METRIC & TOPOLOGICAL SPACES

Exercise sheet 6: Connectedness II

Exercise 1 (Graphs of functions are connected). *Let (X, τ_X) and (Y, τ_Y) be two topological spaces such that (X, τ_X) is connected. Let $f : (X, \tau_X) \to (Y, \tau_Y)$ a continuous function. Consider $X \times Y$ with its product topology τ_Π. Then the set $G = \{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of the function f. Prove that G is connected with the subspace topology. What does this result tells us when we take say $X = (0, 1]$, $Y = \mathbb{R}$, both with the Euclidean topology? Where did we use this result in the course?*

Exercise 2. *Consider $\mathbb{R}^2$ with its standard topology. Prove that the subset $A = \mathbb{R}^2 \backslash S$, where S is a finite set is path-connected. Deduce the same result for $\mathbb{R}^n$ with $n \geq 3$. What happens if S is countable? Give an example of an uncountable S such that $\mathbb{R}^2 \backslash S$ is not connected.*

Exercise 3 ($\mathbb{R}$ vs $\mathbb{R}^n$). *Prove that if (X, τ_X) is path-connected and $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then $f(X)$ is also path-connected. Deduce that path-connectedness is a topological invariant: i.e. if (X, τ_X) is path-connected and $(X, \tau_X) \cong (Y, \tau_Y)$, then (Y, τ_Y) is also path-connected.*

Now, recall from the Exercise sheet 4 that if $f : (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism, it induces also a homeomorphism between $X \backslash \{x\}$ and $f(X \backslash \{x\})$ for any $x \in X$. Argue now that $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.

Exercise 4. *Let (X, τ_X), (Y, τ_Y), (Z, τ_Z) be topological spaces and suppose that $f : (X \times Y, \tau_{X \times Y}) \to (Z, \tau_Z)$ is continuous. Prove that $\forall x \in X$ the function $f_x(y) := f(x, y) : (Y, \tau_Y) \to (Z, \tau_Z)$ is continuous, as is $\forall y \in Y$ the function $f_y(x) := f(x, y) : (X, \tau_X) \to (Z, \tau_Z)$.*

Exercise 5. *Let (X, τ_X) be a topological space and suppose that $A \subseteq X$ is connected. Prove that $cl(A)$ is also connected. What about $int(A)$? Suppose now that $A \subset X$ is path-connected. Is $cl(A)$ necessarily path-connected? What about $int(A)$?*

Exercise 6 (Connected components I). *Prove that connected components are indeed connected, and that any connected component is necessarily closed. Show by example that connected components are not necessarily open. Can you find a criteria for all components to be open?*

Exercise 7 (Connected components II). *Let (X, τ_X) be a topological space. Prove that*

- *any two connected components of X are either disjoint, or coincide;*
- *connected components $(C_i)_{i \in I}$ form a partition of X, i.e. $X = \bigcup_{i \in I} C_i$;*
- *any homeomorphism between topological spaces (X, τ_X) and (Y, τ_Y) induces a bijection between the sets of connected components of X and Y.*

Exercise 8 (Locally path-connected + connected => path-connected). *We say that (X, τ_X) is locally path-connected at x if for any open set $U \subset X$ with $x \in U$, there is some open set $V \subseteq U$ with $x \in V$ such that V is path-connected. Find examples of spaces that are locally path-connected but not path-connected, and of spaces that are path-connected, but not locally path-connected. Prove that a space that is locally path-connected and connected is path-connected.*

0.1. $\star$ For fun (non-examinable) $\star$.

Exercise 9 ($\star$). *Let I be any index set and $((X_i, \tau_{X_i}))_{i \in I}$ a collection of connected topological spaces. Then $\Pi_{i \in I} X_i$ with the product topology is also connected.*

Exercise 10 ($\star$)**.** *A space where every point is a separate connected component is called totally disconnected. For example any topological space with the discrete topology is a totally disconnected space. Prove that the standard Cantor set from Exercise sheet 4 is a totally disconnected space.*

Exercise 11 ($\star$)**.** *A topological group (G, τ_G) is at the same time a topological space and a group such that the group operations (i.e. products and inverses) are continuous w.r.t. the underlying topology. Prove that any open subgroup of G is also closed. What about the converse? What does it say about connectedness of G?*

Exercise 12 ($\star$)**.** *Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ (equipped with the subspace topology). Let $x_0, y_0 \in U$. We wish to show that there exists a homeomorphism $f : U \to U$ such that $f(x_0) = y_0$. To this end, we introduce the following notation. For each $x, y \in U$, we write $x \sim y$ if there exists a homeomorphism $f : U \to U$ such that $f(x) = y$.*

- *Show that $\sim$ is an equivalence relation.*
- *For each $x \in U$, let O_x be the set of points $y \in U$ such that $x \sim y$. Show that O_x is open.*
- *Assume that U is connected. Deduce that for each $x \in U$, $O_x = U$ and conclude.*

[Hint: think of the strategy that we used to prove that connected open sets in $\mathbb{R}^n$ are path-connected]

Exercise 13 ($\star$)**.** *Let $n \in \mathbb{N}$, $n \geq 2$. Let $x_1, \ldots, x_m \in \mathbb{R}^n$ be m distinct points and let $y_1, \ldots, y_m \in \mathbb{R}^n$ be m distinct points. We want to show that there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that for each $i \in \{1, \ldots, m\}$, $f(x_i) = y_i$.*

- *Let $\Omega = \{(x_1, \ldots, x_m) \in (\mathbb{R}^n)^m : \forall i \neq j,\ x_i \neq x_j\}$. Show that Ω is connected. Is this still true if $n = 1$?*
- *Show that the proof and the conclusions of the previous exercise still hold if we define $\sim$ as: $x \sim y$ if and only if there exists a homeomorphism $f : \overline{U} \to \overline{U}$ such that for each $x \in \partial U$, $f(x) = x$.*
- *For each configuration $x = (x_1, \ldots, x_m) \in \Omega$, let O_x be the set of $y = (y_1, \ldots, y_m) \in \Omega$ such that there exists $f : \mathbb{R}^n \to \mathbb{R}^n$ a homeomorphism such that for each $i \in \{1, \ldots, m\}$, $f(x_i) = y_i$. Using the previous point, show that O_x is open.*
- *Reasoning as in exercise 10, deduce that $O_x = \Omega$.*

Exercise 14 ($\star$)**.** *Let $U \subset \mathbb{R}^2$ be a connected open subset. Let $f : U \to \mathbb{R}$ be continuous and **harmonic**, in the sense that for each $\bar{x}_0 = (x_0, y_0) \in U$ and each $r > 0$ such that $B(\bar{x}_0, r) \subset U$,*

$$f(\bar{x}_0) = \frac{1}{2\pi} \int_{B(\bar{x}_0, r)} f(\bar{x}) dx dy \,.$$

Show that if there exists $\bar{x}_1 \in U$ such that $\sup_U f = f(\bar{x}_1)$, then f is constant. [Hint : Show that $f^{-1}(\{\sup_U f\})$ is both open and closed.]