Chapter 3

Functions interpolation

3.1 Introduction

One of the first things that could come to our mind when thinking about function approximation is *Taylor expansion (with Lagrange remainder)*. From the very introductory course of calculus, we know that if $f \in C^{p+1}(I)$, I being a neighborhood of a point x_0, then there exists c, between x_0 and x, such that

$$f(x) = T_p(x) + \frac{f^{(p+1)}(c)}{(p+1)!}(x - x_0)^{p+1},$$

where, let us remind,

$$T_p(x) := \sum_{k=0}^{p} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is the so-called *Taylor polynomial* of degree p centered in x_0. A simple idea would be then to employ T_p for a numerical algorithm in order to approximate a sufficiently regular given function f. However, Taylor polynomials have many drawbacks. First of all, they rely on the computation of the first p derivatives of f, which might be very costly or impossible to do from a numerical point view. Additionally, Taylor expansion provides an accurate approximation of the target function just in a neighborhood of the fixed point x_0 and, in general, can be very bad when we are sufficiently far, namely when outside the convergence radius of the Taylor series. This phenomenon is illustrated in the following numerical example.

Example 3.1. *Let us consider $f(x) = \frac{\sin x}{x}$ and its Taylor polynomial of degree n centered in $x_0 = 0$. From Figure 3.1, we can clearly see that the approximation behavior gets worse as we move away from x_0.*

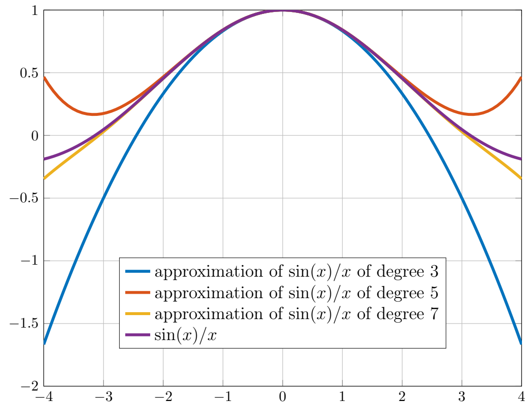


Figure 3.1: Approximation of the function $f(x) = \frac{\sin x}{x}$ by Taylor polynomials for different degrees $n = 3, 5, 7$.

Hence, we need to come up with more sophisticated strategies. In this Chapter we are going to address the so-called *function interpolation*, which is one way of approximating a given function. It consists in constructing "simple" functions, namely polynomials, piecewise polynomials, trigonometric polynomials, that match exactly some information that we have about a real-valued function. Note that the main feature of these "simple" functions is that they can be completely determined by finitely many coefficients, or, more formally, they belong to some finite dimensional vector space.

In this course we will focus on interpolation of functions $f : [a,b] \to \mathbb{R}$, but the main ideas extend to vector-valued functions $f : D \to \mathbb{R}^m$, D being a domain of $\mathbb{R}^n$.

How to choose an interpolation strategy largely depends upon the information we have about the function we aim to approximate. We will make the assumption that f is known at finitely many points:

$$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n),$$

where $y_i = f(x_i)$, $f : [a,b] \to \mathbb{R}$. We look for p, "simple" function, such that $p(x_i) = y_i$ for every $i = 0, \ldots, n$. It is then interesting to study the *interpolation error* $f(x) - p(x)$, and this will be the main theoretical contribution of this Chapter.

In Chapter 3 we are mainly focusing on the case of polynomial and piecewise polynomial interpolation, while the specific case of trigonometric polynomials will be presented in Chapter 4.

In what follows we will denote as $\mathbb{P}_n$ the vector space of polynomials with real coefficients of degree at most n.

Concerning polynomial approximation, let us recall the following classical result due to Weierstrass, which is considered to be the basis of the modern theory of approximation of functions of one real variable. We just need to introduce the notation:

$$\|u\|_\infty := \max_{x \in [a,b]} |u(x)|. \tag{3.1}$$

Theorem 3.1 (Weierstrass, 1885)**.** *Let $f \in C^0([a,b])$. For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ and $p \in \mathbb{P}_n$*

such that

$$\|f - p\|_\infty < \varepsilon.$$

Equivalently, there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}} \subset \mathbb{P}_n$ *such that*

$$\lim_{n\to\infty} \|f - p_n\|_\infty = 0.$$

Weierstrass theorem states the existence, for a given continuous function f in a compact set, of a sequence of polynomials uniformly converging to f. Note that the original proof of the theorem is not constructive and it just shows that there exist such polynomials for a "sufficiently large" degree n. A constructive proof based on the so-called *Bernstein polynomials* was given in 1915. However the uniform convergence of these polynomials to f as n goes to infinity is very slow, hence not useful for computational purposes.

Throughout this Chapter we are going to address the problem about how to construct approximating polynomials in an efficient way from a numerical point of view.

3.2 Lagrange interpolation

In order to address the question of interpolation we first need a basis for the space of polynomials. We all know that the monomials $(x^i)_{i=0}^n$ are a possible choice of such a basis. We will see in the exercise sessions how bad such a choice is, hence here we propose a basis tailored to our needs.

Definition 3.1. *Let* $n \in \mathbb{N}$, $n \geq 1$. *Given* $n+1$ *distinct points* $(x_i)_{i=0}^n \subset [a,b]$, *we define, for every* $0 \leq i, j \leq n$, $\ell_i \in \mathbb{P}_n$ *such that*

$$\ell_i(x_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

The polynomials $(\ell_i)_{i=0}^n$ *are said to be the* Lagrange polynomials *of degree n associated to* $(x_i)_{i=0}^n$.

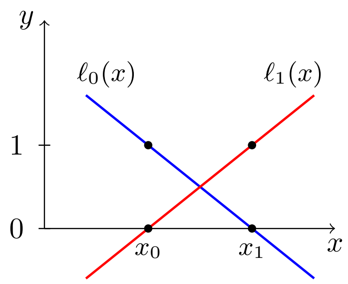
Remark 3.1. *The trivial case* $n = 0$ *can also be included in Definition 3.1 by setting:*

$$\ell_0(x) := 1 \qquad \forall\, x \in \mathbb{R}.$$

Example 3.2. *Let us exhibit the Lagrange polynomials of degree* 1 *and of degree* 2.

- *For* $n = 1$*:*

$$\ell_0(x) = \frac{(x - x_1)}{(x_0 - x_1)},$$
$$\ell_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}.$$



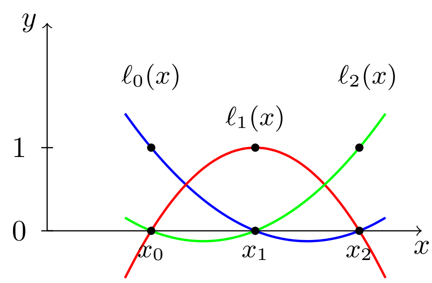


Figure 3.2: Example of Lagrange polynomials of degree 1 (left) and degree 2 (right).

- *For $n = 2$:*

$$\begin{aligned}
\ell_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},\\
\ell_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)},\\
\ell_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_1)(x_2-x_0)}.
\end{aligned}$$

The next result gives an explicit expression for the Lagrange polynomials of degree n, which is useful for computations.

Proposition 3.2. *Let $n \in \mathbb{N}$, $n \geq 1$. The Lagrange polynomials of degree n, associated to $(x_i)_{i=0}^n$, have the following form:*

$$\ell_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)} \qquad \forall\ i = 0,\ldots,n.$$

Proof. For each fixed i, $0 \leq i \leq n$, ℓ_i is required to have n distinct zeros at x_j, $0 \leq j \leq n$, $j \neq i$. Thus ℓ_i is of the form:

$$\ell_i(x) = C_i \prod_{\substack{j=0\\ j\neq i}}^{n} (x-x_j),$$

where $C_i \in \mathbb{R}$ is to be determined. Since $\ell_i(x_i) = 1$, it holds:

$$C_i = \frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{n} (x_i-x_j)},$$

and the proof is finished. □

The following Theorem states that we can safely represent the vector space $\mathbb{P}_n$ using the Lagrange polynomials.

Theorem 3.3. *Let $n \in \mathbb{N}$, $n \geq 0$, and $(x_i)_{i=0}^n$ be distinct points in $[a,b]$. The Lagrange polynomials $(\ell_i)_{i=0}^n$, associated to $(x_i)_{i=0}^n$, form a basis for the vector space $\mathbb{P}_n$.*

Proof. Since we know $\dim \mathbb{P}_n = n+1$, it suffices to show that $(\ell_i)_{i=0}^n$ are linearly independent. Suppose that there exist $(\lambda_i)_{i=0}^n \subset \mathbb{R}$ such that

$$\lambda_0 \ell_0(x) + \cdots + \lambda_0 \ell_n(x) = 0 \qquad \forall\, x \in \mathbb{R}. \tag{3.2}$$

Then, by Definition 3.1, it is enough to evaluate the expression (3.2) in x_i. It holds:

$$\lambda_i = \lambda_i \ell_i(x_i) = 0 \qquad \forall\, i = 0, \ldots, n,$$

which ends the proof. □

Theorem 3.4 (Lagrange Interpolation)**.** *Let $n \in \mathbb{N}$, $n \geq 0$. Let $(x_i)_{i=0}^n \subset \mathbb{R}$ be distinct and $\left(y_i\right)_{i=0}^n \subset \mathbb{R}$. Then there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that*

$$p_n(x_i) = y_i \qquad \forall\, i = 0, \ldots, n.$$

Proof. Thanks to Theorem 3.3, we know that the the following is a polynomial of degree n:

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x),$$

where $(\ell_i)_{i=0}^n$ are the Lagrange polynomials of degree n associated to $(x_i)_{i=0}^n$. By Definition 3.1:

$$p_n(x_i) = y_i \qquad \forall\, i = 0, \ldots, n. \tag{3.3}$$

It remains to show p_n is the unique polynomial of degree n that satisfies (3.3). Indeed, let $q_n \in \mathbb{P}_n$ be such that $q_n(x_i) = y_i$ for all $i = 0, \ldots, n$. Since $p_n - q_n \in \mathbb{P}_n$ and $p_n - q_n$ has $n+1$ distinct roots, namely $p_n(x_i) - q_n(x_i) = 0$ for all $i = 0, \ldots, n$, it holds:

$$p_n(x) - q_n(x) = 0 \qquad \forall\, x \in \mathbb{R},$$

and the proof is finished. □

Definition 3.2. *Let $n \in \mathbb{N}$, $n \geq 0$. Given $(x_i)_{i=0}^n$ distinct real numbers and $\left(y_i\right)_{i=0}^n$ real numbers, the polynomial $p_n \in \mathbb{P}_n$ defined by*

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x),$$

where $(\ell_i)_{i=0}^n$ are the Lagrange polynomials of degree n given by Definition 3.1, is called the Lagrange interpolation polynomial of degree n *with respect to* $\left\{(x_i, y_i)\right\}_{i=0}^n$*. The numbers x_i, $i = 0, \ldots, n$ are called* interpolation points*. In the case $f \in C^0([a,b])$ and $(x_i)_{i=0}^n \subset [a,b]$, the polynomial $p_n \in \mathbb{P}_n$ defined by*

$$p_n(x) = \sum_{i=0}^{n} f(x_i) \ell_i(x)$$

is called the Lagrange interpolation polynomial of degree n for the function f *with respect to the* interpolation points x_i, $i = 0, \ldots, n$.

3.2.1 Interpolation error with Lagrange polynomials

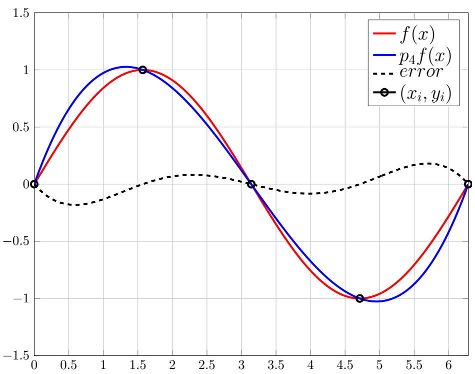
As we have already mentioned in the introduction, we are interested in studying the *interpolation error* between the target function $f \in C^0([a,b])$ and its Lagrange interpolation polynomial p_n, namely $f(x) - p_n(x)$. Indeed, we observe that even if f and p_n agree at the interpolation points, $f(x)$ could be quite "distant" to $p(x)$ when x is not an interpolation point.
Since we are dealing with continuous functions, the natural way to measure this distance is to use the metric induced by the infinity norm $\|\cdot\|_\infty$ (see (3.1)). Hence, the quantity of interest is $\|f - p_n\|_\infty$, and, in particular, we will try to understand how this value depends on the polynomial degree n.
Once for all, the setting is the following: let $f \in C^0([a,b])$ and p_n be its Lagrange interpolation polynomial at the distinct points $x_0, \ldots, x_n \in [a,b]$. Firstly, we remark that $f - p_n \in C^0([a,b])$ and it verifies

$$f(x_i) - p_n(x_i) = 0 \qquad \forall\ i = 0, 1, \ldots, n.$$

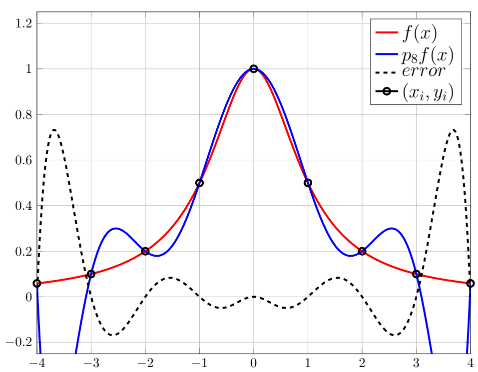
Example 3.3. *Let us illustrate the Lagrange interpolation through a couple of numerical experiences.*

1. *We consider the function $f(x) = \sin(x)$ in $[0, 2\pi]$ and we set $n = 4$. In Figure 3.3a we can see the Lagrange interpolation polynomial p_4 computed with respect to the equispaced interpolation points: $x_i = \frac{\pi}{2} i$ for $i = 0, \ldots, 4$. The error function $f(x) - p_4(x)$ is also plotted and, in particular, it can been seen that it vanishes at the interpolation points. Note that we have $\|f^{(5)}\|_\infty = 1$.*

2. *Let us repeat the previous experience with $f(x) = \frac{1}{1+x^2}$ in $[-4, 4]$ and taking 9 equidistant interpolation points. In Figure 3.3b we can see that this time things did not go as smoothly as before... We notice that $\|f^{(9)}\|_\infty \approx 324256$. In order to understand what is going on here, let us give some preliminary results.*



(a) $f(x) = \sin(x)$.



(b) $f(x) = \frac{1}{1+x^2}$.

Figure 3.3: Lagrange interpolation and error function.

Lemma 3.5. *Let $d \in C^n([a,b])$ such that $d(x_i) = 0$, for $i = 0, \dots, n$. Then, there exists $\xi \in (x_0, x_n)$ such that $d^{(n)}(\xi) = 0$.*

Proof. We need to apply Rolle's Theorem n times. For every $i = 0, \dots, n-1$, by Rolle's Theorem there exists $x_i^{(1)} \in (x_i, x_{i+1})$ such that $d'\left(x_i^{(1)}\right) = 0$. For every $i = 0, \dots, n-2$, again by Rolle's Theorem there exists $x_i^{(2)} \in \left(x_i^{(1)}, x_{i+1}^{(1)}\right)$ such that $d''\left(x_i^{(2)}\right) = 0$. Reiterating this procedure we conclude that there exists $\xi \in \left(x_0^{(n)}, x_1^{(n)}\right)$ such that $d^{(n)}(\xi) = 0$. □

The previous result comes to help for proving the following theorem, which provides a representation formula for the interpolation error.

Theorem 3.6 (Error's representation formula)**.** *Given $n \in \mathbb{N}$, $n \geq 0$. Let $f \in C^{n+1}([a,b])$ and let p_n be its Lagrange interpolation polynomial at the distinct points $(x_i)_{i=0}^n$. Then, for every $x \in [a,b]$ there exists $\xi \in (x_0, x_n)$ such that*

$$f(x) - p_n(x) = f^{(n+1)}(\xi)\,\Pi_n(x), \tag{3.4}$$

where

$$\Pi_n(x) := \frac{1}{(n+1)!}(x - x_0)(x - x_1)\dots(x - x_n).$$

Proof. Let $x \in [a,b]$. We distinguish two cases: either x is an interpolation point or x is *not* an interpolation point.

- Let $x = x_i$ for some $0 \leq i \leq n$. In (3.4) we get $0 = 0$, hence we are done.
- Let $x \neq x_i$ for all $0 \leq i \leq n$, so that $\Pi_n(x) \neq 0$. We consider the auxiliary function

  $$d(x) := f(x) - p_n(x) - \mu\Pi_n(x),$$

  with $\mu \in \mathbb{R}$ to be determined. Notice that d satisfies the hypotheses of Lemma 3.5, guaranteeing the existence of $\xi \in (x_0, x_n)$ such that $d^{(n+1)}(\xi) = 0$. Thus

  $$f^{(n+1)}(\xi) - p_n^{(n+1)}(\xi) - \mu\Pi_n^{(n+1)}(\xi) = 0. \tag{3.5}$$

  We notice that since $p_n \in \mathbb{P}_n$, $p_n^{(n+1)} \equiv 0$. Moreover, by construction of Π_n, it is readily seen that $\Pi_n^{(n+1)} = 1$. Hence, (3.5) implies:

  $$f^{(n+1)}(\xi) = \mu,$$

  which proves (3.4).

□

A straightforward consequence of the representation formula for the Lagrange interpolation error (3.4) is the following:

$$\left\| f - p_n \right\|_\infty \leq \left\| f^{(n+1)} \right\|_\infty \|\Pi_n\|_\infty. \tag{3.6}$$

However, this error bound is not robust. Indeed, on one hand, the term $\left\|f^{(n+1)}\right\|_\infty$ might become arbitrary large as n increases and, on the other, as we will see later on, $\|\Pi_n\|_\infty$ is strongly affected by the position of the interpolation points. This means that the estimate (3.6) does not imply that we have *uniform convergence*, i.e. the limit

$$\lim_{n\to\infty} \|f - p_n\|_\infty$$

may not be zero.

Note that it is usual to pick, among the interpolation points, the extremes of the interval $[a,b]$, namely $x_0 = a$ and $x_n = b$. So, for the sake of simplicity, let us focus on this particular choice in the rest of the section. With this in mind, we state an immediate consequence of Theorem 3.6.

Corollary 3.7. *Give $n \in \mathbb{N}$, $n \geq 0$. Let $f \in C^{n+1}([a,b])$ and p_n be the Lagrange interpolation polynomial of f with respect to the distinct points $(x_i)_{i=0}^n$, with $x_0 = a$ and $x_n = b$. Then*

$$\|f - p\|_\infty \leq \frac{1}{4\,(n+1)!}\,(b-a)^{n+1}\left\|f^{(n+1)}\right\|_\infty.$$

Proof. We have, for any $x \in [a,b]$,

$$\begin{aligned} |\Pi_n(x)| &\leq \frac{1}{(n+1)!}\,|(x-x_0)\,(x-x_1)\cdots(x-x_{n-1})\,(x-x_n)| \\ &= \frac{1}{(n+1)!}\,|(x-a)\,(x-b)|\,|(x-x_1)\cdots(x-x_{n-1})|\,. \end{aligned} \tag{3.7}$$

Notice that we can bound, for any $x \in [a,b]$,

$$|(x-a)\,(x-b)| \leq \frac{(b-a)^2}{4}, \tag{3.8}$$

and, for every $1 \leq i \leq n-1$ and all $x \in [a,b]$,

$$|(x-x_i)| \leq (b-a)\,. \tag{3.9}$$

By Theorem 3.6, for all $x \in [a,b]$,

$$\left|f(x) - p_n(x)\right| \leq |\Pi_n(x)|\left\|f^{(n+1)}\right\|_\infty. \tag{3.10}$$

Hence, by putting together (3.7), (3.8), (3.9), (3.10) we are done. □

Now, we want to investigate in detail under which conditions we have uniform convergence of the Lagrange interpolation polynomial to the target function. As we have already remarked, this strongly depends on two factors (see (3.6)):

- on the term $\left\|f^{(n+1)}\right\|_\infty$, hence on the regularity of the interpolated function f;
- on the term $\|\Pi_n\|_\infty$, so on the choice of the interpolation points $(x_i)_{i=0}^n$.

Let us restrict ourselves, for the moment, to the case of equispaced interpolation points, in order to focus on the regularity issue. We recall the definition of a real analytic function in a given point.

Definition 3.3. *A function $f:\mathbb{R}\to\mathbb{R}$ is said to be* real analytic in $x_0\in\mathbb{R}$ *if $f\in C^\infty(\mathbb{R})$ and there exists $R>0$ such that*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \qquad \forall\, x:|x-x_0|<R.$$

In the next theorem, the target function lives in the whole real line, but we are going to interpolate it just in the symmetric interval $[-\alpha,\alpha]$, $\alpha>0$.

Theorem 3.8. *Fix $n\in\mathbb{N}$, $n\geq 0$. Let $f:\mathbb{R}\to\mathbb{R}$ be real analytic in $x=0$ with convergence radius R, and let p_n be its Lagrange interpolation polynomial at the equidistant points $x_i=-\alpha+\dfrac{2\alpha}{n}i$, $i=0,\dots,n$. If $R>3\alpha$ then p_n converges uniformly to f, i.e.*

$$\lim_{n\to+\infty}\|f-p_n\|_\infty=0.$$

Proof. By Corollary 3.7, it holds:

$$\max_{x\in[-\alpha,\alpha]}\left|f(x)-p_n(x)\right|\leq\frac{1}{4(n+1)!}(2\alpha)^{n+1}\left\|f^{n+1}\right\|_\infty. \tag{3.11}$$

Let us fix x such that $|x|\leq R$ and set $r:=|x|$, so that we can write

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k \qquad \forall\, x:|x|\leq R. \tag{3.12}$$

and, in particular, we can define

$$C_r:=\sum_{k=0}^{\infty}\frac{\left|f^{(k)}(0)\right|}{k!}r^k.$$

Note that there exists $M\in\mathbb{R}$ such that $|C_r|\leq M$ for every $0\leq r\leq R$, since the Taylor series is absolutely convergent. For every fixed $k\in\mathbb{N}$, we can write:

$$\frac{\left|f^{(k)}(0)\right|}{k!}\leq\frac{C_r}{r^k}. \tag{3.13}$$

Let us assume $x\in[0,\alpha]$ so that $\dfrac{d^n}{dx^n}x^k\geq 0$. Taking the n-th derivative of (3.12) and exchanging the differentiation and summation (sign since power series converge uniformly inside their convergence radius), we get:

$$\begin{aligned}\left|f^{(n)}(x)\right|&=\left|\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\frac{d^n}{dx^n}x^k\right|\leq\left|\sum_{k=0}^{\infty}\frac{C_r}{r^k}\frac{d^n}{dx^n}x^k\right|=C_r\left|\sum_{k=0}^{\infty}\frac{d^n}{dx^n}\left(\frac{x}{r}\right)^k\right|\\&\leq C_r\left|\frac{d^n}{dx^n}\sum_{k=0}^{\infty}\left(\frac{x}{r}\right)^k\right|=C_r\frac{d^n}{dx^n}\left(\frac{1}{1-x/r}\right)=\frac{C_r\,r\,n!}{(r-x)^{n+1}},\end{aligned} \tag{3.14}$$

where we have also used (3.13), the sum of a geometric series and $\dfrac{d^n}{dx^n}\left(\dfrac{1}{1-x/r}\right)=\dfrac{n!r}{(r-x)^{n+1}}$

(which can be shown by induction on n). Note that can deal with the case $x \in [-\alpha, 0]$ by setting $y := -x$ and taking the derivative with respect to y. Combining (3.11) and (3.14), we obtain:

$$\max_{x\in[-\alpha,\alpha]} \left|f(x) - p_n(x)\right| \leq \frac{C_r r}{2\alpha}\left|\frac{2\alpha}{r-x}\right|^{n+2},$$

for $|x| < r < R$. Note that $R > 3\alpha$ implies $\left|\frac{2\alpha}{r-x}\right| < 1$, hence $\left\|f - p_n\right\|_\infty \to 0$ for $n \to +\infty$. □

The previous result shows us that, even for the more naive choice of the interpolation points, namely equispaced nodes, the uniform convergence of p_n to f can be always guaranteed at the price of a high regularity request on f. Indeed we need analyticity in an interval much larger (six times more!) than our interpolation interval.

Let us discuss the case of a function which is very regular, but not enough to satisfy the requirements of Theorem 3.8. We consider $f(x) = 1/\left(1+x^2\right)$ and we interpolate it on equidistant nodes in the interval $[-1,1]$. The function f admits a Taylor expansion $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ which converges for every x such that $|x| < 1$. In fact, from Complex Analysis, we know that when we look at f in the complex plane, it has pole singularities at $+i$ and $-i$, hence its Taylor series diverges for $|x| \geq 1$. As a matter of fact, the presence of these poles impacts on the interpolation at equidistant points. In the terminology of Theorem 3.8 we have $R = 1$, hence we are not satisfying its assumptions and it happens that we do not have uniform convergence. The deterioration of the convergence reflects in some oscillations at the neighborhoods of the extremes of the interpolation interval, see Figures 3.3b and 3.4. This problem has been observed for the first time in 1901 by Runge and, for this reason, is called *Runge phenomen.*

3.3 Interpolation on Chebyshev nodes

In the previous section we have seen that, when interpolating a function which is not regular enough, using equispaced interpolation nodes does not seem to be the best idea. In particular, the quantity $\|\prod_n(x)\|_\infty$ depends on the position of the interpolation points.

Our new task is to find the interpolation points $(x_i)_{i=0}^n$ such that the quantity $\|\prod_n(x)\|_\infty$ is minimized, namely

$$\min_{x_0,\ldots,x_n\in[a,b]} \max_{x\in[a,b]} |(x-x_0)(x-x_1)\ldots(x-x_n)|. \tag{3.15}$$

Since $(x-x_0)\ldots(x-x_n) \in \mathbb{P}^1_{n+1}$, the latter being the space of polynomials with unitary leading coefficient, i.e. $\mathbb{P}^1_{n+1} := \{p(x) = x^{n+1} + \sum_{i=0}^n a_i x^i : a_i \in \mathbb{R}, i = 0,\ldots,n\}$, the natural generalization of (3.15) would be the following:

$$\min_{p\in\mathbb{P}^1_{n+1}} L, \quad \text{with} \quad L := \max_{x\in[a,b]} \left|p(x)\right|. \tag{3.16}$$

For the sake of simplicity, let us rescale the min-max problem to the interval $[-1,1]$. Instead of

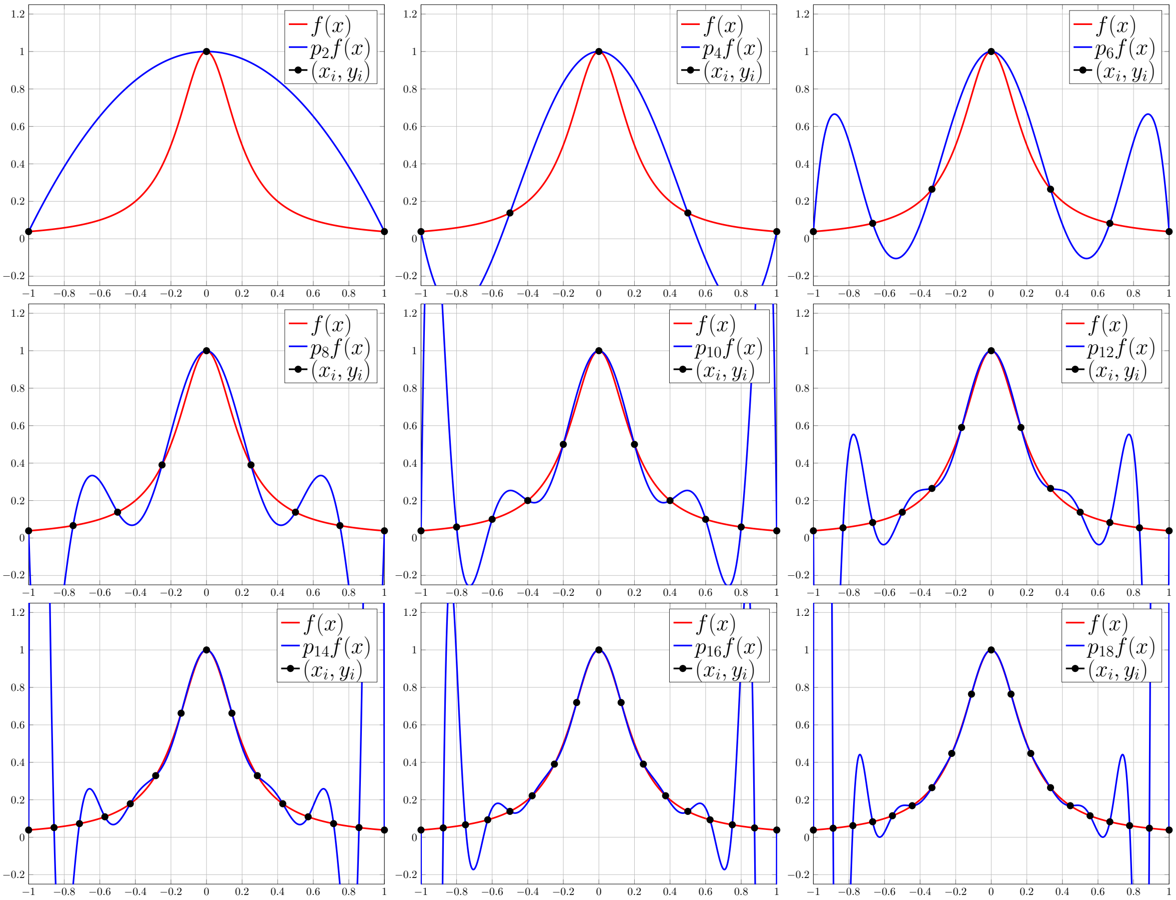


Figure 3.4: The Runge phenomenon for $f(x) = \frac{1}{1+25x^2}$.

(3.16), we consider the following problem:

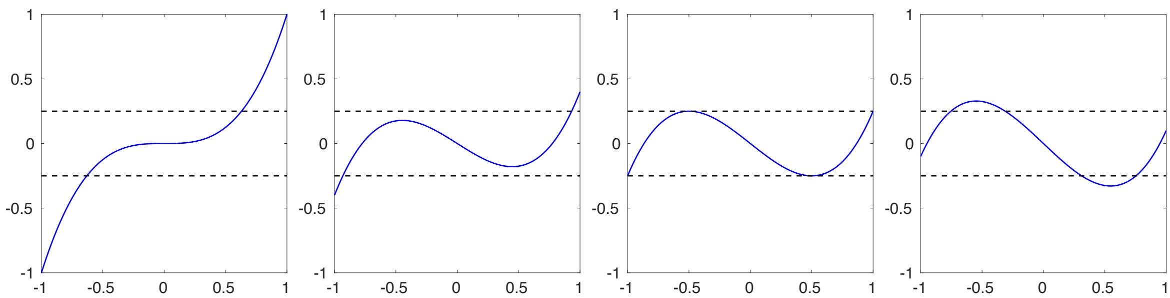
$$\min_{p\in\mathbb{P}^1_{n+1}} L, \quad \text{with} \qquad L := \max_{x\in[-1,1]} \left|p(x)\right|. \tag{3.17}$$

We begin with the case $n = 3$ and set $p(x) = x^3 - ax$ (so that we have symmetry with respect to the origin), where $a \in \mathbb{R}$ needs to be determined. In Figure 3.5 we see some instances of p for different values of a. The optimal solution turns out to be $p(x) = x^3 - \frac{3}{4}x$, which attains alternately the values $+L$ and $-L$ exactly four times.

Chebyshev found the following necessary condition for the optimal polynomial, for every degree n.

Proposition 3.9. *The polynomial $p \in \mathbb{P}^1_{n+1}$ minimizing* (3.17) *takes alternately the values $+L$ and $-L$ exactly $n+2$ times.*

Sketch of the proof. Let us suppose $n = 3$: the general proof exactly follows the same lines. We proceed by contradiction. Assume that $p(x) \in \mathbb{P}^1_3$ minimizes (3.17), but reaches the value $+L$ or $-L$ just 3 times, i.e. there exist $x_1, x_2, x_3 \in [-1,1]$ such that $p(x_1) = +L$, $p(x_2) = -L$, $p(x_3) = +L$. We can construct $q_2 \in \mathbb{P}_2$ such that $q_2(x_1) > 0$, $q_2(x_2) < 0$, $q_2(x_3) > 0$. Then, let us con-



(a) $a = 0.0$, $L = 1.0$. (b) $a = 0.6$, $L = 0.4$. (c) $a = 0.75$, $L = 0.25$. (d) $a = 0.9$, $L = 0.33$.

Figure 3.5: Maximal values of $p(x) = x^3 - ax$, for different values of a.

sider $\tilde{p}(x) := p(x) - \varepsilon q_2(x) \in \mathbb{P}_3^1$. By taking $\varepsilon > 0$ sufficiently small, it is possible to show that $\max_{x\in[-1,1]} |\tilde{p}(x)| < \max_{x\in[-1,1]} |p(x)|$, contradicting the optimality of p. □

Note that for a general n it is not a simple task to find a closed form for the polynomial minimizing (3.17). The so-called *Chebyshev polynomials* come to help.

Definition 3.4. *Given $n \in \mathbb{N}$, we define the n-th* Chebyshev polynomial *as*

$$T_n(x) = \cos(n \arccos x) \qquad \forall\ x \in [-1,1].$$

Remark 3.2. *First of all, let us notice that for any $n \in \mathbb{N}$ and every $x \in [-1,1]$ we have $|T_n(x)| \le 1$. Moreover, in spite of the unusual definition, for every $n \in \mathbb{N}$ it holds $T_n \in \mathbb{P}_{n+1}$. By computing the first few examples, we obtain indeed:*

$$\begin{aligned}
n = 0: &\quad T_0(x) = \cos(0 \arccos x) = \cos(0) = 1,\\
n = 1: &\quad T_1(x) = \cos(1 \arccos x) = x,\\
n = 2: &\quad T_2(x) = \cos(2 \arccos x) = 2\cos^2(\arccos x) - 1 = 2x^2 - 1.
\end{aligned}$$

Proposition 3.10. *The Chebyshev polynomials satisfy the following* recurrence relation*:*

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \forall\ n \in \mathbb{N}, n \ge 1.$$

Proof. For every $n \in \mathbb{N}$, $n \ge 1$, the following trigonometric identity holds:

$$\cos\big((n+1)\varphi\big) + \cos\big((n-1)\varphi\big) = 2\cos\varphi\cos\big(n\varphi\big) \qquad \forall\ \varphi \in \mathbb{R}. \tag{3.18}$$

In particular, (3.18) holds for every $\varphi \in [0,\pi]$. Since $\forall\ x \in [-1,1]$ there exists a unique $\varphi \in [0,\pi]$ such that $\varphi = \arccos x$, equation (3.18) implies:

$$T_{n+1}(x) + T_{n-1}(x) = 2T_1(x)T_n(x) = 2xT_n(x) \qquad \forall\ x \in [-1,1],$$

which ends the proof. □

From the previous Proposition we infer, in particular, that the leading coefficient of T_{n+1} is 2^n. Now, let us look at the points where the Chebyshev polynomials vanish.

Proposition 3.11. *Given $n \in \mathbb{N}$, the roots of the Chebyshev polynomial of degree n, T_n, are*

$$x_k = \cos\left(\frac{(2k+1)\pi}{2n}\right), \qquad k = 0,\dots,n-1,$$

which are called Chebyshev points.

Proof. A direct calculation yields, for every $k \in \mathbb{N}$, $0 \le k \le n-1$,

$$T_n(x_k) = \cos\left(n \arccos\cos\left(\frac{(2k+1)\pi}{2n}\right)\right) = \cos\left(\frac{(2k+1)\pi}{2}\right) = 0.$$

□

The following result shows that Chebyshev polynomials are a good candidate for the solution of problem (3.17), since they satisfy the necessary condition of Proposition 3.9.

Proposition 3.12. *Given $n \in \mathbb{N}$, T_n takes alternately the values $+1$ and -1 exactly $n+1$ times. In particular,*

$$T_n\left(\cos\left(\frac{k\pi}{n}\right)\right) = (-1)^k, \qquad \forall\ k = 0,\dots,n.$$

Proof. It is easy to show that at the points $\cos\left(\frac{k\pi}{n}\right)$, $k = 0,\dots n$, the n-th Chebyshev polynomial attains, respectively, a local minimum for k odd and a local maximum for k even. Since, by construction, $\|T_n\|_\infty \le 1$, these are global extreme points. □

We are now ready to provide the solution to the optimization problem (3.17).

Lemma 3.13. *The Chebyshev polynomial of degree $n+1$ with rescaled coefficients satisfies:*

$$\min_{p \in \mathbb{P}^1_{n+1}} \max_{x \in [-1,1]} |p(x)| = \max_{x \in [-1,1]} \left|2^{-n} T_{n+1}(x)\right|.$$

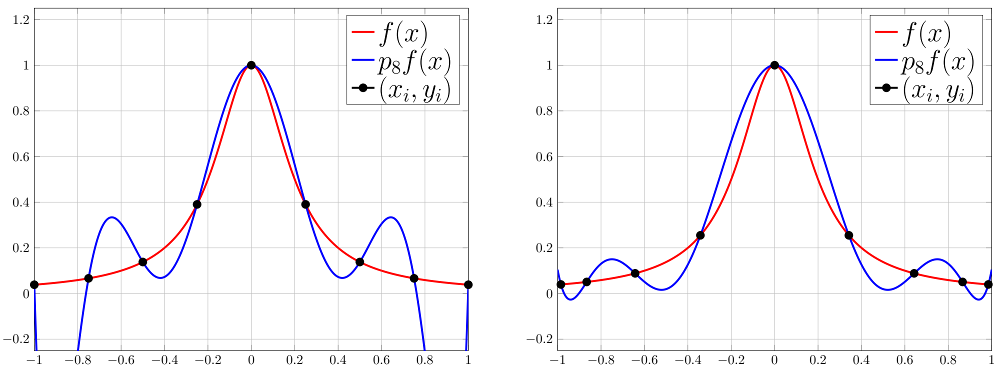
Proof. From Proposition 3.12 we have that $2^{-n}T_{n+1}$ alternates the values $+2^{-n}$ and -2^{-n} exactly $n+2$ times. We assume, by contradiction, that there exists $q \in \mathbb{P}^1_{n+1}$ such that $\|q\|_\infty < 2^{-n}$ and define $p(x) := q(x) - 2^{-n}T_{n+1}(x)$ for every $x \in [-1,1]$. Note that $p \in \mathbb{P}_n$ since the terms x^{n+1} cancel out. Moreover, p changes sign in each interval (z_i, z_{i+1}) for $0 \le i \le n+1$, where $(z_k)_{k=0}^n$ are the points where T_{n+1} attains its extreme values, so that, by the Intermediate Value Theorem, p admits $n+1$ distinct roots, which is not possible. □

We are finally ready to come back to our original task: we are looking for the interpolation points solving (3.15). The next result tells us that the Chebyshev points, rescaled in $[a,b]$, are indeed the interpolation nodes we are looking for.

Theorem 3.14. *The expression* $\max_{x\in[a,b]}|(x-x_0)\dots(x-x_n)|$ *is minimized among all the partitions* $(x_i)_{i=0}^n$ *if and only if*

$$x_k = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{(2k+1)\pi}{2n+2}\right), \qquad k = 0,\dots,n.$$

Proof. Let us fix $n\in\mathbb{N}$. By Lemma 3.13, we have that $\max_{x\in[-1,1]}|(x-x_0)\dots(x-x_n)|$ is minimized if and only if $(x-x_0)\dots(x-x_n) = 2^{-n}T_{n+1}(x)$. In its turn, the latter holds if and only if $(x_i)_{i=0}^n$ are the Chebyshev points of T_{n+1}. At this point, it suffices to rescale them through the affine transformation $x\mapsto \frac{a+b}{2}+\frac{b-a}{2}x$, mapping $[-1,1]$ to $[a,b]$. $\square$



(a) Interpolating on equidistant points, $n = 8$, $f(x)=\frac{1}{1+x^2}$.

(b) Interpolating on Chebyshev nodes, $n = 8$, $f(x)=\frac{1}{1+x^2}$.

Figure 3.6: Comparison Lagrange interpolation with equidistant and Chebyshev points.

Finally, we state (without proving it) a convergence result, telling as that if we suppose the interpolated function f to be reasonably regular, namely Lipschitz, then its Lagrange interpolation polynomial at the Chebyshev points uniformly converges to f.

Proposition 3.15. *Let* $f:[a,b]\to\mathbb{R}$ *be Lipschitz regular. Let* p_n^C *be the Lagrange interpolation polynomial of* f *at the Chebyshev points, then:*

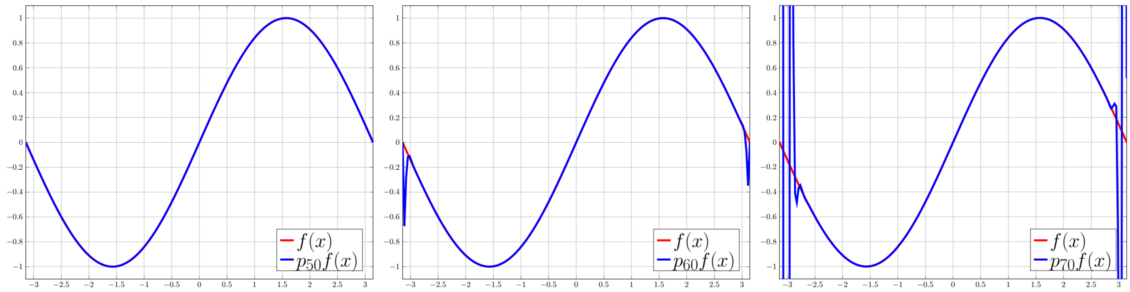
$$\lim_{n\to+\infty}\left\|f-p_n^C\right\|_\infty = 0.$$

In Figure 3.6, a comparison between Lagrange interpolation using equidistant and Chebyshev points is presented in the case of the *Runge function.*

3.4 Effects of rounding errors on the interpolation

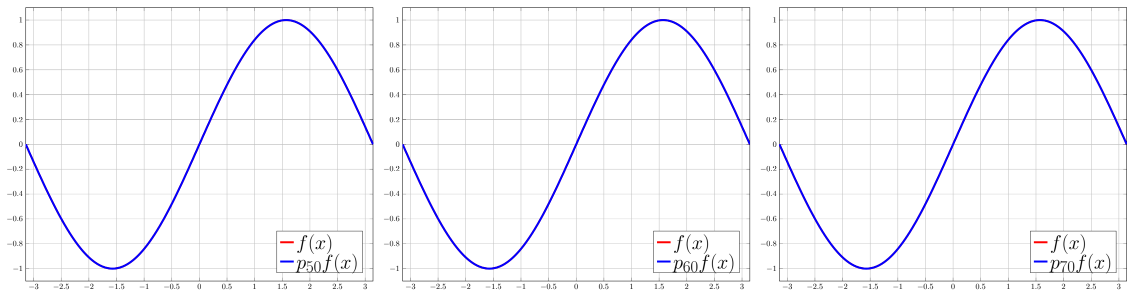
Every year the performance of the computers available on the market almost double, however their precision remains unchanged. This means that modern calculators are able to execute an enormous amount of operations in the blink of an eye at the price of producing loads of rounding errors. With this in mind the readers could ask themselves whether when performing numerical calculations, after all these numerical errors, the results we get are still reliable.

Example 3.4. *Let us consider the target function $f(x) = \sin(x)$ in the interval $[-\pi, \pi]$. Since f satisfies the regularity assumptions of Theorem 3.8, we expect uniform convergence of its Lagrange interpolation polynomial, even at equispaced nodes. However, in Figure 3.7, we can see that some oscillations at the extremes of the interval appear. If we choose, instead, the Chebyshev nodes, then we realize that things go better, see Figure 3.8. In order to explain this phenomenon, we would have to study the effects of the rounding errors on the interpolation process.*



(a) $n = 50$ *(b)* $n = 60$ *(c)* $n = 70$

Figure 3.7: Interpolating on equidistant nodes $f(x) = \sin(x)$.



(a) $n = 50$ *(b)* $n = 60$ *(c)* $n = 70$

Figure 3.8: Interpolating on Chebyshev nodes $f(x) = \sin(x)$.

Let us investigate what is going on in Example 3.4. As usual, let $f \in C^0([a,b])$ and let p_n be its Lagrange interpolation polynomial at $(x_i)_{i=0}^n$, i.e.

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\,\ell_i(x)\,.$$

We assume that there are some mismatches on the data, i.e. we have to perform our calculations with

$$\hat{f}(x_i) = f(x_i)(1+\epsilon)\,, \qquad \text{where} \qquad \epsilon = \text{ machine error, } |\epsilon| \le 10^{-16}.$$

We want to study the error due to the round-off, namely the difference between the polynomial passing through $\left(x_i, f(x_i)\right)_{i=0}^n$ and the one through $\left(x_i, \hat{f}(x_i)\right)_{i=0}^n$ denoted as

$$\hat{p}_n(x) = \sum_{i=0}^n \hat{f}(x_i)\,\ell_i(x). \tag{3.19}$$

We can bound, for every $x \in [a,b]$, the discrepancy error as:

$$\left|p_n(x) - \hat{p}_n(x)\right| = \left|\sum_{i=0}^n \left(f(x_i) - \hat{f}(x_i)\right)\ell_i(x)\right| \leq \sum_{i=0}^n \epsilon\left|f(x_i)\right||\ell_i(x)| \leq \epsilon\left\|f\right\|_\infty \sum_{i=0}^n |\ell_i(x)|. \tag{3.20}$$

Definition 3.5. *The quantity*

$$\Lambda_n := \max_{x\in[a,b]} \sum_{i=0}^n |\ell_i(x)|$$

is called Lebesgue constant *associate to the partition* $(x_i)_{i=0}^n$.

From equation (3.20), we deduce the following result.

Theorem 3.16. *Given $n \in \mathbb{N}$. Let $f \in C^0([a,b])$ and let p_n be its Lagrange interpolation polynomial at the distinct points $(x_i)_{i=0}^n$ and $\hat{p}_n$ be defined as in (3.19). We have:*

$$\left\|p_n - \hat{p}_n\right\|_\infty \leq \epsilon\Lambda_n\left\|f\right\|_\infty.$$

Finally, we state a Theorem (without proof) about the asymptotic behavior of the Lebesgue constant in the case of equispaced and Chebyshev interpolation points, providing an explanation to Example 3.4.

Theorem 3.17.

- *In the case of equispaced nodes, the Lebesgue constant grows exponentially, more precisely*

$$\Lambda_n \approx \frac{2^{n+1}}{en\log(n)} \qquad \textit{as} \quad n \to +\infty.$$

- *For Chebyshev points, the Lebesgue constant grows logarithmically, more precisely*

$$\Lambda_n \approx \frac{2}{\pi}\log(n) \qquad \textit{as} \quad n \to +\infty.$$

3.5 Piecewise polynomial interpolation

So far, we have built one single polynomial p_n to globally approach the target function f in the whole interval $[a,b]$. Note that this strategy leads to an accurate approximation of f at the price of taking the polynomial degree n bigger and bigger. There are various drawbacks linked to this approach. First of all, when n is very large the interpolation problem may suffer the presence of rounding errors as we have seen in Section 3.4. Then, when employing high order polynomials, we are of course constrained to the approximation of very regular target functions.

A much more flexible procedure in order to approximate f is to divide the interval $[a,b]$ into N subintervals, and to look for a piecewise approximation by polynomials of low degree s.

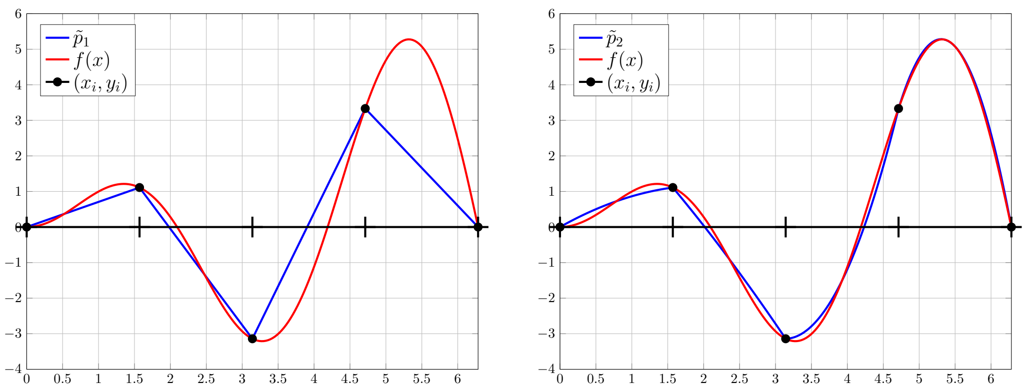
Definition 3.6 (Piecewise polynomial interpolation)**.** *Let us fix $N \in \mathbb{N}$, $N \geq 1$ and $s \in \mathbb{N}$. We consider $f \in C^0([a,b])$ and a partition $(a_i)_{i=0}^N$ of $[a,b]$. For every $0 \leq i \leq N-1$, we construct the local Lagrange interpolation polynomial $p_s^{(i)}$ with respect to $s+1$ chosen points of the local sub-interval $[a_i, a_{i+1})$. We glue together the global interpolant $\tilde{p}_s : [a,b] \to \mathbb{R}$ as follows:*

$$\begin{cases} \tilde{p}_s(x) := p_s^{(i)}(x) & \forall\, x \in [a_i, a_{i+1}), \forall\, 0 \leq i \leq N-1, \\ \tilde{p}_s(b) := p_s^{(N)}(b). \end{cases}$$

Remark 3.3.

- *For every $0 \leq i \leq N$, it holds $\tilde{p}_s|_{[a_i,a_{i+1})} \in \mathbb{P}_s$, i.e. $\tilde{p}_s$ is a* piecewise polynomial.
- *If for every $1 \leq i \leq N-1$ a_i and a_{i+1} are interpolation points for the computation of $p_s^{(i)}$ and $p_s^{(i+1)}$, then $\tilde{p}_s$ is continuous.*
- *In general, we choose $s \ll N$.*

Example 3.5. *Let us consider $f(x) = \sin(\frac{3}{2}x)x$ in the interval $[0, 2\pi]$. We report the piecewise interpolation polynomials of degree s on N subintervals of $[0, 2\pi]$. In Figure 3.9a the case $N = 4$ and $s = 1$ is depicted, while in Figure 3.9b the case $N = 4$ and $s = 2$ is plotted. Note that the piecewise polynomial interpolation for any degree $s \geq 1$ is only C^0-continuous across the sub-intervals.*



(a) Piecewise linear interpolation of f on $N = 4$ subintervals.

(b) Piecewise quadratic interpolation of f on $N = 4$ subintervals.

Figure 3.9: An example of piecewise polynomial interpolation.

Theorem 3.18. *Given $N \in \mathbb{N}$, $N \geq 1$ and $s \in \mathbb{N}$. Let $f \in C^{s+1}([a,b])$ and $\tilde{p}_s$ be its piecewise interpolation polynomial constructed as in Definition 3.6 from a general partition $(a_i)_{i=0}^N$ of $[a,b]$. Then, we have the following error estimate:*

$$\left\| f - \tilde{p}_s \right\|_\infty \leq \frac{H^{s+1}}{4(s+1)!} \left\| f^{(s+1)} \right\|_\infty, \tag{3.21}$$

where $H := \max\limits_{0 \leq i \leq N-1} |a_{i+1} - a_i|$. For an equidistant subdivision $a_i = a_0 + \dfrac{i}{N}(b-a)$, $0 \leq i \leq N$, the *error bound reads as follows:*

$$\left\|f-\tilde{p}_s\right\|_\infty \leq \frac{(b-a)^{s+1}}{4\,(s+1)!} N^{-(s+1)} \left\|f^{(s+1)}\right\|_\infty . \tag{3.22}$$

Proof. It is sufficient to show the error bound in the case of a general subdivision $(a_i)_{i=0}^N$, because the equispaced case follows as a consequence. By applying for each subinterval Corollary 3.7, we have:

$$\left\|f-\tilde{p}_s\right\|_\infty = \max_{0\leq i\leq N-1} \left\|f-p_s^{(i)}\right\|_{\infty,[a_i,a_{i+1})} \leq \frac{1}{4\,(s+1)!} \max_{0\leq i\leq N-1} |a_{i+1}-a_i|^{s+1} \left\|f^{(s+1)}\right\|_\infty ,$$

which is the desired result. □

Remark 3.4.
- *In general the polynomial degree s is fixed a priori, which implies that the term* $\dfrac{(b-a)^{s+1}}{4\,(s+1)!}\left\|f^{(s+1)}\right\|_\infty$ *is bounded.*
- *Note that in both cases* (3.21) *and* (3.22) *we have convergence, namely* $\left\|f-\tilde{p}_s\right\|_\infty \to 0$ *as* $N \to \infty$.

3.6 Approximation in the 2-norm: Legendre interpolation

Let us now go for a different approach in order to approximate a given function. With interpolation we were given a function or a collection of discrete measurements (most likely coming from an unknown function), and we built the polynomial that passed through a given set of data points. This time we want to approximate the target function by the "closest" polynomial with respect to a new notion of distance to be specified. Indeed, so far we have been measuring the interpolation error with respect to the distance induced by the norm $\|\cdot\|_\infty$. For our purpose let us employ, instead, the metric induced by $\|\cdot\|_2$, where

$$\|u\|_2 := \left(\int_a^b |u|^2 \,\mathrm{d}t\right)^{\frac{1}{2}} . \tag{3.23}$$

Notice that (3.23) makes sense even for discontinuous functions (e.g. take $u(x) := \mathrm{sgn}(x)$ in $[-1,1]$). Let us point out that the choice of the norm can significantly influence the outcome of the problem of best approximation: the polynomial of best approximation of a certain fixed degree to a given function in one norm can be very different to the polynomial of best approximation of the same degree in another norm.
Given $f \in C^0([a,b])$, we are interested in the minimization problem:

$$p^* = \operatorname*{arg\,min}_{q_n\in\mathbb{P}_n} \left\|f-q_n\right\|_2^2 , \tag{3.24}$$

where

$$p^* = \sum_{k=0}^{n} \alpha_k p_k, \tag{3.25}$$

and $(p_k)_{k=0}^n$ is the basis of Legendre polynomials which have been defined in (2.11). Note that p^* does not interpolate f in any specific point!

Remark 3.5. *Note that the choice of the Legendre basis in 3.25 is not mandatory, but it will turn out to be useful in the subsequent analysis.*

Theorem 3.19. *The optimal solution p^* to (3.24) is*

$$p^* = \sum_{k=0}^{n} \alpha_k p_k, \qquad \text{with} \qquad \alpha_k = \frac{\int_a^b f p_k \,\mathrm{d}t}{\int_a^b |p_k|^2 \,\mathrm{d}t}, \qquad k = 0, \dots, n.$$

Proof. By definition, p^* satisfies the following variational inequality:

$$\int_a^b |f - p^*|^2 \,\mathrm{d}t \le \int_a^b |f - q_n|^2 \,\mathrm{d}t \qquad \forall\, q_n \in \mathbb{P}_n.$$

Hence, by developing the left hand side, we obtain:

$$\begin{aligned}
\int_a^b |f - p^*|^2 \,\mathrm{d}t &= \int_a^b |f|^2 \,\mathrm{d}t - 2\int_a^b f p^* \,\mathrm{d}t + \int_a^b |p^*|^2 \,\mathrm{d}t \\
&= \int_a^b |f|^2 \,\mathrm{d}t - 2\sum_{k=0}^{n} \alpha_k \int_a^b f p_k \,\mathrm{d}t + \sum_{k=0}^{n}\sum_{k'=0}^{n} \alpha_k \alpha_{k'} \int_a^b p_k p_{k'} \,\mathrm{d}t \\
&= \int_a^b |f|^2 \,\mathrm{d}t - 2\sum_{k=0}^{n} \alpha_k \int_a^b f p_k \,\mathrm{d}t + \sum_{k=0}^{n} \alpha_k^2 \int p_k^2 \,\mathrm{d}t \qquad \forall\, q_n \in \mathbb{P}_n,
\end{aligned}$$

where we have used the orthogonality property of the Legendre polynomials (2.10).
We notice that the application

$$\begin{aligned}
\mathbb{R}^{n+1} &\to \mathbb{R} \\
(\alpha_0, \dots, \alpha_n) &\mapsto \int_a^b |f - p^*|^2 \,\mathrm{d}t
\end{aligned}$$

is convex and differentiable, hence in order to minimize it, we look at its stationary points:

$$\frac{\partial}{\partial \alpha_k} \int_a^b |f - p^*|^2 \,\mathrm{d}t = -2\int_a^b f p_k \,\mathrm{d}t + 2\alpha_k \int_a^b |p_k|^2 \,\mathrm{d}t = 0 \qquad \forall\, k = 0, \dots, n,$$

and we obtain

$$\alpha_k = \frac{\int_a^b f p_k \,\mathrm{d}t}{\int_a^b |p_k|^2 \,\mathrm{d}t} \qquad \forall\, k = 0, \dots, n.$$

□

Remark 3.6. *In order to construct p^* we need to compute $\int_a^b f p_k \,\mathrm{d}t$ for $k = 0, \dots, n$. We know that, in general, we cannot calculate exactly these quantities. However, we can employ a Gauss quadrature* *formula with $n+1$ points in order to approximate them, so that*

$$\tilde{p}^* = \sum_{k=0}^{n} \tilde{\alpha}_k p_k,$$

where the coefficients have been obtained as:

$$\tilde{\alpha}_k = \frac{Q^g_{n+1}\left(f p_k\right)}{\int_a^b \left|p_k\right|^2 \,\mathrm{d}t},$$

since $Q^g_{n+1}\left(p_k^2\right) = \int_a^b \left|p_k\right|^2 \,\mathrm{d}t$.

3.7 Implementational aspects

In this section we provide a possible implementation (written in MATLAB/Octave pseudo-code) of the interpolation method introduced above.

3.7.1 Implementation of the Lagrange interpolation on a single interval

In Listing 3.1 we provide a possible implementation of the Lagrange interpolation p_n of a function $f(x)$. The input parameters are a function handle f, the interval of interest, $[a, b]$, the polynomial degree, denoted by n, and a string that determines whether the interpolation should be performed using equidistant or Chebyshev nodes.

```
% test Lagrange interpolation
f = @(x) sin(x);
a = 0; b = 2*pi;
n = 4;
interpolation_nodes = 'equidistant'; % 'chebyshev'

if(strcmp(interpolation_nodes, 'chebyshev'))
    k = 0:n;
    xnodes = (a + b)/2 + (b-a)/2 .* cos((2*k+1)*pi./(2*n+2));

elseif(strcmp(interpolation_nodes, 'equidistant'))
    xnodes = linspace(a, b, n + 1);

else
    error('Unknown interpolation nodes!')

end

% First, we evaluate the function at the chosen nodes ...
ynodes = f( xnodes );
% then we compute the coefficients of the corresponding
% Lagrange polynomial of degree n ...
```

```
P = polyfit( xnodes, ynodes, n );
% and finally we can evaluate it at some new points
x_new = linspace(a, b, 501);
Pvalues = polyval(P, xnew);
% Plot the true function and its interpolation
plot(x_new, f(x_new), '-r'); hold on;
plot(x_new, Pvalues, '-b');
```

Listing 3.1: Example of computation of the Lagrange interpolation of degree n of a function f, either by using equidistant or Chebyshev nodes.