Analysis IV

David Wiedemann

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Lecture 1: Measure theory

Wed 23 Feb

1 Lebesgue Measure

Motivation

Given a set $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is it possible to integrate f over Ω.
For $n = 1$ and $\Omega = [a, b]$ riemann-integral works, at least for continuous functions.
However, it is not fully satisfactory

1. Extends badly to $\mathbb{R}^n$
2. Stability with limits Take $f_n : [0, 1] \to [0, 1]$ continuous and pointwise decreasing, define $f(x) = \lim f_n(x)$, then the integral over f might not exist.
3. Differentiation and integration.
   What is the biggest class of functions for which the fundamental theorem works ?
   For sure in C_1 but that is not the biggest class.
4. Consider $C^0([0, 1])$ with L^1-distance.
   Then C^0 is not complete, what is the completion of $\bar{C^0}^d$

We want to find a satisfactory theory of integration.
How can we define the length/volume of a subset $\Omega \subset \mathbb{R}^n$?
Ideally to $\Omega \subset \mathbb{R}^n$ associate $m(\Omega) = 0$ with

$0 \leq m(\Omega) \leq \infty \quad m((0, 1)^m) = 1 \quad m(A \cup B) = m(A) + m(B)$ if A and B disjoint.

$$m(A) \leq m(B) \quad m(A + x) = m(A)$$

This is impossible !

1.1 Measurable sets

We can ask that
— (Borel Property) Open and closed are measurable
— Ω measurable $\implies$ Ω^c measurable
— (σ-algebra) We want to take countable intersection of measurable sets

Definition 1 (Lebesgue Measure)
The lebesgue measure $m(\Omega)$ of any measurable set will obey
— $m(\emptyset) = 0$
— $\infty \geq m(\Omega) \geq 0$
— *Monotonicity* $m(\Omega_1) \leq m(\Omega_2)$ *if* $\Omega_1 \subset \Omega_2$

— *If $\Omega_1, \ldots$ are measurable and disjoint, then we want*

$$m(\bigcup_{i=1}^{\infty} \Omega_i) = \sum_{i=1}^{\infty} m(\Omega_i)$$

and with $\leq$ if they are not disjoint.

— *(Normalisation)*

$$m((0,1)^n) = 1$$

— *(Translation invariance)*

$$m(\Omega + x) = m(\Omega) \forall x \in \mathbb{R}^n$$

Remarque

— *From countable subadditivity, finite subadditivity follows*

— *Monotonicity is redundant because, given $\Omega_1 \subset \Omega_2$*

$$m(\Omega_2) = m(\Omega_1 \cup (\Omega_2 \setminus \Omega_1)) = m(\Omega_1) + m(\Omega_2 \setminus \Omega_1)$$

— *The sums above might be infinite*

Remarque

m is a positive measure if the first four conditions above are satisfied

Theorème 3 (Existence of Lebesgue Measure)
There exists a notion of measurable set obeying the conditions of measurable sets and a measure obeying the conditions.

1.2 Outer Measure

We first want to describe a cube and associate a measure to these boxes. Then we will take a more general set, cover it with boxes and define it's measure by the smallest possible covering by boxes.

Definition 2 (Box)
A open box $B \subset \mathbb{R}^n$ is

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

and define the volume of a box

Definition 3 (Volume of a box)
Given $B = \prod_{i=1}^{n} (a_i, b_i)$, we define

$$volB = \prod_{i} (b_i - a_i)$$

Now, how can we cover $\Omega \subset \mathbb{R}^n$?

Definition 4 (Covered set)
Given $\Omega \subset \mathbb{R}^n$ *is covered by* $\{B_j\}_{j\in J}$ *if* $\Omega \subset \bigcup B_j$

Remarque
If m *(the lebesgue measure) exists and* J *is countable, then*

$$m(\Omega) \leq m(\bigcup B_j) \leq \sum m(B_j)$$

Definition 5 (Outer-Measure)
The outer measure of a set Ω *is defined as*

$$m^*(\Omega) = \inf \left\{ \sum volB_j : \{B_j\} \text{ is a countable cover of } \Omega \right\}$$

Remarque
For every Ω *there exists at least one countable cover*

Lemme 6
The outer measure obeys

1. $m^*(\emptyset) = 0$
2. $0 \leq m^*(\Omega) \leq \infty$
3. $m^*(\Omega_1) \leq m^*(\Omega_2)$ *if* $\Omega_1 \subset \Omega_2$
4. $m^*(\Omega + x) = m^*(\Omega)$
5. *Countable subadditivity :* $m^*(\bigcup \Omega_j) \leq \sum m^*(\Omega_j)$

Preuve
- $m^*(\emptyset) = 0$ *because* $\emptyset, \{0\} \subset (-\epsilon, \epsilon)^n \forall \epsilon > 0$
- *All good*
- *Any cover of* Ω_2 *also covers* Ω_1
- *For any cover of* Ω *we can translate it over to* $\Omega + x$
- *For every* $J \in \mathbb{N}$*, let* $\left\{B_i^J\right\}_{i\in I_J}$ *cover* Ω_J*, then* $\Omega_j \subset \bigcup_{i\in I_J} B_i^J$*, then*

$$\sum vol(B_i^J) \leq m^*(\Omega_J) + \frac{\epsilon}{2^J}$$

and since $\left\{B_i^J\right\}_{i,J}$ *covers* $\bigcup \Omega_J$

$$m^*(\bigcup \Omega_J) \leq \sum_{j\in\mathbb{N}} \sum_{i\in I_J} vol(B_i^J) \leq \sum_{j\in\mathbb{N}} (m^*(\Omega_J) + \frac{\epsilon}{2^J}) = \epsilon + \sum m^*(\Omega_J)$$

$\square$

Proposition 7

$$m^*(\overline{B}) = vol(B)$$

Preuve

Clearly $\overline{B}$ *is covered by* $\prod(a_i+\epsilon, b_i+\epsilon)$ *Hence*

$$m^*(\overline{B}) \leq vol(\prod(a_i+\epsilon, b_i+\epsilon)) \to \prod(b_i - a_i)$$

Hence $m^*(\overline{B}) \leq vol(B)$
Now we show that $vol(B) \leq m^*(\overline{B})$.
By Heine-Borel, $\overline{B}$ *is compact.*
Hence we only need to show the result with a finite cover.
In dimension 1, we are given $(a_1, b_1), \ldots$ *covering* $[a,b]$.
Remark that

$$1_{[a,b]} \leq \sum_i 1_{(a_i,b_i)}$$

Integrating (Riemann-integral), we get

$$(b-a) \leq \sum(b_i - a_i)$$

Now, we use induction

$$B_J = \prod_{i=1}^{n}(a_i^s, b_i^s) = \prod_{i=1}^{n-1}(a_i^s, b_i^s) \times (a_n^s, b_n^s)$$

Define

$$f_J(x_m) = vol(A_J)1_{(a_n,b_m)}(x_m)$$

For every x_m*, we get*

$$\left\{A^J : j \in J, x_n \in (a_n^J, b_n^J)\right\} \text{ is a cover of } \overline{A}$$

$$\sum f_j(x_m) = sum_{j\in J,x_n} vol(A_j)1_{(a_n,b_n)} \geq vol\overline{A}$$

□

Lecture 2: Existence of Lebesgue Measure

Thu 24 Feb

Corollaire 8
$m^* * B = vol(B)$ *for every open box* B.

Preuve

For one direction, we use monotonicity, $m^*(B) \leq m^*(\overline{B}) = vol(B)$.
Furthermore, set $B = \prod(a_i, b_i)$*, then for* $\epsilon > 0$*, we get*

$$\prod[a_i+\epsilon, b_i-\epsilon] \subset \prod_i(a_i,b_i) \implies m^*(\prod[a_i+\epsilon, b_i-\epsilon]) \leq \prod_i(b_i-a_i)$$

□

Exemple

- $m^*(\mathbb{R}) = \infty$ *since by monotonicity, we get* $m^*(\mathbb{R}) \geq m^*([0,N]) > N$
- $m^*(\mathbb{Q}) = 0$ *since*

$$m^*(\mathbb{Q}) \leq m^*(\{q\}) = 0$$

  Which proves that the reals are uncountable.

1.3 Measurable sets (again)

We want to know whether $\forall A, E \subset \mathbb{R}^m$, the inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

holds ? This inequality follows directly from countable subadditivity.

Definition 6 (Lebesgue Measurable set)
A set $E \subset \mathbb{R}^m$ *is Lebesgue measurable if*

$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \forall A \subset \mathbb{R}^n$$

Then the lebesgue measure of E *is defined as*

$$m(E) \coloneqq m^*(E)$$

Note that, according to this definition, $\emptyset, \mathbb{R}^n$ are both measurable.

Lemme 10
Half-spaces are measurable

The proof is given as an exercise.
We now establish a few basic facts about measurable sets.

Lemme 11

- *The complement of a measurable set is measurable*
- *The translation of a measurable set is measurable, ie.* E *measurable,* $x \in \mathbb{R}^n$ *implies* $E + x$
- *Finite unions of measurable sets is measurable. (as well as the intersection)*
- *Open (as well as closed) boxes are measurable.*
- *If the outer measure of a set is* 9*, then* E *is measurable.*

Preuve

-

$$m^*(A) = m^*(A \cap E^{c^c}) + m^*(A \cap E^c)$$

- *Given* A *a set and* $x \in \mathbb{R}^n$*, we get*

$$m^*(A-x) = m^*(A-x\cap E)+m^*((A-x)\cap E^c) = m^*(A\cap E+x)+m^*(A\cap E^c+x) = m^*(A)$$

—

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

— *Consider the union of two setsWe now bound $m^*(A)$ by below (the upper bound is always true)*

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

$$= m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \cap E_2^c) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

$$\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

The general result follows immediatly by induction on the number of sets.

— *We get that*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$ □

— *We write boxes as intersections of halfspaces*

Now we want to show that the lebesgue measure is countably additive.

Proposition 12
If $(E_j)_{j \in \mathbb{N}}$ are measurable disjoint sets, then $\bigcup_{i \in \mathbb{N}} E_i$ is measurable and

$$m^*(\bigcup_{j \in \mathbb{N}} E_j) = \sum_{j=1}^{\infty} m^*(E_j)$$

The proof depeds on a lemma

Lemme 13
Let $E_1, \ldots, E_n$ be measurable disjoint sets, $A \subset \mathbb{R}^m$,then

$$m^*(A \cap (\bigcup E)j)) = \sum_{j=1}^{n} m^*(A \cap E_j)$$

As a consequence of this, we get finite additivity.

Preuve
For $n = 2$, we get

$$m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap (E_1 \cup E_2) \cap E_1) + m^*(A \cap (E_1 \cup E_2) \cap E_1^c)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2)$$ □

and the general case follows by induction.

Corollaire 14
$E \subset F$ measurable implies $F \setminus E$ is measurable and

$$m^*(F \setminus E) = m(F) - m(E)$$

Preuve
The set is trivially measurable since $F \setminus E = F \cap E^c$ Using the lemma above, we get

$$m^*(F) = m^*(E) + m^*(F \setminus E)$$

□

We can now prove countable additivity

Preuve
Let $E = \bigcup_{j=1}^{\infty} E_j$.
We claim that $\forall A$

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$$

Indeed note that

$$m^*(A \cap E) \leq \sum_{j=1}^{\infty} m^*(A \cap E_J) = \sup_N \sum_{j=1}^{N} m^*(A \cap E_j)$$

Set $F_n = \bigcup_{j=1}^{N} E_j$, by the lemma, the finite sum above is

$$\sup_N \sum_{j=1}^{N} m^*(A \cap E_j) = m^*(A \cap F_N)$$

Since $F_N \subset E$,

$$m^*(A \setminus E) \leq m^*(A \setminus F_N)$$

Then

$$m^*(A \cap E) + m^*(A \setminus E) < \sup_N m^*(A \cap F_N) + \underbrace{m^*(A \setminus E)}_{\leq m^*(A \setminus F_N)} < \sup_N m^*(A)$$

□

This proves that $m(E) \geq \sup_N m(F_N) = \sup_N \sum_{j=1}^{N} m(E_j) = \sum_{j=1}^{\infty} m(E_J)$

Lemme 15 (Lebesgues sets are a sigma-algebra)
If $(E_J)_J \in \mathbb{N}$ are measurable, then $\bigcup E_j$ and $\bigcap E_j$ are measurable.

Preuve

$$E_1 \cup \ldots = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \ldots$$

and the property about intersections follows from $\bigcap E_J = (\bigcup E_J^c)^c$ □

Lemme 16 (Open sets are measurable)
Every open set is measurable

Preuve
By an exercise, every open set is a countable union of open boxes and a countable union of measurable sets is countable by the lemma above. □

1.4 A glimps on abstract measure theory and theoretical foundations of probability

The idea of Lebesgue was to fix the measure of boxes and then extend the measure to the sigma algebra of measurable sets.

Theorème 17 (Caratheodory theorem)
Given a set Ω, $\mathcal{G}$ an algebra (finite union of boxes), A the smallest algebra containing $\mathcal{G}$.
Let $m_0 : \mathcal{G} \to [0, \infty]$ be a function s.t. $m(\emptyset) = 0, m_0(\bigcup_{i=1}^{\infty} A_i) = \sum_{m=1}^{\infty} m_0(A_m)$ if $A_m \in \mathcal{G}, A_m$ disjoint and $\bigcup A_m \in \mathcal{G}$
Then $\exists$ a measure on A such that $m|_{\mathcal{G}} = m_0$ and, if the measure of $m_0(\Omega) < \infty \implies m$ is unique.

Furthermore

Theorème 18
Every probability $\mathbb{P}$ on $\mathbb{R}^n$ gives rise to a cumulative distribution function, conversely, every cdf gives rise to a (unique) probability measure.

1.5 The cantor set

Definition 7 (Cantor set)
Consider $[1, 1]$, define $P_0 = [0, 1]$, $P_1 = [0, \frac{1}{3},] \cup [\frac{2}{3}, 1]$ and keep going.
By definition $P_0 \supset P_1 \ldots$, the cantor set is the intersection of all of them.

There are a few nice properties of the cantor set

Theorème 19

1. *P is compact*
2. *$m^*(P) = 0$*
3. *P is uncountable*
4. *P is perfect*[a] *and has empty interior.*

a. No point in p is isolated