Algebraic Geometry I

David Wiedemann

Table des matières

1 Presheaves and Sheaves **4**
- 1.1 Presheaves 4
- 1.2 Sheaves 5
- 1.3 Stalks 7
- 1.4 Kernels, cokernels, exactness 12
- 1.5 Direct and inverse image, ringed spaces 15

2 Properties of schemes **22**
- 2.1 Topological properties 22
- 2.2 Scheme-Theoretic Properties 23

3 Open and closed subschemes and immersions **27**

List of Theorems

- 1 Definition (Presheaf) 4
- 2 Definition (Morphism of presheaves) 5
- 3 Definition (Sheaf) 5
- 4 Definition (Morphisms of sheaves) 6
- 5 Definition (Natural sheaf on Spec A) 7
- 6 Definition 7
- 7 Definition 8
- 8 Definition (direct limit) 8
- 9 Definition 9
- 10 Definition (Sheafification) 9
- 20 Proposition 10
- 22 Proposition 10
- 23 Corollary 11
- 11 Definition (Subsheaf) 12
- 12 Definition (Kernel, cokernel of presheaves) 12
- 24 Lemma 12

13 Definition (Cokernel/image of morphisms of sheaves) 12
26 Lemma (cokernels are cokernels) 13
27 Proposition 13
28 Proposition 13
30 Corollary 14
14 Definition (Exact Sequence of sheaves) 14
31 Corollary 15
32 Corollary 15
33 Corollary 15
34 Corollary 15
15 Definition 15
16 Definition (inverse image) 15
35 Lemma 16
36 Proposition 16
37 Corollary 16
38 Corollary 17
17 Definition (Ringed space) 17
18 Definition (Morphism of local rings) 17
19 Definition (Locally ringed space) 18
20 Definition (Affine Scheme) 18
21 Definition (Scheme) 18
46 Lemma 19
47 Proposition 19
48 Corollary 19
49 Corollary 19
50 Corollary 19
22 Definition (Scheme over another scheme) 21
23 Definition 22
56 Lemma 22
24 Definition (Open Subscheme) 23
58 Lemma 23
25 Definition (Reduced Scheme) 23
26 Definition 23
62 Lemma 24
63 Lemma 24
64 Lemma 25
27 Definition 25
28 Definition (Noetherian scheme) 25
29 Definition (Integral scheme) 26
65 Lemma 26
30 Definition (Open Subscheme) 27

66 Lemma 27
31 Definition (Open immersion) 27
32 Definition (Ideal sheaves) 28
33 Definition (Closed Subsceme) 28
34 Definition (Closed immersion) 29
74 Proposition 29

Lecture 1: Intro

Mon 10 Oct

Quick Motivation

We study schemes.
These are objects that "look locally" like $(\operatorname{Spec} A, A)$.
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— $\forall U \subset X$ *open, an object in* C $\mathcal{F}(U)$

— $\forall V \subset U \subset X$ *open, a morphism* $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$

such that

— $\forall U$ *open,* $\rho_{U,U}$ *is the identity on* $\mathcal{F}(U)$

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of* $\mathcal{F}(U)$ *are called sections of* $\mathcal{F}$ *over* U.

— $\mathcal{F}(U)$ *is called the space of sections of* $\mathcal{F}$ *over* U

— *Elements of* $\mathcal{F}(X)$ *are called global sections.*

— *There are alternative notations for* $\mathcal{F}(U)$: $\Gamma(U, F)$ *or* $H_0(F)$

— *The* ρ_{UV} *are called restriction maps, for* $s \in \mathcal{F}(U)$*, we write* $s|_V := \rho_{UV}(s)$ *and is called restriction of* s *to* V.

Example

— *For any object* A *in* C*, we define the constant presheaf* $\underline{A}'$ *defined by* $\underline{A}'(U) = A$ *and with restriction maps the identity.*

- *The presheaf of continuous functions :* C^0.
  We define $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*
- *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

  This example is universal in a certain sense

Remark
Define the category Ouv_X *with*

- *objects* $U \subset X$ *open subsets*
- *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[r, "\rho(U)"] \arrow[d] & \mathcal{F}_2(U) \arrow[d] \\ \mathcal{F}_1(V) \arrow[r, "\rho(V)"'] & \mathcal{F}_2(V) \end{tikzcd}$$

Example

- *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$
- *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
- *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark
There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \ \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$.
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*
— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$

Example

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

— *$PSh_C(X)$ is the category of presheaves of C on X*
— *$Sh_C(X)$ is the category of sheaves of C on X*
If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then Spec A is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6
Let $(I, \leq)$ be a pair where I is a set and $\leq$ is a binary relation.
$(I, \leq)$ is called a preorder if ll is reflexive and transitive.
$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric
$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example

1. *Let $I = \{U \subset X | U \text{ open } \}$ and $U \leq V \iff V \subset U$.*
   Then I is a directed poset.
2. *For $x \in X$, let*
$$I_x = \{U \subset X | U \text{ open } x \in U\}$$
   This is a directed poset.

Definition 7
Let $(I, \leq)$ *be a directed set and* C *a category.*
A direct system in C *indexed by* I *is a pair* $\left(\{A_i\}, \{\rho_{ij}\}_{i,j \in I, i \leq j}\right)$.
Where the A_i *are objects in* C*, the* $\rho_{ij} : A_i \to A_j$ *are morphisms in* C *such that*

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example
If $\mathcal{F}$ *is a presheaf of* C *on* X *and* I_X *as in the second example above, then*

$$\left(\{\mathcal{F}(U_i)_{U_i \in I_X}\}, \{\rho_{U_i,U_j}\}\right)$$

is a direct system.

Definition 8 (direct limit)
Let $(I, \leq)$ *be a directed set,* C *a category.*
Let $\left(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I}\right)$ *be a directed system, then the direct limit is a pair* $(\lim_{i \in I} A_i, \{\rho_i\}_{i \in I})$ *where* $\lim A_i$ *is in* C *and* $\rho_i : A_i \to \lim A_i$ *such that*

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects* A *in* C *and morphisms* $f_i : A_i \to A$ *such that*

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

$\exists! f : \lim_{i \in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

Remark
The direct limit is unique up to unique isomorphism.

Example
Write $(*) = (\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.
Let $*$ *be a direct systement in* Set.
Let $\lim_{i \in I} A_i := A_i / \sim$ *where* $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ *such that* $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
This is the direct limit of $*$.
If $*$ *is a system in* Ab *, let* $\lim A_i := \bigoplus A_i / N$ *with* $N = \langle a_i - \rho_{ij}(a_i) \rangle$.
The natural map $\bigcup A_i / \sim \to \bigoplus A_i / N$ *is a bijection*

Remark
Taking the direct limits in (Ab) *is exact in the following sense :*
$\forall$ *directed sets* I*,* $\forall$ *direct systems* $\{M_i\}, \{N_i\}, \{P_i\}$ *indexed by* I *and for all*

collections of commutative diagrams, we get

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

Definition 9
Let C be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of C on X.
Then the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x.
For $s \in \mathcal{F}(U)$, we write s_x for the image of s in $\mathcal{F}_x$ and call it the germ of s at x.

Remark
A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$

Remark
Let $x \in X, \mathcal{F}$ a presheaf of $\mathrm{Set}, \mathrm{Ab}$

1. $\forall U \subset X$ *open,* $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

Definition 10 (Sheafification)
Let $\mathcal{F}$ be a presheaf of sets (...) on X.
The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

Remark

1. $\mathcal{F}^+$ *is a sheaf*
2. *Sheafification is functorial.*
   For $\rho : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, the collection $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ sending $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. $\exists$ *a natural morphism* $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ *defined by* $\iota_F(U)(s) : x \to s_x$
4. $\forall s \in \mathcal{F}^+(U)$ *there is an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

Proposition 20

$\forall$ *morphisms* $\phi : \mathcal{F} \to \mathcal{G}$ *such that* $\mathcal{G}$ *is a sheaf, there exists a unique morphism* $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ *such that* $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ *open, let* $s \in \mathcal{F}^+(U)$ $\exists$ *an open cover* $U = \bigcup_{i\in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.
Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, *we have to set*

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since $\mathcal{G}$ *is a sheaf and*

$$\phi(U_i)(s_i)|_{U_i\cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i\cap U_j}) = \phi(U_j)(s_i)|_{U_i\cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ *with* $t|_{U_i} = \phi(U_i)(s_i)$.
For ϕ^+ *to be a morphism, we have to set* $\phi^+(U)(s) = t$.
We still have to check that ϕ^+ *is compatible with restriction maps.* $\square$

Remark

The proposition above shows that $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ *naturally in the presheaf and the sheaf* $\mathcal{G}$.
Hence $(-)^+$ *is left-adjoint to the forgetful functor* $Sh(X) \to Psh(X)$

Proposition 22

$X = \operatorname{Spec} A$ $\forall a \in A$ *there exist isomorphisms* $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *such that* $\forall b \in A$ *with* $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow[\sim]{} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *by sending* $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.
Clearly, these make the diagram commute.
This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.
Let $I = Ann(s) = \{r \in A | rs = 0\}$.
Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, *we have* $I \not\subset p$, *hence* $V(I) \subset V(a) \implies a \in \sqrt{I}$.
Thus there exists $m \geq 1$ *such that* $a^m s = 0$, *hece* $\frac{s}{a^n} = 0$.
To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, *by definition of* $\mathcal{O}_X$ *and because* $D(h_i)$ *form a basis, we find* $a_i, g_i, h_i \in A$ *such that*

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

and $s(q) = \frac{a_i}{g_i}$ *for all* $q \in D(h_i)$.

1. *Claim 1 : Can choose* $g_i = h_i$
2. *Claim 2 : Can choose* I *finite*
3. *Claim 3 : Can choose* a_i, h_i *such that* $h_j a_i = h_i a_j$.

Using these claims, since $D(a) = \bigcup D(h_i)$ *, we find* $n > 0, b_j \in A$ *such that* $a^n = \sum b_j h_j$.
Write $c = \sum a_i b_i$.
Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.
Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.
We now prove the claims

1. *We have* $D(h_i) \subset D(g_i)$ *thus* $V(g_i) \subset V(h_i)$ *and thus* $h_i \in \sqrt{(g_i)}$.
   So there exists $c_i \in A$ *and* $n > 1$ *such that* $h_i^n = c_i g_i$.
   Now, we replace h_i *by* h_i^n *and* a_i *by* $a_i c_i$. *Then*
   $$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$
2. *We have* $D(a) \subset \cup D(h_i) \iff V(\sum h_i) = \cap V(h_i) \subset V(a)$.
   This is equivalent to saying that $a \in \sqrt{\sum(h_i)}$.
   Thus there exists $n \geq 1$ *such that* $a^n \in \sum_i (h_i)$.
   So there exist finitely many $b_i \in A$ *such that* $a^n = \sum b_j h_j$
3. *On* $D(h_i) \cap D(h_j) = D(h_i h_j)$, *we have*
   $$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$
   Thus
   $$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$
   Thus, there exists $N_j \geq 1$ *such that* $(h_i h_j)^{N_j}(h_j a_i - h_i a_j) = 0$.
   From claim 2, I *is finite, so we can choose* N *big enough such that* N *works for all* $D(h_i)$.
   Now, we replace h_i *by* h_i^{N+1} *and* a_i *by* $h_i^N a_i$ *and we get* $h_j a_i - h_i a_j = 0 \in A$. □

Corollary 23
Take $X = \operatorname{Spec} A$, *then* $\forall p \in \operatorname{Spec} A \exists$ *isomorphisms* $\phi_p : A_p \to \mathcal{O}_{X,p}$ *such that the appropriate diagram commutes.*

Proof

1. *Observe* $\lim_{a \in A \setminus p} = A_a \simeq A_p$ *(check universal property)*
2. *Observe that* $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$

Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

Definition 11 (Subsheaf)
Let $\mathcal{F}$ be a (pre)sheaf on X.
Then a sub(pre)sheaf of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by $\mathcal{F}$.

Definition 12 (Kernel, cokernel of presheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *The presheaf kernel of ϕ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$*
2. *The presheaf image is defined as $\operatorname{Im}^{pre}(\phi)(U) = \operatorname{Im}(\phi(U))$*
3. *The presheaf cokernel is $\operatorname{coker}^{pre}(\phi)(U) = \operatorname{coker}(\phi(U))$.*

In each case, the restriction maps are induced by those in of $\mathcal{F}$ or $\mathcal{G}$.

Lemma 24
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf kernel is a sheaf.

Proof
Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
Since $\mathcal{F}$ is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.
Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.
Thus

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

Since $\mathcal{G}$ is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$. □

Example
By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are.

Definition 13 (Cokernel/image of morphisms of sheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *sheaf kernel : $\ker^{pre}(\phi)$*

2. *sheaf image* $(\mathrm{Im}^{pre}(\phi))^+$
3. *sheaf cokernel* $(\mathrm{coker}^{pre}(\phi))^+$

Lemma 26 (cokernels are cokernels)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. $\ker \phi \to \mathcal{F}$ *is a categorical kernel in* $Sh(X)$
2. $\mathcal{G} \to \operatorname{coker} \phi$ *is a categorical cokernel in* $Sh(X)$.

Proof

1. *This means that for each commutative diagram with solid arrows, the dotted arrow is unique*
   "Insert cokernel/kernel diagram here"
   This holds for every open U and so the kernel is a sheaf.
2. *The appropriate diagram commutes and we use the universal property of sheafification.* □

Proposition 27
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is a monomorphism in* $Sh(X)$
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

Proof
Recall ϕ is a monomorphism if for every $\psi : \mathcal{F}' \to \mathcal{F}, \phi \circ \psi = 0 \implies \psi = 0$.
The implication $1 \implies 2$ follows by applying the monomorphism property to $\ker \phi \to \mathcal{F}$ $2 \implies 1$ *If $\phi \circ \psi = 0$, then ψ factors through the kernel* $\ker \phi \to \mathcal{F}$ *and so $\psi = 0$*
$2 \iff 3$ *Since* $\ker(\phi)(U) = \ker(\phi(U))$
$3 \implies 4$ *Follows because taking direct limits is exact.*
$4 \implies 3$ *Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.*
So $s_x = 0 \forall x \in U$ and so $s = 0$ □

Proposition 28
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is an epimorphism in* $Sh(X)$
2. $\operatorname{coker}(\phi) = 0$
3. $\operatorname{coker}(\phi_x) = 0$

Proof

Recall that ϕ is an epimorphism if for every $\psi : \mathcal{G} \to \mathcal{G}', \psi \circ \phi = 0 \implies \psi = 0$

$1 \implies 2$ *Apply epimorphism property to $\mathcal{G} \to \operatorname{coker}(\phi)$*

$2 \implies 3$ *We have*

$$\begin{aligned} 0 &= (\operatorname{coker}\phi)_x \\ &= (\operatorname{coker}^{pre}\phi)_x = \operatorname{coker}(\phi_x) \end{aligned}$$

□

$3 \implies 1$

Let $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\psi \circ \phi = 0$, this implies that $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$.

Since ϕ_x is an epimorphism of abelian groups, we get $\psi_x = 0$.

As the hom sheaf is a sheaf, we get that $\psi = 0$

Remark

If $\operatorname{coker}(\phi(U)) = 0 \forall U \subset X \implies \operatorname{coker}(\phi) = 0$ but the converse is not true.

Corollary 30

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent

1. *ϕ is an isomorphism*
2. *$\phi(U)$ is an isomorphism $\forall U \subset X$ open*
3. *ϕ_x is an isomorphism $\forall x \in X$*

Proof

$1 \implies 2$ *since taking sections is a functor*

$2 \implies 3$ *since taking limits is functorial*

$2 \implies 1$ *because $(\phi(U))^{-1}$ defines a morphism of sheaves*

$3 \implies 2$ *Need to show surjectivity of $\phi(U)$.*

Let $t \in \mathcal{G}(U)$, since ϕ_x is an isomorphism $\forall x \in U$, we find $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$.

There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ such that $(s_{V_x})_x = s_x$

Since

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

we can choose $V + x$ sufficiently small such that $\phi(V_x)(s_{V_x}) = t|_{V_x}$.

Since $\phi(V_x \cap V_y)$ is injective and $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$, we have $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$.

Thus there exists $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ and $\phi(U)(s)|_{V_x} = t|_{V_x}$ and thus $\phi(U)(s) = t$. □

Definition 14 (Exact Sequence of sheaves)

A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ is called exact if $\ker\phi_2 = \operatorname{Im}\phi_1$

Corollary 31
A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.

Lecture 4: locally ringed spaces, (affine) Schemes (!)

Fri 21 Oct

Corollary 32
A sequence of sheaves is exact if and only if it is exact on all stalks.

Proof
If $\ker(\phi_{2,x}) = \operatorname{Im}(\phi_{1,x}) \forall x \in X$, *thus* $(\phi_{2,x} \circ \phi_{1,x}) = (\phi_2 \circ \phi_1)_x$.
Thus $\phi_2 \circ \phi_1 = 0$ *because the hom sheaf is a sheaf.*
Thus ϕ_1 *factors as* $\mathcal{F}_1 \to \operatorname{Im} \phi_1 \to \ker \phi_2 \to \mathcal{F}_2$ *as* ψ_x *is an isomorphism,* ψ *is an isomorphism.* □

Corollary 33
Let $\phi : \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves, then* $\operatorname{Im} \phi = \ker(\mathcal{G} to \operatorname{coker} \phi)$

Corollary 34
$Sh(X)$ *is an abelian category.*

1.5 Direct and inverse image, ringed spaces

Definition 15
Let $f : X \to Y$ *be a continuous map.*
We define the direct image of $\mathcal{F}$ *by* f *on* Y *defined by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

We can check that $f_*\mathcal{F}$ is a sheaf with restriction maps induced by $\mathcal{F}$.
If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, then the $(f_*\phi)(X) = \phi(f^{-1}(V))\mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}V)$ define a morphism of sheaves.
Thus we get a functor $f_* : Sh(X) \to Sh(Y)$.

Definition 16 (inverse image)
Let $f : X \to Y$ *be a continuous map and let* $\mathcal{G}$ *be a sheaf on* Y.

The inverse image of $\mathcal{G}$ along f is the sheafification of the presheaf

$$f^{-1,pre}(\mathcal{G})$$

defined by

$$f^{-1,pre}(\mathcal{G})(U) = \varinjlim_{f(U)\subset V} \mathcal{G}(V)$$

We can again check that the if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on Y, we define $f^{-1}\phi : \varinjlim \mathcal{F}(V) \to \varinjlim \mathcal{G}(V)$ using the maps induced by ϕ.
Thus we get a functor $Sh(Y) \to Sh(X)$.

Lemma 35
Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

1. *$\forall y \in Y$ there is a natural isomorphism*

$$(f_*\mathcal{F})_y \simeq \varinjlim_{y\in V\subset Y} \mathcal{F}(f^{-1}(V))$$

In particular $forall x \in X$ there is a natural map $(f_*\mathcal{F})_{f(x)\to\mathcal{F}_x}$

2. *$\forall x \in X$ there is a natural isomorphism $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$*

Proof
The isomorphisms are immediate from the definition.
The morphism $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ is given by

$$(f_*\mathcal{F})_{f(x)} = \varinjlim \mathcal{F}(f^{-1}(V)) = \varinjlim_{x\in f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \to \varinjlim_{x\in U} \mathcal{F}(U) = \mathcal{F}_x$$

$\square$

Proposition 36
If $f : X \to Y$ is a continuous map, then $f_* : Sh(X) \to Sh(Y)$ is right-adjoint to $f^{-1} : Sh(Y) \to Sh(X)$

Corollary 37
$f^{-1} : Sh(Y) \to Sh(X)$ is exact

Proof
Let $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ be exact in $Sh(Y)$.
Thus $\forall y \in Y, 0 \to \mathcal{G}_{1,y} \to \mathcal{G}_{2,y} \to \mathcal{G}_{3,y} \to 0$ is exact.
In particular it is exact at $f(x)\forall x \in X$ and thus the associated inverse image

sequence is exact. □

Corollary 38
$f_* : Sh(X) \to Sh(Y)$ *is left-exact.*

Proof
Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *be exact in* $Sh(X)$.
Recall that the section functor is left-exact, thus $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ *is exact* $\forall U \subset X$.
Thus $0 \to (f_*\mathcal{F}_1)_y \to (f_*\mathcal{F}_2)_y \to (f_*\mathcal{F}_3)_y$ *is exact* $\forall y \in Y$ *and thus* $0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ *is exact.* □

Example
f_* *is usually not right-exact.*
Eg, if $f : X \to \{*\}$ *and* $\mathcal{F}$ *is a sheaf on* X, *then* $(f_*\mathcal{F})(\emptyset) = 0$ *and* $(f_*\mathcal{F})(\{*\}) = \mathcal{F}(X)$ *and taking sections is not exact.*

Definition 17 (Ringed space)
A ringed space is a pair $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X.
A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a pair* $(f, f^\sharp)$ *where* $f : X \to Y$ *is a continous map and* $f^\sharp$ *is a morphism* $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark
Ringed spaces form a category, if $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ *define their composition to be* $(g \circ f, g_*(f^\sharp \circ g^\sharp))$

Example

1. *For every ring* A, $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a ringed space.*
2. *For any field* K *and any topological space* X, *define a sheaf* $Fun_{X,K}(U) = \{s : U \to K\}$.
   There is a functor $\top \to$ (*Ringed spaces*) *sending* $X \mapsto (X, Fun_{X,K})$ *where for* $f : X \to Y$ $f^\sharp$ *is the pullback (precomposition).*
3. (X, C^0_X) *is a ringed space*

Observe that for many of these examples of ringed spaces, the stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 18 (Morphism of local rings)
A morphism of local rings $\phi : A \to B$ *with maximal ideals* m_A *and* m_B *is called local if* $m_A = \phi^{-1}(m_B)$

Example

1. *For all ring homomorphism* $\phi : A \to B$ *and all* $q \in \operatorname{Spec} B$ *the induced map* $A_{\phi^{-1}(q)} \to B_q$ *is local.*
2. *A ring homomorphism* $\phi : A \to K$ *from a local ring* A *to a field iff* $m_A = \ker \phi$

Definition 19 (Locally ringed space)
A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ *such that* $\mathcal{O}_{X,x}$ *is local* $\forall x \in X$.
A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a morphism of ringed spaces such that*

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \xrightarrow{f_x^\sharp} (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$$

is local.

Remark
The category of locally ringed spaces is a subcategory of the category of ringed spaces

Definition 20 (Affine Scheme)
An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ *such that* $X = \operatorname{Spec} A$ *and* $\mathcal{O}_X$ *is the structure sheaf.*

Definition 21 (Scheme)
A scheme is a locally ringed space $(X, \mathcal{O}_X)$ *such that there exists an open cover* $X = \bigcup_{i \in I} U_i$ *such that each* $(U_i, \mathcal{O}_X|_{U_i})$ *is an affine scheme.*
A morphism of schemes is a morphism of the underlying ringed spaces.

Example

1. *If* $(X, \mathcal{O}_X)$ *is a scheme and* $U \subset X$ *is open, then* $(U, \mathcal{O}_X|_U)$ *is not necessarily a scheme (even if* X *is affine).*
2. *If* $(X, \mathcal{O}_X)$ *is a scheme and* $X = \{*\}$*, then* X *is affine.*
   Then $\operatorname{Spec} A = \{*\}$ *iff every* $a \in A$ *is either a unit or nilpotent.*

Lecture 5: Schemes

Mon 24 Oct

Remark
By abuse of notation, we write X *is a scheme with* $\mathcal{O}_X$ *implicit.*

Lemma 46
Let X be a topological space with basiis for the topology $\{v_i\}_{i\in I}$.
Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X.
For any collection of morphisms $\phi_i : \mathcal{F}(V_i) \to \mathcal{G}(V_i)$ such that $\rho_{ij} \circ \phi_i = \phi_j$, then $\exists! \phi : \mathcal{F} \to \mathcal{G}$ which restricts to ϕ_i on the V_i.

Proposition 47
Let $(X, \mathcal{O}_X)$ be a locally ringed space and A a ring, then the map $\hom((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \to \hom(A, \mathcal{O}_X(X))$ which maps $(f, f^\sharp) \to f^\sharp(\operatorname{Spec} A)$ is a natural bijection.

In particular, for all locally ringed spaces $(X, \mathcal{O}_X)$, there is a natural affinization morphism $aff_X : X \to \operatorname{Spec} \mathcal{O}_X(X)$

Corollary 48
Every morphism of locally ringed spaces $(X, \mathcal{O}_X) \to \operatorname{Spec} A$ factors uniquely through aff_X.

Corollary 49
A locally ringed space is an affine scheme iff the affinization is an isomorphism.

Corollary 50
The functor

$$(affSch) \to (Ring)^{op}$$

mapping $(X, \mathcal{O}_X) \to \mathcal{O}_X(X)$ is an equivalence of categories.

Proof
Fully faithful is the proposition above.
Essential surjectiveness is immediate as for any ring, we can look at $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ as $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$. □

We now prove the statement

Proof
We use that there exists a natural isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(a)) \simeq A_a$.
Naturality follows from functoriality of $f^\sharp(-)$.
We have to construct an inverse, let $\phi : A \to \mathcal{O}_X(X)$ be a ring homomorphism, we need to define $(f, f^\sharp) : (X, \mathcal{O}_X) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.
We map $x \mapsto \ker(A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_{X,x}/m_x)$.
We claim that f is continuous.

It suffices to show that $X_{\phi(a)} = f^{-1}(D(a)) = \{x \in X | \phi(a)_x \notin m_x\} \subset X$ *is open.*
Take $x \in X_{\phi(a)}$*, then* $\phi(a)_x \notin m_x \implies \phi(a)_x \in \mathcal{O}_{X,x}^{\times}$.
Thus $\exists x \in V \subset X$ *and* $b \in \mathcal{O}_X(V)$ *such that* $\phi(a)|_V b = 1 \in \mathcal{O}_X(V)$.
Thus $\phi(a)_y b_y = 1 \forall y \in V \implies \phi(A)_y \notin m_y \implies V \subset X_{\phi(a)} \implies X_{\phi(a)}$ *is open.*
To define $f^\sharp$*, observe that* $\forall a \in A, \phi(a)|_{X_{\phi(a)}} \in \mathcal{O}_X(X_{\phi(a)})$ *is a unit in every stalk, hence a unit.*
Thus there is a unique morphism such that $A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{\phi(a)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))} \mathcal{O}_X(X_{\phi(a)})$ *so we get a morphism* $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$.
We still have to show that this map is a morphism of locally ringed spaces.
We claim that $\forall x \in X$*, the map* $f_x^\sharp : A_{f(x)} \to \mathcal{O}_{X,x}$ *is a local homomorphism.*
The diagram induces a commutative diagram

$$A \xrightarrow{\phi}\to \mathcal{O}_X(X) \xrightarrow{\pi_2} \mathcal{O}_{X,x} = A \xrightarrow{\pi_1} A_{f(x)} \xrightarrow{f_x^\sharp} \mathcal{O}_{X,x}$$

Note that $p_1^{-1}(f_x^{\sharp,-1}(m_x)) = \pi_1^{-1} \circ \pi_2^{-1}(m_x) = f(x)$ *by definition.*
Thus $f_x^{\sharp,-1}(m_x) = f(x)A_{f(x)}$.
Now, we need to show that this construction is in fact an inverse.
By construction, if $(f, f^\sharp)$ *comes from* ϕ*, then* $\phi = f^\sharp(\operatorname{Spec} A)$.
Conversely, let $(f, f^\sharp) : X \to \operatorname{Spec} A$ *be a morphism and let* $(f', f'^\sharp) : X \to \operatorname{Spec} A$ *be associated to* $f^\sharp(\operatorname{Spec} A)$.
We need to show that $(f, f^\sharp) = (f', f'^\sharp)$.
$\forall x \in X, \exists$ *a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_{X,x} = A \to A_{f(x)} \to O_{X,x}$$

As $f_x^\sharp$ *and* $f_x'^\sharp$ *are local,* $f(x) = f'(x)$*. Now,* $\forall a \in A$*, there is a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))}= f'^\sharp(D(a)) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A),a})$$

□

Example
For every locally ringed space $(X, \mathcal{O}_X)$*, there exists a unique morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$ *because* $\exists! \mathbb{Z} \to \mathcal{O}_X(X)$.
If $(X, \mathcal{O}_X)$ *is a locally ringed space such that each* $\mathcal{O}_X(U)$ *has characteristic* $p > 0$*, then* $\exists!$ *morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{F}_p, \mathcal{O}_{\operatorname{Spec} \mathbb{F}_p})$.

Definition 22 (Scheme over another scheme)
Let S be a sscheme. The category of schemes over S, Sch/S is the category whose objects are morphisms $X \to S$ and morphisms are commutative triangles.

Example
Let K be a field.
The affine n-space over k is denoted $\mathbb{A}^n_k$ is $\operatorname{Spec} k[x_1, \dots, x_n]$.
If k is algebraically closed, then

$$k^n \simeq \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq \mathbb{A}^n_k \simeq \hom_{k-alg}(k[x_1, \dots, x_n], k)$$

If $\phi : A \to B$ is a surjective ring homomorphism, then the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism onto $V(I)$ where $I = \ker \phi$.
In particular, if $I \subset K[x_1, \dots, x_n]$, $k = \overline{k}$ is an ideal, then $V(I) = \{(a_1, \dots, a_n) \in k^n | f(a_1, \dots, a_n) = 0 \forall f \in I\}$ is the image of $\operatorname{Spec}_{max} k[x_1, \dots, x_n] / I \to \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq k^n$.

Example (glueing two schemes)
If X_1, X_2 are two schemes and $U_i \subset X_i$ are open subsets,

$$(\phi, \phi^\sharp) : (U_1, \mathcal{O}_X|_{U_i}) \simeq (U_2, \mathcal{O}_X|_{U_2})$$

is an isomorphism.
We define the scheme $(X, \mathcal{O}_X)$ by glueing X_1 and X_2 over U_1 as follows.
As a set, $X = X_1 \coprod X_2 / \sim$ where $x_1 \sim \phi(x_1)$.
Note, there are natural maps $\pi_i : X_i \to X$.
We say that a subset $U \subset X$ is open $\iff$ $\pi_i^{-1}(U) \subset X_i$ open for $i = 1, 2$.
We define the structure sheaf as $\mathcal{O}_X(U) = \ker(\mathcal{O}_{X_1}(\pi_1^{-1}(U)) \oplus \mathcal{O}_{X_2}(\pi_2^{-1}(U)) \to \mathcal{O}_{X_1}(\pi_1^{-1}(U) \cap U_1))$.
Then X is a scheme.

Example (Explicit example of glueing)
Take $X_1 = X_2 = \mathbb{A}^1_K$ and $U_1 = U_2 = \mathbb{A}^1_K \setminus 0$.
Notice that $U \simeq \operatorname{Spec} k[x, x^{-1}]$.

1. *Taking the glueing map $\phi = \mathrm{Id}$, we get a line with two origins.*
2. *Taking $\phi^\sharp(U_2) : x \mapsto \frac{1}{x}$, we get the projective line $\mathbb{P}^1_k$.*
   The k-rational points of this scheme are in correspondence with lines in k^2, namely
   $$P^1_k(k) \simeq k^2 \setminus \{0\} / k^\times.$$

2 Properties of schemes

2.1 Topological properties

Definition 23
A scheme $(X, \mathcal{O}_X)$ is called

1. *connected if X is*
2. *irreducible if $\forall U_1, U_2$ open non empty their intersection is non-empty.*
3. *quasi-compact if X is.* [a]
4. *quasi-separated if X is, ie. $\forall U_1, U_2$ open and quasi-compact, their intersection is quasi-compact.*

a. All affine schemes are quasi-compact, but $\mathbb{A}_k^\infty \setminus 0$ is not quasi-compact

Lecture 6: Topological properties

Fri 28 Oct

Remark
$\operatorname{Spec} R \times S = \operatorname{Spec} R \coprod \operatorname{Spec} S$ *but* $\operatorname{Spec} \prod_i R_i \not\simeq \coprod_i \operatorname{Spec} R_i$ *for infinite products*

Lemma 56
Affine schemes are quasi-compact and quasi-separated.

Proof
Let $X = \operatorname{Spec} A$ be an affine scheme.
Quasi-compactness has already been proven.
If $U \subset X$ is open and qc., then $U = \bigcup_{i \in I_U} D(a_i), a_i \in A$ and I_U finite.
For $U_1, U_2 \subset X$ qc. open, then

$$U_1 \cap U_2 = \bigcup_{i \in I_{U_1}, j \in I_{U_2}} D(a_1) \cap D(a_2) = \bigcup D(a_1 a_2)$$

Check that a finite union of qc spaces is qc □

Remark
Let X be a topological space, then $\forall$ subsets $V \subset X$ and $U \subset X$, then

$$U \cap V \neq \emptyset \iff U \cap \overline{V} \neq \emptyset$$

Thus V is irreducible iff it's closure is.

If X is irreducible, then every non-empty open is dense.

2.2 Scheme-Theoretic Properties

Definition 24 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ is a pair $(U, \mathcal{O}_U)$ with U open in X and $\mathcal{O}_U := \mathcal{O}_X|_U$

If P is a property of rings, when do we say that $(X, \mathcal{O}_X)$ satisfies P?

1. $\forall U \subset X, \mathcal{O}_X(U)$ satisfies P (usually too strong)
2. $\forall U \subset X$ open and affine, $\mathcal{O}_X(U)$ satisfies P
3. $\exists$ an open affine cover $U = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P
4. $\forall x \in X \exists x \in U$ open affine such that $\mathcal{O}_X(U)$ satisfies P.
5. $\forall x \in X, \mathcal{O}_{X,x}$ satisfies P.

Observe that $1 \implies 2 \implies 3 \iff 4$.

Lemma 58
For $P =$"reduced ring", then all 5 are equivalent.

Proof
From commutative algebra, we know that a ring A is reduced $\iff$ A_p is reduced $\forall p \in \operatorname{Spec} A$.
This implies that $2 \iff 3 \iff 4 \iff 5$.
Let's show $2 \implies 1$.
Let $U \subset X$ open and $s \in \mathcal{O}_X(U)$ such that $s^n = 0$, then $s^n|_V = 0 \forall V \subset U$ affine.
Thus, $s|_V = 0 \forall V \subset U$ open affine and as $\mathcal{O}_X$ is a sheaf $s = 0$. $\square$

Definition 25 (Reduced Scheme)
A scheme $(X, \mathcal{O}_X)$ is called reduced if $\mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open.

Definition 26
Let P be a property of rings or of open affines $\operatorname{Spec} A \hookrightarrow X$ of a scheme X

- *P is called affine-local if $\forall a_1, \ldots, a_n \in A$ such $(a_1, \ldots, a_n) = A$. A satisfies P every A_{a_i} satisfies P*
- *P is called stalk-local if A satisfies P $\iff$ A_p satisfies $P \forall p \in \operatorname{Spec} A$.*

Remark

Being stalk-local is stronger than being affine local.
This is becauses $A \to A_a$ *induces* $(A_a)_{pA_a} \simeq A_p \forall p \in D(a)$

Example

1. *Reduced is stalk-local*
2. *Normal*
3. *regular*
4. *Cohen-Macaulay*

Example

1. *Integrality is not affine-local (consider* $A = k \times k$ *)*
2. *Factorial is not affine-local*
3. *Noetherian is not stalk-local (consider* $A = \prod_i \mathbb{F}_2$ *)*

Lemma 62
Being Noetherian is affine-local.

Why do we care ?

For affine-local properties, 2 and 4 of our list are equivalent.

Proof
If A *is noetherian, then any quotient and any localization is.*
Assume $(a_1, \ldots, a_n) = A$ *and* A_{a_i} *are Noetherian.*
Let $\phi_i : A \to A_{a_i}$ *be the localization maps.*
Claim : $\forall$ *ideals* $I \subset A$, $I = \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$.
One inclusion is clear.
Let $b \in \cap \phi_i^{-1}(\phi_i(I)A_{a_i})$, *thus there exists* $N > 0$ *and* $b_i \in I$ *such that* $b = \frac{b_i}{a_i^N} \in A_{a_i}$.
Thus there exists an $M > 0$ *such that* $a_i^M(a_i^N b - b_i) = 0$ *in* A.
Set $k = M + N$, *note that* $1 = (a_1^k, \ldots, a_n^k)$.
We can write $1 = \sum_{i=1}^n c_i a_i^k$ *for some* $c_i \in A$.
Thus $b = \sum c_i a_i^k b = \sum c_i a_i^M b_i \in I$.
Let $I_1 \subset \ldots \subset I_n \subset$ *be an ascending chain of ideals in* A, *then we get an ascending chain of ideals* $\phi_1(I_1)A_{a_i} \subset \ldots \subset \phi_i(I_n)A_{a_i}$.
This becomes constant because A_{a_i} *is noetherian and* $\exists N > 0$ *such that* $\phi_i(I_k)A_{a_i} = \phi_i(I_N)A_{a_i} \forall kggN$ □

Lemma 63
Let P *be an affine-local property of rings. Let* $(X, \mathcal{O}_X)$ *be a scheme, then the following are equivalent.*

1. *Every open affine* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P
2. $\exists$ *an open affine cover* $X = \cup \operatorname{Spec} A_i$ *such that each* $\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P.

Proof

$1 \implies 2$ *is clear.*

$2 \implies 1$.

Let $\operatorname{Spec} A \hookrightarrow X$ *open and affine.*

Write $\operatorname{Spec} A = \cup \operatorname{Spec} A_{a_i}$ *with* $a_i \in A$ *such that* $A_{a_i} \simeq (A_i)_{b_i}$ *for some* $b_i \in A_i$.

$\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P, *implies* $(\operatorname{Spec}(A_i)_{b_i}) \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A_{a_i} \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P $\square$

Lemma 64

Let $\operatorname{Spec} A, \operatorname{Spec} B \subset X$ *be open affines, then for every point* $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$ *there exist* $a \in A$ *and* $b \in B$ *such that* $A_a \simeq B_b$ *such that* $x \in D(a) \subset \operatorname{Spec} A$ *and* $x \in D(b) \subset \operatorname{Spec} B$ *and the isomorphism* $\operatorname{Spec} A_a \simeq \operatorname{Spec} B_b$ *commutes with the inclusions to* X.

Proof

$\operatorname{Spec} A \cap \operatorname{Spec} B \subset \operatorname{Spec} A$ *is open.*

Thus, there exists $a \in A$ *with* $x \in D(a) \subset \operatorname{Spec} A \cap \operatorname{Spec} B$.

We can assume wlog that $\operatorname{Spec} A \to X$ *factors throguh* $\operatorname{Spec} B$.

Write $\phi : B \to A$ *for the induced map of rings.*

Since $\operatorname{Spec} A \subset \operatorname{Spec} B$ *is open* $\exists b \in B$ *and* $B \to A \to B_b$ *is just localization of* B *at* b.

THen $A \to B_b$ *satisfies the universal property of* $A \to A_{\phi(b)}$.

So we get a commutative square $B \to A \to A_{\phi(b)}$ *and* $B \to B_b \to A_{\phi(b)}$ *and we get an isomorphism* $B_b \simeq A_{\phi(b)}$. $\square$

Definition 27

Let P *be an affine-local property of rings.*

A scheme $(X, \mathcal{O}_X)$ *is called locally* P *if* $\mathcal{O}_X(U)$ *satisfies* $P \forall U \subset X$ *open affine.*

Definition 28 (Noetherian scheme)

A scheme $(X, \mathcal{O}_X)$ *is called Noetherian if it is locally Noetherian and qc.*

Definition 29 (Integral scheme)
A scheme $(X, \mathcal{O}_X)$ is called integral if $\mathcal{O}_X(U)$ is an integral domain $\forall U \subset X$ open and non-empty.

Lemma 65
For a scheme $(X, \mathcal{O}_X)$, the following are equivalent.

1. *X is integral*
2. *X is reduced and irreducible.*
3. *$\forall U \subset X$ open affine, $\mathcal{O}_X(U)$ is integral.*

Proof
$1 \implies 3$ *is clear.*
$3 \implies 2$.
Reduced is clear.
Let $U_1, U_2 \subset X$ open with $U_1 \cap U_2 = \emptyset$.
Wlog, the U_i are affine.
Then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$.
Thus $\mathcal{O}_X(U_1) = 0$ or $\mathcal{O}_X(U_2) = 0$ which implies U_1 or $U_2 = \emptyset$.
$2 \implies 1$
Let $U \subset X$ be open.
Assume $\exists a, b \in \mathcal{O}_X(U)$ such that $ab = 0$.
Let $U_a = \{x \in U | a_x \notin m_x\}$ and similarly U_b.
Note that $U_a \cap U_b = \emptyset$ since $\forall x \in U_a \cap U_b, a_x$ and b_x are units.
Thus $U_a = \emptyset$ or $U_b = \emptyset$.
If $U_a = \emptyset \forall \operatorname{Spec} A \subset U \forall p \in \operatorname{Spec} A$

$$(a|_{\operatorname{Spec} A})_p \in pA_p$$

thus $a|_{\operatorname{Spec} A} \in p \forall p \in \operatorname{Spec} A$.
Thus $a|_{\operatorname{Spec} A}$ is nilpotent.
But since X is reduced, $a|_{\operatorname{Spec} A} = 0$.
Covering U by affines, $a = 0$ (as A was arbitrary). $\square$

Lecture 7: Open and Closed Subschemes

Mon 31 Oct

3 Open and closed subschemes and immersions

Definition 30 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ is a pair $(U, \mathcal{O}_U)$, with $U \subset X$ open and $\mathcal{O}_U = \mathcal{O}_X|_U$.

Lemma 66
If A is a ring and $a \in A$, then there is an isomorphism of locally ringed spaces $(\operatorname{Spec} A_a, \mathcal{O}_{\operatorname{Spec} A_a}) \simeq (D(a), \mathcal{O}_{\operatorname{Spec} A}|_{D(a)})$.
In particular, open subschemes of schemes are schemes.

Proof
From commutative algebra, localization $A \to A_a$ induces a homeomorphism $\operatorname{Spec} A_a \to D(a) \subset \operatorname{Spec} A$.
On sheaves, we want to give morphisms $\mathcal{O}_{\operatorname{Spec} A}|_{D(a)}(U) \to \mathcal{O}_{\operatorname{Spec} A_a}(f^{-1}(U))$.
If $s : U \to \coprod_{p \in U} A_p \to (f^{-1}(U) \to U \xrightarrow{s} \coprod_{p \in U} A_p \to \coprod_{p \in U} (A_a)_{pA_a})$, using $A_p \simeq (A_a)_{pA_a}$. □

Note that, if $i : U \to X$ is the inclusion of an open, then $(i, i^\sharp) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ with

$$i^\sharp(V) : \mathcal{O}_X(V) \xrightarrow{\rho_{V, V\cap U}} \mathcal{O}_X(V \cap U) = i_* \mathcal{O}_U(V)$$

is a morphism of schemes.

Remark
If $i : U \to X$ is an inclusion of an open, then there are in general many sheaves of rings $\mathcal{F}$ on U such that $\exists i^\sharp$ such that $(i, i^\sharp) : (U, \mathcal{F}) \to (X, \mathcal{O}_X)$ is a morphism of schemes.

FOr example, if $X = \operatorname{Spec} k$, $U = \operatorname{Spec} {}^{k[x]}\!/\!_{x^2}$then $k \subset {}^{k[x]}\!/\!_{x^2}$ induces a morphism $(f, f^\sharp) : U \to X$ such that $f = \operatorname{Id}_X$.

Definition 31 (Open immersion)
An open immersion (or open embedding) is a morphism of schemes $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that f is a homeomorphism onto an open subset $U \subset Y$ $\mathcal{O}_Y|U \simeq (f_* \mathcal{O}_X)|_U$.

Example
Let k be a fieldand let $\iota : \operatorname{Spec} k \to X = \mathbb{A}^n$ be the closed point corresponding to

0.
Then

$$(\mathcal{O}_X)|_{\operatorname{Spec} k}(\operatorname{Spec} k) = (i^{-1}\mathcal{O}_X)(\operatorname{Spec} k) \\ = \varinjlim_{0 \in U \subset \mathbb{A}^n} \mathcal{O}_X(U) = \mathcal{O}_{X,0} = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$$

But $\operatorname{Spec} k[x_i]_{(x_1,\ldots,x_n)}$ *has more than one point.*
Thus, $(\operatorname{Spec} k, (\mathcal{O}_X)|_{\operatorname{Spec} k})$ *is not a scheme.*

Observe : If $Z \subset \operatorname{Spec} A$ is a closed subset, then $Z = V(I)$ for some ideal I. Then the map $\operatorname{Spec} {}^{A}\!/\!_{I} \to \operatorname{Spec} A$ induced by the quotient map is a homeomorphism onto $V(I)$ and this gives a scheme structure on Z (which depends on I!).

Definition 32 (Ideal sheaves)
Let $(X, \mathcal{O}_X)$ *be a scheme, then*

1. *An ideal sheaf on* $(X, \mathcal{O}_X)$ *is a subsheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that* $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ *is an ideal for all* $U \subset X$ *is open.*
2. *For an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$*, the quotient sheaf* ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}}$ *is the cokernel sheaf of the inclusion, namely, the sheafification of the sheaf* $U \mapsto {}^{\mathcal{O}_X(U)}\!/\!_{\mathcal{I}(U)}$.

Definition 33 (Closed Subsceme)
Let $(X, \mathcal{O}_X)$ *be a scheme, then a closed subscheme of* $(X, \mathcal{O}_X)$ *consists of a subset* $Z \subset X$ *and an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that*

1. $Z = \left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\}$
2. $(Z, ({}^{\mathcal{O}_X}\!/\!_{I})|_Z)$ *is a scheme*

Remark
By 1, Z *is closed, indeed, for* $1 \in ({}^{\mathcal{O}_X}\!/\!_{(X)})$*, we have*

$$\left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\} = \operatorname{Supp} 1$$

Remark
The morphism ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} \to i_*(({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})|_Z)$ *is an isomorphism.*

If $Z \subset X$ is a closed subscheme determined by $\mathcal{I}$, then $(i, i^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ where $i : Z \to X$ is the inclusion and $i^\sharp : \mathcal{O}_X \to {}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} = i_*\mathcal{O}_Z$ is a morphism of schemes.

Example

Condition 2 *in the definition of closed subscheme is not automatic, even if* X *is affine.*

Definition 34 (Closed immersion)

A closed immersion (or closed embedding) is a morphism of schemes $(f, f^\sharp) : X \to Y$ *such that* f *is a homeomorphism onto a closed subset and* $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ *is surjective on stalks.*

Remark

The morphism $(i, i^\sharp)$ *of the inclusion of closed subscheme is a closed immersion.*

Example

If A *is a ring and* $I \subset A$ *is an ideal, then the morphism* $\operatorname{Spec} {}^{A}\!/_{I} \to \operatorname{Spec} A$ *is a closed immersion.*

Indeed, by CA, this is a homeomorphism onto $V(I)$.
The map $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A/I}$ *is surjective because* $f^{\sharp_p} : A_p \to ({}^{A}\!/_{I})_{pfaktorAI}$, *which is the localization of a surjective map.*

From now on, $V(I) \subset \operatorname{Spec} A$ for the closed subscheme determined by I.

Proposition 74

If $X = \operatorname{Spec} A$ *is affine, then the map* $I \to V(I)$ *is a bijection between ideals of* A *and closed subschemes.*

Proof

Let $Z \subset X$ *be a closed subscheme determined by* $\mathcal{I}$ $\square$