**Exercise 1.10.3.**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. (1)

Show that $-f$ is a Lyapunov function for the gradient flow. (2)

Show that the trajectories of the gradient flow are orthogonal to the level sets of $f$. (3)

**Proof + reasoning:**

Let's write out the relevant definitions. (4)

The gradient flow is the flow of the differential equation $\dot{x} = \nabla f(x)$. (5)

Denote the time-$t$ gradient flow by $g^t : \mathbb{R}^n \to \mathbb{R}^n$. For all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$, write $g_x(t) := g^t(x)$. (6)

For each $x$ this defines $g_x : \mathbb{R}^+ \to \mathbb{R}^n$. (7)

Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$. Note $(f \circ g_x)(0) = f(x)$ and $(f \circ g_x)(t) = f(g^t(x))$. (8)

By (8), if $(f \circ g_x)'(s) \geq 0$ for all $s \in \mathbb{R}^+$ then $-f$ is Lyapunov. (9)

By (5), $g'_x(t) = \nabla f(g_x(t))$. (10)

By the multivariate chain rule and (10),

$$(f \circ g_x)'(t) = \langle \nabla f(g_x(t)), g'_x(t) \rangle = \langle g'_x(t), g'_x(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. (11)

By definition of inner products, $\langle g'_x(t), g'_x(t) \rangle \geq 0$. (12)

By (11), (12) and (9), $-f$ is Lyapunov. (13)

Next, we want to show statement (2). (14)

To express orthogonality, we need a common inner product space, but this is just $\mathbb{R}^n$ in our case. (15)

Which vectors are we trying to prove are orthogonal? I think, given some point $x \in \mathbb{R}^n$, we should compare the time derivative of the orbit of $x$ at $t = 0$, with a vector in $\mathbb{R}^n$ 'tangent' to the level set of $f$ at $f(x)$. (16)

How can we define the tangent vector? (17)

Let $x \in \mathbb{R}^n$. (18)

Define the level set $C := f^{-1}(f(x))$. (19)

$C$ is a subset of $\mathbb{R}^n$, but I don't think it is necessarily a smooth manifold. Still, we can define tangent vectors in terms of smooth paths in $\mathbb{R}^n$: (20)

Let $T_x = \{\dot{\gamma}(0) : \exists \varepsilon > 0 \text{ s.t. } \gamma : (-\varepsilon, \varepsilon) \to C \text{ is smooth and } \gamma(0) = x\}$. (21)

By (10), $(f \circ g_x)'(0) = \nabla f(g_x(0)) = \nabla f(x) = g'_x(0)$. (22)

Let $V \in T_x$, with corresponding path $\gamma : (-\varepsilon, \varepsilon) \to C$. (23)

We need to show $\langle V, \nabla f(x) \rangle = 0$. (24)

Since $\gamma(t) \in C$ for all $t \in (-\varepsilon, \varepsilon)$, $f(\gamma(t)) = f(\gamma(0))$ for all $t \in (-\varepsilon, \varepsilon)$. (25)

By the multivariate chain rule,

$$\begin{aligned}
\langle \nabla f(x), V \rangle &= \sum_{k=1}^{n} V_k \frac{\partial f}{\partial y_k}(x) \\
&= \sum_{k=1}^{n} V_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\
&= \sum_{k=1}^{n} \dot{\gamma}(0)_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\
&= (f \circ \gamma)'(0). \qquad (26)
\end{aligned}$$

By (25), $(f \circ \gamma)'(0) = 0$, so by (26), $\langle \nabla f(x), V \rangle = 0$, so by (22), the trajectories of the gradient flow are orthogonal to the level sets of $f$. (27)