Algebraic Geometry I

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Table des matières

List of Theorems

Lecture 1: Intro

Mon 10 Oct

Quick Motivation

We study schemes.
These are objects that "look locally" like $(\operatorname{Spec} A, A)$.
Examples include

— A itself

— Varieties in affine or Projective

1 Presheaves and Sheaves

1.1 Presheaves

Let X be a topological space.

Definition 1 (Presheaf)
Let C be a category. A presheaf $\mathcal{F}$ of C on X consists of

— *$\forall U \subset X$ open, an object in C $\mathcal{F}(U)$*

— *$\forall V \subset U \subset X$ open, a morphism $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$*

such that

— *$\forall U$ open, $\rho_{U,U}$ is the identity on $\mathcal{F}(U)$*

— *Restriction maps are compatible*

$$\forall W \subset V \subset U \subset X$$

open, we have $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W}$

Remark
Usually, $C = Set, Ab, Ring, etc.$

In particular, we usually assume the objects in C have elements.

Remark

— *Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.*

— *$\mathcal{F}(U)$ is called the space of sections of $\mathcal{F}$ over U*

— *Elements of $\mathcal{F}(X)$ are called global sections.*

— *There are alternative notations for $\mathcal{F}(U)$: $\Gamma(U, F)$ or $H_0(F)$*

— *The ρ_{UV} are called restriction maps, for $s \in \mathcal{F}(U)$, we write $s|_V := \rho_{UV}(s)$ and is called restriction of s to V.*

Example

— *For any object A in C, we define the constant presheaf $\underline{A}'$ defined by $\underline{A}'(U) = A$ and with restriction maps the identity.*

- *The presheaf of continuous functions :* C^0.
  We define $C^0(U) := \{f : U \to \mathbb{R} | f \text{ continuous } \}$ *and the restriction maps are the natural restrictions.*
- *More generally, if* $\pi : Y \to X$ *is continuous, we can look at the presheaf of continuous sections of* π*, here*

$$\mathcal{F}_\pi(U) := \{s : U \to Y | s \text{ continuous } \pi \circ s = \mathrm{Id}\}$$

  This example is universal in a certain sense

Remark

Define the category Ouv_X *with*
- *objects* $U \subset X$ *open subsets*
- *morphisms* $U \to V$ *are either empty or the inclusion* $U \to V$ *if* $U \subset V$

Then a presheaf of C *on* X *is just a contravariant functor* $\mathrm{Ouv}_X^{op} \to C$

Definition 2 (Morphism of presheaves)
A morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ *of presheaves on* X *consists of a collection of morphisms* $\rho(U) : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ *which are natural.*

$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\rho(U)} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow[\rho(V)]{} & \mathcal{F}_2(V) \end{array}$$

Example
- *Every morphism of objects* $A \to B$ *in* C *yields a morphism* $\underline{A}' \to \underline{B}'$
- *If* $X = \mathbb{R}^n$*, let* C^∞ *be the presheaf of smooth functions, then for every open* U*, there is an inclusion* $C^\infty(U) \to C^0(U)$ *and these inclusions induce a morphism of sheaves* $C^\infty \to C^0$
- *If* $Y_2 \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} X$ *are continuous, we get* $\rho : \mathcal{F}_{\pi_1 \circ \pi_2} \to \mathcal{F}_{\pi_1}$ *by mapping a section* $s \in \mathcal{F}_{\pi_1 \circ \pi_2}(U) \to \pi_2 \circ s$

Remark

There is an equivalence of categories

$$\textit{Presheaves of } C \textit{ on } X \simeq Fun(\mathrm{Ouv}_X^{op}, C)$$

1.2 Sheaves

Definition 3 (Sheaf)
Let $C = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}$.
A sheaf $\mathcal{F}$ *of* $\mathcal{C}$ *on* X *is a presheaf such that* $\forall U \subset X$ *open and all open covers* $U = \bigcup_{i \in I} U_i$

— *$\forall s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$ $\forall i \in I$ then $s = t$*
— *$\forall \{s_i\}$ with $s_i \in \mathcal{F}(U_i)$ and $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ $\forall i, j \in I$, then $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$*

Condition 1 is called locality and condition 2 is the gluing condition.

Remark

— *The section s of the gluability condition is unique by the locality condition.*
— *If C has products, then a presheaf is called a sheaf if*

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram Here the first map is induced by the maps $s_i : \mathcal{F}(U) \to \mathcal{F}(U_i)$, the two second maps are induced by, for each pair $i, j \in I$ the restrictions $\rho_{U_i, U_i \cap U_j}$ resp. $\rho_{U_j, U_i \cap U_j}$

Example

1. *If $\mathcal{F}$ is a sheaf , let $U\emptyset \subset X$ and $I = \emptyset$, then $\mathcal{F}(\emptyset)$ contains at most one element*
2. *C^0 (and C^∞ if $X = \mathbb{R}^n$) are sheaves since $\forall U \subset X$ open*
   — *Two continuous functions $f, g : U \to \mathbb{R}$ that coincide on an open cover are equal*
   — *Given an open cover $U = \bigcup_{i \in I} U_i$ and $f_i : U_i \to \mathbb{R}$, the function $f : U \to \mathbb{R}$ defined in the obvious way is continuous (resp. smooth) because continuity (resp. smoothness) is local.*

Definition 4 (Morphisms of sheaves)
A morphism of sheaves $\rho : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of the underlying presheaves.

Remark

— *$PSh_C(X)$ is the category of presheaves of C on X*
— *$Sh_C(X)$ is the category of sheaves of C on X*
If $C = Ab$, we drop the index.

Remark

There is a forgetful functor $Sh_C(X) \to PSh_C(X)$.
By definition, this functor is fully faithful

Recall

Let A be a commutative ring (with 1), then $\operatorname{Spec} A$ is the set of prime ideals of A.

The closed subsets of the Zariski topology on $\operatorname{Spec} A$ are of the form $V(M) = \{p \in \operatorname{Spec} A | M \subset p\}$.
A basis of this topology is given by $D(a) = \{p \in \operatorname{Spec} A | a \notin p\}$, here $a \in A$

Definition 5 (Natural sheaf on Spec A)
Let A be a ring and $X = \operatorname{Spec} A$, then the structure sheaf $\mathcal{O}_X$ on X is defined by

$$\mathcal{O}_X(U) = \left\{ s : U \to \coprod_{p \in \operatorname{Spec} A} A_p | s \text{ satisfies } i \text{ and } ii \right\}$$

where

1. $\forall p \in U, s(p) \in A_p$
2. $\forall p \in U, \exists a, b \in A$ *and* $V \subset U$ *open with* $p \in V \subset D(b)$ *with* $s(q) = \frac{a}{b} \in A_q \forall q \in V$

and ρ_{UV} are simply the (pointwise) restrictions.

Remark
$\mathcal{O}_X$ is a sheaf of rings :
— *$\mathcal{O}_X(U)$ is a ring with pointwise multiplication and addiiton*

Lecture 2: Stalks

Fri 14 Oct

1.3 Stalks

Let X be a topological space.

Definition 6
Let $(I, \leq)$ be a pair where I is a set and $\leq$ is a binary relation.
$(I, \leq)$ is called a preorder if ll is reflexive and transitive.
$(I, \leq)$ is called a poset if it is preordered and $\leq$ is antisymmetric
$(I, \leq)$ is called a directed set if it is preordered and $\forall i, j \in I \exists k \in I$ such that $i, j \leq k$

Example

1. *Let $I = \{U \subset X | U \text{ open } \}$ and $U \leq V \iff V \subset U$.*
   Then I is a directed poset.
2. *For $x \in X$, let*
$$I_x = \{U \subset X | U \text{ open } x \in U\}$$
   This is a directed poset.

Definition 7
Let $(I, \leq)$ *be a directed set and* C *a category.*
A direct system in C *indexed by* I *is a pair* $\left(\{A_i\}, \{\rho_{ij}\}_{i,j \in I, i \leq j}\right)$.
Where the A_i *are objects in* C*, the* $\rho_{ij} : A_i \to A_j$ *are morphisms in* C *such that*

1. $\rho_{ii} = \mathrm{Id}_{A_i}$
2. $\rho_{ik} = \rho_{jk} \circ \rho_{ij}$

Example
If $\mathcal{F}$ *is a presheaf of* C *on* X *and* I_X *as in the second example above, then*

$$\left(\{\mathcal{F}(U_i)_{U_i \in I_X}\}, \{\rho_{U_i, U_j}\}\right)$$

is a direct system.

Definition 8 (direct limit)
Let $(I, \leq)$ *be a directed set,* C *a category.*
Let $\left(\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I}\right)$ *be a directed system, then the direct limit is a pair* $(\lim_{i \in I} A_i, \{\rho_i\}_{i \in I})$ *where* $\lim A_i$ *is in* C *and* $\rho_i : A_i \to \lim A_i$ *such that*

1. $\rho_j \circ \rho_{ij} = \rho_i$
2. *For all objects* A *in* C *and morphisms* $f_i : A_i \to A$ *such that*

$$f_j \circ \rho_{ij} = f_i \forall i, j \in I, i \leq j$$

$\exists! f : \lim_{i \in I} A_i \to A$ *such that* $f \circ \rho_i = f_i$

Remark
The direct limit is unique up to unique isomorphism.

Example
Write $(*) = (\{A_i\}_{i \in I}, \{\rho_{ij}\}_{i,j \in I, i \leq j})$.
Let $*$ *be a direct systement in* Set.
Let $\lim_{i \in I} A_i \coloneqq A_i / \sim$ *where* $a_i \simeq a_j \iff \exists k \in I, i, j \leq k$ *such that* $\rho_{ik}(a_i) = \rho_{jk}(a_j)$.
This is the direct limit of $*$.
If $*$ *is a system in* Ab *, let* $\lim A_i \coloneqq \bigoplus A_i / N$ *with* $N = \langle a_i - \rho_{ij}(a_i) \rangle$.
The natural map $\bigcup A_i / \sim \to \bigoplus A_i / N$ *is a bijection*

Remark
Taking the direct limits in (Ab) *is exact in the following sense :*
$\forall$ *directed sets* I*,* $\forall$ *direct systems* $\{M_i\}, \{N_i\}, \{P_i\}$ *indexed by* I *and for all*

collections of commutative diagrams, we get

$$0 \to \lim M_i \to \lim N_i \to \lim P_i \to 0$$

Definition 9
Let C be a category with direct limits. Let $x \in X$ be a point, $\mathcal{F}$ a presheaf of C on X.
Then the stalk of $\mathcal{F}$ at x is

$$\mathcal{F}_x = \lim \mathcal{F}(U)$$

where U runs over all open neighbourhoods of x.
For $s \in \mathcal{F}(U)$, we write s_x for the image of s in $\mathcal{F}_x$ and call it the germ of s at x.

Remark
A morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces $\phi_x : \mathcal{F}_x \to \mathcal{G}_x \forall x \in X$

Remark
Let $x \in X, \mathcal{F}$ a presheaf of Set, Ab

1. $\forall U \subset X$ *open,* $x \in U, s, t \in \mathcal{F}(U)$

$$s_x = t_x \iff \exists V \subset U \text{ open such that } s|_V = t|_V$$

2. $\forall s \in \mathcal{F}_x, \exists x \in U$ *open and* $t \in \mathcal{F}(U)$ *such that* $t_x = s$.

Definition 10 (Sheafification)
Let $\mathcal{F}$ be a presheaf of sets (...) on X.
The sheafification of $\mathcal{F}$ is the sheaf $\mathcal{F}^+$ defined by

$$\mathcal{F}^+(U) = \left\{ s : U \to \coprod_{x \in U} \mathcal{F}_x | s \text{ satisfies properties 1 and 2} \right\}$$

1. $\forall x \in U s(x) \in \mathcal{F}_x$
2. $\forall x \in U \exists V \subset U$ *open and* $t \in \mathcal{F}(V) t_u = s(y) \forall y \in V$

Remark

1. *$\mathcal{F}^+$ is a sheaf*
2. *Sheafification is functorial.*
   For $\rho : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves, the collection $\phi^+(U) : \mathcal{F}^+(U) \to \mathcal{G}^+(U)$ sending $s \to (\coprod_{x \in U} \phi_x) \circ s$
3. *$\exists$ a natural morphism $\iota_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}^+$ defined by $\iota_F(U)(s) : x \to s_x$*
4. *$\forall s \in \mathcal{F}^+(U)$ there is an open cover $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $s|_{U_i} = \iota_{\mathcal{F}}(U_i)(s_i)$*

5. $\forall x \in X$, *the map* $\iota_{\mathcal{F},x} : \mathcal{F}_x \to \mathcal{F}_x^+$ *is an isomorphism.*

Proposition 20

$\forall$ morphisms $\phi : \mathcal{F} \to \mathcal{G}$ such that $\mathcal{G}$ is a sheaf, there exists a unique morphism $\phi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\phi = \phi^+ \circ \iota_{\mathcal{F}}$

Proof

Let $U \subset X$ open, let $s \in \mathcal{F}^+(U)$ $\exists$ an open cover $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ such that $\iota_{\mathcal{F}}(U_i)(s_i) = s|_{U_i}$.
Since we want $\phi = \phi^+ \circ \iota_{\mathcal{F}}$, we have to set

$$\phi^+(U_i)(s|_{U_i}) = \phi(U_i)(s_i)$$

Since $\mathcal{G}$ is a sheaf and

$$\phi(U_i)(s_i)|_{U_i \cap U_j} = \phi(U_i \cap U_j)(s_i|_{U_i \cap U_j}) = \phi(U_j)(s_i)|_{U_i \cap U_j}$$

there exists a unique $t \in \mathcal{G}(U)$ with $t|_{U_i} = \phi(U_i)(s_i)$.
For ϕ^+ to be a morphism, we have to set $\phi^+(U)(s) = t$.
We still have to check that ϕ^+ is compatible with restriction maps. $\square$

Remark

The proposition above shows that $\hom_{Sh(X)}(\mathcal{F}^+, \mathcal{G}) \xrightarrow{\sim} \hom_{Psh(X)}(\mathcal{F}, \mathcal{G})$ naturally in the presheaf and the sheaf $\mathcal{G}$.
Hence $(-)^+$ is left-adjoint to the forgetful functor $Sh(X) \to Psh(X)$

Proposition 22

$X = \operatorname{Spec} A$ $\forall a \in A$ there exist isomorphisms $\phi_a : A_a \to \mathcal{O}_X(D(a))$ such that $\forall b \in A$ with $D(b) \subset D(a)$

$$\begin{array}{ccc} A_a & \xrightarrow{\sim} & \mathcal{O}_X(D(a)) \\ \downarrow & & \downarrow \\ A_b & \xrightarrow[\sim]{} & \mathcal{O}_X(D(b)) \end{array}$$

Proof

Define $\phi_a : A_a \to \mathcal{O}_X(D(a))$ by sending $\frac{s}{a^n} \mapsto (p \to \frac{s}{a^n} \in A_p)$.
Clearly, these make the diagram commute.
This map is injective, indeed, suppose $\phi_a(\frac{s}{a^n}) = 0$.
Let $I = Ann(s) = \{r \in A | rs = 0\}$.
Since $\frac{s}{a^n} = 0 \forall p \in D(a)$, we have $I \not\subset p$, hence $V(I) \subset V(a) \implies a \in \sqrt{I}$.
Thus there exists $m \geq 1$ such that $a^m s = 0$, hece $\frac{s}{a^n} = 0$.
To show surjectivity, let $s \in \mathcal{O}_X(D(a))$, by definition of $\mathcal{O}_X$ and because $D(h_i)$ form a basis, we find $a_i, g_i, h_i \in A$ such that

$$D(a) = \bigcup D(h_i), D(h_i) \subset D(g_i)$$

and $s(q) = \frac{a_i}{g_i}$ for all $q \in D(h_i)$.

1. *Claim 1 : Can choose $g_i = h_i$*
2. *Claim 2 : Can choose I finite*
3. *Claim 3 : Can choose a_i, h_i such that $h_j a_i = h_i a_j$.*

Using these claims, since $D(a) = \bigcup D(h_i)$, we find $n > 0, b_j \in A$ such that $a^n = \sum b_j h_j$.
Write $c = \sum a_i b_i$.
Then $h_j = \sum_i a_i b_i h_j = \sum a_j b_i h_i = a^n a_j$.
Thus $\frac{c}{a^n} = \frac{a_j}{h_j} \in A_{h_j} \implies \phi_a(\frac{c}{a^n}) = s$.
We now prove the claims

1. *We have $D(h_i) \subset D(g_i)$ thus $V(g_i) \subset V(h_i)$ and thus $h_i \in \sqrt{(g_i)}$.*
   So there exists $c_i \in A$ and $n > 1$ such that $h_i^n = c_i g_i$.
   Now, we replace h_i by h_i^n and a_i by $a_i c_i$. Then
$$\frac{a_i c_i}{h_i^n} = \frac{a_i}{g_i}$$
2. *We have $D(a) \subset \cup D(h_i) \iff V(\sum h_i) = \cap V(h_i) \subset V(a)$.*
   This is equivalent to saying that $a \in \sqrt{\sum (h_i)}$.
   Thus there exists $n \geq 1$ such that $a^n \in \sum_i (h_i)$.
   So there exist finitely many $b_i \in A$ such that $a^n = \sum b_j h_j$
3. *On $D(h_i) \cap D(h_j) = D(h_i h_j)$, we have*
$$\phi_{h_i h_j}(\frac{a_i}{h_i}) = s|_{D(h_i h_j)} = \phi_{h_i h_j}(\frac{a_j}{h_j})$$
   Thus
$$\frac{a_i}{h_i} = \frac{a_j}{h_j} \in A_{h_i h_j}$$
   Thus, there exists $N_j \geq 1$ such that $(h_i h_j)^{N_j}(h_j a_i - h_i a_j) = 0$.
   From claim 2, I is finite, so we can choose N big enough such that N works for all $D(h_i)$.
   Now, we replace h_i by h_i^{N+1} and a_i by $h_i^N a_i$ and we get $h_j a_i - h_i a_j = 0 \in A$. □

Corollary 23
Take $X = \operatorname{Spec} A$, then $\forall p \in \operatorname{Spec} A \exists$ isomorphisms $\phi_p : A_p \to \mathcal{O}_{X,p}$ such that the appropriate diagram commutes.

Proof

1. *Observe $\lim_{a \in A \setminus p} = A_a \simeq A_p$ (check universal property)*
2. *Observe that $\lim_{p \in D(a)} \mathcal{O}_X(D(a)) \simeq \mathcal{O}_{X,p}$*

Lecture 3: Kernels/cokernels of sheaves

Mon 17 Oct

1.4 Kernels, cokernels, exactness

In this chapter, every (pre)-sheaf is a (pre)sheaf of Abelian groups.

Definition 11 (Subsheaf)
Let $\mathcal{F}$ be a (pre)sheaf on X.
Then a sub(pre)sheaf of $\mathcal{F}$ is a (pre)sheaf $\mathcal{G}$ such that $\mathcal{G}(U) \subset \mathcal{F}(U)$ for every open and the restriction maps are induced by $\mathcal{F}$.

Definition 12 (Kernel, cokernel of presheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *The presheaf kernel of ϕ is the presheaf $\ker^{pre}(\phi)$ defined by $\ker^{pre}(\phi)(U) = \ker(\phi(U))$*
2. *The presheaf image is defined as $\operatorname{Im}^{pre}(\phi)(U) = \operatorname{Im}(\phi(U))$*
3. *The presheaf cokernel is $\operatorname{coker}^{pre}(\phi)(U) = \operatorname{coker}(\phi(U))$.*

In each case, the restriction maps are induced by those in of $\mathcal{F}$ or $\mathcal{G}$.

Lemma 24
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the presheaf kernel is a sheaf.

Proof
Let $U \subset X$ open and $U = \bigcup U_i$ an open cover, $s_i \in \ker^{pre}(\phi)(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
Since $\mathcal{F}$ is a sheaf, $\exists s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.
Since $\ker^{pre}(\phi)(U_i) = \ker(\phi(U_i))$, we have $\phi(U_i)(s_i) = 0$.
Thus

$$\phi(U)(s)|_{U_i} = \phi(U_i)(s|_{U_i}) = 0$$

Since $\mathcal{G}$ is a sheaf, $\phi(U)(s) = 0 \implies s \in \ker^{pre}(\phi)(U)$. □

Example
By an exercise, the image presheaf and cokernel presheaf are, in general, no sheaves, even if $\mathcal{F}$ and $\mathcal{G}$ are.

Definition 13 (Cokernel/image of morphisms of sheaves)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. *sheaf kernel : $\ker^{pre}(\phi)$*

2. *sheaf image* $(\mathrm{Im}^{pre}(\phi))^+$
3. *sheaf cokernel* $(\mathrm{coker}^{pre}(\phi))^+$

Lemma 26 (cokernels are cokernels)
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves

1. $\ker \phi \to \mathcal{F}$ *is a categorical kernel in* $Sh(X)$
2. $\mathcal{G} \to \operatorname{coker} \phi$ *is a categorical cokernel in* $Sh(X)$.

Proof

1. *This means that for each commutative diagram with solid arrows, the dotted arrow is unique*
   "Insert cokernel/kernel diagram here"
   This holds for every open U and so the kernel is a sheaf.
2. *The appropriate diagram commutes and we use the universal property of sheafification.* □

Proposition 27
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is a monomorphism in* $Sh(X)$
2. $\ker(\phi) = 0$
3. $\ker(\phi(U)) = 0$
4. $\ker(\phi_x) = 0$

Proof
Recall ϕ is a monomorphism if for every $\psi : \mathcal{F}' \to \mathcal{F}, \phi \circ \psi = 0 \implies \psi = 0$.
The implication $1 \implies 2$ *follows by applying the monomorphism property to* $\ker \phi \to \mathcal{F}$ $2 \implies 1$ *If $\phi \circ \psi = 0$, then ψ factors through the kernel* $\ker \phi \to \mathcal{F}$ *and so* $\psi = 0$
$2 \iff 3$ *Since* $\ker(\phi)(U) = \ker(\phi(U))$
$3 \implies 4$ *Follows because taking direct limits is exact.*
$4 \implies 3$ *Let $s \in \mathcal{F}(U)$ with $\phi(U)(s) = 0$, then $\phi_x(s_x) = (\phi(U)(s))_x = 0$.*
So $s_x = 0 \forall x \in U$ and so $s = 0$ □

Proposition 28
Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups, then the following are equivalent

1. ϕ *is an epimorphism in* $Sh(X)$
2. $\operatorname{coker}(\phi) = 0$
3. $\operatorname{coker}(\phi_x) = 0$

Proof
Recall that ϕ is an epimorphism if for every $\psi : \mathcal{G} \to \mathcal{G}', \psi \circ \phi = 0 \implies \psi = 0$
$1 \implies 2$ *Apply epimorphism property to* $\mathcal{G} \to \operatorname{coker}(\phi)$
$2 \implies 3$ *We have*

$$\begin{aligned} 0 &= (\operatorname{coker} \phi)_x \\ &= (\operatorname{coker}^{pre} \phi)_x = \operatorname{coker}(\phi_x) \end{aligned}$$

□

$3 \implies 1$
Let $\psi : \mathcal{G} \to \mathcal{G}'$ *such that* $\psi \circ \phi = 0$*, this implies that* $0 = (\psi \circ \phi)_x = \psi_x \circ \phi_x$*.*
Since ϕ_x *is an epimorphism of abelian groups, we get* $\psi_x = 0$*.*
As the hom sheaf is a sheaf, we get that $\psi = 0$

Remark
If $\operatorname{coker}(\phi(U)) = 0 \forall U \subset X \implies \operatorname{coker}(\phi) = 0$ *but the converse is not true.*

Corollary 30
If $\phi : \mathcal{F} \to \mathcal{G}$ *is a morphism of sheaves, then the following are equivalent*

1. ϕ *is an isomorphism*
2. $\phi(U)$ *is an isomorphism* $\forall U \subset X$ *open*
3. ϕ_x *is an isomorphism* $\forall x \in X$

Proof
$1 \implies 2$ *since taking sections is a functor*
$2 \implies 3$ *since taking limits is functorial*
$2 \implies 1$ *because* $(\phi(U))^{-1}$ *defines a morphism of sheaves*
$3 \implies 2$ *Need to show surjectivity of* $\phi(U)$*.*
Let $t \in \mathcal{G}(U)$*, since* ϕ_x *is an isomorphism* $\forall x \in U$*, we find* $s_x \in \mathcal{F}_x$ *such that* $\phi_x(s_x) = t_x$*.*
There exists an open neighbourhood and $s_{V_x} \subset \mathcal{F}(V_x)$ *such that* $(s_{V_x})_x = s_x$
Since

$$(\phi(V_x)(s_{V_x}))_x = t_x$$

we can choose $V + x$ *sufficiently small such that* $\phi(V_x)(s_{V_x}) = t|_{V_x}$*.*
Since $\phi(V_x \cap V_y)$ *is injective and* $\phi(V_x \cap V_y)(s_{V_x}|_{V_x \cap V_y}) = t|_{V_x \cap V_y} = \phi(V_x \cap V_y)(s_{V_y}|_{V_x \cap V_y})$*, we have* $s_{V_x}|_{V_x \cap V_y} = s_{V_y}|_{V_x \cap V_y}$*.*
Thus there exists $s \in \mathcal{F}(U)$ *such that* $s|_{V_x} = s_{V_x}$ *and* $\phi(U)(s)|_{V_x} = t|_{V_x}$ *and thus* $\phi(U)(s) = t$*.* □

Definition 14 (Exact Sequence of sheaves)
A sequence of sheaves $\mathcal{F}_1 \xrightarrow{\phi_1} \mathcal{F}_2 \xrightarrow{\phi_2} \mathcal{F}_3$ *is called exact if* $\ker \phi_2 = \operatorname{Im} \phi_1$

Corollary 31
A sequence of sheaves is exact iff the associated sequence on stalks is exact for all points.

Lecture 4: locally ringed spaces, (affine) Schemes (!)

Fri 21 Oct

Corollary 32
A sequence of sheaves is exact if and only if it is exact on all stalks.

Proof
If $\ker(\phi_{2,x}) = \operatorname{Im}(\phi_{1,x}) \forall x \in X$*, thus* $(\phi_{2,x} \circ \phi_{1,x}) = (\phi_2 \circ \phi_1)_x$.
Thus $\phi_2 \circ \phi_1 = 0$ *because the hom sheaf is a sheaf.*
Thus ϕ_1 *factors as* $\mathcal{F}_1 \to \operatorname{Im}\phi_1 \to \ker\phi_2 \to \mathcal{F}_2$ *as* ψ_x *is an isomorphism,* ψ *is an isomorphism.* □

Corollary 33
Let $\phi : \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves, then* $\operatorname{Im}\phi = \ker(\mathcal{G} to \operatorname{coker}\phi)$

Corollary 34
$Sh(X)$ *is an abelian category.*

1.5 Direct and inverse image, ringed spaces

Definition 15
Let $f : X \to Y$ *be a continuous map.*
We define the direct image of $\mathcal{F}$ *by* f *on* Y *defined by*

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

We can check that $f_*\mathcal{F}$ is a sheaf with restriction maps induced by $\mathcal{F}$.
If $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on X, then the $(f_*\phi)(X) = \phi(f^{-1}(V))\mathcal{F}(f^{-1}(V)) \to \mathcal{G}(f^{-1}V)$ define a morphism of sheaves.
Thus we get a functor $f_* : Sh(X) \to Sh(Y)$.

Definition 16 (inverse image)
Let $f : X \to Y$ *be a continuous map and let* $\mathcal{G}$ *be a sheaf on* Y*.*

The inverse image of $\mathcal{G}$ along f is the sheafification of the presheaf

$$f^{-1,pre}(\mathcal{G})$$

defined by

$$f^{-1,pre}(\mathcal{G})(U) = \varinjlim_{f(U)\subset V} \mathcal{G}(V)$$

We can again check that the if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves on Y, we define $f^{-1}\phi : \varinjlim \mathcal{F}(V) \to \varinjlim \mathcal{G}(V)$ using the maps induced by ϕ.
Thus we get a functor $Sh(Y) \to Sh(X)$.

Lemma 35
Let $f : X \to Y$ be a continuous map, $\mathcal{F}$ a sheaf on X and $\mathcal{G}$ a sheaf on Y.

1. *$\forall y \in Y$ there is a natural isomorphism*

$$(f_*\mathcal{F})_y \simeq \varinjlim_{y\in V\subset Y} \mathcal{F}(f^{-1}(V))$$

In particular $forall x \in X$ there is a natural map $(f_*\mathcal{F})_{f(x)\to\mathcal{F}_x}$

2. *$\forall x \in X$ there is a natural isomorphism $(f^{-1}\mathcal{G})_x \simeq \mathcal{G}_{f(x)}$*

Proof
The isomorphisms are immediate from the definition.
The morphism $(f_*\mathcal{F})_{f(x)} \to \mathcal{F}_x$ is given by

$$(f_*\mathcal{F})_{f(x)} = \varinjlim \mathcal{F}(f^{-1}(V)) = \varinjlim_{x\in f^{-1}(V)} \mathcal{F}(f^{-1}(V)) \to \varinjlim_{x\in U} \mathcal{F}(U) = \mathcal{F}_x$$

□

Proposition 36
If $f : X \to Y$ is a continuous map, then $f_* : Sh(X) \to Sh(Y)$ is right-adjoint to $f^{-1} : Sh(Y) \to Sh(X)$

Corollary 37
$f^{-1} : Sh(Y) \to Sh(X)$ is exact

Proof
Let $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ be exact in $Sh(Y)$.
Thus $\forall y \in Y, 0 \to \mathcal{G}_{1,y} \to \mathcal{G}_{2,y} \to \mathcal{G}_{3,y} \to 0$ is exact.
In particular it is exact at $f(x)\forall x \in X$ and thus the associated inverse image

sequence is exact. □

Corollary 38
$f_* : Sh(X) \to Sh(Y)$ *is left-exact.*

Proof
Let $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *be exact in* $Sh(X)$.
Recall that the section functor is left-exact, thus $0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U)$ *is exact* $\forall U \subset X$.
Thus $0 \to (f_*\mathcal{F}_1)_y \to (f_*\mathcal{F}_2)_y \to (f_*\mathcal{F}_3)_y$ *is exact* $\forall y \in Y$ *and thus* $0 \to f_*\mathcal{F}_1 \to f_*\mathcal{F}_2 \to f_*\mathcal{F}_3$ *is exact.* □

Example
f_* *is usually not right-exact.*
Eg, if $f : X \to \{*\}$ *and* $\mathcal{F}$ *is a sheaf on* X*, then* $(f_*\mathcal{F})(\emptyset) = 0$ *and* $(f_*\mathcal{F})(\{*\}) = \mathcal{F}(X)$ *and taking sections is not exact.*

Definition 17 (Ringed space)
A ringed space is a pair $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of rings on* X.
A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a pair* $(f, f^\sharp)$ *where* $f : X \to Y$ *is a continous map and* $f^\sharp$ *is a morphism* $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Remark
Ringed spaces form a category, if $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y), (g, g^\sharp) : (Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ *define their composition to be* $(g \circ f, g_*(f^\sharp \circ g^\sharp))$

Example
1. *For every ring* A, $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *is a ringed space.*
2. *For any field* K *and any topological space* X*, define a sheaf* $Fun_{X,K}(U) = \{s : U \to K\}$.
   There is a functor $\top \to$ (*Ringed spaces*) *sending* $X \mapsto (X, Fun_{X,K})$ *where for* $f : X \to Y$ $f^\sharp$ *is the pullback (precomposition).*
3. (X, C^0_X) *is a ringed space*

Observe that for many of these examples of ringed spaces, the stalks $\mathcal{O}_{X,x}$ are local rings.

Definition 18 (Morphism of local rings)
A morphism of local rings $\phi : A \to B$ *with maximal ideals* m_A *and* m_B *is called local if* $m_A = \phi^{-1}(m_B)$

Example

1. *For all ring homomorphism* $\phi : A \to B$ *and all* $q \in \operatorname{Spec} B$ *the induced map* $A_{\phi^{-1}(q)} \to B_q$ *is local.*
2. *A ring homomorphism* $\phi : A \to K$ *from a local ring* A *to a field iff* $m_A = \ker \phi$

Definition 19 (Locally ringed space)
A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ *such that* $\mathcal{O}_{X,x}$ *is local* $\forall x \in X$.
A morphism of locally ringed spaces $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *is a morphism of ringed spaces such that*

$$f_x^\sharp : \mathcal{O}_{Y,f(x)} \xrightarrow{f_x^\sharp} (f_*\mathcal{O}_X)_{f(x)} \to \mathcal{O}_{X,x}$$

is local.

Remark
The category of locally ringed spaces is a subcategory of the category of ringed spaces

Definition 20 (Affine Scheme)
An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ *such that* $X = \operatorname{Spec} A$ *and* $\mathcal{O}_X$ *is the structure sheaf.*

Definition 21 (Scheme)
A scheme is a locally ringed space $(X, \mathcal{O}_X)$ *such that there exists an open cover* $X = \bigcup_{i \in I} U_i$ *such that each* $(U_i, \mathcal{O}_X|_{U_i})$ *is an affine scheme.*
A morphism of schemes is a morphism of the underlying ringed spaces.

Example

1. *If* $(X, \mathcal{O}_X)$ *is a scheme and* $U \subset X$ *is open, then* $(U, \mathcal{O}_X|_U)$ *is not necessarily a scheme (even if* X *is affine).*
2. *If* $(X, \mathcal{O}_X)$ *is a scheme and* $X = \{*\}$*, then* X *is affine.*
   Then $\operatorname{Spec} A = \{*\}$ *iff every* $a \in A$ *is either a unit or nilpotent.*

Lecture 5: Schemes

Mon 24 Oct

Remark
By abuse of notation, we write X *is a scheme with* $\mathcal{O}_X$ *implicit.*

Lemma 46
Let X be a topological space with basiis for the topology $\{v_i\}_{i \in I}$.
Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X.
For any collection of morphisms $\phi_i : \mathcal{F}(V_i) \to \mathcal{G}(V_i)$ such that $\rho_{ij} \circ \phi_i = \phi_j$, then $\exists ! \phi : \mathcal{F} \to \mathcal{G}$ which restricts to ϕ_i on the V_i.

Proposition 47
Let $(X, \mathcal{O}_X)$ be a locally ringed space and A a ring, then the map $\hom((X, \mathcal{O}_X), (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})) \to \hom(A, \mathcal{O}_X(X))$ which maps $(f, f^\sharp) \to f^\sharp(\operatorname{Spec} A)$ is a natural bijection.

In particular, for all locally ringed spaces $(X, \mathcal{O}_X)$, there is a natural affinization morphism $aff_X : X \to \operatorname{Spec} \mathcal{O}_X(X)$

Corollary 48
Every morphism of locally ringed spaces $(X, \mathcal{O}_X) \to \operatorname{Spec} A$ factors uniquely through aff_X.

Corollary 49
A locally ringed space is an affine scheme iff the affinization is an isomorphism.

Corollary 50
The functor
$$(affSch) \to (Ring)^{op}$$
mapping $(X, \mathcal{O}_X) \to \mathcal{O}_X(X)$ is an equivalence of categories.

Proof
Fully faithful is the proposition above.
Essential surjectiveness is immediate as for any ring, we can look at $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ as $\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A$. $\square$

We now prove the statement

Proof
We use that there exists a natural isomorphism $\mathcal{O}_{\operatorname{Spec} A}(D(a)) \simeq A_a$.
Naturality follows from functoriality of $f^\sharp(-)$.
We have to construct an inverse, let $\phi : A \to \mathcal{O}_X(X)$ be a ring homomorphism, we need to define $(f, f^\sharp) : (X, \mathcal{O}_X) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$.
We map $x \mapsto \ker(A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_{X,x}/m_x)$.
We claim that f is continuous.

It suffices to show that $X_{\phi(a)} = f^{-1}(D(a)) = \{x \in X | \phi(a)_x \notin m_x\} \subset X$ *is open.*
Take $x \in X_{\phi(a)}$*, then* $\phi(a)_x \notin m_x \implies \phi(a)_x \in \mathcal{O}_{X,x}^\times$.
Thus $\exists x \in V \subset X$ *and* $b \in \mathcal{O}_X(V)$ *such that* $\phi(a)|_V b = 1 \in \mathcal{O}_X(V)$.
Thus $\phi(a)_y b_y = 1 \forall y \in V \implies \phi(A)_y \notin m_y \implies V \subset X_{\phi(a)} \implies X_{\phi(a)}$ *is open.*
To define $f^\sharp$*, observe that* $\forall a \in A, \phi(a)|_{X_{\phi(a)}} \in \mathcal{O}_X(X_{\phi(a)})$ *is a unit in every stalk, hence a unit.*
Thus there is a unique morphism such that $A \xrightarrow{\phi} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{\phi(a)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))} \mathcal{O}_X(X_{\phi(a)})$ *so we get a morphism* $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_X$.
We still have to show that this map is a morphism of locally ringed spaces.
We claim that $\forall x \in X$*, the map* $f_x^\sharp : A_{f(x)} \to \mathcal{O}_{X,x}$ *is a local homomorphism.*
The diagram induces a commutative diagram

$$A \xrightarrow{\phi} \to \mathcal{O}_X(X) \xrightarrow{\pi_2} \mathcal{O}_{X,x} = A \xrightarrow{\pi_1} A_{f(x)} \xrightarrow{f_x^\sharp} \mathcal{O}_{X,x}$$

Note that $p_1^{-1}(f_x^{\sharp,-1}(m_x)) = \pi_1^{-1} \circ \pi_2^{-1}(m_x) = f(x)$ *by definition.*
Thus $f_x^{\sharp,-1}(m_x) = f(x) A_{f(x)}$.
Now, we need to show that this construction is in fact an inverse.
By construction, if $(f, f^\sharp)$ *comes from* ϕ*, then* $\phi = f^\sharp(\operatorname{Spec} A)$.
Conversely, let $(f, f^\sharp) : X \to \operatorname{Spec} A$ *be a morphism and let* $(f', f'^\sharp) : X \to \operatorname{Spec} A$ *be associated to* $f^\sharp(\operatorname{Spec} A)$.
We need to show that $(f, f^\sharp) = (f', f'^\sharp)$.
$\forall x \in X, \exists$ *a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_{X,x} = A \to A_{f(x)} \to O_{X,x}$$

As $f_x^\sharp$ *and* $f_x'^\sharp$ *are local,* $f(x) = f'(x)$*. Now,* $\forall a \in A$*, there is a commutative diagram*

$$A \xrightarrow{f^\sharp(\operatorname{Spec} A)} \mathcal{O}_X(X) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A)}) = A \to A_a \xrightarrow{\exists! f^\sharp(D(a))} = f'^\sharp(D(a)) \to \mathcal{O}_X(X_{f^\sharp(\operatorname{Spec} A),a})$$

$\square$

Example
For every locally ringed space $(X, \mathcal{O}_X)$*, there exists a unique morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$ *because* $\exists! \mathbb{Z} \to \mathcal{O}_X(X)$.
If $(X, \mathcal{O}_X)$ *is a locally ringed space such that each* $\mathcal{O}_X(U)$ *has characteristic* $p > 0$*, then* $\exists!$ *morphism* $(X, \mathcal{O}_X) \to (\operatorname{Spec} \mathbb{F}_p, \mathcal{O}_{\operatorname{Spec} \mathbb{F}_p})$.

Definition 22 (Scheme over another scheme)
Let S be a sscheme. The category of schemes over S, Sch/S is the category whose objects are morphisms $X \to S$ and morphisms are commutative triangles.

Example
Let K be a field.
The affine n-space over k is denoted $\mathbb{A}_k^n$ is $\operatorname{Spec} k[x_1, \dots, x_n]$.
If k is algebraically closed, then

$$k^n \simeq \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq \mathbb{A}_k^n \simeq \hom_{k-alg}(k[x_1, \dots, x_n], k)$$

If $\phi : A \to B$ is a surjective ring homomorphism, then the induced map on spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is a homeomorphism onto $V(I)$ where $I = \ker \phi$.
In particular, if $I \subset K[x_1, \dots, x_n]$, $k = \overline{k}$ is an ideal, then $V(I) = \{(a_1, \dots, a_n) \in k^n | f(a_1, \dots, a_n) = 0 \forall f \in I\}$ is the image of $\operatorname{Spec}_{max} {k[x_1, \dots, x_n]}/{I} \to \operatorname{Spec}_{max} k[x_1, \dots, x_n] \simeq k^n$.

Example (glueing two schemes)
If X_1, X_2 are two schemes and $U_i \subset X_i$ are open subsets,

$$(\phi, \phi^\sharp) : (U_1, \mathcal{O}_X|_{U_i}) \simeq (U_2, \mathcal{O}_X|_{U_2})$$

is an isomorphism.
We define the scheme $(X, \mathcal{O}_X)$ by glueing X_1 and X_2 over U_1 as follows.
As a set, $X = {X_1 \coprod X_2}/{\sim}$ where $x_1 \sim \phi(x_1)$.
Note, there are natural maps $\pi_i : X_i \to X$.
We say that a subset $U \subset X$ is open $\iff$ $\pi_i^{-1}(U) \subset X_i$ open for $i = 1, 2$.
We define the structure sheaf as $\mathcal{O}_X(U) = \ker(\mathcal{O}_{X_1}(\pi_1^{-1}(U)) \oplus \mathcal{O}_{X_2}(\pi_2^{-1}(U)) \to \mathcal{O}_{X_1}(\pi_1^{-1}(U) \cap U_1))$.
Then X is a scheme.

Example (Explicit example of glueing)
Take $X_1 = X_2 = \mathbb{A}_K^1$ and $U_1 = U_2 = \mathbb{A}_K^1 \setminus 0$.
Notice that $U \simeq \operatorname{Spec} k[x, x^{-1}]$.

1. *Taking the glueing map $\phi = \mathrm{Id}$, we get a line with two origins.*
2. *Taking $\phi^\sharp(U_2) : x \mapsto \frac{1}{x}$, we get the projective line $\mathbb{P}_k^1$.*
   The k-rational points of this scheme are in correspondence with lines in k^2, namely
   $$P_k^1(k) \simeq {k^2 \setminus \{0\}}/{k^\times}.$$

2 Properties of schemes

2.1 Topological properties

Definition 23
A scheme $(X, \mathcal{O}_X)$ is called

1. *connected if X is*
2. *irreducible if $\forall U_1, U_2$ open non empty their intersection is non-empty.*
3. *quasi-compact if X is.* [a]
4. *quasi-separated if X is, ie. $\forall U_1, U_2$ open and quasi-compact, their intersection is quasi-compact.*

---

a. All affine schemes are quasi-compact, but $\mathbb{A}_k^\infty \setminus 0$ is not quasi-compact

Lecture 6: Topological properties

Fri 28 Oct

Remark
$\operatorname{Spec} R \times S = \operatorname{Spec} R \coprod \operatorname{Spec} S$ *but* $\operatorname{Spec} \prod_i R_i \not\simeq \coprod_i \operatorname{Spec} R_i$ *for infinite products*

Lemma 56
Affine schemes are quasi-compact and quasi-separated.

Proof
Let $X = \operatorname{Spec} A$ be an affine scheme.
Quasi-compactness has already been proven.
If $U \subset X$ is open and qc., then $U = \bigcup_{i \in I_U} D(a_i), a_i \in A$ and I_U finite.
For $U_1, U_2 \subset X$ qc. open, then

$$U_1 \cap U_2 = \bigcup_{i \in I_{U_1}, j \in I_{U_2}} D(a_1) \cap D(a_2) = \bigcup D(a_1 a_2)$$

Check that a finite union of qc spaces is qc □

Remark
Let X be a topological space, then $\forall$ subsets $V \subset X$ and $U \subset X$, then

$$U \cap V \neq \emptyset \iff U \cap \overline{V} \neq \emptyset$$

Thus V is irreducible iff it's closure is.

If X is irreducible, then every non-empty open is dense.

2.2 Scheme-Theoretic Properties

Definition 24 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ is a pair $(U, \mathcal{O}_U)$ with U open in X and $\mathcal{O}_U := \mathcal{O}_X|_U$

If P is a property of rings, when do we say that $(X, \mathcal{O}_X)$ satisfies P?

1. $\forall U \subset X, \mathcal{O}_X(U)$ satisfies P (usually too strong)
2. $\forall U \subset X$ open and affine, $\mathcal{O}_X(U)$ satisfies P
3. $\exists$ an open affine cover $U = \bigcup U_i$ such that each $\mathcal{O}_X(U_i)$ satisfies P
4. $\forall x \in X \exists x \in U$ open affine such that $\mathcal{O}_X(U)$ satisfies P.
5. $\forall x \in X, \mathcal{O}_{X,x}$ satisfies P.

Observe that $1 \implies 2 \implies 3 \iff 4$.

Lemma 58
For $P =$"reduced ring", then all 5 are equivalent.

Proof
From commutative algebra, we know that a ring A is reduced $\iff$ A_p is reduced $\forall p \in \operatorname{Spec} A$.
This implies that $2 \iff 3 \iff 4 \iff 5$.
Let's show $2 \implies 1$.
Let $U \subset X$ open and $s \in \mathcal{O}_X(U)$ such that $s^n = 0$, then $s^n|_V = 0 \forall V \subset U$ affine.
Thus, $s|_V = 0 \forall V \subset U$ open affine and as $\mathcal{O}_X$ is a sheaf $s = 0$. $\square$

Definition 25 (Reduced Scheme)
A scheme $(X, \mathcal{O}_X)$ is called reduced if $\mathcal{O}_X(U)$ is reduced $\forall U \subset X$ open.

Definition 26
Let P be a property of rings or of open affines $\operatorname{Spec} A \hookrightarrow X$ of a scheme X

- *P is called affine-local if $\forall a_1, \ldots, a_n \in A$ such $(a_1, \ldots, a_n) = A$. A satisfies P every A_{a_i} satisfies P*
- *P is called stalk-local if A satisfies $P \iff A_p$ satisfies $P \forall p \in \operatorname{Spec} A$.*

Remark

Being stalk-local is stronger than being affine local.
This is becauses $A \to A_a$ *induces* $(A_a)_{pA_a} \simeq A_p \forall p \in D(a)$

Example

1. *Reduced is stalk-local*
2. *Normal*
3. *regular*
4. *Cohen-Macaulay*

Example

1. *Integrality is not affine-local (consider* $A = k \times k$ *)*
2. *Factorial is not affine-local*
3. *Noetherian is not stalk-local (consider* $A = \prod_i \mathbb{F}_2$ *)*

Lemma 62
Being Noetherian is affine-local.

Why do we care ?

For affine-local properties, 2 and 4 of our list are equivalent.

Proof
If A *is noetherian, then any quotient and any localization is.*
Assume $(a_1, \dots, a_n) = A$ *and* A_{a_i} *are Noetherian.*
Let $\phi_i : A \to A_{a_i}$ *be the localization maps.*
Claim : $\forall$ *ideals* $I \subset A$, $I = \cap \phi_i^{-1}(\phi_i(I) A_{a_i})$.
One inclusion is clear.
Let $b \in \cap \phi_i^{-1}(\phi_i(I) A_{a_i})$, *thus there exists* $N > 0$ *and* $b_i \in I$ *such that* $b = \frac{b_i}{a_i^N} \in A_{a_i}$.
Thus there exists an $M > 0$ *such that* $a_i^M(a_i^N b - b_i) = 0$ *in* A.
Set $k = M + N$, *note that* $1 = (a_1^k, \dots, a_n^k)$.
We can write $1 = \sum_{i=1}^n c_i a_i^k$ *for some* $c_i \in A$.
Thus $b = \sum c_i a_i^k b = \sum c_i a_i^M b_i \in I$.
Let $I_1 \subset \dots \subset I_n \subset$ *be an ascending chain of ideals in* A, *then we get an ascending chain of ideals* $\phi_1(I_1) A_{a_i} \subset \dots \subset \phi_i(I_n) A_{a_i}$.
This becomes constant because A_{a_i} *is noetherian and* $\exists N > 0$ *such that* $\phi_i(I_k) A_{a_i} = \phi_i(I_N) A_{a_i} \forall k g g N$ $\square$

Lemma 63
Let P *be an affine-local property of rings. Let* $(X, \mathcal{O}_X)$ *be a scheme, then the following are equivalent.*

1. *Every open affine* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P
2. $\exists$ *an open affine cover* $X = \cup \operatorname{Spec} A_i$ *such that each* $\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P.

Proof

$1 \implies 2$ *is clear.*

$2 \implies 1$.

Let $\operatorname{Spec} A \hookrightarrow X$ *open and affine.*

Write $\operatorname{Spec} A = \cup \operatorname{Spec} A_{a_i}$ *with* $a_i \in A$ *such that* $A_{a_i} \simeq (A_i)_{b_i}$ *for some* $b_i \in A_i$.

$\operatorname{Spec} A_i \hookrightarrow X$ *satisfies* P, *implies* $(\operatorname{Spec}(A_i)_{b_i}) \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A_{a_i} \hookrightarrow X$ *satisfies* P *implies* $\operatorname{Spec} A \hookrightarrow X$ *satisfies* P $\square$

Lemma 64

Let $\operatorname{Spec} A, \operatorname{Spec} B \subset X$ *be open affines, then for every point* $x \in \operatorname{Spec} A \cap \operatorname{Spec} B$ *there exist* $a \in A$ *and* $b \in B$ *such that* $A_a \simeq B_b$ *such that* $x \in D(a) \subset \operatorname{Spec} A$ *and* $x \in D(b) \subset \operatorname{Spec} B$ *and the isomorphism* $\operatorname{Spec} A_a \simeq \operatorname{Spec} B_b$ *commutes with the inclusions to* X.

Proof

$\operatorname{Spec} A \cap \operatorname{Spec} B \subset \operatorname{Spec} A$ *is open.*

Thus, there exists $a \in A$ *with* $x \in D(a) \subset \operatorname{Spec} A \cap \operatorname{Spec} B$.

We can assume wlog that $\operatorname{Spec} A \to X$ *factors throguh* $\operatorname{Spec} B$.

Write $\phi : B \to A$ *for the induced map of rings.*

Since $\operatorname{Spec} A \subset \operatorname{Spec} B$ *is open* $\exists b \in B$ *and* $B \to A \to B_b$ *is just localization of* B *at* b.

THen $A \to B_b$ *satisfies the universal property of* $A \to A_{\phi(b)}$.

So we get a commutative square $B \to A \to A_{\phi(b)}$ *and* $B \to B_b \to A_{\phi(b)}$ *and we get an isomorphism* $B_b \simeq A_{\phi(b)}$. $\square$

Definition 27

Let P *be an affine-local property of rings.*

A scheme $(X, \mathcal{O}_X)$ *is called locally* P *if* $\mathcal{O}_X(U)$ *satisfies* $P \forall U \subset X$ *open affine.*

Definition 28 (Noetherian scheme)

A scheme $(X, \mathcal{O}_X)$ *is called Noetherian if it is locally Noetherian and qc.*

Definition 29 (Integral scheme)
A scheme $(X, \mathcal{O}_X)$ is called integral if $\mathcal{O}_X(U)$ is an integral domain $\forall U \subset X$ open and non-empty.

Lemma 65
For a scheme $(X, \mathcal{O}_X)$, the following are equivalent.

1. *X is integral*
2. *X is reduced and irreducible.*
3. *$\forall U \subset X$ open affine, $\mathcal{O}_X(U)$ is integral.*

Proof
$1 \implies 3$ *is clear.*
$3 \implies 2$.
Reduced is clear.
Let $U_1, U_2 \subset X$ open with $U_1 \cap U_2 = \emptyset$.
Wlog, the U_i are affine.
Then $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$.
Thus $\mathcal{O}_X(U_1) = 0$ or $\mathcal{O}_X(U_2) = 0$ which implies U_1 or $U_2 = \emptyset$.
$2 \implies 1$
Let $U \subset X$ be open.
Assume $\exists a, b \in \mathcal{O}_X(U)$ such that $ab = 0$.
Let $U_a = \{x \in U | a_x \notin m_x\}$ and similarly U_b.
Note that $U_a \cap U_b = \emptyset$ since $\forall x \in U_a \cap U_b, a_x$ and b_x are units.
Thus $U_a = \emptyset$ or $U_b = \emptyset$.
If $U_a = \emptyset \forall \operatorname{Spec} A \subset U \forall p \in \operatorname{Spec} A$

$$(a|_{\operatorname{Spec} A})_p \in pA_p$$

thus $a|_{\operatorname{Spec} A} \in p \forall p \in \operatorname{Spec} A$.
Thus $a|_{\operatorname{Spec} A}$ is nilpotent.
But since X is reduced, $a|_{\operatorname{Spec} A} = 0$.
Covering U by affines, $a = 0$ (as A was arbitrary). ∎

Lecture 7: Open and Closed Subschemes

Mon 31 Oct

3 Open and closed subschemes and immersions

Definition 30 (Open Subscheme)
An open subscheme of a scheme $(X, \mathcal{O}_X)$ is a pair $(U, \mathcal{O}_U)$, with $U \subset X$ open and $\mathcal{O}_U = \mathcal{O}_X|_U$.

Lemma 66
If A is a ring and $a \in A$, then there is an isomorphism of locally ringed spaces $(\operatorname{Spec} A_a, \mathcal{O}_{\operatorname{Spec} A_a}) \simeq (D(a), \mathcal{O}_{\operatorname{Spec} A}|_{D(a)})$.
In particular, open subschemes of schemes are schemes.

Proof
From commutative algebra, localization $A \to A_a$ induces a homeomorphism $\operatorname{Spec} A_a \to D(a) \subset \operatorname{Spec} A$.
On sheaves, we want to give morphisms $\mathcal{O}_{\operatorname{Spec} A}|_{D(a)}(U) \to \mathcal{O}_{\operatorname{Spec} A_a}(f^{-1}(U))$.
If $s : U \to \coprod_{p \in U} A_p \to (f^{-1}(U) \to U \xrightarrow{s} \coprod_{p\in U} A_p \to \coprod_{p \in U}(A_a)_{pA_a})$, using $A_p \simeq (A_a)_{pA_a}$. □

Note that, if $i : U \to X$ is the inclusion of an open, then $(i, i^\sharp) : (U, \mathcal{O}_U) \to (X, \mathcal{O}_X)$ with

$$i^\sharp(V) : \mathcal{O}_X(V) \xrightarrow{\rho_{V, V\cap U}} \mathcal{O}_X(V \cap U) = i_*\mathcal{O}_U(V)$$

is a morphism of schemes.

Remark
If $i : U \to X$ is an inclusion of an open, then there are in general many sheaves of rings $\mathcal{F}$ on U such that $\exists i^\sharp$ such that $(i, i^\sharp) : (U, \mathcal{F}) \to (X, \mathcal{O}_X)$ is a morphism of schemes.

FOr example, if $X = \operatorname{Spec} k$, $U = \operatorname{Spec} {}^{k[x]}\!/_{x^2}$then $k \subset {}^{k[x]}\!/_{x^2}$ induces a morphism $(f, f^\sharp) : U \to X$ such that $f = \mathrm{Id}_X$.

Definition 31 (Open immersion)
An open immersion (or open embedding) is a morphism of schemes $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ such that f is a homeomorphism onto an open subset $U \subset Y$ $\mathcal{O}_Y|U \simeq (f_*\mathcal{O}_X)|_U$.

Example
Let k be a fieldand let $\iota : \operatorname{Spec} k \to X = \mathbb{A}^n$ be the closed point corresponding to

0.
Then

$$\begin{aligned}(\mathcal{O}_X)|_{\operatorname{Spec} k}(\operatorname{Spec} k) &= (i^{-1}\mathcal{O}_X)(\operatorname{Spec} k)\\ &= \varinjlim_{0\in U\subset \mathbb{A}^n} \mathcal{O}_X(U) = \mathcal{O}_{X,0} = k[x_1,\dots,x_n]_{(x_1,\dots,x_n)}\end{aligned}$$

But $\operatorname{Spec} k[x_i]_{(x_1,\dots,x_n)}$ *has more than one point.*
Thus, $(\operatorname{Spec} k, (\mathcal{O}_X)|_{\operatorname{Spec} k})$ *is not a scheme.*

Observe : If $Z \subset \operatorname{Spec} A$ is a closed subset, then $Z = V(I)$ for some ideal I. Then the map $\operatorname{Spec} {}^{A}\!/\!_{I} \to \operatorname{Spec} A$ induced by the quotient map is a homeomorphism onto $V(I)$ and this gives a scheme structure on Z (which depends on I!).

Definition 32 (Ideal sheaves)
Let $(X,\mathcal{O}_X)$ *be a scheme, then*

1. *An ideal sheaf on* $(X,\mathcal{O}_X)$ *is a subsheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that* $\mathcal{I}(U) \subset \mathcal{O}_X(U)$ *is an ideal for all* $U \subset X$ *is open.*
2. *For an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$*, the quotient sheaf* ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}}$ *is the cokernel sheaf of the inclusion, namely, the sheafification of the sheaf* $U \mapsto {}^{\mathcal{O}_X(U)}\!/\!_{\mathcal{I}(U)}$.

Definition 33 (Closed Subsceme)
Let $(X,\mathcal{O}_X)$ *be a scheme, then a closed subscheme of* $(X,\mathcal{O}_X)$ *consists of a subset* $Z \subset X$ *and an ideal sheaf* $\mathcal{I} \subset \mathcal{O}_X$ *such that*

1. $Z = \left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\}$
2. $(Z, ({}^{\mathcal{O}_X}\!/\!_{I})|_Z)$ *is a scheme*

Remark
By 1, Z *is closed, indeed, for* $1 \in ({}^{\mathcal{O}_X}\!/\!_{(X)})$*, we have*

$$\left\{ x \in X | ({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})_x \neq 0 \right\} = \operatorname{Supp} 1$$

Remark
The morphism ${}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} \to i_*(({}^{\mathcal{O}_X}\!/\!_{\mathcal{I}})|_Z)$ *is an isomorphism.*

If $Z \subset X$ is a closed subscheme determined by $\mathcal{I}$, then $(i, i^\sharp) : (Z,\mathcal{O}_Z) \to (X,\mathcal{O}_X)$ where $i : Z \to X$ is the inclusion and $i^\sharp : \mathcal{O}_X \to {}^{\mathcal{O}_X}\!/\!_{\mathcal{I}} = i_*\mathcal{O}_Z$ is a morphism of schemes.

Example
Condition 2 in the definition of closed subscheme is not automatic, even if X is affine.

Definition 34 (Closed immersion)
A closed immersion (or closed embedding) is a morphism of schemes $(f, f^\sharp) : X \to Y$ such that f is a homeomorphism onto a closed subset and $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is surjective on stalks.

Remark
The morphism $(i, i^\sharp)$ of the inclusion of closed subscheme is a closed immersion.

Example
If A is a ring and $I \subset A$ is an ideal, then the morphism $\operatorname{Spec} {}^{A}\!/\!_{I} \to \operatorname{Spec} A$ is a closed immersion.

Indeed, by CA, this is a homeomorphism onto $V(I)$.
The map $f^\sharp : \mathcal{O}_{\operatorname{Spec} A} \to \mathcal{O}_{\operatorname{Spec} A/I}$ is surjective because $f^{\sharp_p} : A_p \to ({}^{A}\!/\!_{I})_{pfaktorAI}$, which is the localization of a surjective map.

From now on, $V(I) \subset \operatorname{Spec} A$ for the closed subscheme determined by I.

Proposition 74
If $X = \operatorname{Spec} A$ is affine, then the map $I \to V(I)$ is a bijection between ideals of A and closed subschemes.

Proof
Let $Z \subset X$ be a closed subscheme determined by $\mathcal{I}$.
Let $I_Z = \ker(\mathcal{O}_X(X) \to \mathcal{O}_Z(Z))$.
Note, if $Z = V(I)$ for some ideal, then $I = I_Z$.
So we have to show that $Z = V(I_Z)$.
The morphism $\phi : \mathcal{O}_X(X) \to \mathcal{O}_Z(Z)$ factors through ${}^{\mathcal{O}_X(X)}\!/\!_{I_Z} = {}^{A}\!/\!_{I_Z}$ so $\iota : Z \to X$ factors through $V(I_Z)$.
Replace A by ${}^{A}\!/\!_{I_Z}$, we may assume ϕ is injective and we have to show that ϕ is an isomorphism.

Claim 1

$\forall U \subset Z$ open affine and $s \in A$, we have $D(s) \cap U = D(\phi(s)|_U) \subset U$ and $V(S) \cap U = V(\phi(s)|_U) \subset U$.
It suffices to prove the first equality.
$\forall p \in Z$, the following diagram commutes

$$A \xrightarrow{\phi} \mathcal{O}_Z(Z) \to \mathcal{O}_{Z,p}$$

and $A \to A_p \xrightarrow{\iota_p^\sharp} \mathcal{O}_{Z,p}$ *and* $i_p^\sharp$ *is local.*
Now

$$D(\phi(s)|_U) = \{p \in U | \phi(s)_p \notin m_p \subset \mathcal{O}_{Z,p}\} = \left\{p \in U | i_p^\sharp(s_p) \notin m_p \subset \mathcal{O}_{Z,p}\right\}$$
$$= \{p \in U | s_p \notin m_p \subset A_p\} = D(s) \cap U \square$$

Claim 2

We show $Z \to X$ *is surjective. Since* Z *is closed in* X*, it suffices to show that* $\forall s \in A$ *such that* $Z \subset V(s)$*, we have* $V(s) = X$.
Choose such an $s \in A$.
As closed subspaces of qc. spaces are qc.
We can cover Z *by finitely many open affines* U_i.
By claim 1, $U_i \subset V(\phi(s)|_{U_i}) \subset U_i$.
Thus $\phi(s)|_{U_i} \in p \forall p \in \operatorname{Spec} \mathcal{O}_Z(U_i)$*, thus* $\phi(s)|_{U_i}$ *is nilpotent.*
Thus, there exists $n_i > 1$ *such that* $(\phi(s)|_{U_i})^{n_i} = 0$.
And there exists N *such that* $\phi(s)^N = \phi(s^N) = 0$ *in* $\mathcal{O}_Z(Z)$.
Thus $s^N = 0$ *as* ϕ *is injective by hypothesis, thus* $V(s) = X$.

Claim 3.

$i^\sharp$ *is an isomorphism.*
Since $Z \to X$ *is a closed subscheme,* $i^\sharp$ *is surjective on stalks and thus surjective.*
To show injectivity, it suffices to show that $\forall a \in A$ *such that* $i_p^\sharp(\frac{a}{1}) = 0 \implies \frac{a}{1} = 0 \in A_p$.
Since $i_p^\sharp(\frac{a}{1}) = 0$, $\exists p \in U \subset Z$ *open affine such that* $\phi(a)|_U = 0$.
Choose a finite open affine cover, $Z = U \cup \bigcup_{i=1}^n U_i$.
Choose $s \in A$ *such that* $p \in D(s) \subset U \subset X$.
Then $\phi(sa)|_U = 0$*, thus* $\phi(sa)|_{D(s)\cap U_i} = 0 = \phi(sa)|_{D(\phi(s)|_{U_i})}$.
Thus, there exists $N > 0$ *such that* $\phi(sa)|_{U_i} \cdot (\phi(s)|_{U_i})^N = 0 \in \mathcal{O}_Z(U_i)$.
Thus $\phi(s^{N+1}a)|_{U_i} = 0$*, thus* $\phi(s^{N+1}a) = 0 \implies s^{N+1}a = 0 \in A$ *which implies* $\frac{a}{1} = 0$ *in* A_p.

Lecture 8: Fiber Products

Fri 04 Nov

Corollary 75
Closed subschemes of affine schemes are affine.
Moreover, if $\phi : A \to B$ *is a morphism of rings, then* ϕ *is surjective iff* $\operatorname{Spec} \phi$ *is a closed immersion.*

Remark

For all closed subsets $Z \subset X$ of a scheme X, there is an ideal sheaf $\mathcal{I}$ on X making Z into a closed subscheme.

To prove this, if $U \subset X$ is affine, then $Z \cap U$ is closed in U hence $Z \cap U = V(I)$ for some ideal in $\mathcal{O}_X(U)$.

Then you take radicals and glue them together on a cover of Z.

This structure is called the reduced induced scheme structure on Z.

3.1 Fiber Products

Definition 35 (Fiber product)

Let C be a category, given two morphism $\pi_X : X \to S$ and $\pi_Y : Y \to S$, the fiber product $X \times_S Y$ of X and Y over S is an object together with morphisms p_x to X and p_y to Y which is universal.

Remark

Alternatives names sometimes are fibre product, fibered produc or pullback.

Fiber products are unique up to unique isomorphism.

If S is terminal in C, then the fiber product is just the product.

Remark

If a square is a fiber product, we call the diagram cartesian.

Lemma 79

Assume all fiber products exist.

Let "commutative thingy".

Then

$$(X_1 \times_{S_1} Y_1) \times_{X_0 \times_{S_0} Y_0} (X_2 \times_{S_2} Y_2) = (X_1 \times_{X_0} X_2) \times_{S_1 \times_{S_0} S_2} (Y_1 \times_{Y_0} Y_2)$$

Corollary 80

- *If C admits fiber products, then $X \times_S Y = Y \times_S X$*
- *A composition of two pullback squares is a pullback*
- *For a zigzag $X \to S, Y \to S, Y \to T, Y \to Z$,*

$$(X \times_S Y) \times_T Z = X \times_S (Y \times_T Z)$$

- *For maps $X \to S \to T$ and $Y \to S$*

$$X \times_S Y \to X \times_T Y \to S \times_T S$$

and

$$X \times_S Y \to S \to S \times_T S$$

is a pullback.

Example

1. *If* $\pi_X : X \to S, \pi_Y : Y \to S$ *are in* (Set), *then* $X \times_S Y = \{(x, y) | \pi_X(x) = \pi_Y(y)\} \subset X \times Y$ *together with the two projections.*
2. *If* X *and* Y *are groups (or rings) and* π_X, π_Y *are homomorphisms as above, then* $X \times_S Y$ *is, as a set, the fiber product of the underlying sets, with the obvious groups (resp. ring) structures.*

Goal for today

Theorem 82 (Fiber products of schemes exist)
Fiber products exist in (Sch) *and also in* (Sch/S)

Why do we care ?

Allows us to talk about fibers, graphs, diagonals...
Recall that every point $y \in Y$ of a scheme Y has a natrual scheme structure given by the residue field $\mathcal{O}_{Y,y}/_{y} = k(y)$

Definition 36 (Fibers)
Let $f : X \to Y$ *be a morphism of schemes over* S.

1. *For any* $y \in Y$, *let* $k(y)$ *be the residue field, then the fiber of* f *over* y
$$f^{-1}(y) = X_y = X \times_Y \operatorname{Spec} k(y)$$
2. *The geometric fiber of* f *over* Y *is*
$$X_{\overline{y}} = X \times_Y \operatorname{Spec} \overline{k(y)}$$
3. *a closed fiber is a fiber over a closed point*
4. *For all integral schemes* Y, *there is a unique point* $\eta \in Y$ *such that* $\overline{\{\eta\}} = Y/$
This is called the generic point of Y.
The fiber over the generic point is called the generic fiber of f.
5. *The morphism*
$$\Gamma_f := (\mathrm{Id}, f) : X \to X \times_S Y$$
is called the graph of f.

6. *The morphism*

$$\Delta_{X/Y} = \Gamma_{\mathrm{Id}_X} : X \to X \times_Y X$$

is called the diagonal of X over Y.

Proposition 83

If $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, S = \operatorname{Spec} C$ and $\pi_X : X \to S, \pi_Y : Y \to S$ are morphisms of schemes then $X \times_S Y$ exists in (Sch) and is given by $\operatorname{Spec}(A \otimes_C B)$ together with the maps induced by the natural maps $A \to A \otimes_C B, B \to A \otimes_C B$

Proposition 84

We use the universal property of $A \otimes_C B$ and the equivalence of categories to show that it is a pullback in the category of affine schemes.
For Z a scheme, there is a map from the affinization of Z to $\operatorname{Spec} B$ and $\operatorname{Spec} A$ which then induce a map $affZ \to \operatorname{Spec} A \otimes_C B$.

Example

1. *If $X = Y = \mathbb{A}^1_k$, the fiber product over $\operatorname{Spec} k$, then $X \times_{\operatorname{Spec} K} Y = \mathbb{A}^2_k$.*
2. *If $X - Y = \operatorname{Spec} \mathbb{C}$ and $S = \operatorname{Spec} \mathbb{R}$, then*

$$X \times_S Y = \operatorname{Spec} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \operatorname{Spec} \mathbb{C}[x] \big/ (x^2+1) = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$$

   Note that $X \times_S Y$ has two points but X, Y, S each have only one.

3. *Take $X = \operatorname{Spec} k[x,y,z]/(z^2 - x), Y = \operatorname{Spec} k[z]$, let $f : X \to Y$ induced by mapping $z \to z$.*
   Let $\lambda \in Y$ be the point corresponding to $(z - \lambda)$, then

$$f^{-1}(\lambda) = \operatorname{Spec} k[x,y,z]/(z^2 - xy) \otimes_{k[z]} k[z]/(z - \lambda) = \operatorname{Spec} k[x,y]/(\lambda^2 - xy)$$

$$= \begin{cases} \mathbb{A}^1 \setminus 0 \text{ if } x = 0 \\ V(xy) \subset \mathbb{A}^2 \text{ if } \lambda = 0 \end{cases}$$

4. *If we take a non-rational point in the above example, say $\lambda = (z^2 + 1)$, then*

$$f^{-1}(y) = \operatorname{Spec} \mathbb{R}[x,y,z]/(-1 - xy, z^2 + 1) = \operatorname{Spec} \mathbb{C}[x,y]/(-1 - xy)$$

We now start the proof that fiber products exist.

Proof

Claim 1

If $X \times_S Y$ exists and $U \subset X$ open, then the open subscheme $p_x^{-1}(U) \subset X \times_S Y$ is a fiber product of U and Y over S.
For Z a scheme and two commuting maps $Z \to X \times_S Y$ and $Z \to U$, there

is a map on the topological level $Z \to p_X^{-1}(U)$ and it also exists on the level of schemes.

Claim 2

If $X = \bigcup U_i$ is an open cover such that $U_i \times_S Y$ exists $\forall i \in I$, then $X \times_S Y$ exists.
We postpone the proof of this until monday.

Now, if S is affine, consider the open affine covers $X = \bigcup_i U_i, Y = \bigcup V_i$, then $U_i \times_S V_j$ exists $\forall i, j$, thus $U_i \times_S Y$ exists $\forall i$ and thus $X \times_S Y$ exists.

If S is not affine, let $S = \bigcup_i W_i$ be an open affine cover.
Set $U_i = \pi_X^{-1}(W_i), V_i = \pi_Y^{-1}(W_i)$.
Now, $U_i \times_{W_i} V_i$ exists and now $U_i \times_{W_i} V_i = U_i \times_S Y$ by one of the identities. □

Lecture 9: fiber products exist

Mon 07 Nov

We finish the proof by showing that if $X = \bigcup_{i \in I} U_i$ is an open cover such that each $U_i \times_S Y$ exists, then $X \times_S Y$ exists.

Proof
We know that $p_{U_i}^{-1}(U_i \cap U_j) \simeq (U_i \cap U_j) \times_S Y \simeq p_{U_j}^{-1}(U_i \cap U_j)$ via unique isomorphisms compatible with the projections.
There is a unique scheme T with maps to Y and X such that $p_X^{-1}(U_i) \simeq U_i \times_S Y$.
We claim that T is $X \times_S Y$.
Let Z be a scheme with morphisms $Z \xrightarrow{f_X} X$ and $Z \xrightarrow{f_y} Y$ which commutes with projections to S.
Let $V_i = f_X^{-1}(U_i)$, we get unique morphisms $V_i \to U_i \times_S Y \to T$ which is unique if $p_X \circ f_i = p_Y \circ f_i$.
By claim 1, f_i and f_j coincide on $U_i \cap V_j$ thus they glue to a unique morphism $f : Z \to T$ □

Corollary 86
Let $\pi_X : X \to S, \pi_Y : Y \to S$ be a diagram of schemes, let $S = \bigcup_i W_i, U_i = \pi_X^{-1}(W_i), V_i = \pi_y^{-1}(W_i)$ and $U_i = \bigcup_j U_{ij}, V_i = \bigcup_j V_{ij}$ be open covers.
Then $X \times_S Y = \bigcup_{i \in I} \bigcup_{j,k} U_{ij} \times_{W_i} V_{ik}$ is an open cover.

Proposition 87
Let $f : X \to Y$ be a morphism of schemes.
Then for every $y \in Y$, the map $g : f^{-1}(y) \to X$ is a homeomorphism onto the

set-theoretic fiber $f_{set}^{-1}(y)$*.*

Proof

Without loss of generality, Y *is affine.*
We can also assume that X *is affine, because if* $X = \bigcup_i U_i$ *is an open cover and each* $g|_{g^{-1}(U_i)} : g^{-1}(U_i) \to U_i$ *is a homeomorphism onto* $f_{set}^{-1}(y) \cap U_i$*, then* g *is a homeomorphism onto* $f_{set}^{-1}(y)$*.*

So let $X = \operatorname{Spec} B, Y = \operatorname{Spec} B, y = p \in \operatorname{Spec} A$*, then we claim that* $B \otimes_A k(y) \simeq {}^{S^{-1}B}\!/_{pS^{-1}B}$*.*
Furthermore, the isomorphism is compatible with the maps from B *and* $k(y) = {}^{A_p}\!/_{pA_p}$ *where* $S = \operatorname{Im}(A \setminus p \to B)$*.*
To prove this, we check that ${}^{S^{-1}B}\!/_{pS^{-1}B}$ *satisfies the universal property of* $B \otimes_A k(y)$*.*
Let C *be a ring with morphisms* ${}^{A_p}\!/_{pA_p} \xrightarrow{f_A} C$ *and* $B \xrightarrow{f_B} C$ *compatible with the morphisms from* A π_A, π_B*.*
Notice that $\pi_A(A \setminus p) \subset ({}^{A_p}\!/_{pA_p})^\times$ *and sends* $\pi_A(p) = 0$*.*
Thus $f_B(S) \subset C^\times, f_B(pB) = 0$*.*
Thus there exists a unique $f : {}^{S^{-1}B}\!/_{pS^{-1}B} \to C$ *such that* $f \circ p_B = f_B$*.*
Thus

$$f \circ \pi_A = f_B \circ \pi_B = f_A \circ \pi_A$$

As π_A *is an epimorphism,* $f \circ p_A = f_A$*.*

We now have to check $\operatorname{Spec} {}^{S^{-1}B}\!/_{pS^{-1}B} \to \operatorname{Spec} B$ *is a homeomorphism onto* $f_{set}^{-1}(y)$*.*

We know it's a homeomorphism onto it's image by general commutative algebra.
THe image is $\{q \in \operatorname{Spec} B | S \cap q = \emptyset, pS^{-1}B \subset qS^{-1}B\}$*.*
But this is just the set-theoretic fiber. $\square$

4 Properties of Morphisms

4.1 Properties of properties of morphisms

Remark
If $f : X \to Y$ *and* $g : Y; \to Y$ *are morphisms of schemes, let* $X_{Y'} = X \times_Y Y'$*. Then we call* $f_{y'} : X_{Y'} \to Y'$ *the base change of* f *along* g*.*

Remark
In the following, whenever we say P *is a property of morphisms of schemes, we assume that* P *is satisfied by isomorphisms.*

Definition 37
Let P be a property of morphisms of schemes, we say that P satisfies

1. *(COMP) : P is stable under composition*
2. *(CANC) : if $g \circ f$ satisfies P, then f does*
3. *(BC) : if it is stable under base change that is $\forall f : X \to Y, g : Y' \to Y$ such that f satisfies P, also $f_{Y'}$ satisfies P.*
4. *(LOCT) : If it local on the target, ie. if $\forall f : X \to Y$ and $\forall$ open covers $Y = \bigcup V_I$ f satisfies $P \iff f_{V_i}$ satisfies $P \forall i \in I$*
5. *(LOCS) : If it is local on the source ie. if $\forall f : X \to Y$ and $\forall$ open covers $X = \bigcup_i U_i$ f satisfies $P \iff f|_{U_i}$ satisfies P.*

Definition 38
Let P be a property of morphisms of schemes, then a morphism $f : X \to Y$ is called universally P if $\forall Y' \to Y, f_{Y'}$ satisfies P.

Lemma 90
Let $f : X \to Y$ be a morphism of schemes over S, then the diagram $X \xrightarrow{\Gamma_f} X \times_S Y$ over $Y \to Y \times_S Y$ is cartesian.

Proof
This is a special case of the "magic square" with the isomorphism $X \to X \times_Y Y$ □

Definition 39
Let P be a property of morphisms of schemes, then we say that $f : X \to Y$ satisfies Δ_P if $\Delta_{X/Y}$ satisfies P.

Lemma 91
The following hold

1. *If P satisfies (BC), then Δ_p satisfies (BC)*
2. *If P satisfies (BC) and (COMP), then Δ_P satisfies (COMP).*
3. *If P satisfies (LOCT), then Δ_P satisfies (LOCT)*
4. *If P satisfies (BC) ,(COMP) and $f, g : X \to Z$ satisfy P as morphisms of schemes over S and $X \to S, Z \to S$ satisfy Δ_p, then $(f, g) : X \to Y \times_S Z$ satisfy P*

Lemma 92
Let P be a property of morphisms of schemes.
Assume that P satisfies stability under base change and composition.
Let $f : X \to Y, g : X' \to Y'$ be morphisms of schemes over S satisfying P, then the product $f \times_S g$ satisfies P.

Proof
There are maps

$$X \times_S X' \xrightarrow{\mathrm{Id} \times_S g} X \times_S Y' \xrightarrow{f \times_S \mathrm{Id}} Y \times_S Y'$$

which compose to $f \times_S g$.
But both maps are base changes and thus $f \times_S g$ satisfy P. $\square$

Theorem 93 (Cancellation Theorem)
Let P be a property of morphisms of schemes and P satisfies stability under composition and base change.
Let $f : X \to Y, g : Y \to Z$ two morphisms such that $g \circ f$ satisfies P and $\Delta_{Y/Z}$ satisfies P then f satisfies P.

Proof
Write f as the composition

$$X \xrightarrow{(\mathrm{Id}, f)} X \times_Z Y \to Y$$ $\square$

But (Id, f) is a base change of the diagonal and p_y is a base change of $g \circ f$.

4.2 Topological properties

Definition 40
A morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes

1. *is injective, surjective, bijective if f is.*
2. *is open, resp. closed if f is.*
3. *quasi-compact if $f^{-1}(V)$ is quasi-compact for all open quasi-compact $V \subset Y$.*
4. *quasi-separated if $\Delta_{X/Y}$ is quasi-compact.*
5. *has finite fibers if $f^{-1}(y)$ is finite as a set.*

Lecture 10: geometric meaning of separated and proper morphisms

Mon 14 Nov

5 Valuative Criteria

Definition 41 (Specializations)
Let X be a topological space

1. $x, x' \in X$. *If* $x' \in \overline{\{x\}}$ *we say that* x *specializes to* x' *(or* x' *is a specialization) or* x' *generalizes to* x.
2. *A subset* $V \subset X$ *is called closed under specialization if it contains all the specializations of all it's points.*

Remark
Closed subsets are closed under specialization (the converse is not true in general)

Definition 42 (Relative specialization)
Let $f : X \to Y$ *be a continuous map.*
We say that specializations lift along f *if* $\forall x \in X$ *and any specialization* y *of* $f(x)$ *in* Y*, there exists* $x' \in X$ *mapping to the specialization such that* x' *specializes to* x.