

Exercise 1.1.2

Suppose (X, f) is a factor of (Y, g) by a semi-conjugacy $\pi: Y \rightarrow X$. (1)

Show that if $y \in Y$ is a periodic point, then $\pi(y) \in X$ is periodic. (2)

Give an example to show that the preimage of a periodic point does not necessarily contain a periodic point. (3)

Proof + reasoning:

Let $y \in Y$ be periodic. Then there exists $n \in \mathbb{N}$ such that $g^n(y) = y$. (4)

Since f is a factor of g ,

$$f^n(\pi(y)) = \pi(g^n(y)) = \pi(y). \quad (5)$$

So, $\pi(y)$ is periodic. (6)

Let A and B be sets, and $\pi_A: A \times B \rightarrow A$ the projection. (7)

For all α and β , the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\alpha \times \beta} & A \times B \\ \downarrow \pi_A & & \downarrow \pi_A \\ A & \xrightarrow{\alpha} & A \end{array} \quad (8)$$

Let A be any nonempty set, $B = (0, 1]$, $\alpha = \text{id}_A$ and $\beta: (x \mapsto \frac{1}{2}x)$ (9)

Clearly, β has no periodic points, so $\alpha \times \beta$ has no periodic points, but all points in A are periodic with respect to α . (10)

Exercise 1.2.3

Let G be a topological group. (1)

Prove that for each $g \in G$, the closure $H(g)$ of the set $\{g^n\}_{n=-\infty}^{\infty}$ is a commutative subgroup of G . (2)

Thus, if G has a minimal left translation, then G is abelian. (3)

Proof + reasoning:

Define $\langle g \rangle := \{g^n\}_{n=-\infty}^{\infty}$. (4)

Let $g \in G$. Let $a, b \in \text{cl}(\langle g \rangle)$. (5)

Since the group multiplication $\alpha : G \times G \rightarrow G$ is continuous, $\alpha^{-1}(U)$ is open in $G \times G$. Since $(a, b) \in \alpha^{-1}(U)$, and since sets of the form $A \times B$, where A and B are open, form a basis for the topology on $G \times G$, there exist open V and W such that $a \in V$, $b \in W$, and such that $V \times W \subseteq \alpha^{-1}(U)$. (6)

Since $a, b \in \text{cl}(\langle g \rangle)$ there exist $g^\ell \in V$ and $g^p \in W$. By (6), $g^\ell g^p \in U$, hence $g^{\ell+p} \in U$, so $ab \in \text{cl}(\langle g \rangle) = H(g)$. (7)

By (7), $H(g)$ is closed under taking products. (8)

Let C be a neighborhood of a^{-1} and $U \subseteq C$ an open set such that $a^{-1} \in U$. Since the inverse is continuous, $U' := \{x \in G : x^{-1} \in U\}$ is open, and it contains a . (9)

Since $a \in H(g)$, there exists $g^q \in U'$, where $q \in \mathbb{Z}$. (10)

By (10), $g^{-q} = (g^q)^{-1} \in U$, so $a^{-1} \in H(g)$. (11)

By (11), $H(g)$ is closed under taking inverses. (12)

Let $c, d \in G$ with $cd \neq dc$. (13)

Suppose that G is Hausdorff. (14)

By (14) and (13), there exist open neighborhoods U of cd and U' of dc such that $U \cap U' = \emptyset$. (15)

Suppose $c, d \in H(g)$. (16)

Similarly to (6), $(c, d) \in \alpha^{-1}(U)$ and $(d, c) \in \alpha^{-1}(U')$. (17)

So there are open sets V, V', W, W' such that $(c, d) \in V \times W \subseteq \alpha^{-1}(U)$ and $(d, c) \in V' \times W' \subseteq \alpha^{-1}(U')$. (18)

From (18), $c \in V \cap V'$ and $d \in W \cap W'$, and $V \cap V'$ and $W \cap W'$ are open. (19)

So, by (16), there exist $s, t \in \mathbb{Z}$ such that $g^s \in V \cap V'$ and $g^t \in W \cap W'$. (20)

By (20), $(g^s, g^t) \in V \times W$ and $(g^t, g^s) \in W' \times V'$. (21)

By (21) and (18), $g^s g^t \in U$ and $g^t g^s \in U'$, so $g^{t+s} \in U \cap U'$. (22)

(22) contradicts (14), so (16) is false, hence $c \notin H(g)$ or $d \notin H(g)$, so $H(g)$ is commutative. (23)

By (23), (8) and (12), $H(g)$ is a commutative subgroup of G . (24)

Suppose that G has a minimal left translation $L_h : G \rightarrow G$ where $h \in G$. (25)

By (24), $H(h)$ is a commutative subgroup of G . (26)

By definition, L_h has no proper closed non-empty invariant subsets. (27)

$H(h)$ is a closed non-empty subset of G . (28)

Let $a \in H(h)$. Let C be a neighborhood of ha and U open with $ha \in U \subseteq C$.
 $a \in h^{-1}U$, and $h^{-1}U$ is open, so $\exists q \in \mathbb{Z}$ such that $h^q \in h^{-1}U$. (29)

By (29), $h^{q+1} \in U$, so $H(h)$ is invariant. (30)

By (30), (28) and (27), $H(h) = G$, so G is abelian. (31)

Exercise 1.3.3

For $m \in \mathbb{Z}$, $|m| > 1$, define the times- m map $E_m : S^1 \rightarrow S^1$ by $E_m x = mx \pmod{1}$. Show that the set of points with dense orbits is uncountable. (1)

Proof + reasoning:

As stated in ch1.3, the orbit of a point $0.x_1x_2\dots$ is dense in S^1 iff every finite sequence of elements in $\{0, \dots, m-1\}$ appears in the sequence $(x_i)_{i \in \mathbb{N}}$. (2)

Let U be the set of points in S^1 with a unique base- m expansion. (3)

By the remarks in section 1.3, U is uncountable. (4)

Define $\phi : \Sigma_m \rightarrow S^1$ by $\phi((x_i)_{i \in \mathbb{N}}) := \sum_{i=1}^{\infty} x_i/m^i$ (5)

By the remarks in section 1.3, ϕ is bijective on $\phi^{-1}(U)$. (6)

Let $x \in U$, with base- m expansion $(x_i)_{i \in \mathbb{N}}$. (7)

Let $\mathcal{F}_m = \bigcup_{k=1}^{\infty} \{0, \dots, m-1\}^k$. (8)

Clearly, \mathcal{F}_m is countable, so it can be indexed by $(\omega_i)_{i \in \mathbb{N}}$. (9)

Define $\alpha : U \rightarrow \Sigma_m$ by letting $\alpha(x) = x_1\omega_1x_2\omega_2x_3\omega_3\dots$, and define $\beta = \phi \circ \alpha$. (10)

Since every $y \in U$ has a unique base- m expansion, α is injective, so by (6), β is bijective. By construction, every finite sequence appears in $\alpha(y)$ for every $y \in U$, so by (2), every point in $\beta(U)$ has a dense orbit. (11)

From (12), (11) and (4), we get that the set of all points in S^1 with dense orbits is uncountable. (12)

Exercise 1.4.3.

Verify that the metrics on Σ_m and Σ_m^+ generate the product topology (1)

Proof + reasoning:

Let $C := C_{j_1, \dots, j_k}^{n_1, \dots, n_k} = \{x = (x_\ell) : x_{n_i} = j_i, i = 1, \dots, k\}$ where $n_1 < n_2 < \dots < n_k$ are indices in \mathbb{Z} or \mathbb{N} , and $j_i \in A_m$. (2)

Let $x := (x_i) \in C$. (3)

Let $m = \max\{|n_i| : i \leq k\}$, $y \in B(x, 2^{-m})$, and $l = \min\{|i| : y_i \neq x_i\}$. (4)

We have $2^{-l} = d(x, y) < 2^{-m}$. (5)

From (5), $l > m$, so $x_{n_i} = y_{n_i} \ \forall i \leq k$, hence $y \in C$. (6)

Therefore $B(x, 2^{-m}) \subseteq C$. (7)

By (7), and the fact that the collection of sets such as $C_{j_1, \dots, j_k}^{n_1, \dots, n_k}$ form a basis for the product topology, the metrics generate the product topology. (8)

Exercise 1.5.3.

Suppose p is an attracting fixed point for f . Show that there is a neighborhood U of p such that the forward orbit of every point in U converges to p . (1)

Proof + reasoning:

By assumption, there exists a neighborhood U of p such that \bar{U} is compact, $f(\bar{U}) \subseteq U$, and $\bigcap_{n \geq 0} f^n(\bar{U}) = \{p\}$. (2)

Clearly,

$$U \subset \bar{U}. \quad (3)$$

From (3) and (2)

$$f(U) \subseteq f(\bar{U}) \subseteq U. \quad (4)$$

Therefore,

$$f^{n+1}(U) \subseteq f^n(U) \text{ for all } n \in \mathbb{N} \quad (5)$$

Clearly,

$$\bigcap_{n \geq 0} f^n(U) \subseteq \bigcap_{n \geq 0} f^n(\bar{U}) \quad (6)$$

Conversely,

$$\bigcap_{n \geq 0} f^n(\bar{U}) \subseteq \bigcap_{n \geq 1} f^n(\bar{U}) = \bigcap_{n \geq 0} f^{n+1}(\bar{U}) \subseteq \bigcap_{n \geq 0} f^n(U) \quad (7)$$

So, from (7)

$$\bigcap_{n \geq 0} f^n(U) = \bigcap_{n \geq 0} f^n(\bar{U}) \quad (8)$$

Let $x \in U$. Define $(x_n)_{n \in \mathbb{N}} = (f^n(x))_{n \in \mathbb{N}}$. (9)

Assume (x_n) does not converge. Then $\exists \varepsilon' > 0$ such that $\forall n : \exists k \geq n : d(f^k(x), p) > \varepsilon'$. (10)

From (10), there exists a sequence $f^{m_n}(x)$ such that $d(f^{m_n}(x), p) \geq \varepsilon$ for all $n \geq 0$. By compactness of \bar{U} , this sequence has a convergent subsequence $(f^{z_n}(x))_{n \geq 0}$ with $f^{z_n}(x) \rightarrow z \in \bar{U}$ and $z_n \rightarrow \infty$. (11)

Since f is continuous and \bar{U} compact, $f^n(\bar{U})$ is compact for all $n \geq 0$. (12)

$\forall n \geq 0$ there exists K s.t. $f^{z_K}(x) \in f^n(\bar{U})$, hence $\forall m \geq K$, $f^{z_m}(x) \in f^n(\bar{U})$. (13)

From (13), the limit point z must be in $f^n(\bar{U})$ for all $n \geq 0$. (14)

Therefore, $z \in \bigcap_{n \geq 0} f^n(\bar{U}) = \{p\}$. (15)

So, $z = p$, which contradicts (10), so assumption (10) is false. Hence (x_n) converges. (16)

Suppose $x_n \rightarrow q$ and $q \neq p$. (17)

Let $n \geq 0$. The sequence $(x_i)_{i \geq n}$ is contained in $f^n(\bar{U})$, which is compact, so $q \in f^n(\bar{U})$. Therefore, $q \in \bigcap_{n \geq 0} f^n(\bar{U})$. (18)

By (8), $q \in \bigcap_{n \geq 0} f^n(U) = \{p\}$, which is a contradiction. Hence, (17) is false. (19)

From (19) and (16) it follows that $x_n \rightarrow p$. Therefore, the forward orbit of any point in U converges to p . (20)

Exercise 1.7.3.

Show that the eigenvalues of a two-dimensional hyperbolic toral automorphism are irrational (so the stable and unstable manifolds are dense by exercise 1.11.1). (1)

Proof + reasoning:

Let A be a 2×2 integer matrix such that $\det(A) = 1$ and such that for all eigenvalues λ of A , $|\lambda| \neq 1$. (2)

Let λ be an eigenvalue of A . (3)

Note that $\det(\lambda I - A) = 0$. (4)

Denote $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. (5)

$$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0 \quad (6)$$

$$\lambda^2 - \lambda a_{11} - \lambda a_{22} + a_{11}a_{22} - a_{12}a_{21} = 0 \quad (7)$$

By assumption, $\det(A) = 1$, so $a_{11}a_{22} - a_{12}a_{21} = 1$ (8)

Substituting $1 = a_{11}a_{22} - a_{12}a_{21}$ in (7) gives

$$\lambda^2 - \lambda(a_{11} + a_{22}) + 1 = 0 \quad (9)$$

By the quadratic formula, $\lambda = \frac{1}{2}(a_{11} + a_{22} + ((a_{11} + a_{22})^2 - 4)^{1/2})$ or $\lambda = \frac{1}{2}(a_{11} + a_{22} - ((a_{11} + a_{22})^2 - 4)^{1/2})$ (10)

From (10) we see that $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ if and only if $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R} \setminus \mathbb{Q}$ (11)

By definition of A , $(a_{11} + a_{22}) \in \mathbb{N}$ (12)

Suppose $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{C}$. Then $|\lambda|^2 = \frac{1}{4}((a_{11} + a_{22})^2 + 4 - (a_{11} + a_{22})^2) = 1$ (13)

(13) contradicts (2), hence $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R}$ (14)

From (14) and (12), $a_{11} + a_{22} \geq 3$ (15)

Conjecture 1. $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}\} = \emptyset$

Proof. Suppose for contradiction that $\sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}$. Then $\sqrt{n} = \frac{p}{q}$, where p and q are natural numbers. Since $\sqrt{n} \notin \mathbb{N}$, q does not divide p , so q^2 does not divide p^2 , but $n = \frac{p^2}{q^2}$, hence $nq^2 = p^2$ which is a contradiction. \square

Conjecture 2. $\forall n \in \mathbb{N}, n \geq 3$ implies $(n^2 - 4)^{1/2} \notin \mathbb{N}$

Proof. Suppose $(n^2 - 4)^{1/2} = k$ where $k \in \mathbb{N}$. Then $n^2 - 4 = k^2$, so $n^2 - k^2 = 4$. Clearly, $n > k$. Then $n^2 - k^2 \geq n^2 - (n - 1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1$. So $n^2 - k^2 > 4$, which is a contradiction. \square

Now we can conclude. By (15) and (2), $((a_{11} + a_{22})^2 - 4)^{1/2} \notin \mathbb{N}$ (16)

So, by (16), (14) and (1), $((a_{11} + a_{22})^2 - 4)^{1/2}$ is irrational (17)

So, by (17) and (10), λ is irrational (18)

Let $x \in \mathbb{T}^2$. Without loss of generality, $\lambda > 1 > \lambda^{-1}$, where λ and λ^{-1} are the eigenvalues of A . The stable manifold $W^u(x)$ is the line through x parallel to v where v is the eigenvector corresponding to λ . (19)

Denote $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. A line parallel to v has slope equal to $\frac{v_2}{v_1}$ (20)

$$Av = \lambda v \quad (21)$$

$$\begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \quad (22)$$

$$a_{11}v_1 + a_{12}v_2 = \lambda v_1 \quad (23)$$

$$a_{21}v_1 + a_{22}v_2 = \lambda v_2 \quad (24)$$

$$v_2(\lambda - a_{22}) = a_{21}v_1 \quad (25)$$

$$v_1(\lambda - a_{11}) = a_{12}v_2 \quad (26)$$

Since v is an eigenvector, $v_1 \neq 0$ or $v_2 \neq 0$ (27)

By (18), $(\lambda - a_{22})$ and $(\lambda - a_{11})$ are irrational. (28)

If $v_1 \neq 0$, then by (28) and (26), $v_2 \neq 0$ (29)

If $v_2 \neq 0$, then by (28) and (25), $v_1 \neq 0$ (30)

So, $v_1 \neq 0$ and $v_2 \neq 0$ (31)

By (31) and (25), $v_2 v_1^{-1} = (\lambda - a_{22})^{-1} a_{21} \neq 0$ (32)

By (32) and (28), $v_2 v_1^{-1}$ is irrational (33)

Denote $x = (x_1, x_2)$, let $t \in \mathbb{R}^+$ (34)

Let $\phi_{\frac{v_1}{v_2}}^t(x) := (x_1 + \frac{v_1}{v_2}t, x_2 + t) \mod 1$ (35)

Then $\bigcup_{t \in \mathbb{R}^+} \phi_{\frac{v_1}{v_2}}^t(x) \subseteq W^u(x)$ (36)

By exercise 1.11.1, the orbit of $\phi_{\frac{v_1}{v_2}}$ is dense, so $W^u(x)$ is dense. For the stable manifold, the proof is similar. (37)

Exercise 1.8.3.

Let $\phi : \Sigma_2 = \{0,1\}^{\mathbb{Z}} \rightarrow H$ be the map that assigns to each infinite sequence $\omega = (\omega_i) \in \Sigma_2$ the unique point $\phi(\omega) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{\omega_i})$. (1)

Prove that ϕ is a bijection and that both ϕ and ϕ^{-1} are continuous. (2)

Proof + reasoning:

Suppose $x, y \in \Sigma_2$ with $\phi(x) = \phi(y)$. Then

$$\bigcap_{-\infty}^{\infty} f^{-i}(R_{x_i}) = \phi(x) = \phi(y) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{y_i}). \quad (3)$$

From the description of f , we see that f is injective. (4)

By definition,

$$R_0 = f(D_0) \cap R \quad \text{and} \quad R_1 = f(D_1) \cap R. \quad (5)$$

From (4), (5), and $D_0 \cap D_1 = \emptyset$, we get $R_0 \cap R_1 = \emptyset$. (6)

By (4) and (6), $f^{-i}(R_0) \cap f^{-i}(R_1) = \emptyset$ for all $i \in \mathbb{Z}$. (7)

From (3) and (7), $x_i = y_i$ for all $i \in \mathbb{Z}$, so $x = y$, so ϕ is injective. (8)

Note $f(R) \cap R = R_1 \cup R_0$, and $f^{-1}(R) \subseteq R$. (9)

By (9), $f^{-i}(R) = f^{-i}(R_0) \cup f^{-i}(R_1)$ for all $i \geq 1$. (10)

Clearly, $R_0 \cap R_1 = \emptyset$. (11)

By (4) and (9), for all $i \geq 0$,

$$(f^{-i}(R_0) \cup f^{-i}(R_1)) \cap R = f^{-i}(R_0 \cup R_1) \cap R = f^{-i+1}(R) \cap f^{-i}(R) \cap R. \quad (12)$$

Let $x \in H$ and $j \in \mathbb{Z}$. By (10), (11) and (12), $x \in f^j(R_0)$ or $x \in f^j(R_1)$, but not both. (13)

By (13), we can define $x_j = 0$ if $x \in f^j(R_0)$ and $x_j = 1$ if $x \in f^j(R_1)$. Clearly, this gives a sequence $(x_j)_{j \in \mathbb{Z}} \in \Sigma_2$ such that $\phi((x_j)_{j \in \mathbb{Z}}) = x$. So ϕ is surjective. (14)

By (14) and (8), ϕ is bijective. (15)

Let $\omega \in \phi^{-1}(A \times B)$. For a sequence $\omega \in \{0,1\}^{\mathbb{Z}}$, define

$$R_{\omega_{-m}, \dots, \omega_m} = \bigcap_{i=-m}^m f^{-i}(R_{\omega_i}). \quad (16)$$

Define $\mathcal{C}_m = \{R_{\omega_{-m}, \dots, \omega_m} \times R_{\omega_0, \dots, \omega_m}\}$, $\omega \in \{0,1\}^{\mathbb{Z}}$, $m \in \mathbb{N}$, and define $\mathcal{C} = \bigcup_{m \in \mathbb{N}} \{H \cap C : C \in \mathcal{C}_m\}$. (17)

Conjecture 3. \mathcal{C} is a basis for the topology on H .

Proof: Let $C \in \mathcal{C}$. Then $C = H \cap (R^- \times R^+)$ where $R^- \times R^+ \in \mathcal{C}_m$ for some $m \in \mathbb{N}$. (18)

Note $R^- = [x_1, x_2]$ and $R^+ = [y_1, y_2]$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. (19)

By (11), for all $D \neq D' \in \mathcal{C}_m$, $D \cap D' = \emptyset$. (20)

By (19) and (20), there exist open intervals $I^-, I^+ \subset \mathbb{R}$ such that $R^- \times R^+ \subseteq I^- \times I^+$ and such that $(I^- \times I^+) \cap D = \emptyset$ for all $D \in \mathcal{C}_m$ with $D \neq R^- \times R^+$. (21)

Clearly, \mathcal{C}_m covers H . (22)

By (21) and (22), $H \cap (R^- \times R^+) = H \cap (I^- \times I^+)$. (23)

By (23), C is open in H . (24)

Let A and B be open intervals in \mathbb{R} . (25)

Let $x \in H \cap (A \times B)$. (26)

Let $\varepsilon = \min\{d(x, y) : y \in A \times B\}$. (27)

Because $A \times B$ is open, $\varepsilon > 0$. (28)

Let $k = \min\{n \in \mathbb{N} : \mu^{-n} \leq \varepsilon, \lambda^n \leq \varepsilon\}$. (29)

By (28), and since $\lambda < 1/2$ and $\mu > 2$, $k > 0$. (30)

If $R^- \times R^+ \in \mathcal{C}_m$, then R^- has width equal to $\mu^{-k} \leq \varepsilon$ and R^+ has width equal to $\lambda^k \leq \varepsilon$. (31)

By (31), (27) and (22), there exists an $R^- \times R^+ \in \mathcal{C}_k$ such that $x \in R^- \times R^+$ and $R^- \times R^+ \subseteq A \times B$. (32)

By (24), $H \cap R^- \times R^+$ is open, so \mathcal{C} is a basis for the topology on H . (33)

Let $C \in \mathcal{C}$. $C = H \cap (R_{\omega_{-m}, \dots, \omega_m} \times R_{\omega_0, \dots, \omega_m})$ for $\omega \in \{0, 1\}^{\mathbb{Z}}$. (34)

Let $j \in \{-m, -m+1, \dots, m\}$. (35)

Suppose $z \in \phi^{-1}(C)$. By definition of ϕ ,

$$\phi(z) = \bigcap_{i \in \mathbb{Z}} f^{-i}(R_{z_i}) \subseteq f^j(R_{z_j}). \quad (36)$$

Since $\phi(z) \in R^+ \times R^-$, $\phi(z) \in f^j(R_{\omega_j})$. (37)

By (11), $f^j(R_1) \cap f^j(R_0) = \emptyset$, so $z_j = \omega_j$, so $z \in B(\omega, 2^{-m})$. (38)

Clearly $B(\omega, 2^{-m}) \subseteq \phi^{-1}(C)$. (39)

By (38) and (39), $\phi^{-1}(C) = B(\omega, 2^{-m})$, so $\phi^{-1}(C)$ is open, so, by 3, ϕ is continuous. (40)

Let $B(\gamma, 2^{-n})$ be an open ball in Σ_2 . By the same argument as for (40),

$$B(\gamma, 2^{-n}) = \phi^{-1}(H \cap (R_{\gamma_{-n}, \dots, \gamma_n} \times R_{\gamma_0, \dots, \gamma_n})). \quad (41)$$

So $\phi(B(\gamma, 2^{-n})) = H \cap (R_{\gamma_{-n}, \dots, \gamma_n} \times R_{\gamma_0, \dots, \gamma_n})$. (42)

By (42) and 3, $\phi(B(\gamma, 2^{-n}))$ is open, so ϕ^{-1} is continuous. (43)

Exercise 1.9.3.

Let \mathbb{T} denote the set of sequences $(\phi_i)_{i=0}^\infty$ where $\phi_i \in S^1$ and $\phi_i = 2\phi_{i+1} \pmod 1$ for all i . Let $\alpha : \mathbb{T} \rightarrow \mathbb{T}$ be defined by

$$(\phi_0, \phi_1, \dots) \mapsto (2\phi_1, \phi_1, \phi_2, \dots). \quad (1)$$

Show that \mathbb{T} is a topological group. (2)

Show that α is an automorphism (3)

Proof + reasoning:

Lemma 1. $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, a, b \in \mathbb{N} : x \equiv_a y \Rightarrow bx \equiv_a by$.

Proof. $\exists p \in \mathbb{Z} : x = y + pa$, so $bx = by + bpa$, so $bx \equiv_a by$. □

Lemma 2. $\forall x \in \mathbb{R}, a, b \in \mathbb{N}, a(x \pmod b) \equiv_b ax$.

Proof. Clearly, $x \pmod b \equiv_b x$. By lemma 1, $a(x \pmod b) \equiv_b ax$. □

Lemma 3. $\forall x, y \in \mathbb{R}, \forall k \in \mathbb{N}, (x \pmod k) + (y \pmod k) \equiv_k x + y$.

Proof. $x + y = (x \pmod k) + (y \pmod k) + pk + qk = (x \pmod k) + (y \pmod k) + (p + q)k$.
So, $x + y \equiv_k (x \pmod k) + (y \pmod k)$. □

Given ψ and ϕ in \mathbb{T} , define $(\psi + \phi)_i = (\psi_i + \phi_i) \pmod 1$. (4)

Let ψ and ϕ be elements of \mathbb{T} . It suffices to show $\psi_i + \phi_i \equiv_1 2(\psi + \phi)_{i+1}$. (5)

By (1), lemma 3 and lemma 2,

$$\begin{aligned} \psi_i + \phi_i &= (2\psi_{i+1} \pmod 1) + (2\phi_{i+1} \pmod 1) \\ &\equiv_1 2\psi_{i+1} + 2\phi_{i+1} \\ &= 2(\psi_{i+1} + \phi_{i+1}) \\ &\equiv_1 2((\psi + \phi)_{i+1} \pmod 1) \\ &\equiv_1 2(\psi + \phi)_{i+1} \end{aligned} \quad (6)$$

From (6) and (5), \mathbb{T} is closed under addition. (7)

For $\phi \in \mathbb{T}$, define $(-\phi)_i := -\phi_i$. (8)

Clearly, this is the inverse of ϕ . (9)

Suppose that $\phi^n \rightarrow \phi$ and $\psi^n \rightarrow \psi$ in $(S^1)^{\mathbb{N}_0}$. Let $n \geq 0$. By assumption, $\exists K_n$ s.t. $\forall j \leq n, \phi_j^n = \phi_j, \psi_j^n = \psi_j$, hence $(\phi^n - \psi^n)_j = \phi_j - \psi_j$. (10)

By (10), $\forall \varepsilon > 0, \exists k$ s.t. $d(\phi^i - \psi^i, \phi - \psi) \leq \varepsilon \forall i \geq k$, so $\lim_{n \rightarrow \infty} d(\phi^n - \psi^n, \phi - \psi) = 0$, hence $(\phi, \psi) \mapsto \phi - \psi$ is continuous. (11)

Let $\phi, \psi \in \mathbb{T}$. Show $\alpha(\phi) + \alpha(\psi) = \alpha(\phi + \psi)$. By lemma 3 and lemma 2,

$$\begin{aligned} (\alpha(\phi) + \alpha(\psi))_i &= ((2\phi_i) \pmod 1 + (2\psi_i) \pmod 1) \pmod 1 \\ &\equiv_1 2(\phi_i + \psi_i) \\ &\equiv_1 2((\phi + \psi)_i \pmod 1) \\ &= \alpha(\phi + \psi)_i. \end{aligned} \quad (12)$$

Clearly, α preserves the identity. (13)

Suppose $\alpha(\phi) = \alpha(\psi)$. Then $\phi_{i-1} = \alpha(\phi)_i = \alpha(\psi)_i = \psi_{i-1} \forall i \geq 1$, so $\phi = \psi$, hence α is injective. (14)

Let $\phi \in \mathbb{T}$. Let $(\phi')_i := \phi_{i+1} \forall i \geq 1$. (15)

$\alpha(\phi') = \phi$, so α is surjective. (16)

By (14) and (16), α is bijective. By (12), (13), and (14), α is a group automorphism. (17)

For product topologies the 1-d cylinders form a subbasis. So, to show that α is continuous it suffices to show that $\forall i \in \mathbb{N}, \pi_i \circ \alpha$ is continuous. (18)

Let $i \in \mathbb{N}$. Note $\pi_i \circ \alpha : (\phi_0, \phi_1, \dots) \mapsto \begin{cases} \phi_{i-1} & \text{if } i \geq 1 \\ 2\phi_0 & \text{if } i = 0 \end{cases}$. (19)

The map $r : S^1 \rightarrow S^1 : s \mapsto 2s \pmod{1}$ is clearly continuous. (20)

By (20), if $A \in \mathcal{T}(S^1)$, then

$$(\pi_i \circ \alpha)^{-1}(A) = \begin{cases} (S^1)^{i-1} \times A \times S^1 \times \dots & \text{if } i \geq 1 \\ r^{-1}(A) \times S^1 \times \dots & \text{if } i = 0 \end{cases} \quad (21)$$

By (21) and (20), $\pi_i \circ \alpha$ is continuous, so by (18), α is continuous. (22)

Note $\pi_i \circ \alpha^{-1} : (\phi_0, \phi_1, \dots) \mapsto \phi_{i+1}$. This is clearly continuous, so by (18), α^{-1} is continuous, so by (22) and (17), α is a homeomorphism. (23)

Exercise 1.10.3.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function. (1)

Show that $-f$ is a Lyapunov function for the gradient flow. (2)

Show that the trajectories of the gradient flow are orthogonal to the level sets of f . (3)

Proof + reasoning:

The gradient flow is the flow of the differential equation $\dot{x} = \nabla f(x)$. (4)

Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$. Note $(f \circ g_x)(0) = f(x)$ and $(f \circ g_x)(t) = f(g^t(x))$. (5)

By (5), if $(f \circ g_x)'(s) \geq 0$ for all $s \in \mathbb{R}^+$ then $-f$ is Lyapunov. (6)

By (4), $g'_x(t) = \nabla f(g_x(t))$. (7)

By the multivariate chain rule and (7),

$$(f \circ g_x)'(t) = \langle \nabla f(g_x(t)), g'_x(t) \rangle = \langle g'_x(t), g'_x(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^n . (8)

By definition of inner products, $\langle g'_x(t), g'_x(t) \rangle \geq 0$. (9)

By (8), (9) and (6), $-f$ is Lyapunov. (10)

Let $x \in \mathbb{R}^n$. (11)

Define the level set $C := f^{-1}(f(x))$. (12)

Let $T_x = \{\dot{\gamma}(0) : \exists \varepsilon > 0 \text{ s.t. } \gamma : (-\varepsilon, \varepsilon) \rightarrow C \text{ is smooth and } \gamma(0) = x\}$. (13)

By (7), $(f \circ g_x)'(0) = \nabla f(g_x(0)) = \nabla f(x) = g'_x(0)$. (14)

Let $V \in T_x$, with corresponding path $\gamma : (-\varepsilon, \varepsilon) \rightarrow C$. (15)

Since $\gamma(t) \in C$ for all $t \in (-\varepsilon, \varepsilon)$, $f(\gamma(t)) = f(\gamma(0))$ for all $t \in (-\varepsilon, \varepsilon)$. (16)

By the multivariate chain rule,

$$\begin{aligned} \langle \nabla f(x), V \rangle &= \sum_{k=1}^n V_k \frac{\partial f}{\partial y_k}(x) \\ &= \sum_{k=1}^n V_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\ &= \sum_{k=1}^n \dot{\gamma}(0)_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\ &= (f \circ \gamma)'(0). \end{aligned} \tag{17}$$

By (16), $(f \circ \gamma)'(0) = 0$, so by (17), $\langle \nabla f(x), V \rangle = 0$, so by (14), the trajectories of the gradient flow are orthogonal to the level sets of f . (18)

Exercise 1.11.3.

Suppose $1, s$ and αs are real numbers that are linearly independent over \mathbb{Q} . (1)

Show that every orbit of the time- s map ϕ_α^s is dense in \mathbb{T}^2 . (2)

Proof + reasoning:

Let $x \in \mathbb{T}^2$, $y \in \mathbb{T}^2$, $\varepsilon' > 0$ and $\varepsilon = \frac{\varepsilon'}{2\sqrt{2}}$. (3)

Let \mathcal{P}_ε be a partition of \mathbb{T}^2 into finitely many squares of the form $[a, b)^2$, where $\frac{\varepsilon}{2} < |a - b| < \varepsilon$. (4)

By the pigeonhole principle, there exists a $P \in \mathcal{P}_\varepsilon$ and $k > m$ in \mathbb{Z} such that $\phi_\alpha^{ks}(x)$ and $\phi_\alpha^{ms}(x)$ are in P . (5)

By (5), $d(z, \phi_\alpha^{(k-m)s}(z)) < \sqrt{2}\varepsilon$ for all $z \in \mathbb{T}^2$, where d is the metric on \mathbb{T}^2 . (6)

Conjecture 4. *There exists a $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $y \in \mathbb{T}^2$*

$$\frac{(\phi_\alpha^{(k-m)s}(y))_2 - y_2}{(\phi_\alpha^{(k-m)s}(y))_1 - y_1} = \beta.$$

Proof. Suppose for contradiction that $s = 0$. Then for $p = 1, q = 1, r = 0$ we have $pas + qs + r = 0$, a contradiction, so $s \neq 0$. Similarly, $\alpha s \neq 0$. Suppose for contradiction that $\alpha s \in \mathbb{Q}$. Let $p = 1, q = -\alpha s, r = 0$. Then $pas + qs + r = 0$, a contradiction, so $\alpha s \notin \mathbb{Q}$. Suppose for contradiction that $\frac{1}{\alpha} \in \mathbb{Q}$. Then s is irrational. Let $p = \frac{1}{\alpha}, q = -1, r = 0$. Then $pas + qs + r = 0$, a contradiction, so $\frac{1}{\alpha}$ is irrational. Let $y \in \mathbb{T}^2$. Then

$$\frac{(\phi_\alpha^{(k+m)s}(y))_2 - y_2}{(\phi_\alpha^{(k+m)s}(y))_1 - y_1} = \frac{(k-m)s}{(k-m)\alpha s} = \frac{1}{\alpha}$$

So, with $\beta = \frac{1}{\alpha}$, the statement follows. \square

Let γ be the line in \mathbb{T}^2 starting from x in the direction of $x - \phi_\alpha^{m-k}(x)$. (7)

Let β be the slope of γ , which is finite and in $\mathbb{R} \setminus \mathbb{Q}$ by 4. (8)

By (8), considering γ as a subset of \mathbb{T}^2 , we have

$$\begin{aligned} \gamma \cap (y_1 \times \mathbb{T}) &= \bigcup_{n \geq 0} \{(y_1, (x_2 + \beta(y_1 - x_1) + \beta n) \bmod 1)\} \\ &= \bigcup_{n \geq 0} \{(y_1, R_\beta^n(x_2 + \beta(y_1 - x_1)))\}. \end{aligned} \quad (9)$$

By (8), R_β has dense semiorbits. (10)

By (10) and (9), there exists a $z \in \gamma \cap (y_1 \times (y_2 - \varepsilon, y_2 + \varepsilon))$. (11)

By (6) and Conjecture 4, there exists a $p \in \mathbb{N}$ such that

$$d(\phi_\alpha^{p(k-m)s}(x), z) < \sqrt{2}\varepsilon \quad (12)$$

By (12) and (11),

$$\begin{aligned}
d(\phi_\alpha^{p(k-m)s}(x), y) &\leq d(\phi_\alpha^{p(k-m)s}(x), z) + d(z, y) \\
&\leq \sqrt{2}\varepsilon + \varepsilon \\
&\leq 2\sqrt{2}\varepsilon \\
&\leq \varepsilon'.
\end{aligned} \tag{13}$$

By (13), every orbit of ϕ_α^s is dense in \mathbb{T}^2 . (14)

Exercise 1.12.3.

Compute the Lyapunov exponents for the solenoid. (1)

Proof + reasoning:

Let $F: S^1 \times D^2 \rightarrow S^1 \times D^2$ be the solenoid. Let $x, y \in \mathbb{R}$ and let $\lambda \in (0, \frac{1}{2})$. (2)

Note $F(\phi, x, y) = (2\phi, \lambda x + \frac{1}{2} \cos(2\pi\phi), \lambda y + \sin(2\pi\phi))$. (3)

By writing out the composition, we see that:

$$\begin{aligned}
 F^n(\phi, x, y)_1 &= 2^n \phi \\
 F^n(\phi, x, y)_2 &= \lambda^n x + \frac{1}{2} \lambda^{n-1} \cos(2\pi\phi) + \cdots + \frac{1}{2} \lambda^0 \cos(2^{n-1}\pi\phi) \\
 &= \lambda^n x + \frac{1}{2} \sum_{i=0}^{n-1} \lambda^i \cos(2^{n-1-i}\pi\phi) \\
 F^n(\phi, x, y)_3 &= \lambda^n y + \frac{1}{2} \sum_{i=0}^{n-1} \lambda^i \sin(2^{n-1-i}\pi\phi)
 \end{aligned}
 \tag{4}$$

By (4), denoting $\delta_{ij} := \frac{\partial F_i}{\partial z_j}(\phi, x, y)$, we can express $dF^n(\phi, x, y)$ as follows:

$$\begin{aligned}
 \delta_{11} &= 2^n \\
 \delta_{21} &= -\frac{1}{2} \sum_{i=0}^{n-1} \lambda^i 2^{n-1-i} \pi \sin(2^{n-1-i}\pi\phi) \\
 &= -\frac{\pi}{2} \sum_{i=0}^{n-1} \lambda^i 2^{n-1-i} \sin(2^{n-1-i}\pi\phi) \\
 \delta_{31} &= \frac{\pi}{2} \sum_{i=0}^{n-1} \lambda^i 2^{n-1-i} \cos(2^{n-1-i}\pi\phi) \\
 \delta_{22} &= \lambda^n \\
 \delta_{33} &= \lambda^n \\
 \delta_{ij} &= 0 \quad \text{otherwise}
 \end{aligned}
 \tag{5}$$

The Lyapunov exponent is defined as

$$\chi(\phi, x, y, v) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|dF^n(\phi, x, y)v\|
 \tag{6}$$

By (5),

$$dF^n(\phi, x, y)v = v_1(2^n + \delta_{21} + \delta_{31}) + \lambda^n(v_2 + v_3)
 \tag{7}$$

Note, since $\lambda \in (0, \frac{1}{2})$, that $|\delta_{21}| \leq \pi \cdot n \cdot 2^n$ and $|\delta_{31}| \leq \pi \cdot n \cdot 2^n$. (8)

Suppose $v_1 \neq 0$. By (8) and (7), for n sufficiently large,

$$\begin{aligned}
\frac{1}{n} \log \|dF^n(\phi, x, y)v\| &= \frac{1}{2} \cdot \frac{1}{n} \log (\|dF^n(\phi, x, y)v\|^2) \\
&= \frac{1}{2n} \log (v^2(2^n + \delta_{21} + \delta_{31})^2 + \lambda^{2n}(v_2 + v_3)^2) \\
&\leq \frac{1}{2n} \log ((v_1 \cdot 3\pi \cdot n \cdot 2^n)^2 + \lambda^{2n}(v_2 + v_3)^2) \\
&\leq \frac{1}{2n} \log ((v_1 \cdot 4\pi \cdot n \cdot 2^n)^2) \\
&= \frac{1}{n} \log (v_1 \cdot 4\pi \cdot n \cdot 2^n) \\
&= \frac{1}{n} (\log(v_1 \cdot 4\pi \cdot n) + n \log(2)) \\
&\xrightarrow{n \rightarrow \infty} \log(2)
\end{aligned} \tag{9}$$

By (9), $\chi(\phi, x, y, v) \leq \log(2)$. (10)

For the lower bound, by (11) and (9),

$$\begin{aligned}
\frac{1}{n} \log \|dF^n(\phi, x, y)v\| &= \frac{1}{2n} \log (v_1^2(2^n + \delta_{21} + \delta_{31})^2 + \lambda^{2n}(v_2 + v_3)^2) \\
&\geq \frac{1}{2n} \log (v_1^2 \cdot 2^{2n}) \\
&= \frac{1}{n} \log (v_1 \cdot 2^n) \\
&= \log(2) + \frac{1}{n} \log(v_1) \xrightarrow{n \rightarrow \infty} \log(2)
\end{aligned} \tag{11}$$

By (10) and (11), $\chi(\phi, x, y, v) = \log(2)$. (12)

Suppose $v_1 = 0$. By (13) and (7),

$$\begin{aligned}
\frac{1}{n} \log \|dF^n(\phi, x, y)v\| &= \frac{1}{n} \log (\lambda^n(v_2 + v_3)) \\
&= \log(\lambda) + \frac{1}{n} \log(v_2 + v_3) \\
&\xrightarrow{n \rightarrow \infty} \log(\lambda)
\end{aligned} \tag{13}$$

By (13), $\chi(\phi, x, y, v) = \log(\lambda)$. (14)

By (14) and (12), the Lyapunov exponents are $\log(2)$ and $\log(\lambda)$. \square (15)

Exercise 2.1.3.

Let $f : X \rightarrow X$ be a topological dynamical system. (1)

Show that $\mathcal{R}(f) \subseteq \text{NW}(f)$. (2)

Proof + reasoning:

Let $x \in \mathcal{R}(f)$. (3)

Let U be a neighborhood of x , and V an open set such that $V \subseteq U$ and $x \in V$. (4)

By (3) and (4), there exists a recurrent point z in V . (5)

By (5), there exists an increasing sequence (m_k) such that

$$f^{m_k}(z) \rightarrow z \quad \text{and} \quad m_k \rightarrow \infty. \quad (6)$$

Since V is a neighborhood of z , by (6) there exists an $M \geq 1$ such that $\forall i \geq M$, $f^{m_i}(z) \in V$, so $f^{m_M}(z) \in U$, hence $f^{m_M}(U) \cap U \neq \emptyset$. (7)

By (7), $\mathcal{R}(f) \subseteq \text{NW}(f)$. (8)

Exercise 2.2.3.

Is the product of two topologically transitive systems topologically transitive? (1)

Is a factor of a topologically transitive system topologically transitive? (2)

Proof + reasoning:

Let R_α be the circle translation, where α is irrational. (3)

It is known that R_α is topologically transitive. (4)

Let $(a, b) \in S^1 \times S^1$. (5)

If $a \geq b$, then the orbit of (a, b) under $R_\alpha \times R_\alpha$ is contained in $l_1 \cup l_2$ where

$$\begin{aligned} l_1 &= \{t(a - b, 0) + (1 - t)(1, b - a + 1) : t \in [0, 1]\}, \\ l_2 &= \{t(1, b - a + 1) + (1 - t)(1, b - a + 1) : t \in [0, 1]\}. \end{aligned} \quad (6)$$

If $b \geq a$, then the same holds with

$$\begin{aligned} l_1 &= \{t(0, a - b + 1) + (1 - t)(b - a, 1) : t \in [0, 1]\}, \\ l_2 &= \{t(b - a, 0) + (1 - t)(1, a - b + 1) : t \in [0, 1]\}. \end{aligned} \quad (7)$$

In both cases, l_1 and l_2 are lines contained in $[0, 1) \times [0, 1)$. Since these lines are clearly not dense in $S^1 \times S^1$, the forward orbit of (a, b) is not dense in $S^1 \times S^1$. Hence, $R_\alpha \times R_\alpha$ is not topologically transitive. (8)

Suppose $f : X \rightarrow X$ and $g : Y \rightarrow Y$ are topological dynamical systems, that π is a topological semiconjugacy from f to g , and that f is topologically transitive with point $x \in X$ with dense forward orbit. (9)

Let $U \subseteq Y$ be open. Since π is continuous, $\pi^{-1}(U)$ is open, so by (9), there exists a $k \in \mathbb{N}$ such that $f^k(x) \in \pi^{-1}(U)$. (10)

By (9), $\pi \circ f^k(x) = g^k(\pi(x))$. (11)

By (10) and (11), $g^k(\pi(x)) \in U$, so $\pi(x)$ is dense, hence a factor of a topologically transitive system is topologically transitive. (12)

Exercise 2.3.3.

Show that a factor of a topologically mixing system is also topologically mixing. (1)

Proof + reasoning:

Let $f : X \rightarrow X$ and $g : Y \rightarrow Y$ be topological dynamical systems, and π a topological semiconjugacy from f to g . (2)

Let U and V be nonempty open sets in Y . (3)

Since π is surjective and continuous, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are nonempty and open. (4)

By (2) and (4), there exists an $N \in \mathbf{N}$ such that for all $n \geq N$, $f^n(\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \emptyset$. (5)

By (2),

$$\begin{aligned} \pi(f^n(\pi^{-1}(U)) \cap \pi^{-1}(V)) &\subseteq \pi(f^n(\pi^{-1}(U))) \cap \pi(\pi^{-1}(V)) \\ &= g^n(\pi(\pi^{-1}(U))) \cap \pi(\pi^{-1}(V)) \\ &= g^n(U) \cap V. \end{aligned} \quad (6)$$

By (6) and (5), $g^n(U) \cap V \neq \emptyset$, so g is topologically mixing, hence a factor of a topologically mixing system is topologically mixing. (7)

Exercise 2.5.3.

Let $\{a_n\}$ be a subadditive sequence of non-negative real numbers, i.e. (1)

$$0 \leq a_{m+n} \leq a_m + a_n \text{ for all } m, n \geq 0. \quad (2)$$

$$\text{Show that } \lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 0} \frac{a_n}{n}. \quad (3)$$

Proof + reasoning:

$$\text{Let } k \in \mathbb{N}_+. \quad (4)$$

$$\text{Let } n \geq k. \quad (5)$$

$$\text{By (5), } n = mk + m', \text{ where } m \in \mathbb{N} \text{ and } m' < k. \quad (6)$$

By (6), and the subadditivity of (a_n) ,

$$\begin{aligned} \frac{a_n}{n} - \frac{a_k}{k} &= \frac{a_{mk+m'}}{n} - \frac{a_k}{k} \\ &\leq \frac{a_{mk} + a_{m'}}{n} - \frac{a_k}{k} \\ &\leq \frac{ma_k}{mk + m'} + \frac{ka_1}{n} - \frac{a_k}{k} \\ &\xrightarrow{n \rightarrow \infty} 0 \end{aligned} \quad (7)$$

$$\text{Hence, } \lim_{n \rightarrow \infty} \frac{a_n}{n} \text{ is a lower bound for } \left\{ \frac{a_n}{n} : n \geq 1 \right\}. \quad (8)$$

Additionally, if $C \leq \frac{a_m}{m}$ for all $m \in \mathbb{N}$, then clearly $C \leq \lim_{n \rightarrow \infty} \frac{a_n}{n}$, so by (8),

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 0} \frac{a_n}{n}. \quad (9)$$

Exercise 2.7.3.

Give a non-trivial example of a homeomorphism f of a compact metric space (X, d) such that $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$ for every pair $x, y \in X$. (1)

Proof + reasoning:

Define $f : S^1 \rightarrow S^1$ by

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases} \quad (2)$$

Clearly, f is a homeomorphism such that $d(f^n(x), f^n(y)) \rightarrow 0$ as $n \rightarrow \infty$ for all pairs x, y . (3)

Exercise 2.8.3.

Prove the following generalization of Proposition 2.1.2. If a commutative group G acts by homeomorphisms on a compact metric space X , then there is a non-empty, closed G -invariant subset X' on which G acts minimally. (1)

Proof + reasoning:

Let \mathcal{C} be the collection of non-empty, closed G -invariant subsets of X , with the partial ordering given by inclusion. (2)

Since $X \in \mathcal{C}$, \mathcal{C} is not empty. (3)

Suppose $\mathcal{K} \subseteq \mathcal{C}$ is a totally ordered subset. Then, any finite intersection of elements of \mathcal{K} is nonempty, so by the finite intersection property for compact sets, $\bigcap_{K \in \mathcal{K}} K \neq \emptyset$. Thus, by Zorn's lemma, \mathcal{C} contains a minimal element M . (4)

Suppose that G does not act minimally on M . (5)

Then, there exists a point $b \in M$ and a nonempty open set $C \subseteq M$ such that $Gb \cap C = \emptyset$. (6)

Since $GC = \bigcup_{g \in G} gC$, and each $g \in G$ is a homeomorphism, GC is open. (7)

So, since M is closed, $M \setminus GC$ is closed. (8)

Since $b \in M$ and $Gb \cap C = \emptyset$, $b \in M \setminus GC$, so $M \setminus GC$ is nonempty. (9)

If $m \in M \setminus GC$ and $g \in G$, then $m \neq g^{-1}c$, so $gm \neq C$, so since M is G -invariant, $gm \in M \setminus GC$, so $M \setminus GC$ is G -invariant. (10)

By (8)–(10), $M \setminus GC$ is a closed, nonempty, G -invariant proper subset of M , which contradicts (4), so (5) is false. Hence, G acts minimally on M . (11)

Exercise 3.1.3.

Use a higher block presentation to prove that for any block code $c : X \rightarrow Y$ there is a subshift Z and an isomorphism $f : Z \rightarrow X$ such that $c \circ f : Z \rightarrow Y$ is a $(0, 0)$ -block code. (1)

Proof + reasoning:

Let $c : X \rightarrow Y$ be a block code, with corresponding function $\alpha : W_{a+b+1} \rightarrow \mathcal{A}_m$. (2)

Letting $k = a + b + 1$ and $l = b$, the higher block presentation d of X can be written as

$$d(x)_i = x_{i-a} \dots x_{i+b}, \quad i \in \mathbb{Z} \quad (3)$$

Since $\text{im}(d) \subseteq \Sigma_{W_{a+b+1}(X)}$ we have $W_1(\text{im}(d)) \subseteq W_{a+b+1}(X)$. (4)

If $\omega \in W_{a+b+1}(X)$, then for some sequence $x \in X$ and $i \in \mathbb{Z}$,

$$\omega = x_{i-a} \dots x_{i+b}, \quad \text{so} \quad d(x)_i = \omega,$$

so $\omega \in W_1(\text{im}(d))$. (5)

By (5) and (4), $W_1(\text{im}(d)) = W_{a+b+1}(X)$. (6)

By Exercise 3.1.2, d is an isomorphism onto its image. Let $d^{-1} : \text{im}(d) \rightarrow X$ be its inverse. (7)

If $z \in \text{im}(d)$ and $i \in \mathbb{Z}$, then there exists a unique x such that $d(x) = z$, so

$$\begin{aligned} (c \circ d^{-1})(z)_i &= c(d^{-1}(z))_i \\ &= c(x)_i \\ &= \alpha(x_{i-a} \dots x_{i+b}) \\ &= \alpha(z_i) \end{aligned} \quad (8)$$

By (8) and (5), $c \circ d^{-1} : \text{im}(d) \rightarrow Y$ is a $(0, 0)$ -block code and, by (7), d^{-1} is an isomorphism. (9)

Exercise 3.2.3.

Show that every edge shift is an SFT. (1)

If Σ_B^e is an edge shift with graph Γ_B , then Σ_B^e is precisely the set of sequences that do not contain the words $e'e$ of length 2 in which the target of e is not equal to the source of e' . (2)

Since this collection of words is finite, Σ_B^e is an SFT. (3)

Exercise 3.4.3.

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Calculate the zeta function of Σ_A^e . (1)

Proof + reasoning:

By proposition 3.4.2, $\zeta_A(z) = (\det(I - zA))^{-1}$. (2)

Therefore,

$$\zeta_A(z) = \left(\det \begin{pmatrix} 1-z & -z \\ -z & 1 \end{pmatrix} \right)^{-1} = (1 - 2z - z^2)^{-1}. \quad (3)$$

Exercise 3.7.3.

Show that there are only countably many non-isomorphic sofic shifts. (1)

Conclude that there are subshifts that are not sofic. (2)

Proof + reasoning:

Proposition 3.7.1 states that a subshift is sofic iff it admits a presentation by a finite directed labelled graph. (3)

Note, for each m and n in \mathbb{N} , up to graph isomorphism, there are only a finite number of directed labelled graphs with m vertices and n edges, so there are only countably many non-isomorphic finite directed labelled graphs. (4)

Since a graph isomorphism corresponds to a conjugacy of the corresponding subshifts, by proposition 3.7.1, there are only countably many non-isomorphic sofic subshifts. (5)

By exercise 3.2.2, the collection of all subshifts of Σ_2 is uncountable. (6)

Let $X \subseteq \Sigma_2$ be a subshift. (7)

Let $k, l \in \mathbb{N}$. (8)

Clearly, $W_n(X)$ is finite, so there exist only finitely many functions $W_n(X) \rightarrow A_2$. Therefore, there exist only finitely many (k, l) -block codes $X \rightarrow \Sigma_2$, hence there exist at most countably many block codes $X \rightarrow \Sigma_2$. (9)

By proposition 3.1.2 every code is a block code, so by statement (9), there exist at most countably many codes $X \rightarrow \Sigma_2$. (10)

If there exists a $Y \subseteq \Sigma_2$ such that $X \cong Y$, then by definition there exists an isomorphism $c_Y : X \rightarrow Y$, where c_Y can be considered as a code $c_Y : X \rightarrow \Sigma_2$. (11)

This defines an injective function $\{Y \subseteq \Sigma_2 : Y \cong X\} \rightarrow \{c : X \rightarrow \Sigma_2, c \text{ is a code}\}$. Hence, by (10), $\{Y \subseteq \Sigma_2 : Y \cong X\}$ is at most countable. (12)

Hence, by (12), the set of all isomorphism classes of subshifts of Σ_2 is uncountable. (13)

Hence, by (5), there are subshifts that are not sofic. (14)

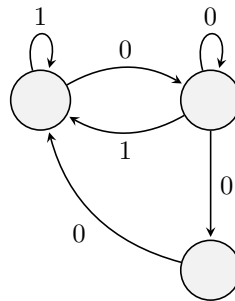
Exercise 3.8.3.

A common coding scheme called modified frequency modulation (MFM) inserts a 0 between each two symbols unless they are both 0's, in which case it inserts a 1. (1)

Prove that the set of sequences produced by the MFM coding is a sofic system. (2)

Proof + reasoning:

Consider the following graph.



(3)

This is a finite directed labeled graph that represents the set of sequences produced by the MFM, so by proposition 3.7.1, this system is sofic. (4)

Exercise 4.2.3

Prove that if T is a measure-preserving transformation, then so are the induced transformations. (1)

Proof + reasoning:

Let $T : (X, \mathcal{A}, \mu) \rightarrow (X, \mathcal{A}, \mu)$ be a measure-preserving transformation. (2)

Let's start with proving that the derivative transformation is measure preserving. First, let's check that it is measurable. (3)

Let \mathcal{E} be the trace σ -algebra with respect to $A \in \mathcal{A}$. (4)

Let $B \in \mathcal{E}$. (5)

By definition, $B = C \cap A$, $C \in \mathcal{A}$. (6)

Let (R_n) and (D_n) be sequences of sets defined inductively by letting

$$\begin{aligned} R_0 &= B, & D_0 &= \emptyset \\ R_1 &= T^{-1}(B) \setminus A, & D_1 &= T^{-1}(B) \cap A \\ R_{n+1} &= T^{-1}(R_n) \setminus A, & D_{n+1} &= T^{-1}(R_n) \cap A \quad \forall n \geq 2 \end{aligned} \quad (7)$$

Let $D := \bigcup_{n \geq 1} D_n$ (8)

Conjecture 5. $D = T_A^{-1}(B)$

Proof. Let $n \geq 2$

$$\begin{aligned} R_n &= T^{-1}(R_{n-1}) \setminus A \\ &= T^{-1}(R_{n-1}) \cap A^c \\ &= T^{-1}(T^{-1}(R_{n-2}) \cap A^c) \cap A^c \\ &= (T^{-2}(R_{n-2}) \cap T^{-1}(A^c)) \cap A^c \\ &= (T^{-n}(B) \cap \dots \cap T^{-1}(A^c)) \cap A^c \\ &= T^{-n}(B) \cap \left(\bigcap_{i=0}^{n-1} T^{-i}(A^c) \right). \end{aligned}$$

This gives

$$D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A.$$

By definition, $T_A^{-1}(B)$ is the set of points $y \in A$ such that $T(y) \in B$ or such that there exists a $k \in \mathbb{N}$ with $k \geq 2$ such that $T^k(y) \in B$ and $T^i(y) \notin A$ for all $i \in \{1, \dots, k-1\}$.

From the above, it follows that $D = T_A^{-1}(B)$. \square

T is \mathcal{A} -measurable, so from $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A$ it follows that $D_n \in \mathcal{E}$. Since D is a countable union of such D_n , $D \in \mathcal{E}$, so by conjecture 5, $T_A^{-1}(B) \in \mathcal{E}$, so T_A is \mathcal{E} -measurable. (9)

Let $i, j \in \mathbb{N}$ with $i > j \geq 1$. (10)

Suppose $D_j \cap D_i \neq \emptyset$. (11)

By (11), there exists $x \in D_j \cap D_i$. By the fact that $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c)\right) \cap A$, it follows that $T^j(x) \in B$. (12)

Since $j < i$, $T^j(x) \in A^c$, but this contradicts (12), so D_j and D_i are disjoint. (13)

From (7), it is clear that $\forall n \in \mathbb{N}$, $D_n \cap R_n = \emptyset$ (14)

$$\mu(R_{n+1}) + \mu(D_{n+1}) = \mu(R_n) \quad \forall n \in \mathbb{N} \quad (15)$$

From (15), $\mu(R_n)$ is decreasing. (16)

By (13),

$$\mu(D) = \sum_{n \geq 1} \mu(D_n) \quad (17)$$

From (15),

$$\sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \mu(R_n) \quad \forall n \in \mathbb{N} \quad (18)$$

By (17), (18) and (16),

$$\mu(D) = \lim_{n \rightarrow \infty} \sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \lim_{n \rightarrow \infty} \mu(R_n) \quad (19)$$

By the Poincaré recurrence theorem,

$$\mu(T_A^{-1}(B)) \geq \mu(B) \quad (20)$$

By conjecture 5 and (20), $\mu(D) \geq \mu(B)$. (21)

By (21) and (19),

$$\mu(B) - \lim_{n \rightarrow \infty} \mu(R_n) = \mu(D) \geq \mu(B) \quad (22)$$

From (22),

$$\lim_{n \rightarrow \infty} \mu(R_n) = 0 \quad (23)$$

From (23), (19) and conjecture 5,

$$\mu(T_A^{-1}(B)) = \mu(D) = \mu(B) \quad (24)$$

By (24), T_A is measure-preserving. (25)

Let $T_f : X_f \rightarrow X_f$ be the primitive transformation, where $f : X \rightarrow \mathbb{N}$ is measurable. (26)

Let $A \in \mathcal{A}$ and $k \in \mathbb{N}$. (27)

Note, $(A \times \{k\}) \cap X_f = (A \cap C_k) \times \{k\}$ where $C_k = f^{-1}(\{n \in \mathbb{N} : n \geq k\})$ (28)

$X_f = \{(x, k) : x \in X, 1 \leq k \leq f(x)\} \subseteq X \times \mathbb{N}$ (29)

Suppose $k > 1$. By (28):

$$T_f^{-1}((A \times \{k\}) \cap X_f) = T_f^{-1}((A \cap C_k) \times \{k\}) = (A \cap C_k) \times \{k-1\} \quad (30)$$

Suppose $k = 1$ (31)

$$T_f^{-1}(A \times \{k\}) = \bigcup_{i \geq 1} (T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \quad (32)$$

$$\text{From (30) and (32), } T_f^{-1}(A \times \{k\}) \cap X_f \in \mathcal{U}_f \quad \forall k \quad (33)$$

If $k > 1$, then by (30) and (28),

$$\begin{aligned} \mu_f(T_f^{-1}((A \times \{k\}) \cap X_f)) &= \mu_f((A \cap C_k) \times \{k-1\}) \\ &= \mu_f((A \cap C_k) \times \{k\} \cap X_f) \end{aligned} \quad (34)$$

If $k = 1$, then by (32) and T being measure-preserving,

$$\begin{aligned} \mu_f(T_f^{-1}(A \times \{1\})) &= \mu_f \left(\bigcup_{i \geq 1} (T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \right) \\ &= \sum_{i \geq 1} \mu_f(T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \\ &= \sum_{i \geq 1} \mu(T^{-1}(f^{-1}(i) \cap A)) \\ &= \mu \left(\bigcup_{i \geq 1} T^{-1}(f^{-1}(i) \cap A) \right) \\ &= \mu(T^{-1}(f^{-1}(\mathbb{N}) \cap A)) \\ &= \mu(T^{-1}(A)) \\ &= \mu(A) \\ &= \mu_f(A \times \{1\}) \end{aligned} \quad (35)$$

By (34) and (35), the primitive transformation is measure-preserving. (36)

Exercise 4.3.3.

A measure-preserving transformation or flow T of a probability space (X, \mathcal{U}, μ) is called *(strong) mixing* if

$$\lim_{t \rightarrow \infty} \mu(T^t(A) \cap B) = \mu(A) \cdot \mu(B)$$

for any two measurable sets $A, B \in \mathcal{U}$.

(1)

Equivalently, T is mixing if

$$\lim_{t \rightarrow \infty} \int_X f(T^t(x))g(x) d\mu = \int_X f d\mu \int_X g d\mu$$

for any two bounded measurable functions.

(2)

Transformation T is called *weak mixing* if $\forall A, B \in \mathcal{U}$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

(3)

Equivalently, T is weak mixing if for all bounded measurable functions,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \int_X f(T^i(x))g(x) d\mu - \int_X f d\mu \int_X g d\mu \right| = 0.$$

(4)

Flow T is called weak mixing if $\forall A, B \in \mathcal{U}$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\mu(T^{-s}(A) \cap B) - \mu(A)\mu(B)| ds = 0.$$

(5)

Equivalently, T is weak mixing if for all bounded measurable functions,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X f(T^s(x))g(x) d\mu - \int_X f d\mu \int_X g d\mu \right| ds = 0.$$

(6)

Show that the two definitions of strong and weak mixing given in terms of sets and bounded measurable functions are equivalent.

(7)

Proof + reasoning:

Let T be a measure-preserving flow on (X, \mathcal{U}, μ) .

(8)

Assume (5) holds.

(9)

Suppose f and g are simple, with

$$f = \sum_{i \leq n} \mathbf{1}_{A_i} a_i, \quad g = \sum_{j \leq n} \mathbf{1}_{A_j} b_j.$$

(10)

Define

$$M := \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X f(T^s(x))g(x) d\mu - \int_X f d\mu \int_X g d\mu \right| ds.$$

(11)

Then

$$M = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X f(T^s(x))g(x) d\mu - \sum_{i,j \leq n} \mu(A_i)\mu(A_j)a_i b_j \right| ds. \quad (12)$$

Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be an arbitrary function. For all $x \in \mathbb{R}$, we have that $f(h(x)) = a_i$ if $h(x) \in A_i$. Hence, for all $x \in \mathbb{R}$,

$$f(h(x)) = \sum_{i \leq n} a_i \mathbf{1}_{h^{-1}(A_i)}(x). \quad (13)$$

By (13),

$$\begin{aligned} \int_X f(T^s(x))g(x) d\mu &= \int_X \left(\sum_{i \leq n} \mathbf{1}_{T^{-s}(A_i)} a_i \right) \left(\sum_{j \leq n} \mathbf{1}_{A_j} b_j \right) d\mu \\ &= \sum_{i,j \leq n} \mu(T^{-s}(A_i) \cap A_j) a_i b_j. \end{aligned} \quad (14)$$

Clearly,

$$\int_X f d\mu \int_X g d\mu = \left(\sum_{i \leq n} \mu(A_i) a_i \right) \left(\sum_{j \leq n} \mu(A_j) b_j \right) = \sum_{i,j \leq n} \mu(A_i)\mu(A_j) a_i b_j. \quad (15)$$

By (14) and (15),

$$M = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \sum_{i,j \leq n} a_i b_j (\mu(T^{-s}(A_i) \cap A_j) - \mu(A_i)\mu(A_j)) \right| ds. \quad (16)$$

By (16),

$$\begin{aligned} M &\leq \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \sum_{i,j \leq n} |a_i b_j| |\mu(T^{-s}(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| ds \\ &= \sum_{i,j \leq n} |a_i b_j| \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\mu(T^{-s}(A_i) \cap A_j) - \mu(A_i)\mu(A_j)| ds \\ &= 0. \end{aligned} \quad (17)$$

Assume that f and g are measurable and bounded by some $C > 0$. (18)

By (36), f and g are the uniform limits of sequences (f_n) and (g_n) respectively, where f_n and g_n are simple functions that are bounded by C . (19)

By the dominated convergence theorem,

$$\begin{aligned} M &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X \lim_{n \rightarrow \infty} f_n(T^s(x)) \lim_{n \rightarrow \infty} g_n(x) d\mu - \int_X \lim_{n \rightarrow \infty} f_n d\mu \int_X \lim_{n \rightarrow \infty} g_n d\mu \right| ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lim_{n \rightarrow \infty} \left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f_n d\mu \int_X g_n d\mu \right| ds \end{aligned} \quad (20)$$

Note,

$$\left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f_n d\mu \int_X g_n d\mu \right| \leq 2C\mu(X). \quad (21)$$

By (20), by the fact that the absolute value is continuous, and by (21) together with the dominated convergence theorem,

$$\begin{aligned} M &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \lim_{n \rightarrow \infty} \left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f_n d\mu \int_X g_n d\mu \right| ds \\ &= \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f_n d\mu \int_X g_n d\mu \right| ds \end{aligned} \quad (22)$$

Let $\mathcal{T} = (0, \delta)$, where $\delta > 0$.

Let $\Delta_1 > 0$.

There exists a $k \in \mathbb{N}$ such that for all $n \geq k$ and all $x \in X$, $f_n(T^s(x))g_n(x) \in B(f(T^s(x))g(x), \Delta_1)$, where $B(x, \epsilon)$ denotes a ball around x of radius ϵ , so

$$\int_X f_n(T^s(x))g_n(x) d\mu \in B\left(\int_X f(T^s(x))g(x) d\mu, \Delta_1\mu(X)\right) \quad (25)$$

Let $\Delta_2 \geq 0$. Then there exists an m such that for $n \geq m$,

$$\int_X f_n d\mu \in B\left(\int_X f d\mu, \Delta_2\mu(X)\right) \quad \text{and} \quad \int_X g_n d\mu \in B\left(\int_X g d\mu, \Delta_2\mu(X)\right). \quad (26)$$

By (25) and (26), for all $n \geq \max(m, k)$,

$$\begin{aligned} &\frac{1}{t} \int_0^t \left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f d\mu \int_X g d\mu \right| ds \\ &\in B\left(\frac{1}{t} \int_0^t \left| \int_X f(T^s(x))g(x) d\mu - \int_X f d\mu \int_X g d\mu \right| ds, \Delta_1\mu(X) + 2\Delta_2\mu(X)\right) \end{aligned} \quad (27)$$

Since Δ_1 and Δ_2 were arbitrary, and (27) does not depend on t ,

$$\|h_n(t) - h(t)\|_{\mathcal{T}} \rightarrow 0. \quad (28)$$

Therefore, by the Moore–Osgood theorem,

$$M = \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left| \int_X f_n(T^s(x))g_n(x) d\mu - \int_X f_n d\mu \int_X g_n d\mu \right| ds = 0. \quad (29)$$

Clearly, (6) implies (5), so (5) and (6) are equivalent.

We will skip the proof of (3) \Leftrightarrow (4), since it is likely very similar to the one for (5) \Leftrightarrow (6).

Next, suppose

$$\lim_{t \rightarrow \infty} \mu(T^{-t}(A) \cap B) = \mu(A) \cdot \mu(B)$$

for any two measurable sets $A, B \in \mathcal{U}$.

By (32) and by dominated convergence,

$$\begin{aligned}
\lim_{t \rightarrow \infty} \int_X f_n(T^{-t}(x)) g_n(x) d\mu &= \lim_{t \rightarrow \infty} \int_X \left(\sum_{i=1}^n \mathbf{1}_{T^{-t}(A_i)} a_i^n \right) \left(\sum_{j=1}^n \mathbf{1}_{A_j} b_j^n \right) d\mu \\
&= \lim_{t \rightarrow \infty} \int_X \left(\sum_{i,j \leq n} \mathbf{1}_{T^{-t}(A_i) \cap A_j}(x) a_i^n b_j^n \right) d\mu \\
&= \sum_{i,j \leq n} \lim_{t \rightarrow \infty} \int_X \mathbf{1}_{T^{-t}(A_i) \cap A_j}(x) a_i^n b_j^n d\mu \\
&= \sum_{i,j \leq n} \lim_{t \rightarrow \infty} \mu(T^{-t}(A_i) \cap A_j) a_i^n b_j^n \\
&= \sum_{i,j \leq n} \mu(A_i) \mu(A_j) a_i^n b_j^n.
\end{aligned} \tag{33}$$

So,

$$\int_X f d\mu \int_X g d\mu. \tag{34}$$

Using (19), (34) and dominated convergence, it follows from an argument similar to the one used to derive (29) that

$$\begin{aligned}
M_s(f, g) &:= \lim_{t \rightarrow \infty} \int_X f(T^t(x)) g(x) d\mu \\
&= \lim_{t \rightarrow \infty} \int_X \lim_{n \rightarrow \infty} f_n(T^t(x)) \lim_{n \rightarrow \infty} g_n(x) d\mu \\
&= \lim_{t \rightarrow \infty} \lim_{n \rightarrow \infty} \int_X f_n(T^t(x)) g_n(x) d\mu \\
&= \lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} \int_X f_n(T^t(x)) g_n(x) d\mu \\
&= \lim_{n \rightarrow \infty} \int_X f_n d\mu \int_X g_n d\mu \\
&= \int_X f d\mu \int_X g d\mu.
\end{aligned} \tag{35}$$

In other words, statement (1) implies statement (2). Since the converse is trivial, (1) and (2) are equivalent. (36)