Serie 3

Analysis IV, Spring semester

EPFL, Mathematics section, Prof. Dr. Maria Colombo

- The exercise series are published every Monday morning on the moodle page of the course. The exercises can be handed in until the following Monday, midnight, via moodle (with the exception of the first exercise which can be handed in until Thursday March 3). They will be marked with 0, 1 or 2 points.
- Starred exercises ($\star$) are either more difficult than other problems or focus on non-core materials, and as such they are non-examinable.

Definition 1 (lower/upper semi-continuity)**.** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a function.

(i) f is *lower semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ such that } |x - x_0| < \delta \ \Rightarrow\ f(x_0) - f(x) \le \varepsilon.$$

(ii) f is *lower semi-continuous* if f is lower semi-continuous in every point $x_0 \in \mathbb{R}^n$.

(iii) f is *upper semi-continuous* in $x_0 \in \mathbb{R}^n$ if

$$\forall\, \varepsilon > 0\ \exists\, \delta = \delta(\varepsilon) > 0 \text{ such that } |x - x_0| < \delta \ \Rightarrow\ f(x) - f(x_0) \le \varepsilon.$$

(iv) f is *upper semi-continuous* if f is upper semi-continuous in every point $x_0 \in \mathbb{R}^n$.

Exercise 1. We show that lower/upper semi-continuity implies measurability.

(i) Show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is lower semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$G_\alpha := \{x \in \mathbb{R} : f(x) \le \alpha\}$$

is closed. Similarly, show that if $f\colon \mathbb{R}^n \to \mathbb{R}$ is upper semi-continuous, then for all $\alpha \in \mathbb{R}$ the set

$$F_\alpha := \{x \in \mathbb{R} : f(x) \ge \alpha\}$$

is closed.

(ii) Deduce that an lower/upper semi-continuous function is measurable.

Exercise 2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be increasing or decreasing, then f is measurable.

Exercise 3. Let $f,\, g\colon \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Show that the functions

$$f^2, \quad fg, \quad |f|$$

are measurable.

Exercise 4. Let φ be measurable and f continuous. Show that $f \circ \varphi$ is measurable. (On the other hand, in general $\varphi \circ f$ is not measurable and we will discuss a counterexample in Serie 5.)

Exercise 5. We want to compute the measure of a intersection of a countable family of decreasing sets.

(i) Show that if $A_1 \supseteq A_2 \supseteq A_3 \ldots$ is a decreasing sequence of measurable sets (that is $A_j \supseteq A_{j+1}$ for every $j \geq 1$) and $\mathrm{m}(A_1) < +\infty$, then

$$\mathrm{m}\left(\bigcap_{j=1}^{\infty} A_j\right) = \lim_{j\to\infty} \mathrm{m}(A_j)$$

(ii) Show that the previous equality fails without the assumption $\mathrm{m}(A_1) < +\infty$.

The following technical exercise will be used, in the next exercise sheet, to give an equivalent definition of the Cantor set and to prove some of its interesting properties.

Exercise 6. For any sequence $\{a_i\}_{i=1}^{\infty}$ with $a_i \in \{0, 1, 2\}$, we denote by $0.a_1a_2a_3\ldots$ the number

$$\sum_{i=1}^{\infty} \frac{a_i}{3^i}.$$

Consider the inductive construction described below: for all $a \in [0, 1]$, we define

$$\begin{cases} a_1 := \lfloor 3a \rfloor, \\ a_{i+1} := \lfloor 3^{i+1}(a - \widetilde{a}_i) \rfloor & i \geq 1, \end{cases}$$

where for $i \geq 1$

$$\widetilde{a}_i := \sum_{n=1}^{i} \frac{a_n}{3^n}\,.$$

Here we set

$$\lfloor y \rfloor := \begin{cases} \max\{n \in \mathbb{N} : n < y\} & \text{if } y > 0\,, \\ 0 & \text{if } y = 0. \end{cases}$$

(i) Show that for all $i \geq 1$

$$0 \leq a - \widetilde{a}_i \leq \frac{1}{3^i}.$$

Deduce that any $a \in [0, 1]$ can be written as $0.a_1a_2a_3\ldots$ with $a_i \in \{0, 1, 2\}$.

(ii) Conversly, assume that $\{a_i\}_{i=1}^{\infty}$ is a sequence with $a_i \in \{0, 1, 2\}$ for all $i \geq 1$. Show that $0.a_1a_2a_3 \ldots \in [0, 1]$.

(iii) The expansion of a number $a \in [0, 1]$ as $0.a_1a_2a_3 \ldots$ is called the ternary expansion. Show that, in general, this expansion is not unique.

(iv) We now adopt the following identification among tenary expansions: if there exists $k \geq 2$ such that $a_i = 2$ for all $i \geq k$ and $a_{i-1} < 2$, then we identify the expansion

$$0.a_1 \cdots a_{k-1} 2 \cdots 2 \ldots$$

with the expansion

$$0.a_1 \cdots (a_{k-1} + 1) 0 \cdots 0 \ldots$$

Prove by contradiction that, modulo this identification, the ternary expansion of a number $a \in [0, 1]$ is unique.

Hint: For (iii), recall that $1 = 0.9999\ldots$

Exercise 7 ($\star$)**.** Let $f \colon \mathbb{R} \to \mathbb{R}$ continuous. Prove that the sets of points $x \in \mathbb{R}$ where f is differentiable is a Lebesgue measurable set.