

## Exercise 10

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As a preliminary result, we will show that, given a graph  $G$  and a walk  $(u_1, \dots, u_k)$  of  $G$ , there exists  $I \subset [n]$ , such that  $(v_1, \dots, v_j)$ ,  $\{v_i, 0 < i \leq j\} \subset \{u_i, 0 < i \leq k\}$  is a path, and such that  $v_1 = u_1, v_j = u_k$ .  
Indeed, consider the walk  $(u_1, \dots, u_k)$  of  $G$ , if  $\forall i, j \in [n], i \neq j : u_i \neq u_j$ , then  $u_1, \dots, u_k$  already is a walk and we are finished.  
Set  $I = [n]$ . Hence, suppose that there exists  $i, j \in [n], i \neq j$  such that  $u_i = u_j$ , without loss of generality we can suppose that  $i < j$ .  
Then, we can redefine  $I = I \setminus \{i + 1, \dots, j\}$ .  
We now obtain a new walk given by  $(v_1, \dots, v_j) = \{u_i\}_{i=1, i \in I}^n$ .  
We now repeat this algorithm until we are left with a path.  
Since a walk is always of finite length and the size of  $I$  decreases by at least 1 at each step, we are guaranteed that the algorithm finishes.  
Also notice that  $v_1$  and  $v_j$  always stay the same at each step of the algorithm, which proves that  $v_1 = u_1, v_j = u_k$ .

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We will write  $V$  for the set of vertices of  $G$ .

Let  $e \in E(G) \setminus E(T)$ , we can write  $e$  as  $e = \{a, b\}, a, b \in V$ .

Since  $E(T)$  is connected, there exists a path in  $T$  of the form  $(a, v_1, \dots, v_n, b), v_i \in G$ .

We define  $K = T + e - \{v_n, b\}$ .

Clearly,  $K$  is still spanning since it still contains  $v_n, a$  and  $b$ <sup>1</sup>.

We now show that  $K$  is a tree.

First, we show that  $K$  still is connected.

Indeed, consider two vertices  $x, y \in V$ .

Consider the path of  $T$  which would connect  $x$  to  $y : x, u_0, \dots, u_n, y$ .

If  $\forall i \in [n], \{u_1, u_{i+1}\} \neq \{v_n, b\}$ , If there exists  $0 \leq i \leq n$  such that  $\{u_i, u_{i+1}\} = \{v_n, b\}$ , replace  $u_i, u_{i+1}$  in the path with the path  $v_n, v_{n-1}, \dots, a, b$ , we are left with a walk contained in  $K$ .

Using the lemma proven, we extract a path from this walk.

This walk being contained in  $K$ , we deduce that  $K$  is connected.

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1.  $v_n$  is still contained in  $K$  because  $\{v_{n-1}, v_n\}$  is contained in  $K$

We now show that  $K$  contains no cycle.

For the sake of contradiction, suppose  $K$  contains a cycle of the form  $c_0, \dots, c_k, c_0$ .

If  $\forall 0 \leq j \leq k, \{c_j, c_{j+1}\} \neq e$ , then the cycle is contained in  $T$  which is a contradiction since  $T$  is a tree.

Hence, suppose there exists a  $j$  such that  $\{c_j, c_{j+1}\} = e$ , without loss of generality, suppose that  $c_j = a$  and  $c_{j+1} = b$ .

If that were the case, we could again create a new walk of the form  $c_0, \dots, c_{j-1}, a, v_1, \dots, v_n, b, c_{j+2}, \dots, c_n, c_0$ , which by definition is contained in  $T$  and in  $K$ .

We now extract a path from this walk, clearly this path is a cycle contained in  $T$ .

Hence, the existence of a cycle in  $K$  implies the existence of a cycle in  $T$ , which is impossible since  $T$  is a tree.

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We deduce that  $K$  also is a spanning tree of  $T$ , and the result follows.