Algebraic Curves

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List of Theorems

Lecture 1: Introduction

Fri 25 Feb

Let K be a field, given a set of polynomials $S = \{f_1, \ldots\}$, we can consider $V(S) = \{(x_1, \ldots) \in K^n | f_i(x_1, \ldots) = 0 \forall i\}$.
Notice that if $a_1, \ldots \in K[x_1, \ldots]$ then also $\sum_i a_i(x) f_i(x) = 0$ only depends on the ideal generated by S.
If $I(S)$ happens to be prime, we call V an algebraic variety.

1 Affine algebraic sets

1.1 Recollection on commutative algebra

All rings are commutative and with unit.
Let R be a ring.

— R is an integral domain, or just domain if there are no zero divisors, ie, $\forall a, b \in R$ s.t.

$$a.b = 0 \implies a = 0 \text{ or } b = 0$$

— Any domain can be embedded into it's quotient ring.

— A proper ideal I is maximal if it's not contained in any other proper ideal

— A proper ideal I is prime if

$$\forall a, b \in R, ab \in I \implies a \in I \text{ or } b \in I$$

— A proper ideal I is radiccal if

$$a^n \in I \implies a \in I$$

— For any ideal $I \subset R$, the radical $\sqrt{I}$ is the smallest radical ideal containing I

Lemme 1
— *$I \subset R$ is maximal $\iff$ R/I is a field*

Lemme 2
— *$I \subset R$ is prime $\iff$ R/I is a domain*

Lemme 3
— *radical $\iff$ R/I has no nilpotent elements.*

Given a subset $S \subset R$ we can consider the ideal generated by S

$$I(S) = \left\{ \sum_i a_i s_i \right\}$$

I is finitely generated if $I = I(S)$ with S finite.

— We say that R is Noetherian $\nexists$ a chain of strictly increasing ideals. Equivalently, every ideal is finitely generated.

> **Theorème 4**
> — *In fact, hilbert's basis theorem says that, if R is Noetherian, then $R[x]$ is noetherian.*

In particular $K[x_1, \ldots, x_n]$ is Noetherian

— I is in principal if it is generated by one element.

— A domain is called a principal ideal domain (PID) if every ideal is principal.

— $a \in R$ is irreducible if a is not a unit, nor zero and if

$$a = b.c$$

then either b or c are units.

— A pid $(a) \subset R$ is prime $\iff$ a is irreducible.

— R is a UFD if R is a domain and elements in R can be factored uniquely up to units and reordering into irreducible elements.

> **Theorème 5**
> *R is a UFD $\implies$ $R[x]$ is a UFD*

And, if R is a PID, then R is a UFD

> **Theorème 6 (Gauss Lemmma)**
> — *R is a UFD and $a \in R[X]$ irreducible, then also $a \in Q(R)[X]$ is irreducible.*

— Localization

Let R be a domain, if $S \subset R$ is a multiplicative subset, then the localization of R at S is defined as

$$S^{-1}R = \left\{ x \in Q(R) | x = \frac{a}{b}, b \in S \right\}$$

If M is an $R-$module, we have similarly

$$S^{-1}M = \left\{ \frac{m}{s} | m \in M, s \in M \right\} / \left\{ \frac{m}{s} = \frac{m'}{s'} \iff ms' = sm' \right\}$$

If $p \subset R$ is a prime ideal, then it's complement is a multiplicative subset and we define
$$R_p = (R \setminus p)^{-1} R$$

— There is a 1-1 correspondence between $p \subset R$ prime and ideals of R_p, furthermore R_p is a local ring

— Localization is exact, in particular, given $I \subset p$ the short exact sequence
$$o \to I \to R \to R/I \to 0$$
gets sent to
$$0 \to I_p \to R_p \to (R/I)_p \to 0$$
ie. localization commutes with taking quotients.

1.2 Polynomial rings

For $a \in \mathbb{N}^n$, we set
$$X^a = X_1^{a_1} \ldots \in k[X_1, \ldots]$$
Thus for any $F \in k[X_1, \ldots, X_n]$, we can write it as
$$F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$$
F is homogeneous or a form of degree d if the coefficients $\lambda_a = 0$ unless $a_1 + \ldots + a_n = d$.
Any F can be written uniquely as $F = F_0 + \ldots + F_d$ where F_i is a form of degree i.
The derivative of $F = \sum_{a \in \mathbb{N}^n} \lambda_a X^a$ with repsect to X_i is $F_{X_i} = \frac{\partial F}{\partial X_i}$.
If F is a form of degree d we have

Theorème 7 (Euler's theorem)
$$\sum_{i=1}^{n} \frac{\partial F}{\partial X_i} X_i = dF$$

Lecture 2: Affine space and algebraic sets

Wed 02 Mar

1.3 Affine spaces and algebraic sets

Let k be a field.

Definition 1
For every $n \geq 0$ the affine n -space $\mathbb{A}_k^n$ the set k^n .

In particular $\mathbb{A}^0$ is a point, $\mathbb{A}^1$ is a line, $\mathbb{A}^2$ the affine plane.
Given a subset $S \subset k[X_1, \ldots, X_n]$ of polynomials, we set

$$V(S) = \{x = (x_1, \ldots, x_n) \in \mathbb{A}^n | f(x_1, \ldots, x_n) = 0 \forall f \in S\}$$

If S is finite, we write $V(f_1, \ldots, f_k)$ for $V(S)$.
If the set S is a singleton, then we call $V(S)$ a hyperplane.
Any subset of $\mathbb{A}^n$ V algebraic if $V = V(S)$ for some subset of polynomials.

Lemme 8

— *Let $S \subset k[X_1, \ldots, X_n]$ and I the ideal generated by S, then $V(S) = V(I)$.*
— *Let $\{I_\alpha\}$ be a collection of ideals, then*

$$V(\bigcup_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$$

— *If $I \subset J$ then $V(J) \subset V(I)$*
— *For polynomials $f, g \in k[x_1, \ldots, x_n]$, then $V(f) \cup V(g) = V(f \cdot g)$ For ideals I, J ideals, then $V(I) \cup V(J) = V(I \cdot J)$ where $IJ = \{fg | f \in I, g \in J\}$*
— *For $a = (a_1, \ldots, a_n) \in \mathbb{A}^n, v(\{x_1 - a_1, \ldots\}) = \{a\}$*

Preuve

1. *Let $h \in \sum_i f_i g_i \subset I$ with $f_i \in S$ and $x \in V(S)$, then $f_i(x) = 0 \forall i$ hence $h(x) = 0 \implies x \in V(I) \implies V(S) \subset V(I)$.*
   Furthermore, if $x \in V(I)$, then in particular $f(x) = 0 \forall f \in S \subset I$, hence $x \in V(S)$ and $V(S) \supset V(I)$
2. *Let $x \in V(\cup I_\alpha)$, then for any α and $f \in I_\alpha$, we must have $f(x) = 0$, hence $x \in V(I_\alpha) \implies x \in \bigcap_\alpha V(I_\alpha)$.*
   Conversely, if $x \in \bigcap_\alpha V(I_\alpha)$ and $f \in \bigcup_\alpha I_\alpha$, then $f \in I_\alpha$ for some α, then $f(x) = 0$ hence $x \in V(\bigcup_\alpha I_\alpha)$ $\square$

By Hilbert's basis theorem $k[x_1, \ldots, x_n]$ is Noetherian hence every ideal is finitely generated.

Corollaire 9

Every algebraic set $V \subset \mathbb{A}^n$ is of the form

$$V = V(f_1, \ldots, f_k) = V(f_1) \cap \ldots \cap V(f_k)$$

1.4 Ideals of a set of points and the nullstellensatz

Using the previous section, we have a map

$$V : \{ \text{ Ideals in } k[X_1, \ldots, X_N]\} \mapsto \{ \text{ algebraic sets in } \mathbb{A}^n\}$$

Conversely, for any subset $X \subset \mathbb{A}^n$ we define

$$I(X) \coloneqq \{f \in k[X_1, \ldots, X_N] | f(x) = 0 \forall x \in X\} \subset k[X_1, \ldots, X_N]$$

Lemme 10

1. *If $X \subset Y$ then $I(X) \supset I(Y)$*
2. *For $J \subset k[X_1, \ldots, X_N]$ an ideal $I(V(J)) \supset J$*
3. *For $W \subset \mathbb{A}^n$ algebraic, $V(I(W)) = W$*

Preuve

1. *Let $f \in I(Y)$, then f vanishes on X and hence f in $I(X)$*
2. $I(V(J)) = \{f \in k[x_1, \ldots, x_n] | f(x) = 0 \forall x \in V(J)\} \supset J$
3. *By definition $V(I(X)) \supset X$ for any X.*
   If in addition, if $X = V(J)$ algebraic, then $V(I(X)) = V((I(V(J)))) \subset V(J) = X$ □

There are essentially two reasons why $I(V(J)) \supsetneq J$ in general

1. $J = (x^n) \subset k[x] \implies V(x^n) = \{0\}$ and $I(\{0\}) = (x)$
2. $(x^2 + 1) \subset \mathbb{R}[x]$ and $I(\emptyset) = \mathbb{R}[X]$

Lemme 11

For any $X \subset \mathbb{A}^n$, $I(X)$ is a radical ideal

Preuve

If $f^n \in I(X)$ for some n, then $f(x)^n = 0$ and hence $f(x) = 0$ □

So the first phenomenon is related to the fact that J is not radical, the second is related to the fact that $\mathbb{R}$ is not algebraically closed.

Theorème 12 (Hilbert's Nullstellensatz)

Let K be algebraically closed, $J \subset k[X_1, \ldots, X_n]$, then

$$I(V(J)) = \sqrt{J}$$

Using this, there is a one to one correspondence

$$\{ \text{ radical ideals in } k[X_1, \ldots, X_n] \ \} \leftrightarrow \{ \text{ algebraic subsets of } \mathbb{A}^n\}$$

Theoreme 13 (Weak Nullstellensatz)
Let K be algebraically closed, every maximal ideal $I \subset K[X_1, \dots, X_n]$ is of the form $I = \{x_1 - a_1, \dots, x_n - a_n\}$ with $a = (a_i) \in \mathbb{A}^n$

Corollaire 14
Let $I \subset K[X_1, \dots, X_n$ be any ideal, then $V(I)$ is a finite set $\iff$ $k[X_1, \dots, X_n]/I$ is a finite dimensional K- vector space.
In this case
$$|V(I)| \leq \dim_k k[X_1, \dots, X_n]/I$$

Preuve
Let $I \subset k[X_1, \dots, X_n]$ be any ideal and $P_1, \dots, P_n \subset V(I)$ distinct.
We can choose (Exercise) $F_1, \dots, F_r \in K[X_1, \dots, X_n]$ s.t. $F_i(P_j) = \delta_{ij}$, then we write $f_1, \dots, f_r$ for the residues of $F_1, \dots, F_r$ in $K[X_1, \dots, X_n]/I$.
We claim $f_1, \dots, f_r$ are linearly independent.
Indeed suppose $\sum_i \lambda_i f_i = 0$, this implies $\sum_i \lambda_i F_i \in I$ hence $0 = \sum \lambda_i F_i(P_j)$ which implies $\lambda_j = 0$, hence the f_i are linearly independent.
It follows that $\dim_k K[X_1, \dots, X_n]/I < \infty \implies |V(I)| < \infty$ and in this case $\dim_k K[X_1, \dots, X_n]/I \geq |V(I)|$.
Now assume $V(I)$ is a finite set $\{P_1, \dots, P_r\} \subset \mathbb{A}^n$ and write $P_i = (a_{i1}, \dots, a_{in})$ and define $F_j = \prod_{i=1}^r (X_j - a_{ij})$.
By construction $F_j \in I(V(I)) = \sqrt{I}$
$\exists N > 0$ such that $F_j^N \in I$.
Hence $f_j^N = 0$ in $K[X_1, \dots, X_n]/I$, but $f_j^N = (x_j^{Nr}) +$ lower order terms .
This means that X_j^{Nr} is a K-linear combination of $\{1, \dots, X_j^{Nr-1}\}$.
This means that X_j^s is a linear combination for any $s > 0$.
Hence taking products for different j's, we see that the set $\{x_1^{m_1}, \dots, x_n^{m_n}\}$ generates $K[X_1, \dots, X_n]/I$ □

Due to these theorems, we'll always suppose K is algebraically closed.

Lecture 3: Irreducible sets

Fri 11 Mar

1.5 Irreducible sets

Definition 2 (Irreducible set)
An algebraic set $V \subset \mathbb{A}^n$ is irreducible if $\forall W_1, W_2 \subset \mathbb{A}^n$ algebraic s.t. $V = W_1 \cup W_2$, then either $W_1 = V$ or $W_2 = V$

Exemple
— *Let $V = \{x_1, \dots, x_n\} \subset \mathbb{A}^n$ is irreducible iff $n = 1$*

— *Let* $f(X,Y) = Y(X^2 - Y), V = V(f) \subset \mathbb{A}^2$ *is not irreducible by taking* $W_1 = V(Y), W_2 = V(X^2 - Y)$

Proposition 16

An algebraic set V *is irreducible iff* $I(V)$ *is prime.*

Preuve

If $I(V)$ *is not prime, let* $F_1, F_2 \notin I(V)$ *s.t.* $F_1, F_2 \in I(V)$*, then we can write* $V = (V \cap V(F_1)) \cup (V \cap V(F_2))$.

Conversely, if $V = W_1 \cup W_2$ *and* $W_i \neq V$*, then* $I(W_i) \supsetneq I(V)$*, pick* $F_i \in I(W_i) \setminus I(V)$*, then* $F_1 F_2 = I(W_1) \cap I(W_2) = I(V)$. □

If $V \subset \mathbb{A}^n$ is irreducible, we can decompose it into a union of irreducible sets. The union is always finite as the polynomial ring is noetherian.

Theorème 17 (Theorem name)

Every $V \subset \mathbb{A}^n$ *algebraic can be written uniquely (up to ordering) as a union of irreducible sets.*

$$V = V_1 \cup \ldots \cup V_k$$

where the V_i*'s are irreducible and* $V_i \not\subset V_j \forall i \neq j$

Definition 3 (Irreducible Components)

The $V_1 \ldots V_k$ *are irreducible components of* V.

Remarque

Applying I *in theorem 1.9, we get*

$$I(V) = I(V_1) \cap \ldots \cap I(V_k)$$

and $I(V_i)$ *is the primary decomposition of* $I(V)$

In general, it is quite difficult to find this decomposition.

For hypersurfaces, it's easy, for $I(F)$, write $F = F_1^{\alpha_1} \cdot \ldots F_k^{\alpha_k}$, then $V(F) = V(F_1) \cup \ldots \cup V(F_k)$.

1.6 Algebraic subsets of $\mathbb{A}^2$

Lemme 19

Let $F, G \in k[X,Y]$ *with no common factors, then* $V(F) \cap V(G)$ *is a finite set of points.*

Preuve
By Gauss's lemma, F, G have no common factors in $k(X)[Y]$. Since $k(x)[Y]$ is a PID $\exists A, B \in k(X)$ such that

$$AF + BG = 1$$

Now there exists $C \in k[X]$ such that $AC, BC \in k[X]$.
Let $(x, y) \in V(F, G)$, then $C(x) = 0$ and hence there are only finitely many x's possible.
By symmetry, the same is true for the Y coordinate, hence $|V(F, G)| < \infty$ $\square$

Using this, we can now classify all algebraic subsets of $\mathbb{A}^2$.

Corollaire 20
The irreducible algebraic subsets of $\mathbb{A}^2$ are $\mathbb{A}^2, V(F)$ with F irreducible or singletons.

2 Affine algebraic varieties

Definition 4 (Affine algebraic variety)
An affine algebraic variety is an irreducible affine algebraic set.

2.1 Zariski topology

Definition 5 (Zariski topology)
The Zariksi-topology on $\mathbb{A}^n$ is the topology whose open sets are complements of algebraic sets.

Lemme 21
This indeed defines a topology on $\mathbb{A}^n$

Preuve
Certainly $\emptyset, \mathbb{A}^n$ are algebraic, hence their complements are open.
Let $\{U_i\}$ be a family of open sets, ie. such that

$$U_i = \mathbb{A}^2 \setminus V(I)$$

Then

$$\bigcup U_i = \bigcup \mathbb{A}^n \setminus V(I_i) = \mathbb{A}^n \setminus \bigcap_i V(I_i) = \mathbb{A}^n \setminus V(\bigcup I)$$

Similarly, if U_1, U_2 are open, then

$$U_1 \cap U_2 = \mathbb{A}^n \setminus I(V_1 V_2)$$

$\square$

is again open.

Exemple
If $n = 1$, then algebraically closed sets are either $\mathbb{A}^n, \emptyset$ are finite union of points so the Zariski topology is the cofinite topology. Hence the open sets are huge.

Definition 6
For $V \subset \mathbb{A}^n$ an algebraic variety or set, the Zariski topology on V is just the subspace topology.

Definition 7 (New definition of irreducibility)
A non-empty subset V of a topological space X is irreducible if it cannot be expressed as $V = W_1 \cup W_2$ where $W_1, W_2 \subsetneq V$ are closed subsets.

Lemme 23
A non-empty open subset of an irreducible topological space is again irreducible and dense.
Furthermore, if $V \subset X$ is irreducible, then so is $\overline{V}$

The proof is an exercise.

Definition 8 (Quasi-affine algebraic variety)
A quasi-affine variety is an open subset of an affine variety.

Remarque
By the lemma above, quasi-affine variety are also irreducible.

2.2 Regular functions and coordinate rings

Regular functions are the natural "continuous" functions on algebraic varieties.

2.2.1 Affine case

Definition 9
Let $V \subset \mathbb{A}^n$ be an affine algebraic variety.
A map

$$f : V \to K = \mathbb{A}^1$$

is regular if $\exists F \in k[X_1, \dots, X_n]$ such that

$$f(X) = F(X) \forall X \in V$$

The set $\Gamma(V)$ of regular functions on V is a ring with the usual pointwise multiplication and addition. and is called the coordinate ring of V.

Lemme 25
If $I = I(V)$ for some prime, then

$$\Gamma(V) \simeq k[X_1, \dots, X_n]/I(V)$$

In particular, $\Gamma(V)$ is a domain.

Preuve
By definition, we have a surjective morphism

$$k[X_1, \dots, X_n] \to \Gamma(V)$$

Now note that $F \in \ker \phi \iff F(X) = 0 \forall x \in V \iff F \in I(V)$ $\square$

Definition 10 (Subobjects)
An affine subvariety of V is an affine variety contained in V.

Lemme 26
There is a one-to-one correspondence between V and $\Gamma(V)$ where

$$\{ \text{ algebraic subsets of } V\} \leftrightarrow \{ \text{ radical ideals of } \Gamma(V)\}$$

$$\{ \text{ algebraic subvarieties of } V\} \leftrightarrow \{ \text{ prime ideals of } \Gamma(V)\}$$

$$\{ \text{ points of } V\} \leftrightarrow \{ \text{ maximal ideals of } \Gamma(V)\}$$

The proof is again an exercise.

Definition 11 (Morphism)
A morphism $\phi : V \to W$ between affine algebraic varieties $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ is a map such that $\exists$ polynomials $T_1, \dots, T_m \in k[X_1, \dots, X_n]$ such that

$$\phi(X) = (T_1(X), \dots, T_m(X))$$

Then ϕ is an isomorphism if there exists a morphism ψ such that $\phi \circ \psi = \mathrm{Id}$ and $\psi \circ \phi = \mathrm{Id}$.

Exemple
Take $V(X^2 - Y) \subset \mathbb{A}^2$ the the projection $p : V(X^2 - Y) \to \mathbb{A}^1$ on the first

coordinate is an isomorphism with inverse $\psi(X) = (X, X^2)$.
A non-example of a bijective map which is not an isomorphism :
$\phi : \mathbb{A}^1 \to V(Y^2 - X^3)$, $\phi(t) = (t^2, t^3)$.
One can check that ϕ *is bijective but not an isomorphism.*

Lecture 4: Morphisms of Affine Varieties

Fri 18 Mar

In general any morphism $\phi : V \to W$ induces a morphism of rings (of k-algebras) $\tilde{\phi} : \Gamma(W) \to \Gamma(V)$ by composition, ie.

$$\tilde{\phi}(f) = f \circ \phi$$

Proposition 28
This defines a one to one correspondence

$$\{ \text{ Morphisms } \phi : V \to W \text{ } \} \leftrightarrow \left\{ \text{ } k\text{-algebra homomorphisms } \tilde{\phi} : \Gamma(W) \to \Gamma(V) \text{ } \right\}$$

In particular ϕ is an isomorphism iff $\tilde{\phi}$ is an isomorphism.

Preuve
Need to construct for any $\alpha : \Gamma(W) \to \Gamma(V)$ *a morphism* $\overline{\alpha} : V \to W$ *s.t.*

$$\tilde{\overline{\alpha}} = \alpha$$

Suppose $V \subset \mathbb{A}^n, W \subset \mathbb{A}^m$ *and write*

$$\Gamma(V) = {}^{k[x_1, \ldots, x_n]}\!\diagup_{I(V)} \text{ and } \Gamma(W) = {}^{k[y_1, \ldots, y_m]}\!\diagup_{I(W)}$$

Choose lifts T_i *of* $\alpha([Y_i])$ *in* $k[x_1, \ldots, x_n]$.
In particular $\forall f \in \Gamma(W)$ *and* F *a lift,then*

$$\alpha(f) = F(T_1, \ldots, T_m) \mod I(V)$$

Then define $T : \mathbb{A}^n \to \mathbb{A}^m : x \mapsto (T_1(x) \ldots T_m(x))$.
We claim that $T(V) \subset W$.
From the diagram, we see that for any $G \in I(W)$, $G(T_1, \ldots, T_m) \in I(V)$, *hence for any* $v \in V$, $0 = G(T_1, \ldots, T_m)(v) = G(T(v))$ *which means that* $T(v) \in W$.
Now

$$\tilde{\overline{\alpha}} : \Gamma(W) \to \Gamma(V)$$

satisfies $\forall v \in V \forall f \in \Gamma(W)$

$$\tilde{\overline{\alpha}}(v) = f(\overline{\alpha}(v)) = f(T(v)) = \alpha(f(v)) \implies \tilde{\overline{\alpha}} = \alpha \qquad \square$$

Definition 12
The quotient field $K(V)$ of $\Gamma(V)$ is called the field of rational function on V.

Let $f \in K(V)$ is defined at a point $p \in V$ if we can write f as the quotient $f = \frac{a}{b}$ and $b(p) \neq 0$.
The pole set of $f \in K(V)$ is the set of points where f is not defined.

Remarque
$\Gamma(V)$ *is not a UFD in general, and so the presentation* $f = \frac{a}{b}$ *is not unique.*

Exemple
$V = (xy - zw) \subset \mathbb{A}^4$ *and let* $\overline{x}, \overline{y}, \overline{z}, \overline{w} \in \Gamma(V)$ *be the respective images.*
Then $f = \frac{\overline{x}}{\overline{y}} = \frac{\overline{z}}{\overline{w}}$.
Hence f *is defined whenever* $Y \neq 0$ *or* $w \neq 0$
Hence the pole set of f *is* $\{Y = 0\} \cap \{W = 0\}$

Definition 13 (Local Ring)
The local ring of V at a point $p \in V$ is a subring $K(V)$ defined by

$$\mathcal{O}_p(V) = \{f \in K(V) | f \text{ defined at } p\ \}$$

We have natural inclusions $\Gamma(V) \subset \mathcal{O}_p(V) \subset K(V)$

Remarque
$\Gamma(V), \mathcal{O}_p(V)$ *and* $K(V)$ *are intrinsic to* V*, ie. if* $V \simeq W$ *then* $\Gamma(V) \simeq \Gamma(W)$ *and* $\mathcal{O}_p(V) \simeq \mathcal{O}_{p'}(W)$

Proposition 32
Let $p \in V$ and $m_p \subset \Gamma(V)$ be the corresponding maximal ideal, then

$$\mathcal{O}_p(V) \simeq \Gamma(V)_{m_p}$$

In particular $\mathcal{O}_p(V)$ is a noetherian local domain and we have that

$$\Gamma(V) = \bigcap_{p \in V} \mathcal{O}_p(V) \subset K(V)$$

Preuve
Recall that $m_p = \{f \in \Gamma(V) | f(p) = 0\}$*, then*

$$\begin{aligned} \Gamma(V)_{m_p} &= \left\{ f \in K(V) | f = \frac{a}{b}, b \notin m_p \right\} \\ &= \mathcal{O}_p(V) \end{aligned}$$

The rest follows from standard properties of localization.

In particular for any domain R we have that

$$R = \bigcap_{m \in R, m \text{ maximal}} R_m$$

□

Notice that the notions of regular functions is sufficient to define morphisms of local rings etc.

How can we extend this to quasi-affine varieties ?

Exemple

Consider $V(XY-1) \subset \mathbb{A}^2$.

There is a natural projection $\phi : V(XY-1) \to x \in \mathbb{A}^1$.

The image of ϕ is $\mathbb{A}^n \setminus \{0\}$ quasi-affine and we'd like ϕ to be an isomorphism, ie.

$$\phi^{-1}(x) = (x, \frac{1}{x})$$

Ie. the map $x \to \frac{1}{x}$ should be a regular function on $\mathbb{A}^1 \setminus \{0\}$.

Definition 14

Let $V \subset \mathbb{A}^n$ be quasi-affine.

A map $f : V \to \mathbb{A}^1 = k$ is called regular if $\forall v \in V$ there exists an open neighbourhood $v \in U \subset V$ and $g, h \in k[x_1, \ldots, x_n]$ s.t. $h(V) \neq 0 \forall x \in U$ and $f(x) = \frac{g(x)}{h(x)}$

Why do we need the U ?

Exemple

Consider again $V = V(XY - ZW) \setminus V(Y, W)$ and consider $f = \frac{x}{w} = \frac{z}{y}$ on V.

None of the two presentations works on V

Definition 15

Let $\mathcal{O}(V)$ be the ring of regular functions on V

Remarque

$f : V\mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 : x \mapsto \frac{1}{x}$ is regular.

Then we may take $U = V$, it is not hard to see that

$$\mathcal{O}(V) = k[x][\frac{1}{x}, \frac{1}{x^2}, \ldots]$$

In particular $\mathcal{O}(V) \supsetneq \Gamma(\mathbb{A}^1)$

If $V \subset \mathbb{A}^n$ is affine, then we have $k[x_1, \ldots, x_n] \to \mathcal{O}(V) : F \mapsto (v \mapsto F(v))$.

Proposition 36

For V affine, we have that $\Gamma(V) \simeq \mathcal{O}(V)$.

Preuve
We have $O(V) \subset O_p(V)$ $\forall p \in V$ *hence* $\Gamma(V) \hookrightarrow O(V) \hookrightarrow \bigcap_{p\in V} O_p(V) = \Gamma(V)$ □

Lemme 37
Let V *be a quasi-affine subset and* $f : V \to \mathbb{A}^1$ *regular, then* f *is continuous (with respect to the Zariski topology)*

Preuve
It is enough to show that $f^{-1}(X)$ *is closed for any closed* X *.*
Without loss of generality $X = \{x\}$.
Let $V = \cup_i U_i i$ *a cover such that* $f|_{U_i} = \frac{g_i}{h_i}$ *and* $h_i \neq 0$ *on* U_i.
Then $f^{-1}(X) \cap U_i = \left\{v \in U_i | f(v) = \frac{g_i(v)}{h_i(v)}\right\} = \{v \in U_i | x \cdot h_i(v) - g_i(v) = 0\}$ *which is an algebraic set.*
Hence $f^{-1}(X) \cap U_i$ *is closed which implies* $f^{-1}(X)$ *is closed.* □

Corollaire 38
Let $f, g \in O(V)$ *and* $U \subset V$ *non empty and open s.t.* $f|_U = g|_U$ *then* $f = g$

Preuve
Using an exercise, open subsets are dense, since f, g *are continuous*

$$f|_U = g|_U \implies f|_{\mathrm{cl}\,U} = f|_{\mathrm{cl}\,V}$$ □

Remarque
Let $U \subset V$ *open, then the restriction of functions induces* $\mathcal{O}(V) \to \mathcal{O}(U)$.
i.e. $\mathcal{O}(-)$ *defines a sheaf of* k*-algebras on* V.
Using this one can define a general algebraic as a topological space X *with some sheaf* $\mathcal{O}_X$ *which locally looks like a quasi-affine variety* V *with* $\mathcal{O}(-)$.
We'll define $\mathcal{O}_p(V)$ *and* $K(V)$ *for* V *quasi-affine, but these depend only on "local structure".*
We can guess $\mathcal{O}_p(V) = \mathcal{O}_p(\mathrm{cl}\,V)$ *and similarly for the quotient field.*

3 (Quasi-)Projective and general algebraic varieties

Affine varieties usually "go to infinity" when we draw them.
This leads to complications in the theory

Exemple
Two distinct lines in $\mathbb{A}^2$ they will intersect in 1 point unless they're parallel

3.1 Projective space

Definition 16 (Projective n-space)
$\mathbb{P}^n$ is the set

$$\mathbb{P}^n = K^{n+1} \setminus \{0\}/\sim$$

Where we identify

$$(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1}) \text{ if } \exists \lambda \in K^* \text{ s.t. } x_i = \lambda y_i$$

Elements in $\mathbb{P}^n$ are called points.

If $p \in \mathbb{P}^n$ is the equivalence classe of $(x_1, \ldots, x_{n+1}) \in \mathbb{A}^{n+1}$ we write

$$p = [x_1 : \ldots : x_n]$$

$x_1, \ldots, x_n$ are the homogenuous coordinates of p.

Remarque
Any point in $\mathbb{A}^n \setminus \{0\}$ defines a line through the origin and $x, y \in \mathbb{A}^n \setminus \{0\}$ define the same line iff $x = \lambda y$

Lecture 5: Projective varieties

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While the i-th coordinate x_i of a point $[x_1 : \ldots : x_{n+1}] \in \mathbb{P}^n$ is not well defined, the equation $x_i = 0$ or $x_i \neq 0$ is well defined.
Hence we can write

$$U_i = \{[x_1 : \ldots : x_n] | x_i \neq 0\}$$

Clearly $\mathbb{P}^n = \cup_i U_i$.
Furthermore for all i, we have a bijection

$$\begin{aligned} \phi_i : \mathbb{A}^n &\to U_i \\ (x_1, \ldots, x_n) &\mapsto [x_1 : \ldots : x_{i-1} : 1 : x_{i+1} : \ldots : x_{n+1}] \end{aligned}$$

And this is clearly a bijection.
We'll see in a bit, that the ϕ_i's provide an open cover of $\mathbb{P}^n$ by $\mathbb{A}^n$

Definition 17
The set

$$H_\infty := \mathbb{P}^n \setminus U_{n+1} = \{x \in \mathbb{P}^n | x_{n+1} = 0\}$$

is called the hyperplane at infinity.

One can identify $H_\infty = \mathbb{P}^{n-1}$
Thus

$$\mathbb{P}^n = U_{n+1} \coprod H_\infty = \mathbb{A}^n \coprod \mathbb{P}^{n-1}$$

Exemple
$\mathbb{P}^0 =$ *point*
$\mathbb{P}^1 = \mathbb{A}^1 \coprod$ *point is called the projective line.*
Similarly $\mathbb{P}^2$ *is called the projective plane.*

3.2 Projective algebraic sets

For a general $F \in k[x_1, \ldots, x_n]$, the equation $F(x) = 0, x \in \mathbb{P}^n$ doesn't make sense.
But it does if F is homogeneous, say of degree d, since then

$$F(\lambda x) = \lambda^d F(x) = 0 \forall x \in \mathbb{A}^{n+1}, \lambda \in k^*$$

Definition 18 (Projective set)
For any set $S \subset k[x_1, \ldots, x_n]$ *of homogeneous polynomials we set*

$$V(S) = \{[x_1 : \ldots : x_n] \in \mathbb{P}^n | F(x_1, \ldots, x_n) = 0 \forall F \in S\}$$

A subset of $\mathbb{P}^n$ *is algebraic if it is of the form* $V(S)$ *as above.*

Exemple
Take $V(X^2 - YZ) \subset \mathbb{P}^2$*, how to draw it ?*
We draw the intersections $V \cap U_i$

Definition 19 (Homogeneous ideal)
An ideal $I \subset k[x_1, \ldots, x_n]$ *is homogeneous if it is generated by homogeneous elements.*
The for I *a homogeneous ideal we set*

$$V(I) = V(T) \subset \mathbb{P}^n$$

where T *is the set of forms in* I*.*

Remarque
Since the ring is noetherian, we can always find a finite number of homogeneous generators.
For $I = (x_1, \ldots, x_{n+1})$ *we have* $V(I) = \emptyset$*, we denote this ideal by* I_+*, it's called the irrelevant ideal.*

Exemple
(x, y^2) *is homogeneous,* $(x+y^2, y^2)$ *is also homogeneous but* $(x+y^2)$ *is not.*

Lemme 46
I *is a homogeneous ideal if and only if for every* $F \in I$*, if we write* $F = \sum_{i\geq 0} F_i$ *with* F_i *homogeneous of degree* i*.*

Preuve
Let $G^{(1)}, \ldots, G^{(k)}$ *be a set of homogeneous generators of* I *with degrees* $d_1, \ldots, d_k$*.*
Any $F = \sum F_i$ *can be written as* $F = \sum A^{(i)} G^{(i)}$ *for some* $A^{(i)}$*.*
Since the degree is additive we get $F_j = \sum A^{(i)}_{j-d_i} G^{(i)}$
For the other direction, let $G^{(1), \ldots, G^{(k)}}$ *any set of generators, then* $G_j^{(i)} \in I$ *and then the set of* $G_j^{(i)}$ *is a set of generators.* □

Furthermore, the sum, the product, the intersection and the radical of homogeneous ideals are homogeneous.
A homogeneous ideal is prime if for any homogeneous $f, g \in k[x_1, \ldots, x_n]$

$$fg \in I \implies f \in I \text{ or } g \in I$$

Definition 20 (Zariski topology)
We define the Zariski topology on $\mathbb{P}^n$ *by taking the open sets to be the complements of algebraic sets.*

This defines a topology using the properties above.

Definition 21
An algebraic set $V \subset \mathbb{P}^n$ *is irreduciable if it is irreducible as a topological space.*

As in the affine case, there is a correspondence

$$\{\text{ Algebraic subsets in} \mathbb{P}^n\} \leftrightarrow \{\text{ Homogeneous ideals in } k[x_1, \ldots, x_{n+1}]\}$$

Where $I(V)$ is the ideal generated by $\{F \in k[x_1, \ldots, x_n] | F \text{ homogeneous }, F(v) = 0 \forall v \in V\}$

Remarque
If we need to distinguish between the affine and projective correspondence we'll write V_a, I_a *and* V_p, I_p *respectively.*

Definition 22 (Cone)
For $V \subset \mathbb{P}^n$ algebraic, we define the conve over V as

$$C(V) = \left\{(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1} | [x_1, \dots, x_{n+1}] \in V\right\} \cup \{(0, \dots, 0)\}$$

Lemme 48

1. *For $V \neq \emptyset$, then*
$$I_p(V) = I_a(C(V))$$
2. *If $I \subsetneq k[x_1, \dots, x_n]$ homogeneous, then*
$$C(V_p(I)) = V_a(I)$$

Preuve

1. *$G \in I_p(V)$ homogeneous and $(x_1, \dots, x_{n+1}) \in C(V)$, then $G(x_1, \dots, x_{n+1}) = 0$*
Conversely, if $G \in I_a(C(V))$ write
$$G = \sum_i G_i, \quad G_i \text{ homogeneous}$$
Then, for every $x \in C(V)$ and $\lambda \in k^*$ we have $\lambda x \in C(V)$ hence
$$0 = G(\lambda x) = \sum_i \lambda^i G_i(x)$$
Let $\tilde{G}(y) = \sum_i y^i G_i(x) \in K[Y]$, this has infinitely many 0's.
Which in turn implies $G_i \in I_p(V)$
2. *Notice for G homogeneous non-constant, then*
$$C(V_p(G)) = V_a(G)$$
Since I is generated by homogeneous polynoials, the satement holds.
$\square$

Proposition 49 (Projective nullstellensatz)
Let I be a homogeneous ideal, then

— *If $V_p(I) = \emptyset$, then $\sqrt{I} = k[x_1, \dots, x_{n+1}]$ or $\sqrt{I} = I_+$*
— *If $V_p(I) = \emptyset$ then $I_p(V_p(I)) = \sqrt{I}$*

Preuve

— *If $V_p(I) = \emptyset \iff V_a(I) \subset \{(0, \dots, 0)\}$ which implies $\sqrt{I} \supset (x_1, \dots, x_{n+1})$.*

— $I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I}$ $\square$

Corollaire 50
There is a one-to-one correspondence between radical homogeneous ideals and projective algebraic sets.
Furthermore $V_p(I)$ is irreducible $\iff$ I is prime.

Remarque
Points in $\mathbb{P}^n$ do not correspond to maximal ideals.

We can also relate affine and projective algebraic sets through the charts

$$\phi_i : \mathbb{A}^n \to U_i$$

We'll focus on $\phi \coloneqq \phi_{n+1} : \mathbb{A}^n \to U \coloneqq U_{n+1}$
For $F \in k[x_1, \ldots, x_n]$ homogeneous, we define

$$F_*(x_1, \ldots, x_n) = F(x_1, \ldots, x_n, 1)$$

Conversely, for $G \in k[x_1, \ldots, x_n]$, we write

$G = \sum_{i=0}^{d} G_i$ and define $G^*(x_1, \ldots, x_{n+1}) = x_{n+1}^d G_0 + \ldots + G_d = X_{n+1}^d G(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}})$

Definition 23 (Homogenization)
$(\cdot)_*$ *and* $(\cdot)^*$ *are called dehomogenisation and homogenization.*

For I an ideal, we denote by I^* be the homogeneous ideal generated by $\{F^* | F \in I\}$.
Conversely, if $V = V_a(I)$, we write

$$V^* = V_p(I^*)$$

V^* is called the projective closudre of V in $\mathbb{P}^n$.
Similarly if I is homogeneous, then

$$I_* = \{F_* | F \in I\}$$

and if $V = V_p(I)$, we set $V_* = V_a(I_*)$

Exemple
Let $F = X_1^2 - X_2$, then

$$F^* = X_1^2 - X_2X_3$$

Lemme 53
If $V \subset \mathbb{A}^n$ is closed, then $\phi(V) = V^* \cap U$
Conversely, if $V \subset \mathbb{P}^n$ is closed then $\phi^{-1}(V \cap U) = V_*$
In particular ϕ is a homeomorphism

Preuve
Recall that $\phi(x_1, \ldots, x_n) = [x_1 : \ldots : x_n : 1]$.
For $V \subset \mathbb{A}^n$ write $V = V_a(F_1, \ldots, F_k)$ then

$$V^* = V_p(F_1^*, \ldots, F_k^*)$$

But $F_i^* = X_{n+1}^d F_i(\frac{x_1}{x_{n+1}, \ldots, \frac{x_n}{x_{n+1}}})$ $F_i(v) = 0 \iff F_i^*(\phi(v)) = 0 \implies \phi(V) = V^* \cap U$ □

Lecture 6: Algebraic varieties

Fri 01 Apr

3.3 Quasi-projective and general varieties

Definition 24
A projective variety is a closed irreducible subset of $\mathbb{P}^n$.
A quasi-projective variety is an open subset of a projective variety.
An algebraic variety is one of the four types we've seen : affine, quasi-affine, projective or quasi-projective.

Remarque
In order to define morphisms between varieties, we need regular functions, ie. $\mathcal{O}(V)$ for V quasi-projective.

Definition 25
Let $V \subset \mathbb{P}^n$ be quasi-projective, a map $f : V \to k$ is regular at $p \in V$ if $\exists$ an open neighbourhood $p \in U \subset V$ and $g, h \in k[x_1, \ldots, x_{n+1}]$ formes of the same degree such that $h(U) \neq 0$ and such that $f(u) = \frac{g(u)}{h(u)}$

Exemple
Set $V = \mathbb{P}^n$ and take also $U = V$ in the definition, then $h(u) \neq 0 \forall u \in \mathbb{P}^n$ is only possible if h is constant.
In fact, constants are the only regular functions on $\mathbb{P}^n$ and in fact on any projective variety.

Definition 26
We again, define $\mathcal{O}(V)$ the ring of regular functions on V.

Remarque
As in the quasi-affine case, regular functions are continuous for the Zariski topology.

3.4 Morphisms

Definition 27 (Morphism)
A morphism between two algebraic varieties V, W is a continuous map $\phi : V \to W$ such that for every open $U \subset W$ and every $f \in \mathcal{O}(U)$ the map $\phi \circ f : \phi^{-1}(U) \to k$ is regular.

ϕ is an isomorphism if $\exists \psi : W \to V$ such that $\phi \circ \psi$ and $\psi \circ \phi$ is the identity.

Remarque
In particular any $\phi : V \to W$ induces a map $\tilde{\phi} : \mathcal{O}(W) \to \mathcal{O}(V)$ a k-algebra homomorphism.
The converse is only true if W is affine.

Proposition 58
The maps $\phi_i : \mathbb{A}^n \to U_i \subset \mathbb{P}^n$ are isomorphisms.
In fact, for every $V \subset \mathbb{A}^n$ quasi-affine, $\phi_i|_V$ is an isomorphism onto it's image $\phi_i(V) \subset \mathbb{P}^n$ which is quasi-projective.

Preuve
We may take $i = n+1$ and write $\phi = \phi_{n+1} : \mathbb{A}^n \to U$
We know that ϕ is a homeomorphism, hence $\phi(V)$ is quasi projective since

$$\phi(V) \subset \phi(\overline{V}) = \overline{\phi(V)} = \overline{\phi(V)}^* \cap U \subset \overline{\phi(V)}^*$$

Let F be some regular function on some open $W \subset \phi(V)$.
By shrinking if necessary we can write $F = \frac{G}{H}$ where G and H are forms of the same degree and $H(w) \neq 0 \forall w \in W$.
Then
$$F \circ \phi = \frac{G \circ \phi}{H \circ \phi} = \frac{G_*}{H_*}$$
and since G_* and H_* don't vanish on $\phi^{-1}(W)$, $F \circ \phi$ is regular.
Thus $\phi|_V$ is a morphism of algebraic varieties.
Conversely, we have $\phi^{-1} : U \to \mathbb{A}^n$, let $W \subset V$, take $F \in \mathcal{O}(W)$, up to shrinking.
Then $F \circ \phi^{-1}([x_1 : \ldots : x_{n+1}]) = \frac{G(\frac{x_1}{x_{n+1}}, \ldots)}{H(\frac{x_1}{x_{n+1}}, \ldots)} = x_{n+1}^{\alpha} \frac{G^*([x_1 : \ldots : x_{n+1}])}{H^*[x_1 : \ldots : x_{n+1}]}$.
Now, if $\alpha \geq 0$ then x_{n+1}^{α} then $x_{n+1}^{\alpha} G^*$ and H^* are of the same degree.

If $\alpha < 0$, then G^* and $x_{n+1}^{-\alpha} H^*$.
Hence $F \circ \phi^{-1} \in \mathcal{O}(\phi(V))$ and hence ϕ^{-1} is a morphism. □

Definition 28
A variety is (quasi-) affine or quasi-projective if V is isomorphic to a quasi-affine or quasi-projective variety.

Corollaire 59
Any variety is quasi-projective.
Every quasi-projective variety admits a finite open cover by quasi-affine varieties namely $V = \bigcup_i V \cap U_i$

Preuve
projective implies quasi-projective and affine implies quasi-affine and the theorem above gives quasi-affine implies quasi-projective.
Furthermore, since $\mathbb{P}^n = \bigcup_i U_i \implies V = \bigcup V \cap U_i$.
If V is projective,$V \cap U_i \subset U_i \simeq \mathbb{A}^n$ is closed and irreducible, hence $V \cap U_i$.
Finally, if $V \subset \overline{V} \subset \mathbb{P}^n$ quasi-projective, then $V \cap U_i \subset \overline{V} \cap U_i$ is quasi-affine. □

Exemple
For any polynomial $f \in k[x_1, \ldots, x_n]$, $\mathbb{A}^n \setminus V(f)$ is affine, but $\mathbb{A}^2 \setminus \{0\}$ is not affine.

Remarque
The above also shows that if V is quasi-affine, then $\mathcal{O}(V) \simeq \mathcal{O}(\phi(V))$, hence all our definitions of $\mathcal{O}(-)$ are compatible.

The definition of a morphism is clean, but difficult to check in practice.
It becomes easier if at least the target is affine :

Lemme 62
Let $\phi : V \to W$ be a map, suppose $W \subset \mathbb{A}^n$ affine, and $x_i : \mathbb{A}^n \to \mathbb{A}^1$ coordinate functions, then ϕ is a morphism $\iff$ $x_i \circ \phi : V \to \mathbb{A}^1$ is regular.

Preuve
If ϕ is a morphism, then $x_i \circ \phi$ is regular by definition.
Conversely, if $x_i \circ \phi$ is regular for all $1 \leq i \leq n$, then

$$f \circ \phi$$

is also regular for any $f \in k[x_1, \ldots, x_n]$.
Since regular functions form a ring.

Hence

$$\begin{aligned}\phi^{-1}(V(f_1,\dots,f_k)\cap W) &= \bigcap \phi^{-1}(V(f_i)\cap W)\\ &= \bigcap (f\circ\phi)^{-1}(0)\end{aligned}$$

Thus ϕ is continuous.
Now let $U \subset W$ be open and $f \in \mathcal{O}(U)$, after shrinking, we may suppose $f = \frac{g}{h}$ where $g, h \in k[x_1,\dots,x_n], h(u) \neq 0 \forall u \in U$.
Then

$$f\circ\phi = \frac{g\circ\phi}{h\circ\phi}$$

where $g\circ\phi, h\circ\phi$ are regular and $h\circ\phi(v) \neq 0 \forall v \in V$.
Thus $\frac{1}{h\circ\phi}$ is regular, therefore $\frac{g\circ\phi}{h\circ\phi}$ is regular. □

Exemple
Now, the map $\mathbb{A}^1 \setminus \{0\} \to V(XY-1) \subset \mathbb{A}^2$ is a morphism.

Corollaire 64
Let V, W be two varieties and W affine, then there is a bijection

$$\hom_{Var}(V,W) \simeq \hom_{k-\ alg}\left(\mathcal{O}(W),\mathcal{O}(V)\right)$$

sending $\phi \to \tilde{\phi}$

Remarque
If V is projective, we claim that $\mathcal{O}(V) \simeq k$

3.5 General rational functions an local rings

Let V be an algebraic variety and $p \in V$.

Definition 29 (Local ring)
The local ring of V at p : $\mathcal{O}_p(V)$ is the set of pairs $\langle U, f\rangle$, where $U \subset V$ is open containing p and f is a regular function on U, modulo the relation

$$[U,f] \sim [U',f'] \iff f = f' \text{ on } U\cap U'$$

Lemme 66
$\sim$ is an equivalence relation and $\mathcal{O}_p(V)$ is a ring with the operations

$$[U,f]+[U',f'] = [U\cap U', f+f']$$

and similarly for the product

$$[U, f] \cdot [U', f'] = [U \cap U', f \cdot f']$$

Furthermore $\mathcal{O}_p(V)$ is a local ring with maximal ideal

$$m = \{[U, f] | f(p) = 0\}$$

Preuve
Reflexivity and identity is obvious, we need to check transitivity, suppose $[U, f] \sim [U', f']$ and $[U'', f''] \sim [U', f']$ then clearly $F = f''$ on $U \cap U' \cap U''$ but

$$U \cap U' \cap U'' \subset U \cap U''$$

is open and dense and since f, f'' are continuous, $f = f''$ on $U \cap U''$.
To show the ring is local, notice that there is an evaluation morphism

$$\mathcal{O}_p(V) \to k$$
$$[U, f] \to f(p)$$

This map is surjective since we have constant functions.
Hence m_p is a maximal ideal.
Finally $\mathcal{O}_p(V)$ is local since $[U, f] \notin m_p$ hence $f(p) \neq 0$.
Thus $\frac{1}{f}$ is regular in some neighbourhood of p.
Thus f is a unit. □

Lecture 7: rational functions and dimension

Fri 08 Apr

3.6 The field of rational functions

Definition 30 (Field of rational functions)
$K(V)$ is the set of pairs (U, f) with $U \subset V$ open and non-empty and f a regular function on U modulo the equivalence relation $(U, f) \sim (U', f')$ iff $f = f'$ on $U \cap U'$.

Remarque
Since V is irreducible, any non-empty open is dense, hence $U \cap U'$ is non-empty and open as well.

Furthermore, note that $[U, f]$ has $\frac{1}{f}$ as an inverse.
As in the affine case, we have inclusions

$$\mathcal{O}(V) \hookrightarrow \mathcal{O}_p(V) \hookrightarrow K(V)$$

Proposition 68

1. *Let V be an algebraic variety and $U \subset V$ open and non-emtpy.*
   Then $K(V) = K(U)$ and $\mathcal{O}_p(V) = \mathcal{O}_p(U)$ for any $p \in U$
2. *For any algebraic variety V and any $p \in V, K(V)$ is the quotient field of $\mathcal{O}_p(V)$.*
3. *If V is affine, then $\mathcal{O}_p(V) = \Gamma(V)_{m_p}$ for anu $p \in V$ and $K(V)$ is the quotient field at $\Gamma(V)$.*
   In particular all definitions of $K(V)$ and $\mathcal{O}_p(V)$ agree for affine varieties.

Remarque

1. *For $V \subset \mathbb{P}^n$ quasi-projective and $P \in V \cap U_i$ for some i, then $\mathcal{O}_p(V) = \mathcal{O}_p(\overline{V}) = \mathcal{O}_p(\overline{V} \cap U_i)$.*
   Since $U_i = \mathbb{A}^n, \overline{V} \cap U_i$ is affine and $\mathcal{O}_p(\overline{V} \cap U_i) = \Gamma(\overline{V} \cap U_i)_{m_p}$.
   Thus $\mathcal{O}_p(V)$ is always an explicit localization(as any variety is quasi-projective).

Preuve

1. *Immediate from the definition : $[W, f] \in \mathcal{O}_p(V)$.*
   We know that $[W, f] \sim [W \cap U, f] \in \mathcal{O}_p(U)$ so there is a bijection between the equivalence classes.
2. *As in the remark, we can reduce to the affine case and then it follows from the third part of the proposition.*
3. *We have a map $\Gamma(V) \to \mathcal{O}_p(V)$ sending $\frac{f}{g} \to [V, \frac{f}{g}]$.*
   It is injective by continuity (as V is open hence dense.).
   And also surjective by definition of $\mathcal{O}_p(V)$.
   Indeed, any $[U, h] \in \mathcal{O}_p(V)$ is of the form $h = \frac{f}{g}$ with $g(u) \neq 0$ on U.
   Since U is quasi-affine , f and g are both quotients of regular functions on $V = \overline{U}$.
   Similarly, we have an inclusion $Q(\Gamma(V)) \hookrightarrow K(V)$ sending $\frac{f}{g} \to [V, \frac{f}{g}]$.
   But any $[U, h] \in K(V)$ is contained in $\mathcal{O}_p(V) \subset Q(\Gamma(V))$ for any $p \in U$ so we also have a reverse inclusion. □

3.7 Dimension of a Variety

Should be a basic invariant, in fact it isn't.

Definition 31 (Dimension of a topological space)
The dimension of a topological space X is the largest integer n such that there exists a chain $Z_n \supset \ldots \supset Z_1 \supset Z_0$ of distinct irreducible closed subsets of X .
Then, the dimension of an algebraic set is it's dimension as a topological

space.

So now we'd like to relate this definition with the dimension theory for rings.
Recall that in any ring A, the height $ht(p)$ of a prime ideal $P \subset A$ is the supremum of all integers n such that $\exists$ a chain $p_0 \subset \ldots \subset p_n = p$ of distinct prime ideals.
The krull dimension of A is defined as

$$\dim A = \sup \{ht(p) | p \subset A \text{ prime } \}$$

Proposition 70
If $V \subset \mathbb{A}^n$ is an affine algebraic variety, then

$$\dim V = \dim \mathcal{O}(V)$$

Preuve
If $V_0 \subset V_1 \ldots \subset V_d = V$ is a maximal chain of irreducible closed subsets of V, then ($d < \infty$ by Noetherianity), applying I gives

$$I(V_1) \supset \ldots \supset I(V_d) = I(V)$$

is a chain of distinct prime ideals containing $I(V)$, so in the quotient we get a chain

$$p_0 \supset \ldots p_d = (0) \text{ in } \mathcal{O}(V)$$ □

So $\dim \mathcal{O}(V) \geq \dim V$, we can of course go the other way to find $\dim \mathcal{O}(V) \leq \dim V$

Computing dimensions is hard, the main tool is the following theorem

Theorème 71
Let K be a field and B a domain, which is a finitely generated K-algebra, then

$$\dim B = \text{ transcendence degree of } {}^{Q(B)}\!/_{K}$$

— *For any prime ideal $p \subset B$ we have*

$$ht(p) + \dim {}^{B}\!/_{p} = \dim B$$

Corollaire 72
$\dim \mathbb{A}^n = n$

Preuve
We have that

$$\dim \mathbb{A}^n = \dim K[x_1, \ldots, x_n] = trdeg K(x_1, \ldots, x_n) = n \qquad \square$$

Corollaire 73
If $V \subset \mathbb{P}^n$ is a quasi-projective variety, then

$$\dim \overline{V} = \dim V$$

To prove this, we need the following lemma

Lemme 74
If X is a topological space and $\{U_i\}$ a family of open subsets covering X, then

$$\dim X = \sup_i \dim U_i$$

Preuve (Of the Corrolary)
Let $U_1, \ldots, U_{n+1}$ be the open charts of $\mathbb{P}^n$, then $V_i = V \cap U_i$ gives an open cover of V by quasi-affine varieties and $\overline{V} \cap U_i = \overline{V_i}$ where the second closure is taken in $\mathbb{U}_i \simeq \mathbb{A}^n$.
If we know the corollary for quasi-affine varieties, we get that

$$\dim \overline{V} = \sup \dim \overline{V_i} = \sup \dim V_i = \dim V$$

Therefore, assume $V \subset \mathbb{A}^n$ is quasi-affine of dimension d and let $Z_0 \subset \ldots \subset Z_d$ be a maximal chain of irreducibles in V .
Hence $Z_d = V$ and $Z_0 = \{x\}$.
Let $m \subset \mathcal{O}(\overline{V})$ be the maximal ideal corresponding to x, we claim that $ht(m) = d$.
If this is true, then we have

$$\dim \overline{V} = \dim \mathcal{O}(\overline{V}) = \dim {}^{\mathcal{O}(\overline{V})}\!/_{m} + ht(m) = d$$

So now we have to prove the claim.
Clearly $ht(m) \geq d$.
Let $p_0 \subset \ldots \subset p_r = m$ be a longer chain of distinct prime ideals in $\mathcal{O}(\overline{V})$.
Set $W_i = V(p_i) \to \overline{V} = W_0 \supset \ldots \supset W_r = \{x\}$.

Intersecting with V gives

$$V = W_0 \cap V \supset \ldots \supset W_r \cap V = \{x\}$$

Since $W_i \cap V \subset W_i$ are open and non-empty, $W_i \cap V$ is irreducible.
By maximality of the initial sequence $\exists i$ such that

$$W_i \cap V = W_{i+1} \cap V$$

But then $W_i = \overline{W_i \cap V} = \overline{W_{i+1} \cap V} = W_{i+1}$ which is a contradiction. $\square$

From linear algebra, we would expect that if V is given by r "independent" equations should have dimension $n - r$.

Proposition 75

- *Let V be an affine variety of $\dim d$ and $H = V(F) \subset \mathbb{A}^n$ a hypersurface such that $V \subsetneq H$. Then every irreducible component of $V \cap H$ has dimension $d - 1$.*

- *Let $I \subset K[x_1, \ldots, x_n]$ be an ideal that can be generated by r polynomials, then every irreducible component of $V(I)$ has dimension $\geq n - r$*

Note that it is <u>not</u> true that if we choose the minimal number of generators, we get equality

Exemple
If $I = (XY, YZ) \subset K[x, y, z]$, then

$$V(I) = V(Y) \cup V(X, Z)$$

Lecture 8: Dimension

Fri 29 Apr

We'll use the following without proof :

Theorème 77 (Krull's Hauptidealsatz)
Let A be a noetherian ring, $f \in A$ neither a 0-divisor nor a unit.
Then any minimal ideal containing f has height 1.

We can now prove the above proposition.

Preuve
$V \not\subset H = V(f)$ which means $f \neq 0$ in $\mathcal{O}(V)$.
Hence, f is not a zero-divisor.
If f is a unit in $\mathcal{O}(V)$, then $V \cap H = \emptyset$ and there is nothing to prove.
Otherwise every irreducible component of $V \cap H \subset V$ corresponds to a prime ideal p in $\mathcal{O}(V)$ and p is minimal .
Hence $ht(p) = 1$ and thus $\dim V(p) = \dim(\mathcal{O}(V)/p) = \dim(\mathcal{O}(V)) - ht(p) =$

$d-1$

To prove the second statement we argue by induction.
For $r=0$, this is trivially true.
Let $I=(f_1,\dots,f_r)$ with $r\geq 1$ and W an irreducible component of $V(I)$.
By induction any irreducible component W' of $V(f_1,\dots,f_{r-1})$ has dimension $n-(r-1)$.
By a every irreducible component of $W'\cap V(f_r)$ has dimension $\geq n-r$.
W is a union of irreducible components of $W'\cap V(f_r)$. □

4 Local Properties of plane curves

Definition 32 (Plane curve)
Let $F,G\in K[x,y]$ are equivalent if $\exists\lambda\in K^\times$ such that $F=\lambda G$.

An affine plane curve is an equivalence class of non constant polynomials in $K[x,y]$.
If $F=\prod F_i e^i$ with F_i irreducible.
We call F_i the components of F ad e_i the multiplicities. Note that we can recover the F_i from $V(F)$ but not the e_i's.
The degree of an affine plane curve is the degree of F as a polynomial.
A line is a curve of degree 1.
If F is irreducible, we write $\Gamma(F),\mathcal{O}_p(F)$ for $\Gamma(V(F))$ etc.

Definition 33 (Singular point)
Let F be a plane curve and $P=(a,b)\in F$.
Then P is called a simple point if either

$$\frac{\partial F}{\partial X}(p)=F_X(p)\neq 0 \text{ or } \frac{\partial F}{\partial Y}(p)\neq 0$$

Ie., if the jacobian of F has full rank. In this case, the line $L(x,y)=F_x(p)(X-a)+F_y(p)(Y-b)$ is the tangent line to F at p .
A point which is not simple is called singular or multiple.
A curve with only simple points is called non-singular or smooth.

We'll usually arrange things such that $p=(0,0)$ is a singular point of F.
Writing $F=F_m+\dots+F_n$ for F_i forms of degree i, $(0,0)\in F \iff m\geq 1$

Definition 34
The integer $m=m_p(F)$ is called the multiplicity of F at $p=(0,0)$

We have that $p = (0,0)$ is simple iff $m = 1$.
And in this case F_1 is the tangent line to F at $(0,0)$.
If $m = 2$ then $p = (0,0)$ is called a double point.
Since F_m is homogeneous and in two variables we can factor it as

$$F = \prod L_i^{r_i}$$

where L_i are distinct lines through the origin.
To see this, notice that the dehomonigenization $(F_m)_* = F_m(X,1)$ factors into linear terms.

Definition 35
The L_i are called the tangent lines to F at $(0,0)$.

L_i is a simple (double, triple,...) tangent if $r_i = 1(2,3)$.
A point P is an ordinary multiple point of F , if F has m distinct tangents at p.
An ordinary double point is called a node. It is in many ways the simplest example of a singular point.

Exemple
Let $F = Y^2 - X^3 - X^2 = F_2 + F_3$.
We can write $F_2 = (Y - X)(Y + X)$

If $F = \prod F_i^{e_i}$ then $m_p(F) = \sum_i e_i m_p(F_i)$.
And if L is a tangent to F_i with multiplicity r_i, then L is a tangent to F with multiplicity $\sum e_i r_i$.
For $p = (a,b)$ let $T(x,y) = (x + a, y + b)$ and set $F^T(x,y) = F(T(x,y))$.
We then define $m_p(F) = m_{(0,0)}(F^T)$ and similarly for tangent lines, multiple points etc.

Theorème 79
Let P be a point on an irreducible plane curve F and $\mathfrak{m}_p(F) \subset \mathcal{O}_p(F)$ the corresponding maximal ideal.
Then for sufficiently large n, we have that

$$m_p(F) = \dim_K \left({\mathfrak{m}_p(F)^n}/{\mathfrak{m}_p(F)^{n+1}} \right)$$

Exemple
Let $F = X$ and $p = (0,0)$, then

$$\mathfrak{m}_P(F) = (Y,X) \subset \left({k[x,y]}/{(x)} \right)_{(x,y)}$$

and

$$\mathfrak{m}_p(F)^n = (Y^n)$$

and thus ${}^{(Y^n)}\!/\!_{(Y^{n+1})}$ *is generated by* Y^n

We'll need the following lemma

> **Lemme 81**
> *Let* $I \subset K[X_1, \ldots, X_n]$ *be an ideal such that* $V(I) = \{P_1, \ldots, P_n\}$ *is finite.*
> *Let* $\mathcal{O}_i = \mathcal{O}_{P_i}(\mathbb{A}^n)$.
> *Then there is an isomorphism*
> $$\Phi : {}^{K[x_1, \ldots, x_n]}\!/\!_{I} \to \prod {}^{\mathcal{O}_i}\!/\!_{I \cdot \mathcal{O}_i}$$

We'll prove the theorem assuming this.
Let's write $\mathcal{O}, \mathfrak{m}$ and m for $\mathcal{O}_p(F), \mathfrak{m}_p(F)$ and $m_p(F)$.
We have the exact sequence

$$0 \to {}^{\mathfrak{m}^n}\!/\!_{\mathfrak{m}^{n+1}} \to {}^{\mathcal{O}}\!/\!_{\mathfrak{m}^{n+1}} \to {}^{\mathcal{O}}\!/\!_{\mathfrak{m}^n} \to 0$$

The theorem follows if $\dim({}^{\mathcal{O}}\!/\!_{\mathfrak{m}^n}) = n \cdot m + s$ and all $n \geq m$ Without loss of generality $P = (0,0)$ and hence $\mathfrak{m}^n = I^n \mathcal{O}$.
Then

$${}^{\mathcal{O}}\!/\!_{\mathfrak{m}^n} = {}^{\mathcal{O}}\!/\!_{I^n \mathcal{O}} = {}^{\mathcal{O}_p(\mathbb{A}^2)}\!/\!_{(F, I^n)} = {}^{K[x,y]}\!/\!_{(F, I^n)}$$

Notice $\forall G \in I^{n-m}$ $GF \in I^n$.
We get a short exact sequence

$${}^{K[x,y]}\!/\!_{I^{n-m}} \xrightarrow{F} {}^{k[x,y]}\!/\!_{I^n} \to {}^{K[x,y]}\!/\!_{(F, I^n)} \to 0$$

Since $\dim_k {}^{K[x,y]}\!/\!_{I^n} = \frac{n(n+1)}{2}$ we have

$$\dim {}^{K[x,y]}\!/\!_{(F, I^n)} = \frac{n(n+1)}{2} - \frac{(n-m)(n-m+1)}{2} = mn - \frac{m(m-1)}{2}$$

Lecture 9: stuff

Fri 06 May

We now show the lemma above.

Preuve
Let $\mathfrak{m}_i = \mathfrak{m}_{p_i}$ *and* $R = {}^{k[x_1, \ldots, x_n]}\!/\!_{I}$ *and* $R_i = {}^{\mathcal{O}_i}\!/\!_{I\mathcal{O}_i}$.
For each i *we have the localisation map*

$$\phi_i : {}^{K[x_1, \ldots, x_n]}\!/\!_{I} \to {}^{\mathcal{O}_i}\!/\!_{I\mathcal{O}_i}$$

And we define ϕ *as the product of the* ϕ_i .
To show that ϕ *is an isomorphism, we construct idempotents :* $e_1, \ldots, e_n \in R$

such that

$$e_i^2 = e_i, e_ie_j = 0 \text{ and } \sum_i e_i = 1$$

Further, we'll want $e_i(P_i) = 1$ *and* $\forall G \in k[x_1, \dots, x_n]$ *with* $G(P_i) \neq 0$, $\exists t \in R$ *such that* $tg = e_i$ $(g = \overline{G})$.
Let's suppose we have such e_i*'s, then*

1. ϕ *is injective.*
   Indeed, if $f \in \ker\phi_i \iff \exists G$ *with* $G(P_i) \neq 0$ *and* $GF \in I$ *Thus,* $f \in \ker\phi \implies \forall i \exists G_i$ *with* $G_i(P_i) \neq 0$ *and* $G_iF = 0$

$$f = \sum_i e_i f = \sum_i t_i g_i f$$

2. ϕ *is surjective.*
   Let $z = (\frac{a_1}{g_1}, \dots, \frac{a_n}{g_n}) \in \prod R_i$ *where* $g_i(P_i) \neq 0$.
   Let $t_i \in R$ *be such that* $t_ig_i = e_i$, *since* $e_i(P_i) = 1$, $\phi_i(e_i) \in R^*$.
   $\phi_i(e_j) = \phi_i(e_je_i)\phi(e_i)^{-1} = 0$
   Thus

$$\phi_i(e_i) = \phi_i(\sum_i e)i) = \phi_i(1) = 1$$

   Hence $\phi_i(t_ig_i) = 1$ *and*

$$\frac{a_i}{g_i} = \phi_i(a_it_i)$$

Let's construct $e_1, \dots, e_n$ *:*
By the nullstellensatz, we have that $\sqrt{I} = I(P_1, \dots, P_n) = \bigcap_{i=1}^N \mathfrak{m}_i$
Thus there exists d *such that* $(\bigcap_i \mathfrak{m}_i)^d \subset I$.
Choose F_i *such that* $F_i(P_j) = \delta_{ij}$ *and set* $E_i = 1 - (1 - F_i^d)^d$.
Then the residues $e_i \in R$ *of* E *satisfy 1 and 2.*
We have $E_i = F_i^d D_i$ *thus* $E_i \in \mathfrak{m}_j^d \forall j \neq i$.
Then $i \neq j$ *implies* $E_iE_j \in \bigcap_k \mathfrak{m}_k^d = (\bigcap \mathfrak{m}_k)^d \subset I$.
Then

$$1 - \sum_i E_i = 1 - E_j = \sum_{i \neq j} E_i \in \bigcap_k \mathfrak{m}_k^d \subset I$$

Further

$$E_i - E_i^2 = E_i(1 - F_i^d)^d \in \bigcap \mathfrak{m}_j^d \mathfrak{m}_i^d \subset I$$

Let $G \in k[x_1, \dots, x_n], G(P_i) \neq 0$, *say* $G(P_i) = 1$, $H = 1 - G \in \mathfrak{m}_i$.
Then $H^dE_i \in \mathfrak{m}_i \bigcap_{j \neq i} \mathfrak{m}_j^d \in I$.
Then

$$g(e_i + he_i + \dots + h^{d-1}e_i) = e_i - h^de_i = e_i \qquad \square$$

4.1 Curves and DVR's

Proposition 82

Let R be a domain that is not a field, then the two following things are equivalent

1. *R is a noetherian, local and the maximal ideal is principal.*
2. *There exists an irreducible element $t \in R$ such that every non zero element $z \in R$ can be written uniquely as $z = U \cdot t^n$, where $U \in R^\times$ and $n \in \mathbb{Z}_{\geq 0}$.*

Definition 36 (Discrete valuation ring)
A ring satisfying these properties is called a discrete valuation ring.

Exemple

1. *$K[[t]]$, with maximal ideal (t)*
2. *$k[x]_{(x)} = \left\{ \frac{f}{g} | g(0) \neq 0 \right\}$*
3. *$\mathbb{Z}_{(p)}$.*

We prove the equivalence above.

Preuve

$1 \implies 2$

Let $\mathfrak{m} \subset R$, let $t \in \mathfrak{m}$ be a generator.
Since $(t) = \mathfrak{m}$, t is irreducible.
Assume $ut^m = vt^n$ with $n \geq m$.
Then

$$u = vt^{n-m} \in R \setminus \mathfrak{m}$$

Assume $\exists z \in R \setminus \{0\}$ which is not of the form ut^n .
Since z is not a unit, $z \in (t)$.
Thus there exists z_1 such that $z = tz_1$.
If z_1 is a unit, we're done, otherwise we get $z_1, \ldots,$ with $z_i = tz_{i+1}$.
Since R is noetherian, the chain

$$(z) \subset (z_1) \subset \ldots$$

stabilizes.
Thus, there exists a $v \in R$ such that $z_{n+1} = vz_n = vtz_{n+1}$.

$2 \implies 1$

Clearly $\mathfrak{m} = (t)$ is the set of non-units, thus R is local with maximal ideal (t).
It's enough to show that every ideal of R is finitely generated.
Let $I \subset R$ be an ideal, then $I \subset \mathfrak{m}$.
Let $n = \min \{n | \exists n \in R^\times$ such that $ut^n \in I\}$.
Then $t^n \in I$, but any $z \in I$ is of the form ut^m and thus I is finitely generated.

The proof also shows that any ideal in R is of the form (t^n), thus DVR $\Longrightarrow$ PID. $\square$

Definition 37 (uniformizer)
A generator t of $\mathfrak{m}$ in DVR is a uniformizer.

If $K = Q(R)$ denotes the quotient field, then any $z \in K$ can be written uniquely as $z = ut^n$ with $u \in R^\times$ and $n \in \mathbb{Z}$.
The integer n is called the order of z and is denoted $ord(z)$.

Corollaire 84
Let F be an irreducible plane curve and $p \in F$, then p is simple iff $\mathcal{O}_p(V)$ is a DVR.

In this case, if $L = aX + bY + c$ is any line through P that is not tangent to F, then it's image in $\mathcal{O}_p(F)$ is a uniformizer.

Definition 38
If $p \in F$ is simple, F irreducible, we write ord_p^F for the order on $K(F)$.

We have $ord_p^F(g) = \dim_K({}^{\mathcal{O}_p(F)}\!/\!_{(g)})$ for any $g \in \mathcal{O}_p(F)$.

Preuve
If $\mathcal{O}_p(F)$ is a DVR, let $\mathfrak{m}_p(F) = (x)$, then

$$\mathfrak{m}_p(F)^n = (x^n) \to \mathcal{O}_p(F) \to k$$

Where we send λx^n to λ.
This map is surjective with kernel (X^{n+1}), thus

$$m_P(F) = \dim_K({}^{(X^n)}\!/\!_{X^{n+1}}) = 1$$ $\square$

Conversely, let P be a simple point, say $P = (0,0)$.
Let T be the unique tangent and $L \neq T$ a line as in the statement.
$\exists A \in GL_2(K)$ such that $AT = \{y = 0\}$ and $AL = \{X = 0\}$.
Since $FA \simeq F$ as varieties, we may assume that $T = Y, L = X$.
Then $F = Y+$ terms of higher order.
We need to show $\mathfrak{m}_p(F) = (x, y) \subset \mathcal{O}_p(F)$ is principal with generator X.
Write $F =$ (All monomial containing Y) $+$ Rest $= Y(1 + H(x, y)) + X^2G(x)$.
The image of $1 + H$ is a unit in $\mathcal{O}_p(F)$.
Thus $Y = x^2G(x)(1 + H)^{-1} \in \mathcal{O}_pF$.
Thus $(x, y) = (x)$ is principal, thus $\mathcal{O}_p(F)$ is a DVR.