École Polytechnique Fédérale de Lausanne
Prof. Marcus May 12, 2022

Math 261 – Discrete Optimization (Spring 2022)

Assignment 10

Problem 1

Let $D = (V, E)$ be a digraph with $|V| = n$ and $|E| = m$ for which the underlying graph is connected. Let T be a collection of edges that form a spanning tree in the underlying graph.

(a) Show that if H is the cycle space of D, then $\dim(H) \geq m - n + 1$.

Hint: Find $m-n+1$ cycles for which the corresponding cycle vectors are linearly independent using the edges not in T.

(b) Show that if B is the cut space of D, then $\dim(B) \geq n - 1$.

Hint: Find $n-1$ cuts for which the corresponding cut vectors are linearly independent using the edges of T.

(c) Show that (as subspaces of the edge space) that $H = B^{\perp}$.

Problem 2

Enthusiastic partying at a prominent Swiss university has unfortunately resulted in the arrival at the university's medical clinic of 169 students in need of emergency treatment. Each of the 169 students requires a transfusion of one unit of blood. The clinic has supplies of 170 units of blood. The number of units of blood available in each of the four major blood groups and the distribution of patients among the groups in summarized below:

Blood type	A	B	O	AB
Supply	46	34	45	45
Demand	39	38	42	50

A patients can only receive A or O blood. B patients can only receive B or O blood. O patients can receive only O blood. AB patients can receive any of the four blood types.

(a) Give a max-flow formulation that determines a distribution that satisfies the demands of a maximum number of patients. You should draw a directed graph with edge capacities such that a feasible flow corresponds to a feasible choice for the transfusion.

(b) Find a cut in your graph of value smaller than 169. Use it to give an explanation of why not all of the patients can receive blood from the available supply. Try to make your explanation understandable to the clinic's staff, who do not know network flow theory.

Problem 3

Construct a network on four vertices for which the Ford-Fulkerson algorithm with a bad pick* function may need more than a million iterations (by picking a bad sequence of augmenting paths).

Problem 4

Let $D = (V, E)$ be a network with source s, sink t, and integer capacities $\mathbf{C}$. Prove or disprove the following statements:

(a) If every C_e is even, then there exists a maximal flow $\mathbf{f}$ such that f_e is even for $e \in E$.

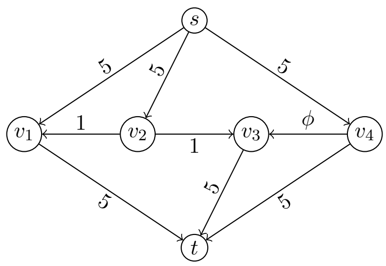
(b) If every C_e is odd, then there exists a maximal flow $\mathbf{f}$ such that f_e is odd for all $e \in E$.

Challenge Problem:

This problem is harder than usual, so please work on it last (and don't worry if you can't do it).

Consider the following network, with edges labeled by their capacities[1] Prove that the Ford-Fulkerson algorithm with a bad pick* function could run forever on this network (by picking a bad sequence of augmenting paths).

Hint: Define the "residual capacity" of an edge e to be $C_e - f_e$ and try to choose augmenting paths that cause the residual capacities on the three horizontal edges to periodically return to having the form $(\phi^k, 0, \phi^{k+1})$ for some k.



[1]The constant ϕ is the "golden ratio" $\frac{\sqrt{5}-1}{2}$. It satisfies the equation $\phi^2 + \phi = 1$.