Exercise 8. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. Let $\mathcal{U}$ be any set (possibly infinite) and let $X = \mathcal{P}_f(\mathcal{U})$ be the set of all finite subsets of $\mathcal{U}$. X is then a poset under the partial ordering given by inclusion.

(1) Show that X is a locally finite poset.
(2) Show that the Mobius function $M(x, y) = (-1)^{|y|-|x|}$ for $x \subseteq y \in X$.

Exercise 2. Consider $\mathbb{Z}_{\geq 1}$ as a poset by the relation "x divides y", or written equivelently, $x \mid y$ for $x, y \in \mathbb{Z}_{\geq 1}$.

(1) Show that this is a locally finite poset.
(2) Show that the Mobius function is $M(x, y) = \mu(x/y)$, whenever $x \mid y$.

Exercise 3. Let $\mathcal{U}$ be a set (not necessarily finite). We call a function $m : U \to \mathbb{Z}_{\geq 0}$ a "multiset". If $\mathrm{supp}(m) = \{u \in \mathcal{U} \mid m(u) > 0\}$ is a finite set, the multiset m is said to be finite. We think of multisets as sets where elements are allowed to repeat, each element u has the multiplicity $m(u)$.

We say that a multiset $m_1 \subseteq m_2$ if for all $u \in \mathcal{U}$, $m_1(u) \leq m_2(u)$. For a finite multiset m, the cardinality of m is $|m| = \sum_{u \in \mathrm{supp}(m)} m(u)$. The empty multiset $\phi : \mathcal{U} \to \mathbb{Z}_{\geq 0}$ is the zero function $\phi(u) = 0$ for all $u \in \mathcal{U}$.

Take X to be the set of all finite multisets on $\mathcal{U}$,

(1) Show that $(X, \subseteq)$ is a locally finite poset.
(2) Show that for $x \subseteq y$ the Mobius function is

$$M(x, y) = \begin{cases} 0 & \text{if } (x \setminus y)(u) \geq 2 \text{ for some } u \in \mathcal{U} \\ (-1)^{|x \setminus y|} & \text{otherwise} \end{cases}.$$

Here $x \setminus y$ is the multiset defined as $(x \setminus y)(u) = x(u) - \min(x(u), y(u))$ for any $x, y \in X$.
(3) Compare the Mobius function of this poset with the result for $(\mathbb{Z}_{\geq 1}, |)$ in the previous question. What do you observe?

Exercise 4. Take $X = \{1, 2, 3 \ldots, n\}$ with the usual ordering $\leq$. Consider the incidence algebra $\mathcal{A}(X)$. Let $M_n(\mathbb{C})$ be the complex matrices.

Consider the following map

$$\begin{aligned} \Psi : \mathcal{A}(X) &\to M_n(\mathbb{C}) \\ f &\mapsto \Psi(f) \\ &\quad \Psi(f)_{ij} = f(i, j) \end{aligned}$$

(1) Show that this map is $\mathbb{C}$-linear and that it is injective by finding the kernel of the map. What is the image of the map?
(2) Show that $\Psi(f * g) = \Psi(f)\Psi(g)$. That is, show that the map Ψ is a $\mathbb{C}$-algebra homomorphism.
(3) What is the $\Psi(Z)$ where $Z \in \mathcal{A}(X)$ is the zeta function? What is $\Psi(\delta)$ where $\delta \in \mathcal{A}(X)$ is the delta function.

(4) Find the Mobius function in $\mathcal{A}(X)$ using the map Ψ.

Exercise 5. Prove that every graph with n vertices and $n-1$ edges that does not contain a cycle is a tree.

Exercise 6. Prove that every connected graph with n vertices and $n-1$ edges is a tree.

Exercise 7. *(Optional)*
Put $\mathcal{U} = \{1, 2, 3, \ldots, n\}$. Consider the poset $X = \mathcal{P}(\mathcal{U})$ ordered by inclusion $\subseteq$. Let A be a finite set and let $A_1, A_2, \ldots, A_n \in \mathcal{P}(A)$. Define $F, f : X \to \mathbb{C}$ as

$$f(I) = \begin{cases} \left|\left(\bigcap_{i\in\mathcal{U}\setminus I} A_i\right) \setminus \left(\bigcup_{i\in I} A_i\right)\right| & \text{if } I \subsetneq \mathcal{U} \\ 0 & \text{if } I = \mathcal{U} \end{cases},$$

$$F(J) = \begin{cases} \left|\bigcap_{i\in\mathcal{U}\setminus J} A_i\right| & \text{if } J \subsetneq \mathcal{U} \\ \left|\bigcup_{i\in J} A_i\right| & \text{if } J = \mathcal{U} \end{cases}.$$

Prove that $F(J) = \sum_{I\subseteq J} f(I)$. Perform a Mobius inversion to write f in terms of F. Use this to prove the inclusion-exclusion principle.