Exercise 7

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We will use, without proof, the identity

$$\sum_{d|n} dM(d,r) = r^d$$

First, notice that the left hand side expands to

$$\begin{aligned}\frac{rz}{1-rz} &= \sum_{i=1}^{\infty} (rz)^i \\ &= rz + (rz)^2 + (rz)^3 + \dots \\ &= rz + r^2z^2 + r^3z^3\end{aligned}$$

We will now develop the right hand side and show the equality :

$$\begin{aligned}\sum_{n=1}^{\infty} nM(n,r)\frac{z^n}{1-z^n} &= \sum_{n=1}^{\infty} nM(n,r)\sum_{i=1}^{\infty} z^{ni} \\ &= \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} nM(n,r)z^{ni} \qquad (1)\end{aligned}$$

Notice that, for each integer $k \in \mathbb{N}_{\geq 1}$, for a fixed n, the coefficient of x^k in

$$\sum_{i=1}^{\infty} nM(n,r)z^{ni}$$

is 1 if and only if $n|k$.
Hence, we can rewrite (1) as

$$\sum_{n=1}^{\infty}\sum_{d|n} dM(d,r)z^n = \sum_{n=1}^{\infty} r^n z^n$$

This proves the equality.