

Exercise 1.11.3.

Suppose $1, s$ and αs are real numbers that are linearly independent over \mathbb{Q} . (1)

Show that every orbit of the time- s map ϕ_α^s is dense in \mathbb{T}^2 . (2)

Proof + reasoning:

Let's try to adapt the proof that R_α has dense semiorbits if α is irrational. (3)

The idea in (3) is to divide S^1 into disjoint ε -sized intervals, and use the pigeonhole principle to show that there exist $k > m$ such that $R_\alpha^k(x)$ and $R_\alpha^m(x)$ belong to the same interval, hence that R_α^{k-m} is a translation by less than ε . In our case, we need to use rectangles instead of intervals, and the translation is not simply defined as the addition of an irrational number. (4)

Let $x \in \mathbb{T}^2$, $y \in \mathbb{T}^2$, $\varepsilon' > 0$ and $\varepsilon = \frac{\varepsilon'}{2\sqrt{2}}$. (5)

Let \mathcal{P}_ε be a partition of \mathbb{T}^2 into finitely many squares of the form $[a, b)^2$, where $\frac{\varepsilon}{2} < |a - b| < \varepsilon$. (6)

By the pigeonhole principle, there exists a $P \in \mathcal{P}_\varepsilon$ and $k > m$ in \mathbb{Z} such that $\phi_\alpha^{ks}(x)$ and $\phi_\alpha^{ms}(x)$ are in P . (7)

By (7), $d(z, \phi_\alpha^{(k-m)s}(z)) < \sqrt{2}\varepsilon$ for all $z \in \mathbb{T}^2$, where d is the metric on \mathbb{T}^2 . (8)

Intuitively, $\phi_\alpha^{(k-m)s}$ is a translation of size less than $\sqrt{2}\varepsilon$, seemingly with an irrational slope. So, if the line from x in the direction of that translation eventually intersects an ε -ball around y , then there should exist an iterate of x that is within ε' of y . Let's make this precise. We can state the conjecture first, and prove it only after we know that it gives the required result. (9)

Conjecture 1. *There exists a $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $y \in \mathbb{T}^2$*

$$\frac{(\phi_\alpha^{(k-m)s}(y))_2 - y_2}{(\phi_\alpha^{(k-m)s}(y))_1 - y_1} = \beta.$$

Proof. Suppose for contradiction that $s = 0$. Then for $p = 1, q = 1, r = 0$ we have $p\alpha s + qs + r = 0$, a contradiction, so $s \neq 0$. Similarly, $\alpha s \neq 0$. Suppose for contradiction that $\alpha s \in \mathbb{Q}$. Let $p = 1, q = -\alpha s, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\alpha s \notin \mathbb{Q}$. Suppose for contradiction that $\frac{1}{\alpha} \in \mathbb{Q}$. Then s is irrational. Let $p = \frac{1}{\alpha}, q = -1, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\frac{1}{\alpha}$ is irrational. Let $y \in \mathbb{T}^2$. Then

$$\frac{(\phi_\alpha^{(k+m)s}(y))_2 - y_2}{(\phi_\alpha^{(k+m)s}(y))_1 - y_1} = \frac{(k-m)s}{(k-m)\alpha s} = \frac{1}{\alpha}$$

So, with $\beta = \frac{1}{\alpha}$, the statement follows. \square

Let γ be the line in \mathbb{T}^2 starting from x in the direction of $x - \phi_\alpha^{m-k}(x)$. (10)

Let β be the slope of γ , which is finite and in $\mathbb{R} \setminus \mathbb{Q}$ by 1. (11)

By (11), considering γ as a subset of \mathbb{T}^2 , we have

$$\begin{aligned} \gamma \cap (y_1 \times \mathbb{T}) &= \bigcup_{n \geq 0} \{(y_1, (x_2 + \beta(y_1 - x_1) + \beta n) \bmod 1)\} \\ &= \bigcup_{n \geq 0} \{(y_1, R_\beta^n(x_2 + \beta(y_1 - x_1)))\}. \end{aligned} \quad (12)$$

By (11), R_β has dense semiorbits. (13)

By (13) and (12), there exists a $z \in \gamma \cap (y_1 \times (y_2 - \varepsilon, y_2 + \varepsilon))$. (14)

By (9) and Conjecture 1, there exists a $p \in \mathbb{N}$ such that

$$d(\phi_\alpha^{p(k-m)s}(x), z) < \sqrt{2}\varepsilon \quad (15)$$

By (15) and (14),

$$\begin{aligned} d(\phi_\alpha^{p(k-m)s}(x), y) &\leq d(\phi_\alpha^{p(k-m)s}(x), z) + d(z, y) \\ &\leq \sqrt{2}\varepsilon + \varepsilon \\ &\leq 2\sqrt{2}\varepsilon \\ &\leq \varepsilon'. \end{aligned} \quad (16)$$

By (16), every orbit of ϕ_α^s is dense in \mathbb{T}^2 . (17)