Exercise 1.1.2

Suppose (X, f) is a factor of (Y, g) by a semi-conjugacy $\pi\colon Y \to X$. (1)

Show that if $y \in Y$ is a periodic point, then $\pi(y) \in X$ is periodic. (2)

Give an example to show that the preimage of a periodic point does not necessarily contain a periodic point. (3)

Proof + reasoning:

Let's start with (2). (4)

Let $y \in Y$ be periodic. (5)

We can just write out the definitions and see if the correct answer follows. (6)

$$f(\pi(y)) \;=\; \pi\big(g(y)\big) \;=\; \pi(y). \tag{7}$$

So, (2) follows from (7). (8)

Now, we want to construct a counterexample. Let's think of any semi-conjugacy and iterate from there. An obvious example is a projection. (9)

Let A and B be sets, and $\pi_A\colon A \times B \to A$ the projection. (10)

For all α and β, the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\alpha\times\beta} & A \times B \\ \downarrow \pi_A & & \downarrow \pi_A \\ A & \xrightarrow{\alpha} & A \end{array} \tag{11}$$

Intuitively, by taking projections, we 'forget' about the effect of β. So, we can simply choose β such that all points in $A \times B$ are non-periodic with respect to β, while choosing α so that all points in A are periodic with respect to α. (12)

Let A be any set, $B = [0, 1]$, $\alpha = \mathrm{id}_A$ and $\beta : (x \mapsto \frac{1}{2}x)$ (13)

Wait, 0 is still a periodic point in the example above. Let's modify it. (14)

Let A be any set, $B = (0, 1]$, $\alpha = \mathrm{id}_A$ and $\beta : (x \mapsto \frac{1}{2}x)$ (15)

Clearly, β has no periodic points, so $\alpha \times \beta$ has no periodic points, but all points in A are periodic with respect to α. (16)

By (16), (3) follows. (17)