Optimisation Discrete

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Lecture 1: Introduction Tue 22 Feb

1 What is optimization ?

Given $f : X \to \mathbb{R}$ find

$$\max \{f(x) : x \in S\} \text{ where } S \subset \mathbb{R}^n$$

f is called the objective function and S the feasible region.
The answer will be a point $x_0 \in S$ such that $f(x_0) \geq f(x)$ for all other $x \in S$.
A solution is anything you can plug into f.
A feasible solution is something in S and the x_0 would be called an optimal solution.
We consider feasible regions described by intersection of equalities and inequalities.
A constraint is "active" or "tight" if f satisfies it with equality.

1.1 New lagrange multipliers

We should try to satisfy $\nabla f = \sum_i \lambda_i \nabla g_i$ where $\lambda_i = 0$ whenever g_i is not active. Feasible regions split into nice ones and not so nice ones.
A region S is convex if $\forall x, y \in S$ and all $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in S$

Lecture 2: Stuff Tue 01 Mar

There are 3 different possibilities for optimisation problems

1. The finite max exists
2. The problem is unbounded
3. The problem is infeasible, ie. $S = \emptyset$

There is a special class of functions f which allow us to solve problems rather easily
— f is linear
— S is made up of linear inequality constraints.
Such an optimisation problem is called a linear program.

1.2 Geometry of linear equalities

What can the feasible region of a LP look like ?
It will look like an intersection of half spaces, we call it a polyhedron.

Proposition 1
Polyhedra are convex

Indeed they are finite intersections of half-spaces, which are all convex. Given vectors $v_1, \dots, v_n$ we can look at

1. linear combinations (span)
2. positive linear combinations $\sum a_i v_i, a_i \geq 0$ (cone)
3. convex combinations $\sum a_i v_i, 0 \leq a_i \leq 1, \sum_i a_i = 1$ (convex hull)

If we have a linear program, only active constraints can be involved in optimal solutions.
Given a linear program, we want to turn it into a standard form.

Definition 1 (Equivalent linear programs)
Two linear programs P and P' are equivalent if for every feasible x in P, there exists a feasible $x' \in P'$ with $value_P(x) = value_{P'}(x')$ and conversely. Furthermore, finding x from x' and x' from x is easy.

What does a general linear program look like ?

$$\max c \cdot x$$

s.t. $Ax \geq b$.

Proposition 2
every linear program can be transformed into a linear program in inequality standard form.

Given an equality constraint $a \cdot x \geq b$, we can turn it into $a \cdot x = b + w$ with $w \geq 0$.
Such a variable w is called a slack variable.
Hence, given the constraint $Ax \geq b$, we can turn it into an equality constraint

$$Ax = b + [w_1, \dots, w_n]^T$$

and ask $w_i \geq 0 \forall i$