Analysis IV

David Wiedemann

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List of Theorems

Lecture 1: Measure theory

1   Lebesgue Measure

Motivation

Given a set $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ is it possible to integrate f over Ω.
For $n = 1$ and $\Omega = [a,b]$ riemann-integral works, at least for continuous functions.
However, it is not fully satisfactory

1. Extends badly to $\mathbb{R}^n$

2. Stability with limits Take $f_n : [0,1] \to [0,1]$ continuous and pointwise decreasing, define $f(x) = \lim f_n(x)$, then the integral over f might not exist.

3. Differentiation and integration.
   What is the biggest class of functions for which the fundamental theorem works ?
   For sure in C_1 but that is not the biggest class.

4. Consider $C^0([0,1])$ with L^1-distance.
   Then C^0 is not complete, what is the completion of $\bar{C^0}^d$

We want to find a satisfactory theory of integration.
How can we define the length/volume of a subset $\Omega \subset \mathbb{R}^n$?
Ideally to $\Omega \subset \mathbb{R}^n$ associate $m(\Omega) = 0$ with

$$0 \leq m(\Omega) \leq \infty \quad m((0,1)^m) = 1 \quad m(A \cup B) = m(A) + m(B) \text{ if } A \text{ and } B \text{ disjoint.}$$

$$m(A) \leq m(B) \quad m(A + x) = m(A)$$

This is impossible !

1.1   Measurable sets

We can ask that
— (Borel Property) Open and closed are measurable
— Ω measurable $\implies \Omega^c$ measurable
— (σ-algebra) We want to take countable intersection of measurable sets

> **Definition 1 (Lebesgue Measure)**
> *The lebesgue measure $m(\Omega)$ of any measurable set will obey*
> — *$m(\emptyset) = 0$*
> — *$\infty \geq m(\Omega) \geq 0$*
> — *Monotonicity $m(\Omega_1) \leq m(\Omega_2)$ if $\Omega_1 \subset \Omega_2$*

— *If $\Omega_1, \ldots$ are measurable and disjoint, then we want*

$$m(\bigcup_{i=1}^{\infty} \Omega_i) = \sum_{i=1}^{\infty} m(\Omega_i)$$

and with $\leq$ if they are not disjoint.
— *(Normalisation)*

$$m((0,1)^n) = 1$$

— *(Translation invariance)*

$$m(\Omega + x) = m(\Omega) \forall x \in \mathbb{R}^n$$

Remarque

— *From countable subadditivity, finite subadditivity follows*
— *Monotonicity is redundant because, given $\Omega_1 \subset \Omega_2$*

$$m(\Omega_2) = m(\Omega_1 \cup (\Omega_2 \setminus \Omega_1)) = m(\Omega_1) + m(\Omega_2 \setminus \Omega_1)$$

— *The sums above might be infinite*

Remarque

m is a positive measure if the first four conditions above are satisfied

Theorème 3 (Existence of Lebesgue Measure)
There exists a notion of measurable set obeying the conditions of measurable sets and a measure obeying the conditions.

1.2 Outer Measure

We first want to describe a cube and associate a measure to these boxes. Then we will take a more general set, cover it with boxes and define it's measure by the smallest possible covering by boxes.

Definition 2 (Box)
A open box $B \subset \mathbb{R}^n$ is

$$B = \prod_{i=1}^{n} (a_i, b_i)$$

and define the volume of a box

Definition 3 (Volume of a box)
Given $B = \prod_{i=1}^{n}(a_i, b_i)$, we define
$$vol B = \prod_{i}(b_i - a_i)$$

Now, how can we cover $\Omega \subset \mathbb{R}^n$?

Definition 4 (Covered set)
Given $\Omega \subset \mathbb{R}^n$ is covered by $\{B_j\}_{j \in J}$ if $\Omega \subset \bigcup B_j$

Remarque
If m (the lebesgue measure) exists and J is countable, then
$$m(\Omega) \le m(\bigcup B_j) \le \sum m(B_j)$$

Definition 5 (Outer-Measure)
The outer measure of a set Ω is defined as
$$m^*(\Omega) = \inf \left\{ \sum vol B_j : \{B_j\} \text{ is a countable cover of } \Omega \right\}$$

Remarque
For every Ω there exists at least one countable cover

Lemme 6
The outer measure obeys

1. *$m^*(\emptyset) = 0$*
2. *$0 \le m^*(\Omega) \le \infty$*
3. *$m^*(\Omega_1) \le m^*(\Omega_2)$ if $\Omega_1 \subset \Omega_2$*
4. *$m^*(\Omega + x) = m^*(\Omega)$*
5. *Countable subadditivity : $m^*(\bigcup \Omega_j) \le \sum m^*(\Omega_j)$*

Preuve
— *$m^*(\emptyset) = 0$ because $\emptyset, \{0\} \subset (-\epsilon, \epsilon)^n \forall \epsilon > 0$*
— *All good*
— *Any cover of Ω_2 also covers Ω_1*
— *For any cover of Ω we can translate it over to $\Omega + x$*
— *For every $J \in \mathbb{N}$, let $\left\{ B_i^J \right\}_{i \in I_J}$ cover Ω_J, then $\Omega_j \subset \bigcup_{i \in I_J} B_i^J$, then*

we can choose the B_i^J in such a way that

$$\sum_i vol(B_i^J) \leq m^*(\Omega_J) + \frac{\epsilon}{2^J}$$

and since $\left\{B_i^J\right\}_{i,J}$ covers $\bigcup_J \Omega_J$

$$m^*(\bigcup \Omega_J) \leq \sum_{j \in \mathbb{N}} \sum_{i \in I_J} vol(B_i^J) \leq \sum_{j \in \mathbb{N}}(m^*(\Omega_J) + \frac{\epsilon}{2^J}) = \epsilon + \sum m^*(\Omega_J)$$

$\square$

Proposition 7

For a closed box $\overline{B}$

$$m^*(\overline{B}) = vol(B)$$

Preuve

Clearly $\overline{B}$ is covered by $\prod(a_i + \epsilon, b_i + \epsilon)$ Hence

$$m^*(\overline{B}) \leq vol(\prod(a_i + \epsilon, b_i + \epsilon)) \rightarrow \prod(b_i - a_i)$$

Hence $m^*(\overline{B}) \leq vol(B)$
Now we show that $vol(B) \leq m^*(\overline{B})$.
By Heine-Borel, $\overline{B}$ is compact.
Hence we only need to show the result with a finite cover.
In dimension 1, we are given $(a_1, b_1), \ldots$ covering $[a, b]$.
Remark that

$$1_{[a,b]} \leq \sum_i 1_{(a_i, b_i)}$$

Integrating (Riemann-integral), we get

$$(b - a) \leq \sum(b_i - a_i)$$

Now, we use induction

$$B_J = \prod_{i=1}^{n}(a_i^s, b_i^s) = \prod_{i=1}^{n-1}(a_i^s, b_i^s) \times (a_n^s, b_n^s)$$

Define

$$f_J(x_m) = vol(A_J)1_{(a_n, b_m)}(x_m)$$

For every x_m, we get

$$\left\{A^J : j \in J, x_n \in (a_n^J, b_n^J)\right\} \text{ is a cover of } \overline{A}$$

$$\sum f_j(x_m) = sum_{j \in J, x_n} vol(A_j)1_{(a_n, b_n)} \geq vol\overline{A}$$

$\square$

Lecture 2: Existence of Lebesgue Measure

> **Corollaire 8**
> $m^*(B) = vol(B)$ for every open box B.

Preuve
For one direction, we use monotonicity, $m^*(B) \leq m^*(\overline{B}) = vol(B)$.
Furthermore, set $B = \prod(a_i, b_i)$, then for $\epsilon > 0$, we get

$$\prod_i [a_i + \epsilon, b_i - \epsilon] \subset \prod_i (a_i, b_i) \implies m^*(\prod[a_i + \epsilon, b_i - \epsilon]) \leq \prod_i (b_i - a_i)$$

$\square$

Exemple
— $m^*(\mathbb{R}) = \infty$ since by monotonicity, we get $m^*(\mathbb{R}) \geq m^*([0, N]) > N$
— $m^*(\mathbb{Q}) = 0$ since

$$m^*(\mathbb{Q}) \leq m^*(\{q\}) = 0$$

Which proves that the reals are uncountable.

1.3 Measurable sets (again)

We want to know whether $\forall A, E \subset \mathbb{R}^m$, the inequality

$$m^*(A) \leq m^*(A \cap E) + m^*(A \setminus E)$$

generalises to an equality ?
The inequality follows directly from countable subadditivity. In fact equality does not hold in general.

> **Definition 6 (Lebesgue Measurable set)**
> A set $E \subset \mathbb{R}^m$ is Lebesgue measurable if
>
> $$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \forall A \subset \mathbb{R}^n$$
>
> Then the lebesgue measure of E is defined as
>
> $$m(E) := m^*(E)$$

Note that, according to this definition, $\emptyset, \mathbb{R}^n$ are both measurable.

> **Lemme 10**
> Half-spaces are measurable

The proof is given as an exercise.

We now establish a few basic facts about measurable sets.

> **Lemme 11**
> — *The complement of a measurable set is measurable*
> — *The translation of a measurable set is measurable, ie. E measurable, $x \in \mathbb{R}^n$ implies $E + x$ measurable*
> — *Finite unions of measurable sets is measurable. (as well as the intersection)*
> — *Open (as well as closed) boxes are measurable.*
> — *If the outer measure of a set is 0, then E is measurable.*

Preuve

—

$$m^*(A) = m^*(A \cap E^{c^c}) + m^*(A \cap E^c)$$

— *Given A a set and $x \in \mathbb{R}^n$, we get*

$$m^*(A-x) = m^*(A-x \cap E) + m^*((A-x) \cap E^c) = m^*(A \cap E + x) + m^*(A \cap E^c + x) = m^*(A)$$

—

$$m^*(A) = m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

— *Consider the union of two sets We now bound $m^*(A)$ by below (the upper bound is always true)*

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

$$= m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \cap E_2^c) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

$$\geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c)$$

The general result follows immediatly by induction on the number of sets.

— *We get that*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \qquad \square$$

— *We write boxes as intersections of halfspaces*

Now we want to show that the lebesgue measure is countably additive.

Proposition 12

If $(E_j)_{j \in \mathbb{N}}$ are measurable disjoint sets, then $\bigcup_{i \in \mathbb{N}} E_i$ is measurable and

$$m^*\left(\bigcup_{j \in \mathbb{N}} E_j\right) = \sum_{j=1}^{\infty} m^*(E_j)$$

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The proof depends on a lemma

> **Lemme 13**
>
> *Let $E_1, \ldots, E_n$ be measurable disjoint sets, $A \subset \mathbb{R}^m$, then*
>
> $$m^*(A \cap (\bigcup E_j)) = \sum_{j=1}^{n} m^*(A \cap E_j)$$
>
> *As a consequence of this, we get finite additivity.*

Preuve

For $n = 2$, we get

$$m^*(A \cap (E_1 \cup E_2)) = m^*(A \cap (E_1 \cup E_2) \cap E_1) + m^*(A \cap (E_1 \cup E_2) \cap E_1^c)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_2) \qquad \square$$

and the general case follows by induction.

> **Corollaire 14**
>
> *$E \subset F$ measurable implies $F \setminus E$ is measurable and*
>
> $$m^*(F \setminus E) = m(F) - m(E)$$

Preuve

The set is trivially measurable since $F \setminus E = F \cap E^c$ Using the lemma above, we get

$$m^*(F) = m^*(E) + m^*(F \setminus E) \qquad \square$$

We can now prove countable additivity

Preuve

Let $E = \bigcup_{j=1}^{\infty} E_j$.
We claim that $\forall A$

$$m^*(A) \geq m^*(A \cap E) + m^*(A \setminus E)$$

Indeed note that

$$m^*(A \cap E) \leq \sum_{j=1}^{\infty} m^*(A \cap E_J) = \sup_N \sum_{j=1}^{N} m^*(A \cap E_j)$$

Set $F_n = \bigcup_{j=1}^{N} E_j$, by the lemma, the finite sum above is

$$\sup_N \sum_{j=1}^{N} m^*(A \cap E_j) = m^*(A \cap F_N)$$

Since $F_N \subset E$,

$$m^*(A \setminus E) \leq m^*(A \setminus F_N)$$

Then

$$m^*(A \cap E) + m^*(A \setminus E) < \sup_N m^*(A \cap F_N) + \underbrace{m^*(A \setminus E)}_{\leq m^*(A \setminus F_N)} < \sup_N m^*(A) \qquad \square$$

This proves that $m(E) \geq \sup_N m(F_N) = \sup_N \sum_{j=1}^{N} m(E_j) = \sum_{j=1}^{\infty} m(E_J)$

Lemme 15 (Lebesgues sets are a sigma-algebra)
If $(E_J)_J \in \mathbb{N}$ are measurable, then $\bigcup E_j$ and $\bigcap E_j$ are measurable.

Preuve

$$E_1 \cup \ldots = E_1 \cup (E_2 \setminus E_1) \cup (E_3 \setminus (E_1 \cup E_2)) \ldots$$

and the property about intersections follows from $\bigcap E_J = (\bigcup E_J^c)^c$ \qquad $\square$

Lemme 16 (Open sets are measurable)
Every open set is measurable

Preuve
By an exercise, every open set is a countable union of open boxes and a countable union of measurable sets is countable by the lemma above. \qquad $\square$

1.4 A glimps on abstract measure theory and theoretical foundations of probability

The idea of Lebesgue was to fix the measure of boxes and then extend the measure to the sigma algebra of measurable sets.

Theorème 17 (Caratheodory theorem)
Given a set Ω, $\mathcal{G}$ an algebra (finite union of boxes), A the smallest algebra containing $\mathcal{G}$.
Let $m_0 : \mathcal{G} \rightarrow [0, \infty]$ be a function s.t. $m(\emptyset) = 0, m_0(\bigcup_{i=1}^{\infty} A_i) = \sum_{m=1}^{\infty} m_0(A_m)$ if $A_m \in \mathcal{G}, A_m$ disjoint and $\bigcup A_m \in \mathcal{G}$
Then $\exists$ a measure on A such that $m|_{\mathcal{G}} = m_0$ and, if the measure of $m_0(\Omega) < \infty \implies m$ is unique.

Furthermore

> **Theorème 18**
>
> *Every probability $\mathbb{P}$ on $\mathbb{R}^n$ gives rise to a cumulative distribution function, conversely, every cdf gives rise to a (unique) probability measure.*

1.5 The cantor set

> **Definition 7 (Cantor set)**
> *Consider $[1,1]$, define $P_0 = [0,1]$, $P_1 = [0, \frac{1}{3},] \cup [\frac{2}{3}, 1]$ and keep going.*
> *By definition $P_0 \supset P_1 \ldots$, the cantor set is the intersection of all of them.*

There are a few nice properties of the cantor set

> **Theorème 19**
> 1. P is compact
> 2. $m^*(P) = 0$
> 3. P is uncountable
> 4. P is perfect[a] and has empty interior.
>
> ---
> a. No point in p is isolated.

Lecture 3: Measurable functions

1.6 Measurable functions

> **Definition 8 (Measurable functions)**
> *Let $\Omega \subset \mathbb{R}^m$ measurable, $f : \Omega \to \mathbb{R}^m$ is measurable if $\forall V$ open, $f^{-1}(V)$ is measurable.*

Remarque

Any function $f : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ is measurable $\iff f^{-1}(B)$ is measurable $\forall B$ open boxes.

> **Preuve**
>
> *Indeed, the implication $\implies$ is immediate.*
> *For the other direction, note that any open set V is a countable union of boxes*
> $$V = \bigcup_i B_i$$
> *and $f^{-1}(V) = \bigcup_i f^{-1}(B_i)$ which is measurable.* $\square$

Remarque

Let $f : \Omega \to \mathbb{R}$ is measurable $\iff f^{-1}((a, \infty))$ are measurable.

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Preuve

By the remark above, it is enough to show that $f^{-1}((a, \infty))$ are measurable $\forall a, b$

$$f^{-1}((a,b)) = f^{-1}((-\infty, b) \cap (a, \infty)) = f^{-1}(a, \infty) \cap f^{-1}([b, \infty))^c$$

Now, rewrite $f^{-1}([b, \infty)) = \bigcap_i f^{-1}((b - \frac{1}{i}, \infty))$ □

> **Definition 9**
> $f : \Omega \to \mathbb{R}^* = \mathbb{R} \cup \{\pm\infty\}$ *is measurable if $f^{-1}((a, \infty])$ is measurable $\forall a \in \mathbb{R}$*

Using the remark above, the definition is compatible with the definition of measurable functions.

Remarque

Consider $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$, f is measurable $\iff$ all projections of f are measurable.

Preuve

To prove this, recall that f is measurable $\iff$ $f^{-1}(B)$ are measurable, we may write $B = B_1 \times \ldots \times B_n$, hence, $f^{-1}(B) = \bigcap_{i=1}^n f_i^{-1}(B_i)$.
Hence the right to left implication follows.
$\implies$ Consider $B = \mathbb{R} \times \ldots \times B_i \times \ldots \times \mathbb{R}$, then $f^{-1}(B) = f_i^{-1}(B_i)$ is measurable □

Remarque

Let $f : \Omega \to W$ and $g : W \to \mathbb{R}^p$, then $g \circ f$ is measurable if g is continuous and f measurable.

> **Lemme 24**
> *Let $\Omega \subset \mathbb{R}^n$ measurable, $f_m : \Omega \to \mathbb{R}^*$ measurable, then the functions*
>
> $$\sup f_m, \inf f_m, \limsup f_m, \liminf f_m$$
>
> *are measurable.*
> *In particular, if $f_m \to f$ pointwise, then f is measurable.*
>
> **Preuve**
>
> *Call $F = \sup f_n$, we want to prove that*
>
> $$F^{-1}((a, \infty]) = \bigcup f_m^{-1}((a, \infty])$$
> □

Lecture 4: Lebesgue Integration

1.7 Lebesgue integration

> **Definition 10 (Simple functions)**
>
> *A measurable function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is simple if (Ω is measurable)*
>
>   1. *$f(\Omega)$ is a finite set*
>   2. *$\exists c_1, \ldots, c_n \in \mathbb{R}$ and $E_1, \ldots, E_n \subset \Omega$ measurable s.t.*
>
> $$f = \sum_{i=1}^{n} c_i 1_{E_i}$$

Preuve

Clearly $\{c_1, \ldots, c_n\} = f(\Omega)$, conversely, if $f(\Omega) = \{c_1, \ldots, c_n\}$, define $E_i = f^{-1}(c_i)$ □

Remarque

Note that simple functions are vector spaces

> **Lemme 26**
>
> *Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be measurable. Then $\exists$ an increasing sequence $\{f_n\}$ converging pointwise to f*

Preuve

Define $f_n(x) = \sup_j \{2^{-n} J \leq \min(f(x), 2^n)\}$.

> **Definition 11**
>
> *Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be a simple function, then the lebesgue integral of f is*
>
> $$\int_\Omega f dx = \sum_{\lambda \in f(\Omega), \lambda \geq 0} \lambda \mu \{x \in \Omega : f(x) = \lambda\}$$
>
> *Note this definition works for general measures.*

Remarque

Let $f = \sum_i c_i 1_{E_i}$, then

$$\int_\Omega f dx = \sum_i c_i \mu(E_i)$$

The integral may be infinite.

Proposition 28 (Properties of simple functions)

Let $f, g : \Omega \to \mathbb{R}_{\geq 0}$ be simple functions

1. *$0 \leq \int_\Omega f \leq \infty$ and $\int_\Omega f = 0 \iff f \equiv 0$ almost everywhere.*
2. *$\int_\Omega f + g d\mu = \int_\Omega f d\mu + \int_\Omega g d\mu$*
3. *$\lambda \int_\Omega f d\mu = c \int_\Omega f$*
4. *if $f \leq g$, then $\int_\Omega f + \int_\Omega g$*

Definition 13 (Lebesgue Integral of non-negative function)

Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be measurable, we define

$$\int_\Omega f := \sup \left\{ \int_\Omega s\, dx : s \leq f, s \text{ simple} \right\}$$

Remarque

In fact, if f is simple both definitions are compatible.

Proposition 30

Let $f, g : \Omega \to \mathbb{R}_{\geq 0}$ be measurable

— *$0 \leq \int_\Omega f \leq \infty$ and $\int_\Omega f = 0 \iff f = 0$ a.e.*
— *$\int_\Omega cf = c \int_\Omega f$*
— *If $f \leq g$ then $\int_\Omega f \leq \int_\Omega g$*
— *If $f = g$ a.e. then $\int_\Omega f = \int_\Omega g$*
— *if $\Omega' \subset \Omega$, then $\int_{\Omega'} f = \int_\Omega (f 1_{\Omega'})$*

We will prove additivity later on

Theorème 31 (Lebesgue Monotone convergence theorem)

Let $\Omega \subset \mathbb{R}^n$ be a measurable set and take f_n an increasing sequence of functions converging pointwise to f.
Then

$$\int_\Omega f = \lim_{m \to +\infty} \int_\Omega f_n$$

Preuve

By definition $f(x) = \lim_{n \to +\infty} f_n(x) = \sup_n f_n(x)$ (since the f_n are increasing).

Using the propositions above, we have that

$$\int_\Omega \sup_m f_m \geq \int_\Omega f_m \quad \forall m$$

Hence $\int_\Omega f \geq \sup \int_\Omega f_m$.
We claim $\int_\Omega \sup f_m \leq \sup \int_\Omega f_m$.
It suffices to show that $\forall \epsilon$

$$(1-\epsilon)\int_\Omega s \leq \sup_m \int_\Omega f_m \quad \forall s \leq \sup f_m \ simple$$

Indeed, note that $\forall x \in \Omega \exists N := N(x)$ s.t. $f_N(x) \geq (1-\epsilon)s(x)$.
Let $E_n = \{x \in \Omega : f_n \geq (1-\epsilon)s\}$.
Since f_n is increasing, $E_1 \subset E_2 \dots$ and $\bigcup E_i = \Omega$, hence we get

$$(1-\epsilon)\int_{E_m} s = \int_{E_m}(1-\epsilon)s \leq \int_{E_m} f_N \leq \int_\Omega f_n$$

Taking the sup yields

$$\sup_n(1-\epsilon)\int_{E_n} s \leq \sup_n \int_\Omega f_n$$

Hence, we only need to show that the left hand side equals $(1-\epsilon)\int_\Omega s$.
Indeed, the inequality $\sup_n(1-\epsilon)\int_{E_n} s \leq (1-\epsilon)\int_\Omega s$.
For the other inequality, write $s = \sum 1_{F_j} c_j$, then

$$\int_{E_n} s = \int_\Omega \sum c_j 1_{E_n \cap F_j} \qquad \square$$

Lecture 5: Monotone Convergence theorem

> **Corollaire 32**
> $f, g : \Omega \to [0, \infty)$ *measurable, then*
>
> $$\int_\Omega f + g = \int_\Omega f + \int_\Omega g$$

Preuve
Let s_n, t_n be simple functions converging pointwise to f respectively g, then $s_n + t_n$ converges pointwise to $f + g$.
Then

$$\int_\Omega f + g = \lim_{n\to+\infty}\int_\Omega s_n + t_n = \lim_{n\to+\infty}\int_\Omega s_n + \int_\Omega t_n = \int f + \int g \qquad \square$$

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Corollaire 33

Let $g_1, \ldots : \Omega \to [0, \infty)$ be measurable functions, then

$$\int_\Omega \sum_{i=1}^\infty g_i = \sum_{i=1}^\infty \int_\Omega g_i$$

Preuve

Let $G_n = \sum_{i=1}^n g_i$, this is a sequence of functions converging to G (from below)

$$\int_\Omega \sum_{i=1}^\infty g_i = \int_\Omega G = \lim_{n \to +\infty} \int_\Omega G_n = \lim_{n \to +\infty} \sum_{i=1}^n \int_\Omega g_i = \sum_{i=1}^\infty \int_\Omega g_i \qquad \square$$

1.8 Fatou's lemma

Theorème 34 (Fatou's lemma)

Let f_i be a sequence of measurable functions $\Omega \to [0, \infty)$, then

$$\int_\Omega \liminf_{m \to \infty} f_m \leq \liminf_{m \to \infty} \int_\Omega f_m$$

Preuve

By definition

$$\liminf f_m = \sup_n \inf_{m \geq n} f_m$$

By monotone convergence theorem

$$\int_\Omega \liminf_n f_n = \sup_n \int_\Omega \inf_{m \geq n} f_m$$

Since $\int_\Omega \inf_{m \geq n} f_m \leq \int_\Omega f_J \forall J \geq m$, hence

$$\int_\Omega \inf_{m \geq n} f_m \leq \inf_{J \geq m} \int_\Omega f_J$$

And finally

$$\int_\Omega \liminf f_m \leq \sup_m \inf_{J \geq m} \int_\Omega f_J = \liminf_{J \to +\infty} \int_\Omega f_J \qquad \square$$

Lemme 35

Let $f : \Omega \to [0, \infty]$ be a measurable function, if $\int_\Omega f < \infty$, then

$$\mu \{x \in \Omega : f(x) = \infty\} = 0$$

Suppose not, let E be this set, then $\forall n$

$$n1_E \leq f \implies n\mu(E) \leq \int_\Omega f$$

$\square$

Exemple (Borel-Cantelli)

Let $\{\Omega_i\}$ be measurable sets such that $\sum \mu(\Omega_i) < \infty$, then

$$\limsup \Omega_i = \{x \in \Omega : x \in \Omega_i \text{ for infinitely many values }\}$$

has measure 0.

Preuve

We claim that $\int_\Omega \sum_i 1_{\Omega_i} < \infty$,, then by the lemma, $f < \infty$ almost everywhere, hence $x \in \Omega_i$ only for finitely many i, hence $x \notin \limsup \Omega_i$.

The proof of the claim follows from the corollary to fatou's lemma :

$$\int_\Omega \sum_i 1_{\Omega_i} = \sum_i \int_\Omega 1_{\Omega_i} = \sum \mu(\Omega_i) < \infty$$

$\square$

Lecture 6: Dominated Convergence Theorem

Wed 16 Mar

1.9 Integration of signed functions

> **Definition 14**
>
> $f : \Omega \to [-\infty, \infty]$ *is absolutely integrable if*
>
> $$\int_\Omega |f| < \infty$$

> **Definition 15 (Integral of a function)**
>
> *Let f be an absolutely integrable function, then*
>
> $$\int_\Omega f = \int_\Omega f^+ - \int_\Omega f^-$$

Remarque

$$|\int_\Omega f| \leq \int_\Omega |f|$$

Proposition 38 (Basic properties)

Let f, g be absolutely integrable functions

— $\forall c \in \mathbb{R}$, cf *is absolutely integrable and* $\int_\Omega cf = c \int_\Omega f$

— $f + g$ is absolutely integrable and $\int_\Omega f + g = \int_\Omega f + \int_\Omega g$
— If $f = g$ almost everywhere then $\int_\Omega f = \int_\Omega g$

> **Theorème 39 (Dominated Convergence Theorem)**
>
> *Let $f_1, f_2, \ldots : \Omega \to [-\infty, \infty]$ be measurable functions. Assume $f_n \to f$ almost everywhere and such that $|f_m(x)| \leq F(x) \forall m, x \in \Omega$ where F is absolutely integrable.*
> *Then*
> $$\lim_{n \to +\infty} \int f_n = \int f$$

Remarque

With the same assumptions, we can conclude that

$$\lim_{n \to +\infty} \int |f_n - f| = 0$$

Indeed, apply the theorem to $g_n = |f_n - f|$.
Then $|g_m| \leq |f_n| + |f| \leq 2F$.
Similarly, let f_m be such that the above condition holds, then $\int f_n \to \int f$, since

$$\left| \int f_n - \int f \right| = \left| \int f_n - f \right| \leq \int |f_n - f| \to 0$$

Preuve

By assumption $|f_n| \leq F$, hence $|f| \leq F$.
Apply Fatou to $F(x) + f_n(x)$, we get

$$\int_\Omega F + f \leq \liminf \int F + f_n \leq \liminf \int f_m + \int f_n$$

Now we apply Fatou to $F - f_n \geq 0$, we get

$$\int_\Omega F - \int_\Omega f \leq \liminf \int_\Omega F - f_n$$

Which in turn implies that

$$\int_\Omega f \leq \liminf_{n \to \infty} \int_\Omega f_n$$

We now apply the same trick to $F - f_n$, noticing again this family of functions is non-negative

$$\int_\Omega F - f \leq \liminf_{n \to \infty} \int_\Omega F - f_n$$

$$\int_\Omega f \geq \limsup_{n \to \infty} \int_\Omega f_n$$

Which implies the limit $\int f_n$ exists and is equal to $\int f$ $\qquad\qquad \square$

Remarque (Differentiation under the integral)

Let $f : \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ be measurable such that

— *$\partial_t f(x,t)$ for almost every x and every t*

— *$|\partial_t f(x,t)| \leq h(x)$ where $h(x)$ is an absolutely integrable function, then*

$$\frac{d}{dt} \int f(x,t)dx = \int \partial_t f(x,t)$$

Preuve

Indeed

$$\frac{d}{dt} \int f(x,t) = \lim_{h \to 0} \int \underbrace{\frac{f(x,t+h) - f(x,t)}{h}}_{\to \partial_t f(x,t)}$$

Now notice that

$$|\frac{f(x,t+h) - f(x,t)}{h}| \leq |\int \partial_t f(x,t+hs)ds| \leq h(x) \qquad \square$$

Definition 16

Let $\Omega \subset \mathbb{R}^m$, f a function (not necessarily measurable).
The upper and lower Lebesgue integrals

$$\overline{\int_\Omega} f = \inf \left\{ \int g : g \text{ measurable }, g \geq f \right\}$$

and similarly the lower integral.

$$\underline{\int_\Omega} f = \inf \left\{ \int g : g \text{ measurable }, g \leq f \right\}$$

1.10   Comparison with Riemann Integral

Theorème 42 (Lebesgue generalizes Riemann)

Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be Riemann integralble, then f is absolutely integrable and

$$\int_I f dx = \text{ Riemann integral of } f \text{ on } I$$

Preuve

f is Riemann integrable if $\forall \epsilon > 0$ there exists p a partition of I such that

$$A - \epsilon \leq \sum |J| \inf_{x \in J} f \leq \sum_{J \in P} |J| \sup f \leq A + \epsilon$$

Since $f_\epsilon^- \leq f \leq f_\epsilon^+$

$$A - \epsilon \leq \int f_\epsilon^- \leq \underline{\int} f \leq \overline{\int} f \leq \int f_\epsilon^+ \leq A + \epsilon$$

Letting $\epsilon \to 0$ yields the result.
Indeed let $f_m^\pm$ be such that $f_m^- \leq f \leq f_m^+$

$$\underline{\int} f - \frac{1}{m} \leq \int f_m^+ \leq \overline{\int} f + m$$

$\square$

Setting $F^- = \sup f_m^-, F^+ = \inf f_m^+$ are measurable.
$F^- \leq f \leq F^+$

1.11 Fubini's Theorem

Theorème 43 (Fubini-Tonelli)
Let $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$. Assume $f \geq 0$ or f absolutely integrable, then
  — *for almost every x , $f(x, \cdot)$ is measurable and*

$$x \mapsto \int f(x,y) dy$$

  is measurable
  — *For almost every y , $f(\cdot, y)$ is measurable and*

$$y \mapsto \int f(x,y) dy$$

  is measurable

$$\int_{\mathbb{R}^n \times \mathbb{R}^m} f \, dxdy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y) dy \right) dx$$

Lecture 7: Fubini's Theorem

Remarque

Tonelli is used on $|f|$ and to show that f is absolutely integrable, then we can apply Fubini.

Preuve
We prove the result under the additional assumptions that $m = n = 1$ and that every function appearing is measurable.
We will prove that

$$\int_{\mathbb{R}^2} f \, dxdy = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) dydx$$

It is enough to prove the above equality when $f \geq 0$.
If not, $f = f^+ - f^-$, then we may apply the above result to f^+ and f^-.

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Notice also that it is sufficient to prove the result for f such that $\operatorname{Supp} f \subset [-N, N]^2$.

Indeed, write $f_n = f 1_{[-n,n]^2}$, then

$$\int_{\mathbb{R}} f_n = \int_{\mathbb{R}} \int_{\mathbb{R}} f_n$$

Now since f_n is monotone, $\int_{\mathbb{R}^2} f_n \to \int_{\mathbb{R}^2} f$ the left hand side yields (again using monotone convergence)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f dy dx$$

We may now reduce the problem even further to simple functions with bounded support.

Indeed for every $f \geq 0$, f is a $\sup$ of simple functions so we can apply monotone convergence.

Now since every simple function is the sum of indicator functions, we only need to prove the result for indicator functions :

$$f = \sum c_i 1_{E_i}$$

$$\int_{\mathbb{R}^2} f dx dy = \sum_i c_i \int_{\mathbb{R}^2} 1_{E_i} dx dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \sum_i c_i 1_{E_i} dx dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} f dy dx$$

It is enough to prove that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_E(x,y) dy dx \leq m(E) \quad E \subset [-N, N]^2$$

Indeed if the above holds, we may apply it to $[-N, N]^2 \setminus E$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_{[-N,N]^2 \setminus E} dy dx \leq m([-N, N]^2 \setminus E)$$

Summing both inequalities yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_E + 1_{[-N,N] \setminus E} dy dx \leq m(E) + m([-N, N]^2 \setminus E) = m([-N, N]^2)$$

Hence all inequalities above are in fact equalities.

So we only need to prove that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} 1_E(x,y) dy dx \leq m(E) \quad E \subset [-N, N]^2$$

Consider a covering $\{B_j\}$ of E s.t. $\sum \operatorname{Vol}(B_j) \leq m(E) + \epsilon$, but this is just

$$\sum \operatorname{Vol}(B_j) = \sum \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{B_j} dy dx$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \underbrace{\sum 1_{B_j}}_{\geq 1_E} dydx \qquad\qquad \square$$

Which concludes the proof.

Lecture 8: Lp spaces

2 L_p spaces

> **Definition 17 (Lp space)**
> *Let $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ and $p \in [1, \infty)$, we define*
>
> $$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f|^p \right)^{\frac{1}{p}}$$
>
> *and*
>
> $$\left\{ f : \Omega \to \mathbb{R} \cup \{\pm\infty\} \, \middle| \, \|f\|_{L_p(\Omega)} < \infty \right\}$$

Remarque

If $p = 1$, then $L^1(\Omega)$ are absolutely integrable functions.
We hope the definition above is a norm, but we need

$$\|f\| = 0 \iff f = 0$$

so we need to ask that $f = 0$ almost everywhere.
We wish to identify in L^p functions that coincide almost everywhere, so we need to identify as follows

$$(L^p(\Omega), \|\cdot\|_{L_p}) = \{f : \Omega \to \mathbb{R} \cup \{\pm\infty\} : \|f\| < \infty\} \, / \sim$$

where $f \sim g \iff f = g$ ae.

> **Definition 18 (L infinity)**
> *Define*
>
> $$\|f\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f| = \inf \{\alpha : f < \alpha \text{ almost everywhere }\}$$

If f is continuous, the sup and ess sup coincide.
Then $L^\infty(\Omega)$ is then defined as above.

Proposition 46

Let $\Omega \subset \mathbb{R}^n$ be measurable and $1 \leq p \leq q \leq \infty$, then
   — *$L^p(\Omega)$ is a vector space*

— If $m(\Omega) < \infty$, then $\|f\|_{L_q} \leq K \|f\|_{L_q} \; \forall f$ where K depends on $m(\Omega), p$ and q.
— if $m(\Omega) < \infty$, then $\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}$
— Minkowski inequality

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

In particular $\|\cdot\|_{L^p}$ is a norm.

> **Theorème 47 (Hoelder inequality)**
> Let Ω be measurable, $p \in [1, \infty]$, then
>
> $$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}$$
>
> where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$

Preuve

The inequality holds iff $\|\lambda_1 f \lambda_2 g\|_{L^p} \leq \|\lambda_1 f\|_{L^p} \|\lambda_2 g\|_{L^{p'}}$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$. So we may reduce ourselves to the case

$$\|f\|_{L^p} = \|g\|_{L^{p'}} = 1$$

Now

$$\int |fg| \leq \int \frac{|f|^p}{p} + \frac{|f|^{p'}}{p'} = 1 \qquad \qquad \square$$

Preuve (Of second point above)

$$\|F\|_{L^P} = \|F^p\|_{L^1}^{\frac{1}{P}} \leq \left(\int F^{P\frac{Q}{P}} \right)^{\frac{1}{Q}} \left(\int 1^{p'} \right)^{\frac{1}{P} - \frac{1}{Q}} \qquad \qquad \square$$

Preuve (Of fourth point)

$$
\begin{aligned}
\|f + g\|_{L^p}^p &= \int |f + g|^p \\
&\leq \int (|f| + |g|)|f + g|^{p-1} \\
&= \int |f||f + g|^{p-1} + \int |g||f + g|^{p-1} \\
&\leq \left(\int |f|^p \right)^{\frac{1}{p}} \left(\int |f + g|^p \right)^{\frac{p-1}{p}} \\
&= (\|f\| + \|g\|) \|f + g\|^{p-1} \qquad \qquad \square
\end{aligned}
$$

2.1 Completeness of L^p

> **Theorème 48 (Lp spaces are complete)**
>
> *Let Ω be measurable, $p \in [1, \infty]$, then $L^p(\Omega)$ is complete, namely if*
>
> $$\lim_{m,n \to +\infty} \|f_n - f_m\|_{L^p} = 0$$
>
> *then $\exists f \in L^p$ s.t. $\lim_{n \to +\infty} \|f_n - f\|_{L^p} = 0$.*
> *Moreover, if the above holds then $\exists$ a subsequence $\{m_k\}$ s.t.*
>
> $$f_{m_k}(x) \to f(x)$$
>
> *Almost everywhere.*

Remarque

Taking the subsequence above is important, see exercises.

Preuve

We prove the result for $p < \infty$, the case $p = \infty$ is an exercise.
We want to prove that $\{f_m\}$ is cauchy in L^p implies there is a subsequence $f_m \to f$ in L^p pointwise.
We look for a speedy converging subsequence.
Indeed, we know from hypothesis that there exists a subsequence $\{m_k\}$ st.

$$\|f_{m_k} - f_{m_{k+1}}\| \leq 2^{-k}$$

Now consider

$$f(x) = f_{m_1}(x) + \sum_k f_{m_{k+1}} - f_{m_k}(x)$$

This is a reasonable definition, but is it well defined.
Namely is the series absolutely converging for almost every x ?
Consider

$$g_h(x) = |f_{m_1}(x)| + \sum_{k=1}^{h} |f_{m_{k+1}}(x) - f_{m_k}(x)|$$

Is $\lim_{h \to +\infty} g_h < \infty$ ae. ? If yes, f is well defined.
Indeed,

$$\|g_j\|_{L^p} \leq \|f_{m_1}\|_{L^p} + \sum \|f_{m_{k+1}} - f_{m_k}\| \leq \|f_{m_1}\| + 1$$

But now

$$\int |g|^p = \lim_{h \to +\infty} \int |g_h|^p < \infty$$

Hence g is finite a.e. and

$$f(x) = \lim f_{m_1}(x) + \sum_k f_{m_{k+1}} - f_m = \lim f_{m_k}(x)$$

And the convergence is dominated by g.

To prove L^p convergence

$$\|f_{m_k} - f\|_{L^p}^p = \int |f_{m_k} - f|^p \to 0$$

$\square$

Lecture 9: Smooth functions are dense

2.2 Approximation of L^p functions with $C_c^\infty(\Omega)$

> **Definition 19 (Compactly supported)**
>
> *If $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$, then $\operatorname{Supp} f = \{x : f(x) \neq 0\}$*
>
> $$C_c^0(\Omega) = \left\{ f \in C^0(\Omega) : \operatorname{Supp} f \subset\subset \Omega \right\}$$
>
> *Where we require $\operatorname{Supp} f$ to be compact. And then we define*
>
> $$C_c^k(\Omega) = C_c^0(\Omega) \cap C^k(\Omega)$$

> **Theorème 50**
>
> *Let Ω be an open set, $1 \le p < \infty, f \in L^p(\Omega)$ then $\exists g_k \in C_k^\infty(\Omega)$ st. $\lim_{k \to +\infty} \|g_k - f\|_{L^p(\Omega)}$*

Preuve

We prove the result for $\Omega = \mathbb{R}^n$, we first find $g_k \in C_c^0(\mathbb{R}^n)$

We prove the result for $f = 1_B$, B a box.

Define $g_\epsilon(x) = \min(1 - \frac{d(x,B)}{\epsilon}, 1)$

Now we want to go from indicator of boxes to indicators of measurable sets.

So assume $f = 1_E, E$ measurable and $\overline{E}$ is compact.

Let $\epsilon > 0$ and $\{B_i\}$ be a cover of E st. $\sum m(B_i) \le m(E) + \epsilon$.

This implies that

$$\int |1_E - \sum 1_{B_i}| = \sum \int |1_{B_i} - 1_E| = \sum m(B_i) - m(E) \le \epsilon$$

Take N st. $\sum_{i=N+1}^\infty m(B_i) < \epsilon$

Using step 1, we find $h^i \in C_c^0(\mathbb{R}^n)$ st. $\left\|h^i - 1_{B_i}\right\| \le \frac{\epsilon}{N}$.

Take $h = \sum_{i=1}^N h^i \in C_c^0(\mathbb{R}^n)$.

Now for $p = 1$, we want to estimate

$$\|1_E - h\|_{L^1} \le \left\|1_E - \sum_{i=1}^N 1_{B_i}\right\|_{L^1} + \left\|\sum (1_{B_i} - h^i)\right\|_{L^1} \le \epsilon + \sum_{i=1}^N \frac{\epsilon}{N} = 2\epsilon$$

If $p > 1$, take $\hat{h} = \max(\min(h, 1), 0)$, now

$$\left\| 1_E - \hat{h} \right\|_{L^p}^p = \int |1_E - \hat{h}|^p \leq \int |1_E - \hat{h}| \to 0$$

Now we prove the statement for f simple, this means that $f = \sum c_i \frac{1}{E_i}$ where the E_i are bounded (by hypothesis on f).
By the step above, take a sequence

$$h_k^i \to 1_{E_i}$$

And look at

$$\left\| \sum c_i h_k^i - f \right\|_{L^p} = \left\| \sum c_i h_k^i - 1_{E_i} \right\| \to 0$$

Now suppose $f \geq 0$ be measurable, then let

$$1_{B_k} \phi_k \to f$$

from below.
Then there exist $g_k \in C_c^0(\mathbb{R}^n)$ such that

$$\| g_k 1_{B_k} \phi_k \|_{L^p} \leq \frac{1}{K}$$

then

$$\| g_k - f \|_{L^p} \to 0 \qquad \square$$

Lecture 10: density of continuous functions

2.3 How to approximate a C_c^0 with C_c^∞ in L^p ?

We will use convolutions.
Let $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi \geq 0, \phi = 0$ outside B_1 such that $\int \phi = 1$.
For instance, we can take $\phi(x) = c e^{\frac{1}{|x|-1}}$ if $|x| < 1$.
The standard convolution kernel is

$$\phi_\epsilon = \epsilon^{-n} \phi(\frac{1}{\epsilon} x) \quad \epsilon > 0$$

so that

$$\int \phi_\epsilon(x) = \int \epsilon^{-n} \phi(\epsilon^{-1} x) = 1$$

Now, let $f \in C_c^0$ and define the convolution of f and ϕ_ϵ as

$$f_\epsilon(x) = f * \phi_\epsilon(x) = \int f(x-y) \phi_\epsilon(y) dy$$

Lemme 51
$\forall \epsilon$ smal, we have that

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1. Supp $f_\epsilon \subset$ Supp $F + B_\epsilon$
2. f_ϵ is smooth
3. $\|f_\epsilon\| \leq \|f\|$ in L^1
4. $f_\epsilon \to f$ uniformly.

Preuve

1. $f_\epsilon(x) = \int_{B_\epsilon} \underbrace{f(x-y)}_{=0} \phi_\epsilon(y) dy = 0$

2. *Observe that*

$$f_\epsilon(x) = \int f(y)\phi_\epsilon(x-y) dy$$

Now we compute

$$\frac{1}{h}(f_\epsilon(x+hv) - f_\epsilon(x)) = \int f(y)\frac{\phi_\epsilon(x+hv-y) - \phi_\epsilon(x-y)}{h})$$

But now note that

$$\partial_v\phi_\epsilon(x-y) = \frac{\phi_\epsilon(x+hv-y) - \phi_\epsilon(x-y)}{h}$$

And this is dominated by $\|\nabla\phi_\epsilon\|$.

Hence the whole integral above is dominated and we get

$$= \int f(y)\partial_v\phi_\epsilon(x-y)$$

Hence $\nabla f_\epsilon = f * \nabla\phi_\epsilon$ and we conclude by induction on the degree of the derivative.

3. *By definition*

$$\int |f_\epsilon| \leq \iint |f(x)|\phi_\epsilon(x-y) dy dx$$

$$= \int |f(y)| \underbrace{\int \phi_\epsilon(x-y) dx}_{=1} dy = \|f\|$$

4. *Since f is uniformly continuous implies that $\forall \epsilon > 0 \exists \delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon$*

$$|f(x) - f_\epsilon(x)| = |f(x) - \int f(x-y)\phi_\epsilon(y) dx dy|$$

$$= |\int (f(x) - f(x-y))\phi_\epsilon(y) dx dy|$$

$$= \int_{B_\epsilon} |f(x) - f(x-y)|\phi_\epsilon(y) dy \leq \epsilon \qquad \square$$

Remarque

L^2 *has a Hilbert Structure.*

Define for $f, g \in L^2(\mathbb{R}^n)$ *a scalar product*

$$\langle f, g \rangle = \int_\Omega f(x)g(\bar{x})dx$$

It has a few properties

- *$\langle f, f \rangle = \int |f|^2$*
- *Hermitian property :* $\langle f, g \rangle = \langle g, \bar{f} \rangle$
- *It is linear in its first component and anti linear in the second one.*
- *Pythagoras theorem : if* $\langle f, g \rangle = 0$, *then*

$$\|f + g\|_{L^2}^2 = \|f\|_{L^2}^2 + \|g\|_{L^2}^2$$

Theorème 53 (Egorov theorem)

Let $\Omega \subset \mathbb{R}^n$ *measurable,* $m(\Omega) < \infty$, *then, if*

$$f_k : \Omega \to \mathbb{R} \longrightarrow f \ ae.$$

Given $\epsilon > 0, \exists C_\epsilon$ *closed contained in* Ω *such that*

$$m(\Omega \setminus C_\epsilon) < \epsilon$$

and $f_k \to f$ *uniformly in* C_ϵ

Preuve

Without loss of generality $f_k(x) \to f(x) \forall x \in \Omega$ *(up to throwing away a set of measure 0).*

$\forall m, k$ *define*

$$E_k^m = \left\{ x \in \Omega : |f_j(x) - f(x)| \leq \frac{1}{m} \forall j \geq K \right\}$$

For fixed m *, we have that*

$$E_k^m \subset E_{k+1}^m$$

and $E_k^n \to \Omega$ *as* $k \to \infty$.

Then

$$m(\Omega \setminus E_k^m) \to 0 \ as \ k \to \infty$$

This means that $\forall n$ *we can fix* k_n

$$m(\Omega \setminus E_{k_n}^n) \leq 2^{-n}$$

Fix ϵ *as in the statement, there exists* N *such that*

$$\sum_N^\infty 2^{-n} \leq \frac{\epsilon}{2}$$

Define $C_\epsilon = \bigcap_{n \geq N} E_{k_n}^n$.
In $C_\epsilon, f_j \to f$ uniformly, indeed

$$\forall \delta > 0 \text{ let } n \text{ such that } \frac{1}{n} < \delta$$

$$|f_j(x) - f(x)| \leq \frac{1}{n} < \delta \qquad \qquad \square$$

Remarque

$\forall E$ measurable $\exists C \subset E$ such that C is closed and $m(E \setminus C) \leq \frac{\epsilon}{2}$

"Littlehood principles"

— Every measurable setis nearly a finite union of balls
— Every pointwise converging sequence of functions is nearly uniformly convergent.
— Every measurable function is nearly continuous.

Lecture 11: Lusin's theorem

> **Theorème 55 (Lusin's theorem)**
> *Let Ω be a measurable set, $m(\Omega) < \infty$ and $f : \Omega \to \mathbb{R}$ measurable/*
> *Then $\forall \epsilon > 0 \ \exists F_\epsilon \subset \Omega$ closed s.t. $m(\Omega \setminus F_\epsilon) \leq \epsilon$ such that $f|_{F_\epsilon}$ is continuous.*

Remarque

F_ϵ cannot be taken open.

Preuve

Using approximation of L^1 functions with smooth functions, $\exists f_n \to f 1_{\{|f| \leq M\}} \in L^\infty(\Omega) \subset L^1(\Omega)$.
And we choose M s.t. $m(\{|f| > M\}) < \frac{\epsilon}{4}$.
Now we can apply Egoroov to make the convergence uniform.
Let $C_{\frac{\epsilon}{4}} \subset \Omega$ s.t. $f_n \to f$ uniformly in C_ϵ and $m(\Omega \setminus C_{\frac{\epsilon}{4}}) \leq \frac{\epsilon}{4}$.
These functions converge $f_n|_{C_{\frac{\epsilon}{4}}} \to f 1_{\{|f| < M\}}$ uniformly on $C_{\frac{\epsilon}{4}}$, hence $f 1_{|f| < M}$ is continuous on $C_{\frac{\epsilon}{4}}$ hence f is continuous on $C_{\frac{\epsilon}{4}} \cap \{|f| < M\}$ $\square$

> **Theorème 57 (Borel sets are strictly included in Measurable sets)**
> — *There exist non-measurable sets.*
> — *There exists a Borel set which is not Borel.*

Preuve

$\forall x \in \mathbb{R}$ consider the coset $x + \mathbb{Q}$.

Note that $x + \mathbb{Q} \cap [0,1] \neq \emptyset$.

$$(x + \mathbb{Q}) \cap (y + \mathbb{Q}) = \begin{cases} x + \mathbb{Q} \text{ if } x - y \in \mathbb{Q} \\ \emptyset \text{ if not} \end{cases}$$

Consider $\mathbb{R}/\mathbb{Q}$.

By the axiom of choice, pick $x_A \in A \cap [0,1] \forall A \in \mathbb{R}/\mathbb{Q}$.

The Vitali set is $\left\{ x_A : A \in \mathbb{R}/\mathbb{Q} \right\}$.

Define

$$X = \bigcup_{q \in [-1,1] \cap \mathbb{Q}} q + V$$

$\square$

Notice that $X \in [-1, 2]$ and $X \supset [0,1]$.

Indeed, let $y \in [0,1]$ and let x_A be a representative, then $|y - x_A|$ is rational and smaller than 1.

If the Vitalli set was measurable, then X would be measurable, then $1 \leq m(X) \leq 3$.

If q_1, q_2 are two different rationals, then $q_1 + V \cap q_2 + V = \emptyset$.

Indeed, if there is a point of intersection, then $q_1 + v_1 = q_2 + v_2$ then $v_1 \sim v_2$ which is a contradiction as the Vitalli set has one element of each coset.

Then $m(X) = \sum_q m(q + V) = \sum_q m(V)$ but then either $m(V) = 0$ which is a contradicition or $m(V) > 0$, then $m(X) = \infty$

Lecture 12: there exist measurable sets which are not Borel

Wed 06 Apr

2.4 $\mathcal{B} \subsetneq M$

Let C be the cantor set.

Let $x \in (0,1)$ and write $x = 0.\epsilon_1 \epsilon_2 \ldots$ where $\epsilon_i \in \{0,1\}$, ie. it's binary expansion.

Write $f(x) = \sum_{k=1}^{\infty} \frac{2\epsilon_i}{3}$

Lemme 58

$f([0,1]) \subset C$, f is strictly monotone and therefore measurable.

Preuve

$f(x)$ in ternary representation has digits $2\epsilon_k = 2$ or 0, hence is in the cantor set.

Let $\sum \frac{a_n}{2^n} = x < y = \sum \frac{b_n}{2^n}$.

Let $k > 1$ such that $a_n = b_n \forall n < k$ and $a_k \neq b_k$.

Then $f(y) - f(x) = \sum_{n \geq k} \frac{2(b_n - a_n)}{3} = \frac{2}{3^k} + \sum_{n \geq k+1} \frac{2(b_n - a_n)}{3^n} > \frac{2}{3^k} - \sum_{n \geq k+1} \frac{2}{3^n} = 0$

$\square$

> **Preuve**
> *Claim : $A_f = \{B \subset \mathbb{R} | f^{-1}(B) \text{ is measurable. }\}$ is a σ-algebra containing intervales.*
> *Then, since $\mathcal{B}$ is the smallest σ-algebra containing intervals, we conclude.* $\square$

Now we can show that there exist measurable sets which are not Borel.

> **Preuve**
> *Let $V \subset [0,1]$ non-measurable and write $B = f(V) \subset f([0,1]) = C$ where f is the lebesgue function.*
> *We claim B is not Borel.*
> *Let's assume by contradiction that B is Borel.*
> *Then $f^{-1}(B)$ is measurable by the lemma above.*
> *However, $f^{-1}(f(V)) = V$ which is not measurable.* $\square$

3 Fourier Analysis

3.1 Derivation of the heat equation

Consider a metal plate $\Omega \subset \mathbb{R}^2$.

We want to study the temperature $u(t, x, y)$.

Newton's cooling law dictates that heat flows from higher to lower temperatures at a rate proportionale to the difference of temperatures.

Consider S a small square, the heat "in S " is defined as $\int_S u(t, x, y)$ and the heat flow in S is $\frac{\partial}{\partial t} \int_S u(t, x, y) = \int_S \partial_t u(t, x, y) \simeq h^2 \partial_t u(t, x_0, y_0)$

Then the heat flow through the boundary ∂S is

$$kh\partial_x u(t, x_0 + \frac{h}{2}, y_0) - kh\partial_x u(x_0 - \frac{h}{2}, y_0) + kh\partial_y u(t, x_0, y_0 + \frac{h}{2}) - kh\partial_y(t, x_0, y_0 - \frac{h}{2}2)$$

$$\simeq kh^2 \partial_{xx} u(t, \xi, y_0) + kh\partial_{yy} u(t, x_0, \xi')$$

Now newton's law implies

$$h^2 \partial_t u(t, x_0, y_0) = kh^2(\partial_{xx} u(t, \xi, y_0) + \partial_{yy} u(t, x_0, \xi'))$$

Now we cancel h^2 and find

$$\partial_t u(t, x, y) = k\partial_{xx} u(t, x, y) + k\partial_{yy} u(t, x, y)$$

So now we consider the Dirichlet problem in $D = \{(x, y) : x^2 + y^2 \leq 1\}$ We fix boundary conditions $u(1, \theta) = f(\theta)$ (where we now have polar coordinates).

We now rewrite the pde in polar coordinates.

$$\Delta u = \partial_{rr} u + \frac{1}{2}\partial_r u + \frac{1}{r^2}\partial_{\theta\theta} u$$

So our PDE reads

$$\begin{cases} r^2 \partial_{rr} u + r \partial_r u = -\partial_{\theta\theta} u \\ u(1, \theta) = f(\theta) \end{cases}$$

For now, we look for solutions of the form

$$u(r, \theta) = F(r)G(\theta)$$

So we get

$$r^2 F''(r)G(\theta) + r F'(r)G(\theta) = F(r)G''(\theta)$$

Hence

$$\frac{1}{F(r)}(r^2 F''(r) + r F'(r)) = -\frac{G''(\theta)}{G(\theta)}$$

So both sides have to be constant, so we get a system

$$\begin{cases} G'' + \lambda G = 0 \\ r^2 F'' + r f' - \lambda F = 0 \end{cases}$$

Solutions of the first ODE are $\cos(\sqrt{\lambda}\theta), \sin(\sqrt{\lambda}\theta)$ if $\lambda \geq 0$ or $e^{\sqrt{-\lambda}\theta}$ if not, but the second kind of solutions are not periodic, so we discard them.
The periodicity constraint also implies that $\lambda = m^2, m \in \mathbb{N}$
So

$$G(\theta) = \tilde{A}\cos(m\theta) + \tilde{B}\sin(m\theta) = Ae^{im\theta} + Be^{-im\theta}$$

The solutions to $F(r)$ are of the form

$$\begin{cases} r^m \\ r^{-m} \\ \text{if } m > 0 \log r \text{ if } m = 0 \end{cases}$$

But we can reject the last two solutions as they blow up in the origin.

Remarque
Note that, if u_1, u_2 are solutions to the equation, then $u_1 + u_2$ is too.

Hence, if $f(\theta) = \sum a_m e^{im\theta}$, then a solution of the heat equation is

$$u(r, \theta) = \sum a_m r^m e^{im\theta}$$

So this motivates the leading question of Fourier analysis, namely :
Given $f : [0, 2\pi] \to \mathbb{R}$, when can we write it as above ?

Lecture 13: Fourier Analysis

3.2 Periodic Functions

> **Definition 20 (Periodic function)**
> *Let $L > 0$, $f : \mathbb{R} \to \mathbb{R}$ is L-periodic if $f(x + L) = f(x)$.*

We look for $f, g \in L^p([0,1])$ and 1-periodic such that

$$\|f - g\|_{L^p} = \left(\int_0^1 |f - g|^p \right)^{\frac{1}{p}}$$

For $p = 2$ there is an associated scalar product given by

$$\langle f, g \rangle = \int_0^1 f\overline{g}dx$$

> **Definition 21 (Space of periodic functions)**
> $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ *is the space of continuous 1-periodic functions.*

> **Lemme 61 (Basic properties)**
> — *If $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, then f is bounded.*
> — $\mathbb{C}^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ *is a vector space and an algebra.*
> — *The space is is closed under uniform limits.*

3.3 Trigonometric polynomials

> **Definition 22**
> $\forall m \in \mathbb{Z}$, *the character with frequency n is*
>
> $$e_n(x) = e^{i2\pi nx}$$

> **Definition 23 (Trigonometric polynomial)**
> *An element $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ is a trigonometric polynomial if*
>
> $$f(x) = \sum_{-N}^{N} c_n e^{i2\pi nx}$$
>
> *for some $N \geq 0$.*

> **Lemme 62**
> *The family of $\{e_n\}$ is an orthonormal system, ie.*
>
> $$\langle e_n, e_m \rangle = \delta_{nm}$$

The proof is an exercise.

> **Corollaire 63**
> Let $f = \sum_{-N}^{N} c_n e_n$, then
> $$c_n = \langle f, e_n \rangle$$
> and
> $$\sum c_n^2 = \|f\|_{L^2}^2$$

Preuve

$$\langle f, e_m \rangle \langle \sum c_n e_n, e_m \rangle = c_m$$

And thus also

$$\|f\|_{L^2}^2 = \sum |\langle f, e_m \rangle|^2$$

$\square$

> **Definition 24 (Fourier Coefficients)**
> Let f be some periodic function, then the n-th fourier coefficient is
> $$\hat{f}(n) = \langle f, e_n \rangle$$

> **Corollaire 64**
> Let f be a trigonometric polynomial, then
> $$f = \sum_{-N}^{N} \langle f, e_n \rangle e_n$$

3.4 Periodic convolutions

> **Theorème 65 (Weierstrass approximation)**
> Let $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and $\epsilon > 0$, then $\exists P$ a trigonometric polynomial such that
> $$\|f_n - P\| \leq \epsilon$$

Definition 25 (Convolution of periodic functions)

Let $f, g \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, then the periodic convolution is

$$f * g(x) = \int_0^1 f(y)g(x-y)dy$$

Remarque

Let $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C}) \cap L^1(\mathbb{R}) \implies f = 0$

Lemme 67 (basic properties)

Let $f, g \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$, then

— $f * g$ is closed in $C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$

— $f * g = g * f$

— $(f + g) * h = f * h + g * h$

Remarque

$$f * e_n = \hat{f}e_n$$

since

$$f * e_n = \int_0^1 f(y)e^{i2\pi n(x-y)}dy = e^{i2\pi nx}\hat{f}(n)$$

Definition 26

Let $\epsilon > 0, \delta \in (0, \frac{1}{2})$, we say that $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ is a periodic $\epsilon - \delta$ approximation of the identity if

1. $f \geq 0, \int_0^1 f = 1$

2. $f(x) < \epsilon \forall x \in [\delta, 1 - \delta]$

Lemme 69

$\forall \epsilon - \delta$ there exists a trigonometric polynomial P which is an $\epsilon - \delta$ approximation of the identity.

Preuve

Consider $F_N(x) = \frac{1}{N}(\sum e_n)^2$, and the rest follows from an exercise. $\qquad\square$

Preuve (Of Weierstrass)

Let $f \in C^0(\mathbb{R}/\mathbb{Z}, \mathbb{C})$ and $\epsilon > 0$.

f is bounded and uniformly continuous hence there exists M such that

$$|f(x)| \leq M$$

$\exists \delta$ such that $|f(x) - f(y)| < \epsilon \forall |x - y| < \delta$.
By lemma, let P be a trigonometric polynomal satisfying the $\epsilon - \delta$ condition.
Then $f * P$ is a trigonometric polynomial.
We claim that $\|f - f * P\|_{L^\infty} < \epsilon$.
Indeed

$$\begin{aligned}
|f(x) - f * P(x)| &= |f(x) - \int_0^1 P(y)f(x - y)dy| \\
&= |\int_0^1 (f(x) - f(x - y))P(y)dy| \\
&\leq \int_0^1 |f(x) - f(x - y)|P(y)dy \\
&\leq \int_\delta^{1-\delta} 2 \max f \epsilon + \int_{[0,\delta] \cup [1-\delta,1]} \epsilon P(y)dy \leq 2M\epsilon + \epsilon \quad \square
\end{aligned}$$

Lecture 14: Fourier Series

Wed 27 Apr

Recall that $f = \sum_{m=-\infty}^{\infty} \hat{f}e^{i2\pi nx}$ when f is a trigonometric polynomial. We will show that in fact

> **Theorème 70**
> $\forall f \in L^2([0,1])$, *the series*
>
> $$\sum_{m=-\infty}^{\infty} \hat{f}e^{i2\pi nx}$$
>
> *converges in L^2 to f.*

In particular $f = \sum_{m=-\infty}^{\infty} \hat{f}e^{i2\pi nx}$ almost everywhere.

Preuve

By weierstrass approximation theoreme $\forall \epsilon \exists P$ a trigonometric polynomial such that
$$\|f - P\|_{L^2} \leq \|f - P\|_{L^\infty} \leq \epsilon$$

In fact we show the following lemma

> **Lemme 71**
> $\|f - F_N\|_{L^2} \leq \|f - P\|_{L^2}$

Hence, we prove this lemma.
Indeed,

$$\langle f - F_n(f), e_m \rangle = 0 \forall m = -N, \ldots, N$$

Since

$$\langle f - F_n f, e_m \rangle = \langle f, e_m \rangle - \langle F_n f, e_m \rangle = 0$$

Furthermore

$$\|f - P\|_{L^2}^2 = \|f - F_n f\|_{L^2}^2 + \|F_n f - P\|_{L^2}^2 + \underbrace{2\langle f - F_n f, F_n f - P \rangle}_{=0}$$

To conclude the proof, for $f \in L^2, \epsilon > 0$, since C^0 is dense in L^2, $\exists g \in C^0$ such that

$$\|f - g\|_{L^2} \leq \frac{\epsilon}{2}$$

By the previous part, for N large enough

$$\|g - F_n g\| \leq \frac{\epsilon}{2}$$

But now,

$$\begin{aligned} \|f - F_n f\|_{L^2} &\leq \|f - F_n g\|_{L^2} \\ &\leq \|f - g\|_{L^2} + \|g - F_n g\|_{L^2} \\ &\leq \epsilon \qquad\qquad\qquad\qquad \square \end{aligned}$$

Theorème 72 (Parseval)
For $f \in L^2$, 1-periodic

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(m)|^2$$

Preuve
Let $f \in L^2, \epsilon > 0$, by the previous theorem, there exists a large enough N such that

$$\|f - F_n f\|_{L^2} < \epsilon$$

Note that

$$\|f\|_{L^2} - \|f - F_n f\| \leq \|F_n f\|_{L^2} = \left(\sum_{-N}^{N} |\hat{f}(m)|^2 \right)^{\frac{1}{2}}$$

We also know that

$$\begin{aligned} \|F_n f\|_{L^2}^2 &\leq \|f\|_{L^2}^2 \\ &= \|f - F_n f\| + \|F_n f\| \qquad\qquad \square \end{aligned}$$

Definition 27

Let $f : \mathbb{R} \to \mathbb{R}$ be a $2l$-periodic function $L^1(0, 2l)$.
The trigonometric fourier coefficients of f are

$$a_n = \frac{1}{l} \int_0^l f(x) \cos(\frac{2\pi x}{l}) \quad b_n = \frac{1}{l} \int_0^l f(x) sin(\frac{\pi n x}{l})$$

Corollaire 73

Let f be $2l$ periodic, then the trigonometric fourier series

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos(\frac{\pi n x}{l}) + b_n \sin(\frac{\pi n x}{l})$$

in $L^2(0, 2l)$

Lecture 15: Pointwise convergence of fourier series

Definition 28

Let $\alpha \in (0, 1)$ and Ω be a bounded set in $\mathbb{R}^n$, then the space of Holder continuous functions is

$$\left\{ f \in C^{0,\alpha}(\Omega) : \sup_{x \, y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|^\sigma} \right\}$$

Theorème 74 (Dirichlet)

— *If f is 1-periodic and piecewise C^1, then $\forall x$*

$$F_n f(x) \to \frac{1}{2}(f^+(x) + f^-(x))$$

— *If f is 1-periodic and $C^{0,\alpha}(-1, 2)$, then*

$$\forall x F_n f(x) \to f(x)$$

— *If $f \in L^1(\mathbb{R}/\mathbb{Z})$ such that for $a \in \mathbb{R} \exists M(a) > 0$ and $\delta(a) > 0$ such that*

$$|f(t + a) - f^+(a)|, |f(a - t) - f^-(a)| \leq M(a)t^a \forall t \in (0, \delta(a))$$

Then $F_n f(a) \to \frac{f^+(a) + f^-(a)}{2}$

Remarque

Note that point 3 trivially implies point 2.

In fact 3 *implies* 1 *as well since*

$$f(a+t) = f(a)^+ + f'(a) + +o(t)$$
$$f(a-t) = f(a)^- + f'(a)^- t + o(t)$$

But then

$$|f(t+a) - f(a)^+| \leq (|f'(a)^+| + 1)|t|$$

Kolmogorov showed that

1. $\exists f$ continuous such that $F_n f(x) \not\to f(x)$

2. $\exists f \in L^1$ such that the fourier series diverges everywhere.

3. Carleson also show that if $f \in L^p(\mathbb{R}/\mathbb{Z})$ then $F_n f(x) \to f(x)$ almost everywhere.

Preuve

Recall that $F_N f(x) = \sum_{-N}^N \hat{f}(n) e^{i2\pi nx} = \sum_{-N}^N \int f(y) e^{i2\pi n(x-y)} dy$.

Then

$$= \int f(y) \sum_{-N}^N e^{i2\pi n(x-y)} dy$$

$$= f * D_N(x)$$

where

$$D_N(y) = \sum_{-N}^N ((e^{i2\pi y})^n) \qquad = \frac{\sin(\pi y(2N+1))}{\sin \pi y}$$

Now let $a \in [0,1]$ *and define*

$$M = \begin{cases} f(a) & \text{if continuous at } a \\ \frac{1}{2}(f^+(a) + f^-(a)) & \text{if not} \end{cases}$$

Then

$$F_n f(a) - M = \int_{-\frac{1}{2}}^{\frac{1}{2}} (f(y) - M) \frac{\sin(\pi y(2N+1))}{\sin \pi y}$$

> **Lemme 76 (Riemann-Lebesgue)**
>
> *Let* $f \in L^1(\mathbb{R})$ *and*
>
> $$\hat{f}(\xi) = \int f(x) e^{-i2\pi \xi x} dx$$
>
> *then* $\lim_{\xi \to +\infty} \hat{f}(\xi) = 0$

Indeed, taking the imaginary part in Riemann-Lebesgue

$$\int F(x) \sin(2\pi \xi x) dx \to 0$$

We apply this to Riemann-Lebesgue, we apply it to $F = \phi_a, \xi = N + \frac{1}{2}$ We check $\phi_a \in L^1$, we do it assuming f continuous

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi_a| = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{f(a+y) - f(a)}{\sin \pi y} \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{My}{cy} = \frac{t^\alpha}{\alpha} \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \qquad \square$$

Lecture 16: truc

> **Theorème 77 (Uniform convergence of Fourier Series)**
> Let $f \in C^1$ and 1-periodic, then
>
> $$\sum |\hat{f}(n)| < \infty$$
>
> and
>
> $$F_n f \to f$$
>
> uniformly on $[0, 1]$.

Remarque
Informally, given $f(x) = \sum_{-\infty}^{\infty} \hat{f}(m) e^{i\pi nx}$ and

$$f'(x) = \sum_{-\infty}^{\infty} \hat{f}(m) i\pi n e^{i\pi nx}$$

And in fact, it is true that $\hat{f}'(m) = i\pi m \hat{f}(m)$.
Rigourously,

$$\hat{f}'(m) = \int f' e^{-i\pi nx} dx = -\int f(x)(-i\pi n) e^{-i\pi nx} dx = i\pi n \hat{f}(n)$$

Preuve
Let's show $f \in C^1 \implies \sum |\hat{f}(m)| < \infty$.
By Plancherel,

$$\infty > \|f'\|_{L^2}^2 = \sum |\hat{f'}^2(m)| = \sum \pi^2 n^2 |\hat{f}(n)^2|$$

Now we show that that $F_n \to f$ uniformly, we know that

$$|f(x) - F_N(x)| = \sum_{|n| \geq N-1} \hat{f}(n) e^{i\pi nx}$$
$$\leq |\hat{f}(n)| \leq \infty \qquad \square$$

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3.5 Fourier Series in sin

Given $f : [0, l] \to \mathbb{R}$

We can extend f oddly on $[-l, l]$ and then extend it by periodicity.

Now we write the fourier series of $\tilde{f}$

$$F\tilde{f} = \sum b_n \sin(\frac{\pi}{l} nx)$$

where b_n are the fourier coefficients which we may rewrite as

$$b_n = \frac{1}{l} \int_{-l}^{l} \tilde{f}(y) \sin(\frac{\pi n y}{l}) dy$$

$$= \frac{2}{l} \int_{0}^{1} f(y) \sin(\frac{\pi n y}{l}) dy$$

Definition 29

The fourier series in sines of f is

$$F^s f = \sum b_n \sin(\frac{\pi}{l} nx)$$

Corollaire 79

If $f \in L^2(0, l)$, then $F_N^s f \to f$ in L^2

Preuve

Trivial. □

We similarly define the fourier series in cosines

Lecture 17: Fourier Transform

4 The Fourier Transform

Definition 30 (Fourier Transform)

Let $f \in L^1(\mathbb{R})$, then the Fourier Transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(y) e^{-i 2\pi y \xi} dy$$

Remarque

Sometimes, the Fourier transform is defined without a 2π in the exponent and a $\frac{1}{\sqrt{2\pi}}$ in front.

Lemme 81 (Basic properties of the Fourier Transform)

Let $f, g \in L^1(\mathbb{R})$, $a, b \in \mathbb{R}$, then

1. *$\hat{f}$ is a continuous function, $\lim_{|\xi| \to \infty} = 0$ and $\left\| \hat{f} \right\|_{L^\infty} \leq \|f\|_{L^1}$*

2. *$\mathcal{F}(af + bg) = a\mathcal{F}f + b\mathcal{F}g$*

3. *If f is differentiable and f' is in L^1, then*

$$\hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi)$$

4. *If $h(x) = xf(x) \in L^1$, then $\hat{f}$ is differentiable*

5. *If $h(x) = f(x + a)$, then $\hat{h}(\xi) = e^{2\pi i \xi a} \hat{f}(\xi)$*

6. *If $h(x) = f(ax)$, then $\hat{h}(\xi) = \frac{1}{a}\hat{f}(\frac{\xi}{a})$.*

7. *Multiplication formula :*

$$\int_{-\infty}^{\infty} \hat{f}(x)g(x)dx = \int_{-\infty}^{\infty} f(x)\hat{g}(x)dx$$

Proposition 82 (Gaussians are good kernels)

Let $f(x) = e^{-\pi x^2}$„ then $\hat{f} = f$

Preuve

$f'(x) = -2\pi x f(x)$ and thus

$$\hat{f}'(\xi) = -2\pi x \hat{f(x)}(\xi)$$

Now applying the lemma above gives

$$2\pi i \xi \hat{f}(\xi) = -i \hat{f}'(\xi)$$

So $\hat{f}$ satisfies the ODE $\hat{f}' = -2\pi \xi \hat{f}$ and since

$$\hat{f}(0) = \int f(y)dy$$

$\square$

Corollaire 83

If $\delta > 0$ and $\kappa_\delta(x) = \delta^{-\frac{1}{2}} e^{-\pi \frac{x^2}{\delta}}$.
Then

$$\hat{\kappa_\delta} = e^{-\pi \xi^2}$$

> **Corollaire 84**
> Let $K_\delta = \delta^{-\frac{1}{2}} e^{-\frac{\pi x^2}{\delta}}$, then $\hat{K}_\delta = e^{-\pi \delta \xi^2}$.
> The following two things hold
>
> 1. $\int K_\delta = \int K_1 = 1$ (this is an exercise)
>
> 2. $\int_{|X|>\eta} K_\delta \to 0$ as $\delta \to 0$

Preuve
Indeed,

$$\int_{|x|>\eta} K_\delta = \int_{|y|>\frac{\eta}{\sqrt{\delta}}} e^{-\pi y^2} dy \to 0$$

$\square$

4.1 Fourier Inversion Formula

Let $f \in L^1(\mathbb{R})$ s.t. $|\hat{f}| \in L^1(\mathbb{R})$, then

$$f(x) = \int \hat{f}(\xi) e^{i 2\pi \xi y}$$

Remarque
Note that it is natural to suppose $\hat{f} \in L^1$ to define the right hand side of the equation above, this implies in particular that f coincides a.e. with a continuous function.
It is natural to ask if there are other conditions for which $f, \hat{f} \in L^1$ is satisfied.
The Schwartz space is the set of all function $f \in C^\infty(\mathbb{R})$ s.t.

$$\sup_x |x|^k ||f^{(l)}(x)| < \infty \quad \forall k, l \in \mathbb{N}$$

The Schwartz space $\mathcal{S}(\mathbb{R})$ contains all smooth compactly supported functions, as well as Gaussians.
Moreover, if $f \in \mathcal{S}(\mathbb{R})$, then $\hat{f} \in \mathcal{S}(\mathbb{R})$, in particular, the Fourier inversion formula applies to $f \in \mathcal{S}(\mathbb{R})$.

> **Corollaire 86**
> *The Fourier Transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an automorphism of vector spaces.*

Preuve
Let $\mathcal{F}^* g = \int g(y) e^{2\pi \xi y} dy$, then

$$\mathcal{F}^* \circ \mathcal{F} = \text{Id} \quad on \ \mathcal{S}(\mathbb{R})$$

$\square$

Furthermore, since $\mathcal{F}^* g = \mathcal{F}(g(-x))$, $\mathcal{F}\mathcal{F}^* = \text{Id}$.

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Now we prove the Fourier Inversion formula

> **Preuve**
>
> *We first prove the result for $f \in C^0 \cap L^\infty$.*
> *We prove the FIF for $\underline{all}$ x.*
> *We first want to show that*
>
> $$f(0) = \int \hat{f}(\xi)d\xi$$
>
> *By lemma 7 ,*
>
> $$\int f(x)K_\delta(x)dx = \int \hat{f}(\xi)G_\delta(\xi)d\xi$$
>
> *As $\delta \to 0$, we want to show that*
>
> $$\int \hat{f}(\xi)G_\delta(\xi)d\xi \to \int \hat{f}(\xi)d\xi$$
>
> *Indeed*
>
> $$|f(0) - \int f(x)K_\delta(x)dx| \leq \int |f(0) - f(x)|K_\delta(x)dx$$
>
> $$\leq \int_{|x|<\eta} |f(0) - f(x)|K_\delta + \|f\|_{L^\infty} \, o(1)$$
>
> *For general x, recall that for $F(y) = f(x + y)$, then*
>
> $$f(x) = F(0) = \int \hat{F}(\xi)d\xi = \int \hat{f}(\xi)e^{2\pi i \xi x}d\xi$$
>
> *Now we prove the result for general functions f satisfying $f, \hat{f} \in L^1(\mathbb{R})$.*
> *For any x, write the multiplication formula applied to $F(y) = f(x + y)$ and K_δ, then we get*
>
> $$\int_{-\infty}^{\infty} F(y)K_\delta(y)dy = \int_{-\infty}^{\infty} \hat{F}(\xi)G_\delta(\xi)d\xi$$
>
> $$= \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i2\pi\xi x}G_\delta(\xi)d\xi$$
>
> *Now we let $\delta \to 0$, then the RHS goes to $\int_{-\infty}^{\infty} \hat{f}(\xi)e^{i2\pi\xi x}$.*
> *Furthermore, we claim that the LHS*
>
> $$\int f(x + y)K_\delta(y)dy \to f(x)$$
>
> *We now prove this claim :*
> *Note that*
>
> $$|f_\delta(x) - f(x)| \leq |\int_{-\infty}^{\infty} (f(x + y) - f(x))K_\delta(y)dy|$$

Thus

$$\int_{-\infty}^{\infty} |f_\delta(x) - f(x)| dx \leq \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} |f(x+y) - f(x)| K_\delta(y) dy dx$$

$$= \int \|f(\cdot + y) - f\|_{L^1} K_\delta(y) dy$$

$$= \int_{-\infty}^{+\infty} \left\| f(\cdot + \delta^{\frac{1}{2}} y) - f \right\|_{L^1} e^{-\pi y^2} \qquad \square$$

Now recall from an exercise that $\forall f \in L^p(\mathbb{R})$, the translations $\|f(\cdot + y) - f\|_{L^p} \to 0$ as $|y| \to 0$.

Thus the quantity above is dominated and the integral goes to 0.

4.2 Plancherel Identity

> **Theorème 87 (Plancherel)**
> *Let $f \in L^1 \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2$ and*
>
> $$\|f\|_{L^2} = \left\| \hat{f} \right\|_{L^2}$$

Let's first prove a preliminary result

Proposition 88
Let $f, g \in L^1(\mathbb{R})$ and consider their convolution

$$f * g = \int f(x - y) g(y) dy$$

Then

1. *$f * g$ is well defined for a.e. and $\|f * g\| \leq \|f\| \|g\|$*
2. *$f * g = g * f$*
3. *$\widehat{f * g} = \hat{f}\hat{g}$*

Preuve
We do not prove measurability.

$$\int \int f(x - y) g(y) dy dx \leq \int \int |f(x - y) g(y)| dy dx$$

$$= \int \|f\|_{L^1} |g(y)| dy$$

$$\leq \|f\|_{L^1} \|f\|_{L^1}$$

Thus $f * g$ is bounded.
It is an exercise to show that $\widehat{f * g} = \hat{f}\hat{g}$ $\qquad \square$

Preuve

Let's first present a "wrong" proof which we'll then fix.

Let $g(x) = f(-x)$, observe that $\hat{g}(x) = \int f(-y)e^{-i2\pi\xi y}dy = -\hat{f}(\xi)$.

Let $h = f * g$, then $\hat{h} = \hat{f}\hat{g} = |\hat{f}|^2$, thus

$$h(0) = \int f(y)g(0-y)dy$$

$$= \int |f|^2(y)dy$$

However, note that FIF can not be applied at $x = 0$ as we only have equality a.e.

And we also need $\hat{h} \in L^1$

Hence, to conclude the proof, we show that if $f \in L^1 \cap L^2(\mathbb{R}) \implies h$ is continuous and $\hat{f} \in L^2$.

To prove 1, notice that

$$|h(x+\epsilon) - h(x)| = |\int f(y)((g(x+\epsilon-y) - g(x-y)))dy$$

$$= \int \int |f(y)||f(y-x-\epsilon) - f(y-x)|dxdy$$

$$\leq \|f(\cdot+\epsilon) - f\|_{L^2} \|f\|_{L^2} \to 0$$

Finally, we have to show that $\hat{f}$ is in L^2.

At this point, we know that Plancherel holds if $f \in L^1 \cap L^2$ and $\hat{f} \in L^2$.

Let $f_\delta = f * K_\delta$, we want to apply the above fact to f_δ and then let $\delta \to 0$.

We know that $\hat{f}_\delta = \hat{f}\hat{K}_\delta$.

We need to verify that f_δ satisfies the right assumptions.

We know that $f_\delta \in L^1$ from the proposition.

Furthermore, since $\hat{f}_\delta = \hat{f}\hat{K}_\delta$, since $\hat{f} \in L^\infty$ and $e^{-\pi\delta x^2} \in L^1$ and thus $\hat{f}_\delta$ is in L^2 (as $\hat{K}_\delta$ is)

We first bound f_δ pointwise

$$\hat{f}_\delta^2(x) = \left[\int_{-\infty}^{\infty} f(y)K_\delta(x-y)dy\right]^2$$

Applying Hoelder inequality yields

$$\leq \int_{-\infty}^{+\infty} f^2(y)K_\delta(x-y)dy \int_{-\infty}^{+\infty} K_\delta(x-y)dy$$

$$= \int_{-\infty}^{+\infty} f^2(y)K_\delta(x-y)dy$$

We now integrate wrt. x, thus

$$\int_{-\infty}^{+\infty} f_\delta^2(x) \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(y)^2 K_\delta(x-y)dydx$$

$$= \int_{-\infty}^{+\infty} f(y)^2 \int_{-\infty}^{+\infty} K_\delta(x-y)dxdy$$

$$= \int_{-\infty}^{+\infty} f(y)^2. \qquad\qquad \square$$

Lecture 19: Applications of Fourier Transforms to PDE's

Thu 12 May

5 Fourier Transforms and PDE's

Exemple

1. *Let $u : \mathbb{R} \to \mathbb{R}$, we search for the solutions to $\partial_{xx} u = 0$*

2. *Ordinary Differential equations*
3.

$$\partial_{xx} u + \partial_{yy} u = 0 \qquad \text{Laplace equation}$$

4. *Poisson equation*

$$\Delta u = f$$

   for some given function f

5. *Heat equation*

$$\partial_t u - \Delta u = 0$$

6. *Wave Equation*

$$\partial_{tt} u - \Delta u = 0$$

7. *Finding u such that*

$$\nabla u = f$$

8. *Burger equation*

$$\partial_t u - \partial_{xx}(u^2) = 0$$

Definition 31 (PDE)

A partial differential equation (PDE) is finding a function u such that

$$F(x, u(x), \nabla u(x), \ldots, \nabla^k u(x)) = 0$$

where $x \in \Omega \subset \mathbb{R}^n$, $u : \Omega \to \mathbb{R}^N$

Remarque

The laplace, heat and wave equations are linear PDE's, namely, if u, v are solution, then $\alpha u + \beta v$ is a solution.

Typically, PDE's come with boundary conditions and/or initial conditions

Lecture 20: Application to PDE's

5.1 The heat equation (on $\mathbb{R}$)

The heat equation has the general form

$$\begin{cases} \partial_t u - \partial_{xx} u = 0 \\ u(0, x) = f(x) \end{cases}$$

We want to solve this for $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$.
We formally derive a solution using the following strategy :

1. Take the fourier transform in x

$$\partial_t \hat{u}(\xi, t) + 4\pi^2 \xi^2 \hat{u}(\xi, t) = 0$$

Now notice that for a fixed ξ, this is an ode in $\hat{u}$, we can compute it's solution :

$$\partial_t[\ln \hat{u}] = -4\pi^2 \xi^2$$
$$\ln \hat{u}(\xi, t) - \ln \hat{u}(\xi, 0) = -4\pi^2 \xi^2 t$$

Taking exponentials

$$\hat{u}(\xi, t) = \hat{u}(\xi, 0) e^{-4\pi^2 \xi^2 t} = \hat{f}(\xi) e^{-4\pi^2 \xi^2 t}$$

Inverting the Fourier Transform yields

$$u(x, t) = f * \widehat{e^{-4\pi^2 t \xi^2}}$$

Theorème 91 (Solution to heat equation)
Let $f \in \mathcal{S}(\mathbb{R})$, define the _heat kernel_

$$H_t(x) := \frac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}$$

Let $u = f * H_t$, then u is

1. C^2 for $x \in \mathbb{R}, t > 0$ and solves the heat equation $\partial_t u - \partial_{xx} u = 0$
2. $u(t, x) \to f(x)$ as $t \to 0$
3. $u(t, \cdot) \to f$ in the L^2 norm as $t \to \infty$

Preuve

1. Take the Fourier transform of u, this gives

$$\hat{u} = \hat{f} \widehat{H_t}$$

The Fourier inversion formula gives

$$u(t,x) = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{-4\pi^2\xi^2 t + 2\pi i \xi x}d\xi$$

*As a remark, notice that if one looks at the heat equation on $(-\infty, 0)$,
the formal computations are the same <u>but</u> the formula for u makes no
sense (as then the heat kernel increases a lot when $t \to \infty$) .
Is u differentiable in x ?
If it is, differentiating under the integral gives*

$$\partial_x u(t,x) = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{-4\pi^2\xi^2 t + 2\pi i \xi x}(2\pi i \xi)d\xi$$

To show this rigorously yields

$$\frac{u(t, x+h) - u(t,x)}{h} = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{-4\pi^2\xi^2 t}e^{2\pi i \xi x}\frac{e^{2\pi i \xi h} - 1}{h}d\xi$$

*Which converges to the same formula as $h \to 0$.
Doing the exact same reasoning again gives*

$$\partial_{xx} u(t,x) = \int_{-\infty}^{+\infty} \hat{f}(\xi)e^{-4\pi^2\xi^2 t}e^{2\pi i x\xi}(2\pi i \xi)^2 d\xi$$

2. *Finally, we want to prove that $u(t,x) - f(x) \to 0$ as $t \to 0$, indeed*

$$u(t,x) - f(x) = \int_{-\infty}^{+\infty} H_t(y)(f(x-y) - f(x))dx$$

We claim that for t sufficiently smal, the integrand is smaller than ϵ

*Fix $R > 0$ such that $|f| < \frac{\epsilon}{4}$ outside $[-R, R]$.
f is uniformly continuous in $[-R-1, R+1]$ thus there exists δ such
that $|f(x) - f(x-y)|\frac{\epsilon}{2}$ if $|y| < \delta$.
Then*

$$\int_{-\infty}^{+\infty} H_t(f(x-y) - f(x))dy \le \int_{|y|<\delta} H_t(f(x-y) - f(x)) + \int_{|y|>\delta} H_t(f(x-y) - f(x))dy$$

$$\le \frac{\epsilon}{2} + 2\sup f \int_{[-\delta,\delta]} H_t dy \le \epsilon$$

for t big enough.

3. *To prove the third point compute*

$$\int_{-\infty}^{+\infty} |u(t,x) - f(x)|^2 dx \underbrace{=}_{Plancherel} |\hat{u} - \hat{f}|^2 d\xi$$

$$= \int_{-\infty}^{+\infty} |\hat{f}||e^{-4\pi^2\xi^2 t} - 1|d\xi \to 0 \qquad \square$$

*as $t \to 0$ by dominated convergence (the integrand is dominanted by
$4|\hat{f}|^2$)*

5.1.1 Discussion : What about uniqueness

Notice that it is sufficient to prove that for $f \equiv 0$, the solution is uniquely equal to 0. We will only sketch the proof of the following theorem :

Theorème 92

If $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is

1. a solution of the heat equation and $u(x, 0) = 0$

2. Continuous in $[0, \infty) \times \mathbb{R}, C^2(\mathbb{R} \times (0, \infty))$

3. $u(\cdot, t)$ is Schwartz uniformly in t

Then $u \equiv 0$

5.2 Heat equation on an interval

Consider the PDE

$$\begin{cases} \partial_t u = c^2 \partial_{xx} u \\ u(0, x) = f(x) \\ u(t, 0) = u(t, L) = 0 \end{cases}$$

where $L, c > 0, f \in C^0([0, L])$ such that $f(0) = f(L) = 0$.

Let's start with a formal computation again.

1. First notice that we may reduce to the case $c = L = 1$, indeed, if u solves the heat equation in this special case, then, $v = u(\frac{L^2 t}{c}, Lx)$ solves the general problem.

2. We can now look for solutions of the form $V(x)W(t)$, then the heat equation rewrites as

$$V(x)W'(t) = V''(x)W(t)$$

Dividing both sides gives

$$\frac{W'(t)}{W(t)} = \frac{V''(x)}{V(x)} = \lambda$$

Hence the heat equation may be rewritten as

$$\begin{cases} W' = \lambda W \\ V'' = \lambda V \\ V(0) = V(1) = 0 \end{cases}$$

Which we can easily solve

Lecture 21: Laplace Equation

In our solution to the heat equation, we saw that a finite sum

$$\sum a_n \sin(n\pi x)e^{-(np)^2 t}$$

is a solution, but we want to take infinite linear combinations, indeed, infinite sums still are (formally) solutions.

Finally, we want to respect the initial condition (at $t = 0$), this can be rewritten as

$$u(x, 0) = \sum a_n sin(n\pi x) = f(x)$$

Recognizing the fourier expansion, we can find

$$a_n = 2 \int_0^1 f(x) \sin(\pi n x) dx$$

Until now, we've derived this result formally (ie. with no respect for actual convergence), let us now prove a rigorous theorem, first we state a general proposition though

Proposition 93
Let $\Omega \subset \mathbb{R}^2$ open and $\{u_N\} \subset C^1(\Omega, \mathbb{R})$.
If $u_N \to u$ pointwise and $\nabla u_N \to v$ locally uniformly in Ω, then

$$u \in C^1(\Omega) \text{ and } \nabla u = v$$

Preuve
First, notice that v is continuous, furthermore

$$u_N(x + he_i) - u_N(x) = h \int \partial_i u_N(x, t + e_i) dt$$

Taking the limit

$$u(x + he_i) - u(x) = h \int_0^1 v_i(x + the_i) dt$$

Now

$$\lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} = \lim_{h \to 0} \int_0^1 v_i(x + the_i) dt = v_i(x) \qquad \square$$

Theorème 94
Let $f \in C^4$, then $u(x, t)$ defined as above satisfies the following

   1. $u \in C^2((0, 1) \times [0, \infty))$ and solves the heat equation.

2. $\lim_{x \to 0 \text{ or } 1} u(x,t) = 0$

3. $f(x) = \lim_{t \to 0} u(x,t) \forall x$

Preuve

— *Let* $u_N(x) = \sum_{n=0}^{N} a_n \sin(n\pi x) e^{-(n\pi)^2 t} \in C^{\infty}((0,1) \times [0,\infty))$ *then*

$$\partial_t u_N = \sum_{n=1}^{N} a_n \sin(n\pi x)(-n^2\pi^2) e^{-(n\pi)^2 t}$$

Recall that, if $f \in L^1$, *then* $\sup |a_n| \leq 2 \|f\|_{L^1}$ *Furthermore, as* $N \to \infty$, $\partial_t u_N \in C^{\infty}$ *converges locally uniformly for* $t > 0$ *to*

$$-\sum_{n=1}^{\infty} a_n \sin(n\pi x) n^2 \pi^2 e^{-n^2\pi^2 t}$$

Similarly, $\partial_x u_N \in C^{\infty}$ *and converges locally uniformly to*

$$\sum_{n=0}^{\infty} a_n (n\pi) \sin(n\pi x) e^{-n^2\pi^2 t}$$

To prove that the limite as $N \to \infty$ *still converges, using the proposition, we get* $u \in C^1$.
Applying the proposition again to $\partial_x u$ *inductively we in fact fidn that* $u \in C^{\infty}((0,1) \times (0,\infty))$ *and*

$$\partial_t^l \partial_x^m u = \sum_{n=1}^{\infty} a_n g_n(x)(n\pi^m) e^{-(\pi n)^2 t}(-\pi n)^{2l}$$

where g_n *is equal to either* $\sin$ *or* $\cos$.

— *Now, since* $u \in C_x^1(0,1)$ *and* $\partial_x u(\cdot, t)$ *is bounded, thus* $u \in C_x^0([0,1])$.
Lets compute

$$|u(x,t) - f(x)| \leq \sum |a_n| |\sin(n\pi x)| \left(e^{-n\pi^2 t} - 1 \right) |$$

The term in parentheses goes to 0 *as* $t \to 0$

$$\leq \pi^2 \sum a_n n^2 t$$

But a_n *decays with order* $\frac{1}{n^4}$ *thus this series converges to 0 as* $t \to 0$

$\square$

5.3 The laplace equation in a box

The Laplace equation reads

$$\begin{cases} \Delta u = 0 \text{ in } (0, L) \times (0, M) \\ u(x, 0) = \alpha(x) \text{ and } u(x, M) = \beta(x) \\ u(0, y) = \gamma(y) \text{ and } u(L, y) = \delta(y) \end{cases}$$

Using separation of variables, we get

$$\phi''(x)\psi(y) = -\phi(x)\psi''(y)$$

$$\frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)}$$

Thus, we get $\phi'' = \lambda\phi$ and $\psi'' = -\lambda\psi$.

Now, if $\lambda < 0$ we get $\phi(x) = \alpha\sin(\sqrt{-\lambda}x) + \beta\cos(\sqrt{-\lambda}x)$ and if $\lambda > 0$ we get $\phi(x) = \gamma e^{\sqrt{\lambda}x} + \delta e^{-\sqrt{\lambda}x}$.

If $\lambda > 0$ we can write ϕ as

$$(\gamma + \delta)\cosh(\sqrt{\lambda}x) + (\gamma - \delta)\sinh(\sqrt{\lambda}x)$$

Overall, solutions will be of the form

$$\left(\alpha\sinh(\sqrt{-\lambda}x) + \beta\cosh(\sqrt{-\lambda}x)\right)\left(\gamma'\cosh(\sqrt{-\lambda}y) + \delta\sinh(\sqrt{-\lambda}y)\right)$$

We can now split this into two simpler problem, suppose we can solve

$$\begin{cases} \Delta v = 0 \\ v(x, 0) = \alpha(x) \text{ and } v(x, M) = \beta(x) \\ v(0, y) = 0 \text{ and } v(L, y) = 0 \end{cases}$$

as well as

$$\begin{cases} \Delta w = 0 \\ w(x, 0) = 0 \text{ and } w(x, M) = 0 \\ v(0, y) = \gamma(y) \text{ and } v(L, y) = \delta(y) \end{cases}$$

Then u, the solution of the original problem is $u = v + w$.

Let's solve the first problem by separation of variables, writing $u = \phi(x)\psi(y)$, we get

$$\frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = \lambda$$

and the initial condition implies $\phi(0) = \phi(L) = 0$, thus $\lambda = -(\frac{n\pi}{L})^2$, thus $\phi = \alpha_n\sin(\frac{n\pi}{L}x)$ and $\psi_n(x) = \xi_n\cosh(\frac{n\pi}{L}y) + \eta_n\sinh(\frac{n\pi}{L}y)$.

Thus, the general solution is

$$\sum_n \left[\xi_n\cosh(\frac{n\pi}{L}y) + \eta_n\sinh(\frac{n\pi}{L}y)\right]\sin(\frac{n\pi}{L}x)$$

Now we impose our boundary solutions $\alpha(x) = v(x,0) = \sum \xi_n \sin(\frac{n\pi}{L}x)$ and so we compute the fourier coefficients of α (in sines), which are given by

$$\xi_n = \frac{2}{L}\int_0^L \alpha(x)\sin(\frac{n\pi}{L}x)dx$$

Similarly, we find

$$\eta_n = \frac{1}{\sinh(\frac{2\pi}{L}M)}\left[\frac{2}{L}\int_0^L \beta(x)\sin(\frac{n\pi}{L}x)dx - \xi_n\cosh(\frac{n\pi}{L}M)\right]$$

Proposition 95
If $\alpha, \beta \in L^1$, then $w \in C^\infty((0,L)\times(0,M))$ and $\Delta w = 0$