

Topology I

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1 Homology Theories

Lecture 1: Introduction

Mon 10 Oct

Aim : Study further algebraic invariants of topological spaces.

We want to assign to pairs of topological spaces abelian groups.

$$h_n : T \rightarrow \text{Ab} \quad \forall n \in \mathbb{Z}$$

and to pairs continuous maps, we want to assign a map $h_n(f) : h_n(X) \rightarrow h_n(Y)$ which is functorial. Here T is the category of pairs of topological spaces $A \subset X$ with morphisms $f : (X, A) \rightarrow (Y, B)$ such that $f(A) \subset B$.

To relate h_n for different $n \in \mathbb{N}$, we will construct connecting morphisms $\partial_n : h_n(X, A) \rightarrow h_{n-1}(A, \emptyset)$.

Axiom 1 (Eilenberg-Steenrod Axiom)

A (generalised) homology theory consists of functors $h_n : T \rightarrow \text{Ab}$ and natural connecting homomorphisms $\partial_n : h_n(X, A) \rightarrow h_{n-1}(A, \emptyset)$ ¹ satisfying

— *Homotopy invariance :*

If $f, g : (X, A) \rightarrow (Y, B)$ are homotopic continuous maps of pairs then the induced maps $h_n(f) = h_n(g)$. Here homotopy of pairs means that there exists $H : X \times [0, 1] \rightarrow Y$ such that $H(A \times [0, 1]) \subset B$

— *Long exact sequence of a pair (LES) :*

Given a pair of topological spaces (X, A) there is a long exact sequence of abelian groups.

Denote $i : (A, \emptyset) \rightarrow (X, \emptyset)$ and $j : (X, \emptyset) \rightarrow (X, A)$, then

$$h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} h_{n-1}(A, \emptyset)$$

— *Excision*

Given $B \subset A \subset X$ subspaces such that $\overline{B} \subset A^\circ$, the inclusion induces a group isomorphism

$$h_n(X \setminus B, A \setminus B) \rightarrow h_n(X, A)$$

We add another axiom to "make things easier"

— *Additivity :*

Given a family of pairs of spaces $(X_i, A_i)_{i \in I}$, the inclusions induce an isomorphism

$$\bigoplus h_n(X_i, A_i) \rightarrow h_n(\coprod X_i, \coprod A_i)$$

This is the end of the axioms for a generalised homology theory, the homology theory is called an ordinary homology theory if the Dimension Axiom holds, namely

$$h_n(pt) = 0 \forall n \neq 0$$

1. From now on, we write $h_n(A) := h_n(A, \emptyset)$

The abelian group $h_0(pt)$ is called the coefficient group of (h_n, ∂_n)

Lemma 2

If $f : X \rightarrow Y$ is a homotopy equivalence, then $\forall n \in \mathbb{Z}$ we obtain $h_n(f) : h_n(X) \rightarrow h_n(Y)$ to be an isomorphism for any homology theory (h_n, ∂_n)

Proof

Choose $g : Y \rightarrow X$ such that $g \circ f \simeq \text{Id}_X$ and $f \circ g \simeq \text{Id}_Y$, then by functoriality and homotopy invariance $\text{Id}_{h_n(X)} = h_n(\text{Id}_X) = h_n(g) \circ h_n(f)$, by symmetry, $h_n(f)$ and $h_n(g)$ are inverses. \square

Similarly, if $f : (X, A) \rightarrow (Y, B)$ is a homotopy equivalence of pairs, then the same result holds.

Example

For any such homology theory

$$h_n(\mathbb{R}^k) \simeq h_n(pt) \simeq h_n(D^k)$$

Lecture 2: Homology Theories

Wed 12 Oct

Recall that the natural homomorphisms ∂_n are natural, in the sense that the compositions

$$h_{n-1}(f) \circ \partial_n : h_n(X, A) \rightarrow h_{n-1}(A) \rightarrow h_{n-1}(B)$$

and

$$\partial_n \circ h_n(f) : h_n(X, A) \rightarrow h_n(Y, B) \rightarrow h_{n-1}(B)$$

coincide.

Today, we compute the homology groups $h_*(S^k)$ for $k \geq 0$ for a given ordinary homology theory h_* . Here, the k -sphere is defined as a subspace of \mathbb{R}^{k+1} .

Recall from the exercises that $h_*(pt \amalg pt) = h_*(pt) \oplus h_*(pt)$ for ordinary homology theories concentrated in degree 0.

There are two maps $\pm : pt \rightarrow S^0$ and one natural map $S^0 \rightarrow pt$ called the "fold" map.

By functoriality, the composition $h_*(pt) \rightarrow h_*(S^0) \rightarrow h_*pt$ is the identity.

To compute $h_*(S^k)$, we use two LES

$$\dots \xrightarrow{\partial_{n+1}} h_n(S^k) \xrightarrow{h_*\iota} h_n(D^{k+1}) = 0 \xrightarrow{h_*\iota} h_n(D^{k+1}, S^k) \rightarrow h_{n-1}(S^k) \rightarrow h_{n-1}(D^{k+1}) = 0 \dots$$

As $h_n(D^{k+1}) = 0$ for $n \neq 0$, we have an isomorphism $\partial_n : h_n(D^{k+1}, S^k) \rightarrow h_{n-1}(S^k)$.

The inclusion $D^k \subset S^k$ (as the upper hemisphere) gives rise to another LES

$$0 = h_n D^k \xrightarrow{h_*\iota} h_n S^k \xrightarrow{h_*\iota} h_n(S^k, D^k) \xrightarrow{\partial_n} h_{n-1} D^k = 0 \rightarrow h_{n-1} S^k \dots$$

And thus we also get an isomorphism $h_n \iota : h_n S^k \rightarrow h_{n-1} D^k$. The inclusion of the north pole $pt \subset D^k \subset S^k$ induces, using excision, the isomorphism $h_n(S^k \setminus pt, D^k \setminus pt) \simeq h_n(S^k, D^k)$ of the following diagram

$$\begin{array}{ccccc} h_n(D^k, S^{k-1}) & \xleftarrow{\simeq} & h_n(S^k \setminus pt, D^k \setminus pt) & \xrightarrow{\simeq} & h_n(S^k, D^k) \\ \simeq \partial_n \downarrow & & \partial_n \downarrow & & \downarrow \partial_n \\ h_{n-1}(S^{k-1}) & \xrightarrow{h_{*}\iota} & h_{n-1}(D^k \setminus pt) & \longrightarrow & h_{n-1}(D^k) \end{array}$$

We know that the bottom row of this diagram is an ES.

In particular $h_n(D^k, S^{k-1}) \simeq h_n(S^k, D^k)$.

The isomorphism $\partial_n : h_n(D^k, S^{k-1}) \rightarrow h_{n-1}(S^{k-1})$ now almost allows us to use induction to find the homology groups.

We now consider the case $n \in \{0, 1\}$ (This part of the proof is not complete yet)

$$h_1(D^k) = 0 \rightarrow h_1 S^k \rightarrow h_1(S^k, D^k) \xrightarrow{\partial_1} h_0 D^k \rightarrow h_0 S^k \rightarrow h_0(S^k, D^k) \rightarrow h_{-1} D^k = 0$$

The case $n \in \{0, 1\}$ gives a split short exact sequence

$$0 \rightarrow h_0 D^k \rightarrow h_0 S^k \rightarrow h_0(S^k, D^k) \simeq h_0(D^k, S^{k-1}) \rightarrow 0$$

The homotopy equivalence $pt \rightarrow D^k$ gives a split of this exact sequence $h_0 S^k \rightarrow h_0 pt \rightarrow h_0 D^k$.

The boundary homomorphism $h_1(S^k, D^k) \rightarrow h_0 D^k$ being 0 using results from the exercise sheet.

Now by induction, $h_n S^k = 0$ for all $n < 0$ and $h_0 S^k = h_0(pt)$ for all $k > 0$.

We also have that $h_n S^1 \simeq h_{n-1} S^0$ for $n \notin \{0, 1\}$.

What about $h_1 S^1$?

$$h_1(D^1, S^0) \rightarrow h_1(S^1, D^1) \rightarrow h_0(D^1)$$

and

$$h_1(D^1, S^0) \rightarrow h_0 S^0 \rightarrow h_0(D^1)$$

Where the last morphism is induced by the fold map, namely $h_0 S^0 = h_0 pt \oplus h_0 pt \rightarrow h_0(pt)$ and $(x, y) \mapsto x + y$.

We have

$$h_1 D^1 \rightarrow h_1(D^1, S^0) \rightarrow h_0 S^0 = h_0 pt \oplus h_0 pt \rightarrow h_0 D^1$$

We were able to show isomorphisms $h_n S^k \simeq h_{n-1} S^{k-1}$ for $n \notin \{0, 1\}$, $h_0 S^k \simeq h_0 pt$ for $k > 0$ and $h_1 S^1 \simeq h_0 pt$.

What about $h_1 S^k$ for $k > 1$?

We have isomorphisms

$$h_1 S^k \rightarrow h_1(S^k, D^k) \xrightarrow{\partial} h_0 D^k \simeq h_0 S^k$$

and

$$h_1(D^k, S^{k-1}) \simeq h_1(S^k, D^k) \rightarrow h_0 S^{k-1} \simeq h_0 D^k$$

and thus $h_1 S^k = 0$ for $k > 1$.

Proposition 4

For any ordinary homology theory (h_*, ∂_*) , the following holds

$$h_n S^k = \begin{cases} h_0 pt \oplus h_0 pt & \text{if } k = 0 = n \\ 0, & k > 0, n \notin \{0, k\} \\ h_0 pt & \text{if } k > 0 \text{ and } n \in \{0, k\} \\ 0, & \text{else} \end{cases}$$

We add one additional assumption, that there exists an ordinary homology theory with coefficient group $h_0 pt \simeq \mathbb{Z}$

Corollary 5

S^k and S^l are not homotopy equivalent for $k \neq l$

Proof

$$h_k S^k \simeq h_0 pt \neq h_k S^l = 0$$

□

Corollary 6 (Brouwer fixed point theorem)

Any continuous map $f : D^n \rightarrow D^n$ has a fixed point.

Proof

Assume $f : D^n \rightarrow D^n$ is a map without a fixed point.

Consider $g : D^n \rightarrow S^{n-1}$ sending $x \mapsto \frac{x-f(x)}{\|x-f(x)\|}$, by assumption, this is continuous.

Next, we claim that $g|_{S^{n-1}}$ is homotopic to $\text{Id}_{S^{n-1}}$ via the map

$$H(x, t) := \frac{x - tf(x)}{\|x - tf(x)\|}$$

If $t = 1$, the denominator is $\neq 0$, if $t < 1$

$$\|tf(x)\| = t\|f(x)\| < \|f(x)\| \leq 1$$

Hence, $\|x - tf(x)\| \neq 0$ and H is a well defined continuous map.

Now, consider

$$h_{n-1} S^{n-1} \xrightarrow{\text{ind}} h_{n-1} D^n \xrightarrow{h_{n-1}(g)} h_{n-1} S^{n-1}$$

By homotopy equivalence $h_{n-1}(g) \circ ind$ is the identity.

For $n > 1$, this implies that the identity factors through 0, which is a contradiction.

The special case $n = 1$ gives

$$h_0 S^0 \rightarrow h_0 D^1 \rightarrow h_0 S^0$$

If the coefficient group is \mathbb{Z} , this is a contradiction. □

2 Constructing singular homology

We want to construct a (ordinary) homology theory.

The idea is to study X by mapping topological simplices into X , here the topological n simplex is defined as

$$\Delta^n = \left\{ (t_0, \dots, t_n) \mid t_i \geq 0 \forall i, \sum_i t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

We define

$$Sing_n(X) = \{ f : \Delta^n \rightarrow X \text{ continuous} \}$$

in general, this set is huge.

Lecture 3: Singular homology

Mon 17 Oct

Goal : Find a way to organise the information in $Sing_n(X)$!

1. Relate $Sing_n(X)$ for different n to each other
2. Linearize!

We'll call $Sing_n(X)$ the n -th component of the singular set.

We think of the edges of the simplices as being ordered.

There are maps $\Delta^1 \rightarrow \Delta^n$ which are inclusions into the edges.

In fact, for every subset $S \subset \{0, \dots, n\}$, there is a continuous injective map $\Delta^k \rightarrow \Delta^n$, where $k = |S|$.

Now, for any $k < n$, we have restriction maps $Sing_n(X) \rightarrow Sing_k(X)$.

Define the category Δ_{inj} , whose objects are $[n]$ for every $n \in \mathbb{N}$ and whose morphisms $[k] \rightarrow [n]$ are order preserving injective maps.

The composition is just the composition of maps.

For X a fixed topological space, we get a contravariant functor $Sing.(X) : \Delta_{inj} \rightarrow \text{Set}$.

Given $\alpha : [k] \rightarrow [n]$ an injective order preserving map, we get

$$Sing_n(X) \rightarrow Sing_k(X)$$

with precomposition by α .

Lemma 7

Δ_{inj} can also be described as the category with objects $[n]$ and generated by maps $d^i : [n] \rightarrow [n+1]$ subject to the relations

$$d^j d^i = d^i d^{j-1}$$

for $0 \leq i < j \leq n$

Proof (Sketch)

This relation is indeed satisfied in Δ_{inj}

$$\{0 < \dots < n-2\} \xrightarrow{d^i} \{0 < \dots < n-1\} \xrightarrow{d^j} \{0 < \dots < n\}$$

Here

$$k \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k \geq i \end{cases} \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k+1 \leq j \\ k+2, k+2 \geq j+1 \end{cases}$$

One can compute that the composition $d^i d^{j-1}$ gives the same map.

What remains to show is that, subject to these relations, any order preserving injective map can be written as a composition of maps d^i .

If α is missing $i_1 < i_2 < \dots < i_{n-k}$, then α can be written as

$$\alpha = d^{i_{n-k}} d^{i_{n-k-1}} \dots d^{i_1}$$

□

We'll call d^i the i -th coface map.

A contravariant functor $\Delta_{inj} \rightarrow \text{Set}$ is called a semi-simplicial set.

Definition 1 (Singular Chain Complex)

A (non-negatively graded) singular chain complex of a space X has as chain groups

$$S_n X = \mathbb{Z} \langle \text{Sing}_n(X) \rangle$$

and differentials $\partial_n : S_n(X) \rightarrow S_{n-1}(X)$ defined on generators as

$$\partial_n(\sigma : \Delta^n \rightarrow X) \mapsto \sum_{i=0}^n (-1)^i \sigma \circ d^i$$

Lemma 8

The singular chain complex of a space is a chain complex.

Proof

By linearity, it is enough to check this on generators $\sigma : \Delta^n \rightarrow X$.

$$\begin{aligned}
\delta_{n-1}\delta_n\sigma &= \delta_{n-1}\left(\sum_{i=0}^n(-1)^i\sigma\circ d^i\right) \\
&= \sum_{i=0}^n(-1)^i\sum_{j=0}^{n-1}(-1)^j\sigma\circ d^i\circ d^j \\
&= \sum_{i=0}^n\sum_{j=0}^{n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&\quad + \sum_{0\leq i\leq j\leq n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&= \sum_{0\leq j<i\leq n}(-1)^{i+j}\sigma\circ d^i\circ d^j \\
&\quad + \sum_{0\leq i<j'\leq n-1}(-1)^{i+j'-1}\sigma\circ d^{j'}\circ d^i \\
&= 0
\end{aligned}$$

□

Lemma 9

We get a functor from chain complexes with chain maps to graded abelian groups, which is just taking homology.

Definition 2 (Singular Homology)

The singular homology $H_\bullet X$ (with integer coefficients) on a space X is the homology of the singular chain complex.

Lecture 4: Homology Theories

Wed 19 Oct

Lemma 10

Homology defines a functor $Ch \rightarrow gr\ Ab$

Proof (Sketch)

Let $f : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$, then $H_n(f) = f_*$ sending $x \in \ker(d_n)/\text{Im}(d_{n+1})$ to $[f(x)]$ □

Example

Let's compute the singular homology of the point.

Clearly $S_* = \mathbb{Z}$ and the maps induced by restriction are the identity.

Hence, the boundary maps will be

$$\dots \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\text{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus $\forall n > 0$, we get $H_n(pt) = 0$ and $H_0(pt) = \mathbb{Z}$.

Now we want to define homology for pairs.

Let $A \subset X$ be a pair of spaces.

We want to associate a singular chain complex $(S_\bullet(X, A), \delta_\bullet)$.

More generally, any continuous map $f : X \rightarrow Y$ induces $Sing_n(X) \rightarrow Sing_n(Y)$ by postcomposition.

Thus we get a functor $Sing_n(-) : \mathcal{T} \rightarrow \text{Set}$.

This in turn defines a chain map by extending $S_n f$ linearly to $S_n X$.

This defines a chain map $C_n X \rightarrow C_n Y$ since

$$\sigma \in S_n X \rightarrow f \circ \sigma \rightarrow \sum_{i=0}^n (-1)^i (f \circ \sigma) \circ d_i$$

and

$$\sigma \in S_n X \rightarrow \sum_{i=0}^n (-1)^i \sigma \circ d^i \rightarrow \sum_{i=0}^n (-1)^i (f \circ \sigma \circ d_i)$$

coincide.

For an inclusion of subspaces $A \subset X$, we get an induced map $S_\bullet(i) : (S_\bullet A, \delta_\bullet) \rightarrow (S_\bullet X, \delta_\bullet)$ which is levelwise injective.

Definition 3 (Singular chain complex of a pair)

The singular chain complex of a pair is defined to be the quotient chain complex $S_\bullet X / S_\bullet A$.

Then the singular homology of the pair (X, A) is the homology of this chain complex.

For any pair (X, A) there is a short exact sequence of chain complexes

$$0 \rightarrow (S_\bullet A, \delta_\bullet) \rightarrow (S_\bullet X, \delta_\bullet) \rightarrow (S_\bullet(X, A), \delta_\bullet) \rightarrow 0$$

(ie. levelwise short exact)

What about coefficient groups $\neq \mathbb{Z}$.

Definition 4

Given a pair of spaces (X, A) and G an abelian group G , define the singular chain complex of (X, A) with coefficient in G as follows

$$S_n(X, A; G) = S_n(X, A) \otimes_{\mathbb{Z}} G$$

with the natural induced differentials. The singular homology of (X, A) with coefficients in G is the homology of this new chain complex.

Proposition 12

For any short exact sequence of chain complexes $0 \rightarrow C_{\bullet} \rightarrow D_{\bullet} \rightarrow E_{\bullet} \rightarrow 0$, we get a long exact sequence of homology groups

$$\dots \rightarrow H_n C_{\bullet} \rightarrow H_n D_{\bullet} \rightarrow H_n E_{\bullet} \rightarrow H_{n-1} C_{\bullet} \rightarrow \dots$$

which is natural in short exact sequences of chain complexes; w

Proof

The definition of the map $\partial_n : H_n E \rightarrow H_{n-1} C$ is a standard diagram chase.

We then prove that :

1. γ is in the kernel of $d_{n-1}^C : C_{n-1} \rightarrow C_{n-2}$

$$f_{n-2} d_{n-1}^C \gamma = d_{n-1}^D f_{n-1} \gamma = 0$$

as f_{n-2} is injective, $d_{n-1}^C \gamma = 0$

2. The choice of β is independent on the choice of γ .

Suppose β' is also such that $g_n \beta = g_n \beta'$.

We want to show that $\gamma - \gamma'$ is in the image of d_n^C .

As $g_n(\beta - \beta') = 0 \exists \tilde{\gamma} : f_n \tilde{\gamma} = \beta - \beta'$

$$f_{n-1} d_n^C \tilde{\gamma} = d_n^D f_n \tilde{\gamma} = d_n^D \beta - d_n^D \beta' = f_{n-1}(\gamma - \gamma').$$

Thus $d_n^D \tilde{\gamma} = \gamma - \gamma'$

3. Independence of the choice of representative α .

We want to show that if $\alpha = d_n^E \tilde{\alpha}$, then $\gamma = 0$.

This again is a standard diagram chase. So we conclude that $\partial_n : H_n E \rightarrow H_{n-1} C$ is a well defined map, it is easy to check that it is linear.

It remains to show that the long sequence above is exact, which is part of the homework. \square

We want to show that the connecting homomorphisms are natural, namely, for two short exact sequences

$$0 \rightarrow C_{\bullet} \rightarrow D_{\bullet} \rightarrow E_{\bullet} \rightarrow 0$$

$$0 \rightarrow C'_{\bullet} \rightarrow D'_{\bullet} \rightarrow E'_{\bullet} \rightarrow 0$$

with $\phi : C_\bullet \rightarrow C'_\bullet, \psi, \eta$ etc which make the diagram commute, we get, for every n a commutative diagram

$$H_n E \xrightarrow{\partial_n} H_{n-1} C_\bullet \rightarrow H_{n-1} C'_\bullet = H_n E \xrightarrow{H_n \eta} H_n E'_\bullet \xrightarrow{\partial'_{n-1}} H_n C'_\bullet$$

Lecture 5: General results about singular homology

Mon 24 Oct

Proposition 13

Relative singular homology with coefficients in an abelian group G defines a functor $H_*(-, -; G) : \text{Top}^{(2)} \rightarrow \text{gr Ab}$ and connecting homomorphisms $\partial_n : H_n(X, A, G) \rightarrow H_{n-1}(A; G)$ such that the LES for homology theories is satisfied.

Proof

$$H_n(X, A; G) = H_n(S_n X / S_n A \otimes G, \bar{\delta}_n).$$

Let $f : (X, A) \rightarrow (Y, B)$ be a map of pairs of spaces.

We have already shown that taking homology is functorial.

We still need to show that $\mathbb{T}^{(2)} \rightarrow \text{Ch}$ mapping $(X, A) \mapsto (S_*(X)/S_*(A), \delta_*)$ is a functor.

Recall that $- \otimes G$ is a functor from chain complexes to chain complexes.

We get a map of short exact sequences

$$0 \rightarrow S_* A \rightarrow S_* X \rightarrow S_* X / S_* A \rightarrow 0$$

to

$$0 \rightarrow S_* B \rightarrow S_* Y \rightarrow S_* Y / S_* B \rightarrow 0 \quad \square$$

induced by f , where the map from $S_* X / S_* A \rightarrow S_* Y / S_* B$ is induced by the universal property.

This map of chain complexes is clearly functorial, by definition of the $f_* : S_* X \rightarrow S_* Y$.

Remark

In general, tensoring with G does not preserve exact sequences but it does preserve split short exact sequences.

Thus, we want to show that, for every n , $0 \rightarrow S_n A \rightarrow S_n X \rightarrow S_n X / S_n A \rightarrow 0$ is split short exact.

Notice that $\text{Sing}_n(X) = \{\sigma : \Delta^n \rightarrow A \rightarrow X\} \coprod \{\sigma : \Delta^n \rightarrow X \mid \text{Im } \sigma \not\subset A\}$.

Thus it is clear that the short exact sequence above is split.

Hence, after tensoring with G , we still obtain a map of short exact sequences as above, thus we have a map of long exact sequences in homology of the form

$$\dots \rightarrow H_n A \rightarrow H_n X \rightarrow H_n(X, A) \xrightarrow{\partial_n^{(X, A)}} H_{n-1} A \rightarrow H_{n-1} X \rightarrow \dots$$

$$\dots \rightarrow H_n B \rightarrow H_n Y \rightarrow H_n(Y, B) \xrightarrow{\partial_n^{(Y, B)}} H_{n-1} B \rightarrow H_{n-1} Y \rightarrow \dots$$

Where the vertical maps are all induced by f_* , (here H_n is homology with coefficients in G)

We now want to show that singular homology is homotopy invariant, namely, if $f, g : (X, A) \rightarrow (Y, B)$ are homotopic maps of pairs, ie. \exists a continuous map $H : X \otimes [0, 1] \rightarrow Y$ such that the restriction to $A \times [0, 1]$ is contained in B which restricts to f (resp. g) on $X \times 0$ (resp. $X \times 1$).

Then, we want to show that $H_n f = H_n g : H_n(X, A; G) \rightarrow H_n(Y, B; G)$ coincide.

Definition 5 (Homotopic chain maps)

Two chain maps $\phi, \psi : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$ are chain homotopic if there exists a family of linear maps $h_n : C_n \rightarrow C'_{n+1} \forall n \geq 0$ such that

$$\phi_n - \psi_n = d'_{n-1} h_n + h_{n-1} d'_n : C_n \rightarrow C'_n$$

The family h_n is then called a chain homotopy.

Proposition 15

Given two homotopic chain maps $\phi, \psi : (C_\bullet, d_\bullet) \rightarrow (C'_\bullet, d'_\bullet)$, the induced maps on homology coincide.

Proof

Pick $[x] \in H_n C$, we want to compare $[\phi(x)]$ and $[\psi(x)]$, we need to show that $\phi(x) - \psi(x) \in \text{Im } d'_{n+1} = [d' h(x) + h d(x)]$.

As $x \in \ker d$, $h d(x) = 0$ and $d' h(x) \in \text{Im } d'_{n+1}$ □

So now we want to show that homotopic maps of pairs induce homotopic maps of chain complexes.

The key idea will be to notice that, in general $\Delta^n \times [0, 1]$ is not a simplex in general but it can be decomposed as a union of $n + 1$ $(n + 1)$ simplices.

We can consider $\Delta^n \times [0, 1] \subset \mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$.

Notice that any convex hull of $n + 2$ linearly independent vectors in \mathbb{R}^{n+2} is homeomorphic to the $n + 1$ simplex which is compatible with the ordering of the vertices.

For $i \in \{0, n\}$, consider $\tau_i = \text{conv}((e_0, 0), (e_1, 0), \dots, (e_i, 0), (e_{i+1}, 1), \dots, (e_n, 0))$.

We want to build a map $S_n(X) \rightarrow S_{n+1}(Y)$, we do this by noticing that a map $\sigma : \Delta^n \rightarrow X$ induces $\Delta^n \times [0, 1] \xrightarrow{\sigma \times \text{Id}} X \times [0, 1] \xrightarrow{H} Y$.

And thus $\tau_i \subset \Delta^n \times [0, 1] \rightarrow X \times [0, 1] \rightarrow Y$ gives an $n + 1$ simplex in Y .

The claim we will prove next time is that this induces a chain homotopy $S_* f, S_* g$

Lecture 6: Homotopy invariance

Wed 26 Oct

If $f, g : X \rightarrow Y$ are homotopic maps, then we construct a map $h_n S_n X \rightarrow S_{n+1} Y$ by defining it

$$h_n(\sigma) = \sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \circ \tau_i$$

We want to show this is a chain homotopy, ie $S_n g - S_n f = h d + d h$.

Notice that $\tau_i \circ d^{i+1} = \tau_{i+1} \circ d^{i+1}$, $\tau_0 \circ d^0 = \text{inclusion of } \Delta^n \times \{1\}$ and $\tau_n \circ d^n = \text{inclusion of } \Delta^n \times \{0\}$.

We need to analyze

$$\begin{aligned} h_{n-1} \delta_n(\sigma) + \delta_{n+1} h_n(\sigma) &= h_{n+1} \left(\sum_{j=0}^n (-1)^j \sigma \circ d^j \right) + \delta_{n+1} \left(\sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \circ \tau_i \right) \\ &= \sum_{j=0}^n (-1)^j h_{n+1}(\sigma \circ d^j) + \sum_{k=0}^{n+1} (-1)^k \sum_{i=0}^n (-1)^i H(\sigma \times \text{Id}) \tau_i \circ d^k \\ &= \sum_{j=0}^n (-1)^j h_{n+1}(\sigma \circ d^j) + \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^n (-1)^{i+k} H(\sigma \times \text{Id}) \tau_i \circ d^k + g(\sigma) - f(\sigma) \end{aligned}$$

Notice that $h_{n-1}(\sigma \circ d^k) = \sum_{i=0}^{k-1} (-1)^i H(\sigma \times \text{Id}) (\tau_i \circ d^{k+1}) + \sum_{i=k+1}^n (-1)^{i+1} (\sigma \times \text{Id}) (\tau_i \circ d^k)$.

Putting everything together yields

$$\begin{aligned} &= \sum_{j=0}^n (-1)^j \left(\sum_{i=0}^{j-1} \right) (-1)^i H(\sigma \times \text{Id}) (\tau_i \circ d^{j+1}) \\ &+ \sum_{i=j-1}^n (-1)^{j+1} H(\sigma \times \text{Id}) (\tau_i \circ d^j) + \sum_{k=0, k \neq \{i, i+1\}}^{n+1} \sum_{i=0}^n (-1)^{i+k} H(\sigma \times \text{Id}) \tau_i \circ d^k \\ &+ g(\sigma) - f(\sigma) \end{aligned}$$

The rest of the proof is cursed and I won't write it down, Dieck's Book seems to have a complete proof.

We still have to show that H_n respects homotopy of pairs

Proposition 16

The functor $H_*(-, -; G) \rightarrow \text{Ab}$ is homotopy invariant

Proof

Let $f, g : (X, A) \rightarrow (Y, B)$ be given homotopic maps via $H : X \times [0, 1] \rightarrow Y$.

This gives us chain homotopies $(f|_A)_*$ to $(g|_A)_*$.

f_*, g_* induce maps $S_* X / S_* A \rightarrow S_* Y / S_* B$ and the chain homotopies between

f_* and g_* induce the same chain homotopies between these chain maps.
 Moreover, tensoring with G maps chain homotopies to chain homotopies. \square

2.1 Excision

Proposition 17

$H_*(-, -; G)$ satisfies the excision axiom.

Proof

Consider the cover $X = (X \setminus B)^\circ \cup A^\circ$ and write $U = \{X \setminus B, A\}$

Let $S_n^U(X) = \mathbb{Z} \langle \sigma : \Delta^n \rightarrow X \mid \text{Im } \sigma \subset X \setminus B \text{ or } A \rangle \subset S_n(X)$ \square

Lecture 7: Excision

Mon 31 Oct

Proof

We want to show that $H_n(S_\bullet^U(X), \delta_\bullet) \rightarrow H_n(X)$ is an isomorphism.

A chain map inducing isomorphisms on all homology groups is also called a quasi-isomorphism.

We'll use barycentric subdivision to make our simplices smaller.

First, let's recall the lebesgue lemma

Lemma 18

Let K be a compact metric space and $(U_i)_{i \in I}$ an open cover of K .

Then, there is an $\epsilon > 0$ s.t. any ϵ -ball around any point in K is contained in a single open set U_i .

To prove this, for every $x \in K$, we choose $\delta(x) > 0$ such that the $\delta(x)$ ball around x is contained in a U_i of the cover.

Now, we look at $\left\{ B_{\frac{\delta(x)}{2}}(x) \right\}_{x \in K}$, this is an open cover of K , so there is a finite subcover $B_{\frac{\delta(x_i)}{2}}(x_i)$.

Set $\epsilon = \min_j \frac{\delta(x_j)}{2}$.

Given any $x \in K$, we want to show that $B_\epsilon(x)$ is completely contained in some U_i .

We can find x_l such that $d(x, x_l) < \frac{\delta(x_l)}{2}$, let $y \in B_\epsilon(x)$ then $\delta(x_l, y) \leq \frac{\delta(x_l)}{2} + \epsilon \leq \delta(x_l)$

We can apply the Lebesgue lemma to the open cover $\{\sigma^{-1}(X \setminus \overline{B}), \sigma^{-1}(A^\circ)\}$ is an open cover of compact metric spaces Δ^n .

Thus, there exists $\epsilon > 0$ such that any open ϵ -ball in Δ^n is mapped by σ either to $X \setminus B$ or to A .

We'll now prove the following proposition

Proposition 19

There is a chain map (called the Barycentric subdivision) $Sd : S_\bullet X \rightarrow S_\bullet X$ which is

- Natural in X
- Chain homotopic to the identity
- If $X = \Delta^n$, then any summand τ of $Sd(\sigma)$ for $\sigma : \Delta^k \rightarrow \Delta^n$ has the following property

$$\text{diam}(\text{Im } \tau) \leq \frac{k}{k+1} \text{diam}(\text{Im } \sigma)$$

Let $v_0, \dots, v_n \in \mathbb{R}^N$, then their barycenter is $\frac{v_0 + \dots + v_n}{n+1}$.

We will consider the following auxiliary "cone" map.

For any $\tau : \Delta^k \rightarrow \Delta^n$ and b the barycenter of Δ^n , define $\rho_b(\tau) : \Delta^{k+1} \rightarrow \Delta^n$ by

$$\rho_b(t_0, \dots, t_{k+1}) \mapsto \begin{cases} t_0 b + \tau(\frac{1}{1-t_0}(t_1, \dots, t_{k+1}))(1-t_0) \\ b \text{ if } t_0 = 1 \end{cases}$$

This is indeed continuous.

What is the relation of the cone construction and the boundary of a simplex?

$$\begin{aligned} \delta_{k+1} \rho_b(\tau) &= \sum_{j=0}^{k+1} (-1)^j \rho_b(\tau) \circ d^j \\ &= \tau + \sum_{j'=0}^k (-1)^{j'+1} \rho_b(\tau \circ d^j) \\ &= \tau - \rho_b(\delta_k \tau) \end{aligned}$$

So we obtain a linear map $\rho_b : S_k(\Delta^n) \mapsto S_{k+1}(\Delta^n)$ with the property

$$\delta_{k+1} \rho = \text{Id} - \rho_b \circ \delta_k$$

We define a map $Sd : S_n(X) \rightarrow S_n X$.

For $n = 0$, $Sd_0 = \text{Id}_{S_0 X}$.

Given some $n > 0$ and $\sigma : \Delta^n \rightarrow X$, define

$$Sd(\sigma) = \sigma_*(\rho_b Sd(\delta_n i_n))$$

where i_n is the identity of the Δ^n simplex, considered as an element of $\text{Sing}_n(\Delta^n)$

We claim that Sd is a chain map.

Given $\sigma : \Delta^n \rightarrow X$, we want to compute

$$\delta_n Sd \sigma = \delta_n \sigma_*(\rho_b Sd(\delta_n i_n))$$

We can switch δ_n and σ_* since σ_* is post composition and δ_n is precomposition.

$$\begin{aligned} &= \sigma_*(\delta_n \rho_b Sd(\delta_n i_n)) \\ &= \sigma_*(Sd(\delta_n i_n) - \rho_b(\delta_{n-1} Sd(\delta_n i_n))) \end{aligned}$$

Notice that

$$\delta_{n-1} Sd(\delta_n i_n) = Sd(\delta_{n-1} \delta_n i_n) = 0$$

Thus

$$\delta_n Sd(\sigma) = \sigma_* Sd(\delta_n i_n) = Sd(\delta_n \sigma) \quad \square$$

For naturality, if $f : X \rightarrow Y$ is a map of topological spaces, one can explicitly check that $Sd \circ f_* = f_* \circ Sd$

Lecture 8: still excision

Wed 02 Nov

I missed the first half of the lecture because I overslept, so there is a part missing here.

Lemma 20

The map of chain complexes $S^U(X) \rightarrow S(X)$ is a quasi-isomorphism

Proof

Let $\sum_k a_k \sigma_k \in S_N X$ be in the kernel of δ_n , ie. $[\sum_k a_k \sigma_k]$ represents an element in $H_n(X)$.

To see surjectivity, notice that

$$Sd(\sum_k a_k \sigma_k) = \sum_k a_k \sigma_k - h(\delta(\sum_k a_k \sigma_k)) - \delta h(\sum_k a_k \sigma_k)$$

The middle term is 0 by our hypothesis, and we see that $Sd(\sum_k a_k \sigma_k)$ is representing the same element in homology.

We can apply Sd arbitrarily many times and hence $Sd^l(\sum_k a_k \sigma_k)$ is too.

Now, for every k , $\sigma_k^{-1} A^\circ, \sigma_k^{-1}(X \setminus \overline{B})$ forms an open covering of Δ_n .

There exists a Lebesgue Number $\epsilon_k > 0$ for this open cover.

There is an $l_k \in \mathbb{N}$ such that $(\frac{n}{n+1})^{l_k} < \frac{\epsilon_k}{\sqrt{2}}$.

Now, for any simplex τ in the barycentric subdivision of Δ_n $\text{diam} \tau \leq \epsilon$.

Now $(\sigma_k)_*(\tau) \subset X \setminus \overline{B}$ or $\subset A^\circ$.

Setting $l = \max_k l_k$, we see that $Sd^l(\sum_k a_k \sigma_k)$ is a preimage for $\sum_k a_k \sigma_k$.

For injectivity, let $\sum_k a_k \sigma_k$ be an element of $S_\bullet^U X$ in $\ker \delta$ which is in the image of δ .

Let $\sum_j b_j \tau_j$ be such that $\delta(\sum_j b_j \tau_j) = \sum_k a_k \sigma_k$.

There exists an m such that $Sd^m(\sum b_j \tau_j) \in S^U X$, now $\delta Sd^m(\sum b_j \tau_j) = Sd^m(\sum a_k \sigma_k)$.

Thus, $\sum a_k \sigma_k$ is a boundary.

This concludes the proof. \square

We can now use that a quasi-isomorphism of complexes of free abelian groups has a homotopy inverse.