Functional Analysis

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Table des matières

List of Theorems

1 Introduction

Lecture 1: Introduction

Wed 12 Oct

Main reference is "Functional Analysis" by H.W. Alt.

1.1 Topological Spaces

Definition 1 (Topological space)
Let X be a set, a topology is a subset $\tau \subset P(X)$ is a topology if
- $\emptyset, X \in \tau$
- *any union of opens is open*
- *Finite intersections of opens are open.*

Definition 2 (Properties)
For $A \subset X$, $\overline{A}$ is the smallest closed set containing A and the interior A^o is the biggest open set contained in A.
Finally, the boundary is $\partial A = \overline{A} \setminus A^o$.
X is separable if $\exists$ a dense countable subset

Definition 3 (Sequences)
Let $x : \mathbb{N} \to X, \overline{x} \in X$, $\lim x_k = \overline{x} \iff$ any neighbourhood $U \in T$ of x eventually contains x_k

Definition 4 (Continuity)
A function $f : X \to Y$ is continuous if $\forall U \in \tau_Y, f^{-1}(U)$.
This is different from sequential continuity $x_n \to \overline{x} \implies f(x_n) \to f(\overline{x})$
.
f is continuous at $x \in X$ if $\forall V \in S$ st $f(x) \in V \implies f^{-1}(V) \in \tau_X$

Lecture 2: More recaps

Fri 14 Oct

1.2 Metric spaces

Definition 5 (Metric space)
X a set, $d : X \times X \to [0, \infty)$ is a matrix

Definition 6

X a set, d_1, d_2 metrics

1. *d_1 is topologically stronger than d_2 if τ_{d_1} is finer.*
2. *d_1 is uniformly stronger than d_2 if $\exists C > 0$ such that $d_2 \leq Cd_1$*
3. *d_1 is uniformly stronger than d_2 if $\exists C > 0$ such that $\frac{1}{C}d_1 \leq d_2 \leq Cd_1$*

Lemma 1

THe following are equivalent

1. *d_1 is topologically stronger than d_2*
2. *$\mathrm{Id} : (X, \tau_{d_1}) \to (X, \tau_{d_2})$ is continuous*
3. *If $x_n \to \overline{x}$ in d_1 then $x_n \to \overline{x}$ in d_2*
4. *$\forall x \in X \forall \epsilon > 0 \exists \delta_{\epsilon,x} > 0$ such that*

$$d(x, y) \leq \delta \implies d_2(x, y) < \epsilon$$

Definition 7

Let (X, d) be a metric space

1. *$A \subset X$ is bounded if $\exists \overline{x} \in X$ such that $\sup_{y \in A} d(x, y) < \infty$ or $A = \emptyset$*
2. *x_n is Cauchy if*
$$\lim_{n \to \infty} \sup_{i,j \geq n} d(x_i, x_j) = 0$$
3. *X complete if x Cauchy $\implies$ x convergent.*
4. *(Y, e) is a matric, $fX \to Y$ is uniformly continuous if $\forall \epsilon > 0 \exists \delta > 0$ such that $d(x, y) < \delta \implies e(f(x), f(y)) < \epsilon$.*

Define $X = \{x : \mathbb{N} \to \mathbb{R}$ such that $\exists N$ such that $x_i = 0$ eventually $\}$. This space, with p-norm is not complete, so we construct the completion.

Proposition 2

Let (X, d) a metric space and (Y, e) a complete metric space, $A \subset X, \phi : A \to Y$ uniformly continuous.
Then $\exists$ unique $\psi : \overline{A} \to Y$ such that ψ is uniformly continuous and $\phi = \psi|_A$.

Proof

If $x : \mathbb{N} \to A$ is Cauchy, then $\phi \circ x$ is also cauchy.
To prove this, let $\epsilon > 0$ and $\partial_\epsilon^\phi > 0$ be such that $d(x, y) < \delta \implies e(\phi(x), \phi(y)) < \epsilon$.

Let $N = N^x_\delta$ *be such that* $i,j \geq N \implies d(x_i, x_j) < \delta$, *then* $e(\phi(x_i), \phi(x_j)) < \epsilon$

Now, let $a \in \overline{A}$, *then* $\exists x_k$ *converging to* a.
x *is d-Cauchy and* $\phi \circ x$ *is e-cauchy.*
$\exists$ *a limit* $b^* = \lim \phi(x_k)$ *So we define* $\psi(a) = b^*$.
We now prove continuity/uniform continuity.
Let $a, b \in \overline{A}, x, y : \mathbb{N} \to A$ *and* $x_i \to b, y_j \to b$.
Then

$$e(\psi(a), \psi(b)) = \lim e(\phi(x_i), \phi(y_j))$$

Now, let $\epsilon > 0$, *then* $\exists \delta > 0$ *such that* $d(x,y) < \delta$.
Thus $e(\phi(x), \phi(y)) < \epsilon$
If $d(a,b) < \delta$ $\exists N$ *such that* $d(x_i, y_j) < \delta \forall i,j > N$

$$e(\phi(x_i), \phi(y_j)) < \epsilon \implies e(\psi(a), \psi(b) \leq \epsilon)$$

□

Theorem 3
If (X,d) *is a metric space, then there exists a complete metric space* (Y,e) *and an isometry* $\phi : X \to Y$ *such that* $Y = \overline{\phi(X)}$.
Both are unique up to a bijective isometry.

Proof
Define $C_X := \{x : \mathbb{N} \to X, x$ *Cauchy* $\}$ *and* $x\tilde{y}$ *if* $\lim_{j\to\infty} d(x_i, y_j) = 0$.
Write $Y = C_X / \sim$.
For $x, y \in Y$, *define* $e(x,y) = \lim_{j\to\infty} d(x_i, x_j)$.
Is this well defined?
If $j, k \geq N$

$$|d(x_i, y_i) - d(x_k, y_k)| \leq d(x_i, x_k) + d(y_j, y_k)$$

And if $x\tilde{x}'$, *then*

$$\lim d(x_i, y_j) = \lim d(x'_j, y_j)$$

because

$$|d(x_i, y_j) - d(x'_j, y_j)| \leq d(x_j, x'_j) \to 0$$

To show that e *is a metric, most properties are obvious.*
We show that if $e(x,y) = 0$ *then* $\lim d(x_j, y_j) = 0 \implies x\tilde{y} \implies x = y$
Triangular equality holds because

$$e(x, y = \lim d(x_j, y_j \leq \limsup d(x_i, z_j) + d(z_j, y_j) = e(x,z) + e(z,y)$$

The isometry $\phi : X \to Y$ *simply sends* $x \mapsto [x]$.
We now show $[x] \in Y$, $\phi(x_k)$ *is a sequence in* Y, *we want to show that*

$\phi(x_k) \to [x]$.

$$\lim_{k\to\infty} e(\phi(x_k),[x]) = \lim_{k\to+\infty}\lim_{j\to\infty} d(x_k,x_j) = 0$$

Which shows $Y = \overline{\phi(X)}$ *Let* y^k *Cauchy* $\forall k \exists x_k \in X$ *such that* $e([y^k],\phi(x_k)) < 2^{-k}$.
We claim $[y^k] \to [x]$

$$d(x^k,x^h = e(\phi(x^k,\phi(x^h)))) \leq 2^{-k} + 2^{-h+e([y^k],[y^h])}$$

Thus $x \in C_X$ $[x] \in Y$

$$e([y^k],[x]) = \lim d(y^k_j,x_j) \leq \lim d(U^k_j,x_k) + d(x_k,x_j) \leq 2^{-k}$$

Finally, to show uniqueness, if (Y,e) *and* (Y',e') *are two completions.*
Let $\psi = \phi \circ (\phi')^{-1} : \phi'(X) \to Y$.
ψ *is an isometry so there is a unique extension* $\psi : Y' \to Y$ *and this is an isometry.* □

1.3 Norms, Banach Spaces

Throughout, $K = \mathbb{R}$ or $\mathbb{C}$

Definition 8 (Normed space)
$\|\cdot\| : X \to [0,\infty)$ *is a norm if*

— $\|x\| = 0 \iff x = 0$
— $\|\lambda x\| = |\lambda| \, \|X\|$
— $\|x+y\| \leq \|x\| + \|y\|$

Definition 9
c_0 *is the space* $c_0 = \{x : \mathbb{N} \to \mathbb{R}$ *s.t.* $\lim x_k = 0\}$ *together with* $\|x\|_{c_0} = \sup |x_k|$
For $p \in [1,\infty)$, $l_p = \left\{x : \mathbb{N} \to \mathbb{R} \text{ s.t. } \sum_{k\in\mathbb{N}} |x_k|^p < \infty\right\}$ *with* $\|x\|_{l_p} = (\sum |x_k|^p)^{\frac{1}{p}}$

Definition 10 (Banach Space)
A Banach space is a complete normed space.

Proposition 4
Any normed space has a completion which is Banach.

Proof

Let (Y, e) be the completion as above, define

$$[x] + [y] := [x + y] \text{ and } \lambda[x] := [\lambda x]$$

□

1.4 Basis of a normed space

Definition 11

Let $A \subset X$.

A is linearly independent if $\forall N \in \mathbb{N}, \forall a_i \in A \forall \lambda_i \in K, \sum_i \lambda_i a_i = 0 \implies \lambda_i = 0$.

We define

$$span(A) = \left\{ \sum(i) \lambda_i a_i, \lambda_i \text{ as above } \right\}$$

A is a Hamel basis if A is linearly independent and $X = span A$

Definition 12 (Schauder Basis)

$e : \mathbb{N} \to X$ is a Schauder basis if $\forall x \in X$ there is a unique $\lambda : \mathbb{N} \to K$ such that $x = \sum_{i=0}^{\infty} \lambda_i e_i \iff \lim \left\| x - \sum^N \lambda_i e_i \right\| = 0$

Lecture 3: Projections onto Hilbert spaces

Wed 19 Oct

Definition 13 (Equivalence of Norms)

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on a vector space X

1. *$\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if the induced metrics are topologically stronger*
2. *$\|\cdot\|_1$ is equivalent to $\|\cdot\|_2$ if the induced metrics are equivalent*

Lemma 5

1. *If $\|\cdot\|_1$ is stronger than $\|\cdot\|_2 \implies \exists C > 0$ such that $\|x\|_2 \leq C \|x\|_1$*
2. *If $\|\cdot\|_1$ is equivalent to $\|\cdot\|_2 \implies \exists C > 0$ such that $\frac{1}{C} \|x\|_1 \leq \|x\|_2 \leq C \|x\|_1$*

Proof

1. *If not, $\forall k \in \mathbb{N}, \exists v_k \in X$ such that $\|v_k\|_2 > k \|v_k\|_1$.*
   Let $w_k = \frac{v_k}{\|v_k\|_2}$ then $1 = \|w_k\|_2 > k \|w_k\|_1$.
   Thus $w_k \to 0 \in \|\cdot\|_1$, thus $w_k \to 0 \in \|\cdot\|_2$ which is a contradiction.
2. *Follows from 1.*

□

1.5 Scalar products and Hilbert spaces

Definition 14
Let H be a K-vector space.
A map $b : H \times H \to K$ is a scalar product if it satisfies

$$b(x, \lambda y + \mu z) = \lambda b(x, y) + \mu b(x, z)$$

$$b(\lambda x + \mu y, x) = \overline{\lambda} b(x, z) + \overline{\mu} b(y, z)$$

$b(x, y) = \overline{b(y, x)}$ *and* $b(x, x) > 0$.
(H, b) *is a pre-Hilbert space*

Example

1. K^d *with the usual scalar product*

2. $\ell^2(\mathbb{R})$

Proposition 7

1. $\|x\|_H = (x, x)^{\frac{1}{2}}$ *is a norm on* H
2. *Cauchy-Schwarz :* $|(x, y)| \leq \|x\| \|y\|$
3. $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Proof
To show Cauchy-Schwarz, note that $(x + ty, x + ty) \geq 0 \forall t \in K$, *thus*

$$(x, x) + t((x, y) + (y, x)) + t^2(y, y) \geq 0 \qquad \square$$

The middle term is $2t \operatorname{Re}(x, y)$, *if the scalar product isn't real, we may rotate* y *to make it real*

Proposition 8
Let $(X, \|\cdot\|)$ *be a normed space.*
If the parallelogram identity holds, then there is a scalar product b *such that* $\|x\| = b(x, x)^{\frac{1}{2}}$

Proof
Define $b(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right)$.
We want to check $b(x, \lambda y + \mu z) = \lambda b(x, y) + \mu b(x, z)$.
First, check $b(x, y + y') + b(x, y - y') = 2b(x, y)$

$$\frac{1}{4}\left[\|x + y + y'\|^2 - \|x - y - y'\|^2 + \|x + y - y'\|^2 - \|x - y - y'\|^2\right] = \frac{1}{2}(\|x + y\|^2 - \|x - y\|^2)$$

From the parallelogram identity, we get that the left hand side is

$$\frac{1}{4}\left[2\|x + y\|^2 + 2\|y'\|^2 - 2\|x - y\|^2 - 2\|y'\|^2\right]$$

and thus the equality above holds.
If $y' = y \implies b(x, 2y) = 2b(x, y)$ *and thus*

$$y' = ny \quad b(x, (n+1)y) = 2b(x, y) - b(x, y - ny)$$

and we conclude by induction that $b(x, ny) = nb(x, y)$.
Thus $b(x, qy) = qb(x, y) \forall q \in \mathbb{Q}$ *and by continuity, they agree on* $\mathbb{R}$.
Pick $v, w \in X$ *and* $y = \frac{v+w}{2}, y' = \frac{v-w}{2}$ *in the above equality, then*

$$b(x, v) + b(x, w) = 2b(x, \frac{v+w}{2}) \qquad \square$$

and we conclude from linearity.
For complex numbers, consider $s(x, y) = b(x, y) - ib(x, iy)$

Definition 15 (Hilbert Space)
(H, b) *is a Hilbert space if it is a complete pre-Hilbert space.*

Lemma 9
Every pre-Hilbert space has a completion, unique up to bijective isometry.

If $M \subset X$, then $p : X \to M$ is a projection if $p^2 = p$ and $p(X) = M$.
M is convex if $x, y \in M, t \in [0, 1]$, then $tx + (1 - t)y \in M$

Theorem 10
Let H *be a Hilbert space,* $M \subset H$ *non-empty, closed, convex, then* $\exists$ *a unique map* $p : H \to M$ *such that*

$$\|x - px\| = d(x, M)$$

Proof
If $x \in M, px = x$.
If $x \notin M$, *let* $d = d(x, M) > 0$.
If $y, z \in M$ *are minimizers, then*

$$\frac{1}{2}\|x - y\|^2 + \frac{1}{2}\|x - z\|^2 = \left\|x - \frac{y+z}{2}\right\| + \left\|\frac{y-z}{2}\right\|^2$$

If $\|x - y\| = \|x - z\| = d$, *then* $\frac{y+z}{2} \in M \implies \|y - z\| \leq 0 \implies y = z \implies$ *uniqueness.*
To show existence, let $d = \inf_{y \in M} \|x - y\|$.
There is a sequence $y : \mathbb{N} \to M$ *such that* $\|x - y_k\| \to d$.

Thus

$$\frac{1}{2}\|x-y_h\|^2+\frac{1}{2}\|x-y_k\|^2=\left\|x-\frac{y_h+y_k}{2}\right\|+\frac{1}{4}\|y_h-y_k\|^2$$

The LHS goes to d^2 *and* $\left\|x-\frac{y_h+y_k}{2}\right\|\geq d^2$. □

Lemma 11
Let everything as above,, then $p: H\to M$ *is an orthogonal projection* $\iff \forall y\in M \forall x\in H, \mathrm{Re}(x-Px, y-px)\leq 0$

Proof
Let $f(t)=\|x-(ty+(1-t)px)\|^2$*, thus* $f(0)=\min f([0,1])$*, thus* $f'(0)\geq 0$

$$f(t)=\|x-Px\|^2-t\left[(x-px,y-px)+(y-px,x-px)\right]+t^2\|y-px\|^2$$

Thus $f'(0)=-Re(x-px,y-px)\geq 0$ □

Corollary 12
If $M\subset H$ *is a closed linear subspace, then*

$$M^\perp=\{x\in H:(x,m)=0\forall m\in M\}$$

is a closed linear subspace, $M\cap M^\perp=\{0\}$, $H=M\oplus M^\perp$ *and* $p:H\to M$ *satisfies* $x-px\in M^\perp$ *and* p *is linear*

Definition 16 (Orthonormal systemes)
Let (X,b) *be a pre-Hilbert space, the family* $(e_i)_{i\in I}$ *of vectors in* X *are orthonormal if* $b(e_i,e_j)=\delta_{ij}$

Lemma 13
If the $e_1,\dots,e_n$ *are orthonormal, then* $\forall x$

$$\sum_i|(e_i,x)|^2\leq\|x\|^2$$

and if $e_1,\dots,$ *are orthonormal, then* $\forall x$

$$\sum_i|(e_i,x)|^2\leq\|x\|^2$$

Proof
Let $y = \sum_i \lambda_i e_i \in span e_i$

$$g(\lambda) \left\| x - \sum_i \lambda_i e_i \right\|^2 = \|x\|^2 - \sum_i (\overline{\lambda_i}(e_i, x) + \lambda_i(x, e_i)) + \sum_i \lambda_i^2$$

Setting $\lambda_i = (e_i, x)$, *we get*

$$g(\lambda) = \|x\|^2 - \sum |\lambda_i|^2$$

□

Lecture 4: Hilbert Spaces

Fri 21 Oct

Lemma 14
Let H *be a hilber space,* $e : \mathbb{N} \to H$ *an orthonormal basis and* $\lambda : \mathbb{N} \to K$, *then*

1. $\sum_{i=0}^{\infty} \lambda_i e_i \iff \sum_i |\lambda_i|^2 < \infty$
2. *If* $\sum_{i=0}^{\infty} \lambda_i e_i$ *converges, then it does not depend on the order*
3. $\forall x, \sum_{i \in \mathbb{N}} (e_i, x) e_i$ *converges.*
   Let $M := \overline{span(\{e_i\})}$, *then the projection map* $p : H \to M$ *may be written* $Px = \sum (e_i, x) e_i$

Proof

1. *for all* j, k *big enough,* $|\sum_{i=j+1}^{k} \lambda_i e_i|^2 \leq \sum_{i=j+1}^{k} |\lambda_i|^2$

□

1.6 Uncountable Orthonormal Systems

If A is a set, $v : A \to H, w \in H$.
What does $\sum_{\alpha \in A} v_\alpha = w$ mean ?
$\sum_{\alpha \in A} v_\alpha = w$ if $\exists \beta : \mathbb{N} \to A$ such that $\sum_i v_{\beta_i} = w$ and $v = 0$ on $A \setminus \beta(\mathbb{N})$ for any ordering of the sum is one possible definition.
$\forall \epsilon > 0 \ \exists T \subset A, \#T < \infty$ such that $\forall S \subset A, \#S < \infty$ such that $T \subset S$ one has $|\sum_{\alpha \in S} v_\alpha - w| < \epsilon$

Proposition 15
Let H *be a hilbert space,* $e : A \to H$ *an orthonormal system.*

1. $\forall x \in H$ *, the set* $A_x = \{\alpha : (e_\alpha, x) \neq 0\}$ *is finite or countable*
2. $\forall x \in H$, $\sum_{\alpha \in A} (e_\alpha, x) e_\alpha$ *is well defined*

Definition 17
Let H *be a Hilbert space and* $e : A \to H$ *an orthonormal system.*
It is a Orthonormal basis if $H = \overline{span(\{e_\alpha\})}$

Theorem 16
If $e : A \to H$ is an orthonormal system, the following are equivalent

1. e_α *is a basis*
2. $\forall x, x = \sum_{\alpha \in A}(e_\alpha, x)e_\alpha$
3. $\forall x, y, (x, y) = \sum_{\alpha \in A}(x, e_\alpha)(e_\alpha, y)$
4. $\|x\|^2 = \sum |(e_\alpha, x)|^2$
5. $(x, e_\alpha)\forall\alpha \implies x = 0$
6. *A is maximal : There is no B with $A \subset B, A \neq B$ and $e : B \to H$ such that the restriction to A is e.*

Theorem 17
If H is an infinite dimensional Hilbert space, the following are equivalent

1. *H is separable*
2. *H has a countable orthonormal basis*
3. *$\exists\phi : \ell^2(\mathbb{N}, K) \to H$ which is a linear isomorphism*

1.7 Projections in Banach spaces

Is there a Banach space X such that $\exists x, y, z \in X, y \neq z\ |x - y| = |x - z|$ and $x - \frac{y+z}{2}| \geq |x - y|$?
Strict inequality is impossible, but we can get equality

$$x(t) = 0, y(t) = \begin{cases} 1, t \leq \frac{1}{2} \\ \frac{3}{2} - t, t \in (\frac{1}{2}, 1] \end{cases} \quad \text{and } z(t) = 1$$

Definition 18
Let $(X, \|\cdot\|)$ be a normed vector space.

1. *X is strictly convex if*

$$\|x\| = \|y\| = 1, x \neq y \implies \left\| \frac{x+y}{2} \right\| < 1$$

2. *X is uniformly convex if $\forall\epsilon > 0, \exists\delta_\epsilon > 0$ such that*

$$\|x\| = \|y\| = 1, \left\| \frac{x+y}{2} > 1 - \delta \right\| \implies \|x - y\| < \epsilon$$

Theorem 18

Let X be a uniformly convex Banach space, $M \subset X$ a non-empty, convex, closed subset.
Then $\exists$ a unique map $p : X \to M$ such that $\|x - Px\| = dist(x, M)$.

Proof

If $x \in M, px = x$.
If $x \notin M, d := dist(x, M) > 0$.
For uniqueness, let $y, z \in M, |x - y| = |x - z| = d$, then $\frac{y+z}{2} \in M$.
Let $v = \frac{y-x}{\|y-x\|}, w = \frac{z-x}{\|z-x\|}$.
If $y \neq z$, then $v \neq w \implies \left\|\frac{v+w}{2}\right\| < 1$.

$$\frac{v+w}{2} = \frac{y-x+z-x}{2d} = \frac{\frac{y+z}{2} - x}{d} \implies \left\|\frac{y+z}{2} - x\right\| < d$$

a contradiction.
To show existence, let $y_i \in M$ be a sequence such that $d_i = \|x - y_i\| \to d$.

$$\frac{y_i + y_k}{2} \in M \implies \left\|x - \frac{y_i + y_i}{2}\right\| \geq d\forall i, k$$

by convexity.
Now let $v = \frac{y_i - x}{d}, w = \frac{y_k - x}{d_2}$, then

$$\frac{v+w}{2} = \frac{y_i - x}{2d} + \frac{y_k - x}{2d_k} = \frac{y_i - x}{2d} + \frac{y_k - x}{2d} + (y_i - x)\left(\frac{1}{2d_i} - \frac{1}{2d} + (y_k - x)(\frac{1}{2d_k} - \frac{1}{2d})\right)$$

Thus $\lim_{i,k\to\infty} \left\|\frac{v_i + w_k}{2}\right\| \geq 1$.
We conclude by uniform convexity. □

Lecture 5: Examples of function spaces

Wed 26 Oct

2 Function Spaces

Definition 19

Let (X, p) be a normed space and $T \neq \emptyset$ a set, then the set

$$B(T, X) = \left\{f : T \to X : \sup_{t \in T} \|f(t)\|_X < \infty\right\}$$

is the set of bounded functions on T.

Lemma 19

$B(T, X)$ is a normed space, it is complete if X is complete.

Lemma 20
If (T, τ) is a topological space, then $C_b(T, X) = C^0(T, X) \cap B(T, X)$ is a closed linear subspace of B.

Remark
$C^0(T, X)$ is a metric space but not a normed space.
We can still metrize uniform convergence by letting

$$d(f, g) = \min(\|f - g\|_\infty, 1)$$

Lemma 22
The continuous image of a compact set is compact.
If $f \in C^0(T, X)$, $K \subset T$ compact and X is normed, then $f(K)$ is bounded.
If $X = \mathbb{R}, K \neq \emptyset, f(K)$ has a maximum and minimum.

Now, let $K \subset T$ be compact and $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
Consider $C^0(K, \mathbb{K})$

Definition 20
A set $\mathbb{A} \subset C^0(K, \mathbb{K})$ is a subalgebra of C^0 if it is a linear subspace and closed under multiplication.
It separates points if $\forall a, b \in K \exists f \in \mathbb{A}$ such that $f(a) \neq f(b)$.

Theorem 23 (Stone-Weierstrass)
If $\mathbb{A} \subset C^0(K, \mathbb{R})$ is a subalgebra that separates points, then either $\overline{\mathbb{A}} = C^0$ or $\exists x_0 \in K$ such that

$$\overline{\mathbb{A}} = \left\{ f \in C^0 : f(x_0) = 0 \right\}$$

Lemma 24
If $\overline{\mathbb{A}}$ is a subalgebra, then $f \in \overline{\mathbb{A}} \implies |f| \in \overline{\mathbb{A}}$, then $\min(f, g), \max(f, g) \in \overline{\mathbb{A}}$.

The proof of this is an exercise.

Lemma 25
Assum $\forall x \in K \exists g \in \mathbb{A}$ such that $g(x) \neq 0$.
$\forall a, b \in K, \forall \lambda, \mu \in \mathbb{K}, \exists g \in \mathbb{A}$ such that $g(a) = \lambda, g(b) = \mu$.

Proof

Let $h \in \mathbb{A}$ *be such that* $h(a) = h(b)$.

Let $g(x) = \alpha h(x) + \beta h^2(x)$, *solving for* $g(\alpha) = \lambda$ *and* $g(\beta) = \mu$, *we get a linear system with determinant* $h(a)h^2(b) - h^2(b)h(a) = h(a)h(b)(h(b) - h(a))$

Suppose $h(a) = 0$, *consider* $g_a \in \mathbb{A}$ *such that* $g_a(a) \neq 0$ *and let* $G(x) = \alpha g_\alpha(x) + \beta h(a)$, *again solving* $G(a) = \lambda, G(b) = \mu$, *we get a linear system with non-zero determinant.* □

We can now prove the Stone-Weierstrass theorem.

Proof

Fix $f \in C^0$.

Pick $a \in K$ *such that* $\exists g \in \mathbb{A}, g(a) \neq 0$, *then* $g(x)\frac{f(a)}{g(a)} \in \mathbb{A}$ *and is equal to* f *at* a.

Assume $\forall a \in \mathbb{K} \exists g_a \in \mathbb{A}$ *such that* $g_a(a) \neq 0$.

Fix $F \in C^0(K)$ *and* $\epsilon > 0$.

We need to show that $\exists f \in \overline{\mathbb{A}}$ *such that* $\|F - f\|_\infty < \epsilon$.

1. $\forall a \in K \exists f_a \in \overline{\mathbb{A}}$ *such that* $F(a) = f_a(a)$ *and* $F(y) < f_a(y) + \epsilon$.
   We prove this later on.
2. *Given 1, notice that* $\forall a \in K$, *let* $V_a = \{y \in K, f_a(y) < F(y) + \epsilon\}$.
   Then for everry $\alpha \in V_a$, V_a *is open and* $K = \cup V_a$ *and we can refine this to a finite cover.*
   Let $g = \min\{f_{a_1}, \dots, f_{a_n}\}$.
   Then $g \geq F - \epsilon; g \leq f_a \leq F + \epsilon$ *on* $V_{a_i} \implies |F - g| \leq \epsilon$

Now, we prove step 1.

Fix $a \in K$.

$\forall b \neq a \exists f_{ab} \in \overline{\mathbb{A}}$ *such that* $f_{ab}(a) = F(a), f_{ab}(b) = F(b)$.

Let $V_{ab} = \{y \in K : F(y) < f_{ab}(y) + \epsilon\}$.

Let $b_1, \dots b_M$ *be such that* $K = \bigcup_{i=1}^M V_{ab_i}$.

Let $f_a = \max\{f_{ab_1}, \dots, f_{ab_M}\} \in \overline{\mathbb{A}}$.

Then $f_a > F - \epsilon$ *on* K.

Assume now $\exists x_0$ *such that* $f \in \mathbb{A} \implies f(x_0) = 0$.

Let $\tilde{\mathbb{A}} = \mathbb{A} + \mathbb{R}$, *then* $\overline{\tilde{\mathbb{A}}} = C^0$.

Let $F \in C^0, F(x_0) = 0$, *then* $\exists f_\epsilon \in \mathbb{A}, \lambda_\epsilon \in \mathbb{R}$ *such that* $|f_\epsilon + \lambda_\epsilon - F|_\infty < \epsilon$.

But $F(x_0) = f_\epsilon(x_0) = 0 \implies |\lambda_\epsilon| < \epsilon \implies |f_\epsilon - F|_\infty < 2\epsilon$ □

Theorem 26

If $\mathbb{A} \subset C^0(K, \mathbb{C})$ *is a subalgebra, separates points and* $f \in \mathbb{A} \implies \overline{f} \in \mathbb{A}$.

Then either $\overline{\mathbb{A}} = C^0$ *or* $\overline{\mathbb{A}} = C^0 \cap \{f(x_0) = 0\}$.

Proof

$\mathbb{A}_{\mathbb{R}} = \mathbb{A} \cap C^0(K, \mathbb{R}), f \in \mathbb{A} \implies \operatorname{Re} f, \operatorname{Im} f \in \mathbb{A}_{\mathbb{R}}$. $\square$

Definition 21

If $\Omega \subset \mathbb{R}^d$ *is open, then* $C^k(\Omega, \mathbb{K}) = \{f : \Omega \to \mathbb{C} | f$ *k-times continuously differentiable.* $\}$.

We define $C^k(\overline{\Omega}, \mathbb{K}) = \left\{f \in C^k(\Omega, \mathbb{K}) | f\right.$ *continuously extends to the boundary.* $\left.\right\}$

Lemma 27

$C_b^k = \left\{f \in C^k : D^\alpha f \in B \forall |\alpha| \le k\right\}$ *is a Banach space.*

Definition 22 (Hoelder Continuity)

Let $f : \Omega \to \mathbb{K}$, $\alpha \in (0, 1]$, *then*

$$[f]_\alpha = \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

$$C^{k,\alpha} = \left\{f \in C^k : D^\beta f \in C^\alpha \forall |\beta| \le k\right\}$$

Lecture 6: Lp spaces

Fri 28 Oct

3 L^p spaces

3.1 Measure spaces

Definition 23

Let X *be a non-empty set*

- *A subset* $\tau \subset P(X)$ *is called a* σ*-algebra on* X *if*
- $\emptyset \in \tau$
- $A \in \tau \implies X \setminus A \in \tau$
- $\forall A_i \in \tau \implies \bigcup A_i \in \tau$

If (X, τ) is a topological space, the borel σ-algebra is the smallest σ-algebra containing τ.

Definition 24 (Meausre space)
A map $\mu : \tau \to [0, \infty]$ is called a measure μ if

— $\mu(\emptyset) = 0$

— *μ is σ additive, namely, if A_k are such that $A_k \cap A_l = \emptyset \forall k \neq l$, then*

$$\mu(\bigcup A_k) = \sum_k \mu(A_k)$$

— *(X, τ, μ) is complete if for any null set A if*

$$B \subset A \implies B \in \tau$$

(in particular, B is also a null-set)

— *μ is σ-finite if there is $A_k \in \tau$ such that $\mu(A_k) < \infty \forall k$ and $X = \bigcup A_k$*

— *A property holds almost everywhere if $\exists N$ a null set such that P holds on $X \setminus N$.*

— *For $E \subset A$ one defines the restricted measure*

$$(\mu|_E)(A) = \mu(E \cap A)$$

— *$\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$ and μ^d denotes the lebesgue measurable sets.*

Remark

— *$(\mathbb{R}^d, \mathcal{L}^d)$ is a complete measure space, $\mathcal{L}^d$ is σ-finite.*

— $B(\mathbb{R}^d) \subsetneq \mu^d \subsetneq P(\mathbb{R}^d)$

In the following, (X, τ, μ) is complete and σ-finite.

3.2 Measurable functions and integrals

Definition 25
Let (X, τ, μ) be a measure space and (Y, τ') a topological space.
A function $f : X \to Y$ is called measurable if $f^{-1}(U) \in \tau \forall U \in \tau'$

Remark

— *f is measurable $\iff$ $f^{-1}(A) \in \tau \forall A \in B(Y)$*

— *$f : X \to [-\infty, \infty]$ is measurable $\iff$ $f^{-1}((a, \infty]) \forall t \in \mathbb{R}$ $\iff$ $f^{-1}([a, \infty] \forall t \in \mathbb{R})$*

— *$f : X \to \mathbb{C}$ is measurable iff $\operatorname{Re} f, \operatorname{Im} f$ are measurable.*

— *$f : X \to \mathbb{R}^d$ is measurable iff every projection is measurable.*

Definition 26 (Integral)
Let (X, τ, μ) *be a measure space.*

- *A function* $f : X \to [0, \infty]$ *is simple if* $\exists \lambda : \mathbb{N} \to [0, \infty]$ *and* $E : \mathbb{N} \to \tau$ *such that* $f = \sum_{n \in \mathbb{N}} \lambda_n \xi_{E_n}$
- *If* f *is simple, define*

$$\int_X f d\mu = \sum_{n \in \mathbb{N}} \lambda_n \mu(E_n) \in [0, \infty]$$

- *For a measurable function* $f : X \to [0, \infty]$*, define*

$$\int_X f d\mu = \sup_{\phi \leq f, \phi \ simple} \int_X \phi d\mu \in [0, \infty]$$

Remark

- *The integral of a simple function is well-defined (ie. independent of the* λ_i *and* E_i *) and is monotone.*
- *One can show* $\int_X f d\mu = \int_{(0,\infty)} \mu(f^{-1}((t, \infty]))d\mathcal{L}^1(t)$

Remark
We'll write dx *or* dx^d *for* $\mathcal{L}^1$ *or* $\mathcal{L}^d$ *respectively*

Definition 27
Let (X, τ, μ) *be a measure space,* (Y, τ') *a topological space,* $f : X \to Y$ *a measurable funcition*

- *If* $Y = [0, \infty]$*, the function is integrable if* $\int_X f d\mu < \infty$
- *Consider* $Y = [-\infty, \infty]$*, the function* f *is integrable if it's positive and negative parts are integrable.*
  In this case, we define $\int_X f d\mu = \int_X f_+ d\mu - \int_X f_- d\mu$*.*
- *A complex valued function is integrable if it's real and imaginary parts are.*
- *A function valued in* $\mathbb{R}^d$ *is integrable if it's components are.*

Remark
f *is integrable* $\iff$ f *is measurable and* $|f|$ *is integrable.*

3.3 The spaces L_p and L^p

Definition 28
Let (X, τ, μ) be a measure space.

- *Let $f : X \to \mathbb{K}$ be a measurable function, we define*

$$\|f\|_p = \|f\|_{L^p} = \begin{cases} (\int_X |f|^p d\mu)^{\frac{1}{p}} \text{ if } p < \infty \\ \operatorname{esssup}_x |f| = \inf \{M \in [0, \infty] : f < M \ a.e.\ \} \end{cases}$$

- *For $p \in [1, \infty]$*

$$L_p(X, \mathbb{K}) = L_p(X, \mu, \mathbb{K}) = \left\{ f : X \to \mathbb{K} : \|f\|_p < \infty \right\}$$

- *We denote L^p the space L_p modded out by functions which are equal a.e.*

Theorem 33
Let $p \in [1, \infty]$

- *$\|\cdot\|_p$ is a semi-norm on L_p and a norm on L^p*
- *$L^p(X, \mathbb{K})$ is a Banach space (Fischer-Riesz)*
- *$L^2(X, \mathbb{K})$ is a Hilbert space with scala product*

$$(f, g) = \int_X \overline{f} g d\mu$$

Lemma 34
The space $L^p(X, \mathbb{K})$ is uniformly convex.

Lecture 7: convolutions and Hausdorff measures

Wed 02 Nov

Theorem 35
Let $p \in [1, \infty)$, $\Omega \subset \mathbb{R}^n$ open, then $C_c^\infty(\Omega, \mathbb{K})$ is dense in $L^p(\Omega, \mathbb{K})$.

3.4 Convolutions

Let $f, g \in L^1(\mathbb{R}^n)$, $(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy$

Lemma 36
Let $\eta \in L^1(\mathbb{R}^n)$, for $r > 0$, let $\eta_r(x) = \frac{1}{r^n} \eta(\frac{x}{r})$.
Then

— $\eta_r \in L^1, \|\eta_r\|_1 = \|\eta\|_1$.
— *If* $f \in L^p(\mathbb{R}^n)$*, then* $\|\eta_r * f\|_p \leq \|\eta_r\|_1 \|f\|_p$
— *If* $p < \infty, \eta \geq 0,\ \|\eta\|_1 = 1$*, then* $f * \eta_r \to f$ *in* $L^p(\mathbb{R}^n)$
— *If* $\eta \in C^\infty$*, then* $\eta_r * f \in C^\infty$*,* $\partial^\alpha(\eta_r * f) = (\partial^\alpha \eta_r) * f$
— *If* $\eta \in C_c(\mathbb{R}^n)$*,* $f \in L^1_{loc}(\mathbb{R}^m)$*, then* $\eta_r * f \in C^0$.

Lemma 37
If $\int_\omega \phi f d\mathcal{L}^n = 0 \forall \phi \in C_c^\infty$*, then* $f = 0$.

Lemma 38
Let $p \in [1, \infty]$ Ω *open and* $f_j \to f$ *in* $L^p(\Omega)$*, then* $\forall \phi \in C^\infty c(\Omega)$

$$\int_\Omega f_j \phi dx \to \int_\omega f \phi dx$$

3.5 Hausdorff Measures

Definition 29
Let (X, d) *be a metric space,* $\delta > 0$ *and* $s \in [0, \infty)$.
Define

$$\mathcal{H}^s_\delta : P(X) \to [0, \infty], \mathcal{H}^s_\delta(E) = \frac{\omega_s}{2^2} \inf \left\{ \sum_{h \in \mathbb{N}} (diam F_h)^s : E \subset \bigcup_{h \in \mathbb{N}} F_h, diam F_h \leq \delta \right\}$$

Where $\omega_s = \frac{\pi^{\frac{s}{2}}}{\Gamma(1 + \frac{s}{2})}$.
Then

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$$

Lemma 39
All Borel sets are $\mathcal{H}^s$*-measurable and* $(X, B, \mathcal{H}^s|_B)$ *is a measure space.*

Theorem 40

$$\mathcal{H}^n|_{m^n} = \mathcal{L}^n$$

Theorem 41
For any $E \subset X$, there is a number $s_E \in [0, \infty]$ such that

$$\mathcal{H}^s(E) = \infty \forall s \in [0, s_E)$$

and

$$\mathcal{H}^s(E) = 0 \forall s \in (s_E, \infty]$$

Definition 30 (Hausdorff Dimension)

$$dim_H(E) =:= s_E$$

Example

$$\dim_H(B(0,1)) = n - 1$$

Theorem 43
If $U \subset \mathbb{R}^k$ and $\psi \in C^1(U, \mathbb{R}^n)$ an injective immersion, then $\psi(U)$ is $\mathcal{H}^k$ measurable, $\mathcal{H}^k|_{\psi(U)}$ is σ-finite and

$$\mathcal{H}^k(\psi(U)) = \int_U (\det D\psi^T D\psi)^{\frac{1}{2}} d\mathcal{L}^k$$

4 Sobolev Spaces

Let $\Omega \subset \mathbb{R}^n$ open, $f \in L^p(\Omega)$.
DOes it have a "derivative" in L^p ?
Notice that

$$\int_\Omega f \partial_i \phi d\mathcal{L}^n = -\int_\Omega (\partial_i f) \phi d\mathcal{L}^n + \int_{\partial\Omega} f \phi \nu_i d\mathcal{H}^{n-1}$$

if $f, \phi \in C^1(\mathbb{R}^n)$, Ω bounded, $\mathcal{H}^{n-1}(\partial\Omega \setminus \partial_r\Omega) = 0$

Morally, the last term is 0, because $\partial\Omega$ is a null-set.

Definition 31
Let $\Omega \subset \mathbb{R}^n$ open, $f \in L^1_{loc}(\Omega)$.
f is weakly differentiable if $\exists g \in L^1_{loc}(\Omega, \mathbb{R}^n)$

$$\int_\omega f \partial_i \phi d\mathcal{L}^n = -\int_\Omega g_i \phi d\mathcal{L}^n \forall \phi \in C^1_c(\Omega)$$

f is k-times weakly differentiable if $\forall \alpha \in \mathbb{N}^n, |\alpha| \leq k$, $\exists g_\alpha \in L^1_{loc}$ such

that

$$\int_\Omega f D^\alpha \phi d\mathcal{L}^n = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi d\mathcal{L}^n$$

We write $g_\alpha = D^\alpha f$

Remark

If $f \in C^k(\Omega)$, *then it is* k*-times weakly differentiable and the derivatives are the classical ones.*

Example

If $\Omega = \mathbb{R}$, *then* $f(x) = |x|$ *has a weak derivative.*
Let $g(x)$ *be the step function, we want to show that for* $\phi \in C_c^1(\mathbb{R})$, *we want to show that*

$$\int_\mathbb{R} |x|\phi'(x)dx = -\int_0^\infty \phi(x)dx + \int_{-\infty}^0 \phi(x)dx$$

Notice that

$$\int_0^\infty \phi(x)dx = x\phi(x)|_0^\infty - \int_0^\infty x\phi'(x)dx$$

The first term is 0 and so the equality holds.

Lecture 8: weak derivatives

Fri 04 Nov

Remark

The definition of weak derivative is equivalent to

$$\int_\Omega f\partial_i\phi dx = -\int_\Omega g_i\phi dx \forall C_c^1(\Omega)$$

similarly, replace $C_c^\infty(\Omega)$ *by* $C_c^k(\Omega)$.
Indeed, let $\phi \in C_c^1(\Omega)$, *then for* $\epsilon < dist(\partial\Omega, \operatorname{Supp}\phi)$ *we have that* $\eta_\epsilon * \phi \in C_c^\infty(\Omega)$ *and it converges to* ϕ *in* C^1.

4.1 Definition of $W^{k,p}, W_0^{k,p}$

Definition 32 (Sobolev space)

Let $\Omega \subset \mathbb{R}^d$ *open,* $1 \leq p \leq \infty, k \in \mathbb{N} \setminus \{0\}$.
The Sobolev space $W^{k,p}(\Omega) = \{f \in L^p(\Omega) | f$ *k-times weakly differentiable and* $\partial^\alpha f \in L^p(\Omega) \forall |\alpha| \leq k\}$.
The norm is defined as

$$\|\cdot\|_{W^{k,p}(\Omega)} = \|\cdot\|_{k,p} = \left(\sum_{0\leq|\alpha|\leq k} \|\partial^\alpha f\|_{L^p}^p\right)^{\frac{1}{p}}$$

Theorem 47 (Sobolev spaces are banach spaces)
Let $\Omega \subset \mathbb{R}^d$ *open,* $1 \leq p \leq \infty$, $k \in \mathbb{N} \setminus 0$.
Then

- $\left(W^{k,p}(\Omega), \|\cdot\|_{W^{k,p}(\Omega)}\right)$ *is Banach*
- $H^k(\Omega) = W^{k,2}(\Omega)$ *is a Hilbert space with inner product*

$$(f, g)_{H^k(\Omega)} = \sum_{k \leq |\alpha|} (\partial^\alpha f, \partial^\alpha g)_{L^2(\Omega)}$$

Proof
It is clear tht $\|\cdot\|_{k,p}$ *is a norm and that* $(\cdot,\cdot)_{H^2}$ *is a scalar product and that for* $p = 2$, *they are compatible.*
We prove completeness.
LLet $(f_j)_j \subset W^{k,p}(\Omega)$ *be Cauchy, then for any multi-index* α *with* $|\alpha| \leq k$, *the sequence*

$$(\partial^\alpha f_j)_j$$

is also Cauchy in L^p, *so there is* $g^{(\alpha)} \in L^p$ *such that*

$$\lim_j \partial^\alpha f_j = g^\alpha$$

in L^p.
In particular, $f_j \xrightarrow{L^p} g^{(0)}$
We want to show that $g^{(0)}$ *is weakly differentiable (*k *times) and* $\partial^\alpha g^{(0)} = g^{(\alpha)}$.
Let $\phi \in C_c^\infty(\Omega)$, *then*

$$\int_\Omega g^{(0)} \partial^\alpha \phi dx = \lim_j \int_\Omega f_j \partial^\alpha \phi dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha f_j \phi dx = (-1)^{|\alpha|} \int_\Omega g^{(\alpha)} \phi dx$$

□

Definition 33
For $1 \leq p < \infty$ *and* $\Omega \subset \mathbb{R}^d$ *open, define*

$$W_0^{k,p}(\Omega) = \left\{ f \in W^{k,p}(\Omega) : (f_j) \subset C_c^\infty(\Omega) \text{ s.t. } f_j \to f \text{ in } W^{k,p} \right\}$$

We define $H_0^k - W_0^{k,2}(\Omega)$.
For $p = \infty$, *the definition is different.*

Remark

$$W_0^{k,p} = \text{ soboolev functions with zero boundary values}$$

If $\Omega = \mathbb{R}^d : W^{k,p}(\mathbb{R}^d) = W_0^{k,p}(\mathbb{R}^d)$

Lemma 49
$W_0^{k,p}$ *is a closed linear subspace of* $W^{k,p}$ *and hence is a Banach space.*

Proof
Take a diagonal sequence.
Let $(f_j)_j \subset W_0^{k,p}$ *be Cauchy, then it converges to some* f^* *in* $W^{k,p}$.
Let $(f_{j,l})_l \in C_c^\infty(\Omega)$ *such that* $f_{j,l} \to f_j$ *in* $W^{k,p}$.
Let $l = l(j) \in \mathbb{N}$ *such that* $\left\| f_{j,l(j)} - f_j \right\|_{k,p} \leq \frac{1}{j}$.
Consider $g_j = f_{j,l(j)}$ $\square$