Serie 4
Analysis IV, Spring semester
EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1 (Properties of signed integrals)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a measureable set and let $f, g \colon \Omega \to \mathbb{R}$ be absolutely integrable functions. Show the following statements:

(i) Prove that

$$\left|\int_\Omega f\,dx\right| \leq \int_\Omega f^+\,dx + \int_\Omega f^-\,dx = \int_\Omega |f|\,dx.$$

(ii) For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and

$$\int_\Omega (cf)\,dx = c\int_\Omega f\,dx\,.$$

(iii) The function $f+g$ is absolutely integrable and

$$\int_\Omega (f+g)\,dx = \int_\Omega f\,dx + \int_\Omega g\,dx\,.$$

(iv) If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have

$$\int_\Omega f\,dx \leq \int_\Omega g\,dx.$$

(v) If $f(x) = g(x)$ for almost every $x \in \Omega$, then

$$\int_\Omega f\,dx = \int_\Omega g\,dx.$$

Hint: For (iii), break f, g and $f+g$ up into positive and negative parts, and try to write everything in terms of integrals of non-negative functions. Then, use the linearity of the integral with respect to non-negative functions.

Exercise 2. Let $\Omega \subseteq \mathbb{R}^n$ measurable and let $f : \Omega \to [0, \infty)$ be a nonnegative and integrable function. If $\alpha > 0$ and $E_\alpha := \{x \in \Omega : f(x) > \alpha\}$, prove that

$$\mathrm{m}(E_\alpha) \leq \frac{1}{\alpha} \int_\Omega f\, dx.$$

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ measurable and let $f \colon \Omega \to \mathbb{R}$ be integrable. Show that for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any measurable set $E \subset \Omega$, it holds that if

$$\mathrm{m}(E) \leq \delta \;\Rightarrow\; \int_E |f(x)|\, dx \leq \varepsilon.$$

Hint: Consider the sequence $f_\nu(x) := \min\{|f(x)|, \nu\}$.

The next exercise introduces the Cantor set - a famous example of a Borel set which has Lebesgue measure 0 and yet the cardinality of the continuum. In order to prove some of its properties, it makes use of the exercise on ternary expansions in the previous exercise sheet.

Exercise 4. We begin by defining

$$P_0 := [0,1]\,,$$

$$P_1 := [0,1/3] \cup [2/3,1]\,,$$

$$P_2 := [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$$

and so on (at each step we eliminate an interval of lenght $1/3^{k+1}$ in the middle of each interval of lenght $1/3^k$) such that

$$\cdots \subset P_{k+1} \subset P_k \subset \cdots \subset P_2 \subset P_1 \subset P_0\,.$$

The Cantor set is defined as

$$P := \bigcap_{k=0}^{\infty} P_k\,.$$

Show the following properties:

(i) P is compact.

(ii) P is measurable of Lebesgue measure 0.

(iii) $P = \{a \in [0,1] : a = 0.a_1a_2\cdots \text{ with } a_i \in \{0,2\}\}$, where $0.a_1a_2\cdots$ denotes **a** possible tenary expansion of $a \in [0,1]$ (remember that this expansion is not unique, see the previous exercise sheet).

(iv) P is uncountable.

Exercise 5 ($\star$)**.** Let $0 < \varepsilon < 1$.

(i) Construct an open dense set $E \subseteq [0,1]$ such that $\mathrm{m}(E) = \varepsilon$.

(ii) Construct a closed set F that does not contain any non-empty open set with $\mathrm{m}(F) = \varepsilon$.