Manifolds

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Lecture 1: Introduction

Wed 12 Oct

1 Recap

Recall theorems about differentiable maps

— Implicit function theorem
For $U \subset \mathbb{R}^p, V \subset \mathbb{R}^q$, $f \in C^k(U \times V, \mathbb{R}^q), 1 \leq k \leq \infty$ and $(a, b) \in U \times V$ st.

$$D_2 f(a, b) = D(f(a, -))(b)$$

is invertible. Then there exists $a \in U_1 \subset U, b \in V_1 \subset V$ and $\phi \in C^k(U_1, V_1)$ such that

$$f(x, x') = y_0$$

iff $x' = \phi(x)$

— Inverse function theorem
If $U \subset \mathbb{R}^p$ is open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty, a \in U$ such that

$$Df(a)$$

is invertible, then there are $a \in U_1 \subset U$ and $f(a) \in V_1 \subset \mathbb{R}^q$ open such that

$$f|_{U_1} : U_1 \to V_1$$

is a diffeomorphism and

$$Df^{-1}|_U(x) = (Df(f^{-1}|_U(x)))^{-1}$$

for all $x \in U$ in particukar f^{-1} is C^k

— Rank theorem
$U \subset \mathbb{R}^p$ open and $f \in C^k(U, \mathbb{R}^q), 1 \leq k \leq \infty$, $a \in U, b \coloneqq f(a), r = rank(Df(a))$ then there are diffeomorphisms

$$\psi : U_\psi \to V_\psi \text{ and } \phi : U_\phi \to V_\psi$$

with $U_\psi, V_\psi \subset \mathbb{R}^p$ and $U_\phi, V_\phi \subset \mathbb{R}^q$ such that

$$\phi \circ f \circ \psi(x_1, \ldots, x_p) = (x_1, \ldots, x_r, \tilde{f}(x_1, \ldots, x_p))$$

If $rk(D(f))$ is contant around r, then we can obtain $\tilde{f} = 0$

2 Manifolds

Definition 1 (Basis)
A basis for a topology on X is a collection B of open sets such that every open set in X is the union of sets in B.

X is called second countable if it has a countable topological basis.

Definition 2 (Chart)
Let X be a topological space

1. *A chart on X is a pair (U, ϕ) where $U \subset X$ open and $\phi : U \to \mathbb{R}^n$ for some n which is a homeomorphism onto an open subset.*
2. *An atlas is a collection of charts $A = \{(U_i, \phi_i) | i \in I\}$ such that $X = \bigcup_{i \in I} U_i$*
3. *A is called smooth (C^k ,continuous, holomorphic, algebraic,...) if and only if for any*

$$(U_i, \phi_i)_{i \in \{1,2\}} \in A$$

   we have $\phi_1 \circ \phi_2^{-1}$ is smooth (C^k ,...) wherever it is defined.

4. *A chart (U, ϕ) is compatible with an atlas A if and only if*

$$A \cup \{(u, \phi)\}$$

   is smooth

5. *An atlas A is maximal if it contains all charts compatible with A. For any atlas A (not necessarily maximal), denote A_{max} the maximal atlas containing it.*
   This maximal atlas is necessarily unique

Definition 3 (Manifold)
A smooth manifold of dimension n is a second countable Hausdorrf space with a maximal smooth atlas of dimension n.

Why Hausdorff ?

Consider $\mathbb{R}/\sim$, $x \sim y \iff |x| = |y| > 1$, this space is locally homeomorphic to $\mathbb{R}$ but the points x and y cannot be separated.

Why second countable ?

Take a disjoint union of infinitely many manifolds.
For a connected example, take $\aleph_1 \times [0,1)$ with the order topology.

2.1 Smooth maps

A function $f : M \to N$ between smooth manifolds is called smooth if for each $p \in M$, there are charts $(U, \phi), (V, \psi)$ $p \in U \subset M, f(p) \in V \subset N$ such that

$$\psi \circ f \circ \phi^{-1}$$

is smooth.

f smooth implies $\tilde{\psi} \circ f \circ \tilde{\phi}^{-1}$ is smooth for any charts $(\tilde{U}, \tilde{\phi}), (\tilde{V}, \tilde{\psi})$ where this is defined.

Lecture 2: Smooth maps

Mon 17 Oct

Example (Projective Spaces)
Let $K = \mathbb{R}$ or $\mathbb{C}$, take $K\mathbb{P}^n = \left\{ \text{ all lines in } K^{n+1} \right\} = K^{n+1} \setminus 0/\sim$.
Then $x \sim y \iff \exists \lambda x = \lambda y l$
We have $\mathbb{RP}^n = S^n / x \sim -x = S^n / {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$
Similarly, $\mathbb{CP}^n = S^{2n+1}/S^1$.
To give projective space a smooth structure, we introduce homogeneous coordinates.
We write $[x] = [x_0 : \ldots : x_n]$ for the equivalence class of x.
For $0 \leq j \leq n$ put

$$U_j = \{[x] \in \mathbb{KP}^n / x_j \neq 0\}$$

and $\phi_j : U_j \to K^n$ is a chart sending $[x] \to (\frac{x_0}{x_j}, \ldots, \frac{x_n}{x_j})$

Remark

1. *Composition of smooth maps are smooth.*
2. *If $M \xrightarrow{f} N$ is a diffeomorphism if it is a smooth map whose inverse is smooth.*

2.2 Partitions of Unity

Theorem 3 (Paracompactness)
Let M be a smooth manifold.
Let (U_α) be an open covering of M.
Then there exists a locally finite refinement $(V_\beta)_{\beta \in B}$ that is

1. *locally finite, ie. each point has an open neighbourhood which meets finitely many* V_β.
2. $\forall \beta \exists \alpha V_\beta \subset U_\alpha$

More precisely, we can choose V_β *such that there exist charts* $\psi_\beta : V_\beta \to \{x \in \mathbb{R}^n | |x| < 3\}$ *and such that* M *is covered by*

$$M = \bigcup_{\beta \in B} \psi_\beta^{-1}(\{x \in \mathbb{R}^n | |x| < 1\})$$

Proof

From the definition, it is clear that any manifold is locally compact. Hence there exist compact sets

$$K_1 \subset K_2^o \subset K_2 \subset K_3^o \dots$$

sich that $M = \bigcup K_j$.
$K_{j+1} \setminus K_j^o$ *is compact, hence for* $p \in K_{j+1} \setminus K_j^o$, *there exists* (V_p, ϕ_p) *with* $\phi_p(V_p) = B(0,3)$, $V_p \subset K_{j+2}^o \setminus K_{j-1}$ *and* $V_p \subset U_\alpha$.
By compactness $\exists p_{j_1}, \dots, p_{jr_j}$ *such thtat*

$$K_{j+1} \setminus K_j^o = \bigcup \phi_{jl}^{-1}(B(0,1))$$

The union of all these charts is a locally finite refinement with the desired properties. $\square$

We can now define a map $f_1 : \mathbb{R} \to \mathbb{R}, f_1(t) = e^{-\frac{1}{t}}$ if $t > 0$ and 0 if not.
f_1 is $C^\infty(\mathbb{R})$.
Now, define $f_2(t) = \frac{f_1(t)}{f_1(t)+f_1(1-t)}$ and then $f_3(t) = f_2(2+t)f_2(2-t)$.
We can now define $f_4 : \mathbb{R}^n \to \mathbb{R}$ as $f_4(x) = f_3(|x|)$

Theorem 4 (Partition of unity)
Let M *be a* C^∞ *manifold and* (U_α) *an open covering.*
There exist $\phi_U \in C^\infty(M)$ *such that*

1. $0 \leq \phi_n \leq 1$
2. $\operatorname{Supp} \phi_n$ *is locally finite*
3. $\forall n \exists \alpha \in A, \operatorname{Supp} \phi_n \subset U_\alpha$
4. $\forall p \in M, \sum_{n=1}^\infty \phi_n(p) = 1$

Lecture 3: Partitions of Unity

Wed 19 Oct

Proposition 5

Let M be a smooth manifold, $A \subset M$ closed, $G \subset M$ open with $A \subset G$, then there exists a smooth function f on M, such that $\operatorname{Im} f \subset [0,1]$ and $f|_A \equiv 1$ and $f|_{G^C} \equiv 0$

Proof

$(M \setminus A, G)$ is an open cover and (ϕ_0, ϕ_1) a partition of unity subordinate to this open cover, then $f = \phi_1$ does the job. □

Theorem 6

Let M be a smooth manifold, (U_α) an open cover, then there exists $\phi_n \in C^\infty(M), n \in \mathbb{N}$ such that

1. $0 \leq \phi_n \leq 1$
2. $\{\operatorname{Supp} \phi_n\}$ *locally finite*
3. $\forall n \operatorname{Supp} \phi_n \subset U_\alpha$
4. $\sum \phi_n = 1$

Proof

By the partition of unity theorem, there are charts (V_n, ψ_n) of M with $\psi_n : V_n \to B(0,3)$.
We let $\tilde{\phi}_n(x) \coloneqq f_4(\psi_n(x)), x \in V_n$ and 0 otherwise.
$\forall x \in M \exists n s.t. \tilde{\phi}_n(x) > 0$, by local finiteness $\tilde{\phi}(x) = \sum \tilde{\phi}_n > 0$ and $\tilde{\phi}$ is non zero and we let $\phi_n = \frac{\tilde{\phi}_n}{\tilde{\phi}}$ □

As an addendum, we claim that if $A \subset \mathbb{N}$, then A can be chosen as index set for the partition, ie. $\phi_n = 0$ if $n \notin A$ and $\operatorname{Supp} \phi_n \subset U_n$ Let

$$J_k \coloneqq \{i \in \mathbb{N} | i \in A \setminus J_0 \cup \ldots \cup J_{k-1}, \operatorname{Supp} \phi_i \subset U_k\}$$

and we let

$$\chi_k = \sum_{i \in J_k} \phi_i$$

3 Tangent Space

If $M \subset \mathbb{R}^n$ is a submanifold, $M = \{x | F(x) = 0\}$, $F : \mathbb{R}^n \to \mathbb{R}$ a submersion, then $T_p M = \nabla F(p)^\perp$.
Let $v \in T_p M$ and choose $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p, \gamma'(0) = v$.
Given $C^\infty M \ni f \mapsto vf$.
This map is a derivation at p.

Definition 4 (Tangent Space)
Let M be a smooth manifold, $p \in M$.
A derivation at p is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ with $X_p(fg) = f(p)X_p g + g(p)X_p f$.
Then T_pM is the set of all derivations at p and it is a subspace of $C^\infty(M)^*$

Remark

1. *If $\phi \in C^\infty(M)$ constant in a neighborhood of p, then $X_p\phi = 0$ for each $X_p \in T_pM$.*
   To prove this, suppose wlog $\phi = 1$ in a neighborhood of p.
   There exists ξ a smooth function on M, constant in a neighborhood of p and 0 outside of the neigborhood.
   Thus $\chi\phi = \chi$.
   Applying the chain rule gives
   $$X_p\chi = \phi(p)X_p\chi + \chi(p)X_p\phi$$
   and thus $X_p\phi = 0$
2. *If $p \neq q$, then $T_pM \cap T_qM = \{0\}$.*
   To prove this, suppose $p \neq q$. Choose $\phi \in C^\infty(M)$ with $\phi \equiv 1$ in a neighborhood of p and $\equiv 0$ in a neighborhood of q. Thus $X\phi = 0$.
   Let $f \in C^\infty M$ such that $f(1-\phi) \equiv 0$ in a neighborhood of p and thus
   $$X(f) = \phi(q)X_q f + f(q)X_q f$$
3. *Given $X \in T_pM, U$ a neigborhood of p, then $X \in T_pU$ by extending $f \in C^\infty(U)$ to a function on M.*
4. *If (U, ϕ) is a chart at p with coordinate functions $x_1, \ldots, x_n$ then we define*
   $$\frac{\partial}{\partial x_i} f|_p \coloneqq \frac{\partial}{\partial r_i} f \circ \phi^{-1}|_{\phi(p)} = D(f \circ \phi^{-1})(\phi(p))[e_i]$$

We want to show that T_pM has dimension n

Lemma 8
Let M be a smooth manifold and $p \in M$. Let (U, ϕ) be a chart centered at p (ie. $\phi(p) = 0$), coordinate functions $x_1, \ldots, x_n$.
Then for $f \in C^\infty(U)$, there exists $f_1, \ldots, f_n \in C^\infty(U)$ such that
$$f = \sum_{i=1}^{n} f_j x_j + f(p)$$

Proof

Without loss of generality $U = (-\epsilon, \epsilon)^n$.
Then

$$\begin{aligned} f(x) &= \left[\sum_{j=1}^{n} f(x_1, \dots, x_j, 0, \dots, 0) - f(x_1, \dots, x_{j-1}, 0, \dots, 0)\right] + f(0) \\ &= f(0) + \left[\sum_{j=1}^{n} \int_0^1 (\partial_j f)(x_1, \dots, x_{j-1}, tx_j) dt x_j\right] \end{aligned}$$

□

Theorem 9

For M *a smooth manifold, let* (U, ϕ) *be a chart centered at* p *taking values in* $\mathbb{R}^n$*, then the dimension of the tangent space is* n.

Lecture 4: Tangent Space

Mon 24 Oct

We prove that the dimension of the tangent space is the dimension of the manifold.

Proof

Without loss of generality, suppose ϕ *is a chart centered at* p.
Let $X \in T_pM, f \in C^\infty(U)$*, writing* $f = f(p) + \sum f_j x_j$*, we get*

$$\begin{aligned} Xf &= \sum_j x_j(p) X f_j + f_j(p) X x_j \\ &= \sum_j f_j(p) X x_j \\ &= \sum_j X x_j \frac{\partial}{\partial x_j}|_p f \end{aligned}$$

Thus X *is a linear combination of* $\frac{\partial}{\partial x_j}$.
Since $\frac{\partial}{\partial x_j} x_i = \delta_{ij}$*, these must be linearly independent.* □

Example

1. $\mathbb{R}^n$*, the vectors* $\frac{\partial}{\partial x_i}|_p$ *form a basis of* $T_p\mathbb{R}^n$.
2. *Polar Coordinatesin* $\mathbb{R}^3$.
   Let $\phi : [0, \infty) \times (0, 2\pi) \times (0, \pi) \to U \subset \mathbb{R}^3$ *Mapping* $(r, \varphi, \theta) \mapsto (x \cos\varphi \sin\theta, r \sin\varphi \sin\theta, r \cos\theta)$.
   Now $\frac{\partial}{\partial r}|_p f = \frac{\partial}{\partial r}(f \circ \phi)(r, \varphi, \theta) = \dots$
3. *If* $x_1, \dots, x_n$ *and* $y_1, \dots, y_n$ *are two coordinate systems named* ϕ *and* ψ.

$$\begin{aligned} \frac{\partial}{\partial x_i}|_p f &= \frac{\partial}{\partial x_i}|_p (f \circ \psi^{-1} \circ \psi)(p) \\ &= \partial_i (f \circ \psi^{-1} \circ \psi \circ \phi^{-1})(\phi(p)) \end{aligned}$$

$$= \sum_{j=1}^{m} \partial_j (f \circ \psi^{-1})(\psi(p)) \frac{\partial \psi_j \circ \phi^{-1}}{\partial x_i}(\phi(p))$$
$$= \sum_{j=1}^{m} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}|_p f$$

Definition 5 (Tangent Map)
Let $f : M \to N$ be a smooth map.
For $p \in M$, we define $T_p f : T_p M \to T_{f(p)} M$.
For $\phi \in C^\infty N$, then
$$T_p f(X)\phi = X(\phi \circ f)$$

It is clear that this map is linear.

Remark

1. *Convince yourself that in charts, this is nothing but the derivative (Jacobi matrix)*
2. *If $M \xrightarrow{g} N \xrightarrow{f} Z$, then $T_p(f \circ g) = T_{g(p)}(f) \circ T_p(g)$*
3. *If (U, ϕ) is a chart, then the tangent map $T_p\phi : T_p M \to T_{\phi(p)}\mathbb{R}^m$*
   If ψ is a second chart around p, then $T_p\psi = T_p(\psi \circ \phi^{-1} \circ \phi) = D_{\phi(p)}(\psi \circ \phi^{-1}) \circ T_p\phi$

Physicists approach to the tangent space

A tangent vector is a family $(\xi^\phi), \xi^\phi \in \mathbb{R}^n$ where ϕ runs through all charts such that $\xi^\psi = D_{\phi(p)}(\psi \circ \phi^{-1})(\xi^\phi)$.

4 Local Properties of smooth maps and submanifolds

We assume all manifolds have constant dimension.

Definition 6
Let M and N of respective dimension m and n.
Let $f : M \to N$ be a smooth map, then

- *$p \in M$ is called critical for f if the rank of $T_p f$ is less that $n = \dim N$*
- *$q \in N$ is a regular value of f if $\forall p \in f^{-1}(q)$, rank$T_p f = n$ Notice that if $q \notin f(M)$, then q is regular.*
- *f is a submersion if $\forall p \in M$, rank$T_p f = n$.*
- *f is called an immersion if $\forall p \in M$, rank$T_p f = M$.*

— f *is called a subimmersion if* $p \mapsto rank T_p M$ *is constant.*

Lecture 5: differentials

Wed 26 Oct

Definition 7 (Submanifold)
Let M^m *be a smooth manifold,* N *a subset.*
N *is called a submanifold if for each chart* (U, ϕ) *of* N, $\phi(N \cap U) \subset \mathbb{R}^m$ *is a submanifold.*
Equivalently, for each $p \in N$ *there is a chart* (U, ϕ) *of* M *centered at* p *such that* $\phi(U \cap N) = \phi(U) \cap \mathbb{R}^n \times \{0\}$.

Clearly, submanifolds are smooth manifolds.
Suppose $N^n \subset M^m$ is a submanifold, then the inclusion map $i : N^n \to M^m$ is an immersion.

Definition 8
$f \in C^\infty(N, M)$ *is called an embedding if*
- f *is an injective immersion*
- $f : N \to f(N)$ *is a homeomorphism when* $f(N)$ *has the relative topology.*

The range of an embedding is a submanifold.

Theorem 12
Let M^m, N^n *be* C^∞ *manifolds and* $f : M \to N$ *be a subimmersion (smooth map with constant rank), then*

1. *For* $q \in N$, *the inverse image* $f^{-1}(q)$ *is a submanifold of dimension* $m - k$
2. *For* $p \in M, q = f(p)$, *there exist neighborhoods* U *of* p *and* V *of* q *such that* $S = f(U) \cap V$ *of dimension* k.

Corollary 13
If $f : M^m \to N^n$ *is a smooth map, then for each regular value* $q : f^{-1}(q)$ *is a submanifold of dimension* $m - n$.

Proof
There is an open neighborhood $U \subset M$ *of* $f^{-1}(q)$ *on which the rank is constant.*
Now, we can apply the theorem. □

To prove the embedding theorem, we use the rank theorem to get charts ϕ, ψ such that $\phi \circ f \circ \psi^{-1}(x_1, \dots, x_p) = (x_1, \dots, x_r, 0)$.
For this map, the two claims are trivial (linear algebra statements).
The statements are clearly invariant under diffeomorphisms.

5 Morse-Sard Theorem

Definition 9 (Null set)
A subset $A \subset M$ is a null set if for any chart (U, ϕ) of M, $\phi(U \cap A)$ is a Lebesgue null set in $\mathbb{R}^m$.

Remark
This is well defined because

1. *A can be covered by countably many charts*
2. *Diffeomorphisms of open subsets of $\mathbb{R}^n$ map null sets to null sets.*

Remark

1. *$\forall p \in M$ $\{p\}$ is a null set, if $\dim M > 0$*
2. *countable unions of null sets are null sets*
3. *If A is a null set, then A° is empty, equivalently, $M \setminus A$ is dense.*

Theorem 16 (Morse-Sard theorem)
If M^m, N^n are smooth manifolds, $n \geq 1$.
Let $f : M^m \to N^n$ be smooth and $C_f = \{p \in M | rank T_p f < M\}$.
Then $f(C_f)$ is a null set in N.

Proof
Wlog $M = \mathbb{R}^m, N = \mathbb{R}^n$.
By induction, if $m = 0$, then range(f) is at most countable, thus a null set.
Assume $m \geq 1$ and that the claim was proved for all dimensions less than m.
Now, let $C_l = \{x \in M | \forall |\alpha| \leq l \partial^\alpha f(x) = 0\} \subset C_f$.
Now we show that $f(C_f \setminus C_1)$ is a null set, $f(C_{l+1} \setminus C_l)$ is a null set and $f(C_l)$ is a null set for l large enough.

$C_f \setminus C_1$ is a null set

Fix $\xi \in C_f \setminus C_1$.
Thus, $\exists i, j$ $\frac{\partial f_i}{\partial x_j}(\xi) \neq 0$ wlog, $i = j = 1$
Let $h(x) = (f_1(x), x_2, \dots, x_m), m \geq 2$ and $f_1(x)$ if $m = 1$.

Let $g = f \circ h^{-1} : V \to V'$, we find $g(t,x) = (t, \tilde{g}(t,x))$.
Wlog $V' = I \times W$, (t,x) is critical for $g \iff x$ is critical for $\tilde{g}(t,\cdot)$.
$g(t,x) = f(h^{-1}(t,x))$ is a critical value for f.
Now, $\lambda^n(f(C_f \cap V)) = \lambda^n(\{g(t,x) | (t,x), x \text{ critical for } \tilde{g}(t,\cdot)\ \})$.

$$= \lambda^n(\{(t,y) \in I \times \mathbb{R}^{n-1} | t \in I \text{ and } y = \tilde{g}(t,x) \text{ critical value of } f \circ \tilde{g}\})$$
$$= \int_I \lambda^{n-1}(\{y \in \mathbb{R}^{n-1} | y \text{ critical value of } \tilde{g}(t,\cdot)\})$$

By induction hypothesis, the integrand is 0.
We now show that $\forall l \geq 1, f(C_l \setminus C_{l+1})$ is a null set, the proof is similar so we omit it.
Let W be a cube of side length d in $\mathbb{R}^m$.
Let $x \in C_k \cap W, y \in W$.
Taylor formula implies that $|f(y) - f(x)| \leq L|x-y|^{k+1}$.
Subdivide W into r^m cubes W_j of side length $\frac{d}{r}$.
If $x \in C_k \cap W_j, y \in W_j, |x-y| \leq \sqrt{m}\frac{d}{r}$.
Then $|f(x) - f(y)| \leq L(\frac{\sqrt{m}d}{r})^{k+1}$.
Thus $f(C_k \cap W_j)$ lies in a cube of side length $2L(\frac{\sqrt{m}d}{r})^{k+1}$.

$$\begin{aligned}\lambda^n(f(C_k \cap W)) &\leq r^m \lambda^n(f(C_k \cap W_{j,max})) \\ &\leq r^m \left\{2L(\frac{\sqrt{m}d}{r})^{k+1}\right\}^n \\ &= r^{m-n(k+1)} \cdot c\end{aligned}$$

This goes to 0 if $k \geq \frac{m}{n}$. □

5.1 Applications of Morse-Sard

1. If $\dim M < \dim N$, then every point is critical and thus the image of f is a null set, thus, there are no smooth space filling curves.
2. Embedding

> **Theorem 17 (Whitney)**
> *If M^m is a smooth manifold, then there is an embedding $f : M^m \to \mathbb{R}^{2m}$.*

We prove this for M compact and $2m+1$ instead of $2m$.

Proof
Strategy :

(a) There is an embedding into some $\mathbb{R}^N$ for some $N \in \mathbb{N}$, so now we suppose $M \subset \mathbb{R}^n$.

(b) *If $N \geq 2m+1$, we find a w such that the projection onto the hyperplane$\langle\omega\rangle^{\perp}$ is an embedding. We will exploit that if M is compact, every injective immersion is an embedding.*

We first prove that injective immersions of compact spaces are embeddings, this is just a topology fact.
Now, we construct the embedding.
Choose charts $(U_j, \phi_j)_{j\in\mathbb{N}}$ with range$(\phi_j) = B(0,3)$ and such that $M = \bigcup_{j=1}^{\infty} \phi_j^{-1}(B(0,1))$.
These form an open cover so, by compactness, $M = \bigcup_{j=1}^{r} \phi_j^{-1}(B(0,1))$.
Pick $g \in C^{\infty}(\mathbb{R}^m)$ such that

$$g(x) = \begin{cases} 1|x| \leq \frac{4}{3} \\ 0, |x| \geq \frac{5}{3} \end{cases}$$

Now, let

$$f_j(p) = \begin{cases} g(\phi_j(p))\phi_j(p), p \in \phi_j \\ 0 \ \textit{otherwise} \end{cases}$$

This is a smooth and furthermore

$$f_j|_{\phi^{-1}(B(0,1))} = \phi_j|_{\phi_j^{-1}(B(0,1))} \qquad \square$$

so f_j is an immersion.
Now, let $F = (f_1, \ldots, f_r, g\circ\phi_1, \ldots, g\circ\phi_r) : M \to \mathbb{R}^{(m+1)r}$, this is an injective immersion, hence an embedding.
Now, we have $M \subset \mathbb{R}^N$ a submanifold, $w \in S^{n-1}, \pi_w(x) = x - \langle x, w\rangle\, w$ is linear.
When is $\pi_w|_{M^m}$ an injective immersion?
$\pi_w(p) = \pi_w(q) \iff p - q||w$.
Now, we map $\phi : M \times M \setminus \{(p,p)|p \in M\} \to S^{N-1}$ mapping $(p,q) \mapsto \frac{p-q}{|p-q|}$.
Now $p - q||W \iff \frac{q-p}{|q-p|} \in$ range(ϕ).
As long as $2m < N-1$ Sard's theorem implies that range(ϕ) is a null set.
π_w *is an immersion if $\forall p \in M \forall v \in T_pM \setminus \{0\}\, \pi_w(v) \neq 0$.*
So now, we introduce $\sigma : TM \setminus \{0_p \in T_pM | p \in M\}, v \mapsto \frac{v}{|v|}$, where TM is the tangent bundle.
Now, π_w is an immersion iff $\forall p \in M \forall w \in T_pM \pm w \notin$ rangeσ.
Thus, $\forall w \in S^{N+1} \setminus A, \pi_w : M \to \mathbb{R}^{N-1}$ is an embedding.

6 Vector Fields and dynamical systems

If (U, ϕ) is a chart of M, then $v \in T_pM$ may be written as $v = \sum_{j=1}^m v_j \frac{\partial}{\partial x_j}|_p$.

Definition 10 (Smooth Vector field)
A smooth vector field on M is a map $X : M \to TM = \coprod_{p \in M} T_pM$ such that

1. $\forall p, X(p) \in T_pM$
2. *For each chart (U, ϕ), $X|_U = \sum X_j^\phi \frac{\partial}{\partial x_j}$ with X_j^ϕ smooth.*

Definition 11 (Integral Curve)
An integral curve to a vector field X is a smooth curve $c : I \to M$ such that

$$\dot{c}(t) = X(c(t))$$

Lecture 6: vector fields

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Theorem 18 (Flow-Box Theorem)
If M^m is a smooth manifold and X a vector field.
Then for $p \in M$, there exists an open neighborhood $U \ni p$ and a smooth map

$$F : (-\epsilon, \epsilon) \times U \to M$$

such that

- $F(0, x) = x$
- $\partial_t F(t, x) = X(F(t, x))$

F is the local flow of the vector field.

Proof
Since the theorem is local, wlog, $M = V \subset \mathbb{R}^n$.
There is a smooth map $f : \overline{B(y_0, r)} \to \mathbb{R}^m$.
f is Lipschitz.
For $y \in B(y_0, r)$ let $F(t, y)$ be the maximal solution of the IVP $\partial_t F(t, y) = f(F(t, y)), F(0, y) = y$ and $a(y) < t < b(y)$.
Now, we recall

Lemma 19 (Gronwall's lemma)
If $[a,b] \subset \mathbb{R}, f, g : [a,b] \to \mathbb{R}_+$.
Assume

$$f(t) \leq C + \int_a^t f(s)g(s)ds$$

Then

$$f(t) \leq Ce^{\int_a^t g(s)ds}$$

To prove this, let

$$\tilde{f}(t) = C + \int_a^t f(s)g(s)ds \geq f(t)$$

and $h(t) = \tilde{f}(t)e^{-\int_a^t g(s)ds}$.
Then

$$h'(t) = (f(t) - \tilde{f}(t))g(t)e^{-\int_a^t g(s)ds} \leq 0$$

□

Note that h *is decreasing and as* $h(a) = 0$, $h'(t) \leq C$ *for* $t \geq a$.
Thus $f(t) \leq \tilde{f}(t) = h(t)e^{\int_a^t g(s)ds} \leq Ce^{\int_a^t g(s)ds}$.