Sheet 3. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. In how many ways can you write $5, 10$ and 15 as a sum of three numbers chosen from the sets $\{1, 2, 3, 5\}$, $\{3, 5, 8, 9\}$ and $\{0, 2, 4\}$ respectively?

Exercise 2. Prove the following using generating polynomials.

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}.$$

Exercise 3. Find the sequence generated by the generating function

(1) $\frac{x^3}{(1+x)^2}$.
(2) $\frac{1+x+x^2}{(1-x)^2}$.

Exercise 4. Determine the generating function of the sequence

(1) $(a_0, a_1, \ldots)$ with $a_k = 2^{\lfloor k/2 \rfloor}$?
(2) $(a_0, a_1, \ldots) = (1, 3, 5, 7, 9, \ldots)$

Exercise 5. For a natural number $n \in \mathbb{Z}^{\geq 1}$, define σ_n as the number of divisors of n. That is

$$\sigma_n = \#\{r \in \{1, 2, \ldots, n\} \mid r \text{ divides } n\}.$$

For example, $\sigma_6 = 4$ and $\sigma_{24} = 8$. Show that for $|x| < 1$, the following holds.

$$\sum_{n=1}^{\infty} \frac{x^n}{1 - x^n} = \sum_{i=1}^{\infty} \sigma_i x^i.$$

Exercise 6.
Suppose $n, k_1, k_2, \ldots, k_r \in \mathbb{Z}^{\geq 0}$ such that $\sum_{i=1}^{r} k_r = n$. We use the following notation for multinomials:

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1!k_2!\ldots k_r!}.$$

Prove that

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r} + \binom{n-1}{k_1, k_2 - 1, k_3, \ldots, k_r} + \cdots \binom{n-1}{k_1, k_2, \ldots, k_r - 1}.$$

Compare this with Proposition 1.11 from the lecture about binomial coefficients.