Notes

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Contents

1 Lie Algebras of Algebraic Groups

Throughout, G is an algebraic group over $k = \overline{k}$ of char. 0.

Definition 1 (Lie Algebra of algebraic group) *We define*

$$\mathfrak{g} = \mathrm{Lie}(G) = T_eG = \mathrm{Der}(\mathcal{O}_{G,e}) = \mathrm{Der}_G(\mathcal{O}_G).$$

Where $\mathrm{Der}_G(\mathcal{O}_G)$ *are the* G*-invariant derivations.*

Let's see that these identifications make sense, first we show the identification

$T_eG = \mathrm{Der}_G(\mathcal{O}_G)$

Let $\delta \in \mathrm{Der}_G(\mathcal{O}_G)$ and $f \in \mathcal{O}(G)$, consider the map $f \mapsto \delta f(e) \in k$, we get an induced map $\mathrm{Der}_G(\mathcal{O}_G) \to T_e(G)$ by mapping $\delta \mapsto (\mathrm{Id} + \delta(-)(e)) \cdot \epsilon)$.
Given $\mathrm{Id} + \alpha\epsilon \in T_eG$, define a derivation $\delta_\alpha \colon A \to A$ defined by $\delta_\alpha(f)(g) = g \cdot \delta_\alpha(f)(e)$

$\mathrm{Der}_G(\mathcal{O}_G) = \mathrm{Der}(\mathcal{O}_{G,e})$

Given $f \in \mathrm{Der}_G(\mathcal{O}_G)$, there is a natural map on stalks $f \in \mathrm{Der}(\mathcal{O}_{G,e})$.
The other way, let $\partial \colon \mathcal{O}_{G,e} \to \mathcal{O}_{G,e}$ be a derivation. Define a derivation $\delta \colon A \mapsto A$ by $\delta(f)(g) = g \cdot \partial f(g)$.

The association $G \mapsto \mathrm{Lie}(G)$ extends to a functor.

Theorem 1

$$\{\, H \subset G \textit{ closed connected} \,\} \to \{\, \mathfrak{h} \subset \mathfrak{g} \,\}$$

is injective.

1.1 Borel and Parabolic subgroups

Let G be an algebraic and $B \subset G$ a Borel subgroup.

Theorem 2 G/B *is projective and all Borel subgroups are conjugate.*

Proof Let $H \subset G$ be a Borel subgroup of maximal dimension. Let V be a G-vector space and $L \subset V$ a line such that $G_L = H$.
Now $H \curvearrowright \mathcal{F}\ell(V/L)$ and because $\mathcal{F}\ell(V/L)$ is complete, there is a fixed point.

Extend this fixed point to a full flag of V by L, denote this flag $F \in \mathcal{F}\ell(V)$, then $G/H \to \mathbb{P}(V)$ sending $gH \mapsto gF$ is bijective, denote the image by GF.
We claim GF is projective, indeed, since H is of maximal dimension, it suffices to show that GF is an orbit of minimal dimension. If $L \subset G$ stabilises a flag, then L is solvable and hence $\dim G/H \geq \dim G/L$. Now GF is an orbit of minimal dimension and hence is closed.
B acts on G/H by left multiplication, since G/H is projective, there is a class gH such that $BgH = gH$, thus $g^{-1}Bg \subset H$, since both groups are Borel, they are equal. □

Theorem 3 *For any parabolic subgroup* $P \subset G$*, we have* $N_G(P) = P$*.*