Linear/Integer Programming Cheat Sheet

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This document is a compilation of all of the definitions and major results that we have talked about in class. The purpose is to make it easier to find things (without having to look them up in the book or the notes). If there is anything that you would like to add (or corrections!), please suggest it on the Piazza page with the `cheat_sheet_additions` tag.

Contents

1 Basic definitions

Given an optimization problem $\max\{f(\mathbf{x}) : \mathbf{x} \in S\}$ for some $S \subseteq \mathbb{R}^n$, we will use the terms

solution
- **solution**: any vector in $\mathbb{R}^n$

feasible solution
- **feasible solution**: any vector in S

optimal solution
- **optimal solution**: a feasible solution $\mathbf{y}$ such that $f(\mathbf{x}) \leq f(\mathbf{y})$ for all $\mathbf{x} \in S$.

Given an optimization problem $\mathcal{P}$, we use

feas($\mathcal{P}$)
- feas($\mathcal{P}$) to denote the set of all feasible solutions

opt($\mathcal{P}$)
- opt($\mathcal{P}$) to denote the optimal value (the value at an optimal solution)

linear program (LP) A **linear program (LP)** is an optimization problem of the form

$$\begin{aligned} \min / \max \ & \mathbf{c}{\cdot}\mathbf{x} \\ \text{subject to} & \\ \mathbf{A}_i \cdot \mathbf{x} \leq b_i & \qquad\qquad i \in I \\ \mathbf{A}_j \cdot \mathbf{x} \geq b_j & \qquad\qquad j \in J \\ \mathbf{A}_k \cdot \mathbf{x} = b_k & \qquad\qquad k \in K \\ x_\ell \geq 0 & \qquad\qquad \ell \in L \\ x_m \leq 0 & \qquad\qquad m \in M \\ x_n \text{ free} & \qquad\qquad n \in N \end{aligned}$$

equality standard form An LP is in **equality standard form** if it has the form

$$\begin{aligned} \min / \max \ & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

inequality standard form and **inequality standard form** if it has the form

$$\begin{aligned} \min / \max \ & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned}$$

equivalent LPs Linear programs $\mathcal{P}$ and $\mathcal{P}'$ are called **equivalent LPs** if there is a way to turn every feasible solution for $\mathcal{P}$ into a feasible solution for $\mathcal{P}'$ with the same cost, *and* there is a way to turn every feasible solution in $\mathcal{P}'$ into a feasible solution for $\mathcal{P}$ with the same cost[1].

Theorem 1. *Every linear program P can be turned into an equivalent LP P'*

- *in equality standard form problem, or*
- *in inequality standard form problem*

by a sequence of reversible transformations.

2 Polyhedra

Given a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ and scalar t,

affine hyperplane
- an **affine hyperplane** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{x} = t\}$

halfspace
- A **halfspace** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{x} \geq t\}$

polyhedron A **polyhedron** is an intersection of a finite number of half spaces. That is, it is a set of the form

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$$

convex where $\mathbf{A}$ is an $m \times n$ matrix. A set S is called **convex** if for any $\mathbf{x}, \mathbf{y} \in S$ and for any $\lambda \in [0,1]$, $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in S$.

convex combination Given a collection of points, $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\} \subseteq \mathbb{R}^n$, a point $\mathbf{y}$ is called a **convex combination** of the vectors in X if there exist $\lambda_1, \ldots, \lambda_k \geq 0$ such that

$$\mathbf{y} = \sum_i \lambda_i \mathbf{x}_i \quad \text{and} \quad \sum_i \lambda_i = 1$$

convex hull The **convex hull** of a collection of points is the set of all convex combinations of the points.

[1] This means they each have the same maximum and minimum value. However, they will *not* in general have the same polytope.

Theorem 2. *The following are convex:*

- *An intersection of convex sets*
- *halfspaces (and therefore polyhedra)*
- *the convex hull of a finite collection of points*

active

An inequality $f(\mathbf{x}) \leq c$ is called **active** or binding (or tight) at a point $\mathbf{y}$ if $f(\mathbf{y}) = c$.

Let $P \subseteq \mathbb{R}^n$ be a polyhedron.

vertex

1. A point $\mathbf{x} \in P$ is called a **vertex** if there exists a linear program for which $\mathbf{x}$ is a unique solution. That is, there exists a vector $\mathbf{c}$ such that $\mathbf{c} \cdot \mathbf{x} > \mathbf{c} \cdot \mathbf{y}$ for all $\mathbf{y} \neq \mathbf{x} \in P$.

extreme point

2. A point $\mathbf{y} \in P$ is called an **extreme point** if it is not the convex combination of any two other points in P. That is, the equation $\mathbf{x} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{z}$ has no solution satisfying $\mathbf{y}, \mathbf{x} \neq \mathbf{x}$ and $\lambda \in [0, 1]$.

basic solution

basic feasible solution (BFS)

3. A point $\mathbf{z} \in \mathbb{R}^n$ is called a **basic solution** if all equality constraints of P are active at $\mathbf{z}$ and a total of n linearly independent (equality or inequality) constraints are active at $\mathbf{z}$. If, in addition, $\mathbf{z} \in P$, then it is called a **basic feasible solution (BFS)**.

Theorem 3. *If P is a polyhedron and $\mathbf{x} \in P$, then the following are equivalent:*

1. $\mathbf{x}$ *is a vertex*
2. $\mathbf{x}$ *is an extreme point*
3. $\mathbf{x}$ *is a basic feasible solution*

contains a line

A polyhedron P **contains a line** if there exists a vector $\mathbf{x}$ and a nonzero vector $\mathbf{d}$ such that $\mathbf{x} + \lambda\mathbf{d} \in P$ for all $\lambda \in \mathbb{R}$.

Theorem 4. *Let $P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\}$ be a nonempty polyhedron. Then the following are equivalent:*

1. *P has at least one extreme point*
2. *$\mathbf{A}$ has rank n*
3. *P does not contain a line*

Theorem 5. *If the feasible set of an LP has at least one extreme point and the LP has a finite optimum, then there exists an extreme point which achieves the optimal value.*

adjacent

degenerate

non-degenerate

Two basic solutions $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are called **adjacent** if there exists a set of $n - 1$ linearly independent constraints which are active at both $\mathbf{x}$ and $\mathbf{y}$. A basic solution $\mathbf{x} \in \mathbb{R}^n$ is called **degenerate** if more than n constraints are active at $\mathbf{x}$. A basic solution with *exactly* n constraints is called **non-degenerate**.

Lemma 6 (Degeneracy for equality standard form)**.** *If A is an $m \times n$ matrix with full row rank, and P is the polyhedron*

$$P = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

then a basic solution $\mathbf{x} \in P$ is degenerate if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ and more than $n - m$ of the variables x_i are 0.

3 Simplex

Let $\mathcal{P}$ be a linear program and $\mathbf{x}$ a feasible solution for P. A vector $\mathbf{d}$ is called a **feasible direction** if there exists $\theta > 0$ for which $\mathbf{x} + \theta\mathbf{d} \in \text{feas}(\mathcal{P})$. If $\mathcal{P}$ is in equality standard form and β is a feasible basis, then

feasible direction

$$\overline{\mathbf{c}}^{\intercal} = \mathbf{c}^{\intercal} - \mathbf{c}_{\beta}^{\intercal}\mathbf{B}^{-1}\mathbf{A}$$

reduced costs

are the **reduced costs** for β.

Theorem 7. *Let P be a minimization problem in equality standard form with feasible solution $\mathbf{x}$.*

- *If all reduced costs are nonnegative, then $\mathbf{x}$ is optimal*
- *If $\mathbf{x}$ is optimal and nondegenerate, then all reduced costs are nonnegative*

3.1 Simplex step

A single step of the simplex method for

- a generic linear programming problem where you have a polyhedron P and a linear objective function f
  1. Start at a vertex/BFS/extreme point v of P
  2. For each edge touching v in the polyhedron, see what happens to f when you move down that edge.
     (a) If none of the edges cause f to improve, stop ($\mathbf{x}$ is optimal)
     (b) Otherwise pick* an edge e_j that, if you were to go down that edge, you would cause f to improve
  3. Go down that edge until you hit the first new BFS
     (a) If you see you will never hit a new BFS, stop (the problem is unbounded)
     (b) Otherwise, move to this new vertex/BFS/extreme point v'
  4. Iterate.

simplex step =

- a minimization problem in equality standard form:

$$\begin{aligned} \min\ & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t. } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

  with $\mathbf{A}$ short and fat.

  1. Start with a (column) basis β corresponding to basis matrix B – the associated BFS is $\mathbf{x}_{\beta} = \mathbf{B}^{-1}\mathbf{b}$ with all nonbasis entries of $\mathbf{x}$ set to 0.
  2. Compute the reduced costs $\overline{\mathbf{c}} = \mathbf{c}^{\intercal} - \mathbf{c}_{\beta}^{\intercal}\mathbf{B}^{-1}\mathbf{A}$.
     (a) If all reduced costs are nonnegative, stop ($\mathbf{x}$ is optimal)
     (b) Otherwise pick* j for which $\overline{c}_j < 0$.
  3. Compute $\mathbf{u} = \mathbf{B}^{-1}\,\text{col}_j(\mathbf{A})$
     (a) If $\mathbf{u} \leq 0$, stop (the problem is unbounded)
     (b) Otherwise set $\ell = \arg\min_{k:u_k>0} \frac{x_{\beta(k)}}{u_k}$ (if there is a tie, then pick* one).

4. Form a new basis matrix by replacing column $\beta(\ell)$ with column j and iterate.

cycling

Note: generic versions of pick* may cause issues with degeneracies. In particular, poor choices of pick* can result in **cycling** — a situation where simplex can consider the same bases over and over again.

Bland's Rule

Bland's Rule: Any time you pick*, pick the one with the smallest subscript.

Theorem 8. *Bland's Rule avoids cycling.*

3.2 Finding an initial BFS

auxiliary LP

Given a linear program $\mathcal{P}$, an **auxiliary LP** is a new linear program $\mathcal{P}'$ that you build in order to find an initial BFS for $\mathcal{P}$.

initial BFS =

- For $\mathcal{P}$ in equality standard form:

  1. Multiply rows by -1 when necessary to get $\mathbf{b} \geq \mathbf{0}$.
  2. Add cheating variables $\mathbf{w}$, and solve

$$\begin{aligned} \mathcal{P}' = \min \quad & \mathbf{1} \cdot \mathbf{w} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{w} = \mathbf{b} \\ & \mathbf{x},\ \mathbf{w} \geq \mathbf{0} \end{aligned}$$

  using simplex with initial BFS $(\mathbf{w}, \mathbf{x}) = (\mathbf{b}, \mathbf{0})$.

     (a) If $\text{opt}(\mathcal{P}') > 0$ stop (original problem is infeasible)

     (b) Otherwise, continue to do simplex steps until only $\mathbf{x}$ in the basis

  3. Use optimal basis for $\mathcal{P}'$ as initial basis in original problem

4 Duality

duality chart

minimize $\mathbf{c} \cdot \mathbf{x}$	maximize $\boldsymbol{\lambda} \cdot \mathbf{b}$
$\text{row}_i(\mathbf{A}) \cdot \mathbf{x} \geq b_i$	$\lambda_i \geq 0$
$\text{row}_i(\mathbf{A}) \cdot \mathbf{x} \leq b_i$	$\lambda_i \leq 0$
$\text{row}_i(\mathbf{A}) \cdot \mathbf{x} = b_i$	λ_i free
$x_j \geq 0$	$\text{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} \leq c_j$
$x_j \leq 0$	$\text{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} \geq c_j$
x_j free	$\text{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} = c_j$

Figure 1: The translation between primal and dual for general LP's.

Important example:

$$\begin{aligned} \mathcal{P} = \min \quad & \mathbf{c} \cdot \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{D} = \max \quad & \mathbf{b} \cdot \boldsymbol{\lambda} \\ \text{s.t.} \quad & \mathbf{A}^{\mathsf{T}}\boldsymbol{\lambda} = \mathbf{c} \\ & \boldsymbol{\lambda} \geq \mathbf{0} \end{aligned}$$

are duals of each other.

duality chart 2

	Finite Optimum	Unbounded	Infeasible
Finite Optimum	YES	NO	NO
Unbounded	NO	NO	YES
Infeasible	NO	YES	YES

Figure 2: Which combinations are possible between Primal and Dual?

Theorem 9. *The duality map is an involution. That is, if $\mathcal{P}$ is a minimization LP and we*

1. *Find $\mathcal{D}$, the (maximization) dual of $\mathcal{P}$.*
2. *Turn $\mathcal{D}$ into a minimization LP $\mathcal{D}'$ by multiplying the cost function by -1.*
3. *Find $\mathcal{Q}$ the (maximization) dual of $\mathcal{D}'$.*
4. *Turn $\mathcal{Q}$ into a minimization LP $\mathcal{Q}'$ by multiplying the cost function by -1.*

Then $\mathcal{Q}' = \mathcal{P}$.

weak duality **Theorem 10** (Weak duality). *Let $\mathcal{P}$ be a minimization LP and $\mathcal{D}$ its (maximization) dual. If $\mathbf{x} \in \mathrm{feas}(\mathcal{P})$ and $\boldsymbol{\lambda} \in \mathrm{feas}(\mathcal{D})$ then*

$$\mathcal{D}(\boldsymbol{\lambda}) \leq \mathcal{P}(\mathbf{x})$$

Corollary 11. *Let $\mathcal{P}$ be a minimization LP and $\mathcal{D}$ its (maximization) dual.*

- *If $\mathcal{P}$ is unbounded, then $\mathcal{D}$ must be infeasible.*
- *If $\mathcal{D}$ is unbounded, then P must be infeasible.*
- *If $\mathbf{x} \in \mathrm{feas}(\mathcal{P})$ and $\boldsymbol{\lambda} \in \mathrm{feas}(\mathcal{D})$ and $\mathcal{D}(\boldsymbol{\lambda}) = \mathcal{P}(\mathbf{x})$, then $\mathbf{x}, \boldsymbol{\lambda}$ are optimal*

associated to β Every basis β determines a pair of solutions **associated to** β.

- A primal solution $\mathbf{x}$:

$$\mathbf{x}_\beta = \mathbf{B}^{-1}\mathbf{b}_\beta \quad \text{and} \quad x_i = 0 \text{ for all other rows}$$

- A dual solution $\boldsymbol{\lambda}$:

$$\boldsymbol{\lambda}_\beta^\intercal = \mathbf{c}_\beta^\intercal \mathbf{B}^{-1} \quad \text{and} \quad \lambda_i = 0 \text{ for all other columns}$$

Note: it is possible for $\mathbf{x}$ to be feasible (or not) in the primal and also possible for $\boldsymbol{\lambda}$ to be feasible (or not) in the dual.

Corollary 12. *If $\mathbf{x}$ and $\boldsymbol{\lambda}$ are the primal/dual solutions associated to a basis β and $\mathbf{x}$ is primal feasible and $\boldsymbol{\lambda}$ is dual feasible, then $\mathbf{x}$ and $\boldsymbol{\lambda}$ are optimal*

strong duality **Theorem 13** (Strong duality). *If a linear program P has a finite optimal solution $\mathbf{x}$, then its dual $\mathcal{D}$ has a finite optimal solution $\boldsymbol{\lambda}$, and $\mathcal{D}(\boldsymbol{\lambda}) = \mathcal{P}(\mathbf{x})$.*

complementary slackness **Theorem 14** (Complementary Slackness). *If $\mathbf{x}$ is primal feasible and $\boldsymbol{\lambda}$ is dual feasible*

- $\lambda_i(\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} - b_i) \geq 0$ *for all i*
- $x_j(c_j - \mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda}) \geq 0$ *for all j*
- $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *are optimal if and only if each inequality holds with equality*

Remark: Weak/strong duality and complementary slackness hold for *all* pairs of optimal solutions — not just ones associated to the same (or any) basis.

4.1 Things equivalent to LP duality

Farkas Lemmas

Farkas Lemmas: A statement using quantifiers that can be proved using optimization techniques (usually duality). Example:

Lemma 15 (Farkas)**.** *For every $m \times n$ matrix $\mathbf{A}$ and vector $\mathbf{b} \in \mathbb{R}^m$, exactly one of the following is true:*

1. *There exists $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$*
2. *There exists $\boldsymbol{\lambda}$ such that $\mathbf{A}^\intercal \boldsymbol{\lambda} \geq \mathbf{0}$ and $\mathbf{b} \cdot \boldsymbol{\lambda} < 0$*

theorem of alternatives

This type of statement is called a **theorem of alternatives** (exactly one must be true).

separating hyperplane

Lemma 16 (separating hyperplane)**.** *Let S be a non-empty, closed, convex set and let $\mathbf{x}$ be a point not in S. Then there exists a vector $\mathbf{c}$ and scalar t such that*

$$\mathbf{c} \cdot \mathbf{x} \geq t \quad \text{and} \quad \mathbf{c} \cdot \mathbf{y} < t$$

for all points $\mathbf{y} \in S$.

4.2 Game Theory

For any function $f(x, y)$, we play a game: player X picks a value of x and Y picks a value of y and then a (possibly negative) amount of $f(x, y)$ clams is transferred from X to Y. So X wants big numbers and Y wants small.

Two natural quantities associated to such games are:

maximin

- the **maximin** value: (if X plays first, Y gets to see it, this is best X can do)[2]

$$\underline{f} = \max_{x \in X} \; \min_{y \in Y} f(x, y)$$

minimax

- the **minimax** value: (if Y plays first, X gets to see it, this is best Y can do)

$$\overline{f} = \min_{y \in Y} \; \max_{x \in X} f(x, y)$$

Lemma 17. *For any function f, $\underline{f} \leq \overline{f}$.*

optimal strategy

A move x_* is an **optimal strategy** for X if

$$\underline{f} = \min_{y \in Y} \; f(x_*, y)$$

and y_* is an optimal strategy for Y if

$$\overline{f} = \max_{x \in X} f(x, y_*).$$

mixed strategy game

A **mixed strategy game** consists of players X and Y picking probability distributions over all possible moves. That is, if X has m possible moves and Y has n possible moves, then X picks $\mathbf{x} \in \Delta_m$, and Y picks $\mathbf{y} \in \Delta_n$ where

$$\Delta_k = \left\{ \mathbf{x} \in \mathbb{R}^k : \sum_i x_i = 1, \mathbf{x} \geq 0 \right\}.$$

payout matrix

The payout of a mixed strategy game is $f_{\mathbf{M}}(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\intercal M \mathbf{y}$ where M is the **payout matrix**.

[2] Remember we read these as if the players are playing from left to right.

Theorem 18 (Von Neumann)**.** *Mixed strategy games have optimal strategies for both players. That is, for any* $m \times n$ *matrix* $\mathbf{M}$

$$\underline{f_{\mathbf{M}}} = \max_{\mathbf{x} \in \Delta_m} \min_{\mathbf{y} \in \Delta_n} \mathbf{x}^\intercal \mathbf{M} \mathbf{y} = \min_{\mathbf{y} \in \Delta_n} \max_{\mathbf{x} \in \Delta_m} \mathbf{x}^\intercal \mathbf{M} \mathbf{y} = \overline{f_{\mathbf{M}}}$$

optimal mixed strategies

The **optimal mixed strategies** for X and Y are the solutions to the LPs:

$$\begin{aligned} \underline{f_{\mathbf{M}}} = \max\ & s \\ \text{s.t.}\ -s\mathbf{1} + \mathbf{M}^\intercal\mathbf{x} &\geq \mathbf{0} \\ \mathbf{1} \cdot \mathbf{x} &= 1 \\ \mathbf{x} &\geq \mathbf{0} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{f_{\mathbf{M}}} = \min\ & t \\ \text{s.t.}\ -t\mathbf{1} + \mathbf{M}\mathbf{y} &\leq \mathbf{0} \\ \mathbf{1} \cdot \mathbf{y} &= 1 \\ \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

Remark: These are almost duals (see the Class Notes)

- if $(\boldsymbol{\lambda}^*, u^*)$ is an optimal solution to the dual of $\overline{f_{\mathbf{M}}}$ then $(-\boldsymbol{\lambda}^*, u^*)$ will be an optimal solution to $\underline{f_{\mathbf{M}}}$.
- if $(\boldsymbol{\lambda}^*, u^*)$ is an optimal solution to the dual of $\underline{f_{\mathbf{M}}}$ then $(\boldsymbol{\lambda}^*, -u^*)$ will be an optimal solution to $\overline{f_{\mathbf{M}}}$.

This can be used to build a solution of one from the other (using complementary slackness).

5 Integer Programming

The following are generalizations of linear programs:

1. A **binary integer program** (BIP) is the same as a linear program, but with the added constraint $\mathbf{z} \in \{0, 1\}^n$.

integer program

2. An **integer program** (IP) is the same as a linear program, but with added constraint $\mathbf{z} \in \mathbb{Z}^n$
3. A **mixed integer linear program** (MILP) is the same as linear programming, but with some variables allowed to be real (as in a linear program) and others forced to be integer (as in an integer program).

Given a mixed integer program

$$\begin{aligned} \mathcal{P} = \min / \max\ & \mathbf{c} \cdot \mathbf{x} + \mathbf{d} \cdot \mathbf{z} \\ \text{s.t. } & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b} \\ & \mathbf{x}, \mathbf{z} \geq 0 \\ & \mathbf{z} \text{ integers} \end{aligned}$$

LP relaxation

the **LP relaxation** is

$$\begin{aligned} \mathcal{P}_{\mathbb{R}} = \min / \max\ & \mathbf{c} \cdot \mathbf{x} + \mathbf{d} \cdot \mathbf{z} \\ \text{s.t. } & \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{z} = \mathbf{b} \\ & \mathbf{x}, \mathbf{z} \geq 0 \end{aligned}$$

formulations

stronger formulation

Two integer programs $\mathcal{P}$ and $\mathcal{P}'$ for which $\text{feas}(\mathcal{P}) = \text{feas}(\mathcal{P}')$ but $\text{feas}(\mathcal{P}_{\mathbb{R}}) \neq \text{feas}(\mathcal{P}'_{\mathbb{R}})$ are said to be different **formulations**. In the case that $\text{feas}(\mathcal{P}_{\mathbb{R}}) \subset \text{feas}(\mathcal{P}'_{\mathbb{R}})$, we say that P is the **stronger formulation**. The strongest possible formulation occurs when

$$\text{feas}(\mathcal{P}_{\mathbb{R}}) = \text{conv}(\text{feas}(\mathcal{P})).$$

In this case, every solution of the LP relaxation will be a valid solution to $\mathcal{P}$.

Remark: Since integer programs can be NP-hard and linear programs always finish in polynomial time, we can conclude that finding the strongest formulation is (in general) NP-hard.

cutting planes

Given a formulation $\mathcal{P}$, one can build a stronger formulation by adding cuts (or **cutting planes**). These are new constraints which, when added to $\mathcal{P}$, changes $\text{feas}(\mathcal{P}_{\mathbb{R}})$ while keeping $\text{feas}(\mathcal{P})$ constant.

Theorem 19. *For any formulation $\mathcal{P}$ of a mixed integer linear program, there exists a finite sequence of cutting planes (each of which is achieved by either rounding or disjunction) after which the new formulation $\mathcal{P}'$ is the strongest possible: that is,* $\text{feas}(\mathcal{P}'_{\mathbb{R}}) = \text{conv}(\text{feas}(\mathcal{P}))$.

Remark: Even if such a sequence exists, one cannot (in general) find it in polynomial time[3] (see previous Remark).

[3] Either because it is hard to find the sequence or the sequence itself is non-polynomial in length. Both can happen.

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