Exercise 2.1.3.

Let $f : X \to X$ be a topological dynamical system. (1)

Show that $\mathcal{R}(f) \subseteq \mathrm{NW}(f)$. (2)

Proof + reasoning:

Let $x \in \mathcal{R}(f)$. (3)

Let U be a neighborhood of x, and V an open set such that $V \subseteq U$ and $x \in V$. (4)

We need to show that there exists an $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. (5)

By (3) and (4), there exists a recurrent point y in U. (6)

By (6), there exists an increasing sequence (n_k) such that

$$f^{n_k}(y) \to y \quad \text{and} \quad n_k \to \infty. \tag{7}$$

Intuitively, the sequence of sets $f^{n_k}(U)$ should eventually intersect with U, giving our required result. (8)

A possible problem in the proof so far is that y may not be in the interior of U, so x is not necessarily in a neighborhood of y, so it is possible that $f^{n_k}(U)$ comes arbitrarily close to y but never intersects with U. (9)

I think we can avoid this problem by making a stronger statement than (6), using V instead of U: (10)

By (3) and (4), there exists a recurrent point z in V. (11)

By (11), there exists an increasing sequence (m_k) such that

$$f^{m_k}(z) \to z \quad \text{and} \quad m_k \to \infty. \tag{12}$$

Since V is a neighborhood of z, by (12) there exists an $M \geq 1$ such that $\forall i \geq M$, $f^{m_i}(z) \in V$, so $f^{m_M}(z) \in U$, hence $f^{m_M}(U) \cap U \neq \emptyset$. (13)

By (13), $\mathcal{R}(f) \subseteq \mathrm{NW}(f)$. (14)