

**Exercise 1.7.3.**

Show that the eigenvalues of a two-dimensional hyperbolic toral automorphism are irrational (so the stable and unstable manifolds are dense by exercise 1.11.1). (1)

**Proof + reasoning:**

Are there hyperbolic toral automorphisms that aren't represented by a matrix? No, not in this context. (2)

Let  $A$  be a  $2 \times 2$  integer matrix such that  $\det(A) = 1$  and such that for all eigenvalues  $\lambda$  of  $A$ ,  $|\lambda| \neq 1$ . (3)

Let  $\lambda$  be an eigenvalue of  $A$ . (4)

Let's try the following: first, relate the eigenvalues to the determinant. Then, conclude from the first step and the given assumptions that the eigenvalues are irrational. (5)

Note that  $\det(\lambda I - A) = 0$ . (6)

Denote  $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ . (7)

$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$  (8)

$\lambda^2 - \lambda a_{11} - \lambda a_{22} + a_{11}a_{22} - a_{12}a_{21} = 0$  (9)

By assumption,  $\det(A) = 1$ , so  $a_{11}a_{22} - a_{12}a_{21} = 1$  (10)

Substituting  $1 = a_{11}a_{22} - a_{12}a_{21}$  in (9) gives:  $\lambda^2 - \lambda a_{11} - \lambda a_{22} + 1 = 0$  (11)

$\lambda^2 - \lambda(a_{11} + a_{22}) + 1 = 0$  (12)

That was the first step. How can we conclude? (13)

What given assumption have I not used yet? (14)

I haven't used that  $|\lambda| \neq 1$ . From (12) it may be possible to conclude that  $\lambda$  is equal to 1 or irrational, which together with (15) gives that  $\lambda$  is irrational. (15)

We can factorize the left-hand side of (12). (16)

However, (12) could have complex solutions for certain values of  $a_{11} + a_{22}$ . (17)

If the discriminant of (12) is greater than 0, (12) only has real roots. (18)

I'm not sure how to proceed. Let's take a few steps back. (19)

Equation (12) is quadratic in  $\lambda$ , so all solutions are given by the quadratic formula. (20)

By the quadratic formula,  $\lambda = \frac{1}{2}(a_{11} + a_{22} + ((a_{11} + a_{22})^2 - 4)^{1/2})$  or  $\lambda = \frac{1}{2}(a_{11} + a_{22} - ((a_{11} + a_{22})^2 - 4)^{1/2})$  (21)

From (21) we see that  $\lambda \in \mathbb{R} \setminus \mathbb{Q}$  if and only if  $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R} \setminus \mathbb{Q}$  (22)

By definition of  $A$ ,  $(a_{11} + a_{22}) \in \mathbb{N}$  (23)

Let's show that  $\lambda$  cannot be in  $\mathbb{C}$ . My guess is that if  $\lambda$  were in  $\mathbb{C}$ , its magnitude would be equal to 1. (24)

$$\text{Suppose } ((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{C}. \text{ Then } |\lambda|^2 = \frac{1}{4}((a_{11} + a_{22})^2 + 4 - (a_{11} + a_{22})^2) = 1 \quad (25)$$

$$(25) \text{ contradicts (3), hence } ((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R} \quad (26)$$

$$\text{From (26) and (23), } a_{11} + a_{22} \geq 3 \quad (27)$$

We just need to show that  $\lambda \notin \mathbb{Q}$ . My intuition is that  $((a_{11} + a_{22})^2 - 4)^{1/2}$  is always irrational, because it seems  $\forall n \in \mathbb{N}$ ,  $n^{1/2}$  is rational only if  $n$  is a square number, and subtracting 4 makes it no longer square, given  $\sqrt{n} \geq 3$ . Let's prove the first part. (28)

$$\text{Conjecture: } \{n \in \mathbb{N} : \sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}\} = \emptyset \quad (29)$$

$$\text{Proof of (29): Suppose } \sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}. \text{ Then } \sqrt{n} = \frac{p}{q}, \text{ where } p \text{ and } q \text{ are natural numbers.} \quad (30)$$

$$\text{Since } \sqrt{n} \notin \mathbb{N}, q \text{ does not divide } p, \text{ so } q^2 \text{ does not divide } p^2 \quad (31)$$

$$\text{From (30), } n = \frac{p^2}{q^2}, \text{ hence } nq^2 = p^2 \quad (32)$$

$$(31) \text{ contradicts (32), so (30) is false } \square_{c23} \quad (33)$$

Now let's prove the second part, that subtracting 4 makes the number no longer square. Intuitively, this seems true because the distance between consecutive squares will eventually be greater than any fixed number, like 4, so by subtracting a fixed number from large enough squares, we end up in between squares. (34)

$$\text{Conjecture: } \forall n \in \mathbb{N}, n \geq 3 \text{ implies } (n^2 - 4)^{1/2} \notin \mathbb{N} \quad (35)$$

$$\text{Proof of (35): Suppose } (n^2 - 4)^{1/2} = k \text{ where } k \in \mathbb{N} \quad (36)$$

$$\text{Then } n^2 - 4 = k^2 \quad (37)$$

$$\text{So } n^2 - k^2 = 4 \quad (38)$$

$$\text{Clearly, } n > k \quad (39)$$

$$\text{Then } n^2 - k^2 \geq n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1 \quad (40)$$

$$\text{So } n^2 - k^2 > 4 \quad (41)$$

$$(41) \text{ contradicts (38), so } \square_{c34} \quad (42)$$

$$\text{Now we can conclude. By (27) and (35), } ((a_{11} + a_{22})^2 - 4)^{1/2} \notin \mathbb{N} \quad (43)$$

$$\text{So, by (43), (26) and (29), } ((a_{11} + a_{22})^2 - 4)^{1/2} \text{ is irrational} \quad (44)$$

$$\text{So, by (44) and (21), } \lambda \text{ is irrational} \quad (45)$$

We still need to show that the (un)stable manifolds are dense by exercise 1.11.1. (46)

$$\text{Let } x \in \mathbb{T}^2. \text{ Without loss of generality, } \lambda > 1 > \lambda^{-1}, \text{ where } \lambda \text{ and } \lambda^{-1} \text{ are the eigenvalues of } A. \text{ The stable manifold } W^s(x) \text{ is the line through } x \text{ parallel to } v \text{ where } v \text{ is the eigenvector corresponding to } \lambda. \quad (47)$$

I think the intended proof is to show that the slope of a line parallel to  $v$  is irrational, hence the flow defined in section 1.11 has dense orbits, from which it follows  $W^u(x)$  is dense. (48)

$$\text{Denote } v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \text{ A line parallel to } v \text{ has slope equal to } \frac{v_2}{v_1} \quad (49)$$

My guess is that the slope is irrational because  $\lambda$  is irrational and because  $v$  is an eigenvector of an integer-valued matrix. Let's write out the defining equation for eigenvectors.

(50)

$$Av = \lambda v \quad (51)$$

$$\begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \quad (52)$$

$$a_{11}v_1 + a_{12}v_2 = \lambda v_1 \quad (53)$$

$$a_{21}v_1 + a_{22}v_2 = \lambda v_2 \quad (54)$$

$$v_2(\lambda - a_{22}) = a_{21}v_1 \quad (55)$$

$$v_1(\lambda - a_{11}) = a_{12}v_2 \quad (56)$$

If  $a_{21} = 0$ , then  $v_2v_1^{-1} = 0$ . Why is this not possible? Probably because  $v$  is nonzero, by assumption.

(57)

Since  $v$  is an eigenvector,  $v_1 \neq 0$  or  $v_2 \neq 0$  (58)

By (45),  $(\lambda - a_{22})$  and  $(\lambda - a_{11})$  are irrational. (59)

If  $v_1 \neq 0$ , then by (59) and (56),  $v_2 \neq 0$  (60)

If  $v_2 \neq 0$ , then by (59) and (55),  $v_1 \neq 0$  (61)

So,  $v_1 \neq 0$  and  $v_2 \neq 0$  (62)

By (62) and (55),  $v_2v_1^{-1} = (\lambda - a_{22})^{-1}a_{21} \neq 0$  (63)

By (63) and (59),  $v_2v_1^{-1}$  is irrational (64)

Denote  $x = (x_1, x_2)$ , let  $t \in \mathbb{R}^+$  (65)

Let  $\phi_{\frac{v_1}{v_2}}^t(x) := (x_1 + \frac{v_1}{v_2}t, x_2 + t) \mod 1$  (66)

Then  $\bigcup_{t \in \mathbb{R}^+} \phi_{\frac{v_1}{v_2}}^t(x) \subseteq W^u(x)$  (67)

By exercise 1.11.1, the orbit of  $\phi_{\frac{v_1}{v_2}}$  is dense, so  $W^u(x)$  is dense. For the stable manifold, the proof is similar. (68)