Color Coding

Speeding up Network Searches

858L

Efficient Algorithms for Detecting Signaling Pathways in Protein Interaction Networks

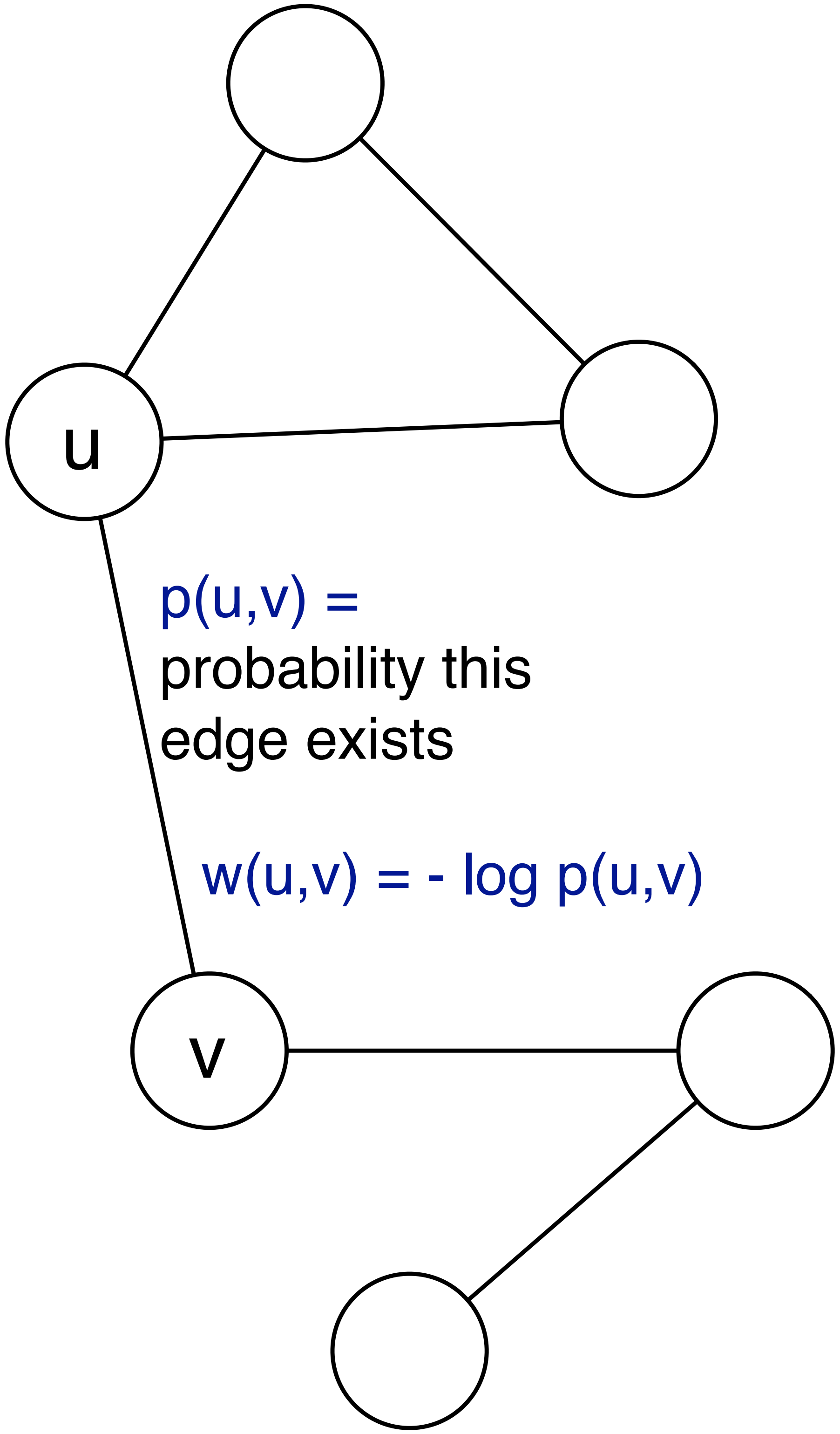
Scott, Ideker, Karp, Sharan

RECOMB 2005

- Color Coding: Alon et al, 1995.

Searching for High Scoring Paths

Weighted network G:



G might be an alignment graph, a PPI network, metabolic network, etc...

P = simple path

Weight(P) = sum of *w(u,v)* values along its edges

Length(P) = number of nodes in P

Goal: Low-weight, simple, length-k paths

Given: Graph G, a subset of nodes I, and a node v.

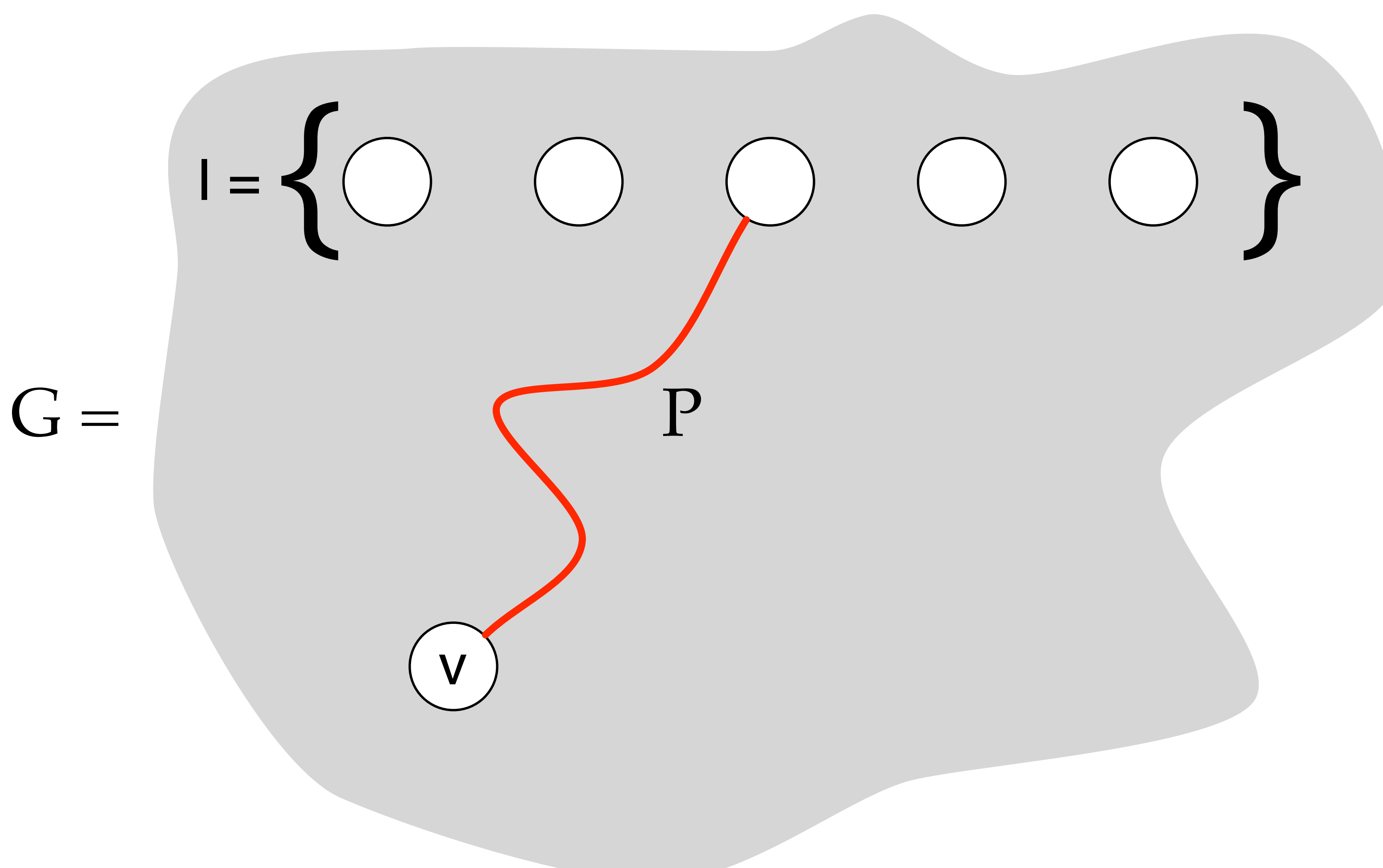
Find: The lowest-weight path P that:

(1) starts at some vertex in I

(2) ends at v

(3) is of length k and is simple (doesn't use any vertex twice)

Set I let's us specify, e.g., that the path should start at a surface receptor protein.



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Given: Graph G, a subset of nodes I, and a node v.

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Without the simple condition or length-k condition, the problem is easy.

Dynamic Programming Algorithm

v ∈ S   Set of ≤ k vertices

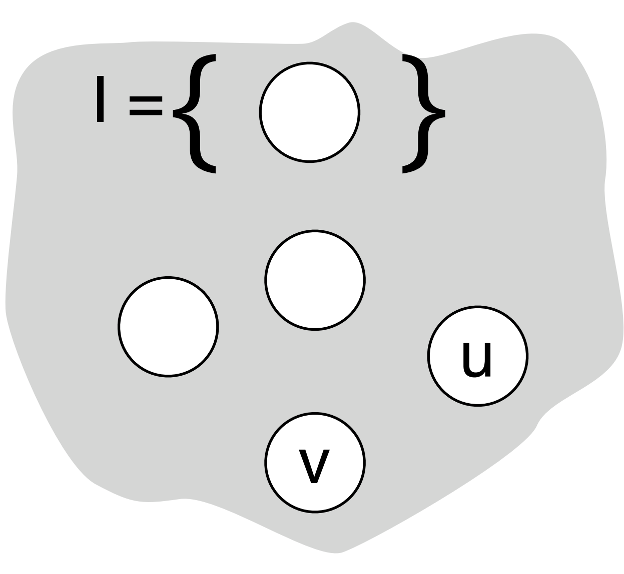
W(v, S) := minimum weight of a simple path that starts at I, visits each vertex in S, and ends at v, and is of length |S|.

W(v, S) := ∞ if no such path exists.

$$W(v, \{v\}) = \begin{cases} 0 & \text{if } v \in I \\ \infty & \text{if } v \notin I \end{cases}$$

$$W(v, S) = \min_{u \in S - \{v\}} W(u, S - \{v\}) + w(u, v)$$

Smaller size "S" set, so we can compute W(•, •) in order of increasing size of S.



Ok, So:

$$\mathrm{OPT}(I, v) = \min_{S:|S|=k} W(v, S)$$

Note how “simple” this algorithm is: try all possible sets of *k* nodes, compute their optimal order, and return the best set.

What’s the running time?

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What's the running time?

Number of sets we will consider = all possible subsets of nodes of size ≤ k =

$$\sum_{i=0}^{k} \binom{n}{i} = n^k$$

For each set, computing the min takes at most O(k) steps.

Therefore: Running time = $O(kn^k)$.

Color Coding

- $O(kn^k)$ is too slow for any interesting k.

- Can we do better?

- Idea: rather than keeping track of all of S, we'll keep track of less information about which nodes we've already visited.

- This will introduce a problem: we may miss the optimum path...

Color Coding

Main Step: Randomly color each node with a color from $\{1,2,\ldots,k\}$. Let $c(u)$ be the color of node u.

Define: a path is "colorful" if it contains exactly 1 vertex of each color.

Note: any colorful path is simple.

So, we consider this modified problem:

Given: Graph G, a subset of nodes I, and a node v.
Find: The lowest-weight, colorful, length-k path between I and v.

Color Coding DP Algorithm

$c(v) \in C$ Set of $\leq k$ colors

$\overline{W}(v, C)$:= minimum weight of a path that starts at I, visits a vertex of each color in C, ends at v, and is of length $|C|$.

$\overline{W}(v, C) := \infty$ if no such path exists.

$$\bar{W}(v, C) = \min_{u:c(u) \in C - \{c(v)\}} \bar{W}(u, C - \{c(v)\}) + w(u, v)$$

"C" keeps track of the remaining allowed colors.

Intuition for faster run time: we must consider only 2^k possible sets "C" instead of $O(n^k)$

$$\sum_{i=0}^{k} \binom{k}{i} = 2^k$$

Alternative View of Color Coding Algorithm

Let *I* be the given *starting* node set
Let colorings(u, j) be the set of valid path colorings for a path of length j-1 from *I* to u

```
For all u in I: colorings(u,1) = {c(u)}
For j = 1, ..., k:
  For every edge (u, w):
    For every C in colorings(u, j):
      If c(w) not in C:
        Add C U {c(w)} to colorings(w, j+1).
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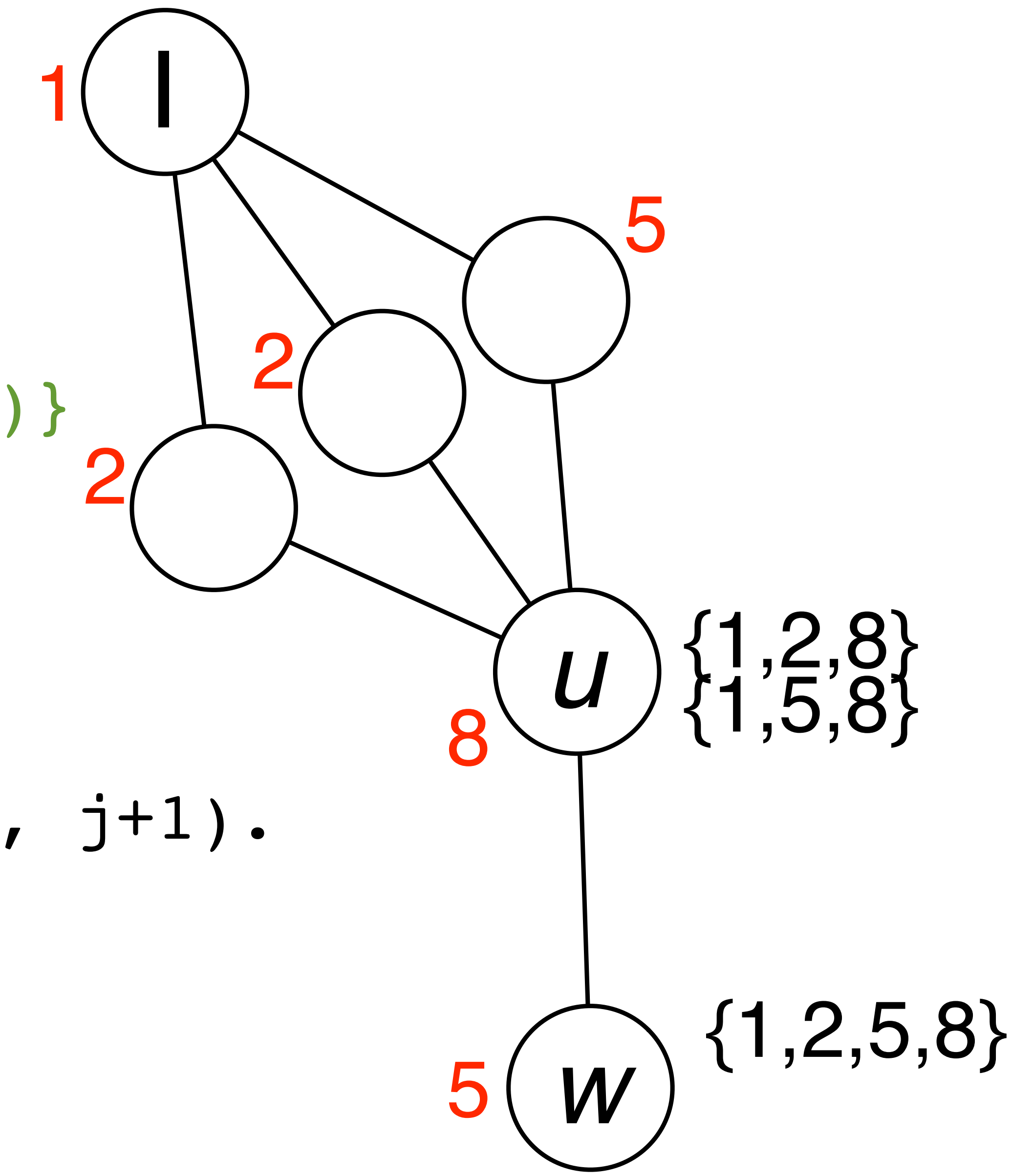

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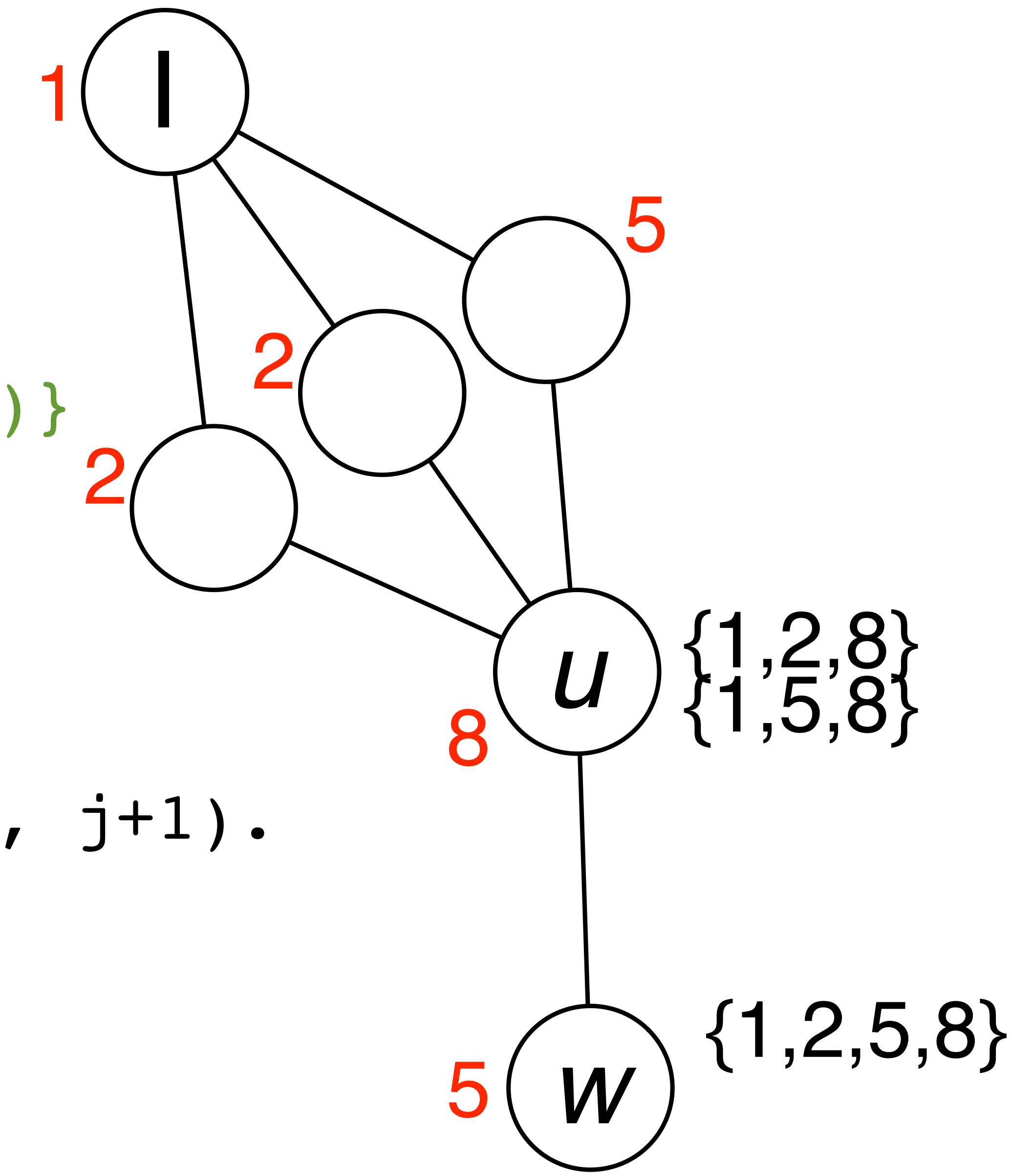


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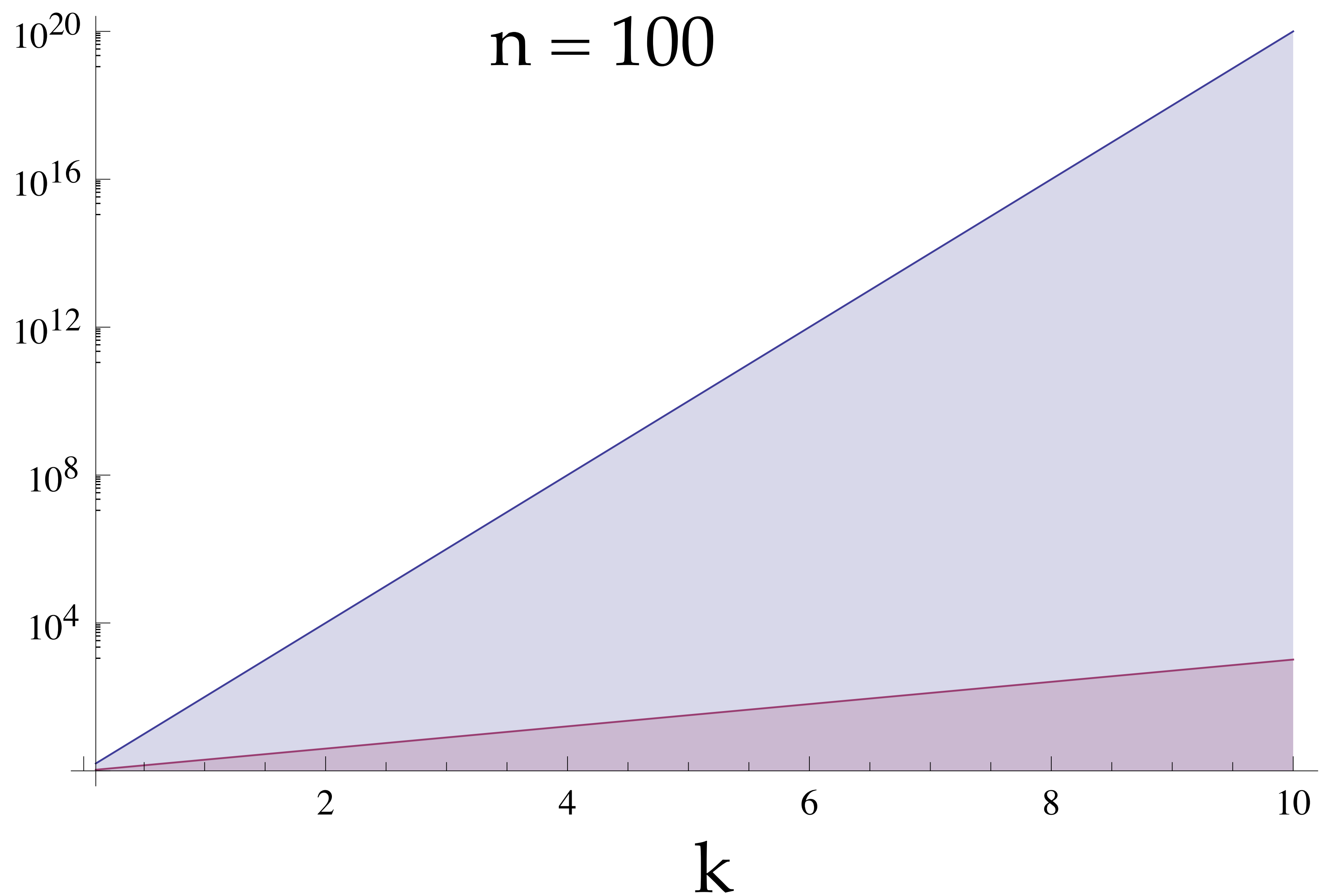
Running time:

$$\sum_{j=0}^{k}\left[|E|\binom{k}{j}j\right] = O(2^k k|E|)$$

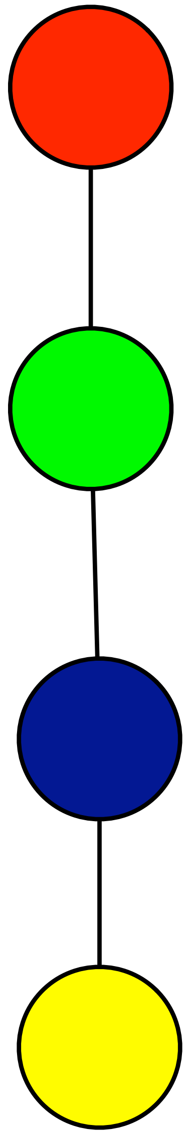
So:

We had an algorithm that was ≈ $O(n^k)$
We converted it into an ≈ $O(2^k)$ algorithm,
but with an ε probability we'll miss the optimal answer.



What if the optimal path is not colorful?

Have to repeat this procedure enough times so that the probability that that happens is low.



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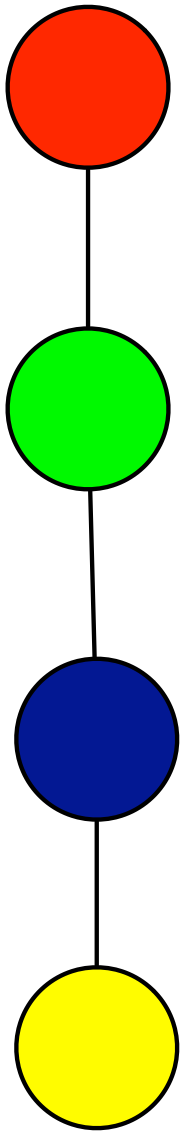

k! ways to make a path colorful.

k^k ways to color a path.

Pr[Path is colorful] = $k!/k^k \geq e^{-k}$.

Pr[OPT is colorful] $\geq e^{-k}$.
Pr[OPT is not colorful] $< (1-e^{-k})$

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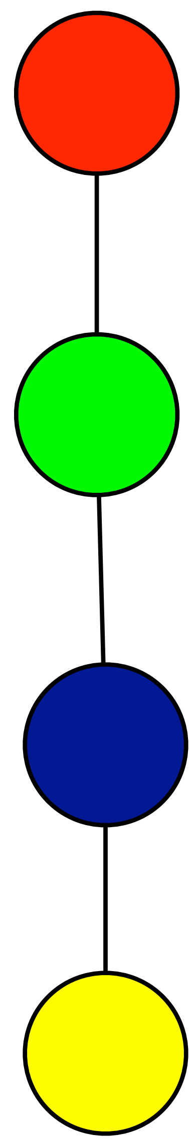


Repeat algorithm

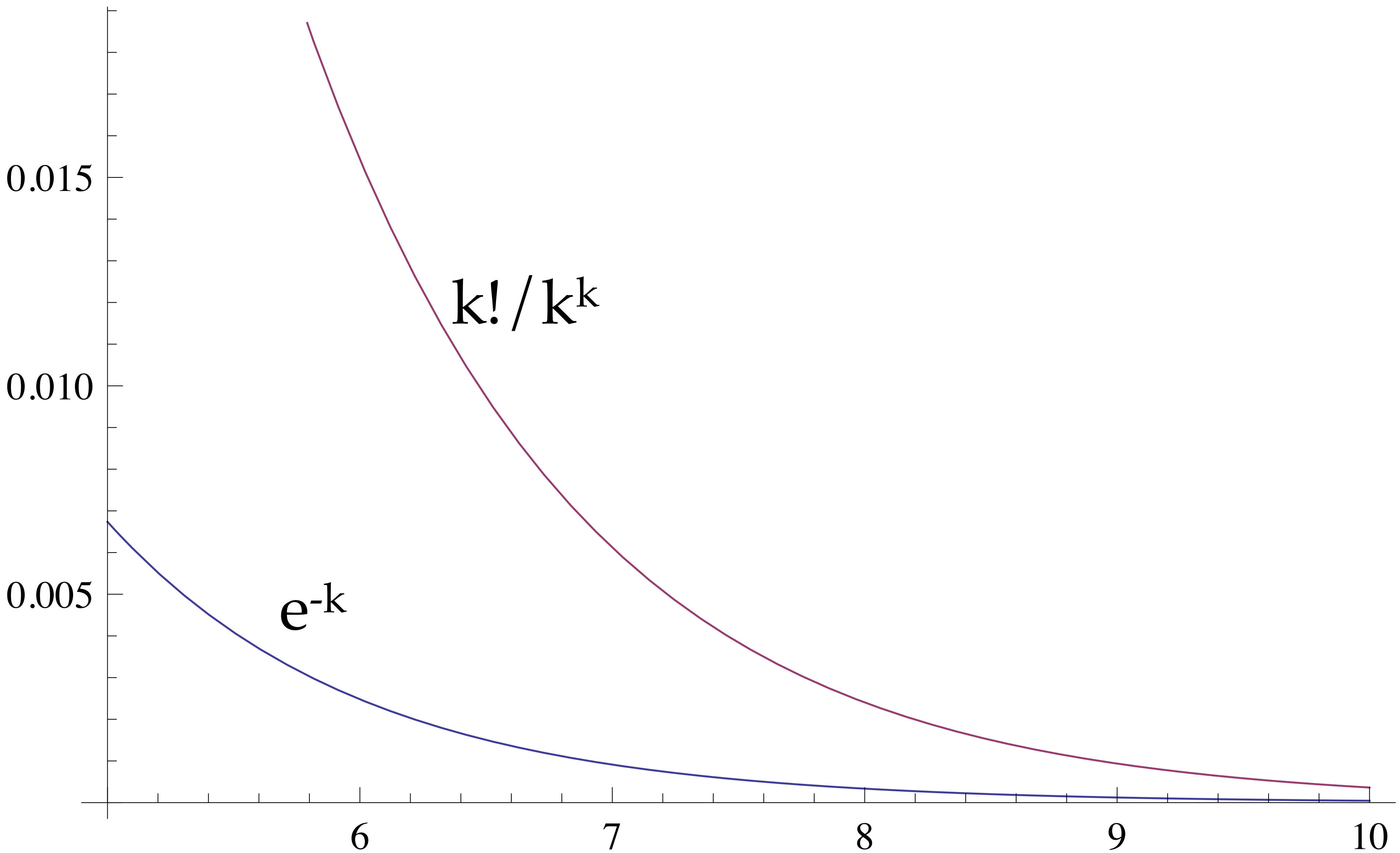
$$-e^k \ln \epsilon$$

times.

$$\Pr[\text{OPT is never colorful}] \leq$$

$$\left(1 - e^{-k}\right)^{-e^k \ln \epsilon} = \left[\left(1 + \frac{1}{-e^k}\right)^{-e^k}\right]^{\ln \epsilon}$$

$$\leq e^{\ln \epsilon} = \epsilon$$

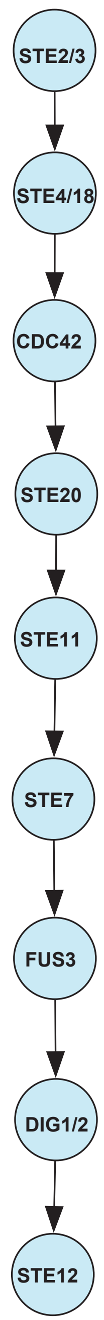

k!/k^k
e^-k
0.015
0.010
0.005
6
7
8
9
10

Running Times

Yeast Network with ~4,500 nodes and ~14,500 edges:

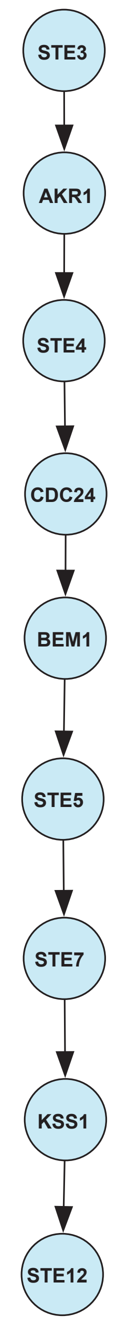
Path length	Success probability	#Paths	Time (sec)
10	99.9%	100	5613
9	99.9%	100	1241
8	99.9%	500	322
8	99.9%	300	297
8	99.9%	100	294
8	90%	100	99
8	80%	100	75
8	70%	100	61
8	50%	100	42
7	99.9%	100	86
6	99.9%	100	36

Pheromone Response Pathway



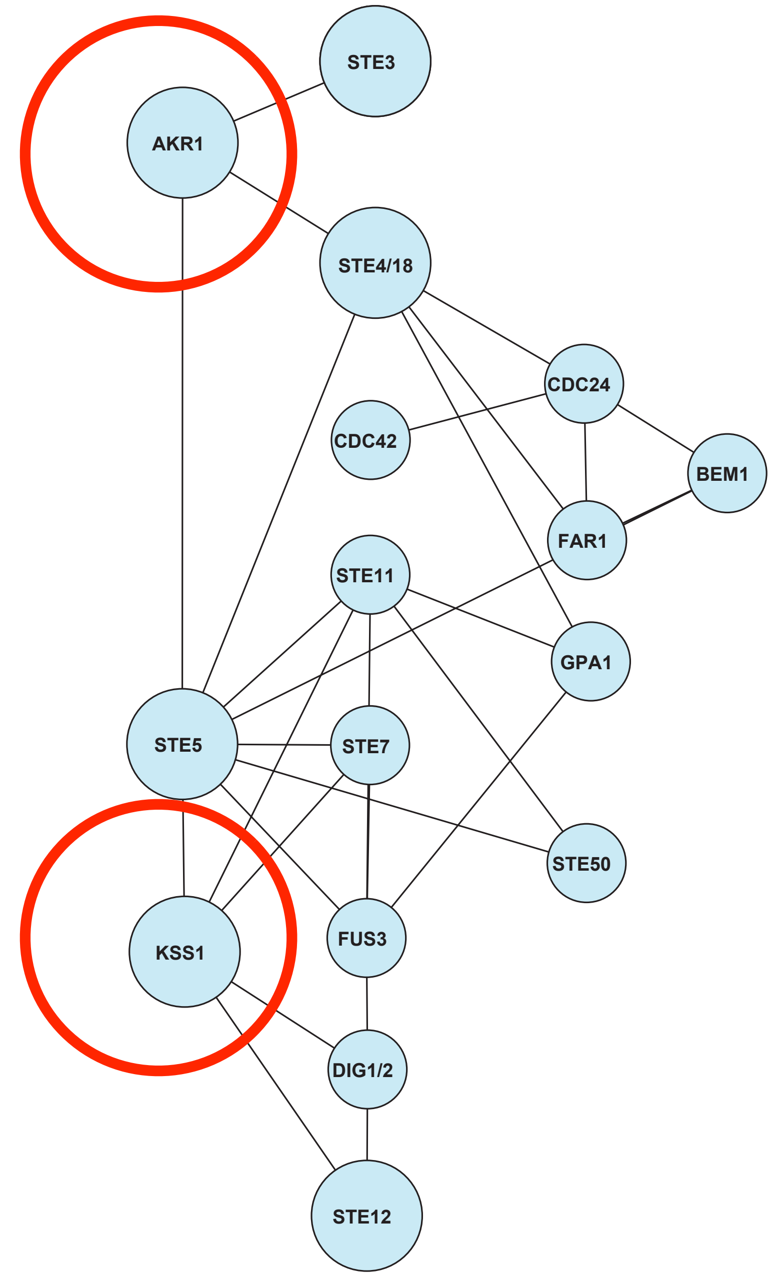
(a)

Known pathway



(b)

Best length-9 pathway between STE3 and STE12



(c)

Collection of all low-weight paths between STE3 and STE12

Color Coding Summary

- Turned a slow, $O(n^k)$ algorithm into a less-slow $O(2^k)$ algorithm that is correct with high probability.
- Used on yeast to identify signaling pathways.
- Directly extends to finding good-scoring pathways in the alignment graph of PathBLAST.

- Color Coding: Alon et al, 1995.