

# Algebraic Curves

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## Lecture 1: Introduction

Fri 25 Feb

Let  $K$  be a field, given a set of polynomials  $S = \{f_1, \dots\}$ , we can consider  $V(S) = \{(x_1, \dots) \in K^n \mid f_i(x_1, \dots) = 0 \forall i\}$ .

Notice that if  $a_1, \dots \in K[x_1, \dots]$  then also  $\sum_i a_i(x) f_i(x) = 0$  only depends on the ideal generated by  $S$ .

If  $I(S)$  happens to be prime, we call  $V$  an algebraic variety.

## 1 Affine algebraic sets

### 1.1 Recollection on commutative algebra

All rings are commutative and with unit.

Let  $R$  be a ring.

- $R$  is an integral domain, or just domain if there are no zero divisors, ie,  $\forall a, b \in R$  s.t.

$$a \cdot b = 0 \implies a = 0 \text{ or } b = 0$$

- Any domain can be embedded into its quotient ring.
- A proper ideal  $I$  is maximal if it's not contained in any other proper ideal
- A proper ideal  $I$  is prime if

$$\forall a, b \in R, ab \in I \implies a \in I \text{ or } b \in I$$

- A proper ideal  $I$  is radical if

$$a^n \in I \implies a \in I$$

- For any ideal  $I \subset R$ , the radical  $\sqrt{I}$  is the smallest radical ideal containing  $I$

#### Lemme 1

- $I \subset R$  is maximal  $\iff R/I$  is a field

#### Lemme 2

- $I \subset R$  is prime  $\iff R/I$  is a domain

#### Lemme 3

- radical  $\iff R/I$  has no nilpotent elements.

Given a subset  $S \subset R$  we can consider the ideal generated by  $S$

$$I(S) = \left\{ \sum_i a_i s_i \right\}$$

$I$  is finitely generated if  $I = I(S)$  with  $S$  finite.

- We say that  $R$  is Noetherian  $\iff$   $\nexists$  a chain of strictly increasing ideals. Equivalently, every ideal is finitely generated.

**Theorème 4**

- *In fact, hilbert's basis theorem says that, if  $R$  is Noetherian, then  $R[x]$  is noetherian.*

In particular  $K[x_1, \dots, x_n]$  is Noetherian

- $I$  is in principal if it is generated by one element.
- A domain is called a principal ideal domain (PID) if every ideal is principal.
- $a \in R$  is irreducible if  $a$  is not a unit, nor zero and if

$$a = b.c$$

then either  $b$  or  $c$  are units.

- A pid  $(a) \subset R$  is prime  $\iff a$  is irreducible.
- $R$  is a UFD if  $R$  is a domain and elements in  $R$  can be factored uniquely up to units and reordering into irreducible elements.

**Theorème 5**

$R$  is a UFD  $\implies R[x]$  is a UFD

And, if  $R$  is a PID, then  $R$  is a UFD

**Theorème 6 (Gauss Lemmma)**

- *$R$  is a UFD and  $a \in R[X]$  irreducible, then also  $a \in Q(R)[X]$  is irreducible.*

- Localization

Let  $R$  be a domain, if  $S \subset R$  is a multiplicative subset, then the localization of  $R$  at  $S$  is defined as

$$S^{-1}R = \left\{ x \in Q(R) \mid x = \frac{a}{b}, b \in S \right\}$$

If  $M$  is an  $R$ -module, we have similarly

$$S^{-1}M = \left\{ \frac{m}{s} \mid m \in M, s \in S \right\} / \left\{ \frac{m}{s} = \frac{m'}{s'} \iff ms' = sm' \right\}$$

If  $p \subset R$  is a prime ideal, then its complement is a multiplicative subset and we define

$$R_p = (R \setminus p)^{-1}R$$

- There is a 1-1 correspondence between  $p \subset R$  prime and ideals of  $R_p$ , furthermore  $R_p$  is a local ring
- Localization is exact, in particular, given  $I \subset p$  the short exact sequence

$$0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$$

gets sent to

$$0 \rightarrow I_p \rightarrow R_p \rightarrow (R/I)_p \rightarrow 0$$

ie. localization commutes with taking quotients.

## 1.2 Polynomial rings

For  $a \in \mathbb{N}^n$ , we set

$$X^a = X_1^{a_1} \dots \in k[X_1, \dots]$$

Thus for any  $F \in k[X_1, \dots, X_n]$ , we can write it as

$$F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$$

$F$  is homogeneous or a form of degree  $d$  if the coefficients  $\lambda_a = 0$  unless  $a_1 + \dots + a_n = d$ .

Any  $F$  can be written uniquely as  $F = F_0 + \dots + F_d$  where  $F_i$  is a form of degree  $i$ .

The derivative of  $F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$  with respect to  $X_i$  is  $F_{X_i} = \frac{\partial F}{\partial X_i}$ .

If  $F$  is a form of degree  $d$  we have

**Théorème 7 (Euler's theorem)**

$$\sum_{i=1}^n \frac{\partial F}{\partial X_i} X_i = dF$$

## Lecture 2: Affine space and algebraic sets

Wed 02 Mar

### 1.3 Affine spaces and algebraic sets

Let  $k$  be a field.

**Definition 1**

For every  $n \geq 0$  the affine  $n$ -space  $\mathbb{A}_k^n$  is the set  $k^n$ .

In particular  $\mathbb{A}^0$  is a point,  $\mathbb{A}^1$  is a line,  $\mathbb{A}^2$  the affine plane.  
 Given a subset  $S \subset k[X_1, \dots, X_n]$  of polynomials, we set

$$V(S) = \{x = (x_1, \dots, x_n) \in \mathbb{A}^n \mid f(x_1, \dots, x_n) = 0 \forall f \in S\}$$

If  $S$  is finite, we write  $V(f_1, \dots, f_k)$  for  $V(S)$ .

If the set  $S$  is a singleton, then we call  $V(S)$  a hyperplane.

Any subset of  $\mathbb{A}^n$  is algebraic if  $V = V(S)$  for some subset of polynomials.

### Lemme 8

- Let  $S \subset k[X_1, \dots, X_n]$  and  $I$  the ideal generated by  $S$ , then  $V(S) = V(I)$ .
- Let  $\{I_\alpha\}$  be a collection of ideals, then

$$V\left(\bigcup_{\alpha} I_{\alpha}\right) = \bigcap_{\alpha} V(I_{\alpha})$$

- If  $I \subset J$  then  $V(J) \subset V(I)$
- For polynomials  $f, g \in k[x_1, \dots, x_n]$ , then  $V(f) \cup V(g) = V(f \cdot g)$   
 For ideals  $I, J$  ideals, then  $V(I) \cup V(J) = V(I \cdot J)$  where  $IJ = \{fg \mid f \in I, g \in J\}$
- For  $a = (a_1, \dots, a_n) \in \mathbb{A}^n$ ,  $v(\{x_1 - a_1, \dots\}) = \{a\}$

### Preuve

1. Let  $h \in \sum_i f_i g_i \in I$  with  $f_i \in S$  and  $x \in V(S)$ , then  $f_i(x) = 0 \forall i$  hence  $h(x) = 0 \implies x \in V(I) \implies V(S) \subset V(I)$ .  
 Furthermore, if  $x \in V(I)$ , then in particular  $f(x) = 0 \forall f \in S \subset I$ , hence  $x \in V(S)$  and  $V(S) \supset V(I)$
2. Let  $x \in V(\bigcup_{\alpha} I_{\alpha})$ , then for any  $\alpha$  and  $f \in I_{\alpha}$ , we must have  $f(x) = 0$ , hence  $x \in V(I_{\alpha}) \implies x \in \bigcap_{\alpha} V(I_{\alpha})$ .  
 Conversely, if  $x \in \bigcap_{\alpha} V(I_{\alpha})$  and  $f \in \bigcup_{\alpha} I_{\alpha}$ , then  $f \in I_{\alpha}$  for some  $\alpha$ , then  $f(x) = 0$  hence  $x \in V(\bigcup_{\alpha} I_{\alpha})$   $\square$

By Hilbert's basis theorem  $k[x_1, \dots, x_n]$  is Noetherian hence every ideal is finitely generated.

### Corollaire 9

Every algebraic set  $V \subset \mathbb{A}^n$  is of the form

$$V = V(f_1, \dots, f_k) = V(f_1) \cap \dots \cap V(f_k)$$

## 1.4 Ideals of a set of points and the nullstellensatz

Using the previous section, we have a map

$$V : \{ \text{Ideals in } k[X_1, \dots, X_N] \} \mapsto \{ \text{algebraic sets in } \mathbb{A}^n \}$$

Conversely, for any subset  $X \subset \mathbb{A}^n$  we define

$$I(X) := \{ f \in k[X_1, \dots, X_N] \mid f(x) = 0 \forall x \in X \} \subset k[X_1, \dots, X_N]$$

### Lemme 10

1. If  $X \subset Y$  then  $I(X) \supset I(Y)$
2. For  $J \subset k[X_1, \dots, X_N]$  an ideal  $I(V(J)) \supset J$
3. For  $W \subset \mathbb{A}^n$  algebraic,  $V(I(W)) = W$

### Preuve

1. Let  $f \in I(Y)$ , then  $f$  vanishes on  $X$  and hence  $f \in I(X)$
2.  $I(V(J)) = \{ f \in k[x_1, \dots, x_n] \mid f(x) = 0 \forall x \in V(J) \} \supset J$
3. By definition  $V(I(X)) \supset X$  for any  $X$ .  
If in addition, if  $X = V(J)$  algebraic, then  $V(I(X)) = V(I(V(J))) \subset V(J) = X$   $\square$

There are essentially two reasons why  $I(V(J)) \supsetneq J$  in general

1.  $J = (x^n) \subset k[x] \implies V(x^n) = \{0\}$  and  $I(\{0\}) = (x)$
2.  $(x^2 + 1) \subset \mathbb{R}[x]$  and  $I(\emptyset) = \mathbb{R}[X]$

### Lemme 11

For any  $X \subset \mathbb{A}^n$ ,  $I(X)$  is a radical ideal

### Preuve

If  $f^n \in I(X)$  for some  $n$ , then  $f(x)^n = 0$  and hence  $f(x) = 0$   $\square$

So the first phenomenon is related to the fact that  $J$  is not radical, the second is related to the fact that  $\mathbb{R}$  is not algebraically closed.

### Theorème 12 (Hilbert's Nullstellensatz)

Let  $K$  be algebraically closed,  $J \subset k[X_1, \dots, X_n]$ , then

$$I(V(J)) = \sqrt{J}$$

Using this, there is a one to one correspondence

$$\{ \text{radical ideals in } k[X_1, \dots, X_n] \} \leftrightarrow \{ \text{algebraic subsets of } \mathbb{A}^n \}$$



**Theorème 13 (Weak Nullstellensatz)**

Let  $K$  be algebraically closed, every maximal ideal  $I \subset K[X_1, \dots, X_n]$  is of the form  $I = \{x_1 - a_1, \dots, x_n - a_n\}$  with  $a = (a_i) \in \mathbb{A}^n$

**Corollaire 14**

Let  $I \subset K[X_1, \dots, X_n]$  be any ideal, then  $V(I)$  is a finite set  $\iff K[X_1, \dots, X_n]/I$  is a finite dimensional  $K$ -vector space.

In this case

$$|V(I)| \leq \dim_k K[X_1, \dots, X_n]/I$$

**Preuve**

Let  $I \subset K[X_1, \dots, X_n]$  be any ideal and  $P_1, \dots, P_n \subset V(I)$  distinct.

We can choose (Exercise)  $F_1, \dots, F_r \in K[X_1, \dots, X_n]$  s.t.  $F_i(P_j) = \delta_{ij}$ , then we write  $f_1, \dots, f_r$  for the residues of  $F_1, \dots, F_r$  in  $K[X_1, \dots, X_n]/I$ .

We claim  $f_1, \dots, f_r$  are linearly independent.

Indeed suppose  $\sum_i \lambda_i f_i = 0$ , this implies  $\sum_i \lambda_i F_i \in I$  hence  $0 = \sum \lambda_i F_i(P_j)$  which implies  $\lambda_j = 0$ , hence the  $f_i$  are linearly independent.

It follows that  $\dim_k K[X_1, \dots, X_n]/I < \infty \implies |V(I)| < \infty$  and in this case  $\dim_k K[X_1, \dots, X_n]/I \geq |V(I)|$ .

Now assume  $V(I)$  is a finite set  $\{P_1, \dots, P_r\} \subset \mathbb{A}^n$  and write  $P_i = (a_{i1}, \dots, a_{in})$  and define  $F_j = \prod_{i=1}^r (X_j - a_{ij})$ .

By construction  $F_j \in I(V(I)) = \sqrt{I}$

$\exists N > 0$  such that  $F_j^N \in I$ .

Hence  $f_j^N = 0$  in  $K[X_1, \dots, X_n]/I$ , but  $f_j^N = (x_j^{Nr}) + \text{lower order terms}$ .

This means that  $x_j^{Nr}$  is a  $K$ -linear combination of  $\{1, \dots, x_j^{Nr-1}\}$ .

This means that  $x_j^s$  is a linear combination for any  $s > 0$ .

Hence taking products for different  $j$ 's, we see that the set  $\{x_1^{m_1}, \dots, x_n^{m_n}\}$  generates  $K[X_1, \dots, X_n]/I$   $\square$

Due to these theorems, we'll always suppose  $K$  is algebraically closed.

**Lecture 3: Irreducible sets**

Fri 11 Mar

**1.5 Irreducible sets****Definition 2 (Irreducible set)**

An algebraic set  $V \subset \mathbb{A}^n$  is irreducible if  $\forall W_1, W_2 \subset \mathbb{A}^n$  algebraic s.t.  $V = W_1 \cup W_2$ , then either  $W_1 = V$  or  $W_2 = V$

**Example**

— Let  $V = \{x_1, \dots, x_n\} \subset \mathbb{A}^n$  is irreducible iff  $n = 1$

- Let  $f(X, Y) = Y(X^2 - Y)$ ,  $V = V(f) \subset \mathbb{A}^2$  is not irreducible by taking  $W_1 = V(Y)$ ,  $W_2 = V(X^2 - Y)$

**Proposition 16**

An algebraic set  $V$  is irreducible iff  $I(V)$  is prime.

**Preuve**

If  $I(V)$  is not prime, let  $F_1, F_2 \notin I(V)$  s.t.  $F_1, F_2 \in I(V)$ , then we can write  $V = (V \cap V(F_1)) \cup (V \cap V(F_2))$ .

Conversely, if  $V = W_1 \cup W_2$  and  $W_i \neq V$ , then  $I(W_i) \subsetneq I(V)$ , pick  $F_i \in I(W_i) \setminus I(V)$ , then  $F_1 F_2 \in I(W_1) \cap I(W_2) = I(V)$ .  $\square$

If  $V \subset \mathbb{A}^n$  is irreducible, we can decompose it into a union of irreducible sets. The union is always finite as the polynomial ring is noetherian.

**Theorème 17 (Theorem name)**

Every  $V \subset \mathbb{A}^n$  algebraic can be written uniquely (up to ordering) as a union of irreducible sets.

$$V = V_1 \cup \dots \cup V_k$$

where the  $V_i$ 's are irreducible and  $V_i \not\subset V_j \forall i \neq j$

**Definition 3 (Irreducible Components)**

The  $V_1 \dots V_k$  are irreducible components of  $V$ .

**Remarque**

Applying  $I$  in theorem 1.9, we get

$$I(V) = I(V_1) \cap \dots \cap I(V_k)$$

and  $I(V_i)$  is the primary decomposition of  $I(V)$

In general, it is quite difficult to find this decomposition.

For hypersurfaces, it's easy, for  $I(F)$ , write  $F = F_1^{\alpha_1} \cdot \dots \cdot F_k^{\alpha_k}$ , then  $V(F) = V(F_1) \cup \dots \cup V(F_k)$ .

## 1.6 Algebraic subsets of $\mathbb{A}^2$

**Lemme 19**

Let  $F, G \in k[X, Y]$  with no common factors, then  $V(F) \cap V(G)$  is a finite set of points.

**Preuve**

By Gauss's lemma,  $F, G$  have no common factors in  $k(X)[Y]$ . Since  $k(X)[Y]$  is a PID  $\exists A, B \in k(X)$  such that

$$AF + BG = 1$$

Now there exists  $C \in k[X]$  such that  $AC, BC \in k[X]$ .

Let  $(x, y) \in V(F, G)$ , then  $C(x) = 0$  and hence there are only finitely many  $x$ 's possible.

By symmetry, the same is true for the  $Y$  coordinate, hence  $|V(F, G)| < \infty$   $\square$

Using this, we can now classify all algebraic subsets of  $\mathbb{A}^2$ .

**Corollaire 20**

The irreducible algebraic subsets of  $\mathbb{A}^2$  are  $\mathbb{A}^2, V(F)$  with  $F$  irreducible or singletons.

## 2 Affine algebraic varieties

**Definition 4 (Affine algebraic variety)**

An affine algebraic variety is an irreducible affine algebraic set.

### 2.1 Zariski topology

**Definition 5 (Zariski topology)**

The Zariski-topology on  $\mathbb{A}^n$  is the topology whose open sets are complements of algebraic sets.

**Lemme 21**

This indeed defines a topology on  $\mathbb{A}^n$

**Preuve**

Certainly  $\emptyset, \mathbb{A}^n$  are algebraic, hence their complements are open.

Let  $\{U_i\}$  be a family of open sets, ie. such that

$$U_i = \mathbb{A}^n \setminus V(I_i)$$

Then

$$\bigcup U_i = \bigcup \mathbb{A}^n \setminus V(I_i) = \mathbb{A}^n \setminus \bigcap_i V(I_i) = \mathbb{A}^n \setminus V\left(\bigcup_i I_i\right)$$

Similarly, if  $U_1, U_2$  are open, then

$$U_1 \cap U_2 = \mathbb{A}^n \setminus V(I_1 \cup I_2)$$

$\square$

| is again open.

### Example

If  $n = 1$ , then algebraically closed sets are either  $\mathbb{A}^1$ ,  $\emptyset$  or finite union of points so the Zariski topology is the cofinite topology. Hence the open sets are huge.

#### Definition 6

For  $V \subset \mathbb{A}^n$  an algebraic variety or set, the Zariski topology on  $V$  is just the subspace topology.

#### Definition 7 (New definition of irreducibility)

A non-empty subset  $V$  of a topological space  $X$  is irreducible if it cannot be expressed as  $V = W_1 \cup W_2$  where  $W_1, W_2 \subsetneq V$  are closed subsets.

#### Lemme 23

A non-empty open subset of an irreducible topological space is again irreducible and dense.

Furthermore, if  $V \subset X$  is irreducible, then so is  $\overline{V}$

The proof is an exercise.

#### Definition 8 (Quasi-affine algebraic variety)

A quasi-affine variety is an open subset of an affine variety.

### Remarque

By the lemma above, quasi-affine variety are also irreducible.

## 2.2 Regular functions and coordinate rings

Regular functions are the natural "continuous" functions on algebraic varieties.

### 2.2.1 Affine case

#### Definition 9

Let  $V \subset \mathbb{A}^n$  be an affine algebraic variety.

A map

$$f : V \rightarrow K = \mathbb{A}^1$$

is regular if  $\exists F \in k[X_1, \dots, X_n]$  such that

$$f(X) = F(X) \forall X \in V$$

The set  $\Gamma(V)$  of regular functions on  $V$  is a ring with the usual pointwise multiplication and addition. and is called the coordinate ring of  $V$ .

**Lemme 25**

If  $I = I(V)$  for some prime, then

$$\Gamma(V) \simeq k[X_1, \dots, X_n] / I(V)$$

In particular,  $\Gamma(V)$  is a domain.

**Preuve**

By definition, we have a surjective morphism

$$k[X_1, \dots, X_n] \rightarrow \Gamma(V)$$

Now note that  $F \in \ker \phi \iff F(X) = 0 \forall x \in V \iff F \in I(V)$  □

**Definition 10 (Subobjects)**

An affine subvariety of  $V$  is an affine variety contained in  $V$ .

**Lemme 26**

There is a one-to-one correspondence between  $V$  and  $\Gamma(V)$  where

$$\begin{aligned} \{ \text{algebraic subsets of } V \} &\leftrightarrow \{ \text{radical ideals of } \Gamma(V) \} \\ \{ \text{algebraic subvarieties of } V \} &\leftrightarrow \{ \text{prime ideals of } \Gamma(V) \} \\ \{ \text{points of } V \} &\leftrightarrow \{ \text{maximal ideals of } \Gamma(V) \} \end{aligned}$$

The proof is again an exercise.

**Definition 11 (Morphism)**

A morphism  $\phi : V \rightarrow W$  between affine algebraic varieties  $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$  is a map such that  $\exists$  polynomials  $T_1, \dots, T_m \in k[X_1, \dots, X_n]$  such that

$$\phi(X) = (T_1(X), \dots, T_m(X))$$

Then  $\phi$  is an isomorphism if there exists a morphism  $\psi$  such that  $\phi \circ \psi = \text{Id}$  and  $\psi \circ \phi = \text{Id}$ .

**Example**

Take  $V(X^2 - Y) \subset \mathbb{A}^2$  the the projection  $p : V(X^2 - Y) \rightarrow \mathbb{A}^1$  on the first

coordinate is an isomorphism with inverse  $\psi(X) = (X, X^2)$ .

A non-example of a bijective map which is not an isomorphism :

$\phi : \mathbb{A}^1 \rightarrow V(Y^2 - X^3), \phi(t) = (t^2, t^3)$ .

One can check that  $\phi$  is bijective but not an isomorphism.

## Lecture 4: Morphisms of Affine Varieties

Fri 18 Mar

In general any morphism  $\phi : V \rightarrow W$  induces a morphism of rings ( of  $k$ -algebras)  $\tilde{\phi} : \Gamma(W) \rightarrow \Gamma(V)$  by composition, ie.

$$\tilde{\phi}(f) = f \circ \phi$$

### Proposition 28

This defines a one to one correspondence

$$\{ \text{Morphisms } \phi : V \rightarrow W \} \leftrightarrow \{ k\text{-algebra homomorphisms } \tilde{\phi} : \Gamma(W) \rightarrow \Gamma(V) \}$$

In particular  $\phi$  is an isomorphism iff  $\tilde{\phi}$  is an isomorphism.

#### Preuve

Need to construct for any  $\alpha : \Gamma(W) \rightarrow \Gamma(V)$  a morphism  $\bar{\alpha} : V \rightarrow W$  s.t.

$$\tilde{\bar{\alpha}} = \alpha$$

Suppose  $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$  and write

$$\Gamma(V) = k[x_1, \dots, x_n] / I(V) \text{ and } \Gamma(W) = k[y_1, \dots, y_m] / I(W)$$

Choose lifts  $T_i$  of  $\alpha([Y_i])$  in  $k[x_1, \dots, x_n]$ .

In particular  $\forall f \in \Gamma(W)$  and  $F$  a lift, then

$$\alpha(f) = F(T_1, \dots, T_m) \mod I(V)$$

Then define  $T : \mathbb{A}^n \rightarrow \mathbb{A}^m : x \mapsto (T_1(x) \dots T_m(x))$ .

We claim that  $T(V) \subset W$ .

From the diagram, we see that for any  $G \in I(W)$ ,  $G(T_1, \dots, T_m) \in I(V)$ , hence for any  $v \in V$ ,  $0 = G(T_1, \dots, T_m)(v) = G(T(v))$  which means that  $T(v) \in W$ .

Now

$$\tilde{\bar{\alpha}} : \Gamma(W) \rightarrow \Gamma(V)$$

satisfies  $\forall v \in V \forall f \in \Gamma(W)$

$$\tilde{\bar{\alpha}}(v) = f(\bar{\alpha}(v)) = f(T(v)) = \alpha(f(v)) \implies \tilde{\bar{\alpha}} = \alpha \quad \square$$

**Definition 12**

The quotient field  $K(V)$  of  $\Gamma(V)$  is called the field of rational function on  $V$ .

Let  $f \in K(V)$  is defined at a point  $p \in V$  if we can write  $f$  as the quotient  $f = \frac{a}{b}$  and  $b(p) \neq 0$ .

The pole set of  $f \in K(V)$  is the set of points where  $f$  is not defined.

**Remarque**

$\Gamma(V)$  is not a UFD in general, and so the presentation  $f = \frac{a}{b}$  is not unique.

**Example**

$V = (xy - zw) \subset \mathbb{A}^4$  and let  $\bar{x}, \bar{y}, \bar{z}, \bar{w} \in \Gamma(V)$  be the respective images.

Then  $f = \frac{\bar{x}}{\bar{y}} = \frac{\bar{z}}{\bar{w}}$ .

Hence  $f$  is defined whenever  $Y \neq 0$  or  $w \neq 0$

Hence the pole set of  $f$  is  $\{Y = 0\} \cap \{W = 0\}$

**Definition 13 (Local Ring)**

The local ring of  $V$  at a point  $p \in V$  is a subring  $\mathcal{O}_p(V)$  defined by

$$\mathcal{O}_p(V) = \{f \in K(V) | f \text{ defined at } p\}$$

We have natural inclusions  $\Gamma(V) \subset \mathcal{O}_p(V) \subset K(V)$

**Remarque**

$\Gamma(V), \mathcal{O}_p(V)$  and  $K(V)$  are intrinsic to  $V$ , ie. if  $V \simeq W$  then  $\Gamma(V) \simeq \Gamma(W)$  and  $\mathcal{O}_p(V) \simeq \mathcal{O}_{p'}(W)$

**Proposition 32**

Let  $p \in V$  and  $m_p \subset \Gamma(V)$  be the corresponding maximal ideal, then

$$\mathcal{O}_p(V) \simeq \Gamma(V)_{m_p}$$

In particular  $\mathcal{O}_p(V)$  is a noetherian local domain and we have that

$$\Gamma(V) = \bigcap_{p \in V} \mathcal{O}_p(V) \subset K(V)$$

**Preuve**

Recall that  $m_p = \{f \in \Gamma(V) | f(p) = 0\}$ , then

$$\begin{aligned} \Gamma(V)_{m_p} &= \left\{ f \in K(V) | f = \frac{a}{b}, b \notin m_p \right\} \\ &= \mathcal{O}_p(V) \end{aligned}$$

The rest follows from standard properties of localization.

In particular for any domain  $R$  we have that

$$R = \bigcap_{m \in R, m \text{ maximal}} R_m$$

□

Notice that the notions of regular functions is sufficient to define morphisms of local rings etc.

How can we extend this to quasi-affine varieties?

### Example

Consider  $V(XY - 1) \subset \mathbb{A}^2$ .

There is a natural projection  $\phi : V(XY - 1) \rightarrow x \in \mathbb{A}^1$ .

The image of  $\phi$  is  $\mathbb{A}^1 \setminus \{0\}$  quasi-affine and we'd like  $\phi$  to be an isomorphism, ie.

$$\phi^{-1}(x) = (x, \frac{1}{x})$$

I.e. the map  $x \rightarrow \frac{1}{x}$  should be a regular function on  $\mathbb{A}^1 \setminus \{0\}$ .

### Definition 14

Let  $V \subset \mathbb{A}^n$  be quasi-affine.

A map  $f : V \rightarrow \mathbb{A}^1 = k$  is called regular if  $\forall v \in V$  there exists an open neighbourhood  $v \in U \subset V$  and  $g, h \in k[x_1, \dots, x_n]$  s.t.  $h(V) \neq 0 \forall x \in U$  and  $f(x) = \frac{g(x)}{h(x)}$

Why do we need the  $U$ ?

### Example

Consider again  $V = V(XY - ZW) \setminus V(Y, W)$  and consider  $f = \frac{x}{w} = \frac{z}{y}$  on  $V$ .

None of the two presentations works on  $V$

### Definition 15

Let  $\mathcal{O}(V)$  be the ring of regular functions on  $V$

### Remarque

$f : V \setminus \{0\} \rightarrow \mathbb{A}^1 : x \mapsto \frac{1}{x}$  is regular.

Then we may take  $U = V$ , it is not hard to see that

$$\mathcal{O}(V) = k[x] \left[ \frac{1}{x}, \frac{1}{x^2}, \dots \right]$$

In particular  $\mathcal{O}(V) \supsetneq \Gamma(\mathbb{A}^1)$

If  $V \subset \mathbb{A}^n$  is affine, then we have  $k[x_1, \dots, x_n] \rightarrow \mathcal{O}(V) : F \mapsto (v \mapsto F(v))$ .

### Proposition 36

For  $V$  affine, we have that  $\Gamma(V) \simeq \mathcal{O}(V)$ .



**Preuve**

We have  $O(V) \subset O_p(V) \forall p \in V$  hence  $\Gamma(V) \hookrightarrow O(V) \hookrightarrow \bigcap_{p \in V} O_p(V) = \Gamma(V)$   $\square$

**Lemme 37**

Let  $V$  be a quasi-affine subset and  $f : V \rightarrow \mathbb{A}^1$  regular, then  $f$  is continuous (with respect to the Zariski topology)

**Preuve**

It is enough to show that  $f^{-1}(X)$  is closed for any closed  $X$ .

Without loss of generality  $X = \{x\}$ .

Let  $V = \bigcup_i U_i$  a cover such that  $f|_{U_i} = \frac{g_i}{h_i}$  and  $h_i \neq 0$  on  $U_i$ .

Then  $f^{-1}(X) \cap U_i = \left\{ v \in U_i \mid f(v) = \frac{g_i(v)}{h_i(v)} \right\} = \{v \in U_i \mid x \cdot h_i(v) - g_i(v) = 0\}$  which is an algebraic set.

Hence  $f^{-1}(X) \cap U_i$  is closed which implies  $f^{-1}(X)$  is closed.  $\square$

**Corollaire 38**

Let  $f, g \in O(V)$  and  $U \subset V$  non empty and open s.t.  $f|_U = g|_U$  then  $f = g$

**Preuve**

Using an exercise, open subsets are dense, since  $f, g$  are continuous

$$f|_U = g|_U \implies f|_{\text{cl } U} = g|_{\text{cl } U} \implies f|_{\text{cl } V} = g|_{\text{cl } V} \implies f = g \quad \square$$

**Remarque**

Let  $U \subset V$  open, then the restriction of functions induces  $\mathcal{O}(V) \rightarrow \mathcal{O}(U)$ .

i.e.  $\mathcal{O}(-)$  defines a sheaf of  $k$ -algebras on  $V$ .

Using this one can define a general algebraic as a topological space  $X$  with some sheaf  $\mathcal{O}_X$  which locally looks like a quasi-affine variety  $V$  with  $\mathcal{O}(-)$ .

We'll define  $\mathcal{O}_p(V)$  and  $K(V)$  for  $V$  quasi-affine, but these depend only on "local structure".

We can guess  $\mathcal{O}_p(V) = \mathcal{O}_p(\text{cl } V)$  and similarly for the quotient field.

### 3 (Quasi-)Projective and general algebraic varieties

Affine varieties usually "go to infinity" when we draw them.

This leads to complications in the theory

### Example

Two distinct lines in  $\mathbb{A}^2$  they will intersect in 1 point unless they're parallel

## 3.1 Projective space

### Definition 16 (Projective n-space)

$\mathbb{P}^n$  is the set

$$\mathbb{P}^n = K^{n+1} \setminus \{0\} / \sim$$

Where we identify

$$(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1}) \text{ if } \exists \lambda \in K^* \text{ s.t. } x_i = \lambda y_i$$

Elements in  $\mathbb{P}^n$  are called points.

If  $p \in \mathbb{P}^n$  is the equivalence classe of  $(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1}$  we write

$$p = [x_1 : \dots : x_n]$$

$x_1, \dots, x_n$  are the homogenous coordinates of  $p$ .

### Remarque

Any point in  $\mathbb{A}^n \setminus \{0\}$  defines a line through the origin and  $x, y \in \mathbb{A}^n \setminus \{0\}$  define the same line iff  $x = \lambda y$

## Lecture 5: Projective varieties

Fri 25 Mar

While the  $i$ -th coordinate  $x_i$  of a point  $[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n$  is not well defined, the equation  $x_i = 0$  or  $x_i \neq 0$  is well defined.

Hence we can write

$$U_i = \{[x_1 : \dots : x_n] | x_i \neq 0\}$$

Clearly  $\mathbb{P}^n = \cup_i U_i$ .

Furthermore for all  $i$ , we have a bijection

$$\begin{aligned} \phi_i : \mathbb{A}^n &\rightarrow U_i \\ (x_1, \dots, x_n) &\mapsto [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_{n+1}] \end{aligned}$$

And this is clearly a bijection.

We'll see in a bit, that the  $\phi_i$ 's provide an open cover of  $\mathbb{P}^n$  by  $\mathbb{A}^n$

### Definition 17

The set

$$H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{x \in \mathbb{P}^n | x_{n+1} = 0\}$$

is called the hyperplane at infinity.

One can identify  $H_\infty = \mathbb{P}^{n-1}$   
Thus

$$\mathbb{P}^n = U_{n+1} \coprod H_\infty = \mathbb{A}^n \coprod \mathbb{P}^{n-1}$$

### Exemple

$\mathbb{P}^0 = \text{point}$

$\mathbb{P}^1 = \mathbb{A}^1 \coprod \text{point}$  is called the projective line.

Similarly  $\mathbb{P}^2$  is called the projective plane.

## 3.2 Projective algebraic sets

For a general  $F \in k[x_1, \dots, x_n]$ , the equation  $F(x) = 0, x \in \mathbb{P}^n$  doesn't make sense.

But it does if  $F$  is homogeneous, say of degree  $d$ , since then

$$F(\lambda x) = \lambda^d F(x) = 0 \forall x \in \mathbb{A}^{n+1}, \lambda \in k^*$$

### Definition 18 (Projective set)

For any set  $S \subset k[x_1, \dots, x_n]$  of homogeneous polynomials we set

$$V(S) = \{[x_1 : \dots : x_n] \in \mathbb{P}^n \mid F(x_1, \dots, x_n) = 0 \forall F \in S\}$$

A subset of  $\mathbb{P}^n$  is algebraic if it is of the form  $V(S)$  as above.

### Exemple

Take  $V(X^2 - YZ) \subset \mathbb{P}^2$ , how to draw it?

We draw the intersections  $V \cap U_i$

### Definition 19 (Homogeneous ideal)

An ideal  $I \subset k[x_1, \dots, x_n]$  is homogeneous if it is generated by homogeneous elements.

Then for  $I$  a homogeneous ideal we set

$$V(I) = V(T) \subset \mathbb{P}^n$$

where  $T$  is the set of forms in  $I$ .

### Remarque

Since the ring is noetherian, we can always find a finite number of homogeneous generators.

For  $I = (x_1, \dots, x_{n+1})$  we have  $V(I) = \emptyset$ , we denote this ideal by  $I_+$ , it's called the irrelevant ideal.

**Exemple**

$(x, y^2)$  is homogeneous,  $(x + y^2, y^2)$  is also homogeneous but  $(x + y^2)$  is not.

**Lemme 46**

$I$  is a homogeneous ideal if and only if for every  $F \in I$ , if we write  $F = \sum_{i \geq 0} F_i$  with  $F_i$  homogeneous of degree  $i$ .

**Preuve**

Let  $G^{(1)}, \dots, G^{(k)}$  be a set of homogeneous generators of  $I$  with degrees  $d_1, \dots, d_k$ .

Any  $F = \sum F_i$  can be written as  $F = \sum A^{(i)} G^{(i)}$  for some  $A^{(i)}$ .

Since the degree is additive we get  $F_j = \sum A_{j-d_i}^{(i)} G^{(i)}$

For the other direction, let  $G^{(1)}, \dots, G^{(k)}$  any set of generators, then  $G_j^{(i)} \in I$  and then the set of  $G_j^{(i)}$  is a set of generators.  $\square$

Furthermore, the sum, the product, the intersection and the radical of homogeneous ideals are homogeneous.

A homogeneous ideal is prime if for any homogeneous  $f, g \in k[x_1, \dots, x_n]$

$$fg \in I \implies f \in I \text{ or } g \in I$$

**Definition 20 (Zariski topology)**

We define the Zariski topology on  $\mathbb{P}^n$  by taking the open sets to be the complements of algebraic sets.

This defines a topology using the properties above.

**Definition 21**

An algebraic set  $V \subset \mathbb{P}^n$  is irreducible if it is irreducible as a topological space.

As in the affine case, there is a correspondence

$$\{ \text{Algebraic subsets in } \mathbb{P}^n \} \leftrightarrow \{ \text{Homogeneous ideals in } k[x_1, \dots, x_{n+1}] \}$$

Where  $I(V)$  is the ideal generated by  $\{F \in k[x_1, \dots, x_n] \mid F \text{ homogeneous}, F(v) = 0 \forall v \in V\}$

**Remarque**

If we need to distinguish between the affine and projective correspondence we'll write  $V_a, I_a$  and  $V_p, I_p$  respectively.

**Definition 22 (Cone)**

For  $V \subset \mathbb{P}^n$  algebraic, we define the cone over  $V$  as

$$C(V) = \{(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1} \mid [x_1, \dots, x_{n+1}] \in V\} \cup \{(0, \dots, 0)\}$$

**Lemme 48**

1. For  $V \neq \emptyset$ , then

$$I_p(V) = I_a(C(V))$$

2. If  $I \subsetneq k[x_1, \dots, x_n]$  homogeneous, then

$$C(V_p(I)) = V_a(I)$$

**Preuve**

1.  $G \in I_p(V)$  homogeneous and  $(x_1, \dots, x_{n+1}) \in C(V)$ , then

$$G(x_1, \dots, x_{n+1}) = 0$$

Conversely, if  $G \in I_a(C(V))$  write

$$G = \sum_i G_i, \quad G_i \text{ homogeneous}$$

Then, for every  $x \in C(V)$  and  $\lambda \in k^*$  we have  $\lambda x \in C(V)$  hence

$$0 = G(\lambda x) = \sum_i \lambda^i G_i(x)$$

Let  $\tilde{G}(y) = \sum_i y^i G_i(x) \in K[Y]$ , this has infinitely many 0's.

Which in turn implies  $G_i \in I_p(V)$

2. Notice for  $G$  homogeneous non-constant, then

$$C(V_p(G)) = V_a(G)$$

Since  $I$  is generated by homogeneous polynomials, the statement holds.

□

**Proposition 49 (Projective nullstellensatz)**

Let  $I$  be a homogeneous ideal, then

- If  $V_p(I) = \emptyset$ , then  $\sqrt{I} = k[x_1, \dots, x_{n+1}]$  or  $\sqrt{I} = I_+$
- If  $V_p(I) \neq \emptyset$  then  $I_p(V_p(I)) = \sqrt{I}$

**Preuve**

- If  $V_p(I) = \emptyset \iff V_a(I) \subset \{(0, \dots, 0)\}$  which implies  $\sqrt{I} \supset (x_1, \dots, x_{n+1})$ .

$$— I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I} \quad \square$$

**Corollaire 50**

*There is a one-to-one correspondence between radical homogeneous ideals and projective algebraic sets.*

*Furthermore  $V_p(I)$  is irreducible  $\iff I$  is prime.*

**Remarque**

*Points in  $\mathbb{P}^n$  do not correspond to maximal ideals.*

We can also relate affine and projective algebraic sets through the charts

$$\phi_i : \mathbb{A}^n \rightarrow U_i$$

We'll focus on  $\phi := \phi_{n+1} : \mathbb{A}^n \rightarrow U := U_{n+1}$

For  $F \in k[x_1, \dots, x_n]$  homogeneous, we define

$$F_*(x_1, \dots, x_n) = F(x_1, \dots, x_n, 1)$$

Conversely, for  $G \in k[x_1, \dots, x_n]$ , we write

$$G = \sum_{i=0}^d G_i \text{ and define } G^*(x_1, \dots, x_{n+1}) = x_{n+1}^d G_0 + \dots + G_d = X_{n+1}^d G\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right)$$

**Definition 23 (Homogenization)**

$(\cdot)_*$  and  $(\cdot)^*$  are called dehomogenisation and homogenization.

For  $I$  an ideal, we denote by  $I^*$  be the homogeneous ideal generated by  $\{F^* | F \in I\}$ .

Conversely, if  $V = V_a(I)$ , we write

$$V^* = V_p(I^*)$$

$V^*$  is called the projective closure of  $V$  in  $\mathbb{P}^n$ .

Similarly if  $I$  is homogeneous, then

$$I_* = \{F_* | F \in I\}$$

and if  $V = V_p(I)$ , we set  $V_* = V_a(I_*)$

**Example**

Let  $F = X_1^2 - X_2$ , then

$$F^* = X_1^2 - X_2 X_3$$

**Lemme 53**

If  $V \subset \mathbb{A}^n$  is closed, then  $\phi(V) = V^* \cap U$

Conversely, if  $V \subset \mathbb{P}^n$  is closed then  $\phi^{-1}(V \cap U) = V_*$

In particular  $\phi$  is a homeomorphism

**Preuve**

Recall that  $\phi(x_1, \dots, x_n) = [x_1 : \dots : x_n : 1]$ .

For  $V \subset \mathbb{A}^n$  write  $V = V_a(F_1, \dots, F_k)$  then

$$V^* = V_p(F_1^*, \dots, F_k^*)$$

But  $F_i^* = X_{n+1}^d F_i(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}})$   $F_i(v) = 0 \iff F_i^*(\phi(v)) = 0 \implies \phi(V) = V^* \cap U$  □

**Lecture 6: Algebraic varieties**

Fri 01 Apr

**3.3 Quasi-projective and general varieties****Definition 24**

A projective variety is a closed irreducible subset of  $\mathbb{P}^n$ .

A quasi-projective variety is an open subset of a projective variety.

An algebraic variety is one of the four types we've seen : affine, quasi-affine, projective or quasi-projective.

**Remarque**

In order to define morphisms between varieties, we need regular functions, ie.  $\mathcal{O}(V)$  for  $V$  quasi-projective.

**Definition 25**

Let  $V \subset \mathbb{P}^n$  be quasi-projective, a map  $f : V \rightarrow k$  is regular at  $p \in V$  if  $\exists$  an open neighbourhood  $p \in U \subset V$  and  $g, h \in k[x_1, \dots, x_{n+1}]$  formes of the same degree such that  $h(U) \neq 0$  and such that  $f(u) = \frac{g(u)}{h(u)}$

**Example**

Set  $V = \mathbb{P}^n$  and take also  $U = V$  in the definition, then  $h(u) \neq 0 \forall u \in \mathbb{P}^n$  is only possible if  $h$  is constant.

In fact, constants are the only regular functions on  $\mathbb{P}^n$  and in fact on any projective variety.

**Definition 26**

We again, define  $\mathcal{O}(V)$  the ring of regular functions on  $V$ .

### Remarque

As in the quasi-affine case, regular functions are continuous for the Zariski topology.

## 3.4 Morphisms

### Definition 27 (Morphism)

A morphism between two algebraic varieties  $V, W$  is a continuous map  $\phi : V \rightarrow W$  such that for every open  $U \subset W$  and every  $f \in \mathcal{O}(U)$  the map  $\phi \circ f : \phi^{-1}(U) \rightarrow k$  is regular.

$\phi$  is an isomorphism if  $\exists \psi : W \rightarrow V$  such that  $\phi \circ \psi$  and  $\psi \circ \phi$  is the identity.

### Remarque

In particular any  $\phi : V \rightarrow W$  induces a map  $\tilde{\phi} : \mathcal{O}(W) \rightarrow \mathcal{O}(V)$  a  $k$ -algebra homomorphism.

The converse is only true if  $W$  is affine.

### Proposition 58

The maps  $\phi_i : \mathbb{A}^n \rightarrow U_i \subset \mathbb{P}^n$  are isomorphisms.

In fact, for every  $V \subset \mathbb{A}^n$  quasi-affine,  $\phi_i|_V$  is an isomorphism onto its image  $\phi_i(V) \subset \mathbb{P}^n$  which is quasi-projective.

### Preuve

We may take  $i = n + 1$  and write  $\phi = \phi_{n+1} : \mathbb{A}^n \rightarrow U$

We know that  $\phi$  is a homeomorphism, hence  $\phi(V)$  is quasi projective since

$$\phi(V) \subset \phi(\overline{V}) = \overline{\phi(V)} = \overline{\phi(V)}^* \cap U \subset \overline{\phi(V)}^*$$

Let  $F$  be some regular function on some open  $W \subset \phi(V)$ .

By shrinking if necessary we can write  $F = \frac{G}{H}$  where  $G$  and  $H$  are forms of the same degree and  $H(w) \neq 0 \forall w \in W$ .

Then

$$F \circ \phi = \frac{G \circ \phi}{H \circ \phi} = \frac{G_*}{H_*}$$

and since  $G_*$  and  $H_*$  don't vanish on  $\phi^{-1}(W)$ ,  $F \circ \phi$  is regular.

Thus  $\phi|_V$  is a morphism of algebraic varieties.

Conversely, we have  $\phi^{-1} : U \rightarrow \mathbb{A}^n$ , let  $W \subset V$ , take  $F \in \mathcal{O}(W)$ , up to shrinking.

$$\text{Then } F \circ \phi^{-1}([x_1 : \dots : x_{n+1}]) = \frac{G(\frac{x_1}{x_{n+1}}, \dots)}{H(\frac{x_1}{x_{n+1}}, \dots)} = x_{n+1}^\alpha \frac{G^*([x_1 : \dots : x_{n+1}])}{H^*([x_1 : \dots : x_{n+1}])}.$$

Now, if  $\alpha \geq 0$  then  $x_{n+1}^\alpha$  then  $x_{n+1}^\alpha G^*$  and  $H^*$  are of the same degree.



If  $\alpha < 0$ , then  $G^*$  and  $x_{n+1}^{-\alpha} H^*$ .

Hence  $F \circ \phi^{-1} \in \mathcal{O}(\phi(V))$  and hence  $\phi^{-1}$  is a morphism.  $\square$

### Definition 28

A variety is (quasi-) affine or quasi-projective if  $V$  is isomorphic to a quasi-affine or quasi-projective variety.

### Corollaire 59

Any variety is quasi-projective.

Every quasi-projective variety admits a finite open cover by quasi-affine varieties namely  $V = \bigcup_i V \cap U_i$

### Preuve

projective implies quasi-projective and affine implies quasi-affine and the theorem above gives quasi-affine implies quasi-projective.

Furthermore, since  $\mathbb{P}^n = \bigcup_i U_i \implies V = \bigcup V \cap U_i$ .

If  $V$  is projective,  $V \cap U_i \subset U_i \simeq \mathbb{A}^n$  is closed and irreducible, hence  $V \cap U_i$ .

Finally, if  $V \subset \bar{V} \subset \mathbb{P}^n$  quasi-projective, then  $V \cap U_i \subset \bar{V} \cap U_i$  is quasi-affine.  $\square$

### Exemple

For any polynomial  $f \in k[x_1, \dots, x_n]$ ,  $\mathbb{A}^n \setminus V(f)$  is affine, but  $\mathbb{A}^2 \setminus \{0\}$  is not affine.

### Remarque

The above also shows that if  $V$  is quasi-affine, then  $\mathcal{O}(V) \simeq \mathcal{O}(\phi(V))$ , hence all our definitions of  $\mathcal{O}(-)$  are compatible.

The definition of a morphism is clean, but difficult to check in practice.

It becomes easier if at least the target is affine :

### Lemme 62

Let  $\phi : V \rightarrow W$  be a map, suppose  $W \subset \mathbb{A}^n$  affine, and  $x_i : \mathbb{A}^n \rightarrow \mathbb{A}^1$  coordinate functions, then  $\phi$  is a morphism  $\iff x_i \circ \phi : V \rightarrow \mathbb{A}^1$  is regular.

### Preuve

If  $\phi$  is a morphism, then  $x_i \circ \phi$  is regular by definition.

Conversely, if  $x_i \circ \phi$  is regular for all  $1 \leq i \leq n$ , then

$$f \circ \phi$$

is also regular for any  $f \in k[x_1, \dots, x_n]$ .

Since regular functions form a ring.

Hence

$$\begin{aligned}\phi^{-1}(V(f_1, \dots, f_k) \cap W) &= \bigcap \phi^{-1}(V(f_i) \cap W) \\ &= \bigcap (f \circ \phi)^{-1}(0)\end{aligned}$$

Thus  $\phi$  is continuous.

Now let  $U \subset W$  be open and  $f \in \mathcal{O}(U)$ , after shrinking, we may suppose  $f = \frac{g}{h}$  where  $g, h \in k[x_1, \dots, x_n]$ ,  $h(u) \neq 0 \forall u \in U$ .

Then

$$f \circ \phi = \frac{g \circ \phi}{h \circ \phi}$$

where  $g \circ \phi, h \circ \phi$  are regular and  $h \circ \phi(v) \neq 0 \forall v \in V$ .

Thus  $\frac{1}{h \circ \phi}$  is regular, therefore  $\frac{g \circ \phi}{h \circ \phi}$  is regular.  $\square$

### Example

Now, the map  $\mathbb{A}^1 \setminus \{0\} \rightarrow V(XY - 1) \subset \mathbb{A}^2$  is a morphism.

### Corollaire 64

Let  $V, W$  be two varieties and  $W$  affine, then there is a bijection

$$\text{hom}_{\text{Var}}(V, W) \simeq \text{hom}_{k\text{-alg}}(\mathcal{O}(W), \mathcal{O}(V))$$

sending  $\phi \rightarrow \tilde{\phi}$

### Remarque

If  $V$  is projective, we claim that  $\mathcal{O}(V) \simeq k$

## 3.5 General rational functions and local rings

Let  $V$  be an algebraic variety and  $p \in V$ .

### Definition 29 (Local ring)

The local ring of  $V$  at  $p : \mathcal{O}_p(V)$  is the set of pairs  $\langle U, f \rangle$ , where  $U \subset V$  is open containing  $p$  and  $f$  is a regular function on  $U$ , modulo the relation

$$[U, f] \sim [U', f'] \iff f = f' \text{ on } U \cap U'$$

### Lemme 66

$\sim$  is an equivalence relation and  $\mathcal{O}_p(V)$  is a ring with the operations

$$[U, f] + [U', f'] = [U \cap U', f + f']$$

and similarly for the product

$$[U, f] \cdot [U', f'] = [U \cap U', f \cdot f']$$

Furthermore  $\mathcal{O}_p(V)$  is a local ring with maximal ideal

$$m = \{[U, f] | f(p) = 0\}$$

### Preuve

Reflexivity and identity is obvious, we need to check transitivity, suppose  $[U, f] \sim [U', f']$  and  $[U'', f''] \sim [U', f']$  then clearly  $f = f''$  on  $U \cap U' \cap U''$  but

$$U \cap U' \cap U'' \subset U \cap U''$$

is open and dense and since  $f, f''$  are continuous,  $f = f''$  on  $U \cap U''$ .

To show the ring is local, notice that there is an evaluation morphism

$$\begin{aligned} \mathcal{O}_p(V) &\rightarrow k \\ [U, f] &\rightarrow f(p) \end{aligned}$$

This map is surjective since we have constant functions.

Hence  $m_p$  is a maximal ideal.

Finally  $\mathcal{O}_p(V)$  is local since  $[U, f] \notin m_p$  hence  $f(p) \neq 0$ .

Thus  $\frac{1}{f}$  is regular in some neighbourhood of  $p$ .

Thus  $f$  is a unit. □

## Lecture 7: rational functions and dimension

Fri 08 Apr

### 3.6 The field of rational functions

#### Definition 30 (Field of rational functions)

$K(V)$  is the set of pairs  $(U, f)$  with  $U \subset V$  open and non-empty and  $f$  a regular function on  $U$  modulo the equivalence relation  $(U, f) \sim (U', f')$  iff  $f = f'$  on  $U \cap U'$ .

#### Remarque

Since  $V$  is irreducible, any non-empty open is dense, hence  $U \cap U'$  is non-empty and open as well.

Furthermore, note that  $[U, f]$  has  $\frac{1}{f}$  as an inverse.

As in the affine case, we have inclusions

$$\mathcal{O}(V) \hookrightarrow \mathcal{O}_p(V) \hookrightarrow K(V)$$

**Proposition 68**

1. Let  $V$  be an algebraic variety and  $U \subset V$  open and non-empty.  
Then  $K(V) = K(U)$  and  $\mathcal{O}_p(V) = \mathcal{O}_p(U)$  for any  $p \in U$ .
2. For any algebraic variety  $V$  and any  $p \in V$ ,  $K(V)$  is the quotient field of  $\mathcal{O}_p(V)$ .
3. If  $V$  is affine, then  $\mathcal{O}_p(V) = \Gamma(V)_{m_p}$  for any  $p \in V$  and  $K(V)$  is the quotient field at  $\Gamma(V)$ .  
In particular all definitions of  $K(V)$  and  $\mathcal{O}_p(V)$  agree for affine varieties.

**Remarque**

1. For  $V \subset \mathbb{P}^n$  quasi-projective and  $P \in V \cap U_i$  for some  $i$ , then  $\mathcal{O}_p(V) = \mathcal{O}_p(\overline{V}) = \mathcal{O}_p(\overline{V} \cap U_i)$ .  
Since  $U_i = \mathbb{A}^n$ ,  $\overline{V} \cap U_i$  is affine and  $\mathcal{O}_p(\overline{V} \cap U_i) = \Gamma(\overline{V} \cap U_i)_{m_p}$ .  
Thus  $\mathcal{O}_p(V)$  is always an explicit localization (as any variety is quasi-projective).

**Preuve**

1. Immediate from the definition :  $[W, f] \in \mathcal{O}_p(V)$ .  
We know that  $[W, f] \sim [W \cap U, f] \in \mathcal{O}_p(U)$  so there is a bijection between the equivalence classes.
2. As in the remark, we can reduce to the affine case and then it follows from the third part of the proposition.
3. We have a map  $\Gamma(V) \rightarrow \mathcal{O}_p(V)$  sending  $\frac{f}{g} \rightarrow [V, \frac{f}{g}]$ .  
It is injective by continuity (as  $V$  is open hence dense. ).  
And also surjective by definition of  $\mathcal{O}_p(V)$ .  
Indeed, any  $[U, h] \in \mathcal{O}_p(V)$  is of the form  $h = \frac{f}{g}$  with  $g(u) \neq 0$  on  $U$ .  
Since  $U$  is quasi-affine,  $f$  and  $g$  are both quotients of regular functions on  $V = \overline{U}$ .  
Similarly, we have an inclusion  $Q(\Gamma(V)) \hookrightarrow K(V)$  sending  $\frac{f}{g} \rightarrow [V, \frac{f}{g}]$ .  
But any  $[U, h] \in K(V)$  is contained in  $\mathcal{O}_p(V) \subset Q(\Gamma(V))$  for any  $p \in U$  so we also have a reverse inclusion.  $\square$

**3.7 Dimension of a Variety**

Should be a basic invariant, in fact it isn't.

**Definition 31 (Dimension of a topological space)**

The dimension of a topological space  $X$  is the largest integer  $n$  such that there exists a chain  $Z_n \supset \dots \supset Z_1 \supset Z_0$  of distinct irreducible closed subsets of  $X$ .

Then, the dimension of an algebraic set is its dimension as a topological

space.

So now we'd like to relate this definition with the dimension theory for rings.

Recall that in any ring  $A$ , the height  $ht(p)$  of a prime ideal  $P \subset A$  is the supremum of all integers  $n$  such that  $\exists$  a chain  $p_0 \subset \dots \subset p_n = p$  of distinct prime ideals.

The krull dimension of  $A$  is defined as

$$\dim A = \sup \{ ht(p) | p \subset A \text{ prime} \}$$

**Proposition 70**

If  $V \subset \mathbb{A}^n$  is an affine algebraic variety, then

$$\dim V = \dim \mathcal{O}(V)$$

**Preuve**

If  $V_0 \subset V_1 \dots \subset V_d = V$  is a maximal chain of irreducible closed subsets of  $V$ , then ( $d < \infty$  by Noetherianity), applying  $I$  gives

$$I(V_1) \supset \dots \supset I(V_d) = I(V)$$

is a chain of distinct prime ideals containing  $I(V)$ , so in the quotient we get a chain

$$p_0 \supset \dots \supset p_d = (0) \text{ in } \mathcal{O}(V) \quad \square$$

So  $\dim \mathcal{O}(V) \geq \dim V$ , we can of course go the other way to find  $\dim \mathcal{O}(V) \leq \dim V$

Computing dimensions is hard, the main tool is the following theorem

**Theorème 71**

Let  $K$  be a field and  $B$  a domain, which is a finitely generated  $K$ -algebra, then

$$\dim B = \text{transcendence degree of } Q(B)/K$$

— For any prime ideal  $p \subset B$  we have

$$ht(p) + \dim B/p = \dim B$$

**Corollaire 72**

$$\dim \mathbb{A}^n = n$$

**Preuve**

We have that

$$\dim \mathbb{A}^n = \dim K[x_1, \dots, x_n] = \text{trdeg} K(x_1, \dots, x_n) = n \quad \square$$

**Corollaire 73**

If  $V \subset \mathbb{P}^n$  is a quasi-projective variety, then

$$\dim \overline{V} = \dim V$$

To prove this, we need the following lemma

**Lemme 74**

If  $X$  is a topological space and  $\{U_i\}$  a family of open subsets covering  $X$ , then

$$\dim X = \sup_i \dim U_i$$

**Preuve**

Let  $U_1, \dots, U_{n+1}$  be the open charts of  $\mathbb{P}^n$ , then  $V_i = V \cap U_i$  gives an open cover of  $V$  by quasi-affine varieties and  $\overline{V} \cap U_i = \overline{V}_i$  where the second closure is taken in  $U_i \simeq \mathbb{A}^n$ .

If we know the corollary for quasi-affine varieties, we get that

$$\dim \overline{V} = \sup \dim \overline{V}_i = \sup \dim V_i = \dim V$$

Therefore, assume  $V \subset \mathbb{A}^n$  is quasi-affine of dimension  $d$  and let  $z_0 \subset \dots \subset z_d$  be a maximal chain of irreducibles in  $V$ .

Hence  $Z_d = V$  and  $Z_0 = \{x\}$ .

Let  $m \subset \mathcal{O}(\overline{V})$  be the maximal ideal corresponding to  $x$ , we claim that  $\text{ht}(m) = d$ .

If this is true, then we have

$$\dim \overline{V} = \dim \mathcal{O}(\overline{V}) = \dim \mathcal{O}(\overline{V}/_m) + (m) = d$$

So now we have to prove the claim.

Clearly  $\text{ht}(m) \geq d$ .

Let  $p_0 \subset \dots \subset p_r = m$  be a longer chain of distinct prime ideals in  $\mathcal{O}(\overline{V})$ .

Set  $W_i = V(p_i) \rightarrow \overline{V} = W_0 \supset \dots \supset W_r = \{x\}$ .

*Intersecting with  $V$  gives*

$$V = W_0 \cap V \supset \dots \supset W_r \cap V = \{x\}$$

*Since  $W_i \cap V \subset W_i$  are open and non-empty,  $W_i \cap V$  is irreducible.*

*By maximality of the initial sequence  $\exists i$  such that*

$$W_i \cap V = W_{i+1} \cap V$$

*But then  $W_i = \overline{W_i \cap V} = \overline{W_{i+1} \cap V} = W_{i+1}$  which is a contradiction.  $\square$*