Exercise for Submission 5

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We will show this result by using the binomial theorem.
We will denote by $\lfloor x \rfloor$ the biggest integer satisfying $\lfloor x \rfloor \leq x$, ie. the floor of the real number x.

$$\begin{aligned}
\frac{1}{2}\left((1+\sqrt{2})^n + (1-\sqrt{2})^n\right) &= \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}\sqrt{2} + \sum_{k=0}^{n}\binom{n}{k}(-\sqrt{2})\right) \\
&= \frac{1}{2}\left(\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}\sqrt{2}^{2k} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k-1}\sqrt{2}^{2k-1}\right. \\
&\left. + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}(-\sqrt{2})^{2k} + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k-1}(-\sqrt{2})^{2k-1}\right)
\end{aligned}$$

Notice that all the uneven powers simplify and that $(-\sqrt{2}^{2k}) = \sqrt{2}^{2k}$, thus we are left with

$$\frac{1}{2}\left(2 \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}\sqrt{2}^{2k}\right) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2k}2^k$$

Which proves that $\frac{1}{2}((1+\sqrt{2})^n + (1-\sqrt{2})^n)$ always is a whole number.