Ahlfors Exercises

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Contents

Chapter 1

Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1.1.1.1

$$(1+2i)^3 = 1 + 6i - 12 - 8i = \boxed{-11 - 2i}$$

$$\frac{5}{-3+4i} = \frac{-15-20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{4+7i}{13}\right)^2 = \boxed{-\frac{33}{169} + \frac{56}{169}i}$$

From the binomial expansion of the LHS, and cancelling odd powers of i,

$$(1+i)^n + (1-i)^n = 2\sum_{m=0}^{n/2} \binom{n/2}{2m} (-1)^m$$

1.1.1.2

$$\operatorname{Re} z^4 = x^4 - 6x^2y^2 + y^4$$

$$\operatorname{Re} \frac{1}{z} = \frac{x}{x^2+y^2}$$

$$\operatorname{Re} \frac{z-1}{z+1} = \frac{x^2-1}{(x+1)^2+y^2}$$

$$\operatorname{Re} \frac{1}{z^2} = \operatorname{Re} \frac{1}{x^2-y^2+2xyi} = \frac{x^2-y^2}{(x^2+y^2)^4}$$

1.1.1.3

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = -\frac{1}{8} \pm \frac{3\sqrt{3}}{8}i + \frac{9}{8} \mp \frac{3\sqrt{3}}{8}i = 1$$
$$\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = \frac{1}{64} + \frac{6\sqrt{3}}{64}i - \frac{45}{64} - \frac{60\sqrt{3}}{64}i + \frac{135}{64} + \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$
$$\left(\frac{\pm 1 \mp i\sqrt{3}}{2}\right)^6 = \frac{1}{64} - \frac{6\sqrt{3}}{64}i - \frac{45}{64} + \frac{60\sqrt{3}}{64}i + \frac{135}{64} - \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$

1.1.2 Square Roots

1.1.2.1

(a)
$$a^2 - b^2 = 0 \quad 2ab = 1 \implies a = b = \pm\frac{1}{\sqrt{2}} \implies \sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

(b)
$$a^2 - b^2 = 0 \quad 2ab = -1 \implies a = b = \pm\frac{i}{\sqrt{2}} \implies \sqrt{-i} = \pm\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

(c) sub $b = 1/2a$ so
$$a^2 = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$
$$b^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2}$$
enforcing the condition that
$$ab = \frac{1}{2}$$
we obtain
$$\pm\left(\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}} + i\sqrt{-\frac{1}{2} + \frac{1}{\sqrt{2}}}\right)$$

(d) I really cannot be bothered to do this...
$$a^2 - b^2 = \frac{1}{2} \qquad ab = \frac{\sqrt{3}}{4}$$
$$(a^2 - b^2)^2 = \frac{1}{4} \qquad (a^2 + b^2)^2 = 1$$
$$a^2 = \frac{3}{4} \qquad b^2 = \frac{1}{4}$$
Thus,
$$\sqrt{\frac{1 - i\sqrt{3}}{2}} = \pm\left(\frac{\sqrt{3}}{2} - \sqrt{i}2\right)$$

1.1.2.2

Cuz i'm lazy:

$$\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}},\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

1.1.2.3

do i really have to

using the fact that $\sqrt{i}=\pm\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$, $\sqrt{i}=\mp\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$, and $i^4=1$,

$$\sqrt{\sqrt{\pm i}}=a_\pm+ib_\pm \implies a_\pm^2-b_\pm^2=\frac{1}{\sqrt{2}} \qquad a_\pm b_\pm=\frac{1}{2\sqrt{2}}$$

$$a_\pm^4-2a_\pm^2b_\pm^2+b_\pm^4=\frac{1}{2} \implies (a_\pm^2+b_\pm^2)^2=1 \implies a_\pm^2+b_\pm^2=1$$

$$\implies a_\pm^2=\frac{1}{2}+\frac{1}{2\sqrt{2}} \qquad b_\pm^2=\frac{1}{2}-\frac{1}{2\sqrt{2}}$$

$$\begin{aligned}\sqrt{\sqrt{i}}=&\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}},-\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}},\\&-\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}},\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\\\sqrt{\sqrt{-i}}=&-\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}},-\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}},\\&\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}},\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\end{aligned}$$

1.1.2.4

are you serious

Plugging into the quadratic formula,

$$z=\frac{-\alpha-i\beta\pm\sqrt{\alpha^2-\beta^2+i2\alpha\beta-4\gamma-i4\delta}}{2}$$

Taking the square root,

$$a^2-b^2=\alpha^2-\beta^2-4\gamma$$

$$ab=\alpha\beta-2\delta$$

$$(a^2-b^2)^2 = \alpha^4+\beta^4+16\gamma^2-2\alpha^2\beta^2-8\alpha^2\gamma+8\beta^2\gamma$$
$$a^2b^2 = \alpha^2\beta^2+4\delta^2-4\alpha\beta\delta$$
$$(a^2+b^2)^2 = \alpha^4+\beta^4+16\gamma^2+2\alpha^2\beta^2-8\alpha^2\gamma+8\beta^2\gamma+16\delta^2-16\alpha\beta\delta$$

$$a = \frac{\sqrt{\alpha^2-\beta^2-4\gamma+\sqrt{(\alpha^2+\beta^2)^2+8\gamma(2\gamma-a^2+\beta^2)+16\delta(\delta-\alpha\beta)}}}{2}$$
$$b = \frac{\sqrt{-\alpha^2+\beta^2+4\gamma+\sqrt{(\alpha^2+\beta^2)^2+8\gamma(2\gamma-a^2+\beta^2)+16\delta(\delta-\alpha\beta)}}}{2}$$

for

$$z = \frac{-\alpha\pm 2a}{2} - i\frac{\beta\pm 2b}{2}$$

where i literally cannot be bothered to try and fit the above in one single expression.

1.1.3 Justification

1.1.3.1

Let capital members denote matrices and lower case members denote complex numbers. For a relation $f : Z \mapsto z$ to be a homomomorphism it must obey $f(E_+) = e_+$ and $f(E_\times) = e_\times$, that is, we must have

$$f\left[\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right] = 0 \quad f\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right] = 1$$

Further, we use the fact that

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

to fix

$$f\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] = i$$

arbitrarily. We can thus identify

$$f\left[\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}\right] = \alpha + i\beta$$

with the inverse map

$$f^{-1}[\alpha+i\beta] = \begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$$

We show that this is a field homomorphism. Let $f(Z) = \alpha+i\beta$ and $f(W) = \gamma+i\delta$. Then,

$$f(Z+W) = f\left[\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix} + \begin{pmatrix}\gamma & \delta\\ -\delta & \gamma\end{pmatrix}\right]$$

$$\begin{aligned}
&= f\left[\begin{pmatrix} \alpha+\gamma & \beta+\delta \\ -(\beta+\delta) & \alpha+\delta \end{pmatrix}\right] \\
&= (\alpha+\gamma) + i(\beta+\delta) \\
&= (\alpha+i\beta) + (\gamma+i\delta) \\
&= f(Z) + f(W)
\end{aligned}$$

thus addition is respected. Similarly,

$$\begin{aligned}
f(ZW) &= f\left[\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}\right] \\
&= f\left[\begin{pmatrix} \alpha\gamma-\beta\delta & \alpha\delta+\beta\gamma \\ -(\alpha\gamma+\beta\delta) & \alpha\gamma+\beta\gamma \end{pmatrix}\right] \\
&= (\alpha\gamma-\beta\delta) + i(\alpha\delta+\beta\gamma) \\
&= (\alpha+i\beta)(\gamma+i\delta) \\
&= f(Z)f(W)
\end{aligned}$$

thus multiplication is also respected. Thus, f is a field homomorphism.

We can further see that f is a bijection; it is a surjection because all $z \in \mathbb{C}$ can be written $z = \alpha+i\beta$ for $\alpha, \beta \in \mathbb{R}$, and it clearly an injection because $f(Z) = f(W) \implies f(Z)-f(W) = 0 \implies f(Z-W) = 0 \implies Z-W = 0 \implies Z = W$.

Thus, as f is a field homomorphism and a bijection, it is an isomorphism, and these matrices equiped with matrix addition and matrix multiplication is isomorphic to the complex field.

1.1.3.2

I have no idea what this means

1.1.4 Conjugation, Absolute Value

1.1.4.1

$$\begin{aligned}
\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+i2xy+1} \\
&= \frac{x^3-xy^2-2xy^2+x+i(x^2y-y^3-2x^2y+y)}{x^4-6x^2y^2+y^4+x^2-y^2+1} \\
\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2-i2xy+1} \\
&= \frac{x^3-xy^2-2xy^2+x-i(x^2y-y^3-2x^2y+y)}{x^4-6x^2y^2+y^4+x^2-y^2+1} \\
&= \overline{\left(\frac{z}{z^2+1}\right)}
\end{aligned}$$

1.1.4.2

Splitting into terms,

(a) $2 \cdot \sqrt{10} \cdot \sqrt{20}\sqrt{2} = 40$

(b) $5 \cdot \sqrt{5}/\sqrt{2}\sqrt{10} = \frac{5}{2}$

1.1.4.3

$$\begin{aligned}\left|\frac{a-b}{1-\bar{a}b}\right| &= \frac{a-b}{1-\bar{a}b}\frac{\bar{a}-\bar{b}}{1-a\bar{b}} \\ &= \frac{\bar{a}a+\bar{b}b-\bar{a}b-a\bar{b}}{1-\bar{a}b-a\bar{b}+\bar{a}a\bar{b}b} \\ &= \frac{|a|^2+|b|^2-2\operatorname{Re}\bar{a}b}{1+|a|^2|b|^2-2\operatorname{Re}\bar{a}b}\end{aligned}$$

We see that if either $|a| = 1$ or $|b| = 1$ that the numerator equals the denominator and the fraction cancels. In the case where $|a| = |b| = 1$, the expression still holds so long as $\operatorname{Re}\bar{a}b \neq 1$, that is $a \neq b$.

1.1.4.4

Make the substitution

$$\alpha = a+b \qquad \beta = a-b \qquad \Longrightarrow \qquad a = \frac{\alpha+\beta}{2} \qquad b = \frac{\alpha-\beta}{2}$$

so

$$\begin{aligned}\alpha \operatorname{Re} z + i\beta \operatorname{Im} z &= -c \\ \bar{\alpha} \operatorname{Re} z - i\bar{\beta} \operatorname{Im} z &= -\bar{c}\end{aligned}$$

Adding,

$$\operatorname{Re}\alpha \operatorname{Re} z - \operatorname{Im}\beta \operatorname{Im} z = -\operatorname{Re} c$$

and subtracting,

$$\operatorname{Im}\alpha \operatorname{Re} z + \operatorname{Re}\beta \operatorname{Im} z = -\operatorname{Im} c$$

Cases: if α is real,

$$\operatorname{Im} z = -\frac{\operatorname{Im} c}{\operatorname{Re}\beta}$$

if α is imaginary,

$$\operatorname{Im} z = -\frac{\operatorname{Re} c}{\operatorname{Im}\beta}$$

if β is real,

$$\operatorname{Re} z = -\frac{\operatorname{Re} c}{\operatorname{Re} \alpha}$$

if β is imaginary,

$$\operatorname{Re} z = -\frac{\operatorname{Im} c}{\operatorname{Im} \alpha}$$

Note that there is no solution if one of α, β is real and the other is purly imaginary. Further, if either α or β is zero, we have either infinitely many solutions, characterized by two parallel lines, or no solutions.

Finally, consider the case where α, β are nonzero and have both imaginary and complex components. Solving,

$$\operatorname{Re} z = -\frac{\operatorname{Re} \beta \operatorname{Re} c + \operatorname{Im} \beta \operatorname{Im} c}{\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta}$$

$$\operatorname{Im} z = -\frac{\operatorname{Re} \alpha \operatorname{Im} c - \operatorname{Im} \alpha \operatorname{Re} c}{\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta}$$

We see there is a unique solution so long as $\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta \neq 0$, or if α, β are both real or both imaginary.

1.1.4.5

Trivially, Lagrange's identity holds for $n = 1$:

$$|a_1 b_1| = |a_1| \, |b_1| + 0$$

First, note that through multiplying conjugates, we obtain

$$\sum_{i=1}^{n+1} \left| a_i \bar{b}_{n+1} - a_{n+1} \bar{b}_i \right|^2 = |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 + |a_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 - 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n+1} a_i b_i \right]$$

Thus, if Lagrange's identity holds for some n,

$$\begin{aligned} \left| \sum_{i=1}^{n+1} a_i b_i \right|^2 &= \left| \sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1} \right|^2 \\ &= \left| \sum_{i=1}^{n} a_i b_i \right|^2 + |a_{n+1}|^2 \, |b_{n+1}|^2 - 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n} a_i b_i \right] \end{aligned}$$

$$
\begin{aligned}
\left|\sum_{i=1}^{n+1} a_i b_i\right|^2 &= \left|\sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1}\right|^2 \\
&= 2\left|\sum_{i=1}^{n} a_i b_i\right|^2 + 2\left|a_{n+1}\right|^2\left|b_{n+1}\right|^2 - \left|\sum_{i=1}^{n} a_i b_i - a_{n+1} b_{n+1}\right|^2 \\
&= 2\sum_{i=1}^{n}\left|a_i\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 + 2\left|a_{n+1}\right|^2\left|b_{n+1}\right|^2 - 2\sum_{1\leq i<j\leq n}\left|a_i\bar{b}_j - a_i\bar{b}_j\right|^2 \\
&\qquad - \left|\sum_{i=1}^{n} a_i b_i - a_{n+1} b_{n+1}\right|^2 \\
&= \sum_{i=1}^{n}\left|a_i\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 + \left|a_{n+1}\right|^2\sum_{i=1}^{n}\left|b_n\right|^2 + \left|b_{n+1}\right|^2\sum_{i=1}^{n+1}\left|a_i\right|^2 - 2\sum_{1\leq i<j\leq n}\left|a_i\bar{b}_j - a_j\bar{b}_i\right|^2 \\
&\quad + \sum_{i=1}^{n}\left|a_i\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 - \left|a_{n+1}\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 - \left|b_{n+1}\right|^2\sum_{i=1}^{n}\left|a_i\right|^2 - \left|\sum_{i=1}^{n} a_i b_i - a_{n+1} b_{n+1}\right|^2 \\
&\quad + \left|a_{n+1}\right|^2\left|b_{n+1}\right|^2 \\
&= \sum_{i=1}^{n+1}\left|a_i\right|^2\sum_{i=1}^{n+1}\left|b_i\right|^2 + \left|a_{n+1}\right|^2\left|b_{n+1}\right|^2 - 2\sum_{1\leq i<j\leq n}\left|a_i\bar{b}_j - a_j\bar{b}_i\right|^2 \\
&\quad + \sum_{i=1}^{n}\left|a_i\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 - \left|a_{n+1}\right|^2\sum_{i=1}^{n}\left|b_i\right|^2 - \left|b_{n+1}\right|^2\sum_{i=1}^{n}\left|a_i\right|^2 - \left|\sum_{i=1}^{n} a_i b_i - a_{n+1} b_{n+1}\right|^2
\end{aligned}
$$

1.1.5 Inequalities

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1.2.2 The Binomial Equation

1.2.3 Analytic Geometry

1.2.4 The Spherical Representation