Assignment 3

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1

We first show that $\mathbb{Z}_{(p)}$ is a ring, to show this, we show that it is in fact a subring of $\mathbb{Q}$.
Clearly $1 = \frac{1}{1} \in \mathbb{Z}_{(p)}$.
Furthermore, let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Z}_{(p)}$, then

$$\frac{ac}{bd} \in \mathbb{Z}_{(p)}$$

since $p \not| b, b \not| d \implies p \not| bd$, where we used that p is prime.
Similarly,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \in \mathbb{Z}_{(p)}$$

By the same argument as above.

Now, suppose $\mathbb{Z}_{(p)}$ is a finitely generated ring, then there exist $c_1, \ldots, c_n \in \mathbb{Z}_{(p)}$ which generate $\mathbb{Z}_{(p)}$.
Write $\forall i \quad c_i = \frac{a_i}{b_i}$ where $a_i, b_i \in \mathbb{Z}$.
Now, since there exist an infinite number of prime, choose a prime q different from p and such that $(q, b_i) = 1 \forall 1 \leq i \leq n$.
We now pretend that $\frac{1}{q} \in \mathbb{Z}_{(p)}$ is not contained in the subring generated by $c_1, \ldots, c_n$.
Indeed, suppose there exists a polynomial in $p \in \mathbb{Z}[x_1, \ldots, x_n]$ such that

$$ev_c(p) = p(c_1, \ldots, c_n) = \frac{1}{q}$$

We note that for ν_q the q-adic valuation on $\mathbb{Q}$, we get that

$$\nu_q(p(c_1, \ldots, c_n)) \geq 0$$

This follows from the fact ν_q is indeed a valuation on $\mathbb{Q}$ (as shown in previous problem sheets) . Indeed, $p(c_1, \ldots, c_n)$ simply is the multiplication and addition of elements of positive valuation.
But $\nu_q(\frac{1}{q}) = -1$, implying $p(c_1, \ldots, c_n) \neq \frac{1}{q}$, yielding a contradiction.

2

First, we show again that $\mathbb{Z}_p$ is a ring by showing that it is a subring of $\mathbb{Q}$ (it clearly is included in $\mathbb{Q}$).
Again, note that $1 = \frac{1}{p^0} \in \mathbb{Z}_p$, furthermore, for $\frac{a}{p^j}, \frac{b}{p^l} \in \mathbb{Z}_p$, we get that

$$\frac{a}{p^j} \cdot \frac{b}{p^l} = \frac{ab}{p^{j+l}} \in \mathbb{Z}_p$$

Furthermore,

$$\frac{a}{p^j} + \frac{b}{p^l} = \frac{ap^l + bp^j}{p^{j+l}} \in \mathbb{Z}_p$$

Hence $\mathbb{Z}_p$ is a ring.

We now show that it is indeed generated by $\frac{1}{p}$ by showing that the evaluation map

$$ev_{\frac{1}{p}} : \mathbb{Z}[x] \to \mathbb{Z}_p$$

is surjective.
Indeed, let $\frac{a}{p^i} \in \mathbb{Z}_p$, then the polynomial ax^i clearly is a preimage for $\frac{a}{p^i}$ implying that $\mathbb{Z}_p$ is finitely generated by $\frac{1}{p}$.

3

Let $A \subset \mathbb{Z}_p$ be a subring.
Suppose $A \neq \mathbb{Z}$, then there exists an element $\frac{b}{p^i} \in A, i \neq 0, (b,p) = 1$.
Now, since A is a subring, it is closed under addition, hence adding $\frac{b}{p^i}$ p^{i-1} times to itself implies that $\frac{b}{p} \in A$.
We may suppose that b is coprime to p, if it wasn't, the fraction could be reduced further.
This implies in particular that $p - b$ is coprime to b.
Indeed, suppose they aren't coprime, then there exists $a \in \mathbb{N}$ such that $a|p-b$ and $a|b \implies a|p - b + b = p$ so a would also be a divisor of p and b hence $a = 1$.
Let $c, d \in \mathbb{Z}$ be such that $cb + d(p - b) = 1$, then note that

$$c\frac{b}{p} + (1 - \frac{b}{p})d = c\frac{b}{p} + \frac{p-b}{p}d = \frac{1}{p}$$

Hence A contains $\frac{1}{p}$ and since $\mathbb{Z}_p$ is generated by $\frac{1}{p}$, this implies that $A = \mathbb{Z}_p$.

4

Indeed, $\mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$ is, by definition, finitely generated as it is the subring of $\mathbb{Q}$ generated by those two elements.

Hence, suppose that $\phi : \mathbb{Z}_{(p)} \to \mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$ is an isomorphism, then we claim $\phi^{-1}(\frac{1}{p}), \phi^{-1}(\frac{1}{q})$ would generate all of $\mathbb{Z}_{(p)}$ which contradicts part 1.
We prove the claim, indeed, if $a \in \mathbb{Z}_{(p)}$, then we may write $\phi(a) = k_a \frac{1}{p} + l_a \frac{1}{q} \implies a = k_a \phi^{-1}(\frac{1}{p}) + l_a \phi^{-1}(\frac{1}{q})$ which would imply $\mathbb{Z}_{(p)}$ is finitely generated.

5

We pretend that in fact $\mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$ is generated by exactly one element.
First, we show that $\mathbb{Z}[\frac{1}{pq}] = \mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$.
Indeed, it is clear that $\mathbb{Z}[\frac{1}{pq}] \subset \mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$.
Furthermore, note that $\frac{1}{p} \in \mathbb{Z}\left[\frac{1}{pq}\right]$ since $\frac{1}{pq} \cdot q = \frac{1}{p}$ and similarly $\frac{1}{q} \in \mathbb{Z}\left[\frac{1}{pq}\right]$, which implies that $\mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right] \subset \mathbb{Z}\left[\frac{1}{pq}\right]$.

Furthermore, we show that $\mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$ cannot be generated by 0 elements, ie. is not isomorphic to $\mathbb{Z}$.
Indeed, note that $\mathbb{Z}_p = \mathbb{Z}\left[\frac{1}{p}\right] \subset \mathbb{Z}\left[\frac{1}{pq}\right]$ implying in particular that $\mathbb{Z}\left[\frac{1}{p}, \frac{1}{q}\right]$ has at least one non-trivial subring.
But $\mathbb{Z}$ has no non-trivial subring, hence $\mathbb{Z}$ cannot be isomorphic to $\mathbb{Z}\left[\frac{1}{p}\right]$.