École Polytechnique Fédérale de Lausanne
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Apr 7, 2022

Math 261 – Discrete Optimization (Spring 2022)

Midterm Exam Solutions

NAME: ______________________________
SCIPER #: _______________

Duration: 90 minutes

Rules:

- There are 4 problems on this exam. The first is worth 10 points and the others 5 points (each). Some of the problems may be longer or more difficult than others, so please plan your time and effort wisely. It is quite possible that you will not have time to finish all of the problems, so it is important that you solve the ones that you can as well as possible.
- You should show your work and explain what you are doing as much as possible, but do not waste time writing more than you need. Brief explanations with arrows are fine (you do not need to write paragraphs).
- You should not use any materials on this exam other than the provided Cheat Sheet.
- Please write all of your answers on this test booklet (you can use the reverse side of the page). If you need extra paper, it can be given to you, but please make sure that you inform the instructor at the end of the exam that there are external pages you wish to have graded.
- If you have any questions during the exam about wording or whether something is missing or whether you are allowed to assume something that isn't there — PLEASE ASK.
- GOOD LUCK!

Problem 1

Consider the linear program:

$$\begin{array}{rrcrcrcl} \mathcal{P} = \max & x_1 & + & x_2 & + & 2x_3 & & \\ \text{s.t.} & x_1 & & & - & 2x_3 & \leq & 2 \\ & 2x_1 & + & x_2 & + & 3x_3 & \leq & 2 \\ & 6x_1 & + & 2x_2 & - & 2x_3 & \geq & 5 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

(a) Write $\mathcal{P}$ in equality standard form.

Solution:
We need to add slack variables and to make things easier we move from a min to a max.

$$\begin{array}{rrcrcrcrcrcrcl} \min & -x_1 & - & x_2 & - & 2x_3 & & & & & & & & \\ \text{s.t.} & x_1 & & & - & 2x_3 & + & s_1 & & & & & = & 2 \\ & 2x_1 & + & x_2 & + & 3x_3 & & & + & s_2 & & & = & 2 \\ & 6x_1 & + & 2x_2 & - & 2x_3 & & & & & - & s_3 & = & 5 \\ & x_1, & & x_2, & & x_3, & & s_1, & & s_2, & & s_3 & \geq & 0 \end{array}$$

(b) In your answer to part (a), find a column basis β that corresponds to the solution

$$(x_1, x_2, x_3) = (1, 0, 0).$$

in $\mathcal{P}$. List the variables that form β (with explanation).

Solution:
Plugging in $(1, 0, 0)$ and solving for s_1, s_2, s_3 gives

$$s_1 = 1 \qquad s_2 = 0 \qquad s_3 = 1$$

where we need three variables to form the basis, and those will be the ones which are nonzero, so $\beta = \{x_1, s_1, s_3\}$.

(c) Assume that for the basis β you found in part (b) that you have computed

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{-7}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 11 & 0 & 3 & 1 \end{bmatrix}$$

where $\mathbf{B}$ is the basis matrix and $\mathbf{A}$ is the constraint matrix. Determine which variable will leave the basis and which variable will enter the basis in this round of simplex.

Note: the original problem had a $1/2$ here instead of $-7/2$. However it should not have affected your solution at all.

Solution:
We first compute the reduced costs $\mathbf{c}^\intercal - \mathbf{c}_\beta^\intercal \mathbf{B}^{-1}\mathbf{A}$ where

$$\mathbf{c}_\beta^\intercal \mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 11 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{2} & -\frac{3}{2} & 0 & -\frac{1}{2} & 0 \end{bmatrix}$$

and so the reduced costs are $[0, -\frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, 0]^\intercal$. Since x_2 and x_3 are negative numbers, we choose x_2 to enter the basis (via Bland's rule).[1] To see what to remove, we use the direction vector (see the CheatSheet)

$$\mathbf{u} = \mathbf{B}^{-1}\operatorname{col}_2(\mathbf{A}) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

The positive values corresponds to x_1 and s_3 and for these we get the ratio

$$\frac{x_{\beta(i)}}{u_i} = \begin{cases} 2 & \text{for } x_1 \\ 1 & \text{for } s_3 \end{cases}$$

so the smallest one is s_3 with $\theta = 1$. Hence we remove s_3 and add x_2.

(d) Find a certificate that proves $(x_1, x_2, x_3) = (\frac{1}{2}, 1, 0)$ is an optimal solution for $\mathcal{P}$. To receive full credit, your certificate needs to work directly on $\mathcal{P}$ (not your solution to (a)).

Solution:
First we compute the value of $(\frac{1}{2}, 1, 0)$ in $\mathcal{P}$ is $\frac{1}{2} + 1 + 0 = \frac{3}{2}$. Therefore we should look for a certificate that says

$$x_1 + x_2 + 2x_3 \leq \frac{3}{2}.$$

Plugging in $(\frac{1}{2}, 1, 0)$ into $\mathcal{P}$, we see that the active constraints are

$$2x_1 + x_2 + 3x_3 \leq 2 \quad \text{and} \quad 6x_1 + 2x_2 - 2x_3 \geq 5 \quad \text{and} \quad x_3 \geq 0$$

In order to find the weightings, we need to solve

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 6 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

so, as linear equations, we have

$$2a_1 + 6a_2 = 1 \quad \text{and} \quad a_1 + 2a_2 = 1 \quad \text{and} \quad 3a_1 - 2a_2 + a_3 = 2.$$

which we can easily solve to get $(a_1, a_2, a_3) = (2, -1/2, -5)$. So now if we take this linear combination of the constraints, we get

$$\begin{array}{rcccccccc} 2\times & (& 2x_1 & + & x_2 & + & 3x_3 &) & \leq & 2\times(2) \\ -\frac{1}{2}\times & (& 6x_1 & + & 2x_2 & - & 2x_3 &) & \leq & -\frac{1}{2}\times(5) \\ -5\times & (& & & & & x_3 &) & \leq & -5\times(\ 0\ ) \\ \hline = & & x_1 & + & x_2 & + & 2x_3 & & \leq & \frac{9}{2} \end{array}$$

just as we wanted.

Another option here was to compute the dual and find a dual feasible solution.

[1] Note that if we had kept the problem a maximization problem, we would want the first *positive* number. This was a *very* common mistake.

Problem 2

Consider the linear programs

$$\begin{array}{rlll} \mathcal{P}_1 = \max & \mathbf{c}\cdot\mathbf{x} & & \\ \text{s.t.} & \mathbf{Ax} & \leq & \mathbf{b} \end{array} \qquad \text{and} \qquad \begin{array}{rlll} \mathcal{P}_2 = \max & \mathbf{c}\cdot\mathbf{x} & & \\ \text{s.t.} & \mathbf{Ax} & \leq & \mathbf{0} \\ & \mathbf{c}\cdot\mathbf{x} & \leq & 1 \end{array}$$

where $\mathcal{P}_1$ is known to have a feasible solution. Show that $\mathcal{P}_1$ has a finite maximum if and only if $\mathcal{P}_2$ has optimal value of 0.

Solution:

Because $\mathcal{P}_1$ has a feasible solution (as guaranteed by the problem), the only remaining possibilities are that $\mathcal{P}_1$ has a finite optimal solution or that it is unbounded. There were two ways proceed:

1. The easier way (in my opinion) is to look at the duals. Note that $\mathcal{P}_2$ has one row more than $\mathcal{P}_1$, to which I am going to associate dual variable μ (the others I will call $\boldsymbol{\lambda}$ as usual):

$$\begin{array}{rlll} \mathcal{D}_1 = \min & \boldsymbol{\lambda}\cdot\mathbf{b} & & \\ \text{s.t.} & \boldsymbol{\lambda}^\mathsf{T}\mathbf{A} & = & \mathbf{c}^\mathsf{T} \\ & \boldsymbol{\lambda} & \geq & \mathbf{0} \end{array} \qquad \text{and} \qquad \begin{array}{rlllll} \mathcal{D}_2 = \min & & & \mu & & \\ \text{s.t.} & \boldsymbol{\lambda}^\mathsf{T}\mathbf{A} & + & \mu\mathbf{c}^\mathsf{T} & = & \mathbf{c}^\mathsf{T} \\ & \boldsymbol{\lambda} & & & \geq & \mathbf{0} \\ & & & \mu & \geq & 0 \end{array}$$

   By looking at $\mathcal{D}_1$ and $\mathcal{D}_2$, it is easy to conclude that the following are equivalent:

   (a) $\mathcal{D}_1$ is feasible
   (b) There exists a feasible solution to $\mathcal{D}_2$ with $\mu = 0$
   (c) $\mathcal{D}_2$ has optimal value 0

   However we also know

   (a) $\mathcal{D}_1$ is feasible if and only if $\mathcal{P}_1$ has a finite optimal solution (by weak duality, since $\mathcal{P}_1$ being infeasible is not an option)
   (b) $\mathcal{P}_2$ has optimal value 0 if and only if $\mathcal{D}_2$ has optimal value 0 (by strong duality).

   Hence $\mathcal{P}_1$ has a finite maximum if and only if $\mathcal{P}_2$ has optimal value 0.

2. The other way was to consider what it means for $\mathcal{P}_1$ to be unbounded (or not). Since there exists a feasible solution $\mathbf{y}$ to $\mathcal{P}_1$, $\mathcal{P}_1$ is unbounded if and only if there is a direction vector $\mathbf{d}$ for which

   (a) $\mathbf{y} + \theta\mathbf{d}$ is feasible for all $\theta > 0$ — in other words, $\mathbf{Ad} \leq \mathbf{0}$.
   (b) The cost along this ray is increasing — in other words, $\mathbf{c}\cdot\mathbf{d} > 0$

   If $\mathcal{P}_1$ is unbounded, then we can use this $\mathbf{d}$ to build a solution $\mathbf{x}' = \mathbf{d}/\mathbf{c}\cdot\mathbf{d}$ that is feasible in $\mathcal{P}_2$ and is witness to the fact that $\mathcal{P}_2$ has optimal value > 0.

   On the other hand, if $\mathcal{P}_1$ is bounded, it means any direction vector satisfying $\mathbf{Ad} \leq 0$ must have cost $\mathbf{c}\cdot\mathbf{d} \leq 0$. This shows the optimal value of $\mathcal{P}_2$ is at most 0, and then it is easy to see that the optimal value is at least 0 by noting that $\mathbf{x} = \mathbf{0}$ is a feasible solution for $\mathcal{P}_2$ (with cost 0).

Problem 3

Let $P \subseteq \mathbb{R}^n$ be a bounded polyhedron, $\mathbf{c} \in \mathbb{R}^n$ a vector and $t \in \mathbb{R}$ a scalar. Consider the polyhedron

$$Q_P = \{\mathbf{x} \in P : \mathbf{c} \cdot \mathbf{x} = t\}$$

Show that each vertex of Q_P is either a vertex of P or a convex combination of two adjacent vertices of P.

Solution:

Let $\mathbf{x}_*$ be a vertex of Q_P. Since any vertex is also a basic feasible solution, there exists a set of n linearly independent constraints (forming Q_P) that are active at $\mathbf{x}_*$ (call them β). There are two possibilities:

1. The constraint $\{\mathbf{c} \cdot \mathbf{x} = t\}$ is not one of the ones that appears in β. Then $\vec{x}_*$ is a BFS of P (and therefore a vertex of P).
2. The constraint $\{\mathbf{c} \cdot \mathbf{x} = t\}$ is one of the constraints in β. Then the remaining $n-1$ linearly independent constraints in β define an edge in P, which (because P is bounded) has two end points $\mathbf{y}$ and $\mathbf{z}$ which are vertices in P. Then $\mathbf{x}_*$ is a convex combination of these.

Note: the statement in the problem is *not true* if P is not a bounded polyhedron (our proof used it when asserting the existence of $\mathbf{y}$ and $\mathbf{z}$), so solutions that did not explicitly use this property did not earn full credit.

Problem 4
Consider the linear program

$$\begin{aligned} \mathcal{P} = \max \quad & x + y \\ \text{s.t.} \quad 2x + y &\le 6 \\ x + 2y &\le 8 \\ 3x + 4y &\le 22 \\ x + 5y &\le 23 \end{aligned}$$

Show that $(4/3, 10/3)$ is an optimal solution.

Solution:
First we check feasibility:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 3 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4/3 \\ 10/3 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 52/3 \\ 54/3 \end{bmatrix} \le \begin{bmatrix} 6 \\ 8 \\ 22 \\ 23 \end{bmatrix}$$

so it satisfies the constraints. Furthermore, we can see that only the first two constraints are tight and (since we are in $\mathbb{R}^2$ and those rows are linearly independent) these constitute a basis. We can therefore use the basis matrix

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

to find a dual solution by solving $\boldsymbol{\lambda}_\beta^\mathsf{T}\mathbf{B} = \mathbf{c}$, or (substituting for $\mathbf{B}$ and $\mathbf{c}$)

$$\boldsymbol{\lambda}_\beta^\mathsf{T} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

You can then solve for $\boldsymbol{\lambda}$ any number of ways (it is actually not hard to guess that the solution is $\boldsymbol{\lambda}_\beta^\mathsf{T} = (1/3, 1/3)$, which you can then check). Adding these two rows together with these weights then gives

$$\begin{array}{rcccccl} \frac{1}{3} \times & (& 2x & + & y &) & \le \frac{1}{3} \times (6) \\ +\frac{1}{3} \times & (& x & + & 2y &) & \le \frac{1}{3} \times (8) \\ \hline = & & x & + & y & & \le \frac{14}{3} \end{array}$$

which matches the value of $\mathcal{P}$ at $(4/3, 10/3)$, making it optimal.