Exercise 1.1.2

Suppose (X, f) is a factor of (Y, g) by a semi-conjugacy $\pi \colon Y \to X$. $\hspace{1cm}$ (1)

Show that if $y \in Y$ is a periodic point, then $\pi(y) \in X$ is periodic. $\hspace{1cm}$ (2)

Give an example to show that the preimage of a periodic point does not necessarily contain a periodic point. $\hspace{1cm}$ (3)

Proof + reasoning:

Let's start with (2). $\hspace{1cm}$ (4)

Let $y \in Y$ be periodic. $\hspace{1cm}$ (5)

We can just write out the definitions and see if the correct answer follows. $\hspace{1cm}$ (6)

$$f(\pi(y)) \;=\; \pi\big(g(y)\big) \;=\; \pi(y).$$ $\hspace{1cm}$ (7)

So, (2) follows from (7). $\hspace{1cm}$ (8)

Now, we want to construct a counterexample. Let's think of any semi-conjugacy and iterate from there. An obvious example is a projection. $\hspace{1cm}$ (9)

Let A and B be sets, and $\pi_A \colon A \times B \to A$ the projection. $\hspace{1cm}$ (10)

For all α and β, the following diagram commutes:

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\;\alpha \times \beta\;} & A \times B \\[4pt]
\downarrow \pi_A & & \downarrow \pi_A \\[4pt]
A & \xrightarrow{\;\alpha\;} & A
\end{array}
$$ $\hspace{1cm}$ (11)

Intuitively, by taking projections, we 'forget' about the effect of β. So, we can simply choose β such that all points in $A \times B$ are non-periodic with respect to β, while choosing α so that all points in A are periodic with respect to α. $\hspace{1cm}$ (12)

Let A be any set, $B = [0, 1]$, $\alpha = \mathrm{id}_A$ and $\beta : (x \mapsto \frac{1}{2}x)$ $\hspace{1cm}$ (13)

Wait, 0 is still a periodic point in the example above. Let's modify it. $\hspace{1cm}$ (14)

Let A be any set, $B = (0, 1]$, $\alpha = \mathrm{id}_A$ and $\beta : (x \mapsto \frac{1}{2}x)$ $\hspace{1cm}$ (15)

Clearly, β has no periodic points, so $\alpha \times \beta$ has no periodic points, but all points in A are periodic with respect to α. $\hspace{1cm}$ (16)

By (16), (3) follows. $\hspace{1cm}$ (17)