Exercise 6. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. How many permutations of the set $[n]$ exist with the following conditions?

(1) Such that 1 and n are adjacent.
(2) Such that 1 and n are separated by exactly one element.

Exercise 2. Let q_n be the number of **nonconsecutive** permutations of $[n]$, i.e. permutations of the set $[n]$ such that no consecutive numbers appear consecutively. For example, for $n = 4$, 1324 is nonconsecutive but 1243 is not.
Show that

$$q_n = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-k)!.$$

Exercise 3. Prove the following identity for $n \geq 1$.

$$\sum_{A \subseteq [n]} \sum_{B \subseteq [n]} |A \cap B| = n4^{n-1}$$

Exercise 4. Determine the generating function of triangular numbers $a_n = \binom{n}{2}$.

Exercise 5. If F_n is the nth Fibonacci number, prove that

$$F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n-1}.$$

Exercise 6. (*Motzkin numbers*)
Let m_n be the number of ways the vector $w = (n, 0)$ can be written as a sum

$$w = v_1 + v_2 + \cdots + v_n,$$

where each $v_i \in \{(1,0), (1,1), (1,-1)\}$ and for any $k \leq n$, the y-coordinate of $v_1 + v_2 + \cdots + v_k$ is non-negative. Such arrangements are called Motzkin paths. Can you see how similar or different they are to Dyck paths?

Set $m_0 = 1$. Find m_1, m_2, m_3, m_4. If $m(x) = \sum_{i=0}^{\infty} m_i x^i$ is their generating function, show that

$$x^2 m(x)^2 + (x-1)m(x) + 1 = 0$$

Exercise 7. *Optional exercise, for fun*
Let $m \leq n$. Let $c_{m,n}$ be the number of ways in which n coins of 1 franc and m coins of 2 franc can be distributed among $m+n$ people standing in a coffee machine queue in the following way. Each person gets exactly one coin, and when they start buying a coffee in the order they are standing, the coffee machine never runs out of change. The coffee costs 1 franc and the machine has no coins inside to begin with.

Show that $c_{m,n} = \frac{n-m+1}{n+1}\binom{m+n}{n}$.