

PROBA

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1 Some historical models

1.1 Laplace Model

Definition 1 (Laplace Model)

Ω finite set, $|\Omega| = n$ is the set of outcomes.

We can observe whether $E \subset \Omega$ happens, and we define it's probability

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Question

Why should this have any meaning/content ?

Proposition 1

Consider laplace model for n coin tosses \Rightarrow every sequence has probability 2^{-n}

Denote by H_n the number of heads in n tosses

$$\mathbb{P}\left(\left|\frac{H_n}{n} - \frac{1}{2}\right| > \epsilon\right) \rightarrow 0$$

More generally

Proposition 2

If you have a laplace model for some event E , and look at n repetitions, then

$$\forall \epsilon > 0 \mathbb{P}\left(\left|\frac{E_n}{n} - \mathbb{P}(E)\right| > \epsilon\right) \rightarrow 0$$

Limitations of Laplace Model

- All outcomes have equal probability ?
- Need $|\Omega| < \infty$, so what about infinite sets ?

What next ?

Definition 2 (Intermediate model)

Let Ω to be any set and $P : \Omega \rightarrow [0, 1]$, s.t. $\sum_{\omega \in \Omega} p(\omega) = 1$

Event : $E \subset \Omega$ and

$$\mathbb{P}(E) := \sum_{\omega \in E} p(\omega)$$

- More freedom
- If you take Ω finite, $p(\omega) = \frac{1}{|\Omega|} \Rightarrow$ Laplace model
- Price ? How to choose $p : \Omega \rightarrow [0, 1] \rightarrow$ collect data, do statistics
- keeps many nice properties

- For countable sets, this is equivalent to the standard model.
- For uncountable Ω ?
- Problem 1 : There is no function s.t.

$$p(\omega) > 0 \forall \omega \in \Omega \text{ and } \sum p(\omega) = 1$$

This intermediate model is in essence only for countable sets.

What about uncountable sets ?

- What about a random point in $[0, 1]$ or $[0, 1]^n$?

Intuitively, consider $[0, 1]$, then we can set

$$\mathbb{P}(A) = \text{length}(A)$$

Definition 3 (Geometric probability)

Take $f : \mathbb{R} \rightarrow (0, \infty)$ to be a riemann-integrable function with total mass 1.

For any $A \subset \mathbb{R}$, s.t. 1_A riemann-integrable, we set $\mathbb{P}(A) = \int_A f(x)dx$

- In general quite ok
BUT
- You would expect there is one framework for uncountable and countable sets.
- What about more complicated spaces (eg. space of continuous functions)
- $\mathbb{P}(\mathbb{Q})$ is undefined

2 Basic Formalism

2.1 Measure spaces : A notion of area

- Set + structure
- General setting to talk about area

Definition 4 (Measure space)

$(\Omega, \mathcal{F}, \mu)$ is called a measure space if :

- Ω is some set
- $\mathcal{F} \subset P(\Omega)$ called a σ -algebra
 - $\emptyset \in \mathcal{F}$
 - $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$
 - $F_1, F_2, \dots \in \mathcal{F}$, then $\bigcup_{i \geq 1} F_i \in \mathcal{F}$ each F is called a measurable set.
- $\mu : \mathcal{F} \rightarrow [0, \infty)$ called the measure
 - $\mu(\emptyset) = 0$

— If F_1, \dots , are disjoint sets of the σ -algebra, then

$$\mu\left(\bigcup_{i \geq 1} F_i\right) = \sum_{i \geq 1} \mu(F_i)$$

— Defined by Borel 1898 and Lebesgue 1901-1903

2.2 Probability spaces

Given by Kolmogorov in 1933

Definition 5 (Probability space)

A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space if it is a measure space and $\mathbb{P}(\Omega) = 1$

Interpretation

- Ω state space/universe
- \mathcal{F} is the set of events you can observe/have access to
- $\mathbb{P}(E)$ is the probability of E

Lemme 3

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space

- $\Omega \in \mathcal{F}$
- $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \setminus F_2 \in \mathcal{F}$
- $F_1, \dots \in \mathcal{F} \Rightarrow \bigcap F_i \in \mathcal{F}$
- $F_1, F_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{i \geq 1} F_i \in \mathcal{F}$

Let us compare this definition with the prior ones

- Ω finite set, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}(F) = \frac{|F|}{|\Omega|}$ this is a probability space and a laplace model.
- For Ω countable, $\mathcal{F} = \mathcal{P}(\Omega)$, $\mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$
- The really new part is \mathcal{F} which restricts the sets we can measure

Lecture 2: ...

Wed 29 Sep

2.3 Basic properties

- $F_1, F_2, \dots \in \mathcal{F}$ disjoint

$$\mu\left(\bigcup F_i\right) = \sum \mu(F_i)$$

- $F_1 \subset F_2 \in \mathcal{F} \Rightarrow \mu(F_1) \leq \mu(F_2)$
- $F_1 \subset F_2 \subset \dots \in \mathcal{F}$

$$\mu(F_n) \rightarrow \mu\left(\bigcup F_i\right)$$

— $F_1, F_2, \dots, \mathcal{F}$

$$\mu(\bigcup F_i) \leq \sum \mu(F_i)$$

In addition, in probability spaces

— $\mathcal{P}(F^c) = 1 - \mathcal{P}(F)$

— $F_1 \supset F_2 \supset \dots \Rightarrow \mathcal{P}(F_n) \rightarrow \mathcal{P}(\bigcap F_i)$

2.4 Measurable and measure preserving maps

Definition 6

Let $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ two measure spaces.

$f : \Omega_1 \rightarrow \Omega_2$ is called measurable if for every $F \in \mathcal{F}_2$, $f^{-1}(F) \in \mathcal{F}_1$

A measurable function $f : (\Omega_1, \mathcal{F}_1) \rightarrow (\Omega_2, \mathcal{F}_2)$ is called measure preserving if $\forall F \in \mathcal{F}_2 \mu_1(f^{-1}(F)) = \mu_2(F)$.

Lemme 4 (Push-Forward measure)

Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), (\Omega_2, \mathcal{F}_2)$ be two measure spaces, and f measurable, then $\mathbb{P}_2(F) = \mathbb{P}_1(f^{-1}(F))$ is a probability measure.

3 Probability spaces

— Discrete probability spaces : Ω countable

— Continuous probability spaces : Ω uncountable.

3.1 Discrete probability spaces

Does introducing a σ -algebra \mathcal{F} enlargen the generality?

Proposition 5

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a discret probability space, $\exists \Omega_2$ countable, $\mathbb{P}_2 : \mathcal{P}(\Omega_2) \rightarrow [0, 1]$ s.t. $(\Omega_2, \mathcal{P}(\Omega_2), \mathbb{P}_2)$ is a probability space and $\exists f : (\Omega_1, \mathcal{F}_1) \rightarrow (\Omega_2, \mathcal{F}_2)$ is measure preserving

Still \mathcal{F} is useful :

— can sequentially study a model/situation by taking $\mathcal{F}_1 \subset \mathcal{F}_2 \dots$

Lemme 6

There is no shift-invariant probability measure on $(\mathbb{Z}, \mathcal{P}(\mathbb{Z}))$

Preuve

$$\mathbb{P}(\mathbb{Z}) = \mathbb{P}(\bigcup_n \{n\}) = \sum \mathbb{P}(\{n\}) = \infty$$

\Rightarrow cannot treat everyone on an equal ground !

□

3.1.1 Symmetric simple random walk

A simple walk of length n s.t. $|s_n - s_{n-1}| = 1$.

Let Ω be the set of all walks of length n , and consider $(\Omega, P(\Omega), \mathbb{P})$.

What is the probability that S hits 0?

What does it look like, what is its max?

3.2 Continuous probability spaces

Can we define a probability measure on S^1 s.t. $(S^1, P(S^1))$ that is rotation invariant?

Similarly to the countable case, but not the same as Ω is uncountable and setting $P(\{\omega\}) = 0$ gives no contradiction.

Proposition 7

You can not.

Preuve

Idea : decompose S^1 into countable many sets A_n st $\bigcup A_n = S^1$, they are disjoint and rotations of each other.

$\forall x \in S^1$, define S_x as $\{\dots, T^{-2}x, T^{-1}x, x, Tx, \dots\}$.

Note that either $S_x = S_y$ or $S_x \cap S_y = \emptyset$.

Lecture 3: Measurable maps

Wed 06 Oct

3.3 Borel σ -algebra

- Cannot define shift-invariant probability measure on $([0, 1], \mathcal{P}([0, 1]))$.
- What σ -algebra to choose on (X, τ) ?
- Want to know the size of all open-sets

Definition 7 (Borel sigma-algebra)

On (X, τ) the borel σ -algebra \mathcal{F}_τ is the smallest σ -algebra containing τ .

This is well defined because, given a collection of σ -algebras, their intersection is too.

Two nice properties

- Continuous functions on a Borel σ -algebra are also measurable.

Preuve

Suffices to check that $f^{-1}(U) \in \mathcal{F}_{\tau_1}$ for $U \in \tau_2$ but this is immediate since f is continuous.

In (\mathbb{R}^n, τ_E) , the Borel σ -algebra \mathcal{F}_E is generated by $(a_1, b_1) \times \dots \times (a_n, b_n)$.
 \mathcal{F}_E is the smallest σ -algebra containing open intervals. \square

3.4 Probability Measures on \mathbb{R}^n

Theorème 8 (Existence of Lebesgue-measure)

There exists a unique measure λ on $(\mathbb{R}^n, \mathcal{F}_E)$ s.t. $\lambda((a_1 \times b_1) \times \dots \times (a_n, b_n)) = \prod_i |b_i - a_i|$

Theorème 9 (Uniform Measure)

There exists a unique \mathbb{P} measure on $([0, 1]^n, \mathcal{F}_E)$ with the same property.

Both λ and \mathbb{P} are shift-invariant in fact only shift invariant measures on \mathbb{R} (up to a constant)

Preuve

Consider the case of $(\mathbb{R}^n, \mathcal{F}_E)$ and $f_r : x \rightarrow x + \tau, \tau \in \mathbb{R}^n$.

- f_r continuous \Rightarrow measurable
- $\tilde{\mathbb{P}}(A) = \mathbb{P}(f^{-1}(A))$ is a probability measure
- All boxes have the same measure \square

3.5 Probability measures on $(\mathbb{R}, \mathcal{F}_E)$

We saw that we can put a uniform measure on $[0, 1]$.

All probability measures on $(\mathbb{R}, \mathcal{F}_E)$

1. $\mathbb{P} : \mathcal{F}_E \rightarrow [0, 1]$
2. These are actually only characterized by $\mathbb{P}((-\infty, x))$

Definition 8 (Cumulative distribution function)

$F : \mathbb{R} \rightarrow [0, 1]$ is called a c.d.f if

- F is non-decreasing
- $F(x_n) \rightarrow 0$ then $x_n \rightarrow -\infty$
- $F(x_n) \rightarrow 1$ if $x_n \rightarrow 1$
- F is right-continuous.

Theorème 10

Given a probability measure \mathbb{P} on $(\mathbb{R}, \mathcal{F}_E)$, then $f(x) := \mathbb{P}((-\infty, x))$ is a c.d.f

Given a c.d.f, there exists a unique probability measure s.t. $\mathbb{P}((-\infty, x)) = F(x)$

Preuve

Given \mathbb{P} on $(\mathbb{R}, \mathcal{F}_E)$.

Let's show that $F(x) = \mathbb{P}((-\infty, x))$ is a c.d.f.

- $x < y \quad F(x) = \mathbb{P}((-\infty, x)) \leq \mathbb{P}((-\infty, y)) = F(y)$
- $x_n \rightarrow -\infty \quad F(x_n) = \mathbb{P}((-\infty, x_n)) \rightarrow \mathbb{P}(\bigcap_n (-\infty, x_n)) = 0$
- $x_n \rightarrow \infty \Rightarrow F(x_n) \rightarrow 1$ is similar
- Also for right continuous $x_n \rightarrow x$, we have that $[x_n, \infty) \subset [x_{n+1}, \infty)$

How do we construct \mathbb{P} given F ?

Trick using push-forward measure.

Define $f : (0, 1) \rightarrow \mathbb{R}$, define

$$f(x) = \inf_{y \in \mathbb{R}} \{F(y) \geq x\}$$

□

Define $\mathbb{P}(A) := \mathbb{P}_U(f^{-1}(A)) \forall A \in \mathcal{F}_E$ Why is f measurable?

If f is increasing $\Rightarrow f$ is measurable

Lecture 4: ...

Wed 13 Oct

Each c.d.f gives rise to a unique \mathbb{P} .

A priori $\mathbb{P}_1 = \mathbb{P}_2$ means $\forall F \in \mathcal{F}_E \mathbb{P}_1(F) = \mathbb{P}_2(F)$.

We show that it suffices to show that $\mathbb{P}_1((-\infty, x]) = \mathbb{P}_2((-\infty, x]) \forall x \in \mathbb{R}$.

Lemme 11

Given $(\mathbb{R}, \mathcal{F}_E, \mathbb{P})$ then $\forall B \in \mathcal{F}_E, \forall \epsilon > 0$ one can find disjoint intervals I_1, \dots, I_n s.t. $\mathbb{P}(B \Delta (I_1 \cup \dots \cup I_n)) < \epsilon$

Preuve

Consider the collection H of all subsets $H \in \mathcal{F}_E$ s.t. the property above holds.

We know that H contains all intervals, hence $\sigma(H) = \mathcal{F}_E$.

So we only need to show that H is a σ -algebra

1. $\emptyset \in H$: Know that $\forall x (-\infty, x] \in H$

2. If $B \in H \Rightarrow B^C \in H$.

Given $\epsilon > 0$, choose I_1, \dots, I_n s.t. $\mathbb{P}(B \Delta (I_1 \cup \dots)) < \epsilon$, but $(B \Delta A) = B^C \Delta A^C$, hence

$$\mathbb{P}(B^C \Delta (I_1 \cup \dots)) < \epsilon$$

3. $H_1, \dots \in H$, we want $\bigcup_i H_i \in H \exists n \in \mathbb{N}$

$$\mathbb{P}((\bigcup_{i=0}^m H_i) \Delta (\bigcup_i H_i)) < \frac{\epsilon}{2}$$

$\forall i = 1, \dots, m$, we have disjoint $I_{i,1}, \dots, I_{i,m_i}$ s.t.

$$\mathbb{P}(H_i \Delta (I_{i,1} \cup \dots)) < \frac{\epsilon}{2m}$$

Now use that

$$(\bigcup_{i=1}^m H_i) \Delta (\bigcup_{i=1}^m \bigcup_{j=1}^{m_i} I_{i,j}) \subseteq \bigcup_{i=1}^m (H_i \Delta \bigcup_{j=1}^{m_i} I_{i,j})$$

Finally, we can write a finite union of disjoint intervals

□

Corollaire 12

$\mathbb{P}_1, \mathbb{P}_2$ probability measure on $(\mathbb{R}, \mathcal{F}_E)$, then $\mathbb{P}_1 = \mathbb{P}_2$ as soon as

$$\mathbb{P}_1((-\infty, x]) = \mathbb{P}_2((-\infty, x])$$

or

$$\mathbb{P}_1(x, y) = \mathbb{P}_2(x, y)$$

Preuve

Notice $(-\infty, x)$ can be written as

$$(-\infty, x) = \left(\bigcup_n (x, x+n) \right)^C$$

So it suffices to prove the first point.

Observe, for all intervals $\mathbb{P}_1(I) = \mathbb{P}_2(I)$ since

$$\mathbb{P}_i(y, x) = \mathbb{P}_i(-\infty, x) - \mathbb{P}_i(-\infty, y)$$

The condition holds for B if $\forall \epsilon > 0$, we can pick I_1, \dots, I_n s.t.

$$\mathbb{P}_1(B \Delta (I_1 \cup \dots)) < \epsilon$$

and

$$\mathbb{P}_2(B \Delta (I_1 \cup \dots)) < \epsilon$$

So we need to check again that this is a σ -algebra and we are done.

Now we can conclude that

$$|\mathbb{P}_1(B) - \mathbb{P}_1(I_1 \cup \dots)| = |\mathbb{P}_1(B) - \mathbb{P}_2(I_1 \cup \dots)| < \epsilon$$

and

$$|\mathbb{P}_2(B) - \mathbb{P}_1(I_1 \cup \dots)| = |\mathbb{P}_2(B) - \mathbb{P}_2(I_1 \cup \dots)| < \epsilon$$

□

An abstract uniqueness result follows from a similar strategy.

Theorème 13 (Dynkin)

\mathbb{P}_1 and \mathbb{P}_2 two probability measures on (Ω, \mathcal{F}) , suppose $\mathbb{P}_1(H) = \mathbb{P}_2(H)$ for all $H \in \mathcal{H} \subset \mathcal{F}$ and

- $\sigma(\mathcal{H}) = \mathcal{F}$
- $H_1 \in \mathcal{H}, H_2 \in \mathcal{H} \Rightarrow H_1 \cap H_2 \in \mathcal{H}$

Then $\mathbb{P}_1 = \mathbb{P}_2$

3.6 Probability measures on \mathbb{R}^n **Definition 9 (Joint c.d.f.)**

$$F : \mathbb{R}^n \rightarrow [0, 1]$$

- F non-decreasing in each coordinate
- $F(x_1, \dots, x_n) \rightarrow 1$ if all $x_i \rightarrow -\infty$
- right-continuous

Theorème 14

Joint c.d.f $\iff \mathbb{P}$ on $(\mathbb{R}^n, \mathcal{F}_E)$

3.7 Product probability measures on $\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}$

- Related to independence
- Natural mathematically

2 steps

- product σ -algebra
- product measure

3.7.1 Product σ -algebra

Definition 10 (Product algebra)

Let $(\Omega_i, \mathcal{F}_i)_{i \geq 1}$ measurable spaces, then the product σ -algebra \mathcal{F}_π on $\prod_i \Omega_i$ is the σ -algebra generated by sets $F = E_1 \times \dots \times E_n \times \Omega_{n+1} \times \dots$, $E_i \in \mathcal{F}_i$

Remarque

- Projections are measurable
- In fact, product σ -algebra s.t. all projections are measurable

Notice on \mathbb{R}^n , we now have two ways to define a σ -algebra.

- Take (\mathbb{R}^n, τ_E) and induce a Borel σ -algebra
- Take n copies of $(\mathbb{R}, \mathcal{F}_E)$ and consider \mathcal{F}_π on \mathbb{R}^n