Série 7 Exercice 8

David Wiedemann

24 avril 2022

1

We show the double implication.
First, suppose there exists $s \in A$ such that $s^2 = b^2 - 4ac$.
Since $\gcd(a, b, c) = 1$, the polynomial $ax^2 + bx + c$ is primitive and we may apply Gauss's lemma which states that $ax^2 + bx + c$ is irreducible in $A[x]$ if and only if it is irreducible in $K[x]$.
Note that, in $K[x]$, we may write

$$\begin{aligned}
a(x - \frac{-b+s}{2a})(x - \frac{-b-s}{2a}) &= a\left(x^2 - \frac{-b-s}{2a}x - \frac{-b+s}{2a}x + \frac{(-b+s)(-b-s)}{4a^2}\right) \\
&= ax^2 + bx + a\frac{b^2 - s^2}{4a^2} \\
&= ax^2 + bx + c
\end{aligned}$$

Hence, $ax^2 + bx + c$ is not irreducible in $K[x]$ and thus also not in $A[x]$.

Now suppose $ax^2 + bx + c$ is not irreducible in $A[x]$, then it is also not irreducible in $K[x]$ by Gauss's lemma (as $ax^2 + bx + c$ is primitive by hypothesis) .
We now use the fact that a polynomial of degree two over a field is not irreducible if and only if it's zero set is non-empty (example 3.4.7.4 from the course notes).
Thus, rewrite (in $K[x]$)

$$\begin{aligned}
ax^2 + bx + c &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\
&= a\left[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right] \\
&= a\left[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}\right]
\end{aligned}$$

Thus, if

$$a\left[(x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

has a non-empty zero-set, then, there exists $\frac{s'}{d'} \in K, s', d' \in A$ such that

$$(\frac{s'}{d'} + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2} = 0$$

In particular, define $s = s'2a + bd'$ then we have that

$$\frac{s^2}{4a^2d'^2} = \frac{b^2 - 4ac}{4a^2} \iff \frac{s^2}{d'^2} = b^2 - 4ac$$

Thus $\frac{s}{d'}$ is an element of K satisfying the condition.
In fact $\frac{s}{d'}$ is in A and we show this general fact below.
This concludes the proof.

Claim :
If $a, b \in A$ and $\frac{a^2}{b^2} \in A$, then in fact $\frac{a}{b} \in A$.
Indeed, suppose $\frac{a}{b} \notin A$, then writing $a = v \prod_{i=1}^n a_i, b = u \prod_{j=1}^m b_i$ implies there exists an element b_k which is not associated to any $a_i, 1 \leq i \leq n$.
Indeed, if it was associated, we could simplify the two terms.
But then $\frac{a^2}{b^2} = \frac{v^2 \prod_{i=1}^n a_i^2}{u^2 b_k^2 \prod_{j=1, j\neq k}^m b_j^2} \notin A$ as b_k still has no associated element in the a_j^2, this follows from the fact that in a UFD, factorization into irreducibles is unique up to units. [1]

2

a)

We view $x^2 + 2yx + 1$ as an element of $(\mathbb{C}[y])[x]$ and use the criteria established above.
Indeed, $x^2+2yx+1$ is primitive as a polynomial over $\mathbb{C}[y]$ as $\gcd(1, 2y, 1) = 1$.
Furthermore, $4y^2 - 4$ (the discrimant $b^2 - 4ac$ of the polynomial) may be rewritten as $(2y - 2)(2y + 2)$, and we claim that there does not exist a polynomial f such that $f^2 = (2y - 2)(2y + 2)$.
Indeed, this would mean that $\deg f = 1$, but then f is linear and thus has exactly one 0, however f^2 has two distinct zero's, a contradiction.
Hence, the polynomial is irreducible.

b)

Simply write

$$y^2x^2 + yx^2 + yx + y^2 = y(yx^2 + x^2 + x + y)$$

1. This result only holds if A is a UFD, if A is not a UFD the result is false and I believe taking $A = \mathbb{C}[x, y] / (y^2 - x^2(x + 1))$ and $(\frac{y}{x})^2 = x + 1$ is a counterexample.

Thus, the polynomial is not irreducible, to find it's irreducible form, note that, looking at $(y+1)x^2+x+y$ as a polynomial in $(\mathbb{C}[y])[x]$, yx^2+x^2+x+y is primitive since $\gcd(y+1, 1, y) = 1$.
Furthermore,

$$1 - 4(y+1)y = 1 - 4y^2 - 4y$$

As $1 - 4y^2 - 4y$ has two distinct roots ($-\frac{1}{2} - \frac{1}{\sqrt{2}}$ and $-\frac{1}{2} + \frac{1}{\sqrt{2}}$), it cannot be the square of an element of $\mathbb{C}[y]$ (by the same argument as above).
Thus the factorization we have found is the decomposition into irreducibles.

c)

We use the same trick as in a) and consider it as a polynomial over $\mathbb{C}[y]$. Hence the discrimant is $y^2 - 4y^2 = -3y^2$ which is the square of $\sqrt{3}iy$.
Thus, we may write (this formula follows from our general computations in part 1)

$$\left(x - \frac{-y - \sqrt{3}iy}{2}\right)\left(x - \frac{-y + \sqrt{3}iy}{2}\right) = x^2 + yx + y^2.$$

As both these polynomials are irreducible over $\mathbb{C}[x, y]$ (they are primitive as their leading coefficient is 1 and of degree 1), we have found the decomposition of $x^2 + yx + y^2$ into irreducibles.

For completeness sake, we still prove that polynomials of degree 1 are irreducible. [2]
Indeed, if $f, g \in \mathbb{C}[x, y]$ such that $f \cdot g = ax + by + c$, then $\deg f + \deg g = 1$ implying either f or g is constant and thus invertible, hence $ax + by + c$ is irreducible.

2. I'm not showing that the degree is multiplicative for multivariate polynomials over $\mathbb{C}$