# Algebraic Geometry I

David Wiedemann

## Table des matières



## List of Theorems

**Lecture 1: Intro**

Mon 10 Oct

# Quick Motivation

We study schemes.
These are objects that "look locally" like $(\operatorname{Spec} A, A)$.
Examples include

— $A$ itself

— Varieties in affine or Projective

# 1 Presheaves and Sheaves

## 1.1 Presheaves

Let $X$ be a topological space.

**Definition 1 (Presheaf)**
*Let $C$ be a category. A presheaf $\mathcal{F}$ of $C$ on $X$ consists of*

— $\forall U \subset X$ *open, an object in* $C$ $\mathcal{F}(U)$

— $\forall V \subset U \subset X$ *open, a morphism* $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$

*such that*

— $\forall U$ *open,* $\rho_{U,U}$ *is the identity on* $\mathcal{F}(U)$

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

*open, we have* $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

**Remark**
*Usually,* $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in $C$ have elements.

**Remark**

— *Elements of* $\mathcal{F}(U)$ *are called sections of* $\mathcal{F}$ *over* $U$*.*

— $\mathcal{F}(U)$ *is called the space of sections of* $\mathcal{F}$ *over* $U$

— *Elements of* $\mathcal{F}(X)$ *are called global sections.*

— *There are alternative notations for* $\mathcal{F}(U)$ : $\Gamma(U, F)$ *or* $H_0(F)$

— *The* $\rho_{UV}$ *are called restriction maps, for* $s \in \mathcal{F}(U)$*, we write* $s|_V := \rho_{UV}(s)$ *and is called restriction of* $s$ *to* $V$*.*

**Example**

— *For any object* $A$ *in* $C$*, we define the constant presheaf* $\underline{A}'$ *defined by* $\underline{A}'(U) = A$ *and with restriction maps the identity.*

— *The presheaf of continuous functions :* $C^0$.
*We define* $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*

— *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* $\pi$*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

*This example is universal in a certain sense*

**Remark**
*Define the category* $\mathrm{Ouv}_X$ *with*

— *objects* $U \subset X$ *open subsets*
— *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

*Then a presheaf of* $C$ *on* $X$ *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

**Definition 2 (Morphism of presheaves)**
*A morphism* $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* $X$ *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho(U)} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow[\rho(V)]{} & \mathcal{F}_2(V) \end{array}$$

**Example**

— *Every morphism of objects* $A \to B$ *in* $C$ *yields a morphism* $\underline{A}' \to \underline{B}'$
— *If* $X = \mathbb{R}^n$*, let* $C^\infty$ *be the presheaf of smooth functions, then for every open* $U$*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
— *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

**Remark**
*There is an equivalence of categories*

$$\text{Presheaves of } C \text{ on } X \ \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

## 1.2 Sheaves

**Definition 3 (Sheaf)**
*Let* $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$ .
*A sheaf* $\mathcal{F}$ *of* $\mathcal{C}$ *on* $X$ *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s, t \in \mathcal{F}(U)$ , if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

**Remark**

— *The section $s$ of the gluability condition is unique by the locality condition.*
— *If $C$ has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

*is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$*

**Example**

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *$C^0$ ( and $C^\infty$ if $X = \mathbb{R}^n$ ) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous ( resp. smooth) because continuity ( resp. smoothness) is local.*

**Definition 4 (Morphisms of sheaves)**
*A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.*

**Remark**

— *$PSh_C(X)$ is the category of presheaves of $C$ on $X$*
— *$Sh_C(X)$ is the category of sheaves of $C$ on $X$*
*If $C = Ab$, we drop the index.*

**Remark**

*There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.*
*By definition, this functor is fully faithful*

## Recall

Let $A$ be a commutative ring ( with 1), then $\operatorname{Spec} A$ is the set of prime ideals of $A$.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

**Definition 5 (Natural sheaf on Spec A)**
*Let $A$ be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on $X$ is defined by*

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

*where*

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

*and $\rho_{UV}$ are simply the ( pointwise) restrictions.*

**Remark**
*$\mathcal{O}_X$ is a sheaf of rings :*
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

## Lecture 2: Stalks

Fri 14 Oct

## 1.3 Stalks

Let $X$ be a topological space.

**Definition 6**
*Let $(I, \leq)$ be a pair where $I$ is a set and $\leq$ is a binary relation.*
*$(I, \leq)$ is called a preorder if ll is reflexive and transitive.*
*$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric*
*$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$*

**Example**
1. *Let $I = \{U \subset X | U \text{ open } \}$ and $U \leq V \iff V \subset U$.*
   *Then $I$ is a directed poset.*
2. *For $x \in X$, let*
$$I_x = \{U \subset X | U \text{ open } x \in U\}$$
   *This is a directed poset.*

**Definition 7**
*Let* $(I, \leq)$ *be a directed set and* $C$ *a category.*
*A direct system in* $C$ *indexed by* $I$ *is a pair* $\left(\{A_i\}, \{\rho_{ij}\}_{i,j\in I, i\leq j}\right)$.
*Where the* $A_i$ *are objects in* $C$*, the* $\rho_{ij} : A_i \to A_j$ *are morphisms in* $C$ *such that*

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

**Example**
*If* $\mathcal{F}$ *is a presheaf of* $C$ *on* $X$ *and* $I_X$ *as in the second example above, then*

$$\left(\{\mathcal{F}(U_i)_{U_i\in I_X}\}, \{\rho_{U_i,U_j}\}\right)$$

*is a direct system.*

**Definition 8 (direct limit)**
*Let* $(I, \leq)$ *be a directed set,* $C$ *a category.*
*Let* $\left(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I}\right)$ *be a directed system, then the direct limit is a pair* $(\lim_{i\in I} A_i, \{\rho_i\}_{i\in I})$ *where* $\lim A_i$ *is in* $C$ *and* $\rho_i : A_i \to \lim A_i$ *such that*

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects* $A$ *in* $C$ *and morphisms* $f_i : A_i \to A$ *such that*

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

$\exists! f : \lim_{i\in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

**Remark**
*The direct limit is unique up to unique isomorphism.*

**Example**
*Write* $(*) = (\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I,i\leq j})$.
*Let* $*$ *be a direct systement in* Set.
*Let* $\lim_{i\in I} A_i \coloneqq A_i/\sim$ *where* $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ *such that* $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
*This is the direct limit of* $*$.
*If* $*$ *is a system in* Ab *, let* $\lim A_i \coloneqq \bigoplus A_i/N$ *with* $N = \langle a_i - \rho_{ij}(a_i)\rangle$.
*The natural map* $\bigcup A_i/\sim \to \bigoplus A_i/N$ *is a bijection*

**Remark**
*Taking the direct limits in* (Ab) *is exact in the following sense :*
$\forall$ *directed sets* $I$, $\forall$ *direct systems* $\{M_i\}, \{N_i\}, \{P_i\}$ *indexed by* $I$ *and for all*

*collections of commutative diagrams, we get*

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

**Definition 9**
*Let $C$ be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of $C$ on $X$.*
*Then the stalk of $\mathcal{F}$ at $x$ is*

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

*where $U$ runs over all open neighbourhoods of $x$.*
*For $s \in \mathcal{F}(U)$, we write $s_x$ for the image of $s$ in $\mathcal{F}_x$ and call it the germ of $s$ at $x$.*

**Remark**
*A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$*

**Remark**
*Let $x \in X, \mathcal{F}$ a presheaf of* $\mathrm{Set}, \mathrm{Ab}$

1. $\forall U \subset X$ *open,* $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

**Definition 10 (Sheafification)**
*Let $\mathcal{F}$ be a presheaf of sets ( ... ) on $X$.*
*The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by*

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

**Remark**

1. $\mathcal{F}^+$ *is a sheaf*
2. *Sheafification is functorial.*
   *For* $\rho : \mathcal{F} \to \mathcal{G}$ *a morphism of presheaves, the collection* $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ *sending* $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. $\exists$ *a natural morphism* $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ *defined by* $\iota_F(U)(s) : x \to s_x$
4. $\forall s \in \mathcal{F}^+(U)$ *there is an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

**Proposition 20**

$\forall$ *morphisms* $\phi : \mathcal{F} \to \mathcal{G}$ *such that* $\mathcal{G}$ *is a sheaf, there exists a unique morphism* $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ *such that* $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

**Proof**

*Let* $U \subset X$ *open, let* $s \in \mathcal{F}^+(U)$ $\exists$ *an open cover* $U = \bigcup_{i\in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.
*Since we want* $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, *we have to set*

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

*Since* $\mathcal{G}$ *is a sheaf and*

$$\phi(U_i)(s_i)|_{U_i\cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i\cap U_j}) = \phi(U_j)(s_i)|_{U_i\cap U_j}$$

*there exists a unique* $t \in \mathcal{G}(U)$ *with* $t|_{U_i} = \phi(U_i)(s_i)$.
*For* $\phi^+$ *to be a morphism, we have to set* $\phi^+(U)(s) = t$.
*We still have to check that* $\phi^+$ *is compatible with restriction maps.* □

**Remark**

*The proposition above shows that* $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ *naturally in the presheaf and the sheaf* $\mathcal{G}$.
*Hence* $(-)^+$ *is left-adjoint to the forgetful functor* $Sh(X) \to Psh(X)$

**Proposition 22**

$X = \operatorname{Spec} A$ $\forall a \in A$ *there exist isomorphisms* $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *such that* $\forall b \in A$ *with* $D(b) \subset D(a)$

$$\begin{tikzcd}
A_a \arrow[r, "\sim"] \arrow[d] & \mathcal{O}_X(D(a)) \arrow[d] \\
A_b \arrow[r, "\sim"'] & \mathcal{O}_X(D(b))
\end{tikzcd}$$

**Proof**

*Define* $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *by sending* $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.
*Clearly, these make the diagram commute.*
*This map is injective, indeed, suppose* $\phi_a(\frac{s}{a^n}) = 0$.
*Let* $I = Ann(s) = \{r \in A | rs = 0\}$.
*Since* $\frac{s}{a^n} = 0 \forall p \in D(a)$, *we have* $I \not\subset p$, *hence* $V(I) \subset V(a) \implies a \in \sqrt{I}$.
*Thus there exists* $m \geq 1$ *such that* $a^m s = 0$, *hece* $\frac{s}{a^n} = 0$.
*To show surjectivity, let* $s \in \mathcal{O}_X(D(a))$, *by definition of* $\mathcal{O}_X$ *and because* $D(h_i)$ *form a basis, we find* $a_i, g_i, h_i \in A$ *such that*

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

*and* $s(q) = \frac{a_i}{g_i}$ *for all* $q \in D(h_i)$.

1. *Claim 1 : Can choose $g_i = h_i$*
2. *Claim 2 : Can choose $I$ finite*
3. *Claim 3 : Can choose $a_i, h_i$ such that $h_j a_i = h_i a_j$.*

*Using these claims, since $D(a) = \bigcup D(h_i)$ , we find $n > 0, b_j \in A$ such that $a^n = \sum b_j h_j$.*
*Write $c = \sum a_i b_i$.*
*Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.*
*Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.*
*We now prove the claims*

1. *We have $D(h_i) \subset D(g_i)$ thus $V(g_i) \subset V(h_i)$ and thus $h_i \in \sqrt{(g_i)}$.*
   *So there exists $c_i \in A$ and $n > 1$ such that $h_i^n = c_i g_i$.*
   *Now, we replace $h_i$ by $h_i^n$ and $a_i$ by $a_i c_i$. Then*
   $$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$
2. *We have $D(a) \subset \cup D(h_i) \iff V(\sum h_i) = \cap V(h_i) \subset V(a)$.*
   *This is equivalent to saying that $a \in \sqrt{\sum (h_i)}$.*
   *Thus there exists $n \geq 1$ such that $a^n \in \sum_i (h_i)$.*
   *So there exist finitely many $b_i \in A$ such that $a^n = \sum b_j h_j$*
3. *On $D(h_i) \cap D(h_j) = D(h_i h_j)$, we have*
   $$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$
   *Thus*
   $$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$
   *Thus, there exists $N_j \geq 1$ such that $(h_i h_j)^{N_j}(h_j a_i - h_i a_j) = 0$.*
   *From claim 2, $I$ is finite, so we can choose $N$ big enough such that $N$ works for all $D(h_i)$.*
   *Now, we replace $h_i$ by $h_i^{N+1}$ and $a_i$ by $h_i^N a_i$ and we get $h_j a_i - h_i a_j = 0 \in A$.* $\square$

**Corollary 23**
*Take $X = \operatorname{Spec} A$, then $\forall p \in \operatorname{Spec} A \exists$ isomorphisms $\phi_p : A_p \to \mathcal{O}_{X,p}$ such that the appropriate diagram commutes.*

**Proof**

1. *Observe $\lim_{a \in A \setminus p} = A_a \simeq A_p$ (check universal property)*
2. *Observe that $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$*

## Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

## 1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

**Definition 11 (Subsheaf)**
*Let $\mathcal{F}$ be a (pre)sheaf on $X$.*
*Then a sub(pre)sheaf of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by $\mathcal{F}$.*

**Definition 12 (Kernel, cokernel of presheaves)**
*Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves*

1. *The presheaf kernel of $\phi$ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$*
2. *The presheaf image is defined as $\mathrm{Im}^{pre}(\phi)(U) = \mathrm{Im}(\phi(U))$*
3. *The presheaf cokernel is $\mathrm{coker}^{pre}(\phi)(U) = \mathrm{coker}(\phi(U))$.*

*In each case, the restriction maps are induced by those in of $\mathcal{F}$ or $\mathcal{G}$.*

**Lemma 24**
*If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf kernel is a sheaf.*

**Proof**
*Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.*
*Since $\mathcal{F}$ is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.*
*Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.*
*Thus*

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

*Since $\mathcal{G}$ is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$.* □

**Example**
*By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are.*

**Definition 13 (Cokernel/image of morphisms of sheaves)**
*Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves*

1. *sheaf kernel : $\ker^{pre}(\phi)$*

2. *sheaf image* $(\mathrm{Im}^{pre}(\phi))^+$
3. *sheaf cokernel* $(\mathrm{coker}^{pre}(\phi))^+$

**Lemma 26 (cokernels are cokernels)**
*Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves*

1. $\ker \phi \to \mathcal{F}$ *is a categorical kernel in* $Sh(X)$
2. $\mathcal{G} \to \operatorname{coker} \phi$ *is a categorical cokernel in* $Sh(X)$.

**Proof**

1. *This means that for each commutative diagram with solid arrows, the dotted arrow is unique*
   *"Insert cokernel/kernel diagram here"*
   *This holds for every open $U$ and so the kernel is a sheaf.*
2. *The appropriate diagram commutes and we use the universal property of sheafification.* □

**Proposition 27**
*Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent*

1. $\phi$ *is a monomorphism in* $Sh(X)$
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

**Proof**
*Recall $\phi$ is a monomorphism if for every $\psi : \mathcal{F}' \to \mathcal{F}, \phi \circ \psi = 0 \implies \psi = 0$ .*
*The implication* $1 \implies 2$ *follows by applying the monomorphism property to* $\ker \phi \to \mathcal{F}$ $2 \implies 1$ *If $\phi \circ \psi = 0$, then $\psi$ factors through the kernel* $\ker \phi \to \mathcal{F}$ *and so $\psi = 0$*
$2 \iff 3$ *Since* $\ker(\phi)(U) = \ker(\phi(U))$
$3 \implies 4$ *Follows because taking direct limits is exact.*
$4 \implies 3$ *Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.*
*So $s_x = 0 \forall x \in U$ and so $s = 0$* □

**Proposition 28**
*Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent*

1. $\phi$ *is an epimorphism in* $Sh(X)$
2. $\operatorname{coker}(\phi) = 0$
3. $\operatorname{coker}(\phi_x) = 0$

**Proof**
*Recall that $\phi$ is an epimorphism if for every $\psi : \mathcal{G} \to \mathcal{G}', \psi \circ \phi = 0 \implies \psi = 0$*
$1 \implies 2$ *Apply epimorphism property to $\mathcal{G} \to \operatorname{coker}(\phi)$*
$2 \implies 3$ *We have*

$$\begin{aligned} 0 &= (\operatorname{coker} \phi)_x \\ &= (\operatorname{coker}^{pre} \phi)_x = \operatorname{coker}(\phi_x) \end{aligned}$$

□

$3 \implies 1$
*Let $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\psi \circ \phi = 0$, this implies that $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$.*
*Since $\phi_x$ is an epimorphism of abelian groups, we get $\psi_x = 0$.*
*As the hom sheaf is a sheaf, we get that $\psi = 0$*

**Remark**
*If $\operatorname{coker}(\phi(U)) = 0 \forall U \subset X \implies \operatorname{coker}(\phi) = 0$ but the converse is not true.*

**Corollary 30**
*If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent*

1. *$\phi$ is an isomorphism*
2. *$\phi(U)$ is an isomorphism $\forall U \subset X$ open*
3. *$\phi_x$ is an isomorphism $\forall x \in X$*

**Proof**
$1 \implies 2$ *since taking sections is a functor*
$2 \implies 3$ *since taking limits is functorial*
$2 \implies 1$ *because $(\phi(U))^{-1}$ defines a morphism of sheaves*
$3 \implies 2$ *Need to show surjectivity of $\phi(U)$.*
*Let $t \in \mathcal{G}(U)$, since $\phi_x$ is an isomorphism $\forall x \in U$, we find $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$.*
*There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ such that $(s_{V_x})_x = s_x$*
*Since*

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

*we can choose $V + x$ sufficiently small such that $\phi(V_x)(s_{V_x}) = t|_{V_x}$.*
*Since $\phi(V_x \cap V_y)$ is injective and $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$, we have $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$.*
*Thus there exists $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ and $\phi(U)(s)|_{V_x} = t|_{V_x}$ and thus $\phi(U)(s) = t$.* □

**Definition 14 (Exact Sequence of sheaves)**
*A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ is called exact if $\ker \phi_2 = \operatorname{Im} \phi_1$*

**Corollary 31**

*A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.*