Analysis IV

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List of Theorems

Lecture 1: Measure theory

Wed 23 Feb

1 Lebesgue Measure

Motivation

Given a set $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is it possible to integrate f over Ω.
For $n = 1$ and $\Omega = [a, b]$ riemann-integral works, at least for continuous functions.
However, it is not fully satisfactory

1. Extends badly to $\mathbb{R}^n$
2. Stability with limits Take $f_n : [0, 1] \to [0, 1]$ continuous and pointwise decreasing, define $f(x) = \lim f_n(x)$, then the integral over f might not exist.
3. Differentiation and integration.
   What is the biggest class of functions for which the fundamental theorem works ?
   For sure in C_1 but that is not the biggest class.
4. Consider $C^0([0, 1])$ with L^1-distance.
   Then C^0 is not complete, what is the completion of $\bar{C^0}^d$

We want to find a satisfactory theory of integration.
How can we define the length/volume of a subset $\Omega \subset \mathbb{R}^n$?
Ideally to $\Omega \subset \mathbb{R}^n$ associate $m(\Omega) = 0$ with

$$0 \leq m(\Omega) \leq \infty \quad m((0, 1)^m) = 1 \quad m(A \cup B) = m(A) + m(B) \text{ if } A \text{ and } B \text{ disjoint.}$$

$$m(A) \leq m(B) \quad m(A + x) = m(A)$$

This is impossible !

1.1 Measurable sets

We can ask that
— (Borel Property) Open and closed are measurable
— Ω measurable $\implies$ Ω^c measurable
— (σ-algebra) We want to take countable intersection of measurable sets

Definition 1 (Lebesgue Measure)
The lebesgue measure $m(\Omega)$ of any measurable set will obey
— $m(\emptyset) = 0$
— $\infty \geq m(\Omega) \geq 0$
— *Monotonicity* $m(\Omega_1) \leq m(\Omega_2)$ *if* $\Omega_1 \subset \Omega_2$

— *If $\Omega_1, \dots$ are measurable and disjoint, then we want*

$$m(\bigcup_{i=1}^{\infty} \Omega_i) = \sum_{i=1}^{\infty} m(\Omega_i)$$

and with $\leq$ if they are not disjoint.

— *(Normalisation)*

$$m((0,1)^n) = 1$$

— *(Translation invariance)*

$$m(\Omega + x) = m(\Omega) \forall x \in \mathbb{R}^n$$

Remarque

— *From countable subadditivity, finite subadditivity follows*

— *Monotonicity is redundant because, given $\Omega_1 \subset \Omega_2$*

$$m(\Omega_2) = m(\Omega_1 \cup (\Omega_2 \setminus \Omega_1)) = m(\Omega_1) + m(\Omega_2 \setminus \Omega_1)$$

— *The sums above might be infinite*

Remarque

m is a positive measure if the first four conditions above are satisfied

Theorème 3 (Existence of Lebesgue Measure)
There exists a notion of measurable set obeying the conditions of measurable sets and a measure obeying the conditions.

1.2 Outer Measure

We first want to describe a cube and associate a measure to these boxes. Then we will take a more general set, cover it with boxes and define it's measure by the smallest possible covering by boxes.

Definition 2 (Box)
A open box $B \subset \mathbb{R}^n$ is

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

and define the volume of a box

Definition 3 (Volume of a box)
Given $B = \prod_{i=1}^{n}(a_i, b_i)$, *we define*

$$volB = \prod_i (b_i - a_i)$$

Now, how can we cover $\Omega \subset \mathbb{R}^n$?

Definition 4 (Covered set)
Given $\Omega \subset \mathbb{R}^n$ *is covered by* $\{B_j\}_{j\in J}$ *if* $\Omega \subset \bigcup B_j$

Remarque
If m *(the lebesgue measure) exists and* J *is countable, then*

$$m(\Omega) \leq m(\bigcup B_j) \leq \sum m(B_j)$$

Definition 5 (Outer-Measure)
The outer measure of a set Ω *is defined as*

$$m^*(\Omega) = \inf\left\{\sum volB_j : \{B_j\} \text{ is a countable cover of } \Omega \right\}$$

Remarque
For every Ω *there exists at least one countable cover*

Lemme 6
The outer measure obeys

1. $m^*(\emptyset) = 0$
2. $0 \leq m^*(\Omega) \leq \infty$
3. $m^*(\Omega_1) \leq m^*(\Omega_2)$ *if* $\Omega_1 \subset \Omega_2$
4. $m^*(\Omega + x) = m^*(\Omega)$
5. *Countable subadditivity :* $m^*(\bigcup \Omega_j) \leq \sum m^*(\Omega_j)$

Preuve

— $m^*(\emptyset) = 0$ *because* $\emptyset, \{0\} \subset (-\epsilon, \epsilon)^n \forall \epsilon > 0$
— *All good*
— *Any cover of* Ω_2 *also covers* Ω_1
— *For any cover of* Ω *we can translate it over to* $\Omega + x$
— *For every* $J \in \mathbb{N}$, *let* $\left\{B_i^J\right\}_{i\in I_J}$ *cover* Ω_J, *then* $\Omega_j \subset \bigcup_{i\in I_J} B_i^J$, *then*

we can choose the B_i^J in such a way that

$$\sum_i vol(B_i^J) \leq m^*(\Omega_J) + \frac{\epsilon}{2^J}$$

and since $\left\{B_i^J\right\}_{i,J}$ covers $\bigcup_J \Omega_J$

$$m^*(\bigcup \Omega_J) \leq \sum_{j\in\mathbb{N}} \sum_{i\in I_J} vol(B_i^J) \leq \sum_{j\in\mathbb{N}} (m^*(\Omega_J) + \frac{\epsilon}{2^J}) = \epsilon + \sum m^*(\Omega_J)$$

□

Proposition 7
For a closed box $\overline{B}$

$$m^*(\overline{B}) = vol(B)$$

Preuve
Clearly $\overline{B}$ is covered by $\prod(a_i + \epsilon, b_i + \epsilon)$ Hence

$$m^*(\overline{B}) \leq vol(\prod(a_i + \epsilon, b_i + \epsilon)) \to \prod(b_i - a_i)$$

Hence $m^*(\overline{B}) \leq vol(B)$
Now we show that $vol(B) \leq m^*(\overline{B})$.
By Heine-Borel, $\overline{B}$ is compact.
Hence we only need to show the result with a finite cover.
In dimension 1, we are given $(a_1, b_1), \ldots$ covering $[a, b]$.
Remark that

$$1_{[a,b]} \leq \sum_i 1_{(a_i,b_i)}$$

Integrating (Riemann-integral), we get

$$(b - a) \leq \sum(b_i - a_i)$$

Now, we use induction

$$B_J = \prod_{i=1}^{n}(a_i^s, b_i^s) = \prod_{i=1}^{n-1}(a_i^s, b_i^s) \times (a_n^s, b_n^s)$$

Define

$$f_J(x_m) = vol(A_J)1_{(a_n,b_m)}(x_m)$$

For every x_m, we get

$$\left\{A^J : j \in J, x_n \in (a_n^J, b_n^J)\right\} \text{ is a cover of } \overline{A}$$

$$\sum f_j(x_m) = sum_{j\in J,x_n} vol(A_j)1_{(a_n,b_n)} \geq vol\overline{A}$$

□

Lecture 2: Existence of Lebesgue Measure

Thu 24 Feb

Corollaire 8
$m^*(B) = vol(B)$ for every open box B.

Preuve
For one direction, we use monotonicity, $m^*(B) \leq m^*(\overline{B}) = vol(B)$.
Furthermore, set $B = \prod(a_i, b_i)$, then for $\epsilon > 0$, we get

$$\prod[a_i+\epsilon, b_i-\epsilon] \subset \prod_i (a_i, b_i) \implies m^*(\prod[a_i+\epsilon, b_i-\epsilon]) \leq \prod_i (b_i - a_i)$$

□

Exemple

— $m^*(\mathbb{R}) = \infty$ *since by monotonicity, we get* $m^*(\mathbb{R}) \geq m^*([0,N]) > N$

— $m^*(\mathbb{Q}) = 0$ *since*

$$m^*(\mathbb{Q}) \leq m^*(\{q\}) = 0$$

Which proves that the reals are uncountable.

1.3 Measurable sets (again)

We want to know whether $\forall A, E \subset \mathbb{R}^m$, the inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

generalises to an equality ?
The inequality follows directly from countable subadditivity. In fact equality does not hold in general.

Definition 6 (Lebesgue Measurable set)
A set $E \subset \mathbb{R}^m$ is Lebesgue measurable if

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \forall A \subset \mathbb{R}^n$$

Then the lebesgue measure of E is defined as

$$m(E) \coloneqq m^*(E)$$

Note that, according to this definition, $\emptyset, \mathbb{R}^n$ are both measurable.

Lemme 10
Half-spaces are measurable

The proof is given as an exercise.
We now establish a few basic facts about measurable sets.

Lemme 11

- *The complement of a measurable set is measurable*
- *The translation of a measurable set is measurable, ie. E measurable, $x \in \mathbb{R}^n$ implies $E + x$ measurable*
- *Finite unions of measurable sets is measurable. (as well as the intersection)*
- *Open (as well as closed) boxes are measurable.*
- *If the outer measure of a set is 0, then E is measurable.*

Preuve

- $$m^*(A) = m^*(A \cap E^{c^c}) + m^*(A \cap E^c)$$
- *Given A a set and $x \in \mathbb{R}^n$, we get*
  $$m^*(A-x) = m^*(A-x \cap E) + m^*((A-x) \cap E^c) = m^*(A \cap E+x) + m^*(A \cap E^c+x) = m^*(A)$$
- $$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$
- *Consider the union of two setsWe now bound $m^*(A)$ by below (the upper bound is always true)*
  $$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$
  $$= m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \cap E_2^c) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$
  $$\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$
  The general result follows immediatly by induction on the number of sets.
- *We get that*
  $$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$ □
- *We write boxes as intersections of halfspaces*

Now we want to show that the lebesgue measure is countably additive.

Proposition 12
If $(E_j)_{j \in \mathbb{N}}$ are measurable disjoint sets, then $\bigcup_{i \in \mathbb{N}} E_i$ is measurable and

$$m^*(\bigcup_{j \in \mathbb{N}} E_j) = \sum_{j=1}^{\infty} m^*(E_j)$$

The proof depends on a lemma

Lemme 13

Let $E_1, \ldots, E_n$ be measurable disjoint sets, $A \subset \mathbb{R}^m$,then

$$m^*(A \cap (\bigcup E_j)) = \sum_{j=1}^{n} m^*(A \cap E_j)$$

As a consequence of this, we get finite additivity.

Preuve

For $n = 2$, we get

$$m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap (E_1 \cup E_2) \cap E_1) + m^*(A \cap (E_1 \cup E_2) \cap E_1^c)$$
$$= m^*(A \cap E_1) + m^*(A \cap E_2) \qquad \square$$

and the general case follows by induction.

Corollaire 14

$E \subset F$ measurable implies $F \setminus E$ is measurable and

$$m^*(F \setminus E) = m(F) - m(E)$$

Preuve

The set is trivially measurable since $F \setminus E = F \cap E^c$ Using the lemma above, we get

$$m^*(F) = m^*(E) + m^*(F \setminus E) \qquad \square$$

We can now prove countable additivity

Preuve

Let $E = \bigcup_{j=1}^{\infty} E_j$.
We claim that $\forall A$

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$$

Indeed note that

$$m^*(A \cap E) \leq \sum_{j=1}^{\infty} m^*(A \cap E_J) = \sup_N \sum_{j=1}^{N} m^*(A \cap E_j)$$

Set $F_n = \bigcup_{j=1}^{N} E_j$, by the lemma, the finite sum above is

$$\sup_N \sum_{j=1}^{N} m^*(A \cap E_j) = m^*(A \cap F_N)$$

Since $F_N \subset E$,

$$m^*(A \setminus E) \leq m^*(A \setminus F_N)$$

Then

$$m^*(A \cap E) + m^*(A \setminus E) < \sup_N m^*(A \cap F_N) + \underbrace{m^*(A \setminus E)}_{\leq m^*(A \setminus F_N)} < \sup_N m^*(A)$$

□

This proves that $m(E) \geq \sup_N m(F_N) = \sup_N \sum_{j=1}^N m(E_j) = \sum_{j=1}^\infty m(E_J)$

Lemme 15 (Lebesgues sets are a sigma-algebra)
If $(E_J)_J \in \mathbb{N}$ are measurable, then $\bigcup E_j$ and $\bigcap E_j$ are measurable.

Preuve

$$E_1 \cup \ldots = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \ldots$$

and the property about intersections follows from $\bigcap E_J = (\bigcup E_J^c)^c$ □

Lemme 16 (Open sets are measurable)
Every open set is measurable

Preuve
By an exercise, every open set is a countable union of open boxes and a countable union of measurable sets is countable by the lemma above. □

1.4 A glimps on abstract measure theory and theoretical foundations of probability

The idea of Lebesgue was to fix the measure of boxes and then extend the measure to the sigma algebra of measurable sets.

Theorème 17 (Caratheodory theorem)
Given a set Ω, $\mathcal{G}$ an algebra (finite union of boxes), A the smallest algebra containing $\mathcal{G}$.
Let $m_0 : \mathcal{G} \to [0, \infty]$ be a function s.t. $m(\emptyset) = 0, m_0(\bigcup_{i=1}^\infty A_i) = \sum_{m=1}^\infty m_0(A_m)$ if $A_m \in \mathcal{G}, A_m$ disjoint and $\bigcup A_m \in \mathcal{G}$
Then $\exists$ a measure on A such that $m|_{\mathcal{G}} = m_0$ and, if the measure of $m_0(\Omega) < \infty \implies m$ is unique.

Furthermore

Theorème 18
Every probability $\mathbb{P}$ on $\mathbb{R}^n$ gives rise to a cumulative distribution function, conversely, every cdf gives rise to a (unique) probability measure.

1.5 The cantor set

Definition 7 (Cantor set)
Consider $[1, 1]$, define $P_0 = [0, 1]$, $P_1 = [0, \frac{1}{3},] \cup [\frac{2}{3}, 1]$ and keep going. By definition $P_0 \supset P_1 \ldots$, the cantor set is the intersection of all of them.

There are a few nice properties of the cantor set

Theorème 19

1. *P is compact*
2. *$m^*(P) = 0$*
3. *P is uncountable*
4. *P is perfect*[a] *and has empty interior.*

a. No point in p is isolated.

Lecture 3: Measurable functions

Thu 03 Mar

1.6 Measurable functions

Definition 8 (Measurable functions)
Let $\Omega \subset \mathbb{R}^m$ measurable, $f : \Omega \to \mathbb{R}^m$ is measurable if $\forall V$ open, $f^{-1}(V)$ is measurable.

Remarque
Any function $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ is measurable $\iff$ $f^{-1}(B)$ is measurable $\forall B$ open boxes.

Preuve
Indeed, the implication $\implies$ is immediate.
For the other direction, note that any open set V is a countable union of boxes

$$V = \bigcup_i B_i$$

and $f^{-1}(V) = \bigcup_i f^{-1}(B_i)$ which is measurable. □

Remarque
Let $f : \Omega \to \mathbb{R}$ is measurable $\iff$ $f^{-1}((a, \infty))$ are measurable.

Preuve
By the remark above, it is enough to show that $f^{-1}((a,\infty))$ *are measurable* $\forall a, b$

$$f^{-1}((a,b)) = f^{-1}((-\infty,b)\cap(a,\infty)) = f^{-1}(a,\infty)\cap f^{-1}([b,\infty))^c$$

Now, rewrite $f^{-1}([b,\infty)) = \bigcap_i f^{-1}((b-\frac{1}{i},\infty))$ $\square$

Definition 9
$f:\Omega\to\mathbb{R}^* = \mathbb{R}\cup\{\pm\infty\}$ *is measurable if* $f^{-1}((a,\infty])$ *is measurable* $\forall a\in\mathbb{R}$

Using the remark above, the definition is compatible with the definition of measurable functions.

Remarque
Consider $f:\Omega\subset\mathbb{R}^n\to\mathbb{R}^m$, f *is measurable* $\iff$ *all projections of* f *are measurable.*

Preuve
To prove this, recall that f *is measurable* $\iff$ $f^{-1}(B)$ *are measurable, we may write* $B = B_1\times\ldots\times B_n$, *hence,* $f^{-1}(B) = \bigcap_{i=1}^n f_i^{-1}(B_i)$.
Hence the right to left implication follows.
$\implies$ *Consider* $B = \mathbb{R}\times\ldots\times B_i\times\ldots\times\mathbb{R}$, *then* $f^{-1}(B) = f_i^{-1}(B_i)$ *is measurable* $\square$

Remarque
Let $f:\Omega\to W$ *and* $g:W\to\mathbb{R}^p$, *then* $g\circ f$ *is measurable if* g *is continuous and* f *measurable.*

Lemme 24
Let $\Omega\subset\mathbb{R}^n$ *measurable,* $f_m:\Omega\to\mathbb{R}^*$ *measurable, then the functions*

$$\sup f_m, \inf f_m, \limsup f_m, \liminf f_m$$

are measurable.
In particular, if $f_m\to f$ *pointwise, then* f *is measurable.*

Preuve
Call $F = \sup f_n$, *we want to prove that*

$$F^{-1}((a,\infty]) = \bigcup f_m^{-1}((a,\infty])$$ $\square$

Lecture 4: Lebesgue Integration

Wed 09 Mar

1.7 Lebesgue integration

Definition 10 (Simple functions)
A measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is simple if (Ω is measurable)

1. *$f(\Omega)$ is a finite set*
2. *$\exists c_1, \dots, c_n \in \mathbb{R}$ and $E_1, \dots, E_n \subset \Omega$ measurable s.t.*

$$f = \sum_{i=1}^{n} c_i 1_{E_i}$$

Preuve
Clearly $\{c_1, \dots, c_n\} = f(\Omega)$, conversely, if $f(\Omega) = \{c_1, \dots, c_n\}$, define $E_i = f^{-1}(c_i)$ □

Remarque
Note that simple functions are vector spaces

Lemme 26
Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be measurable. Then $\exists$ an increasing sequence $\{f_n\}$ converging pointwise to f

Preuve
Define $f_n(x) = \sup_j \{2^{-n} J \leq \min(f(x), 2^n)\}$.

Definition 11
Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be a simple function, then the lebesgue integral of f is

$$\int_\Omega f dx = \sum_{\lambda \in f(\Omega), \lambda \geq 0} \lambda \mu \{x \in \Omega : f(x) = \lambda\}$$

Note this definition works for general measures.

Remarque
Let $f = \sum_i c_i 1_{E_i}$, then

$$\int_\Omega f dx = \sum_i c_i \mu(E_i)$$

The integral may be infinite.

Definition 12 (Almost everywhere)
A property $P(x)$ holds almost everywhere if $P(x)$ holds for every x except a set of measure 0.

Proposition 28 (Properties of simple functions)
Let $f, g : \Omega \to \mathbb{R}_{\geq 0}$ be simple functions

1. $0 \leq \int_\Omega f \leq \infty$ *and* $\int_\Omega f = 0 \iff f \equiv 0$ *almost everywhere.*
2. $\int_\Omega f + g d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$
3. $\lambda \int_\Omega f d\mu = c \int_\Omega f$
4. *if* $f \leq g$, *then* $\int_\Omega f + \int_\Omega g$

Definition 13 (Lebesgue Integral of non-negative function)
Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable, we define

$$\int_\Omega f := \sup \left\{ \int_\Omega s dx : s \leq f, s \text{ simple} \right\}$$

Remarque
In fact, if f is simple both definitions are compatible.

Proposition 30
Let $f, g : \Omega \to \mathbb{R}_{\geq 0}$ be measurable

- $0 \leq \int_\Omega f \leq \infty$ *and* $\int_\Omega f = 0 \iff f = 0$ *a.e.*
- $\int_\Omega cf = c \int_\Omega f$
- *If* $f \leq g$ *then* $\int_\Omega f \leq \int_\Omega g$
- *If* $f = g$ *a.e. then* $\int_\Omega f = \int_\Omega g$
- *if* $\Omega' \subset \Omega$, *then* $\int_{\Omega'} f = \int_\Omega (f 1_{\Omega'})$

We will prove additivity later on

Theorème 31 (Lebesgue Monotone convergence theorem)
Let $\Omega \subset \mathbb{R}^n$ be a measurable set and take f_n an increasing sequence of functions converging pointwise to f.
Then

$$\int_\Omega f = \lim_{m \to +\infty} \int_\Omega f_n$$

Preuve
By definition $f(x) = \lim_{n \to +\infty} f_n(x) = \sup_n f_n(x)$ (since the f_n are increasing).

Using the propositions above, we have that

$$\int_\Omega \sup_m f_m \geq \int_\Omega f_m \quad \forall m$$

Hence $\int_\Omega f \geq \sup \int_\Omega f_m$.
We claim $\int_\Omega \sup f_m \leq \sup \int_\Omega f_m$.
It suffices to show that $\forall \epsilon$

$$(1-\epsilon)\int_\Omega s \leq \sup_m \int_\Omega f_m \quad \forall s \leq \sup f_m \text{ simple}$$

Indeed, note that $\forall x \in \Omega \exists N := N(x)$ *s.t.* $f_N(x) \geq (1-\epsilon)s(x)$.
Let $E_n = \{x \in \Omega : f_n \geq (1-\epsilon)s\}$.
Since f_n *is increasing,* $E_1 \subset E_2 \ldots$ *and* $\bigcup E_i = \Omega$*, hence we get*

$$(1-\epsilon)\int_{E_m} s = \int_{E_m} (1-\epsilon)s \leq \int_{E_m} f_N \leq \int_\Omega f_n$$

Taking the sup yields

$$\sup_n (1-\epsilon)\int_{E_n} s \leq \sup_n \int_\Omega f_n$$

Hence, we only need to show that the left hand side equals $(1-\epsilon)\int_\Omega s$.
Indeed, the inequality $\sup_n (1-\epsilon)\int_{E_n} s \leq (1-\epsilon)\int_\Omega s$.
For the other inequality, write $s = \sum 1_{F_j} c_j$*, then*

$$\int_{E_n} s = \int_\Omega \sum c_j 1_{E_n \cap F_j} \qquad \square$$

Lecture 5: Monotone Convergence theorem

Thu 10 Mar

Corollaire 32
$f, g : \Omega \to [0, \infty)$ *measurable, then*

$$\int_\Omega f + g = \int_\Omega f + \int_\Omega g$$

Preuve
Let s_n, t_n *be simple functions converging pointwise to* f *respectively* g*, then* $s_n + t_n$ *converges pointwise to* $f + g$.
Then

$$\int_\Omega f + g = \lim_{n \to +\infty} \int_\Omega s_n + t_n = \lim_{n \to +\infty} \int_\Omega s_n + \int_\Omega t_n = \int f + \int g \qquad \square$$

Corollaire 33
Let $g_1, \ldots : \Omega \to [0, \infty)$ be measurable functions, then

$$\int_\Omega \sum_{i=1}^{\infty} g_i = \sum_{i=1}^{\infty} \int_\Omega g_i$$

Preuve
Let $G_n = \sum_{i=1}^{n} g_i$, this is a sequence of functions converging to G (from below)

$$\int_\Omega \sum_{i=1}^{\infty} g_i = \int_\Omega G = \lim_{n \to +\infty} \int_\Omega G_n = \lim_{n \to +\infty} \sum_{i=1}^{n} \int_\Omega g_i = \sum_{i=1}^{\infty} \int_\Omega g_i$$

□

1.8 Fatou's lemma

Theorème 34 (Fatou's lemma)
Let f_i be a sequence of measurable functions $\Omega \to [0, \infty)$, then

$$\int_\Omega \liminf_{m \to \infty} f_m \leq \liminf_{m \to \infty} \int_\Omega f_m$$

Preuve
By definition

$$\liminf f_m = \sup_n \inf_{m \geq n} f_m$$

By monotone convergence theorem

$$\int_\Omega \liminf_n f_n = \sup_n \int_\Omega \inf_{m \geq n} f_m$$

Since $\int_\Omega \inf_{m \geq n} f_m \leq \int_\Omega f_J \forall J \geq m$, hence

$$\int_\Omega \inf_{m \geq n} f_m \leq \inf_{J \geq m} \int_\Omega f_J$$

And finally

$$\int_\Omega \liminf f_m \leq \sup_m \inf_{J \geq m} \int_\Omega f_J = \liminf_{J \to +\infty} \int_\Omega f_J$$

□

Lemme 35
Let $f : \Omega \to [0, \infty]$ be a measurable function, if $\int_\Omega f < \infty$, then

$$\mu \{x \in \Omega : f(x) = \infty\} = 0$$

Preuve
Suppose not, let E be this set, then $\forall n$

$$n 1_E \leq f \implies n\mu(E) \leq \int_\Omega f$$ □

Exemple (Borel-Cantelli)
Let $\{\Omega_i\}$ *be measurable sets such that* $\sum \mu(\Omega_i) < \infty$*, then*

$$\limsup \Omega_i = \{x \in \Omega : x \in \Omega_i \text{ for infinitely many values } \}$$

has measure 0.

Preuve
We claim that $\int_\Omega \sum_i 1_{\Omega_i} < \infty$*,, then by the lemma,* $f < \infty$ *almost everywhere, hence* $x \in \Omega_i$ *only for finitely many* i*, hence* $x \notin \limsup \Omega_i$*.*
The proof of the claim follows from the corollary to fatou's lemma :

$$\int_\Omega \sum_i 1_{\Omega_i} = \sum_i \int_\Omega 1_{\Omega_i} = \sum \mu(\Omega_i) < \infty$$ □

Lecture 6: Dominated Convergence Theorem

Wed 16 Mar

1.9 Integration of signed functions

Definition 14
$f : \Omega \to [-\infty, \infty]$ *is absolutely integrable if*

$$\int_\Omega |f| < \infty$$

Definition 15 (Integral of a function)
Let f be an absolutely integrable function, then

$$\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$$

Remarque

$$|\int_\Omega f| \leq \int_\Omega |f|$$

Proposition 38 (Basic properties)
Let f, g be absolutely integrable functions
— $\forall c \in \mathbb{R}$*,* cf *is absolutely integrable and* $\int_\Omega cf = c\int_\Omega f$

— *$f+g$ is absolutely integrable and* $\int_\Omega f+g = \int_\Omega f + \int_\Omega g$
— *If $f=g$ almost everywhere then* $\int_\Omega f = \int_\Omega g$

Theorème 39 (Dominated Convergence Theorem)
Let $f_1, f_2, \ldots : \Omega \to [-\infty, \infty]$ be measurable functions. Assume $f_n \to f$ almost everywhere and such that $|f_m(x)| \leq F(x) \forall m, x \in \Omega$ where F is absolutely integrable.
Then

$$\lim_{n\to+\infty} \int f_n = \int f$$

Remarque
With the same assumptions, we can conclude that

$$\lim_{n\to+\infty} \int |f_n - f| = 0$$

Indeed, apply the theorem to $g_n = |f_n - f|$.
Then $|g_m| \leq |f_n| + |f| \leq 2F$.
Similarly, let f_m be such that the above condition holds, then $\int f_n \to \int f$, since

$$|\int f_n - \int f| = |\int f_n - f| \leq \int |f_n - f| \to 0$$

Preuve
By assumption $|f_n| \leq F$, hence $|f| \leq F$.
Apply Fatou to $F(x) + f_n(x)$, we get

$$\int_\Omega F + f \leq \liminf \int F + f_n \leq \liminf \int f_m + \int f_n$$

Now we apply Fatou to $F - f_n \geq 0$, we get

$$\int_\Omega F - \int_\Omega f \leq \liminf \int_\Omega F - f_n$$

Which in turn implies that

$$\int_\Omega f \leq \liminf_{n\to\infty} \int_\Omega f_n$$

We now apply the same trick to $F - f_n$, noticing again this family of functions is non-negative

$$\int_\Omega F - f \leq \liminf_{n\to\infty} \int_\Omega F - f_n$$
$$\int_\Omega f \geq \limsup_{n\to\infty} \int_\Omega f_n$$

Which implies the limit $\int f_n$ exists and is equal to $\int f$ □

Remarque (Differentiation under the integral)

Let $f : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ *be measurable such that*

- $\partial_t f(x,t)$ *for almost every* x *and every* t
- $|\partial_t f(x,t)| \leq h(x)$ *where* $h(x)$ *is an absolutely integrable function, then*

$$\frac{d}{dt}\int f(x,t)dx = \int \partial_t f(x,t)$$

Preuve

Indeed

$$\frac{d}{dt}\int f(x,t) = \lim_{h\to 0}\int \underbrace{\frac{f(x,t+h)-f(x,t)}{h}}_{\to \partial_t f(x,t)}$$

Now notice that

$$\left|\frac{f(x,t+h)-f(x,t)}{h}\right| \leq \left|\int \partial_t f(x,t+hs)ds\right| \leq h(x)$$

$\square$

Definition 16

Let $\Omega \subset \mathbb{R}^m$, f *a function (not necessarily measurable).*

The upper and lower Lebesgue integrals

$$\overline{\int_\Omega} f = \inf\left\{\int g : g \text{ measurable }, g \geq f\right\}$$

and similarly the lower integral.

$$\underline{\int_\Omega} f = \inf\left\{\int g : g \text{ measurable }, g \leq f\right\}$$

1.10 Comparison with Riemann Integral

Theorème 42 (Lebesgue generalizes Riemann)

Let $I \subset \mathbb{R}$ *be an interval,* $f : I \to \mathbb{R}$ *be Riemann integralble, then* f *is absolutely integrable and*

$$\int_I f dx = \text{ Riemann integral of } f \text{ on } I$$

Preuve

f *is Riemann integrable if* $\forall \epsilon > 0$ *there exists* p *a partition of* I *such that*

$$A - \epsilon \leq \sum |J| \inf_{x\in J} f \leq \sum_{J\in P} |J| \sup f \leq A + \epsilon$$

Since $f_\epsilon^- \leq f \leq f_\epsilon^+$

$$A - \epsilon \leq \int f_\epsilon^- \leq \underline{\int} f \leq \overline{\int} f \leq \int f_\epsilon^+ \leq A + \epsilon$$

Letting $\epsilon \to 0$ yields the result.
Indeed let $f_m^\pm$ be such that $f_m^- \leq f \leq f_m^+$

$$\underline{\int} f - \frac{1}{m} \leq \int f_m^+ \leq \overline{\int} f + m$$

□

Setting $F^- = \sup f_m^-, F^+ = \inf f_m^+$ are measurable.
$F^- \leq f \leq F^+$

1.11 Fubini's Theorem

Theorème 43 (Fubini-Tonelli)
Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Assume $f \geq 0$ or f absolutely integrable, then

— *for almost every x , $f(x, \cdot)$ is measurable and*

$$x \mapsto \int f(x, y) dy$$

is measurable

— *For almost every y , $f(\cdot, y)$ is measurable and*

$$y \mapsto \int f(x, y) dy$$

is measurable

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f dx dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x, y) dy \right) dx$$

Lecture 7: Fubini's Theorem

Thu 17 Mar

Remarque
Tonelli is used on $|f|$ and to show that f is absolutely integrable, then we can apply Fubini.

Preuve
We prove the result under the additional assumptions that $m = n = 1$ and that every function appearing is measurable.
We will prove that

$$\int_{\mathbb{R}^2} f dx dy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y) dy dx$$

It is enough to prove the above equality when $f \geq 0$.
If not, $f = f^+ - f^-$, then we may apply the above result to f^+ and f^-.

Notice also that it is sufficient to prove the result for f such that $\operatorname{Supp} f \subset [-N,N]^2$.
Indeed, write $f_n = f1_{[-n,n]^2}$, then

$$\int_{\mathbb{R}} f_n = \int_{\mathbb{R}}\int_{\mathbb{R}} f_n$$

Now since f_n is monotone, $\int_{\mathbb{R}^2} f_n \to \int_{\mathbb{R}^2} f$ the left hand side yields (again using monotone convergence)

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f dydx$$

We may now reduce the problem even further to simple functions with bounded support.
Indeed for every $f \geq 0$, f is a sup *of simple functions so we can apply monotone convergence.*
Now since every simple function is the sum of indicator functions, we only need to prove the result for indicator functions :

$$\begin{aligned} f &= \sum c_i 1_{E_i} \\ \int_{\mathbb{R}^2} f dxdy &= \sum_i c_i \int_{\mathbb{R}^2} 1_{E_i} dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \sum_i c_i 1_{E_i} dxdy \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f dydx \end{aligned}$$

It is enough to prove that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 1_E(x,y)dydx \leq m(E) \quad E \subset [-N,N]^2$$

Indeed if the above holds, we may apply it to $[-N,N]^2 \setminus E$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 1_{[-N,N]^2\setminus E} dydx \leq m([-N,N]^2 \setminus E)$$

Summing both inequalities yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 1_E + 1_{[-N,N]\setminus E} dydx \leq m(E) + m([-N,N]^2 \setminus E) = m([-N,N]^2)$$

Hence all inequalities above are in fact equalities.
So we only need to prove that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} 1_E(x,y)dydx \leq m(E) \quad E \subset [-N,N]^2$$

Consider a covering $\{B_j\}$ of E s.t. $\sum \operatorname{Vol}(B_j) \leq m(E) + \epsilon$, but this is just

$$\sum \operatorname{Vol}(B_j) = \sum \int_{\mathbb{R}}\int_{\mathbb{R}} 1_{B_j} dydx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{\sum 1_{B_j}}_{\geq 1_E} dydx \qquad \square$$

Which concludes the proof.

Lecture 8: Lp spaces

Wed 23 Mar

2 L_p spaces

Definition 17 (Lp space)
Let $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ and $p \in [1, \infty)$, we define

$$\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f|^p \right)^{\frac{1}{p}}$$

and

$$\left\{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \,|\, \|f\|_{L_p(\Omega)} < \infty \right\}$$

Remarque
If $p = 1$, then $L^1(\Omega)$ are absolutely integrable functions.
We hope the definition above is a norm, but we need

$$\|f\| = 0 \iff f = 0$$

so we need to ask that $f = 0$ almost everywhere.
We wish to identify in L^p functions that coincide almost everywhere, so we need to identify as follows

$$(L^p(\Omega), \|\cdot\|_{L_p}) = \{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} : \|f\| < \infty \} \,/ \sim$$

where $f \sim g \iff f = g$ ae.

Definition 18 (L infinity)
Define

$$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f| = \inf \{ \alpha : f < \alpha \text{ almost everywhere } \}$$

If f is continuous, the sup and ess sup coincide.
Then $L^\infty(\Omega)$ is then defined as above.

Proposition 46
Let $\Omega \subset \mathbb{R}^n$ be measurable and $1 \leq p \leq q \leq \infty$, then

- *$L^p(\Omega)$ is a vector space*

— *If* $m(\Omega) < \infty$, *then* $\|f\|_{L_q} \leq K \|f\|_{L_q} \forall f$ *where* K *depends on* $m(\Omega), p$ *and* q.
— *if* $m(\Omega) < \infty$, *then* $\lim_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$
— *Minkowski inequality*

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

In particular $\|\cdot\|_{L^p}$ *is a norm.*

Theorème 47 (Hoelder inequality)
Let Ω *be measurable,* $p \in [1,\infty]$, *then*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}$$

where p' *satisfies* $\frac{1}{p} + \frac{1}{p'} = 1$

Preuve
The inequality holds iff $\|\lambda_1 f \lambda_2 g\|_{L^p} \leq \|\lambda_1 f\|_{L^p} \|\lambda_2 g\|_{L^{p'}}$ *for all* $\lambda_1, \lambda_2 \in \mathbb{R}$. *So we may reduce ourselves to the case*

$$\|f\|_{L^p} = \|g\|_{L^{p'}} = 1$$

Now

$$\int |fg| \leq \int \frac{|f|^p}{p} + \frac{|f|^{p'}}{p'} = 1$$

□

Preuve (Of second point above)

$$\|F\|_{L^P} = \|F^p\|_{L^1}^{\frac{1}{P}} \leq (\int F^{P\frac{Q}{P}})^{\frac{1}{Q}} (\int 1^{p'})^{\frac{1}{P}-\frac{1}{Q}}$$

□

Preuve (Of fourth point)

$$\begin{aligned}
\|f+g\|_{L^p}^p &= \int |f+g|^p \\
&\leq \int (|f|+|g|)|f+g|^{p-1} \\
&= \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \\
&\leq (\int |f|^p)^{\frac{1}{p}} (\int |f+g|^p)^{\frac{p-1}{p}} \\
&= (\|f\| + \|g\|) \|f+g\|^{p-1}
\end{aligned}$$

□

2.1 Completeness of L^p

Theorème 48 (Lp spaces are complete)
Let Ω be measurable, $p \in [1, \infty]$, then $L^p(\Omega)$ is complete, namely if

$$\lim_{m,n \to +\infty} \|f_n - f_m\|_{L^p} = 0$$

then $\exists f \in L^p$ s.t. $\lim_{n\to+\infty} \|f_n - f\|_{L^p} = 0$.
Moreover, if the above holds then $\exists$ a subsequence $\{m_k\}$ s.t.

$$f_{m_k}(x) \to f(x)$$

Almost everywhere.

Remarque
Taking the subsequence above is important, see exercises.

Preuve
We prove the result for $p < \infty$, the case $p = \infty$ is an exercise.
We want to prove that $\{f_m\}$ is cauchy in L^p implies there is a subsequence $f_m \to f$ in L^p pointwise.
We look for a speedy converging subsequence.
Indeed, we know from hypothesis that there exists a subsequence $\{m_k\}$ st.

$$\left\|f_{m_k} - f_{m_{k+1}}\right\| \leq 2^{-k}$$

Now consider

$$f(x) = f_{m_1}(x) + \sum_k f_{m_{k+1}} - f_{m_k}(x)$$

This is a reasonable definition, but is it well defined.
Namely is the series absolutely converging for almost every x ?
Consider

$$g_h(x) = |f_{m_1}(x)| + \sum_{k=1}^{h} |f_{m_{k+1}}(x) - f_{m_k}(x)|$$

Is $\lim_{h\to+\infty} g_h < \infty$ ae. ? If yes, f is well defined.
Indeed,

$$\|g_j\|_{L^p} \leq \|f_{m_1}\|_{L^p} + \sum \left\|f_{m_{k+1}} - f_{m_k}\right\| \leq \|f_{m_1}\| + 1$$

But now

$$\int |g|^p = \lim_{h\to+\infty} \int |g_h|^p < \infty$$

Hence g is finite a.e. and

$$f(x) = \lim f_{m_1}(x) + \sum_k f_{m_{k+1}} - f_m = \lim f_{m_k}(x)$$

And the convergence is dominated by g.
To prove L^p convergence

$$\|f_{m_k} - f\|_{L^p}^p = \int |f_{m_k} - f|^p \to 0 \qquad \square$$

Lecture 9: Smooth functions are dense

Thu 24 Mar

2.2 Approximation of L^p functions with $C_c^\infty(\Omega)$

Definition 19 (Compactly supported)
If $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, then $\operatorname{Supp} f = \{x : f(x) \neq 0\}$

$$C_c^0(\Omega) = \left\{f \in C^0(\Omega) : \operatorname{Supp} f \subset\subset \Omega\right\}$$

Where we require $\operatorname{Supp} f$ *to be compact. And then we define*

$$C_c^k(\Omega) = C_c^0(\Omega) \cap C^k(\Omega)$$

Theorème 50
Let Ω be an open set, $1 \leq p < \infty, f \in L^p(\Omega)$ then $\exists g_k \in C_k^\infty(\Omega)$ st. $\lim_{k\to+\infty} \|g_k - f\|_{L^p(\Omega)}$

Preuve
We prove the result for $\Omega = \mathbb{R}^n$, we first find $g_k \in C_c^0(\mathbb{R}^n)$
We prove the result for $f = 1_B$, B a box.
Define $g_\epsilon(x) = \min(1 - \frac{d(x,B)}{\epsilon}, 1)$
Now we want to go from indicator of boxes to indicators of measurable sets.
So assume $f = 1_E, E$ measurable and $\overline{E}$ is compact.
Let $\epsilon > 0$ and $\{B_i\}$ be a cover of E st. $\sum m(B_i) \leq m(E) + \epsilon$.
This implies that

$$\int |1_E - \sum 1_{B_i}| = \sum \int |1_{B_i} - 1_E| = \sum m(B_i) - m(E) \leq \epsilon$$

Take N st. $\sum_{i=N+1}^\infty m(B_i) < \epsilon$
Using step 1, we find $h^i \in C_c^0(\mathbb{R}^n)$ st. $\left\|h^i - 1_{B_i}\right\| \leq \frac{\epsilon}{N}$.
Take $h = \sum_{i=1}^N h^i \in C_c^0(\mathbb{R}^n)$.
Now for $p = 1$, we want to estimate

$$\|1_E - h\|_{L^1} \leq \left\|1_E - \sum_{i=1}^N 1_{B_i}\right\|_{L^1} + \left\|\sum (1_{B_i} - h^i)\right\|_{L^1} \leq \epsilon + \sum_{i=1}^N \frac{\epsilon}{N} = 2\epsilon$$

If $p > 1$, take $\hat{h} = \max(\min(h, 1), 0)$ $\square$