Exercise 1.3.3

For $m \in \mathbb{Z}$, $|m| > 1$, define the times-m map $E_m : S^1 \to S^1$ by $E_m x = mx \mod 1$. Show that the set of points with dense orbits is uncountable. (1)

Proof + reasoning:

My first idea is to show that for any irrational x, the orbit under E_m is dense. How would we show this? We could use the semiconjugacy from (Σ_m, σ) to (S^1, E_m). As stated in ch1.3, the orbit of a point $0.x_1x_2\dots$ is dense in S^1 iff every finite sequence of elements in $\{0, \dots, m-1\}$ appears in the sequence $(x_i)_{i \in \mathbb{N}}$. (2)

Let's try proof by contradiction using (2). (3)

Let x be an irrational number. (4)

x has a base-m expansion $0.x_1x_2\dots$ (5)

Suppose that the orbit of x is not dense. There exists a finite sequence $a_1 \dots a_n$ of elements in $\{0, \dots, m-1\}$ that does not occur anywhere in $x_1x_2\dots$ (6)

Is the statement that any irrational x has a dense orbit true? Let's try a different approach. (7)

A more direct way to prove the statement is to construct an injective function from an uncountable set to the set of points in S^1 with dense orbits. It is noted in chapter 1.3 that we can construct a point in S^1 with dense orbit by simply concatenating all finite sequences. It seems likely that we can do something similar to construct the needed function. (8)

Let U be the set of points in S^1 with a unique base-m expansion. (9)

By the remarks in section 1.3, U is uncountable. (10)

Define $\phi : \Sigma_m \to S^1$. by $\phi((x_i)_{i \in \mathbb{N}}) := \sum_{i=1}^{\infty} x_i/m^i$ (11)

By the remarks in section 1.3, ϕ is bijective on $\phi^{-1}(U)$. (12)

Let $x \in U$, with base-m expansion $(x_i)_{i \in \mathbb{N}}$. (13)

Let $\mathcal{F}_m = \bigcup_{k=1}^{\infty} \{0, \dots, m-1\}^k$. (14)

Clearly, $\mathcal{F}_m$ is countable, so it can be indexed by $(\omega_i)_{i \in \mathbb{N}}$. (15)

Define $\alpha : U \to \Sigma_m$ by letting $\alpha(x) = x_1\omega_1x_2\omega_2x_3\omega_3\dots$, and define $\beta = \phi \circ \alpha$. (16)

Since every $y \in U$ has a unique base-m expansion, α is injective, so by (12), β is bijective. By construction, every finite sequence appears in $\alpha(y)$ for every $y \in U$, so by (2), every point in $\beta(U)$ has a dense orbit. (17)

From (18), (17), (10), we get that the set of all points in S^1 with dense orbits is uncountable. (18)