Exercise to submit 2

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We will denote by $F(k,n)$ the set of all maps from $[k]$ to $[n]$. We also denote by $S(k,n)$ the set of all surjective maps from $[k]$ to $[n]$.
To compute the set of all surjective maps, we simply substract the number of non-surjective maps from $F(k,n)$.
Let $A_i \subset F(k,n)$ denote the set of all maps from $[k] \to [n]$ whose image does not contain a subset of cardinality i, ie.

$$A_i = \{f \in F(k,n) | \exists I \text{ such that } |I| = i \text{ and } f(A) \cap I = \emptyset\}$$

Hence, we can write

$$S(k,n) = F(k,n) \setminus \left(\bigcup_{i \in [n]} A_i \right)$$

Let us now compute the cardinality of $A_{\{i\}}$.
First we choose i elements from $[n]$, these elements will not be contained in the image.
For the $n-j$ remaining elements, we build a k-letter words. Hence, the cardinality of A_i is given by

$$|A_i| = \binom{n}{i} \cdot (n-j)^k$$

We can now apply the inclusion-exclusion formula, which yields

$$\left| \bigcup_{i \in [n]} A_I \right| = \sum_{i=1}^{n} (-1)^{i+1} |A_i| = \sum_{i=1}^{n} (-1)^{i+1} \cdot \binom{n}{i} \cdot (n-j)^k$$

Finally, we can compute

$$\begin{aligned} |S(k,n)| &= |F(k,n)| - |\bigcup_{i \in [n]} A_i| \\ &= n^k - \sum_{i=1}^{n} (-1)^{i+1} |A_i| \\ &= \sum_{i=0}^{n} (-1)^i \cdot \binom{n}{i} \cdot (n-j)^k \end{aligned}$$

Which is the desired result.