Math Methods You'll Actually Use

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Introduction

While physics students are often required to take Math Methods or upper level mathematics coursework, these often focus moreso on the abstract nature and usefulness of advanced math concepts rather than tricks to make computation easier or circumvent their use altogether. This text is a collection of some typical math methods, as well as some particular tricks that I've found useful that weren't emphasized as much in the curriculum I took at CMU, as well as some topics that I find interesting and may be useful.

Note, that while this book has some semblance of rigorous-ness, it is for, first and foremost, *physicists*, and not mathematicians. Most objects we deal with in physics are nice and well-behaved (except the Dirac Delta, of course), and so, many of the caveats that you'd have to deal with in a typical analysis course will be glossed over or completely ignored.

This book assumes familiarity with the concepts of multivariate calculus and high-school algebra. It will begin with showing the usefulness of complex numbers and their associated properties, then continue on with the introduction of linear algebra techniques. Finally, it will conclude with a variety of miscellaneous topics, such as orthonormal coordinates and differential equations.

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Part I

Math on the Complex Plane

Chapter 1

Complex Numbers

1.1 The Imaginary Unit

As you may recall, there is no real solution to the polynomial

$$x^2 + 1 = 0$$

Rather, we define a new number, the imaginary unit i, such that

$$i^2 \equiv -1$$

This is a number just like any real number, but doesn't add directly to real numbers. Instead, when added to a real number, it forms a *complex number*. The set of all complex numbers is denoted $\mathbb{C}$.

$$\mathbb{C} \equiv \{z | z = a + ib \text{ where } a, b \in \mathbb{R}\} \tag{1.1}$$

Examples of complex numbers are: $1 + 2i, 4, 3i, \pi + i\sqrt{e}$.

In general, arithmetic with complex numbers is very similar to the arithmetic of algebraic variables, albeit with one caveat: we replace any occurences of i^2 with -1. For instance, let $w, z \in \mathbb{C}$ be complex numbers. We may write, for real coefficients $a, b, c, d \in \mathbb{R}$ that $z = a+ib, w = c+id$ for suitable coefficients. We can then perform addition (or subtraction) as normal:

$$z \pm w = (a + ib) \pm (c + id) = (a \pm c) + i(b \pm d) \tag{1.2}$$

Similarly, we can multiply using FOIL:

$$zw = (a + ib)(c + id) = ac + ibc + iad + i^2bd = (ac - bd) + i(ad + bc) \tag{1.3}$$

Following from the difference of two perfect squares

$$(a + b)(a - b) = a^2 - b^2$$

we find that

$$(a + ib)(a - ib) = a^2 + b^2$$

The quantity

$$z^* = a - ib \tag{1.4}$$

is the *complex conjugate* of $z = a + ib$. In standard math texts, you will also see the notation $\bar{z} = a - ib$. Examples of complex conjugates are $4 + 3i$ and $4 - 3i$, or $-2 - 3i$ and $-2 + 3i$.

The complex conjugate is useful in that it allows us to now divide. Consider

$$\frac{z}{w} = \frac{a + ib}{c + id}$$

it is not immediately obvious how we can simplify this. However strategically multiplying by one[1] will make this much simpler:

$$\begin{aligned}\frac{a + ib}{c + id} &= \frac{a + ib}{c + id} \cdot \frac{c - id}{c - id} \\ &= \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i \end{aligned} \tag{1.5}$$

Exercise 1.1.1. Let $w = 4 - 3i$, $x = -2 - 1i$, $y = -3 - i$ and $z = 1 + 5i$. Compute the following

(a) $w + z$

(b) $w + x + y$

(c) $z - w$

(d) xy

(e) wx

(f) w^*

(g) $x(y^*)$

(h) $(xy)^*$

(i) w/z

(j) $(y + 3)/(x - 4)$

(k) wx/yz

(l) $(wx)^*/yz$

Compare your results for part (g) to part (h) and for part (k) to part (l).

Exercise 1.1.2. Show that multiplication by complex numbers is commutative; that is, $zw = wz, \forall w, z \in \mathbb{C}$.

Exercise 1.1.3. Properties of the complex conjugate:

(a) Convince yourself that $(z^*)^* = z$

(b) Show that

 (i) $(w \pm z)^* = w^* \pm z^*$

 (ii) $(wz)^* = w^* z^*$

 (iii) $(w/z)^* = w^*/z^*$

 or, that the conjugate distributes over the four basic arithmetic operations

[1] See Section 15.2.1

1.2 Euler's Formula

Euler's formula is by far the most useful result of complex numbers in computation. It states that

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.6}$$

We can show that this is true by comparing Taylor Series expansions of both sides. Recall:

$$\begin{aligned}
\exp(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 \dots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \dots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots
\end{aligned}$$

Computing the LHS of Equation 1.6,

$$\exp(ix) = 1 + ix - \frac{1}{2}x^2 - \frac{i}{3!}x^3 + \frac{1}{4!}x^4 + \frac{i}{5!}x^5 + \dots$$

Similarly, the terms on the RHS can be computed as

$$\begin{aligned}
\cos(x) &= 1 \quad && - \frac{1}{2}x^2 && + \frac{1}{4!}x^4 && + \dots \\
i\sin(x) &= \quad ix && - \frac{i}{3!}x^3 && + \frac{i}{5!}x^5 && + \dots
\end{aligned}$$

As you should be able to see, the terms in the Taylor expansions of both sides matches. This equation will be very useful in deriving trigonometric identities, as will be seen in the next chapter, but truly shows its power in quantum mechanics.

Exercise 1.2.1. Use the above argument to write $e^{-i\theta}$ in terms of $\sin\theta$ and $\cos\theta$. How then, are $e^{i\theta}$ and $e^{-i\theta}$ related?

Exercise 1.2.2. Use Euler's formula to write $\sin\theta$ and $\cos\theta$ in terms of the complex exponentials $e^{i\theta}$ and $e^{-i\theta}$.

1.3 The Complex Plane

A complex number can be separated into its real and imaginary components as follows

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z) \tag{1.7}$$

where $\mathrm{Re}(z)$ is called the real part (as it is not attached to the imaginary unit) and $\mathrm{Im}(z)$ is called the imaginary part (as it is attached to the imaginary unit). For example, if we consider $z = a + ib$, we have the real part of z to be $\mathrm{Re}(z) = a$ and the imaginary part to be $\mathrm{Im}(z) = b$. Note that while the imaginary part is called the imaginary part, it is a *real* number.

Exercise 1.3.1. Write $\mathrm{Re}(z), \mathrm{Im}(z)$ in terms of z, z^*. Compare this to your expressions for $\sin\theta, \cos\theta$ in terms of complex exponentials.

We can plot the complex number on the 2D plane by using the mapping

$$a + ib \mapsto (a, b)$$

That is, we treat the real component as the x position and the imaginary component as the y position.

[someone make a nice TikZ'd/inkscaped diagram for this ps tysm]

This visualization lends itself nicely to an interpretation of the size of a complex number. Just as the size of a real number is given by how far it is from 0, the size of a complex number is given by how far it is from $0 + i0 = 0$. We can do this of course using the Pythagorean theorem:

$$|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} \tag{1.8}$$

The magnitude of the complex number $z = a+ib$ would then be $|z| = \sqrt{a^2 + b^2}$, for instance. An important property of the complex magnitude is that it is positive semidefinite; that is, it obeys

$$|z| \geq 0, \forall z \in \mathbb{C} \tag{1.9}$$

with equality if and only if[2].

$$|z| = 0 \iff z = 0 \tag{1.10}$$

This makes sense, as we do not expect a number to have a "negative size."

Exercise 1.3.2. Some properties of the complex conjugate

(a) Show that if $z^* = z$ then $z \in \mathbb{R}$; that is, z is purely real.

(b) Show that if $z^* = -z$, then z can be written in the form ia where $a \in \mathbb{R}$. That is, z is purely imaginary.

(c) Using (a), show that $z^* z$ is real.

(d) Show that we can write $|z| = \sqrt{z^* z}$.

Exercise 1.3.3. Show that $\left|e^{i\theta}\right| = 1, \forall \theta \in \mathbb{R}$.

Exercise 1.3.4. Properties of the magnitude of a complex number

(a) Show that $|zw| = |z|\,|w|$.

(b) Show that $|z + w| \leq |z| + |w|$. This is known as the *triangle inequality.*

[2] If and only if will be henceforth abbreviated *iff*

1.3.1 Polar Form and Complex Exponents

Euler's formula lends itself nicely to an additional way to denote complex numbers, which is the *polar form*. As mentioned before, we can graphically represent the complex numbers on the complex plane. However, just like a cartesian plane, we can also represent each point with a different set of variables, in *polar coordinates*. In this set of coordinates, we transform by

$$x = r\cos\theta$$
$$y = r\sin\theta$$

How can we rewrite complex numbers in polar form? Consider the number

$$z = re^{i\theta}$$

Expanding the complex exponential using Euler's formula, we see that we can rewrite it as

$$z = r\cos\theta + ir\sin\theta$$

thus, we see that

$$\mathrm{Re}(z) = r\cos\theta \tag{1.11a}$$
$$\mathrm{Im}(z) = r\sin\theta \tag{1.11b}$$

[another diagram pls]

Recalling how we displayed the complex numbers on the plane, we see that $z = re^{i\theta}$ is *exactly* just z in terms of the polar coordinates r, θ. This transformation is given

$$r = |z| \tag{1.12a}$$
$$\theta \equiv \arg(z) = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) \tag{1.12b}$$

where the angle θ is called the *argument* of the complex number. By definition, we require that $r \geq 0$, and $0 \leq \theta < 2\pi$.

The polar form of a complex number allows us now to easily multiply and exponentiate complex numbers. This is because the exponential, e^z behaves very similarly when applied to complex numbers as to real numbers[3]. Namely, it obeys the rules we expect it to:

$$e^z e^w = e^{z+w} \tag{1.13a}$$

$$(e^z)^w = (e^w)^z = e^{wz} \tag{1.13b}$$

We can then see that if we have two complex numbers $z = re^{i\theta}$ and $w = se^{i\phi}$,

$$wz = sre^{i(\theta+\phi)} \tag{1.14}$$

[3]Why is beyond the scope of this text.

that is, we multiply the magnitudes and add the angle.

If we take the exponential of an arbitrary complex number $z = a + ib$, we see

$$e^z = e^a e^{ib}$$

so

$$|e^z| = e^{\operatorname{Re} z} \tag{1.15a}$$

$$\arg e^z = \operatorname{Im} z \tag{1.15b}$$

Exercise 1.3.5. Compute the following.

(a) Write $1 + i\sqrt{3}$ in polar form.

(b) Write $1 - i\sqrt{3}$ in polar form.

(c) Evaluate $(\sqrt{2} - i\sqrt{2})^4$.

(d) Evaluate $(1 + i\sqrt{3})^3$

(e) Evaluate $(1 - i\sqrt{3})^{2+i\pi}$.

(f) Evaluate i^i. Are you surprised?

1.3.2 Roots of Unity

Note that just like the equation $x^2 = 1$ has two solutions, $x = \pm 1$, the equation $z^2 = -1$ also has two solutions, $z = \pm i$. In fact, by the Fundamental Theorem of Algebra, an n degree polynomial has n (not necessarily disticty) roots. Thus, there are n solutions to the equation $z^n = 1$. The n solutions for z are known as the nth roots of unity.

Why are there so many solutions for $z^n = 1$? Let us return to the polar form of a complex number. Recall that when we multiply (or indeed exponentiate) complex numbers, we multiply the magnitudes and add the arguments. Rewriting,

$$z^n = r^n e^{in\theta} = 1$$

taking the magnitude, we find that

$$|r^n| = |r|^n = 1$$

since $0 < r \in \mathbb{R}$, $r = 1$. Thus, our problem reduces to

$$e^{in\theta} = 1 = e^{i0}$$

However, since $e^{i\theta}$ is periodic with period 2π (think back to Euler's formula), we see that this equation is valid for all $n\theta = 2\pi m$, or

$$\theta = \frac{2\pi}{n} m, \qquad m, n \in \mathbb{N} \text{ and } m < n$$

Exercise 1.3.6. Find the 4th roots of unity

Exercise 1.3.7. Find the 6th roots of unity

Solutions to Chapter 1

Solution 1.1.1 (a) I'm too lazy to compute these rn

Note that they do not equal.

Solution 1.1.2 Too lazy, but follows from commutation in reals.

Solution 1.1.3 (a) Let $z = a + ib$. Then, $z^* = a - ib$ and $(z^*)^* = a - (-ib) = a + ib = z$.

(b) (i) too lazy

Solution 1.2.1

$$e^{-i\theta} = \boxed{\cos\theta - i\sin\theta}$$

Thus, they are $\boxed{\text{conjugates}}$.

Solution 1.2.2 just do it

Solution 1.3.1

$$\mathrm{Re}(z) = \frac{z + z^*}{2} \qquad \mathrm{Im}(z) = \frac{z - z^*}{2}$$

Solution 1.3.2 (a) Let $z = a + ib$ where $a, b \in \mathbb{R}$. Substituting and cancelling,

$$\begin{aligned} z &= z^* \\ a + ib &= a - ib \\ ib &= -ib \\ b &= -b \\ b &= 0 \end{aligned}$$

thus,

$$z = a \in \mathbb{R}$$

(b) We have

$$(z^*z)^* = (z^*)^*z^* = zz^* = z^*z$$

by commutivity. Thus, since z^*z is its on conjugate, it must be purely real.

Solution 1.3.3

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Solution 1.3.4 (a) too lazy

Solution 1.3.5 (a) i don't want to

Solution 1.3.6 $\boxed{1, i, -1, -i}$

Solution 1.3.7 $\boxed{1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}}$

Chapter 2

Trigonometric Functions

2.1 Trig Identities

Pretty much all useful trig identities can be derived from just Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.6}$$

and all of the familiar properties of the exponential function. Firstly, we see that we can write[1]

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta\frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{2.1}$$

As seen in problem 1.2.1, we have

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

so

$$e^{-i\theta} = \left(e^{i\theta}\right)^*$$

thus, we have that

$$\left|e^{i\theta}\right| = \sqrt{e^{i\theta}e^{-i\theta}} = \sqrt{e^0} = 1$$

and we obtain the Pythagorean identity

$$\left|e^{i\theta}\right|^2 = \cos^2\theta + \sin^2\theta = 1 \tag{2.2}$$

Additionally, using the fact that

$$e^{-i\theta} = e^{i(-\theta)}$$

we can find the parity (evenness or oddness) of sine and cosine:

$$\cos(-\theta) = \cos\theta \qquad \sin(-\theta) = -\sin\theta \tag{2.3}$$

[1] Or, in fact, *define*

By multiplying the Euler formula,

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$$

and matching coefficients, (i.e., setting real parts equal and imaginary parts equal), we can obtain the angle addition formulae

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{2.4a}$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{2.4b}$$

Similarly, we can find the angle subtraction formulae, either by multiplying $e^{i(\theta-\phi)} = e^{i\theta}e^{-i\phi}$, or by using the parity of sine and cosine.

Exercise 2.1.1. Do just that—derive the angle subtraction formulae

Combining the angle addition formulae with the pythagorean identity, we obtain the angle reduction formulae

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{2.5a}$$

$$\cos 2\theta = \frac{1}{2}\cos^2\theta - \frac{1}{2} \tag{2.5b}$$

$$= \frac{1}{2} - \frac{1}{2}\sin^2\theta \tag{2.5c}$$

Exercise 2.1.2. Verify the above are true.

2.1.1 de Moivre's Theorem

de Moivre's theorem relies on a very basic premise:

$$(e^{i\theta}) = e^{in\theta} = e^{i(n\theta)}$$

Using Euler's formula, we can expand the leftmost and rightmost terms to obtain de Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{2.6}$$

From de Moivre's theorem, we can obtain angle reduction formulae by expanding the LHS and

2.2 Hyperbolic Trig Functions

2.3 Substitutions

I legitimately cannot remember what examples to use here but the only thing i can think of are boosts

Solutions to Chapter 2

Solution 2.1.1 literally just do what I said

Solution 2.1.2 again, just do what I said.

Chapter 3

Complex Functions

After Ahlfors

Solutions to Chapter 3

Chapter 4

Integration Tricks

These are integration tricks beyond those that have been discussed before.

4.1 Gamma Function and Gaussian Integrals

4.2 Feynman's Integration Trick

Problems: 1. Use Feynman's Integration trick to prove that $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

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Part IV

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Chapter 15

Simplifying Algebra

15.1 Definition and Substitution

Never carry around bulky expressions; chances are you'll mess up copying it down at some point and end up with a fraction that doesn't reduce or a term that doesn't cancel. Rather, you should always define a placeholder variable and carry that around instead.

15.2 The Power of a Unit

15.2.1 Strategically Multiplying by One

15.2.2 Strategically Adding Zero

A similar technique can be accomplished in addition;

Lagrange Multipliers

One well known example of strategically adding zero is of course the method of *Lagrange multipliers* in the optimization of a function subject to a constraint.

15.2.3 Strategically using the Identity

We can extend this idea of strategically multiplying by one to strategically multiplying by the identity.

15.2.4 Strategically Differentiating WRT One

Perhaps one of the most egregious abuse of a unit is to differentiate with respect to the number one. As we can arbitrarily insert a 1 anywhere, we can insert a variable $\xi \equiv 1$, take the derivative ∂_ξ, and evaluate it at $\xi = 1$. In fact, this can be used to derive the Gibbs-Duhem relation in thermodynamics. For example, say I wish to calculate

$$\int_0^\infty x^2 e^{-x^2}\,\mathrm{d}x$$

Obviously this is a gamma function in disguise (just substitute $u = x^2$), but say we don't know that.

15.3 Superposition Principle

Remember that the superposition principle works with both addition and subtraction. Sometimes it is easier to consider the lack of an object rather than compute a wider sum or the whole. For example, consider a disc radius r and mass density ρ, with a hole drilled out of it at $r/2$ with radius $r/2$. If we want to calculate the centre of mass, we could

$$r_{cm} =$$

however, we could instead view it as the sum of the larger disc and a smaller "anti-disc." This turns the calculation from a tedious integral to a simple weighted sum. We can compute the inertia tensor in the same way. [excersize: do this]

Moment of Inertia and CoM of disk with missing disk

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Appendix A

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