Exercise for Week 4

David Wiedemann

28 mars 2021

1

Let us first count the case $n = 3$.
We denote each point by it's index, here are all the possibilities to pair two indices together :

$$\{(1,2),(3,4),(5,6)\}$$
$$\{(1,2),(3,6),(4,5)\}$$
$$\{(1,4),(2,3),(5,6)\}$$
$$\{(1,6),(2,3),(4,5)\}$$
$$\{(1,6),(2,5),(3,4)\}$$

Hence, there are 5 possibilities for the case $n = 3$.

2

Let us search a recursive formula.
We will denote by $a(n)$ the number of ways to do this for a circle with $2n$ labelled points.
We define $a(0) = 1$.

We first index all of the $2n$ points from 1 to $2n$.

First we choose what point to connect with point 1, there are exactly n points to choose, since choosing a point with an uneven index will force 2 lines to intersect.
Hence, the coordinates of the chosen point have to be even, suppose we choose the point with coordinates $2k$.
Now we are left with two subset of points, one contains $2n - 2k$ points, the other one contains $2k - 2$ points. Hence, the formula of arrangements for a fixed k is given by

$$a(n-k) \cdot a(k-1)$$

Summing over all possibilities for k yields

$$a(n) = \sum_{k=1}^{n} a(n-k) \cdot a(k-1)$$

This however, simply is the definition of the Catalan Numbers and hence we conclude that there are b_n ways to choose n pairs of points on a circle such that their segments do not intersect.