Exercise 9

David Wiedemann

4 mai 2021

As shown in the course, to each unlabeled tree T with n vertices, we can associate a (not necessarily unique) sequence $S \in \{1, -1\}^{2n-2}$. [1]
T can then be uniquely reconstructed from this sequence.
Let us show that any sequence in S corresponding to such a tree satisfies

$$\sum_{i=1}^{k} S_i \geq 0 \quad \forall 1 \leq k \leq 2n-2$$

and

$$\sum_{i=1}^{2n-2} S_i = 0.$$

First, notice that the second equality follows immediatly from the way we define the contouring path of an unlabelled tree.

To show that $\sum_{i=1}^{k} S_i \geq 0$, we use induction.
For $k = 1$, it does not make sense to reduce the distance to the root vertex, so $S_1 > 0$.
Suppose shown for $1 \leq k < 2n-2$, we will now show the result for $k+1$.
If $\sum_{i=1}^{k} S_i > 0$, then S_{k+1} can take any value and the result will still hold.
If $\sum_{i=1}^{k} S_i = 0$, the distance to the root vertex is 0 and hence it is impossible to further reduce the distance to the root node, this implies that $S_{k+1} = 1$, which concludes the proof.

As shown in the fourth exercise sheet, the number of sequences satisfying these properties is given by b_{n-1}.
Since there is a surjection from the set of such sequences to the set of unlabelled graphs, we deduce that

$$b_{n-1} \geq T^n.$$

1. We simply substituted $+$ with 1 and $-$ with -1.