Serie 13

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Exercise 1. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : y > 0\}$. Use the Fourier transform to solve the following initial value problem

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega, \\ u(x,0) = \dfrac{8x^2}{(1+x^2)^2} & x \in \mathbb{R}, \\ \lim_{y\to+\infty} u(x,y) = 0 & x \in \mathbb{R}\,. \end{cases}$$

Hint: Use that for $\omega \neq 0$ we have

$$f(x) = \frac{1}{x^2+\omega^2} \quad \Rightarrow \quad \hat{f}(\xi) = \pi\frac{e^{-2\pi|\xi\omega|}}{|\omega|},$$
$$g(x) = \frac{4x^2}{(x^2+\omega^2)^2} \quad \Rightarrow \quad \hat{g}(\xi) = 2\pi\Big(\frac{1}{|\omega|} - 2\pi|\xi|\Big)e^{-2\pi|\xi\omega|}.$$

Exercise 2.

(i) Let $f \in L^1(0,2\pi)$. Find, formally, the solution $u = u(x,t)$ of the initial value problem

$$\begin{cases} u_t(x,t) = u_{xx}(x,t) & (x,t) \in (0,2\pi)\times(0,\infty), \\ u(0,t) = u(2\pi,t) & t > 0, \\ u_x(0,t) = u_x(2\pi,t) & t > 0, \\ u(x,0) = f(x) & x \in (0,2\pi)\,. \end{cases}$$

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic. Discuss under which additional assumptions on f , we have that

$$\lim_{t\to 0} u(x,t) = f(x) \qquad \text{uniformly in } x.$$

Exercise 3. Let $f \in L^1(\mathbb{R})$ and assume that $\hat{f} \geq 0$ pointwise and f is continuous in a neighbourhood of 0.

(i) Show that $\hat{f} \in L^1(\mathbb{R})$.

(ii) Give an example showing that $\hat{f}$ doesn't need to be in $L^1(\mathbb{R})$ if we drop the assumption $\hat{f} \geq 0$.

Hint: Choose $\varphi \in C^\infty(\mathbb{R})$ to be a standard Gaussian and consider $\varphi_\varepsilon(x) = \varepsilon^{-1}\varphi\left(\frac{x}{\varepsilon}\right)$. Use and prove that $\lim_{\varepsilon\to 0}(f * \varphi_\varepsilon)(0) = f(0)$.

Exercise 4. Consider the 1-dimensional wave equation

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0 & (x,t) \in \mathbb{R}\times(0,\infty) \\ u(x,0) = g(x), \partial_t u(x,0) = h(x) & x \in \mathbb{R}\,. \end{cases}$$

with $f, g, h \in L^2(\mathbb{R})$.

(i) Find, formally, a representation formula for u.

(ii) Assume that $u \in C^2(\mathbb{R}\times(0,\infty))$ such that $u(\cdot,t)$ has compact support in space for every fixed time $t > 0$, that $g \in C^1_c(\mathbb{R})$ and that $h \in C_c(\mathbb{R})$. Define the total energy at time t, that is the sum of the kinetic energy and the potential energy,

$$E(t) := \frac{1}{2}\int_{\mathbb{R}^n} (u_t(x,t))^2 + (u_x(x,t))^2\, dx$$

for $t \geq 0$. Show that $E(t) = E(0)$ for $t > 0$.

(iii) ($\star$) Under the assumptions of (ii), show that asymptotically as $t \to \infty$, the total energy splits equally into its kinetic and potential parts; that is

$$\lim_{t\to\infty}\int (u_x(x,t))^2\, dx = \lim_{t\to\infty}\int (u_t(x,t))^2\, dx = E(0).$$

Hint: Show that for every $f \in C_c^\infty(\mathbb{R})$, it holds

$$\lim_{t\to\infty}\int_{\mathbb{R}} \cos(t2\pi|\xi|)\sin(t2\pi|\xi|)f(\xi)\, d\xi = 0\,. \tag{10}$$