Exercise 1.11.3.

Suppose $1, s$ and αs are real numbers that are linearly independent over $\mathbb{Q}$. (1)

Show that every orbit of the time-s map ϕ_α^s is dense in $\mathbb{T}^2$. (2)

Proof + reasoning:

Let's try to adapt the proof that R_α has dense semiorbits if α is irrational. (3)

The idea in (3) is to divide S^1 into disjoint ε-sized intervals, and use the pigeonhole principle to show that there exist $k > m$ such that $R_\alpha^k(x)$ and $R_\alpha^m(x)$ belong to the same interval, hence that R_α^{k-m} is a translation by less than ε. In our case, we need to use rectangles instead of intervals, and the translation is not simply defined as the addition of an irrational number. (4)

Let $x \in \mathbb{T}^2$, $y \in \mathbb{T}^2$, $\varepsilon' > 0$ and $\varepsilon = \frac{\varepsilon'}{2\sqrt{2}}$. (5)

Let $\mathcal{P}_\varepsilon$ be a partition of $\mathbb{T}^2$ into finitely many squares of the form $[a, b)^2$, where $\frac{\varepsilon}{2} < |a - b| < \varepsilon$. (6)

By the pigeonhole principle, there exists a $P \in \mathcal{P}_\varepsilon$ and $k > m$ in $\mathbb{Z}$ such that $\phi_\alpha^{ks}(x)$ and $\phi_\alpha^{ms}(x)$ are in P. (7)

By (7), $d(z, \phi_\alpha^{(k-m)s}(z)) < \sqrt{2}\varepsilon$ for all $z \in \mathbb{T}^2$, where d is the metric on $\mathbb{T}^2$. (8)

Intuitively, $\phi_\alpha^{(k-m)s}$ is a translation of size less than $\sqrt{2}\varepsilon$, seemingly with an irrational slope. So, if the line from x in the direction of that translation eventually intersects an ε-ball around y, then there should exist an iterate of x that is within ε' of y. Let's make this precise. Note that we can state the conjecture first, and prove it only after we know that it gives the required result. (9)

Conjecture 1. *There exists a $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $y \in \mathbb{T}^2$*

$$\frac{(\phi_\alpha^{(k-m)s}(y))_2 - y_2}{(\phi_\alpha^{(k-m)s}(y))_1 - y_1} = \beta.$$

Proof. Suppose for contradiction that $s = 0$. Then for $p = 1, q = 1, r = 0$ we have $p\alpha s + qs + r = 0$, a contradiction, so $s \neq 0$. Similarly, $\alpha s \neq 0$. Suppose for contradiction that $\alpha s \in \mathbb{Q}$. Let $p = 1, q = -\alpha s, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\alpha s \notin \mathbb{Q}$. Suppose for contradiction that $\frac{1}{\alpha} \in \mathbb{Q}$. Then s is irrational. Let $p = \frac{1}{\alpha}, q = -1, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\frac{1}{\alpha}$ is irrational. Let $y \in \mathbb{T}^2$. Then

$$\frac{(\phi_\alpha^{(k+m)s}(y))_2 - y_2}{(\phi_\alpha^{(k+m)s}(y))_1 - y_1} = \frac{(k-m)s}{(k-m)\alpha s} = \frac{1}{\alpha}$$

So, with $\beta = \frac{1}{\alpha}$, the statement follows. $\square$

Let γ be the line in $\mathbb{T}^2$ starting from x in the direction of $x - \phi_\alpha^{m-k}(x)$. (10)

Let β be the slope of γ, which is finite and in $\mathbb{R} \setminus \mathbb{Q}$ by 1. (11)

By (11), considering γ as a subset of $\mathbb{T}^2$, we have

$$\begin{aligned}\gamma \cap (y_1 \times \mathbb{T}) &= \bigcup_{n \geq 0} \{(y_1, (x_2 + \beta(y_1 - x_1) + \beta n) \bmod 1)\} \\ &= \bigcup_{n \geq 0} \left\{\left(y_1, R_\beta^n(x_2 + \beta(y_1 - x_1))\right)\right\}.\end{aligned} \tag{12}$$

By (11), R_β has dense semiorbits. (13)

By (13) and (12), there exists a $z \in \gamma \cap (y_1 \times (y_2 - \varepsilon, y_2 + \varepsilon))$. (14)

By (9) and Conjecture 1, there exists a $p \in \mathbb{N}$ such that

$$d(\phi_\alpha^{p(k-m)s}(x), z) < \sqrt{2}\varepsilon \tag{15}$$

By (15) and (14),

$$\begin{aligned}d(\phi_\alpha^{p(k-m)s}(x), y) &\leq d(\phi_\alpha^{p(k-m)s}(x), z) + d(z, y) \\ &\leq \sqrt{2}\varepsilon + \varepsilon \\ &\leq 2\sqrt{2}\varepsilon \\ &\leq \varepsilon'.\end{aligned} \tag{16}$$

By (16), every orbit of ϕ_α^s is dense in $\mathbb{T}^2$. (17)