Exercise 2.7.3.

Give a non-trivial example of a homeomorphism f of a compact metric space (X, d) such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for every pair $x, y \in X$. (1)

Proof + reasoning:

Let's recall simple examples of continuous maps of compact metric spaces, and see if they satisfy (1). (2)

The translation R_α on S^1 is not valid, as it preserves distances. (3)

The map $h : x \mapsto \frac{1}{2}x$ on S^1 has the property that $d(h^n(x), h^n(y)) \to 0$ as $n \to \infty$ for every pair $x, y \in X$, but it is not surjective. (4)

To solve this issue, we can define a map that is the piecewise combination of a contraction (such as f) on one half of the circle and an expansion on the other half of the circle. (5)

Define $f : S^1 \to S^1$ by

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases} \tag{6}$$

Clearly, f is a homeomorphism such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for all pairs x, y. (7)