Exercise 10

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10 mai 2021

First, note that we can suppose that G is not a tree, and thus contains a cycle.
Indeed, if G is a tree, it's spanning tree is G, and hence $E(G) \setminus E(T) = \emptyset$.
Thus, suppose that G is a graph with at least one cycle.
We will now show that, if $e \in E(G) \setminus E(T)$, then e is contained in a cycle of G.
For the sake of contradiction, suppose that $e = \{a, b\}$ [1] is not contained in a cycle.
Then there exists $c \in V$ such that there only is one path $v_0, \ldots, v_k$ such that $v_0 = a, v_k = c$.
Since the spanning tree contains c as a vertice, we deduce that the spanning tree must contain a as a vertex.

Hence, choose an edge $e' \in E(T)$ of the form $\{a, d\}$ and which is contained in the same cycle as e, hence e' is not a leaf.
Remove e' and add e from T, note that the resulting graph is still a tree since we have not added any cycles and that it still contains all the edges of G since b is not a leaf (it is contained in a cycle).

1. $a, b \in V$