Algebraic Geometry I

David Wiedemann

Table des matières

List of Theorems

Lecture 1: Intro

Mon 10 Oct

Quick Motivation

We study schemes.
These are objects that "look locally" like $(\operatorname{Spec} A, A)$.
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— $\forall U \subset X$ *open, an object in* C $\mathcal{F}(U)$

— $\forall V \subset U \subset X$ *open, a morphism* $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$

such that

— $\forall U$ *open,* $\rho_{U,U}$ *is the identity on* $\mathcal{F}(U)$

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of* $\mathcal{F}(U)$ *are called sections of* $\mathcal{F}$ *over* U.

— $\mathcal{F}(U)$ *is called the space of sections of* $\mathcal{F}$ *over* U

— *Elements of* $\mathcal{F}(X)$ *are called global sections.*

— *There are alternative notations for* $\mathcal{F}(U)$: $\Gamma(U, F)$ *or* $H_0(F)$

— *The* ρ_{UV} *are called restriction maps, for* $s \in \mathcal{F}(U)$*, we write* $s|_V :=$ $\rho_{UV}(s)$ *and is called restriction of* s *to* V.

Example

— *For any object* A *in* C*, we define the constant presheaf* $\underline{A}'$ *defined by* $\underline{A}'(U) = A$ *and with restriction maps the identity.*

- *The presheaf of continuous functions :* C^0.
  We define $C^0(U) := \{f : U \to \mathbb{R} | f$ *continuous* $\}$ *and the restriction maps are the natural restrictions.*
- *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

  This example is universal in a certain sense

Remark
Define the category Ouv_X *with*
- *objects* $U \subset X$ *open subsets*
- *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho(U)} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow[\rho(V)]{} & \mathcal{F}_2(V) \end{array}$$

Example
- *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$
- *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
- *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark
There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \ \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$.
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— $\forall s, t \in \mathcal{F}(U)$, *if* $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ *then* $s = t$

— $\forall \{s_i\}$ *with* $s_i \in \mathcal{F}(U_i)$ *and* $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, *then* $\exists s \in \mathcal{F}(U)$ *such that* $s|_{U_i} = s_i$

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*

— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, *the two second maps are induced by, for each pair* $i, j \in I$ *the restrictions* $\rho_{U_i, U_i \cap U_j}$ *resp.* $\rho_{U_j, U_i \cap U_j}$

Example

1. *If* $\mathcal{F}$ *is a sheaf , let* $U\emptyset \subset X$ *and* $I = \emptyset$, *then* $\mathcal{F}(\emptyset)$ *contains at most one element*
2. C^0 *(and* C^∞ *if* $X = \mathbb{R}^n$ *) are sheaves since* $\forall U \subset X$ *open*
   — *Two continuous functions* $f, g : U \to \mathbb{R}$ *that coincide on an open cover are equal*
   — *Given an open cover* $U = \bigcup_{i \in I} U_i$ *and* $f_i : U_i \to \mathbb{R}$, *the function* $f : U \to \mathbb{R}$ *defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ *is a morphism of the underlying presheaves.*

Remark

— $PSh_C(X)$ *is the category of presheaves of C on X*

— $Sh_C(X)$ *is the category of sheaves of C on X*

If $C = Ab$, *we drop the index.*

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then Spec A is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6
Let $(I, \leq)$ be a pair where I is a set and $\leq$ is a binary relation.
$(I, \leq)$ is called a preorder if ll is reflexive and transitive.
$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric
$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example

1. *Let $I = \{U \subset X | U \text{ open } \}$ and $U \leq V \iff V \subset U$.*
   Then I is a directed poset.
2. *For $x \in X$, let*
   $$I_x = \{U \subset X | U \text{ open } x \in U\}$$
   This is a directed poset.

Definition 7
Let $(I, \leq)$ *be a directed set and* C *a category.*
A direct system in C *indexed by* I *is a pair* $\left(\{A_i\}, \{\rho_{ij}\}_{i,j\in I, i\leq j}\right)$.
Where the A_i *are objects in* C*, the* $\rho_{ij} : A_i \to A_j$ *are morphisms in* C *such that*

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example
If $\mathcal{F}$ *is a presheaf of* C *on* X *and* I_X *as in the second example above, then*

$$\left(\{\mathcal{F}(U_i)_{U_i\in I_X}\}, \{\rho_{U_i,U_j}\}\right)$$

is a direct system.

Definition 8 (direct limit)
Let $(I, \leq)$ *be a directed set,* C *a category.*
Let $\left(\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I}\right)$ *be a directed system, then the direct limit is a pair* $(\lim_{i\in I} A_i, \{\rho_i\}_{i\in I})$ *where* $\lim A_i$ *is in* C *and* $\rho_i : A_i \to \lim A_i$ *such that*

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects* A *in* C *and morphisms* $f_i : A_i \to A$ *such that*

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

$\exists! f : \lim_{i\in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

Remark
The direct limit is unique up to unique isomorphism.

Example
Write $(*) = (\{A_i\}_{i\in I}, \{\rho_{ij}\}_{i,j\in I,i\leq j})$.
Let $*$ *be a direct systement in* Set.
Let $\lim_{i\in I} A_i := A_i/\sim$ *where* $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ *such that* $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
This is the direct limit of $*$.
If $*$ *is a system in* Ab *, let* $\lim A_i := \bigoplus A_i/N$ *with* $N = \langle a_i - \rho_{ij}(a_i)\rangle$.
The natural map $\bigcup A_i/\sim \to \bigoplus A_i/N$ *is a bijection*

Remark
Taking the direct limits in (Ab) *is exact in the following sense :*
$\forall$ *directed sets* I*,* $\forall$ *direct systems* $\{M_i\}, \{N_i\}, \{P_i\}$ *indexed by* I *and for all*

collections of commutative diagrams, we get

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

Definition 9
Let C be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of C on X.
Then the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x.
For $s \in \mathcal{F}(U)$, we write s_x for the image of s in $\mathcal{F}_x$ and call it the germ of s at x.

Remark
A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$

Remark
Let $x \in X, \mathcal{F}$ a presheaf of $\mathrm{Set}, \mathrm{Ab}$

1. $\forall U \subset X$ *open,* $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

Definition 10 (Sheafification)
Let $\mathcal{F}$ be a presheaf of sets (...) on X.
The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

Remark

1. $\mathcal{F}^+$ *is a sheaf*
2. *Sheafification is functorial.*
   For $\rho : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, the collection $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ sending $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. $\exists$ *a natural morphism* $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ *defined by* $\iota_F(U)(s) : x \to s_x$
4. $\forall s \in \mathcal{F}^+(U)$ *there is an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

Proposition 20

$\forall$ morphisms $\phi : \mathcal{F} \to \mathcal{G}$ such that $\mathcal{G}$ is a sheaf, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ open, let $s \in \mathcal{F}^+(U)$ $\exists$ an open cover $U = \bigcup_{i\in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.
Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, we have to set

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since $\mathcal{G}$ is a sheaf and

$$\phi(U_i)(s_i)|_{U_i\cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i\cap U_j}) = \phi(U_j)(s_i)|_{U_i\cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ with $t|_{U_i} = \phi(U_i)(s_i)$.
For ϕ^+ to be a morphism, we have to set $\phi^+(U)(s) = t$.
We still have to check that ϕ^+ is compatible with restriction maps. $\square$

Remark

The proposition above shows that $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ naturally in the presheaf and the sheaf $\mathcal{G}$.
Hence $(-)^+$ is left-adjoint to the forgetful functor $Sh(X) \to Psh(X)$

Proposition 22

$X = \operatorname{Spec} A$ $\forall a \in A$ there exist isomorphisms $\phi_a : A_a \to \mathcal{O}_X(D(a))$ such that $\forall b \in A$ with $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow[\sim]{} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \to \mathcal{O}_X(D(a))$ by sending $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.
Clearly, these make the diagram commute.
This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.
Let $I = Ann(s) = \{r \in A | rs = 0\}$.
Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, we have $I \not\subset p$, hence $V(I) \subset V(a) \implies a \in \sqrt{I}$.
Thus there exists $m \geq 1$ such that $a^m s = 0$, hece $\frac{s}{a^n} = 0$.
To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, by definition of $\mathcal{O}_X$ and because $D(h_i)$ form a basis, we find $a_i, g_i, h_i \in A$ such that

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

and $s(q) = \frac{a_i}{g_i}$ for all $q \in D(h_i)$.

1. *Claim 1 : Can choose* $g_i = h_i$
2. *Claim 2 : Can choose* I *finite*
3. *Claim 3 : Can choose* a_i, h_i *such that* $h_j a_i = h_i a_j$.

Using these claims, since $D(a) = \bigcup D(h_i)$ *, we find* $n > 0, b_j \in A$ *such that* $a^n = \sum b_j h_j$.
Write $c = \sum a_i b_i$.
Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.
Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.
We now prove the claims

1. *We have* $D(h_i) \subset D(g_i)$ *thus* $V(g_i) \subset V(h_i)$ *and thus* $h_i \in \sqrt{(g_i)}$.
   So there exists $c_i \in A$ *and* $n > 1$ *such that* $h_i^n = c_i g_i$.
   Now, we replace h_i *by* h_i^n *and* a_i *by* $a_i c_i$. *Then*
   $$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$
2. *We have* $D(a) \subset \cup D(h_i) \iff V(\sum h_i) = \cap V(h_i) \subset V(a)$.
   This is equivalent to saying that $a \in \sqrt{\sum (h_i)}$.
   Thus there exists $n \geq 1$ *such that* $a^n \in \sum_i (h_i)$.
   So there exist finitely many $b_i \in A$ *such that* $a^n = \sum b_j h_j$
3. *On* $D(h_i) \cap D(h_j) = D(h_i h_j)$, *we have*
   $$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$
   Thus
   $$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$
   Thus, there exists $N_j \geq 1$ *such that* $(h_i h_j)^{N_j}(h_j a_i - h_i a_j) = 0$.
   From claim 2, I *is finite, so we can choose* N *big enough such that* N *works for all* $D(h_i)$.
   Now, we replace h_i *by* h_i^{N+1} *and* a_i *by* $h_i^N a_i$ *and we get* $h_j a_i - h_i a_j = 0 \in A$. □

Corollary 23
Take $X = \operatorname{Spec} A$*, then* $\forall p \in \operatorname{Spec} A \exists$ *isomorphisms* $\phi_p : A_p \to \mathcal{O}_{X,p}$ *such that the appropriate diagram commutes.*

Proof

1. *Observe* $\lim_{a \in A \setminus p} = A_a \simeq A_p$ *(check universal property)*
2. *Observe that* $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$

Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

Definition 11 (Subsheaf)
Let $\mathcal{F}$ be a (pre)sheaf on X.
Then a sub(pre)sheaf of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by $\mathcal{F}$.

Definition 12 (Kernel, cokernel of presheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *The presheaf kernel of ϕ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$*
2. *The presheaf image is defined as $\mathrm{Im}^{pre}(\phi)(U) = \mathrm{Im}(\phi(U))$*
3. *The presheaf cokernel is $\mathrm{coker}^{pre}(\phi)(U) = \mathrm{coker}(\phi(U))$.*

In each case, the restriction maps are induced by those in of $\mathcal{F}$ or $\mathcal{G}$.

Lemma 24
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf kernel is a sheaf.

Proof
Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
Since $\mathcal{F}$ is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.
Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.
Thus

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

Since $\mathcal{G}$ is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$. □

Example
By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are.

Definition 13 (Cokernel/image of morphisms of sheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *sheaf kernel : $\ker^{pre}(\phi)$*

2. *sheaf image* $(\mathrm{Im}^{pre}(\phi))^+$
3. *sheaf cokernel* $(\mathrm{coker}^{pre}(\phi))^+$

Lemma 26 (cokernels are cokernels)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. $\ker \phi \to \mathcal{F}$ *is a categorical kernel in* $Sh(X)$
2. $\mathcal{G} \to \mathrm{coker}\, \phi$ *is a categorical cokernel in* $Sh(X)$.

Proof

1. *This means that for each commutative diagram with solid arrows, the dotted arrow is unique*
   "Insert cokernel/kernel diagram here"
   This holds for every open U and so the kernel is a sheaf.
2. *The appropriate diagram commutes and we use the universal property of sheafification.* □

Proposition 27
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is a monomorphism in* $Sh(X)$
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

Proof
Recall ϕ is a monomorphism if for every $\psi : \mathcal{F}' \to \mathcal{F}, \phi \circ \psi = 0 \implies \psi = 0$.
The implication $1 \implies 2$ *follows by applying the monomorphism property to* $\ker \phi \to \mathcal{F}$ $2 \implies 1$ *If $\phi \circ \psi = 0$, then ψ factors through the kernel* $\ker \phi \to \mathcal{F}$ *and so $\psi = 0$*
$2 \iff 3$ *Since* $\ker(\phi)(U) = \ker(\phi(U))$
$3 \implies 4$ *Follows because taking direct limits is exact.*
$4 \implies 3$ *Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.*
So $s_x = 0 \forall x \in U$ and so $s = 0$ □

Proposition 28
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is an epimorphism in* $Sh(X)$
2. $\mathrm{coker}(\phi) = 0$
3. $\mathrm{coker}(\phi_x) = 0$

Proof

Recall that ϕ is an epimorphism if for every $\psi : \mathcal{G} \to \mathcal{G}', \psi \circ \phi = 0 \implies \psi = 0$

$1 \implies 2$ *Apply epimorphism property to $\mathcal{G} \to \operatorname{coker}(\phi)$*

$2 \implies 3$ *We have*

$$\begin{aligned} 0 &= (\operatorname{coker} \phi)_x \\ &= (\operatorname{coker}^{pre} \phi)_x = \operatorname{coker}(\phi_x) \end{aligned}$$

□

$3 \implies 1$

Let $\psi : \mathcal{G} \to \mathcal{G}'$ such that $\psi \circ \phi = 0$, this implies that $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$.

Since ϕ_x is an epimorphism of abelian groups, we get $\psi_x = 0$.

As the hom sheaf is a sheaf, we get that $\psi = 0$

Remark

If $\operatorname{coker}(\phi(U)) = 0 \forall U \subset X \implies \operatorname{coker}(\phi) = 0$ but the converse is not true.

Corollary 30

If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then the following are equivalent

1. *ϕ is an isomorphism*
2. *$\phi(U)$ is an isomorphism $\forall U \subset X$ open*
3. *ϕ_x is an isomorphism $\forall x \in X$*

Proof

$1 \implies 2$ *since taking sections is a functor*

$2 \implies 3$ *since taking limits is functorial*

$2 \implies 1$ *because $(\phi(U))^{-1}$ defines a morphism of sheaves*

$3 \implies 2$ *Need to show surjectivity of $\phi(U)$.*

Let $t \in \mathcal{G}(U)$, since ϕ_x is an isomorphism $\forall x \in U$, we find $s_x \in \mathcal{F}_x$ such that $\phi_x(s_x) = t_x$.

There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ such that $(s_{V_x})_x = s_x$

Since

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

we can choose $V + x$ sufficiently small such that $\phi(V_x)(s_{V_x}) = t|_{V_x}$.

Since $\phi(V_x \cap V_y)$ is injective and $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$, we have $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$.

Thus there exists $s \in \mathcal{F}(U)$ such that $s|_{V_x} = s_{V_x}$ and $\phi(U)(s)|_{V_x} = t|_{V_x}$ and thus $\phi(U)(s) = t$. □

Definition 14 (Exact Sequence of sheaves)

A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ is called exact if $\ker \phi_2 = \operatorname{Im} \phi_1$

Corollary 31
A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.

Lecture 4: locally ringed spaces, (affine) Schemes (!)

Fri 21 Oct

Corollary 32
A sequence of sheaves is exact if and only if it is exact on all stalks.

Proof
If $\ker(\phi_{2,x}) = \operatorname{Im}(\phi_{1,x}) \forall x \in X$*, thus* $(\phi_{2,x} \circ \phi_{1,x}) = (\phi_2 \circ \phi_1)_x$.
Thus $\phi_2 \circ \phi_1 = 0$ *because the hom sheaf is a sheaf.*
Thus ϕ_1 *factors as* $\mathcal{F}_1 \to \operatorname{Im}\phi_1 \to \ker\phi_2 \to \mathcal{F}_2$ *as* ψ_x *is an isomorphism,* ψ *is an isomorphism.* □

Corollary 33
Let $\phi : \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves, then* $\operatorname{Im}\phi = \ker(\mathcal{G} to \operatorname{coker}\phi)$

Corollary 34
$Sh(X)$ *is an abelian category.*

1.5 Direct and inverse image, ringed spaces

Definition 15
Let $f : X \to Y$ *be a continuous map.*
We define the direct image of $\mathcal{F}$ *by* f *on* Y *defined by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

We can check that $f_*\mathcal{F}$ is a sheaf with restriction maps induced by $\mathcal{F}$.
If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, then the $(f_*\phi)(X) = \phi(f^{-1}(V))\mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}V)$ define a morphism of sheaves.
Thus we get a functor $f_* : Sh(X) \to Sh(Y)$.

Definition 16 (inverse image)
Let $f : X \to Y$ *be a continuous map and let* $\mathcal{G}$ *be a sheaf on* Y.

The inverse image of $\mathcal{G}$ along f is the sheafification of the presheaf

$$f^{-1,pre}(\mathcal{G})$$

defined by

$$f^{-1,pre}(\mathcal{G})(U) =_{f(U)\subset V} \mathcal{G}(V)$$

We can again check that the if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on Y, we define $f^{-1}\phi : \mathcal{F}(V) \to \mathcal{G}(V)$ using the maps induced by ϕ.
Thus we get a functor $Sh(Y) \to Sh(X)$.

Lemma 35
Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

1. *$\forall y \in Y$ there is a natural isomorphism*

$$(f_*\mathcal{F})_y \simeq_{y\in V\subset Y} \mathcal{F}(f^{-1}(V))$$

   In particular $forall x \in X$ there is a natural map $(f_*\mathcal{F})_{f(x)\to\mathcal{F}_x}$

2. *$\forall x \in X$ there is a natural isomorphism $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$*

Proof
The isomorphisms are immediate from the definition.
The morphism $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ is given by

$$(f_*\mathcal{F})_{f(x)} = \mathcal{F}(f^{-1}(V)) =_{x\in f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \to_{x\in U} \mathcal{F}(U) = \mathcal{F}_x \qquad \square$$

Proposition 36
If $f : X \to Y$ is a continuous map, then $f_* : Sh(X) \to Sh(Y)$ is right-adjoint to $f^{-1} : Sh(Y) \to Sh(X)$

Corollary 37
$f^{-1} : Sh(Y) \to Sh(X)$ is exact

Proof
Let $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ be exact in $Sh(Y)$.
Thus $\forall y \in Y, 0 \to \mathcal{G}_{1,y} \to \mathcal{G}_{2,y} \to \mathcal{G}_{3,y} \to 0$ is exact.
In particular it is exact at $f(x)\forall x \in X$ and thus the associated inverse image sequence is exact. $\square$

Corollary 38

$f_* : Sh(X) \to Sh(Y)$ is left-exact.

Proof

Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ be exact in $Sh(X)$.
Recall that the section functor is left-exact, thus $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ is exact $\forall U \subset X$.
Thus $0 \to (f_*\mathcal{F}_1)_y \to (f_*\mathcal{F}_2)_y \to (f_*\mathcal{F}_3)_y$ is exact $\forall y \in Y$ and thus $0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ is exact. $\square$

Example

f_* is usually not right-exact.
Eg, if $f : X \to \{*\}$ and $\mathcal{F}$ is a sheaf on X, then $(f_*\mathcal{F})(\emptyset) = 0$ and $(f_*\mathcal{F})(\{*\}) = \mathcal{F}(X)$ and taking sections is not exact.

Definition 17 (Ringed space)

A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.
A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continous map and $f^\sharp$ is a morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark

Ringed spaces form a category, if $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ define their composition to be $(g \circ f, g_*(f^\sharp \circ g^\sharp))$

Example

1. *For every ring A, $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ is a ringed space.*
2. *For any field K and any topological space X, define a sheaf $Fun_{X,K}(U) = \{s : U \to K\}$.*
   There is a functor $\top \to$ (Ringed spaces) sending $X \mapsto (X, Fun_{X,K})$ where for $f : X \to Y$ $f^\sharp$ is the pullback (precomposition).
3. *(X, C^0_X) is a ringed space*

Observe that for many of these examples of ringed spaces, the stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 18 (Morphism of local rings)

A morphism of local rings $\phi : A \to B$ with maximal ideals m_A and m_B is called local if $m_A = \phi^{-1}(m_B)$

Example

1. *For all ring homomorphism $\phi : A \to B$ and all $q \in \operatorname{Spec} B$ the induced*

map $A_{\phi^{-1}(q)} \to B_q$ is local.

2. *A ring homomorphism $\phi : A \to K$ from a local ring A to a field iff $m_A = \ker \phi$*

Definition 19 (Locally ringed space)
A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,x}$ is local $\forall x \in X$.
A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces such that

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \xrightarrow{f_x^\sharp} (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$$

is local.

Remark
The category of locally ringed spaces is a subcategory of the category of ringed spaces

Definition 20 (Affine Scheme)
An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that $X = \operatorname{Spec} A$ and $\mathcal{O}_X$ is the structure sheaf.

Definition 21 (Scheme)
A scheme is a locally ringed space $(X, \mathcal{O}_X)$ such that there exists an open cover $X = \bigcup_{i \in I} U_i$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme.
A morphism of schemes is a morphism of the underlying ringed spaces.

Example

1. *If $(X, \mathcal{O}_X)$ is a scheme and $U \subset X$ is open, then $(U, \mathcal{O}_X|_U)$ is not necessarily a scheme (even if X is affine).*
2. *If $(X, \mathcal{O}_X)$ is a scheme and $X = \{*\}$, then X is affine.*
   Then $\operatorname{Spec} A = \{*\}$ iff every $a \in A$ is either a unit or nilpotent.