Discrete Mathematics

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List of Theorems

Lecture 1: Introduction

Mon 22 Feb

1 Counting

1.1 Finite sets

Let A be a finite set. We denote by $|A|$ the cardinality of A.

Definition 1 (First Numbers)
We denote by $[n]$ the set of n first natural numbers.

1.2 Bijections

Theorème 1
If there exists a bijection between finite sets A and B then $|A| = |B|$.

1.3 Operations with finite sets

— union
— intersection
— product
— exponentiation
— quotient

Definition 2 (Cartesion product)

$$A \times B = \{(a, b) | a \in A, b \in B\}$$

Theorème 2

$$|A \times B| = |A||B|$$

Definition 3 (Disjoint union)
Define

$$A \sqcup B = A \times \{0\} \cup B \times \{1\}$$

Theorème 3

$$|A \sqcup B| = |A| + |B|$$

Definition 4 (Exponential object)

$$A^B = \{f | f \textit{ is a function from } A \textit{ to } B \}$$

Theorème 4

$$|A^B| = |A|^{|B|}$$

Definition 5 (Binomial coefficient)

A binomial coefficient $\binom{n}{k}$ *is the number of ways in which one can choose* k *objects out of* n *distinct objects assuming order doesn't matter.*

Proposition 5

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Proposition 6

The following identities hold :

1. $$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

2. $\binom{n}{k}$ *is the* k*-th element in the* n*-th line of Pascal's triangle.*

Preuve

Each subset of $[n+1]$ *either contains* $n+1$ *or not.*
Number of $(k+1)$*-element subsets containing* $n+1$ *is* $\binom{n}{k}$
Number of $(k+1)$*-element subsets not containing* $n+1$ *is* $\binom{n}{k+1}$ □

Proposition 7

The number of subsets of an n*-element set is* 2^n*, since we have*

$$2^n = \sum \binom{n}{i}$$

Proposition 8

The number of subsets of even cardinality is the same as even cardinality : 2^{n-1}

Preuve

Consider

$$\phi : 2^{[n]} \to 2^{[n]}$$

defined by

$$\phi(A) = A\Delta\{1\} = \begin{cases} A \setminus \{1\}, & \textit{if } 1 \in A \\ A \cup \{1\}, & \textit{otherwise} \end{cases}$$ □

The cardinality of subsets A and $\phi(A)$ always have different parity.
Since $\phi \circ \phi = \mathrm{Id}$ we deduce that ϕ is a bijection between the set of odd and even subsets is the same.

Theorème 9

$$(1+x)^n = \sum \binom{n}{i} x^i$$

Preuve

In lecture notes. □

Proposition 10

Assume we have k identical objects and n different persons. Then ne number of ways in which one can distribute this k objects among the n persons equals

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

Equivalently, it is the number of solutions of the equation $x_1 + ... + x_n = k$

Preuve

Let $\mathcal{A}$ be the set of all solutions of the equation. Let $\mathcal{B}$ be the set of all subsets of cardinality $n-1$ in $k+n-1$.
we construct a bijection $\psi : \mathcal{A} \to \mathcal{B}$ in the following way

$$A = (x_1, \dots, x_n) \mapsto B = \{x_1 + 1, x_1 + x_2 + 2 \dots\}$$

It suffices to show that ψ is invertible. Let $B \in \mathcal{B}$. Suppose that $b_1 \dots, b_{n-1}$ are the elements of B, ordered. Then the preimage is an n-tuple of integers $(x_1, \dots)$ defined by

$$\begin{aligned} x_1 &= b_1 - 1 \\ x_i &= b_i - b_{i-1} \\ x_n &= k + n - 1 - b_{n-1} \end{aligned}$$ □

It is easy to see from these equations that the x_i are non-negative and their sums yield k.

Lecture 2: factorials and birthday paradox

Sat 27 Feb

Theorème 11 (Stirling's formula)

$$n! \sqrt{2\pi n} n^n e^{-n}$$

meaning the ration goes to 1.

Preuve

Euler's integral for $n!$ *gives*

$$n! = \int_0^\infty x^n e^{-x} dx$$

This is proven by induction on n*.*

The base case $n = 0$ *simply gives 1.*

For the integration step, we integrate by parts, giving

$$\int_0^\infty x^n e^{-x} = \int_0^\infty e^{-x} \frac{d}{dx} x^n dx$$

To prove Stirlings formula, we take

$$xt^n e^{-x} = \exp(n \log x - x)$$

We now taylor expand around the maximum, this yields

$$n \log x - x = n \log n - n - \frac{1}{2n}(x-n)^2 + \dots$$ □

integrating this gives the desired formula.

Lecture 3: Inclusion-Exclusion Induction

Sat 06 Mar

Let A, B be two sets, we want to compute $|A \cup B| = |A| + |B| - |A \cap B|$. What happens if we have n sets $A_1, \dots, A_n$.

Theorème 12 (Inclusion-Exclusion Formula)

Let $A_1, \dots, A_n$ *be finite sets, then*

$$|\bigcup A_i| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \dots$$

Let $B_1, \dots, B_m$ *and* $w_1 \dots, w_m$*, then*

$$\sum_i w_i |B_i| = \sum_i \sum_{b \in B_i} w_i = \sum_{b \in B} \sum_{indices\ i\ such\ that\ b \in B_i} w_i$$

where $B = \bigcup B_i$

Lecture 4: Combinatorial applications of polynomials and generating series

Sun 14 Mar

We note that arithmetic operations with finite sets have similarities.

$$(a+b) \cdot c = a \cdot c + b \cdot c$$

$$(A \cup B) \cap C = A \cap C \cup B \cap C$$

Exemple
Prove the identity

$$\sum \binom{n}{i}^2 = \binom{2}{n} n$$

Consider

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n}$$

By computing the coefficients of x^n, we find the desired equality.

Theorème 14 (Multinomial theorem)

$$(x_1 + \ldots + x_n)^k = \sum_{i_1, \ldots, \geq 0, i_1 + i_2 + \ldots = k} \frac{k!}{i_1! \ldots i_n!} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

Preuve
Note that

$$\frac{k!}{i_1! \ldots i_n!}$$

is the number of sequences of length k from the letters "$x_1, x_2, \ldots$ " such that x_j is used i_j times. □

Definition 6 (Generating series)
Let a_n be a sequence of complex numbers, then the generating series of this sequence is

$$a(x) = \sum_{n=0}^{\infty} a_n x^n$$

Definition 7 (Formal power series)
A formal power series is an infinite sum

$$a(x) = \sum a_n x^n$$

where a_n is a sequence of complex numbers and x is a formal variable.

Proposition 15
Let $a(x) = \sum a_n x^n$ be a formal power series. Suppose that there exists a positive real number K such that $|a_n| < K^n$ for all n. Then the series converges absolutely for all $x \in]-\frac{1}{k}, \frac{1}{k}[$.
Moreover, the function $a(x)$ as derivatives of all orders at 0.

We can add and multiply formal power series.
However, in general, substitution is not well defined

$$a(b(x)) = \sum_{n=0}^{\infty} a_n b(x)^n = \sum_{n=0}^{\infty} a_n (\sum_{m=0}^{\infty} b_m x^m)^n$$

It is only well defined if $b_0 = 0$.
We can also differentiate, resp. integrate formal power series.

Theorème 16 (Generalized binomial theorem)
For every $r \in \mathbb{R}$, we have

$$(1+x)^r = \binom{r}{0} + \binom{r}{1} x \dots$$

where

$$\binom{r}{k} = \frac{r(r-1)\dots(r-k+1)}{k!}$$