Exercise 6

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1

We will suppose that $q \neq 0$, indeed, if the recurrence relation is of depth k and $q = 0$, the characteristic polynomial wouldn't have a constant term and thus the recurrence relation wouldn't be of depth k.
Let

$$p(x) = x^k - \alpha_1 x^{k-1} - \ldots - \alpha_k$$

be the characteristic polynomial of the linear recurrence.
By hypothesis q is a root with multiplicity m, we will use without proof that, for $0 \leq i < m$, q will be a root of $\frac{d^i}{dx^i} p(x)$.
First, notice that the case $i = 0$ is clear, indeed, we have

$$\begin{aligned} 0 &= q^k - \alpha_1 q^{k-1} - \ldots - \alpha_k \\ q^k &= \alpha_1 q^{k-1} + \ldots + \alpha_k \\ q^n &= \alpha_1 q^{n-1} + \ldots + \alpha_k q^{n-k} \end{aligned}$$

Where, in the last step, we have simply multiplied by q^{n-k}.
Since this holds for all $n > k$, we have shown that q^n is a solution of the linear recurrence.
We will now prove the result for $i < m$.
Notice that, if $m > 1$, the result cited above implies in particular that

$$x^n - \alpha_1 x^{n-1} - \ldots - \alpha_k x^{n-k} = 0$$

Taking the derivative yields

$$\begin{aligned} nx^{n-1} - \alpha_1(n-1)x^{n-2} - \ldots - (n-k)x^{n-k-1} &= 0 \\ nx^n - \alpha_1(n-1)x^{n-1} - \ldots - (n-k)x^{n-k} &= 0 \\ nq^n - \alpha_1(n-1)q^{n-1} - \ldots - \alpha_k(n-k)q^{n-k} &= 0 \end{aligned}$$

Thus, nq^n also satisfies the linear recurrence relation.
Note that we can substitute x by q since we assumed that $q \neq 0$.

In general, for $i < m$, repeating this process i times (ie. differentiating with respect to x and then multiplying by x) gives the equality

$$n^i q^n - \alpha_1 (n-1)^i q^{n-1} - \ldots - \alpha_k (n-k)^i q^{n-k} = 0$$

And thus, $n^i q^n$ is a solution to the linear recurrence if $i < m$, since for $i \geq m$, q will no longer be a root of the equation.

2

Suppose there exist factors $x_0, \ldots, x_{m-1} \in \mathbb{R}$ satisfying

$$x_0 \left\{q^n\right\}_{n=1}^{\infty} + \ldots + x_{m-1} \left\{n^{m-1} q^n\right\}_{n=1}^{\infty} = \left\{0\right\}_{n=1}^{\infty}$$

Then, taking the $m-1$ first terms of each sequence, we get the linear system

$$\begin{cases} x_0 q + x_1 q + \ldots x_{m-1} q = 0 \\ x_0 q^2 + x_1 2 q^2 + \ldots + x_{m-1} 2^{m-1} q^2 = 0 \\ \vdots \\ x_0 q^{m-1} + x_1 (m-1) q^{m-1} + \ldots + x_{m-1} (m-1)^{m-1} q^{m-1} = 0 \end{cases}$$

Which simplifies to

$$\begin{cases} x_0 + x_1 + \ldots x_{m-1} = 0 \\ x_0 + x_1 2 + \ldots + x_{m-1} 2^{m-1} = 0 \\ \vdots \\ x_0 + x_1 (m-1) + \ldots + x_{m-1} (m-1)^{m-1} = 0 \end{cases}$$

Putting the system into matrix form, we get a Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 2^{m-1} \\ 1 & 3 & \ldots & 3^{m-1} \\ \vdots & & \ddots & \vdots \\ 1 & (m-1) & \ldots & (m-1)^{m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \ldots \\ x_{m-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

As shown in our linear algebra course, the determinant of this matrix is given by the formula

$$\prod_{1 \leq i,j \leq m-1, i \neq j} (i-j)$$

Which implies that the determinant of the matrix is non-zero since none of the terms in the product are zero.
Using a fundamental result of linear algebra, this implies that $x_i = 0 \quad \forall 0 \leq i \leq m-1$ and thus the sequences are linearly independent.