrow{\phi} \mathcal{O}_X(X) \xrightarrow{\pi_2} \mathcal{O}_{X,x} = A \xrightarrow{\pi_1} A_{f(x)} \xrightarrow{f_x^\sharp} \mathcal{O}_{X,x}$$

Note that $p_1^{-1}(f_x^{\sharp,-1}(m_x)) = \pi_1^{-1} \circ \pi_2^{-1}(m_x) = f(x)$ by definition.

Thus $f_x^{\sharp,-1}(m_x) = f(x)A_{f(x)}$.

Now, we need to show that this construction is in fact an inverse.

By construction, if (f, f^\sharp) comes from ϕ , then $\phi = f^\sharp(\text{Spec } A)$.

Conversely, let $(f, f^\sharp) : X \rightarrow \text{Spec } A$ be a morphism and let $(f', f'^\sharp) : X \rightarrow \text{Spec } A$ be associated to $f^\sharp(\text{Spec } A)$.

We need to show that $(f, f^\sharp) = (f', f'^\sharp)$.

$\forall x \in X, \exists$ a commutative diagram

$$A \xrightarrow{f^\sharp(\text{Spec } A)} \mathcal{O}_X(X) \rightarrow \mathcal{O}_{X,x} = A \rightarrow A_{f(x)} \rightarrow \mathcal{O}_{X,x}$$

As f_x^\sharp and $f'_x{}^\sharp$ are local, $f(x) = f'(x)$. Now, $\forall a \in A$, there is a commutative diagram

$$A \xrightarrow{f^\sharp(\text{Spec } A)} \mathcal{O}_X(X) \rightarrow \mathcal{O}_X(X_{f^\sharp(\text{Spec } A)}) = A \rightarrow A_a \xrightarrow{\exists! f^\sharp(D(a))} \mathcal{O}_X(X_{f^\sharp(\text{Spec } A), a})$$

□

Example

For every locally ringed space (X, \mathcal{O}_X) , there exists a unique morphism $(X, \mathcal{O}_X) \rightarrow (\text{Spec } \mathbb{Z}, \mathcal{O}_{\text{Spec } \mathbb{Z}})$ because $\exists! \mathbb{Z} \rightarrow \mathcal{O}_X(X)$.

If (X, \mathcal{O}_X) is a locally ringed space such that each $\mathcal{O}_X(U)$ has characteristic $p > 0$, then $\exists!$ morphism $(X, \mathcal{O}_X) \rightarrow (\text{Spec } \mathbb{F}_p, \mathcal{O}_{\text{Spec } \mathbb{F}_p})$.

Definition 22 (Scheme over another scheme)

Let S be a sscheme. The category of schemes over S , Sch/S is the category whose objects are morphisms $X \rightarrow S$ and morphisms are commutative triangles.

Example

Let K be a field.

The affine n -space over k is denoted \mathbb{A}_k^n is $\text{Spec } k[x_1, \dots, x_n]$.

If k is algebraically closed, then

$$k^n \simeq \text{Spec}_{max} k[x_1, \dots, x_n] \simeq \mathbb{A}_k^n \simeq \text{hom}_{k\text{-alg}}(k[x_1, \dots, x_n], k)$$

If $\phi : A \rightarrow B$ is a surjective ring homomorphism, then the induced map on spectra $\text{Spec } B \rightarrow \text{Spec } A$ is a homeomorphism onto $V(I)$ where $I = \ker \phi$.

In particular, if $I \subset K[x_1, \dots, x_n]$, $k = \bar{k}$ is an ideal, then $V(I) = \{(a_1, \dots, a_n) \in k^n \mid f(a_1, \dots, a_n) = 0 \forall f \in I\}$ is the image of $\text{Spec}_{max} k[x_1, \dots, x_n] / I \rightarrow \text{Spec}_{max} k[x_1, \dots, x_n] \simeq k^n$.

Example (glueing two schemes)

If X_1, X_2 are two schemes and $U_i \subset X_i$ are open subsets,

$$(\phi, \phi^\sharp) : (U_1, \mathcal{O}_X|_{U_1}) \simeq (U_2, \mathcal{O}_X|_{U_2})$$

is an isomorphism.

We define the scheme (X, \mathcal{O}_X) by glueing X_1 and X_2 over U_1 as follows.

As a set, $X = X_1 \amalg X_2 / \sim$ where $x_1 \sim \phi(x_1)$.

Note, there are natural maps $\pi_i : X_i \rightarrow X$.

We say that a subset $U \subset X$ is open $\iff \pi_i^{-1}(U) \subset X_i$ open for $i = 1, 2$.

We define the structure sheaf as $\mathcal{O}_X(U) = \ker(\mathcal{O}_{X_1}(\pi_1^{-1}(U)) \oplus \mathcal{O}_{X_2}(\pi_2^{-1}(U)) \rightarrow \mathcal{O}_{X_1}(\pi_1^{-1}(U) \cap U_1))$.

Then X is a scheme.

Example (Explicit example of glueing)

Take $X_1 = X_2 = \mathbb{A}_K^1$ and $U_1 = U_2 = \mathbb{A}_K^1 \setminus 0$.

Notice that $U \simeq \text{Spec } k[x, x^{-1}]$.

1. Taking the glueing map $\phi = \text{Id}$, we get a line with two origins.

2. Taking $\phi^\sharp(U_2) : x \mapsto \frac{1}{x}$, we get the projective line \mathbb{P}_k^1 .

The k -rational points of this scheme are in correspondence with lines in k^2 , namely

$$P_k^1(k) \simeq k^2 \setminus \{0\} /_{k^\times}.$$

2 Properties of schemes

2.1 Topological properties

Definition 23

A scheme (X, \mathcal{O}_X) is called

1. *connected* if X is
2. *irreducible* if $\forall U_1, U_2$ open non empty their intersection is non-empty.
3. *quasi-compact* if X is.^a
4. *quasi-separated* if X is, ie. $\forall U_1, U_2$ open and quasi-compact, their intersection is quasi-compact.

^a. All affine schemes are quasi-compact, but $\mathbb{A}_k^\infty \setminus 0$ is not quasi-compact

Lecture 6: Topological properties

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Remark

$\text{Spec } R \times S = \text{Spec } R \coprod \text{Spec } S$ but $\text{Spec } \prod_i R_i \not\cong \coprod_i \text{Spec } R_i$ for infinite products

Lemma 56

Affine schemes are quasi-compact and quasi-separated.

Proof

Let $X = \text{Spec } A$ be an affine scheme.

Quasi-compactness has already been proven.

If $U \subset X$ is open and qc., then $U = \bigcup_{i \in I_U} D(a_i)$, $a_i \in A$ and I_U finite.

For $U_1, U_2 \subset X$ qc. open, then

$$U_1 \cap U_2 = \bigcup_{i \in I_{U_1}, j \in I_{U_2}} D(a_i) \cap D(a_j) = \bigcup D(a_i a_j)$$

Check that a finite union of qc spaces is qc □

Remark

Let X be a topological space, then \forall subsets $V \subset X$ and $U \subset X$, then

$$U \cap V \neq \emptyset \iff U \cap \overline{V} \neq \emptyset$$

Thus V is irreducible iff its closure is.

If X is irreducible, then every non-empty open is dense.

2.2 Scheme-Theoretic Properties

Definition 24 (Open Subscheme)

An open subscheme of a scheme (X, \mathcal{O}_X) is a pair (U, \mathcal{O}_U) with U open in X and $\mathcal{O}_U := \mathcal{O}_X|_U$

If P is a property of rings, when do we say that (X, \mathcal{O}_X) satisfies P ?

1. $\forall U \subset X, \mathcal{O}_X(U)$ satisfies P (usually too strong)
2. $\forall U \subset X$ open and affine, $\mathcal{O}_X(U)$ satisfies P
3. \exists an open affine cover $U = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P
4. $\forall x \in X \exists x \in U$ open affine such that $\mathcal{O}_X(U)$ satisfies P .
5. $\forall x \in X, \mathcal{O}_{X,x}$ satisfies P .

Observe that $1 \implies 2 \implies 3 \iff 4$.

Lemma 58

For $P = \text{"reduced ring"}$, then all 5 are equivalent.

Proof

From commutative algebra, we know that a ring A is reduced $\iff A_p$ is reduced $\forall p \in \text{Spec } A$.

This implies that $2 \iff 3 \iff 4 \iff 5$.

Let's show $2 \implies 1$.

Let $U \subset X$ open and $s \in \mathcal{O}_X(U)$ such that $s^n = 0$, then $s^n|_V = 0 \forall V \subset U$ affine.

Thus, $s|_V = 0 \forall V \subset U$ open affine and as \mathcal{O}_X is a sheaf $s = 0$. \square

Definition 25 (Reduced Scheme)

A scheme (X, \mathcal{O}_X) is called reduced if $\mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open.

Definition 26

Let P be a property of rings or of open affines $\text{Spec } A \hookrightarrow X$ of a scheme X

- P is called affine-local if $\forall a_1, \dots, a_n \in A$ such $(a_1, \dots, a_n) = A$.
 A satisfies P every A_{a_i} satisfies P
- P is called stalk-local if A satisfies $P \iff A_p$ satisfies $P \forall p \in \text{Spec } A$.

Remark

Being stalk-local is stronger than being affine local.

This is because $A \rightarrow A_a$ induces $(A_a)_{pA_a} \simeq A_p \forall p \in D(a)$

Example

1. Reduced is stalk-local
2. Normal
3. regular
4. Cohen-Macaulay

Example

1. Integrality is not affine-local (consider $A = k \times k$)
2. Factorial is not affine-local
3. Noetherian is not stalk-local (consider $A = \prod_i \mathbb{F}_2$)

Lemma 62

Being Noetherian is affine-local.

Why do we care?

For affine-local properties, 2 and 4 of our list are equivalent.

Proof

If A is noetherian, then any quotient and any localization is.

Assume $(a_1, \dots, a_n) = A$ and A_{a_i} are Noetherian.

Let $\phi_i : A \rightarrow A_{a_i}$ be the localization maps.

Claim : \forall ideals $I \subset A$, $I = \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$.

One inclusion is clear.

Let $b \in \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$, thus there exists $N > 0$ and $b_i \in I$ such that $b = \frac{b_i}{a_i^N} \in A_{a_i}$.

Thus there exists an $M > 0$ such that $a_i^M(a_i^N b - b_i) = 0$ in A .

Set $k = M + N$, note that $1 = (a_1^k, \dots, a_n^k)$.

We can write $1 = \sum_{i=1}^n c_i a_i^k$ for some $c_i \in A$.

Thus $b = \sum c_i a_i^k b = \sum c_i a_i^M b_i \in I$.

Let $I_1 \subset \dots \subset I_n \subset A$ be an ascending chain of ideals in A , then we get an ascending chain of ideals $\phi_1(I_1)A_{a_1} \subset \dots \subset \phi_n(I_n)A_{a_n}$.

This becomes constant because A_{a_i} is noetherian and $\exists N > 0$ such that $\phi_i(I_k)A_{a_i} = \phi_i(I_N)A_{a_i} \forall k \geq N$ \square

Lemma 63

Let P be an affine-local property of rings. Let (X, \mathcal{O}_X) be a scheme, then the following are equivalent.

1. Every open affine $\text{Spec } A \hookrightarrow X$ satisfies P
2. \exists an open affine cover $X = \cup \text{Spec } A_i$ such that each $\text{Spec } A_i \hookrightarrow X$ satisfies P .

Proof

1 \implies 2 is clear.

2 \implies 1.

Let $\text{Spec } A \hookrightarrow X$ open and affine.

Write $\text{Spec } A = \cup \text{Spec } A_{a_i}$ with $a_i \in A$ such that $A_{a_i} \simeq (A_i)_{b_i}$ for some $b_i \in A_i$.

$\text{Spec } A_i \hookrightarrow X$ satisfies P , implies $(\text{Spec } (A_i)_{b_i}) \hookrightarrow X$ satisfies P implies $\text{Spec } A_{a_i} \hookrightarrow X$ satisfies P implies $\text{Spec } A \hookrightarrow X$ satisfies P \square

Lemma 64

Let $\text{Spec } A, \text{Spec } B \subset X$ be open affines, then for every point $x \in \text{Spec } A \cap \text{Spec } B$ there exist $a \in A$ and $b \in B$ such that $A_a \simeq B_b$ such that $x \in D(a) \subset \text{Spec } A$ and $x \in D(b) \subset \text{Spec } B$ and the isomorphism $\text{Spec } A_a \simeq \text{Spec } B_b$ commutes with the inclusions to X .

Proof

$\text{Spec } A \cap \text{Spec } B \subset \text{Spec } A$ is open.

Thus, there exists $a \in A$ with $x \in D(a) \subset \text{Spec } A \cap \text{Spec } B$.

We can assume wlog that $\text{Spec } A \rightarrow X$ factors through $\text{Spec } B$.

Write $\phi : B \rightarrow A$ for the induced map of rings.

Since $\text{Spec } A \subset \text{Spec } B$ is open $\exists b \in B$ and $B \rightarrow A \rightarrow B_b$ is just localization of B at b .

Then $A \rightarrow B_b$ satisfies the universal property of $A \rightarrow A_{\phi(b)}$.

So we get a commutative square $B \rightarrow A \rightarrow A_{\phi(b)}$ and $B \rightarrow B_b \rightarrow A_{\phi(b)}$ and we get an isomorphism $B_b \simeq A_{\phi(b)}$. \square

Definition 27

Let P be an affine-local property of rings.

A scheme (X, \mathcal{O}_X) is called locally P if $\mathcal{O}_X(U)$ satisfies $P \forall U \subset X$ open affine.

Definition 28 (Noetherian scheme)

A scheme (X, \mathcal{O}_X) is called Noetherian if it is locally Noetherian and qc.

Definition 29 (Integral scheme)

A scheme (X, \mathcal{O}_X) is called integral if $\mathcal{O}_X(U)$ is an integral domain $\forall U \subset X$ open and non-empty.

Lemma 65

For a scheme (X, \mathcal{O}_X) , the following are equivalent.

1. X is integral
2. X is reduced and irreducible.
3. $\forall U \subset X$ open affine, $\mathcal{O}_X(U)$ is integral.

Proof

1 \implies 3 is clear.

3 \implies 2.

Reduced is clear.

Let $U_1, U_2 \subset X$ open with $U_1 \cap U_2 = \emptyset$.

Wlog, the U_i are affine.

Then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$.

Thus $\mathcal{O}_X(U_1) = 0$ or $\mathcal{O}_X(U_2) = 0$ which implies U_1 or $U_2 = \emptyset$.

2 \implies 1

Let $U \subset X$ be open.

Assume $\exists a, b \in \mathcal{O}_X(U)$ such that $ab = 0$.

Let $U_a = \{x \in U \mid a_x \notin m_x\}$ and similarly U_b .

Note that $U_a \cap U_b = \emptyset$ since $\forall x \in U_a \cap U_b, a_x$ and b_x are units.

Thus $U_a = \emptyset$ or $U_b = \emptyset$.

If $U_a = \emptyset \forall \text{Spec } A \subset U \forall p \in \text{Spec } A$

$$(a|_{\text{Spec } A})_p \in pA_p$$

thus $a|_{\text{Spec } A} \in p \forall p \in \text{Spec } A$.

Thus $a|_{\text{Spec } A}$ is nilpotent.

But since X is reduced, $a|_{\text{Spec } A} = 0$.

Covering U by affines, $a = 0$ (as A was arbitrary). □

3 Open and closed subschemes and immersions

Definition 30 (Open Subscheme)

An open subscheme of a scheme (X, \mathcal{O}_X) is a pair (U, \mathcal{O}_U) , with $U \subset X$ open and $\mathcal{O}_U = \mathcal{O}_X|_U$.

Lemma 66

If A is a ring and $a \in A$, then there is an isomorphism of locally ringed spaces $(\text{Spec } A_a, \mathcal{O}_{\text{Spec } A_a}) \simeq (D(a), \mathcal{O}_{\text{Spec } A}|_{D(a)})$.

In particular, open subschemes of schemes are schemes.

Proof

From commutative algebra, localization $A \rightarrow A_a$ induces a homeomorphism $\text{Spec } A_a \rightarrow D(a) \subset \text{Spec } A$.

On sheaves, we want to give morphisms $\mathcal{O}_{\text{Spec } A}|_{D(a)}(U) \rightarrow \mathcal{O}_{\text{Spec } A_a}(f^{-1}(U))$.

If $s : U \rightarrow \coprod_{p \in U} A_p \rightarrow (f^{-1}(U) \rightarrow U \xrightarrow{s} \coprod_{p \in U} A_p \rightarrow \coprod_{p \in U} (A_a)_{pA_a})$, using $A_p \simeq (A_a)_{pA_a}$. \square

Note that, if $i : U \rightarrow X$ is the inclusion of an open, then $(i, i^\#) : (U, \mathcal{O}_U) \rightarrow (X, \mathcal{O}_X)$ with

$$i^\#(V) : \mathcal{O}_X(V) \xrightarrow{\rho_{V, V \cap U}} \mathcal{O}_X(V \cap U) = i_* \mathcal{O}_U(V)$$

is a morphism of schemes.

Remark

If $i : U \rightarrow X$ is an inclusion of an open, then there are in general many sheaves of rings \mathcal{F} on U such that $\exists i^\#$ such that $(i, i^\#) : (U, \mathcal{F}) \rightarrow (X, \mathcal{O}_X)$ is a morphism of schemes.

For example, if $X = \text{Spec } k$, $U = \text{Spec } k[x]_{(x)}$ then $k \subset k[x]_{(x)}$ induces a morphism $(f, f^\#) : U \rightarrow X$ such that $f = \text{Id}_X$.

Definition 31 (Open immersion)

An open immersion (or open embedding) is a morphism of schemes $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ such that f is a homeomorphism onto an open subset $U \subset Y$ $\mathcal{O}_Y|_U \simeq (f_* \mathcal{O}_X)|_U$.

Example

Let k be a field and let $\iota : \text{Spec } k \rightarrow X = \mathbb{A}^n$ be the closed point corresponding to

0.

Then

$$\begin{aligned} (\mathcal{O}_X)|_{\mathrm{Spec} k}(\mathrm{Spec} k) &= (i^{-1}\mathcal{O}_X)(\mathrm{Spec} k) \\ &= \varinjlim_{0 \in U \subset \mathbb{A}^n} \mathcal{O}_X(U) = \mathcal{O}_{X,0} = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)} \end{aligned}$$

But $\mathrm{Spec} k[x_1, \dots, x_n]$ has more than one point.

Thus, $(\mathrm{Spec} k, (\mathcal{O}_X)|_{\mathrm{Spec} k})$ is not a scheme.

Observe : If $Z \subset \mathrm{Spec} A$ is a closed subset, then $Z = V(I)$ for some ideal I .

Then the map $\mathrm{Spec} A/I \rightarrow \mathrm{Spec} A$ induced by the quotient map is a homeomorphism onto $V(I)$ and this gives a scheme structure on Z (which depends on I !).

Definition 32 (Ideal sheaves)

Let (X, \mathcal{O}_X) be a scheme, then

1. An ideal sheaf on (X, \mathcal{O}_X) is a subsheaf $\mathcal{I} \subset \mathcal{O}_X$ such that $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ is an ideal for all $U \subset X$ is open.
2. For an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, the quotient sheaf $\mathcal{O}_X/\mathcal{I}$ is the cokernel sheaf of the inclusion, namely, the sheafification of the sheaf $U \mapsto \mathcal{O}_X(U)/\mathcal{I}(U)$.

Definition 33 (Closed Subscheme)

Let (X, \mathcal{O}_X) be a scheme, then a closed subscheme of (X, \mathcal{O}_X) consists of a subset $Z \subset X$ and an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that

1. $Z = \{x \in X \mid (\mathcal{O}_X/\mathcal{I})_x \neq 0\}$
2. $(Z, (\mathcal{O}_X/\mathcal{I})|_Z)$ is a scheme

Remark

By 1, Z is closed, indeed, for $1 \in (\mathcal{O}_X/\mathcal{I}(X))$, we have

$$\{x \in X \mid (\mathcal{O}_X/\mathcal{I})_x \neq 0\} = \mathrm{Supp} 1$$

Remark

The morphism $\mathcal{O}_X/\mathcal{I} \rightarrow i_*((\mathcal{O}_X/\mathcal{I})|_Z)$ is an isomorphism.

If $Z \subset X$ is a closed subscheme determined by \mathcal{I} , then $(i, i^\#) : (Z, \mathcal{O}_Z) \rightarrow (X, \mathcal{O}_X)$ where $i : Z \rightarrow X$ is the inclusion and $i^\# : \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I} = i_*\mathcal{O}_Z$ is a morphism of schemes.

Example

Condition 2 in the definition of closed subscheme is not automatic, even if X is affine.

Definition 34 (Closed immersion)

A closed immersion (or closed embedding) is a morphism of schemes $(f, f^\#) : X \rightarrow Y$ such that f is a homeomorphism onto a closed subset and $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is surjective on stalks.

Remark

The morphism $(i, i^\#)$ of the inclusion of closed subscheme is a closed immersion.

Example

If A is a ring and $I \subset A$ is an ideal, then the morphism $\text{Spec } A/I \rightarrow \text{Spec } A$ is a closed immersion.

Indeed, by CA, this is a homeomorphism onto $V(I)$.

The map $f^\# : \mathcal{O}_{\text{Spec } A} \rightarrow \mathcal{O}_{\text{Spec } A/I}$ is surjective because $f^\#_p : A_p \rightarrow (A/I)_p$ is the localization of a surjective map, which is the localization of a surjective map.

From now on, $V(I) \subset \text{Spec } A$ for the closed subscheme determined by I .

Proposition 74

If $X = \text{Spec } A$ is affine, then the map $I \rightarrow V(I)$ is a bijection between ideals of A and closed subschemes.

Proof

Let $Z \subset X$ be a closed subscheme determined by \mathcal{I} .

Let $I_Z = \ker(\mathcal{O}_X(X) \rightarrow \mathcal{O}_Z(Z))$.

Note, if $Z = V(I)$ for some ideal, then $I = I_Z$.

So we have to show that $Z = V(I_Z)$.

The morphism $\phi : \mathcal{O}_X(X) \rightarrow \mathcal{O}_Z(Z)$ factors through $\mathcal{O}_X(X)/I_Z = A/I_Z$ so $\iota : Z \rightarrow X$ factors through $V(I_Z)$.

Replace A by A/I_Z , we may assume ϕ is injective and we have to show that ϕ is an isomorphism.

Claim 1

$\forall U \subset Z$ open affine and $s \in A$, we have $D(s) \cap U = D(\phi(s)|_U) \subset U$ and $V(S) \cap U = V(\phi(s)|_U) \subset U$.

It suffices to prove the first equality.

$\forall p \in Z$, the following diagram commutes

$$A \xrightarrow{\phi} \mathcal{O}_Z(Z) \rightarrow \mathcal{O}_{Z,p}$$

and $A \rightarrow A_p \xrightarrow{i_p^\#} \mathcal{O}_{Z,p}$ and $i_p^\#$ is local.

Now

$$\begin{aligned} D(\phi(s)|_U) &= \{p \in U \mid \phi(s)_p \notin m_p \subset \mathcal{O}_{Z,p}\} = \{p \in U \mid i_p^\#(s_p) \notin m_p \subset \mathcal{O}_{Z,p}\} \\ &= \{p \in U \mid s_p \notin m_p \subset A_p\} = D(s) \cap U \square \end{aligned}$$

Claim 2

We show $Z \rightarrow X$ is surjective. Since Z is closed in X , it suffices to show that $\forall s \in A$ such that $Z \subset V(s)$, we have $V(s) = X$.

Choose such an $s \in A$.

As closed subspaces of qc. spaces are qc.

We can cover Z by finitely many open affines U_i .

By claim 1, $U_i \subset V(\phi(s)|_{U_i}) \subset U_i$.

Thus $\phi(s)|_{U_i} \in p \forall p \in \text{Spec } \mathcal{O}_Z(U_i)$, thus $\phi(s)|_{U_i}$ is nilpotent.

Thus, there exists $n_i > 1$ such that $(\phi(s)|_{U_i})^{n_i} = 0$.

And there exists N such that $\phi(s)^N = \phi(s^N) = 0$ in $\mathcal{O}_Z(Z)$.

Thus $s^N = 0$ as ϕ is injective by hypothesis, thus $V(s) = X$.

Claim 3.

$i^\#$ is an isomorphism.

Since $Z \rightarrow X$ is a closed subscheme, $i^\#$ is surjective on stalks and thus surjective.

To show injectivity, it suffices to show that $\forall a \in A$ such that $i_p^\#(\frac{a}{1}) = 0 \implies \frac{a}{1} = 0 \in A_p$.

Since $i_p^\#(\frac{a}{1}) = 0$, $\exists p \in U \subset Z$ open affine such that $\phi(a)|_U = 0$.

Choose a finite open affine cover, $Z = U \cup \bigcup_{i=1}^n U_i$.

Choose $s \in A$ such that $p \in D(s) \subset U \subset X$.

Then $\phi(sa)|_U = 0$, thus $\phi(sa)|_{D(s) \cap U_i} = 0 = \phi(sa)|_{D(\phi(s)|_{U_i})}$.

Thus, there exists $N > 0$ such that $\phi(sa)|_{U_i} \cdot (\phi(s)|_{U_i})^N = 0 \in \mathcal{O}_Z(U_i)$.

Thus $\phi(s^{N+1}a)|_{U_i} = 0$, thus $\phi(s^{N+1}a) = 0 \implies s^{N+1}a = 0 \in A$ which implies $\frac{a}{1} = 0$ in A_p .

Lecture 8: Fiber Products

Fri 04 Nov

Corollary 75

Closed subschemes of affine schemes are affine.

Moreover, if $\phi : A \rightarrow B$ is a morphism of rings, then ϕ is surjective iff $\text{Spec } \phi$ is a closed immersion.

Remark

For all closed subsets $Z \subset X$ of a scheme X , there is an ideal sheaf \mathcal{I} on X making Z into a closed subscheme.

To prove this, if $U \subset X$ is affine, then $Z \cap U$ is closed in U hence $Z \cap U = V(I)$ for some ideal in $\mathcal{O}_X(U)$.

Then you take radicals and glue them together on a cover of Z .

This structure is called the reduced induced scheme structure on Z .

3.1 Fiber Products**Definition 35 (Fiber product)**

Let C be a category, given two morphism $\pi_X : X \rightarrow S$ and $\pi_Y : Y \rightarrow S$, the fiber product $X \times_S Y$ of X and Y over S is an object together with morphisms p_x to X and p_y to Y which is universal.

Remark

Alternatives names sometimes are fibre product, fibered product or pullback.

Fiber products are unique up to unique isomorphism.

If S is terminal in C , then the fiber product is just the product.

Remark

If a square is a fiber product, we call the diagram cartesian.

Lemma 79

Assume all fiber products exist.

Let "commutative thingy".

Then

$$(X_1 \times_{S_1} Y_1) \times_{X_0 \times_{S_0} Y_0} (X_2 \times_{S_2} Y_2) = (X_1 \times_{X_0} X_2) \times_{S_1 \times_{S_0} S_2} (Y_1 \times_{Y_0} Y_2)$$

Corollary 80

- If C admits fiber products, then $X \times_S Y = Y \times_S X$
- A composition of two pullback squares is a pullback
- For a zigzag $X \rightarrow S, Y \rightarrow S, Y \rightarrow T, Y \rightarrow Z$,

$$(X \times_S Y) \times_T Z = X \times_S (Y \times_T Z)$$

- For maps $X \rightarrow S \rightarrow T$ and $Y \rightarrow S$

$$X \times_S Y \rightarrow X \times_T Y \rightarrow S \times_T S$$

and

$$X \times_S Y \rightarrow S \rightarrow S \times_T S$$

is a pullback.

Example

1. If $\pi_X : X \rightarrow S, \pi_Y : Y \rightarrow S$ are in (Set) , then $X \times_S Y = \{(x, y) | \pi_X(x) = \pi_Y(y)\} \subset X \times Y$ together with the two projections.
2. If X and Y are groups (or rings) and π_X, π_Y are homomorphisms as above, then $X \times_S Y$ is, as a set, the fiber product of the underlying sets, with the obvious groups (resp. ring) structures.

Goal for today

Theorem 82 (Fiber products of schemes exist)

Fiber products exist in (Sch) and also in (Sch/S)

Why do we care?

Allows us to talk about fibers, graphs, diagonals...

Recall that every point $y \in Y$ of a scheme Y has a natural scheme structure given by the residue field $\mathcal{O}_{Y,y}/\mathfrak{m}_y = k(y)$

Definition 36 (Fibers)

Let $f : X \rightarrow Y$ be a morphism of schemes over S .

1. For any $y \in Y$, let $k(y)$ be the residue field, then the fiber of f over y

$$f^{-1}(y) = X_y = X \times_Y \text{Spec } k(y)$$

2. The geometric fiber of f over Y is

$$X_{\overline{y}} = X \times_Y \text{Spec } \overline{k(y)}$$

3. a closed fiber is a fiber over a closed point
4. For all integral schemes Y , there is a unique point $\eta \in Y$ such that $\overline{\{\eta\}} = Y$

This is called the generic point of Y .

The fiber over the generic point is called the generic fiber of f .

5. The morphism

$$\Gamma_f := (\text{Id}, f) : X \rightarrow X \times_S Y$$

is called the graph of f .

6. The morphism

$$\Delta_{X/Y} = \Gamma_{\text{Id}_X} : X \rightarrow X \times_Y X$$

is called the diagonal of X over Y .

Proposition 83

If $X = \text{Spec } A, Y = \text{Spec } B, S = \text{Spec } C$ and $\pi_X : X \rightarrow S, \pi_Y : Y \rightarrow S$ are morphisms of schemes then $X \times_S Y$ exists in (Sch) and is given by $\text{Spec}(A \otimes_C B)$ together with the maps induced by the natural maps $A \rightarrow A \otimes_C B, B \rightarrow A \otimes_C B$

Proposition 84

We use the universal property of $A \otimes_C B$ and the equivalence of categories to show that it is a pullback in the category of affine schemes.

For Z a scheme, there is a map from the affinization of Z to $\text{Spec } B$ and $\text{Spec } A$ which then induce a map $\text{aff } Z \rightarrow \text{Spec } A \otimes_C B$.

Example

1. If $X = Y = \mathbb{A}_k^1$, the fiber product over $\text{Spec } k$, then $X \times_{\text{Spec } K} Y = \mathbb{A}_k^2$.
2. If $X = Y = \text{Spec } \mathbb{C}$ and $S = \text{Spec } \mathbb{R}$, then

$$X \times_S Y = \text{Spec } \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \text{Spec } \mathbb{C}[x]/(x^2 + 1) = \text{Spec}(\mathbb{C} \times \mathbb{C})$$

Note that $X \times_S Y$ has two points but X, Y, S each have only one.

3. Take $X = \text{Spec } k[x, y, z]/(z^2 - x), Y = \text{Spec } k[z]$, let $f : X \rightarrow Y$ induced by mapping $z \rightarrow z$.

Let $\lambda \in Y$ be the point corresponding to $(z - \lambda)$, then

$$\begin{aligned} f^{-1}(\lambda) &= \text{Spec } k[x, y, z]/(z^2 - xy) \otimes_{k[z]} k[z]/(z - \lambda) = \text{Spec } k[x, y]/(\lambda^2 - xy) \\ &= \begin{cases} \mathbb{A}^1 \setminus 0 & \text{if } \lambda \neq 0 \\ V(xy) \subset \mathbb{A}^2 & \text{if } \lambda = 0 \end{cases} \end{aligned}$$

4. If we take a non-rational point in the above example, say $\lambda = (z^2 + 1)$, then

$$f^{-1}(\lambda) = \text{Spec } \mathbb{R}[x, y, z]/(-1 - xy, z^2 + 1) = \text{Spec } \mathbb{C}[x, y]/(-1 - xy)$$

We now start the proof that fiber products exist.

Proof

Claim 1

If $X \times_S Y$ exists and $U \subset X$ open, then the open subscheme $p_X^{-1}(U) \subset X \times_S Y$ is a fiber product of U and Y over S .

For Z a scheme and two commuting maps $Z \rightarrow X \times_S Y$ and $Z \rightarrow U$, there

is a map on the topological level $Z \rightarrow p_X^{-1}(U)$ and it also exists on the level of schemes.

Claim 2

If $X = \bigcup U_i$ is an open cover such that $U_i \times_S Y$ exists $\forall i \in I$, then $X \times_S Y$ exists.

We postpone the proof of this until monday.

Now, if S is affine, consider the open affine covers $X = \bigcup_i U_i, Y = \bigcup_j V_j$, then $U_i \times_S V_j$ exists $\forall i, j$, thus $U_i \times_S Y$ exists $\forall i$ and thus $X \times_S Y$ exists.

If S is not affine, let $S = \bigcup_i W_i$ be an open affine cover.

Set $U_i = \pi_X^{-1}(W_i), V_i = \pi_Y^{-1}(W_i)$.

Now, $U_i \times_{W_i} V_i$ exists and now $U_i \times_{W_i} V_i = U_i \times_S Y$ by one of the identities. \square

Lecture 9: fiber products exist

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We finish the proof by showing that if $X = \bigcup_{i \in I} U_i$ is an open cover such that each $U_i \times_S Y$ exists, then $X \times_S Y$ exists.

Proof

We know that $p_{U_i}^{-1}(U_i \cap U_j) \simeq (U_i \cap U_j) \times_S Y \simeq p_{U_j}^{-1}(U_i \cap U_j)$ via unique isomorphisms compatible with the projections.

There is a unique scheme T with maps to Y and X such that $p_X^{-1}(U_i) \simeq U_i \times_S Y$.

We claim that T is $X \times_S Y$.

Let Z be a scheme with morphisms $Z \xrightarrow{f_X} X$ and $Z \xrightarrow{f_Y} Y$ which commutes with projections to S .

Let $V_i = f_X^{-1}(U_i)$, we get unique morphisms $V_i \rightarrow U_i \times_S Y \rightarrow T$ which is unique if $p_X \circ f_i = p_Y \circ f_i$.

By claim 1, f_i and f_j coincide on $U_i \cap U_j$ thus they glue to a unique morphism $f : Z \rightarrow T$ \square

Corollary 86

Let $\pi_X : X \rightarrow S, \pi_Y : Y \rightarrow S$ be a diagram of schemes, let $S = \bigcup_i W_i, U_i = \pi_X^{-1}(W_i), V_i = \pi_Y^{-1}(W_i)$ and $U_i = \bigcup_j U_{ij}, V_i = \bigcup_j V_{ij}$ be open covers.

Then $X \times_S Y = \bigcup_{i \in I} \bigcup_{j,k} U_{ij} \times_{W_i} V_{ik}$ is an open cover.

Proposition 87

Let $f : X \rightarrow Y$ be a morphism of schemes.

Then for every $y \in Y$, the map $g : f^{-1}(y) \rightarrow X$ is a homeomorphism onto the

set-theoretic fiber $f_{\text{set}}^{-1}(y)$.

Proof

Without loss of generality, Y is affine.

We can also assume that X is affine, because if $X = \bigcup_i U_i$ is an open cover and each $g|_{g^{-1}(U_i)} : g^{-1}(U_i) \rightarrow U_i$ is a homeomorphism onto $f_{\text{set}}^{-1}(y) \cap U_i$, then g is a homeomorphism onto $f_{\text{set}}^{-1}(y)$.

So let $X = \text{Spec } B, Y = \text{Spec } B, y = p \in \text{Spec } A$, then we claim that $B \otimes_A k(y) \simeq S^{-1}B / pS^{-1}B$.

Furthermore, the isomorphism is compatible with the maps from B and $k(y) = A_p / pA_p$ where $S = \text{Im}(A \setminus p \rightarrow B)$.

To prove this, we check that $S^{-1}B / pS^{-1}B$ satisfies the universal property of $B \otimes_A k(y)$.

Let C be a ring with morphisms $A_p / pA_p \xrightarrow{f_A} C$ and $B \xrightarrow{f_B} C$ compatible with the morphisms from A π_A, π_B .

Notice that $\pi_A(A \setminus p) \subset (A_p / pA_p)^\times$ and sends $\pi_A(p) = 0$.

Thus $f_B(S) \subset C^\times, f_B(pB) = 0$.

Thus there exists a unique $f : S^{-1}B / pS^{-1}B \rightarrow C$ such that $f \circ p_B = f_B$.

Thus

$$f \circ \pi_A = f_B \circ \pi_B = f_A \circ \pi_A$$

As π_A is an epimorphism, $f \circ p_A = f_A$.

We now have to check $\text{Spec } S^{-1}B / pS^{-1}B \rightarrow \text{Spec } B$ is a homeomorphism onto $f_{\text{set}}^{-1}(y)$.

We know it's a homeomorphism onto its image by general commutative algebra.

The image is $\{q \in \text{Spec } B \mid S \cap q = \emptyset, pS^{-1}B \subset qS^{-1}B\}$.

But this is just the set-theoretic fiber. □

4 Properties of Morphisms

4.1 Properties of properties of morphisms

Remark

If $f : X \rightarrow Y$ and $g : Y' \rightarrow Y$ are morphisms of schemes, let $X_{Y'} = X \times_Y Y'$.

Then we call $f_{Y'} : X_{Y'} \rightarrow Y'$ the base change of f along g .

Remark

In the following, whenever we say P is a property of morphisms of schemes, we assume that P is satisfied by isomorphisms.

Definition 37

Let P be a property of morphisms of schemes, we say that P satisfies

1. (COMP) : P is stable under composition
2. (CANC) : if $g \circ f$ satisfies P , then f does
3. (BC) : if it is stable under base change that is $\forall f : X \rightarrow Y, g : Y' \rightarrow Y$ such that f satisfies P , also $f_{Y'}$ satisfies P .
4. (LOCT) : If it local on the target, ie. if $\forall f : X \rightarrow Y$ and \forall open covers $Y = \bigcup V_i$ f satisfies $P \iff f|_{V_i}$ satisfies $P \forall i \in I$
5. (LOCS) : If it is local on the source ie. if $\forall f : X \rightarrow Y$ and \forall open covers $X = \bigcup U_i$ f satisfies $P \iff f|_{U_i}$ satisfies P .

Definition 38

Let P be a property of morphisms of schemes, then a morphism $f : X \rightarrow Y$ is called universally P if $\forall Y' \rightarrow Y, f_{Y'}$ satisfies P .

Lemma 90

Let $f : X \rightarrow Y$ be a morphism of schemes over S , then the diagram $X \xrightarrow{\Gamma_f} X \times_S Y$ over $Y \rightarrow Y \times_S Y$ is cartesian.

Proof

This is a special case of the "magic square" with the isomorphism $X \rightarrow X \times_Y Y$ □

Definition 39

Let P be a property of morphisms of schemes, then we say that $f : X \rightarrow Y$ satisfies Δ_P if $\Delta_{X/Y}$ satisfies P .

Lemma 91

The following hold

1. If P satisfies (BC), then Δ_P satisfies (BC)
2. If P satisfies (BC) and (COMP), then Δ_P satisfies (COMP).
3. If P satisfies (LOCT), then Δ_P satisfies (LOCT)
4. If P satisfies (BC), (COMP) and $f, g : X \rightarrow Z$ satisfy P as morphisms of schemes over S and $X \rightarrow S, Z \rightarrow S$ satisfy Δ_P , then $(f, g) : X \rightarrow Y \times_S Z$ satisfy P

Lemma 92

Let P be a property of morphisms of schemes.

Assume that P satisfies stability under base change and composition.

Let $f : X \rightarrow Y, g : X' \rightarrow Y'$ be morphisms of schemes over S satisfying P , then the product $f \times_S g$ satisfies P .

Proof

There are maps

$$X \times_S X' \xrightarrow{\text{Id} \times_S g} X \times_S Y' \xrightarrow{f \times_S \text{Id}} Y \times_S Y'$$

which compose to $f \times_S g$.

But both maps are base changes and thus $f \times_S g$ satisfy P . \square

Theorem 93 (Cancellation Theorem)

Let P be a property of morphisms of schemes and P satisfies stability under composition and base change.

Let $f : X \rightarrow Y, g : Y \rightarrow Z$ two morphisms such that $g \circ f$ satisfies P and $\Delta_{Y/Z}$ satisfies P then f satisfies P .

Proof

Write f as the composition

$$X \xrightarrow{(\text{Id}, f)} X \times_Z Y \rightarrow Y$$

But (Id, f) is a base change of the diagonal and p_Y is a base change of $g \circ f$. \square

4.2 Topological properties**Definition 40**

A morphism $(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of schemes

1. is injective, surjective, bijective if f is.
2. is open, resp. closed if f is.
3. quasi-compact if $f^{-1}(V)$ is quasi-compact for all open quasi-compact $V \subset Y$.
4. quasi-separated if $\Delta_{X/Y}$ is quasi-compact.
5. has finite fibers if $f^{-1}(y)$ is finite as a set.

Lecture 10: geometric meaning of separated and proper morphisms

Mon 14 Nov

5 Valuative Criteria

Definition 41 (Specializations)

Let X be a topological space

1. $x, x' \in X$. If $x' \in \overline{\{x\}}$ we say that x specializes to x' (or x' is a specialization) or x' generalizes to x .
2. A subset $V \subset X$ is called closed under specialization if it contains all the specializations of all its points.

Remark

Closed subsets are closed under specialization (the converse is not true in general)

Definition 42 (Relative specialization)

Let $f : X \rightarrow Y$ be a continuous map.

We say that specializations lift along f if $\forall x \in X$ and any specialization y of $f(x)$ in Y , there exists $x' \in X$ mapping to the specialization such that x' specializes to x .

Remark

Both concepts are stable under composition and local on the target.

Example

Let K be an alg. closed field, let X be the affine line with two origins over K .

1. Look at the morphism $f : \mathbb{A}^1 \rightarrow X$ mapping to the upper origin.
The generic point corresponding to the upper point of X lifts along f , but the other generic point does not.
2. The map $g : X \rightarrow \mathbb{A}^1$ lifts the generic point of \mathbb{A}^1 non-uniquely.

Lemma 97

Let $f : X \rightarrow Y$ be a quasi-compact morphism. Then f is closed if and only if specializations lift along f .

Proof

If f is closed, let $x \in X$ and $f(x)y'$.

Then $y' \in \overline{\{f(x)\}} = \overline{\{f(\overline{\{x\}})\}} = f(\overline{\{x\}})$.

So $\exists x' \in \overline{\{x\}}$ such that $f(x') = y'$.

Conversely, assume specializations lift.

Let $Z \subset X$ be closed, equipped with the reduced induced scheme structure.

As $\text{Im}(X \rightarrow Y) = \text{Im}(Z \rightarrow X \rightarrow Y)$, we may assume wlog that $Z = X$.

Then $f(X)$ is stable under specialization.

We have to show $f(X)$ is closed.

Closed, qc, liftability of specializations are LOCT so wlog Y is affine.

Since f is qc. \exists a surjection $\coprod \text{Spec } A_i \rightarrow X$ where the disjoint union is affine.

But $\coprod \text{Spec } A_i \simeq \text{Spec}(\prod A_i)$, so we can assume that X is affine.

We have reduced to a commutative algebra claim :

Let $\phi : A \rightarrow B$ be a morphism of rings. If the image T of $\text{Spec } B \rightarrow \text{Spec } A$ is closed under specializations, then it is closed.

Wlog, ϕ is injective.

Recall $p \in \text{Spec } A$ is called minimal if it is minimal wrt inclusion.

Every $p' \in \text{Spec } A$ is a specialization of a minimal prime.

So it suffices to show that T contains all minimal primes of A .

Let $p \in \text{Spec } A$ be a minimal prime, consider the fiber product $\text{Spec } B_p = \text{Spec } A_p \times_{\text{Spec } A} \text{Spec } B$.

Since localization is exact, if $A \rightarrow B$ is an inclusion, $A_p \rightarrow B_p$ is too, so p has a preimage in B .

Hence $T = \text{Spec } A$ □

Definition 43 (Valuation Rings)

Let A be a local integral domain with maximal ideal m_A contained in a field K

1. If $B \subset K$ is a local ring with maximal ideal m_B , we say that B dominates A if $A \subset B$ and $m_B \cap A = m_A$
2. The ring A is a valuation ring if it is maximal w.r.t. domination
3. The ring A is called discrete valuation ring if it is a Noetherian valuation ring.

Lemma 98

Let $A \subset K$ be a local subring, then A is dominated by a valuation ring with fraction field K .

Proof

Apply Zorn's lemma to the appropriate set $M = \{A_k \subset K, A_k \text{ local}, A_k \text{ dominated by } A\}$.

Let $\{A_i\}_{i \in I}$ be a totally ordered subset of M .

Let $B = \bigcup_i A_i, m_B = \bigcup_i m_{A_i}$.

B is a ring, $m_B \subset B$ is an ideal with $m_B \cap A_i = m_{A_i}$.
 $b \in B$ is a unit $\iff \exists c \in B$ such that $bc = 1 \iff \exists i \in I, bc = 1 \in A_i \iff$
 $\exists i$ such that $b \in A_i \setminus m_{A_i} \iff b \notin m_B$.
 $B \in M$ and dominates all A_i .
 So M has a maximal element. \square

Lemma 99

Let X be a scheme and let $x, x' \in X$ with $x \rightarrow x'$.
 Then there exists a valuation ring A and a morphism $f : \text{Spec } A \rightarrow X$
 such that $f(\eta_A), f(m_A) = x'$.
 More precisely, given a field extension $g : k(x) \rightarrow K$, we may assume A
 has fraction field K .

Proof

Since $x' \in \overline{\{x\}}$, we have $\mathcal{O}_{X,x'} \rightarrow k(x) \xrightarrow{g} K$.
 By the lemma above, there is a valuation ring A with fraction field K domi-
 nating $\phi(\mathcal{O}_{X,x'})$.
 This map induces $\text{Spec } A \rightarrow X$ with the desired properties. \square

Definition 44

Let $f : X \rightarrow Y$ be a morphism of schemes.
 We say that f satisfies the existence part of the valuative criterion if for
 every commutative diagram of solid arrows $\text{Spec } A \rightarrow Y \leftarrow X$ commuting
 with $A \leftarrow \text{Spec } K \rightarrow X$ where A is a valuation ring with field of fractions
 K , there is a lift $\text{Spec } A \rightarrow X$ making the diagram commute.
 f satisfies the uniqueness part of the valuative criterion if there is at most
 one such map.

Theorem 100 (Valuative criterion)

Let $f : X \rightarrow Y$ be a morphism of schemes, TFAE

1. Specializations lift along any base change
2. f satisfies the existence part of the valuative criterion.

Proof

1 \implies 2

Wlog $Y = \text{Spec } A$.

Let $x \in X$ be the image of g .

As specializations lift, $\exists x' \in X$ such that $x \rightarrow x'$ and $f(x') = m_A$.

We get a diagram of rings $A \rightarrow K \xleftarrow{g_{x'}} \mathcal{O}_{X,x'} \xleftarrow{f_{x'}} A$ commuting with $A \rightarrow K$.

Since $f_{x'}^\sharp$ is local, $g_{x'}^\sharp(\mathcal{O}_{X,x'})$ dominates A , hence $g_{x'}^\sharp(\mathcal{O}_{X,x'}) = A$.

We get a dotted arrow $\mathcal{O}_{X,x'} \rightarrow A$.

$2 \implies 1$

Let $Y' \rightarrow Y$ be any morphism, then there is a lift $\text{Spec } A \rightarrow X_{Y'}$ by universal property of the fiber product.

So it suffices to show that specializations lift along f .

Let $x \in X$ and $f(x) \rightarrow y'$.

Let $K = k(x)$ and consider $g : k(f(x)) \rightarrow k(x)$ induced by f .

By the lemma, we get a lift $\text{Spec } A \rightarrow X$ and the image of m_A precisely is the preimage of the specialization. \square

Now

1. If $f : X \rightarrow Y$ is qc.
 f universally closed $\iff f$ satisfies existence in valuative criterion
2. $f : X \rightarrow Y$ is qs
 f is separated $\iff f$ satisfies uniqueness in the valuative criterion
3. $f : X \rightarrow Y$ is qs and of finite type
 f is proper $\iff f$ satisfies existence and uniqueness in valuative criterion.

Lecture 11: quasi-coherent sheaves

Fri 18 Nov

6 Quasi-coherent sheaves

Recall that affine schemes are equivalent to rings, what about modules?

Definition 45

Let (X, \mathcal{O}_X) be a ringed space

1. A presheaf of \mathcal{O}_X modules is a presheaf \mathcal{F} on X such that
 - (a) $\forall U \subset X$ open, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$ -module
 - (b) $\forall V \subset U \subset X$ open, the restriction map $\mathcal{F}(U) \rightarrow \mathcal{F}(V)$ is an $\mathcal{O}_X(U)$ module homomorphism, where $\mathcal{F}(V)$ is considered a $\mathcal{O}_X(U)$ module via the restriction map.
2. A morphism of presheaves of \mathcal{O}_X -modules is a morphism of presheaves $\phi : \mathcal{F} \rightarrow \mathcal{G}$ such that $\forall U \subset X$ open, $\phi(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$ -module homomorphism.
3. For two sheaves of \mathcal{O}_X -modules, the presheaf $\text{hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an \mathcal{O}_X -module, called the hom-sheaf.
4. The category of \mathcal{O}_X -modules is denoted by $\text{Mod}(X, \mathcal{O}_X)$.

Remark

There is a forgetful functor $\text{Mod}(X, \mathcal{O}_X) \rightarrow \text{Sh}(X)$.

The sheafification of a presheaf of \mathcal{O}_X -modules is an \mathcal{O}_X module.
 $\text{Mod}(X, \mathcal{O}_X)$ is an abelian category

Definition 46

Let \mathcal{F}, \mathcal{G} be \mathcal{O}_X -modules. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

If $A \rightarrow B$ is a ring homomorphism, there is a restriction of scalars functor $A- : \text{Mod}(B) \rightarrow \text{Mod}(A)$ and extension of scalars $- \otimes_A B : \text{Mod}(A) \rightarrow \text{Mod}(B)$, we can do the same for morphisms of sheaves of rings.

Definition 47

Let $f : X \rightarrow Y$ be a morphism of ringed spaces

1. The direct image functor $f_* : \text{Mod}(X, \mathcal{O}_X) \rightarrow \text{Mod}(Y, \mathcal{O}_Y)$ is the composition $\text{Mod}(X, \mathcal{O}_X) \xrightarrow{f_*} \text{Mod}(Y, f_* \mathcal{O}_X) \xrightarrow{\mathcal{O}_Y^-} \text{Mod}(Y, \mathcal{O}_Y)$.
2. The inverse image functor

$$f^* : \text{Mod}(Y, \mathcal{O}_Y) \rightarrow \text{Mod}(X, \mathcal{O}_X)$$

$$\text{is the composition } \text{Mod}(Y, \mathcal{O}_Y) \xrightarrow{f^{-1}} \text{Mod}(X, f^{-1} \mathcal{O}_Y) \xrightarrow{- \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X} \text{Mod}(X, \mathcal{O}_X).$$

Definition 48

Let (X, \mathcal{O}_X) be a ringed space.

A sheaf of \mathcal{O}_X -modules is

1. free if $\mathcal{F} = \mathcal{O}_X^{\oplus I}$ for some set I
2. locally free if \exists an open cover $\bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ is free.
3. locally free of finite rank if \exists an open cover $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i} \simeq \mathcal{O}_{U_i}^{\oplus r_i}$ for all $i \in I$ and some $r_i \in \mathbb{N}$
4. invertible (or line bundle) if its locally free of rank 1.

Definition 49

Let (X, \mathcal{O}_X) be a ringed space.

A sheaf of \mathcal{O}_X -modules is called

1. quasi-coherent if there exists an open cover $X = \bigcup U_i$ such that each $\mathcal{F}|_{U_i}$ appears in an exact sequence

$$\mathcal{O}_{U_i}^{\oplus I_i} \rightarrow \mathcal{O}_{U_i}^{\oplus J_i} \rightarrow \mathcal{F}|_{U_i} \rightarrow 0$$

2. coherent if

— there exists an open cover $X = \bigcup U_i$ and $n_i \in \mathbb{N}$ and a surjection

$$\mathcal{O}|_{U_i}^{\oplus n_i} \rightarrow \mathcal{F}|_{U_i} \rightarrow 0$$

— $\forall U \subset X$ open, $n \in \mathbb{N}$ and all surjections

$$\phi : \mathcal{O}_U^{\oplus n} \rightarrow \mathcal{F}|_U$$

$\ker \phi$ satisfies a.

We get categories $Coh(X, \mathcal{O}_X)$ and $QCoh(X, \mathcal{O}_X)$ of coherent and quasi-coherent sheaves.

In general, $QCoh$ is not abelian, though for schemes it is.

Definition 50

Let $X = \text{Spec } A$ be an affine scheme.

Let M be an A -module.

The sheaf of \mathcal{O}_X -modules associated to M is $\tilde{M} : U \mapsto \{s : U \rightarrow \coprod M_p | s \text{ satisfies i) and ii) } \}$

1. $\forall p \in U, s(p) \in M_p$

2. $\forall p \in U \exists m \in M, a \in A$ and $V \subset U$ open such that

$$p \in V \subset D(a) \text{ such that } s(q) = \frac{m}{a} \in M_q \forall q \in V$$

As in the affine case, we get that

Proposition 102

Let $X = \text{Spec } A$ and M an A -module

1. $\forall a \in A$, there exist natural isomorphisms

$$\phi_A : M_a \rightarrow \tilde{M}(D(a))$$

and these isomorphisms commute with the localization maps $M_a \rightarrow M_b$.

2. $\forall p \in \text{Spec } A$, there exist natural isomorphisms $\phi_p : M_p \rightarrow \tilde{M}_p$ which are also natural wrt the maps $M_a \rightarrow M_p \forall a \in A \setminus p$

We get a functor $A\text{-mod} \rightarrow \text{Mod}(X, \mathcal{O}_X)$ if $X = \text{Spec } A$ where the maps are induced by postcomposition with the obvious map $\coprod M_p \rightarrow \coprod N_p$.

Lemma 103

Let $X = \text{Spec } A \forall A$ -modules M, N , the maps $\text{hom}(M, N) \leftrightarrow \text{hom}(\tilde{M}, \tilde{N})$ sending $\phi : M \rightarrow N$ to $\tilde{\phi}$ and $f : \tilde{M} \rightarrow \tilde{N}$ to $f(X)$ are mutually inverse.

The proof of this will follow from an exercise.

Proof

Let $X = \text{Spec } A$ be an affine scheme

1. A sequence of modules $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ is exact iff the associated sequence of sheaves is exact
 2. If $\phi : M \rightarrow N$ is a morphism of A -modules, then $\widetilde{\ker \phi} = \ker(\tilde{\phi})$ and similarly for the cokernel.
 3. If $\{M_i\}$ is a family of A -modules, then $\widetilde{\bigoplus M_i} = \bigoplus \tilde{M}_i$.
 4. If $\{M_i\}, \{\phi_{ij}\}$ is a directed system, then direct limits commute with \sim
-

Theorem 104 (Criterion for quasi-coherence)

Let X be a scheme and \mathcal{F} an \mathcal{O}_X module, then the following are equivalent

1. $\forall \text{Spec } A \subset X$ open affine, $\mathcal{F}|_{\text{Spec } A} \simeq \tilde{M}$ for some A -module M .
2. The same for some cover
3. \mathcal{F} is quasi-coherent
4. $\forall \text{Spec } A = U \subset X$ open affine and every $a \in A$, the natural map $\mathcal{F}(U)_a \rightarrow \mathcal{F}(D(a))$ is an isomorphism.

Lecture 12: criterion for quasi-coherence and graded rings

Mon 21 Nov

Proof

1 \implies 2 is clear.

2 \implies 3 Choose a presentation $A_i^{\oplus I_i} \rightarrow A_i^{\oplus J_i} \rightarrow M_i \rightarrow 0$.

Taking \sim , this is the same as $\mathcal{O}_{U_i}^{\oplus I_i} \rightarrow \mathcal{O}_{U_i}^{\oplus J_i} \rightarrow \tilde{M}_i \rightarrow 0$.

This is exact as \sim preserves exact sequences.

4 \implies 1 Write $U = \text{Spec } A$ and $M = \mathcal{F}(U)$.

We have compatible isomorphisms $M_a \simeq \mathcal{F}(D(a)) \forall a \in A$.

By some lemma, these glue to an isomorphism $\tilde{M} \simeq \mathcal{F}|_U$.

3 \implies 4 Wlog $X = \text{Spec } A$, so X is qc.

Choose finitely many b_i such that $X = \bigcup_{i=1}^n D(b_i)$ and exact sequences

$$\widetilde{A_{b_i}^{\oplus I_i}} \rightarrow \widetilde{A_{b_i}^{\oplus J_i}} \rightarrow \mathcal{F}|_{D(b_i)}$$

call the first map $\tilde{\phi}$.

There exists $\phi : A_{b_i}^{\oplus I_i} \rightarrow A_{b_i}^{\oplus J_i}$ inducing $\tilde{\phi}$.

Thus $\mathcal{F}|_{D(b_i)} = \text{coker}(\tilde{\phi}) = \widetilde{\text{coker } \phi}$.

Let $a \in A$.

Since \mathcal{F} is a sheaf, we have a map of exact sequences from

$$0 \rightarrow \mathcal{F}(X) \rightarrow \prod_{i=1}^n \mathcal{F}(D(b_i)) \rightarrow \prod_{i,j} \mathcal{F}(D(b_i b_j))$$

to

$$0 \rightarrow \mathcal{F}(X)_a \rightarrow \prod_{i=1}^n \mathcal{F}(D(b_i))_a \rightarrow \prod_{i,j} \mathcal{F}(D(b_i b_j))$$

to

$$0 \rightarrow \mathcal{F}(D(a)) \rightarrow \prod_i \mathcal{F}(D(ab_i)) \rightarrow \prod_{i,j} \mathcal{F}(D(ab_i b_j))$$

Since $\mathcal{F}|_{D(b_i)}$ is \widetilde{M} for some A_{b_i} -module, M , the map $\prod \mathcal{F}(D(b_i))_a \rightarrow \prod_i \mathcal{F}(D(ab_i))$ is an isomorphism.

Similarly, the map $\prod_{i,j} \mathcal{F}(D(b_i b_j))_a \rightarrow \prod_{i,j} \mathcal{F}(D(ab_i b_j))$ is an isomorphism.

By the five-lemma, the map $\mathcal{F}(X)_a \rightarrow \mathcal{F}(D(a))$ is an isomorphism. \square

Corollary 105

Let $X = \text{Spec } A$.

Then $\widetilde{} : A\text{-mod} \rightarrow \text{Mod}(X, \mathcal{O}_X)$ induces an equivalence between $A\text{-mod}$ and $QCoh(X, \mathcal{O}_X)$

Remark

Let $X = \text{Spec } A$

1. If $\widetilde{M} \in \text{Coh}(X, \mathcal{O}_X)$, then M is finitely generated.

2. If A is Noetherian and M is a f.g. A -module, then $\widetilde{M} \in \text{Coh}(X, \mathcal{O}_X)$.

Moreover, if X is a locally Noetherian scheme and $\mathcal{F} \in \text{Mod}(X, \mathcal{O}_X)$, then the following are equivalent

$\mathcal{F} \in \text{Coh}(X, \mathcal{O}_X) \iff \forall U \subset X$ open affine there is a f.g. A -module M such that $\widetilde{M} \simeq \mathcal{F}|_U$

Corollary 107

If (X, \mathcal{O}_X) is a scheme, then $QCoh(X, \mathcal{O}_X)$ is abelian

Proof (Sketch)

If X is affine, this is clear using the equivalence of categories.

If not, we need to check that kernels, cokernels, finite direct sums of qcoh sheaves are qcoh.

As all these conditions are local, we conclude. \square

Proposition 108

Let $\phi : A \rightarrow B$ be a morphism of rings and let $f : X = \text{Spec } B \rightarrow \text{Spec } A = Y$ be the induced map on spectra.

Let $\mathcal{F} = \widetilde{M} \in QCoh(X, \mathcal{O}_X), \mathcal{G} = \widetilde{N} \in QCoh(Y, \mathcal{O}_Y)$, then there is a natural isomorphism

$$f_* \widetilde{M} = (\widetilde{{}_A M})$$

and

$$f^* \widetilde{N} = (\widetilde{N \otimes_A B})$$

In particular, $f_* \mathcal{F}$ is qcch and $f^* \mathcal{G}$ is too.

Proof

Take $a \in A$, we have

$$\begin{aligned} (f_* \widetilde{M})(D(a)) &= \widetilde{M}(D(\phi(a))) \\ &= M_{\phi(A)} \\ &= ({}_A M)_a \\ &= (\widetilde{{}_A M})(D(a)) \end{aligned}$$

These morphisms glue to a natural isomorphism $f_* \widetilde{M} \simeq (\widetilde{{}_A M})$.

We have a chain of natural isomorphisms $\text{hom}_{\mathcal{O}_X}(f^* \widetilde{N}, \mathcal{F}) \simeq \text{hom}_{\mathcal{O}_Y}(\widetilde{N}, f_* \mathcal{F}) \simeq \text{hom}_{\mathcal{O}_Y}(\widetilde{N}, (\widetilde{{}_A M})) \simeq \text{hom}_A(N, {}_A M) \simeq \text{hom}_B(N \otimes_A B, M) \simeq \text{hom}_{\mathcal{O}_X}(\widetilde{N \otimes_A B}, \mathcal{F})$.

By the Yoneda lemma, this implies that $f^* \widetilde{N} \simeq \widetilde{N \otimes_A B}$. \square

Proposition 109

Let $f : X \rightarrow Y$ be a morphism of schemes.

Then,

1. $\forall \mathcal{G} \in QCoh(Y, \mathcal{O}_Y), f^* \mathcal{G} \in (X, \mathcal{O}_X)$
2. If X is locally noetherian, then $\forall \mathcal{G} \in Coh(Y, \mathcal{O}_Y), f^* \mathcal{G} \in Coh(X, \mathcal{O}_X)$
3. If f is qc. and qs. $\forall \mathcal{F} \in QCoh(X, \mathcal{O}_X), f_* \mathcal{F} \in QCoh(Y, \mathcal{O}_Y)$
4. If f is proper and Y is locally Noetherian, then $\forall \mathcal{F} \in Coh(X, \mathcal{O}_X), f_* \mathcal{F} \in Coh(Y, \mathcal{O}_Y)$

Proof

Wlog. $Y = \text{Spec } A$ affine since quasi-coherence is local.

For 1 and 2, cover X by open affines and assume wlog that X is affine.

Then $f^* \widetilde{M} = (\widetilde{M \otimes_A B})$.

Since f is qc., $X = \bigcup_{i=1}^n U_i$ with $U_i = \text{Spec } B_i$.

Since f is qs, we can write $U_i \cap U_j = \bigcup_k U_{ijk}$ with $U_{ijk} = \text{Spec } B_{ijk}$.

Since \mathcal{F} is a sheaf, we have an exact sequence

$$0 \rightarrow f_* \mathcal{F} \rightarrow \prod_{i=1}^n (f|_{U_i})_*(\mathcal{F}|_{U_i}) \rightarrow \prod_{ij} \prod_k (f|_{U_{ijk}})_*(\mathcal{F}|_{U_{ijk}})$$

This is exact as for every $V \subset Y$ open, giving $s \in f_* \mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$ which is the same as giving $s_i \in \mathcal{F}(f^{-1}(V) \cap U_i) = \mathcal{F}(f|_{U_i}^{-1}(V)) =$

| $((f|_{U_i})_*(\mathcal{F}|_{U_i}))(V)$ such that the s_i agree on intersections. □

Proposition 110

Let (X, \mathcal{O}_X) be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals.

Let $Z = \text{Supp}(\mathcal{O}_X/\mathcal{I}) = \{x \in X | (\mathcal{O}_X/\mathcal{I})_x \neq 0\}$ and $\mathcal{O}_Z = (\mathcal{O}_X/\mathcal{I})|_Z$.

Then (Z, \mathcal{O}_Z) is a scheme iff \mathcal{I} is quasi-coherent.

Z is ascheme and \mathcal{I} is quasi-coherent are local on X , so wlog $X = \text{Spec } A$.

If \mathcal{I} is qcch., then $\mathcal{I} = \tilde{I}$ for some ideal $I \subset A$.

Lecture 13: Proj construction

Fri 25 Nov

7 The proj construction

7.1 Algebraic preliminaries

Definition 51 (Graded rings)

1. A (\mathbb{N}) -graded ring is a ring A together with a decomposition $A = \bigoplus_{d=0}^{\infty} A_d$ of it's underlying additive group such that $A_d \cdot A_{d'} \subset A_{d+d'}$.
2. A graded morphism of graded rings is a ring morphism $\phi : A \rightarrow B$ such that $\phi(A_d) \subset B_d \forall d \geq 0$
3. A graded module over a graded ring is an A -module M together with a decomposition $M = \bigoplus_{i=0}^{\infty} M_d$ which is compatible with the grading from A .
4. A graded morphism of graded modules is what you'd expect
5. The irrelevant ideal of a graded ring is $A_+ = \bigoplus_{d=1}^{\infty} A_d$.
6. An element $a \in A$ is called homogeneous if $a \in A_d$ for some $d \geq 0$
7. An ideal $I \subset A$ is homogeneous if $I = \bigoplus_d (I \cap A_d)$
8. The homogenization of an ideal $I \subset A$

Example

For any ring $A_0, A = A_0[x_0, \dots, x_n]$ becomes a graded ring if we set A_d to be the subset of polynomials of degree d .

Lemma 112

Let A be a graded ring, then

1. A_0 is a subring
2. If $\mathfrak{p} \subset A$ is a prime, then so is $\mathfrak{p}^h \subset A$
3. An ideal is homogeneous iff it can be generated by homogeneous

elements

4. A homogeneous ideal $I \subset A$ is prime iff $\forall a, b \in A$ homogeneous such that $ab \in I$, either $a \in I$ or $b \in I$
5. Being homogeneous is stable under taking radicals

Definition 52 (Localizations of graded rings)

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a graded ring and M a graded A -module

1. Let $S \subset A$ be a multiplicatively closed subset consisting of homogeneous elements then the homogeneous localization of M at S is $(S^{-1}M)_0 = \{ \frac{a}{b} \in S^{-1}M \mid \deg a = \deg b \}$
2. If $a \in A$ is homogeneous, then $S = \{a^i\}$ consists of homogeneous elements and we let $M_{(a)} = (S^{-1}M)_0$ is the homogeneous localization of M at a .
3. If $p \subset A$ is a homogeneous prime ideal, we set $S = \{a \in A \mid a \notin p\}$ and let $M_{(p)} = (S^{-1}M)_0$ be the homogeneous localization of M at p .

7.2 The projective spectrum

Definition 53

Let A be a graded ring.

1. The projective spectrum of A is $\text{Proj} A = \{p \in \text{Spec} A \mid p \text{ homogeneous, } A_+ \not\subset p\}$
2. The Zariski topology on $\text{Proj} A$ is defined by defining the closed subsets $V_+(I) = \{p \in \text{Proj} A \mid I \subset p\}$ for subsets $I \subset A$.
3. The principal open subsets associated to a homogeneous element $a \in A_+$ is $D_+(a) = \text{Proj}(A) \setminus V_+(\{a\}) = \{p \in \text{Proj} A \mid a \notin p\}$.

Lemma 113

The following identities hold

1. For any subset $I \subset A$, we have $V_+(I) = V_+((I)) = V_+((I)^h)$
2. For any two subsets $I, J \subset A$,

$$V_+(I) \cup V_+(J) = V_+(I \cap J)$$

3. For abritrarily many J_i

$$\cap_i V_+(J_i) = V_+(\sum_i J_i)$$

4.

$$V_+(A_+) = V_+(A) = \emptyset$$

5.

$$V_+((0)) = Proj A$$

6. If $a = \sum a_i \in A$ is an element and the a_i are homogeneous of degree i , then $D(a) \cap Proj(A) = (D(a_0) \cap Proj A) \cup (\bigcup D_+(a_i))$.

7. If $a = a_0 \in A$, then $D(a_0) \cap Proj A = \bigcup_{b \in A_d, d \geq 1} D_+(a_0 b)$

8. In particular, the Zariski topology on $Proj A$ is a topology by the above properties

Lemma 114

Let I, J be homogeneous ideals in a graded ring, then $V_+(I) \subset V_+(J) \iff J \cap A_+ \subset \sqrt{I}$.

In particular, if $a, b \in A_+$ are homogeneous, then $D_+(b) \subset D_+(a) \iff D(b) \subset D(a)$.

Proof

If $J \cap A_+ \subset \sqrt{I}$ and let $p \in V_+(I) \implies I \subset p \implies \sqrt{I} \subset p \implies J \cap A_+ \subset p \implies p \in V_+(J)$

Assume $V_+(I) \subset V_+(J)$.

Let $p \in Spec A$ with $I \subset p$, then $I \subset p^h$.

If $A_+ \not\subset p^h$, then $p^h \in V_+(I) \subset V_+(J) \implies J \subset p^h \implies J \subset p$.

If $A_+ \subset p^h$, then $J \cap A_+ \subset p$.

Finally, $D_+(b) \subset D_+(a) \iff V_+(a) \subset V_+(b) \iff (b) \cap A_+ \subset \sqrt{(a)} \iff (b) \subset \sqrt{(a)} \iff D(b) \subset D(a)$ \square

Corollary 115

Let A be a graded ring, then $Proj A = \emptyset$ iff $A_+ \subset \sqrt{0}$.

7.3 The structure sheaf on $Proj A$ and quasi-coherent sheaves

If M is a graded A -module and $D_+(b) \subset D_+(a) (\iff D(b) \subset D(a))$, then there is a map $M_{(a)} \rightarrow M_{(b)}$ which commutes with the maps $M_{(a)} \rightarrow M_a$

Lemma 116

If $A = \bigoplus_{d=0}^{\infty} A_d$ a graded ring, $M = \bigoplus_{d=0}^{\infty} M_d$ is a graded A -module and $a, b \in A_+$ are homogeneous with $D_+(b) \subset D_+(a)$.

There is a map $M_{(a)} \rightarrow M_a \rightarrow M_b$ which factors through an $A_{(a)}$ linear map $M_{(a)} \rightarrow M_{(b)}$ via the map $(M_{(a)})_{b^{\deg a} / a^{\deg b}} \simeq M_{(b)}$.

Proof

$D(b) \subset D(a) \implies \exists n > 0, c \in A$ such that $b^n = ac$.

Since b^n is homogeneous and a is homogeneous, we can choose c homogeneous.

Take $\frac{e}{a^m} \in M_{(a)}$ with $\deg e = m \deg a$.

Then, in M_b , we have

$$\frac{e}{a^m} = \frac{c^m e}{c^m a^m} = \frac{c^m e}{b^{nm}} \in M_{(b)} \quad \square$$

In particular, we get homeomorphisms $\phi_a : D_+(a) \rightarrow \text{Spec } A_{(a)}$

Proof

Note that $\phi_a(p) = pA_a \cap A_{(a)}$.

Define $\psi_a : \text{Spec } A_{(a)} \rightarrow D_+(a)$ by sending q to $\bigoplus_d \left\{ b \in A_d \mid \frac{b^{\deg a}}{a^d} \in q \right\} \quad \square$

Definition 54

Let A be a graded ring, M a graded A -module.

Let $X = \text{Proj } A$.

1. The structure sheaf \mathcal{O}_X on X is the unique sheaf of rings on X which agrees with $\widetilde{A_{(a)}}$ on $D_+(a)$

Lecture 14: Twisting sheaves

Mon 28 Nov

Definition 55

Let A be a graded ring and M a graded A -module.

Let $X = \text{Proj } A$

1. The structure sheaf \mathcal{O}_X on X is the unique sheaf of rings on $\text{Proj } A$ such that $\mathcal{O}_X|_{D_+(a)} = \widetilde{A_{(a)}}$
2. The sheaf of \mathcal{O}_X -modules associated to M is the unique sheaf \widetilde{M} such that $\widetilde{M}|_{D_+(a)} = \widetilde{M_{(a)}}$
3. The n -th (Serre) twisting sheaf on $\text{Proj } A$ is $\widetilde{A(n)}$ where $A(n) = \bigoplus_{d \in \mathbb{Z}} A(n)_d$, where $A(n)_d = A_{n+d}$
4. The n -th twist $\widetilde{M(n)}$ of \widetilde{M} is defined as $\widetilde{M(n)}$ where we shift the

grading as above.

$\text{Proj} A$ is a scheme and $\widetilde{M} \in \text{QCoh}(\text{Proj} A)$

Corollary 117

Let A be a graded ring, M a graded A -module, $X = \text{Proj} A$.

For every $p \in D_+(b) \subset D_+(a) \subset \text{Proj} A$.

We find that $\widetilde{M}_p \simeq M_{(p)}$

Proof

By definition of \widetilde{M} we have $\widetilde{M}|_{D_+(a)} = \widetilde{M_{(a)}}$.

We need to check that $M(p) = \lim_{p \in D_+(a)} M_{(a)}$ □

Remark

Thinking of $\mathcal{O}_{\text{Proj} A}$ and \widetilde{M} as their own sheafification, we obtain descriptions $\mathcal{O}(U)$ and $\widetilde{M}(U)$ similar to the description of the structure sheaf.

Definition 56

Let A_0 be a ring, $n \geq 1$, $A = A_0[x_0, \dots, x_n]$.

Equip A with the standard grading.

The scheme $\text{Proj} A$ is called *Projective n -space over A* and is denoted $\mathbb{P}_{A_0}^n$

Remark

For any ring A_0 , $\mathbb{P}_{A_0}^n \simeq \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} \text{Spec } A_0$.

If X is any scheme, we define $\mathbb{P}_X^n = \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} X$

Remark

Consider $\mathbb{P}_{A_0}^n = \text{Proj} A_0[x_0, \dots, x_n]$.

1. $D_+(x_i) \simeq \text{Spec } A_{(x_i)} = \text{Spec } A_0[\frac{x_0}{x_i}, \dots] \simeq \mathbb{A}_{A_0}^n$
2. If $A_0 = K$ is a field, then $V_+(x_i) \subset \mathbb{P}_{A_0}^n$ with its reduced induced scheme structure is isomorphic to \mathbb{P}_K^{n-1}

Definition 57

Let A be a graded ring, then the *affine cone over $\text{Proj} A$* is $\text{Spec } A$.

Definition 58

A graded ring A is *finitely generated in degree 1* if it is generated by finitely many elements of A_1 as an A_0 algebra.

Lemma 121

Assume A is f.g. in degree 1, then there exists $n > 0$ and a closed immersion $\text{Proj} A \rightarrow \mathbb{P}_{A_0}^n$ over $\text{Spec } A_0$.

Proof

There exists a graded surjection $A_0[x_1, \dots, x_n] \rightarrow A$ and then apply an exercise. \square

Remark

If A is f.g. over A_0 , then $A^{(n)} = \bigoplus A_d^{(n)}$ with $A_d^{(n)} = A_{nd}$ is f.g. in degree 1 for a suitable n .

By an exercise on the next sheet $\text{Proj} A \simeq \text{Proj} A^{(n)}$.

Theorem 123

Let A be a graded ring, f.g. in degree 1 then $\pi_A : \text{Proj} A \rightarrow \text{Spec } A_0$ is proper.

Proposition 124

We know that $\mathbb{P}_{A_0}^n \simeq \mathbb{P}_{\mathbb{Z}}^n \times_{\text{Spec } \mathbb{Z}} \text{Spec } A_0$ and that there is a closed immersion $\text{Proj} A \rightarrow \mathbb{P}_{A_0}^n$.

Since proper is stable under base change and (COMP) and closed immersions are proper, it suffices to show $\pi : \mathbb{P}_{\mathbb{Z}}^n \rightarrow \text{Spec } \mathbb{Z}$ is proper.

We want to show that π is separated, of finite type and satisfies the existence in the valuative criterion.

 π is separated

Suffices to show that $D_+(x_i) \cap D_+(x_j) \rightarrow D_+(x_i) \times_{\text{Spec } \mathbb{Z}} D_+(x_j)$ is a closed immersion.

We have to show that $\mathbb{Z}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}] \otimes \mathbb{Z}[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}] \rightarrow \mathbb{Z}[\frac{x_0^2}{x_i x_j}, \frac{x_0 x_1}{x_i x_j}, \dots]$ is surjective, which is obvious.

 π is of finite type

$\mathbb{P}_{\mathbb{Z}}^n = \bigcup D_+(x_i) = \bigcup \text{Spec } \mathbb{Z}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]$ and these specs are of finite type.

Lecture 15: main theorem of elimination theory

Fri 02 Dec

Remark

We proved that $\pi : \text{Proj } A_0[x_0, \dots, x_n] \rightarrow \text{Spec } A_0$ is closed \forall rings A_0 .

In particular, given homogeneous polynomials $f_1, \dots, f_n \in A_0[x_0, \dots, x_n]$.

In particular, there exists an ideal $I \subset A_0$ such that $\phi : A_0 \rightarrow K$ where K is a

field with induced map $\phi' : A_0[x_0, \dots, x_n] \rightarrow K[x_0, \dots, x_n]$, then $\phi'(f_1), \dots, \phi'(f_m)$ have a common solution in \overline{K} (ie., the fiber $V_+(f_1, \dots, f_m) \rightarrow \text{Spec } A_0$ over the K -rational point corresponding to ϕ is non-empty.) if and only if $\phi(I) = 0$ (ie. the K -rational point corresponding to ϕ factors through $V(I)$)

8 Quasi-Coherent sheaves on $\text{Proj } A$