EPFL - Fall 2021	Prof. Zs. Patakfalvi
Rings and modules	**Exercises**
Sheet 12	16 Decembre 2021

Exercise 1. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Prove that $\sqrt{I} \subseteq \sqrt{J}$ if and only if $V(J) \subseteq V(I)$.

Exercise 2. Let F be an algebraically closed field, and let I, J be ideals of $R = F[x_1, ..., x_n]$. Show that

(1) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$
(2) $V(I) \cap V(J) = V(I + J)$

Exercise 3. Prove that $Z = \{(u^3, u^2v, uv^2, v^3) : u, v \in \mathbb{C}\} \subset \mathbb{C}^4$ is an algebraic set (i.e. there exists an ideal I of $\mathbb{C}[x_1, x_2, x_3, x_4]$ such that $Z = V(I)$). Find $I(Z)$.
[*Hint:* Make sure you have everything!]

Exercise 4. Let F be an algebraically closed field, and $X \subseteq F^m$ an algebraic set with ideal $I = I(X)$. Define the coordinate ring $A(X)$ of X to be $A(X) := F[x_1, \dots, x_m] \big/ I$. Notice that every element of $A(X)$ naturally defines a set-map from X to F, and thus one may think of $A(X)$ as the set of global algebraic functions on X.

(1) If $X = V(I) \subseteq F^m$, and $Y = V(J) \subseteq F^n$ are algebraic sets with ideals $I = I(X)$ and $J = I(Y)$, then a morphism $f : X \to Y$ is defined to be a set-map from the points of X to the points of Y, for which the following holds: there exists a vector $(h_1, \dots, h_n)$ of polynomials $h_i \in F[x_1, \dots, x_m]$, such that for every $\underline{a} \in X$ we have $f(\underline{a}) = (h_1(\underline{a}), h_2(\underline{a}), \dots, h_n(\underline{a})) \in Y$.
Show that whenever there is a morphism $f : X \to Y$ of algebraic sets as defined above, there is a unique homomorphism of F-algebras $\lambda_f : A(Y) \to A(X)$, such that the following diagram commutes.

$$\begin{array}{ccc} F[y_1, \dots, y_n] & \xrightarrow{y_i \mapsto h_i} & F[x_1, \dots, x_m] \\ \downarrow & & \downarrow \\ A(Y) & \xrightarrow{\lambda_f} & A(X) \end{array}$$

Here the vertical arrows are the quotient maps stemming from the definition of $A(X)$ and $A(Y)$, and the top horizontal map is given by sending y_i to $h_i(x_1, ..., x_m)$.

(2) With setup as above, show that if there is a homomorphism of F-algebras $\lambda : A(Y) \to A(X)$, then there is a morphism $f : X \to Y$ such that $\lambda = \lambda_f$. Furthermore, all choices of f are the same (as set-maps from the points of X to the points of Y).

(3) Show that $R_1 := F[x, y] \big/ (y^2 - x^3 - x^2)$ is an integral domain, and compute the integral closure S_1 of R_1 in the fraction field of R_1.

(4) Let R_1 and S_1 be as above. In Example 6.2.9 of the printed course notes it was shown that $R_2 := F[x, y, z] \big/ (x^2 - y^2 z)$ is an integral domain, and the integral closure S_2 of R_2 inside its field of fractions was computed.
For $i = 1, 2$, define the conductor ideal $\mathcal{I}_i$ to be the ideal in R_i which is the annihilator of the R_i-module $S_i \big/ R_i$. Calculate $\mathcal{I}_i$ for $i = 1, 2$.

(5) With the notation as above, let X_i be the algebraic set corresponding to R_i for $i = 1, 2$ (that is, X_i is the algebraic set corresponding to the ideal in the quotient defining R_i). Assuming that $F = \mathbb{C}$, draw the real points of the X_i. Draw also $V(\mathcal{I}_i + I(X_i))$[1]. What do you notice about $V(\mathcal{I}_i + I(X_i)) \subseteq X_i$?

Exercise 5. Let F be an algebraically closed field. Let X be an algebraic set in F^n with ideal $I(X) = I$. Prove that points of F^n contained in X are naturally in bijection with maximal ideals of the coordinate ring $A(X) = F[x_1, ..., x_n]/I$.

Exercise 6. Let R be a ring which is the quotient of a polynomial ring over an algebraically closed field F by a radical ideal. This naturally determines an algebraic set X whose coordinate ring is R. Noether normalisation says there is a subring $S \subseteq R$ such that $S \cong F[t_1, ..., t_r]$ and R is an integral extension of S. Give a geometric interpretation of Noether normalisation. That is, the inclusion $S \to R$ corresponds to a morphism f of algebraic sets. Prove that the fibres of f are finite, i.e. the preimage of any point in F^r under f consists of a finite set of points in X.
[*Hint:* Use Exercise 5 to describe the morphism of algebraic sets induced by an F-algebra morphism (provided by Exercise 4) purely in terms of the maximal ideals of the respective coordinate rings.]

Exercise 7. Let F be an algebraically closed field. Calculate the Krull dimension of the ring

$$F[w, x, y, z]/(x^2 - wy, y^2 - xz, wz - xy).$$

[1]This is equal to the subset of X_i in F^n which is the vanishing locus of the functions in $\mathcal{I}_i$