Exercise 11

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Without loss of generality, we can suppose that $e = \{1, n\}$.
This means that we can write $L(G)$ in the following way

$$L = \begin{pmatrix} n-2 & -1 & -1 & \dots & 0 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \dots & n-2 \end{pmatrix}$$

Now we remove the last line and last column to obtain L_0, we also take the determinant of both sides

$$\det L_0 = \begin{vmatrix} n-2 & -1 & -1 & \dots & -1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix} \quad \text{Note that we now have a } n-1 \times n-1 \text{ matrix.}$$

We now add each row to the top row, this does not change the value of the determinant.

$$\det L_0 = \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ -1 & n-1 & -1 & \dots & -1 \\ -1 & -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & n-1 \end{vmatrix}$$

Then we exchange the first and last row, as well as the first and last collumn, this, again, does not change the value of the determinant.

$$\det L_0 = \begin{vmatrix} n & 0 & 0 & \dots & -1 \\ 0 & n & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$

Now we add each row $\frac{-1}{n}$ times to the $n-1$-th row.

$$\det L_0 = \begin{vmatrix} n & 0 & 0 & \dots & -1 \\ 0 & n & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{n-2}{n} \end{vmatrix}$$

Since this now is a upper triangular matrix, we can easily compute it's determinant.

$$\det L_0 = n^{n-2}\frac{n-2}{n} = n^{n-3}(n-2)$$

We can now use Kirchhof's theorem to see that

$$\# \textbf{ number of spanning trees of } G = n^{n-3}(n-2)$$

Note that if $n \leq 3$, then removing an edge yields a disconnected graph, which does not contain a spanning tree.

Alternatively, one could notice that the number of spanning trees of G is the number of spanning trees of K_n minus the number of spanning trees which do not contain $\{1\}$ minus the number of spanning trees which do not contain $\{n\}$, but why do that when you can have fun with row operations.