

Manifolds

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1 Recap

Recall theorems about differentiable maps

— Implicit function theorem

For $U \subset \mathbb{R}^p, V \subset \mathbb{R}^q, f \in C^k(U \times V, \mathbb{R}^q), 1 \leq k \leq \infty$ and $(a, b) \in U \times V$ st.

$$D_2 f(a, b) = D(f(a, -))(b)$$

is invertible. Then there exists $a \in U_1 \subset U, b \in V_1 \subset V$ and $\phi \in C^k(U_1, V_1)$ such that

$$f(x, x') = y_0$$

iff $x' = \phi(x)$

— Inverse function theorem

If $U \subset \mathbb{R}^p$ is open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty, a \in U$ such that

$$Df(a)$$

is invertible, then there are $a \in U_1 \subset U$ and $f(a) \in V_1 \subset \mathbb{R}^q$ open such that

$$f|_{U_1} : U_1 \rightarrow V_1$$

is a diffeomorphism and

$$Df^{-1}|_U(x) = (Df(f^{-1}|_U(x)))^{-1}$$

for all $x \in U$ in particular f^{-1} is C^k

— Rank theorem

$U \subset \mathbb{R}^p$ open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty, a \in U, b := f(a), r = \text{rank}(Df(a))$ then there are diffeomorphisms

$$\psi : U_\psi \rightarrow V_\psi \text{ and } \phi : U_\phi \rightarrow V_\psi$$

with $U_\psi, V_\psi \subset \mathbb{R}^p$ and $U_\phi, V_\phi \subset \mathbb{R}^q$ such that

$$\phi \circ f \circ \psi(x_1, \dots, x_p) = (x_1, \dots, x_r, \tilde{f}(x_1, \dots, x_p))$$

If $\text{rk}(D(f))$ is constant around r , then we can obtain $\tilde{f} = 0$

2 Manifolds

Definition 1 (Basis)

A basis for a topology on X is a collection B of open sets such that every open set in X is the union of sets in B .

X is called second countable if it has a countable topological basis.

Definition 2 (Chart)

Let X be a topological space

1. *A chart on X is a pair (U, ϕ) where $U \subset X$ open and $\phi : U \rightarrow \mathbb{R}^n$ for some n which is a homeomorphism onto an open subset.*
2. *An atlas is a collection of charts $A = \{(U_i, \phi_i) | i \in I\}$ such that $X = \bigcup_{i \in I} U_i$*
3. *A is called smooth (C^k , continuous, holomorphic, algebraic, ...) if and only if for any*

$$(U_i, \phi_i)_{i \in \{1, 2\}} \in A$$

we have $\phi_1 \circ \phi_2^{-1}$ is smooth (C^k , ...) wherever it is defined.

4. *A chart (U, ϕ) is compatible with an atlas A if and only if*

$$A \cup \{(u, \phi)\}$$

is smooth

5. *An atlas A is maximal if it contains all charts compatible with A . For any atlas A (not necessarily maximal), denote A_{max} the maximal atlas containing it.
This maximal atlas is necessarily unique*

Definition 3 (Manifold)

A smooth manifold of dimension n is a second countable Hausdorff space with a maximal smooth atlas of dimension n .

Why Hausdorff?

Consider \mathbb{R}/\sim , $x \sim y \iff |x| = |y| > 1$, this space is locally homeomorphic to \mathbb{R} but the points x and y cannot be separated.

Why second countable?

Take a disjoint union of infinitely many manifolds.
For a connected example, take $\mathbb{N}_1 \times [0, 1)$ with the order topology.

2.1 Smooth maps

A function $f : M \rightarrow N$ between smooth manifolds is called smooth if for each $p \in M$, there are charts $(U, \phi), (V, \psi)$ $p \in U \subset M, f(p) \in V \subset N$ such that

$$\psi \circ f \circ \phi^{-1}$$

is smooth.

f smooth implies $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is smooth for any charts $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi})$ where this is defined.

Lecture 2: Smooth maps

Mon 17 Oct

Example (Projective Spaces)

Let $K = \mathbb{R}$ or \mathbb{C} , take $K\mathbb{P}^n = \{ \text{all lines in } K^{n+1} \} = K^{n+1} \setminus 0 / \sim$.

Then $x \sim y \iff \exists \lambda x = \lambda y$

We have $\mathbb{RP}^n = S^n / x \sim -x = S^n / \mathbb{Z}/2\mathbb{Z}$

Similarly, $\mathbb{CP}^n = S^{2n+1} / S^1$.

To give projective space a smooth structure, we introduce homogeneous coordinates.

We write $[x] = [x_0 : \dots : x_n]$ for the equivalence class of x .

For $0 \leq j \leq n$ put

$$U_j = \{[x] \in K\mathbb{P}^n / x_j \neq 0\}$$

and $\phi_j : U_j \rightarrow K^n$ is a chart sending $[x] \rightarrow (\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j})$

Remark

1. Composition of smooth maps are smooth.
2. If $M \xrightarrow{f} N$ is a diffeomorphism if it is a smooth map whose inverse is smooth.

2.2 Partitions of Unity

Theorem 3 (Paracompactness)

Let M be a smooth manifold.

Let (U_α) be an open covering of M .

Then there exists a locally finite refinement $(V_\beta)_{\beta \in B}$ that is

1. locally finite, ie. each point has an open neighbourhood which meets finitely many V_β .

2. $\forall \beta \exists \alpha V_\beta \subset U_\alpha$

More precisely, we can choose V_β such that there exist charts $\psi_\beta : V_\beta \rightarrow \{x \in \mathbb{R}^n \mid |x| < 3\}$ and such that M is covered by

$$M = \bigcup_{\beta \in B} \psi_\beta^{-1}(\{x \in \mathbb{R}^n \mid |x| < 1\})$$

Proof

From the definition, it is clear that any manifold is locally compact.

Hence there exist compact sets

$$K_1 \subset K_2^o \subset K_2 \subset K_3^o \dots$$

such that $M = \bigcup K_j$.

$K_{j+1} \setminus K_j^o$ is compact, hence for $p \in K_{j+1} \setminus K_j^o$, there exists (V_p, ϕ_p) with $\phi_p(V_p) = B(0, 3)$, $V_p \subset K_{j+2}^o \setminus K_{j-1}$ and $V_p \subset U_\alpha$.

By compactness $\exists p_{j_1}, \dots, p_{j_{r_j}}$ such that

$$K_{j+1} \setminus K_j^o = \bigcup \phi_{j_l}^{-1}(B(0, 1))$$

The union of all these charts is a locally finite refinement with the desired properties. \square

We can now define a map $f_1 : \mathbb{R} \rightarrow \mathbb{R}$, $f_1(t) = e^{-\frac{1}{t}}$ if $t > 0$ and 0 if not.

f_1 is $C^\infty(\mathbb{R})$.

Now, define $f_2(t) = \frac{f_1(t)}{f_1(t) + f_1(1-t)}$ and then $f_3(t) = f_2(2+t)f_2(2-t)$.

We can now define $f_4 : \mathbb{R}^n \rightarrow \mathbb{R}$ as $f_4(x) = f_3(|x|)$

Theorem 4 (Partition of unity)

Let M be a C^∞ manifold and (U_α) an open covering.

There exist $\phi_U \in C^\infty(M)$ such that

1. $0 \leq \phi_n \leq 1$
2. $\text{Supp } \phi_n$ is locally finite
3. $\forall n \exists \alpha \in A, \text{Supp } \phi_n \subset U_\alpha$
4. $\forall p \in M, \sum_{n=1}^\infty \phi_n(p) = 1$