Exercise 1.4.3.

Verify that the metrics on Σ_m and Σ_m^+ generate the product topology (1)

Proof (including reasoning)

Firstly, what product topology is being referred to? $\Sigma_m = A_m^{\mathbb{Z}}$, which is the space of sequences of elements in $\{1, \ldots, m\}$ indexed by $\mathbb{Z}$, which can be seen as a product $\prod_{z \in \mathbb{Z}} A_m$. (2)

The product topology is generated by products of open sets. Is there a general result that countable product topologies are generated by cylinders, instead of just by countable products of open sets? (3)

Yes, the result is that the n-dimensional cylinders form a basis for the product topology. This is exactly the topology on Σ_m and Σ_m^+. (4)

How does (4) help us? From a metric we can generate the collection of open balls around all points. A metric generates a topology when, given a basis set B and any point $x \in B$, there is an open ball containing x that is contained in B. (5)

Let $C := C_{j_1,\ldots,j_k}^{n_1,\ldots,n_k} = \{x = (x_\ell) : x_{n_i} = j_i, i = 1, \ldots, k\}$ where $n_1 < n_2 < \cdots < n_k$ are indices in $\mathbb{Z}$ or $\mathbb{N}$, and $j_i \in A_m$. (6)

Let $x := (x_i) \in C$. (7)

We want to show that C contains an open ball B such that $x \in B$. (8)

Recall that the metrics on Σ_m and Σ_m^+ are given by $d(x, x') = 2^{-l}$, where $l = \min\{|i| : x_i \neq x'_i\}$. If we pick ε small enough, we can construct $B(x, \varepsilon)$ such that all points in $B(x, \varepsilon)$ agree with x up to the 'largest' index in the definition of C. (9)

Let $m = \max\{|n_i| : i \leq k\}$,$y \in B(x, 2^{-m})$, and $l = \min\{|i| : y_i \neq x_i\}$. (10)

We have $2^{-l} = d(x, y) < 2^{-m}$. (11)

From (11), $l > m$, so $x_{n_i} = y_{n_i}$ $\forall i \leq k$, hence $y \in C$. (12)

Therefore $B(x, 2^{-m}) \subseteq C$. (13)

By (13), and the fact that the collection of sets such as $C_{j_1,\ldots,j_k}^{n_1,\ldots,n_k}$ form a basis for the product topology, the metrics generate the product topology. (14)