SEMAINE 4

Exercice 30. Pour chaque i, la fonction de densité de X_i est

$$f_{X_i}(x_i;\theta) = \frac{1}{\theta}\mathbf{1}\{x_i \in [0,\theta]\}.$$

Ainsi, les X_i étant indépendantes, la fonction de densité conjointe est

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n;\theta) = \frac{1}{\theta^n}\prod_{i=1}^{n}\mathbf{1}\{x_i \in [0,\theta]\} = \frac{1}{\theta^n}\mathbf{1}\{x_{(n)} \leq \theta\}\mathbf{1}\{x_{(1)} \geq 0\}.$$

Par le théorème 2.3 (p. 48), nous avons que $T(X_1,\dots,X_n) = X_{(n)}$ est une statistique exhaustive pour θ.
Il est évident que $\mathbb{P}(X_{(n)} \leq 0) = 0$ et $\mathbb{P}(X_{(n)} \leq \theta) = 1$. Pour $0 < t < \theta$, X_i étant indépendantes, on a

$$F_T(t;\theta) = \mathbb{P}\left(X_{(n)} \leq t\right) = \mathbb{P}\left(\bigcap_{i=1}^{n}\{X_i \leq t\}\right) = \prod_{i=1}^{n}\mathbb{P}(X_i \leq t) = \left(\frac{t}{\theta}\right)^n.$$

En prenant la dérivée, il s'en suit que $T = X_{(n)}$ est une variable aléatoire continue avec densité

$$f_T(t;\theta) = n\frac{t^{n-1}}{\theta^n}, \qquad t \in [0,\theta].$$

Exercice 31. Pour chaque i, la fonction de masse de X_i est

$$f_{X_i}(x_i;\lambda) = \frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\mathbf{1}\{x_i \in \mathcal{X}\}, \qquad \mathcal{X} = \{0,1,2,\dots\}.$$

Ainsi, les X_i étant indépendantes, la fonction de masse conjointe est

$$f_{X_1,\dots,X_n}(x_1,\dots,x_n;\lambda) = \prod_{i=1}^{n}\frac{\lambda^{x_i}}{x_i!}e^{-\lambda}\mathbf{1}\{x_i \in \mathcal{X}\} = \lambda^{\sum_{i=1}^n x_i}e^{-n\lambda}\left(\prod_{i=1}^{n}\frac{1}{x_i!}\right)\mathbf{1}\{x_i \in \mathcal{X} \ \ \forall i\}.$$

Par le théorème 2.3 (p. 48), nous avons que $T(X_1,\dots,X_n) = \sum_{i=1}^n X_i$ est une statistique exhaustive pour λ.
D'après l'exercice 4, série 1, la distribution de T est $Poisson(n\lambda)$, c'est-à-dire $f_T(t;\lambda) = e^{-n\lambda}(n\lambda)^t/t!$ pour $t = 0,1,2,\dots$.

Exercice 32. Let

$$T = \frac{\bar{X}-\mu}{S/\sqrt{n}}$$

We need to show that $f_T(t)$, the density function of T, is as in Definition 2.10 (p.54). Observe that T can be written as a function of two independent random variables $Z = (\bar{X}-\mu)/(\sigma/\sqrt{n})$ and $V = (n-1)S^2/\sigma^2$ (by Proposition 2.7, p.51) :

$$T = \frac{Z}{\sqrt{\frac{V}{n-1}}}$$

Since,

$$f_{Z,V}(z,v) = f_Z(z)f_V(v) = \frac{1}{2^{\frac{n}{2}}\pi^{\frac{1}{2}}\Gamma(\frac{n-1}{2})}v^{\frac{n-1}{2}-1}e^{-\frac{1}{2}(v+z^2)}$$

we can apply Theorem 1.33 (p.28) to the transformation

$$g : (Z, V) \mapsto (T, V) = \left(\frac{Z}{\sqrt{V/(n-1)}}, V \right)$$

to get

$$f_{T,V}(t, v) = \frac{1}{2^{n/2}\sqrt{\pi(n-1)}\Gamma(\frac{n-1}{2})} \cdot v^{\frac{n-2}{2}} e^{-\frac{v}{2}(1+\frac{t^2}{n-1})}$$

And as a consequence,

$$f_T(t) = \frac{1}{2^{n/2}\sqrt{\pi(n-1)}\Gamma(\frac{n-1}{2})} \cdot \int_0^\infty v^{\frac{n-2}{2}} e^{-\frac{v}{2}\left(1+\frac{t^2}{n-1}\right)} \, dv$$

Substitute

$$y = \frac{v}{2} \left(\frac{t^2}{n-1} + 1 \right)$$

for v and integrate to obtain the marginal distribution of T. The conclusion follows.

Exercice 33.

(i) It is easy to show that $\mathbf{a}_1'\mathbf{a}_1 = 1$. Observe that for any $2 \leq j \leq n$, the lth term of $\mathbf{a}_j$ equals $[j(j-1)]^{-1/2}$ if $l = 1, 2, \ldots, (j-1)$, equals $-[(j-1)/j]^{1/2}$ if $l = j$, and equals 0 if $l > j$. So, direct calculation shows that

$$\mathbf{a}_j'\mathbf{a}_j = 1 \quad \& \quad \mathbf{a}_j'\mathbf{a}_1 = [nj(j-1)]^{-1/2} \left\{ \sum_{l=1}^{j-1} 1 - (j-1) \right\} = 0$$

for all $j = 2, 3, \ldots, n$. Further, for any $2 \leq j < k \leq n$,

$$\mathbf{a}_j'\mathbf{a}_k = [jk(j-1)(k-1)]^{-1/2} \left\{ \sum_{l=1}^{j-1} 1 - (j-1) \right\} = 0.$$

Thus, $\mathbf{A}^T\mathbf{A} = \mathbf{I}_n$. Since $\mathbf{A}$ is a $n \times n$ matrix, this also implies that $\mathbf{A}^T = \mathbf{A}^{-1}$ and $\mathbf{A}\mathbf{A}^T = \mathbf{I}_n$.

(ii) In matrix notation, $\mathbf{Y} = \mathbf{A}^T(\mathbf{X} - \mathbf{m})$, where $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)'$. Thus, the inverse transformation is given by $\mathbf{X} = \mathbf{A}\mathbf{Y} + \mathbf{m}$, which is a linear transformation. So, $X_i = \mathbf{b}_i'\mathbf{Y} + \mu$ for all $i = 1, 2, \ldots, n$, where $\mathbf{b}_i'$ is the ith row of $\mathbf{A}$. Also, the Jacobian of the inverse transformation is $\mathbf{A}$. Since $\mathbf{A}$ is an orthogonal matrix, $|\det(\mathbf{A})| = 1$. Define

$\mathbf{y} = (y_1, y_2, \dots, y_n)'$. The joint distribution of $Y_1, Y_2, \dots, Y_n$ is given by

$$
\begin{aligned}
f_{Y_1,Y_2,\dots,Y_n}(y_1, y_2, \dots, y_n) &= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (\mathbf{b}_i'\mathbf{y} + \mu - \mu)^2\right\} \times |\det(\mathbf{A})| \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n \mathbf{y}'\mathbf{b}_i\mathbf{b}_i'\mathbf{y}\right\} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\mathbf{y}'\left(\sum_{i=1}^n \mathbf{b}_i\mathbf{b}_i'\right)\mathbf{y}\right\} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\mathbf{y}'\left(\mathbf{A}^T\mathbf{A}\right)\mathbf{y}\right\} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\mathbf{y}'\mathbf{y}\right\} \\
&= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n y_i^2\right\}.
\end{aligned}
$$

So,

$$f_{Y_1,Y_2,\dots,Y_n}(y_1, y_2, \dots, y_n) = \prod_{i=1}^n \left(\frac{1}{2\pi\sigma^2}\right)^{1/2} \exp\left\{-\frac{y_i^2}{2\sigma^2}\right\},$$

which implies that $Y_1, Y_2, \dots, Y_n$ are independent random variables. Further, $Y_i \sim N(0, \sigma^2)$ for each $i = 1, 2, \dots, n$.

(iii) By definition of Y_1, it follows that $Y_1 = n^{-1/2}\sum_{i=1}^n (X_i - \mu) = \sqrt{n}(\bar{X} - \mu)$.
Observe that

$$\sum_{i=1}^n Y_i^2 = \mathbf{Y}'\mathbf{Y} = (\mathbf{X} - \mathbf{m})'\mathbf{A}\mathbf{A}^T(\mathbf{X} - \mathbf{m}) = (\mathbf{X} - \mathbf{m})'(\mathbf{X} - \mathbf{m}) = \sum_{i=1}^n (X_i - \mu)^2.$$

Thus,

$$\sum_{i=2}^n Y_i^2 = \sum_{i=1}^n Y_i^2 - Y_1^2 = \sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 = (n-1)S^2.$$

(iv) Note that $\bar{X}$ is a function of Y_1 only and S^2 is a function of $Y_2, Y_3, \dots, Y_n$. Since Y_1 is independent of $Y_2, Y_3, \dots, Y_n$, it follows that $\bar{X}$ and S^2 are independent.
Since $Y_1 \sim N(0, \sigma^2)$, it follows that $\bar{X} \sim N(\mu, \sigma^2/n)$. Further, $(n-1)S^2/\sigma^2 = \sum_{i=2}^n (Y_i/\sigma)^2$. As $Y_i/\sigma \overset{iid}{\sim} N(0, 1)$ for $i = 2, 3, \dots, n$, the latter sum is that of $(n-1)$ independent χ_1^2 random variables. So, $(n-1)S^2/\sigma^2 \sim \chi_{n-1}^2$.

Exercice 34.
First observe that the random variables $Y_i = X_i^2, i = 1, 2, \dots, n$, are independent, and each one of them have a χ_1^2 distribution. So, the joint density of $Y_1, Y_2, \dots, Y_n$ is given by

$$f_{Y_1,Y_2,\dots,Y_n} = \frac{c_n \exp\{-\sum_{i=1}^n y_i/2\}}{(y_1 y_2 \dots y_n)^{1/2}},$$

where c_n is a constant depending on n.
Define the transformation $Z_1 = \sum_{i=1}^n Y_i, Z_2 = Y_2/Z_1, \ldots, Z_n = Y_n/Z_1$. The inverse transformation is given by $Y_1 = Z_1(1 - \sum_{i=2}^n Z_i), Y_2 = Z_1 Z_2, \ldots, Y_n = Z_1 Z_n$. The Jacobian of the inverse transformation is given by

$$\begin{bmatrix} 1-\sum_{i=2}^n z_i & -z_1 & -z_1 & -z_1 & \ldots & -z_1 \\ z_2 & z_1 & 0 & 0 & \ldots & 0 \\ z_3 & 0 & z_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ z_n & 0 & 0 & 0 & \ldots & z_1 \end{bmatrix} = \begin{bmatrix} 1-\sum_{i=2}^n z_i & \mathbf{a}' \\ \mathbf{b} & z_1 \mathbf{I}_{n-1} \end{bmatrix},$$

where $\mathbf{a} = (-z_1, -z_1, \ldots, -z_1)'$ and $\mathbf{b} = (z_2, z_3, \ldots, z_n)'$. Using the formula for the determinant of a partitioned matrix, it follows that the determinant of the Jacobian of the inverse transformation is $\det(z_1 \mathbf{I}_{n-1}) \times \{1 - \sum_{i=2}^n z_i - \mathbf{a}' z_1^{-1} \mathbf{I}_{n-1} \mathbf{b}\} = z_1^{n-1}$.
So, the joint density of $Z_1, Z_2, \ldots, Z_n$ is given by

$$\begin{aligned} f_{Z_1,Z_2,\ldots,Z_n}(z_1, z_2, \ldots, z_n) &= \frac{c_n \exp\{-z_1/2\} z_1^{n-1}}{\left(z_1^n [1 - \sum_{i=2}^n z_i] z_2 z_3 \ldots z_n\right)^{1/2}} \\ &= \frac{c_n \, z_1^{n/2-1} \exp\{-z_1/2\}}{\left([1 - \sum_{i=2}^n z_i] z_2 z_3 \ldots z_n\right)^{1/2}}. \end{aligned}$$

Thus, Z_1 is independent of $Z_2, Z_3, \ldots, Z_n$. Since $Y_1/Z_1 = 1 - \sum_{i=2}^n Y_i/Z_1 = 1 - \sum_{i=2}^n Z_i$, it follows that Y_1/Z_1 is also independent of Z_1.