Geometry

affine hyperplane: a set of the form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} = t\}$

halfspace: a set of the form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \geq t\}$

polyhedron: an intersection of a finite number of half spaces, has form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$

polytope: a bounded polyhedron

convex hull: $\mathrm{conv}(a_1, \ldots, a_k) = \{\sum_i \lambda_i a_i : \sum_i \lambda_i = 1, \lambda_i \geq 0\}$

vertex: there exists a linear program for which $\mathbf{x}$ is the unique optimal solution.

extreme point: is not the convex combination of any two other points in $\mathcal{P}$.

basic solution: in $\mathbb{R}^n$ all equality constraints and a total of n linearly independent constraints active

Linear Programming ($\mathbf{A}$ is an $m \times n$ matrix)

equality standard form: $\min / \max\{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, col. basis m vars. allowed to be $\neq 0$ $(m \leq n)$

inequality standard form: $\min / \max\{\mathbf{c} \cdot \mathbf{x} : \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$, row basis: n active constraints $(n \leq m)$

degenerate solution: a basic solution in $\mathbb{R}^n$ with more than n constraints active

complexity: Depends on algorithm — ellipsoid method is poly-time, but simplex is (in general) not.

Simplex (equality standard form, min problem, "basis" refers to column basis)

BFS for a basis β: $\mathbf{x}_\beta = \mathbf{B}^{-1}\mathbf{b}$ with nonbasis entries 0 (where $\mathbf{B} = \mathbf{A}_\beta$)

reduced cost: $c_j - \mathbf{c}_\beta^\intercal \mathbf{B}^{-1} \mathrm{col}_j(\mathbf{A})$ [Bland: pick smallest index from all good[1] reduced costs to enter basis]

what exits? for $\mathbf{u} = \mathbf{B}^{-1} \mathrm{col}_j(\mathbf{A})$, $\ell = \arg\min_{k: u_k > 0} \frac{x_{\beta(k)}}{u_k}$ exits[2] basis [Bland: if tie, pick smallest index]

certificate: $\boldsymbol{\lambda}^\intercal = \mathbf{c}_\beta^\intercal \mathbf{B}^{-1}$ (requires β to be optimal)

Phase I: Mult. by -1 to get $\mathbf{b} \geq \mathbf{0}$, solve $\min\{\mathbf{1} \cdot \mathbf{w} : \mathbf{A}\mathbf{x} + \mathbf{w} = \mathbf{b}; \mathbf{x} \geq \mathbf{0}; \mathbf{w} \geq 0\}$, start BFS: $\mathbf{w} = \mathbf{b}, \mathbf{x} = \vec{0}$.

Duality ($\mathcal{P}$ denotes the min LP and $\mathcal{D}$ denotes the dual max LP for this entire section)

Weak duality: If $\mathbf{x}$ is feasible in $\mathcal{P}$ and $\boldsymbol{\lambda}$ feasible in $\mathcal{D}$. Then $\boldsymbol{\lambda} \cdot \mathbf{b} \leq \mathbf{x} \cdot \mathbf{c}$.

Strong duality: If $\mathcal{P}$ has a finite optimal solution $\mathbf{x}$, then $\mathcal{D}$ has a finite optimal solution $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda} \cdot \mathbf{b} = \mathbf{x} \cdot \mathbf{c}$.

Complementary Slackness 1: If $\mathbf{x}$ is feasible in $\mathcal{P}$ and $\boldsymbol{\lambda}$ is feasible in $\mathcal{D}$, then

$$\lambda_i(\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} - b_i) \geq 0 \text{ for all } i \quad \text{and} \quad x_j(c_j - \mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda}) \geq 0 \text{ for all } j$$

Complementary Slackness 2: If $\mathbf{x}$ is optimal in $\mathcal{P}$ and $\boldsymbol{\lambda}$ is optimal in $\mathcal{D}$, then

$$\lambda_i(\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} - b_i) = 0 \text{ for all } i \quad \text{and} \quad x_j(c_j - \mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda}) = 0 \text{ for all } j. \text{ In particular,}$$

$$\lambda_i \neq 0 \to \mathrm{row}_i(\mathbf{A}) \text{ active in } \mathcal{P} \qquad \text{and} \qquad x_j \neq 0 \to \mathrm{row}_j(\mathbf{A}^\intercal) = \mathrm{col}_j(\mathbf{A}) \text{ active in } \mathcal{D} \qquad (+ \text{ contrapositives})$$

Farkas: Exactly one of the following is true: (1) $\exists \mathbf{x} \geq \mathbf{0}$ s.t. $\mathbf{A}\mathbf{x} = \mathbf{b}$, (2) $\exists \boldsymbol{\lambda}$ s.t. $\mathbf{A}^\intercal \boldsymbol{\lambda} \geq \mathbf{0}$ and $\mathbf{b} \cdot \boldsymbol{\lambda} < 0$

minimize $\mathbf{c} \cdot \mathbf{x}$	maximize $\boldsymbol{\lambda} \cdot \mathbf{b}$
$\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} \geq b_i$	$\lambda_i \geq 0$
$\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} \leq b_i$	$\lambda_i \leq 0$
$\mathrm{row}_i(\mathbf{A}) \cdot \mathbf{x} = b_i$	λ_i free
$x_j \geq 0$	$\mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} \leq c_j$
$x_j \leq 0$	$\mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} \geq c_j$
x_j free	$\mathrm{col}_j(\mathbf{A}) \cdot \boldsymbol{\lambda} = c_j$

	Finite Opt	Unbounded	Infeasible
Finite Opt	YES	NO	NO
Unbounded	NO	NO	YES
Infeasible	NO	YES	YES

[1] What is "good" will depend on whether LP is min (< 0) or max (> 0). If none are "good", you are at optimum.
[2] If all are nonpositive, problem is unbounded.