Ahlfors Exercises

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Contents

Chapter 1

Complex Numbers

1.1 The Algebra of Complex Numbers

1.1.1 Arithmetic Operations

1.1.1.1

$$(1+2i)^3 = 1 + 6i - 12 - 8i = \boxed{-11 - 2i}$$

$$\frac{5}{-3+4i} = \frac{-15-20i}{25} = \boxed{-\frac{3}{5} - \frac{4}{5}i}$$

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{4+7i}{13}\right)^2 = \boxed{-\frac{33}{169} + \frac{56}{169}i}$$

From the binomial expansion of the LHS, and cancelling odd powers of i,

$$(1+i)^n + (1-i)^n = 2\sum_{m=0}^{n/2} \binom{n/2}{2m} (-1)^m$$

1.1.1.2

$$\operatorname{Re} z^4 = x^4 - 6x^2y^2 + y^4$$

$$\operatorname{Re}\frac{1}{z} = \frac{x}{x^2+y^2}$$

$$\operatorname{Re}\frac{z-1}{z+1} = \frac{x^2-1}{(x+1)^2+y^2}$$

$$\operatorname{Re}\frac{1}{z^2} = \operatorname{Re}\frac{1}{x^2-y^2+2xyi} = \frac{x^2-y^2}{(x^2+y^2)^4}$$

1.1.1.3

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = -\frac{1}{8} \pm \frac{3\sqrt{3}}{8}i + \frac{9}{8} \mp \frac{3\sqrt{3}}{8}i = 1$$
$$\left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = \frac{1}{64} + \frac{6\sqrt{3}}{64}i - \frac{45}{64} - \frac{60\sqrt{3}}{64}i + \frac{135}{64} + \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$
$$\left(\frac{\pm 1 \mp i\sqrt{3}}{2}\right)^6 = \frac{1}{64} - \frac{6\sqrt{3}}{64}i - \frac{45}{64} + \frac{60\sqrt{3}}{64}i + \frac{135}{64} - \frac{54\sqrt{3}}{64}i - \frac{27}{64}$$

1.1.2 Square Roots

1.1.2.1

(a)
$$a^2 - b^2 = 0 \quad 2ab = 1 \implies a = b = \pm\frac{1}{\sqrt{2}} \implies \sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

(b)
$$a^2 - b^2 = 0 \quad 2ab = -1 \implies a = b = \pm\frac{i}{\sqrt{2}} \implies \sqrt{-i} = \pm\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

(c) sub $b = 1/2a$ so
$$a^2 = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$$
$$b^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2}$$
enforcing the condition that
$$ab = \frac{1}{2}$$
we obtain
$$\pm\left(\sqrt{\frac{1}{2} + \frac{1}{\sqrt{2}}} + i\sqrt{-\frac{1}{2} + \frac{1}{\sqrt{2}}}\right)$$

(d) I really cannot be bothered to do this...
$$a^2 - b^2 = \frac{1}{2} \qquad ab = \frac{\sqrt{3}}{4}$$
$$(a^2 - b^2)^2 = \frac{1}{4} \qquad (a^2 + b^2)^2 = 1$$
$$a^2 = \frac{3}{4} \qquad b^2 = \frac{1}{4}$$
Thus,
$$\sqrt{\frac{1 - i\sqrt{3}}{2}} = \pm\left(\frac{\sqrt{3}}{2} - \sqrt{i}2\right)$$

1.1.2.2

Cuz i'm lazy:

$$\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$$

1.1.2.3

do i really have to

using the fact that $\sqrt{i}=\pm\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$, $\sqrt{i}=\mp\frac{1}{\sqrt{2}}\pm\frac{i}{\sqrt{2}}$, and $i^4=1$,

$$\sqrt{\sqrt{\pm i}}=a_\pm+ib_\pm \implies a_\pm^2-b_\pm^2=\frac{1}{\sqrt{2}} \qquad a_\pm b_\pm=\frac{1}{2\sqrt{2}}$$

$$a_\pm^4-2a_\pm^2b_\pm^2+b_\pm^4=\frac{1}{2} \implies (a_\pm^2+b_\pm^2)^2=1 \implies a_\pm^2+b_\pm^2=1$$

$$\implies a_\pm^2=\frac{1}{2}+\frac{1}{2\sqrt{2}} \qquad b_\pm^2=\frac{1}{2}-\frac{1}{2\sqrt{2}}$$

$$\begin{aligned}\sqrt{\sqrt{i}}=&\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}, -\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}},\\ &-\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}, \sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\\ \sqrt{\sqrt{-i}}=&-\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}, -\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}},\\ &\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}-i\sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}, \sqrt{\frac{1}{2}-\frac{1}{2\sqrt{2}}}+i\sqrt{\frac{1}{2}+\frac{1}{2\sqrt{2}}}\end{aligned}$$

1.1.2.4

are you serious

Plugging into the quadratic formula,

$$z=\frac{-\alpha-i\beta\pm\sqrt{\alpha^2-\beta^2+i2\alpha\beta-4\gamma-i4\delta}}{2}$$

Taking the square root,

$$a^2-b^2=\alpha^2-\beta^2-4\gamma$$

$$ab=\alpha\beta-2\delta$$

$$(a^2-b^2)^2 = \alpha^4+\beta^4+16\gamma^2-2\alpha^2\beta^2-8\alpha^2\gamma+8\beta^2\gamma$$
$$a^2b^2 = \alpha^2\beta^2+4\delta^2-4\alpha\beta\delta$$
$$(a^2+b^2)^2 = \alpha^4+\beta^4+16\gamma^2+2\alpha^2\beta^2-8\alpha^2\gamma+8\beta^2\gamma+16\delta^2-16\alpha\beta\delta$$

$$a = \frac{\sqrt{\alpha^2-\beta^2-4\gamma+\sqrt{(\alpha^2+\beta^2)^2+8\gamma(2\gamma-a^2+\beta^2)+16\delta(\delta-\alpha\beta)}}}{2}$$
$$b = \frac{\sqrt{-\alpha^2+\beta^2+4\gamma+\sqrt{(\alpha^2+\beta^2)^2+8\gamma(2\gamma-a^2+\beta^2)+16\delta(\delta-\alpha\beta)}}}{2}$$

for
$$z = \frac{-\alpha\pm 2a}{2} - i\frac{\beta\pm 2b}{2}$$
where i literally cannot be bothered to try and fit the above in one single expression.

1.1.3 Justification

1.1.3.1

Let capital members denote matrices and lower case members denote complex numbers. For a relation $f : Z \mapsto z$ to be a homomomorphism it must obey $f(E_+) = e_+$ and $f(E_\times) = e_\times$, that is, we must have
$$f\left[\begin{pmatrix}0&0\\0&0\end{pmatrix}\right] = 0 \quad f\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\right] = 1$$
Further, we use the fact that
$$\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$
to fix
$$f\left[\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right] = i$$
arbitrarily. We can thus identify
$$f\left[\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}\right] = \alpha+i\beta$$
with the inverse map
$$f^{-1}[\alpha+i\beta] = \begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}$$
We show that this is a field homomorphism. Let $f(Z) = \alpha+i\beta$ and $f(W) = \gamma+i\delta$. Then,
$$f(Z+W) = f\left[\begin{pmatrix}\alpha&\beta\\-\beta&\alpha\end{pmatrix}+\begin{pmatrix}\gamma&\delta\\-\delta&\gamma\end{pmatrix}\right]$$

$$
\begin{aligned}
&= f\left[\begin{pmatrix} \alpha+\gamma & \beta+\delta \\ -(\beta+\delta) & \alpha+\delta \end{pmatrix}\right] \\
&= (\alpha+\gamma) + i(\beta+\delta) \\
&= (\alpha+i\beta) + (\gamma+i\delta) \\
&= f(Z) + f(W)
\end{aligned}
$$

thus addition is respected. Similarly,

$$
\begin{aligned}
f(ZW) &= f\left[\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\begin{pmatrix} \gamma & \delta \\ -\delta & \gamma \end{pmatrix}\right] \\
&= f\left[\begin{pmatrix} \alpha\gamma-\beta\delta & \alpha\delta+\beta\gamma \\ -(\alpha\gamma+\beta\delta) & \alpha\gamma+\beta\gamma \end{pmatrix}\right] \\
&= (\alpha\gamma-\beta\delta) + i(\alpha\delta+\beta\gamma) \\
&= (\alpha+i\beta)(\gamma+i\delta) \\
&= f(Z)f(W)
\end{aligned}
$$

thus multiplication is also respected. Thus, f is a field homomorphism.

We can further see that f is a bijection; it is a surjection because all $z \in \mathbb{C}$ can be written $z = \alpha+i\beta$ for $\alpha, \beta \in \mathbb{R}$, and it clearly an injection because $f(Z) = f(W) \implies f(Z)-f(W) = 0 \implies f(Z-W) = 0 \implies Z-W = 0 \implies Z = W$.

Thus, as f is a field homomorphism and a bijection, it is an isomorphism, and these matrices equiped with matrix addition and matrix multiplication is isomorphic to the complex field.

1.1.3.2

I have no idea what this means

1.1.4 Conjugation, Absolute Value

1.1.4.1

$$
\begin{aligned}
\frac{z}{z^2+1} &= \frac{x+iy}{x^2-y^2+i2xy+1} \\
&= \frac{x^3-xy^2-2xy^2+x+i(x^2y-y^3-2x^2y+y)}{x^4-6x^2y^2+y^4+x^2-y^2+1} \\
\frac{\bar{z}}{\bar{z}^2+1} &= \frac{x-iy}{x^2-y^2-i2xy+1} \\
&= \frac{x^3-xy^2-2xy^2+x-i(x^2y-y^3-2x^2y+y)}{x^4-6x^2y^2+y^4+x^2-y^2+1} \\
&= \overline{\left(\frac{z}{z^2+1}\right)}
\end{aligned}
$$

1.1.4.2

Splitting into terms,

(a) $2 \cdot \sqrt{10} \cdot \sqrt{20}\sqrt{2} = 40$

(b) $5 \cdot \sqrt{5}/\sqrt{2}\sqrt{10} = \frac{5}{2}$

1.1.4.3

$$\begin{aligned}\left|\frac{a-b}{1-\bar{a}b}\right| &= \frac{a-b}{1-\bar{a}b}\frac{\bar{a}-\bar{b}}{1-a\bar{b}} \\ &= \frac{\bar{a}a+\bar{b}b-\bar{a}b-a\bar{b}}{1-\bar{a}b-a\bar{b}+\bar{a}a\bar{b}b} \\ &= \frac{|a|^2+|b|^2-2\operatorname{Re}\bar{a}b}{1+|a|^2|b|^2-2\operatorname{Re}\bar{a}b}\end{aligned}$$

We see that if either $|a| = 1$ or $|b| = 1$ that the numerator equals the denominator and the fraction cancels. In the case where $|a| = |b| = 1$, the expression still holds so long as $\operatorname{Re}\bar{a}b \neq 1$, that is $a \neq b$.

1.1.4.4

Make the substitution

$$\alpha = a+b \qquad \beta = a-b \qquad \implies \qquad a = \frac{\alpha+\beta}{2} \qquad b = \frac{\alpha-\beta}{2}$$

so

$$\begin{aligned}\alpha\operatorname{Re}z + i\beta\operatorname{Im}z &= -c \\ \bar{\alpha}\operatorname{Re}z - i\bar{\beta}\operatorname{Im}z &= -\bar{c}\end{aligned}$$

Adding,

$$\operatorname{Re}\alpha\operatorname{Re}z - \operatorname{Im}\beta\operatorname{Im}z = -\operatorname{Re}c$$

and subtracting,

$$\operatorname{Im}\alpha\operatorname{Re}z + \operatorname{Re}\beta\operatorname{Im}z = -\operatorname{Im}c$$

Cases: if α is real,

$$\operatorname{Im}z = -\frac{\operatorname{Im}c}{\operatorname{Re}\beta}$$

if α is imaginary,

$$\operatorname{Im}z = -\frac{\operatorname{Re}c}{\operatorname{Im}\beta}$$

if β is real,

$$\operatorname{Re} z = -\frac{\operatorname{Re} c}{\operatorname{Re} \alpha}$$

if β is imaginary,

$$\operatorname{Re} z = -\frac{\operatorname{Im} c}{\operatorname{Im} \alpha}$$

Note that there is no solution if one of α, β is real and the other is purly imaginary. Further, if either α or β is zero, we have either infinitely many solutions, characterized by a line, or no solutions.

Finally, consider the case where α, β are nonzero and have both imaginary and complex components. Solving,

$$\operatorname{Re} z = -\frac{\operatorname{Re} \beta \operatorname{Re} c + \operatorname{Im} \beta \operatorname{Im} c}{\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta}$$
$$\operatorname{Im} z = -\frac{\operatorname{Re} \alpha \operatorname{Im} c - \operatorname{Im} \alpha \operatorname{Re} c}{\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta}$$

We see there is a unique solution so long as $\operatorname{Re} \alpha \operatorname{Re} \beta + \operatorname{Im} \alpha \operatorname{Im} \beta \neq 0$, or if α, β are both real or both imaginary.

1.1.4.5

Trivially, Lagrange's identity holds for $n = 1$:

$$|a_1 b_1| = |a_1| \, |b_1| + 0$$

First, note that through multiplying conjugates, we obtain

$$\sum_{i=1}^{n+1} \left| a_i \bar{b}_{n+1} - a_{n+1} \bar{b}_i \right|^2 = |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 + |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 - 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n+1} a_i b_i \right]$$

Thus, if Lagrange's identity holds for some n,

$$\begin{aligned}
\left| \sum_{i=1}^{n+1} a_i b_i \right|^2 &= \left| \sum_{i=1}^{n} a_i b_i + a_{n+1} b_{n+1} \right|^2 \\
&= \left| \sum_{i=1}^{n} a_i b_i \right|^2 + |a_{n+1}|^2 \, |b_{n+1}|^2 + 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n} a_i b_i \right] \\
&= \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leq i < j \leq n} \left| a_i \bar{b}_j - a_j \bar{b}_i \right|^2 + |a_{n+1}|^2 \, |b_{n+1}|^2 \\
&\quad + 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n} a_i b_i \right] + |a_{n+1}|^2 \sum_{i=1}^{n} |b_i| + |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 \\
&\quad - |a_{n+1}|^2 \sum_{i=1}^{n} |b_i| - |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 - |a_{n+1}|^2 \, |b_{n+1}|^2 + |a_{n+1}|^2 \, |b_{n+1}|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 + |a_{n+1}|^2 \sum_{i=1}^{n} |b_i|^2 + |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 \\
&\quad - \sum_{1 \le i < j \le n} \left| a_i \bar{b}_j - a_j \bar{b}_i \right|^2 \\
&\quad - |a_{n+1}|^2 |b_{b+1}|^2 - |a_{n+1}|^2 \sum_{i=1}^{n} |b_i|^2 - |b_{n+1}|^2 \sum_{i=1}^{n+1} |a_i|^2 \\
&\quad + 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n} a_i b_i \right] + \underbrace{2 |a_{n+1}|^2 |b_{n+1}|^2}_{= 2 \operatorname{Re}\left[\bar{a}_{n+1} \bar{b}_{n+1} a_{n+1} b_{n+1} \right]} \\
&= \sum_{i=1}^{n+1} |a_i|^2 \sum_{i=1}^{n+1} |b_i|^2 - \sum_{1 \le i < j \le n} \left| a_i \bar{b}_j - a_j \bar{b}_i \right|^2 \\
&\quad - |a_{n+1}|^2 \sum_{i=1}^{n+1} |b_i|^2 - |b_{n+1}|^2 \sum_{i=1}^{n+1} |b_i|^2 - 2 \operatorname{Re} \left[\bar{a}_{n+1} \bar{b}_{n+1} \sum_{i=1}^{n+1} a_i b_i \right] \\
&= \sum_{i=1}^{n+1} |a_i|^2 \sum_{i=1}^{n+1} |b_i|^2 - \sum_{1 \le i < j \le n} \left| a_i \bar{b}_j - a_j \bar{b}_i \right|^2 - \sum_{i=1}^{n+1} \left| a_i \bar{b}_{n+1} - a_{n+1} \bar{b}_i \right|^2 \\
&= \sum_{i=1}^{n+1} |a_i|^2 \sum_{i=1}^{n+1} |b_i|^2 - \sum_{1 \le i < j \le n+1} \left| a_i \bar{b}_j - a_j \bar{b}_i \right|^2
\end{aligned}
$$

it also holds for $n+1$. Thus, Lagrange's identity holds for all $n \in \mathbb{N}$.

1.1.5 Inequalities

1.1.5.1

From 1.1.4.3, we can write

$$
\begin{aligned}
\left| \frac{a-b}{1-\bar{a}b} \right| &= \frac{|a|^2 + |b|^2 - 2 \operatorname{Re} \bar{a}b}{1 + |a|^2 |b|^2 - 2 \operatorname{Re} \bar{a}b} \\
&=
\end{aligned}
$$

1.1.5.2

Cauchy's inequality holds as an equality for $n = 1$:

$$
|a_1 b_1|^2 = |a_1|^2 |b_1|^2
$$

Suppose Cauchy's inequality holds for n. Then,

$$
\left| \sum_{i=1}^{n+1} a_i b_i \right|^2 \le \left| \sum_{i=1}^{n} a_i b_i \right|^2 + |a_{n+1}|^2 |b_{n+1}|^2
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 + |a_{n+1}|^2 |b_{n+1}|^2 \\
&\leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 + |a_{n+1}|^2 |b_{n+1}|^2 + |a_{n+1}| \sum_{i=1}^{n} |b_i|^n + |b_{n+1}|^2 \sum_{i=1}^{n} |a_i|^2 \\
&= \sum_{i=1}^{n+1} |a_i|^2 \sum_{i=1}^{n+1} |b_i|^2
\end{aligned}
$$

so it also holds for $n+1$. Thus, Cauchy's inequality holds for all $n \in \mathbb{N}$.

1.1.5.3

$$
\begin{aligned}
\left| \sum_{i=1}^{n} \lambda_i a_i \right| &\leq \sum_{i=1}^{n} |\lambda_i a_i| \\
&= \sum_{i=1}^{n} \lambda_i |a_i| \\
&\leq \sum_{i=1}^{n} \lambda_i \\
&= 1
\end{aligned}
$$

1.1.5.4

By the the parallelogram rule and triangle inequalities,

$$4|c|^2 = |z-a|^2 + |z+a|^2 + 2\left|z^2 - a^2\right| = 2|z|^2 + 2|a|^2 + 2\left|z^2 - a^2\right| \geq 4|a|^2$$

or

$$|c| \geq |a|$$

Thus, there are only solutions for $|a| \leq |c|$. There are two solutions given by the pair of equations

$$|z-a| = |c| \qquad\qquad |z+a| = |c|$$

which yield

$$z = \pm i \frac{a}{|a|} \sqrt{|c|^2 - |a|^2}$$

thus, for $|a| \leq |c|$ solutions exist. $|z|$ is bounded above by $|c|$ like $|a|$, but can go to zero.

The actual bounds seem too hard to think about right now.

1.2 The Geometric Representation of Complex Numbers

1.2.1 Geometric Addition and Multiplication

1.2.1.1

1.2.1.2

1.2.1.3

1.2.1.4

1.2.2 The Binomial Equation

1.2.2.1

From de Moivre's,

$$\cos 3\phi = \cos^3\phi - 3\cos\phi\sin^2\phi$$

$$\cos 4\phi = \cos^4\phi - 6\cos^2\phi\sin^2\phi + \sin^4\phi$$

$$\sin 5\phi = \sin^5\phi - 10\sin^3\phi\cos^2\phi + 5\sin\phi\cos^4\phi$$

1.2.2.2

Note we can add the two terms,

$$\Sigma = 1 + \cos\varphi + i\sin\varphi + \cdots = 1 + e^{i\varphi} + e^{2i\varphi} + \cdots + e^{ni\varphi}$$

yielding

$$\Sigma = \frac{1 - e^{in\varphi}}{1 - e^{i\phi}} = e^{i\frac{n-1}{2}\phi}\frac{e^{i\frac{n}{2}\phi} - e^{-i\frac{n}{2}\phi}}{e^{-i\phi/2} - e^{-i\phi/2}} = e^{i\frac{n-1}{2}\phi}\frac{\sin\frac{n\phi}{2}}{\sin\frac{\phi}{2}}$$

The cosine terms are the real part of this,

$$\sum_{m=1}^{n}\cos m\phi = \operatorname{Re}\Sigma = \cos\frac{n-1}{2}\phi\frac{\sin\frac{n\phi}{2}}{\sin\frac{\phi}{2}}$$

and the sine terms are the imaginary part of this,

$$\sum_{m=1}^{n}\sin m\phi = \operatorname{Im}\Sigma = \sin\frac{n-1}{2}\phi\frac{\sin\frac{n\phi}{2}}{\sin\frac{\phi}{2}}$$

1.2.2.3

1.2.2.4

$$1+\omega^h+\cdots+\omega^{(n-1)h}=\frac{1-\omega^{nh}}{1-\omega^h}=\frac{1-1^h}{1-\omega^h}=0$$

if h is not divisible by n.

1.2.2.5

$$1-\omega^h+\cdots+(-1)^{n-1}\omega^{(n-1)h}=\frac{1-(-1)^n\omega^{nh}}{1+\omega^h}=\frac{1-(-1)^n1^h}{1+\omega^h}=\frac{1-(-1)^n}{1+\omega^h}$$

1.2.3 Analytic Geometry

1.2.3.1

When $a=\pm b$. See 1.1.4.4.

1.2.3.2

1.2.3.3

1.2.3.4

Fix the circle to be at the origin, and set the chords to be between pairs $z,\bar{z}$. The midpoints lie on the line $\mathrm{Im}(z)=0$, which is a diameter, and perpendicular to the chords.

1.2.3.5

1.2.4 The Spherical Representation

1.2.4.1

Let

$$z=\frac{x_1+ix_2}{1-x_3} \qquad z'=\frac{-x_1-ix_2}{1-x_3}$$

then,

$$z\bar{z}'=\frac{-x_1^2-x_2^2}{1-x_3^2}=\frac{-x_1^2-x_2^2}{x_1^2+x_2^2}=-1$$

1.2.4.2

1.2.4.3

1.2.4.4

1.2.4.5

Chapter 2

Complex Functions

2.1 Introduction to the Concept of Analytic Function

2.1.1 Limits and Continuity

2.1.2 Analytic Functions

2.1.2.1

2.1.2.2

2.1.2.3

2.1.2.4 do this

Suppose $f(z)$ is analytic and WLOG that $|f(z)| = 1\,\forall z \in \mathbb{C}$. Write

$$f(z) = u(x,y) + v(x,y) \qquad\qquad z = x + iy$$

Then, we have

$$u^2 + v^2 = 1$$
$$\nabla^2 u = \nabla^2 v = 0$$

and the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Taking the derivatives of $|f(z)|^2 = 1$, we find

$$\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 = 0$$

$$\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2} + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2 = 0$$

Adding together and using the harmonicity of u and v to simplify,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = 0$$

Usign the Cauchy-Riemann equations, we have

$$\left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right) \qquad \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2$$

so we can simplify to find that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 0$$

so

$$|f'(z)|^2 = 0 \implies f'(z) = 0$$

or $f(z)$ must be a constant.

2.1.2.5 do this

Suppose $f(z)$ is analytic. It has a derivative characterized by the existence of

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

which satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

we can thus write

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Let

$$\bar{f}(\bar{z}) = u - iv = u + iv'(x, y')$$

where $v' = -v$ and $y' = -y$. To show that $\bar{f}(\bar{z})$ is analytic, we need only show that u and $v'(x, y')$ are harmonic conjugates. First, we show that v' is harmonic:

$$\nabla^2 v' = \frac{\partial^2 v'}{\partial x^2} + \frac{\partial}{\partial y}\frac{\partial v'}{\partial y} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial y}\frac{\partial v}{\partial y} = \nabla^2 v = 0$$

now we must show that v' is conjugate to u; that is it satisfies the Cauchy-Riemann equations. We saw that we can write

$$\frac{\partial v'}{\partial x} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$
$$\frac{\partial v'}{\partial y} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

thus $f(z)$ is analytic iff $\bar{f}(\bar{z})$ is analytic.

2.1.2.6

2.1.2.7 optional

Suppose $u(x,y)$ is harmonic; that is $\partial_x^2 u + \partial_y^2 u = 0$. We write the "change of coordinates"

$$2x = z + \bar{z} \qquad 2iy = z - \bar{z}$$

so

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{i}{2}\frac{\partial u}{\partial y}$$

and similarly,

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y}$$

we then have

$$\begin{aligned}\frac{\partial}{\partial z}\frac{\partial u}{\partial \bar{z}} &= \frac{\partial}{\partial z}\frac{1}{2}\frac{\partial u}{\partial x} + \frac{\partial}{\partial z}\frac{i}{2}\frac{\partial u}{\partial y} \\ &= \frac{1}{4}\frac{\partial^2 u}{\partial x^2} - \frac{i}{2}\frac{\partial^2 u}{\partial y\,\partial x} + \frac{i}{2}\frac{\partial^2 u}{\partial x\,\partial y} + \frac{1}{4}\frac{\partial^2 u}{\partial y^2} \\ &= 0\end{aligned}$$

because mixed partials are probably symmetric for harmonic functions.

2.1.3 Polynomials

2.1.4 Rational Functions

2.1.4.1

2.1.4.2

2.1.4.3

2.1.4.4

2.1.4.5

2.1.4.6

2.2 Elementary Theory of Power Series

2.2.1 Sequences

2.2.2 Series

2.2.3 Uniform Convergence

2.2.3.1 do this

A sequence $\{a_n\}$ converges to a iff $\forall \varepsilon > 0 \, \exists N$ s.t. $|a - a_n| < \varepsilon \, \forall n \geq N$. Fix N such that it satisfies $\varepsilon = 1$. The sequence $\{a_n\}$ is then bounded by

$$r = \max(1, |a - a_1|, \ldots, |a - a_N|)$$

2.2.3.2 optional

2.2.3.3

2.2.3.4

2.2.3.5

2.2.3.6 do this

Let

$$U_n = \sum_{i=1}^{n} u_i \qquad V_n = \sum_{i=1}^{n} v_i$$

and WLOG let U converge absolutely. Let

$$P_n = U_n V_n = \sum_{i=1}^{n} p_i$$

where

$$p_n = \sum_{i=1}^{n-1} u_i v_{n-i}$$

we thus want to show that P_n converges to some P.

Let $\delta_n = V_n - V$. Then, we can rewrite

$$\begin{aligned} P_n &= u_1 v_1 + (u_1 v_2 + u_2 v_1) + (u_1 v_3 + u_2 v_2 + u_3 v_1) + \cdots \\ &= u_1 V_n + u_2 V_{n-1} + \cdots + u_n V_1 \\ &= u_1(V + \delta_n) + \cdots + u_n(V + \delta_1) \\ &= U_n V + u_1 \delta_n + u_2 \delta_{n-1} + \cdots u_n \delta_1 \end{aligned}$$

Let $R = u_1\delta_n + \cdots + u_n\delta_1$. Because V converges, $\delta_n \to 0$. Choose N such that $\delta_n < \epsilon$ for $n \geq N$. Then,

$$|R| = |u_n \delta_1 + \cdots + u_N|$$

2.2.4 Power Series

2.2.4.1

2.2.4.2

2.2.4.3 do this

2.2.4.4 do this

2.2.4.5 optional

2.2.4.6

2.2.4.7

2.2.4.8

2.2.4.9

2.2.5 Abel's Limit Theorem

2.3 The Exponential and Trigonometric Functions

2.3.1 The Exponential

2.3.2 The Trigonometric Functions

2.3.2.1

2.3.2.2

2.3.2.3

2.3.2.4

2.3.3 The Periodicity

2.3.4 The Logarithm

2.3.4.1

2.3.4.2 optional

2.3.4.3

2.3.4.4

2.3.4.5