Exercise 1.2.3

Let G be a topological group. (1)

Prove that for each $g \in G$, the closure $H(g)$ of the set $\{g^n\}_{n=-\infty}^{\infty}$ is a commutative subgroup of G. (2)

Thus, if G has a minimal left translation, then G is abelian. (3)

Proof + reasoning:

First, let's show the closure of $\{g^n\}_{n=-\infty}^{\infty}$ is a subgroup of G, starting with showing closure under the group operation. (4)

Define $\langle g \rangle := \{g^n\}_{n=-\infty}^{\infty}$. (5)

Let $g \in G$. Let $a, b \in \mathrm{cl}(\langle g \rangle)$. (6)

What do I know about closures? The closure of A is the set X of points such that any neighborhood of $x \in X$ contains a point in A. (7)

My intuition is that the required proof will resemble the one I would follow if G were a metric space. In a metric space, if a and b are limits of g^{k_n} and g^{l_n} we should have $g^{k_n} g^{l_n} \to ab$. Here we do not have a metric, so there is no notion of convergent sequences, but instead of neighborhoods: x is a limit point of A if every neighborhood of x contains a point in A other than x itself. (8)

Intuitively, since a and b are in $\langle g \rangle$ or limit points of $\langle g \rangle$, the product of the two points that 'witness' this property should be the point that witnesses ab being a limit point. (9)

Let C be a neighborhood of ab, and $U \subseteq C$ an open set containing ab. (10)

$a^{-1}U$ and Ub^{-1} are open. (11)

$b \in a^{-1}U$ and $a \in Ub^{-1}$. Since a and b are limit points of $\langle g \rangle$, $\exists k, m \in \mathbb{Z}$ such that $g^m \in a^{-1}U$ and $g^k \in Ub^{-1}$. (12)

$g^k g^m$ should be in U, but I can't show why. What tools can I give myself to help prove $g^k g^m \in U$? (13)

Well, g^m and g^k are homeomorphisms, so $g^k g^m \in g^k a^{-1}U$, $g^k g^m \in Ub^{-1}g^m$, and $(g^k a^{-1}U) \cup (Ub^{-1}g^m)$ is open. (14)

Now I am stuck. (15)

What given assumptions have I not used? (16)

I have not used the fact that the group operation $G \times G \to G$ is continuous. I only used that, $\forall g \in G$, left and right multiplication by g is a continuous function $G \to G$, which seems to be a weaker statement. Using the 'joint' continuity should work. (17)

Since the group multiplication $\alpha : G \times G \to G$ is continuous, $\alpha^{-1}(U)$ is open in $G \times G$. Since $(a, b) \in \alpha^{-1}(U)$, and since sets of the form $A \times B$, where A and B are open, form a basis for the topology on $G \times G$, there exist open V and W such that $a \in V$, $b \in W$, and such that $V \times W \subseteq \alpha^{-1}(U)$. (18)

Since $a, b \in \mathrm{cl}(\langle g \rangle)$ there exist $g^\ell \in V$ and $g^p \in W$. By (18), $g^\ell g^p \in U$, hence $g^{\ell+p} \in U$, so $ab \in \mathrm{cl}(\langle g \rangle) = H(g)$. (19)

By (19), $H(g)$ is closed under taking products. (20)

Now we need to show that $H(g)$ has inverses, by showing $a^{-1} \in H(g)$. (21)

Let C be a neighborhood of a^{-1} and $U \subseteq C$ an open set such that $a^{-1} \in U$. Since the inverse is continuous, $U' := \{x \in G : x^{-1} \in U\}$ is open, and it contains a. (22)

Since $a \in H(g)$, there exists $g^q \in U'$, where $q \in \mathbb{Z}$. (23)

By (23), $g^{-q} = (g^q)^{-1} \in U$, so $a^{-1} \in H(g)$. (24)

By (24), $H(g)$ is closed under taking inverses. (25)

Now to prove that $H(g)$ is commutative. (26)

We need to show that $ab = ba$. If G were a metric space, the proof would follow from the fact that the limits of convergent sequences are unique. Is there something like uniqueness of limit points in a general topological space? The answer seems to be no, only when adding separation properties. (27)

Let's take a few steps back and try again. Note, the product in $H(g)$ is just the restriction of the one in G, so if $ab \neq ba$ in G, $ab \neq ba$ in $H(g)$. So, the only way in which $H(g)$ can be commutative is if it excludes at least all non-commutative elements in G. (28)

So, $H(g)$ must be a proper subgroup if G is not abelian. Considering (28), I think we should try to prove the contrapositive instead, i.e. prove if two elements of G are not commutative, then at least one of them is not in $H(g)$. (29)

Let $c, d \in G$ with $cd \neq dc$. (30)

Why is $(c, d) \notin H(g) \times H(g)$? I am stuck here. (31)

Why has my best attempt not worked? To show (31), we need to show that there exists a neighborhood of (c, d) containing no element of $\langle g \rangle$, but I can't find any obvious neighborhood. There is no given neighborhood from the definitions. I think the exercise is not correct without adding a separation property, so let's add it ourselves. (32)

Suppose that G is Hausdorff. (33)

By (33) and (30), there exist open neighborhoods U of cd and U' of dc such that $U \cap U' = \emptyset$. (34)

Suppose $c, d \in H(g)$. (35)

Similarly to (18), $(c, d) \in \alpha^{-1}(U)$ and $(d, c) \in \alpha^{-1}(U')$. (36)

So there are open sets V, V', W, W' such that $(c, d) \in V \times W \subseteq \alpha^{-1}(U)$ and $(d, c) \in V' \times W' \subseteq \alpha^{-1}(U')$. (37)

From (37), $c \in V \cap V'$ and $d \in W \cap W'$, and $V \cap V'$ and $W \cap W'$ are open. (38)

So, by (35), there exist $s, t \in \mathbb{Z}$ such that $g^s \in V \cap V'$ and $g^t \in W \cap W'$. (39)

By (39), $(g^s, g^t) \in V \times W$ and $(g^t, g^s) \in W' \times V'$. (40)

By (40) and (37), $g^s g^t \in U$ and $g^t g^s \in U'$, so $g^{t+s} \in U \cap U'$. (41)

(41) contradicts (33), so (35) is false, hence $c \notin H(g)$ or $d \notin H(g)$, so $H(g)$ is commutative. (42)

By (42), (20) and (25), $H(g)$ is a commutative subgroup of G. (43)

We still need to prove that if G has a minimal left translation, then G is Abelian. (44)

Suppose that G has a minimal left translation $L_h : G \to G$ where $h \in G$. (45)

By (43), $H(h)$ is a commutative subgroup of G. (46)

By definition, L_h has no proper closed non-empty invariant subsets. (47)

$H(h)$ is a closed non-empty subset of G. (48)

Is $H(h)$ invariant with respect to L_h? (49)

Let $a \in H(h)$. Let C be a neighborhood of ha and U open with $ha \in U \subseteq C$. $a \in h^{-1}U$, and $h^{-1}U$ is open, so $\exists q \in \mathbb{Z}$ such that $h^q \in h^{-1}U$. (50)

By (50), $h^{q+1} \in U$, so $H(h)$ is invariant. (51)

By (51), (48) and (47), $H(h) = G$, so G is abelian. (52)