# General Relativity Notes

Charles Yang

Last updated: November 27, 2021

# Contents

# Part I

# idk

# Chapter 1

# Introduction

## 1.1 Why GR?

Consider the following conundrum:

$$m_I \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \mathbf{F}$$

we see that the inertial mass term coincides with the mass term in Newton's law of gravitation:

$$F = \frac{GM}{r^2} m_g$$

This is known as the Weak Equivalence Principle. There is no apparent reason why these two masses need be related so directly. Yet, numerous experiments show that these two masses to be identical to at least one part in $10^{13}$ .

Another unintuitive observation is that contrary to the Newtonian view, an object at rest on a table experiences an acceleration while a falling object experiences no acceleration. In an inertial frame, in which the object *experiences* no acceleration (such as an object in a parabolic trajectory), we see that gravity vanishes; gravity is not a true force but a fictitious force. Rather than gravity, the acceleration of an object that we observe is dictated by the curvature of space-time, and the requirement to travel along a "straight-line path." Even on earth, we can see that a laser pointer is deflected by a gravitation field, *despite having no mass*. We will see that mass defines the curvature of space-time, while the curvature determines how the masses will move.

# Chapter 2

# Geometry

In order to describe GR properly, we need to have a proper grasp of geometry. Rather than viewing a sphere as a round surface, we need rather understand it as a 2D surface rather than an object embedded in 3D space; we need to understand the differential geometry of space.

Consider an ant living on a sphere of radius $r$. If the ant is at the polar angle $\theta$ and determines a circle of constant $\theta$, the ant can infer the curvature of the surface it lives on. The radius of the circle can be found

$$r = a\theta$$

while the circumference of the circle is given

$$C = 2\pi(a \sin\theta) = 2\pi a \sin\left(\tfrac{r}{a}\right)$$

In the limit $a \gg r$, we can expand the sine about small angles as

$$C \to 2\pi r - \frac{1}{3}\pi r \left(\tfrac{r}{a}\right)^2 + \mathscr{O}\left[\left(\tfrac{r}{a}\right)^5\right]$$

## 2.1 Line Elements

Such geometric tests can be extended to the infinitessimal neighbourhood of a point via the *line element*. If we consider two points in 2D Euclidian space, $p, q$, we can find the distance between the two points as

$$\Delta s^2 = \Delta x^2 + \Delta y^2$$

Of course, we can do the same in the infinitessimal limit of pythagoran theorem:

$$\mathrm{d}s^{\,2} = \mathrm{d}x^{\,2} + \mathrm{d}y^{\,2}$$

However, this is only true in flat euclidian space. For instance, if we instead use polar coordinates $(r, \theta)$, our line element becomes

$$\mathrm{d}s^{\,2} = \mathrm{d}r^{\,2} + r^2\,\mathrm{d}\theta^{\,2}$$

We can integrate over these line elements to find the lengths of curves:

$$L = \int_\gamma \mathrm{d}s$$

for the case of a circle:

$$C = \oint_\gamma \mathrm{d}s = \oint_\gamma \mathrm{d}x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \oint_\gamma \mathrm{d}\theta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}$$

However, in the second case, we have $\frac{\mathrm{d}r}{\mathrm{d}\theta} = 0$ for the circular curve $\gamma$ and thus it reduces to

$$C = \oint_\gamma R \,\mathrm{d}\theta = 2\pi R$$

We see that different coordinate systems lend themselves to different problems.

Let us now consider the line element on a 2-sphere. We leave it as an exercise to show

$$\mathrm{d}s^2 = r^2 \,\mathrm{d}\theta^2 + (r \sin\theta)^2 \,\mathrm{d}\phi^2$$

We see that the line element is dependent on the size of the 2-sphere.

The most important feature of a line element is that it is *invariant*; its magnitude is the same no matter what coordinate system is used. Let us return to the simple coordinate systems of cartesian and polar. We have the the relations

$$x = r\cos\theta \qquad y = r\sin\theta$$

Trivially, we have

$$\begin{aligned} \mathrm{d}x &= \cos\theta \,\mathrm{d}r - r\sin\theta \,\mathrm{d}\theta \\ \mathrm{d}y &= \sin\theta \,\mathrm{d}r + r\cos\theta \,\mathrm{d}\theta \end{aligned}$$

we can then show

$$\begin{aligned} \mathrm{d}x^2 + \mathrm{d}y^2 &= (\cos\theta \,\mathrm{d}r - r\sin\theta \,\mathrm{d}\theta)^2 + (\sin\theta \,\mathrm{d}r + r\cos\theta \,\mathrm{d}\theta)^2 \\ &= \mathrm{d}r^2 + r^2 \,\mathrm{d}\theta^2 \end{aligned}$$

## 2.2 Invariance and Geometry

Consider a point $P$ with a frame $(x, y)$ and a rotated frame $(y', x')$. These two coordinates are related by a rotation, which is a linear transformation. From simple trigonometry we can write

$$\begin{aligned} x' &= \cos\theta x + \sin\theta y \\ y' &= -\sin\theta x + \cos\theta y \end{aligned}$$

Naturally, we can write this as a matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Thus, we can write

$$R(\theta) \simeq \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

so

$$\mathbf{r}' = R(\theta)\mathbf{r} \tag{2.2}$$

if we consider a new point $Q$ at $\mathbf{r}_Q \simeq \langle \tilde{x}, \tilde{y} \rangle$ we see that it transforms just as $P$. Finding the distance between these two points, we find

$$(\mathbf{r}'_Q - \mathbf{r}'_P) = R(\theta)[\mathbf{r}_Q - \mathbf{r}_P]$$

so

$$\mathrm{d}s^2 = \left\|\mathbf{r}'_Q - \mathbf{r}'_P\right\|^2 = [\mathbf{r}_Q - \mathbf{r}_P]^\intercal R^\intercal(\theta) R(\theta)[\mathbf{r}_Q - \mathbf{r}_P] = \|\mathbf{r}_Q - \mathbf{r}_P\|^2$$

so that under a rotation the line element is unchanged.

More generally, if we consider an inner product between two vectors:

$$\mathbf{p}^\intercal \mathbf{q} = p^1 q^1 + p^2 q^2$$

we see that this is preserved under rotations:

$$\mathbf{p}'^\intercal \mathbf{q}' = \mathbf{p}^\intercal R^\intercal R \mathbf{q} = \mathbf{p}^\intercal \mathbf{q}$$

Operators that satisfy $O^\intercal O$ are orthogonal, and are defined by $\det O = \pm 1$. For rotations, we consider the negative determinant, and are left with $SO(2)$, or the special orthogonal group in dimension 2.

## 2.3 Vectors

We now consider the line element. Consider

$$\mathrm{d}\mathbf{x} = \begin{bmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{bmatrix} \tag{2.3}$$

It has the property where

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{x}^\intercal \, \mathrm{d}\mathbf{x}$$

This provides a basis for our coordinate system. In this sense, $\mathrm{d}\mathbf{x}$ is the prototypical template for a vector; a vector is only a vector if it transforms like $\mathrm{d}\mathbf{x}$.

Consider two observers; one in a rotated frame and relative to another. The vector field is given

$$\mathbf{v}(\mathbf{x}') = R\mathbf{v}(\mathbf{x})$$

Physics should not depend on the observer; the laws of physics should be invariant. If we apply a rotation to Newton's law

$$mR\mathbf{a} = R\mathbf{F}$$

and so

$$m\mathbf{a}' = \mathbf{F}'$$

or Newton's law maintains its form. Thus, we say the equation is *covariant*, as it transforms the same way as the vectors.

<u>$N$-Dimensions</u>

Consider an $N$ dimensional space. We can write the line element for euclidean space as

$$\mathrm{d}s^2 = \sum_{i=1}^{D} (\mathrm{d}x^i)^2$$

Similarly, a matrix can be represented as

$$M^i_j$$

Where we begin to use Einstein summation. We can write a transformation as

$$\mathbf{u} = M\mathbf{v} \implies u^i = M^i_j v^j$$

A useful mnemonic is "upper indices go up and down, lower indices go left right."

Note that if we have a vector $\mathbf{p}' = R\mathbf{p}$, that a vector

$$\mathbf{q} = \begin{bmatrix} ap^1 \\ bp^2 \end{bmatrix}$$

it is only a vector if $a = b$. This is because if we write $\mathbf{q} = A\mathbf{p}$, if we transform $\mathbf{p}$ then convert it to $\mathbf{q}$, we do not in general get the same result as if we converted $\mathbf{p}$ to $\mathbf{q}$ then transformed it. Thus, $\mathbf{q} = A\mathbf{p}$ does *not* transform like a vector. However, note one important subtlety. The transformation that defines what a vector is in this case is not in fact represented by $R$; rather, $R$ *restricted* to $p$ is a representation of the transformation law. In general, this law need not be linear.

For our purposes, an object $V^\mu$ is a vector if it transforms like

$$V'^\nu = \Lambda^\nu_\mu V^\mu \tag{2.4}$$

### 2.3.1 Contravariance

Consider a general change of variable, where we obtain

$$\mathrm{d}x'^i = \frac{\partial x'^i}{\partial x^j} \mathrm{d}x^j$$

we have

$$S^i_j = \frac{\partial x'^i}{\partial x^j}$$

More generally,

$$\partial_i = \frac{\partial}{\partial x^i}$$

and

$$\partial'_i = \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i}\frac{\partial}{\partial x_j} = (S^{-1})\frac{\partial}{\partial x^j} = (S^{-1})^j_i \partial_j$$

We see that the *contravariant* $\partial_i$ transforms with the inverse of the transform that the *covariant* $\mathrm{d}x^i$ transforms with.

## 2.4 Metric Tensor

Equipped with our new-found knowledge of tensors we can redefine the line element in terms of the *metric tensor* $g_{\mu\nu}$:

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\ \mathrm{d}x^\nu \tag{2.5}$$

For example, the metric in euclidian spacetime is given $g_{\mu\nu} = \delta_{\mu\nu}$, while the metric for the 2-sphere is given

$$g_{\mu\nu} \simeq \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

and for polar coordinates the metric is given

$$g_{\mu\nu} \simeq \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

In GR, we will have a symmetric metric—that is, we will have

$$g_{\mu\nu} = g_{\nu\mu} \tag{2.6}$$

This is the result of a few key assumptions we will discuss later.

Given a metric in one coordinate system and a change of coordinates to another system, we can transform the metric to gain the metric in the new system. Consider an arbitrary transformation

$$x^\mu \to x'^\mu$$

The line element is invariant under such a transformation. Thus,

$$\begin{aligned} \mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\ \mathrm{d}x^\nu &= g'_{\mu\nu}\,\mathrm{d}x'^\mu\ \mathrm{d}x'^\nu \\ &= g'_{\mu\nu}(\partial_\rho x'^\mu)(\partial_\sigma x'^\nu)\,\mathrm{d}x^\sigma\ \mathrm{d}x^\nu \end{aligned}$$

or alternatively

$$g'_{\rho\sigma} = g_{\mu\nu}(\partial'_\rho x^\sigma)(\partial'_\sigma x^\rho) \tag{2.7}$$

## 2.5 Tensors

An object $T^{\mu\nu}$ is a tensor if it transforms like a tensor:

$$T'^{\mu\nu} = \Lambda^{\mu}_{\sigma}\Lambda^{\nu}_{\rho}T^{\sigma\rho} \tag{2.8}$$

More generally, we can have tensors such as $W^{\mu\nu}_{\sigma\xi\rho}$.

## 2.6 Coordinate Transformations

In general, we have

$$\begin{aligned} \mathrm{d}x'^{\mu} &= \frac{\partial x'^{\mu}}{\partial x^{\nu}}\,\mathrm{d}x^{\nu} \\ \mathrm{d}x^{\mu} &= \frac{\partial x^{\mu}}{\partial x'^{\nu}}\,\mathrm{d}x'^{\nu} \end{aligned}$$

giving us a representation for our transformation as

$$S^{\mu}_{\nu}(x) = \frac{\partial x'^{\mu}}{\partial x^{\nu}}$$

$$(S^{-1})^{\mu}_{\nu}(x) = \frac{\partial x^{\mu}}{\partial x'^{\nu}}$$

And so trivially, we find

$$(S^{-1})^{\mu}_{\rho}S^{\rho}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\rho}}\frac{\partial x'^{\rho}}{\partial x^{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\nu}} = \delta^{\mu}_{\nu}$$

## 2.7 Area and Volume

For a diagonal metric, We can write an area element

$$\mathrm{d}A = \mathrm{d}\ell^1\ \mathrm{d}\ell^2 = \sqrt{g_{11}g_{22}}\,\mathrm{d}x^1\ \mathrm{d}x^2$$

the 3-volume element

$$\mathrm{d}V = \sqrt{g_{11}g_{22}g_{33}}\,\mathrm{d}x^1\ \mathrm{d}x^2\ \mathrm{d}x^3$$

and 4-volume element

$$\mathrm{d}\mathcal{V} = \sqrt{g}\,\mathrm{d}x^0\ \mathrm{d}x^1\ \mathrm{d}x^2\ \mathrm{d}x^3$$

where

$$g = \det(g_{\alpha\beta})$$

# Chapter 3

# Action Principle

## 3.1 Variational Principle

In ordinary calculus, we extremize a function with respect to variables; in variational calculus, we extremize a functional with respect to its function parameters. For example, consider a string hanging between two points. The potential energy of a particular shape of the string $\phi$ with boundary conditions $\phi(\pm L/2) = 0$ can be given by the functional $E[\phi(x)]$

$$E[\phi(x)] = \int_{-L/2}^{L/2} \mathrm{d}x \, \left[ \frac{T}{2} \left( \frac{\mathrm{d}\phi}{\mathrm{d}x} \right)^2 - \sigma g \phi \right]$$

we will consider small deviations from an assumed extrema

$$\phi \to \phi + \delta\phi$$

Then

$$\delta E = E(\phi + \delta\phi) - E(\phi)$$

Taylor expanding, we obtain

$$E(\phi + \delta\phi) = \int \mathrm{d}x \, \left[ \frac{T}{2} \left( \frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{\mathrm{d}\delta\phi}{\mathrm{d}x} \right)^2 - \sigma g(\phi + \delta\phi) \right]$$

so, keeping only first order terms[1],

$$\delta E = \int \mathrm{d}x \, T \left( \frac{\mathrm{d}\phi}{\mathrm{d}x} \frac{\mathrm{d}\delta\phi}{\mathrm{d}x} \right) - \sigma g \delta\phi$$

Integrating by parts, and discarding boundary conditions, we find

$$\delta E = \int \mathrm{d}x \, \left( -T \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \sigma g \right) \delta\phi$$

[1] While second order terms in $\mathrm{d}\delta\phi \, / \, \mathrm{d}x$ don't necessarily vanish, we will assume they do.

Thus, we have

$$\frac{\delta E}{\delta \phi} = -T\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \sigma g = 0$$

The solution to this shows that the energy is extremized with $\phi$ a parabola.

<u>Sidenote of Variations</u>

Note that we can compute

$$\delta\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\phi + \delta\phi) - \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\phi}{\mathrm{d}x} + \frac{\mathrm{d}\delta\phi}{\mathrm{d}x} - \frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\delta\phi}{\mathrm{d}x}$$

so we can show that

$$\delta\frac{\mathrm{d}\phi}{\mathrm{d}x} = \frac{\mathrm{d}\delta\phi}{\mathrm{d}x}$$

As another example, from the energy functional for the gravitational potential and a mass density $\rho$,

$$E[\phi] = \int \mathrm{d}^3x \left[\frac{1}{8\pi G}(\nabla\phi)^2 + \rho\phi\right]$$

we can take a variation to obtain Poisson's equation.

$$\nabla^2\phi = 4\pi G\rho$$

### 3.1.1 Newton's Law

Recall that we showed that Newton's Law is invariant under translation and rotation, but not under constant rotation. Rewriting as a covariant equation,

$$m\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = F^i$$

Further, we can write the force in terms of the potential as

$$F^i = -\partial^i V$$

Multiplying by velocity, we obtain

$$m\ddot{x}^i\dot{x}^i = -\partial_i V\dot{x}^i$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}m\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + V(x)\right] = 0$$

so energy is conserved.

## 3.2 Action Principle

Newton's law can be derived from the action functional

$$S[q] = \int_0^T L(q, \dot{q}, t)\, \mathrm{d}t \tag{3.1}$$

That is, we can determine the trajectory of a particle by extremizing its action; this is the *principle of least action*.

$$\delta S = 0 \tag{3.2}$$

Take the Lagrangian to be that of a particle in a gravitational field:

$$L = \frac{1}{2} m\dot{q}^2 - mgq$$

We have already solved this problem in our hanging string. Substituting variables, we obtain Newton's law:

$$m\ddot{q} = -mg = -V'$$

When we take the variation of an arbitrary Lagrangian as in Equation 3.1, we obtain the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \tag{3.3}$$

# Chapter 4

# Symmetry and Relativity

## 4.1 Symmetry and Conservation

Recall the lagrangian can be written

$$L(\dot{q}, q, t) = \frac{1}{2}\dot{q}^2 - V(q)$$

The Lagrangian exhibits a symmetry if it is invariant under a transformation. Consider a translation $q = q + \delta q$. Then, for the Lagrangian to exhibit symmetry, we must have

$$0 = \delta L = \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\partial L}{\partial q}\delta q$$

This comes from the total differential of $L$. Substituting Euler-Lagrange into the second term,

$$0 = \frac{\partial L}{\partial \dot{q}}\delta\dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]$$

Thus, the quantity

$$Q = \frac{\partial L}{\partial \dot{q}}\delta q$$

is conserved. This is *Noether's Theorem*—for every symmetry in the lagrangian, there is a corresponding conserved quantity.

### 4.1.1 Gallilean Group

Newton's laws are invariant under the Gallilean group—the group generated by time translation, 3 spatial translations, 3 rotations, and 3 Gallilean boosts. Recall a gallilean boost is given

$$x'^i = x^i + v^i t \qquad t' = t$$

and that the line element is given

$$\mathrm{d}s^2 = \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

Trivially, we can indeed show that Newton's law is invariant under such a transformation.

However, we immediately see one new relation: velocities transform as

$$u'^i = u^i + v^i$$

thus, the speed of light does not transform as we observe; it is not invariant under a gallilean boost. Similarly, Maxwell's equations are not invariant under such a transformation.

## 4.2 Special Relativity

Using the invariance of the speed of light, we can derive the new invariant line element using Einstein's Clock as well as a new symmetry group for the laws of physics.

Consider two mirrors separated by a distance $L$. We bounce light back and forth, perpendicular to the two surfaces. The time it takes light for a round trip is trivially $\Delta t = 2L/c$. However, when an obsever moves parallel to the surface of the mirrors at a velocity $v$, as an external observer, they no longer see light travelling perpendicularly; it rather zigzags between the two. The new round-trip length the light must take is now $L' = 2\sqrt{L^2 + (v\Delta t'/2)^2}$. Thus, we can show easily that

$$-(c\Delta t)^2 + (\Delta x')^2 = -(c\Delta t)^2 + (\Delta x)^2$$

More generally, we find the invariant line element becomes

$$\mathrm{d}s^2 \equiv -(c\,\mathrm{d}t)^2 + (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2 \tag{4.1}$$

We will often use natural units in which $c = 1$. Note, we are using the metric signature $(-+++)$. We define the *Minkowski metric* for flat space-time as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{4.2}$$

where

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

### 4.2.1 Lorentz Boost

Consider a boost

$$x = vt + x'$$

Because $x'$ is some constant, we find that we must have some propotionality constant such that

$$x' = \gamma(x - vt)$$

By symmetry, we also find that because $x' = x - vt$ we have

$$x = \gamma(x' + vt')$$

Using these two expressions, we find that

$$t' = \gamma \left[ t + \frac{1-\gamma^2}{v\gamma^2} x \right]$$

and enforcing the invariant interval, we find that

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c} \tag{4.4}$$

and

$$ct' = \gamma(ct - \beta x) \tag{4.5a}$$
$$x' = \gamma(x - vt) \tag{4.5b}$$

Setting $c = 1$, we ifnd that we can rewrite this as

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \tag{4.6}$$

Because the line element is preserved, the determinant of the matrix is one, and indeed, we have

$$\gamma^2 - v^2\gamma^2 = \frac{1}{1-v^2} - \frac{v^2}{1-v^2} = 1$$

We define $\gamma = \cosh\psi$ for some *rapidity* $\psi$, which shows us that $\sinh\psi = -v\gamma$, using the hyperbolic trig identity

$$\cosh^2\psi - \sinh^2\psi = 1$$

Thus, we obtain a representation of the lorentz boost in terms of a hyperbolic rotation:

$$\Lambda^\mu_\nu = \begin{bmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{bmatrix} \tag{4.7}$$

or, in Einstein notation,

$$x'^\mu = \Lambda^\mu_\nu x^\nu \tag{4.8}$$

Inserting this into the inner product, we find

$$\eta = \Lambda^\intercal \eta \Lambda$$

or

$$\eta_{\rho\sigma} = \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta'_{\mu\nu} \tag{4.9}$$

just like what we had for rotations earlier.

### 4.2.2 Light cones and Causality

Recall our metric. A line element is time-like if $\mathrm{d}s^{\,2} < 0$, time-like if $\mathrm{d}s^{\,2} > 0$, and null if $\mathrm{d}s\, s^2 = 0$. Finite intervals can more simply be determined by examining the light cone, given by

$$(ct)^2 = x^2 + y^2 + z^2$$

Time-like separation is contained within the light cone and space-like separation is outside of the light cone. Objects can only interact with objects within their past or future light cones; if two objects have a space-like separation, they can never interact, and there is no causal connection. In this view, Gallilean space-time is the limit where $c \to \infty$.

We define the proper time to be defined by the invariant line element, as we can arbitrarily fix the space-dimensions $\mathrm{d}x^i\, = 0$.

$$\mathrm{d}\tau^{\,2} = -\,\mathrm{d}s^{\,2} \tag{4.10}$$

This value is then the same in all frames. Thus, $\tau$ gives a natural parametrization for a worldline. We can write

$$\tau = \int_A^B \mathrm{d}\tau\; = \int \sqrt{\mathrm{d}t^{\,2} - \mathrm{d}\mathbf{x}^{\,2}} = \int \mathrm{d}\tau \sqrt{\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}\right)^2} = \int \mathrm{d}t \sqrt{1 - \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2}$$

It is important to note that in flat space-time *a straight-line path maximises proper time.* Further, for light, $\mathrm{d}\tau^{\,2} = 0$.

# Chapter 5

# Mechanics in SR

## 5.1 Vectors

**Lower Indices**

Before we begin studying mechanics in special relativity, we will expand the toolkit with which we have to deal with vectors.

$$\mathrm{d}s^{2} = \eta_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$$

From this, we define the notion of *lowering an index* as

$$\mathrm{d}s^{2} = \mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} = \mathrm{d}x_{\mu}\,\mathrm{d}x^{\mu} = (\eta_{\mu\nu}\,\mathrm{d}x^{\nu})\,\mathrm{d}x^{\mu}$$

or more generally

$$\mathrm{d}x_{\mu} = \eta_{\mu\nu}\,\mathrm{d}x^{\nu} \tag{5.1a}$$
$$\mathrm{d}x^{\mu} = \eta^{\mu\nu}\,\mathrm{d}x_{\nu} \tag{5.1b}$$

where

$$\eta^{\mu\kappa}\eta_{\kappa\nu} = \delta^{\mu}_{\nu} \tag{5.2}$$

Note: this definition enforces that $\eta^{\mu}_{\nu}$ *does not exist.*

### 5.1.1 Vector Quantities

The position 4-vector is given

$$x^{\mu} = (t, x, y, z) \tag{5.3}$$

From the definition of proper time, we see that we can find a velocity 4-vector given

$$\mathrm{d}\tau^{2} = -\eta_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}$$

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{5.4a}$$

we find that

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\mathrm{d}t}{\sqrt{\mathrm{d}t^{\,2} - \mathrm{d}\mathbf{x}^{\,2}}} = \frac{1}{\sqrt{1 - \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2}}$$

We recognize $\mathrm{d}\mathbf{x}\,/\,\mathrm{d}t\ = \mathbf{v}_N$ the newtonian velocity. Thus,

$$u^\mu = (\gamma, \gamma\mathbf{v}_N) \tag{5.4b}$$

We trivially find a normalization condition[1]

$$u_\mu u^\mu = -1 \tag{5.4c}$$

Importantly, we found that

$$\boxed{\gamma = \frac{\mathrm{d}t}{\mathrm{d}\tau}} \tag{5.5}$$

Multiplying $u^\mu$ by mass, we obtain the 4-momentum

$$p^\mu = mu^\mu \tag{5.6a}$$

The first term[2] we recognize as the energy

$$p^0 = \gamma m \to mc^2 + \frac{1}{2}m\mathbf{v}_N^2 + \mathscr{O}(\mathbf{v}_N^4) = E$$

From the normalization of 4-velocity, we trivially find[3]

$$p_\mu p^\mu = -m^2 \tag{5.6b}$$

which, expanding out (and fixing $\gamma = 1$), we recover Einstein's famous energy-mass equivalence

$$E^2 - \mathbf{p}^2 = m^2$$

[1] If we include factors of $c$, we find that $u^\mu = (\gamma c, \gamma\mathbf{v}_N)$ and $u_\mu u^\mu = -c^2$
[2] and that $p^0 = \gamma mc = E/c$
[3] and that $p_\mu p^\mu = -m^2c^2$

<u>Light</u>

Consider a reparametrization of light in terms of $\lambda$ (we must do this because $\mathrm{d}\tau = 0$) as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$$

Then we have

$$u_\mu u^\mu = 0$$

and

$$E = \hbar\omega \tag{5.7a}$$

$$\mathbf{p} = \hbar\mathbf{k} \tag{5.7b}$$

$$p^\mu = \hbar k^\mu \tag{5.7c}$$

so

$$k_\mu k^\mu = 0 \tag{5.7d}$$

Consider the collision of a photon $k^\mu$ with a stationary particle $p^\mu$. Because momentum is conserved

$$k^\mu + p^\mu = k'^\mu + p'^\mu$$

In the lab frame, the particle has 4-momentum

$$p^\mu = (m, 0)$$

To make this a lorentz invariant, we consider

$$k'_\mu p^\mu = -m\omega'$$

Substituting our conservation of momentum,

$$-m\omega' = k'_\mu(k'^\mu + p'^\mu - k^\mu)$$

The first term goes to zero. Squaring momentum conservation, we additionally find

$$k_\mu k^\mu + 2k_\mu p^\mu + p_\mu p^\mu = k'_\mu k'^\mu + 2k'\mu p'^\mu - p'_\mu p'^\mu$$

or substituting invariants for momentum and wavevector,

$$k_\mu p^\mu = k'_\mu p'^\mu$$

Substitiuting into our invariant,

$$k'_\mu p^\mu = k_\mu p^\mu - k'_\mu k^\mu$$

Evaluating (and negating),

$$m\omega' = m\omega - (\omega'\omega - \mathbf{k}' \cdot \mathbf{k})$$

or

$$m\omega' = m\omega - \omega\omega f(1 - \cos\theta)$$

note that the second term results from $c = \omega/k = 1$. Thus, we obtain

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}$$

## 5.2 Relativistic Action

First, consider the case where there is no potential; the free particle. Consider the item with the closest form to the action—the proper time:

$$\begin{aligned} S &\stackrel{?}{=} \int \mathrm{d}\tau \\ &= \int \mathrm{d}t \sqrt{1 - \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2} \\ &\to -\int \mathrm{d}t\ -1 + \frac{1}{2}\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2 + \dots \end{aligned}$$

Comparing to the classical Lagrangian, we note we are missing a mass term. Multiplying,

$$\begin{aligned} S &= -mc \int \mathrm{d}t \sqrt{c^2 - \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2} \\ &= \int \mathrm{d}t \left[\frac{1}{2}mv^2 - mc^2 + \dots\right] \end{aligned}$$

And so we recover the classical lagrangian with a rest mass energy term.

We thus define the action of a free particle as

$$S = -m\int \mathrm{d}\tau\ = -m\int \sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\ \mathrm{d}x^\nu} \tag{5.8}$$

Rewritting,

$$S = -m\int \mathrm{d}\sigma \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}$$

and so our Lagrangian can be written in terms of the invariant

$$L = -m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}} \tag{5.9}$$

for some parameter $\sigma$. Fixing $\sigma = \tau$, we find

$$L = \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = 1$$

Applying Euler's equation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\tau}\eta_{\mu\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) = 0$$

so

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} = 0$$

or the worldline of a free particle is a straight line.

For a massless particle, such as light, we can reparametrize the action

$$\tilde{S} = \int \mathrm{d}\xi \sqrt{\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\xi}\frac{\mathrm{d}x^\nu}{\mathrm{d}\xi}}$$

### 5.2.1 Potentials

When we wish to add a potential, we can add it in one of two places; either inside the square root of the action or as second term in the lagrangian. The former provides more GR forces, while the latter more EM forces.

#### Gravity-like

Consider a perturbative term to the time component:

$$S = -m\int\sqrt{\left(1+\frac{2V}{m}\right)\mathrm{d}t^{\,2} - \mathrm{d}\mathbf{x}^{\,2}}$$

We can assume the separation is timelike; in the non-relativistic limit, we find $\mathrm{d}t \gg \|\mathrm{d}\mathbf{x}\,\|$. Thus,

$$\to -m\int\sqrt{1+\frac{2V}{m}}\,\mathrm{d}t - \frac{\mathrm{d}\mathbf{x}^{\,2}}{2\sqrt{1+\frac{2V}{m}}\,\mathrm{d}t}$$

Further, with $V \ll c$ we can expand $\sqrt{1+2V/m} \to 1+V/m$. Keeping only first order terms,

$$\begin{aligned}&\to -m\int\left[1+\frac{V}{m}\right]\mathrm{d}t - \frac{\mathrm{d}\mathbf{x}^{\,2}}{2\,\mathrm{d}t}\\ &= \int \mathrm{d}t\left[\frac{1}{2}m\left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\right)^2 - V - m\right]\end{aligned}$$

which yields us our classical lagrangian. We note that the form of our perturbation can be written as

$$g(x)\,\mathrm{d}t^{\,2} - \tilde{g}(x)\,\mathrm{d}\mathbf{x}^{\,2}$$

however, *this does not transform like a tensor*; the lorentz boost would necessarily create cross terms with cross terms $g\tilde{g}\,\mathrm{d}x\ \mathrm{d}t$.

Rather, we promote this object to a true metric

$$g_{00} = -\left(1 + \frac{2V}{m}\right) \quad g_{i0} = 0 = g_{0i} \quad g_{ij} = \delta_{ij}$$

which keeps the action lorentz invariant. Note that now, we can easily see that the stronger a potential

$$V \to -\frac{GM}{r}$$

the more the proper time is affected. Indeed, noticing that for a stationary particle $\mathrm{d}\mathbf{x} = 0$, we find

$$\mathrm{d}\tau = \sqrt{1 - \frac{2GM}{r}}\,\mathrm{d}t$$

and so clocks in a gravitational potential run slow; i.e. $\mathrm{d}\tau < \mathrm{d}t$.

From this we can derive graviational redshift. To first order, we have

$$\mathrm{d}\tau = (1 + \phi(x))\,\mathrm{d}t$$

and so we find

$$\omega(x) \sim \frac{1}{1+\phi} \sim 1 - \phi$$

so

$$\frac{\omega_f}{\omega_i} \sim \frac{1-\phi_f}{1-\phi_i} \sim 1 - \phi_f + \phi_i$$

and thus

$$\frac{\omega_f - \omega_i}{\omega_i} = -\frac{\phi_f - \phi_i}{c^2}$$

**EM-like**

We now consider a potential outside the square root. We promote the potential to a lorentz invariant quantity by converting it to a 4-potential and obtain the action

$$S = -m \int \sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\ \mathrm{d}x^\nu} + A_\mu(x)\,\mathrm{d}x^\mu \tag{5.10}$$

where $A_0(x) = -V(x)$ to maintain consistency with the non-relativistic action. When we vary $x^\mu \to x^\mu + \delta x^\mu$ we recover the lorentz force law; when we vary $A_\mu \to A_\mu + \delta A_\mu$ we obtain Maxwell's Field Equations.

# Chapter 6

# General Relativity

## 6.1 Equivalence Principle

Einstein's Equivalence principle states that a (small enough) frame in a gravitational field is indistinguishable from a frame with the same acceleration as the gravitational field's[1]. More rigourously, locally in spacetime, the laws of physics are invariant under the Poincaré group; locally, spacetime looks minkowskian. The weak equivalence principle (equivalence of inertial and gravitational mass) follows naturally from this equivalence principle.

Two ways to test this are the effects of gravity on light. If a gravitational frame is equivalent to a uniformly accelerating frame, we should expect light perpendicular to the field to deflect downward, and the light parallel to the field to be doppler shifted. Indeed, these two effects have been measured and confirmed experimentally. We interpret this as rather than gravity acting on the light, the light follows a straight-line path in spacetime.

## 6.2 Curved Spacetime

Recall our gravitational action

$$S = -m \int \sqrt{-\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}$$

In general relativity, we replace the constant Minkowski metric with one that depends on the position in spacetime—a general metric tensor

$$S = -m \int \sqrt{-g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} \tag{6.1}$$

[1]However, if the frame is large enough you can begin to measure the variation in the field, namely tidal forces

**Free Particle in 1D**

Consider a particle in an accelerating frame. In newtonian mechanics, we transform the frames via

$$y = x - \frac{1}{2}at^2$$

so

$$\ddot{y} = \ddot{x} - a$$

We observe a fictitious force due to the acceleration of the frame.

If we instead consider the same system in relativity, we have

$$\mathrm{d}\tau^{\,2} = -\eta_{\rho\sigma}\,\mathrm{d}y^\rho\ \mathrm{d}y^\sigma$$

so

$$1 = \frac{\mathrm{d}y^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}y_\rho}{\mathrm{d}\tau}$$

and

$$\frac{\mathrm{d}^2 y^\rho}{\mathrm{d}\tau^2} = 0$$

If we consider

$$y^\rho = y^\rho(x^\mu)$$

we find

$$\frac{\mathrm{d}^2 y^\rho}{\mathrm{d}\tau^2} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

Plugging in $\mathrm{d}^2 y^\rho\,\mathrm{d}\tau^2$ , and using

$$\frac{\partial x^\lambda}{\partial x^\mu} = \frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial y^\rho}{\partial x^\mu} = \delta^\lambda_\mu$$

we find

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \left(\frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{6.2}$$

From the proper time, we find the metric of a constantly accelerating frame:

$$\mathrm{d}\tau^{\,2} = \eta_{\rho\sigma}\,\mathrm{d}y^\rho\ \mathrm{d}y^\sigma\ = \eta_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu}\,\mathrm{d}x^\mu\ \mathrm{d}x^n u$$

$$g_{\mu\nu} = \eta_{\rho\sigma}\frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} \tag{6.3}$$

Thus, we see that the metric is locally just a coordinate transformation from the minkowski metric.

Further, we see that the observer sees a fictituous force in the accelerating frame.

From Equation 6.2, we define the Christoffel symbol such that

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\rho\nu}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{6.4}$$

so

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial y^\rho}\frac{\partial^2 y^\rho}{\partial x^\mu \partial x^\nu} \tag{6.5}$$

Equation 6.4 is the *geodesic equation*, and gives us our equations of motion. Recall we can derive this equation from applying the Euler equation to the Lagrangian,

$$L = -m\sqrt{g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}$$

In doing so, we see another expression for the Christoffel symbol in terms of derivatives of the metric:

$$\Gamma^\lambda_{\rho\nu} = \frac{1}{2}g^{\lambda\alpha}\left(\partial_\nu g_{\alpha\rho} + \partial_\rho g_{\alpha\nu} - \partial_\alpha g_{\rho\nu}\right) \tag{6.6}$$

The Christoffel symbol offers a mathematical description of the equivalence principle.

**2-Sphere**

Consider a 2-sphere where the line element is given

$$\mathrm{d}s^2 = a\left(\mathrm{d}\theta^2 \sin^2\theta\, \mathrm{d}\phi^2\right)$$

To find a transformation to a local inertial frame, we need to have a diagonal metric with vanishing first derivative. We consider the *Riemann normal coordinates*

$$\begin{aligned} x &= a\theta\cos\phi \\ y &= a\theta\sin\phi \end{aligned}$$

so

$$\theta = \frac{\sqrt{x^2+y^2}}{a} \qquad\qquad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

and

$$\begin{aligned} \mathrm{d}\theta &= \frac{x}{a\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{y}{a\sqrt{x^2+y^2}}\,\mathrm{d}y \\ \mathrm{d}\phi &= -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \end{aligned}$$

so our metric becomes (to leading order)

$$g(x,y) = \begin{bmatrix} 1 - \frac{2y^2}{3a^2} & \frac{2xy}{3a^2} \\ \frac{2xy}{3a^2} & 1 - \frac{2x^2}{3a^2} \end{bmatrix}$$

At $\theta = 0$, we have $x = y = 0$ and our metric reduces to the identity.

## 6.3 Lightcones and Worldlines

We can give each point on a worldline a local light cone. These light cones are at 45° angles to the worldline as given by the local metric. The light cones are found by setting

$$\mathrm{d}s^2 = 0$$

## 6.4 Alcubierre Metric

The line element in the Alcubierre Metric is given

$$\mathrm{d}s^2 = -\mathrm{d}t^2 \left[\mathrm{d}x - V_s(t) f(r_s)\,\mathrm{d}t\right]^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{6.7}$$

The function $f(r_s)$ is the "shape" of a warp bubble, with the properties

$$f(0) = 1$$

$$f(r_s > R) = 0$$

where $r_s$ is the distance to the centre of the bubble

$$r_s^2 = (x - x_s(t))^2 + y^2 + z^2$$

and $V_s(t)$ is the speed of the ship. The light cone is given

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \pm 1 + V(t) f(r_s)$$

which can be greater than $c = 1$. However,

$$V_s - 1 < V_s < V_s + 1$$

and so the ship only travels within the light cone.

If we consider the light cone inside the bubble, the light cone is tipped forward. However, if light is inside the bubble, we find that the light cannot touch the entire bubble; someone inside the bubble cannot be the one who creates the bubble.

Writing out the metric, we find

$$g_{\alpha\beta} = \begin{bmatrix} V_s^2(t) - 1 & -V_s(t) & 0 & 0 \\ -V_s(t) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a constant velocity $V_s$, we further have

$$\partial_i g_{jk} = 0$$

so the ship feels no force; it sits still as the bubble carries it forward.

Further, considering the proper time and $V_s(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = V_s$ near the centre of the bubble, $f \to 0$, we find

$$\tau = \int_0^T \sqrt{\mathrm{d}t^2 - \left[\mathrm{d}x - V_s\,\mathrm{d}t\right]^2}$$

$$= \int_0^T \sqrt{\mathrm{d}t^2}$$

$$= T$$

and so the proper time isn't altered.

# Chapter 7

# Geodesics

Recall the Lagrangian

$$L = \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}$$

If we work through the Euler equation,

$$0 = \frac{\mathrm{d}}{\mathrm{d}\sigma} \left[ \frac{\partial L}{\partial \left( \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right)} \right] - \frac{\partial L}{\partial x^\alpha}$$

we can find

$$\begin{aligned} \frac{\partial L}{\partial \left( \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma} \right)} &= \frac{1}{2} \frac{1}{\sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}}} \left( -g_{\mu\nu} \delta^\mu_\alpha x^\nu - g_{\mu\nu} x^\mu \delta^\nu_\alpha \right) \\ &= \frac{1}{L} g_{\mu\alpha} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \end{aligned}$$

Noticing $\frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = \frac{\sqrt{-g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}}{\sigma} = L$, we find can rewrite as

$$= -g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$$

Similarly, we can find

$$\begin{aligned} \frac{\partial L}{\partial x^\alpha} &= -\frac{1}{2L} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma} \\ &= -\frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \end{aligned}$$

Thus, plugging into the Euler equation, we can obtain the geodesic equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau} \left( g_{\alpha\beta} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \right) - \frac{1}{2} \partial_\alpha g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \tag{7.1}$$

Expanding the first term, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\alpha\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}\right) = \partial_\gamma g_{\alpha\beta}\frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} + g_{\alpha\beta}\frac{\mathrm{d}^2x^\beta}{\mathrm{d}\tau^2}$$

renaming indices and multiplying by the inverse metric, we obtain

$$0 = \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau} + g^{\gamma\alpha}\left(\partial_\mu g_{\alpha\nu} - \frac{1}{2}\partial_\alpha g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

Exploiting the symmetry of

$$\partial_\mu g_{\alpha\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

we obtain the more familiar expression of the geodesic equation:

$$\frac{\mathrm{d}^2x^\beta}{\mathrm{d}\tau^2} + \frac{1}{2}g^{\gamma\alpha}\left(\partial_\mu g_{nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{7.2}$$

which we recognize as Equation 6.4

$$\frac{\mathrm{d}^2x^\gamma}{\mathrm{d}\tau^2} + \Gamma^\gamma_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} = 0 \tag{6.4}$$

The Christoffel symbol is not a tensor, but does obey the symmetry

$$\Gamma^\gamma_{\mu\nu} = \Gamma^\gamma_{\nu\mu}$$

Unit Sphere

Consider the unit two-sphere given

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

and define $x_1 = \theta$ and $x_2 = \phi$. The lagrangian can be written

$$L = \sqrt{\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2}$$

The Euler equations can be computed as above to obtain

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + 2\cot\theta\frac{\mathrm{d}\theta}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$$

Thus, we obtain the Cristoffel symbols

$$\Gamma^1_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & -\sin\theta\cos\theta \end{pmatrix}$$

$$\Gamma^2_{ij} = \begin{pmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{pmatrix}$$

or, more clearly,

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

with all other terms zero.

## 7.1 Embedding

We can embed a $D$ dimensional manifold onto the Euclidean space $E^N$ by a mapping

$$y^A(x^1, \ldots, x^D)$$

for example, a 2-sphere $S^2$ is given $x^\mu(x^1, x^2) = (\theta\phi)$. We can embed this 2-sphere onto $E^3$ by

$$y^A(y^1, y^2, y^3) = (x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

If we examine the neighbourhood of a point

$$x^\mu + \mathrm{d}x^\mu$$

we expect the neighbourhood of the embedded point to behave by the chain rule

$$y^A + \frac{\partial y^A}{\partial x^\mu}\,\mathrm{d}x^\mu$$

The line element should be preserved

$$\mathrm{d}s^2 = \sum_A \mathrm{d}y^{A\,2} = \sum_A \frac{\partial y^A}{\partial x^\mu}\frac{\partial y^A}{\partial x^\nu}\,\mathrm{d}x^\nu = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

and so, we find

$$g_{\mu\nu} = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\delta_{\alpha\beta}$$

and so the embedding transforms the Euclidean metric $\delta_{\mu\nu}$. Embedding often restricts $y^A$ to some constraint. For example, for

$$1 = x^2 + y^2 + z^2$$

we can eliminate $z$. Thus, we can rewrite the line element

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 = \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)^2}{1 - x^2 - y^2}$$

The rotational invariance enforces

$$x = r\cos\phi \qquad\qquad y = r\sin\phi$$

so

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-r^2} + r^2\,\mathrm{d}\phi^2$$

however, when we fix $r = \sin\theta$ we recover the line element

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

### 7.1.1 Wormholes

The wormhole metric is given by the line element

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + \mathrm{d}r^2 + (b^2 + r^2)\left(\mathrm{d}\theta^2 + \sin\theta^2\,\mathrm{d}\phi^2\right) \tag{7.3}$$

Note that this just a slight modification to flat spacetime in spherical coordinates.

Because the metric is time-independent, we can take a constant time slice. If we consider the slice with constant $t$ and $\theta = \pi/2$ we restrict our surface to

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + (b^2 + r^2)\,\mathrm{d}\phi^2$$

Due to the cylindrical symmetry, we wish to embed into the cylidrical coordinates

$$\mathrm{d}s = \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\psi^2 + \mathrm{d}z^2$$

and so we wish to find the functions $z(r), \rho(r), \psi = \phi$ which will give our embedding.

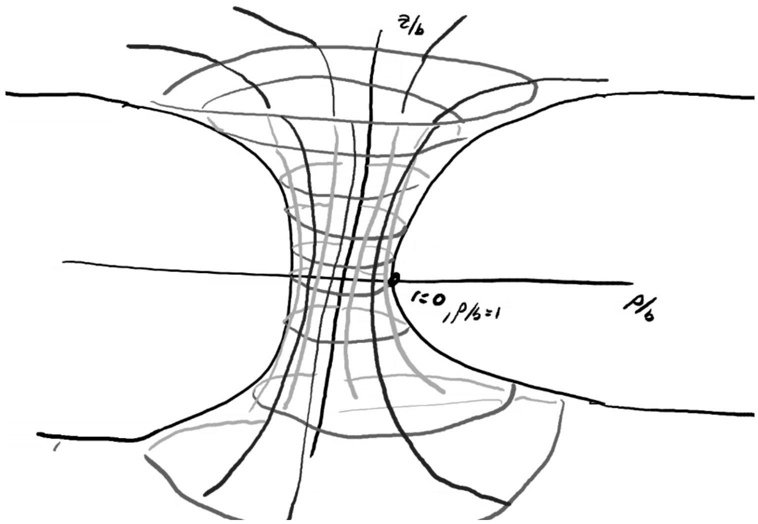


Figure 7.1: Wormhole Embedding

Using our equation for the metric transformation, we find

$$g_{rr} = \left(\frac{\partial \rho}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = 1$$

$$g_{\phi\phi} = \left(\frac{\partial \psi}{\partial \phi}\right)^2 \rho^2 = b^2 + r^2 \implies b^2 + r^2 = \rho^2$$

from this, we then find

$$\frac{\partial \rho}{\partial r} = \frac{r}{\sqrt{r^2 + b^2}}$$
$$\frac{\partial z}{\partial r} = \frac{b}{\sqrt{r^2 + b^2}}$$

solving, we find

$$z(r) = b \sinh^{-1}\left(\frac{r}{b}\right)$$
$$\rho(z) = b \cosh\left(\frac{z}{b}\right)$$

plotting $\rho(z)$, we obtain our embedding, and see that it looks like a hyperbola of one sheet, similar to the idea of a wormhole connecting two planes.

We note that when $r = 0$ we have $\rho/b = 1$ and we are at the neck of the wormhole

**Geodesics**

We can of course compute the geodesic equations by taking the variation of the lagrangian:

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0 \tag{7.4a}$$

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = r\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2\right] \tag{7.4b}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(b^2+r^2\right)\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right] = \left(b^2+r^2\right)\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \tag{7.4c}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(b^2+r^2\right)\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right] = 0 \tag{7.4d}$$

Consider a particle in free motion radially, that is

$$u^\alpha = (u^t, u^r, u^\theta, u^\phi) = (C, U, 0, 0)$$

Normalizing the 4-velocity, we find

$$u^\alpha = \left((1-U^2)^{1/2}, U, 0, 0\right)$$

Because $u^\theta = u^\phi = 0$ we find Equation 7.4b becomes

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\tau^2} = \frac{\mathrm{d}u^r}{\mathrm{d}\tau} = 0$$

so

$$r(\tau) = U\tau + R$$

and so to get from one side of the wormhole to the other,

$$\Delta\tau = \frac{2R}{U}$$

## 7.2 Symmetries and Conservation

Recall Noether's theorem: suppose $L$ remains invariant under a certain coordinate transformation. Then, every such transformation there exists a conserved quantity. However, our lagrangian is in terms of the metric, and so we must find a new way to extract symmetries.

## 7.3 Killing Vectors

We can reformulate Noether's theorem using the metric and geodesic equation. Consider the line element

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$$

for flat spacetime. This metric doesn't depend on any of $x, y, z$, and so our metric has *translational symmetry.*

We define a *Killing vector field* which is a vector field that points along a direction where the metric remains invariant. Three components of the Killing field correspond to the translational symmetry:

$$\xi^1 = (1, 0, 0)$$

$$\xi^2 = (0,1,0)$$
$$\xi^3 = (0,0,1)$$

More explicity, for a symmetry along $x^\sigma$, we have a Killing vector

$$\Sigma^\mu = (\partial_\sigma)^\mu$$

and three components correspond to the rotational symmetry. This metric is *maximally symmetric* because these six degrees of freedom are the maximum number of symmetries in 3D space. In 4D space time, there are a maximum of 10 symmetries, corresponding to the 10 degrees of freedom in the Poincaré group.

We see that if the metric is independent of a coordinate, say $x^1$, we find

$$\frac{\partial L}{\partial x^1} = 0$$

so

$$\frac{\partial L}{\partial\left(\frac{\mathrm{d}x^1}{\mathrm{d}\sigma}\right)} = \text{const}$$

Rewriting the LHS, we have

$$\frac{\partial L}{\partial\left(\frac{\mathrm{d}x^1}{\mathrm{d}\sigma}\right)} = -g_{1\beta}\frac{1}{L}\frac{\mathrm{d}x^\beta}{\mathrm{d}\sigma} = -g_{1\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}$$

We see that because $\xi^1 = (1,0,0)$, we can insert it into the equation to find

$$-\xi^1 \cdot u = \text{const}$$

more generally,

$$\xi \cdot p = \text{const} \tag{7.5}$$

or momentum along a killing vector is conserved. In cartesian coordinates, we have then the momentum along translations is conserved, or linear momentum, and that momentum along rotations is conserved, or angular momentum.

Another example, is using polar coordinates

$$g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

we find a killing vector

$$\xi = (0,1)$$

We find then, that we have a conserved quantity

$$\ell = \xi \cdot u = g_{AB}\xi^A u^B = r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}$$

From the normalization of the velocity,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 = 1 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 1 - \frac{1}{r^2}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 1 - \frac{\ell^2}{r^2}$$

With an appropriate choice of coordinates, we can then extract a geodesic equation.

## 7.4 Motion in a static isotropic spacetime

Consider the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,\mathrm{d}t)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right) \tag{7.6}$$

This is a *static* metric, meaning there is no time dependence. Further, we impose spherical symmetry, given by $g_{\theta\theta}$ and $g_{\phi\phi}$. Note, the spherical symmetry is sometimes written using

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\mathrm{d}\phi^2$$

We can thus extract the two killing vectors

$$\xi^\mu = (1, 0, 0, 0)$$

$$\eta^\mu = (0, 0, 0, 1)$$

Our first conserved quantity is given (in planck natural units $c = G = \hbar = k_B = 1$)

$$e = -\xi \cdot u = g_{tt}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}$$

$$\ell = -\eta \cdot u = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\tau}$$

$$u \cdot u = -1$$

We denote these $e$ and $\ell$, as they are akin to an energy and angular momentum, each per unit mass.

### 7.4.1 Effective Potential

Just like in the two-body problem in classical mechanics, we will consider the orbits in an effective potential. Note that from conservation of $\ell$, we know that the particle must have a planar orbit. We can fix then $\theta = \pi/2$ and $u^\theta = 0$. Using the normalization of velocity,

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2$$

Plugging in our conserved quantities,

$$1 = -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{\ell^2}{r^2}$$

Rearranging, we find

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2} - 1\right)\right]$$

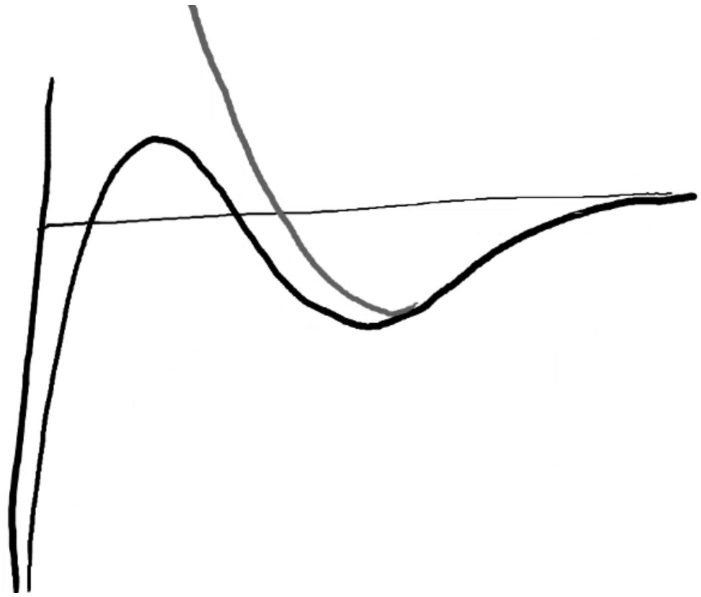

Figure 7.2: Effective Potential

Note that because the LHS is a constant, the RHS is also constant. We redefine $\mathcal{E} = \frac{e^2-1}{2}$. We identify the first term on the RHS as the kinetic energy per unit mass, and the second term as $V_{eff}$, the effective potential. Expanding, we find

$$V_{eff} = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

We then identify the first term in the effective potential as the gravitational potential and the second as the centrifugal barrier. Howerver, we obtain a third, new term in the effective potential. Thus we obtain

$$\mathcal{E} = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{eff}(r)$$

which we can solve as in classical mechanics.

If we plot the effective potential, as in Figure 7.2 we see there is a new unstable circular orbit. Less than this unstable orbit, the particle is pulled to the origin. Additionally, the minimum bound orbit is closer to the centre than classically expected.

Finding the extrema, we find

$$Mr^2 - \ell^2 r + 3M\ell = 0$$

so

$$r_{\min,\max} = \frac{\ell}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{\ell}\right)^2}\right]$$

We see there is then a critical $\ell_c$ where the stable and unstable orbit coincide:

$$\frac{\ell_c}{M} = \sqrt{12}$$

We also have a special radius, the Schwarzschild radius, where the metric becomes singular:

$$r_s = 2M$$

these two values are related by

$$\frac{\ell_c}{r_s} = \sqrt{3}$$

### 7.4.2 Plunging Orbits

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{e}{1 - \frac{r_s}{r}}$$

### 7.4.3 Ellipirical Orbit

Recall by definition that

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\ell}{r^2}$$

so

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\ell}{2}$$

thus, we can integrate

$$\phi(r) = \pm \int \mathrm{d}r \, \frac{\ell}{r^2\sqrt{\mathcal{E}^2 - 2V(r)}}$$

Upon integration, we find that the perihelion processes.

Consider the tangent of the embedding of the schwarzschild metric into $\mathbb{R}^3$. If we revolve this tangent around, it forms a cone; however to form this cone into flat spacetime, we need to add an angular sliver; this angular sliver corresponds to the precession. Define

$$\tan\alpha = \frac{\mathrm{d}z}{\mathrm{d}r}$$

we then have

$$(2\pi - \delta)R = 2\pi r = 2\pi R\cos\alpha$$

for $r \gg 2M$, we can use the small angle approximation, so

$$\delta = \frac{2\pi M}{r} = \frac{1}{3}\frac{6\pi M}{r}$$

which is about a third of the orbit in flat space.

### 7.4.4 Bound Orbits

Just as in classical mechanics, we make the substitution

$$u = r^{-1}$$

so we can find

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \frac{1}{u^2}\,\mathrm{d}u\,\phi$$

and

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \ell u^2$$

so we can rewrite our orbital equation as

$$\mathcal{E} = \frac{1}{2}\left(-\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 \ell u^4 - Mu + \frac{1}{2}\ell^2 u^2 - M\ell^2 u^3$$

There is an innermost stable circular orbit, given by

$$r_{\mathrm{ISCO}} = 6M = 3r_2$$

which is were the minimum coincides with the maximum in the potential.

We find that if we send a particle from $r = \infty$ to the innermost stable circular orbit, a particle will release 6% of its energy (eventually to radiation)[1]. Compared to nuclear fusion, which releasees only 0.7%, this is immense[2].

## 7.5 Bending of Light

Recall that light follows null geodesics, so

$$u \cdot u = g_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda} = 0$$

This allows us to rewrite conserved quantities

$$e = -\xi \cdot u = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\lambda}$$

$$\ell = -\eta \cdot u = r^2 \sin^2\theta \frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$$

$$u \cdot u = -\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right) = 0$$

[1] Quadrupole moments are required to generate gravitational waves, however, so perfectly circular orbits will not radiate away

[2] Rotating black holes can further increase this efficiency to around 40%!

so

$$\frac{1}{b^2} \equiv \frac{e^2}{\ell^2} = \frac{1}{\ell^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

for the *impact parameter*, $b$. We can the write

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{eff}$$

for the effective potential $W_{eff}$. Now, our effective potential for light is slightly different than for a particle.

$$W_{eff} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

we find there is a single circular orbit at

$$r = 3M$$

but no stable bound orbits. At this point, the impact parameter is given

$$\frac{1}{b^2} = \frac{1}{27M^2}$$

for $r > 3M$, we have *scattering*, where

$$\frac{1}{b^2} < \frac{1}{27M^2}$$

and the photon is deflected, and for $r < 3M$ we have the photon captured by the singularity.

For $r = 2M$, we are at the *event horizon*, where light can only escape if it travels radially outward.

Once again rewriting in terms of $u = 1/r$, we find

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = 3Mu^2$$

and can compute the bending of light around a mass.

# Chapter 8

# Causality

Consider the schwarzschild metric as $r \to r_s$. The gravitational redshift of a photon at $r_e$ can be found

$$\frac{\omega(r)}{\omega(r_e)} = \sqrt{\frac{g_{tt}(r_e)}{g_{tt}(r)}}$$

as $r \to r_s$ we find that $\omega(r)/\omega(r_e) \to \infty$, or equivalently, if the photon is emitted at $r$ and detected at $r_d$, we find $\omega(r_d)/\omega(r) \to 0$, or that the light emitted at the event horizon is redshifted to zero always.

If we consider the radial lightcones

$$\mathrm{d}s^2 = 0 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r}{r_s}\right)^{-1}\mathrm{d}r^2$$

$$\mathrm{d}t = \pm\frac{r}{r - r_s}\,\mathrm{d}r$$

for $+$, we have the outward lightcone, as an increase in $\mathrm{d}t$ leads to an increase in $\mathrm{d}r$; for $-$, we have correspondingly the inward lightcone.

As $r \to \infty$, we recover the minkowski light cones. As we approach $r_s$, the cone gets narrower and narrower, until we reach $r_s$, and the lightcone collapses, and inside $r_s$, our coordinates become meaningless

We change our coordinates by

$$\mathrm{d}\bar{t} = \mathrm{d}t + \frac{r_s}{r - r_s}\,\mathrm{d}r$$

allowing us rewrite

$$\mathrm{d}s^2 = \left(\frac{r - r_s}{r}\right)(\mathrm{d}\bar{t} + \mathrm{d}r)\left(\mathrm{d}\bar{t} - \frac{r + r_s}{r - r_s}\,\mathrm{d}r\right)$$

we then have two lightcones—the incoming light cone

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}r} = -1$$

and the outgoing

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}r} = \frac{r + r_s}{r - r_s}$$

we can then find a family of lightcone solutions, shown in Fig. 8.1

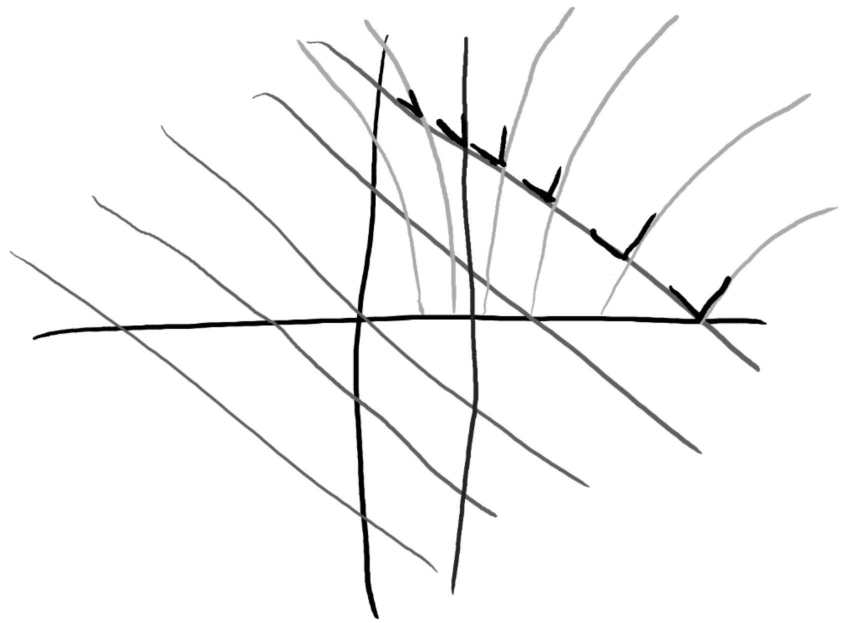

Figure 8.1: Lightcones for Black Hole

We then see that as one goes toward the singularity of the event horizon, the light cones pinch off and tilt toward the singularity, and once the particle passes the event horizon, the particle can no longer escape, as no part of the light cone points outward.

However, this isn't very elucidating in the $r$ direction at the event horizon; what happens to a photon moving radially outward starting at the event horizon?

## 8.1 Eddington-Finkelstein Coordinates

Note that we can factor the Schwarzschild metric as

$$\mathrm{d}s^2 = -\left(\frac{r - r_s}{r}\right)\left(\mathrm{d}t + \frac{r}{r - r_s}\,\mathrm{d}r\right)\left(\mathrm{d}t + \frac{r}{r - r_s}\,\mathrm{d}r\right) + r^2\,\mathrm{d}\Omega^2$$

We can refine the coordinates by using

$$\mathrm{d}v == \mathrm{d}t + \frac{r}{r - r_s}\,\mathrm{d}r$$

so

$$\mathrm{d}s^2 = -\left(\frac{r - r_s}{r}\right)\mathrm{d}v^2 - 2\,\mathrm{d}v\;\mathrm{d}r + r^2\,\mathrm{d}\Omega^2$$

we find radial light rays follow

$$\left(\frac{r-r_s}{r}\right)\mathrm{d}v\ = 2\,\mathrm{d}r$$

integrating, we find

$$v = t + r + r_s \ln\left|\frac{r-r_s}{r}\right| = \bar{t} + r$$

## 8.2 Kruskal-Szekeres Coordinates

Note that there is one more term that seems very evocative of a subsitution. Thus,

$$\mathrm{d}p\ = \mathrm{d}t\ + \frac{r}{r-r_s}\,\mathrm{d}r \qquad\qquad \mathrm{d}q\ = \mathrm{d}t\ - \frac{r}{r-r_s}\,\mathrm{d}r$$

so

$$\mathrm{d}s\ = -\left(\frac{r-r_s}{r}\right)\mathrm{d}p\ \mathrm{d}q\ + r^2\,\mathrm{d}\Omega$$

However, notice that

$$\mathrm{d}(p+q)\ = 2\,\mathrm{d}t \qquad \mathrm{d}(p-q)\ = 2\left(1+\frac{r_s}{r-r_s}\right)\mathrm{d}r$$

or

$$p+q = 2t \qquad\qquad p-q = 2r + r_s \log\left|\frac{r-r_s}{r}\right|$$

we see that for $r \gg r_s$ the second term vanishes and we recover minkowski space, as we expect.

Changing coordinates once more,

$$P = e^{p/2r_s} \qquad\qquad Q = e^{-q/2r_s}$$

we find

$$\mathrm{d}s^{\,2} = -\frac{4r_s}{r}e^{-r/r_s}\,\mathrm{sgn}(r-r_s)\,\mathrm{d}P\ \mathrm{d}Q\ + r^2\,\mathrm{d}\Omega^{\,2}$$

Recall that

$$\mathrm{sgn}(x) \equiv \begin{cases} -1 & x<0 \\ 0 & x=0 \\ 1 & x>0 \end{cases}$$

This reveals that the only *true* singularity is $r=0$, while the $r=r_s$ singularity earlier was due only to a coordinate singularity.

Once again transforming coordinates,

$$V = \frac{1}{2}(P+Q) \qquad\qquad U = \frac{1}{2}(P-Q)$$

we find

$$\mathrm{d}V^{\,2} + \mathrm{d}Q^{\,2} = \mathrm{d}P\;\mathrm{d}Q$$

$$\mathrm{d}V^{\,2} - \mathrm{d}Q^{\,2} = \mathrm{sgn}(r - r_s)\,\mathrm{d}P\;\mathrm{d}Q$$

so

$$\mathrm{d}s^{\,2} = -\frac{4r_s}{r}e^{-r/r_s}\left(\mathrm{d}V^{\,2} + \mathrm{d}U^{\,2}\right) + r^2\,\mathrm{d}\Omega^{\,2} \tag{8.1}$$

and that radial light rays are given

$$\mathrm{d}U = \pm\,\mathrm{d}V$$

however, we note that because $\mathrm{d}V^{\,2} - \mathrm{d}U^{\,2}$ has the signum function in it, we have within the schwarzschild radius that $V$ and $U$ switch; that is

$$V(r < r_s) = \frac{1}{2}(P - Q) \qquad\qquad U(r < r_s) = \frac{1}{2}(P + Q)$$

from some algebra, we find

$$V^2 + U^2 = \left(1 - \frac{r}{r_s}\right)e^{r/r_s}$$

and that outside the event horizon

$$\frac{V}{U} = \frac{P+Q}{P-Q} = \tanh\frac{t}{2r_s}$$

and inside it

$$\frac{V}{U} = \frac{P-Q}{P+Q} = \coth\frac{t}{2r_s}$$

allowing us to write (outside the horizon)

$$V = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s}\sinh\frac{t}{2r_s}$$

$$U = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s}\cosh\frac{t}{2r_s}$$

and reversed inside the horizon.

Lines of constant $t$ give radial lines, while $t = \infty$ corresponds to $V = \pm U$. Similarly, lines of constant $r$ give hyperbolae, which face along the $U$ direction outside $r_s$, and point along $V$ inside $r_s$. At $r_s$ we have $V = \pm U$ once again, and at $r = 0$ we obtain the parabolae

$$V = \sqrt{U^2 + 1}$$

recall the lightcone structure is given $\mathrm{d}V = \pm\,\mathrm{d}U$

We define the the regions, separated by the event horizon as I through IV going counterclockwise from the +ve U axis. I is the space outside the black hole, II is inside the schwarzschild black hole, while III is a wormhole, and IV is a white hole region.

Note that objects within the *anti-horizon*, the white hole's $r = 2M$ plane, can causally impact region I but cannot be in turn causally affected; this is the opposite of the black hole. Similarly, because of the causal structure, when you cross the horizon, you see all the light that ever passed through the event horizon.

Further, note that due to the causal structure, it is impossible for a massive particle to reach the other side of the wormhole. However, if we take the metric for $t = 0 \implies V = 0$, we find the metric becomes

$$\mathrm{d}s^2 = \frac{4r_s^3}{r} e^{-r/r_s} \, \mathrm{d}U^2 + r^2 \, \mathrm{d}\Omega^2$$

further, our slice gives

$$U = \left(\frac{r}{r_s} - 1\right)^{1/2} e^{r/2r_s}$$

which upon substitution yields

$$\mathrm{d}s^2 = \frac{1}{1 - \frac{r_s}{r}} \, \mathrm{d}r + r^2 \, \mathrm{d}\Omega^2$$

Varying $V$ shows us that the wormhole is "pinched off" by intersection with the singularity and separated soon after formation; the wormhole is not static but has a dynamic evolution.

$$\mathrm{d}s^2 = \frac{1}{1 - \frac{r_s}{r}\left(1 - V_0 e^{-r/r_s}\right)} \, \mathrm{d}r^2 + r^2 \, \mathrm{d}\Omega^2$$

The timescale wher the wormhole is open is on the order of $r_s = 2M$

# Chapter 9

# Math!

## 9.1 Vectors and Manifolds

Loosely speaking, a vector is all the ways you can travel through a point on a manifold. More strictly, consider an arbitrary function $f(t)$ that has the property of passing through a point $p$ on the manifold $M$; its tangent vector is given

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\frac{\mathrm{d}f}{\mathrm{d}x^\mu}$$

We wish to consider all such tangent vectors, for any vector that passes through $p$, thus, we instead consider

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\partial_\mu$$

we consider $\mathrm{d}x^\mu \,/\, \mathrm{d}x$ to be the component of an arbitrary vector $\mathrm{d} \,/\, \mathrm{d}t$ and the basis to be $e_\mu = \partial_\mu$. The space of all of these vectors $\mathrm{d} \,/\, \mathrm{d}t$ forms the *tangent space* $T_pM$, and forms a tangent bundle $TM$ over the entire manifold $M$.

From our definition of a vector, we see trivially that the transformation for the basis

$$\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\nu}\partial_\mu$$

gields the transformation law of the components

$$V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}V^\nu$$

to keep the vector constant.

## 9.2 Dual Vectors, 1-Forms or Covectors

Of course, for each tangent space there is an associated *dual space* $T_p^*$ which are linear maps from vectors in $T_p$ to scalars.

$$T_p^* \ni \omega : T_p \to \mathbb{R}$$

that is,

$$\omega(V) = V^\mu \omega(\partial_\mu) \equiv V^\mu \omega_\mu$$

More concretely, we can consider the gradient of a function $f$, $\mathrm{d}f$, acting along a vector, $\mathrm{d}\,/\,\mathrm{d}t$, yielding the directional derivative:

$$\mathrm{d}f\,\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\partial_\mu f$$

We have a natural basis for the dual vectors coming from

$$\partial_\mu x^\nu = \delta^\nu_\mu = \mathrm{d}x^\nu\,\partial_\mu$$

we can thus write

$$\omega = \omega_\nu\,\mathrm{d}x^\nu \equiv \omega_\nu f^\nu$$

We can similarly extract the transformation law for the components of $\omega$ by

$$\delta^\mu_\nu = \mathrm{d}x^\mu\,\partial_\nu = \mathrm{d}x'^\mu\,\partial'_\nu = \mathrm{d}x'^\mu\,\frac{\partial x'^\rho}{\partial x^\nu}\partial_\rho$$

so

$$\mathrm{d}x'^\mu = \frac{\partial x^\mu}{\partial x'^\nu}\,\mathrm{d}x^\nu$$

and

$$\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\omega_\mu$$

We can relate the action of a 1-form on a vector by

$$\omega(V) = V^\mu \omega_\nu f^\nu(e_\mu) = V^\mu \omega_\nu \delta^\nu_\mu = V^\mu \omega_\mu = V \cdot \omega = g_{\mu\nu} V^\mu \omega^\nu$$

and thus, we can raise and lower indices by contracting with the metric:

$$\omega_\mu = g_{\mu\nu}\omega^\nu$$

## 9.3 Tensors

A tensor is just another name for a *multilinear map*. We define a tensor of rank $(r, s)$ as a multilinear map

$$T \in \mathcal{L}(\underbrace{T^*_p M \times \cdots \times T^*_p M}_{r \text{ times}} \times \underbrace{T_p M \times \cdots \times T_p M}_{s \text{ times}}, \mathbb{R})$$

Thus, we see that vectors are rank $(1, 0)$ vectors, while 1-forms are $(0, 1)$ tensors, and the metric is a rank $(0, 2)$ tensor.

We can write the components of this map along some basis $f^\mu, e_\nu$ as:

$$T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \cdots \nu_s} = T(f^{\mu_1}, \ldots, f^{\mu_r}, e_{\nu_1}, \ldots, e_{\nu_s})$$

It is important to know that because $T$ in general, not a symmetric tensor, the location of the indices is important, as it indicates which argument of the tensor is being used for which vector. For example, consider the application of a rank $(2,1)$ tensor $A$ on three vectors $\eta, \omega, x$:

$$A(\eta,\omega,x) = A(\eta_\mu f^\mu \omega_\nu f^\mu, x^\rho e_\rho) = \eta_\mu\omega_\nu x^\rho A(f^\mu, f^\nu, e_\rho) = A^{\mu\nu}{}_\rho \eta_\mu\omega_\nu x^\rho$$

Tensors inherit their transformation laws from the vectors that define their basis. For example, $A$ above transforms as

$$A'^{\mu\nu}{}_\rho = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta}\frac{\partial x'^\gamma}{\partial x^\rho}A^{\alpha\beta}{}_\gamma$$

We define indices that transform like the basis $e_\mu$ to be *covariant*, while bases which transform like the coordinates to be *contravariant*. For example, vectors are contravariant, while 1-forms are covariant.

## 9.4 Derivatives and connections

We unfortunately run into an issue when taking the derivative of a vector, $\omega_\nu V^\mu$. If we transform the basis,

$$\begin{aligned}\partial'_\nu V'^\mu &= \frac{\partial x^\alpha}{\partial x'^\nu}\partial_\alpha\left(\frac{\partial x'^\mu}{\partial x^\beta}V^\beta\right)\\ &= \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\beta}(\partial_\alpha V^\beta) + \frac{\partial^2 x'^\mu}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\alpha}{\partial x'^\mu}V^\beta\end{aligned}$$

while the first term is as expected, the second term does not, in general, vanish! Thus, our "derivative" $\partial_\mu$ is not a true derivative; we must define a new derivative which does transform properly—the *covariant derivative.*

$$\nabla_\mu V^\nu = \nabla_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda$$

The object

$$[\Gamma_\mu]^\nu_\lambda$$

is known as the *connection.*

To see why we need this connection, we find that if we embed a manifold into flat space and parallel transport a vector along a curve, the vector no longer lies in the tangent space on a different point along the curve; the connection "removes" this component.

We should expect, geometrically, that

$$\nabla_\mu V = \frac{\partial}{\partial x^\mu}(v^\lambda e_\lambda) = \frac{\partial V^\lambda}{\partial x_\mu}e_\lambda + V^\lambda\frac{\partial e_\lambda}{\partial x^\mu}$$

where the first term is from how the component changes, and the second is from the change of basis. Matching this to our definition of the covariant derivative, we obtain the *affine connection*

$$\Gamma^\nu_{\mu\lambda}e_\nu = \frac{\partial e_\lambda}{\partial x^\mu}$$

plugging this in, we do obtain the original definition:

$$\nabla_\mu V = \partial_\mu V^\lambda e_\lambda + V^\lambda \Gamma^\nu_{\mu\lambda} e_\nu$$

relabelling indices,

$$\begin{aligned} &= \partial_\mu V^\nu e_\nu + V^\lambda \Gamma_{\mu\lambda^\nu} e_\nu \\ &= \left(\partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}\right) e_\nu \\ \nabla_\mu V^\nu &= \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \end{aligned}$$

as expected.

We find that the covariant derivative is a derivative, in that it obeys linearity and the leibniz rule:

$$\nabla_\nu (V^\mu \omega_\mu) = (\nabla_\nu V^\mu)\omega_\mu + V^\mu (\nabla_\nu \omega_\mu)$$

Further, we expect

$$\nabla_\nu (V^\mu \omega_\mu) = \partial_\nu (V^\mu \omega_\nu)$$

so the covariant derivative of a 1-form is

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda$$

It can easily be verified that the covariant derivative transforms properly:

$$\nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu V^\nu$$

From this, we can then see that the affine connection is not a true tensor, and transforms as

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} + \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu x^\lambda}$$

We can however consider the *torsion tensor* which is an antisymmetrization of the connection

$$2\Gamma^\lambda_{[\mu\nu]} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}$$

General relativity is a torsion-free theory, so

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} = \Gamma^\lambda_{(\mu\nu)}$$

and further, it has metric compatibility,

$$\nabla_\rho g_{\mu\nu} = 0$$

The combination of being torsion-free and having metric compatibility means that the connection is unique; this connection is the Christoffel connection which appears in Equation 6.4. We thus have for a locally inertial frame (LIF) that

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} 0$$

Evaluating the derivative,

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda}$$

and cycling the indices, we can obtain the expression for the Cristoffel symbol we have previously learned:

$$0 = \nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

We can further define a *directional covariant derivative* as

$$\frac{\mathrm{D}}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \nabla_\mu = u^\mu \nabla_\mu$$

For example, the directional derivative of the unit tangent vector to the geodesic

$$u^\mu \nabla_\mu u^\beta = u^\mu \left[\partial_\mu u^\beta + \Gamma^\beta_{\mu\gamma} u^\gamma\right] = \frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\tau^2} + \Gamma^\beta_{\mu\gamma} u^\mu u^\gamma = 0$$

where the last equality is due to the geodesic equation. Thus, parallel transport along a geodesic keeps a vector parallel to the unit tangent vector; we define paralle transport of a vector $X$ to be such that

$$\nabla_\mu X = 0$$

## 9.5 Curvature

Consider a triangle in flat space, where the interior angle adds to *pi*. If we parallel transport a vector about this curve, the vector gains no angle.

However, if we parallel transport the vector along a triangle in curved space, the vector now differs from the original vector by an angle. Using the commutator of the covariant derivative, we can obtain an expression for the *Riemann Curvature Tensor*

$$[\nabla_\mu, \nabla_\nu]_\sigma V^\rho$$

$$(\nabla_u \nabla_u \xi)^\alpha = -R^\alpha{}_{\beta\gamma\delta} u^\beta \chi^\gamma u^\delta$$

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_\nu \sigma - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$