Exercise 1. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. Proof the following equations.

(1)
$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

(2)
$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}$$

Exercise 2. How many functions $f : [n] \to [n]$ are there that are monotone. That is, they satisfy $i < j \Rightarrow f(i) \leq f(j)$.

Exercise 3. Show that $\left(7n^2 - 3n + 29\right)\left(n^7 + 3n^3 + 256\right) = O(n^9)$

Exercise 4. Prove the following equations

(1) $n^a = O(a^n)$ for any $a > 1$.
(2) $n^a = O(n^b)$ for any $a \leq b$.
(3) $2^n + n^2 = O(2^n)$.
(4) $\frac{n}{\log n} = o(n)$.

Exercise 5. Suppose there are 20 students and each student has to choose a number in [100]. What are the chances that all students choose a different number? If the number of students is increased, does this probablity increase or decrease?

How many students should there be so that the probabilty that two students have chosen the same number is at least 50%?

Exercise 6. *(Optional exercise, for fun)*
Find a combinatorial (or any other!) proof of the identity

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n+k}{m} = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k} 2^k$$