PROBA

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List of Theorems

Lecture 1: Introduction

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1 Some historical models

1.1 Laplace Model

Definition 1 (Laplace Model)
Ω finite set, $|\Omega| = n$ is the set of outcomes.
We can observe whether $E \subset \Omega$ happens, and we define it's probability

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|}$$

Question

Why should this have any meaning/content ?

Proposition 1
Consider laplace model for n coint tosses $\Rightarrow$ every sequence has probability 2^{-n}
Denote by H_n the number of heads in n tosses

$$\mathbb{P}(|\frac{H_n}{n} - \frac{1}{2}| > \epsilon) \to 0$$

More generally

Proposition 2
If you have a laplace modelfor some event E, and look at n repetitions, then

$$\forall \epsilon > 0 \mathbb{P}(|\frac{E_n}{n} - \mathbb{P}(E)| > \epsilon) \to 0$$

Limitations of Laplace Model

- All outcomes have equal probability ?
- Need $|\Omega| < \infty$, so what about infinite sets ?

What next ?

Definition 2 (Intermediate model)
Let Ω to be any set and $P : \Omega \to [0, 1]$, s.t. $\sum_{\omega\in\Omega} p(\omega) = 1$

Event : $E \subset \Omega$ and

$$\mathbb{P}(E) := \sum_{\omega\in E} p(\omega)$$

- More freedom
- If you take Ω finite, $p(\omega) = \frac{1}{|\Omega|} \Rightarrow$ Laplace model
- Price ? How to choose $p : \Omega \to [0, 1] \to$ collect data, do statistics
- keeps many nice properties

- For contable sets, this is equivalent to the standard model.
- For uncountable Ω ?
- Problem 1 : There is no function s.t.

$$p(\omega) > 0 \forall \omega \in \Omega \text{ and } \sum p(\omega) = 1$$

  This intermediate model is in essence only for countable sets.

What about uncountable sets ?

- What about a random point int $[0, 1]$ or $[0, 1]^n$?

Intuitively, consider $[0, 1]$, then we can set

$$\mathbb{P}(A) = \text{length}(A)$$

Definition 3 (Geometric probability)
Take $f : \mathbb{R} \to (0, \infty)$ to be a riemann-integrable function with total mass 1.
For any $A \subset \mathbb{R}$,s.t. 1_A riemann-integrable, we set $\mathbb{P}(A) = \int_A f(x)dx$

- In general quite <u>ok</u>
  BUT

- You would expect there is one framework for uncountable and countable sets.
- What about more complicated spaces (eg. space of continuous functions)
- $\mathbb{P}(\mathbb{Q})$ is undefined

2 Basic Formalism

2.1 Measure spaces : A notion of area

- Set + structure
- General setting to talk about area

Definition 4 (Measure space)
$(\Omega, \mathcal{F}, \mu)$ is called a measure space if :
- *Ω is some set*
- *$\mathcal{F} \subset P(\Omega)$ called a σ-algebra*
  - *$\in \mathcal{F}$*
  - *$F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$*
  - *$F_1, F_2, \ldots, \in \mathcal{F}$, then ${}_{i\geq 1}F_i \in \mathcal{F}$ each F is called a measurable set.*
- *$\mu : \mathcal{F} \to [0, \infty)$ called the measure*
  - *$\mu() = 0$*

— *If $F_1, \dots,$ are disjoints sets of the σ-algebra, then*

$$\mu(\bigcup_{i \geq 1} F_i) = \sum_{i \geq 1} \mu(F_i)$$

— *Defined by Borel 1898 and Lebesgue 1901-1903*

2.2 Probability spaces

Given by Kolmogorov in 1933

Definition 5 (Probability space)
A triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space if it is a measure space and $\mathbb{P}(\Omega) = 1$

Interpretation

— Ω state space/universe
— $\mathcal{F}$ is the set of events you can observe/have access to
— $\mathbb{P}(E)$ is the probability of E

Lemme 3
Let $(\Omega, \mathcal{F}, \mu)$ be a measure space
— $\Omega \in \mathcal{F}$
— $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \setminus F_2 \in \mathcal{F}$
— $F_1, \dots \in \mathcal{F} \Rightarrow \bigcap F_i \in \mathcal{F}$
— $F_1, F_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{i \geq 1} F_i$

Let us compare this definition with the prior ones
— Ω finite set, $\mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(F) = \frac{|F|}{|\Omega|}$ this is a probability space and a laplace model.
— For Ω countable, $\mathcal{F} = \mathcal{P}(\Omega), \mathbb{P}(E) = \sum_{\omega \in E} \mathbb{P}(\omega)$
— The really new part is $\mathcal{F}$ which restricts the sets we can measure