Derived Categories of Special Cubic Fourfolds

David Wiedemann

Abstract

These are notes prepated for my talk on derived categories of cubic hypersurfaces in the corresponding seminar in Bonn during the wintersemester 2023/24. The main goal is to illustrate the Kuznetsov conjectures, and more generally rationality problems for cubic fourfolds via two key examples, mainly following the book [Huy23].

Contents

1 Cubics and Rationality Problems

Throughout, let $k = \mathbb{C}$ and let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d.
In the last talk, we defined the Kuznetsov component of a cubic hypersurface, this is a specified **admissible subcategory** of the bounded derived category of X which can be thought of as the non-trivial part of $D^\flat(X)$. Given a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d, a version of Bott vanishing shows that the longest exceptional sequence of twisting sheaves on X one can get is $\langle \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n+1-d)\rangle$. We define the Kuznetsov component to be the right orthogonal to this:

$$\mathcal{A}_X := \langle \mathcal{O}_X, \dots, \mathcal{O}_X(n+1-d)\rangle^\perp.$$

As the subcategories $\langle \mathcal{O}_X(j) \rangle \subset D^b(X)$ are all equivalent[1] to $D^b(\operatorname{Spec} k)$, $\mathcal{A}_X$ can be fruitfully thought of as the non-trivial part.
Today, we will use derived categories to study the rationality of cubic fourfolds, as a refresher, here is what we know about rationality of (smooth) cubics in lower dimension:

- If C is a smooth cubic curve, it is an elliptic curve and hence never rational.
- If X is a smooth cubic surface, then it is the blowup of $\mathbb{P}^2$ in 6 points. In particular, it is always rational.
- If X is a smooth cubic threefold, then, by a result of Clemens and Griffiths, it is never rational.

The situation for cubic fourfolds is still unclear and a largely open problem, in fact, not a single cubic fourfold is known to be non-rational.
Our goal today is to shed light on a conjecture due to Kuznetsov

A smooth cubic fourfold X is rational if and only if there is a K3 surface S such that there is an equivalence $\mathcal{A}_X \simeq D^b(S)$.

From now on, X is a smooth cubic fourfold and S a K3 surface.
We first make the observation that $\mathcal{A}_X$ shares some striking similarities with $D^b(S)$. Recall the general result from last time

Proposition 1 (Kuznetsov) *The Kuznetsov component $\mathcal{A}_X \subset D^b(X)$ has a Serre functor $\mathcal{S}$ given by $\mathcal{S} = [2]$. Furthermore $\mathcal{A}_X$ is indecomposable, ie. there do not exist subcategories $A, B \subset \mathcal{A}_X$ such that $\langle A, B \rangle = \langle B, A \rangle$ are semi-orthogonal decompositions.*

Much like the case of K3 surfaces! Moreover, certain numerical invariants of $\mathcal{A}_X$ coincide with those of a K3 surface, indeed, we find isomorphisms in the *Hochschild homology* $HH^\bullet(\mathcal{A}_X) \simeq HH^\bullet(D^b(S))$.

2 Pfaffian Cubic Fourfolds

We illustrate the conjecture in a case where we know X to be rational. Remember the following theorem from a previous talk.

Theorem 2 (Pfaffian cubics are rational) *A Pfaffian cubic fourfold is rational.*

[1] equivalent as k-linear triangulated categories

We start by recalling what Pfaffian cubic fourfolds are and defining there associated K3 surfaces.
Let W be a six dimensional vector space, then $\Lambda^2 W$ is 15 dimensional and we consider it's projectivization $\mathbb{P}(\Lambda^2 W) \simeq \mathbb{P}^{14}$. The Pfaffian is the subvariety $\mathrm{Pf}(W) \subset \mathbb{P}(\Lambda^2 W)$ given by $\mathrm{Pf}(W) = \{\omega \in \mathbb{P}(\Lambda^2 W) | \omega \wedge \omega \wedge \omega = 0\}$.

Definition 1 (Pfaffian Cubic Fourfold) *A smooth cubic fourfold is a Pfaffian Cubic fourfold if it is isomorphic to $X_V := \mathrm{Pf}(W^*) \cap \mathbb{P}(V)$ where $V \subset \Lambda^2 W^*$ is a 6 dimensional sub vector space.*

Definition 2 (Associated K3 surface) *Let $V \subset \Lambda^2 W^*$ be a subspace as above and consider the Grassmanian of lines $\mathbb{G}(1, \mathbb{P}(W))$ as a closed subscheme of $\mathbb{P}(\Lambda^2 W)$ via the Plücker embedding. The associated K3 surface to V is defined as $S_V := \{p \in \mathbb{G}(1, \mathbb{P}(W)) | \omega|_p = 0$ for all $\omega \in V\}$.*

It was proven in a previous talk that S_V is indeed a K3 surface that does not contain any lines and that X_V is a smooth cubic fourfold. Today, we will see that the K3 surface S_V is the "associated K3" to X_V. Our main first theorem is

Theorem 3 *If X is a Pfaffian cubic fourfold, then there is an equivalence of categories*

$$D^b(S_V) \simeq \mathcal{A}_X.$$

In the first part of this talk, we will sketch the proof of this theorem. An important ingredient is the correspondence

$$\begin{tikzcd} \Sigma_V \subset S_V \times X_V \arrow[r, "p_X"] \arrow[d, "p_S"'] & X_V \\ S_V & \end{tikzcd}$$

where $\Sigma_V = \{(p, \omega) \in S_V \times X_V | p \cap \ker \omega \neq 0\}$. We give Σ_V the reduced induced scheme structure.
Going forward, we will drop the subscript V from our notation.

Let us sketch the structure of the proof:

1. Using Σ, we construct a (exact, k-linear) functor $\Psi\colon D^b(S) \to D^b(X)$
2. We show that the image of Ψ is contained in $\mathcal{A}_X$.
3. We show that the restriction $\Psi\colon D^b(S) \to \mathcal{A}_X$ is fully faithful
4. We show that Ψ is an equivalence

Constructing Ψ

Define

$$\mathcal{I}(-1) := \mathcal{I}_\Sigma \otimes p_X^*\mathcal{O}_X(-1),$$

and let

$$\Phi_{\mathcal{I}(-1)} \colon D^b(X) \to D^b(S)$$

be the associated Fourier-Mukai functor. As seen in the previous talk, Fourier-Mukai functors have right-adjoints which are themselves Fourier-Mukai. Explicitly, the right adjoint Ψ to $\Phi_{\mathcal{I}(-1)}$ has kernel $\mathcal{I}(-1)^\vee \otimes p_X^*\mathcal{O}_X(-3)[4]$.

Ψ lands in the Kuznetsov component

To prove this, we need a result describing the geometry of the correspondence $\Sigma \subset S \times X$.

Proposition 4 1. *The fibers of $\Sigma \to S$ are either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$ and the embedding $\Sigma_p \to X \to \mathbb{P}^5$ describes Σ_p as a quartic normal scroll.* [2]
In particular, the restriction maps $H^0(\mathbb{P}^5, \mathcal{O}(n)) \to H^0(\Sigma_p, \mathcal{O}_{\Sigma_p}(n))$ are bijections.

2. *Let $P \in S$ and let Σ_P be the corresponding fiber, then*

$$\chi(\mathrm{RHom}_{D^b(X)}(\mathcal{O}_{\Sigma_P}, \mathcal{O}_{\Sigma_P})) = 10$$

Recall that $\mathcal{A}_X$ is defined as the right orthogonal of the category spanned by $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$, hence it suffices to show that $\hom_{D^b(X)}(\mathcal{O}_X(j), \Psi(E)) = 0$ for all $j = 0, 1, 2$ and for all $E \in D^b(S)$. As Ψ is right-adjoint, this is equivalent (by Yoneda) to showing that $\Phi_{\mathcal{I}(-1)}(\mathcal{O}_X(j)) = 0$ for all $j = 0, 1, 2$.
Consider the short exact sequence of sheaves on $S \times X$

$$0 \to \mathcal{I}_\Sigma \to \mathcal{O}_{X \times S} \to \mathcal{O}_\Sigma \to 0$$

Twisting by $\mathcal{O}_X(-1)$, we see that to show $\Phi_{\mathcal{I}(-1)}(\mathcal{O}_X) = 0$, it suffices to show the vanishing $p_{S*}(\mathcal{O}_{X\times S}(-1)) = p_{S*}(\mathcal{O}_\Sigma(-1)) = 0$. Using cohomology and base change, it suffices to show that the cohomology of the fibers vanishes

- By [Huy23, lemma 1.1.7] and Serre duality, $H^\bullet(X, \mathcal{O}_X(-1)) = 0$.

[2] See [Huy23, Sec. 6.2.6] for a more in depth discussion on these

- By [Huy23, lemma 6.2.20], the fibers of the projection $\Sigma \to S$ are quartic normal scrolls so either isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_2$.
  If the fiber over p is $\Sigma_p = \mathbb{P}^1 \times \mathbb{P}^1$, we can use Künneth and the fact that $p_X^*\mathcal{O}_X(-1)|_{\Sigma_p} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,-1)$ to obtain the vanishing

$$H^\bullet(\Sigma_p, \mathcal{O}(-2,-1)) = 0.$$

The cases $j = 1,2$ are proven similarly using normality of $\Sigma_p \subset \mathbb{P}^5$ and we omit them here.
The case where the fibers are $\mathbb{F}_2$ is the exercise for today's talk.

Ψ is fully faithful

To prove this, we will use the following general result due to Bondal and Orlov.

Theorem 5 *Let X, Y be smooth projective varieties and let $\Phi_P \colon D^b(X) \to D^b(Y)$ be the Fourier-Mukai Functor associated to $P \in D^b(X \times Y)$. Then Φ_P is fully faithful if an only if for any two points $x, y \in X$, we have*

$$\hom(\Phi_P(k(x)), \Phi_P(k(y))[i]) = \begin{cases} k \text{ if } x = y \text{ and } i = 0 \\ 0 \text{ if } x \neq y \text{ or } i < 0 \text{ or } i > \dim X \end{cases}.$$

We know that Ψ is the Fourier-Mukai functor whose associated kernel is $\mathcal{I}(-1)^\vee \otimes p_X^*(\mathcal{O}_X(-3))[4]$, hence for $P \in S$, we find

$$\Psi(k(P)) = \mathcal{I}_{\Sigma_P}(-1)(-3)[4]$$

Why is $\mathcal{I}_{\Sigma_P}$ simple?

Let $P_1, P_2 \in S$, we compute

$$\mathrm{Ext}^i_{\mathcal{A}_X}(\Psi(k(P_1)), \Psi(k(P_2))) = \mathrm{Ext}^i_{\mathcal{A}_X}(\mathcal{I}_{\Sigma_{P_2}}, \mathcal{I}_{\Sigma_{P_1}})$$

Now clearly, $\mathrm{Ext}^{\bullet<0}_{\mathcal{A}_X}(\mathcal{I}_{\Sigma_{P_2}}, \mathcal{I}_{\Sigma_{P_1}}) = 0$ and hence, because $\mathcal{A}_X$ is 2-Calabi-Yau, $\mathrm{Ext}^{\bullet>0}_{\mathcal{A}_X}(\mathcal{I}_{\Sigma_{P_2}}, \mathcal{I}_{\Sigma_{P_1}}) = 0$.
Since $\Sigma_{P_1}, \Sigma_{P_2}$ are disjoint, $\hom(\mathcal{I}_{\Sigma_{P_1}}, \mathcal{I}_{\Sigma_{P_2}}) = 0$ Why? Maybe write out ses for P_1 and take les in ext Hence, by Serre duality $\mathrm{Ext}^2_{\mathcal{A}_X}(\mathcal{I}_{\Sigma_{P_1}}, \mathcal{I}_{\Sigma_{P_2}}) = 0$.
Hence, it suffices to show that $\dim \mathrm{Ext}^1(\mathcal{I}_{\Sigma_{P_2}}, \mathcal{I}_{\Sigma_{P_1}}) = 0$.
To prove this, we consider the Euler characteristic of the complex $R\hom(\mathcal{I}_{\Sigma_{P_2}}, \mathcal{I}_{\Sigma_{P_1}})$, by

constancy of the Euler characteristic in flat families

$$\begin{aligned}\chi(\mathcal{I}_{\Sigma_{P_1}}, \mathcal{I}_{\Sigma_{P_2}}) &= \chi(\mathcal{I}_{\Sigma_P}, \mathcal{I}_{\Sigma_P}) \\ &= \chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(\mathcal{O}_X, \mathcal{O}_{\Sigma_P}) - \chi(\mathcal{O}_{\Sigma_P}, \mathcal{O}_X) + \chi(\mathcal{O}_{\Sigma_P}, \mathcal{O}_{\Sigma_P}) \\ &= 0\end{aligned}$$

Should I present this?

2.1 Ψ is essentially surjective

Recall that $\mathcal{A}_X$ is indecomposable and suppose that Ψ is not essentially surjective, then the image is a full admissible subcategory of $\mathcal{A}_X$ and there is a natural semi-orthogonal decomposition

$$\mathcal{A}_X = \left\langle {}^{\perp}\Psi(D^b(S)), \Psi(D^b(S)) \right\rangle = \left\langle \Psi(D^b(S)), {}^{\perp}\Psi(D^b(S))[2] \right\rangle.$$

This contradicts the indecomposability of $\mathcal{A}_X$, since Ψ is not trivial, it must be essentially surjective.

3 Twisted Derived Categories

We will now introduce a new triangulated category called the *twisted derived category* of sheaves. This category: $D^b(S, \alpha)$ will depend on the choice of a projective variety S and on a cohomology class $\alpha \in H^2_{ét}(S, \mathbb{G}_m)$.
$D^b(S, \alpha)$ will be the derived bounded category of an abelian category $\mathrm{Coh}(S, \alpha)$ which we now construct.
There are two different but equivalent ways of defining $\mathrm{Coh}(S, \alpha)$.

Definition 3 (Twisted coherent sheaves) *Let $\alpha \in H^2_{ét}(S, \mathbb{G}_m)$, let $U \to S$ be an étale cover such that α is represented by a cocycle $a \in \Gamma(U \times_S U \times_S U, \mathbb{G}_m)$.*
A α-twisted sheaf $(\mathcal{F}, \phi)$ on S is a sheaf $\mathcal{F}$ on U together with an isomorphism

$$\phi\colon \mathrm{pr}_1^* \mathcal{F} \to \mathrm{pr}_2^* \mathcal{F}$$

satisfying the cocycle conditions "up to" α, ie. such that

$$\mathrm{pr}_{1,2}^* \phi \circ \mathrm{pr}_{2,3}^* \phi = a \cdot \mathrm{pr}_{1,3}^* \phi.$$

One would now have to check that this construction does not depend on the choice of cocycle/étale cover.

Definition 4 (Morphisms of twisted sheaves) *Let $(\mathcal{F}, \phi_{\mathcal{F}}), (\mathcal{G}, \phi_{\mathcal{G}})$ be α-twisted sheaves on S and suppose that they are defined on the same étale cover $U \to S$, a morphism from $(\mathcal{F}, \phi_{\mathcal{F}}) \to (\mathcal{G}, \phi_{\mathcal{G}})$ is a morphism of sheaves $\mathcal{F} \to \mathcal{G}$ on U such that*

$$\phi_{\mathcal{G}} \circ \mathrm{pr}_1^* \mathcal{F} = \mathrm{pr}_2^* \circ \phi_{\mathcal{F}}.$$

Let $\mathcal{E}$ be a locally-free α-twisted and let $\mathcal{A} = \mathcal{E} \otimes \mathcal{E}^{\vee} = \mathrm{End}\,(\mathcal{E})$, this has the natural structure of a sheaf of algebras.
Then $\mathcal{A}$ is no longer twisted and descends to a sheaf of algebras on S which we also call $\mathcal{A}$. There is an equivalence of categories

$$\mathrm{Coh}(S, \alpha) \simeq \mathrm{Coh}(S, \mathcal{A}),$$

where the left hand side is the category of right $\mathcal{A}$-modules.
Finally, we can associate to the twisted locally free sheaf $\mathcal{E}$, it's projectivisation $\mathbb{P}(\mathcal{E})$.
The isomorphism ϕ then gives étale descent data for $\mathbb{P}(\mathcal{E})$ and we call $\mathbb{P}(\mathcal{E})$ the corresponding scheme over S. In fact, there are bijections

$$H^2_{\text{ét}}(S, \mathbb{G}_m) \simeq \{\text{ equivalence classes of Azumaya algebras }\} \simeq \{\text{ equivalence classes of } \mathbb{P}^n\text{-fibrations }\}$$

4 Cubic Fourfolds containing a Plane

Let X be a general cubic fourfold containing a plane $P \subset X$, in this last part we will sketch a proof of the following theorem.

Theorem 6 *There exists a K3-surface S and a Brauer class $\alpha \in \mathrm{Br}(S)$ such that*

$$D^{\flat}(S, \alpha) \simeq \mathcal{A}_X.$$

Let us start with the geometric setup that was already covered in a previous talk. Let $p \colon X \dashrightarrow \mathbb{P}^2$ be the projection from P, this induces a map $\tilde{X} \coloneqq \mathrm{Bl}_P X \to \mathbb{P}^2$ which is a fibration in 2-dimensional quartics.
We make the simplifying assumption here that the discriminant divisor of p is smooth[3].
Let $\tilde{F}$ be the relative Fano variety of $\tilde{X} \to \mathbb{P}^2$, ie. $\tilde{F}$ is a closed subscheme of $F(\tilde{X})$ containing only the points corresponding to lines contained in fibers of p.
There is a natural map $q \colon \tilde{F} \to \mathbb{P}^2$ sending the class of a line $[L] \in \tilde{F} \to p([L]) \in \mathbb{P}^2$.

Proposition 7 *The $\mathcal{O}$-connected part of the Stein factorization of q corresponds to a $\mathbb{P}^1$-fibration $\tilde{F} \to S$ with S a K3-surface.*

[3] Ie. the locus of non-smooth fibers

We can now infer the general strategy of proof of theorem 6, the $\mathbb{P}^1$-fibration determines a Brauer class $\alpha \in \mathrm{Br}(S)$ and we should use the two maps $S \to \mathbb{P}^2$ and $\tilde{X} \to \mathbb{P}^2$ to relate $D^\flat(S, \alpha)$ to $\mathcal{A}_X$.

Proof ((Sketch) of theorem 6) The Orlov formula for blow-ups gives a semi-orthogonal decomposition of $D^\flat(\tilde{X})$

$$D^\flat(\tilde{X}) = \left\langle D^\flat(P), \mathcal{A}_X, \mathcal{O}(-2E), \mathcal{O}(-E), \mathcal{O} \right\rangle$$

where E is the exceptional divisor.
A general formula for quadric fibrations gives a second semi-orthogonal decomposition

$$D^\flat(\tilde{X}) = \left\langle D^\flat(\mathbb{P}^2, \mathcal{B}), D^\flat(\mathbb{P}^2) \otimes \mathcal{O}(-E), D^\flat(\mathbb{P}^2) \right\rangle .$$

Where $\mathcal{B}$ is the pushforward to $\mathbb{P}^2$ of the Azumaya algebra on S corresponding to α. Since the map $S \to \mathbb{P}^2$ is finite, it induces an equivalence of categories $D^\flat(S, \alpha) \simeq D^\flat(\mathbb{P}^2, \mathcal{B})$, concluding the proof. $\square$