École Polytechnique Fédérale de Lausanne
Prof. Marcus May 12, 2022

Math 261 – Discrete Optimization (Spring 2022)

Assignment 10

Problem 1

Let $D = (V, E)$ be a digraph with $|V| = n$ and $|E| = m$ for which the underlying graph is connected. Let T be a collection of edges that form a spanning tree in the underlying graph.

(a) Show that if H is the cycle space of D, then $\dim(H) \geq m - n + 1$.

Hint: Find $m-n+1$ cycles for which the corresponding cycle vectors are linearly independent using the edges not in T.

Solution:

We need to pick $m - n + 1$ cycles that are linearly independent. For each edge e which is not in T, consider the subgraph with edges $T \cup e$. As we have seen, this must have a cycle C^e, so consider the cycle vector for that cycle: $\mathbf{h}^{C^e}$. There are $m - n + 1$ edges not in T, so that gives us $m - n + 1$ vectors. Furthermore, for each edge $e \notin T$, the vector $\mathbf{h}^{C^e}$ will be the only one for which e gets a nonzero value. Hence these will be linearly independent and so the space has dimension at least $m - n + 1$.

(b) Show that if B is the cut space of D, then $\dim(B) \geq n - 1$.

Hint: Find $n-1$ cuts for which the corresponding cut vectors are linearly independent using the edges of T.

Solution:

We need to pick $n - 1$ cuts that are linearly independent. For each edge e which is in T, removing e will result in T being cut into two pieces $T_1(e), T_2(e)$. Let K^e be the cut formed by the vertices that are touched by T_1, and consider the cut vector $\mathbf{r}^{K^e}$. Since there are $n - 1$ edges in T, this gives us $n - 1$ vectors and it should be clear that the edge e crosses cut K^e but not any of the other $K^{e'}$. This means $\mathbf{r}^{K^e}$ will be the only vector for which e is nonzero, and so these vectors will be linearly independent. Hence the dimension of B is at least $n - 1$.

(c) Show that (as subspaces of the edge space) that $H = B^\perp$.

Solution:

Since H and B are orthogonal (as we showed in the previous problem set), we have

$$\dim(H \oplus B) = \dim(H) + \dim(B).$$

So parts (a) and (b) imply $\dim(H \oplus B) \geq m$. On the other hand, H and B are linear subspaces of $\mathbb{R}^E$, so $H \oplus B$ will be a linear subspace of $\mathbb{R}^E$ as well. This means $\dim(H \oplus B) \leq m$. Taken together, this implies that all of the inequalities must be equalities — that is, $\dim(H) = m - n + 1$ and $\dim(B) = n - 1$. But two orthogonal spaces that (together) span the entire space are orthogonal complements, so $H = B^\perp$.

Problem 2

Enthusiastic partying at a prominent Swiss university has unfortunately resulted in the arrival at the university's medical clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of blood. The clinic has supplies of 170 units of blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups in summarized below:

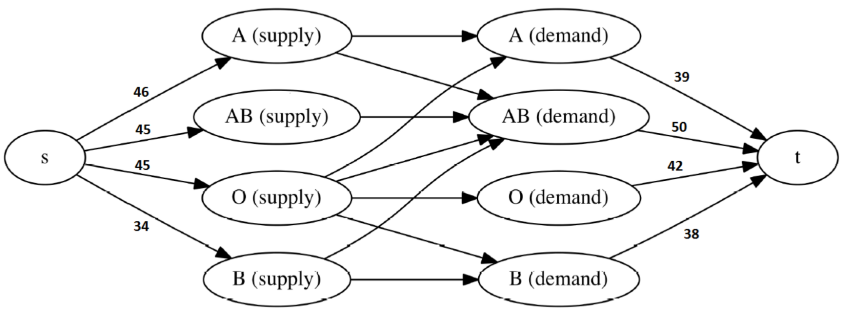
Blood type	A	B	O	AB
Supply	46	34	45	45
Demand	39	38	42	50

A patients can only receive A or O blood. B patients can only receive B or O blood. O patients can receive only O blood. AB patients can receive any of the four blood types.

(a) Give a max-flow formulation that determines a distribution that satisfies the demands of a maximum number of patients. You should draw a directed graph with edge capacities such that a feasible flow corresponds to a feasible choice for the transfusion.

Solution:

We formulate the problem as follows: there are two nodes for each blood type, representing the supply and the demand, and there is a source node s and a sink node t. The arcs represent which type of blood in the supply can be donated to which group. The capacities represent the supply or the demand of a blood type. Note that in the edges between a supply and a demand node the capacity is set to ∞. This is an arbitrary choice. It represents the fact that we don't constrain the amount of blood that can be given to a group. But this will also ensure that the minimum cut will not cross any of these arcs, which makes it easier when we have to go looking for where the small cuts are in the graph.

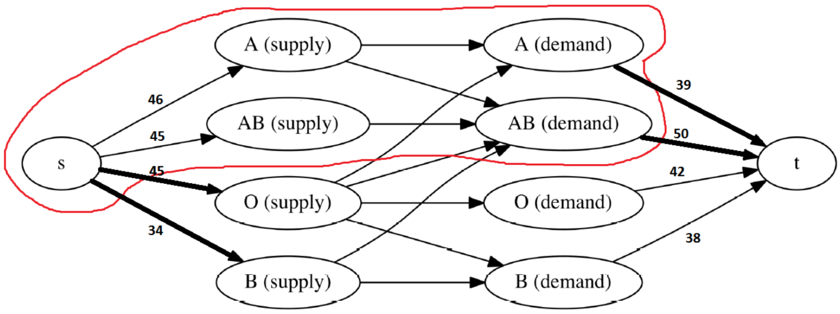


(b) Find a cut in your graph of value smaller than 169. Use it to give an explanation of why not all of the patients can receive blood from the available supply. Try to make your explanation understandable to the clinic's staff, who do not know network flow theory.

Solution:

To find such a cut we can use the algorithm seen in class to compute the maximum flow (of value 168) and then find a cut of the same capacity by looking at the saturated edges.

The cut shows why we cannot satisfy the totality of the demand. The problem can be easily conveyed by combining the blood types O and B (the ones outside of the cut) together into a single group (which we will call O/B). The total supply of O/B type blood is 79 and the demand is 80. Since no other blood types can be used for O/B patients, there will be a shortage.

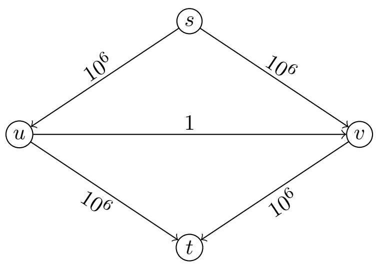


Problem 3

Construct a network on four vertices for which the Ford-Fulkerson algorithm with a bad pick* function may need more than a million iterations (by picking a bad sequence of augmenting paths).

Solution:

Consider the following network:



Clearly, the maximum value of a flow is 2 million, but the Ford-Fulkerson algorithm might always choose a path that uses the edge uv by alternately choosing the augmenting paths(s, u, v, t) and (s, v, u, t). This way the value of the flow increases by 1 in every step, so it takes 2 million steps to reach a maximum flow.

Problem 4

Let $D = (V, E)$ be a network with source s, sink t, and integer capacities $\mathbf{C}$. Prove or disprove the following statements:

(a) If every C_e is even, then there exists a maximal flow $\mathbf{f}$ such that f_e is even for $e \in E$.

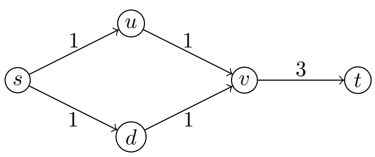
Solution:

Dividing all capacities by two, we obtain a network with integral capacities. Therefore, it has an integral maximum flow. Multiplying this flow by 2, we get an even maximum flow on the original network.

(b) If every C_e is odd, then there exists a maximal flow $\mathbf{f}$ such that f_e is odd for all $e \in E$.

Solution:

Counterexample:

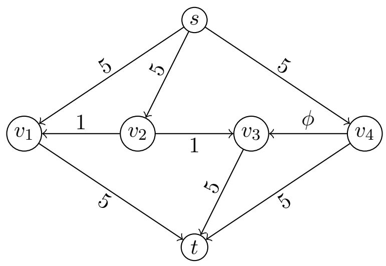


Challenge Problem:

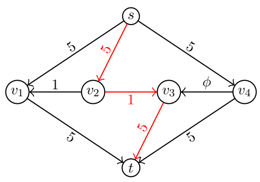
This problem is harder than usual, so please work on it last (and don't worry if you can't do it).

Consider the following network, with edges labeled by their capacities[1] Prove that the Ford-Fulkerson algorithm with a bad pick* function could run forever on this network (by picking a bad sequence of augmenting paths).

Hint: Define the "residual capacity" of an edge e to be $C_e - f_e$ and try to choose augmenting paths that cause the residual capacities on the three horizontal edges to periodically return to having the form $(\phi^k, 0, \phi^{k+1})$ for some k.

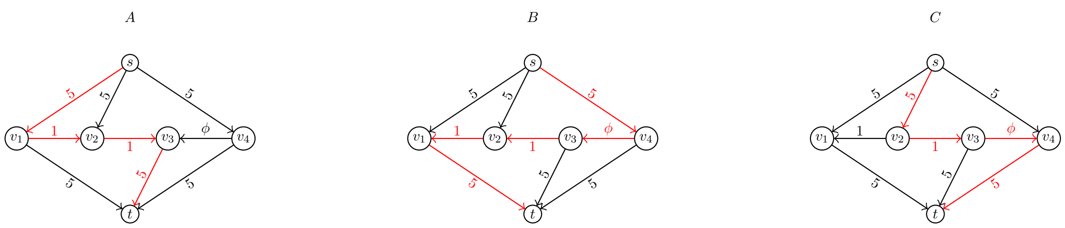


Solution:
The goal will be to find a sequence of augmenting paths which never reaches a maximum (and so can be repeated infinitely). We start by choosing the central augmenting path, shown in the figure below.



After adding 1 unit of flow along this path, the three horizontal edges will have residual capacities 1, 0, and ϕ (in order from left to right). Suppose inductively that we have a flow $\mathbf{f}_k$ for which the horizontal edges have residual capacities $(\phi^{k-1},\ 0,\ \phi^k)$ for some positive integer $k \geq 1$. Add the following sequence of augmented paths (shown in red):

1. B: this adds ϕ^k to the flow and the residual capacities become $(\phi^{k+1}, \phi^k, 0)$.
2. C: this adds ϕ^k to the flow and the residual capacities become $(\phi^{k+1}, 0, \phi^k)$.
3. B: this adds ϕ^{k+1} to the flow and the residual capacities become $(0, \phi^{k+1}, \phi^{k+2})$.
4. A: this adds ϕ^{k+1} to the flow and the residual capacities become $(\phi^{k+1}, 0, \phi^{k+2})$.



After doing this sequence, we have increased the total flow value and are back in a state satisfying the inductive hypothesis, so this can be continued forever.

[1]The constant ϕ is the "golden ratio" $\frac{\sqrt{5}-1}{2}$. It satisfies the equation $\phi^2 + \phi = 1$.