Practice Midterm

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This is a practice exam to give you an indication as to the level of difficulty and length to expect on the final exam. You should use no additional aids other than the provided CheatSheet (no calculators, books, or notes). You should show your work and give explanations where necessary.

NOTE: To give you more practice, this exam has been designed to take 3 hours. The actual midterm will only be 1.5 hours. Hence you should except the actual midterm to be approximately half the length of this one.

Problem 1

A factory produces two different products: X and Y from raw materials A, B, C

1. one unit of X requires one unit of A and one unit of B
2. one unit of Y requires one unit of B and two units of C
3. raw material B needs to be washed before it can be used — washing one unit of B takes one minute
4. there is only one washer, and it can only run 16 hours in one day
5. at most 1200 kilograms of raw material can be imported each day
6. one unit of $A/B/C$ weighs 4/3/2 kilograms (respectively)
7. at most 130 units of X and 100 units of Y can be exported each day
8. one unit of X sells for 6 clams, and one unit of Y sells for 9 clams

Formulate (with explanation!) the problem of maximizing the total number of clams one can earn in a single day as a linear program in two variables.

Problem 2

Let $\mathcal{P}$ and $\mathcal{D}$ be the dual linear programs

$$\mathcal{P} = \min\{\mathbf{c}_1 \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}_1, \mathbf{x} \geq \mathbf{0}\} \quad \text{and} \quad \mathcal{D} = \max\{\mathbf{b}_1 \cdot \boldsymbol{\lambda} : \boldsymbol{\lambda}^\mathsf{T}\mathbf{A} \leq \mathbf{c}_1^\mathsf{T}\}$$

where $\mathcal{P}$ has a finite optimal solution $\mathbf{x}_1$ and $\mathcal{D}$ has a finite optimal solution $\boldsymbol{\lambda}_1$.

(a) Show that if $\mathbf{x}_2$ is an optimal solution to the linear program

$$\mathcal{P}' = \min\{\mathbf{c}_2 \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}_1, \mathbf{x} \geq \mathbf{0}\}$$

then $(\mathbf{c}_2 - \mathbf{c}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1) \leq 0$.

(b) Show that if $\mathbf{x}_3$ is the optimal solution to the linear program

$$\mathcal{P}'' = \min\{\mathbf{c}_1 \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}_2, \mathbf{x} \geq \mathbf{0}\}$$

then $\boldsymbol{\lambda}_1 \cdot (\mathbf{b}_2 - \mathbf{b}_1) \leq \mathbf{c}_1 \cdot (\mathbf{x}_3 - \mathbf{x}_1)$.

Problem 3

Consider the linear program in *inequality* standard form

$$\mathcal{P} = \min \{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Now assume that we are given a solution $\mathbf{z}$ which is feasible for P. We can use $\mathbf{z}$ to form a new linear program

$$\mathcal{Q}(\mathbf{z}) = \min \left\{\mathbf{c} \cdot \mathbf{y} : \mathbf{A}_{\mathrm{act}(\mathbf{z})}\mathbf{y} \geq \mathbf{0}\right\}$$

where $\mathbf{y} \in \mathbb{R}^n$ and $\mathbf{A}_{\mathrm{act}(\mathbf{z})}$ consists of the submatrix of $\mathbf{A}$ whose rows are active at $\mathbf{z}$.

(a) Show that $\mathcal{Q}(\mathbf{z})$ always has at least one feasible solution.

(b) Show that if the optimal value of $\mathcal{Q}(\mathbf{z})$ is 0 then $\mathbf{z}$ is an optimal solution for $\mathcal{P}$.

Hint: Assume (for contradiction) $\mathbf{z}$ is not optimal in $\mathcal{P}$ and consider the vector $\mathbf{y} = \mathbf{x} - \mathbf{z}$ where $\mathbf{x}$ is a solution that is better than $\mathbf{z}$.

(c) Show that if $\mathbf{z}$ is an optimal solution for $\mathcal{P}$ then the optimal value of $\mathcal{Q}(\mathbf{z})$ is 0.

Problem 4

For a set S, denote as $\operatorname{conv} S$ the convex hull of the points in S. Let $X = \{\mathbf{x}_1, \ldots \mathbf{x}_k\}$ be a collection of points in $\mathbb{R}^n$ with $k > n$ and let $\mathbf{y} \in \operatorname{conv} X$

1. Show that there exists a subset $K \subseteq X$ with $|K| \leq n + 1$ such that $\mathbf{y} \in \operatorname{conv} K$.
2. For all n, give an example of a set X and a point $\mathbf{y} \in \operatorname{conv} X$ for which *no subset* of X with size n contains $\mathbf{y}$ in the convex hull.

Problem 5

Consider the following LP:

$$
\begin{aligned}
\min \quad & 6x - 2y - 3z \\
\text{s.t.} \quad & x - 4y - 2z \le 5 \\
& -2x + 6y + z \le 2 \\
& -2x + 3y - 4z \le 1 \\
& x \ge 0 \\
& y \ge 0 \\
& z \ge 0
\end{aligned}
$$

Solve this problem using the simplex algorithm with Bland's rule. Your initial BFS should be the one that corresponds to the solution $x = y = z = 0$. For each iteration of the simplex algorithm, indicate the current basis and the corresponding BFS.

At the end, you should provide the optimal value of $\mathbf{x}$, the value of the objective function, and a certificate that shows that this value is optimal.

Here is a list of some of the invertible submatrices and their inverses (so you don't have to compute them). Note that you may not need all of them.

$$
\begin{bmatrix} 1 & -4 & 1 \\ -2 & 6 & 0 \\ -2 & 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/2 & -1 \\ 0 & 1/3 & -1/3 \\ 1 & 5/6 & -1/3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} -4 & -2 & 1 \\ 6 & 1 & 0 \\ 3 & -4 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 4/27 & 1/27 \\ 0 & 1/9 & -2/9 \\ 1 & 22/27 & -8/27 \end{bmatrix} \qquad \begin{bmatrix} -4 & 1 & 0 \\ 6 & 0 & 1 \\ 3 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1/3 \\ 1 & 0 & 4/3 \\ 0 & 1 & -2 \end{bmatrix}
$$

$$
\begin{bmatrix} -4 & 1 & 0 \\ 6 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/6 & 0 \\ 1 & 2/3 & 0 \\ 0 & -1/2 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ -4 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 4 & 1 \end{bmatrix}
$$