ETCS, ORDINALS, AND CHOICE

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Abstract. In this paper we introduce the basic axiomatization of the Elementary Theory of the Category Sets and discuss it in relationship to standard notions of Set Theory, describing ordinal numbers in the theory and exploring consequences of the axiom of choice.

Contents

1. Introduction

Lawvere's Elementary Theory of the Category of sets is a classification of all categories sufficiently similar to **Sets** as derived from a set-theoretic foundation. The guiding philosophy was to have the foundations of mathematics be more in-tune with contemporary research, with an emphasis on structure and morphisms. Specifically, Lawvere sought to replace the Zermelo-Franklin emphasis on membership with an updated notion of structures invariant under morphisms. It is an oversimplification, but not a terrible one, to think of Lawvere's argument as follows: elements of a set must be distinct, but their properties beyond that are unimportant. From this perspective we can consider bijection an isomorphism of sets and cardinality the defining quality of sets.

Remark 1.1. This paper aspires to be approachable for those with and those without familiarity with category theory. To this end, the appendix was written to provide a minimal but sufficient introduction to the field's terminology. It is suggested that those not familiar with notions of limits, cones, etc. start with the appendix.

2. An Introduction to ETCS

In this section we aim to introduce the axioms of ETCS through discussion of properties we would like our category of sets to have.

Objects and their Morphisms. Clearly, if our category is to be satisfactory, objects must be be roughly analogous to sets, morphisms to functions.

One requirement that should intuitively be placed on a prototypical category of sets is an analogue to empty set. We should expect that there should be (at least one) object representing a singleton. Categorically, these objects are best described in terms of their related morphisms.

Later, we will axiomatize the existence of limits and colimits, guaranteeing the existence of terminal and initial objects. They will be unique up to a unique isomorphism.

An initial object 0 satisfies our need for an empty set due to the existence and uniqueness of $0 \to X$ for all X. In terms of ordinary set theory, this is representative of the fact that there is only one function $\phi : \emptyset \to X$, the empty function.

Similarly, a terminal object 1 satisfies our need for a singleton due to the existence and uniqueness of $X \to 1$ for all X. This is analogous to the fact that for every single-element set there is a unique function $\psi : X \to \{a\}$ for all sets X.

Definition 2.1. a is an element of $(\in)$ X if it is a morphism of the form $1 \xrightarrow{a} X$.

The motivation for this definition is that every unique mapping of a single element set into X represents a unique element in X. We can now begin to discuss important types of morphisms in the category. One might have already guessed that in our category, monomorphisms and epimorphisms, defined in the appendix, will play the roles of injections and surjections, respectively. Further, injections will take on another important role:

Definition 2.2. A morphism $X \xrightarrow{a} Y$ is a **subset** of Y if it is a monomorphism. Then every morphism from 1 to X determines a unique element of Y when composed with a.

It is helpful to consider membership and subset relations between morphisms themselves.

Definition 2.3. Let y be an element of Y and a be a subset $X \xrightarrow{a} Y$ of Y. Then $y \in a$ if there exists $1 \xrightarrow{\overline{y}} X$ such that $y = a\overline{y}$.

$$\begin{tikzcd} X \arrow[r, "a"] & Y \\ 1 \arrow[u, dashed, "\exists!\overline{y}"] \arrow[ur, "y"'] & \end{tikzcd}$$

By the monomorphism property of a, $\overline{y}$ is a unique element of Y. Here the notion of membership is extended to define any element of Y in the image of a subset morphism as an element of that morphism.

Definition 2.4. Let a, b be subsets of X. Then we say $a \subseteq b$ if there exists a morphism h from the domain of a to the domain of b such that $bh = a$. Note that h is uniquely determined by a due to the monomorphism property of b. The following proposition illuminates the motivation behind this definition.

Proposition 2.5. *If a, b are subsets of X and $a \subseteq b$, then for all $x \in X$ if $x \in a$ then $x \in b$.*

Proof. Because $x \in a$ there exists a unique $\overline{x}$ from 1 to $dom(a)$ such that $x = a\overline{x}$. But then there exists h such that $a = bh$ giving $x = \overline{x}hb$. But then the composition $\overline{x}h$ gives $x \in b$. □

Currently, the category

$$0 \to 1$$

satisfies the ETCS axioms. In order to prevent this, we impose:

Axiom 2.6. *There exists an object with more than one element.*

Axiom 2.7. *Every object other than 0 (and objects isomorphic to it) has at least one element.*

Proposition 2.8. *1 has a single element and 0 has no elements.*

Proof. $1 \to 1$ is unique. If there then exists $1 \to 0$ then because $0 \to 1$ is unique $1 = 0$.

$$\begin{tikzcd} 0 \arrow[r, shift left] \arrow[dr, "\exists!"'] & 1 \arrow[l, shift left] \arrow[d, dashed, "\exists!"] \\ & X \end{tikzcd}$$

Therefore, by the initiality of 0, every object has one element, a contradiction. □

One of the central goals of a theory of sets is to provide adequate ability to construct new sets from old sets. The following axioms work with that goal in mind.

Axiom 2.9. *The category is complete and cocomplete (has limits and colimits indexed by sets).*

From this axiom and the (co)limit definitions of terminal and initial objects, which can be found in the appendix, it follows that the category of sets has an initial and a terminal object.

Axiom 2.10. *If S is a set, then*

$$\coprod_{s \in S} 1$$

is isomorphic to S via the canonical map. Equivalently, every element of the coproduct factors through an inclusion.

Proposition 2.11. *If f, g are two morphisms from X to Y, then $f \neq g$ if an only if there exists $x \in X$ such that $fx \neq gx$.*

Proof. By the universal property of coproducts, taking $\coprod_{x \in X} 1 \to X$ gives that f, g are determined fully by the inclusions and the canonical map. □

Proposition 2.12. *Let $y \in Y$ and $X \xrightarrow{f} Y$. Then there exists an object $f^{-1}(y)$ such that $x \in f^{-1}(y)$ if and only if $fx = y$.*

Proof. Take the equalizer of f and $X \to 1 \xrightarrow{y} Y$

$$\begin{tikzcd} X \arrow[r] \arrow[rr, "f"'] & 1 \arrow[r, "y"] & Y \\ E \arrow[u, "k"] & 1 \arrow[ul, "x"'] \arrow[l, dashed, "\exists! h_x"] & \end{tikzcd}$$

For every element of $x \in X$ there exists and element h_x of the equalizer. Then the subset k of X is the inverse of f on y. $\square$

Definition 2.13. If $X \xrightarrow{x} Z$ and $Y \xrightarrow{y} Z$ are subsets of Z, then their **intersection** $X \cap Y$ is the pullback:

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d, "x"] \\ Y \arrow[r, "y"'] & Z \end{tikzcd}$$

Proposition 2.14. *The elements of the intersection $X \cap Y$ correspond to the elements of X and Y such that their subset images are equal.*

Proof. The subset morphisms from X and Y into Z commute when composed with the pullback $X \cap Y$, meaning that the composures agree on all elements of the intersection. $\square$

Definition 2.15. If $X \xrightarrow{x} Z$ and $Y \xrightarrow{y} Z$ are subsets of Z, then their **union** $X \cup Y$ is the pushout on them and their intersection.

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr, "x"] & \\ Y \arrow[r] \arrow[drr, "y"'] & X \cup Y \arrow[dr] & \\ & & Z \end{tikzcd}$$

Proposition 2.16. *The elements of the union are precisely the elements of Z that are elements of X or Y.*

Proof. The proposition follows from the universal property of the colimit, with each element of the union filtering through X and Y into Z. $\square$

Axiom 2.17. *For any two objects A, B the exponential object B^A exists, and comes equipped with an evaluation morphism e such that for any morphism $X \times A \xrightarrow{f} B$ their exists a unique morphism $X \xrightarrow{h} B^A$, and vice versa, such that following commutes.*

$$\begin{tikzcd} X \arrow[d, dashed, "h"] & X \times A \arrow[d, "h \times id_A"'] \arrow[dr, "f"] & \\ B^A & B^A \times A \arrow[r, "e"'] & B \end{tikzcd}$$

Here h assigns to an element of X a "function" in B^A.

Remark 2.18. This axiom, along with the existence of terminal objects and products, classify what are known as Cartesian Closed Categories. Specifically, it states that the endofunctor $- \times Y$ has a right-adjoint $-^Y$ for all objects Y.

In set theory, all subsets of a set X correspond to functions $\chi : X \to \{0, 1\}$. That notion is characterized by a subobject classifier.

Definition 2.19. A subobject classifier 2 is an object with a designated element $t \in 2$ such that if X is a set and $A \xrightarrow{a} X$ is a subset of x, then there exists a morphism $\chi : X \to 2$ such that a is an inverse image of t under χ_A:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & 1 \arrow[d, "t"] \\ X \arrow[r, "\chi_A"'] & 2 \end{tikzcd}$$

Axiom 2.20. *There exists a subobject classifier. In fact, by Lawvere* [1]*, it is a nontrivial result that our axioms require that* 2 *have exactly two elements.*

Definition 2.21. A Dedekind-Pierce [1] object is an object $\mathbb{N}$ with an element 0 and "successor" morphism $\mathbb{N} \xrightarrow{s} \mathbb{N}$ with the following universal mapping property

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x_0"'] & \mathbb{N} \arrow[r, "s"] \arrow[d, "x"] & \mathbb{N} \arrow[d, "x"] \\ & X \arrow[r, "t"'] & X \end{tikzcd}$$

For any object X with a "transition" morphism $X \xrightarrow{t} X$ and a specified element x_0 there exists a morphism $\mathbb{N} \xrightarrow{x} \mathbb{N}$ such that the diagram commutes.

The mapping x is said to define a sequence in X by simple recursion, starting with x_0 and using the transition rule t. By [1], the triple $(\mathbb{N}, 0, s)$ satisfies the Peano postulates.

Axiom 2.22. *There exists a Dedekind-Pierce object.*

Axiom 2.23. *The Axiom of Choice: Every morphism $X \xrightarrow{f} Y$ has a* **generalized inverse** $Y \xrightarrow{g} X$*, namely, $fgf = f$.*

It is this final axiom upon which much of the following discussion will be based. This is equivalent to every epimorphism splitting, having a **section**, or a right inverse giving identity on composition.

Proposition 2.24. *Any set with no elements is isomorphic to 0.*

Proof. Let X be a set with no elements. Then by the initial property there exists unique $0 \to X$ and by choice there exists a section $X \to 0$. Due to the uniqueness of arrows $1 \to 1$ and $0 \to 0$ their compositions must be identities. $\square$

Proposition 2.25. *A morphism a is an epimorphism $X \xrightarrow{a} Y$ if and only if for all $y \in Y$ there exists $x \in X$ such that $ax = y$.*

$$\begin{tikzcd} X \arrow[r, "a"] & Y \\ 1 \arrow[u, dashed, "\exists x"] \arrow[ur, "y"'] & \end{tikzcd}$$

Proof. Let $A \xrightarrow{f} B$ be an epimorphism and s a section given by choice. Take $y \in B$.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "s"] \\
1 \arrow[u, dashed, "sy"] \arrow[ur, "y"'] &
\end{tikzcd}$$

Then $fsy = y$ so we may take $x = sy$.

In the opposite direction, let $X \xrightarrow{f} Y$ satisfy the surjective definition. Then let a, b be distinct morphisms from Y to Z. Then there exists $y \in Y$ such that $ay \neq by$. By surjectivity, there exists $x \in X$ such that $afx \neq bfx$ giving $af \neq bf$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, shift left, "b"] \arrow[r, shift right, "a"'] & Z \\
1 \arrow[u, dashed, "\exists x"] \arrow[ur, "y"'] & &
\end{tikzcd}$$

Thus we have demonstrated the contrapositive of our statement, and we are done. $\square$

3. Order Categories

Remark 3.1. Currently, the language we have used can say little about sets with additional structures, such as an ordering. The following definitions will allow us to discuss such sets. Let S be an object in some category. Then the elements $1 \to S$ of S can be used as the objects in order categories, where morphisms $x \to y$ represent $x \leq y$.

Definition 3.2. Let S be a set with the structure of a category, where its objects are its elements. Then S is partially ordered if for all objects $x, y \in S$ there is at most one morphism $x \to y$

Further, if we require for all objects that either $x \to y$ or $y \to x$ or both, then S is totally ordered. (Equivalently, all objects must be comparable).

Definition 3.3. If S is a totally ordered set, then it is also a well ordered set if for every subcategory S' that is a set, there exists an initial object, called the least element. A functor $F : W \to U$ from one well ordered set to another is called an **initial segment embedding** of W in U if it preserves order morphisms and for any $x \to F(y)$ there exists $z \in W$ such that $F(z) = x$.

Proposition 3.4. *An initial segment embedding, viewed as a morphism in a category of well-ordered sets, is a monomorphism.*

Proof. Let $X \xrightarrow{f} Y$ be an initial embedding. Assume there exist distinct elements of X such that $x \to y$. Then if $fx = fy$, we have contradicted the order preserving requirement on f. $\square$

4. Ordinal Numbers

Definition 4.1. The notion of order types of well-ordered sets, **ordinal numbers**, can now be defined. If we consider the category of well ordered sets with initial embeddings as morphisms, then taking the skeleton of this category, denote it Ω, yields the ordinal numbers as representatives of isomorphism classes of well-ordered sets.

Theorem 4.2. *Principle of Transfinite Induction. Let W be a well-ordered set and let P be a property such that if Py is true for all $y \to x$, $y \neq x$, then Px. If we know that $P0$ holds (where 0 is initial in W) then for all $x \in W$ we know Px holds.*

Proof. Take the subset of elements U of W for which P does not hold. Take the least element of $u \in U$. Then clearly for all $v \to u$ we have Pv. But then we know Pu, so $U = \emptyset$. □

Lemma 4.3. *Morphisms $\alpha \to \gamma$ in Ω are unique.*

Proof. Suppose not. Let f, g be two initial embeddings from $\alpha \to \gamma$. By construction, f, g satisfy the transfinite induction hypothesis in the sense that if $f(y) = g(y)$ for all $y \to x$ then $f(x) = f(y)$. Further, they clearly agree on the least element of α. Thus $f = g$. □

Definition 4.4. A morphism $\alpha \to \beta$ in Ω is a **strict embedding** if it is not an isomorphism.

Definition 4.5. If W is a totally ordered set and $y \in W$, denote by $y^<$ the set $\{x \in W \mid x \to y, x \neq y\}$.

Lemma 4.6. *Principle of transfinite recursion. If we have a mapping*

$$F : \bigcup_{w \in W} X^{w^<} \to X$$

then there exists a unique function $f : W \to X$ with $f(w) = F(f\mid_{w^<})$ for all $w \in W$.

Proof. We demonstrate that f is well defined by transfinite recursion. Clearly, we can explicitly define such a function for the least element of W. Then for any element $v \in W$, if we have defined for all $u \in v^<$ such a function f_u with the desired property, then we can take the colimit over this diagram to produce a function $f : v^< \to X$, which satisfies the properties we desired. Thus by induction such a function exists for all elements of W. □

Lemma 4.7. *If α and γ are ordinals, they are either equal, or one admits a strict embedding into the other.*

Proof. An initial embedding between α and γ can be defined by mapping the least element of α to the least element of γ, and proceeding by transfinite induction. For arbitrary $x \in \alpha$ map x via f to the least element $y \in \{z \in \gamma \mid \forall a \in \alpha, a \to x, f(a) \neq z\}$. By transfinite induction, we see that f is defined on all of α. Note that if the inductive step fails before α is exhausted then inverting f gives a strict embedding in the opposite direction.

If γ is equal to the image of the embedding, we have an isomorphism, which can be seen by taking a section of the embedding (which will be unique). Otherwise we have shown that α embeds strictly into γ. □

Lemma 4.8. *Every set of objects of Ω has a least element.*

Proof. Let S be a set of ordinals. Take an arbitrary $\alpha \in S$. As S is totally ordered by comparability, it suffices to show that there is a minimal element amongst those less than α. Let W be the set of the images of all strict initial embeddings from elements of S less than α into α.

Then by the well-ordering of α we have that W is well ordered by inclusion, so there is a least element of W, which is isomorphic to an element of S. Thus S has a minimal element.

□

Definition 4.9. An ordinal α is a limit ordinal if it satisfies

$$\alpha = \bigcup_{\beta \xrightarrow{f} \alpha} f\beta$$

where f is a strict embedding. It is a successor ordinal if for some ordinal β we have $\beta \xrightarrow{f} \alpha$ and there is a unique element of α not in the image of $f\beta$.

Theorem 4.10. *Every ordinal $\alpha \in \Omega$ is either a successor or a limit ordinal.*

Proof. Assume α is not a limit ordinal. Then

$$\bigcup_{\beta \xrightarrow{f} \alpha} f\beta \subsetneq \alpha$$

and there exists $x \in \alpha$ such that x is not in the union of strict initial embeddings into α. If there is another such element, call it x'. Then both $x \to x'$ are not in the union, but $\bigcup_{\beta \xrightarrow{f} \alpha} f\beta \cup \{x\}$ (where x is maximal) can be strictly embedded into α, a contradiction. Thus there is a unique element not in the image of the strict embedding, and α is a successor. □

Proposition 4.11. *Ω is not a set.*

Proof. If it were, then for every ordinal α there would be an element $1 \xrightarrow{\alpha} \Omega$. By transfinite recursion, we construct for every α a strict embedding in Ω.

The base case is simple, map the one-element ordinal to the least element of Ω (namely, that ordinal itself). Then take $\alpha \in \Omega$ and let $F_\gamma : \gamma \to \Omega$ be a strict embedding defined for all $\gamma \xrightarrow{g} \alpha$ where g is also a strict embedding.

If α is a successor $\beta + 1$ then let $x \in \alpha$ be the element not in the image of $\beta \xrightarrow{f} \alpha$. For all $y \in \alpha$, $y \neq x$, there exists a unique $y' \in \beta$ such that $fy' = y$. Let F_α map y to $F_\beta(y')$. Then we map x to the first element not yet in the image of some F_γ.

If α is a limit ordinal then for all $a \in \alpha$ map let β be the ordinal isomorphic to $a^<$. Then map a to the first element not in the image of F_β.

Thus every ordinal can be strictly embedded into α. As α is by construction a limit ordinal, it's strictly less than its successor, a contradiction to α's maximality.

□

5. Consequences of Choice

Zorn's Lemma.

Definition 5.1. If P is a partially ordered set and T is totally ordered, then a functor $F : T \to P$ is called a **chain** in P. In the following proof, we will refer to a chain F and its image in P without distinction.

Definition 5.2. In an order category P a strict upper bound on a full subcategory P' of P is an object $x \in P$ such that for all $y \in P'$ we have $y \to x$ but not $x \to y$.

Theorem 5.3. *Zorn's Lemma: If P is a poset category and every chain admits a cocone, then there is an object in P whose every outward morphism is an isomorphism, also known as quasi-terminal object.*

Proof. Then every element χ_A of 2^X corresponds to a subset A of X. We can then construct the coproduct

$$\mathcal{A} = \coprod_{1 \xrightarrow{\chi_A} 2^X} A$$

requiring that A be nonempty. We also have the morphism $\mathcal{A} \xrightarrow{f} 2^X$ that takes an element $a \in A$ to A. By choice we have the section $\tilde{f}$ such that the composition

$$2^X \xrightarrow{\tilde{f}} \mathcal{A} \xrightarrow{i} X$$

chooses an element from every subset of X. We will denote this composition c.

Then let P be a poset category that satisfies the hypothesis of Zorn's Lemma. Proceeding by contradiction, assume P has no maximal element.

Let α be an ordinal. For every nonempty chain define $g(C) = c(\{x \in P \mid \forall y \in C, y \to x, y \neq x\})$, so that g chooses some strict upper bound of C.

Define a mapping F

$$F : \bigcup_{w \in \alpha} X^{w^<} \to P$$

$$\left(w^< \xrightarrow{\psi} X\right) \mapsto g(\psi(w^<))$$

. Then by transfinite recursion 4.6 there exists $f_\alpha : \alpha \to P$ such that $f_\alpha(w) = F(f \mid w^<)$ Then f_α is an order-preserving monomorphism from α to P. Then for every ordinal we have a strict embedding into a subset of P. Because the f_α are unique, these embeddings agree up to restriction, showing that there is a subset of P receiving an embedding from every ordinal. But this well-ordered subset would be isomorphic to some ordinal β. Since β would be maximal we have a contradiction. □

The Well-Ordering Theorem.

Theorem 5.4. *Well Ordering Theorem. There exists for any set S a well ordered set with the same objects as S.*

Proof. Let P be the poset (ordered by initial embeddings) of all well-ordered subsets of S. By Zorn's Lemma there is a maximal well ordered subset W. Assume $W \neq S$ and take $x \in S$ such that $x \notin W$. A larger well ordered subset can be made by appending x to the end of W, a contradiction to the maximality of W. □

An immediate corollary of this theorem is that Ω is a skeleton for the category of sets in that for any set X, there exists an ordinal α such that α without its ordering is isomorphic to X.

6. Conclusions and discussions of ETCS

From the perspective of an incoming sophomore, viewing foundational set theory through ETCS is certainly enlightening and has been a positive learning experience for the author. However, whether it is due to the existence of better techniques than those used in this paper or the nature of ETCS, most material past the basic axioms seems less intuitive than its set-theoretic counterpart. For example, the Von Neumann construction of ordinal numbers, which relies heavily on a membership-based approach to sets, yields much simpler and more natural proofs of some theorems and lemmas discussed in this paper. While the added generality of ETCS may prove useful in the future, it is difficult for the author to see many advantages to its usage over traditional set theory, especially with regards to pedagogy.

That said, challenging oneself to take the morphism-invariance approach to sets provided by Lawvere's work is instructive for any student interested in set theory and structure and understanding orderings in categorial terms provides powerful ways to better understand generalized orderings such as topological inclusion or logical deduction that are less transparent with only a set-theoretical background.

In conclusion, it is the author's opinion that it is somewhat clear why usage of ETCS in modern mathematics has been limited. Given the equality of the two systems, it is unsurprising that instructors seem to opt for the more intuitive axioms of ZFC. However, ETCS provides an alternative lens on the set-theoretic construction of mathematics that undoubtedly has both its advantages and challenges. This alternative view, when learned in addition to ZFC, does provide a fuller understanding of the foundations of mathematics in this author's current (and ever-changing) view.

7. Appendix

Important categorical terms. These terms will be familiar to those familiar with category theory. A reader who feels comfortable with this sort of material can most likely skip this section.

Definition 7.1. A Category $\mathcal{C}$ is a collection of objects along with a collection of morphisms (sometimes called arrows) between those objects. A category comes equipped with functions that assign to each morphism a domain and codomain and satisfies the following properties:

- If $\mathcal{C}$ has two morphisms in $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that the codomain of f is the domain of g, then there exists a morphism gf denoting their composition.
- For every object X of $\mathcal{C}$ a morphism $X \xrightarrow{id_X} X$ serving as the identity morphism on X.

Definition 7.2. A functor F is a mapping from one category to another taking objects to objects and morphisms to morphisms that preserves identity morphisms and respects composition:

For a covariant functor (often, the term covariant is dropped when there is no fear of ambiguity)

$$F(g \circ f) = F(g) \circ F(f)$$

For a contravariant functor

$$F(g \circ f) = F(f) \circ F(g)$$

Definition 7.3. If $\mathcal{C}$ is a category, then the category $\mathcal{C}^{op}$ is the image of C under the contravariant functor that is identity on objects and inversion on morphisms. More succinctly, $\mathcal{C}^{op}$ is $\mathcal{C}$ with its arrows inverted.

$\mathcal{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

$\mathcal{C}^{op}$

$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ & Z \arrow[u, "g"'] \arrow[ul, "fg"] \end{tikzcd}$$

The notion of the opposite category is central to category theory. For most important objects $-$ (where $-$ serves as a placeholder for some important concept) there exists a "dual" concept (normally called a co$-$) such that a $-$ in $\mathcal{C}^{op}$ is a co$-$ in $\mathcal{C}$. Examples of this will come shortly.

Definition 7.4. A **diagram** of type $\mathcal{J}$ in $\mathcal{C}$ is a covariant functor $F : \mathcal{J} \to \mathcal{C}$.

Intuitively, one can think of a diagram as one of potentially many possible ways of taking objects and arrow in the shape of $\mathcal{J}$ to objects and arrows with the same shape in C.

Definition 7.5. A **cone** from an object N of $\mathcal{C}$ to a diagram F in $\mathcal{C}$ is a family of morphisms $L \xrightarrow{n_X} F(X)$ for every object X in the domain of F satisfying:

for any morphism $X \xrightarrow{f} Y$ in the domain of F, the following commutes:

$$\begin{tikzcd} & N \arrow[dl, "n_X"'] \arrow[dr, "n_Y"] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

The obvious dual notion is known as a **cocone**.

Definition 7.6. A **limit** in a category $\mathcal{C}$ is a universal cone in the sense that if L is a limit and N is some other cone over the same diagram, there exists a unique morphism $N \xrightarrow{h} L$ such that the following commutes:

$$\begin{tikzcd} & L \arrow[ddl, bend right, "l_X"'] \arrow[ddr, bend left, "l_Y"] & \\ & N \arrow[u, dashed, "h"] \arrow[dl, "n_X"] \arrow[dr, "n_Y"'] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

A **colimit** is the dual of a limit. Alternatively, a colimit is a universal cocone.

Proposition 7.7. *(Co)limits are unique up to a unique isomorphism.*

Proof. If L, L' are (co)limits of a diagram D, then let h, h' be the morphisms guaranteed by the universal property of (co)limits. Then there is a unique correspondence between L, L' in both directions, and they are isomorphic. $\square$

Definition 7.8. An object 1 is called a **terminal object** if for all objects X of $\mathcal{C}$ there is a unique morphism $X \to 1$.

Proposition 7.9. *A terminal object is precisely a limit over an empty diagram.*

Proof. Note that every object is a cone over an empty diagram, as no extra morphisms are required. Then by the terminal property of 1, every cone (and thus every object) has a unique morphism to 1. $\square$

Definition 7.10. An object 0 is called an **initial object** if for all objects X of $\mathcal{C}$ there is a unique morphism $0 \to X$. An initial object is the dual of a terminal object.

Proposition 7.11. *An initial object is precisely a colimit under an empty diagram.*

This can be shown through duality or simply by modifying the previous argument for 1.

Definition 7.12. A **product** of two objects X, Y is another object $X \times Y$ equipped with morphisms p_x, p_y (called projections) from the product to X, Y respectively with the following universal mapping property:

$$\begin{tikzcd}
& & X \\
Z \arrow[urr, "x"] \arrow[drr, "y"'] \arrow[r, dashed, "\exists ! h"] & X \times Y \arrow[ur, "p_x"'] \arrow[dr, "p_y"] & \\
& & Y
\end{tikzcd}$$

if Z is an object with morphisms x, y to X, Y respectively, then there exists a unique morphism $Z \xrightarrow{h} X \times Y$ such that $hp_x = x$ and $hp_y = y$. In traditional set theory, this product is completely analogous to the cross product of two sets.

Definition 7.13. A **coproduct** $X \coprod Y$ of two objects X, Y is dual to their product.

$$\begin{tikzcd}
X \arrow[dr, "i_x"'] \arrow[drr, "x"] & & \\
& X \coprod Y \arrow[r, dashed, "\exists ! h"] & Z \\
Y \arrow[ur, "i_y"] \arrow[urr, "y"'] & &
\end{tikzcd}$$

If X, Y have morphisms x, y respectively to Z, then there is a unique mapping $X + Y \xrightarrow{h} Z$ such that $i_x h = x$ and $i_y h = y$. This is equivalent to the disjoint union in set-theoretic language.

Proposition 7.14. *There exists a terminal and an initial object.*

Definition 7.15. An **equalizer** over a parallel diagram is an object E (called the equalizer) equipped with a morphism k such that $fk = gk$ with the following universal mapping property:

$$\begin{tikzcd}
X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y \\
E \arrow[u, "k"] & Z \arrow[ul, "z"'] \arrow[l, dashed, "\exists! h"]
\end{tikzcd}$$

if $Z \xrightarrow{z} X$ exists and $fz = gz$, then there exists a unique $Z \xrightarrow{h} E$ such that $kh = z$. If $Z = 1$ then the elements of the equalizer are precisely the elements of X on which f, g agree.

Definition 7.16. A **coequalizer** $E*$ is the dual to an equalizer

$$\begin{tikzcd}
X \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] & Y \arrow[r, "k"] \arrow[d, "z"'] & E* \arrow[dl, dashed, "h"] \\
 & Z &
\end{tikzcd}$$

such that $kf = kg$ and for any Z with a similarly commuting morphism z, there exists a unique morphism $E* \xrightarrow{h} Z$ with $hk = z$. The coequalizer is a generalization of a partition under the equivalence relationship $f(x) \sim g(x)$.

As (co)products and (co)equalizers are examples of (co)limits they exist in any category satisfying the ETCS axioms.

Definition 7.17.

$$\begin{tikzcd}
X \arrow[r, shift left, "b'"] \arrow[r, shift right, "b"'] & Y \arrow[r, "a"] & Z
\end{tikzcd}$$

If for all b, b' we have that $ab = ab'$ implies $b = b'$, then a is a **monomorphism**.

Definition 7.18. The dual of a monomorphism is an epimorphism.

$$\begin{tikzcd}
Z \arrow[r, "a"] & X \arrow[r, shift left, "b'"] \arrow[r, shift right, "b"'] & Y
\end{tikzcd} \qquad\qquad b = b' \iff ba = b'a$$

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