Algebraic Curves

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Table des matières

List of Theorems

Lecture 1: Introduction

Fri 25 Feb

Let K be a field, given a set of polynomials $S = \{f_1, \dots\}$, we can consider $V(S) = \{(x_1, \dots) \in K^n | f_i(x_1, \dots) = 0 \forall i\}$.
Notice that if $a_1, \dots \in K[x_1, \dots]$ then also $\sum_i a_i(x) f_i(x) = 0$ only depends on the ideal generated by S.
If $I(S)$ happens to be prime, we call V an algebraic variety.

1 Affine algebraic sets

1.1 Recollection on commutative algebra

All rings are commutative and with unit.
Let R be a ring.

— R is an integral domain, or just domain if there are no zero divisors, ie, $\forall a, b \in R$ s.t.

$$a.b = 0 \implies a = 0 \text{ or } b = 0$$

— Any domain can be embedded into it's quotient ring.

— A proper ideal I is maximal if it's not contained in any other proper ideal

— A proper ideal I is prime if

$$\forall a, b \in R, ab \in I \implies a \in I \text{ or } b \in I$$

— A proper ideal I is radiccal if

$$a^n \in I \implies a \in I$$

— For any ideal $I \subset R$, the radical $\sqrt{I}$ is the smallest radical ideal containing I

Lemme 1
— *$I \subset R$ is maximal $\iff$ R/I is a field*

Lemme 2
— *$I \subset R$ is prime $\iff$ R/I is a domain*

Lemme 3
— *radical $\iff$ R/I has no nilpotent elements.*

Given a subset $S \subset R$ we can consider the ideal generated by S

$$I(S) = \left\{ \sum_i a_i s_i \right\}$$

I is finitely generated if $I = I(S)$ with S finite.

— We say that R is Noetherian $\not\exists$ a chain of strictly increasing ideals. Equivalently, every ideal is finitely generated.

> **Theorème 4**
> — *In fact, hilbert's basis theorem says that, if R is Noetherian, then $R[x]$ is noetherian.*

In particular $K[x_1, \ldots, x_n]$ is Noetherian

— I is in principal if it is generated by one element.

— A domain is called a principal ideal domain (PID) if every ideal is principal.

— $a \in R$ is irreducible if a is not a unit, nor zero and if

$$a = b.c$$

then either b or c are units.

— A pid $(a) \subset R$ is prime $\iff$ a is irreducible.

— R is a UFD if R is a domain and elements in R can be factored uniquely up to units and reordering into irreducible elements.

> **Theorème 5**
> *R is a UFD $\implies$ $R[x]$ is a UFD*

And, if R is a PID, then R is a UFD

> **Theorème 6 (Gauss Lemmma)**
> — *R is a UFD and $a \in R[X]$ irreducible, then also $a \in Q(R)[X]$ is irreducible.*

— Localization

Let R be a domain, if $S \subset R$ is a multiplicative subset, then the localization of R at S is defined as

$$S^{-1}R = \left\{ x \in Q(R) | x = \frac{a}{b}, b \in S \right\}$$

If M is an $R-$module, we have similarly

$$S^{-1}M = \left\{ \frac{m}{s} | m \in M, s \in M \right\} / \left\{ \frac{m}{s} = \frac{m'}{s'} \iff ms' = sm' \right\}$$

If $p \subset R$ is a prime ideal, then it's complement is a multiplicative subset and we define

$$R_p = (R \setminus p)^{-1} R$$

— There is a 1-1 correspondence between $p \subset R$ prime and ideals of R_p, furthermore R_p is a local ring

— Localization is exact, in particular, given $I \subset p$ the short exact sequence

$$o \to I \to R \to R/I \to 0$$

gets sent to

$$0 \to I_p \to R_p \to (R/I)_p \to 0$$

ie. localization commutes with taking quotients.

1.2 Polynomial rings

For $a \in \mathbb{N}^n$, we set

$$X^a = X_1^{a_1} \ldots \in k[X_1, \ldots]$$

Thus for any $F \in k[X_1, \ldots, X_n]$, we can write it as

$$F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$$

F is homogeneous or a form of degree d if the coefficients $\lambda_a = 0$ unless $a_1 + \ldots + a_n = d$.
Any F can be written uniquely as $F = F_0 + \ldots + F_d$ where F_i is a form of degree i.
The derivative of $F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$ with repsect to X_i is $F_{X_i} = \frac{\partial F}{\partial X_i}$.
If F is a form of degree d we have

Theorème 7 (Euler's theorem)

$$\sum_{i=1}^{n} \frac{\partial F}{\partial X_i} X_i = dF$$

Lecture 2: Affine space and algebraic sets

Wed 02 Mar

1.3 Affine spaces and algebraic sets

Let k be a field.

Definition 1
For every $n \geq 0$ the affine n -space $\mathbb{A}_k^n$ the set k^n .

In particular $\mathbb{A}^0$ is a point, $\mathbb{A}^1$ is a line, $\mathbb{A}^2$ the affine plane.
Given a subset $S \subset k[X_1, \ldots, X_n]$ of polynomials, we set

$$V(S) = \{x = (x_1, \ldots, x_n) \in \mathbb{A}^n | f(x_1, \ldots, x_n) = 0 \forall f \in S\}$$

If S is finite, we write $V(f_1, \ldots, f_k)$ for $V(S)$.
If the set S is a singleton, then we call $V(S)$ a hyperplane.
Any subset of $\mathbb{A}^n$ V algebraic if $V = V(S)$ for some subset of polynomials.

Lemme 8

— *Let $S \subset k[X_1, \ldots, X_n]$ and I the ideal generated by S, then $V(S) = V(I)$.*
— *Let $\{I_\alpha\}$ be a collection of ideals, then*

$$V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$$

— *If $I \subset J$ then $V(J) \subset V(I)$*
— *For polynomials $f, g \in k[x_1, \ldots, x_n]$, then $V(f) \cup V(g) = V(f \cdot g)$ For ideals I, J ideals, then $V(I) \cup V(J) = V(I \cdot J)$ where $IJ = \{fg | f \in I, g \in J\}$*
— *For $a = (a_1, \ldots, a_n) \in \mathbb{A}^n, v(\{x_1 - a_1, \ldots\}) = \{a\}$*

Preuve

1. *Let $h \in \sum_i f_i g_i \subset I$ with $f_i \in S$ and $x \in V(S)$, then $f_i(x) = 0 \forall i$ hence $h(x) = 0 \implies x \in V(I) \implies V(S) \subset V(I)$.*
   Furthermore, if $x \in V(I)$, then in particular $f(x) = 0 \forall f \in S \subset I$, hence $x \in V(S)$ and $V(S) \supset V(I)$
2. *Let $x \in V(\cup I_\alpha)$, then for any α and $f \in I_\alpha$, we must have $f(x) = 0$, hence $x \in V(I_\alpha) \implies x \in \bigcap_\alpha V(I_\alpha)$.*
   Conversely, if $x \in \bigcap_\alpha V(I_\alpha)$ and $f \in \bigcup_\alpha I_\alpha$, then $f \in I_\alpha$ for some α, then $f(x) = 0$ hence $x \in V(\bigcup_\alpha I_\alpha)$ □

By Hilbert's basis theorem $k[x_1, \ldots, x_n]$ is Noetherian hence every ideal is finitely generated.

Corollaire 9

Every algebraic set $V \subset \mathbb{A}^n$ is of the form

$$V = V(f_1, \ldots, f_k) = V(f_1) \cap \ldots \cap V(f_k)$$

1.4 Ideals of a set of points and the nullstellensatz

Using the previous section, we have a map

$$V : \{\text{ Ideals in } k[X_1, \ldots, X_N]\} \mapsto \{\text{ algebraic sets in } \mathbb{A}^n\}$$

Conversely, for any subset $X \subset \mathbb{A}^n$ we define

$$I(X) \coloneqq \{f \in k[X_1, \ldots, X_N] | f(x) = 0 \forall x \in X\} \subset k[X_1, \ldots, X_N]$$

Lemme 10

1. *If $X \subset Y$ then $I(X) \supset I(Y)$*
2. *For $J \subset k[X_1, \ldots, X_N]$ an ideal $I(V(J)) \supset J$*
3. *For $W \subset \mathbb{A}^n$ algebraic, $V(I(W)) = W$*

Preuve

1. *Let $f \in I(Y)$, then f vanishes on X and hence f in $I(X)$*
2. *$I(V(J)) = \{f \in k[x_1, \ldots, x_n] | f(x) = 0 \forall x \in V(J)\} \supset J$*
3. *By definition $V(I(X)) \supset X$ for any X.*
   If in addition, if $X = V(J)$ algebraic, then $V(I(X)) = V((I(V(J)))) \subset V(J) = X$ □

There are essentially two reasons why $I(V(J)) \supsetneq J$ in general

1. $J = (x^n) \subset k[x] \implies V(x^n) = \{0\}$ and $I(\{0\}) = (x)$
2. $(x^2 + 1) \subset \mathbb{R}[x]$ and $I(\emptyset) = \mathbb{R}[X]$

Lemme 11

For any $X \subset \mathbb{A}^n$, $I(X)$ is a radical ideal

Preuve

If $f^n \in I(X)$ for some n, then $f(x)^n = 0$ and hence $f(x) = 0$ □

So the first phenomenon is related to the fact that J is not radical, the second is related to the fact that $\mathbb{R}$ is not algebraically closed.

Theorème 12 (Hilbert's Nullstellensatz)

Let K be algebraically closed, $J \subset k[X_1, \ldots, X_n]$, then

$$I(V(J)) = \sqrt{J}$$

Using this, there is a one to one correspondence

$$\{\text{ radical ideals in } k[X_1, \ldots, X_n]\ \} \leftrightarrow \{\text{ algebraic subsets of } \mathbb{A}^n\}$$

Theorème 13 (Weak Nullstellensatz)
Let K be algebraically closed, every maximal ideal $I \subset K[X_1, \dots, X_n]$ is of the form $I = \{x_1 - a_1, \dots, x_n - a_n\}$ with $a = (a_i) \in \mathbb{A}^n$

Corollaire 14
Let $I \subset K[X_1, \dots, X_n$ be any ideal, then $V(I)$ is a finite set $\iff$ $k[X_1, \dots, X_n]/_I$ is a finite dimensional K- vector space.
In this case

$$|V(I)| \leq \dim_k k[X_1, \dots, X_n]/_I$$

Preuve
Let $I \subset k[X_1, \dots, X_n]$ be any ideal and $P_1, \dots, P_n \subset V(I)$ distinct.
We can choose (Exercise) $F_1, \dots, F_r \in K[X_1, \dots, X_n]$ s.t. $F_i(P_j) = \delta_{ij}$, then we write $f_1, \dots, f_r$ for the residues of $F_1, \dots, F_r$ in $K[X_1, \dots, X_n]/_I$.
We claim $f_1, \dots, f_r$ are linearly independent.
Indeed suppose $\sum_i \lambda_i f_i = 0$, this implies $\sum_i \lambda_i F_i \in I$ hence $0 = \sum \lambda_i F_i(P_j)$ which implies $\lambda_j = 0$, hence the f_i are linearly independent.
It follows that $\dim_k K[X_1, \dots, X_n]/_I < \infty \implies |V(I)| < \infty$ and in this case $\dim_k K[X_1, \dots, X_n]/_I \geq |V(I)|$.
Now assume $V(I)$ is a finite set $\{P_1, \dots, P_r\} \subset \mathbb{A}^n$ and write $P_i = (a_{i1}, \dots, a_{in})$ and define $F_j = \prod_{i=1}^r (X_j - a_{ij})$.
By construction $F_j \in I(V(I)) = \sqrt{I}$
$\exists N > 0$ such that $F_j^N \in I$.
Hence $f_j^N = 0$ in $K[X_1, \dots, X_n]/_I$, but $f_j^N = (x_j^{Nr}) +$ lower order terms .
This means that X_j^{Nr} is a K-linear combination of $\{1, \dots, X_j^{Nr-1}\}$.
This means that X_j^s is a linear combination for any $s > 0$.
Hence taking products for different j's, we see that the set $\{x_1^{m_1}, \dots, x_n^{m_n}\}$ generates $K[X_1, \dots, X_n]/_I$ $\square$

Due to these theorems, we'll always suppose K is algebraically closed.

Lecture 3: Irreducible sets

Fri 11 Mar

1.5 Irreducible sets

Definition 2 (Irreducible set)
An algebraic set $V \subset \mathbb{A}^n$ is irreducible if $\forall W_1, W_2 \subset \mathbb{A}^n$ algebraic s.t. $V = W_1 \cup W_2$, then either $W_1 = V$ or $W_2 = V$

Exemple

— *Let $V = \{x_1, \dots, x_n\} \subset \mathbb{A}^n$ is irreducible iff $n = 1$*

— *Let $f(X,Y) = Y(X^2 - Y), V = V(f) \subset \mathbb{A}^2$ is not irreducible by taking $W_1 = V(Y), W_2 = V(X^2 - Y)$*

Proposition 16
An algebraic set V is irreducible iff $I(V)$ is prime.

Preuve
If $I(V)$ is not prime, let $F_1, F_2 \notin I(V)$ s.t. $F_1, F_2 \in I(V)$, then we can write $V = (V \cap V(F_1)) \cup (V \cap V(F_2))$.
Conversely, if $V = W_1 \cup W_2$ and $W_i \neq V$, then $I(W_i) \supsetneq I(V)$, pick $F_i \in I(W_i) \setminus I(V)$, then $F_1 F_2 = I(W_1) \cap I(W_2) = I(V)$. □

If $V \subset \mathbb{A}^n$ is irreducible, we can decompose it into a union of irreducible sets. The union is always finite as the polynomial ring is noetherian.

Theorème 17 (Theorem name)
Every $V \subset \mathbb{A}^n$ algebraic can be written uniquely (up to ordering) as a union of irreducible sets.

$$V = V_1 \cup \ldots \cup V_k$$

where the V_i's are irreducible and $V_i \not\subset V_j \forall i \neq j$

Definition 3 (Irreducible Components)
The $V_1 \ldots V_k$ are irreducible components of V.

Remarque
Applying I in theorem 1.9, we get

$$I(V) = I(V_1) \cap \ldots \cap I(V_k)$$

and $I(V_i)$ is the primary decomposition of $I(V)$

In general, it is quite difficult to find this decomposition.
For hypersurfaces, it's easy, for $I(F)$, write $F = F_1^{\alpha_1} \cdot \ldots F_k^{\alpha_k}$, then $V(F) = V(F_1) \cup \ldots \cup V(F_k)$.

1.6 Algebraic subsets of $\mathbb{A}^2$

Lemme 19
Let $F, G \in k[X,Y]$ with no common factors, then $V(F) \cap V(G)$ is a finite set of points.

Preuve

By Gauss's lemma, F, G have no common factors in $k(X)[Y]$. Since $k(x)[Y]$ is a PID $\exists A, B \in k(X)$ such that

$$AF + BG = 1$$

Now there exists $C \in k[X]$ such that $AC, BC \in k[X]$.
Let $(x, y) \in V(F, G)$, then $C(x) = 0$ and hence there are only finitely many x's possible.
By symmetry, the same is true for the Y coordinate, hence $|V(F, G)| < \infty$ $\square$

Using this, we can now classify all algebraic subsets of $\mathbb{A}^2$.

Corollaire 20
The irreducible algebraic subsets of $\mathbb{A}^2$ are $\mathbb{A}^2, V(F)$ with F irreducible or singletons.

2 Affine algebraic varieties

Definition 4 (Affine algebraic variety)
An affine algebraic variety is an irreducible affine algebraic set.

2.1 Zariski topology

Definition 5 (Zariski topology)
The Zariksi-topology on $\mathbb{A}^n$ is the topology whose open sets are complements of algebraic sets.

Lemme 21
This indeed defines a topology on $\mathbb{A}^n$

Preuve

Certainly $\emptyset, \mathbb{A}^n$ are algebraic, hence their complements are open.
Let $\{U_i\}$ be a family of open sets, ie. such that

$$U_i = \mathbb{A}^2 \setminus V(I)$$

Then

$$\bigcup U_i = \bigcup \mathbb{A}^n \setminus V(I_i) = \mathbb{A}^n \setminus \bigcap_i V(I_i) = \mathbb{A}^n \setminus V(\bigcup I)$$

Similarly, if U_1, U_2 are open, then

$$U_1 \cap U_2 = \mathbb{A}^n \setminus I(V_1 V_2)$$

$\square$

is again open.

Exemple

If $n = 1$, then algebraically closed sets are either $\mathbb{A}^n, \emptyset$ are finite union of points so the Zariski topology is the cofinite topology. Hence the open sets are huge.

Definition 6

For $V \subset \mathbb{A}^n$ an algebraic variety or set, the Zariski topology on V is just the subspace topology.

Definition 7 (New definition of irreducibility)

A non-empty subset V of a topological space X is irreducible if it cannot be expressed as $V = W_1 \cup W_2$ where $W_1, W_2 \subsetneq V$ are closed subsets.

Lemme 23

A non-empty open subset of an irreducible topological space is again irreducible and dense.
Furthermore, if $V \subset X$ is irreducible, then so is $\overline{V}$

The proof is an exercise.

Definition 8 (Quasi-affine algebraic variety)

A quasi-affine variety is an open subset of an affine variety.

Remarque

By the lemma above, quasi-affine variety are also irreducible.

2.2 Regular functions and coordinate rings

Regular functions are the natural "continuous" functions on algebraic varieties.

2.2.1 Affine case

Definition 9

Let $V \subset \mathbb{A}^n$ be an affine algebraic variety.
A map

$$f : V \to K = \mathbb{A}^1$$

is regular if $\exists F \in k[X_1, \ldots, X_n]$ such that

$$f(X) = F(X) \forall X \in V$$

The set $\Gamma(V)$ of regular functions on V is a ring with the usual pointwise multiplication and addition. and is called the coordinate ring of V.

Lemme 25
If $I = I(V)$ for some prime, then

$$\Gamma(V) \simeq {k[X_1, \ldots, X_n]}\big/{I(V)}$$

In particular, $\Gamma(V)$ is a domain.

Preuve
By definition, we have a surjective morphism

$$k[X_1, \ldots, X_n] \to \Gamma(V)$$

Now note that $F \in \ker \phi \iff F(X) = 0 \forall x \in V \iff F \in I(V)$ $\square$

Definition 10 (Subobjects)
An affine subvariety of V is an affine variety contained in V.

Lemme 26
There is a one-to-one correspondence between V and $\Gamma(V)$ where

$$\{\text{ algebraic subsets of } V\} \leftrightarrow \{\text{ radical ideals of } \Gamma(V)\}$$

$$\{\text{ algebraic subvarieties of } V\} \leftrightarrow \{\text{ prime ideals of } \Gamma(V)\}$$

$$\{\text{ points of } V\} \leftrightarrow \{\text{ maximal ideals of } \Gamma(V)\}$$

The proof is again an exercise.

Definition 11 (Morphism)
A morphism $\phi : V \to W$ between affine algebraic varieties $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ is a map such that $\exists$ polynomials $T_1, \ldots, T_m \in k[X_1, \ldots, X_n]$ such that

$$\phi(X) = (T_1(X), \ldots, T_m(X))$$

Then ϕ is an isomorphism if there exists a morphism ψ such that $\phi \circ \psi = \mathrm{Id}$ and $\psi \circ \phi = \mathrm{Id}$.

Exemple
Take $V(X^2 - Y) \subset \mathbb{A}^2$ the the projection $p : V(X^2 - Y) \to \mathbb{A}^1$ on the first

coordinate is an isomorphism with inverse $\psi(X) = (X, X^2)$.
A non-example of a bijective map which is not an isomorphism :
$\phi : \mathbb{A}^1 \to V(Y^2 - X^3)$, $\phi(t) = (t^2, t^3)$.
One can check that ϕ *is bijective but not an isomorphism.*

Lecture 4: Morphisms of Affine Varieties

Fri 18 Mar

In general any morphism $\phi : V \to W$ induces a morphism of rings (of k-algebras) $\tilde{\phi} : \Gamma(W) \to \Gamma(V)$ by composition, ie.

$$\tilde{\phi}(f) = f \circ \phi$$

Proposition 28
This defines a one to one correspondence

$$\{ \text{ Morphisms } \phi : V \to W \ \} \leftrightarrow \left\{ \ k\text{-algebra homomorphisms } \tilde{\phi} : \Gamma(W) \to \Gamma(V) \ \right\}$$

In particular ϕ is an isomorphism iff $\tilde{\phi}$ is an isomorphism.

Preuve
Need to construct for any $\alpha : \Gamma(W) \to \Gamma(V)$ *a morphism* $\overline{\alpha} : V \to W$ *s.t.*

$$\tilde{\overline{\alpha}} = \alpha$$

Suppose $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ *and write*

$$\Gamma(V) = {k[x_1, \dots, x_n]}\big/{I(V)} \text{ and } \Gamma(W) = {k[y_1, \dots, y_m]}\big/{I(W)}$$

Choose lifts T_i *of* $\alpha([Y_i])$ *in* $k[x_1, \dots, x_n]$.
In particular $\forall f \in \Gamma(W)$ *and* F *a lift,then*

$$\alpha(f) = F(T_1, \dots, T_m) \mod I(V)$$

Then define $T : \mathbb{A}^n \to \mathbb{A}^m : x \mapsto (T_1(x) \dots T_m(x))$.
We claim that $T(V) \subset W$.
From the diagram, we see that for any $G \in I(W)$, $G(T_1, \dots, T_m) \in I(V)$, *hence for any* $v \in V$, $0 = G(T_1, \dots, T_m)(v) = G(T(v))$ *which means that* $T(v) \in W$.
Now

$$\tilde{\overline{\alpha}} : \Gamma(W) \to \Gamma(V)$$

satisfies $\forall v \in V \forall f \in \Gamma(W)$

$$\tilde{\overline{\alpha}}(v) = f(\overline{\alpha}(v)) = f(T(v)) = \alpha(f(v)) \implies \tilde{\overline{\alpha}} = \alpha$$

$\square$

Definition 12
The quotient field $K(V)$ of $\Gamma(V)$ is called the field of rational function on V.

Let $f \in K(V)$ is defined at a point $p \in V$ if we can write f as the quotient $f = \frac{a}{b}$ and $b(p) \neq 0$.
The pole set of $f \in K(V)$ is the set of points where f is not defined.

Remarque
$\Gamma(V)$ *is not a UFD in general, and so the presentation* $f = \frac{a}{b}$ *is not unique.*

Exemple
$V = (xy - zw) \subset \mathbb{A}^4$ *and let* $\overline{x}, \overline{y}, \overline{z}, \overline{w} \in \Gamma(V)$ *be the respective images.*
Then $f = \frac{\overline{x}}{\overline{y}} = \frac{\overline{z}}{\overline{w}}$.
Hence f *is defined whenever* $Y \neq 0$ *or* $w \neq 0$
Hence the pole set of f *is* $\{Y = 0\} \cap \{W = 0\}$

Definition 13 (Local Ring)
The local ring of V at a point $p \in V$ is a subring $K(V)$ defined by

$$\mathcal{O}_p(V) = \{f \in K(V) | f \text{ defined at } p\ \}$$

We have natural inclusions $\Gamma(V) \subset \mathcal{O}_p(V) \subset K(V)$

Remarque
$\Gamma(V), \mathcal{O}_p(V)$ *and* $K(V)$ *are intrinsic to* V*, ie. if* $V \simeq W$ *then* $\Gamma(V) \simeq \Gamma(W)$ *and* $\mathcal{O}_p(V) \simeq \mathcal{O}_{p'}(W)$

Proposition 32
Let $p \in V$ *and* $m_p \subset \Gamma(V)$ *be the corresponding maximal ideal, then*

$$\mathcal{O}_p(V) \simeq \Gamma(V)_{m_p}$$

In particular $\mathcal{O}_p(V)$ *is a noetherian local domain and we have that*

$$\Gamma(V) = \bigcap_{p \in V} \mathcal{O}_p(V) \subset K(V)$$

Preuve
Recall that $m_p = \{f \in \Gamma(V) | f(p) = 0\}$*, then*

$$\begin{aligned} \Gamma(V)_{m_p} &= \left\{ f \in K(V) | f = \frac{a}{b}, b \notin m_p \right\} \\ &= \mathcal{O}_p(V) \end{aligned}$$

The rest follows from standard properties of localization.

In particular for any domain R we have that

$$R = \bigcap_{m \in R, m \text{ maximal}} R_m$$

$\square$

Notice that the notions of regular functions is sufficient to define morphisms of local rings etc.
How can we extend this to quasi-affine varieties ?

Exemple
Consider $V(XY-1) \subset \mathbb{A}^2$.
There is a natural projection $\phi : V(XY-1) \to x \in \mathbb{A}^1$.
The image of ϕ is $\mathbb{A}^n \setminus \{0\}$ quasi-affine and we'd like ϕ to be an isomorphism, ie.

$$\phi^{-1}(x) = (x, \frac{1}{x})$$

Ie. the map $x \to \frac{1}{x}$ should be a regular function on $\mathbb{A}^1 \setminus \{0\}$.

Definition 14
Let $V \subset \mathbb{A}^n$ be quasi-affine.
A map $f : V \to \mathbb{A}^1 = k$ is called regular if $\forall v \in V$ there exists an open neighbourhood $v \in U \subset V$ and $g, h \in k[x_1, \ldots, x_n]$ s.t. $h(V) \neq 0 \forall x \in U$ and $f(x) = \frac{g(x)}{h(x)}$

Why do we need the U ?

Exemple
Consider again $V = V(XY - ZW) \setminus V(Y, W)$ and consider $f = \frac{x}{w} = \frac{z}{y}$ on V.
None of the two presentations works on V

Definition 15
Let $\mathcal{O}(V)$ be the ring of regular functions on V

Remarque
$f : V\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 : x \mapsto \frac{1}{x}$ is regular.
Then we may take $U = V$, it is not hard to see that

$$\mathcal{O}(V) = k[x][\frac{1}{x}, \frac{1}{x^2}, \ldots]$$

In particular $\mathcal{O}(V) \supsetneq \Gamma(\mathbb{A}^1)$

If $V \subset \mathbb{A}^n$ is affine, then we have $k[x_1, \ldots, x_n] \to \mathcal{O}(V) : F \mapsto (v \mapsto F(v))$.

Proposition 36
For V affine, we have that $\Gamma(V) \simeq \mathcal{O}(V)$.

Preuve
We have $O(V) \subset O_p(V)$ $\forall p \in V$ *hence* $\Gamma(V) \hookrightarrow O(V) \hookrightarrow \bigcap_{p\in V} O_p(V) = \Gamma(V)$ $\square$

Lemme 37
Let V *be a quasi-affine subset and* $f : V \to \mathbb{A}^1$ *regular, then* f *is continuous (with respect to the Zariski topology)*

Preuve
It is enough to show that $f^{-1}(X)$ *is closed for any closed* X *.*
Without loss of generality $X = \{x\}$*.*
Let $V = \cup_i U_i i$ *a cover such that* $f|_{U_i} = \frac{g_i}{h_i}$ *and* $h_i \neq 0$ *on* U_i*.*
Then $f^{-1}(X) \cap U_i = \left\{ v \in U_i | f(v) = \frac{g_i(v)}{h_i(v)} \right\} = \{v \in U_i | x \cdot h_i(v) - g_i(v) = 0\}$
which is an algebraic set.
Hence $f^{-1}(X) \cap U_i$ *is closed which implies* $f^{-1}(X)$ *is closed.* $\square$

Corollaire 38
Let $f, g \in O(V)$ *and* $U \subset V$ *non empty and open s.t.* $f|_U = g|_U$ *then* $f = g$

Preuve
Using an exercise, open subsets are dense, since f, g *are continuous*

$$f|_U = g|_U \implies f|_{\mathrm{cl}\, U} = f|_{\mathrm{cl}\, V}$$ $\square$

Remarque
Let $U \subset V$ *open, then the restriction of functions induces* $\mathcal{O}(V) \to \mathcal{O}(U)$*.*
i.e. $\mathcal{O}(-)$ *defines a sheaf of* k*-algebras on* V*.*
Using this one can define a general algebraic as a topological space X *with some sheaf* $\mathcal{O}_X$ *which locally looks like a quasi-affine variety* V *with* $\mathcal{O}(-)$*.*
We'll define $\mathcal{O}_p(V)$ *and* $K(V)$ *for* V *quasi-affine, but these depend only on "local structure".*
We can guess $\mathcal{O}_p(V) = \mathcal{O}_p(\mathrm{cl}\, V)$ *and similarly for the quotient field.*

3 (Quasi-)Projective and general algebraic varieties

Affine varieties usually "go to infinity" when we draw them.
This leads to complications in the theory

Exemple
Two distinct lines in $\mathbb{A}^2$ they will intersect in 1 point unless they're parallel

3.1 Projective space

Definition 16 (Projective n-space)
$\mathbb{P}^n$ is the set

$$\mathbb{P}^n = K^{n+1} \setminus \{0\} / \sim$$

Where we identify

$$(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1}) \text{ if } \exists \lambda \in K^* \text{ s.t. } x_i = \lambda y_i$$

Elements in $\mathbb{P}^n$ are called points.

If $p \in \mathbb{P}^n$ is the equivalence classe of $(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1}$ we write

$$p = [x_1 : \dots : x_n]$$

$x_1, \dots, x_n$ are the homogenuous coordinates of p.

Remarque
Any point in $\mathbb{A}^n \setminus \{0\}$ defines a line through the origin and $x, y \in \mathbb{A}^n \setminus \{0\}$ define the same line iff $x = \lambda y$

Lecture 5: Projective varieties

Fri 25 Mar

While the i-th coordinate x_i of a point $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n$ is not well defined, the equation $x_i = 0$ or $x_i \neq 0$ is well defined.
Hence we can write

$$U_i = \{[x_1 : \dots : x_n] | x_i \neq 0\}$$

Clearly $\mathbb{P}^n = \cup_i U_i$.
Furthermore for all i, we have a bijection

$$\begin{aligned} \phi_i : \mathbb{A}^n &\to U_i \\ (x_1, \dots, x_n) &\mapsto [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_{n+1}] \end{aligned}$$

And this is clearly a bijection.
We'll see in a bit, that the ϕ_i's provide an open cover of $\mathbb{P}^n$ by $\mathbb{A}^n$

Definition 17
The set

$$H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{x \in \mathbb{P}^n | x_{n+1} = 0\}$$

is called the hyperplane at infinity.

One can identify $H_\infty = \mathbb{P}^{n-1}$
Thus

$$\mathbb{P}^n = U_{n+1} \coprod H_\infty = \mathbb{A}^n \coprod \mathbb{P}^{n-1}$$

Exemple
$\mathbb{P}^0 =$ *point*
$\mathbb{P}^1 = \mathbb{A}^1 \coprod$ *point is called the projective line.*
Similarly $\mathbb{P}^2$ *is called the projective plane.*

3.2 Projective algebraic sets

For a general $F \in k[x_1, \dots, x_n]$, the equation $F(x) = 0, x \in \mathbb{P}^n$ doesn't make sense.
But it does if F is homogeneous, say of degree d, since then

$$F(\lambda x) = \lambda^d F(x) = 0 \forall x \in \mathbb{A}^{n+1}, \lambda \in k^*$$

Definition 18 (Projective set)
For any set $S \subset k[x_1, \dots, x_n]$ *of homogeneous polynomials we set*

$$V(S) = \{[x_1 : \dots : x_n] \in \mathbb{P}^n | F(x_1, \dots, x_n) = 0 \forall F \in S\}$$

A subset of $\mathbb{P}^n$ *is algebraic if it is of the form* $V(S)$ *as above.*

Exemple
Take $V(X^2 - YZ) \subset \mathbb{P}^2$*, how to draw it ?*
We draw the intersections $V \cap U_i$

Definition 19 (Homogeneous ideal)
An ideal $I \subset k[x_1, \dots, x_n]$ *is homogeneous if it is generated by homogeneous elements.*
The for I *a homogeneous ideal we set*

$$V(I) = V(T) \subset \mathbb{P}^n$$

where T *is the set of forms in* I*.*

Remarque
Since the ring is noetherian, we can always find a finite number of homogeneous generators.
For $I = (x_1, \dots, x_{n+1})$ *we have* $V(I) = \emptyset$*, we denote this ideal by* I_+*, it's called the irrelevant ideal.*

Exemple
(x, y^2) is homogeneous, $(x+y^2, y^2)$ is also homogeneous but $(x+y^2)$ is not.

Lemme 46
I is a homogeneous ideal if and only if for every $F \in I$, if we write $F = \sum_{i \geq 0} F_i$ with F_i homogeneous of degree i.

Preuve
Let $G^{(1)}, \ldots, G^{(k)}$ be a set of homogeneous generators of I with degrees $d_1, \ldots, d_k$.
Any $F = \sum F_i$ can be written as $F = \sum A^{(i)} G^{(i)}$ for some $A^{(i)}$.
Since the degree is additive we get $F_j = \sum A^{(i)}_{j-d_i} G^{(i)}$
For the other direction, let $G^{(1), \ldots, G^{(k)}}$ any set of generators, then $G^{(i)}_j \in I$ and then the set of $G^{(i)}_j$ is a set of generators. $\square$

Furthermore, the sum, the product, the intersection and the radical of homogeneous ideals are homogeneous.
A homogeneous ideal is prime if for any homogeneous $f, g \in k[x_1, \ldots, x_n]$

$$fg \in I \implies f \in I \text{ or } g \in I$$

Definition 20 (Zariski topology)
We define the Zariski topology on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets.

This defines a topology using the properties above.

Definition 21
An algebraic set $V \subset \mathbb{P}^n$ is irreduciable if it is irreducible as a topological space.

As in the affine case, there is a correspondence

$$\{ \text{ Algebraic subsets in} \mathbb{P}^n \} \leftrightarrow \{ \text{ Homogeneous ideals in } k[x_1, \ldots, x_{n+1}] \}$$

Where $I(V)$ is the ideal generated by $\{F \in k[x_1, \ldots, x_n] | F \text{ homogeneous }, F(v) = 0 \forall v \in V\}$

Remarque
If we need to distinguish between the affine and projective correspondence we'll write V_a, I_a and V_p, I_p respectively.

Definition 22 (Cone)
For $V \subset \mathbb{P}^n$ algebraic, we define the conve over V as

$$C(V) = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1} | [x_1, \dots, x_{n+1}] \in V\right\} \cup \{(0, \dots, 0)\}$$

Lemme 48

1. *For $V \neq \emptyset$, then*
$$I_p(V) = I_a(C(V))$$
2. *If $I \subsetneq k[x_1, \dots, x_n]$ homogeneous, then*
$$C(V_p(I)) = V_a(I)$$

Preuve

1. $G \in I_p(V)$ *homogeneous and* $(x_1, \dots, x_{n+1}) \in C(V)$, *then* $G(x_1, \dots, x_{n+1}) = 0$
Conversely, if $G \in I_a(C(V))$ *write*
$$G = \sum_i G_i, \quad G_i \text{ homogeneous}$$
Then, for every $x \in C(V)$ *and* $\lambda \in k^*$ *we have* $\lambda x \in C(V)$ *hence*
$$0 = G(\lambda x) = \sum_i \lambda^i G_i(x)$$
Let $\tilde{G}(y) = \sum_i y^i G_i(x) \in K[Y]$, *this has infinitely many* 0*'s.*
Which in turn implies $G_i \in I_p(V)$
2. *Notice for* G *homogeneous non-constant, then*
$$C(V_p(G)) = V_a(G)$$
Since I *is generated by homogeneous polynoials, the satement holds.*
□

Proposition 49 (Projective nullstellensatz)
Let I *be a homogeneous ideal, then*

— *If* $V_p(I) = \emptyset$, *then* $\sqrt{I} = k[x_1, \dots, x_{n+1}]$ *or* $\sqrt{I} = I_+$
— *If* $V_p(I) = \emptyset$ *then* $I_p(V_p(I)) = \sqrt{I}$

Preuve

— *If* $V_p(I) = \emptyset \iff V_a(I) \subset \{(0, \dots, 0)\}$ *which implies* $\sqrt{I} \supset (x_1, \dots, x_{n+1})$.

— $I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I}$ $\square$

Corollaire 50
There is a one-to-one correspondence between radical homogeneous ideals and projective algebraic sets.
Furthermore $V_p(I)$ is irreducible $\iff$ I is prime.

Remarque
Points in $\mathbb{P}^n$ do not correspond to maximal ideals.

We can also relate affine and projective algebraic sets through the charts

$$\phi_i : \mathbb{A}^n \to U_i$$

We'll focus on $\phi \coloneqq \phi_{n+1} : \mathbb{A}^n \to U \coloneqq U_{n+1}$
For $F \in k[x_1, \ldots, x_n]$ homogeneous, we define

$$F_*(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 1)$$

Conversely, for $G \in k[x_1, \ldots, x_n]$, we write

$G = \sum_{i=0}^{d} G_i$ and define $G^*(x_1, \ldots, x_{n+1}) = x_{n+1}^d G_0 + \ldots + G_d = X_{n+1}^d G(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$

Definition 23 (Homogenization)
$(\cdot)_*$ *and* $(\cdot)^*$ *are called dehomogenisation and homogenization.*

For I an ideal, we denote by I^* be the homogeneous ideal generated by $\{F^* | F \in I\}$.
Conversely, if $V = V_a(I)$, we write

$$V^* = V_p(I^*)$$

V^* is called the projective closudre of V in $\mathbb{P}^n$.
Similarly if I is homogeneous, then

$$I_* = \{F_* | F \in I\}$$

and if $V = V_p(I)$, we set $V_* = V_a(I_*)$

Exemple
Let $F = X_1^2 - X_2$, then

$$F^* = X_1^2 - X_2X_3$$

Lemme 53

If $V \subset \mathbb{A}^n$ *is closed, then* $\phi(V) = V^* \cap U$

Conversely, if $V \subset \mathbb{P}^n$ *is closed then* $\phi^{-1}(V \cap U) = V_*$

In particular ϕ *is a homeomorphism*

Preuve

Recall that $\phi(x_1, \ldots, x_n) = [x_1 : \ldots : x_n : 1]$.

For $V \subset \mathbb{A}^n$ *write* $V = V_a(F_1, \ldots, F_k)$ *then*

$$V^* = V_p(F_1^*, \ldots, F_k^*)$$

But $F_i^* = X_{n+1}^d F_i(\frac{x_1}{x_{n+1}, \ldots, \frac{x_n}{x_{n+1}}})$ $F_i(v) = 0 \iff F_i^*(\phi(v)) = 0 \implies \phi(V) = V^* \cap U$ □