Topology I

Course by Viktoryia Ozornova

Table des matières

List of Theorems

1 Homology Theories

Lecture 1: Introduction

Mon 10 Oct

Aim : Study further algebraic invariants of topological spaces.
We want to assign to pairs of topological spaces abelian groups.

$$h_n : T \to \mathrm{Ab} \quad \forall n \in \mathbb{Z}$$

and to pairs continuous maps, we want to assign a map $h_n(f) : h_n(X) \to h_n(Y)$ which is functorial. Here T is the category of pairs of topological spaces $A \subset X$ with morphisms $f : (X, A) \to (Y, B)$ such that $f(A) \subset B$.
To relate h_n for different $n \in \mathbb{N}$, we will construct connecting morphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$.

Axiom 1 (Eilenberg-Steenrod Axiom)
A (generalised) homology theory consists of functors $h_n : T \to \mathrm{Ab}$ and natural connecting homomorphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$ [1] satisfying

— *Homotopy invariance :*
If $f, g : (X, A) \to (Y, B)$ are homotopic continous maps of pairs then the induced maps $h_n(f) = h_n(g)$. Here homotopy of pairs means that there exists $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subset B$

— *Long exact sequence of a pair (LES) :*
Given a pair of topological spaces (X, A) there is a long exact sequence of abelian groups.
Denote $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$, then

$$h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A, \emptyset)$$

— *Excision*
Given $B \subset A \subset X$ subspaces such that $\overline{B} \subset A^o$, the inclusion induces a group isomorphism

$$h_n(X \setminus B, A \setminus B) \to h_n(X, A)$$

We add another axiom to "make things easier"

— *Additivity :*
Given a family of pairs of spaces $(X_i, A_i)_{i \in I}$, the inclusions induce an isomorphism

$$\bigoplus h_n(X_i, A_i) \to h_n(\coprod X_i, \coprod A_i)$$

This is the end of the axioms for a generalised homology theory, the homology theory is called an ordinary homology theory if the <u>*Dimension Axiom*</u> *holds, namely*

$$h_n(pt) = 0 \forall n \neq 0$$

1. From now on, we write $h_n(A) \coloneqq h_n(A, \emptyset)$

The abelian group $h_0(\ pt\ )$ is the called the coefficient group of (h_n, ∂_n)

Lemma 2
If $f : X \to Y$ is a homotopy equivalence, then $\forall n \in \mathbb{Z}$ we obtain $h_n(f) : h_n(X) \to h_N(Y)$ to be an isomorphism for any homology theory (h_n, ∂_n)

Proof
Choose $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$, then by functoriality and homotopy invariance $\mathrm{Id}_{h_n(X)} = h_n(\mathrm{Id}_X) = h_n(g) \circ h_n(f)$, by symmetry, $h_n(f)$ and $h_n(g)$ are inverses. □

Similarly, if $f : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs, then the same result holds.

Example
For any such homology theory

$$h_n(\mathbb{R}^k) \simeq h_n(\ pt\ ) \simeq h_n(D^k)$$

Lecture 2: Homology Theories

Wed 12 Oct

Recall that the natural homomorphisms ∂_n are natural, in the sense that the compositions

$$h_{n-1}(f) \circ \partial_n : h_n(X, A) \to h_{n-1}(A) \to h_{n-1}(B)$$

and

$$\partial_n \circ h_n(f) : h_n(X, A) \to h_n(Y, B) \to h_{n-1}(B)$$

coincide.
Today, we compute the homology groups $h_*(S^k)$ for $k \geq 0$ for a given ordinary homology theory h_* Here, the k-sphere is defined as a subspace of $\mathbb{R}^{k+1}$.
Recall from the exercises that $h_*(pt \coprod pt) = h_*(pt) \oplus h_*(pt)$ for ordinary homology theories concentrated in degree 0.
There are two maps $\pm : pt \to S^0$ and one natural map $S^0 \to pt$ called the "fold" map.
By functoriality, the composition $h_*(pt) \to h_*(S^0) \to h_*pt$ is the identity.
To compute $h_*(S^k)$, we use two LES

$$\ldots \xrightarrow{\partial_{n+1}} h_n(S^k) \xrightarrow{h_*\iota} h_n(D^{k+1}) = 0 \xrightarrow{h_*\iota} h_n(D^{k+1}, S^k) \to h_{n-1}(S^k) \to h_{n-1}(D^{k+1}) = 0 \ldots$$

As $h_n(D^{k+1}) = 0$ for $n \neq 0$, we have an isomorphism $\partial_n : h_n(D^{k+1}, S^k) \to h_{n-1}(S^k)$.
The inclusion $D^k \subset S^k$ (as the upper hemisphere) gives rise to another LES

$$0 = h_n D^k \xrightarrow{h_*\iota} h_n S^k \xrightarrow{h_*\iota} h_n(S^k, D^k) \xrightarrow{\partial_n} h_{n-1} D^k = 0 \to h_{n-1} S^k \ldots$$

And thus we also get an isomorphism $h_n\iota : h_n S^k \to h_{n-1} D^k$ The inclusion of the north pole $pt \subset D^k \subset S^k$ induces, using excision, the isomorphism $h_n(S^k \setminus pt, D^k \setminus pt) \simeq h_n(S^k, D^k)$ of the following diagram

$$\begin{tikzcd} h_n(D^k, S^{k-1}) \arrow[d, "\simeq\partial_n"'] & h_n(S^k \setminus pt, D^k \setminus pt) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "\partial_n"'] & h_n(S^k, D^k) \arrow[d, "\partial_n"] \\ h_{n-1}(S^{k-1}) \arrow[r, "h_*\iota"'] & h_{n-1}(D^k \setminus pt) \arrow[r] & h_{n-1}(D^k) \end{tikzcd}$$

We know that the bottom row of this diagram is an ES.
In particular $h_n(D^k, S^{k-1}) \simeq h_n(S^k, D^k)$.
The isomorphism $\partial_n : h_n(D^k, S^{k-1}) \to h_{n-1}(S^{k-1})$ now almost allows us to use induction to find the homology groups.
We now consider the case $n \in \{0, 1\}$ (This part of the proof is not complete yet)

$$h_1(D^k) = 0 \to h_1 S^k \to h_1(S^k, D^k) \xrightarrow{\partial_1} h_0 D^k \to h_0 S^k \to h_0(S^k, D^k) \to h_{-1} D^k = 0$$

The case $n \in \{0, 1\}$ gives a split short exact sequence

$$0 \to h_0 D^k \to h_0 S^k \to h_0(S^k, D^k) \simeq h_0(D^k, S^{k-1}) \to 0$$

The homotopy equivalence $pt \to D^k$ gives a split of this exact sequence $h_0 S^k \to h_0 pt \to h_0 D^k$.
The boundary homomorphism $h_1(S^k, D^k) \to h_0 D_k$ being 0 using results from the exercise sheet.

Now by induction, $h_n S^k = 0$ for all $n < 0$ and $h_0 S^k = h_0(pt)$ for all $k > 0$.
We also have that $h_n S^1 \simeq h_{n-1} S^0$ for $n \notin \{0, 1\}$.
What about $h_1 S^1$?

$$h_1(D^1, S^0) \to h_1(S^1, D^1) \to h_0(D^1)$$

and

$$h_1(D^1, S^0) \to h_0 S^0 \to h_0(D^1)$$

Where the last morphism is induced by the fold map, namely $h_0 S^0 = h_0 pt \oplus h_0 pt \to h_0(pt)$ and $(x, y) \mapsto x + y$.
We have

$$h_1 D^1 \to h_1(D^1, S^0) \to h_0 S^0 = h_0 pt \oplus h_0 pt \to h_0 D^1$$

We were able to show isomorphisms $h_n S^k \simeq h_{n-1} S^{k-1}$ for $n \notin \{0, 1\}$, $h_0 S^k \simeq h_0 pt$ for $k > 0$ and $h_1 S^1 \simeq h_0 pt$.
What about $h_1 S^k$ for $k > 1$?
We have isomorphisms

$$h_1 S^k \to h_1(S^k, D^k) \xrightarrow{\partial} h_0 D^k \simeq h_0 S^k$$

and
$$h_1(D^k, S^{k-1}) \simeq h_1(S^k, D^k) \to h_0 S^{k-1} \simeq h_0 D^k$$
and thus $h_1 S^k = 0$ for $k > 1$.

Proposition 4

FOr any ordinary homology theory (h_*, ∂_*), the following holds
$$h_n S^k = \begin{cases} h_0 pt \oplus h_0 pt \text{ if } k = 0 = n \\ 0, k > 0, n \notin \{0, k\} \\ h_0 pt \text{ if } k > 0 \text{ and } n \in \{0, k\} \\ 0, else \end{cases}$$
We add one additional assumption, that there exists an ordinary homology theory with coefficient group $h_0 pt \simeq \mathbb{Z}$

Corollary 5

S^k and S^l are not homotopy equivalent for $k \neq l$

Proof

$$h_k S^k \simeq h_0 pt \neq h_k S^l = 0$$
□

Corollary 6 (Brouwer fixed point theorem)

Any continuous map $f : D^n \to D^n$ has a fixed point.

Proof

Assume $f : D^n \to D^n$ is a map without a fixed point.
Consider $g : D^n \to S^{n-1}$ sending $x \mapsto \frac{x - f(x)}{\|x - f(x)\|}$, by assumption, this is continuous.
Next, we claim that $g|_{S^{n-1}}$ is homotopic to $\mathrm{Id}_{S^{n-1}}$ via the map
$$H(x, t) := \frac{x - tf(x)}{\|x - tf(x)\|}$$
If $t = 1$, the denominator is $\neq 0$, if $t < 1$
$$\|tf(x)\| = t \|f(x)\| < \|f(x)\| \leq 1$$
Hence, $\|x - tf(x)\| \neq 0$ and H is a well defined continuous map.
Now, consider
$$h_{n-1} S^{n-1} \xrightarrow{ind} h_{n_1} D^n \xrightarrow{h_{n-1}(g)} h_{n-1} S^{n-1}$$

By homotopy equivalence $h_{n-1}(g) \circ ind$ is the identity.
For $n > 1$, this implies that the identity factors through 0, which is a contradiction.
The special case $n = 1$ gives

$$h_0 S^0 \to h_0 D^1 \to h_0 S^0$$

If the coefficient group is $\mathbb{Z}$, this is a contradiction. $\square$

2 Constructing singular homology

We want to construct a (ordinary) homology theory.
The idea is to study X by mapping topological simplices into X, here the topological n simplex is defined as

$$\Delta^n = \left\{ (t_0, \dots, t_n) | t_i \geq 0 \forall i, \sum_i t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

We define

$$Sing_n(X) = \{ f : \Delta^n \to X \text{ continuous } \}$$

in general, this set is huge.

Lecture 3: Singular homology

Mon 17 Oct

Goal : Find a way to organise the information in $Sing_n(X)$!

1. Relate $Sing_n(X)$ for different n to each other
2. Linearize!

We'll call $Sing_n(X)$ the n-th component of the singular set.
We think of the edges of the simplices as being ordered.
There are maps $\Delta^1 \to \Delta^n$ which are inclusions into the edges.
In fact, for every subset $S \subset \{0, \dots, n\}$, there is a continuous injective map $\Delta^k \to \Delta^n$, where $k = |S|$.
Now, for any $k < n$, we have restriction maps $Sing_n(X) \to Sing_k(X)$.
Define the category Δ_{inj}, whose onjects are $[n]$ for every $n \in \mathbb{N}$ and whose morphisms $[k] \to [n]$ are order preserving injective maps.
The composition is just the composition of maps.
For X a fixed topological space, we get a contravariant functor $Sing.(X) : \Delta_{inj} \to \text{Set}$.
Given $\alpha : [k] \to [n]$ an injective order preserving map, we get

$$Sing_n(X) \to Sing_k(X)$$

with precomposition by α.

Lemma 7

Δ_{inj} can also be described as the category with objects $[n]$ and generated by maps $d^i : [n] \to [n+1]$ subject to the relations

$$d^j d^i = d^i d^{j-1}$$

for $0 \leq i < j \leq n$

Proof (Sketch)

This relation is indeed satisfied in Δ_{inj}

$$\{0 < \ldots < n-2\} \xrightarrow{d^i} \{0 < \ldots < n-1\} \xrightarrow{d^j} \{0 < \ldots < n\}$$

Here

$$k \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k \geq i \end{cases} \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k+1 \leq j \\ k+2, k+2 \geq j+1 \end{cases}$$

One can compute that the composition $d^i d^{j-1}$ gives the same map.
What remains to show is that, subject to these relations, any order preserving injective map can be written as a composition of maps d^i.
If α is missing $i_1 < i_2 < \ldots < i_{n-k}$, then α can be written as

$$\alpha = d^{i_{n-k}} d^{i_{n-k-1}} \ldots d^{i_1}$$

□

We'll call d^i the i-th coface map.
A contravariant functor $\Delta_{inj} \to \mathrm{Set}$ is called a semi-simplicial set.

Definition 1 (Singular Chain Complex)

A (non-negatively graded) singular chain complex of a space X has as chain groups

$$S_n X = \mathbb{Z} \langle Sing_n(X) \rangle$$

and differentials $\delta_n : S_n(X) \to S_{n-1}(X)$ defined on generators as

$$\partial_n (\sigma : \Delta^n \to X) \mapsto \sum_{i=0}^{n} (-1)^i \sigma \circ d^i$$

Lemma 8

The singular chain complex of a space is a chain complex.

Proof

By linearity, it is enough to check this on generators $\sigma : \Delta^n \to X$.

$$\begin{aligned}
\delta_{n-1}\delta_n\sigma &= \delta_{n-1}\left(\sum_{i=0}^{n}(-1)^i\sigma\circ d^i\right)\\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j\sigma\circ d^i\circ d^j\\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&+ \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&+ \sum_{0\le i<j'\le n-1}(-1)^{i+j'-1}\sigma\circ d^{j'}\circ d^i\\
&= 0
\end{aligned}$$

□

Lemma 9

We get a functor from chain complexes with chain maps to graded abelian groups, which is just taking homology.

Definition 2 (Singular Homology)

The singular homology $H_\bullet X$ (with integer coefficients) on a space X is the homology of the singular chain complex.

Lecture 4: Homology Theories

Wed 19 Oct

Lemma 10

Homology defines a functor $Ch \to gr\,\mathrm{Ab}$

Proof (Sketch)

Let $f : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$, then $H_n(f) = f_*$ sending $x \in \ker(d_n)/\operatorname{Im}(d_{n+1})$ to $[f(x)]$ □

Example

Let's compute the singular homology of the point.

Clearly $S_* = \mathbb{Z}$ and the maps induced by restriction are the identity.

Hence, the boundary maps will be

$$\dots \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus $\forall n > 0$, we get $H_n(pt) = 0$ and $H_0(pt) = \mathbb{Z}$.

Now we want to define homology for pairs.
Let $A \subset X$ be a pair of spaces.
We want to associate a singular chain complex $(S_\bullet(X, A), \delta_\bullet)$.
More generally, any continuous map $f : X \to Y$ induces $Sing_n(X) \to Sing_n(Y)$ by postcomposition.
Thus we get a functor $Sing_n(-) : \top \to \mathrm{Set}$.
This in turn defines a chain map by extending $S_n f$ linearly to $S_n X$.
This defines a chain map $C_n X \to C_n Y$ since

$$\sigma \in S_n X \to f \circ \sigma \to \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ d_i$$

and

$$\sigma \in S_n X \to \sum_{i=0}^{n} (-1)^i \sigma \circ d^i \to \sum_{i=0}^{n} (-1)^i (f \circ \sigma \circ d_i)$$

coincide.
For an inclusion of subspaces $A \subset X$, we get an induced map $S_\bullet(i) : (S_\bullet A, \delta_\bullet) \to (S_\bullet X, \delta_\bullet)$ which is levelwise injective.

Definition 3 (Singular chain complex of a pair)
The singular chain complex of a pair is defined to be the quotient chain complex $S_\bullet X / S_\bullet A$.
Then the singular homology of the pair (X, A) is the homology of this chain complex.

For any pair (X, A) there is a short exact sequence of chain complexes

$$0 \to (S_\bullet A, \delta_\bullet) \to (S_\bullet X, \delta_\bullet) \to (S_\bullet(X, A), \delta_\bullet) \to 0$$

(ie. levelwise short exact)
What about coefficient groups $\neq \mathbb{Z}$.

Definition 4

Given a pair of spaces (X, A) and G an abelian group G, define the singular chain complex of (X, A) with coefficient in G as follows

$$S_n(X, A; G) = S_n(X, A) \otimes_{\mathbb{Z}} G$$

with the natural induced differentials. The singular homology of (X, A) with coefficients in G is the homology of this new chain complex.

Proposition 12

For any short exact sequence of chain complexes $0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$, we get a long exact sequence of homology groups

$$\ldots \to H_n C_\bullet \to H_n D_\bullet \to H_n E_\bullet \to H_{n-1} C_\bullet \to \ldots$$

which is natural in short exact sequences of chain complexesj;w

Proof

The definition of the map $\partial_n : H_n E \to H_{n-1} C$ is a standard diagram chase. We then prove that :

1. *γ is in the kernel of $d^C_{n-1} : C_{n-1} \to C_{n-2}$*

$$f_{n-2} d^C_{n-1} \gamma = d^D_{n-1} f_{n-1} \gamma = 0$$

   as f_{n-2} is injective, $d^C_{n-1}\gamma = 0$

2. *The choice of β is inddependent on the choice of γ.*
   Suppose β' is also such that $g_n \beta = g_n \beta'$.
   We want to show that $\gamma - \gamma'$ is in the image of d^C_n.
   As $g_n(\beta - \beta') = 0 \exists \tilde{\gamma} : f_n \tilde{\gamma} = \beta - \beta'$
   $f_{n-1} d^C_n \tilde{\gamma} = d^D_n f_n \tilde{\gamma} = d^D_n \beta - d^D_n \beta' = f_{n-1}(\gamma - \gamma')$.
   Thus $d^D_n \tilde{\gamma} = \gamma - \gamma'$

3. *Independence of the choice of representative α.*
   We want to show that if $\alpha = d^E_n \tilde{\alpha}$, then $\gamma = 0$.
   This again is a standard diagram chase. So we conclude that $\partial_n : H_n E \to H_{n-1} C$ is a well defined map, it is easy to check that it is linear.
   It remains to show that the long sequence above is exact, which is part of the homework. □

We want to show that the connecting homomorphisms are natural, namely, for thwo short exact sequences

$$0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$$

$$0 \to C'_\bullet \to D'_\bullet \to E'_\bullet \to 0$$

with $\phi : C_\bullet \to C'_\bullet, \psi, \eta$ etc which make the diagram commute, we get, for every n a commutative diagram

$$H_n E \xrightarrow{\partial_n} H_{n-1} C_\bullet \to H_{n-1} C'_\bullet = H_n E \xrightarrow{H_n \eta} H_n E'_\bullet \xrightarrow{\partial'_{n-1}} H_n C'_\bullet$$

Lecture 5: General results about singular homology

Mon 24 Oct

Proposition 13

Relative singular homology with coefficients in an abelian group G defines a functor $H_*(-,-;G) : Top^{(2)} \to gr\,\mathrm{Ab}$ and connecting homomorphisms $\partial_n : H_n(X, A, G) \to H_{n-1}(A; G)$ such that the LES for homology theories is satisfied.

Proof

$H_n(X, A; G) = H_n({S_n X}/{S_n A} \otimes G, \overline{\delta_n})$.

Let $f : (X, A) \to (Y, B)$ be a map of pairs of spaces.

We have already shown that taking homology is functorial.

We still need to show that $\top^{(2)} \to Ch$ mapping $(X, A) \mapsto ({S_*(X)}/{S_*(A)}, \delta_*)$ is a functor.

Recall that $- \otimes G$ is a functor from chain complexes to chain complexes.

We get a map of short exact sequences

$$0 \to S_* A \to S_* X \to {S_* X}/{S_* A} \to 0$$

to

$$0 \to S_* B \to S_* Y \to {S_* Y}/{S_* B} \to 0$$

□

induced by f, where the map from ${S_* X}/{S_* A} \to {S_* Y}/{S_* B}$ is induced by the universal property.

This map of chain complexes is clearly functorial, by definition of the $f_* : S_* X \to S_* Y$.

Remark

In general, tensoring with G does not preserve exact sequences <u>*but*</u> *it does preserve split short exact sequences.*

Thus, we want to show that, for eveyr n, $0 \to S_n A \to S_n X \to {S_n X}/{S_n A} \to 0$ is split short exact.

Notice that $Sing_n(X) = \{\sigma : \Delta^n \to A \to X\} \coprod \{\sigma : \Delta^n \to X | \operatorname{Im} \sigma \not\subset A\}$.

Thus it is clear that the short exact sequence above is split.

Hence, after tensoring with G, we still obtain a map of short exact sequences as above, thus we a map of long exact sequences in homology of the form

$$\ldots \to H_n A \to H_n X \to H_n(X, A) \xrightarrow{\partial_n^{(X,A)}} H_{n-1} A \to H_{n-1} X \to \ldots$$

$$\ldots \to H_n B \to H_n Y \to H_n(Y,B) \xrightarrow{\partial_n^{(Y,B)}} H_{n-1}B \to H_{n-1}Y \to \ldots$$

Where the vertical maps are all induced by f_*, (here H_n is homology with coefficients in G)

We now want to show that singular homology is homotopy invariant, namely, if $f, g : (X, A) \to (Y, B)$ are homotopic maps of pairs, ie. $\exists$ a continuous map $H : X \otimes [0, 1] \to Y$ such that the restriction to $A \times [0, 1]$ is contained in B which restricts to f (resp. g) on $X \times 0$ (resp. $X \times 1$).
Then, we want to show that $H_n f = H_n g : H_n(X, A; G) \to H_n(Y, B; G)$ coincide.

Definition 5 (Homotopic chain maps)
Two chain maps $\phi, \psi : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$ are chain homotopic if there exists a family of linear maps $h_n : C_n \to C'_{n+1} \forall n \geq 0$ such that

$$\phi_n - \psi_n = d'_{n-1}h_n + h_{n-1}d'_n : C_n \to C'_n$$

The family h_n is then called a chain homotopy.

Proposition 15
Given two homotopic chain maps $\phi, \psi : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$, the induced maps on homology coincide.

Proof
Pich $[x] \in H_n C$, we want to compar $[\phi(x)]$ and $[\psi(x)]$, we need to show that $\phi(x) - \psi(x) \in \operatorname{Im} d'_{n+1} = [d'h(x) + hd(x)]$.
As $x \in \ker d, hd(x) = 0$ and $d'h(x) \in \operatorname{Im} d'_{n+1}$ □

So now we want to show that homotopic maps of pairs induce homotopic maps of chain complexes.
The key ideay will be to notice that, in general $\Delta^n \times [0, 1]$ is not a simplex in general but it can be decomposed as a union of $n + 1$ $(n + 1)$ simplices.

We can consider $\Delta^n \times [0, 1] \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$.
Notice that any convex hull of $n + 2$ linearly independent vectors in $\mathbb{R}^{n+2}$ is homeomorphic to the $n + 1$ simplex which is compatible with the ordering of the vertices.
FOr $i \in \{0, n\}$, consider $\tau_i = conv((e_0, 0), (e_1, 0), \ldots, (e_i, 0), (e_{i+1}, 1), \ldots, (e_n, 0))$.
We want to build a map $S_n(X) \to S_{n+1}(Y)$, we do this by noticing that a map $\sigma : \Delta^n \to X$ induces $\Delta^n \times [0, 1] \xrightarrow{\sigma \times \mathrm{Id}} X \times [0, 1] \xrightarrow{H} Y$.
And thus $\tau_i \subset \Delta^n \times [0, 1] \to X \times [0, 1] \to Y$ gives an $n + 1$ simplex in Y.
The claim we will prove next time is that this induces a chain homotopy $S_* f, S_* g$

Lecture 6: Homotopy invariance

Wed 26 Oct

If $f, g : X \to Y$ are homotopic maps, then we construct a map $h_n S_n X \to S_{n+1} Y$ by defining it

$$h_n(\sigma) = \sum_{i=0}^{n} (-1)^i H(\sigma \times \mathrm{Id}) \circ \tau_i$$

We want to show this is a chain homotopy, ie $S_n g - S_n f = hd + dh$.
Notice that $\tau_i \circ d^{i+1} = \tau_{i+1} \circ d^{i+1}$, $\tau_0 \circ d^0 =$ inclusion of $\Delta^n \times \{1\}$ and $\tau_n \circ d^n =$ inclusion of $\Delta^n \times \{0\}$.
We need to analyze

$$\begin{aligned} h_{n-1}\delta_n(\sigma) + \delta_{n+1}h_n(\sigma) &= h_{n+1}(\sum_{j=0}^{n} (-1)^i \sigma \circ d^j) + \delta_{n+1}(\sum_{i=0}^{n} (-1)^i H(\sigma \times \mathrm{Id}) \circ \tau_i) \\ &= \sum_{j=0}^{n} (-1)^i h_{n+1}(\sigma \circ d^j) + \sum_{k=0}^{n+1} (-1)^k \sum_{i=0}^{n} (-1)^i H(\sigma \times \mathrm{Id}) \tau_i \circ d^k \\ &= \sum_{j=0}^{n} (-1)^i h_{n+1}(\sigma \circ d^j) + \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^{n} (-1)^{i+k} H(\sigma \times \mathrm{Id}) \tau_i \circ d^k + g(\sigma) - f(\sigma) \end{aligned}$$

Notice that $h_{n-1}(\sigma \circ d^k) = \sum_{i=0}^{k-1} (-1)^i H(\sigma \times \mathrm{Id})(\tau_i \circ d^{k+1}) + \sum_{i=k+1}^{n} (-1)^{i+1} (\sigma \times \mathrm{Id})(\tau_i \circ d^k)$.
Putting everythink together yields

$$\begin{aligned} &= \sum_{j=0}^{n} (-1)^j (\sum_{i=0}^{j-1}) (-1)^j H(\sigma \times \mathrm{Id})(\tau_i \circ d^{j+1}) \\ &+ \sum_{i=j-1}^{n} (-1)^{j+1} H(\sigma \times \mathrm{Id})(\tau_i \circ d^j) \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^{n} (-1)^{i+k} H(\sigma \times \mathrm{Id}) \tau_i \circ d^k \\ &+ g(\sigma) - f(\sigma) \end{aligned}$$

The rest of the proof is cursed and I won't write it down, Diecks Book seems to have a complete proof.
We still have to show that H_n respects homotopy of pairs

Proposition 16
The functor $H_*(-, -; G) \to \mathrm{Ab}$ is homotopy invariant

> **Proof**
> *Let $f, g : (X, A) \to (Y, B)$ be given homotopic maps via $H : X \times [0, 1] \to Y$.*
> *This gives us chain homotopies $(f|_A)_*$ to $(g|_A)_*$.*
> *f_*, g_* induce maps ${S_*X}/{S_*A} \to {S_*Y}/{S_*B}$ and the chain homotopies between*

f_* and g_* induce the same chain homotopies between these chain maps.
Moreover, tensoring with G maps chain homotopies to chain homotopies. □

2.1 Excision

Proposition 17

$H_*(-,-;G)$ satisfies the excision axiom.

Proof

Consider the cover $X = (X \setminus B)^\circ \cup A^\circ$ and write $U = \{X \setminus B, A\}$
Let $S_n^U(X) = \mathbb{Z}\langle \sigma : \Delta^n \to X | \operatorname{Im} \sigma \subset X \setminus B$ or $A\rangle \subset S_n(X)$ □

Lecture 7: Excision

Mon 31 Oct

Proof

We want to show that $H_n(S_\bullet^U(X), \delta_\bullet) \to H_n(X)$ is an isomorphism.
A chain map inducing isomorphisms on all homology groups is also called a quasi-isomorphism.
We'll use barycentric subdivision to make our simplices smaller.
First, let's recall the lebesgue lemma

Lemma 18

Let K be a compact metric space and $(U_i)_{i \in I}$ an open cover of K.
Then, there is an $\epsilon > 0$ s.t. any ϵ-ball around any point in K is contained in a single open set U_i.

To prove this, for every $x \in K$, we choose $\delta(x) > 0$ such that the $\delta(x)$ ball around x is contained in a U_i of the cover.
Now, we look at $\left\{B_{\frac{\delta(x)}{2}}(x)\right\}_{x \in K}$, this is an open cover of K, so there is a finite subcover $B_{\frac{\delta(x_i)}{2}}(x_i)$.
Set $\epsilon = \min_j \frac{\delta(x_j)}{2}$.
Given any $x \in K$, we want to show that $B_\epsilon(x)$ is completely contained in some U_i.
We can find x_l such that $d(x, x_l) < \frac{\delta(x_l)}{2}$, let $y \in B_\epsilon(x)$ then $\delta(x_l, y) \leq \frac{\delta(x_l)}{2} + \epsilon \leq \delta(x_l)$

We can apply the Lebesgue lemma to the open cover $\left\{\sigma^{-1}(X \setminus \overline{B}), \sigma^{-1}(A^\circ)\right\}$ is an open cover of compact metric spaces Δ^n.
Thus, there exists $\epsilon > 0$ such that any open $\epsilon-$ball in Δ^n is mapped by σ either to $X \setminus B$ or to A.
We'll now prove the following proposition

Proposition 19

There is a chain map (called the Barycentric subdivision) $Sd : S_\bullet X \to S_\bullet X$ which is

— *Natural in X*
— *Chain homotopic to the the identity*
— *If $X = \Delta^n$, then any summand τ of $Sd(\sigma)$ for $\sigma : \Delta^k \to \Delta^n$ has the following property*

$$diam(\operatorname{Im}\tau) \leq \frac{k}{k+1} diam(\operatorname{Im}\sigma)$$

Let $v_0, \ldots, v_n \in \mathbb{R}^N$, then their barycenter is $\frac{v_0+\ldots+v_n}{n+1}$.
We will consider the following auxiliary "cone" map.
For any $\tau : \Delta^k \to \Delta^n$ and b the barycenter of Δ^n, define $\rho_b(\tau) : \Delta^{k+1} \to \Delta^n$ by

$$\rho_b(t_0, \ldots, t_{k+1}) \mapsto \begin{cases} t_0 b + \tau(\frac{1}{1-t_0}(t_1, \ldots, t_{k+1}))(1-t_0) \\ b \text{ if } t_0 = 1 \end{cases}$$

This is indeed continuous.

What is the relation of the cone construction and the boundary of a simplex ?

$$\begin{aligned} \delta_{k+1}\rho_b(\tau) &= \sum_{j=0}^{k+1}(-1)^j \rho_b(\tau) \circ d^j \\ &= \tau + \sum_{j'=0}^{k}(-1)^{j'+1}\rho_b(\tau \circ d^j) \\ &= \tau - \rho_b(\delta_k \tau) \end{aligned}$$

So we obtain a linear map $\rho_b : S_k(\Delta^n) \mapsto S_{k+1}(\Delta^n)$ with the property

$$\delta_{k+1}\rho = \operatorname{Id} - \rho_b \circ \delta_k$$

We define a map $Sd : S_n(X) \to S_n X$.
For $n = 0, Sd_0 = \operatorname{Id}_{S_0 X}$.
Given some $n > 0$ and $\sigma : \Delta^n \to X$, define

$$Sd(\sigma) = \sigma_*(\rho_b Sd(\delta_n i_n))$$

where i_n is the identity of the Δ^n simplex, considered as an element of $Sing_n(\Delta^n)$
We claim that Sd is a chain map.
Given $\sigma : \Delta^n \to X$, we want to compute

$$\delta_n Sd\sigma = \delta_n \sigma_*(\rho_b Sd(\delta_n i_n))$$

We can switch δ_n and σ_* since σ_* is post composition and δ_n is precomposition.

$$\begin{aligned} &= \sigma_*(\delta_n \rho_b Sd(\delta_n i_n)) \\ &= \sigma_*(Sd(\delta_n i_n) - \rho_b(\delta_{n-1} Sd(\delta_n i_n))) \end{aligned}$$

Notice that

$$\delta_{n-1} Sd(\delta_n i_n) = Sd(\delta_{n-1}\delta_n i_n) = 0$$

Thus

$$\delta_n Sd(\sigma) = \sigma_* Sd(\delta_n i_n) = Sd(\delta_n \sigma)$$ □

For naturality, if $f : X \to Y$ is a map of topological spaces, one can explicitly check that $Sd \circ f_* = f_* \circ Sd$

Lecture 8: still excision

Wed 02 Nov

I missed the first half of the lecture because I overslept, so there is a part missing here.

Lemma 20

The map of chain complexes $S^U(X) \to S(X)$ is a quasi-isomorphism

Proof

Let $\sum_k a_k\sigma_k \in S_N X$ be in the kernel of δ_n, ie. $[\sum_k a_k\sigma_k]$ representes an element in $H_n(X)$.
To see surjectivity, notice that

$$Sd(\sum_k a_k\sigma_k) = \sum_k a_k\sigma_k - h(\delta(\sum a_k\sigma_k)) - \delta h(\sum a_k\sigma_)$$

The middle term is 0 by our hypothesis, and we see that $Sd(\sum_k a_k\sigma_k)$ is representing the same element in homology.
We can apply Sd arbitrarily many times and hence $Sd^l(\sum_k a_k\sigma_k)$ is too.
Now, for every k, $\sigma_k^{-1}A^\circ, \sigma_k^{-1}(X \setminus \overline{B})$ forms an open covering of Δ_n.
There exists a Lebesgue Number $\epsilon_k > 0$ for this open cover.
There is an $l_k \in \mathbb{N}$ such that $(\frac{n}{n+1})^{l_k} < \frac{\epsilon_k}{\sqrt{2}}$.
Now, for any simplex τ in the barycentric subdivision of Δ_n $\operatorname{diam}\tau \leq \epsilon$.
Now $(\sigma_k)_*(\tau) \subset X \setminus \overline{B}$ or $\subset A^\circ$.
Setting $l = \max_k l^k$, we see that $Sd^l(\sum a_k\sigma_k)$ is a preimage for $\sum a_k\sigma_k$.
For injectivity, let $\sum a_k\sigma_k$ be an element of $S^U_\bullet X$ in $\ker\delta$ which is in the image of δ.
Let $\sum b_j\sigma_j$ be such that $\delta(\sum b_j\tau_j) = \sum a_k\sigma_k$.

There exists an m such that $Sd^m(\sum b_j\tau_j) \in S^U X$, now $\delta Sd^m(\sum b_j\tau_j) = Sd^m(\sum a_k\sigma_k)$.
Thus, $\sum a_k\sigma_k$ is a boundary.
This concludes the proof. $\square$

We can now use that a quasi-isomorphism of complexes of free abelian groups has a homotopy inverse.

Lecture 9: Conclusion of Excision

Mon 07 Nov

We proved last week that $S_*^{\{X\setminus B,A\}X\to S_*X}$ is in fact a quasi-isomorphism. In homework, we proved that any quasi-isomorphism between chain complexes of levelwise free abelian groups is a chain homotopy.
With this, we can now prove excision

Proof
There are maps between the two following SES

$$0 \to S_*(A\setminus B) \to S_*(X\setminus B) \to S_*(X\setminus B, A\setminus B) \to 0$$

$$0 \to S_*(B) \to S_*(X) \to S_*(X,B) \to 0$$

which are induced by inclusion.
There are maps $S_*A \to S_*^U X$ and $S_*(X\setminus B) \to S_*^U(X)$ and a quotient map $S_*^U X \to S_*(X\setminus B, A\setminus B)$.
As the map of chain complexes is a quasi-isomorphism, they are all chain homotopic and thus tensoring with G preserves the quasi-isomorphism.
Applying the LES in homology, we get maps between the two following short exact sequences

$$H_n(A;G) \to H_n(S_*^{\{X\setminus B,A\}}(X)\otimes G) \to H_n(X\setminus B, A\setminus B;G) \xrightarrow{\partial} H_{n-1}(A;G) \to H_{n-1}(S_*^{\{X\setminus B,A\}}(X)\otimes G)$$

$$H_n(A;G) \to H_n(X;G) \to H_n(X,A;G) \xrightarrow{\partial} H_{n-1}(A;G) \to H_{n-1}(X;G)$$

Now, applying the five lemma, we see that $H_n(X\setminus B, A\setminus B;G) \simeq H_n(X,A;G)$ $\square$

It remains to show additivity.
Let $\{(X_i,A_i)\}_{i\in I}$ be a family, we need to show that the inclusions induce a map

$$\bigoplus_{i\in I} H_n(X_i,A_i) \to H_n\left(\coprod_{i\in I} X_i, \coprod_{i\in I} A_i\right)$$

Notice that the homology functor $H_n : Ch_{\geq 0} \to gr\,\mathrm{Ab}$ commutes with arbitrary direct sums.
Furthermore, from the definition, it is clear that $S_n(\coprod_i X_i, \coprod_i A_i;G) \simeq \bigoplus_i S_n(X_i,A_i;G)$.
It is easy to check that $H_n(pt;\mathbb{Z}) = 0$ for all $n \neq 0$, tensoring the chain complex

(of singular chains of a point)

$$\dots \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z} \dots$$

with G, it is clear that $H_n(pt; G) = 0 \forall n \geq 1$ and $H_0(pt, G) = G$

Lecture 10: computations

Wed 09 Nov

$h_n(D^k, S^{k-1})$

Example
We know that

$$h_n(D^k, S^{k-1}) = \begin{cases} h_0(pt), n = k \\ 0 \text{ else} \end{cases}$$

for any ordinary homology theory.
This is almost $h_n S^k$ *but* S^k *is the collapse* D^k/S^{k-1}.

Example
Consider $D^k \setminus 0 \to D^k$.
Using homotopy equivalence, we get

$$h_n(D^k) \to h_n(D^k, D^k \setminus \{0\}) \to h_{n-1}(D^k \setminus 0) \to h_{n-1}(D^k)$$

Thus

$$h_n(D^k, D^k \setminus 0) = \begin{cases} h_0(pt) \text{ if } n = k \\ 0 \end{cases}$$

Compare this to the quotient $D^k/D^k \setminus 0$ *which has two points, with exactly one of them being open.*
This is called the Sierpinski space $\mathcal{S}$*, call it's points* a *and* b *with* a *open.*
Notice that there is a bijection between continuous maps $X \to \mathcal{S} \leftrightarrow \{U \subset X \text{ open }\}$.
We claim that $\mathcal{S}$ *is contractible, there is a unique map* $\mathcal{S} \to pt$ *and we define a map* $pt \to a$.
We define the map $\mathcal{S} \times [0,1] \to \mathcal{S}$ *to be the one induced by the open set* $\{a\} \times [0,1] \cup \{a,b\} \times [0,1)$.
Thus, in this case relative homology and homology of the quotient don't agree.

Our next goal will be to make the difference between these pairs of spaces precise.

> **Definition 6**
> *A subspace* $A \subset X$ *is called neighborhood deformation retract (NDR) if* $A \subset X$ *is closed and there is a neighborhood* $A \subset V \subset X$ *such that* A *is a deformation retract of* V.

Ie. there exists a continuous map $r : V \to A$ such that $r|_A = \mathrm{Id}_A$ and there exists a continuous map $H : V \times [0,1] \to V$ such that $H_0 = \mathrm{Id}_V$, $H_1 = \iota \circ r$ and A is fixed by H .

Example
$S^{k-1} \to D^k$ is a NDR, choosing $r : D^k \setminus 0 \to S^{k-1}$ sending $x \mapsto \frac{x}{\|x\|}$ with the obvious homotopy.

Example
$D^k \setminus 0 \to D^k$ is not closed so in particular not a NDR.
A slightly more elaborate example is $A = \left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup 0 \subset \mathbb{R}$.
This is a closed subset but not a deformation retract as $V \xrightarrow{r} A$ can no be both continuous and the identity on A for any neighborhood V of A in $X = \mathbb{R}$.

Theorem 25
Let $A \subset X$ be a NDR and let (h_*, ∂_*) be any ordinary homology theory. Then the quotient map $q : X \to X/A$ induces isomorphisms

$$h_n(X, A) \to h_n(X/A, A/A) \forall n$$

2.2 Pushouts of topological spaces

Proposition 26
Given $C \leftarrow A \to B$, it can always be completed to a pushout.

Lecture 11: pushouts

Mon 14 Nov

Lemma 27
Let $X \hookleftarrow A \xrightarrow{f} B$ be a pushout diagram with pushout $B \xrightarrow{j} Y \xleftarrow{g} X$. Then if i is an embedding, open embedding or closed embedding, then j is too.

Definition 7 (Locally compact)
A topological space is called locally compact if for all $z \in Z$ and any neighborhood $U \subset Z$, there exists a compact neigborhood $z \in C \subset U$

Example
Any compact Hausdorff space Z is locally compact.

Proposition 29
Let $X \xrightarrow{p} Y$ be a quotient map and Z a locally compact topological space, then $X \times Z \xrightarrow{p \times \mathrm{Id}} Y \times Z$ is a quotient.

Lecture 12: how much of a homology theory is determined by the axioms ?

Wed 16 Nov

Corollary 30

Let D be the pushout of $C \leftarrow A \rightarrow B$ and Z a locally compact space, then $D \times Z$ is the pushout of $C \times Z \leftarrow A \times Z \rightarrow B \times Z$.

Proposition 31

Pushouts of NDR are NDR.

Proof

Let (X, A) be a NDR and let Y be the pushout of $A \leftarrow A \rightarrow B$.
Let V be a neighborhood of A and $r : V \rightarrow A$ be a retraction and $H : V \times [0,1] \rightarrow V$ with $H_0 = \mathrm{Id}, H_1 = \iota \circ r$ and which fixes A.
Take the pushout W of $V \leftarrow A \rightarrow B$, we get and induced map to Y which is an ebedding and W is a neighborhood of B in Y.
Universal properties give a retract $s : W \rightarrow B$.
To check this is a NDR, we use the above proposition to get an induced map $W \times [0,1] \rightarrow W$, this turns out to be the desired homotopy. □

Remark

W can be chosen as a pushout of $V \leftarrow A \rightarrow B$.

Now, we want to show that for $A \rightarrow X$ a NDR and h_* an ordinary homology theory, the map $X \rightarrow X/A$ induces isomorphisms on homology.

Proof

Let V be the neighborhood of A, there are maps induced by inclusion

$$h_n(X, A) \rightarrow h_n(X, V) \xleftarrow{\simeq} h_n(X \setminus A, V \setminus A)$$

The quotient map induces maps to

$$h_n(X/A, A/A) \rightarrow h_n(X/A, V/A) \leftarrow h_n((X \setminus A)/A, (V \setminus A)/A)$$

We show $h_n(X, A) \rightarrow h_n(X, V)$ is an isomorphism.
Compare the LES of

$$h_n(A) \rightarrow h_n X \rightarrow h_n(X, A) \rightarrow h_{n-1} A \rightarrow h_{n-1} X$$

and

$$h_n(V) \rightarrow h_n X \rightarrow h_n(X, V) \rightarrow h_{n-1} V \rightarrow h_{n-1} X$$

the inclusions induces isomorphisms on all groups except $h_n(X, A) \rightarrow h_n(X, V)$ and we conclude by the five lemma. We can also show that the maps $h_n(X/A, A/A) \rightarrow h_n(X/A, V/A)$ are isomorphisms by notice that $A/A \rightarrow V/A$ is a NDR. □

We now want to identify nice spaces on which h_*X is already determined by it's coefficient group.

Definition 8 (CW complexes)
Let (X, A) be a pair of spaces.
The pair (X, A) together with a filtration $X_{-1} = A \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset\subset X$ is a relative CW-complex if

1. *There exist pushout diagrams $\forall n \geq 0$ such that X_n is the pushout of*
$$\coprod_{i \in I_n} D_n \xleftarrow{\Phi_i} \coprod_{i \in I_n} S^{n-1} \xrightarrow{\phi_i} X_{n-1}$$
The ϕ_i are called attaching maps and Φ_i are the "characteristic maps".
2. *$X \simeq \varinjlim_{n \in \mathbb{N}} X_n$ where X carries the weak topology.*
If $A = \emptyset$, X together with a filtration is called a CW-complex.

Lecture 13: stuff

Mon 21 Nov

Remark
If $X = X_n$ for some $n \in \mathbb{N}_0$, the CW-complex is finite dimensional.
For a fixed filtration of a space, the images $\Phi_i(D^{\circ,n})$ are well defined and identified with path-connected components of $X_n \setminus X_{n-1}$ and they are called n-cells.

Definition 9 (Cellular map)
A map of CW-complexes $f : X \to Y$ is called cellular if $f(X_n) \subset Y_n \forall n$.

Lecture 14: dimension of cw-complexes

Wed 23 Nov