Math Methods You'll Actually Use

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Introduction

While physics students are often required to take Math Methods or upper level mathematics coursework, these often focus moreso on the abstract nature and usefulness of advanced math concepts rather than tricks to make computation easier or circumvent their use altogether. This text is a collection of some typical math methods, as well as some particular tricks that I've found useful that weren't emphasized as much in the curriculum I took at CMU, as well as some topics that I find interesting and may be useful.

Note, that while this book has some semblance of rigorous-ness, it is for, first and foremost, *physicists*, and not mathematicians. Most objects we deal with in physics are nice and well-behaved (except the Dirac Delta, of course), and so, many of the caveats that you'd have to deal with in a typical analysis course will be glossed over or completely ignored.

This book assumes familiarity with the concepts of multivariate calculus and high-school algebra. It will begin with showing the usefulness of complex numbers and their associated properties, then continue on with the introduction of linear algebra techniques. Finally, it will conclude with a variety of miscellaneous topics, such as orthonormal coordinates and differential equations.

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Part I

Math on the Complex Plane

Chapter 1

Complex Numbers

1.1 The Imaginary Unit

As you may recall, there is no real solution to the polynomial

$$x^2 + 1 = 0$$

Rather, we define a new number, the imaginary unit i, such that

$$i^2 \equiv -1$$

This is a number just like any real number, but doesn't add directly to real numbers. Instead, when added to a real number, it forms a *complex number*. The set of all complex numbers is denoted $\mathbb{C}$. Examples of complex numbers are: $1 + 2i, 4, 3i, \pi + i\sqrt{e}$.

In general arithmetic with complex numbers is very similar to arithmetic of algebraic variables, albeit with one caveat: we replace any occurences of i^2 with -1. For instance, let $w, z \in \mathbb{C}$ be complex numbers. We may write, for real coefficients $a, b, c, d \in \mathbb{R}$ that $z = a + ib, w = c + id$ for suitable coefficients. We can then perform addition as normal:

$$z + w = a + ib + c + id = (a + c) + i(b + d)$$

Similarly, we can multiply using FOIL:

$$zw = (a + ib)(c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(ad + bc)$$

1.1.1 The Complex Plane

A complex number can be separated into its real and imaginary components as follows

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z) \tag{1.1}$$

where $\mathrm{Re}(z)$ is called the real part (as it is not attached to the imaginary unit) and $\mathrm{Im}(z)$ is called the imaginary part (as it is attached to the imaginary unit). For example, if we consider $z = a + ib$, we have the real part of z to be $\mathrm{Re}(z) = a$ and the imaginary part to be $\mathrm{Im}(z) = b$. Note that while the imaginary part is called the imaginary part, it is a *real* number.

We can plot the complex number on the 2D plane by using the mapping

$$a + ib \mapsto (a, b)$$

That is, we treat the real component as the x position and the imaginary component as the y position.

This visualization lends itself nicely to an interpretation of the size of a complex number. Just as the size of a real number is given by how far it is from 0, the size of a complex number is given by how far it is from $0 + i0 = 0$. We can do this of course using pythagorean theorem:

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \tag{1.2}$$

The magnitude of the complex number $z = a + ib$ would then be $|z| = \sqrt{a^2 + b^2}$, for instance

1.2 Euler's Formula

Euler's formula is by far the most useful result of complex numbers in computation. It states that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.3}$$

We can show that this is true by comparing Taylor Series expansions of both sides. Recall:

$$\begin{aligned}
\exp(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \dots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots
\end{aligned}$$

Computing the LHS of Equation 1.3,

$$\exp(ix) = 1 + ix - \frac{1}{2}x^2 - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 + \frac{i}{5!}x^5 + \dots$$

Similarly, the terms on the RHS can be computed as

$$\begin{aligned}
\cos(x) &= 1 \quad && - \frac{1}{2}x^2 && && + \frac{1}{4!}x^4 && && + \dots \\
i\sin(x) &= && ix && - \frac{i}{3!}x^3 && && + \frac{i}{5!}x^5 && + \dots
\end{aligned}$$

As you should be able to see, the terms in the Taylor expansions of both sides matches. This equation will be very useful in deriving trigonometric identities, as will be seen in the next chapter, but truly shows its power in quantum mechanics.

Note that for any theta, we then have

$$\left|e^{i\theta}\right| = \cos^2\theta + \sin^2\theta = 1$$

that is, $e^{i\theta}$ has unit magnitude. We can thus consider $e^{i\theta}$ as a unit vector.

1.2.1 Polar Complex Numbers

Euler's formula lends itself nicely to an additional way to denote complex numbers, which is the *polar form*. As mentioned before, we can graphically represent the complex numbers on the complex plane. However, just like a cartesian plane, we can also represent each point with a different set of variables, in *polar coordinates*. In this set of coordinates, we transform by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

How can we rewrite complex numbers in polar form? Consider the number

$$z = re^{i\theta}$$

Expanding the complex exponential using Euler's formula, we see that we can rewrite it as

$$z = r\cos\theta + ir\sin\theta$$

thus, we see that

$$\mathrm{Re}(z) = r\cos\theta \tag{1.4a}$$
$$\mathrm{Im}(z) = r\sin\theta \tag{1.4b}$$

Recalling how we displayed the complex numbers on the plane, we see that $z = re^{i\theta}$ is *exactly* just z in terms of the polar coordinates r, θ. This transformation is given

$$r = |z| \tag{1.5a}$$
$$\theta = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) \tag{1.5b}$$

Chapter 2

Trigonometric Functions

Recall the elementary trigonometric functions, sine and cosine.

2.1 Trig Identities

2.2 Trig Substitutions

2.3 Hyperbolic Trig Functions

Chapter 3

Complex Functions

Chapter 4

Integration Tricks

These are integration tricks beyond those that have been discussed before.

4.1 Gamma Function and Gaussian Integrals

4.2 Feynman's Integration Trick

Problems: 1. Use Feynman's Integration trick to prove that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

4.3 Contour Integration

Part II

Linear Algebra

Chapter 5

Vectors and Matricies

5.1 Vector Spaces

5.2 Matrices

5.3 Inner products

5.3.1 Gram-Schmidt

5.4 Linear Algebra

5.5 Linearity

5.6 Representations

5.7 Abstract Vector Spaces

Chapter 6

Inner Product Spaces

6.1 Inner Products

generalization of scalar dot product

6.2 ON Bases

6.2.1 Gram-Schmidt

Polynonmial wrt kernels

6.2.2 Spectral Decomposition

Chapter 7

Operator Spaces

7.1 Hilbert Spaces

7.2 Dirac Notation

7.3 Commutation

7.4 Spectral Theory

7.4.1 Operator Functions

7.4.2 Det, Tr

Part III

Advanced Topics

Chapter 8

Multilinearity

8.1 Bilinearity

8.1.1 Symmetric

Examples

8.1.2 Alternating

Examples

8.2 Multilinear Forms

8.2.1 Symmetric Forms

8.2.2 Alternating Forms

Determinant Revisited

8.3 Tensors

8.3.1 Einstein Notation

Chapter 9

Special Functions

Chapter 10

Orthonormal Coordinate Systems

10.1 Scale Factors

10.2 Cylindrical Coordinates

10.3 Spherical Coordinates

Chapter 11

Useful Series

11.1 Taylor Series

11.1.1 Buckingham Pi

11.2 Orthogonal Polynomials

11.2.1 Legendre

Spherical Harmonics

11.2.2 Hermite

11.3 Fourier Series

11.3.1 Discrete

11.3.2 Continuous

Chapter 12

Differential Equations

12.1 Differential Operator

12.2 Complex Extensions

$$\cos \omega t = \mathrm{Re}(e^{i\omega t})$$

12.3 Series Solutions

Linear operator -> superposition principle

12.4 Laplace Transforms

12.5 Green's Functions

Part IV

Miscellaneous Topics

Chapter 13

Simplifying Algebra

13.1 Definition and Substitution

Never carry around bulky expressions; chances are you'll mess up copying it down at some point and end up with a fraction that doesn't reduce or a term that doesn't cancel. Rather, you should always define a placeholder variable and carry that around instead.

13.2 The Power of a Unit

13.2.1 Strategically Multiplying by One

13.2.2 Strategically Adding Zero

A similar technique can be accomplished in addition;

Lagrange Multipliers

One well known example of strategically adding zero is of course the method of *Lagrange multipliers* in the optimization of a function subject to a constraint.

13.2.3 Strategically using the Identity

We can extend this idea of strategically multiplying by one to strategically multiplying by the identity.

13.2.4 Strategically Differentiating WRT One

Perhaps one of the most egregious abuse of a unit is to differentiate with respect to the number one. As we can arbitrarily insert a 1 anywhere, we can insert a variable $\xi \equiv 1$, take the derivative ∂_ξ, and evaluate it at $\xi = 1$. In fact, this can be used to derive the Gibbs-Duhem relation in thermodynamics. For example, say I wish to calculate

$$\int_0^\infty x^2 e^{-x^2}\,\mathrm{d}x$$

Obviously this is a gamma function in disguise (just substitute $u = x^2$), but say we don't know that.

13.3 Superposition Principle

Remember that the superposition principle works with both addition *and subtraction*. Sometimes it is easier to consider the lack of an object rather than compute a wider sum or the whole. For example, consider a disc radius r and mass density ρ, with a hole drilled out of it at $r/2$ with radius $r/2$. If we want to calculate the centre of mass, we could

$$r_{cm} =$$

however, we could instead view it as the sum of the larger disc and a smaller "anti-disc." This turns the calculation from a tedious integral to a simple weighted sum. We can compute the inertia tensor in the same way. [excersize: do this]

Moment of Inertia and CoM of disk with missing disk

Chapter 14

Differentials