Serie 7

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Derive the following formula by expanding part of the integrand into a series and justify the term-by-term integration.

$$\int_0^\infty e^{-sx}\frac{\sin x}{x}\,dx = \arctan(1/s) \qquad \text{for } s > 1\,.$$

Exercise 2. Consider the convolution of two measurable functions $f, g : \mathbb{R}^d \to \mathbb{R}$ defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy\,.$$

Recall the properties of the convolution that we have established in Exercise 4 of Serie 6.

(i) Show that $f * g$ is well-defined for every $x \in \mathbb{R}^d$ and that $f * g$ is uniformly continuous, if f is integrable and g is bounded.

(ii) If in addition g is integrable, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Exercise 3. Let $\Omega \subset \mathbb{R}^d$ be measurable. Prove that $L^\infty(\Omega)$ is complete. In other words, if $f_n\colon \Omega \to \mathbb{R}$, $n \in \mathbb{N}$ is a Cauchy sequence in $L^\infty(\Omega)$, then there is $f \in L^\infty(\Omega)$ such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty} = 0.$$

Hint: Note that $\{x \in \Omega : |f_n(x) - f_m(x)| \geq \|f_n - f_m\|_{L^\infty}\}$ has measure 0. Find a set E of measure 0 such that for any $x \in E^c$, $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy sequence. Then call the limit $f(x) = \lim_{n\to\infty} f_n(x)$, for all $x \in E^c$. We get a function f defined a.e. Finally show that $f \in L^\infty(\Omega)$ and $f_n \to f$ in L^∞.

Exercise 4 (Layer cake representation)**.** Let $f : \mathbb{R}^d \to \mathbb{R}$ be integrable. For $\alpha > 0$, we set $E_\alpha := \{x \in \mathbb{R}^d : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty \mathrm{m}(E_\alpha)\,d\alpha.$$

Exercise 5 ($*$)**.** We show that there exist many non-measurable sets. More precisely, let $A \subseteq \mathbb{R}$ with $\mathrm{m}^*(A) > 0$. Show that then there exists $B \subseteq A$ such that B is not measurable.
Hint: Consider rational translations of the Vitali set V and use and prove the following
Claim: For any measurable set E such that $\mathrm{m}^*(E) > 0$ the difference set $D_E := \{x - y : x, y \in E\}$ contains an interval around the origin.