Topology I

Course by Viktoryia Ozornova

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1 Homology Theories

Lecture 1: Introduction

Mon 10 Oct

Aim : Study further algebraic invariants of topological spaces.
We want to assign to pairs of topological spaces abelian groups.

$$h_n : T \to \mathrm{Ab} \quad \forall n \in \mathbb{Z}$$

and to pairs continuous maps, we want to assign a map $h_n(f) : h_n(X) \to h_n(Y)$ which is functorial. Here T is the category of pairs of topological spaces $A \subset X$ with morphisms $f : (X, A) \to (Y, B)$ such that $f(A) \subset B$.
To relate h_n for different $n \in \mathbb{N}$, we will construct connecting morphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$.

Axiom 1 (Eilenberg-Steenrod Axiom)
A (generalised) homology theory consists of functors $h_n : T \to \mathrm{Ab}$ and natural connecting homomorphisms $\partial_n : h_n(X, A) \to h_{n-1}(A, \emptyset)$ [1] satisfying

— *Homotopy invariance :*
If $f, g : (X, A) \to (Y, B)$ are homotopic continous maps of pairs then the induced maps $h_n(f) = h_n(g)$. Here homotopy of pairs means that there exists $H : X \times [0, 1] \to Y$ such that $H(A \times [0, 1]) \subset B$

— *Long exact sequence of a pair (LES) :*
Given a pair of topological spaces (X, A) there is a long exact sequence of abelian groups.
Denote $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$, then

$$h_n(A, \emptyset) \xrightarrow{h_n(i)} h_n(X, \emptyset) \xrightarrow{h_n(j)} h_n(X, A) \xrightarrow{\partial_n} \to h_{n-1}(A, \emptyset)$$

— *Excision*
Given $B \subset A \subset X$ subspaces such that $\overline{B} \subset A^o$, the inclusion induces a group isomorphism

$$h_n(X \setminus B, A \setminus B) \to h_n(X, A)$$

We add another axiom to "make things easier"

— *Additivity :*
Given a family of pairs of spaces $(X_i, A_i)_{i \in I}$, the inclusions induce an isomorphism

$$\bigoplus h_n(X_i, A_i) \to h_n(\coprod X_i, \coprod A_i)$$

This is the end of the axioms for a generalised homology theory, the homology theory is called an ordinary homology theory if the <u>*Dimension Axiom*</u> *holds, namely*

$$h_n(pt) = 0 \forall n \neq 0$$

1. From now on, we write $h_n(A) \coloneqq h_n(A, \emptyset)$

The abelian group $h_0(\ pt\ )$ is the called the coefficient group of (h_n, ∂_n)

Lemma 2
If $f : X \to Y$ is a homotopy equivalence, then $\forall n \in \mathbb{Z}$ we obtain $h_n(f) : h_n(X) \to h_N(Y)$ to be an isomorphism for any homology theory (h_n, ∂_n)

Proof
Choose $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$, then by functoriality and homotopy invariance $\mathrm{Id}_{h_n(X)} = h_n(\mathrm{Id}_X) = h_n(g) \circ h_n(f)$, by symmetry, $h_n(f)$ and $h_n(g)$ are inverses. $\square$

Similarly, if $f : (X, A) \to (Y, B)$ is a homotopy equivalence of pairs, then the same result holds.

Example
For any such homology theory

$$h_n(\mathbb{R}^k) \simeq h_n(\ pt\ ) \simeq h_n(D^k)$$

Lecture 2: Homology Theories

Wed 12 Oct

Recall that the natural homomorphisms ∂_n are natural, in the sense that the compositions

$$h_{n-1}(f) \circ \partial_n : h_n(X, A) \to h_{n-1}(A) \to h_{n-1}(B)$$

and

$$\partial_n \circ h_n(f) : h_n(X, A) \to h_n(Y, B) \to h_{n-1}(B)$$

coincide.
Today, we compute the homology groups $h_*(S^k)$ for $k \geq 0$ for a given ordinary homology theory h_* Here, the k-sphere is defined as a subspace of $\mathbb{R}^{k+1}$.
Recall from the exercises that $h_*(pt \coprod pt) = h_*(pt) \oplus h_*(pt)$ for ordinary homology theories concentrated in degree 0.
There are two maps $\pm : pt \to S^0$ and one natural map $S^0 \to pt$ called the "fold" map.
By functoriality, the composition $h_*(pt) \to h_*(S^0) \to h_* pt$ is the identity.
To compute $h_*(S^k)$, we use two LES

$$\ldots \xrightarrow{\partial_{n+1}} h_n(S^k) \xrightarrow{h_*\iota} h_n(D^{k+1}) = 0 \xrightarrow{h_*\iota} h_n(D^{k+1}, S^k) \to h_{n-1}(S^k) \to h_{n-1}(D^{k+1}) = 0 \ldots$$

As $h_n(D^{k+1}) = 0$ for $n \neq 0$, we have an isomorphism $\partial_n : h_n(D^{k+1}, S^k) \to h_{n-1}(S^k)$.
The inclusion $D^k \subset S^k$ (as the upper hemisphere) gives rise to another LES

$$0 = h_n D^k \xrightarrow{h_*\iota} h_n S^k \xrightarrow{h_*\iota} h_n(S^k, D^k) \xrightarrow{\partial_n} h_{n-1} D^k = 0 \to h_{n-1} S^k \ldots$$

And thus we also get an isomorphism $h_n\iota : h_nS^k \to h_{n-1}D^k$ The inclusion of the north pole $pt \subset D^k \subset S^k$ induces, using excision, the isomorphism $h_n(S^k \setminus pt, D^k \setminus pt) \simeq h_n(S^k, D^k)$ of the following diagram

$$\begin{tikzcd}
h_n(D^k, S^{k-1}) \arrow[d, "\simeq\partial_n"'] & h_n(S^k \setminus pt, D^k \setminus pt) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, "\partial_n"'] & h_n(S^k, D^k) \arrow[d, "\partial_n"] \\
h_{n-1}(S^{k-1}) \arrow[r, "h_*\iota"'] & h_{n-1}(D^k \setminus pt) \arrow[r] & h_{n-1}(D^k)
\end{tikzcd}$$

We know that the bottom row of this diagram is an ES.
In particular $h_n(D^k, S^{k-1}) \simeq h_n(S^k, D^k)$.
The isomorphism $\partial_n : h_n(D^k, S^{k-1}) \to h_{n-1}(S^{k-1})$ now almost allows us to use induction to find the homology groups.
We now consider the case $n \in \{0, 1\}$ (This part of the proof is not complete yet)

$$h_1(D^k) = 0 \to h_1S^k \to h_1(S^k, D^k) \xrightarrow{\partial_1} h_0D^k \to h_0S^k \to h_0(S^k, D^k) \to h_{-1}D^k = 0$$

The case $n \in \{0, 1\}$ gives a split short exact sequence

$$0 \to h_0D^k \to h_0S^k \to h_0(S^k, D^k) \simeq h_0(D^k, S^{k-1}) \to 0$$

The homotopy equivalence $pt \to D^k$ gives a split of this exact sequence $h_0S^k \to h_0pt \to h_0D^k$.
The boundary homomorphism $h_1(S^k, D^k) \to h_0D_k$ being 0 using results from the exercise sheet.

Now by induction, $h_nS^k = 0$ for all $n < 0$ and $h_0S^k = h_0(pt)$ for all $k > 0$.
We also have that $h_nS^1 \simeq h_{n-1}S^0$ for $n \notin \{0, 1\}$.
What about h_1S^1 ?

$$h_1(D^1, S^0) \to h_1(S^1, D^1) \to h_0(D^1)$$

and

$$h_1(D^1, S^0) \to h_0S^0 \to h_0(D^1)$$

Where the last morphism is induced by the fold map, namely $h_0S^0 = h_0pt \oplus h_0pt \to h_0(pt)$ and $(x, y) \mapsto x + y$.
We have

$$h_1D^1 \to h_1(D^1, S^0) \to h_0S^0 = h_0pt \oplus h_0pt \to h_0D^1$$

We were able to show isomorphisms $h_nS^k \simeq h_{n-1}S^{k-1}$ for $n \notin \{0, 1\}$, $h_0S^k \simeq h_0pt$ for $k > 0$ and $h_1S^1 \simeq h_0pt$.
What about h_1S^k for $k > 1$?
We have isomorphisms

$$h_1S^k \to h_1(S^k, D^k) \xrightarrow{\partial} h_0D^k \simeq h_0S^k$$

and

$$h_1(D^k, S^{k-1}) \simeq h_1(S^k, D^k) \to h_0 S^{k-1} \simeq h_0 D^k$$

and thus $h_1 S^k = 0$ for $k > 1$.

Proposition 4

FOr any ordinary homology theory (h_*, ∂_*), the following holds

$$h_n S^k = \begin{cases} h_0 pt \oplus h_0 pt \ \textit{if}\ k = 0 = n \\ 0, k > 0, n \notin \{0, k\} \\ h_0 pt \ \textit{if}\ k > 0 \ \textit{and}\ n \in \{0, k\} \\ 0, else \end{cases}$$

We add one additional assumption, that there exists an ordinary homology theory with coefficient group $h_0 pt \simeq \mathbb{Z}$

Corollary 5

S^k and S^l are not homotopy equivalent for $k \neq l$

Proof

$$h_k S^k \simeq h_0 pt \neq h_k S^l = 0$$

□

Corollary 6 (Brouwer fixed point theorem)

Any continuous map $f : D^n \to D^n$ has a fixed point.

Proof

Assume $f : D^n \to D^n$ is a map without a fixed point.

Consider $g : D^n \to S^{n-1}$ sending $x \mapsto \frac{x - f(x)}{\|x - f(x)\|}$, by assumption, this is continuous.

Next, we claim that $g|_{S^{n-1}}$ is homotopic to $\mathrm{Id}_{S^{n-1}}$ via the map

$$H(x,t) := \frac{x - tf(x)}{\|x - tf(x)\|}$$

If $t = 1$, the denominator is $\neq 0$, if $t < 1$

$$\|tf(x)\| = t\,\|f(x)\| < \|f(x)\| \leq 1$$

Hence, $\|x - tf(x)\| \neq 0$ and H is a well defined continuous map.

Now, consider

$$h_{n-1} S^{n-1} \xrightarrow{ind} h_{n_1} D^n \xrightarrow{h_{n-1}(g)} h_{n-1} S^{n-1}$$

By homotopy equivalence $h_{n-1}(g) \circ ind$ is the identity.
For $n > 1$, this implies that the identity factors through 0, which is a contradiction.
The special case $n = 1$ gives

$$h_0 S^0 \to h_0 D^1 \to h_0 S^0$$

If the coefficient group is $\mathbb{Z}$, this is a contradiction. □

2 Constructing singular homology

We want to construct a (ordinary) homology theory.
The idea is to study X by mapping topological simplices into X, here the topological n simplex is defined as

$$\Delta^n = \left\{ (t_0, \ldots, t_n) | t_i \geq 0 \forall i, \sum_i t_i = 1 \right\} \subset \mathbb{R}^{n+1}$$

We define

$$Sing_n(X) = \{ f : \Delta^n \to X \text{ continuous } \}$$

in general, this set is huge.

Lecture 3: Singular homology

Mon 17 Oct

Goal : Find a way to organise the information in $Sing_n(X)$!

1. Relate $Sing_n(X)$ for different n to each other
2. Linearize !

We'll call $Sing_n(X)$ the n-th component of the singular set.
We think of the edges of the simplices as being ordered.
There are maps $\Delta^1 \to \Delta^n$ which are inclusions into the edges.
In fact, for every subset $S \subset \{0, \ldots, n\}$, there is a continuous injective map $\Delta^k \to \Delta^n$, where $k = |S|$.
Now, for any $k < n$, we have restriction maps $Sing_n(X) \to Sing_k(X)$.
Define the category Δ_{inj}, whose onjects are $[n]$ for every $n \in \mathbb{N}$ and whose morphisms $[k] \to [n]$ are order preserving injective maps.
The composition is just the composition of maps.
For X a fixed topological space, we get a contravariant functor $Sing.(X)$: $\Delta_{inj} \to \text{Set}$.
Given $\alpha : [k] \to [n]$ an injective order preserving map, we get

$$Sing_n(X) \to Sing_k(X)$$

with precomposition by α.

Lemma 7
Δ_{inj} can also be described as the category with objects $[n]$ and generated by maps $d^i : [n] \to [n+1]$ subject to the relations

$$d^j d^i = d^i d^{j-1}$$

for $0 \leq i < j \leq n$

Proof (Sketch)
This relation is indeed satisfied in Δ_{inj}

$$\{0 < \ldots < n-2\} \xrightarrow{d^i} \{0 < \ldots < n-1\} \xrightarrow{d^j} \{0 < \ldots < n\}$$

Here

$$k \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k \geq i \end{cases} \quad \mapsto \begin{cases} k, k \leq i-1 \\ k+1, k+1 \leq j \\ k+2, k+2 \geq j+1 \end{cases}$$

One can compute that the composition $d^i d^{j-1}$ gives the same map.
What remains to show is that, subject to these relations, any order preserving injective map can be written as a composition of maps d^i.
If α is missing $i_1 < i_2 < \ldots < i_{n-k}$, then α can be written as

$$\alpha = d^{i_{n-k}} d^{i_{n-k-1}} \ldots d^{i_1}$$

□

We'll call d^i the i-th coface map.
A contravariant functor $\Delta_{inj} \to$ Set is called a semi-simplicial set.

Definition 1 (Singular Chain Complex)
A (non-negatively graded) singular chain complex of a space X has as chain groups

$$S_n X = \mathbb{Z} \langle Sing_n(X) \rangle$$

and differentials $\delta_n : S_n(X) \to S_{n-1}(X)$ defined on generators as

$$\partial_n (\sigma : \Delta^n \to X) \mapsto \sum_{i=0}^{n} (-1)^i \sigma \circ d^i$$

Lemma 8
The singular chain complex of a space is a chain complex.

Proof

By linearity, it is enough to check this on generators $\sigma : \Delta^n \to X$.

$$\begin{aligned}
\delta_{n-1}\delta_n\sigma &= \delta_{n-1}\left(\sum_{i=0}^{n}(-1)^i\sigma\circ d^i\right)\\
&= \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j\sigma\circ d^i\circ d^j\\
&= \sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&+ \sum_{0\le i\le j\le n-1}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&= \sum_{0\le j<i\le n}(-1)^{i+j}\sigma\circ d^i\circ d^j\\
&+ \sum_{0\le i<j'\le n-1}(-1)^{i+j'-1}\sigma\circ d^{j'}\circ d^i\\
&= 0
\end{aligned}$$

□

Lemma 9

We get a functor from chain complexes with chain maps to graded abelian groups, which is just taking homology.

Definition 2 (Singular Homology)

The singular homology $H_\bullet X$ (with integer coefficients) on a space X is the homology of the singular chain complex.

Lecture 4: Homology Theories

Wed 19 Oct

Lemma 10

Homology defines a functor $Ch \to gr\,\mathrm{Ab}$

Proof (Sketch)

Let $f : (C_\bullet, d_\bullet) \to (C'_\bullet, d'_\bullet)$, then $H_n(f) = f_*$ sending $x \in \ker(d_n)/\operatorname{Im}(d_{n+1})$ to $[f(x)]$

□

Example
Let's compute the singular homology of the point.
Clearly $S_* = \mathbb{Z}$ *and the maps induced by restriction are the identity.*
Hence, the boundary maps will be

$$\dots \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \xrightarrow{\mathrm{Id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus $\forall n > 0$, *we get* $H_n(pt) = 0$ *and* $H_0(pt) = \mathbb{Z}$.

Now we want to define homology for pairs.
Let $A \subset X$ be a pair of spaces.
We want to associate a singular chain complex $(S_\bullet(X, A), \delta_\bullet)$.
More generally, any continuous map $f : X \to Y$ induces $Sing_n(X) \to Sing_n(Y)$ by postcomposition.
Thus we get a functor $Sing_n(-) : \top \to \mathrm{Set}$.
This in turn defines a chain map by extending $S_n f$ linearly to $S_n X$.
This defines a chain map $C_n X \to C_n Y$ since

$$\sigma \in S_n X \to f \circ \sigma \to \sum_{i=0}^{n} (-1)^i (f \circ \sigma) \circ d_i$$

and

$$\sigma \in S_n X \to \sum_{i=0}^{n} (-1)^i \sigma \circ d^i \to \sum_{i=0}^{n} (-1)^i (f \circ \sigma \circ d_i)$$

coincide.
For an inclusion of subspaces $A \subset X$, we get an induced map $S_\bullet(i) : (S_\bullet A, \delta_\bullet) \to (S_\bullet X, \delta_\bullet)$ which is levelwise injective.

Definition 3 (Singular chain complex of a pair)
The singular chain complex of a pair is defined to be the quotient chain complex $S_\bullet X / S_\bullet A$.
Then the singular homology of the pair (X, A) *is the homology of this chain complex.*

For any pair (X, A) there is a short exact sequence of chain complexes

$$0 \to (S_\bullet A, \delta_\bullet) \to (S_\bullet X, \delta_\bullet) \to (S_\bullet(X, A), \delta_\bullet) \to 0$$

(ie. levelwise short exact)
What about coefficient groups $\neq \mathbb{Z}$.

Definition 4
Given a pair of spaces (X, A) and G an abelian group G, define the singular chain complex of (X, A) with coefficient in G as follows

$$S_n(X, A; G) = S_n(X, A) \otimes_{\mathbb{Z}} G$$

with the natural induced differentials. The singular homology of (X, A) with coefficients in G is the homology of this new chain complex.

Proposition 12
For any short exact sequence of chain complexes $0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$, we get a long exact sequence of homology groups

$$\ldots \to H_n C_\bullet \to H_n D_\bullet \to H_n E_\bullet \to H_{n-1} C_\bullet \to \ldots$$

which is natural in short exact sequences of chain complexesj;w

Proof
The definition of the map $\partial_n : H_n E \to H_{n-1} C$ is a standard diagram chase. We then prove that :

1. *γ is in the kernel of $d^C_{n-1} : C_{n-1} \to C_{n-2}$*

   $$f_{n-2} d^C_{n-1} \gamma = d^D_{n-1} f_{n-1} \gamma = 0$$

   as f_{n-2} is injective, $d^C_{n-1}\gamma = 0$
2. *The choice of β is inddependent on the choice of γ.*
   Suppose β' is also such that $g_n \beta = g_n \beta'$.
   We want to show that $\gamma - \gamma'$ is in the image of d^C_n.
   As $g_n(\beta - \beta') = 0 \exists \tilde{\gamma} : f_n \tilde{\gamma} = \beta - \beta'$
   $f_{n-1} d^C_n \tilde{\gamma} = d^D_n f_n \tilde{\gamma} = d^D_n \beta - d^D_n \beta' = f_{n-1}(\gamma - \gamma')$.
   Thus $d^D_n \tilde{\gamma} = \gamma - \gamma'$
3. *Independence of the choice of representative α.*
   We want to show that if $\alpha = d^E_n \tilde{\alpha}$, then $\gamma = 0$.
   This again is a standard diagram chase. So we conclude that $\partial_n : H_n E \to H_{n-1} C$ is a well defined map, it is easy to check that it is linear.
   It remains to show that the long sequence above is exact, which is part of the homework. □

We want to show that the connecting homomorphisms are natural, namely, for thwo short exact sequences

$$0 \to C_\bullet \to D_\bullet \to E_\bullet \to 0$$

$$0 \to C'_\bullet \to D'_\bullet \to E'_\bullet \to 0$$

with $\phi : C_\bullet \to C'_\bullet, \psi, \eta$ etc which make the diagram commute, we get, for every n a commutative diagram

$$H_n E \xrightarrow{\partial_n} H_{n-1} C_\bullet \to H_{n-1} C'_\bullet = H_n E \xrightarrow{H_n \eta} H_n E'_\bullet \xrightarrow{\partial'_{n-1}} H_n C'_\bullet$$