Exercise 10

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16 mai 2021

As a preliminary result, we will show that, given a graph G and a walk $U = (u_1, \ldots, u_k)$ of G, there exists $I \subset [k]$, such that $V = \{u_i, i \in I\} \subset \{u_i, 0 < i \leq k\}$ is a path, and such that the first and last elements of V and U coincide.

Indeed, consider the walk $(u_1, \ldots, u_k)$ of G, if $\forall i, j \in [n], i \neq j : u_i \neq u_j$, then $u_1, \ldots, u_k$ already is a path and we are finished.
Hence, suppose that there exists $i, j \in [k], i \neq j$ such that $u_i = u_j$, without loss of generality we can suppose that $i < j$.
Set $I = [k]$. Then, we can redefine $I = I \setminus \{i+1, \ldots, j\}$.
We now obtain a new walk given by $(v_1, \ldots, v_j) = \{u_i\}_{i=1, i \in I}^k$.
We now repeat this algorithm until we are left with a path.
Since a walk is always of finite length and the size of I decreases by at least 1 at each step, we are guaranteed that the algorithm finishes.
Also notice that the first and last elements always stay the same at each step of the algorithm, which proves that $v_1 = u_1, v_j = u_k$.

We will write V for the set of vertices of G.
Let $e \in E(G) \setminus E(T)$, we can write e as $e = \{a, b\}, a, b \in V$.
Since T is connected, there exists a path in T of the form $(a, v_1, \ldots, v_n, b), v_i \in G$.
We define $K = T + e - \{v_n, b\}$.
Clearly, K is still spanning since it still contains v_n, a and b [1] .
We now show that K is a tree.

First, we show that K still is connected.
Indeed, consider two vertices $x, y \in V$.
Consider the path of T which would connect x to y : $x, u_0, \ldots, u_n, y$.
If $\forall i \in [n]$, $\{u_1, u_{i+1}\} \neq \{v_n, b\}$, the path is still contained in K and we have finished.
If there exists $0 \leq i \leq n$ such that $\{u_i, u_{i+1}\} = \{v_n, b\}$, replace u_i, u_{i+1} in the path with the path $v_n, v_{n-1}, \ldots, a, b$, we are left with a walk contained

1. v_n is still contained in K because $\{v_{n-1}, v_n\}$ is contained in K

in K.
Using the lemma proven above, we extract a path from this walk.
This walk being contained in K, we deduce that K is connected.

We now show that K contains no cycle.
For the sake of contradiction, suppose K contains a cycle of the form $c_0, \dots, c_k, c_0$.
If $\forall 0 \leq j \leq k, \{c_j, c_{j+1}\} \neq e$, then the cycle is contained in T which is a contradiction since T is a tree.
Hence, suppose there exists a j such that $\{c_j, c_{j+1}\} = e$, without loss of generality, suppose that $c_j = a$ and $c_{j+1} = b$.
If that were the case, we could again create a new walk of the form $c_0, \dots, c_{j-1}$, $a, v_1, \dots, v_n, b, c_{j+2}, \dots, c_n, c_0$, which by definition is contained in T.
We now extract a path from this walk, clearly this path is a cycle contained in T.
Hence, the existence of a cycle in K implies the existence of a cycle in T, which is impossible since T is a tree.

We deduce that K also is a spanning tree of T, and the result follows.