École Polytechnique Fédérale de Lausanne
Prof. Marcus

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Math 261 – Discrete Optimization (Spring 2022)

Problem Set 4 Solutions

Problem 1

The purpose of this problem is to prove Theorem 7 on the CheatSheet. Let

$$\mathcal{P} = \min\{\mathbf{c}\cdot\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

be a linear program, and let β a feasible column basis for a BFS $\mathbf{x}$ with reduced costs $\overline{\mathbf{c}}$ and $\mathrm{col}_\beta(\mathbf{A}) = \mathbf{B}$.

(a) Show that $\overline{c}_i x_i = 0$ for all indices i.

Solution:
By definition, we have

$$\overline{c}_j = c_j - \mathbf{c}_\beta^\intercal \mathbf{B}^{-1}\,\mathrm{col}_j(\mathbf{A}).$$

and so for the collection of vectors in the basis, we have $\mathrm{col}_\beta(\mathbf{A}) = \mathbf{B}$

$$\overline{\mathbf{c}}_\beta^\intercal = \mathbf{c}_\beta^\intercal - \mathbf{c}_\beta^\intercal \mathbf{B}^{-1}\mathbf{B} = \mathbf{c}_\beta - \mathbf{c}_\beta = 0.$$

Hence $\overline{c}_i x_i = 0$ whenever i is in the basis. On the other hand, we know $x_i = 0$ for all $i \notin \beta$, so $\overline{c}_i x_i = 0$ for all $i \notin \beta$ as well.

(b) Show $\overline{\mathbf{c}}\cdot\mathbf{d} = \mathbf{c}\cdot\mathbf{d}$ for any feasible direction $\mathbf{d}$.

Solution:
Using the formula, we have

$$\overline{\mathbf{c}}^\intercal\mathbf{d} = \mathbf{c}^\intercal\mathbf{d} - \mathbf{c}_\beta^\intercal\mathbf{B}^{-1}\mathbf{A}\mathbf{d}.$$

However, if $\mathbf{d}$ is a feasible direction, it means that $\mathbf{x} + \theta\mathbf{d}$ is feasible for some $\theta > 0$. Hence

$$\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{b}$$

for some $\theta > 0$, which means $\mathbf{A}\mathbf{d} = 0$. Hence $\overline{\mathbf{c}}\cdot\mathbf{d} = \mathbf{c}\cdot\mathbf{d}$.

(c) Show that if $\overline{\mathbf{c}}$ is nonnegative, then $\mathbf{x}$ must be an optimal solution.

Solution:
Let $\mathbf{y} \neq \mathbf{x}$ be an arbitrary feasible solution for $\mathcal{P}$. In order to show that $\mathbf{x}$ is optimal, we need to show that $\mathbf{c}\cdot\mathbf{y} \geq \mathbf{c}\cdot\mathbf{x}$. Or, if we let $\mathbf{d} = \mathbf{y} - \mathbf{x}$, then we need to show that $\mathbf{d}\cdot\mathbf{c} \geq 0$. However $\mathbf{d}$ is a feasible direction (by convexity) and so by part (b) this is equivalent to showing $\mathbf{d}\cdot\overline{\mathbf{c}} \geq 0$.

However, we claim that $d_i\overline{c}_i \geq 0$ for all i. The fact that $\mathbf{y}$ is feasible means $\mathbf{y} \geq 0$, so if $\overline{\mathbf{c}} \geq 0$ (as we have assumed), then $y_i\overline{c}_i \geq 0$ for all i. On the other hand, we showed in part (a) that $x_i\overline{c}_i = 0$ for all i. Hence

$$\begin{aligned} d_i\overline{c}_i &= y_i\overline{c}_i - x_i\overline{c}_i \\ &= y_i\overline{c}_i - 0 \\ &\geq 0 \end{aligned}$$

and since this is true for all i, we have $\mathbf{c}\cdot\mathbf{d} \geq 0$ (making $\mathbf{x}$ an optimal solution).

(d) Finish the proof of CheatSheet Theorem 7 by showing that if $\mathbf{x}$ is an optimal, nondegenenerate solution, then $\overline{\mathbf{c}} \geq 0$.

Solution:
Assume $\overline{c}_j < 0$ for some j. By what we saw in (a), $\overline{c}_i = 0$ for all $i \in \beta$ and so it must be that $j \notin \beta$. Since $\mathbf{x}$ is nondegenerate, the direction associated to adding variable x_j to the basis is a feasible direction and the cost in that direction decreases. Hence $\mathbf{x}$ is not an optimal solution.

Problem 2

In this problem, we would like to show how to use linear programs to solve linear algebra problems[1]

(a) Let $\mathbf{a} \in \mathbb{R}^n$ and let $b \geq 0$, and let Q be the linear equation

$$\mathbf{a} \cdot \mathbf{x} = b$$

and $\mathcal{P}$ the linear program

$$\begin{array}{lr} \min & w \\ \text{s.t.} & \mathbf{a} \cdot \mathbf{x} + w = b \\ & w \geq 0 \end{array}$$

The book calls the new w variable an *artificial variable*, but I like to call it a *cheating variable*[2]. Show the following:

i. the point $(\mathbf{x}, w) = (\mathbf{0}, b)$ is a feasible solution for $\mathcal{P}$

ii. Q has a feasible solution if and only if the optimal value of $\mathcal{P}$ is 0.

iii. If $(\mathbf{y}, 0)$ is an optimal solution for $\mathcal{P}$ then $\mathbf{y}$ is a feasible solution to Q.

Solution:
We address each one:

i. Plug it in (notice that $b \geq 0$ is necessary).

ii. There exists a solution $\mathbf{y}$ to Q if and only if $(\mathbf{y}, 0)$ is feasible for $\mathcal{P}$ if and only if the optimal value of $\mathcal{P}$ is 0.

iii. If $(\mathbf{y}, 0)$ is an optimal solution for $\mathcal{P}$ then (plugging in $w = 0$), we must have $\mathbf{y}$ is a solution to Q.

(b) Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Construct a linear program[3] $\mathcal{P}$ (using cheating variables) which satisfies the following:

i. it is easy to find a feasible point in $\mathcal{P}$

ii. $\mathbf{Ax} = \mathbf{b}$ has a solution if and only if the optimal value of $\mathcal{P}$ is 0.

iii. If the optimal value of $\mathcal{P}$ is 0, then the optimal solution for $\mathcal{P}$ gives you a solution to $\mathbf{Ax} = \mathbf{b}$.

Solution:
Let $\mathbf{a}_i \cdot \mathbf{x} = b_i$ be a row of $\mathbf{Ax} = \mathbf{b}$. We would like to add a cheating variable w_i for each one, however the way we do this will depend on the value of b_i. That is,

$$\mathbf{a}_i \cdot \mathbf{x} + w_i = b_i \quad \text{for } b_i \geq 0 \quad \text{and} \quad \mathbf{a}_i \cdot \mathbf{x} - w_i = b_i \quad \text{for } b_i \leq 0$$

[1]This may seem counter-intuitive, but it brings about a useful idea.

[2]Because I feel like I am cheating by finding a way to get a solution to Q without ever needing to actually solve Q.

[3]Note: it does not have to be in equality standard form.

OR, to make life easier, we can take any row that has $b_i \leq 0$ and multiply it by -1 (since equalities stay the same). This way we can assume $\mathbf{b} \geq \mathbf{0}$ and then use the linear program

$$P = \min \{\mathbf{1} \cdot \mathbf{w} : \mathbf{A}\mathbf{x} + \mathbf{I}_m \mathbf{w} = \mathbf{b}, \mathbf{w} \geq \mathbf{0}\}.$$

where $\mathbf{1}$ is a vector of 1's and $\mathbf{I}_m$ is the identity. It is clear that $\mathcal{P}$ has a feasible solution because $(\mathbf{0}, \mathbf{b})$ is feasible. Furthermore, $\mathcal{P}$ has optimal value 0 if and only if there exists a feasible solution with $\mathbf{w} = \mathbf{0}$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution. Finally, any solution with optimal value 0 will look like $(\mathbf{y}, \mathbf{0})$ where $\mathbf{A}\mathbf{y} = \mathbf{b}$.

Problem 3

Let

$$\mathcal{P} = \min \{\mathbf{c}^\mathsf{T}\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\} \tag{1}$$

where $\mathbf{A}$ has full row rank. An *auxiliary linear program* $\mathcal{P}'$ is a linear program whose sole purpose is to find a BFS for $\mathcal{P}$ (using simplex). In particular $\mathcal{P}'$ must have the properties:

- $\mathcal{P}'$ is in the same form as $\mathcal{P}$ (either equality standard or inequality standard form)
- $\mathcal{P}'$ has an obvious BFS (with an obvious column basis) for simplex to start at
- $\mathcal{P}'$ has optimal value 0 if and only if $\mathcal{P}$ has a feasible solution
- In the case that $\mathcal{P}'$ has optimal value 0, the optimal basis for $\mathcal{P}'$ can be used to construct a feasible basis for $\mathcal{P}$.

(a) Construct an auxiliary linear program for $\mathcal{P}$ and show that it has all of the necessary properties. Be sure to show how to construct a feasible *column basis* for $\mathcal{P}$, not just a feasible point. You may assume (in this part) that the optimal solution to your auxiliary linear program is not degenerate.

Solution:
We use cheating variables as in Problem 2. First we multiply all of the rows of $\mathbf{A}\mathbf{x} = \mathbf{b}$ by ± 1 to make $\mathbf{b}$ have nonnegative entries. Then we use the linear program

$$\mathcal{P}' = \min \{\mathbf{1} \cdot \mathbf{w} \colon \mathbf{A}\mathbf{x} + \mathbf{I}_m \mathbf{w} = \mathbf{b},\ \mathbf{w} \geq \mathbf{0}, \mathbf{x} \geq \mathbf{0}\}.$$

which is in equality standard form. Let n be the dimension of $\mathbf{x}$ and m be the dimension of $\mathbf{w}$ (which is also the length of $\mathbf{b}$).

An obvious feasible solution for $\mathcal{P}'$ is $(\mathbf{x}, \mathbf{w}) = (\mathbf{0}, \mathbf{b})$ and we claim this is a BFS. The matrix of equality constraints here has the form $[\mathbf{A}\ \mathbf{I}_m]$ (where I_m is the identity matrix) and so it should be clear that the last m columns (corresponding to the variables $\mathbf{w}$) are linearly independent (and therefore form a column basis). Since $\mathbf{w} \geq \mathbf{0}$, $\mathcal{P}'$ can have optimal value 0 if and only if $\mathbf{w} = \mathbf{0}$ which happens if and only if $\mathcal{P}$ has a feasible solution.

What remains is to show that when $\mathcal{P}'$ has optimal value of 0, we can use this to find a feasible basis for $\mathcal{P}$. In general, this can be tricky (see part (b)). On one hand, if $\mathcal{P}'$ has optimal value 0 then the optimal solution must have all of the $w_i = 0$. However we assumed here that the optimal solution is nondegenerate — this means there cannot be variables set to 0 in the basis. In other words, $w_i = 0$ means that w_i is not in the optimal basis. Instead, m of the x_i variables will have to be in the basis and so the same basis in $\mathcal{P}$ will be feasible.

(b) Show how one can construct a feasible column basis for $\mathcal{P}$ in the case that the optimal solution to your auxiliary linear program is degenerate.

Solution:
Here things get a bit trickier because the w_i could be 0 but also still be in the basis. Then

you can't simply move the optimal basis over to $\mathcal{P}$ because $\mathcal{P}$ doesn't have w_i variables. But the fact that the optimal solution to $\mathcal{P}'$ is 0 means that $\mathcal{P}$ is feasible — and this means $\mathcal{P}'$ must have an optimal basis which doesn't have any w_i in it. To find it, you need to do what the book calls "driving the artificial variables out of the basis." Essentially this constitutes doing more rounds of simplex — the difference being that we will allow ourselves to move in the wrong direction (in terms of the reduced cost) as long as the distance we move is 0 (this is something we would normally not need to do). In most cases, however, you can simply try replacing the w_i variables with x_i variables that are not in the basis (but are already set to 0). If there are not many combinations, it is easy to test each one and see which ones give you a BFS (of course they will be feasible, but there might be some issues with linear independence).

If you are using a computer solver, another option would be to try to play the "kick the linear program" game like was mentioned in the notes (add random noise to **b** and **c** of your auxiliary linear program). However, for solving by hand this makes things more complicated (it replaces nice whole numbers with horrible decimals).

Problem 4

Solve the linear program

$$\begin{array}{rrcrcrcrcr}
\max & 6a & + & 9b & + & 2c & + & 3d & & \\
\text{s.t.} & a & + & 3b & + & c & + & 2d & = & -4 \\
& & & b & + & c & - & d & \leq & -1 \\
& 3a & + & 3b & - & c & & & \leq & 1 \\
& a & & & & & & & \leq & 0 \\
& & & b & & & & & \leq & 0 \\
& & & & & c & & & \leq & 0
\end{array}$$

using the *two phase method*. That is,[4]

Phase 0: Put the problem in the form you want (simplify if you can!)

Phase 1: Find an initial BFS for the output of Phase 0 (or show that it is infeasible). Often this requires constructing an auxiliary linear program, but sometimes you can simply guess.

Phase 2: Run simplex method on the output of Phase 0 starting at BFS found in Phase 1.

Solution:
We do the three phases:

Phase 0: We can simplify this a bit and make life much easier. In particular, there is an equality constraint which contains a free variable (d), so we can solve for that free variable, substitute everywhere, and get rid of both the constraint and the free variable. So we plug in $d = -2 - c/2 - 3b/2 - a/2$ to get the new program

$$\begin{array}{rrcrcrcr}
-6 + \max & 9/2a & + & 9/2b & + & 1/2c & & \\
\text{s.t.} & 1/2a & + & 5/2b & + & 3/2c & \leq & -1 \\
& 3a & + & 3b & - & c & \leq & 1 \\
& a & & & & & \leq & 0 \\
& & & b & & & \leq & 0 \\
& & & & & c & \leq & 0
\end{array}$$

[4]So technically what I am about to write has three phases, but the books calls it "The 2-phase method" because it assumes Phase 0 has already happened and the LP is in a nice form. Since that might not be the case, and this is often a very easy way to make your life easier, I thought it should be listed separately.

and let me pull out the 1/2 in the objective and multiply the second constraint by 2 just to get rid of the fractions

$$\begin{array}{rrcrcrcr}
-6+(1/2)\max & 9a & + & 9b & + & c & & \\
\text{s.t.} & a & + & 5b & + & 3c & \leq & -6 \\
& 3a & + & 3b & - & c & \leq & 1 \\
& a & & & & & \leq & 0 \\
& & & b & & & \leq & 0 \\
& & & & & c & \leq & 0
\end{array}$$

Now I will

- substitute $x = -a$, $y = -b$, $z = -c$
- add slack variables s, t
- switch from max to min

to get

$$\begin{array}{rl}
-6-(1/2)\min & 9x+9y+z \\
\text{subject to} & x+5y+3z-s \quad = 6 \\
& 3x+3y-z \quad -t = -1 \\
& x, \ y, \ z, \ s, \ t, \geq 0
\end{array} \tag{1}$$

Phase 1: The first thing we should do is see if there is an easy BFS we can guess — if there is, we can skip the auxiliary LP and move to Phase 2. And there *almost* is — consider the solution

$$(x, y, z, s, t) = (0, 0, 0, -6, 1)$$

which almost works (it only breaks one constraint). That might seem like useless information, but it's not — it means that we could try to avoid putting a cheating variable on the second constraint. Sometimes this happens, and it makes life a bit easier, at least when you are doing it by hand — if you did add a cheating variable to the second constraint, it should still work (it will just take an extra turn of simplex). When you are using a computer, you can just add the cheating variable anyway because "Who cares?" The computer is doing all the simplex work anyway. However when you are doing it by hand, this kind of observation can sometimes make your computations shorter.

So let us try adding a cheating variable w to the first constraint and see what happens. We get

$$\begin{array}{rl}
\min & w \\
\text{subject to} & x+5y+3z-s \quad +w = 6 \\
& 3x+3y-z \quad -t \quad = -1 \\
& x, \ y, \ z, \ s, \ t, \ w \geq 0
\end{array}$$

and (as we suspected) there is an easy-to-find initial feasible solution

$$(x, y, z, s, t, w) = (0, 0, 0, 0, 1, 6)$$

corresponding to the basis $\{t, w\}$. Now when we run simplex, we should finish after one round with the optimal basis $\{x, t\}$, corresponding to solution

$$x = 6 \quad y = 0 \quad z = 0 \quad s = 0 \quad t = 19 \quad w = 0$$

and optimal cost 0. So this tells us two things: firstly, that (1) has a feasible solution and secondly, that the columns $\{x, t\}$ form a BFS.

Phase 2: Now if we run simplex on the original problem starting with the basis $\{x, t\}$ we should follow the path

$$\{x, t\} \to \{y, t\} \to \{y, z\}$$

and find that $\{y, z\}$ is optimal. So now we can compute the solution associated to this basis:

$$x = 0 \quad y = 3/14 \quad z = 23/14 \quad s = 0 \quad t = 0$$

which when we turn back into the original coordinates is

$$a = 0 \quad b = -3/14 \quad c = -23/14 \quad d = -6/7$$

and an optimal value of $-109/14$. Finally, we can see that the solution is correct by using the certificate[5]

$$\begin{bmatrix} 27/14 & 6/7 & 11/14 & 12/7 & 0 & 0 \end{bmatrix}$$

applied to the 6 rows of

$$\begin{array}{rrrrrrrrl}
\text{max} & 6a & + & 9b & + & 2c & + & 3d & \\
\text{subject to} & a & + & 3b & + & c & + & 2d & = -4 \\
 & & & b & + & c & - & d & \leq -1 \\
 & 3a & + & 3b & - & c & & & \leq 1 \\
 & a & & & & & & & \leq 0 \\
 & & & b & & & & & \leq 0 \\
 & & & & & c & & & \leq 0
\end{array}$$

gives the inequality

$$6a + 9b + 2c + 3d \leq -109/14$$

matching the optimal value.

[5] Note you were not expected to this — it is just to see that we are correct.