Manifolds

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List of Theorems

Lecture 1: Introduction

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1 Recap

Recall theorems about differentiable maps

— Implicit function theorem

For $U \subset \mathbb{R}^p, V \subset \mathbb{R}^q$, $f \in C^k(U \times V, \mathbb{R}^q), 1 \leq k \leq \infty$ and $(a,b) \in U \times V$ st.

$$D_2 f(a,b) = D(f(a,-))(b)$$

is invertible. Then there exists $a \in U_1 \subset U, b \in V_1 \subset V$ and $\phi \in C^k(U_1, V_1)$ such that

$$f(x, x') = y_0$$

iff $x' = \phi(x)$

— Inverse function theorem

If $U \subset \mathbb{R}^p$ is open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty, a \in U$ such that

$$Df(a)$$

is invertible, then there are $a \in U_1 \subset U$ and $f(a) \in V_1 \subset \mathbb{R}^q$ open such that

$$f|_{U_1} : U_1 \to V_1$$

is a diffeomorphism and

$$Df^{-1}|_U(x) = (Df(f^{-1}|_U(x)))^{-1}$$

for all $x \in U$ in particukar f^{-1} is C^k

— Rank theorem

$U \subset \mathbb{R}^p$ open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty$, $a \in U, b := f(a), r = rank(Df(a))$ then there are diffeomorphisms

$$\psi : U_\psi \to V_\psi \text{ and } \phi : U_\phi \to V_\psi$$

with $U_\psi, V_\psi \subset \mathbb{R}^p$ and $U_\phi, V_\phi \subset \mathbb{R}^q$ such that

$$\phi \circ f \circ \psi(x_1, \ldots, x_p) = (x_1, \ldots, x_r, \tilde{f}(x_1, \ldots, x_p))$$

If $rk(D(f))$ is contant around r, then we can obtain $\tilde{f} = 0$

2 Manifolds

Definition 1 (Basis)
A basis for a topology on X is a collection B of open sets such that every open set in X is the union of sets in B.

X is called second countable if it has a countable topological basis.

Definition 2 (Chart)
Let X be a topological space

1. *A chart on X is a pair (U, ϕ) where $U \subset X$ open and $\phi : U \to \mathbb{R}^n$ for some n which is a homeomorphism onto an open subset.*
2. *An atlas is a collection of charts $A = \{(U_i, \phi_i) | i \in I\}$ such that $X = \bigcup_{i \in I} U_i$*
3. *A is called smooth (C^k ,continuous, holomorphic, algebraic,...) if and only if for any*
$$(U_i, \phi_i)_{i \in \{1,2\}} \in A$$
we have $\phi_1 \circ \phi_2^{-1}$ is smooth (C^k ,...) wherever it is defined.
4. *A chart (U, ϕ) is compatible with an atlas A if and only if*
$$A \cup \{(u, \phi)\}$$
is smooth
5. *An atlas A is maximal if it contains all charts compatible with A. For any atlas A (not necessarily maximal), denote A_{max} the maximal atlas containing it.*
This maximal atlas is necessarily unique

Definition 3 (Manifold)
A smooth manifold of dimension n is a second countable Hausdorrf space with a maximal smooth atlas of dimension n.

Why Hausdorff ?

Consider $\mathbb{R}/\sim$, $x \sim y \iff |x| = |y| > 1$, this space is locally homeomorphic to $\mathbb{R}$ but the points x and y cannot be separated.

Why second countable ?

Take a disjoint union of infinitely many manifolds.
For a connected example, take $\aleph_1 \times [0,1)$ with the order topology.

2.1 Smooth maps

A function $f : M \to N$ between smooth manifolds is called smooth if for each $p \in M$, there are charts $(U, \phi), (V, \psi)$ $p \in U \subset M, f(p) \in V \subset N$ such that

$$\psi \circ f \circ \phi^{-1}$$

is smooth.

f smooth implies $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is smooth for any charts $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi})$ where this is defined.