Exercise to submit 2

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We will denote by $F(k,n)$ the set of all maps from $[k]$ to $[n]$. We also denote by $S(k,n)$ the set of all surjective maps from $[k]$ to $[n]$.
To compute the number of all surjective maps, we simply substract the number of non-surjective maps from $F(k,n)$.
Let $I \subset [n], A_I \subset F(k,n)$ denote the set of all maps from $[k] \to [n]$ whose image does not contain elements of I, ie.

$$A_I = \{f \in F(k,n) | f(A) \cap I = \emptyset\}$$

Hence, we can write

$$S(k,n) = F(k,n) \setminus \left(\bigcup_{I \subset [n]} A_I \right)$$

We can now notice that for $I, J \subset [n]$, $A_I \cap A_j = A_{I \cup J}$. We can now apply the inclusion-exclusion formula, which yields

$$\left| \bigcup_{I \subset [n]} A_I \right| = \sum_{i=1}^{n} \sum_{I \subset [n], |I| = j} (-1)^{i-1} |A_I|$$

Note that, we sum from 1 to n, the case $n = 0$ would count surjective functions.
It is clear that the cardinality of A_I is given by

$$|A_I| = (n - |I|)^k$$

The number of subsets of $[n]$ of cardinality j is given by $\binom{n}{j}$.

Finally, we can compute

$$\begin{aligned} |S(k,n)| &= |F(k,n)| - | \bigcup_{I \subset [n]} A_i | \\ &= n^k - \sum_{i=1}^{i} (-1)^{i+1} \binom{n}{i} (n-i)^k \\ &= \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot (n-j)^k \end{aligned}$$

Which is the desired result.