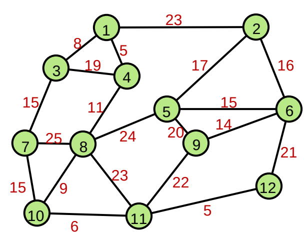
Exercise 10. Exercises for the course "Discrete Mathematics" (2021)

Exercise 1. Let G be a connected weighted graph. Assume it has at least one cycle, and let edge e be an edge that has strictly greater cost than all other edges in that cycle. Show that e does not belong to any minimal weight spanning tree of G.

Exercise 2. Prove that any connected graph with distinct weights assigned to the edges has a unique minimal weight spanning tree.

Exercise 3. Apply Kruskal's algorithm to the following graph to obtain a minimum spanning tree:



Exercise 4. The following is called Prim's algorithm: Given a weighted graph G

- Initialize a tree with a single vertex, chosen arbitrarily from the graph.
- Grow the tree by one edge: of the edges that connect the tree to vertices not yet in the tree, find the minimum-weight edge, and add it to the tree.
- Repeat step 2 (until all vertices are in the tree).

Prove that the output of Prim's algorithm is a minimum spanning tree of G. Apply Prim's algorithm to the graph from exercise 2.

Exercise 5. Let $G = (V, E)$ be a weighted graph with the weight function $f : E \to \mathbb{R}$. What can you say about the output of Kruskal's algorithm for G if

(1) f is a constant function. That is for all $e \in E$, $f(e) = C$ for some fixed $C \in \mathbb{R}$?
(2) f is an injective function?
(3) f is replaced by the function $-f$ defined as $e \mapsto -f(e)$?
(4) f is replaced by a weight function $f' : E \to \mathbb{R}$ such that for all $e_1, e_2 \in E, f(e_1) \leq f(e_2) \Leftrightarrow f'(e_1) \leq f'(e_2)$?