Série 3 Exercice 8

David Wiedemann

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1

Indeed, let $\frac{a}{b} \in \mathbb{Q}$ in reduced form such that $\nu_p(\frac{a}{b}) = 0$. By the definition of p-adic, this means that we may suppose both a and b share no common factors with p, then $\frac{b}{a}$ also shares no common factor with p and hence $\nu_p(\frac{b}{a}) = 0$, implying $\frac{b}{a} \in R_\nu$.
Finally, $\frac{a}{b} \cdot \frac{b}{a} = \frac{1}{1}$ which finally implies that $\frac{a}{b}$ is invertible in R_ν.

2

First we show that all (p^n) are distinct ideals of R, indeed suppose there exists $a, b \in \mathbb{N}$ such that $(p^a) = (p^b)$, without loss of generality suppose $a < b$.
Hence, there exists an element $\frac{a}{b} \in \mathbb{Q}$ with $\nu_p(\frac{a}{b}) \geq 0$ such that $\frac{a}{b} p^b = p^a$.
As $\mathbb{Q}$ is a field, this implies that $\frac{a}{b} = p^{a-b}$ which means $\frac{a}{b}$ has a negative valuation which contradicts our hypothesis.

Now we show that the ideals mentionned in the exercise are indeed all the ideals of R.
Let I be an non-zero ideal of R.
Define $a = \inf_{x \in I \setminus \{0\}} \{\nu(x)\}$. Since $\nu|_{I \setminus \{0\}}$ has codomain $\mathbb{N}$, this infimum exists and is attained by some element $y \in I$.
Since we may write $y = p^a \frac{d}{c}$ where d and c are coprime to p.
By part 1, we know that $\frac{d}{c}$ is invertible, hence implying that (since I is an ideal) $p^a \in I$.

We pretend that $I = (p^a)$, to do this, we show the double inclusion.
First, note that, since by definition $p^a \in I$, we immediatly get that $(p^a) \subset I$ since (p^a) is the smallest ideal containing p^a.
Furthermore, let $x \in I$, then by definition of a, $\nu(x) \geq a$.
Since we may then write $x = p^{\nu(x)} \frac{d}{c} = p^a p^{\nu(x)-a} \frac{d}{c}$ where d and c are coprime to p, this implies that $x \in (p^a)$.

Hence, if I is a non-zero ideal, I is of the form p^n for some n and since these ideals are disjoint, we have characterised all of them.

3

Using the exercise of week 2, we know that $\mathbb{Z} \subset R$.
Hence consider the composition $\mathbb{Z} \overset{\iota}{\hookrightarrow} R \overset{q_R}{\longmapsto} R/(p^n)$ where ι is the inclusion morphism and q_R is the canonical projection morphism.
Furthermore define $q : \mathbb{Z} \to \mathbb{Z}/(p^n)$ to be the canonical projection.
We now pretend that $\ker(q_R \circ \iota) = \ker q = (p^n)$, indeed if $a \in \ker q = (p^n)$, then $p^n|a$ hence $p^n|\iota(a) \implies q_R(a) = 0$.
Similarly, if $r \in \ker(q_R \circ \iota)$, then $p^n|r$, ie. there exists $\frac{a}{b} \in R$ (in reduced form) such that $p^n \frac{a}{b} = r$ since $\nu(\frac{a}{b}) \geq 0$, in particular we may suppose b is coprime to p.
Hence, since $p^n \frac{a}{b}$ is an integer, $b|a$ implying $b = 1$.
Finally, this means that there exists an integer a such that $p^n a = r$ which means that $a \in (p^n) = \ker q$.
Hence applying the universal property of the quotient ring, we get an induced morphism as such :

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "q_R \circ \iota"] \arrow[d, "q"'] & R/(p^n) \\
\mathbb{Z}/(p^n) \arrow[ur, dotted, "\exists! \varphi"'] &
\end{tikzcd}$$

We now show that $q_R \circ iota$ is surjective. Let $[p^i \frac{a}{b}] \in R/(p^n)$, where, as always, we have assumed $\frac{a}{b}$ is in reduced form and shares no factors with p.
Now we pretend that $q_R \circ \iota(p^i a) = [p^i \frac{a}{b}]$, indeed, notice that

$$p^i \frac{a}{b} - p^i a = \frac{p^i (b-1) a}{b}$$