Manifolds

David Wiedemann

Table des matières

List of Theorems

Lecture 1: Introduction

Wed 12 Oct

1 Recap

Recall theorems about differentiable maps

— Implicit function theorem

For $U \subset \mathbb{R}^p, V \subset \mathbb{R}^q$, $f \in C^k(U \times V, \mathbb{R}^q), 1 \leq k \leq \infty$ and $(a, b) \in U \times V$ st.

$$D_2 f(a, b) = D(f(a, -))(b)$$

is invertible. Then there exists $a \in U_1 \subset U, b \in V_1 \subset V$ and $\phi \in C^k(U_1, V_1)$ such that

$$f(x, x') = y_0$$

iff $x' = \phi(x)$

— Inverse function theorem

If $U \subset \mathbb{R}^p$ is open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty, a \in U$ such that

$$Df(a)$$

is invertible, then there are $a \in U_1 \subset U$ and $f(a) \in V_1 \subset \mathbb{R}^q$ open such that

$$f|_{U_1} : U_1 \to V_1$$

is a diffeomorphism and

$$Df^{-1}|_U(x) = (Df(f^{-1}|_U(x)))^{-1}$$

for all $x \in U$ in particukar f^{-1} is C^k

— Rank theorem

$U \subset \mathbb{R}^p$ open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty$, $a \in U, b := f(a), r = rank(Df(a))$ then there are diffeomorphisms

$$\psi : U_\psi \to V_\psi \text{ and } \phi : U_\phi \to V_\psi$$

with $U_\psi, V_\psi \subset \mathbb{R}^p$ and $U_\phi, V_\phi \subset \mathbb{R}^q$ such that

$$\phi \circ f \circ \psi(x_1, \ldots, x_p) = (x_1, \ldots, x_r, \tilde{f}(x_1, \ldots, x_p))$$

If $rk(D(f))$ is contant around r, then we can obtain $\tilde{f} = 0$

2 Manifolds

Definition 1 (Basis)
A basis for a topology on X is a collection B of open sets such that every open set in X is the union of sets in B.

X is called second countable if it has a countable topological basis.

Definition 2 (Chart)
Let X be a topological space

1. *A chart on X is a pair (U, ϕ) where $U \subset X$ open and $\phi : U \to \mathbb{R}^n$ for some n which is a homeomorphism onto an open subset.*
2. *An atlas is a collection of charts $A = \{(U_i, \phi_i) | i \in I\}$ such that $X = \bigcup_{i \in I} U_i$*
3. *A is called smooth (C^k ,continuous, holomorphic, algebraic,...) if and only if for any*
$$(U_i, \phi_i)_{i \in \{1,2\}} \in A$$
we have $\phi_1 \circ \phi_2^{-1}$ is smooth (C^k ,...) wherever it is defined.
4. *A chart (U, ϕ) is compatible with an atlas A if and only if*
$$A \cup \{(u, \phi)\}$$
is smooth
5. *An atlas A is maximal if it contains all charts compatible with A. For any atlas A (not necessarily maximal), denote A_{max} the maximal atlas containing it.*
This maximal atlas is necessarily unique

Definition 3 (Manifold)
A smooth manifold of dimension n is a second countable Hausdorrf space with a maximal smooth atlas of dimension n.

Why Hausdorff?

Consider $\mathbb{R}/\sim$, $x \sim y \iff |x| = |y| > 1$, this space is locally homeomorphic to $\mathbb{R}$ but the points x and y cannot be separated.

Why second countable?

Take a disjoint union of infinitely many manifolds.
For a connected example, take $\aleph_1 \times [0, 1)$ with the order topology.

2.1 Smooth maps

A function $f : M \to N$ between smooth manifolds is called smooth if for each $p \in M$, there are charts $(U, \phi), (V, \psi)$ $p \in U \subset M, f(p) \in V \subset N$ such that

$$\psi \circ f \circ \phi^{-1}$$

is smooth.

f smooth implies $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is smooth for any charts $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi})$ where this is defined.

Lecture 2: Smooth maps

Mon 17 Oct

Example (Projective Spaces)
Let $K = \mathbb{R}$ or $\mathbb{C}$, take $K\mathbb{P}^n = \{$ all lines in $K^{n+1}\} = K^{n+1} \setminus 0/ \sim$.
Then $x \sim y \iff \exists \lambda x = \lambda y l$
We have $\mathbb{R}\mathbb{P}^n = S^n / x \sim -x = S^n / {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$
Similarly, $\mathbb{C}\mathbb{P}^n = S^{2n+1}/S^1$.
To give projective space a smooth structure, we introduce homogeneous coordinates.
We write $[x] = [x_0 : \ldots : x_n]$ for the equivalence class of x.
For $0 \leq j \leq n$ put

$$U_j = \{[x] \in \mathbb{K}\mathbb{P}^n / x_j \neq 0\}$$

and $\phi_j : U_j \to K^n$ is a chart sending $[x] \to (\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j})$

Remark

1. *Composition of smooth maps are smooth.*
2. *If $M \xrightarrow{f} N$ is a diffeomorphism if it is a smooth map whose inverse is smooth.*

2.2 Partitions of Unity

Theorem 3 (Paracompactness)
Let M be a smooth manifold.
Let (U_α) be an open covering of M.
Then there exists a locally finite refinement $(V_\beta)_{\beta \in B}$ that is

1. *locally finite, ie. each point has an open neighbourhood which meets finitely many* V_β.
2. $\forall \beta \exists \alpha V_\beta \subset U_\alpha$

More precisely, we can choose V_β *such that there exist charts* $\psi_\beta : V_\beta \to \{x \in \mathbb{R}^n | |x| < 3\}$ *and such that* M *is covered by*

$$M = \bigcup_{\beta \in B} \psi_\beta^{-1}(\{x \in \mathbb{R}^n | |x| < 1\})$$

Proof

From the definition, it is clear that any manifold is locally compact.
Hence there exist compact sets

$$K_1 \subset K_2^o \subset K_2 \subset K_3^o \dots$$

sich that $M = \bigcup K_j$.
$K_{j+1} \setminus K_j^o$ *is compact, hence for* $p \in K_{j+1} \setminus K_j^o$, *there exists* (V_p, ϕ_p) *with* $\phi_p(V_p) = B(0,3)$, $V_p \subset K_{j+2}^o \setminus K_{j-1}$ *and* $V_p \subset U_\alpha$.
By compactness $\exists p_{j_1}, \dots, p_{jr_j}$ *such thtat*

$$K_{j+1} \setminus K_j^o = \bigcup \phi_{jl}^{-1}(B(0,1))$$

The union of all these charts is a locally finite refinement with the desired properties. □

We can now define a map $f_1 : \mathbb{R} \to \mathbb{R}, f_1(t) = e^{-\frac{1}{t}}$ if $t > 0$ and 0 if not.
f_1 is $C^\infty(\mathbb{R})$.
Now, define $f_2(t) = \frac{f_1(t)}{f_1(t)+f_1(1-t)}$ and then $f_3(t) = f_2(2+t)f_2(2-t)$.
We can now define $f_4 : \mathbb{R}^n \to \mathbb{R}$ as $f_4(x) = f_3(|x|)$

Theorem 4 (Partition of unity)
Let M *be a* C^∞ *manifold and* (U_α) *an open covering.*
There exist $\phi_U \in C^\infty(M)$ *such that*

1. $0 \leq \phi_n \leq 1$
2. $\operatorname{Supp} \phi_n$ *is locally finite*
3. $\forall n \exists \alpha \in A, \operatorname{Supp} \phi_n \subset U_\alpha$
4. $\forall p \in M, \sum_{n=1}^{\infty} \phi_n(p) = 1$

Lecture 3: Partitions of Unity

Wed 19 Oct

Proposition 5

Let M be a smooth manifold, $A \subset M$ closed, $G \subset M$ open with $A \subset G$, then there exists a smooth function f on M, such that $\operatorname{Im} f \subset [0,1]$ and $f|_A \equiv 1$ and $f|_{G^C} \equiv 0$

Proof

$(M \setminus A, G)$ is an open cover and (ϕ_0, ϕ_1) a partition of unity subordinate to this open cover, then $f = \phi_1$ does the job. □

Theorem 6

Let M be a smooth manifold, (U_α) an open cover, then there exists $\phi_n \in C^\infty(M), n \in \mathbb{N}$ such that

1. $0 \leq \phi_n \leq 1$
2. $\{\operatorname{Supp} \phi_n\}$ *locally finite*
3. $\forall n \operatorname{Supp} \phi_n \subset U_\alpha$
4. $\sum \phi_n = 1$

Proof

By the partition of unity theorem, there are charts (V_n, ψ_n) of M with $\psi_n : V_n \to B(0,3)$.
We let $\tilde{\phi}_n(x) := f_4(\psi_n(x)), x \in V_n$ and 0 otherwise.
$\forall x \in M \exists n s.t. \tilde{\phi}_n(x) > 0$, by local finiteness $\tilde{\phi}(x) = \sum \tilde{\phi}_n > 0$ and $\tilde{\phi}$ is non zero and we let $\phi_n = \frac{\tilde{\phi}_n}{\tilde{\phi}}$ □

As an addendum, we claim that if $A \subset \mathbb{N}$, then A can be chosen as index set for the partition, ie. $\phi_n = 0$ if $n \notin A$ and $\operatorname{Supp} \phi_n \subset U_n$ Let

$$J_k := \{i \in \mathbb{N} | i \in A \setminus J_0 \cup \ldots \cup J_{k-1}, \operatorname{Supp} \phi_i \subset U_k\}$$

and we let

$$\chi_k = \sum_{i \in J_k} \phi_i$$

3 Tangent Space

If $M \subset \mathbb{R}^n$ is a submanifold, $M = \{x | F(x) = 0\}$, $F : \mathbb{R}^n \to \mathbb{R}$ a submersion, then $T_p M = \nabla F(p)^\perp$.
Let $v \in T_p M$ and choose $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p, \gamma'(0) = v$.
Given $C^\infty M \ni f \mapsto vf$.
This map is a derivation at p.

Definition 4 (Tangent Space)
Let M be a smooth manifold, $p \in M$.
A derivation at p is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ with $X_p(fg) = f(p)X_pg + g(p)X_pf$.
Then T_pM is the set of all derivations at p and it is a subspace of $C^\infty(M)^*$

Remark

1. *If $\phi \in C^\infty(M)$ constant in a neighborhood of p, then $X_p\phi = 0$ for each $X_p \in T_pM$.*
   To prove this, suppose wlog $\phi = 1$ in a neighborhood of p.
   There exists ξ a smooth function on M, constant in a neighborhood of p and 0 outside of the neigborhood.
   Thus $\chi\phi = \chi$.
   Applying the chain rule gives
   $$X_p\chi = \phi(p)X_p\chi + \chi(p)X_p\phi$$
   and thus $X_p\phi = 0$
2. *If $p \neq q$, then $T_pM \cap T_qM = \{0\}$.*
   To prove this, suppose $p \neq q$. Choose $\phi \in C^\infty(M)$ with $\phi \equiv 1$ in a neighborhood of p and $\equiv 0$ in a neighborhood of q. Thus $X\phi = 0$.
   Let $f \in C^\infty M$ such that $f(1-\phi) \equiv 0$ in a neighborhood of p and thus
   $$X(f) = \phi(q)X_qf + f(q)X_qf$$
3. *Given $X \in T_pM, U$ a neigborhood of p, then $X \in T_pU$ by extending $f \in C^\infty(U)$ to a function on M.*
4. *If (U,ϕ) is a chart at p with coordinate functions $x_1,\ldots,x_n$ then we define*
   $$\frac{\partial}{\partial x_i}f|_p := \frac{\partial}{\partial r_i}f \circ \phi^{-1}|_{\phi(p)} = D(f\circ\phi^{-1})(\phi(p))[e_i]$$

We want to show that T_pM has dimension n

Lemma 8
Let M be a smooth manifold and $p \in M$. Let (U,ϕ) be a chart centered at p (ie. $\phi(p) = 0$), coordinate functions $x_1,\ldots,x_n$.
Then for $f \in C^\infty(U)$, there exists $f_1,\ldots,f_n \in C^\infty(U)$ such that
$$f = \sum_{i=1}^n f_jx_j + f(p)$$

Proof

Without loss of generality $U = (-\epsilon, \epsilon)^n$.

Then

$$\begin{aligned} f(x) &= \left[\sum_{j=1}^{n} f(x_1, \ldots, x_j, 0, \ldots, 0) - f(x_1, \ldots, x_{j-1}, 0, \ldots, 0)\right] + f(0) \\ &= f(0) + \left[\sum_{j=1}^{n} \int_0^1 (\partial_j f)(x_1, \ldots, x_{j-1}, tx_j) dt x_j\right] \end{aligned} \qquad \square$$

Theorem 9

For M *a smooth manifold, let* (U, ϕ) *be a chart centered at* p *taking values in* $\mathbb{R}^n$*, then the dimension of the tangent space is* n*.*