Series 2 Exercise 7

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$\nu(1) = 0$ and $\nu(-1) = 0$

Indeed, note that

$$\nu(1 \cdot 1) = \nu(x) + \nu(1) = \nu(1) \iff 2\nu(1) = \nu(1) \iff \nu(1) = 0$$

Now for the second part, notice that since $-1 \cdot -1 = 1$ we get

$$\nu(-1 \cdot -1) = \nu(-1) + \nu(-1) = \nu(1) = 0 \iff 2\nu(-1) = 0 \iff \nu(-1) = 0$$

R_ν is a subring of K

To show R_ν is a subring, we have to show that $1, 0 \in R_\nu$ and that R_ν is closed under addition and multiplication.

Using the first part of the exercise, we immediatly get that $1 \in R_\nu$ since $\nu(1) \geq 0$ and by definition $0 \in R_\nu$.

R_ν is closed under multiplication

Let $x, y \in R_\nu \setminus \{0\}$, we get that $\nu(x \cdot y) = \nu(x) + \nu(y) \geq 0$ since $\nu(x), \nu(y) \geq 0$ by hypothesis.
If either x or y is equal to 0, then clearly $x \cdot y = 0 \in R_\nu$.
Hence R_ν is closed under multiplication.

R_ν is closed under addition

Indeed, let $x, y \in R_\nu$, now $\nu(x + y) \geq \min(\nu(x), \nu(y)) \geq 0$ hence $x + y \in R_\nu$.

This show that R_ν is a subring of K.

K is the fraction field of R_ν

Before proving the result, we notice two things.

- Given $x \in K$, we have that $\nu(x^{-1}) = -\nu(x)$, this follows immediatly using part 1.
- If ν is a non-trivial valuation, we have that $\forall N \in \mathbb{N} \exists x \in K$ such that $\nu(x) \geq N$.
  Indeed, by the above point we can find $x \in K, \nu(x) > 0$(which exists since the valuation is non-trivial), simply note that there exists $n \in \mathbb{N}$ such that $\nu(x^n) = n \cdot \nu(x) \geq 0$

First, suppose the valuation is non-trivial and fix $x \in K$ such that $\nu(x) \geq 0$
We now show that K satisfies the universal property of the fraction field.
Let L be a field and $j : R_\nu \to L$ a ring homomorphism.
Given $a \in K$, let $n \in \mathbb{N}$ such that $\nu(x^n \cdot a) \geq 0$, then we define

$$f(a) = f(\frac{a \cdot x^n}{x^n}) = \frac{j(a \cdot x^n)}{j(x^n)}$$

We now show that f is well defined (ie. doesn't depend on x and the choice of n), that f is a ring homomorphism and that it is the unique function $f : K \to L$ with the universal property of the fraction field.