Ahlfors Exercises

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Contents

Chapter 1

Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1.1.1.1

$$(1+2i)^3 = 1 + 6i - 12 - 8i = \boxed{-11 - 2i}$$

$$\frac{5}{-3+4i} = \frac{-15-20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{4+7i}{13}\right)^2 = \boxed{-\frac{33}{169} + \frac{56}{169}i}$$

From the binomial expansion of the LHS, and cancelling odd powers of i,

$$(1+i)^n + (1-i)^n = 2\sum_{m=0}^{n/2} \binom{n/2}{2m} (-1)^m$$

1.1.1.2

$$\operatorname{Re} z^4 = x^4 - 6x^2y^2 + y^4$$

$$\operatorname{Re} \frac{1}{z} = \frac{x}{x^2+y^2}$$

$$\operatorname{Re} \frac{z-1}{z+1} = \frac{x^2-1}{(x+1)^2+y^2}$$

$$\operatorname{Re} \frac{1}{z^2} = \operatorname{Re} \frac{1}{x^2-y^2+2xyi} = \frac{x^2-y^2}{(x^2+y^2)^4}$$

1.1.1.3

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = -\frac{1}{8} \pm \frac{3\sqrt{3}}{8}i + \frac{9}{8} \mp \frac{3\sqrt{3}}{8}i = 1$$

$$\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = \frac{1}{64} + \frac{6\sqrt{3}}{64}i - \frac{45}{64} - \frac{60\sqrt{3}}{64}i + \frac{135}{64} + \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$

$$\left(\frac{\pm 1 \mp i\sqrt{3}}{2}\right)^6 = \frac{1}{64} - \frac{6\sqrt{3}}{64}i - \frac{45}{64} + \frac{60\sqrt{3}}{64}i + \frac{135}{64} - \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$

1.1.2 Square Roots

1.1.2.1

$$a^2 - b^2 = 0 \quad 2ab = 1 \implies a = b = \pm\frac{1}{\sqrt{2}} \implies \sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$a^2 - b^2 = 0 \quad 2ab = -1 \implies a = b = \pm\frac{i}{\sqrt{2}} \implies \sqrt{-i} = \pm\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

sub $b = 1/2a$ so

$$a^2 = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$

$$b^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2}$$

enforcing the condition that

$$ab = \frac{1}{2}$$

we obtain

$$\pm\left(\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}} + i\sqrt{-\frac{1}{2} + \frac{1}{\sqrt{2}}}\right)$$

I really cannot be bothered to do this…

1.1.2.2

Cuz i'm lazy:

$$\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$$

1.1.2.3

do i really have to using the fact that $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$ and $i^4 = 1$,

$$\sqrt{\sqrt{i}} = \sqrt{\frac{1}{2\sqrt{2}} + \frac{1}{2}} + + i\sqrt{-\frac{1}{2\sqrt{2}} + \frac{1}{2}}$$

multiplying by i, i^2, i^3, we obtain the other three solutions.

$$-\sqrt{\frac{1}{2\sqrt{2}} + \frac{1}{2}} + + i\sqrt{-\frac{1}{2\sqrt{2}} + \frac{1}{2}}, -\sqrt{\frac{1}{2\sqrt{2}} + \frac{1}{2}} + - i\sqrt{-\frac{1}{2\sqrt{2}} + \frac{1}{2}}, \sqrt{\frac{1}{2\sqrt{2}} + \frac{1}{2}} + - i\sqrt{-\frac{1}{2\sqrt{2}} + \frac{1}{2}}$$

1.1.3 Justification

1.1.4 Conjugation, Absolute Value

1.1.5 Inequalities

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1.2.2 The Binomial Equation

1.2.3 Analytic Geometry

1.2.4 The Spherical Representation