Exercise 1.7.3.

Show that the eigenvalues of a two-dimensional hyperbolic toral automorphism are irrational (so the stable and unstable manifolds are dense by exercise 1.11.1). (1)

Proof + reasoning:

Are there hyperbolic toral automorphisms that aren't represented by a matrix? No, not in this context. (2)

Let A be a 2×2 integer matrix such that $\det(A) = 1$ and such that for all eigenvalues λ of A, $|\lambda| \neq 1$. (3)

Let λ be an eigenvalue of A. (4)

Let's try the following: first, relate the eigenvalues to the determinant. Then, conclude from the first step and the given assumptions that the eigenvalues are irrational. (5)

Note that $\det(\lambda I - A) = 0$. (6)

Denote $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. (7)

$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$ (8)

$\lambda^2 - \lambda a_{11} - \lambda a_{22} + a_{11}a_{22} - a_{12}a_{21} = 0$ (9)

By assumption, $\det(A) = 1$, so $a_{11}a_{22} - a_{12}a_{21} = 1$ (10)

Substituting $1 = a_{11}a_{22} - a_{12}a_{21}$ in (9) gives: $\lambda^2 - \lambda a_{11} - \lambda a_{22} + 1 = 0$ (11)

$\lambda^2 - \lambda(a_{11} + a_{22}) + 1 = 0$ (12)

That was the first step. How can we conclude? (13)

What given assumption have I not used yet? (14)

I haven't used that $|\lambda| \neq 1$. From (12) it may be possible to conclude that λ is equal to 1 or irrational, which together with (15) gives that λ is irrational. (15)

We can factorize the left-hand side of (12). (16)

However, (12) could have complex solutions for certain values of $a_{11}+a_{22}$. (17)

If the discriminant of (12) is greater than 0, (12) only has real roots. (18)

I'm not sure how to proceed. Let's take a few steps back. (19)

Equation (12) is quadratic in λ, so all solutions are given by the quadratic formula. (20)

By the quadratic formula, $\lambda = \frac{1}{2}(a_{11} + a_{22} + ((a_{11} + a_{22})^2 - 4)^{1/2})$ or $\lambda = \frac{1}{2}(a_{11} + a_{22} - ((a_{11} + a_{22})^2 - 4)^{1/2})$ (21)

From (21) we see that $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ if and only if $((a_{11}+a_{22})^2 - 4)^{1/2} \in \mathbb{R} \setminus \mathbb{Q}$ (22)

By definition of A, $(a_{11} + a_{22}) \in \mathbb{N}$ (23)

Let's show that λ cannot be in $\mathbb{C}$. My guess is that if λ were in $\mathbb{C}$, its magnitude would be equal to 1. (24)

Suppose $((a_{11}+a_{22})^2-4)^{1/2} \in \mathbb{C}$. Then $|\lambda|^2 = \frac{1}{4}((a_{11}+a_{22})^2+4-(a_{11}+a_{22})^2) = 1$ (25)

(25) contradicts (3), hence $((a_{11}+a_{22})^2-4)^{1/2} \in \mathbb{R}$ (26)

From (26) and (23), $a_{11}+a_{22} \geq 3$ (27)

We just need to show that $\lambda \notin \mathbb{Q}$. My intuition is that $((a_{11}+a_{22})^2-4)^{1/2}$ is always irrational, because it seems $\forall n \in \mathbb{N}$, $n^{1/2}$ is rational only if n is a square number, and subtracting 4 makes it no longer square, given $\sqrt{n} \geq 3$. Let's prove the first part. (28)

Conjecture: $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}\} = \emptyset$ (29)

Proof of (29): Suppose $\sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}$. Then $\sqrt{n} = \frac{p}{q}$, where p and q are natural numbers. (30)

Since $\sqrt{n} \notin \mathbb{N}$, q does not divide p, so q^2 does not divide p^2 (31)

From (30), $n = \frac{p^2}{q^2}$, hence $nq^2 = p^2$ (32)

(31) contradicts (32), so (30) is false $\square_{c23}$ (33)

Now let's prove the second part, that subtracting 4 makes the number no longer square. Intuitively, this seems true because the distance between consecutive squares will eventually be greater than any fixed number, like 4, so by subtracting a fixed number from large enough squares, we end up in between squares. (34)

Conjecture: $\forall n \in \mathbb{N}, n \geq 3$ implies $(n^2-4)^{1/2} \notin \mathbb{N}$ (35)

Proof of (35): Suppose $(n^2-4)^{1/2} = k$ where $k \in \mathbb{N}$ (36)

Then $n^2 - 4 = k^2$ (37)

So $n^2 - k^2 = 4$ (38)

Clearly, $n > k$ (39)

Then $n^2 - k^2 \geq n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1$ (40)

So $n^2 - k^2 > 4$ (41)

(41) contradicts (38), so $\square_{c34}$ (42)

Now we can conclude. By (27) and (35), $((a_{11}+a_{22})^2-4)^{1/2} \notin \mathbb{N}$ (43)

So, by (43), (26) and (29), $((a_{11}+a_{22})^2-4)^{1/2}$ is irrational (44)

So, by (44) and (21), λ is irrational (45)

We still need to show that the (un)stable manifolds are dense by exercise 1.11.1. (46)

Let $x \in \mathbb{T}^2$. Without loss of generality, $\lambda > 1 > \lambda^{-1}$, where λ and λ^{-1} are the eigenvalues of A. The stable manifold $W^u(x)$ is the line through x parallel to v where v is the eigenvector corresponding to λ. (47)

I think the intended proof is to show that the slope of a line parallel to v is irrational, hence the flow defined in section 1.11 has dense orbits, from which it follows $W^u(x)$ is dense. (48)

Denote $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. A line parallel to v has slope equal to $\frac{v_2}{v_1}$ (49)

My guess is that the slope is irrational because λ is irrational and because v is an eigenvector of an integer-valued matrix. Let's write out the defining equation for eigenvectors. (50)

$$Av = \lambda v \tag{51}$$

$$\begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \tag{52}$$

$$a_{11}v_1 + a_{12}v_2 = \lambda v_1 \tag{53}$$

$$a_{21}v_1 + a_{22}v_2 = \lambda v_2 \tag{54}$$

$$v_2(\lambda - a_{22}) = a_{21}v_1 \tag{55}$$

$$v_1(\lambda - a_{11}) = a_{12}v_2 \tag{56}$$

If $a_{21} = 0$, then $v_2 v_1^{-1} = 0$. Why is this not possible? Probably because v is nonzero, by assumption. (57)

Since v is an eigenvector, $v_1 \neq 0$ or $v_2 \neq 0$ (58)

By (45), $(\lambda - a_{22})$ and $(\lambda - a_{11})$ are irrational. (59)

If $v_1 \neq 0$, then by (59) and (56), $v_2 \neq 0$ (60)

If $v_2 \neq 0$, then by (59) and (55), $v_1 \neq 0$ (61)

So, $v_1 \neq 0$ and $v_2 \neq 0$ (62)

By (62) and (55), $v_2 v_1^{-1} = (\lambda - a_{22})^{-1} a_{21} \neq 0$ (63)

By (63) and (59), $v_2 v_1^{-1}$ is irrational (64)

Denote $x = (x_1, x_2)$, let $t \in \mathbb{R}^+$ (65)

Let $\phi^t_{\frac{v_1}{v_2}}(x) := (x_1 + \frac{v_1}{v_2}t, x_2 + t) \mod 1$ (66)

Then $\bigcup_{t \in \mathbb{R}^+} \phi^t_{\frac{v_1}{v_2}}(x) \subseteq W^u(x)$ (67)

By exercise 1.11.1, the orbit of $\phi_{\frac{v_1}{v_2}}$ is dense, so $W^u(x)$ is dense. For the stable manifold, the proof is similar. (68)