Exercise 2.5.3.

Let $\{a_n\}$ be a subadditive sequence of non-negative real numbers, i.e. (1)

$0 \leq a_{m+n} \leq a_m + a_n$ for all $m, n \geq 0$. (2)

Show that $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 0} \frac{a_n}{n}$. (3)

Proof + reasoning:

We need to show $\lim_{n\to\infty} \frac{a_n}{n}$ is the infimum, which we can do by checking that it is a lower bound of $\left\{\frac{a_n}{n}\right\}$, and that it is greater than or equal to any lower bound of $\left\{\frac{a_n}{n}\right\}$. (4)

Let $k \in \mathbb{N}_+$. (5)

We need to show $\lim_{n\to\infty} \frac{a_n}{n} \leq \frac{a_k}{k}$. To do that, we could find $f(n) \to 0$ such that, for n sufficiently large,

$$\frac{a_n}{n} \leq \frac{a_k}{k} + f(n).$$

What natural choice for n can we make? (6)

Let $n \geq k$. (7)

To prove (6), we will need to show that $\frac{a_n}{n} - \frac{a_k}{k} \to 0$ as $n \to \infty$. This should follow in some way from the nonnegativity and the subadditivity of (a_i). (8)

Observe that for any natural number m, the value a_{mk} is bounded by ma_k, therefore,

$$\frac{a_{mk}}{mk} \leq \frac{ma_k}{mk} = \frac{a_k}{k}.$$

In other words, for the subsequence of a_i where i is a multiple of k we have the required convergence. And, again by subadditivity, any other element is at most ka_1 away from this subsequence. (9)

By (7), $n = mk + m'$, where $m \in \mathbb{N}$ and $m' < k$. (10)

By (10), and the subadditivity of (a_n),

$$\begin{aligned}\frac{a_n}{n} - \frac{a_k}{k} &= \frac{a_{mk+m'}}{n} - \frac{a_k}{k} \\ &\leq \frac{a_{mk} + a_{m'}}{n} - \frac{a_k}{k} \\ &\leq \frac{ma_k}{mk+m'} + \frac{ka_1}{n} - \frac{a_k}{k} \\ &\xrightarrow{n\to\infty} 0\end{aligned} \tag{11}$$

Hence, $\lim_{n\to\infty} \frac{a_n}{n}$ is a lower bound for $\left\{\frac{a_n}{n} : n \geq 1\right\}$. (12)

Additionally, if $C \leq \frac{a_m}{m}$ for all $m \in \mathbb{N}$, then clearly $C \leq \lim_{n\to\infty} \frac{a_n}{n}$, so by (12),

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 0} \frac{a_n}{n}. \tag{13}$$