SUBGROUPS OF FREE GROUPS AND FREE PRODUCTS

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Abstract. The aim of this paper is to present algebraic and topological approaches to prove two fundamental theorems in the study of free groups and free products - The Nielsen - Schreier Theorem and the Kurosh Theorem.

Contents

1. Introduction

The study of group theory provides structures and tools with which we observe the mathematical world. Our understanding of topology, geometry and logic is built upon several fundamental algebraic structures, one of which is the free group. This paper explores two fundamental results in the study of free groups: The Nielsen-Schreier Theorem and The Kurosh Theorem. Both make claims about subgroups - the former states that subgroups of free groups are themselves free, while the latter argues that subgroups of free products can be presented as free products of related groups. This paper will outline a topological and then algebraic proof of the Nielsen-Schreier theorem, followed by a topological case for the Kurosh theorem. Sections 2 and 3 will lay the groundwork for the topological arguments for both the theorems, presenting important results with regards to covering spaces, fundamental groups and trees. Section 4 will compile theorems and arguments from sections 2 and 3 to present a proof of the Nielsen-Schreier theorem.

Section 5, which provides an algebraic alternative to the topological proof, is a stand alone section that utilizes group actions to prove Nielsen-Schreier. Finally, Section 6 introduces arguments about free products, which are utilized in a proof of the Kurosh theorem. The topological proofs are mostly a mix of theorems and arguments presented in Rotman ([1]) and Hatcher ([2]), while the algebraic proof of Nielsen-Schreier draws from a cited paper ([3]). This paper assumes a basic understanding of free groups and (simplicial/CW) complexes ([2] Section 1 or [1] Chapter 11.1) and homotopy theoretic notions such as fundamental groups and covering spaces.

2. Covering Complexes and Fundamental Groups

The following sections of the paper explore a topological proof of the Nielsen-Schreier theorem. Most arguments in sections 2-4 follow from [1] and [2], and the individual theorems are marked accordingly.

The Nielsen-Schreier theorem states that any subgroup of a free group is also free. In order to understand this theorem in a topological context, we must be able to find some representation of a subgroup in terms of spaces. This is where we make use of a fundamental result in the covering space theory, the Galois correspondence of covering spaces and subgroups:

Theorem 2.1. *([2]) Given a space X with basepoint x_0 that is path connected, locally path connected and semi-locally simply-connected, for any subgroup H of $\pi_1(X, x_0)$, there is a covering complex $p : X_H \to X$, such that the induced map $p_\# : \pi_1(X_H, x_H) \to G$ is an isomorphism onto $H \subseteq G$.*

This paper shall make arguments using CW complexes, and therefore it is necessary to show that a connected CW complex satisfies the criteria stated in Theorem 2.1. It follows from [2] (pg 519) that a connected CW complex has these required properties.

3. The Fundamental Group of a Graph

A **graph** is a 1 dimensional CW complex and a **tree** is a graph with no cycles. Equivalently, any two points in a tree are connected by a unique path. As we shall see, graphs are rather useful to us as they act as a topological presentation of a free group.

However, before making such arguments, we must first prove the following result which is referenced several times in this paper.

Theorem 3.1. *Every graph X has a maximal tree.*

Proof. If X is a finite graph, then we can inductively show that there must be a maximal tree. If X is connected, then we can remove edges from every cycle in X until there are no cycles left. This leaves a connected graph with no cycles - by definition a tree. This tree is maximal as it contains every vertex in X. If X is not connected, it can be partitioned into disjoint, connected subgraphs, $\{X_i\}$, and a subgraph with the most vertices contains a maximal tree.

For an infinite tree, a Zorn's lemma argument must be used. Given a graph X, we can define a set T of trees in, X. This set is partially ordered by the subgraph relation. Every chain in this poset consists of a sequence of trees, each a subgraph of the next. Each tree is contained in the next, so if there exists some finite loop in the union, it exists in every tree $T_i, T_{i+1}...$ for some finite i, which is a contradiction as a tree contains no loop by definition. The union of all such trees is contained in T. Therefore, by Zorn's lemma, the poset contains a maximal element and X contains a maximal tree.

□

A very interesting property of trees is the fact that they are **contractible** (homotopy equivqlent to a point). This means that the fundamental group of a graph X can be generated by the edge set X/T, where T is a maximal tree in X.

Theorem 3.2. *Trees are contractible.*

Proof. For the case of a finite tree, a simple induction argument is sufficient. Informally, consider the leaves of a tree in a complex. A leaf is an edge, one end of which is not connected to any other point. This means that the leaf can be contracted. Once every leaf of the tree has been contracted, the tree has a new set of leaves. This is repeated inductively until the tree is contracted to a single point. The argument for infinite trees is based on the fact that in a tree, there is a unique path between any two points and we can use these paths in our deformation retraction. However, this process must be continuous for every point in the tree. By properties of CW complexes, such a contraction is possible, as outlined in [2](pg 84). □

We finally come to a complex that represents a free group. It can be shown that the wedge sum of indexed circles (i.e. a bouquet of circles) has a free fundamental group.

Corollary 3.3. ([2]) *Given a bouquet of circles $B = \vee_I S^1$, $\pi_1(B, b)$ is a free group of rank $|I|$, where b is the intersection point of the circles.*

Proof. We use Van Kampen's Theorem. Consider U_i, an open neighbourhood of S_i, a circle in the bouquet. This open neighbourhood consists of S_i along with every other circle with one point missing from each such that the missing point is not the intersection point b of the bouquet. A circle missing a point is homotopic to a point, and all $S_{i \in I}$ intersect at a single point b. Therefore, these 'broken' circles all deformation retract to the singular point b, leaving S_i as the deformation retract of its own open neighbourhood U_i. Given two open neighbourhoods U_i and U_j where $i \neq j$, $U_i \cap U_j$ contains no complete circles, allowing us to deformation retract it to the intersection point b. Thus, by the Van Kampen Theorem, $p : \ast\pi_1(S_1, b) \to \pi_1(B, b)$ is an isomorphism. Thus $\pi_1(B, b)$ is isomorphic to $\ast_I\mathbb{Z}$, which is free. $\square$

This notion can now be generalized to graphs by combining the two previous arguments.

Theorem 3.4. *The fundamental group of a connected graph is free*

Proof. Let T be a maximal tree in X. By Theorem 3.2, T is contractible. This implies that the spaces X and X/T are homotopy equivalent. By definition of homotopy equivalent, these two spaces have the same fundamental group, so $\pi_1(X, x_0) = \pi_1(X/T, x_0)$. But X/T is the bouquet of $|I|$ circles where $|I|$ is the number of edges in X that are not in T (all vertices of X are in T hence are identified together, and the edges not in T become circles). Therefore, by Corollary 3.3, $\pi_1(X, x_0)$ is free. $\square$

4. The Nielsen-Schreier Theorem

Now we have all the tools we need to prove the Nielsen-Schreier Theorem.

Theorem 4.1. *([1]) Every subgroup of H of a free group F is free.*

Proof. If F is a free group with rank $|I|$ and X is a bouquet of I circles, then by the previous corollary, $\pi_1(X, x_0)$ is a free group and $\pi_1(X, x_0) \cong F$. We know that $H \leqslant \pi_1(X, x_0)$, so by Theorem 2.1, there exists a covering space $p : X_H \to X$ such that p is an isomorphism $\pi_1(X_H, x_H) \to H$. It is a known fact that the covering space of a graph is itself a graph (see [2] pg 85). By Theorem 3.4, $\pi_1(X_H, x_H)$ is a free group. Therefore H is free. $\square$

5. An Algebraic Approach to Nielsen-Schreier

This section will look at an alternate proof of the Nielsen-Schreier theorem, a proof more rooted in fundamental concepts of group theory rather than topology. This section assumes a basic understanding of group actions.

In order to study free groups in the context of group actions, we need to rewrite the universal property of a free group.

Theorem 5.1. *([3]) A group F is free generated by $X \subseteq F$ if and only if given any set A and any map $\sigma : X \to S_A$, there exists a unique right action of F on A such that for $x \in X \subseteq F$,*

$$ax = \sigma(x)(a)$$

where S_A is the group of bijections on A.

The universal property of free groups states that a group F is free on a set X if given any group G with map $\phi : X \to G$, there exists a unique homomorphism $\Phi : F \to G$ extending ϕ (definition as stated in [1]).

Informally, we can intuit the above theorem as follows. An action of a group F on A is given by a homomorphism $\Phi : F \to S_A$. The action $ax = \sigma(x)(a)$ can be viewed as an extension of the map $\sigma : X \to S_A$. Therefore S_A acts as the group G in the universal definition of free groups.

Consider a subgroup H of a group G, where H acts on a set A. We can then define a right action of H on $A \times G$ by $(a, g)h = (ah, h^{-1}g)$. The following notation will be useful in the proof of Nielsen-Schreier.

Notation 5.2. Given a tranversal T of G/H (a set of representatives of the orbits), $\overline{g}$ denotes the representative of the coset Hg in T, for $g \in G$.

We shall need the following:

Proposition 5.3. *Let H be a subgroup of G, T a transversal of G/H and A a set. Then any action of H on A determines an action of G on $A \times G/H$ through the rule:*

$$(a, Hg)g' = (a\overline{g}g'(\overline{gg'})^{-1}, Hgg') \tag{5.4}$$

The action of H on A is recovered from this by restricting to H and projecting to the first coordinate:

$$(a, H)h = (ah, H) \tag{5.5}$$

Proof. A straightforward computation. □

In order to show that a subgroup $H \leqslant F$ is free using the universal property, we must construct a generating set for H. The Schreier transversal is used to construct such a set.

Definition 5.6. Given a free group F and a subgroup $H \leqslant F$, a **Schreier transversal** T is a transversal on H in F such that the set T is closed under the operation of taking prefixes. So if a word $w_1w_2....w_n$ is an element of T, then so are e, w_1, $w_1w_2....$ and $w_1w_2...w_{n-1}$.

The existence of Schreier transversals follows from a Zorn's lemma argument. Consider all the prefix closed sets of reduced words generated by $X \cup X^{-1}$. We will define a collection of such sets P such that every element of P intersects each coset of H at most once. It is clear that P is non empty as it at least contains $\{1\}$. We can let the binary relation of inclusion partially order P (Given sets A and B, they are ordered by inclusion if $A \subseteq B$). This means that given a totally ordered subset of P, the union of the elements of this subset lie in P, so by Zorn's Lemma, P contains a maximal element T, which we call the Schreier transversal. To show T is a transversal, it must be shown that each coset of H has a singular representative in T. A computational argument is presented in [3]. Given the existence of Schreier transversals, we can now construct a generating set for H.

Theorem 5.7. *([3]) Given a free group F on X with subgroup H and a Schreier transversal T of H, the set $B = \{tx(\overline{tx})^{-1} | t \in T, x \in X, tx(\overline{tx})^{-1} \neq 1\}$ freely generates H*

Proof. First we need to check that $B \subseteq H$. For any $tx(\overline{tx})^{-1}$ in B, $\overline{tx}$ is by definition the representative of coset Htx in T, so $\overline{tx} = htx$, where $h \in H$. Therefore, $tx(\overline{tx})^{-1} = txx^{-1}t^{-1}h^{-1} = h^{-1} \in H$.

Following from Theorem 5.1, it suffices to show that given any set A and map $\sigma : B \to S_A$, there is a unique action of H on A such that $ab = \sigma(b)(a)$ for $b \in B$. We must show that such an action a) is unique and b) exists. Note that uniqueness is proved first as it provides a template for an action that is used in the existence argument.

a Uniqueness: Assume we have two different actions of H on A. Then Proposition 5.3 gives two different actions of F on $A \times F/H$ via

$$(a, Hw)w' = (a\overline{w}w'(\overline{ww'})^{-1}, Hww') \tag{5.8}$$

However if we take $x \in X$ we see that x acts identically in both actions:

$$(a, Hw)x = (\sigma(\overline{w}x(\overline{wx})^{-1})(a), Hwx) \tag{5.9}$$

But since X generates F, the two actions agreeing on X implies that they must agree on F, contradiction.

b Existence: Given $\sigma : B \to S_A$, equation (5.9) specifies a map $X \to S_{A\times F/H}$ which by Theorem 5.1 is the same as an action of

F on $A \times F/H$. Finally, by Proposition 5.3 we get an action of H on A and we need to check that given $b \in B$

$$(a, H)b = (\sigma(b)(a), H) \tag{5.10}$$

i.e. that

$$(a, H)tx(\overline{tx})^{-1} = (\sigma(tx(\overline{tx})^{-1})(a), H) \tag{5.11}$$

for any $x \in X$. But from (5.9) we already know that

$$(a, H)tx = (\sigma(tx(\overline{tx})^{-1})(a), H)(\overline{tx})^{-1}$$

If we apply to both sides of this equation the next Lemma 5.12 we get (5.11).

Lemma 5.12. *([3]) Given $t \in T$, $(a, H)t = (a, Ht)$*

Proof. We use induction to prove this lemma. If t is the trivial word, then it is clear that the lemma is true. Given $t = ux$, where x is a reduced word, $u \in T$ as it is a prefix of t. By the inductive hypothesis, $(a, H)t = (a, H)ux = (a, Hu)x$. By definition of the action of F on $A \times F/H$, $(a, Hu)x = (\sigma(\overline{u}x(\overline{ux})^{-1})(a), Hux)$. Given $u, ux \in T$, $\overline{ux} = \overline{u}x = ux$. Therefore $(a, Hu)x = (\sigma(e)(a), Ht) = (a, Ht)$. □

□

6. The Kurosh Theorem

While the Nielsen-Schreier theorem is concerned with subsets of free groups, one can ask the same question with regards to the free product of groups. The Kurosh theorem answers this question and we shall present a topological proof of it. This section assumes a basic understanding of free products.

We must first determine a way to represent a group as a free product, and the fundamental group provides a convenient means to do so.

Theorem 6.1. *([1]) Given a 2 dimensional CW complex X that is the union of connected subcomplexes X_i, if there exists a tree T such that $T = X_i \cap X_j$ for all $i \neq j$, then $\pi_1(X, x_0) = *_{i \in I} \pi_1(X_i, x_i)$.*

Proof. From every subcomplex X_i, we can choose a maximal tree T_i. By Lemma 6.2, $T = \cup T_i$ is a maximal tree, as $\cup T_i$ contains all points in X. Let the fundamental group $\pi_1(X_i, x_i)$ have a presentation of the form $(G_i|R_i)$, where the generating set G_i is the set of all edges in X_i not in T_i and R_i the set of relations, one for every 2 dimensional cell in X_i. Informally, a 2 dimensional cell in a complex determines a relation as follows: Its boundary is a loop in our complex, say $g_1 \cdots g_n$ where the g_i are the edges in the loop, and since we are filling in the interior of the loop we can contract it to a point and thus have $g_1 \cdots g_n = 1$ in π_1; see [2] (pg 84) for a more formal description.

The complex X has presentation $(G|R)$, where G is the set of all edges in X not in T and R is the set of 2 dimensional cells in X. It is clear that $G = \bigcup G_i$, as $\bigcup T_i = T$, so the set of all edges in X not in T is the union of the set of all edges in each X_i not in T_i for all $i \in I$. A 2 dimensional cell in X must be a 2 dimensional cell in some X_i, resulting in $R = \cup R_i$. Therefore $(G|R) = (\cup G_i | \cup R_i)$ is a presentation of $*_{i \in I} \pi_1(X_i, x_i)$. □

Lemma 6.2. *Given a family of trees $\{T_i\}$ in a complex, if the intersection of any two trees is also a tree, then $\cup T_i$ is a tree.*

Proof. To show that $L = \cup T_i$ is a tree, one must show that it is connected graph with no cycles. Clearly L is a graph and since the intersection of two trees is a tree, $\cup T_i$ is connected. Assume the union of two distinct trees T_i and T_j contains a cycle. Given a finite cycle, inductively we can find a smallest subcycle, which we shall consider for this proof. Such a subcycle cannot be completely contained in T_i or T_j, as both are trees. Given two points x_0 and x_1 in this subcycle, there exist distinct paths $\alpha \in T_i$ and $\beta \in T_j$ from x_0 to x_1 such that $\alpha \notin T_j$ and $\beta \notin T_i$. It follows that x_0 and x_1 are contained in $T_i \cap T_j$, however there is no path connecting the two in the intersection. This contradicts the assumption that the intersection is a tree. Inductively, we can extend this argument to L. □

Theorem 6.3. *([1]) If $H \leqslant *_{i \in I} A_i$, then $H = F * (*_{k \in K} H_k)$, where F is a free group and each H_k is a conjugate of a subgroup of some A_i.*

Proof. By Corollary 1.28 in [2], any group can be presented as the fundamental group of some connected 2-dimensional CW complex, so $A_i \cong \pi_1(X_i, x_i)$, where X_i is a connected 2 dimensional complex and the basepoints are 0 cells. Take another 0 cell x_0, form the space $x_0 \coprod \coprod_i X_i$ and further connect x_0 to the basepoint x_i of every complex X_i by an edge (x_0, x_i). This forms a connected CW complex X.

Let T be the tree generated by the added edges $\{(x_0, x_i) | i \in I\}$. Since the X_i are disjoint we have $(X_i \cup T) \cap (X_j \cup T) = T$ for $i \neq j$ so by Theorem 6.1,

$$\pi_1(X, x_0) \cong *_{i \in I} \pi_1(X_i \cup T, x_i)$$

By Theorem 3.2 $\pi_1(X_i \cup T, x_i) \cong \pi_1(X_i, x_i) \cong A_i$. Consider a subgroup $H \leqslant *_{i \in I} A_i$. By Theorem 2.1, X has a cover $p : X_H \to X$ such that $p_\#$ is an isomorphism $\pi_1(X_H, x_H) \to H$. $p : X_H \to X$ is a covering complex, therefore by definition $p^{-1}(X_i)$ is the disjoint union of components $\tilde{X_{ij}}$ in $\tilde{X}$.

Consider a maximal tree of this subcomplex, $\tilde{T_{i,j}}$. Let $\tilde{L}$ be the tree $\tilde{L} = \bigcup(\tilde{T_{ij}} \cup p^{-1}(T))$. Notice that the union of the subcomplexes $\tilde{L}$ and $\{\tilde{X_{ij}} \cup \tilde{T} | i \in I, j \in J\}$ is the covering complex X_H, while the intersection of any two of these subcomplexes is $\tilde{T}$, as $\tilde{T_{ij}} \in \tilde{T}$.This allows us to use Theorem 6.1 to present a free product of the fundamental groups of these subcomplexes.

$$\pi_1(X_H, x_H) \cong \pi_1(\tilde{L}, \tilde{x_0}) * (\ast \pi_1(\tilde{X_{ij}} \cup \tilde{T}, \tilde{x_{ij}}))$$

Recall that $\tilde{L}$ is a tree, and therefore by Theorem 3.4 $\pi_1(\tilde{L}, \tilde{w})$ is a free group. By Theorem 3.2 trees are contractible, so $\pi_1(\tilde{X_{ij}} \cup \tilde{T}, \tilde{x_{ij}}) \cong \pi_1(\tilde{X_{ij}}, \tilde{x_{ij}})$.

Observe that by restricting p to the complex $\tilde{X_{ij}}$ we get that covering complex $p : \tilde{X_{ij}} \to X_i$ such that the induced homomorphism between the two fundamental groups, $p_\#$, is injective (follows from [1], Theorem 11.36). Thus $\pi_1(\tilde{X_{ij}}, \tilde{x_{ij}})$ is isomorphic to a subgroup of $\pi_i(X_i, x_i) \cong A_i$. This means that $p(\tilde{x_{ij}}) = x_i$ for all $j \in J$. Thus it follows from Theorem 6.4 that this subgroup is conjugate to another subgroup in that A_i. □

Theorem 6.4. *([1]) Let $p : \tilde{X} \to X$ be a cover of X with basepoint $\tilde{x_0} \in \tilde{X}$ with $p(\tilde{x_0}) = x_0$. If $p(\tilde{y_0}) = x_0$, then the groups $p_\#(\pi_1(\tilde{X}, \tilde{x_0}))$ and $p_\#(\pi_1(\tilde{X}, \tilde{y_0}))$ are conjugate subgroups of $\pi_1(X, x_0)$.*

Proof. We know that $\tilde{X}$ is a connected complex, therefore there exists a path, $\tilde{\alpha}$, between $\tilde{x_0}$ and $\tilde{y_0}$. Given that $p(\tilde{x_0})$ and $p(\tilde{y_0})$ are both x_0, $p(\tilde{\alpha})$ is a closed path $[\alpha]$ in the fundamental group of X. We can say that

$$\tilde{\alpha} \pi_1(\tilde{X}, \tilde{x_0}) \tilde{\alpha}^{-1} = \pi_1(\tilde{X}, \tilde{y_0}).$$

Applying the map p, we get

$$[\alpha] p_\#(\pi_1(\tilde{X}, \tilde{x_0}))[\alpha^{-1}] = p_\#(\pi_1(\tilde{X}, \tilde{y_0})).$$

□

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