Algebraic Geometry I

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List of Theorems

Lecture 1: Intro

Mon 10 Oct

Quick Motivation

We study schemes.
These are objects that "look locally" like ($\operatorname{Spec} A, A$).
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— *$\forall U \subset X$ open, an object in C $\mathcal{F}(U)$*

— *$\forall V \subset U \subset X$ open, a morphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$*

such that

— *$\forall U$ open, $\rho_{U,U}$ is the identity on $\mathcal{F}(U)$*

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*

— *$\mathcal{F}(U)$ is called the space of sections of $\mathcal{F}$ over U*

— *Elements of $\mathcal{F}(X)$ are called global sections.*

— *There are alternative notations for $\mathcal{F}(U)$: $\Gamma(U, F)$ or $H_0(F)$*

— *The ρ_{UV} are called restriction maps, for $s \in \mathcal{F}(U)$, we write $s|_V := \rho_{UV}(s)$ and is called restriction of s to V.*

Example

— *For any object A in C, we define the constant presheaf $\underline{A}'$ defined by $\underline{A}'(U) = A$ and with restriction maps the identity.*

- *The presheaf of continuous functions :* C^0.
  We define $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*
- *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

  This example is universal in a certain sense

Remark
Define the category Ouv_X *with*
- *objects* $U \subset X$ *open subsets*
- *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{tikzcd} \mathcal{F}_1(U) \arrow[r, "\rho(U)"] \arrow[d] & \mathcal{F}_2(U) \arrow[d] \\ \mathcal{F}_1(V) \arrow[r, "\rho(V)"'] & \mathcal{F}_2(V) \end{tikzcd}$$

Example
- *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$
- *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
- *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark
There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \ \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$.
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s,t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i,j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*
— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i,j \in I$ the restrictions $\rho_{U_i,U_i\cap U_j}$ resp. $\rho_{U_j,U_i\cap U_j}$

Example

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i\in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

— *$PSh_C(X)$ is the category of presheaves of C on X*
— *$Sh_C(X)$ is the category of sheaves of C on X*
If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then Spec A is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6
Let $(I, \leq)$ be a pair where I is a set and $\leq$ is a binary relation.
$(I, \leq)$ is called a preorder if ll is reflexive and transitive.
$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric
$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example
1. *Let $I = \{U \subset X | U \text{ open }\}$ and $U \leq V \iff V \subset U$.*
   Then I is a directed poset.
2. *For $x \in X$, let*
$$I_x = \{U \subset X | U \text{ open } x \in U\}$$
   This is a directed poset.

Definition 7
Let $(I, \leq)$ *be a directed set and* C *a category.*
A direct system in C *indexed by* I *is a pair* $\left(\{A_i\}, \{\rho_{ij}\}_{i,j \in I, i \leq j}\right)$.
Where the A_i *are objects in* C, *the* $\rho_{ij} : A_i \to A_j$ *are morphisms in* C *such that*

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example
If $\mathcal{F}$ *is a presheaf of* C *on* X *and* I_X *as in the second example above, then*

$$\left(\{\mathcal{F}(U_i)_{U_i \in I_X}\}, \{\rho_{U_i, U_j}\}\right)$$

is a direct system.

Definition 8 (direct limit)
Let $(I, \leq)$ *be a directed set,* C *a category.*
Let $\left(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I}\right)$ *be a directed system, then the direct limit is a pair* $(\lim_{i \in I} A_i, \{\rho_i\}_{i \in I})$ *where* $\lim A_i$ *is in* C *and* $\rho_i : A_i \to \lim A_i$ *such that*

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects* A *in* C *and morphisms* $f_i : A_i \to A$ *such that*

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

   $\exists! f : \lim_{i \in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

Remark
The direct limit is unique up to unique isomorphism.

Example
Write $(*) = (\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.
Let $*$ *be a direct systement in* Set.
Let $\lim_{i \in I} A_i \coloneqq A_i / \sim$ *where* $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ *such that* $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
This is the direct limit of $*$.
If $*$ *is a system in* Ab , *let* $\lim A_i \coloneqq \bigoplus A_i / N$ *with* $N = \langle a_i - \rho_{ij}(a_i) \rangle$.
The natural map $\bigcup A_i / \sim \to \bigoplus A_i / N$ *is a bijection*

Remark
Taking the direct limits in (Ab) *is exact in the following sense :*
$\forall$ *directed sets* I, $\forall$ *direct systems* $\{M_i\}, \{N_i\}, \{P_i\}$ *indexed by* I *and for all*

collections of commutative diagrams, we get

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

Definition 9
Let C be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of C on X.
Then the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x.
For $s \in \mathcal{F}(U)$, we write s_x for the image of s in $\mathcal{F}_x$ and call it the germ of s at x.

Remark
A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$

Remark
Let $x \in X, \mathcal{F}$ a presheaf of $\mathrm{Set}, \mathrm{Ab}$

1. $\forall U \subset X$ *open,* $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

Definition 10 (Sheafification)
Let $\mathcal{F}$ be a presheaf of sets (...) on X.
The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

Remark

1. $\mathcal{F}^+$ *is a sheaf*
2. *Sheafification is functorial.*
   For $\rho : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, the collection $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ sending $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. $\exists$ *a natural morphism* $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ *defined by* $\iota_F(U)(s) : x \to s_x$
4. $\forall s \in \mathcal{F}^+(U)$ *there is an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

Proposition 20

$\forall$ *morphisms* $\phi : \mathcal{F} \to \mathcal{G}$ *such that* $\mathcal{G}$ *is a sheaf, there exists a unique morphism* $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ *such that* $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ *open, let* $s \in \mathcal{F}^+(U)$ $\exists$ *an open cover* $U = \bigcup_{i \in I} U_i$ *and* $s_i \in \mathcal{F}(U_i)$ *such that* $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.
Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, *we have to set*

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since $\mathcal{G}$ *is a sheaf and*

$$\phi(U_i)(s_i)|_{U_i \cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i \cap U_j}) = \phi(U_j)(s_i)|_{U_i \cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ *with* $t|_{U_i} = \phi(U_i)(s_i)$.
For ϕ^+ *to be a morphism, we have to set* $\phi^+(U)(s) = t$.
We still have to check that ϕ^+ *is compatible with restriction maps.* $\square$

Remark

The proposition above shows that $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ *naturally in the presheaf and the sheaf* $\mathcal{G}$.
Hence $(-)^+$ *is left-adjoint to the forgetful functor* $Sh(X) \to Psh(X)$

Proposition 22

$X = \operatorname{Spec} A$ $\forall a \in A$ *there exist isomorphisms* $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *such that* $\forall b \in A$ *with* $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow[\sim]{} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \to \mathcal{O}_X(D(a))$ *by sending* $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.
Clearly, these make the diagram commute.
This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.
Let $I = Ann(s) = \{r \in A | rs = 0\}$.
Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, *we have* $I \not\subset p$, *hence* $V(I) \subset V(a) \implies a \in \sqrt{I}$.
Thus there exists $m \geq 1$ *such that* $a^m s = 0$, *hece* $\frac{s}{a^n} = 0$.
To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, *by definition of* $\mathcal{O}_X$ *and because* $D(h_i)$ *form a basis, we find* $a_i, g_i, h_i \in A$ *such that*

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

$\square$

and $s(q) = \frac{a_i}{g_i}$ *for all* $q \in D(h_i)$.

1. *Claim 1 : Can choose* $g + i = h_i$
2. *Claim 2 : Can choose* I *finite*
3. *Claim 3 : Can choose* a_i, h_i *such that* $h_j a_i = h_i a_j$.

Using these claims, since $D(a) = \bigcup D(h_i)$ *, we find* $n > 0, b_j \in A$ *such that* $a^n = \sum b_j h_j$.
Write $c = \sum a_i b_i$.
Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.
Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$