École Polytechnique Fédérale de Lausanne
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Math 261 – Discrete Optimization (Spring 2022)

Problem Set 4

Problem 1

The purpose of this problem is to prove Theorem 7 on the CheatSheet. Let

$$\mathcal{P} = \min \{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

be a linear program, and let β a feasible column basis for a BFS $\mathbf{x}$ with reduced costs $\overline{\mathbf{c}}$ and $\mathrm{col}_\beta(\mathbf{A}) = \mathbf{B}$.

(a) Show that $\overline{c}_i x_i = 0$ for all indices i.

(b) Show $\overline{\mathbf{c}} \cdot \mathbf{d} = \mathbf{c} \cdot \mathbf{d}$ for any feasible direction $\mathbf{d}$.

(c) Show that if $\overline{\mathbf{c}}$ is nonnegative, then $\mathbf{x}$ must be an optimal solution.

(d) Finish the proof of CheatSheet Theorem 7 by showing that if $\mathbf{x}$ is an optimal, nondegenenerate solution, then $\overline{\mathbf{c}} \geq 0$.

Problem 2

In this problem, we would like to show how to use linear programs to solve linear algebra problems[1]

(a) Let $\mathbf{a} \in \mathbb{R}^n$ and let $b \geq 0$, and let Q be the linear equation

$$\mathbf{a} \cdot \mathbf{x} = b$$

and $\mathcal{P}$ the linear program

$$\begin{array}{rr} \min & w \\ \text{s.t.} & \mathbf{a} \cdot \mathbf{x} + w = b \\ & w \geq 0 \end{array}$$

The book calls the new w variable an *artificial variable*, but I like to call it a *cheating variable*[2]. Show the following:

i. the point $(\mathbf{x}, w) = (\mathbf{0}, b)$ is a feasible solution for $\mathcal{P}$

ii. Q has a feasible solution if and only if the optimal value of $\mathcal{P}$ is 0.

iii. If $(\mathbf{y}, 0)$ is an optimal solution for $\mathcal{P}$ then $\mathbf{y}$ is a feasible solution to Q.

(b) Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Construct a linear program[3] $\mathcal{P}$ (using cheating variables) which satisfies the following:

i. it is easy to find a feasible point in $\mathcal{P}$

ii. $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution if and only if the optimal value of $\mathcal{P}$ is 0.

iii. If the optimal value of $\mathcal{P}$ is 0, then the optimal solution for $\mathcal{P}$ gives you a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$.

[1]This may seem counter-intuitive, but it brings about a useful idea.

[2]Because I feel like I am cheating by finding a way to get a solution to Q without ever needing to actually solve Q.

[3]Note: it does not have to be in equality standard form.

Problem 3

Let

$$\mathcal{P} = \min\{\mathbf{c}^\intercal \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq \mathbf{0}\} \tag{1}$$

where $\mathbf{A}$ has full row rank. An *auxiliary linear program* $\mathcal{P}'$ is a linear program whose sole purpose is to find a BFS for $\mathcal{P}$ (using simplex). In particular $\mathcal{P}'$ must have the properties:

- $\mathcal{P}'$ is in the same form as $\mathcal{P}$ (either equality standard or inequality standard form)
- $\mathcal{P}'$ has an obvious BFS (with an obvious column basis) for simplex to start at
- $\mathcal{P}'$ has optimal value 0 if and only if $\mathcal{P}$ has a feasible solution
- In the case that $\mathcal{P}'$ has optimal value 0, the optimal basis for $\mathcal{P}'$ can be used to construct a feasible basis for $\mathcal{P}$.

(a) Construct an auxiliary linear program for $\mathcal{P}$ and show that it has all of the necessary properties. Be sure to show how to construct a feasible *column basis* for $\mathcal{P}$, not just a feasible point. You may assume (in this part) that the optimal solution to your auxiliary linear program is not degenerate.

(b) Show how one can construct a feasible column basis for $\mathcal{P}$ in the case that the optimal solution to your auxiliary linear program is degenerate.

Problem 4

Solve the linear program

$$\begin{array}{rrcrcrcrcr}
\max & 6a & + & 9b & + & 2c & + & 3d & & \\
\text{s.t.} & a & + & 3b & + & c & + & 2d & = & -4 \\
 & & & b & + & c & - & d & \leq & -1 \\
 & 3a & + & 3b & - & c & & & \leq & 1 \\
 & a & & & & & & & \leq & 0 \\
 & & & b & & & & & \leq & 0 \\
 & & & & & c & & & \leq & 0
\end{array}$$

using the *two phase method.* That is,[4]

Phase 0: Put the problem in the form you want (simplify if you can!)

Phase 1: Find an initial BFS for the output of Phase 0 (or show that it is infeasible). Often this requires constructing an auxiliary linear program, but sometimes you can simply guess.

Phase 2: Run simplex method on the output of Phase 0 starting at BFS found in Phase 1.

[4] So technically what I am about to write has three phases, but the books calls it "The 2-phase method" because it assumes Phase 0 has already happened and the LP is in a nice form. Since that might not be the case, and this is often a very easy way to make your life easier, I thought it should be listed separately.