École Polytechnique Fédérale de Lausanne
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Discrete Optimization (Spring 2022)

Assignment 1

Problem 1

Consider the optimization problem

$$\min\{f(x,y) = x^2 + y : g(x,y) = y^3 - x^2 \geq 0\}$$

(a) Try to solve it using the method of Lagrange multipliers.

(b) Find the true solution (try graphing it).

(c) Why did Lagrange multipliers fail? Can you come up with a modification of the method that gives the correct solution?

Problem 2

Picture yourself in the role of a production manager. There are n production tasks to be executed, and a task j requires p_j working hours to be completed. You have m employees at your disposal that can each, due to his or her qualifications, work on a subset of the tasks. Denote by S_i the set of jobs that employee i can work on. A work allocation plan has to ensure that all tasks are completed. You want to create an allocation which is also fair: the maximum number of working hours assigned to an employee is to be minimized. Model this as an optimization problem with the following properties:

- The objective function $f(\mathbf{x})$ is linear (that is, $f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$ for some vector $\vec{c}$)
- The feasible region is defined by linear constraints (that is, they all have the form $\mathbf{a}_i \cdot \mathbf{x} \geq b_i$ for some vectors $\mathbf{a}_i$ and real numbers b_i).

Problem 3

Suppose we are given a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ that form a basis and we wish to express a vector $\mathbf{x}$ as a linear combination of the basis vectors. Find a method (formula) for doing this.

Problem 4

Let $H_0, H_1, \ldots, H_k$ be a collection of hyperplanes in $\mathbb{R}^{k+1}$ with normal vectors $\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_k$.

(a) Show that the intersection $H_1 \cap \cdots \cap H_k$ forms a line if and only if the vectors $\mathbf{n}_1, \ldots, \mathbf{n}_k$ are linearly independent.

(b) Assume $H_1 \cap \cdots \cap H_k$ forms a line L in such a way that $L \cap H_0$ is a single point. Show that $\{\mathbf{n}_0, \mathbf{n}_1, \ldots, \mathbf{n}_k\}$ form a basis for $\mathbb{R}^{k+1}$.

Problem 5

Show that if $C_1, C_2, \ldots, C_k$ are convex sets, then $C_1 \cap C_2 \cap \cdots \cap C_k$ is also a convex set.

Problem 6

Show that $g(n) = O(f(n))$ and $f(n) = O(g(n))$ for the following two functions[1] :

$$f(n) = \log(n!) \quad \text{and} \quad g(n) = n \log n$$

[1] Note that one benefit of big-O notation is that log can be taken to be any basis, so pick whichever one is convenient.