Physical Mechanics Notes

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Contents

Chapter 1

Newtonian Mechanics

1.1 Introduction

1.1.1 Notation

Rodriguez uses different notation compared to the textbook:

	T&M	Rodriguez
scalars	x, r	x, r
vectors	$\mathbf{x}$, $\mathbf{r}$	$\vec{x}$, $\vec{r}$
unit vectors	e_r, e_θ	$\hat{r}$, $\hat{\theta}$
time derivatives	$\dot{x}$, $\ddot{\mathbf{x}}$	$\frac{\mathrm{d}x}{\mathrm{d}t}$, $\ddot{\vec{x}}$, $\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2}$

1.1.2 Partial vs Total Derivative

Partial Derivative:

$$\frac{\partial f}{\partial t}$$

Total Derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial t}$$

1.1.3 Physical Mechanics

In other schools, this course would be called classical mechanics. As opposed to Newtonian mechanics, which relies on forces and acceleration, Classical Mechanics relies on Energy with the calculus of variations to compute time evolution.

1.2 Newtonian Mechanics

1.2.1 First Law

A body remains at rest, or in uniform motion, unles acted upon by a force.

The first law is a bit vague, as force is not properly defined yet. Further, this idea of motion is deeply connected to the idea of inertial reference frames, and is easiest in cartesian coordinates. Any motion (without acceleration) can be transformed to a different frame by $\vec{x}' = \vec{x} + \vec{v}t$. Physics between these two inertial reference frames must obey the same laws; the same is not true of non-inertial reference frames, which require the introduction of pseudo-forces (e.g. centrifugal force).

1.2.2 Second Law

A body acted upon by a force moves in such a manner that the time rate of change of momentum equals the force.

Momentum is defined to be $\vec{p} = m\vec{v}$. Thus, the second law can be written as:

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\vec{v} + m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \tag{1.1}$$

In the limiting case of $\frac{\mathrm{d}m}{\mathrm{d}t} = 0$ (or constant mass), this equation reduces to the famous

$$\vec{F} = m\vec{a}$$

Constant Acceleration

Given a constant acceleration,

$$F = ma = m\ddot{x}$$
$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$
$$v = v_0 + at$$
$$v_0 + at = \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Additionally,

$$\ddot{x} = a$$
$$2\dot{x}\ddot{x} = 2\dot{x}a$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x})^2 = 2\frac{\mathrm{d}x}{\mathrm{d}t}a$$
$$v^2 - v_0^2 = 2a(x - x_0)$$

Parabolic Motion: Brute Force

A man is standing at the top of a h high cliff and throws a javelin at angle θ above the horizontal with an initial velocity of v_0 off the cliff. The initial conditions are given:

$$x_0 = \left\langle 0, h \right\rangle, \qquad v_0 = \left\langle v_0 \cos\theta, v_0 \sin\theta \right\rangle$$

Because the javelin is considered to be in freefall, the constant acceleration is given:

$$a = \left\langle 0, -g \right\rangle$$

The horizontal velocity remains constant. Thus, to find the vertical component:

$$v_y^2 = v_0^2 \sin^2\theta = 2(-g)(0 - h)$$

$$v_y = \sqrt{2hg + v_0^2 \sin^2\theta}$$

Thus, the total velocity when hitting the ground, is:

$$\|v_f\| = \sqrt{v_0^2 + 2gh}$$

Parabolic Motion: Easier Way

The easier way to solve this problem is to examine it from an energetics problem. The energy of the system at the beginning of the problem is

$$E = T + U = \frac{1}{2}mv_0^2 + mgh$$

by conservation of energy, the final energy must equal the initial velocity, and the potential energy must go to zero. Thus,

$$\frac{1}{2}mv_0^2 + mgh = \frac{1}{2}mv^2$$

$$v^2 = v_0^2 + 2gh$$

$$v = \sqrt{v_0^2 + 2gh}$$

which is the same as what was obtained using the previous method.

1.2.3 Third Law

If two bodies exert forces on each other, these forces are equal in magnitude and opposite in direction.

This statement results in the conservation of momentum, after integrating wrt. time. However, this is only true of *central forces*, or forces that are applied along the line connecting the two bodies.

A notable exception to this is the magnetic force. Imagine two wires, which lie in parallel planes, but are perpendicular to each other. In this system, the infinitessimal elements $I\,\mathrm{d}\vec{\ell}$ apply a force on each other, but these forces are *at a perpendicular angle*, not anti-parallel. The resolution to this issue is the comparison of momenta; the momenta can be dumped into the electric and magnetic field.

1.3 Non-Cartesian Reference Frames

1.3.1 Polar Coordinates

The position vector is given specified by the pair (r, θ). The coordinates can be transformed between cartesian and polar coordinates as follows:

$$x = r\cos\theta \tag{1.2a}$$

$$y = r\sin\theta \tag{1.2b}$$

$$r = \sqrt{x^2 + y^2} \tag{1.2c}$$

$$\theta = \tan^{-1}(y/x) \tag{1.2d}$$

However, the basis vectors depend on the position. The unit vectors $\hat{r}$ and $\hat{\theta}$ are given (in cartesian coordinates):

$$\hat{r} = \big\langle \cos\theta, \sin\theta \big\rangle \tag{1.3a}$$

$$\hat{\theta} = \big\langle -\sin\theta, \cos\theta \big\rangle \tag{1.3b}$$

These unit vectors also change in time:

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \big\langle -\sin\theta, \cos\theta \big\rangle \frac{\mathrm{d}\theta}{\mathrm{d}t} = \boxed{\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}} \tag{1.4a}$$

$$\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \big\langle -\cos\theta, -\sin\theta \big\rangle \frac{\mathrm{d}\theta}{\mathrm{d}t} = \boxed{-\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r}} \tag{1.4b}$$

Velocity and Acceleration

The position vector is given:

$$\vec{r} = r\hat{r}$$

Then, the velocity is obtained through differentiation:

$$\begin{aligned} \dot{\vec{r}} &= \dot{r}\hat{r} + r\dot{\hat{r}} \\ &= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \end{aligned} \tag{1.5a}$$

The acceleration is similarly obtained:

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} \tag{1.5b}$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \tag{1.5c}$$

Of interest are the $-r\dot{\theta}^2$ and the $2\dot{r}\dot{\theta}$, which correspond to the centrifugal and coriolis forces.

Simple Pendulum Motion

A block is on a semi-circular half-pipe track. What is the period of oscillation of the block?
The polar coordinate system is chosen such that the origin is the centre of the half-pipe and the lowest point of the track is $\theta = 0$. The normal force that the box experiences is always in the $-\hat{r}$ direction, and is written:

$$\vec{F}_N = -F_N\hat{r}$$

1.4 Conservation Laws

Total linear momentum of a particle is conserved when the force on it is zero

This follows immediately from Equation 1.1. If instead there are multiple particles, following from Newton's third law, the total linear momentum of the system, if there are no external forces, is conserved.

Total angular momentum of a particle is conserved when the torque on it is zero

The angular momentum is defined:

$$\vec{L} = \vec{r} \times \vec{p} \tag{1.6}$$

and torque is defined

$$\vec{N} = \vec{\tau} = \vec{r} \times \vec{F} \tag{1.7}$$

The torque is to angular momentum as force is to linear momentum; that is:

$$\vec{N} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} \tag{1.8}$$

This can be seen from

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p}) = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \cancel{m(\vec{v} \times \vec{v})} + \vec{r} \times \vec{F} = \vec{N}$$

Conservation of Energy Work is defined

$$W = \int_\gamma \vec{F} \cdot \mathrm{d}\vec{r} \tag{1.9}$$

For a straight line,

$$\begin{aligned} W = \int_A^B F \cdot \mathrm{d}r &= \int_A^B m \frac{\mathrm{d}v}{\mathrm{d}t} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_A^B \frac{1}{2} m \cdot 2v \cdot \frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_A^B \frac{1}{2} m \frac{\mathrm{d}}{\mathrm{d}t} (v \cdot v) \, \mathrm{d}t \\ &= \int_A^B \frac{1}{2} m \frac{\mathrm{d}}{\mathrm{d}t} (v^2) \, \mathrm{d}t \\ &= \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 \end{aligned}$$

From this, the work energy theorem (for a conservative force), is obtained:

$$W \equiv \int_A^B \vec{F} \cdot \mathrm{d}r = T_B - T_A \tag{1.10}$$

where $T \equiv \frac{1}{2} m v^2$ is the kinetic energy.

1.4.1 Conservative Forces

A conservative force is *path independent*; that is, any path between two points yields the same amount of work. Thus, an integral over any closed loop is zero:

$$W_{\text{loop}} = \oint_\gamma \vec{F} \cdot \mathrm{d}\vec{r} = 0 \tag{1.11}$$

A very important result can be obtained using this fact.

$$\oint_{\partial S} \vec{F} \cdot \mathrm{d}\vec{r} = \int_S \nabla \times \vec{F} \cdot \mathrm{d}\vec{A} \tag{1.12}$$

Equation 1.12 is *Stoke's Theorem*. By applying Stoke's theorem to any closed loop with a conservative force, the force is *curl free*:

$$\nabla \times \vec{F} = 0$$

Thus, TFAE:

1. $\nabla \times \vec{F} = 0$ everywhere
2. $\int_a^b \vec{F} \cdot \mathrm{d}\vec{r}$ is path independent
3. $\oint \gamma \vec{F} \cdot \mathrm{d}\vec{r} = 0$ for every loop γ
4. There exists a scalar potential U such that $\vec{F} = -\nabla U$

Point 4 implies the others through the Fundamental Theorem of Line Integrals:

$$\int_\gamma -\nabla U \cdot \mathrm{d}vvr = \int_{\gamma[A]}^{\gamma[B]} -\mathrm{d}U = U(\gamma[A]) - U(\gamma[B]) \tag{1.13}$$

However, this means that any transformation $U' = U + C$ has an equivalent gradient. Thus, potentials can only be defined relatively; there is no absolute potential. A reference point must be defined for a potential, and is chosen to make calculations easier[1].

If we define a total energy

$$E = T + U$$

We can get (for a constant mass):

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{m}{2}\vec{v}\cdot\vec{v}\right) + \nabla U \cdot \frac{\mathrm{d}r}{\mathrm{d}t} + \frac{\partial U}{\partial t} \\
&= m\vec{a}\cdot\vec{v} + \nabla U \cdot \vec{v} + \frac{\partial U}{\partial t} \\
&= (m\vec{a} + \nabla U)\cdot\vec{v} + \frac{\partial U}{\partial t} \\
&= (m\vec{a} - F)\cdot\vec{v} + \frac{\partial U}{\partial t} \\
&= \frac{\partial U}{\partial t}
\end{aligned} \tag{1.14}$$

Thus, as long as the potential U is constant with time, then energy is conserved.

1.5 Equations of motion from energy

Using the fact that $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, we can rewrite energy as

$$E = \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + U$$

then solve the differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m}(E-U)}$$

$$\mathrm{d}t = \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E-U)}}$$

[1]In higher physics, this becomes the choice of a *gauge*

$$\Delta t = \int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E-U)}} \tag{1.15}$$

While this equation is difficult to gain an exact form, it can be solved numerically. Further, from the denominator, you can determine where a particle would be able to reach, and where it would be forbidden (zero/complex denominator)

1.5.1 Simple Harmonic Oscillator

One case where the integral can be solved is a quadratic potential:

$$U = \frac{1}{2}kx^2$$

Plugging this into Equation 1.15, the integral can be solved with inverse trig functions:

$$\begin{aligned}
\sqrt{\frac{2}{m}}\Delta t &= \sqrt{\frac{2}{k}}\sin^{-1}\left[x\sqrt{\frac{k}{2E}}\right]\Bigg|_{x_0}^{x} \\
\sqrt{\frac{k}{m}}(t-t_0) &= \sin^{-1}\left[\sqrt{\frac{k}{2E}}x\right] - \underbrace{\sin^{-1}\left[\sqrt{\frac{k}{2E}}x_0\right]}_{\equiv C} \\
\sin^{-1}\left[\sqrt{\frac{k}{2E}}x\right] &= \sqrt{\frac{k}{m}}(t-t_0) + C
\end{aligned}$$

$$x(t) = \sqrt{\frac{2E}{k}}\sin\left[\sqrt{\frac{k}{m}}(t-t_0)+C\right] \tag{1.16a}$$

$$= A\sin\left[\omega(t-t_0)\right] \tag{1.16b}$$

Stable Equilibrium

We can expand a potential using a taylor series. Using the second derivative test, the an extrema of the potential energy is a stable equilibrium if the second derivative is greater than zero. If it is zero, then higher order derivatives need to be examined. If the lowest nonzero derivative is of an odd order or is negative, it is unstable. If the lowest nonzero derivative is both of even order and positive, then it is a stable equilibrium.

Phase Space

It is useful to view the system in what is called *phase space*. For the simple harmonic oscillator, it is useful to plot velocity against position. We have the velocity and position parametrized with time as:

$$x = \sqrt{\frac{2E}{k}}\sin\left[\sqrt{\frac{k}{m}}(t-t_0)+c\right]$$

$$v = \sqrt{\frac{2E}{m}} \cos\left[\sqrt{\frac{k}{m}}(t - t_0) + c\right]$$

Plotting the points (x, v) as a function of t yields an ellipse. At higher energies, the size of the ellipse increases; at higher energies, the particle can go higher up the well.

We also see the typical behaviour of the harmonic oscillator: the particle moves fastest when $x = 0$, and the particle is furthest from the minimum when $v = 0$.

A more interesting phase space is that of $U = -A\cos(x)$, as it has bound and free states.

1.6 Gravitation

Given two particles of mass m and M, the gravitational attraction between the two particles is

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{GmM}{r^3}\vec{r} \tag{1.17}$$

This force does not depend on time, and it is easy to verify that this force is curl-free; the gravitational force is conservative. In fact, we can write the gravitational force's scalar potential as:

$$U_g = -\frac{GMm}{r} \tag{1.18}$$

Here, we implicitly define a point off at ∞ as the reference potential of $U(\infty) = 0$.

Something else that is interesting to note, is that you can rewrite the gravitational force as a vector field of acceleration, named the *gravitational field*[2]

$$\vec{a} \equiv \vec{g} = -\frac{GM}{r^2}\hat{r} \tag{1.19}$$

Another consequence of the fact that the gravitational field is an acceleration field, it is typical to use the *specific potential* rather than the scalar potential:

$$\Phi \equiv U/m = -\frac{GM}{r} \tag{1.20}$$

Collection of Particles

If instead we have multiple particles, we claim that the force obeys the superposition principle. Thus,

$$\vec{F} = \sum_i \vec{F_i}$$

$$\implies \vec{g} = \sum_i \vec{g_i} = \sum_i -\nabla\Phi_i = -\nabla\left(\sum_i \Phi_i\right)$$

[2]This is one statement of the *equivalence principle* of GR, which states that a gravitational field is indistinguishable from an accelerating frame.

Therefore,[3]

$$\vec{g} = \sum_i -\frac{GM_i}{\|\vec{r} - \vec{r}_i\|^3}(\vec{r} - \vec{r}_i) \tag{1.21a}$$

$$\Phi = \sum_i -\frac{GM_i}{\|\vec{r} - \vec{r}_i\|} \tag{1.21b}$$

Continuous Limit

For a solid object, we can define an infinitessimal mass element $\mathrm{d}m$. Then, to compute the gravitational field from the object, we can integrate:

$$\vec{g} = -\int \frac{G\,\mathrm{d}m}{\|\vec{r} - \vec{r}_i\|^3}(\vec{r} - \vec{r}_i)$$

More commonly, this is rewritten with $\mathrm{d}m = \rho(\vec{r}')\,\mathrm{d}V \equiv \rho(\vec{r}')\,\mathrm{d}\vec{r}'$, such that:[4]

$$\vec{g}(\vec{r}) = -\int_V \frac{G\rho(\vec{r}')}{\|\vec{r} - \vec{r}'\|^3}(\vec{r} - \vec{r}')\,\mathrm{d}\vec{r}' \tag{1.22}$$

Once again, this can be written in terms of a potential:

$$\Phi = -\int_V \frac{G\rho(\vec{r}')}{\|\vec{r} - \vec{r}'\|}\,\mathrm{d}\vec{r}' \tag{1.23}$$

In terms of a surface mass, this equation becomes:

$$\Phi = -G\int_A \frac{\sigma(\vec{r}')}{\|\vec{r} - \vec{r}'\|}\,\mathrm{d}A'$$

and so forth for a line mass.

1.7 Spherical Shell

We have a spherical shell centred at the origin, with radius R and constant density σ. We want to find the gravitaional potential at a point r outside of the shell.

The total mass of the sphere is

$$M = \int_a \sigma(\vec{r}')\,\mathrm{d}A' = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, \sigma r^2 \sin\theta = \sigma 4\pi R^2$$

[3]Griffiths defines a variable cursive r such that $\mathscr{r} = \vec{r} - \vec{r}_i$

[4]the differentials $\mathrm{d}V = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z = \mathrm{d}\vec{r} = \mathrm{d}^3 r = \mathrm{d}\tau$ are different notations for a volume element

To calculate the potential,

$$\begin{aligned}
\Phi &= -G \int_A \frac{\sigma}{\|\vec{r} - \vec{r}'\|}\, \mathrm{d}A' \\
&= -G \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, \frac{\sigma R^2 \sin\theta}{\eta}
\end{aligned}$$

Using the law of cosines, we can rewrite the η

$$\eta^2 = r^2 + r'^2 - 2rr'\cos\theta$$

Differentiating wrt θ allows us to rewrite it as

$$\begin{aligned}
2\eta \frac{\mathrm{d}\eta}{\mathrm{d}\theta} &= 2rR\sin\theta \\
\implies \frac{\sin\theta\, \mathrm{d}\theta}{\eta} &= \frac{\mathrm{d}\eta}{Rr}
\end{aligned}$$

Thus, the integral becomes

$$\begin{aligned}
\Phi &= -G\sigma \int_0^{2\pi} \mathrm{d}\phi \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, \frac{R}{r} \\
&= -\frac{2\pi G\sigma R}{r}[\eta_{\max} - \eta_{\min}] \\
&= -\frac{2\pi G\sigma R}{r}[(r+R) - (r-R)] \\
&= -\frac{2\pi G\sigma R}{r} \cdot 2R \\
&= -\frac{G \cdot 4\pi R^2 \sigma}{r} \\
&= -\frac{GM}{r}
\end{aligned}$$

This result is known as *Newton's Second Shell Theorem* What happens instead if $r < R$? Then,

$$\begin{aligned}
\Phi &= -G\sigma \int_0^{2\pi} \mathrm{d}\phi \int_{\eta_{\min}}^{\eta_{\max}} \mathrm{d}\eta\, \frac{R}{r} \\
&= -\frac{2\pi G\sigma R}{r}[\eta_{\max} - \eta_{\min}] \\
&= -\frac{2\pi G\sigma R}{r}[(R+r) - (R-r)] \\
&= -\frac{4\pi G\sigma Rr}{r} \\
&= -4\pi G\sigma R \\
&= -\frac{GM}{R}
\end{aligned}$$

This is *Newton's First Shell Theorem.*

Conceqently, the force from opposite sides with a given solid angle Ω cancels out:

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$$\delta g_i = \frac{G\delta m_i}{r_i^2} = \frac{G\rho(r_i^2\Omega)}{r_i^2} = G\rho\Omega$$

The force on the test mass is only a function of the

If we consider the inverse, that is, the flux from the test mass going through a solid angle, we get[5]

$$\begin{aligned}\phi_i &= \int_S \vec{g} \cdot \mathrm{d}\vec{a} \\ &= \int_S -\frac{GM}{r_i^2} r_i^2 \,\mathrm{d}\Omega \\ &= -GM \int_S \mathrm{d}\Omega\end{aligned}$$

For the whole sphere, this becomes:

$$\begin{aligned}\phi &= -GM \oint_S \mathrm{d}\Omega \\ &= -4\pi GM \\ &= -4\pi G \int_V \rho \,\mathrm{d}V \qquad (1.24a) \\ \underset{\partial V}{\oiint} g \cdot \mathrm{d}A &= -4\pi G \iiint_V \rho \,\mathrm{d}V \qquad (1.24b)\end{aligned}$$

Using Equation 1.24a, which is the integral form of *Gauss's Law* for gravitation, along with the symmetries of the problem, deriving Newton's Shell Theorems becomes trivial. Outside the shell, we choose a concentric spherical shell. Because of the spherical symmetry, the flux lines are parallel to the radius, or normal to the surface. Thus,

$$-4\pi GM = g4\pi r^2 \implies g = \frac{-GM}{r^2}$$

Inside the shell, we choose a surface that is contained wholey by the sphere:

$$0 = 4\pi r^2 g \implies g = 0$$

Using the divergence Theorem, we can rewrite Gauss' law into a differential form:

$$\begin{aligned}\nabla \cdot g &= -4\pi G\rho \\ \nabla^2\Phi &= 4\pi G\rho\end{aligned}$$

[5] This abuses the fact that the area S is so small that all the radii are (approximately) parallel.

1.8 Ring mass

We have a ring of radius R, mass density λ, centered on the origin. The differential potential element along the z axis:

$$\mathrm{d}\Phi = \frac{-G\,\mathrm{d}m}{\|\mathscr{r}\|} = -\frac{G\lambda R\,\mathrm{d}\phi}{\sqrt{z^2 + R^2}}$$

Thus, the potential along the z axis is

$$\Phi(z) = \frac{-GM}{\sqrt{Z^2 + R^2}} \tag{1.26}$$

This linear potential has an equilibrium point at $z = 0$, and it is stable.

However, pertubations in the xy-plane do not return to equilibrium. The differential potential element is once again

$$\mathrm{d}\Phi = \frac{-G\lambda R\,\mathrm{d}\phi}{\|\mathscr{r}\|} = \frac{-G\lambda R\,\mathrm{d}\phi}{\sqrt{R^2 + r^2 - 2rR\cos\phi}} = \frac{-G\lambda\,\mathrm{d}\phi}{\sqrt{1 + \left(\frac{r}{R}\right)^2 - 2\left(\frac{r}{R}\right)\cos\phi}}$$

Thus,

$$\int \mathrm{d}\Phi = \int_0^{2\pi} \frac{-G\lambda\,\mathrm{d}\phi}{\sqrt{1 + \left(\frac{r}{R}\right) - 2\frac{r}{R}\cos\phi}}$$

Defining

$$\epsilon = \frac{r}{R}\left(\frac{r}{R} - 2\cos\phi\right)$$

with $r \ll R$. Thus, $\epsilon \ll 1$. We can then taylor expand the integrand:

$$\int \mathrm{d}\Phi \approx -G\lambda \int_0^{2\pi} 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + O(\epsilon^4)\,\mathrm{d}\phi$$

Expanding, and keeping terms up to $O[(r/R)^3]$, the integrand becomes:

$$1 + \frac{r}{R}\cos\phi + \left(\frac{r}{R}\right)^2 \frac{3\cos^2\phi - 1}{2} + \left(\frac{r}{R}\right)^3 \frac{5\cos^2\cos\phi - 3\cos\phi}{2}$$

The coefficients of (r/R) actually end up giving Legendre polynomials of $\cos\phi$.

$$\frac{1}{\sqrt{1 - 2\left(\frac{r}{R}\right)\cos\phi + \left(\frac{r}{R}\right)^2}} = \sum_{n=0}^{\infty} P_n(\cos\phi)\left(\frac{r}{R}\right)^n \tag{1.27}$$

Recall that

$$\int_0^2 \pi\cos^{2n+1}\phi\,\mathrm{d}\phi = 0 \qquad \int_0^2 \pi\cos^2\phi\,\mathrm{d}\phi = \pi$$

Then, the integral (to third order) becomes:

$$\Phi = -G\lambda \int_0^{2\pi} 1 + \left(\frac{r}{R}\right) \frac{3\cos^2\phi - 1}{2}\,\mathrm{d}\phi$$

$$= -G\lambda \left[2\pi + \left(\frac{r}{R} \right) \frac{3\pi - 2\pi}{2} \right]$$

$$= -G\lambda 2\pi \left[1 + \frac{1}{4} \left(\frac{r}{R} \right)^2 \right]$$

$$= -\frac{GM}{R} \left[1 + \frac{1}{4} \left(\frac{r}{R} \right)^2 \right]$$

The equilibrium point once again occurs at $r = 0$. However, when we test the stability,

$$\frac{\partial \Phi}{\partial r2} = -\frac{GM}{2R^3} < 0$$

which is unstable.

Chapter 2

Calculus of Variations

2.1 Hamilton's Principle

Hamilton's Principle is a way to reformulate mechanics. Rather than using kinematics and dynamics, it searches all paths for the path that has the minimum "action." A way to visualize this is imagine you are on a beach, and you want to get to a point in the ocean. What is the fastest way to get between the two points? By varying the path between the two points, you can minimize the time to find the optimal path.

2.1.1 Paths

Imagine two points on a (not necessarily flat) plane. What is the shortest distance between two points? We know how to minimize a differentiable function; just take derivativs and check the neighbourhood around the zeroes.

We can do something similar by making a function of the path:

$$D = \int_0^1 \mathrm{d}\ell \tag{2.1}$$

Writing the path in terms of a functional $y(x)$, we can rewrite the differential length as:

$$\begin{aligned} \mathrm{d}\ell &= \sqrt{\mathrm{d}x^{\,2} + \mathrm{d}y^{\,2}} \\ &= \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \\ D &= \int_{x[0]}^{x[1]} \mathrm{d}x\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \end{aligned}$$

Plugging in the path $y(x) = mx$, we obtain the expected result by pythagorean theorem. However, what if we don't have a pre-determined path?

We introduce the concept of a *functional*. A function is an object that maps an element of one set to another set. For a functional, the domain is a space of functions.

One such functional is:

$$J[y] = \int_{x_0}^{x_1} F(y, y', x) \, \mathrm{d}x \tag{2.2}$$

In this sense, we defined before the distance functional for a 2D path.

We can minimize a functional in much the same way as we minimize a function. There exists a function y_{min} such that all functions within its neighbourhood have a larger value of the functional.

What does the neighbourhood of y look like? We can add an arbitrary (differentiable) function $\eta(x)$ such that $\eta(x_1) = \eta(x_0) = 0$, and redefine y such that $y(x, \alpha) = y(x, 0) + \alpha\eta(x)$. This allows us to reparametrize the functional as:

$$J(\alpha) = \int_{x_0}^{x_1} F\left[y(x, \alpha), \frac{\mathrm{d}y(x, \alpha)}{\mathrm{d}x}, x\right] \mathrm{d}x$$

so that the minimum occurs at

$$\left.\frac{\mathrm{d}J}{\mathrm{d}\alpha}\right|_{\alpha=0} = 0$$

Functional Example

Define a function $\eta(x) = \sin(x)$ and the function $F = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ We can show that $y = x$ is an extremum by defining

$$y(x, \alpha) = x + \alpha \sin x$$

This function has the derivative

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \alpha \cos x$$

Thus, the functional becomes

$$\begin{aligned} J(\alpha) &= \int_0^{2\pi} 1 + 2\alpha \cos x + \alpha^2 \cos^2 x \, \mathrm{d}x && (2.3) \\ &= 2\pi\alpha^2\pi && (2.4) \end{aligned}$$

Clearly, this is minimized by $\alpha = 0$.

The reason that we contrain η to be differentiable is that we can instead evaluate:

$$\begin{aligned} 0 &= \frac{\partial J}{\partial \alpha} \\ &= \frac{\partial}{\partial \alpha} \int_{x_0}^{x_1} \mathrm{d}x \, F(y, y', x) \end{aligned}$$

$$
\begin{aligned}
&= \int_{x_0}^{x_1} \mathrm{d}x \, \frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} \\
&= \int_{x_0}^{x_1} \mathrm{d}x \, \frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'
\end{aligned}
$$

Integrating by parts and throwing away boundary conditions

$$
\begin{aligned}
&= \cancel{\left[\frac{\partial F}{\partial y'}\eta(x)\Big|_{x_1}^{x_2}\right]} - \int_{x_0}^{x_1} \mathrm{d}x \, \frac{\partial F}{\partial x}\eta + \eta\frac{\partial}{\partial x}\frac{\partial F}{\partial y'} \\
&= \int_{x_0}^{x_1} \mathrm{d}x \left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)\right]\eta \\
\implies 0 &= \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right) \qquad (2.5)
\end{aligned}
$$

Equation 2.5 follows from the fact that the equation must be independent of perturbing path η. This equation is known as the *Euler Equation*, and latter will be applied as the *Euler Lagrange Equation*

2.2 Brachistochrone

Given two points A, B and a constant gravitational field g, what is the path of least time?

Define the coordinate axis as $+y$ pointing right and x pointing down. We can write a time functional

$$
T = \int_{x_0}^{x_1} \mathrm{d}t = \int_{x_1}^{x_2} \frac{\mathrm{d}\ell}{v}
$$

Assuming no friction and a stationary initial condition, the velocity of the particle can be written in terms of the height fallen. Further, writing out the differential path length element,

$$
T = \int_{x_1}^{x_2} \frac{1}{\sqrt{2gx}}\sqrt{1 + (y')^2}\,\mathrm{d}x
$$

Thus, we have

$$
F = \frac{1}{2gx}\sqrt{1 + (y')^2}
$$

Applying the Euler equation,

$$
\frac{\partial F}{\partial y} = 0
$$

$$
\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{2gx}}\frac{y'}{\sqrt{1 + (y')^2}}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2gx}}\frac{y'}{\sqrt{1 + (y')^2}} = 0
$$

$$\frac{1}{\sqrt{x}} \frac{y'}{\sqrt{1+(y')^2}} = k \equiv \frac{1}{\sqrt{2a}}$$
$$\frac{(y')^2}{x(1+(y')^2)} = \frac{1}{2a}$$

Using the substitution $x = a(1-\cos\theta)$,

$$y = \int \sqrt{\frac{a(1-\cos\theta)}{2a - a(1-\cos\theta)}} a \sin\theta \, \mathrm{d}\theta$$

and several trig identities,

$$\begin{aligned} &= a \int 1 - \cos\theta \, \mathrm{d}\theta \\ \implies \; y &= a(\theta - \sin\theta) \\ x &= a(1-\cos\theta) \end{aligned}$$

This parametrization gives a cycloid. This is the shape that occurs when following a point on circle of radius a as it rolls along the line $x = A_x$. The choice of coordinates, while unorthodox, yields a nice form; while using a standard x, y plane would seem no different, the resulting differential equation is *much* more difficult to solve.

2.3 Geodesic on a Sphere

The differential line element in spherical coordinates is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \, \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\phi^2$$

thus,

$$\mathrm{d}s = \mathrm{d}\theta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) + r^2 + r^2 \sin^2\theta \left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)}$$

Note however, that the radius is a constant $r = R$. Thus, we can rewrite the line element as

$$\mathrm{d}s = \mathrm{d}\theta \, R \sqrt{1 + \sin^2\theta (\phi')^2}$$

Anticipating the use of Euler's Equation, we write the functional

$$J = R \int_{\theta_0}^{\theta_1} \mathrm{d}\theta \sqrt{1 + \sin^2\theta (\phi')^2}$$

Thus, we use

$$F = \sqrt{1 + \sin^2\theta (\phi')^2}$$
$$\frac{\partial F}{\partial \phi} = 0$$

$$\frac{\partial F}{\partial \phi'} = \frac{\sin^2\theta\phi'}{\sqrt{1+\sin^2\theta(\phi')^2}}$$

$$\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0 - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = 0$$

$$\sin^2\theta\phi' = C\sqrt{1+\sin^2\theta(\phi')^2}$$

$$(\phi')^2 = \frac{C^2}{\sin^2\theta(\sin^2\theta - C^2)}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\theta} = \frac{C\csc^2\theta}{\sqrt{1-C^2\csc^2\theta}}$$

$$\Delta\phi = \sin^{-1}\left(\frac{\cot\theta}{\beta}\right)$$

for $\beta \equiv \frac{\sqrt{1-c^2}}{c}$ Rearranging,

$$\sin(\phi-\alpha) = \frac{\cot\theta}{\beta}$$

$$\sin\phi\cos\alpha - \cos\phi\sin\alpha = \frac{\cos\theta}{\beta\sin\theta}$$

$$\sin\phi\cos\alpha\sin\theta - \cos\phi\sin\alpha\sin\theta = \cos\theta$$

Defining $A \equiv \beta\cos\alpha$ and $B \equiv \beta\sin\alpha$, and multiplying both sides by the radius,

$$R\cos\theta = AR\sin\theta\sin\phi - BR\sin\theta\cos\phi$$

$$z = Ay - Bx$$

Or, the shortest path is the intersection of the sphere with the plane defined by the centre of the sphere and the two endpoints. This is a segment of an object called the *great circle*. Note, that there are actually two solutions to the euler equation—one going clockwise and the other going counterclockwise. This is because the euler equation gives extrema, not just minima. In fact, the two solutions are the shortest straight line path and longest straight line path.

2.4 Soap Film Problem

What curve $y(x)$, when revolved around the x-axis, yields the surface with minimal area? This problem actually happens to have an issue that makes it difficult to solve with the Euler Equation. To see why, let us try to compute this curve.

We can take the differential area element to be the surface area of an infinitessimally short, hollow cylinder:

$$\mathrm{d}A = 2\pi rh = 2\pi y\,\mathrm{d}s = 2\pi y\,\mathrm{d}x\sqrt{1}+(y')^2$$

Computing,

$$\frac{\partial F}{\partial y} = \sqrt{1+(y')^2}$$

$$\frac{\partial F}{\partial y'} = \frac{yy'}{\sqrt{1+(y')^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} = \frac{(y')^2}{\sqrt{1+(y')^2}} + \frac{yy''}{\sqrt{1+(y')^2}} - \frac{y(y')^2}{[1+(y')^2]^{3/2}}$$

Putting these together,

$$0 = \sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}} - \frac{yy''}{\sqrt{1+(y')^2}} + \frac{y(y')^2}{[1+(y')^2]^{3/2}}$$

which is not an easy equation to solve.

While we can swap the axes to make the integral easier, we can also use an alternate formulation of the Euler equation.

2.5 Alternate Form of Euler Eqn

We can write the total derivative of F as:

$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial y'}\frac{\mathrm{d}y'}{\mathrm{d}x} + \frac{\partial F}{\partial x}$$

$$\frac{\mathrm{d}F}{\mathrm{d}x} = y'\frac{\partial F}{\partial y} + y''\frac{\partial F}{\partial y'} + \frac{\partial F}{\partial x}$$

Using the fact that

$$\frac{\partial}{\partial x}\left(y'\frac{\partial F}{\partial y'}\right) = y''\frac{\partial F}{\partial y'} + y'\frac{\partial}{\partial x}\frac{\partial F}{\partial y'}$$

we can substitute the total differential in

$$= \frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\partial F}{\partial x} - y'\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y;}\right]$$

However, because we know the bracketed term is the original statement of the euler equation, which is equal to zero, and because of the linearity of the derivative operator, we obtain

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \tag{2.6}$$

Using this statement of the euler equation, we can revisit the soap film problem. Once again, using

$$F = \sqrt{1+(y')^2}$$

we obtain:

$$F - y'\frac{\partial F}{\partial y'} = c$$

as F doesn't have explicit dependence on x. Solving this differential equation,

$$\begin{aligned} C = y\sqrt{1+(y')^2} - y'\frac{yy'}{\sqrt{1+(y')^2}} \\ \implies y = C\sqrt{1+(y')^2} \\ \implies y' = \sqrt{\left(\frac{y}{c}\right)^2 - 1} \\ \implies x + A = C\cosh^{-1}(y/c) \\ \implies y = C\cosh\left(\frac{x+a}{C}\right) \end{aligned}$$

2.6 Euler in Higher Dimensions

Currently, we have only been applying the Euler Equation to functions of a single variable. However, in mechanics, we will often encounter functions of multiple variables. What we can do is we can parametrize the path as a vector valued function:

$$x(u), y(u)$$

and rewrite the action integral as:

$$J = \int_{u_0}^{u_1} \mathrm{d}u\, F(x, x', y, y', u) \tag{2.7}$$

However, this is predicated on the fact that x and y are independent—there are no constraints restricting one to a function of the other.

Similar to the 1D case, we assume we have a function $\vec{r}(u)$ that is a solution to the minimization problem. We can then write a nearby point in terms of a perturbing function:

$$\vec{r}(u, \alpha) = \vec{r}(u, 0) + \alpha\vec{\eta}(u) \tag{2.8}$$

Thus,

$$\begin{aligned} \frac{\partial J}{\partial \alpha} &= \int_{u_0}^{u_1} \frac{\partial F}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial F}{\partial x'}\frac{\partial x'}{\partial \alpha} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha}\,\mathrm{d}u \\ &= \int_{u_0}^{u_1} \eta_1\left[\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial x'}\right] + \eta_2\left[\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}y}\frac{\partial F}{\partial y'}\right]\mathrm{d}u \end{aligned} \tag{2.9}$$

Because we have assumed that x and y are independent variables, then we must have that each of the bracketed terms satisfies zero independently.

2.6.1 Shortest Path on a Plane

We revisit the problem of the shortest path on a plane using the euler equation we just derived for higher dimensions. The line elements is given:

$$\begin{aligned} \mathrm{d}s &= \sqrt{\mathrm{d}x^{\,2} + \mathrm{d}y^{\,2}} \\ &= \mathrm{d}u\,\sqrt{(x')^2 + (y')^2} \end{aligned}$$

Then, using

$$F = \sqrt{(x')^2 + (y')^2}$$

we can solve the euler equation for each of the components:

$$\begin{aligned} 0 &= \frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial x'} \\ &= 0 - \frac{\mathrm{d}}{\mathrm{d}u}\frac{x'}{\sqrt{(x')^2 + (y')^2}} \\ C_1 &= \frac{x'}{\sqrt{(x')^2 + (y')^2}} \end{aligned}$$

similarly,

$$\begin{aligned} C_2 &= \frac{y'}{\sqrt{(x')^2 + (y')^2}} \\ m &\equiv \frac{y'}{x'} = \frac{\mathrm{d}y}{\mathrm{d}x} \\ \mathrm{d}y &= m\,\mathrm{d}x \\ \Delta y &= m\Delta x \end{aligned}$$

which is once again the equation of a line.

2.7 Multiple Dimensions with Constraints

Say we want to constrian a problem to the surface of a circle. We impose a *constraint equation*

$$g(x, y) = x^2 + y^2 - R^2 = 0$$

which constrains the values of x and y to lie on the circle.

Differentiating the constraint equation,

$$\frac{\mathrm{d}g}{\mathrm{d}\alpha} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \alpha}$$

and substituting the perturbing function η,

$$\frac{\mathrm{d}g}{\mathrm{d}\alpha} = \frac{\partial g}{\partial x}\eta_1 + \frac{\partial g}{\partial y}\eta_2 = 0$$

Thus,

$$\frac{\partial g}{\partial x}\eta_1 = -\frac{\partial g}{\partial y}\eta_2 \tag{2.10}$$

or

$$\nabla g \cdot \eta = 0$$

The action integral can be then written:

$$\frac{\partial J}{\partial \alpha} = \int_{u_0}^{u_1} \left[\left(\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial x'} \right) - \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial y'} \right) \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} \right] \eta_1 \,\mathrm{d}u$$

Once again, the quantity in the square brackets must be equally zero. Thus,

$$\left(\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial x'} \right) \left(\frac{\partial g}{\partial x} \right)^{-1} = \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial y'} \right) \left(\frac{\partial g}{\partial y} \right)^{-1} \equiv -\lambda(u) \tag{2.11}$$

where the -1 refers to the multiplicative inverse, not the function inverse.

We can then split the equation into one of each variable, to determine an equation with *lagrange multipliers*

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}u}\frac{\partial F}{\partial x'} + \lambda(u)\frac{\partial g}{\partial x} = 0 \tag{2.12}$$

A similar equation results from the y variable, but the function $\lambda(u)$ is the same for both equations.

2.7.1 Constraint example

Let us apply this to the action integral

$$J = \int_1^2 \mathrm{d}\ell = \int_{u_0}^{u_1} \sqrt{x'^2 + y'^2}\,\mathrm{d}u$$

subjected to the constraint

$$g = x^2 + y^2 - R^2$$

This results in

$$\frac{\partial F}{\partial x} = 0$$

$$\frac{\partial F}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2}}$$

Thus,

$$-\frac{\mathrm{d}}{\mathrm{d}u}\frac{x'}{\sqrt{x'^2 + y'^2}} + 2\lambda(u) = 0$$

Following a similar procedure, and setting the λ's equal to each other,

$$\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}u}\frac{x'}{\sqrt{x'^2 + y'^2}} = \frac{1}{y}\frac{\mathrm{d}}{\mathrm{d}u}\frac{y'}{\sqrt{x'^2 + y'^2}}$$

differentiating wrt u and simplifying, we obtain the equation

$$(yy' - xx')(x''y' + x'y'') = 0$$

Recognizing that

$$\frac{\mathrm{d}g}{\mathrm{d}u} = xx' + yy' = 0 \implies -xx' = yy'$$

we get

$$2yy'(x''y' + x'y'') = 0$$

2.8 Action

In most literature, the shorthand

$$\delta J \,\mathrm{d}\alpha \equiv \frac{\partial J}{\partial \alpha}\,\mathrm{d}\alpha \tag{2.13}$$

is introduced. This is equivalent to:

$$\delta J = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \right) \delta y \,\mathrm{d}x \tag{2.14}$$

Where once again

$$\delta y \equiv \frac{\partial y}{\partial \alpha}\,\mathrm{d}\alpha \sim \eta(x)$$

<u>"Derivation"</u>

$$\begin{aligned}
\delta J &= \delta \int_{x_0}^{x_1} F \,\mathrm{d}x \\
&= \int_{x_0}^{x_1} \delta F \,\mathrm{d}x \\
&= \int_{x_0}^{x_1} \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \,\mathrm{d}x \\
&= \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial y'} \right) \delta y \,\mathrm{d}x
\end{aligned}$$

Chapter 3

Lagrangian Mechanics

Now with knowledge of teh calculus of variations, we are equipped to revisit Hamilton's Principle. Hamilton's principle states that the evolution of a system is a stationary point of the action functional

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t) \,\mathrm{d}t \tag{3.1}$$

where the function

$$L \equiv T - U \tag{3.2}$$

is known as the *Lagrangian*. Assuming the kinetic and potential terms can be written:

$$T = T(\dot{X}) \qquad U = U(x)$$

we can write the action as

$$\begin{aligned} \delta S = \delta \int_{t_0}^{t_1} L \,\mathrm{d}t \ &= \int_{t_0}^{t_1} \delta L \,\mathrm{d}t \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \right) \delta x \,\mathrm{d}t \end{aligned} \tag{3.3}$$

The application of the Euler equation to the Lagrangian yields the *Euler-Lagrange Equation*:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \tag{3.4}$$

Notice that force does not play a role in the Euler-Lagrange equation; rather, we rely on the *principle of least action.* Hamilton's principle is a quantifiable to the philosophical idea of least action. Because force is not required to solve the equations of motion, we can solve a more diverse systems where discussing forces does not make sense; for example, we can use this approach in the Standard Model of particle physics (Lagrangian omitted for brevity). Similarly, general relativity can be calculated using a Lagrangian:

$$L_{GR} = R\sqrt{-g}$$

3.1 Equivalence with Newton's Laws

Consider the motion of a particle in 1D with a conservative potential $U(x)$. The kinetic energy can be states

$$T = \frac{1}{2}m\dot{x}^2 \tag{3.5}$$

Thus, the Lagrangian can be states

$$L = \frac{1}{2}m\dot{x}^2 - U \tag{3.6}$$

Plugging into the Euler Lagrange equation, we obtain:

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p$$

Thus,

$$F - \frac{\mathrm{d}}{\mathrm{d}t}p = -$$

or

$$F = \frac{\mathrm{d}p}{\mathrm{d}t}$$

While more difficult, it is possible to show that Hamilton's principles leads to Newton's laws in all cases, and even more so, that Newton's laws leads to the Lagrangian (after applying D'Alembert's principle).

3.2 Simple Harmonic Motion

The lagrangian for this system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Using the Euler Lagrange equation,

$$0 = -kx - m\ddot{x} \implies m\ddot{x} = -kx$$

3.3 Plane Pendulum

We do not apply the small angle approximation. We examine a pendulum with length ℓ and the angle θ from the stable equilibrium, with pivot at point (ℓ, ℓ)

The position of the pendulum can be seen to be

$$x = \ell(1 + \sin\theta) \tag{3.7a}$$
$$y = \ell(1 - \cos\theta) \tag{3.7b}$$

Thus, the kinetic term can be found

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ &= \frac{1}{2}m\ell^2\dot{\theta}^2 \end{aligned} \tag{3.8}$$

The potential term is

$$\begin{aligned} U &= mgy \\ &= mg\ell(1 - \cos\theta) \end{aligned} \tag{3.9}$$

The lagrangian is then

$$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1 - \cos\theta) \tag{3.10}$$

Plugging in the Euler-Lagrange equation,

$$0 = \frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}}$$

$$0 = -mg\ell\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}m\ell^2\dot{\theta}$$

$$\implies \ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

which is the expected result from newtonian mechanics.

Note that we were able to define the lagrangian in terms of the polar coordinates, and were able to use the euler-lagrange equation without having to implement any jacobians or scale factors.

3.4 Atwood's Machine

The atwood machine is two masses on either end of a rope hung over a pulley. We define the coordinates such that x is the distace from the pulley to mass1, and the total length to be ℓ, neglecting the size of the pulley.

The kinetic term is

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

and the potential term is

$$U = -m_1 g x - m_2 g(\ell - x)$$

Thus, the lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\ddot{x}^2 + m_1 g x + m_2 g(\ell - x)$$

Applying the Euler Lagrange equation,

$$0 = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}$$

$$0 = (m_1 - m_2)g - \frac{\mathrm{d}}{\mathrm{d}t}(m_1 + m_2)\dot{x}$$

$$\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g$$

The lagrangian allowed us to ignore the effects of tension against the force of gravity.

3.5 Double Pendulum

We can define a double pendulum by two angles ϕ_1, ϕ_2, lengths ℓ_1, ℓ_2, and masses m_1, m_2.

Note: we never finished this example.

3.6 Configuration Spaces

3.6.1 Degrees of Freedom

For a single particle in 3D space, then to uniquely describe the particle, we need a position vector, for a total of 3 degrees of freedom.

For a collection of N particles, with no constraints, the degrees of freedom naturally scale extensively, with $3N$ degrees of freedom. Constraints can reduce the degrees of freedom. In general each constraint reduces the degrees of freedom by one. For configuration (no velocity), the degrees of freedom S is:

$$S = ND - m \tag{3.11}$$

where N is the number of particles, D is the number of dimensions, m is the number of constaints.

Note we don't consider the velocity vector. Consider a 2D pendulum. While we can describe the system uniquely by θ and $\dot{\theta}$, the only mechanical degree of freedom is θ; the "velocity" $\dot{\theta}$ is how the degree of freedom is being used.

3.6.2 Generalized Coordinates

We can create a set of generalized coordinates q with one coordinate q_i for each degree of freedom. In general, we can index these as:

$$x_{\alpha,i} = x_{\alpha,i}(q_{\alpha,1}, q_{\alpha,2}, \ldots, q_{\alpha,s}, t) \tag{3.12}$$

for an object α. For example, the x_1 (x) position of a 2D pendulum can be written $x_1 = \ell \sin\theta$ and the x_2 (y) position would be $x_2 = \ell\cos\theta$

Typically, for shorthand, we write

$$x_\alpha = x_\alpha(q_\alpha, t) \tag{3.13}$$

Thus, the time derivative is

$$\dot{x}_\alpha = x_\alpha(q_\alpha, \dot{q}_\alpha, t) \tag{3.14}$$

Naturally, we can also write the generalized coordinates in terms of the cartesian, as:

$$q_\alpha = q_\alpha(x_\alpha, t) \tag{3.15a}$$
$$\dot{q}_\alpha = \dot{q}_\alpha(x_\alpha, \dot{x}_\alpha, t) \tag{3.15b}$$

Something to note is that the cartesian coordinates implicitly include equations of contraints within the function $x_\alpha(q_\alpha, t)$, as while there are ND $x_{\alpha,i}$, there are only $ND - m$ $q_{\alpha,j}$. We include these as *holonomic constraints*, or constraints that do not depend on velocity. These contraints can be written as

$$g(q_\alpha, t) = 0 \tag{3.16}$$

3.7 Mass on an inclined plane

A block slides down a wedge that is lying on the ground. There is no friction between the wedge and the block, nor between the wedge and the ground. There are only two degrees of freedom—the wedge has a constraint restricting its y position, and the block has a constraint to remain on the incline on the wedge. Thus, the degrees of freedom is $2 \cdot 2 - 2 = 2$. Label the objects 1 for the wedge and 2 for the block.

We denote the generalized coordinates d being how far the block has slid down the wedge, and x how far the wedge has slid along the ground.

From this, we see that

$$x_1 = x \quad \dot{x}_1 = \dot{x}$$
$$x_2 = \left\langle x, h\right\rangle + d\left\langle \cos\alpha, \sin\alpha\right\rangle \quad \dot{x}_2 = \left\langle \dot{x} + \dot{d}\cos\alpha, -\dot{\sin}\alpha\right\rangle$$

Thus, the kinetic energy is given

$$\begin{aligned} T &= T_1 + T_2 \\ &= \frac{1}{2}m_1\dot{x} + \frac{1}{2}m_2\left(\dot{x}^2 + \dot{d}^2 + 2\dot{x}\dot{d}\cos\alpha\right) \end{aligned}$$

The potential is similarly

$$\begin{aligned} U &= U_1 + U_2 \\ &= 0 + m_2 g(h - d\sin\alpha) \end{aligned}$$

Thus, the full Lagrangian is:

$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\left(\dot{x}^2 + \dot{d}^2 + 2\dot{x}\dot{d}\cos\alpha\right) - m_2 g(h - d\sin\alpha)$$

Applying the Euler Lagrange equation,

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \\ &= 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(m_1\dot{x} + m_2\dot{x} + m_2\dot{d}\cos\alpha\right) \\ k_1 &= m_1\dot{x} + m_2\left(\dot{x} + \dot{d}\cos\alpha\right) \end{aligned}$$

Note that this can also be written as $p_1 + p_2 = k_1$; this is because momentum is an integral of the motion.

$$\begin{aligned} 0 &= \frac{\partial L}{\partial d} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{d}} \\ &= m_2 g\sin\alpha - \frac{\mathrm{d}}{\mathrm{d}t}m_2\left(\dot{d} + \dot{x}\cos\alpha\right) \\ &= m_2 g\sin\alpha - m_2\left(\ddot{d} + \ddot{x}\cos\alpha\right) \end{aligned}$$

We can take the time derivative of the x equation of motion and plug into the d equation of motion to obtain:

$$\begin{aligned} &= m_2 g\sin\alpha + m_2\left(\ddot{d} - \frac{m_2}{m_1 + m_2}\ddot{d}\cos^2\alpha\right) \\ g\sin\alpha &= \ddot{d}\left(1 - \frac{m_2}{m_1 + m_2}\cos^2\alpha\right) \end{aligned}$$

Defining $\beta \equiv \frac{m_2}{m_1+m_2}$

$$\ddot{d} = \frac{g\sin\alpha}{1 - \beta\cos^2\alpha}$$

Similarly, we can obtain

$$\ddot{x} = \frac{-\beta g\sin\alpha\cos\alpha}{1 - \beta\cos^2\alpha}$$

Thus, the acceleration of both the block and the mass are constant.

Checking limiting cases, when $\alpha = 0$, we see the acceleration of the block is $\ddot{d} = 0$ and the acceleration of the wedge $\ddot{x} = 0$. Thus, we reobtain free 1D motion.

Additionally, when $\alpha = \pi/2$, we get the acceleration of the block to be $\ddot{d} = g$ and the acceleration of the wedge $\ddot{x} = 0$, which is identical to a free-falling mass.

Perturbing masses, in the regime $m_1 \gg m_2$, we have $\beta \to 0$. Thus, $\ddot{=} g \sin\alpha$ and $\ddot{x} = 0$ which reduces the problem to that of a block sliding down a fixed ramp.

Finally, if we take $m_2 \gg m_1$, we have $\beta \to 1$. Thus, $\ddot{d} = \frac{g}{\sin\alpha}$ and $\ddot{x} = -g\cot\alpha$. Interstingly, the acceleration of the block down the wedge is strictly greater than gravity alone! This is because the wedge is getting shot out from under the block at the same time.

3.8 Pendulum revisited

We reparametrize the pendulum in terms of the distance from the pendulum to the vertical, $d = \ell \sin\theta$. Then,

$$x = \ell + d \qquad y = \ell - \sqrt{\ell^2 + d^2}$$

Further,

$$\dot{x} = \dot{d} \qquad \dot{y} = \frac{d\dot{d}}{\sqrt{\ell^2 - d^2}}$$

The lagrangian is then

$$\mathcal{L} = \frac{1}{2}m\left(\dot{d}^2 + \frac{d^2\dot{d}^2}{\ell^2 - d^2}\right) + mg\ell - mg\sqrt{\ell^2 + d^2}$$

Plugging into mathematica, we can plot a numerical solution as a function of time, and it yields the same solution as the angular coordinate system. This demonstrates that the lagrangian formulation is coordinate independent, and the real art of lagrangian mechanics is choosing convenient coordinate systems.

3.9 Atwood Machine Revisited

We can work the Atwood machine considering the force on each of the two masses. We can consider the length of the rope a constraint equation, with

$$g(y_1, y_2) = y_1 + y_2 + \pi R - \ell = 0$$

The Lagrangian is simple for this case:

$$\mathcal{L} = \frac{1}{2}m_1\dot{y}_1^1 + \frac{1}{2}m_2\dot{y}_2^2 + m_1 g y_1 + m_2 g y_2$$

The Euler Lagrange equations can be written:

$$\frac{\partial L}{\partial y_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}_i} + \lambda\frac{\partial g}{\partial y_1} = 0$$

This then becomes:

$$m_i g - m_i \ddot{y}_i + \lambda = 0$$

setting the two λ's equal,

$$(m_1 - m_2)g - m_1\ddot{y}_1 + m_2\ddot{y}_2 = 0$$

From the constraint equation, we have

$$\ddot{g} = \ddot{y}_1 + \ddot{y}_2 = 0$$

Thus,

$$(m_1 - m_2)g - (m_1 + m_2)\ddot{y}_1 = 0$$

$$\ddot{y}_1 = \frac{m_1 - m_2}{m_1 + m_2} g$$

Similarly,

$$\ddot{y}_2 = \frac{m_2 - m_1}{m_1 + m_2} g$$

Plugging these back into the euler lagrange equation, we can find the lagrange multiplier

$$m_1 g - m_1 \frac{m_1 - m_2}{m_1 + m_2} g + \lambda \frac{\partial g}{\partial y_1} = 0$$

$$\lambda \frac{\partial g}{\partial y_1} = \frac{2m_1 m_2}{m_1 + m_2} g$$

note that this equal to the tension force from newtonian mechanics. The term $\lambda \frac{\partial g}{\partial y_1}$ is known as a *generalized force.*

3.10 Euler Lagrange with Constraints

We can write the action of a constrained lagrangian as

$$\delta S = \delta \int_{t_0}^{t_1} \mathcal{L} - \sum_i \lambda_i f_i \, \mathrm{d}t$$

where the constraint functions f_j are defined as

$$f_j = f_j(q, \dot{q}, t) = 0$$

We can add these to the action integral because they are equally zero.

Varying the constaints,

$$\delta f_i = \frac{\partial f_i}{\partial \alpha} = \frac{\partial f_j}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial f_j}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha} = 0$$

Summing the contraints and integrating,

$$\int_{t_0}^{t_1} \mathrm{d}t \sum_j \frac{\partial f_j}{\partial q}\frac{\partial q}{\partial \alpha} + \frac{\partial f}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial \alpha} = 0$$

Integrating the second term by parts,

$$\int_{t_0}^{t_1} \mathrm{d}t\, \frac{\partial f_j}{\partial \dot q} = \cancel{\left[\frac{\partial f}{\partial \dot q}\frac{\partial q}{\partial \alpha}\bigg|_{t_0}^{t_1} \right]} - \int_{t_0}^{t_1} \mathrm{d}t\, \frac{\partial q}{\partial \alpha}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_j}{\partial \dot q}$$

Recall that $\frac{\partial q}{\partial \alpha} \equiv \delta q$, which by definition is zero at the endpoints. Thus,

$$0 = \int_{t_0}^{t_1} \sum_j \left[\frac{\partial f_j}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_j}{\partial \dot q} \right] \delta q\, \mathrm{d}t \tag{3.17}$$

Defining

$$\chi_j \equiv \frac{\partial f_j}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_j}{\partial \dot q} \tag{3.18}$$

we can rewrite the integral as

$$\int_{t_0}^{t_1} \sum_j \chi_j \delta q\, \mathrm{d}t \tag{3.19}$$

We can then define a space with the inner product

$$\langle a, b\rangle \equiv \int_{t_0}^{t_1} ab\, \mathrm{d}t \tag{3.20}$$

Thus, we can use this inner product to show that the sum of the constraints is orthogonal to the variations:

$$\langle \sum_j \chi_j, \delta q\rangle = 0 \tag{3.21}$$

However, the above holds for all j, including $j = 1$, so

$$\langle \chi_j, \delta q\rangle = 0$$

We can similarly define

$$\Lambda \equiv \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q}$$

thus

$$\langle \Lambda, \delta q\rangle = 0$$

We can define a subspace of functions orthogonal to δq. We claim that the set χ_j spans the space[1], and thus

$$\Lambda = \sum_j \lambda_j \chi_j \tag{3.22}$$

or, more verbosely,

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q} = \sum_j \frac{\partial f_j}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_j}{\partial \dot q}$$

[1] no idea how the χ's span the subspace but

Making use of linearity, we can rewrite this as

$$\frac{\partial}{\partial q}\left(L+\sum_j \lambda_j f_j\right)-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\left(L+\sum_j \lambda j f_j\right)=0 \tag{3.23}$$

Thus, for holonomic constraints, the problem becomes the determination of

$$\mathcal{L}=L+\sum_j \lambda_j f_j \tag{3.24}$$

$$\frac{\partial \mathcal{L}}{\partial q}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}}=0 \tag{3.25}$$

Note that if the constraints are not holonomic (i.e., they have a dependence on $\dot{q}$), then the second term becomes difficult to determine. Plugging in holonomic constraints, we obtain:

$$0=\frac{\partial L}{\partial q}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}+\sum \lambda_j \frac{\partial f_j}{\partial q} \tag{3.26}$$

Non-holonomic constraints

There are some functions that are easier to treat in a constrained problem. Such a function is one of the form

$$\tilde{f}(x,\dot{x},t)=\sum_i \frac{\partial f}{\partial x_i}\dot{x}_i+\frac{\partial f}{\partial t}$$

for some holonomic constraint $f(x,t)=0$. It can be written as

$$\tilde{f}=\frac{\mathrm{d}f}{\mathrm{d}t}$$

This type of function is known as a *semi-holonomic constraint*

3.10.1 Constraints and D'alembert's Principle

In Newtonian mechanics, we consider the force to be broken up as:

$$\overrightarrow{F}_{ext}+\overrightarrow{N}=m\ddot{\overrightarrow{x}}$$

The normal force $\overrightarrow{N}$ is akin to a constraint force

Take a box going down a ramp defined by

$$f(x,y)=y+\frac{3}{4}x-3=0$$

Note that the gradient

$$\frac{\nabla f}{\|\nabla f\|} = \frac{3}{5}\hat{x} + \frac{4}{5}\hat{y}$$

points in the same direction as the normal force; more generally,

$$\vec{N} = \lambda(t)\nabla f$$

for a constraint $f(x,t) = 0$.

Introducing a vector $\tau \perp \nabla f$, we can multiply the newtonian law

$$(m\ddot{\vec{x}} - \vec{F} - \lambda\nabla f)\cdot\vec{\tau} = 0 \tag{3.27}$$

Thus,

$$(m\ddot{\vec{x}} - \vec{F})\cdot\vec{\tau} = 0$$

This can be expanded to multiple constraints as:

$$\vec{N_i} = \lambda(x,t)\nabla f_i$$

Thus, for multiple particles, we have:

$$\sum_i \vec{\tau_i}\cdot\nabla_i f_j = 0$$

and our analogue to Equation 3.27 becomes:

$$\sum_i\left(m_i\ddot{\vec{x}}_i - \vec{F_i} - \sum_j \lambda_j\nabla_i f_j\right)\cdot\vec{\tau_i} = 0 \tag{3.28}$$

Because we know the contrains are orthogonal to τ, we obtain D'alembert's Principle:

$$\sum_i\left(m\ddot{\vec{x}}_i - \vec{F_i}\right)\cdot\vec{\tau_i} = 0 \tag{3.29}$$

Because the dot product is independent of coordinate system, we can rewrite the the vector τ as:

$$\vec{\tau_i} = \sum_\alpha \epsilon^\alpha \frac{\partial\vec{x}_i}{\partial q^\alpha}$$

This definition is taken to satisfy the chain rule:

$$\sum_i \tau_i\nabla_i f_j = \epsilon^\alpha\sum_i\frac{\partial\vec{x}}{\partial q^\alpha}\nabla_i f_j = \epsilon^\alpha\sum_i\frac{\partial f_j}{\partial q^\alpha} = 0$$

Thus, we rewrite d'Alembert's principle for generalized coordinatesas

$$\sum_i\left(m_i\ddot{\vec{x}}_i - \vec{F_i}\right)\cdot\frac{\partial\vec{x}_i}{\partial q^\alpha} = 0 \tag{3.30}$$

From here, we can use the product rule to rewrite:

$$m_i\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\left[m_i\dot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha}\right] - m_i\dot{\vec{x}}_i \cdot \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \vec{x}_i}{\partial q^\alpha}$$

using

$$\dot{\vec{x}}_i = \sum_\alpha \frac{\partial \vec{x}_i}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial \vec{x}_i}{\partial t}$$

and

$$\frac{\partial \dot{\vec{x}}_i}{\partial \dot{q}^\alpha} = \frac{\partial \vec{x}_i}{\partial q^\alpha}$$

we can rewrite the second term as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \vec{x}_i}{\partial q^\alpha} &= \sum_\beta \frac{\partial^2 \vec{x}_i}{\partial q^\alpha \partial q^\beta}\dot{q}^\beta + \frac{\partial}{\partial t}\left(\frac{\partial \vec{x}_i}{\partial q^\alpha}\right) \\
&= \frac{\partial}{\partial q^\alpha}\left[\sum_\beta \frac{\partial \vec{x}_i}{\partial q^\beta}\dot{q}^\beta + \frac{\partial \vec{x}_i}{\partial t}\right] \\
&= \frac{\partial}{\partial q^\alpha}\frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t} \\
&= \frac{\partial \dot{\vec{x}}_i}{\partial q^\alpha}
\end{aligned}$$

Thus,

$$\sum_i m_i\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_i\dot{\vec{x}}_i \cdot \frac{\partial \dot{\vec{x}}}{\partial \dot{q}^\alpha}\right) - m_i\dot{\vec{x}}_i \cdot \frac{\partial \dot{\vec{x}}}{\partial q^\alpha}\right] \tag{3.31}$$

From kinetic energy,

$$T = \frac{1}{2}\sum_i m_i\dot{\vec{x}}_i^2$$

$$\frac{\partial T}{\partial \dot{q}^\alpha} = \sum_i m_i\dot{\vec{x}}_i \cdot \frac{\partial \dot{\vec{x}}}{\partial \dot{q}^\alpha}$$

$$\frac{\partial T}{\partial q^\alpha} = \sum_i m_i\dot{\vec{x}}_i \cdot \frac{\partial \dot{\vec{x}}}{\partial q^\alpha}$$

so

$$\sum_i m_i\ddot{\vec{x}}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}^\alpha}\right) - \frac{\partial T}{\partial q^\alpha} \tag{3.32}$$

plugging into the original equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} - \sum_i \vec{F}_i \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = 0$$

using $\vec{F}_i = \nabla U$, we see

$$-\sum_i \nabla_i U \cdot \frac{\partial \vec{x}_i}{\partial q^\alpha} = -\frac{\partial U}{\partial q_\alpha}$$

so,

$$\frac{\mathrm{d}}{\mathrm{d}T}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\partial T}{\partial q^\alpha} + \frac{\partial U}{\partial q^\alpha} = 0$$

$$\frac{\partial T}{\partial q^\alpha} - \frac{\partial U}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\alpha}} = 0$$

Considering $U(q^\alpha)$ independent of $\dot{q}^\alpha$,

$$\frac{\partial T}{\partial q^\alpha} - \frac{\partial U}{\partial q^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}^\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{\partial U}{\partial \dot{q}^\alpha} = 0\right]$$

$$\frac{\partial}{\partial q^\alpha}(T-U) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}^\alpha}(T-U) = 0$$

where $L = T - U$ is the familiar Lagrangian

Chapter 4

Hamiltonian Mechanics

We start from the second form of the euler equation, where

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - \dot{y}\frac{\partial f}{\partial \dot{y}}\right) = 0$$

Converting to the language of mechanics,

$$\frac{\partial L}{\partial t} - \frac{\mathrm{d}}{\mathrm{d}t}\left(L - \dot{q}\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{4.1}$$

Considering only closed systems, where $\frac{\partial L}{\partial t} = 0$, we define the hamiltonian to be

$$H \equiv \sum_i \dot{q}p_i - L \tag{4.2}$$

where the conjugate variable p_i is the generalized momentum:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{4.3}$$

Note that the Hamiltonian is the Legendre transform of the Lagrangian with respect to the variables $\dot{q}_i$.

$$H(q_i, p_i, t) = \sum_i p_i\dot{q}_i(q_i, p_i, t) - L(q_i, \dot{q}_i(q_i, p_i, t), t) \tag{4.4}$$

Using the fact that

$$\frac{\partial L}{\partial q_i} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = \frac{\mathrm{d}}{\mathrm{d}t}p_i$$

we then have

$$\frac{\partial L}{\partial q_i} = \dot{p}_i \tag{4.5}$$

4.1 Canonical Equations of Motion

We write the differential of the Hamiltonan as

$$\mathrm{d}H = \sum_j (P_j \,\mathrm{d}\dot{q}_j + \dot{q}_j \,\mathrm{d}p_j) - \sum_j \left(\frac{\partial L}{\partial q_j} \,\mathrm{d}q_j + \frac{\partial L}{\partial \dot{q}_j} \,\mathrm{d}\dot{q}_j \right) - \frac{\partial L}{\partial t} \,\mathrm{d}t$$

Substituting in the generalized momentum,

$$\mathrm{d}H = \sum_j (p_j \,\mathrm{d}\dot{q}_j + \dot{q}_j \,\mathrm{d}p_j) - \sum_j (\dot{p}_j \,\mathrm{d}q_j + p_j \,\mathrm{d}\dot{q}_j) - \frac{\partial L}{\partial t} \,\mathrm{d}t$$

$$\mathrm{d}H = \sum_j (\dot{q}_j \,\mathrm{d}p_j - \dot{p}_j \,\mathrm{d}q_j) - \frac{\partial L}{\partial t} \,\mathrm{d}t$$

Because $\mathrm{d}H$ is an exact differential, we naturally have that:

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \tag{4.6a}$$

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j \tag{4.6b}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{4.6c}$$

Equations 4.6a and 4.6b are the *canonical equations of motion*. Equation 4.6c is not considered part of the canonical equations of motion, but for a time independent potential, it is zero.

Note that Lagrangian mechanics provides $s = 3N - m$ second order differential equations, while Hamiltonian mechnics provides $2s$ first order differential equations.

4.2 Particle on a Cylinder with a Central Force

Consider a particle constrained on a cylinder with a force

$$\vec{F} = -k\vec{r}$$

pulling it toward the origin. The corresponding potential is

$$U = \frac{1}{2}kr^2 = \frac{1}{2}k\left(R^2 + z^2\right)$$

and kinetic term

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\cancel{\dot{R}^2} + R^2\dot{\phi}^2 + \dot{z}^2\right)$$

Finally, the constraint equation is given

$$f = x^2 + y^2 - R^2 = 0$$

but is implicitly included in choice of coordinates. Rewriting in terms of generalized momenta,

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}$$

Note that the momentum p_z corresponds to the linear momentum along the z direction, while p_ϕ corresponds to the angular momentum along the z direction.

While we can find the Hamiltonian as the legendre transfrom of the Lagrangian, it is simpler to note the Hamiltonian is the total energy

$$H = T + U \tag{4.7}$$

Thus, in terms of the generalized momenta,

$$H = \frac{1}{2m}\left(p_\phi^2 + p_z^2\right) + \frac{1}{2}k(z^2 + R^2)$$

The canonical equations are then

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$
$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz = m\ddot{z}$$
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2}$$
$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

From the first equation we have constant angular momentum. From the second equation, we then have $z = A\cos\omega t$, which is simple harmonic oscillation.

For the systems treated in classical mechanics, the Hamiltonian is not often easier/simpler to use than the Lagrangian. The Lagrangian works in configuration space, $(q, \dot{q}, t)$; q and $\dot{q}$ are not independent. However, the Hamiltonian works in phase space (q, p, t); the quantities q and p are independent, but constrained by the problem.

The phase space for this problem is defined as:

$$z = A\cos(\omega t)$$
$$p_z = m\dot{z} = -Am\omega\sin(\omega t)$$
$$\phi = \frac{p_\phi}{mR^2}T + A$$

Note that the trajectory on phase space is given by a helix, constrained to a cylindrical subspace defined by $E = H$. Interestingly, different trajectories in phase space can never cross.

Another useful feature of the hamiltonian is that if a generalized momentum is constant, it is as a *cyclic coordinate*:

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$p = mR^2\dot{\phi} = k$$

$$\dot{\phi} = \frac{p_\phi}{mR^2} = \frac{\partial H}{\partial p_\phi}$$

Thus,

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \omega_\phi$$

$$\implies \phi = \int \omega_\phi \, \mathrm{d}t$$

This integral is where the term "integral of the motion" comes from. Every cyclic coordinate removes 2 equations of motion from the Canonical equations. In Hamilton/Jacobi Theory, there are certain problems that can be transformed into only cyclic coordinates—a typical example is the the solar system.

<u>Legendre Transform</u>

Any convex/concave function may be reparametrized in terms of its derivative. The envelope of the set of all tangent lines can be used to define the line. Very simply, the legendre transform maps (x, y) onto $(m \equiv \frac{\mathrm{d}y}{\mathrm{d}x}, b)$ by

$$y = mx + b$$

$$-b = mx - y$$

The hamiltonian, for example, is

$$H = p\dot{q} - L$$

4.3 Hamiltonian

When the coordinates and potential are independent of time, the system is considered a closed system. This gives

$$\frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

so the hamiltonian is constant wrt time. We know that the definition of the hamiltonian is given

$$H = \sum_j p_j \dot{q}_j - L$$

$$
\begin{aligned}
&= \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q} j} - L \\
&= \sum_j \dot{q}_j \left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial U}{\partial \dot{q}} \right) - L
\end{aligned}
$$

Because we typically hold that U is independent of velocity, this simplifies to:

$$
= \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L
$$

The first term can be rewritten

$$
\sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} = 2T
$$

this is using the fact that in arbitrary coordinates that

$$
T = \frac{1}{2} m \sum_i \dot{x}_i^2 = \frac{1}{2} m \left(\sum_{ij} \frac{\partial x_i}{\partial q_j} \dot{q}_j \right)^2
$$

where once again, the coordinates are independent of time, and the cartesian coordinates are reparametrized in terms of generalized coordinates q_j. Expanding out,

$$
T = \sum_{ijk} \left(\frac{1}{2} m \frac{\partial x_i}{\partial q_j} \frac{\partial x_i}{\partial q_k} \right) \dot{q}_j \dot{q}_k = a_{jk}^{(i)} \dot{q}_j \dot{q}_k
$$

taking the derivative wrt $\dot{q}_\ell$,

$$
\frac{\partial T}{\partial \dot{q}_\ell} = \sum_{i,\ell} \dot{q}_\ell \left(\sum_k a_{\ell k}^{(i)} \dot{q}_k + \sum_j a_{j\ell}^{(i)} \dot{q}_j \right)
$$

Because multiplication is commutative, $a_{jk}^{(i)} = a_{kj}^{(i)}$:

$$
\begin{aligned}
&= \sum_{i,\ell} \dot{q}_\ell \left(\sum_k a_{k\ell}^{(i)} \dot{q}_k + \sum_j a_{j\ell}^{(i)} \dot{q}_j \right) \\
&= \sum_{i,\ell} \dot{q}_\ell \left(\sum_j a_{j\ell}^{(i)} \dot{q}_j + \sum_j a_{j\ell}^{(i)} \dot{q}_j \right) \\
&= 2 \sum_{ij\ell} a_{j\ell}^{(i)} \dot{q}_\ell \dot{q}_j \\
&= 2T
\end{aligned}
$$

Thus,

$$
H = \sum_j \dot{q}_j \frac{\partial T}{\partial \dot{q}_j} - L = 2T - (T - U)
$$

$$H = T + U \tag{4.8}$$

In summary, the constraints that the potential and coodinates are time independent state that the hamiltonian is constant. The constraints that the coordinates are time-independent and the potential is velocity-independent state that the hamiltonian is the total energy. Combining all three states that energy is constant.

The first two constraints imply that the system has the symmetry $t \to t+\Delta t$. This symmetry (homogeneity of time) corresponds to the conservation of energy. This is a specific case of Noether's theorem.

4.4 3D Pendulum

Take the coordinates to be spherical coordinates, wrt the $-z$ axis. The velocities are given

$$v_\theta = b\dot{\theta}$$

$$v_\phi = b\sin\theta\dot{\phi}$$

The kinetic energy is then

$$T = \frac{1}{2}m(b^2\dot{\theta}^2 + b^2\sin^2\theta\dot{\phi}^2)$$

and the potential

$$U = -mgb\cos\theta$$

from the lagrangian, the generalized momenta are given

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mb^2\dot{\theta} \qquad {}_\phi = \frac{\partial L}{\partial\dot{\phi}} = mb^2\sin^2\theta\dot{\phi}$$

Further, for this system we have that

$$H = E = T + U$$

$$H = \frac{1}{2}\left(\frac{p_\theta^2}{mb^2} + \frac{p_\phi^2}{mb^2\sin^2\theta}\right) - mgb\cos\theta$$

where the kinetic term was rewritten in terms of the generalized momenta. The canonical equations are then given:

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mb^2\sin^2\theta}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial\theta} = \frac{p_\theta^2\cos\theta}{mb^2\sin^2\theta} - mgb\sin\theta$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial\phi} = 0$$

from the lagrangian, we also see that

$$\cancel{\frac{\partial L}{\partial \phi}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = 0$$

in fact, it is the symmetry $\phi \to \phi + \delta\phi$ that leads to the conservation of angular momentum about z. The θ dependence is not conserved, as the gravity makes it such that there is no such symmetry.

4.5 Noether's Theorem

Noether's Theorem states that every symmetry of the action of a system has an associated conserved quantity. The main symmetries considered in classical mechanics are as follows:

- The *Homogeneity of Time* is the symmetry $t \to t + \delta t$, which leads to the Conservation of Energy.
- The *Homogeneity of Space* is the symmetry $x \to \times + \delta x$, and leads to the Conservation of Linear Momentum.
- The *Isotropy of Space* is the symmetry $\vec{x} \to \vec{x} \times \delta\vec{\theta}$ and leads to the Conservation of Angular Momentum.

Other fields introduce other symmetries; much of quantum mechanics obeys Charge ($q \to -q$), Parity ($r \to -r$), and Time ($t \to -t$).

4.6 Liouville's Theorem

Phase space has dimension $2N$ where N is the number of degrees of freedom of the system. For a kilogram of an ideal gas, this is on the order of 6×10^{23} generalized coordinates p_i, q_i. Similar to Thermal Physics, while we can't analyze every trajectory, we can examine the statistical behaviour of the phase space.

From the general theory of ODEs, two paths on phase space cannot intersect or join together. In fact, Liouville's Theorem states that they cannot even grow closer together or farther apart.

From E&M, we have the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{\jmath} \tag{4.9}$$

where ρ is the density, and $\vec{\jmath}$ is the flux defined by the motion of the density. We can then write the flux as $\vec{\jmath} = \rho\vec{v}$ so that

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{v}) \tag{4.10}$$

In fact, we can apply this equation to the flux of paths through phase space. The area element becomes

$$\mathrm{d}A = \mathrm{d}q\ \mathrm{d}p$$

and has a thickness of $\mathrm{d}t$ in the third dimension.Thus, the density becomes

$$\mathrm{d}N = \rho\,\mathrm{d}A$$

WLOG, we can assume the paths go in from the negative values of the coordinates and go toward positive values. From the below q, the volume flowing in becomes

$$\rho\frac{\mathrm{d}q}{\mathrm{d}t}\,\mathrm{d}t\ \mathrm{d}p = \rho\dot{q}\,\mathrm{d}t\ \mathrm{d}p$$

and the volume flowing in from below p becomes

$$\rho\dot{p}\,\mathrm{d}t\ \mathrm{d}q$$

so, the increase in the density is

$$\left.\frac{\partial\rho}{\partial t}\right|_{\text{in}} = \rho(\dot{q}\,\mathrm{d}p + \dot{p}\,\mathrm{d}q\,)$$

Taylor expanding the flux into the volume, we can calculate the flux going out as

$$\rho\dot{q}\,\mathrm{d}p \approx \rho\dot{q}\,\mathrm{d}p + \frac{\partial}{\partial q}(\rho\dot{q})\,\mathrm{d}q\ \mathrm{d}p$$

Thus, the decrease in density is

$$\left.\frac{\partial\rho}{\partial t}\right|_{\text{out}} = -\left(p\dot{q} + \frac{\partial}{\partial q}(\rho\dot{q})\,\mathrm{d}q\right)\mathrm{d}p - \left(\rho\dot{p} + \frac{\partial}{\partial p}(\rho\dot{p})\,\mathrm{d}p\right)\mathrm{d}q$$

Thus, time derivative of the density becomes

$$\begin{aligned}\frac{\partial\rho}{\partial t}\,\mathrm{d}q\ \mathrm{d}p &= \left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\text{in}} - \left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\text{out}} = -\left[\frac{\partial}{\partial q}(\rho\dot{q}) + \frac{\partial}{\partial p}(\rho\dot{p})\right]\mathrm{d}p\ \mathrm{d}q\\ \frac{\partial\rho}{\partial t} &= -\left[\frac{\partial}{\partial q}(\rho\dot{q}) - \frac{\partial}{\partial p}(\rho\dot{p})\right]\end{aligned}$$

generalizing to many particles,

$$\begin{aligned}&= -\sum_j \frac{\partial}{\partial q_j}(\rho\dot{q}_j) + \frac{\partial}{\partial p_j}(\rho\dot{p}_j)\\ &= -\nabla\cdot(\rho\dot{x})\end{aligned}$$

which is analogous to the continuity equation for electric charge.

Thus, if we select a volume in phase space $\rho(t = 0)$, we can follow its trajectory to a time $\rho(t)$. Expanding out our newly-derived continuity equation for phase space,

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\sum_j \frac{\partial \rho}{\partial q_j}\dot{q}_j + \rho\frac{\partial \dot{q}_j}{\partial q_j} + \frac{\partial \rho}{\partial p_j}\dot{p}_j + \rho\frac{\partial \dot{p}_j}{\partial p_j} \\
&= -\sum_j \frac{\partial \rho}{\partial q_j}\dot{q}_j + \frac{\partial \rho}{\partial p_j}\dot{p}_j + \rho\frac{\partial}{\partial q_j}\frac{\partial H}{\partial p_j} - \rho\frac{\partial}{\partial p_j}\frac{\partial H}{\partial q_j} \\
&= -\sum_j \frac{\partial \rho}{\partial q_j}\frac{\mathrm{d}q_j}{\mathrm{d}t} + \frac{\partial \rho}{\partial p_j}\frac{\mathrm{d}p_j}{\mathrm{d}t} \\
&= -\frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\partial \rho}{\partial t} \\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= 0 \qquad (4.11)
\end{aligned}$$

Equation 4.11 is known as Liouville's theorem, and shows how a collection of states evolves similarly to an incompressible fluid.

Chapter 5

Central Force Motion

5.0.1 Newtonian

For a two-body problem, a central force is one that acts only along the direction of the radial vector between the two bodies:

$$F = F_{12}\frac{r_1 - r_2}{\|r_1 - r_2\|^2}$$

Gravity, for example, gives this force as

$$F = -\frac{Gm_1m_2}{\mathscr{r}^2}\hat{\mathscr{r}}$$

From newton's third law, we additionally have

$$m_1\ddot{r}_1 + m_2\ddot{r}_2 = 0$$

integrating,

$$m_1r_1 + m_2r_2 = At + B$$
$$\frac{m_1r_1 + m_2r_2}{m_1 + m_2} = \frac{At + B}{m_1 + m_2}$$

We define the LHS vector as the centre of mass, $\overrightharpoon{R}$, while the RHS provides a velocity $\vec{v}_{cm}$ We typically transform to the frame $\overrightharpoon{R} = 0$ when using the Newtonian approach.

5.0.2 Lagrangian

The Lagrangian of this system can be written

$$L = \frac{1}{2}(m_1 + m_2)\dot{\overrightharpoon{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r) \tag{5.1}$$

where

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

is the reduced mass and little $\vec{r}$ is the distance between the two particles. As there is no dependence on R, it is a cyclic parameter

$$p_R = \frac{\partial L}{\partial \dot{\vec{R}}}$$

Thus, we transform to the frame where $\dot{\vec{R}} = 0$ yielding

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

It can easily be seen that the motion of the 2-body system is limited to a plane:

$$\vec{r} \times \left(m\ddot{\vec{r}} - F_r\hat{r}\right) = \vec{r} \times 0 = 0$$

$$\implies \vec{r} \times \ddot{\vec{r}} = 0$$

Using product rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{r} \times \dot{\vec{r}}\right) = \cancel{\dot{\vec{r}} \times \dot{\vec{r}}} + \vec{r} \times \ddot{\vec{r}} = 0$$

and so, the motion of the particle is restricted such that

$$\vec{r} \times \dot{\vec{r}} = \vec{h}$$

or the motion is in a plane normal to the vector $\vec{h}$. Thus, in polar coordinates

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \tag{5.2}$$

This doesn't depend on θ, so it is a cyclic coodinate. Thus, angular momentum

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = \ell$$

is a conserved quantity.

Areal Velocity

If we imagine the particle as orbiting an ellipse with a focus at the origin, the area swept by the orbit over a given time is constant. For a small time δt, the area can be approximated

$$A = \frac{1}{2}ab\sin\theta$$

$$\delta A = \frac{1}{2}r(r+\delta r)\sin\delta\theta$$

$$\mathrm{d}A = \frac{1}{2}r^2\,\mathrm{d}\theta$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2}r^2\frac{\mathrm{d}\theta}{\mathrm{d}t} = \ell$$

and so, the amount of area sewpt out in a time t is independent on the location on the orbit. This is known as Kepler's Second Law: an orbit will sweep an equal area for an equal time. Interestingly, this applies for *any* central force—it results from the conservation of angular momentum.

5.0.3 Hamiltonian

Because the potential and coordinates are independent of time and the potential also of velocity, we can write the hamiltonian as:

$$E = H = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)$$

or

$$H = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \tag{5.3}$$

To characterize the motion, we want to know, obviously, the quantities $r(t), \theta(t)$, but also it is useful to know $r(\theta)$ and $\theta(r)$. From the total energy,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} - U(r)\right)}$$

$$\mathrm{d}t = \frac{\mathrm{d}r}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} - U(r)\right)}}$$

While difficult it may be difficult to compute, we will obtain an equation $t(r)$ that we can invert to find $r(t)$

From the conservation of angular momentum, we aditionally have

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\theta(t) = \frac{\ell}{\mu}\int_{t_0}^{t}\frac{\mathrm{d}t}{r^2}$$

Finally, we can use chain rule to find

$$\frac{\mathrm{d}\theta}{\mathrm{d}r} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\dot{\theta}}{\dot{r}} = \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} - U(r)\right)}}$$

integrating, we will find $\theta(r)$. However, we are usually more interested in the orbit equation, $r(\theta)$.

5.0.4 Orbit Equation

Returning to the Lagrangian

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U$$

we can apply Euler Lagrange as

$$\begin{aligned}
0 &= \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial R}{\partial \dot{r}} \\
&= \mu r\dot{\theta}^2 - \frac{\partial U}{\partial r} + \frac{\mathrm{d}}{\mathrm{dt}}\mu\dot{r} \\
F(r) &= \mu\ddot{r} - \frac{\ell^2}{\mu r^3}
\end{aligned}$$

making the substitution $r = u^{-1}$ such that $r = r(u(\theta(t)))$,

$$\frac{\mathrm{d}r}{\mathrm{dt}} = \frac{\mathrm{d}r}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{dt}} = \frac{\mathrm{d}}{\mathrm{d}u}\left[\frac{1}{u}\right]\frac{\mathrm{d}u}{\mathrm{d}\theta}\dot{\theta} = -\frac{1}{u^2}\cdot\frac{\ell}{\mu r^2}\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{\ell}{\mu}\frac{\mathrm{d}u}{\mathrm{d}\theta}$$

Then, the second derivative is

$$\frac{\mathrm{d}r}{\mathrm{dt}} = -\frac{\ell}{\mu}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}\frac{\mathrm{d}\theta}{\mathrm{dt}} = -\frac{\ell^2u^2}{\mu^2}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2}$$

The equation of motion then becomes

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{\mu}{\ell^2}\cdot\frac{1}{u^2}F(\tfrac{1}{u})$$

This equation can be used to verify if an orbit equation is caused by a central force, and if so, by what. More interestingly, For certain forces, this equation is integrable, and an exact solution may be obtained.

<u>Force from an orbit</u>

Take an orbit with $r = k\theta^2$. We obviously have $u = \frac{1}{k\theta^2}$. Thus,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = \frac{6}{k\theta^4} = 6ku^2$$

The orbit equation becomes

$$6ku^2 + u = -\frac{\mu}{\ell^2}\cdot\frac{1}{u^2}F(u^{-1})$$

Solving,

$$\begin{aligned}
F(u^{-1}) &= -\frac{\ell^2}{\mu}(6ku^4 + u^3) \\
F(r) &= -\frac{\ell^2}{\mu}\left(\frac{6k}{r^4} + \frac{1}{r^3}\right)
\end{aligned}$$

The equation of motion becomes

$$\dot{\theta} = \frac{\ell}{ur^2} = \frac{\ell}{uk^2\theta^4} \implies \boxed{\theta = \left(\frac{5\ell}{\mu k^2}t + C\right)^{1/5}} \implies \boxed{r = k\theta^2 = k\left(\frac{5\ell}{\mu k^2}t + C\right)^{2/5}}$$

If the force, for example, follows an inverse square law

$$F = -\frac{k}{r^2}$$

the orbit equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\mu k}{\ell^2}$$

This equation has solutions of the form

$$u = \frac{\mu k}{\ell^2}\left[1 + e\cos(\theta - \theta_0)\right]$$

or

$$r = \frac{\ell^2/\mu k}{1 + e\cos(\theta - \theta_0)} \tag{5.4}$$

for some coefficent e. This corresponds to an elliptical orbit whose properties can be easily seen from the equation.

Effective Potential Energy

The total energy can be written

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

If we combine the centrifugal term $\ell^2/2\mu r^2$ with the potetial $U(r)$, we obtain the "effective potential"

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r)$$

where

$$V = \frac{\ell}{2\mu r^2} + U \tag{5.5}$$

Inserting the gravitational potential,

$$V = \frac{\ell^2}{2\mu}\cdot\frac{1}{r^2} - k\cdot\frac{1}{r}$$

When $E < 0$, the radial distance will oscillate in the well defined by V. When the particle is at the minimum of the effective potential, the particle will have a perfectly circular orbit.

The turning point for $E_{\min} < E < 0$ determines the minimum and maxima radia for Equation 5.4. Fixing $r_{\min} = r(\theta_0)$, and solving the energy,

$$r_{\min} = \frac{1}{A + \frac{\mu k}{\ell^2}}$$

$$E = \frac{\ell^2}{2\mu r_{\min}^2} - \frac{k}{r_{\min}}$$

$$r(\theta) = \frac{\alpha}{1 + e\cos\theta} \tag{5.6}$$

where

$$\alpha = \frac{\ell^2}{\mu k} \qquad e = \sqrt{1 + \frac{2E\alpha}{k}}$$

For the the eccentricity, e to be real, the minimum energy is

$$E_{\text{min}} = -\frac{k}{2\alpha} = -\frac{\mu k^2}{2\ell^2}$$

The eccentricity is a property of an ellipse that denotes how much it "deviates" from being circular—$e = 0$ is a circle while $0 < e < 1$ causes the ellipse to be more elongated. It is important to note that the particle will alway orbit around a focus of the ellipse. Once the energy becomes 0, the eccentricity becomes $e = 1$ and the orbit is a parabola. When the energy grows greater and the eccentricity exceeds $e > 1$, the orbit takes the shape of a hyperbola. Collectively, these shapes are conic sections, and the higher energies can be interpreted as taking a steeper cut on a cone of position.

The quantity α has a different interpretation for each type of orbit; for a circle, α corresponds to the radius, for an ellipse the semimajor axis, and for a parabola twice the closest approach.

Returning to elliptical orbits, a typical ellipse can be written

$$\frac{x^2}{a^2} + \frac{y^2}{b^2}$$

In a geometry sense, the ellipse is the locus of all points such the perimeter of the triangle defined by two given points (foci) and the third point is a constant. The semiminor axis is half the shortest axis of the ellipse, while the semimajor axis is the half the longest axis of the ellipse. For an elliptical orbit, the semimajor axis will always be represented by a, and the semimajor axis by a. The focus is a point defined such that

$$f^2 = a^2 - b^2$$

Transforming the ellipse to be centered on one focus $+f$, the ellipse can be rewritten

$$1 = \frac{(x+f)^2}{a^2} + \frac{y^2}{b^2}$$

This transformation shows that the orbit is defined by one particle at the focus of the elliptical orbit of the other; this is Kepler's first law. Inserting polar coordinates and solving,

$$r(\theta) = \frac{b^2/a}{1 + \frac{f}{a}\cos\theta} \tag{5.7}$$

Thus,

$$\alpha = \frac{b^2}{a} \qquad \frac{\sqrt{a^2 - b^2}}{a}$$

solving,

$$a = \frac{\alpha}{1-e^2} = \frac{k}{2\,|E|}$$
$$b = \frac{\alpha}{\sqrt{1-e^2}} = \frac{\ell}{\sqrt{2\mu\,|E|}}$$

Thus, the semimajor axis depends solely on the conserved quantity of total energy, while the semiminor (and thus eccentricity) also depends on the conserved quantity angular momentum

One final way to rewrite the orbit equation, in terms of the semi-major axis a and the eccentricity e,

$$r(\theta) = \frac{a(1-e^2)}{1+e\cos\theta} \tag{5.9}$$

5.1 Kepler's Third Law

Using the fact that the area of an ellipse is given $A = \pi ab$, we can obtain the area of encircled by the orbit is given

$$A = \frac{\pi k \ell}{\sqrt{8\mu\,|E|^3}}$$

Further, knowing that $\mathrm{d}A\,t = \frac{\ell}{2\mu}$, we can integrate to obtain

$$A = \int_0^A \mathrm{d}A\ = \int_0^\tau \frac{\ell}{2\mu}\,\mathrm{d}t\ = \frac{\ell}{2\mu}\tau$$

for the orbital period τ. Setting the two equations equal, we obtain

$$\frac{\pi^2 k^2}{2\,|E|^3} = \frac{\tau^2}{\mu}$$

Substituting the expression for the semi-major axis, we obtain Kepler's third law,

$$\tau^2 = \frac{4\pi^2\mu}{k}a^3 = \frac{4\pi^2}{G(m_1+m_2)}a^3 \tag{5.10}$$

5.2 Kepler's Laws and Useful Relations

We have show, from first principles, each of Kepler's laws:

I. Planets move in elliptical orbits around the Sun with the Sun at one focus.

II. The area per unit time swept out by a radius vector from the Sun to a planet is constant.

III. The square of the radius is proportional to the cube of the semi-major axis.

5.2.1 Useful Points

The point of closest approach to the ellipse is known as the pericenter of the ellipse. From the orbit equaiton, we can clearly see that this occurs at

$$r(\theta = 0) = \frac{a(1-e^2)}{1+e} = a(1-e)$$

Similarly, the farthest point, the apocenter, is given

$$r(\theta = \pi) = a(1+e)$$

5.2.2 Angular Momentum

Recall that

$$e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}} \qquad a = -\frac{k}{2E}$$

Inverting these relations, we obtain

$$E = -\frac{k}{2a} \qquad \ell = \sqrt{\mu k a(1-e^2)}$$

From this, we see that as $e \to 1$, $\ell \to 0$, so eccentric orbits have less angular momentum than a circular orbit, explaining why elliptical orbits are more stable than circular orbits.

From kepler's second law, and also from the conservation of energy, as the pericenter has the lowest potential energy and thus must also have the highest kinetic energy:

$$E = \frac{1}{2}\mu v^2 - \frac{k}{r} \implies v = \sqrt{\frac{2}{\mu}\left(E - \frac{k}{r}\right)} = \sqrt{\frac{k}{\mu}\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Plugging in the positions of the peri-/apocenter,

$$v_{peri} = \sqrt{\frac{k}{\mu a}\left(\frac{1+e}{1-e}\right)}$$

$$v_{apo} = \sqrt{\frac{k}{\mu a}\left(\frac{1-e}{1+e}\right)}$$

Transfer Orbit

Suppose we consider the system of the Earth and Mars orbiting the Sun, which has solar mass $M_\odot$. While it might seem more reasonable to wait for Mars and Earth to be closest, it is actually easier if Mars and Earth are on opposite sides of the Sun, so that the probe's perihelion is at the Earth, and the apohelion is at Mars. Thus, one velocity boost is required to go from the "circular" Earth orbit to the elliptical orbit, then another velocity boost to go from the elliptical orbit the "circular" Martian orbit. This transfer orbit is one of the most energy efficient maneouvres between the two bodies.

The velocity of the earth orbit is given

$$v_e = \sqrt{\frac{k}{\mu}\left(\frac{2}{r_e} - \frac{1}{a_e}\right)} \approx \sqrt{\frac{GM_\odot}{r_e}}$$

For the transfer orbit, the semimajor axis can be found

$$a = \frac{r_m + r_e}{2}$$

and so the velocity at the perihelion of the transfer orbit can be found

$$v_p = \sqrt{GM_\odot\left(\frac{2}{r_e} - \frac{2}{r_m + r_e}\right)} = \sqrt{\frac{2GM_\odot}{r_e}\left(\frac{r_m}{r_e + r_m}\right)}$$

This gives a Δ-v of

$$v_{boost} = v_p - v_e$$

Then, at the apoheliun, the velocity is initially

$$v_a = \sqrt{\frac{2GM_\odot}{r_m}\left(\frac{r_e}{r_2 + r_m}\right)}$$

with

$$v_m = \sqrt{\frac{GM_\odot}{r_m}}$$

giving a Δv of

$$v_{circ} = v_m - v_a$$

The time taken to transfer from earth orbit to mars orbit is half of the period of the transfer orbit, yielding

$$t = \frac{\tau}{2} = \frac{1}{2}\sqrt{\frac{4\pi^2 a^2}{GM_\odot}} = \pi\sqrt{\frac{(r_m + r_m)^3}{2^3 GM_\odot}} \approx 0.7\,\text{years}$$

A similar estimation puts the transfer orbit to go to pluto yields 47 years.

The transit time for Curiosity to Mars was about the same that was computed, while New Horizons took only about 10 years to make the transit. This is because New Horizons took a boost orbit around Jupiter

5.2.3 Boost Orbit

A boost orbit is akin to a hyperbolic orbit, albeit with the centre planet also moving. In the rest frame of, say Jupiter (in the case of New Horizons), the probe has incoming and outgoing velocity v_0; however, transfering to the rest frame of the solar system, Jupiter would have velocity v_p, so the incoming velocity of the probe would be $v_i = v_0 - v_p$ and $v_f = v_0 + v_p$, so the probe gains velocity $2v_p$. This was assuming the incoming and outgoing velocities were parallel; if they are not, considerations must be taken to correct for the angle

5.3 Stability of Circular Orbits

Recall that the radius of a circular orbit (for the Kepler Problem) is the minimum defined by the effective potential

$$V = U + \frac{\ell^2}{2\mu r^2}$$

Given a force $F = -k/r^n$, the effective potential is given

$$V = -\frac{k}{n-1} \cdot \frac{1}{r^{n-1}} + \frac{\ell^2}{2\mu} \cdot \frac{1}{r^2}$$

The minimum radius can then be found

$$0 = \frac{\partial V}{\partial r} = \frac{k}{r_c^n} - \frac{\ell^2}{\mu} \cdot \frac{1}{r_c^3} \implies r_c^{n-3} = \frac{\mu k}{\ell^2}$$

To evaluate the stability, we examine the second derivative of the effective potential:

$$\left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_c} = -nk\frac{1}{r_c^{n+1}} + \frac{3\ell^2}{\mu} \cdot r_c^4 = \frac{1}{r_c^4}\left(-nk\frac{1}{r_c^{n-3}} + \frac{3\ell^2}{\mu}\right) = \frac{\ell^2}{\mu r_c^4} \cdot (3-n)$$

Thus, for $n < 3$, the circular orbits are stable.

5.3.1 Closed Orbits

Take a perturbed orbit

$$r = r_c + \eta$$

Taylor expanding the effective potential, we obtain

$$V(r) \approx V(r_c) + \eta \cancelto{0}{\left.\frac{\partial V}{\partial r}\right|_{r=r_c}} + \frac{1}{2}\eta^2 \left.\frac{\partial^2 V}{\partial r^2}\right|_{r=r_c} = V(r_c) + \frac{1}{2}k\eta^2$$

where

$$k \equiv \frac{\ell^2}{\mu r_c^4}(3-m)$$

This gives an oscillation frequency

$$\omega_r = \frac{\ell}{\mu r_c^2}\sqrt{3-n}$$

This frequency is the radial oscillation of the orbit. Recall we defined $\ell = \mu r_c^2 \dot{\theta}$, allowing us to rewrite the radial frequency as

$$\omega_r = \sqrt{3-n}\,\omega_\theta$$

From this, we can interpret an elliptical orbit for an inverse square law as having $\omega_\theta = \omega_r$. so *any* perturbation off of a circular orbit makes an elliptical orbit.

If we instead have a hookean force $F = -k(r - r_c)$ we have

$$\omega_r = 2\omega_\theta$$

which gives a peanut-shaped orbit.

These two orbits are *closed orbits*, In general, because $\sqrt{3-n}$ is usually not an integer, the orbit precesses an amount $\Delta\theta$ each orbit:

$$\Delta\theta = 2\int_{r_{\min}}^{r_{\max}} \frac{\ell/r^2}{\sqrt{2\mu\left(E - U - \frac{\ell^2}{2\mu r}\right)}}\,\mathrm{d}r \tag{5.12}$$

An orbit is considered closed if there exists a rational number $\frac{a}{b}$ such that

$$\Delta\theta = \frac{a}{b}2\pi$$

For examle, the Kepler problem has

$$\Delta\theta = \frac{1}{1}\cdot 2\pi$$

while a pentagram-like orbit would have

$$\Delta\theta = \frac{2}{5}\cdot 2\pi$$

For a general power-law force $F = -kr^{-n}$, closed orbits are given by

$$\frac{a}{b} = \sqrt{3-n} \tag{5.13}$$

This gives rise to *Bertrand's Theorem*, which states that the only two central forces where every bound orbit is a closed orbit are $n = 2$ and $n = -1$.

5.4 Virial Theorem

Let there be bodies at r_i with momentum p_i. We can define the quantity

$$S = \int_i r_i \cdot p_i \tag{5.14}$$

Assuming S is bounded, we can take the derivative wrt time

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \sum_i \dot{r}_i \cdot p_i + r_i \cdot \dot{p}_i \\ &= \sum_i \frac{p_i^2}{m} + r \cdot F_i\end{aligned}$$

$$= \sum_i 2T_i + F_i \cdot r_i$$

Taking the time average, $\langle x \rangle = \frac{1}{\tau} \int_0^\tau x \, \mathrm{d}t$

$$\left\langle \frac{\mathrm{d}S}{\mathrm{d}t} \right\rangle = \left\langle \sum_i 2T_i \right\rangle + \left\langle \sum_i F_i \cdot r_i \right\rangle$$

Applying the fundamental theorem of calculus,

$$\left\langle \frac{\mathrm{d}S}{\mathrm{d}t} \right\rangle = \frac{1}{\tau} \left[S(\tau) - S(0) \right]$$

Because we take the assumption that S is bounded, as $\tau \to \infty$, $\langle \dot{S} \rangle \to 0$. Thus,

$$\langle T \rangle = -\frac{1}{2} \left\langle \sum_i F_i \cdot r_i \right\rangle \tag{5.15}$$

where the RHS is known as the *virial* of the system.

Ideal Gas

Let there be an ideal gas inside a box. The force on a small element of the box is given

$$\mathrm{d}F_{wall} = -\,\mathrm{d}F_{atom} = P \, \mathrm{d}A$$

The virial is then

$$-\frac{1}{2} \left\langle \sum_i F_i \cdot r_i \right\rangle = -\frac{1}{2} \left\langle \oint \mathrm{d}F_{atom} \cdot r \right\rangle = \frac{1}{2} \left\langle \oiint_{\partial V} P \, \mathrm{d}a \cdot r \right\rangle = \frac{P}{2} \oiint_{\partial V} r \cdot \mathrm{d}a$$

Using virial theorem,

$$\langle T \rangle = \frac{P}{2} \oiint_{\partial V} r \cdot \mathrm{d}a = \frac{P}{V} \iiint_V \nabla \cdot r \, \mathrm{d}V = \frac{1}{2} P \iiint 3V = \frac{3}{2} PV$$

We also know, by Equipartition Theorem in thermodynamics, that

$$\langle T \rangle = \frac{3}{2} N k_B T_{emp}$$

Combininge these two equations,

$$PV = N k_B T_{emp}$$

yielding the Ideal Gas Law.

5.4.1 Virial theorem for specific forces

Assume the force take the form of a central power law

$$F_i = -\frac{k}{r_i^n}\hat{r}_i$$

Thus, the potential is given

$$F_i - \nabla_i U_i \implies U_i = -\frac{k}{n-1} \cdot \frac{1}{r_i^{n-1}}$$

Calculating the virial

$$\sum_i F_i \cdot r_i = \sum_i -\frac{k}{r_i^{n-1}} = \sum_i (n-1)U_i = (n-1)U$$

Thus,

$$\langle T \rangle = -\frac{n-1}{2} \langle U \rangle \tag{5.16}$$

Equation 5.16 is known as the virial theorem for the $F = -k/r^n$ force. For example, the virial theorem fot a simple harmonic oscilliator, with $n = -1$, we have

$$\langle T \rangle = \langle U \rangle$$

For an inverse-square law,

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle$$

Thus, for gravity,

$$E = \langle T \rangle + \langle U \rangle = - \langle T \rangle$$

5.4.2 Orbits

Assume the earth orbits with $e = 0, a = r_e, m_e \gg M_\odot$. Thus,

$$T = \frac{1}{2}m_e v^2 = \frac{1}{2}m_2 \left[GM_\odot \left(\frac{2}{r} - \frac{1}{a}\right)\right] = \frac{Gm_eM_\odot}{2r}$$

The potential is given

$$U = -\frac{Gm_eM_\odot}{r_e} = -2T$$

so the virial equation is satisified by circular orbits. If instead we choose a satelite with perihelion $r_p = a(1-e), a = 2r_e, e = \frac{1}{2}$, the kinetic energy at the perihelion is instead given

$$T = \frac{Gm_eM\odot}{2}\left(\frac{2}{r_e} - \frac{1}{2r_e}\right) = \frac{3Gm_eM_\odot}{4r_e}$$

with

$$T = -\frac{3}{4}U$$

which does not satisfy the virial theorem; however that is because these are instantaneous values of T, U rather than time average. To calculate the time average, we can take the orbital period as the value for τ to get the average.

$$\langle U \rangle = \frac{1}{\tau}\int_0^\tau \frac{-Gm_e M_\odot}{r(t)}\,\mathrm{d}t$$

making a change of variable,

$$\begin{aligned}
&= -\frac{Gm_e M_\odot \mu}{\gamma \ell}\int_0^{2\pi} r\,\mathrm{d}\theta \\
&= -Gm_e M_\odot a\left(1 - e^2\right)^2
\end{aligned}$$

Similarly, the average kinetic energy can be found

$$\langle T \rangle =$$

The virial theorem was used to show the existence of dark matter; the Coma Cluster had a higher effective mass than would be predicted by intensity alone.

Chapter 6

Multi-particle Systems

In the appendix, we discuss three-particle systems. This discussion is extended to a system of n particles. There are two cases we can consider: the discrete case and the continuous case. As it is impossible to describe the exact motion, we instead focus on statistical properties of these systems.

6.1 Properties

6.1.1 Linear Momentum

The center of mass of a system is the mass-weighted average of the positions in the system. For a discrete system, this is given by

$$\vec{R}_{cm} = \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} \tag{6.1}$$

while for the continuous case, it is given

$$\vec{R}_{cm} = \frac{\int \vec{r}\,\mathrm{d}m}{\int \mathrm{d}m} = \frac{1}{M}\int r\rho\,\mathrm{d}\tau \tag{6.2}$$

For a 3-body system, we saw that the internal interactions cancelled out and did not affect the centre of mass. However, forces due to an external force can have such an effect. This is true of the N-body system as well. The internal force on particle α can be written as

$$\vec{f}_\alpha = \sum_{\beta \neq \alpha} f_{\alpha\beta}$$

The total force on a particle is then

$$\vec{F}_\alpha = \vec{f}_\alpha + \vec{f}^{ext}_\alpha = \dot{\vec{p}}_\alpha$$

The total force on every particle is then given

$$\vec{F}^{tot} = \sum_\alpha \vec{F}_\alpha = \sum_{\alpha,\beta\neq\alpha} \vec{f}_{\alpha\beta} + \sum_\alpha \vec{f}^{ext}_\alpha$$

By newton's third law, we have $f_{\alpha\beta} = -f_{\beta\alpha}$, so the total force reduces to

$$\vec{F}^{tot} = \sum_\alpha \vec{f}^{ext}_\alpha \tag{6.3}$$

or, the total force on a system is that applied to it by external forces. Further, we have

$$\vec{F}^{tot} = \dot{\vec{p}}^{tot} = \sum_\alpha \dot{\vec{p}}_\alpha = \sum_\alpha m_\alpha \ddot{\vec{r}}_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_\alpha m_\alpha \vec{r}_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}t^2} M\vec{R}_{cm}$$

or, the centre of mass behaves like a single particle with mass $M = \sum_\alpha m_\alpha$ acted upon by the sum of all external forces.

$$\vec{F}^{tot} = M\ddot{\vec{R}}_{cm}$$

The total linear momentum of the system further is the same as that of a single particle of mass M located at the centre of mass.

$$\vec{p}^{tot} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha m_\alpha \vec{r}_\alpha = M\dot{\vec{R}}_{cm}$$

Thus, in the absense of foreces, the total linear momentum of a system is equal to the linear momentum of the cenre of mass.

$$\dot{\vec{p}}^{tot} = \vec{F}^{tot} = 0$$

6.1.2 Angular Momentum

Similar results may be obtained for the angular momentum of a system, keeping in mind that the angular momentum depends on the choice of origin. We provide all particles a vector $\vec{r}'_\alpha$ such that the positions of each particles $\vec{r}_\alpha$ may be specified:

$$\vec{r}_\alpha = \vec{R}_{cm} + \vec{r}'_\alpha$$

Computing the angular momentum about the orgin, we have

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha$$

and total angular momentum

$$\vec{L}^{tot} = \sum_\alpha \vec{L}_\alpha = \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}_\alpha = \sum_\alpha m_\alpha \left(\vec{R}_{cm} + \vec{r}'_\alpha\right) \times \left(\dot{\vec{R}}_{cm} + \dot{\vec{r}}'_\alpha\right)$$

Expanding,

$$\vec{L}^{tot} = \sum_\alpha m_\alpha \left(\vec{R}_{cm} \times \dot{\vec{R}}_{cm} + \vec{r}'_\alpha \times \dot{\vec{R}}_{cm} + \vec{R}_{cm} \times \dot{\vec{r}}'_\alpha + \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha\right)$$

Using the bilinearity of the cross product, the second term may be rewritten

$$\left(\sum_\alpha m_\alpha \vec{r}'_\alpha\right) \times \dot{\vec{R}}_{cm} = \sum_\alpha m_\alpha \left(\vec{r}_\alpha - \vec{R}_{cm}\right) \times \dot{\vec{R}}_{cm} = \left(M\vec{R}_{cm} - M\vec{R}_{cm}\right) \times \dot{\vec{R}}_{cm} = 0$$

This similarly applies to the third term. Thus,

$$\vec{L}^{tot} = M\vec{R}_{cm} \times \dot{\vec{R}}_c m + \sum_\alpha m_\alpha \vec{r}'_\alpha \times \dot{\vec{r}}'_\alpha$$

or the total angular momentum can be written as the angular momentum of the centre of mass added to the angular momentum of the particles around the centre of mass

$$\vec{L}^{tot} = \vec{L}_{cm} + \sum_\alpha \vec{L}'_\alpha \tag{6.4}$$

6.1.3 Moment of Inertia

If instead of a discrete system of particles, we consider a rigid, continous body, the individual angular momenta can no longar point in an arbitrary direction; all angular momenta must be parallel. Considering an infinitessimal mass dm with displacement $\vec{r}'_\alpha$ from the center of mass with angle θ_α to the angular frequency vector $\vec{\omega}$, the velocity can be written

$$\dot{\vec{r}}'_\alpha = \omega \times \vec{r}'_\alpha = \omega r'_\alpha \sin(\theta_\alpha)\hat{\phi}_\alpha$$

Summing the total angular momentum

$$\sum_\alpha \vec{L}'_\alpha = \sum_\alpha m_\alpha \vec{r}'_\alpha \times (\vec{\omega} \times \vec{r}'_\alpha) = \omega \sum_\alpha m_\alpha r'_\alpha \sin\theta_\alpha \left(\vec{r}'_\alpha \times \hat{\phi}_\alpha\right)$$

If we restrict our consideration of the angular momentum parallel to the rotation,

$$L_\parallel = \omega \sum_\alpha m_\alpha r'_\alpha \sin\theta_\alpha \hat{\omega} \cdot \left(\vec{r}'_\alpha \times \hat{\phi}_\alpha\right)$$

Using the cyclic property of the scalar triple product, we can rewrite it as $\hat{\phi}_\alpha \cdot (\hat{\omega} \times \vec{r}'_\alpha) = \hat{\phi}_\alpha \cdot (r'_\alpha \sin\theta_\alpha \hat{\phi}_\alpha) = r' \sin\theta_\alpha$ so the parallel component of the angular momentum can be written

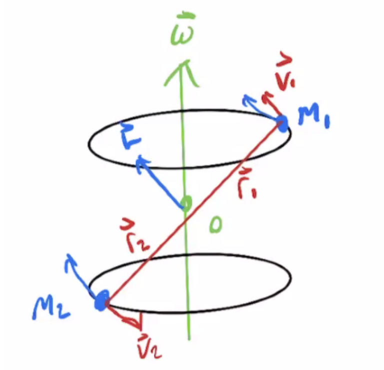
$$L_\parallel = \omega \sum_\alpha m_\alpha r'^2_\alpha \sin^2\theta_\alpha$$

$$L_\parallel = \omega \sum_\alpha m_\alpha r'^2_{\alpha,\perp} \tag{6.5}$$

or, for a continuous distribution,

$$L_\parallel = \omega \int r'^2_{\alpha,\perp}\, \mathrm{d}m \tag{6.6}$$

Interestingly, $\vec{L}$ need not be parallel to the rotation $\vec{\omega}$. As an example, we consider two particle restricted to motion on two parallel circular tracks. The angular momentum goes through the centres of these tracks, perpendicular to the surfaces. If we take m_1 to be opposite m_2, the angular momentum is actually at an angle to the rotation $\vec{\omega}$.



We recognize

$$L_{\parallel} = \omega I \tag{6.7}$$

as the definition of the moment of inertia.

6.1.4 Torque

The torque on an individual particle is defined to be:

$$\overrightharpoon{N}_\alpha = \dot{\overrightharpoon{L}}_\alpha = \vec{r}_\alpha \times \overrightharpoon{F}_\alpha \tag{6.8}$$

This can be written more verbosely as

$$\overrightharpoon{N}_\alpha = \vec{r}_\alpha \overrightharpoon{f}_\alpha^{ext} + \vec{r}_\alpha \times \overrightharpoon{f}_\alpha$$

as the sum of the internal and external torques.

$$\overrightharpoon{N}^{tot} = \sum_\alpha \overrightharpoon{N}_\alpha = \sum_\alpha \overrightharpoon{N}_\alpha^{ext} + \sum_\alpha \vec{r}_\alpha \times \left(\sum_{\beta \neq \alpha} \overrightharpoon{f}_{\alpha\beta} \right)$$

We can once again rewrite the last term as

$$\sum_\alpha \sum_{\beta > \alpha} \left(\vec{r}_\alpha \times \overrightharpoon{f}_{\alpha\beta} + \vec{r}_\beta \times \overrightharpoon{f}_{\beta\alpha} \right) = 0$$

which follows from the "strong" Newton's third law.[1], This law states that for central forces,

$$\overrightharpoon{f}_{\alpha\beta} = f_{\alpha\beta} \left(\vec{r}_\alpha - \vec{r}_\beta \right)$$

Thus,

$$\overrightharpoon{N}^{tot} = \sum_\alpha \overrightharpoon{N}_\alpha^{ext} \tag{6.9}$$

Additionally, we can relate the torque to the angular momentum as follows:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = m\vec{v} \times \vec{v} + \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{\tau}$$

[1] The "weak" Newton's third law only states that $f_{\alpha\beta} = -f_{\beta\alpha}$ and may be false

6.1.5 Kinetic Energy

For an arbitrary collection of particles, the work done one a particle α between time t_1, t_2 is

$$W_{12\alpha} = \int_1^2 \vec{F}_\alpha \cdot \mathrm{d}\vec{r}_\alpha = \int_1^2 M_\alpha \ddot{\vec{r}}_\alpha \cdot \dot{\vec{r}}_\alpha \, \mathrm{d}t = \int_1^2 m_\alpha \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\dot{r}_\alpha^2 \, \mathrm{d}t = \int_1^2 \mathrm{d}T_\alpha$$

Thus, for the total system,

$$W_{12}^{tot} = \sum_\alpha \int_1^2 \mathrm{d}T_\alpha = \Delta T^{tot}$$

where

$$T^{tot} = \sum_\alpha \frac{1}{2} m_\alpha \dot{r}_\alpha^2$$

Transforming to $\vec{r}_\alpha = \vec{R}_{cm} + \vec{r}'_\alpha$, we obtain

$$T^{tot} = \sum_\alpha \frac{1}{2} m_\alpha \dot{r}'^2_\alpha + \frac{1}{2} m_\alpha \dot{R}_{cm}^2 + m_\alpha \dot{\vec{R}}_{cm} \cdot \dot{\vec{r}}'$$

however, we know that $\sum m_\alpha \dot{\vec{r}}'{}_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha \vec{r}'_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} 0 = 0$, so

$$T^{tot} = \frac{1}{2} M \dot{R}_{cm}^2 + \sum_\alpha \frac{1}{2} m_\alpha \dot{r}'^2_\alpha \tag{6.10}$$

or, the kinetic energy is the sum of the internal kinetic energy, and the tranlational kinetic energy of the system

Rotating Rigid Body

If we imagine a rigid body rotating around an axis with $\vec{\omega}$. We can write the kinetic energy wrt to a point along the axis of rotation

$$\begin{aligned} T &= \frac{1}{2} \int v^2 \, \mathrm{d}m \\ &= \frac{1}{2} \iiint \rho(r) v^2(r) \, \mathrm{d}^3 r \\ &= \frac{1}{2} \iiint \rho r_\perp^2 \omega^2 \, \mathrm{d}^3 r \\ &= \frac{1}{2} \left(\iiint \rho r_\perp^2 \, \mathrm{d}^3 r \right) \omega^2 \\ &= \frac{1}{2} I \omega^2 \end{aligned}$$

If we instead choose the kinetic energy in terms of the centre of mass, we obtain

$$T = \frac{1}{2} M V_{cm}^2 + \sum_\alpha \frac{1}{2} m_\alpha v'^2_\alpha$$

Using $v' = |\vec{\omega} \times \vec{r}'| = \omega r'_\perp$ we see that this becomes

$$T = \frac{1}{2}MV_{cm}^2 + \frac{1}{2}I'\omega^2$$

if D is the distance to the centre of mass from the axis of rotation, we obtain $V_{cm} = \left|\vec{\omega} \times \vec{D}\right| = \omega D$, so

$$T = \frac{1}{2}MD^2\omega^2 + \frac{1}{2}I'\omega^2$$

and thus we recover the *parallel axis theorem*

$$I' = MD^2 + I_{cm} \tag{6.11}$$

6.2 Scattering and Collisions

A particle 1 is shot at a stationary particle 2, albeit with a vertical displacement b, called the *impact parameter*. The particles start at infinity, and end at infinity. We are interested in the deflection angle ϕ between the initial velocity of particle 1 $\vec{u_1}$ and the final velocity of $\vec{v}_2$. The second particle starts at rest and leaves with velocity $\vec{v}_2$ at angle ξ.

Transforming to the COM frame, both particles move toward each other at angle 0, and leave at opposite angles θ from their original motion. We can easily relate the velocities in the COM frame by

$$\vec{v}_1' = -\frac{m_2}{m_1}\vec{v}_2'$$
$$\vec{u}_1' = -\frac{m_2}{m_1}\vec{u}_2'$$

We can convert to the lab frame by boosting the centre of mass:

$$\vec{u}_2 = 0 \implies \vec{u}_2' = -\vec{V}_{cm}$$

and so forth.

There are multiple types of collisions, based on which conservation laws are satisfied. We typically consider only inelastic collision. For if $v_1', v_2' > V_{cm}$ we can access any angle ϕ, However, if one of v_1', v_2' is $< V_{cm}$, there is a maximum deflection angle, for when $\vec{v}_1' \perp \vec{v}_1$. This yields $\phi_{\max} = \sin^{-1}\left(\frac{v_1'}{v_c m}\right)$. If $\phi \neq \phi_{\max}$, there are actually two available θ values for each ϕ.

If, however, we consider elastic collisions, we will obtain $v_2' = u_2' = V_{cm}$. In this case, we will have $\xi_{\max} = \pi/2$.

6.2.1 Elastic Collisions

In the lab frame, we have the total momentum

$$\vec{p}_{tot} = M\vec{V} = m_1 u_1$$

so

$$\vec{V} = \frac{m_1}{m_1 + m_2}\vec{u}_1 = -\vec{u}_2'$$

Further, in the centre of mass frame, we have

$$\vec{u}_1' = -\frac{m_2}{m_1}\vec{u}_2'$$

so

$$\vec{u}_1' = \frac{m_2}{m_1 + m_2}\vec{u}_1$$

We can then write the following

$$\begin{aligned}
\vec{u}_2' &= -\vec{V}_{cm} \\
&= \frac{m_1}{m_1 + m_2}\vec{u}_1 \\
\vec{u}_1' &= \frac{m_2}{m_1 + m_2}\vec{u}_1
\end{aligned}$$

$$\begin{aligned}
\vec{u}_1' &= -\frac{m_2}{m_1}\vec{u}_2' \\
\vec{v}_1' &= -\frac{m_2}{m_1}\vec{v}_2'
\end{aligned}$$

Thus far, we have only used conservation of linear momentum. Using conservation of energy, we additionally have

$$\frac{1}{2}m_1 u_1'^2 + \frac{1}{2}m_2 u_2'^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

We can then substitute and obtain

$$u_1' = v_1'$$

and so

$$u_2' = v_2'$$

Using the fact that

$$\vec{v}_1 = \vec{V}_{cm} + \vec{v}_1'$$

we can then relate the lab frame angle ϕ and the centre of mass frame angle θ as

$$\tan\phi = \frac{\sin\theta}{\frac{m_1}{m_2} + \cos\theta} \tag{6.12}$$

In the range $m_1 \ll m_2$, we then get $\phi \approx \theta$. If $m_1 = m_2$, we obtain $\phi = \frac{\theta}{2}$. In this case, if we further consider

$$V_{cm} = \frac{1}{2}u_1 = u_1' = u_2' = v_1' = v_2'$$

(which is always true) we obtain that $2\xi + \theta = \pi$ so

$$\xi + \phi = \frac{\pi}{2}$$

or that in the lab frame the two outgoing particles are at 90° angles.

Kinetic Energy

Considering the kinetic energies, the initial kinetic energy in the lab frame is given

$$T_0 = \frac{1}{2}m_1u_1^2$$

while in the COM frame, we have

$$T_0' = \frac{1}{2}m_1u_1'^2 + \frac{1}{2}m_2u_2'^2 = \frac{1}{2}\mu u_1^2$$

so

$$fracT_0m_1 = \frac{T_0'}{\mu} \implies \frac{T_0'}{T_0} = \frac{m_2}{m_1+m_2}$$

We can see in the case $m_1 = m_2$ using the law of cosines

$$v_1'^2 = v_{cm}^2 = v_1^2 + v_{cm}^2 - 2v_1v_{cm}\cos\phi$$

$$v_1 = u_1\cos\phi$$

$$T_1 = \frac{1}{2}m_1v_1^2 = T_0\cos^2\phi$$

$$T_2 = T_0\sin^2\phi = T_0\cos^2\xi$$

for the final kinetic energies of the particles m_1 and m_2 respectively. In the general case, the dependence is much more complicated and has the form

$$T_1 = T_0\left[\cdots\cos\phi \pm \sqrt{m\cdots\sin^2\phi}\right]$$

which returns to the fact that ϕ, ξ have two solutions for each θ if $m_1 > m_2$ for ϕ and if $m_2 > m_1$ for ξ.

6.2.2 Inelastic Collision

An inelastic collision has $T_i \neq T_f$, but the momentum is still conserved. In 1D, the coefficient of restitution is defined

$$\epsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$$

6.2.3 Scattering Trajectories

The impact parameter b is defined to be the displacement of m_1 and m_2 along the axis of the collision.

Hard Sphere Scatterings

We define an intensity, defined as a number of particles per unit area per time

$$[I] = \frac{1}{L^2 T}$$

The cross section is defined to be

$$\Sigma = \pi(R + r)^2$$

where R is the radius of the target. The particles deflect off the target at an angle θ. We then define the scattering rate as

$$S = I\Sigma$$

where Σ is the total area of interactions. The differential cross section for a polar angle θ and azimuthal angle ϕ is given

$$\sigma(\theta, \phi)\,\mathrm{d}\Omega$$

Then, the differential scattering rate would be given

$$\mathrm{d}S = I\sigma\,\mathrm{d}\Omega$$

Similarly,

$$\Sigma = \int \sigma\,\mathrm{d}\Omega$$

Note, that for central forces, the system is symmetric about the ϕ coordinate, so

$$\Sigma = 2\pi \int \sigma(\theta) \sin\theta\,\mathrm{d}\theta$$

similarly, for the scattering rate,

$$\mathrm{d}S = 2\pi I\sigma(\theta) \sin\theta\,\mathrm{d}\theta$$

rearranging,

$$\frac{\mathrm{d}S}{I} = 2\pi\sigma(\theta) \sin\theta\,\mathrm{d}\theta$$

For a given impact parameter b, the deflection angle is $\theta(b)$. If the force is a repulsive central force, we then have $\theta(b + \mathrm{d}b) < \theta(b)$. Considering the cross sectional area of the region between b and $b + \mathrm{d}b$,

$$\mathrm{d}A = 2\pi b\,\mathrm{d}b$$

Thus,

$$\mathrm{d}S = I\,\mathrm{d}A$$

$$\frac{\mathrm{d}S}{I} = 2\pi b\,\mathrm{d}b = -2\pi\sigma(\theta) \sin\theta\,\mathrm{d}\theta$$

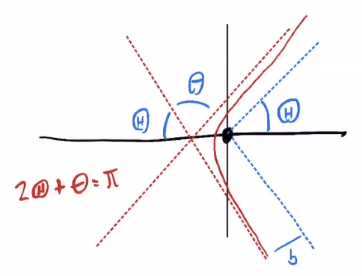
where the negative is inculded because $+\,\mathrm{d}b \implies -\,\mathrm{d}\theta$. Finally, we obtain

$$\sigma(\theta) = -\frac{b}{\sin\theta} \cdot \frac{\mathrm{d}b}{\mathrm{d}\theta} \tag{6.13}$$

We can use deflection as a function of the impact parameter to determine the force law of the interaction. We can use the orbit equation

$$\Theta(r) = \int \frac{\ell/\mu r^2 \,\mathrm{d}r}{\sqrt{\frac{2}{\mu}\left(E - \frac{\ell^2}{2\mu r^2} - U(r)\right)}}$$

Examining a sketch of the orbits,



we obtain

$$2\Theta + \theta = \pi$$

The energy is given

$$E = \frac{1}{2}\mu v^2$$

and the angular momentum by

$$\ell = |r \times p| = \mu v b$$

so we can substitute in to obtain

$$\theta(b) = \pi - 2\pi \int_{r_0}^{\infty} \frac{\mathrm{d}r}{r^2\sqrt{1 - \frac{v(r)}{E - \frac{b^2}{r^2}}}}$$

where r_0 is the distance of closest approach. Once we obtain $\theta(b)$, we can compute $\sigma(\theta)$ and compare to experiment.

Rutherford Scattering

While it is possible to use the previously derived equations to derive Rutherford scattering fromt the force, we can instead recognize that the potential is of the same form as gravity, and so we can use our trajectories we have previously derived. We can start from

$$U = -\frac{\alpha}{r}$$

$$E = \frac{1}{2}\mu r^2$$

$$\ell = \mu v b$$

$$2\Theta + \theta = \pi$$

$$e = \sqrt{1 + \frac{2E\ell^2}{\mu\alpha^2}}$$

for hyperbolic orbits, we must further have

$$e \cos\Theta = 1$$

Using these equations, we can show

$$b(\theta = \frac{\alpha}{\mu v^2} \cot\frac{\theta}{2}$$

$$\sigma(\theta) = \frac{\alpha^2}{4\mu^2 v^2} \cdot \frac{1}{\cos^4\theta/2}$$

Chapter 7

Noninertial Reference Frames

7.1 Rigid Body Dynamics

Consider an arbitrary body rotating about an axis through its centre of mass. Define the axis to be z, and consider the cyllindrical coordinates defined accordingly. We define a rigid body to be one such that for any mass element $\mathrm{d}m_i$ at point $\vec{\pi}_i$, the point moves such that $\dot{r}_i = \dot{z}_i = 0$.

The velocity of the mass element is given

$$\vec{v}_i = \dot{\vec{\pi}}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left[r_i\,\hat{e}_r + z_i\,\hat{e}_z\right] = \dot{r}_i\,\hat{e}_r + r_i\dot{\hat{r}} + \dot{z}_i\,\hat{e}_z + z_i\dot{\hat{z}}$$

Substituting our definition of a rigid body, this reduces simply to

$$\vec{v}_i = r_i\dot{\theta}\,\hat{e}_\theta$$

In a slightly different form, we know

$$\vec{\omega} = \dot{\theta}\hat{z}$$

Computing the cross product,

$$\vec{v}_i = \vec{\omega}\times\vec{\pi}_i = r_i\dot{\theta}(\,\hat{e}_z\times\,\hat{e}_r) + z_i\dot{\theta}(\,\hat{e}_z\times\,\hat{e}_z) = r_i\dot{\theta}\,\hat{e}_\theta$$

$$\vec{v}_i = \vec{\omega}\times\vec{\pi}_i \tag{7.1}$$

we can equivalently write this as

$$\vec{v}_i = \vec{\omega}\times\vec{r}_i$$

as the parallel component doesn't contribute anything[1].

Similarly, we compute the acceleration

$$\vec{a}_i = \dot{\vec{v}}_i = r_i\ddot{\theta}\,\hat{e}_\theta - r_i\dot{\theta}^2\hat{r}$$

[1] $\vec{r}_i = r_i\,\hat{e}_r$

We can rewrite the first term using cross products to obtain

$$\vec{a}_i = \vec{\alpha} \times \vec{\pi} + \vec{\omega} \times (\vec{\omega} \times \vec{\pi}_i) \tag{7.2}$$

where $\vec{\alpha} = \dot{\vec{\omega}}$.

We can find the rotational kinetic energy by substituting $\vec{v}_i = \vec{\omega} \times \vec{r}_i$

$$\begin{aligned} T &= \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i) \\ &= \frac{1}{2} \sum_i m_i \left[(\vec{\omega} \cdot \vec{\omega})(\vec{r}_i \cdot \vec{r}_i) - \cancelto{0}{(\vec{r}_i \cdot \vec{\omega})} \cancelto{0}{(\vec{\omega} \cdot \vec{r}_i)} \right] \\ &= \frac{1}{2} \omega^2 \sum_i m_i r_i^2 \end{aligned} \tag{7.3}$$

The quantitiy in the summation is defined to be the *moment of inertia* about an axis.

$$I = \sum_i m_i r_i^2 = \int \mathrm{d}m \, r^2 \tag{7.4}$$

For example, we can compute the moment of inertia of a ring about its axis using

$$\mathrm{d}m = \left(\frac{m}{2\pi a} \right) \mathrm{d}\ell$$

where $\mathrm{d}\ell = a \, \mathrm{d}\theta$ Thus,

$$I = \int \mathrm{d}m \, r^2 = \int_0^{2\pi} \frac{m}{2\pi a} a^2 \cdot a \, \mathrm{d}\theta = ma^2$$

Similarly, we can compute the moment of inertial of a solid disk through its axis as

$$I = \frac{1}{2} ma^2$$

Finally, consider the moment of inertial of a ring about a diameter. The mass element can be written

$$\mathrm{d}m = \frac{m}{2\pi a} \mathrm{d}s$$

where

$$\mathrm{d}s = a \, \mathrm{d}\theta$$

Thus, the integral becomes

$$I = \int_0^{2\pi} r^2 \frac{m \, \mathrm{d}\theta}{2\pi} = \int_0^{2\pi} a^2 \sin^2\theta \frac{m \, \mathrm{d}\theta}{2\pi} = \frac{1}{2} ma^2$$

Other useful moments of inertia are a spherical shell about its centre

$$I = \frac{2}{3} ma^2$$

and of a solid sphere about its centre

$$I = \frac{2}{5} ma^2$$

7.1.1 Angular Momentum

The angular momentum of a specific mass element is given

$$\vec{L}_i = \vec{\pi}_i \times \vec{p}_i$$

which yields

$$\vec{L} = \sum_i m_i \vec{\pi}_i \times (\vec{\omega}_i \times \vec{\pi}_i) = \sum_i m_i \vec{\pi}^2 \vec{\omega} - (\vec{\pi}_i \times \vec{\omega}) \pi_i$$

Where we used the BAC-CAB rule of the triple vector product. Simplifying, we obtain

$$\vec{L} = \left[\sum_i m_i r_i^2\right] \vec{\omega} - \left[\sum_i m_i r_i z_i \,\hat{e}_r\right] \omega$$

We recognize the first term as simply $I\vec{\omega}$, which we expected. The second term can be rewritten by expanding out $r_i \,\hat{e}_r = x_i \,\hat{e}_x + y_i \,\hat{e}_y$

$$-\sum_i m_i x_i z_i \,\hat{e}_x - \sum_i m_i y_i z_i \,\hat{e}_y$$

Essentially, if an object is symmetric about its axis of rotation, this secondary term vanishes; otherwise, it makes a contribution which causes the angular momentum to be on a different axis from the rotation.

7.2 Non-Inertial Frames

Consider a fixed system defined by the axes x_i' and an origin O', as well as a a moving system x_i with origin O. Define a vector

$$\vec{R}(t) = O - O'$$

First, assume that there is neither rotation nor acceleration. The position of a stationary point in the moving frame, P, is given in each frame as

$$\vec{r} = P - O = \sum x_i \,\hat{e}_i$$

For ease of computation, we restrict $\vec{R}(t) = \vec{v}t$. Thus, we have the position of the point in the fixed basis as

$$\vec{r}' = \vec{R} + \vec{r} = \sum_i v_i t \,\hat{e}_i' + x_i \,\hat{e}_i$$

However, we have assumed no rotation; this fixes $\hat{e}_i' = \hat{e}_i$, so we can write

$$\vec{r}'(t) = \sum_i (v_i t + x_i) \,\hat{e}_i'$$

Taking the time derivative of this position, we get a constant vector:

$$\dot{\vec{r}}' = \sum_i v_i \,\hat{e}_i' = \vec{v}$$

Both of these frames are inertial, as a constant velocity is maintained in the absence of forces.

7.2.1 Accelerating Frame

Now, let the un-primed frame accelerate with

$$\vec{a} = \sum_i a_i \, \hat{e}'_i$$

again, for simplicity, we let

$$\vec{R}(t) = \frac{1}{2}\vec{a}t^2$$

Once again, this yields

$$\vec{r}' = \sum_i (\frac{1}{2}a_i t^2 + x_i) \, \hat{e}'_i$$

Consider a point P' which is stationary in the fixed system. In the accelerating system, this point can be denoted by

$$\vec{r} = \vec{r}' - \vec{R}$$

When we take the derivatives, we find that although no forces are applied to the point,

$$\ddot{\vec{r}} = -\vec{a}$$

so the moving frame is non-inertial.

If instead, we consider P' to be the position of a mass that has a force being applied to it. Then, we see

$$\vec{F}_{eff} = m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = m\frac{\mathrm{d}^2\vec{r}'}{\mathrm{d}t^2} - m\vec{a}$$

or, more succinctly

$$\vec{F}_{eff} = \vec{F} - m\vec{a}$$

We consider the quantity $\vec{F} = m\vec{a}'$ a real force, while $m\vec{a}$ is a fictitous force arising entirely due to the accelerating reference frame.

7.2.2 Rotating Frame

Assume the origins of the two systems coincide. Clearly, we no longer have $\hat{e}_i = \hat{e}'_i$. We can obtain coefficients of expansion by taking dot products:

$$x'_i = \hat{e}'_i\vec{r} = \lambda_{i1}x_1 + \lambda_{i2}x_2 + \lambda_{i3}x_3$$

until we obtain a matrix

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Consider a rotation of with angle θ going from $\hat{e}_1 \to \hat{e}'_1$ about $\hat{e}_3$. The corresponding transformation matrix is

$$\lambda = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7.2.3 Translation and Rotation

Consider a frame that is both translating and rotating. We wish to determine the fictitous forces, and consequently the effective forces. Once again, consider a fixed frame defined by $\hat{e}_i'$ and a rotating frame defined by $\hat{e}_i$, with origins separated by $\vec{R}$. Let the rotating frame have angular velocity $\vec{\omega}$ as viewed from the fixed frame.

Consider a position vector in the rotating system, $\vec{r}$. We can write it in the fixed frame as

$$\vec{r}' = \vec{R} + \vec{r}$$

we want to find (wrt the fixed system)

$$\left.\frac{\mathrm{d}\vec{r}'}{\mathrm{d}t}\right|_{\text{rotating}} = \cancelto{0}{\frac{\mathrm{d}\vec{R}}{\mathrm{d}t}} + \sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t}\,\hat{e}_i + x_i\dot{\hat{e}}_i$$

The quantity

$$\sum_i \frac{\mathrm{d}x_i}{\mathrm{d}t}\,\hat{e}_i = \left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\text{rotating}}$$

We then need to compute the second term. Consider an infinitessimal rotation $\mathrm{d}\vec{\theta}$. We then have

$$\mathrm{d}\hat{e}_i = \mathrm{d}\vec{\theta} \times \hat{e}_i$$

thus, we have

$$\dot{\hat{e}}_i = \frac{\mathrm{d}\hat{e}_i}{\mathrm{d}t} = \frac{\mathrm{d}\vec{\theta}}{\mathrm{d}t} \times \hat{e}_i = \vec{\omega} \times \hat{e}_i$$

so, we obtain

$$\left.\frac{\mathrm{d}\vec{r}'}{\mathrm{d}t}\right|_{\text{fixed}} = \left.\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right|_{\text{rotating}} + \vec{\omega} \times \vec{r} \tag{7.5}$$

We can easily change this to include translating, rotating frames as

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r} \tag{7.6}$$

where $\vec{V} = \dot{\vec{R}}$

Thus, we can find fictitous forces by differentiating wrt to the fixed frame:

$$\begin{aligned}\left.\frac{\mathrm{d}\vec{v}_f}{\mathrm{d}t}\right|_{\text{fixed}} &= \left.\frac{\mathrm{d}\vec{V}}{\mathrm{d}t}\right|_{\text{fixed}} + \left.\frac{\mathrm{d}\vec{v}_r}{\mathrm{d}t}\right|_{\text{fixed}} + \left.\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\omega} \times \vec{r}\right)\right|_{\text{fixed}} \\ &= \ddot{\vec{R}}_f + \vec{a}_r + 2\omega \times \vec{v}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})\end{aligned}$$

Thus, we can write

$$\vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + m\dot{\vec{\omega}} \times \vec{r} + m\vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2m\vec{\omega} \times \vec{v}_r$$

where the subscript denotes which frame the non-position quantity is being measured in. Identifying $m\vec{a}_r = \vec{F}_{eff}$, we can rewrite this as

$$\vec{F}_{eff} = m\vec{a}_r = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}} \times \vec{r} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r$$

We identify the fictitious forces as follows. The term $-m\ddot{\vec{R}}_f$ is the fictitous force from translation, $-m\dot{\vec{\omega}} \times \vec{r}$ is the angular acceleration force, $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is the centrifugal force, and $-2m\vec{\omega} \times \vec{v}_r$ is the Coriolis force.

Centrifugal Force

Let $\vec{\omega} = \omega\hat{z}$ and $\vec{r} = r\,\hat{e}_r$. We can compute the direction of the centrifugal force by

$$\vec{\omega} \times \vec{r} = \omega r(\hat{z} \times \hat{e}_r) = \omega r\,\hat{e}_\theta$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \omega^2 r(\hat{z} \times \hat{e}_\theta) = \omega^2 r(-\hat{e}_r)$$

$$F_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) = m\omega^2 r\,\hat{e}_r \tag{7.7}$$

so the particle experiences an outward force.

Let us consider a space station that is a rotating ring. If we neglect the Coriolis force, then we would want the ring to be spinning with

$$m\omega^2 r = mg \implies \omega = \sqrt{\frac{g}{r}}$$

Coriolis Force

The Coriolis force is interesting as it is dependent on the motion of the particle. Consider again $\vec{\omega} = \omega\hat{z}$. If the motion is such that $\vec{v}_r = v_r\,\hat{e}_r$, then the Coriolis force is $\vec{F}_{cor} = -2m\omega v_r\,\hat{e}_\theta$. Similarly, if we have $\vec{v}_r = v_r\,\hat{e}_\theta$, thent the coriolis force is in the $+\hat{e}_r$ direction; simply, the coriolis force always pulls you to the right (if $\vec{\omega}$ is upward and $\vec{\omega} \cdot \vec{r} \neq 0$).

If we consider the coriolis force in our rotating ring space ship, then we see that the coriolis force plays a large impact if you move around. For example, if you are in an elevator, the coriolis force will push you one way on the way up, and the opposite on the way down.

7.3 Reference Frame on Planet

Consider the reference frame with respect to an observer standing at latitude λ on a rotating planet. The observer experiences a force $\mathbf{g}_0 = -g\hat{z}$ downward due to gravitation. Additionally, there is a force due to the centrifugal effect pointing to

$$\mathbf{F}_{cent} = +\omega^2 R\cos^2\lambda\hat{z} + \omega^2 R\sin\lambda\cos\lambda\hat{x}$$

where $\hat{x}$ points southward, parallel to the surface of the planet.

Let

$$\mathbf{g} = \mathbf{g}_0 + \mathbf{F}_{cent}$$

The surface of the planet is normal to $\mathbf{g}$, *not* $\mathbf{g}_0$. A quick calculation shows that the angle between gravity and the normal is

$$\sin\varepsilon = \sin\lambda\cos\lambda\left(\frac{\omega^2 R}{g}\right)$$

Because $\varepsilon \ll 1$, we can approximate

$$\varepsilon = \frac{\omega^2 R}{2g}\sin 2\lambda$$

so we see that the equator and the poles are indeed normal to gravity, as expected. The greatest deviation occurs at $\lambda = 45°$, which is about 1.73 milliradian $\approx 0.1°$. We should instead set $\hat{z} \parallel \mathbf{g}$ rather than $\mathbf{g}_0$.

Similarly, if you climb a mountain of height h, the period increases by around

$$T' = T\frac{h}{R}$$

7.4 Foucault Pendulum

The angular momentum of the earth points roughly from the south to the north pole. Consider a pendulum at a latitude $\lambda \neq 0, \pi$. The attachment point of the pendulum is not inertial as it is rotating around with the earth. Pick $\hat{z}$ such that $\mathbf{g} = g\hat{z}$, so we don't need to consider the centrifugal term. The bob on the pendulum is acted on by the tension force, gravity, and the coriolis force. Define θ to be the polar angle and ϕ be the azimuthal angle of the pendulum with respect to $\hat{z}$. The tension of the string applies a force

$$T_z = T\cos\theta\hat{z}$$

$$T_x = -T\sin\theta\cos\phi$$

$$T_y = -T\sin\theta\sin\phi$$

From the definition of spherical coordinates, it then follows easily that

$$T_x = -T\frac{x}{\ell}$$

$$T_y = -T\frac{y}{\ell}$$

The net acceleration of the mass is given

$$\ddot{\mathbf{r}} = \mathbf{g} + \frac{\mathbf{T}}{m} - 2\vec{\omega}\times\dot{\mathbf{r}}$$

As per usual, we assume $\ell \gg 1 \implies \theta \ll 1$. Thus, $z = \ell(1-\cos\theta) \approx 0$ is a about a constant, so we set $\dot{z} = 0 \implies T = mg$ and obtain $\dot{\mathbf{r}} = \dot{x}\hat{x} + \dot{y}\hat{y}$. Then, the coriolis dependence can be written

$$\vec{\omega} \times \dot{\mathbf{r}} = -\omega \sin\lambda \dot{y}\hat{x} + \omega \sin\lambda \dot{x}\hat{y} - \omega\cos\lambda\dot{y}\hat{z}$$

Once again, we ignore the z contribution. Thus, we obtain the equations of motion

$$\ddot{x} = -\frac{T}{m\ell}x + 2\omega\sin\lambda\dot{y}$$
$$\ddot{y} = -\frac{T}{m\ell}y - 2\omega\sin\lambda\dot{x}$$
$$\ddot{z} = -g + \frac{T}{m} + 2\omega\cos\lambda\dot{y} \approx 0$$

Neglecting the coriolis terms, we obviously obtain simple harmonic motion:

$$r_i = A_i e^{\pm i\sqrt{\frac{T}{m\ell}}t}$$

In polar coordinates, we can write

$$r(t) = \sqrt{x_0^2 + y_0^2}\cos\left(\sqrt{\frac{T}{m\ell}}t\right)$$

$$\phi(t) = \arctan\left(\frac{y_0}{x_0}\right) = \phi_0$$

for initial conditions x_0, y_0, and $\dot{r}(t=0) = 0$.

We can "decouple" the differential equations by defining a complex variable

$$q = x + iy$$

so then the differential equations be come a single equation

$$\ddot{q} - 2i\omega\sin(\lambda)\dot{q} + \frac{T}{m\ell}q = 0$$

This equation is that of the damped harmonic oscillator. Let$\omega_\lambda = \omega\sin\lambda$ and $\alpha^2 = \frac{T}{m\ell}$. From the auxiliary equation, we substitute $q = e^{\Omega t}$ and obtain the quadratic

$$\Omega^2 + 2i\omega_\lambda\Omega + \alpha^2 = 0$$

or

$$\Omega_\pm = -i\omega_\lambda \pm i\sqrt{\omega_\lambda^2 + \alpha^2}$$

The general solution is then

$$q = A_\pm e^{\Omega_\pm t}$$

If $\omega_\lambda = 0$, we know that this solution should reduce to that of the simple harmonic oscillator. Indeed, we obtain

$$q_0 = Ae^{i\alpha t} + Be^{i\alpha t}$$

Assuming the pendulum is much smaller than the earth, we have the natural frequency of the pendulum, α much smaller than the frequency of the earth ω_λ. Taking the approximation $\alpha \gg \omega_\lambda$, we can write the general solution as

$$q \approx e^{-i\omega_\lambda t} q_0$$

Let $q_0 = x' + iy'$. Then,

$$\begin{aligned} x &= \cos\omega_\lambda t x' + \sin\omega_\lambda t y' \\ y &= -\sin\omega_\lambda t x' + \cos\omega\lambda t y' \end{aligned}$$

Rewriting in terms of matrix multiplication,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\omega_\lambda t & \sin\omega_\lambda t \\ -\sin\omega_\lambda t & \cos\omega_\lambda t \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

we see that the transformation is a rotation matrix. Thus, we can write

$$\mathbf{r} = R(\omega_\lambda t)\mathbf{r}'$$

In the northern hemisphere, we see that the precession will be clockwise whereas it will be counterclockwise in the southern hemisphere:

$$\mathbf{r}' = R(-\omega_\lambda t)\mathbf{r}$$

The period of this precession is given

$$\frac{2\pi}{\omega_\lambda} = T \sim 35\,\mathrm{hr}$$

7.4.1 Hurricanes

In a hurricane, the air comes in from high pressure toward the low pressure eye, whereupon the coriolis force causes it to rotate with a velocity v. The centripetal force on the air is then given by the interplay between the coriolis force of the rotational v and the force of pressure pushing it inward. Thus, we have

$$\frac{v^2}{r} = \frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - 2\omega v \sin\lambda$$

This is quadratic in v. Solving, we can find the windspeed of the hurricane:

$$v = -\omega r \sin\lambda \pm \sqrt{(\omega r \sin\lambda)^2 + \frac{r}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}}$$

Interestingly, an anti-cyclone (where air rushes out from a high-pressure eye) is not likely to occur

Chapter 8

Rigid Body Dynamics

8.1 Inertia Tensor

Consider the rotation of a rigid body about a point $O \neq \mathbf{r}_{cm}$, which is separated from the origin in a fixed frame by $\mathbf{R}$. The position of a mass element can be written as $\mathbf{r}_\alpha$. Because it is a rigid body, in the frame of the rigid body, we have $\frac{\mathrm{d}\mathbf{r}_\alpha}{\mathrm{d}t}\Big|_{rot} = 0$. The velocity of the mass element in the fixed frame,

$$\mathbf{v}'_\alpha = \dot{\mathbf{R}} + \vec{\omega} \times \mathbf{r}_\alpha$$

The angular momentum about the fixed origin, O' can be given

$$\begin{aligned}
\mathbf{L}'_T &= \sum_\alpha m_\alpha (\mathbf{R} + \mathbf{r}_\alpha) \times \mathbf{v}'_\alpha \\
&= \sum_\alpha m_\alpha \mathbf{R} \times \mathbf{V} + \left(\sum_\alpha m_\alpha \mathbf{r}_\alpha \right) \times V + \mathbf{R} \times \left(\vec{\omega} \times \sum_\alpha m_\alpha \mathbf{r}_\alpha \right) + \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha) \\
&= M(\mathbf{R} + \mathbf{r}_{cm}) \times \mathbf{V} + M\mathbf{R} \times \mathbf{v}_{cm} + \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\vec{\omega} \times \mathbf{r}_\alpha)
\end{aligned}$$

The first term is the angular momentum due to the object veloctiy wrt the centre of mass, the second term the angular momentum of due to the motion of the centre of mass, while the last term is the angular momentum wrt the origin O. Thus, we define

$$\mathbf{L}_{body} \equiv \mathbf{L} = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\omega \times \mathbf{r}_\alpha)$$

Using BAC-CAB, can rewrite this as

$$\mathbf{L} = \sum_\alpha m_\alpha r_\alpha^2 \omega - \sum_\alpha m_\alpha (\mathbf{r}_\alpha \cdot \vec{\omega}) \mathbf{r}_\alpha \tag{8.1}$$

Let $\mathbf{r}_\alpha = x_i^{(\alpha)} \hat{e}_i$ and $\vec{\omega} = \omega_i e_i$. Then,[1]

$$L_j = \sum_\alpha m_\alpha \left[\left(\sum_{k=1}^{3} x_k^{(\alpha)} x_k^{(\alpha)} \right) \omega_j - \left(\sum_i x_i^{(\alpha)} \omega_i \right) x_j^{(\alpha)} \right]$$

[1] In class, we use the notation $x_{\alpha i} = x_i^{(\alpha)}$

We see from the second term that $\mathbf{L}$ is not always parallel to $\vec{\omega}$. Let $\omega_j = \sum_i \delta_{ij}\omega_i$. Then, we can rewrite L_j as

$$\begin{aligned} L_j &= \sum_\alpha m_\alpha \sum_i \left[\left(\sum_k x_k^{(\alpha)} x_k^{(\alpha)} \delta_{ij} \right) - x_i^{(\alpha)} x_j^{(\alpha)} \right] \omega_j \\ &= \sum_i \omega_i \sum_\alpha m_\alpha \left[\left(\sum_k x_k^{(\alpha)} x_k^{(\alpha)} \delta_{ij} \right) - x_i^{(\alpha)} x_j^{(\alpha)} \right] \end{aligned}$$

We thus define the *inertia tensor*[2]

$$I_{ij} = \sum_\alpha m_\alpha \left[\left(\sum_k x_k^{(\alpha)} x_k^{(\alpha)} \right) \delta_{ij} - x_i^{(\alpha)} x_j^{(\alpha)} \right] \tag{8.2}$$

such that

$$\mathbf{L} = \overset{\leftrightarrow}{\mathbf{I}}\, \vec{\omega}$$

or

$$L_i = \sum_j I_{ij}\omega_j$$

It is easy to see that $I_{ij} = I_{ji}$, so the tensor is *symmetric*. Thus, there are only 6 independent components to the tensor, rather than $3 \times 3 = 9$.

The diagonal elements are known as the moments of inertia, where I_{ii} gives the moment about the x_i axis, while the off-diagonal elements are the products of inertia.

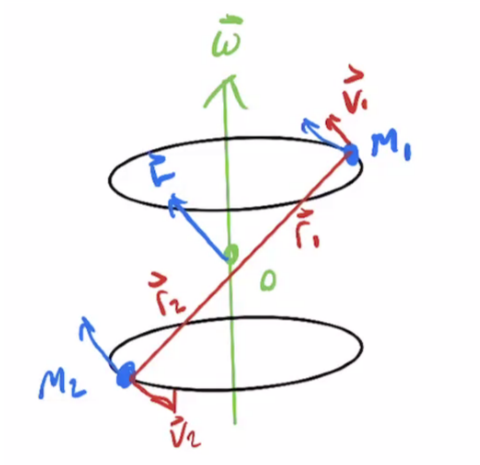
We can extend our definition of the inertia tensor to allow *continuous* rigid bodies in the natural way:

$$I_{ij} = \int_V \mathrm{d}m \, \left[x^2\delta_{ij} - x_i x_j \right] = \int_V \rho \, \mathrm{d}^3 x \, \left[x^2\delta_{ij} - x_i x_j \right] \tag{8.3}$$

Spinning Dumbell

Consider a spinning dumbell, which consists of two point masses m connected by a massless rod length $2a$and spinning about an axis through the centre of mass, but is not necessarily at a angle aligned to the rod.

[2]Note, that this is not the *moment of inertia tensor*.



Let $\vec{\omega} = \omega\hat{z}$, and the rod at angle θ to $\hat{z}$. Then, we can write $\mathbf{a}_1 = a(\sin\theta\hat{y} + \cos\theta\hat{z})$ and $\mathbf{a}_2 = -a(\sin\theta\hat{y} + \cos\theta\hat{z})$ where at a time $t = 0$, we fix x such that $a_x = 0$. Then, the angular momentum of the system is given

$$\mathbf{L} = 2m\omega a^2 \sin\theta(\sin\theta\hat{z} - \cos\theta\hat{y})$$

Note, that $\mathbf{L} \parallel \vec{\omega}$ iff $\cos\theta = 0 \implies \theta = \frac{\pi}{2}$.

As we allow the dumbell to evolve in time, we need to consider the inertia tensor. For consistency, let ϕ be the angle from the y axis, rather than from the x axis, as would be expected from normal spherical coordinates. We then have

$$-\mathbf{a}_2 = \mathbf{a}_1 = a\left\langle \sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta \right\rangle$$

as the positions of the two masses. Our inertia tensor then becomes

$$I = 2ma^2 \begin{bmatrix} \sin^2\theta\cos^2\phi + \cos^2\theta & -\sin^2\theta\sin\phi\cos\phi & -\sin\theta\cos\theta\sin\phi \\ -\sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi + \cos^2\theta & -\sin\theta\cos\theta\cos\phi \\ -\sin\theta\cos\theta\sin\phi & -\sin\theta\cos\theta\cos\phi & \sin^2\theta \end{bmatrix}$$

If $\dot{\phi} = 0$, then we are in the body system, using *body coordinates*. Thus, the problem reduces to what we had previously considered. As before, fix $\phi = 0$, so

$$I = 2ma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\theta & -\sin\theta\cos\theta \\ 0 & -\sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Thus, we see that

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}}\,\vec{\omega} = 2ma^2\omega\sin\theta\,(\sin\theta\hat{z} - \cos\theta\hat{y})$$

which matches our previous result.

If we, however, take $\dot{\phi} \neq 0$, such as in the fixed frame, the angular momentum instead rotates around the z axis. This rotation is called *precesion*. Because we see that the angular momentum is changing, there must be a torque being applied to maintain the precession.

Using the relation between the time derivative of vectors in moving frames and fixed frames, we obtain that the torque becomes

$$\vec{\tau} = \underbrace{\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{\text{fixed}}}_{=0} + \vec{\omega} \times \mathbf{L}$$

$$\vec{\tau} = 2\omega^2 m a^2 \sin\theta \cos\theta \hat{x}$$

Trivially, we see there is no torque when $\theta = 0, \pi/2$.

8.1.1 Principle Axes of Inertia

Note, that because the inertia tensor has a symmetric matrix representation, that there exists an orthonormal basis such that the inertia tensor is diagonal; these axes are known as the *principle axes.* We then write, along the principle axes, that

$$L = I_i \omega_i$$

Finding these principle axes is a standard eigenvalue problem.

$$\det(\overleftrightarrow{\mathbf{I}} - I_i \mathbb{1}) = 0$$

The eigenvalues I_i are the *principle moments*, and the eigenvectors are the *principle axes.*

8.1.2 Kinetic Energy

Recall that the kinetic energy is defined

$$T = \frac{1}{2} \sum_\alpha m_\alpha v_\alpha^2$$

Substituting $v_\alpha = \omega \times r_\alpha$, we obtain

$$\begin{aligned} T_{\text{rot}} &= \frac{1}{2} \sum_\alpha m_\alpha (\omega \times r_\alpha) \times (\omega \times r_\alpha) \\ &= \frac{1}{2} m_\alpha \omega \cdot (r_\alpha \times (\omega \times r_\alpha)) \\ &= \frac{1}{2} \omega \cdot \sum_\alpha m_\alpha (r_\alpha \times (\omega \times r_\alpha)) \end{aligned}$$

Recognizing the final sum, we see that

$$T_{\text{rot}} = \frac{1}{2} \vec{\omega} \times \mathbf{L} = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{\mathbf{I}} \vec{\omega} \tag{8.4}$$

Expanding the second representation, we obtain

$$T_{\text{rot}} = \frac{1}{2} \sum_{ij} \omega_i I_{ij} \omega_j \tag{8.5}$$

8.1.3 Steiner's Parallel Axis Theorem

Consider a stationary coordinate system at a distance **a** away from the CoM coordinate system and with parallel. Trivially, if X is the second coordinate system, and x is the first coordiate system, we can write $X = x + a$. When we plug this into the formula for the inertia tensor in X, we can simplify to obtain

$$J_{ij} = I_{ij} + M(a^2\delta_{ij} - a_i a_j) \tag{8.6}$$

where I_{ij} is the CoM inertia tensor, and the second term is the inertia tensor of a point mass M in the displaced coordinate system.

8.1.4 Changing Axes

Let $x' = \Lambda x$. Then, we can transform the inertia tensor via

$$I' = \Lambda I \Lambda^{-1}$$

What systems satisfy $I' = I$? We see trivially that with principle axes x, y, z, we can write

$$I = \left(\sum_\alpha m_\alpha r_\alpha^2\right)\mathbb{1} - \left(\sum_\alpha m_\alpha r \otimes r\right)$$

Or,

$$I = I_0 \mathbb{1} - \sum_\alpha m_\alpha r^{\otimes 2}$$

Define an arbitrary scaling transformation $x, y, z \to a_x x, a_y y, a_z z$. Then,

$$I'_{ii} = I'_0 - a_i^2 \sum_\alpha m_\alpha r_{\alpha,i}^2$$

Where I_0 doesn't necessarily equal I'_0. Choosing a, b, c such that $I' = I'\mathbb{1}$, we see that *any body* can be "squashed" until they have a spherical inertia tensor.

8.2 Euler Angles

Similar to how a plane has pitch, roll, and yaw, a set of three *Eulerian Angles* which completely define the orientation of the object. These angles, ϕ, θ, ψ, along with the Euler-Lagrange equations will allow us to find the equations of motion of a "tumbling" object.

Force-Free

Consider the case wherer there is no translational kinetic energy. We will define our system by composing three rotations[3] First, we will rotate by ϕ about the x_3''' axis. We then obtain

[3] In lecture, the x''' and x' labels are flipped. I chose to start with x''' rather than x' for a nice progression of each transformation "deleting" a prime.

our first transformation as

$$\begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1''' \\ x_2''' \\ x_3''' \end{bmatrix}$$

We make our next rotation by θ about x_1''. This gives

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \\ x_3'' \end{bmatrix}$$

Finally, we rotate by ψ about x_3', so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

Note that the three vectors $\dot{\phi}, \dot{\theta}, \dot{\psi}$ are *not* orthogonal. However, there is an interesting line called the *line of nodes* which points along the $\dot{\theta}$ direction, and is the intersection between the x_1x_2 and $x_1'''x_2'''$ planes. When we compound these three rotations, we obtain the transformation matrix from $x''' \to x$ as

$$\begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\phi \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \tag{8.7}$$

8.2.1 Euler's Equations of Motion

To use the Euler-Lagrange equations, we must know how the coordinates change with time. Trivially, we have

$$\dot{\psi} = \dot{\psi} x_3 \tag{8.8}$$

The second velocity $\dot{\theta}$ lies along the line of nodes; it is the intersection of the x_1x_2 and $x_1'''x_2'''$ axes. It is *in* the x_1x_2 plane, but is offset at an angle ψ down from x_1. Thus,

$$\dot{\theta} = \dot{\theta}\cos\phi\hat{x}_1 - \dot{\theta}\sin\theta\hat{x}_2 \tag{8.9}$$

Some work needs to be done for our final angle, however. After compounding the two rotations $R_\psi \circ R_\theta$, we obtain

$$\dot{\phi} = \dot{\phi}\sin\theta\sin\psi\hat{x} + \dot{\phi}\sin\theta\cos\psi + \dot{\phi}\cos\theta \tag{8.10}$$

Note that our first angle is akin to a 1D spherical (linear) coordinates, the second akin to 2D spherical (polar) coordinates with angle θ, and the third akin to 3D spherical coordinates, with polar angle θ and azimuthal angle ψ.

Our overall angular velocity is given

$$\omega = \dot{\psi} + \dot{\theta} + \dot{\phi} \tag{8.11}$$

To go from Lagrange's equations of motions to Euler's equations, we use our three angles as generalized coordinates. First, we set our reference frame with respect to the principle axes, so our inertia tensor is diagonal. Further, considering force-free motion, we can fix $U = 0$. We can consequently write our Lagrangian as

$$L = T = \frac{1}{2}\sum_i I_i \omega_i^2$$

Beginning with ψ,

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\psi}} \\ &= \frac{\partial T}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\psi}} \\ &= \sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \psi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}} \\ &= \left(\sum_i L_i \cdot \frac{\partial \omega_i}{\partial \psi}\right) - \frac{\mathrm{d}}{\mathrm{d}t} L_3 \end{aligned}$$

Plugging in our values for ω, we obtain the Euler Equations of Motion

$$\begin{aligned} (I_2 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 - I_2\dot{\omega}_2 &= 0 \\ (I_1 - I_2)\omega_1\omega_2 - I_3\dot{\omega}_3 &= 0 \end{aligned}$$

However, when we have torque, we have

$$N = \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{\text{fixed}} = \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{\text{body}} + \omega \times L$$

Because we chose I to be diagonal, we know

$$L_i = I_i\omega_i \implies \dot{L}_i = I_i\dot{\omega}_i$$

we can then rewrite our Euler equations of motion as

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1 \tag{8.13a}$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2 \tag{8.13b}$$
$$I_3\dot{\omega}_3 - (I_2 - I_1)\omega_1\omega_2 = \tau_3 \tag{8.13c}$$

or, in general,

$$(I_i - I_j)\omega_i\omega_j - \sum_k (I_k\dot{\omega}_k - N_k)\varepsilon_{ijk} = 0 \tag{8.14}$$

8.3 Symmetric Top

Consider a top with

$$I_1 = I_2 \neq I_3$$

In the absense of force, we can set $\tau = 0$. Plugging into the Euler EoM,

$$\begin{aligned} 0 &= I_3\dot{\omega}_3 \\ 0 &= (I_1 - I_3)\omega_2\omega_3 - I_1\dot{\omega}_1 \\ 0 &= (I_3 - I_1)\omega_1\omega_3 - I_1\dot{\omega}_2 \end{aligned}$$

Simplifying, we obtain

$$\dot{\omega}_1 = -\frac{I_3 - I_1}{I_1}\omega_3\omega_2 \equiv -\Omega\omega_2$$

$$\dot{\omega}_2 = +\frac{I_3 - I_1}{I_1}\omega_3\omega_1 \equiv +\Omega\omega_1$$

where $\Omega = \frac{I_3 - I_1}{I_1}\omega_3$. A prolate object (such as a can of red bull) will have $\Omega < 0$, while an oblate object (such as a frisbee) will have $\Omega > 0$. Let $q = \omega_1 + i\omega_2$. Then, our differential equation becomes

$$\dot{q} - i\Omega q = 0$$

or

$$q = Ae^{i\Omega t}$$

with

$$\omega_1 = A\cos\Omega t$$

$$\omega_2 = A\sin\Omega t$$

Thus, we see that $\vec{\omega}$ precesses about x_3 with angular velocity Ω.

Further, because there is no torque or force, the kinetic energy is constant, so the angle β between $\mathbf{L}$ and $\vec{\omega}$ is constant. Further, we can show that

$$\mathbf{L} \cdot (\vec{\omega} \times \hat{e}_3) = 0$$

so these three vectors are all coplanar.

{diagram}

We can imagine this scenario as two cones rolling about each other. The axis of one cone is the fixed x_3' direction, or the $\mathbf{L}$ direction, while the other is the body x_3 direction. Of course, the angle between these two axes is the Eulerian angle θ. Finally, the point of contact of these cones is along the angular velocity vector, $\vec{\omega}$. We can find the angle α between the angular velocity $\vec{\omega}$ and the body axis x_3 by using the relation $L = I\omega$ in body coordinates. Thus, we obtain the equations

$$L\sin\theta = I_1\omega\sin\alpha$$

$$L\cos\theta = I_3\omega\cos\alpha$$

so we obtain

$$\tan\theta = \frac{I_1}{I_3}\tan\alpha$$

Recall that if $I_1 = I_2 > I_3$ our object is prolate and $\theta > \alpha$. If $I_1 = I_2 < I_3$, our object is oblate and $\theta < \alpha$. We rotate the body cone along x_3' to obtain the precession; we see that if $\alpha > \theta$ then this is akin to rolling an object inside of another object, and the precession is in the opposite direction—this is consistent with the angular velocity of the precession Ω having negative sign for an oblate object.

{diagram}

Now we wish to consider the precession of x_3 about $\mathbf{L}$ (this is in contrast to what we calculated earlier, the precession of $\vec{\omega}$ about x_3'). The precession turns out to have angular velocity $\dot{\phi}$, as this is the angular velocity about the x_3' axis.

Fixing $\psi = 0$, we see that when we plug into the expansion of ω, we have

$$\omega_2 = \dot{\phi}\sin\theta$$

so

$$\dot{\phi} = \frac{\omega_2}{\sin\theta} = \frac{\omega\sin\alpha}{\sin\theta} = \frac{\omega\frac{L_2}{I_1\omega}}{\frac{L_2}{L}} = \frac{L}{I_1}$$

where we substituted $\sin\alpha = L_2/I_1\omega$ and $\sin\theta = L_2/L$ from earlier.

8.3.1 Fixed Point

Consider a top of mass M at an angle θ with its tip fixed at the origin in a gravitational field. The centre of mass is located at a vector $\mathbf{h}$ from the origin. The top rotates at a rate $\dot{\psi}\hat{x}_3$ and precesses with $\dot{\phi}\hat{x}_3'$.

By substituting what we know about the components of ω, we can compute the kinetic energy as

$$T = \frac{1}{2}I_1(\dot{\phi}^2\sin\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2$$

Trivially, the potential energy is

$$U = Mgh\cos\theta$$

so

$$L = \frac{1}{2}I_1(\dot{\phi}^2\sin\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}\cos\theta + \dot{\psi})^2 - Mgh\cos\theta$$

From Euler-Lagrange, we see that if $\frac{\partial L}{\partial x} = 0$ then $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0$ and so that coodinate is a constant of the motion. Trivially, we see that this is the case for both ϕ and ψ, so p_ϕ and p_ψ are conserved momenta. This corresponds to no torque along x_3 or x_3'. These angular momenta can be written explicitly as

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = (I_1\sin^2\theta + I_3\cos^2\theta)\dot{\phi} + I_3\dot{\psi}\cos\theta$$

$$p_\psi = I_3(\dot{\psi} + \dot{\phi}\cos\theta)$$

Solving for $\dot{\psi}$,

$$\dot{\psi} = \frac{p_\psi - I_3\dot{\phi}\cos\theta}{I_3}$$

Using this expression we can then find $\dot{\phi}$,

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2\theta}$$

and reinsert to into $\dot{\psi}$ to get

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1\sin^2\theta}$$

However, we see that both $\dot{\psi}$ and $\dot{\phi}$ are not constant, as they are functions of $\theta(t)$; the variation in θ is known as *nutation*, In principle, we can solve for θ directly, but it is easier to consider an effective potential instead. In the toal energy,

$$E = \frac{1}{2}\omega \cdot I\omega + U$$

we see that the term $\frac{1}{2}I_3\omega_3^2 = \frac{p_\psi^2}{2I_3}$ is a constant of the motion, E is conserved, and thus we can define $E' = E - \frac{1}{2}I_3\omega_3^2$ as a conserved quantity. Substituting in our expression of $\dot{\phi}$ into $\omega_1^2 + \omega_2^2 = \dot{\phi}^2\sin^2\theta + \dot{\theta}^2$ to obtain

$$E' = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

so we obtain the effective potential that governs θ as

$$V(\theta) = \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta} + Mgh\cos\theta$$

Of course, we can use the equation

$$t(\theta) = \int \frac{\mathrm{d}\theta}{2/I_1[E' - V(\theta)]}$$

and $\theta(t)$ can be obtained by inverting.

If we have $E' = V_{\min}$, then we see that $\theta(t) = \dot{\theta}$, and the precession $\dot{\phi}$ is a constant. Minimising $V(\theta)$, we find that

$$0 = \frac{\partial V}{\partial \theta} = \frac{-\cos\theta_0(p_\phi - p_\psi\cos\theta_0)^2 + p_\psi(p_\phi - p_\psi\cos\theta_0)\sin^2\theta_0}{I_1\sin^3\theta_0} - Mgh\sin\theta_0$$

Substituting

$$\beta = \dot{\phi}I_1\sin^2\theta = p_\phi - p_\psi\cos\theta_0$$

we can simplify as a quadratic in β:

$$0 = (-\cos\theta_0)\beta^2 + p_\psi \sin^2\theta_0 \beta - MghI_1 \sin^4\theta_0$$

so we can use the quadratic formula to find β. In particular, the top is only stable if β is real; thus our stability condition is given by the discriminant.

$$\beta = \frac{p_\psi \sin^2\theta_0}{2\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4MghI_1\cos\theta_0}{p_\psi^2}}\right)$$

thus, our stability condition is given

$$p_\psi^2 \geq 4MghI_2 \cos\theta_0$$

This is always satisfied when $\theta_0 \geq \pi/2$. Note that this corresponds to the top dangling from the ceiling. If this is true, then the top will always precess; there is no minimum condition for p_ψ.

Considering the more restricive case that $\theta_0 \leq \pi/2$ we can substitute our value for $p_\psi = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3$. Thus, we see that this stability condition translates to

$$\omega_3 \geq \frac{\sqrt{4MghI_1\cos\theta_0}}{I_3}$$

The precession $\dot\phi$ about x' is given

$$\dot\phi_0 = \frac{\beta}{I_1 \sin^2\theta_0}$$

Now, recall that β has two roots. If we have a *very* fast spinning top, we can taylor expand the radicand to see

$$\beta^2 \cdot \frac{2\cos\theta_0}{p_\psi \sin^2\theta_0} \approx 1 \pm \frac{2MghI_1\cos\theta}{p_\psi^2}$$

Substituting into $\dot\phi_0$, we get two different precession rates, $\Phi_0^\pm$ corresponding to the value of $\beta^\pm$. We get the fast precession to be

$$\Phi_0^+ = \frac{I_3\omega_3}{I_1\cos\theta_0}$$

$$\Phi_0^- = \frac{Mgh}{I_3\omega_3}$$

Both of these precessions are ein the same direction as ω_3. By inspection, we see that Φ_0^+ is similar to free precession.

If we relax the condition that $E' = V_{\text{min}}$, then we see nutation between θ_1 and θ_2, where $V(\theta_1) = V(\theta_2) = E'$. Because we see that $\dot\phi$ is dependent on $\cos\theta$, we see that there are even some circumstances where $\dot\phi$ can even change sign as θ varies, depending on the magnitude of p_ϕ and p_ψ

Chapter 9

Coupled Oscillators

Coupled oscillators can exchange energy between each other. An example is the Harmonic Solid model for phonons.

9.1 Two coupled oscillators

Consider two equal masses, connected to each other and to the walls with three springs, with displacements x_1, x_2, and spring constants $k_1, k_2 = k$ and a couping springk_{12}. The kinetic energy of the system is given

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$

and similarly, the potential energy is given

$$U = \frac{1}{2}kx_1^2 + \frac{1}{2}k_{12}(x_1 - x_2)^2 + \frac{1}{2}kx_2$$

Thus, the lagrangian can be writen

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k_{12}(x_1 - x_2)^2 \tag{9.1}$$

Our equations of motion are then given

$$m\ddot{x}_1 = -(k + k_{12})x_1 + k_{12}x_2$$

$$m\ddot{x}_2 = +k_{12}x_1 - (k + k_{12})x_2$$

If there is no coupling, i.e. $k_{12} = 0$, we trivially have harmonic oscillator solutions (duh.). Thus, we use test solutions

$$x_j = B_j e^{i\omega t}$$

Plugging into our EoM and rewriting as a matrix equation,

$$\begin{bmatrix} -m\omega^2 + (k + k_{12}) & -k_{12} \\ -k_{12} & -m\omega^2 + (k + k_{12}) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0$$

For there to be a nontrivial solution, the determinant of the matrix must be zero. This gives us a characteristic equation for ω. Rather than computing the determinant, we notice that instead we need only consider

$$-m\omega^2 + (k + k_{12}) = \mp k_{12}$$

so

$$\omega^2 = \frac{(k + k_{12}) \pm k_{12}}{m} = \frac{k}{m}, \frac{k + 2k_{12}}{m} \tag{9.2}$$

solving for the eigenvectors is routine computation. We see that our solutions have

$$B_1^{\pm} = \mp B_2^{\pm} \tag{9.3}$$

or, the lower frequency mode is a translational mode, while the higher frequency mode is a streching mode, when focusing only on spring k_{12}. Our solutions are then linear combinations of these two modes, or

$$x_1 = B_1^+ e^{i\omega_+ t} + B_1^- e^{-i\omega_+ t} + B_2^+ e^{i\omega_- t} + B_2^- e^{i\omega_- t} \tag{9.4a}$$

$$x_2 = -B_1^+ e^{i\omega_+ t} - B_1^- e^{-i\omega_+ t} + B_2^+ e^{i\omega_- t} - B_2^- e^{i\omega_- t} \tag{9.4b}$$

The physical solutions to our equations are of course the real projections of x_1, x_2.

9.2 Normal Coordinates

The previous way we solved for the motion is not the most elegant or easiest way to do so. Let us define define the normal coodinates

$$\eta_1 = x_1 + x_2 \qquad\qquad \eta_2 = x_1 - x_2 \tag{9.5}$$

Plugging these into our equation of motion and multiplying through by 2,

$$M(\ddot{\eta}_1 + \ddot{\eta}_2) + (k + k_{12})\eta_1 + k\eta_2 = 0$$
$$M(\ddot{\eta}_1 - \ddot{\eta}_2) + (k + k_{12})\eta_1 - k\eta_2 = 0$$

These are easily decoupled into

$$0 = m\ddot{\eta}_1 + k\eta_1$$
$$0 = m\ddot{\eta}_2 + (k + 2k_{12})\eta_2$$

Trivially, the are harmonic oscillators with $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{(k + 2k_{12})/m}$. In general, the normal coordinates for a system of coupled oscillators obey

$$0 = m\ddot{\eta}_i + k\eta_i$$

These two normal coordinates correspond to the *normal modes* of osciilation; in particular, η_1 corresponds to the *symmetric mode* and η_2 corresponds to the *antisymmetric mode*, as the phase of the two masses is symmetric or antisymmetric respectively. We see the antisymmetric mode has a higher frequency of oscillation, because the middle spring is stretched and applies a greater restoring force. Typically, antisymmetric modes will be higher in frequency.

If we hold one mass fixed, we see the effective spring constant is $k + k_{12}$ and so the frequency is given

$$\omega_0 = \sqrt{\frac{k + k_{12}}{m}}$$

This we define as the base frequency of the harmonic oscillators, as this is the frequency they would oscillate at if alone. Thus, we observe frequency splitting, as ω_0 splits into a higher frequency $\omega_0 < \omega_2$ and a lower frequency $\omega_1 < \omega_0$ The magnitude of this splitting is dependent on the size of the coupling, k_{12}

Weak Coupling

Consider $k_{12} \ll k$. We can then taylor expand and write

$$\omega_0 \approx \omega_1(1 + \epsilon)$$

$$\omega_2 \approx \omega_1(1 + 2\epsilon)$$

or, in terms of ω_0 (discarding higher order terms,)

$$\omega_1 \approx \omega_0(1 - \epsilon)$$

$$\omega_2 \approx \omega_0(1 + \epsilon)$$

9.3 General Oscillations

Consider a system of p masses in 3D, for a total of $n = 3p$ degrees of freedom. In rectilinear coordinates, we can define the position

$$x_{\alpha,i} \qquad \alpha \in \{1, \ldots, p\} \quad i \in \{1, 2, 3\}$$

Further, let's assume that there's no time dependence in the transformation between rectilinear and generalized coordinates:

$$x_{\alpha,i} = x_{\alpha,i}(q_i)$$

Using the fact that at equilibrium we have

$$q_k = A \quad \dot{q}_k = 0 \quad \ddot{q}_k = 0$$

we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\sum_i a_i \dot{q}^i$$

If $i < 1$, because $\frac{\mathrm{d}a_i}{\mathrm{d}t} = 0$ is independent of time, there is zero contribution; further for $i > 1$ we can substitute $\dot{q} = 0$ and $\ddot{q} = 0$. Thus, inserting into euler lagrange, we have

$$\frac{\partial L}{\partial q_k} = 0$$

or

$$\frac{\partial T}{\partial q_k} = \frac{\partial U}{\partial q_k} \tag{9.7}$$

at equilibrium

Trivialy, we have

$$T = \frac{1}{2}\sum_{\alpha=1}^{p}\sum_{i=1}^{3} m_\alpha \dot{x}_{\alpha,i}^2$$

Expanding with the chain rule, we can write T in terms of the generalized coordinates:

$$\dot{x}_{\alpha,i} = \sum_j \frac{\partial x_{\alpha,i}}{\partial q_j}\dot{q}_j$$

$$\dot{x}_{\alpha,i}^2 = \sum_{jk} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial k}\dot{q}_j\dot{q}_k$$

so

$$T = \frac{1}{2}\sum_{\alpha=1}^{p} m_\alpha \sum_{ijk} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}\dot{q}_j\dot{q}_k$$

We will instead condense the p, i sum into a tensor so we can write

$$T = \frac{1}{2}\sum_{jk} m_{jk}\dot{q}_j\dot{q}_k \tag{9.8}$$

with

$$m_{jk} = \sum_{\alpha=1}^{p} m_\alpha \sum_{i=1}^{3} \frac{\partial x_{\alpha,i}}{\partial q_j}\frac{\partial x_{\alpha,i}}{\partial q_k}$$

Thus, applying equilibrium, we see that

$$\left.\frac{\partial T}{\partial q_k}\right|_0 = 0$$

Similarly, from Eq. 9.7, we see that we must have

$$\left.\frac{\partial U}{\partial q_k}\right|_0 = 0$$

WLOG, we set the equilibrium value of $\mathbf{q} = 0$. If we taylor expand about equilibrium, we see that
$$U(\mathbf{q}) \approx U(0) + \nabla U'(0) \cdot \mathbf{q} + \frac{1}{2}\mathbf{q} \cdot D^2 U(0)\mathbf{q}$$
We can further fix $U(0) = 0$ to write
$$U = \frac{1}{2}\sum_{jk} A_{jk} q_j q_k$$
where
$$A_{jk} = \frac{\partial^2 U}{\partial q_j \partial q_k}$$
We will now verify that our leading order approximation for U is valid for T. Expanding m_{jk} about equilibrium, we see
$$m_{jk} = m_{jk}(0) + \sum_{\ell} \left.\frac{\partial m_{jk}}{\partial q_\ell}\right|_0 q_\ell + \dots$$
We see that beyond leading order terms, when we plug this into T, we get 3rd order terms, and thus can neglect them. Thus, we need only consider $m_{jk} = m_{jk}(0)$

Plugging these into the lagrangian, we obtain
$$-\frac{\partial U}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{q}_k} = 0$$
so
$$\sum_j A_{jk} q_j + m_{jk}\ddot{q}_j = 0 \tag{9.9}$$
which is n second order differential equations. Trivially, we see that
$$q_j(t) = a_j e^{i(\omega t - \delta)}$$
Plugging in our guess and dividing out the time dependence,
$$\sum_j (A_{jk} - \omega^2 m_{jk}) a_j = 0 \tag{9.10}$$
Once again, the determinant of this matrix equation should be zero:
$$\det[A - \omega^2 m] = 0 \tag{9.11}$$
This is the *characteristic* or *secular* equation. There are n solutions for ω, which are our eigenfrequencies. Of couse, these need not be unique; we can have degenerate modes. Using the principle of superposition, any solution is a linear combination of these trial solutions:
$$q_j(t) = \sum_{r=1}^{n} a_{jr} e^{i(\omega_r t - \delta_r)} \tag{9.12}$$
where our physical solution is the real part of this solution.

9.3.1 Generalized Normal Coordinates

Of course, as both our kinetic energy and potential energy are written as quadratic forms:

$$T = \frac{1}{2}\langle \dot{q}|m|\dot{q}\rangle$$

$$U = \frac{1}{2}\langle q|A|q\rangle$$

we can find a basis with which they are diagonal. In fact, it is true that they are diagonalwr to the same basis, the *normal coordinates*. Because we can diagonalize with real eigenvalues, distinct eigenspaces are orthogonal, and we can normalize subject to the constraint

$$\sum_{jk} m_{jk} a_{jr} a_{ks} = \delta_{rs}$$

This can be obtained by plugging Eq 9.12 into Eq 9.9 and summing over with a second a_{ks}.

$$\omega_s^2 \sum_k m_{jk} a_{ks} = \sum_k A_{jk} a_{ks}$$

$$\omega_r^2 \sum_{jk} m_{jk} a_{jr} a_{ks} = \sum_{jk} A_{jk} A_{jk} a_{jr} a_{ks} = \omega_s^2 \sum_{jk} m_{jk} a_{jr} a_{ks} \tag{9.13}$$

In general, then we can write

$$q_j = \sum_r \gamma_r a_{jr} e^{i(\omega_r t - \delta_r)}$$

absorbing the phase into γ_r, we find that

$$q_j = \sum_r a_{jr} \beta_r e^{i\omega_r t}$$

and so we find our normal coordinates to be

$$\eta_r = \beta_r e^{i\omega_r t} \tag{9.14}$$

or

$$q_j = \sum_r a_{jr} \eta_r$$

Plugging into the Lagrangian and enforcing our orthonormality constraint, we find that

$$T \to \frac{1}{2} \sum_r \dot{\eta}_r^2$$

similarly, plugging into the porential energy and applying Eq.9.13

$$U = \frac{1}{2} \sum_r \omega_r^2 \eta_r^2$$

so

$$\ddot{\eta}_r^2 = -\omega_r^2 \eta_r$$

9.4 Molecular vibrations

An individual atom has three degrees of freedom in space—one for each translational degree of freedom. For a molecule with $N = 2$, there are 6 degrees of freedom—however, rather than 6 translational degrees, we have 3 translational degrees, 2 rotational degrees, and one vibrational degree.

In particular we which to consider CO_2, which is a linear $N = 3$ molecule. Again, it has 3 translational degrees of freedom and 2 rotational degrees of freedom. However, it is more difficult to determin how many vibrational degrees there are. It turns out there are 2 scissor modes, an antisymmetric stretch, and a symetric stretch. We can measure these modes via laser light.

Considering the longitudinal stretching modes, we see that we can model the system as two coupled harmonic oscillators. The normal modes are given by the symmetric and antisymmetric motion of the oxygen atoms. However, the antisymmetric stretch corresponds to a motion of the carbon, which causes a time-dependent dipole moment, and thus radates.

Similarly, we can model the scissoring modes as a angle-dependent restorign force. Once again, there is a time-dependent dipole moment, and thus we expect radiation.

Comparing the frequencies of the vibration to the frequency of light emitted, and because we know the mass of the atoms, we can find the spring constants associated with each mode. It turns out each of these modes corresponds to infra red light.

Appendix A

A brief survey of interesting problems

A.1 Three-Body Problem

We began mechanics by examining the behaviour of a single particle through the Newtonian, Lagrangian, and Hamiltonian formalism, moved on to the 2-particle central force problem and N particles. However, the three body problem is formally unsolvable, and is still an area of active research. We will look at special cases which have solutions.

If we consider three particles in space and denote them $1, 2, 3$, we can easily denote the distances and forces on each problem. Focusing on particle 1, the total force is given

$$F_1 = F_{12} + F_{13} = m\ddot{r}_1$$

and so forth for the other forces. Further, from Newton's third law, we know that

$$F_{ij} = -F_{ji}$$

thus, we see that

$$\sum_i F_i = \sum_i m_i \ddot{r}_i = \sum_{ij} F_{ij} = 0$$

or,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\sum_i m_i \vec{r}_i\right] = \left(\sum_i m_i\right) \ddot{R}_{cm} = 0$$

The three-body problem is generally split into three classes of problem. The first is the restricted three-body problem, which is the case of the moon orbiting the earth, which in turn orbits the sun; $m_m \ll m_e \ll m_\odot$. The hierarchical three-body problem rather has all masses of the same order, but with the orbit split into two sub-binaries. Finally the non-hierarchical does not assume any binaries.

A.1.1 Restricted Three-body problem

We assume that the orbits are coplanar, and that binaries are circular orbits. We consider the non-inertial frame where the sun and moon are fixed, and examine where we can place the moon. The rate the earth rotates around the sun is

$$\Omega = \frac{G(M_\odot + m_e)}{R^3}$$

The gravitational potential is given

$$U(x) = -\frac{GM_\odot m}{R-x} - \frac{Gm_e m}{X}$$

adding the centrifugal term, the effective potential can be written

$$V_{eff}(x) = -\frac{GM_\odot m}{R-x} - \frac{Gm_e m}{X} - \frac{1}{2}m\frac{G(M_\odot + m_e)}{R^3}(R-x)^2 \tag{A.1}$$

We can find stable points by

$$0 = \left.\frac{\partial V_{eff}}{\partial x}\right|_{r_L}$$

if $M_\odot \gg m_e$ and $r_L \ll R$, we see that

$$r_L \approx \left(\frac{m_e}{3M_\odot}\right)^{1/3} R$$

More generally, we can find points in all of 2D space for these points; these points are known as the Lagrange points. The points L1–3 are semi-stable equilibria (which are stable along the angular direction and unstable along the radial direction), while L4–5 are unstable. As one of the stars in a binary pair becomes a giant, its radius can expand beyond the Lagrange points, causing the hydrogen to be siphoned toward the second star.

A.1.2 Hierarchical Three-body Problem

We consider two binaries, where masses 1 and 2 form an elliptical system that forms an additional binary with mass 3. We assume that the semimajor axis of the inner binary a_1 is much less than the outer binary a_2. We can write the hamiltonian of the total system as the sum of the inner binary H_1, the outer binary H_2, and an interaction hamiltonian;

$$H_{LK} = H_1 + H_2 + H_{int} \tag{A.2}$$

The interaction entropy looks akin to a multipole expansion from E&M. Averaging the system over the two orbital periods, we obtain

$$\left\langle \left\langle H_{LK} \right\rangle_1 \right\rangle_2 = \left\langle \left\langle H_{int} \right\rangle_1 \right\rangle_2$$

We can then understand the time evolution of the eccentricity through the Poisson brackets. The eccetricity has an interesting periodic evolution where it swings between extremes of eccentricity.

Such time evolutions lead to exoplanets known as "hot Jupiters," where a a gas giant orbits are closer to their stars than mercury is to the sun. They also have the interesting property where their "spins" are anti-aligned with that of the sun, which is something we do not observe in our solar system. The influence of a third body can cause the orbit of a planet to flip and go retrograde to their normal orbit.

A.1.3 Non-Hierarchical Three-body Problem

The non-hierarchical three-body problem is a version of the hierarchical three-body problem when $a_1 \sim a_2$. It cannot solved analytically, and in fact forms a chaotic system. As such, statistical methods must be taken to generate solutions. The approach uses the *ergodicity*, which in a sense describes the available paths in phase space.

Returning to the phase space of a double pendulum, the phase space paths change drastically with energy, until becoming almost completely random at a certain energy. The Ergodic Hypothesis compares this phase space to that of a thermodynamic system, assuming that every microstate is equally likely at any given time.

Analyzing the phase space volume of a 3-body system with ergodic theory generates predictions of chaos similar to those observed in numerical simulations. This is of interest because we believe most gravitational waves are due to these sorts of systems