Exercise 2.2.3.

Is the product of two topologically transitive systems topologically transitive? (1)

Is a factor of a topologically transitive system topologically transitive? (2)

Proof + reasoning:

I think the statement is false. Let's construct a counterexample, by starting with the simplest non-trivial product of topologically transitive systems, and iterating from there. (3)

Let R_α be the circle translation, where α is irrational. (4)

It is known that R_α is topologically transitive. (5)

$R_\alpha \times R_\alpha$ already seems to form a counterexample, since $R_\alpha \times R_\alpha$ preserves the distance between the components of points in $S^1 \times S^1$. (6)

Let $(a, b) \in S^1 \times S^1$. (7)

If $a \geq b$, then the orbit of (a, b) under $R_\alpha \times R_\alpha$ is contained in $l_1 \cup l_2$ where

$$\begin{aligned} l_1 &= \{t(a-b, 0) + (1-t)(1, b-a+1) : t \in [0,1]\}, \\ l_2 &= \{t(1, b-a+1) + (1-t)(1, b-a+1) : t \in [0,1]\}. \end{aligned} \tag{8}$$

If $b \geq a$, then the same holds with

$$\begin{aligned} l_1 &= \{t(0, a-b+1) + (1-t)(b-a, 1) : t \in [0,1]\}, \\ l_2 &= \{t(b-a, 0) + (1-t)(1, a-b+1) : t \in [0,1]\}. \end{aligned} \tag{9}$$

In both cases, l_1 and l_2 are lines contained in $[0,1) \times [0,1)$. Since these lines are clearly not dense in $S^1 \times S^1$, the forward orbit of (a, b) is not dense in $S^1 \times S^1$. Hence, $R_\alpha \times R_\alpha$ is not topologically transitive. (10)

Suppose $f : X \to X$ and $g : Y \to Y$ are topological dynamical systems, that π is a topological semiconjugacy from f to g, and that f is topologically transitive with point $x \in X$ with dense forward orbit. (11)

We want to show that the forward orbit of $\pi(x)$ is dense. (12)

Let $U \subseteq Y$ be open. Since π is continuous, $\pi^{-1}(U)$ is open, so by (11), there exists a $k \in \mathbb{N}$ such that $f^k(x) \in \pi^{-1}(U)$. (13)

By (11), $\pi \circ f^k(x) = g^k(\pi(x))$. (14)

By (13) and (14), $g^k(\pi(x)) \in U$, so $\pi(x)$ is dense, hence a factor of a topologically transitive system is topologically transitive. (15)