Exercise 12

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1

We prove the double implication.
$\Rightarrow$
Let us denote by A the adjacency matrix.
First suppose that G contains a cycle of length three, without loss of generality, we can suppose that the three vertices of the cycle are numbered by $1, 2$ and 3.
Hence, $(1, 2) \in E(G)$, and we deduce that the $(1, 2)$ entry of the adjacency matrix is different from 0.
Now consider the $(1, 2)$ entry of A^2, applying the formula for matrix multiplication yields

$$\left(A^2\right)_{1,2} = \sum_{i=1}^{n} A_{1,i} A_{i,2}$$

Note that if $i = 3$, by defintion $A_{1,3}A_{3,1} = 1$, and, since all other terms of the sum are nonnegative, $(A^2)_{1,2} \geq 1$.
$\Leftarrow$
Now suppose that G is a graph such that $A_{i,j} \neq 0$ and $(A^2)_{i,j} \neq 0$.
This implies that the vertices i and j are adjacent.
Furthermore, this implies that

$$\sum_{k=1}^{n} A_{i,k} A_{k,j} \neq 0$$

This implies that there exists $l \in [n]$ such that $A_{i,l} = A_{l,j} = 1$, and hence $(i, j), (i, l), (l, j) \in E(G)$, which means G contains a triangle.

2

We will proceed by induction on the number n of vertices of T which are not leafs.
If $n = 1$, let v be the vertice of degree different to 1 and $l_1, \ldots, l_k$ the set of all leafs.

Since $f(l_i) = g(l_i) \forall i \in [k]$ and since f and G are bijections on the set of vertices, we immediatly deduce that $f(v) = g(v)$.
Suppose the result shown for n, we will now show it for $n + 1$.