École Polytechnique Fédérale de Lausanne
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Math 261 – Discrete Optimization (Spring 2022)

Assignment 6

Problem 1

Recall that for a vector $\mathbf{v} \in \mathbb{R}^m$,

$$\|\mathbf{v}\|_1 = \sum_{i=1}^{m} |v_i| \quad \text{and} \quad \|\mathbf{v}\|_\infty = \max\{|v_i| : 1 \le i \le m\}$$

and for an $m \times n$ matrix $\mathbf{A}$ and vector $\mathbf{b}$, consider the problems

$$\mathcal{P} = \inf\{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty : \mathbf{x} \in \mathbb{R}^n\}$$

and

$$\mathcal{Q} = \sup\{\boldsymbol{\lambda} \cdot \mathbf{b} : \boldsymbol{\lambda}^\intercal \mathbf{A} = \mathbf{0}^\intercal, \|\boldsymbol{\lambda}\|_1 \le 1\}$$

(a) Show that P and Q provide certificates for each other — that is,

$$\boldsymbol{\lambda} \cdot \mathbf{b} \le \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty$$

whenever $\mathbf{x}$ is feasible in $\mathcal{P}$ and $\boldsymbol{\lambda}$ is feasible in $\mathcal{Q}$.

(b) Find a linear program $\mathcal{P}'$ which has the same optimal solution as $\mathcal{P}$ and a linear program $\mathcal{Q}'$ which has the same optimal solution as $\mathcal{Q}$ such that $\mathcal{P}'$ and $\mathcal{Q}'$ are duals of each other.

(c) Part (b) implies that $\mathcal{P}$ and $\mathcal{Q}$ provide optimal certificates for each other — that is, there exists a $\mathbf{x}^*$ feasible in P and $\boldsymbol{\lambda}^*$ feasible in $\mathcal{Q}$ for which

$$\boldsymbol{\lambda}^* \cdot \mathbf{b} = \|\mathbf{A}\mathbf{x}^* - \mathbf{b}\|_\infty.$$

What do the complementary slackness conditions from part (b) say?

Problem 2

Let $\mathcal{P}$ be the linear program

$$\mathcal{P} = \max\{\mathbf{0} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}.$$

(a) Find the dual of $\mathcal{P}$ and show that it is always feasible.

(b) Use your answer to part (a) to prove the following lemma, one of the many versions of *Farkas' Lemma*

Lemma (Farkas)**.** *Let $\mathbf{A}$ be a matrix of dimension $m \times n$ and let $\mathbf{b} \in \mathbb{R}^m$. Then exactly one of the following holds:*

(I) There exists a vector $\mathbf{x} \ge \mathbf{0}$ satisfying $\mathbf{A}\mathbf{x} = \mathbf{b}$.

(II) There exists a vector $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}^\intercal \mathbf{A} \ge \mathbf{0}^\intercal$ and $\boldsymbol{\lambda} \cdot \mathbf{b} < 0$.

Problem 3

In this problem, we will consider how we can get certificates of geometric statements. Two sets $X, Y \subseteq \mathbb{R}^n$ are said to be *separated by a hyperplane* if there exists a vector $\mathbf{v}$ and a real number c such that

$$\mathbf{v} \cdot \mathbf{x} < c \quad \text{for all } \mathbf{x} \in X \quad \text{and} \quad \mathbf{v} \cdot \mathbf{y} \ge c \quad \text{for all } \mathbf{y} \in Y$$

(a) Consider the regions

$$X = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\} \quad \text{and} \quad Y = \left\{(x,y) \in \mathbb{R}^2 : (x+3/2)y \geq 2, y \geq 0\right\}$$

Show that X and Y can be separated by a hyperplane (find a valid $\mathbf{v}$ and c).

Note: You do not have to prove this formally — it would suffice to show a picture.

(b) Let $\{\mathbf{u}_i\}_{i=1}^m$ be a collection of vectors in $\mathbb{R}^n$ and let

$$X = \text{cone}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\} = \left\{\sum_i \alpha_i \mathbf{u}_i : \alpha_i \geq 0\right\}.$$

Show that, for any point $\mathbf{y} \in \mathbb{R}^n$, the following are equivalent (if and only if)

- $\mathbf{y} \notin X$
- $\mathbf{y}$ and X can be separated by a hyperplane that goes through the origin