Rigid Analytic Geometry

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Lecture 1: Covering sieves

Wed 05 Apr

Theorem 1

Every sieve τ containing a covering sieve τ' of X is itself covering.
The intersection of two covering sieves is covering.

Proof

If $(v : V \to X)$ is a morphism in τ' then $v^*\tau = v\tau'^*$.
Let τ, τ' be covering sieves of X and $v : V \to X \in \tau$, then $v^*(\tau \cap \tau') = v^*\tau'$.
This covers V by GTTrans, by GTLoc, $\tau \cap \tau'$ covers X. $\square$

Remark

We are mostly interested in the case where the category C is the poset of open subsets of a topological space.
Then a sieve in V is just a set of open subsets of V such that $V \in \tau$, $W \subset V, W$ open implies $W \in \tau$.
The pullback along the (unique if it exists) morphism $V \to U$ are just the open subsets of V.
We write $\tau / = V$ if τ is a sieve oveoverV which covers V.
If several grothendieck topologies must be distinguished, I will write $\tau / =_\pi V$

Definition 1

We will write $[V_i | i \in I]$ for the sieve generated by the family V_i of open subsets of V. We have $[V_i] = \{V \in O_X | \exists i \in I \ st \ V \subset V_i\} = \bigcap_{\tau \text{ sieve in } X \text{ containing } V_i} \tau$.
A sieve is finitely generated if it can be written as $[V_i]$ for finitely many V_i

Remark

More generally, we consider Grothendieck topologies on B, a topology base for X, considered as posets.

Definition 2

Let $[\Omega_i]_B$ be the B-sieve generated by the Ω_i, ie.

$$\{\theta \in B | \theta \subset \Omega_i \text{ for at least one } i \in I\}$$

The subscript B will always be used when $B \subsetneq O_X$.

Proposition 4

Let X be a topological space and B a topology base for X.
Then we have a bijection between

— *Grothendieck topologices T_B on B*

— *Grothendieck topologies T on O_X st. $[B_V]$ covers V.*

If T_B is given, T is defined by $\tau / =_T V, \tau \cap B_\Omega / =_{T_B} \Omega$ for all $\Omega \in B_V$.

When T is given, T_B is defined by

$$\tau / =_{T_B} \Omega \text{ iff } [\tau] / =_T \Omega$$

Lecture 2: idk

Wed 12 Apr

Definition 3

Let B be a topology base on X. By a G_+ -topology on B, we understand a Grothendieck topology on B with the following additional assumptions.

- *If $\mathcal{S} / = \Omega$ then $\Omega = \bigcup_{\theta \in \mathcal{S}} \theta$*
- *$\emptyset / = \emptyset$*
- *The topology generated by B is T_0.*

If $B = \mathbb{O}_X$ for a topological space X then we speak of a G_+-topology on X.

Remark

Under a coarser pretopology, we call an open covering $\mathcal{U}$ of V admissible for the G_+-topology under consideration if $[\mathcal{U}] / = V$

Example

- *Ordinary Topologies : $\mathcal{S} / = V$ iff $\bigcup_{v \in \mathcal{S}} v = V$*
- *If V is an open neighbourhood of a G_+-toopological space X, then it carries an induced G_+-topology*
- *Let B have the additional property that $\Omega, \theta \in B \implies \Omega \cap \theta \in B$.*
  Let $\mathcal{S}$ be a covering sieve for Ω iff there is an $n \in \mathbb{N}$ and $\theta_1, \dots, \theta_n$ such that $\Omega = \bigcup_i \theta_i$.
  To verify the three axioms, note thaat GTtriv is trivial.
  We check transitivity, let $\mathcal{S} / = \Omega$ and θ_i as above, $\Theta \in B_\Omega$, then $\Theta = \bigcup_i (\theta_i \cap \Theta)$.
  To see locality, let $\mathcal{S}$ cover Ω, θ_i as above and T another seive st $T//\theta$ then $\theta_i \bigcup \theta_{ij}$ with $\theta_{ij} \in T$, hence $\Omega = \bigcup_i \bigcup_j \theta_{ij}$, hence $T / = \Omega$

Definition 4

Let X be a G_+ topological space, a G_+-topology base for X is an ordinary topology base for the undelrying ordinary topology satisfying the additional assumption that $[B_V] / = V$ for all $V \in \mathbb{O}_X$.

Thhe topology base is called G_{++} if, in addition, membership in B is local in the following sense

- *If $\Omega, \theta \in B$, then $\Omega \cap \theta \in B$*
- *If $\Omega \in B$ and $V \in \mathcal{O}_\Omega$ such that $\{\theta \in B_\Omega | \theta \cap V \in B\} / = \Omega$*

Corollary 7

- *If B is a topology base on the topological space X, then there is a bijection between the G_+ topologies on B and the G_+ topologies on X for which B is a G_+ topology base.*
- *If in addition B is closed under intersections, then there is a unique G_+ topology on X with the property that a covering of an element of B is admissible iff it has a finite subcovering*

Definition 5

This G_+-topology is called the G_+ topology obtained by forcing the elements of B to be quasicompact.

Definition 6

Let B be a topology base closed under arbitrary finite intersections.
A seive $\mathcal{S}$ is called a prime sieve if $N \in \mathbb{N}$, $(\Omega_i) \in B$ and $\bigcap_i \Omega_i \in B$ implies there is $i \in [1, n]$ s.t. $\Omega_i \in \mathcal{S}$.

Remark

Obviously, $\mathcal{S}$ is a prime sieve iff

- $\Omega, \theta \in B$ *and* $\Omega \cap \theta \in \mathcal{S} \iff \Omega \in \mathcal{S}$ *or* $\theta \in \mathcal{S}$
- $X \notin \mathcal{S}$

Proposition 9

Let B be a G_+ topology base for a G_+-topological space X which is closed under taking intersections.
Ten the following conditions on a subset $\xi \subset B$ are equivalent

- — *If $U \in \xi$ and $U \subset V$, then $U \subset V$, then $V \in \xi$*
  - *A finite intersection in X of elements of ξ is $\in \xi$*
  - *If $\Omega \in \xi$ and $\mathcal{S}/ = \Omega$ then $\mathcal{S} \cap \xi \neq \emptyset$*
- $\mathcal{S} = B \setminus \xi$ *is a prime sieve containing every element Ω of B with $\mathcal{S}//\Omega$*

Definition 7 (Vander Put point)

Let B be a G_+ topology base such that $\Omega, \Theta \in B \implies \Omega \cap \Theta \in B$.
A Van der Put-point for B is a subset $\xi \subset B$ such that

- $\Omega \in \xi, \Theta \in B, \Omega \subset \Theta \implies \Theta \in \xi$
- *If Ω, Θ then $\Omega \cap \Theta \in \xi$*
- $\xi \neq \emptyset$
- *If $\Omega \in \xi$ and $\mathcal{S}/ = \Omega$ then $\mathcal{S} \cap \xi \neq \emptyset$*

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Lecture 3: stuff

Wed 19 Apr

Corollary 10

If X is an ordinary topological space, then $X \to X^*$ iff X is sober.

Example

Let F be an ordered field. Equip $\mathbb{A}^1_F = F$ with it's order topology and the G_+-topology forcing the elements of $B = \emptyset \cup \{(a,b)_F | -\infty \leq a < b \leq \infty\}$ to be quasi-compact.
To describe the $\mathbb{A}^{1,*}_F$ of van der Put points, let a generalized Dedekind cut of F be a decomposition $F = A \cup B$ such that

1. $a \in A, \alpha \in (-\infty, a]_F \implies \alpha \in A$
2. $b \in B, \beta \in [b, \infty) \implies \beta \in B$
3. $|A \cap B| \leq 1$

There is a bijection $\mathbb{A}^{1,*}_F \leftrightarrow \{$ generalized Dedekind cuts $\}$ given by sending a Van der Put point ξ to the cut $F = A \cup B$ with $A = \{a \in F | (-\infty, a) \notin \xi\}$ and $B = \{b \in F | (b, \infty) \notin \xi\}$.
THe inverse sends a cut $F = A \cup B \mapsto \xi = \{(a,b) | a \notin B, b \notin A\}$.
Indeed, if $f \in F \setminus (A \cup B)$ (with ξ given), then $(-\infty, f)$ and (f, ∞) are both $\in \xi$ hence there intersection is empty and still contained in ξ, contradicting the fact that ξ is a Van der Put point.
If $a < b$ are elements of F then $\mathbb{A}^1_F = (-\infty, b) \cup (a, \infty)$ is an admissible covering, but $\mathbb{A}^1_F \in \xi$ and hence $(-\infty, b) \in \xi$ or (a, ∞) hence $b \notin A$ or $a \notin B$, hence $\{a, b\} \not\subset A \cap B$ showing that a Van der Put point gives a cut.
We leave out the proof of the remaining properties.
The map $F = \mathbb{A}^1_F \to \mathbb{A}^{1,*}_F$ sends $f \in F$ to $F = A \cup B$ to the cut with center f.
There are also the related "Neighbouring" cuts $f_- : (-\infty, f) \cup [f, \infty)$ and f_+ defined similarly.
In addition to this, one has a point of $\mathbb{A}^{1*}_F$ for each Dedekind cut not belonging to an element of F, including the improper cuts $F = \emptyset \cup F$ (giving the point $-\infty$) and similarly $F = F \cup \infty$.
One can order $\mathbb{A}^{1*}_F$ by $(A, B) \leq (\tilde{A}, \tilde{B})$ iff $A \subset \tilde{A}$ and $\tilde{B} \subset B$.
Then a topology base on $\mathbb{A}^{1*}_F$ is given by $\{(a,b) | \infty \leq a < b \leq \infty\}$.

Remark

Recall $(U \cap V)^* = (U^*) \cap (V^*)$.
We may however have $U^* \cup V^* \subsetneq (U \cup V)^*$, for instance $F = \mathbb{Q}$ in example 3 and the dedekind cut determined by π.
This is related to the fact that the covering $\mathbb{Q} = U \cup V$ is not admissible.

Notice that if $U^* = \cup_{V \in \mathcal{S}} V^*$ when $\mathcal{S} / = U$.

Definition 8
A G_+-topological space X has sufficiently many Van der Put points if the converse to the above fact holds, ie. :

$$(P)\mathcal{S}/=U \iff U^* = \bigcup_{V\in\mathcal{S}}(V^*)$$

Example

1. *Every ordinary T_0 space has sufficiently many van der Put points*
2. *Let $X=[0,1]$ with the discrete topology, then the following G_+-topology :*

   $\mathcal{S}/=U \iff$ *there are $(X_i)\in\mathcal{S}$ such that $U\setminus\bigcup V_i$ has Lebesgue measure 0.*

   Then one can show that $X\to X^*$ is bijective, but $X=\bigcup_{x\in X}\{x\}$ is obviously not admissible.

One can show that there is a bijection $X^* = \{$ Topos points of the topos of sheaves of sets on X $\}$. This is related to Delignes example in SGA4 (IV.7.4).

Definition 9
An open subset U is called G_+-quasi compact iff every covering sieve of U contains a finitely generated covering sieve.

Proposition 14
Let X be a topological space with sufficiently many Van der Put points. Then $U\in\mathcal{O}_X$ is G_+-qc iff $U^*\in\mathcal{O}_{X^*}$ is t-qc (ie. quasi-compact in the usual sense, t stands for topological).

Proof
Assume U^* is t-qc and let $\mathcal{S}/=U$, then $U^*=\cup_{V\in\mathcal{S}}V^*$ by (P), which has a finite subcover $U^*=\bigcup_{i=1}^N V_i^*$, thus $\tilde{\mathcal{S}}^*/=U$ (by (P)) where $[V_1,\dots,V_N]\subset\mathcal{S}$ is finitely generated.
Let U be G_+-qc and $U^*=\bigcup_{i\in I}W_i$.
Without loss of generality, $W_i=U_i^*$ (as elements of $\mathcal{O}_{X^*}$ of this form form a topology base).
Then $\mathcal{S}/=U$ (by (P)) where $\mathcal{S}=[U_i]$.
As U is G_+-qc there is $\tilde{\mathcal{S}}\subset\mathcal{S}$ s.t. $\tilde{\mathcal{S}}=[V_1,\dots,V_n]$ is finitely generated, $V_i\subset U_{j_i}$.
Then (by P) $U^*=\bigcup_{V\in\tilde{\mathcal{S}}}V^*=\bigcup_{i=1}^N V_i^*\subset\bigcup_{i=1}^N U_{j_i}^*$ showing the existence of a finite subcovering. □

Remark
If $\Omega\in B$ for some basis B with a Grothendieck topology $\mathbb{T}$, then Ω is $\mathbb{T}$-qc iff every sieve $\mathcal{S}\in B$ with $\mathcal{S}/=_{\mathbb{T}_B}\Omega$ has a finitely generated subsieve $\tilde{\mathcal{S}}\subset\mathcal{S}$ such

that $\tilde{\mathcal{S}}/=_{\mathbb{T}_B}\Omega$

Remark
As a consequence, if $\mathbb{T}$ *is obtained by enforcing the qc-ness of elements of* B*, then the elements of* B *are* G_+*-qc in the sense of definition 13 such that the following proposition can be applied.*

Proposition 17
Let X *be a* G_+*-topological space which has a* G_+*-topology base* B *whose elements are* G_+*-qc, then* X *has sufficiently many Van der Put points.*

Remark
In addition, if $\Omega \in B$ *and* $\mathcal{S}$ *does not cover* Ω*, then there is* $\omega \in \Omega \setminus_{V \in \mathcal{S}} V^*$*.*
If in addition B *is closed under finite intersections in* X*, then* X^* *is a spectral space.*