Homework 1

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Exercise 1

1

True.
Suppose X is connected, and suppose by contradiction that there exists $U \subset X$ s.t. $cl(U) = int(U)$.
Then $cl(U)$ is both open and closed, contradicting the fact that X is connected.
Now suppose that either $U = \emptyset, U = X$ or $cl(U) = int(U)$ but suppose that X is not connected.
Then there exist open sets V and U s.t. $V \cap U = \emptyset$ and $V \cup U = X$.
Then V is both closed and open and hence $cl(V) = int(V)$.

2

False, the empty set and X are closed.
However, if the union is non-trivial, the result is true since the closed sets will also be open (as they are complements of closed sets).

3

False, consider the topologists sine curve.

Exercise 2

1

Let (X, τ_X) be a topological space, X is Hausdorff if and only if for all points $x, y \in X, x \neq y$, there exist open sets $U_x \ni x, U_y \ni y$ such that $U_x \cap U_y = \emptyset$.

Hausdorff Topologies

Consider $\mathbb{Q}$ with the discrete topology, this is Hausdorff, since all singletons are contained in the discrete topology.

Consider $\mathbb{Q}$ with the induced Euclidean topology, this is Hausdorff, since $\forall x \neq y \in \mathbb{Q}, \left(B(x, \frac{|x-y|}{3}) \cap \mathbb{Q}\right) \cap (B(y, \frac{|x-y|}{3}) \cap \mathbb{Q}) = \emptyset$

Non-Hausdorff Topologies

Consider $\mathbb{Q}$ with the indiscrete topology, this is not Hausdorff since you trivially can't separate points.

Consider $\mathbb{Q}$ with the cofinite topology, this is not Hausdorff since for any two open sets U and V $\exists q \in \mathbb{Q}$ s.t. $\forall i > q, i \in U \cap V$.

2

Define the set

$$\tau^B = \left\{ \prod_i U_i | U_i \in \tau_{X_i} \right\}$$

We show this forms a basis.
First, it is clear that τ_B spans $\prod_i X_i$ since $\prod_i X_i \in \tau^B$.
Now let $V = \prod_i V_i \in \tau_B$ and $U = \prod_i U_i \in \tau_B$.
Suppose $V \cap U \neq \emptyset$, let $x = (x_1, \ldots, x_n) \in V \cap U$.
Note that $x_i \in V_i \cap U_i \in \tau_{X_i}$, hence $x \in \prod_i (V_i \cap U_i)$.

We now show that $\prod_i X_i$ is Hausdorff if and only if X_i is Hausdorff for all i.
First suppose X_i is Hausdorff for all i, let $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$ be two different elements in $\prod_i X_i$.
Hence, there exists at least one $j \in [i]$ such that $a_j \neq b_j$.
Since X_j is Hausdorff, there exist disjoint $A_j, B_j \in \tau_{X_j}$ such that $a_j \in A_j, b_j \in B_j$, considering $A = X_1 \times \ldots \times A_j \times X_{j+1} \times \ldots X_n$ and $B = X_1 \times \ldots \times B_j \times X_{j+1} \times \ldots X_n$, note that $a \in A, b \in B$ and $A \cap B = \emptyset$.

Now suppose that $\prod_i X_i$ is Hausdorff, without loss of generality, we will show that X_1 is Hausdorff.
Let $a, b \in X_1$, for $2 \leq i \leq n$ fix some $x_i \in X_i$, consider $(a, x_2, \ldots, x_n), (b, x_2, \ldots, x_n)$.
There exist open sets $\bigcup_j \prod_i U_{ji}, \bigcup_j \prod_i V_{ji}$ separating $(a, x_2, \ldots)$ and $(b, x_2, \ldots)$.
Note that $\forall i \geq 2$ $U_{ji} \cap V_{ji} = \emptyset$, hence forcing $U_{j1} \cap V_{j1} = \emptyset$.
However $a \in U_{j1}, b \in V_{j1}$, showing that X_1 is Hausdorff.

3

We'll denote the diagonal of X as ΔX.
First suppose that X is Hausdorff, we show that the complement is open. Let $(a, b) \in \Delta X^C$, by the Hausdorff property, there exist open sets $U \ni a, V \ni b$ separating a and b in X, hence $U \times V \ni (a, b)$.
Furthermore, since the intersection of U and V is empty, $U \times V \cap \Delta X^C = \emptyset$, showing that ΔX is closed.

Now suppose that ΔX is closed, we show that X is Hausdorff.
Let $a, b \in X$ be distinct points, consider $(a, b) \in \Delta X^C$, by hypothesis, there exists a open set $\bigcup_j U_j \times V_j$ [1] such that $\left(\bigcup_j U_j \times V_j\right) \cap \Delta X = \emptyset$.
Now there exists at least one i such that $(a, b) \in U_i \times V_i$, note that $U_i \cap V_i = \emptyset$, since if not, the intersection with the diagonal would be non-empty.
Hence, U_i and V_i separate a and b in X.

1. Here j runs over some index set and U_j resp. V_j are open sets in X