École Polytechnique Fédérale de Lausanne
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Math 261 – Discrete Optimization (Spring 2022)

Assignment 3

Problem 1

Let P be the polyhedron given by all $(x, y, z) \in \mathbb{R}^3$ such that

$$\begin{array}{rcrcrcr} -7x & + & y & + & 2z & \leq & 4 \\ 5x & - & 11y & + & 2z & \leq & 4 \\ -x & + & 7y & - & 10z & \leq & 4 \\ x & + & y & + & 2z & \leq & 4 \end{array}$$

Find all vertices of P.

Problem 2

Given a vertex $\mathbf{v}$ of a polyhedron P, show that $\mathbf{v}$ must also be an extreme point of P.

Problem 3

Given $\mathbf{x}^* = (0\ 1\ 1)^\intercal \in \mathbb{R}^3$ and the vector $\mathbf{d} = (1\ 1\ {-1})^\intercal \in \mathbb{R}^3$ decide if the ray $\{\mathbf{x}^* + \lambda \mathbf{d} : \lambda \in \mathbb{R}_{\geq 0}\}$ intersects the following hyperplanes while moving in the direction of $\mathbf{d}$. Give the order in which the trajectory passes the planes.

$$P_1 = \left\{\mathbf{x} \in \mathbb{R}^3 : (1\ 2\ 3)\mathbf{x} = 0\right\} \qquad P_2 = \left\{\mathbf{x} \in \mathbb{R}^3 : (3\ 2\ 1)\mathbf{x} = 4\right\}\}$$
$$P_3 = \left\{\mathbf{x} \in \mathbb{R}^3 : (1\ 1\ 1)\mathbf{x} = 2\right\} \qquad P_4 = \left\{\mathbf{x} \in \mathbb{R}^3 : (0\ 1\ 3)\mathbf{x} = -1\right\}$$

Problem 4

(a) Given an $m \times n$ matrix $\mathbf{A}$ and a vector $\mathbf{b} \in \mathbb{R}^m$, assume that there exists a vector $\boldsymbol{\lambda} \in \mathbb{R}^m$ such that $\boldsymbol{\lambda}^\intercal \mathbf{A} = \mathbf{0}$ and $\boldsymbol{\lambda} \cdot \mathbf{b} > 0$. Show that the system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

has no solution.

(b) Prove that the system of linear equations

$$\begin{bmatrix} 2 & 1 & 0 \\ 5 & 4 & 1 \\ 7 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

has no solution by finding a vector $\boldsymbol{\lambda}$ that satisfies part (a) of this problem.

(c) Show that if you can find a $\boldsymbol{\lambda}$ like the one in part (a) which *also* satisfies $\boldsymbol{\lambda} \geq \mathbf{0}$, then the polyhedron $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ must be empty.

Problem 5

Consider the following linear program in inequality standard form:

$$\begin{array}{llrcrcrcrl} \mathcal{P} = & \min & 5x & - & 3y & - & 2z & & & \\ & \text{s.t.} & 2x & - & 4y & + & 2z & \geq & -6, & (1) \\ & & -5x & + & 3y & - & 5z & \geq & -1, & (2) \\ & & -4x & + & 2y & - & z & \geq & -3, & (3) \\ & & x & & & & & \geq & 0, & (4) \\ & & & & y & & & \geq & 0, & (5) \\ & & & & & & z & \geq & -1 & (6) \end{array}$$

(a) Show that the point $\mathbf{v} = (0, 16/7, 11/7)^\intercal$ is a vertex of the feasible region of $\mathcal{P}$.

(b) Rewrite $\mathcal{P}$ in equality standard form.

(c) Let $\mathcal{P}'$ be the linear program you found in (b). Find a solution $\mathbf{v}'$ in $\mathcal{P}'$ that has the same cost value as $\mathbf{v}$ does in $\mathcal{P}$ (the $\mathbf{v}$ and $\mathcal{P}$ from part (a)). Determine which of the nonnegativity constraints in $\mathcal{P}'$ are active at $\mathbf{v}'$.

(d) Try to prove that $\mathbf{v}$ is an optimal solution to $\mathcal{P}$.

Hint: We haven't discussed how to do this in class yet (we will), but the only thing missing at this point is a bit of cleverness (so try to do something clever). If you are stuck, try to get some inspiration from your solution to 4(c) and from the physical interpretation of Lagrange multipliers we talked about in class.