PROBABILITY

Exercise sheet 5

Exercise 1. *For $i \geq 1$, let $(\mathbb{R}, \mathcal{F}_E, \mathbb{P}_i)$ be probability measures. Consider the product probability measure $\mathbb{P}_\Pi$ on $(\mathbb{R}^\mathbb{N}, \mathcal{F}_\Pi)$ of the collection $((\mathbb{R}, \mathcal{F}_E, \mathbb{P}_i))_{i \geq 1}$.*

Further, for $n \in \mathbb{N}$, consider the projection $\pi : \mathbb{R}^\mathbb{N} \to \mathbb{R}^n$ to first $n-$coordinates, i.e. the map $(x_1, x_2, \dots) \to (x_1, \dots, x_n)$. Show that the pushforward measure of $\mathbb{P}_\Pi$ induced on $(\mathbb{R}^n, \mathcal{F}_E)$ by this projection is characterized by the c.d.f.

$$F(x_1, \dots, x_n) = \Pi_{i=1}^n \mathbb{P}_i((-\infty, x_i]).$$

Answer. We first note that for any Borel subset $B \subset \mathbb{R}^n$, its preimage under the projection map π is $\pi^{-1}(B) = B \times \mathbb{R} \times \mathbb{R} \times \dots \subset \mathbb{R}^\mathbb{N}$. By definition the pushforward measure $\pi_* \mathbb{P}_\Pi$ of $\mathbb{P}_\Pi$ induced by π satisfies

$$\pi_* \mathbb{P}_\Pi(B) = \mathbb{P}_\Pi(\pi^{-1}(B)) = \mathbb{P}_\Pi(B \times \mathbb{R} \times \mathbb{R} \times \dots) \quad \text{for all } B \in \mathcal{F}_E(\mathbb{R}^n).$$

In particular, we get that the c.d.f. of $\pi_* \mathbb{P}_\Pi$ is given by

$$\begin{aligned} F_*(x_1, \dots, x_n) &:= \pi_* \mathbb{P}_\Pi((-\infty, x_1] \times \dots \times (-\infty, x_n]) \\ &= \mathbb{P}_\Pi((-\infty, x_1] \times \dots \times (-\infty, x_n] \times \mathbb{R} \times \mathbb{R} \times \dots) \\ &= \prod_{i=1}^n \mathbb{P}_i((-\infty, x_i]) = F(x_1, \dots, x_n) \end{aligned}$$

as desired. In the first equality of the last line we have used definition of the product measure. □

Exercise 2. *The French, Swiss and German decide to elect the greatest mathematician of all time. The French propose Poincaré, the Swiss propose Euler and the German Gauss. Each country has one vote, and the candidate with most votes wins. In case of equal votes, the winner is chosen uniformly randomly. Now Mathematico, an organization that predicts elections, forecasts that*

- *the French will give their vote with probability $1/2$ to Poincaré and equally with probability $1/4$ to Euler or Gauss;*
- *the Swiss will give their vote with probability $1/2$ to Euler and equally with probability $1/4$ to Poincaré or Gauss;*
- *the German will give their vote with probability $1/2$ to Gauss and equally with probability $1/4$ to Poincaré or Euler.*

Moreover, Mathematico thinks that none of the countries cares about the opinion of the others.

Build a probabilistic model to be able to predict the winner. What assumptions are you making? In this model, what is the probability that Euler wins? What is the probability that Euler gets at least 2 votes? Now, surprisingly it comes out that the Swiss have elected

Gauss instead of Euler. How would you now estimate the probability that Euler still wins the election?

Answer. The probability distribution for any event is invariant under the exchanges Euler $\leftrightarrow$ Poincaré. $\leftrightarrow$ Gauss. Since the probability that either mathematician wins the election adds up to 1, we have that $\mathbb{P}(\text{Euler wins}) = \frac{1}{3}$ by symmetry.

For the next question, we define $\Omega := M^3$ where $M = \{P, E, G\}$ (the mathematicians are represented by their initials). The sequences $(j_1, j_2, j_3) \in \Omega$ (with $j_i \in \{P, E, G\}$) are taken to mean that the French elect mathematician j_1, the Swiss elect j_2 and the German elect j_3. The probability of the event "Euler gets at least 2 votes" is given by

$$\mathbb{P}(\underbrace{\{(E,E,E)\}}_{\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}}, \underbrace{\{(E,E,\bar{E})\}}_{\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{3}{4}}, \underbrace{\{(E,\bar{E},E)\}}_{\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}}, \underbrace{\{(\bar{E},E,E)\}}_{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{4}}) = \frac{1}{4}\ ,$$

where $\bar{E}$ stands for P or G.

For the final part, we have to augment Ω by replacing (j_1, j_2, j_3) with

$$\{(j_1, j_2, j_3; E), (j_1, j_2, j_3; P), (j_1, j_2, j_3; G)\}$$

for all the sequences with $j_1 \neq j_2 \neq j_3$. This accounts for the fact that the winner is drawn randomly for the case of equal number of votes. The event that the Swiss elect Gauss is denoted by A. We then have

$$\mathbb{P}(\text{Euler wins}|A) = \frac{\mathbb{P}(\text{Euler wins} \cap A)}{\mathbb{P}(A)}$$

where

$$\mathbb{P}(\text{Euler wins} \cap A) = \underbrace{\mathbb{P}(\{E,G,E\})}_{\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}} + \underbrace{\mathbb{P}\{(P,G,E;E)\}}_{\frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}} + \underbrace{\mathbb{P}\{(E,G,P;E)\}}_{\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}} = \frac{1}{32}$$

and $\mathbb{P}(A) = \frac{1}{4}$, which gives $\mathbb{P}(\text{Euler wins}|A) = \frac{1}{8}$. $\square$

Exercise 3. *Roger Federer is now* 70 *years old and still playing. He is a bit tired of running and has limited his strategy in his serve game: he either serves an ace with probability* $1/2$ *and obtains a point, or with the same probability makes a double fault and his opponent gains a point. The game has also been simplified and the player who first obtains 3 points wins. Build a probabilistic model (or several) to answer the following questions and answer them:*

- *What is the probability that Roger wins his serve game?*
- *What is the probability that Roger won his serve game, given that he hit at least two aces?*
- *What is the probability that he will win his serve game, given that he started by hitting two aces?*

Answer. The probability that Roger wins his serve game is $\frac{1}{2}$ by symmetry. The simplified tennis game described above is equivalent to playing "best of five rallies", so the space of outcomes can be viewed as sequences of 0's and 1's of length 5: $\Omega = \{0,1\}^5$ (where $1 \equiv$ ace).

For the second part, we have

$$\begin{aligned}\mathbb{P}(\text{win}|\text{at least 2 aces}) &= \frac{\mathbb{P}(\text{win} \cap \text{at least 2 aces})}{\mathbb{P}(\text{at least 2 aces})} \\ &= \frac{\mathbb{P}(\text{win})}{\mathbb{P}(\text{at least 2 aces})}\end{aligned}$$

The last equality follows from the fact that win $\subseteq$ at least 2 aces. Now $\mathbb{P}(\text{at least 2 aces}) = 1 - \underbrace{\mathbb{P}(0 \text{ aces})}_{\left(\frac{1}{2}\right)^3} - \underbrace{\mathbb{P}(1 \text{ ace})}_{3\cdot\left(\frac{1}{2}\right)^4} = \frac{11}{16}$, so $\mathbb{P}(\text{win}|\text{at least 2 aces}) = \frac{8}{11}$. Note that 0 aces means that no aces were served before the game was terminated (in this situation the game was terminated after Roger served three double faults).

For the final part,

$$\mathbb{P}(\text{win}|\text{start with 2 aces}) = \frac{\mathbb{P}(\text{win} \cap \text{start with 2 aces})}{\mathbb{P}(\text{start with 2 aces})}$$

with

$$\mathbb{P}(\text{win} \cap \text{start with 2 aces}) = \mathbb{P}(\underbrace{\{(1,1,0,1,\cdot)\}}_{\frac{1}{2^4}}, \underbrace{\{(1,1,0,0,1)\}}_{\frac{1}{2^5}}, \underbrace{\{(1,1,1,\cdot,\cdot)\}}_{\frac{1}{2^3}}) = \frac{7}{32}$$

and $\mathbb{P}(\text{start with 2 aces}) = \frac{1}{4}$, giving $\mathbb{P}(\text{win}|\text{start with 2 aces}) = \frac{7}{8}$. $\square$

Exercise 4 (Another look at Erdos-Renyi)**.** *Consider the model of uniform random graphs from the notes. Let $E_{i,j}$ be the event that the edge $\{i,j\}$ is present. What is $\mathbb{P}(E_{i,j})$? Prove that the events $E_{i,j}$ are independent. Find the appropriate product space to model uniform random graphs. How could you tweak this model to allow $\mathbb{P}(E_{i,j})$ to take an arbitrary value $p \in (0,1)$?*

Answer. Fix $i < j$. Then each simple graph exists in 2 versions, one when $(v_i, v_j) \in E$, and one when $(v_i, v_j) \notin E$, thus $\mathbb{P}(E_{i,j}) = \frac{1}{2}$. Then fix $i < j$ and $i' < j'$, such that $i \neq i'$ or $j \neq j'$. Then again, each simple graph exists in 4 versions, depending on the fact that (v_i, v_j) and $(v_{i'}, v_{j'})$ are in E or not, thus $\mathbb{P}(E_{i,j} \cap E_{i',j'}) = \frac{1}{4} = \mathbb{P}(E_{i,j})\mathbb{P}(E_{i',j'})$. Finally, if we define, for each $i < j$ the probability space $(\Omega_{i,j} := \{(v_i, v_j) \in E \ ; \ (v_i, v_j) \notin E\} \ , \ \mathcal{P}(\Omega_{i,j}) \ , \ \mathbb{P})$ where $\mathbb{P}((v_i, v_j) \in E) = \frac{1}{2} = \mathbb{P}((v_i, v_j) \notin E)$, then the product probability space $\prod_{i<j} \Omega_{i,j}$ models the uniform random graph. And if we had chosen $\mathbb{P}((v_i, v_j) \in E) = p = 1 - \mathbb{P}((v_i, v_j) \notin E)$, this would allow $\mathbb{P}(E_{i,j})$ to take the value p.

Proof using notation from the lecture notes:
Our original set Ω from the lecture notes is $\{0,1\}^E$ with 1 on the ith position corresponding to the event that ith edge in E is present (we start by enumerating all the edges of the complete graph on vertices $\{1, \ldots, n\}$ in an arbitrary but fixed way). In particular, the event $E_{i,j}$ corresponds to all $0-1$ sequences of length $|E|$ with kth component 1, where k is the unique number of edge (i,j) in our fixed enumeration. Note also that to each such sequence there exists exactly one sequence with 0 at the kth position and other components being the same. And since all sequences $\in \{0,1\}^E$ have the same probability, we conclude that $\mathbb{P}(E_{i,j}) = 1/2$. By a similar argument to the other version of the proof of this exercise we conclude that $E_{i,j}$'s are independent. Moreover, working with the same set $\{0,1\}^E$ and its power set as the sigma-algebra (recall that a finite product of power sets of some finite spaces

is a power set of the corresponding product space) we can define a new probability measure by setting $\tilde{\mathbb{P}} = \mathbb{P}_\Pi$, where $\mathbb{P}_\Pi$ is a product measure on the space $\prod_{k=1}^{|E|}(\{0,1\}, \mathcal{P}(\{0,1\}), \mathbb{P}_k)$ where $\mathbb{P}_k(1) = 1 - \mathbb{P}_k(0) = p$. Note that $\tilde{\mathbb{P}}$ has the desired property, namely, $\tilde{\mathbb{P}}(E_{i,j}) = \mathbb{P}_\Pi(\{0,1\}^{k-1} \times \{1\} \times \{0,1\}^{|E|-k}) = \mathbb{P}_k(1) = p$. □

Exercise 5 (Pairwise independent but not mutually independent)**.** *Consider the probability space for two independent coin tosses. Let E_1 denote the event that the first coin comes up heads, E_2 the event that the second coin comes up heads and E_3 the event that both coin come up on the same side. Show that E_1, E_2, E_3 are pairwise independent but not mutually independent.*

Answer. $E_1 = \{(H,H),(H,T)\}$, $E_2 = \{(H,H),(T,H))\}$ and $E_3 = \{(H,H),(T,T)\}$. Now $\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(\{(H,H)\}) = \frac{1}{4}$. We also have $\mathbb{P}(E_1) \cdot \mathbb{P}(E_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, so E_1 and E_2 are pairwise independent. Now $\mathbb{P}(E_1 \cup E_3) = \mathbb{P}(\{(H,H)\}) = \frac{1}{4} = \mathbb{P}(E_1) \cdot \mathbb{P}(E_3)$, so E_1 and E_3 are independent. Likewise, E_2 and E_3 are pairwise independent. Thus, E_1, E_2 and E_3 are pairwise independent. However, $\mathbb{P}(E_1 \cap E_2 \cap E_3) = \mathbb{P}(\{(H,H)\}) = \frac{1}{4}$ whereas $\mathbb{P}(E_1)\mathbb{P}(E_2)\mathbb{P}(E_3) = \frac{1}{8}$, so E_1, E_2 and E_3 are not mutually independent. □

Exercise 6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E_1, E_2, E_3 pairwise independent events with positive probability. Show that if E_1 and E_2 are conditionally independent, given E_3, then E_1, E_2, E_3 are mutually independent.*

Answer. Since the events E_i are pairwise independent, it suffices to check that the probability of their intersection is equal to the product of their three probabilities. Since E_1 and E_2 are independent given E_3,

$$\mathbb{P}[E_1 \cap E_2 \cap E_3] = \mathbb{P}[E_3]\mathbb{P}[E_1 \cap E_2 | E_3] = \mathbb{P}[E_3]\mathbb{P}[E_1|E_3]\mathbb{P}[E_2|E_3]\,.$$

Let $i \in \{1,2\}$. Since E_i is independent from E_3,

$$\mathbb{P}[E_i|E_3] = \mathbb{P}[E_i \cap E_3]/\mathbb{P}[E_3] = \mathbb{P}[E_i]\,.$$

Hence,

$$\mathbb{P}[E_1 \cap E_2 \cap E_3] = \mathbb{P}[E_3]\mathbb{P}[E_1|E_3]\mathbb{P}[E_2|E_3] = \mathbb{P}[E_3]\mathbb{P}[E_2]\mathbb{P}[E_1]$$

as announced. □

Exercise 7 (Bayes' rule and positive test results)**.** *In late spring 2020 several antibody tests to see whether your body has produced antibodies against SARS-CoV-2 and thus whether you could be immune to COVID at least that moment. Their preciseness was a good-sounding 95%, meaning that both false-positives (the test tells that you have antibodies when you actually don't) and false-negatives (the test tells that you don't have antibodies, but you actually do) would only appear in 5% of the tests taken. However, despite this good preciseness, caution was recommended in interpreting your result. Let's try to understand why:*

- *You hear someone claim that, when some tests positive they have 95% chance of actually having antibodies. Is this statement correct?*
- *Now, consider this additional information: in late spring 2020 it was estimated that 5% of the population have actually been in contact with SARS-CoV-2. Which probability space would you now build to estimate the probability that you have antibodies after a positive test? What is this probability? What if you take two independent tests on the same day and both come up positive?*

- *Suppose now that 50% of the population have been in contact with SARS-CoV-2. How does this change the result?*

Answer. We start by defining S ('sick') to be the event where a person has antibodies present, and P ('positive') the event where the test for antibodies is positive.

In any probabilistic framework, the probability of being sick, given a positive test, would be represented by $\mathbb{P}(S|P)$. On the other hand, the reliability of the test tells you that $\mathbb{P}(P|S) = 0.95$.

For the first part, this someone claims that $\mathbb{P}(S|P) = 0.95$. However, notice that we are given instead the information that $\mathbb{P}(P|S) = 0.95$. Thus, the claim is equivalent to saying that $\mathbb{P}(S|P) = \mathbb{P}(P|S)$. Is there any reason for this to be true? From Bayes' law, we have that $\mathbb{P}(S|P) = \mathbb{P}(P|S) \cdot \frac{\mathbb{P}(S)}{\mathbb{P}(P)}$. Therefore, the statement is only correct under the assumption $\mathbb{P}(S) = \mathbb{P}(P)$. But a priori, there is no clear reason why this assumption should be true.

For the second, we can model the problem using the outcome space $\Omega = \{P, \bar{P}\} \times \{S, \bar{S}\}$ with $\mathcal{F} = \mathcal{P}(\Omega)$, $S = \{(P, S) \cup (\bar{P}, S)\}$, $P = \{(P, S) \cup (P, \bar{S})\}$, $\mathbb{P}(P|S) = \mathbb{P}(\bar{P}|\bar{S}) = \frac{95}{100}$ and $\mathbb{P}(S) = \frac{5}{100}$. Applying Bayes' law gives

$$\begin{aligned}
\mathbb{P}(S|P) &= \frac{\mathbb{P}(P|S) \cdot \mathbb{P}(S)}{\mathbb{P}(P)} \\
&= \frac{\mathbb{P}(P|S) \cdot \mathbb{P}(S)}{\mathbb{P}(P|S)\mathbb{P}(S) + (1 - \mathbb{P}(\bar{P}|\bar{S}))(1 - \mathbb{P}(S))} \\
&= \frac{1}{2}
\end{aligned}$$

where the first equality comes from $\mathbb{P}(P) = P(P \cap S) + \mathbb{P}(P \cap \bar{S}) = \mathbb{P}(P|S)\mathbb{P}(S) + \mathbb{P}(P|\bar{S})\mathbb{P}(\bar{S}) = \mathbb{P}(P|S)\mathbb{P}(S) + (1 - \mathbb{P}(\bar{P}|\bar{S}))(1 - \mathbb{P}(S))$.

If we take two independent tests on the same day, and both come up positive, we can model this using $\Omega = \{P_1, \bar{P}_1\} \times \{P_2, \bar{P}_2\} \times \{S, \bar{S}\}$. Proceeding as above:

$$\begin{aligned}
\mathbb{P}(S|P_1 \cap P_2) &= \frac{\mathbb{P}(P_1 \cap P_2|S)\mathbb{P}(S)}{\mathbb{P}(P_1 \cap P_2)} \\
&= \frac{\mathbb{P}(P_1|S) \cdot \mathbb{P}(P_2|S) \cdot \mathbb{P}(S)}{\mathbb{P}(P_1 \cap P_2|S)\mathbb{P}(S) + \mathbb{P}(P_1 \cap P_2|\bar{S})\mathbb{P}(\bar{S})} \\
&= \frac{\mathbb{P}(P_1|S)^2 \cdot \mathbb{P}(S)}{\mathbb{P}(P_1|S)^2\mathbb{P}(S) + (1 - \mathbb{P}(\bar{P}_1|\bar{S}))^2(1 - \mathbb{P}(S))} \\
&= \frac{95}{100}
\end{aligned}$$

where we have used the $1 \leftrightarrow 2$ symmetry and the fact that P_1 and P_2 are independent conditional on S (in fact P_1 and P_2 are *not* independent without conditioning on S – to see this, think of whether you expect a second covid test to be positive given that the first is positive).

For the third part, we set $\mathbb{P}(S) = \frac{1}{2}$ and get $\mathbb{P}(S|P) = \frac{95}{100}$. □

0.1. ⋆ **For fun (non-examinable)** ⋆.

Exercise 8 ($\star$ Erdos-Renyi random graphs)**.** *Consider the probability model of random graphs, where each edge $\{i, j\}$ is included in the graph independently with probability p. (See Exercise 4 above)*

- *Show that if we take $p = \frac{1}{\log n}$, then the probability that the graph is connected converges to 1 as $n \to \infty$.*
- *Show that if we take $p = \frac{1}{n^2}$, then the probability that the graph is connected converges to 0 as $n \to \infty$.*