PROBA

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List of Theorems

Lecture 1: Introduction

Wed 22 Sep

1 Some historical models

1.1 Laplace Model

Definition 1 (Laplace Model)
Ω finite set, $|\Omega| = n$ is the set of outcomes.
We can observe whether $E \subset \Omega$ happens, and we define it's probability

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Question

Why should this have any meaning/content ?

Proposition 1
Consider laplace model for n coint tosses $\Rightarrow$ every sequence has probability 2^{-n}
Denote by H_n the number of heads in n tosses

$$\mathbb{P}(|\frac{H_n}{n} - \frac{1}{2}| > \epsilon) \to 0$$

More generally

Proposition 2
If you have a laplace modelfor some event E, and look at n repetitions, then

$$\forall \epsilon > 0 \mathbb{P}(|\frac{E_n}{n} - \mathbb{P}(E)| > \epsilon) \to 0$$

Limitations of Laplace Model

— All outcomes have equal probability ?
— Need $|\Omega| < \infty$, so what about infinite sets ?

What next ?

Definition 2 (Intermediate model)
Let Ω to be any set and $P : \Omega \to [0, 1]$, s.t. $\sum_{\omega \in \Omega} p(\omega) = 1$

Event : $E \subset \Omega$ and

$$\mathbb{P}(E) := \sum_{\omega \in E} p(\omega)$$

— More freedom
— If you take Ω finite, $p(\omega) = \frac{1}{|\Omega|} \Rightarrow$ Laplace model
— Price ? How to choose $p : \Omega \to [0, 1] \to$ collect data, do statistics
— keeps many nice properties

— For contable sets, this is equivalent to the standard model.
— For uncountable Ω ?
— Problem 1 : There is no function s.t.

$$p(\omega) > 0 \forall \omega \in \Omega \text{ and } \sum p(\omega) = 1$$

This intermediate model is in essence only for countable sets.

What about uncountable sets ?

— What about a random point int $[0,1]$ or $[0,1]^n$?

Intuitively, consider $[0,1]$, then we can set

$$\mathbb{P}(A) = \text{length}(A)$$

Definition 3 (Geometric probability)
Take $f : \mathbb{R} \to (0, \infty)$ to be a riemann-integrable function with total mass 1.
For any $A \subset \mathbb{R}$,s.t. 1_A riemann-integrable, we set $\mathbb{P}(A) = \int_A f(x)dx$

— In general quite <u>ok</u>
BUT

— You would expect there is one framework for uncountable and countable sets.
— What about more complicated spaces (eg. space of continuous functions)
— $\mathbb{P}(\mathbb{Q})$ is undefined

2 Basic Formalism

2.1 Measure spaces : A notion of area

— Set + structure
— General setting to talk about area

Definition 4 (Measure space)
$(\Omega, \mathcal{F}, \mu)$ is called a measure space if :
— *Ω is some set*
— *$\mathcal{F} \subset P(\Omega)$ called a σ-algebra*
  — $\emptyset \in \mathcal{F}$
  — $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$
  — *$F_1, F_2, \ldots, \in \mathcal{F}$, then $\bigcup_{i \geq 1} F_i \in \mathcal{F}$ each F is called a measurable set.*
— *$\mu : \mathcal{F} \to [0, \infty)$ called the measure*
  — $\mu(\emptyset) = 0$

— *If $F_1, \ldots,$ are disjoints sets of the σ-algebra, then*

$$\mu(\bigcup_{i\geq 1} F_i) = \sum_{i\geq 1} \mu(F_i)$$

— *Defined by Borel 1898 and Lebesgue 1901-1903*

2.2 Probability spaces

Given by Kolmogorov in 1933

Definition 5 (Probability space)
A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space if it is a measure space and $\mathbb{P}(\Omega) = 1$

Interpretation

— Ω state space/universe
— $\mathcal{F}$ is the set of events you can observe/have access to
— $\mathbb{P}(E)$ is the probability of E

Lemme 3
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space
— $\Omega \in \mathcal{F}$
— $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \setminus F_2 \in \mathcal{F}$
— $F_1, \ldots \in \mathcal{F} \Rightarrow \bigcap F_i \in \mathcal{F}$
— $F_1, F_2, \ldots \in \mathcal{F} \Rightarrow \bigcap_{i\geq 1} F_i$

Let us compare this definition with the prior ones
— Ω finite set, $\mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(F) = \frac{|F|}{|\Omega|}$ this is a probability space and a laplace model.
— For Ω countable, $\mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$
— The really new part is $\mathcal{F}$ which restricts the sets we can measure

Lecture 2: ...

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2.3 Basic properties

— $F_1, F_2, \ldots, \in \mathcal{F}$ disjoint

$$\mu(F_i) = \sum \mu(F_i)$$

— $F_1 \subset F_2 \in \mathcal{F}$ $\mu(F_1) \leq \mu(F_2)$
— $F_1 \subset F_2 \ldots \in \mathcal{F}$

$$\mu(F_n) \to \mu(\bigcup F_i)$$

— $F_1, F_2, \ldots, \mathcal{F}$

$$\mu(\bigcup F_i) \leq \sum \mu(F_i)$$

In addition, in probability spaces

— $\mathcal{P}(F^c) = 1 - \mathcal{P}(F)$

— $F_1 \supset F_2 \supset \ldots \Rightarrow \mathcal{P}(F_n) \to \mathcal{P}(\bigcap F_i)$

2.4 Measurable and measure preserving maps

Definition 6

Let $(\Omega_1, \mathcal{F}_1, \mu_1), (\Omega_2, \mathcal{F}_2, \mu_2)$ *two measure spaces.*
$f : \Omega_1 \to \Omega_2$ *is called measurable if for every* $F \in \mathcal{F}_2$, $f^{-1}(F) \in \mathcal{F}_1$
A measurable function $f : (\Omega_1, \mathcal{F}_1) \to (\Omega_2, \mathcal{F}_2)$ *is called measure preserving if* $\forall F \in \mathcal{F}_2$ $\mu_1(f^{-1}(F)) = \mu_2(F)$.

Lemme 4 (Push-Forward measure)
Let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), (\Omega_2, \mathcal{F}_2)$ *be two measure spaces, and* f *measurable, then* $\mathbb{P}_2(F) = \mathbb{P}_1(f^{-1}(F))$ *is a probability measure.*