

Assignment 3

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8 avril 2022

1

We first show that $\mathbb{Z}_{(p)}$ is a ring, to show this, we show that it is in fact a subring of \mathbb{Q} .

Clearly $1 = \frac{1}{1} \in \mathbb{Z}_{(p)}$.

Furthermore, let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Z}_{(p)}$, then

$$\frac{ac}{bd} \in \mathbb{Z}_{(p)}$$

since $p \nmid b, p \nmid d \implies p \nmid bd$, where we used that p is prime.

Similarly,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Z}_{(p)}$$

By the same argument as above.

Now, suppose $\mathbb{Z}_{(p)}$ is a finitely generated ring, then there exist $c_1, \dots, c_n \in \mathbb{Z}_{(p)}$ which generate $\mathbb{Z}_{(p)}$.

Write $\forall i \quad c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{Z}$.

Now, since there exist an infinite number of prime, choose a prime q different from p and such that $(q, b_i) = 1 \forall 1 \leq i \leq n$.

We now pretend that $\frac{1}{q} \in \mathbb{Z}_{(p)}$ is not contained in the subring generated by c_1, \dots, c_n .

Indeed, suppose there exists a polynomial in $p \in \mathbb{Z}[x_1, \dots, x_n]$ such that

$$ev_c(p) = p(c_1, \dots, c_n) = \frac{1}{q}$$

We note that for ν_q the q -adic valuation on \mathbb{Q} , we get that

$$\nu_q(p(c_1, \dots, c_n)) \geq 0$$

This follows from the fact ν_q is indeed a valuation on \mathbb{Q} (as shown in previous problem sheets). Indeed, $p(c_1, \dots, c_n)$ simply is the multiplication and addition of elements of positive valuation.

But $\nu_q(\frac{1}{q}) = -1$, implying $p(c_1, \dots, c_n) \neq \frac{1}{q}$, yielding a contradiction.

2

First, we show again that \mathbb{Z}_p is a ring by showing that it is a subring of \mathbb{Q} (it clearly is included in \mathbb{Q}).

Again, note that $1 = \frac{1}{p^0} \in \mathbb{Z}_p$, furthermore, for $\frac{a}{p^j}, \frac{b}{p^l} \in \mathbb{Z}_p$, we get that

$$\frac{a}{p^j} \cdot \frac{b}{p^l} = \frac{ab}{p^{j+l}} \in \mathbb{Z}_p$$

Furthermore,

$$\frac{a}{p^j} + \frac{b}{p^l} = \frac{ap^l + bp^j}{p^{j+l}} \in \mathbb{Z}_p$$

Hence \mathbb{Z}_p is a ring.

We now show that it is indeed generated by $\frac{1}{p}$ by showing that the evaluation map

$$ev_{\frac{1}{p}} : \mathbb{Z}[x] \rightarrow \mathbb{Z}_p$$

is surjective.

Indeed, let $\frac{a}{p^i} \in \mathbb{Z}_p$, then the polynomial ax^i clearly is a preimage for $\frac{a}{p^i}$ implying that \mathbb{Z}_p is finitely generated by $\frac{1}{p}$.

3

Let $A \subset \mathbb{Z}_p$ be a subring.

Suppose $A \neq \mathbb{Z}_p$, then there exists an element $\frac{b}{p^i} \in A, i \neq 0, (b, p) = 1$.

Now, since A is a subring, it is closed under addition, hence adding $\frac{b}{p^i} p^{i-1}$ times to itself implies that $\frac{b}{p} \in A$.

We may suppose that b is coprime to p , if it wasn't, the fraction could be reduced further.

This implies in particular that $p - b$ is coprime to b .

Indeed, suppose they aren't coprime, then there exists $a \in \mathbb{N}$ such that $a|p-b$ and $a|b \implies a|p-b+b=p$ so a would also be a divisor of p and b hence $a=1$.

Let $c, d \in \mathbb{Z}$ be such that $cb + d(p-b) = 1$, then note that

$$c\frac{b}{p} + (1 - \frac{b}{p})d = c\frac{b}{p} + \frac{p-b}{p}d = \frac{1}{p}$$

Hence A contains $\frac{1}{p}$ and since \mathbb{Z}_p is generated by $\frac{1}{p}$, this implies that $A = \mathbb{Z}_p$.

4

Indeed, $\mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$ is, by definition, finitely generated as it is the subring of \mathbb{Q} generated by those two elements.

Hence, suppose that $\phi : \mathbb{Z}_{(p)} \rightarrow \mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$ is an isomorphism, then we claim $\phi^{-1}(\frac{1}{p}), \phi^{-1}(\frac{1}{q})$ would generate all of $\mathbb{Z}_{(p)}$ which contradicts part 1.

We prove the claim, indeed, if $a \in \mathbb{Z}_{(p)}$, then we may write $\phi(a) = k_a \frac{1}{p} + l_a \frac{1}{q} \implies a = k_a \phi^{-1}(\frac{1}{p}) + l_a \phi^{-1}(\frac{1}{q})$ which proves the claim.

5

We pretend that in fact $\mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$ is generated by exactly one element.

First, we show that $\mathbb{Z}[\frac{1}{pq}] = \mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$.

Indeed, it is clear that $\mathbb{Z}[\frac{1}{pq}] \subset \mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$.

Furthermore, note that $\frac{1}{p} \in \mathbb{Z} \left[\frac{1}{pq} \right]$ since $\frac{1}{pq} \cdot q = \frac{1}{p}$ and similarly $\frac{1}{q} \in \mathbb{Z} \left[\frac{1}{pq} \right]$, which implies that $\mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right] \subset \mathbb{Z} \left[\frac{1}{pq} \right]$.

Furthermore, we show that $\mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$ cannot be generated by 0 elements, ie. is not isomorphic to \mathbb{Z} .

Indeed, note that $\mathbb{Z}_p = \mathbb{Z} \left[\frac{1}{p} \right] \subset \mathbb{Z} \left[\frac{1}{pq} \right]$ implying in particular that $\mathbb{Z} \left[\frac{1}{p}, \frac{1}{q} \right]$ has at least one non-trivial subring.

But \mathbb{Z} has no non-trivial subring, hence \mathbb{Z} cannot be isomorphic to $\mathbb{Z} \left[\frac{1}{p} \right]$.