Exercise 1.1.2

Suppose (X, f) is a factor of (Y, g) by a semi-conjugacy $\pi\colon Y \to X$. (1)

Show that if $y \in Y$ is a periodic point, then $\pi(y) \in X$ is periodic. (2)

Give an example to show that the preimage of a periodic point does not necessarily contain a periodic point. (3)

Proof + reasoning:

Let $y \in Y$ be periodic. (4)

$$f(\pi(y)) \;=\; \pi\big(g(y)\big) \;=\; \pi(y). \tag{5}$$

So, (2) follows from (5). (6)

Let A and B be sets, and $\pi_A\colon A \times B \to A$ the projection. (7)

For all α and β, the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{\alpha \times \beta} & A \times B \\ \downarrow \pi_A & & \downarrow \pi_A \\ A & \xrightarrow{\alpha} & A \end{array} \tag{8}$$

Let A be any nonempty set, $B = (0, 1]$, $\alpha = \mathrm{id}_A$ and $\beta : (x \mapsto \frac{1}{2}x)$ (9)

Clearly, β has no periodic points, so $\alpha \times \beta$ has no periodic points, but all points in A are periodic with respect to α. (10)

By (10), (3) follows. (11)

Exercise 1.2.3

Let G be a topological group. (1)

Prove that for each $g \in G$, the closure $H(g)$ of the set $\{g^n\}_{n=-\infty}^{\infty}$ is a commutative subgroup of G. (2)

Thus, if G has a minimal left translation, then G is abelian. (3)

Proof + reasoning:

Define $\langle g \rangle := \{g^n\}_{n=-\infty}^{\infty}$. (4)

Let $g \in G$. Let $a, b \in \mathrm{cl}(\langle g \rangle)$. (5)

Since the group multiplication $\alpha : G \times G \to G$ is continuous, $\alpha^{-1}(U)$ is open in $G \times G$. Since $(a, b) \in \alpha^{-1}(U)$, and since sets of the form $A \times B$, where A and B are open, form a basis for the topology on $G \times G$, there exist open V and W such that $a \in V$, $b \in W$, and such that $V \times W \subseteq \alpha^{-1}(U)$. (6)

Since $a, b \in \mathrm{cl}(\langle g \rangle)$ there exist $g^\ell \in V$ and $g^p \in W$. By (6), $g^\ell g^p \in U$, hence $g^{\ell+p} \in U$, so $ab \in \mathrm{cl}(\langle g \rangle) = H(g)$. (7)

By (7), $H(g)$ is closed under taking products. (8)

Let C be a neighborhood of a^{-1} and $U \subseteq C$ an open set such that $a^{-1} \in U$. Since the inverse is continuous, $U' := \{x \in G : x^{-1} \in U\}$ is open, and it contains a. (9)

Since $a \in H(g)$, there exists $g^q \in U'$, where $q \in \mathbb{Z}$. (10)

By (10), $g^{-q} = (g^q)^{-1} \in U$, so $a^{-1} \in H(g)$. (11)

By (11), $H(g)$ is closed under taking inverses. (12)

Let $c, d \in G$ with $cd \neq dc$. (13)

Suppose that G is Hausdorff. (14)

By (14) and (13), there exist open neighborhoods U of cd and U' of dc such that $U \cap U' = \emptyset$. (15)

Suppose $c, d \in H(g)$. (16)

Similarly to (6), $(c, d) \in \alpha^{-1}(U)$ and $(d, c) \in \alpha^{-1}(U')$. (17)

So there are open sets V, V', W, W' such that $(c, d) \in V \times W \subseteq \alpha^{-1}(U)$ and $(d, c) \in V' \times W' \subseteq \alpha^{-1}(U')$. (18)

From (18), $c \in V \cap V'$ and $d \in W \cap W'$, and $V \cap V'$ and $W \cap W'$ are open. (19)

So, by (16), there exist $s, t \in \mathbb{Z}$ such that $g^s \in V \cap V'$ and $g^t \in W \cap W'$. (20)

By (20), $(g^s, g^t) \in V \times W$ and $(g^t, g^s) \in W' \times V'$. (21)

By (21) and (18), $g^s g^t \in U$ and $g^t g^s \in U'$, so $g^{t+s} \in U \cap U'$. (22)

(22) contradicts (14), so (16) is false, hence $c \notin H(g)$ or $d \notin H(g)$, so $H(g)$ is commutative. (23)

By (23), (8) and (12), $H(g)$ is a commutative subgroup of G. (24)

Suppose that G has a minimal left translation $L_h : G \to G$ where $h \in G$. (25)

By (24), $H(h)$ is a commutative subgroup of G. (26)

By definition, L_h has no proper closed non-empty invariant subsets. (27)

$H(h)$ is a closed non-empty subset of G. (28)

Let $a \in H(h)$. Let C be a neighborhood of ha and U open with $ha \in U \subseteq C$. $a \in h^{-1}U$, and $h^{-1}U$ is open, so $\exists q \in \mathbb{Z}$ such that $h^q \in h^{-1}U$. (29)

By (29), $h^{q+1} \in U$, so $H(h)$ is invariant. (30)

By (30), (28) and (27), $H(h) = G$, so G is abelian. (31)

Exercise 1.3.3

For $m \in \mathbb{Z}$, $|m| > 1$, define the times-m map $E_m : S^1 \to S^1$ by $E_m x = mx \mod 1$. Show that the set of points with dense orbits is uncountable. (1)

Proof + reasoning:

As stated in ch1.3, the orbit of a point $0.x_1x_2\ldots$ is dense in S^1 iff every finite sequence of elements in $\{0, \ldots, m-1\}$ appears in the sequence $(x_i)_{i\in\mathbb{N}}$. (2)

Let U be the set of points in S^1 with a unique base-m expansion. (3)

By the remarks in section 1.3, U is uncountable. (4)

Define $\phi : \Sigma_m \to S^1$. by $\phi((x_i)_{i\in\mathbb{N}}) := \sum_{i=1}^{\infty} x_i/m^i$ (5)

By the remarks in section 1.3, ϕ is bijective on $\phi^{-1}(U)$. (6)

Let $x \in U$, with base-m expansion $(x_i)_{i\in\mathbb{N}}$. (7)

Let $\mathcal{F}_m = \bigcup_{k=1}^{\infty}\{0, \ldots, m-1\}^k$. (8)

Clearly, $\mathcal{F}_m$ is countable, so it can be indexed by $(\omega_i)_{i\in\mathbb{N}}$. (9)

Define $\alpha : U \to \Sigma_m$ by letting $\alpha(x) = x_1\omega_1x_2\omega_2x_3\omega_3\ldots$, and define $\beta = \phi \circ \alpha$. (10)

Since every $y \in U$ has a unique base-m expansion, α is injective, so by (6), β is bijective. By construction, every finite sequence appears in $\alpha(y)$ for every $y \in U$, so by (2), every point in $\beta(U)$ has a dense orbit. (11)

From (12), (11) and (4), we get that the set of all points in S^1 with dense orbits is uncountable. (12)

Exercise 1.4.3.

Verify that the metrics on Σ_m and Σ_m^+ generate the product topology (1)

Proof + reasoning:

Let $C := C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k} = \{x = (x_\ell) : x_{n_i} = j_i, i = 1, \ldots, k\}$ where $n_1 < n_2 < \cdots < n_k$ are indices in $\mathbb{Z}$ or $\mathbb{N}$, and $j_i \in A_m$. (2)

Let $x := (x_i) \in C$. (3)

Let $m = \max\{|n_i| : i \leq k\}$,$y \in B(x, 2^{-m})$, and $l = \min\{|i| : y_i \neq x_i\}$. (4)

We have $2^{-l} = d(x, y) < 2^{-m}$. (5)

From (5), $l > m$, so $x_{n_i} = y_{n_i}$ $\forall i \leq k$, hence $y \in C$. (6)

Therefore $B(x, 2^{-m}) \subseteq C$. (7)

By (7), and the fact that the collection of sets such as $C^{n_1,\ldots,n_k}_{j_1,\ldots,j_k}$ form a basis for the product topology, the metrics generate the product topology. (8)

Exercise 1.5.3.

Suppose p is an attracting fixed point for f. Show that there is a neighborhood U of p such that the forward orbit of every point in U converges to p. (1)

Proof + reasoning:

By assumption, there exists a neighborhood U of p such that $\overline{U}$ is compact, $f(\overline{U}) \subseteq U$, and $\bigcap_{n \geq 0} f^n(\overline{U}) = \{p\}$. (2)

Clearly,

$$U \subset \overline{U}. \tag{3}$$

From (3) and (2)

$$f(U) \subseteq f(\overline{U}) \subseteq U. \tag{4}$$

Therefore,

$$f^{n+1}(U) \subseteq f^n(U) \text{ for all } n \in \mathbb{N} \tag{5}$$

Clearly,

$$\bigcap_{n \geq 0} f^n(U) \subseteq \bigcap_{n \geq 0} f^n(\overline{U}) \tag{6}$$

Conversely,

$$\bigcap_{n \geq 0} f^n(\overline{U}) \subseteq \bigcap_{n \geq 1} f^n(\overline{U}) = \bigcap_{n \geq 0} f^{n+1}(\overline{U}) \subseteq \bigcap_{n \geq 0} f^n(U) \tag{7}$$

So, from (7)

$$\bigcap_{n \geq 0} f^n(U) = \bigcap_{n \geq 0} f^n(\overline{U}) \tag{8}$$

Let $x \in U$. Define $(x_n)_{n \in \mathbb{N}} = (f^n(x))_{n \in \mathbb{N}}$. (9)

Assume (x_n) does not converge. Then $\exists \varepsilon' > 0$ such that $\forall n : \exists k \geq n : d(f^k(x), p) > \varepsilon'$. (10)

From (10), there exists a sequence $f^{m_n}(x)$ such that $d(f^{m_n}(x), p) \geq \varepsilon$ for all $n \geq 0$. By compactness of $\overline{U}$, this sequence has a convergent subsequence $(f^{z_n}(x))_{n \geq 0}$ with $f^{z_n}(x) \to z \in \overline{U}$ and $z_n \to \infty$. (11)

Since f is continuous and $\overline{U}$ compact, $f^n(\overline{U})$ is compact for all $n \geq 0$. (12)

$\forall n \geq 0$ there exists K s.t. $f^{z_K}(x) \in f^n(\overline{U})$, hence $\forall m \geq K$, $f^{z_m}(x) \in f^n(\overline{U})$. (13)

From (13), the limit point z must be in $f^n(\overline{U})$ for all $n \geq 0$. (14)

Therefore, $z \in \bigcap_{n \geq 0} f^n(\overline{U}) = \{p\}$. (15)

So, $z = p$, which contradicts (10), so assumption (10) is false. Hence (x_n) converges. (16)

Suppose $x_n \to q$ and $q \neq p$. (17)

Let $n \geq 0$. The sequence $(x_i)_{i \geq n}$ is contained in $f^n(\overline{U})$, which is compact, so $q \in f^n(\overline{U})$. Therefore, $q \in \bigcap_{n \geq 0} f^n(\overline{U})$. (18)

By (8), $q \in \bigcap_{n \geq 0} f^n(U) = \{p\}$, which is a contradiction. Hence, (17) is false. (19)

From (19) and (16) it follows that $x_n \to p$. Therefore, the forward orbit of any point in U converges to p. (20)

Exercise 1.7.3.

Show that the eigenvalues of a two-dimensional hyperbolic toral automorphism are irrational (so the stable and unstable manifolds are dense by exercise 1.11.1). (1)

Proof + reasoning:

Let A be a 2×2 integer matrix such that $\det(A) = 1$ and such that for all eigenvalues λ of A, $|\lambda| \neq 1$. (2)

Let λ be an eigenvalue of A. (3)

Note that $\det(\lambda I - A) = 0$. (4)

Denote $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. (5)

$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0$ (6)

$\lambda^2 - \lambda a_{11} - \lambda a_{22} + a_{11}a_{22} - a_{12}a_{21} = 0$ (7)

By assumption, $\det(A) = 1$, so $a_{11}a_{22} - a_{12}a_{21} = 1$ (8)

Substituting $1 = a_{11}a_{22} - a_{12}a_{21}$ in (7) gives: $\lambda^2 - \lambda a_{11} - \lambda a_{22} + 1 = 0$ (9)

$\lambda^2 - \lambda(a_{11} + a_{22}) + 1 = 0$ (10)

By the quadratic formula, $\lambda = \frac{1}{2}(a_{11} + a_{22} + ((a_{11} + a_{22})^2 - 4)^{1/2})$ or $\lambda = \frac{1}{2}(a_{11} + a_{22} - ((a_{11} + a_{22})^2 - 4)^{1/2})$ (11)

From (11) we see that $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ if and only if $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R} \setminus \mathbb{Q}$ (12)

By definition of A, $(a_{11} + a_{22}) \in \mathbb{N}$ (13)

Suppose $((a_{11}+a_{22})^2-4)^{1/2} \in \mathbb{C}$. Then $|\lambda|^2 = \frac{1}{4}((a_{11}+a_{22})^2+4-(a_{11}+a_{22})^2) = 1$ (14)

(14) contradicts (2), hence $((a_{11} + a_{22})^2 - 4)^{1/2} \in \mathbb{R}$ (15)

From (15) and (13), $a_{11} + a_{22} \geq 3$ (16)

Conjecture 1 (stm:ex1.7.3-c23)**.** $\{n \in \mathbb{N} : \sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}\} = \emptyset$

Proof. Suppose $\sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}$. Then $\sqrt{n} = \frac{p}{q}$, where p and q are natural numbers. Since $\sqrt{n} \notin \mathbb{N}$, q does not divide p, so q^2 does not divide p^2. From (17), $n = \frac{p^2}{q^2}$, hence $nq^2 = p^2$. This contradicts (18), so (17) is false. $\square_{c23}$ $\square$

Proof of (??): Suppose $\sqrt{n} \in \mathbb{Q} \setminus \mathbb{N}$. Then $\sqrt{n} = \frac{p}{q}$, where p and q are natural numbers. (17)

Since $\sqrt{n} \notin \mathbb{N}$, q does not divide p, so q^2 does not divide p^2 (18)

From (17), $n = \frac{p^2}{q^2}$, hence $nq^2 = p^2$ (19)

(18) contradicts (19), so (17) is false $\square_{c23}$ (20)

Conjecture: $\forall n \in \mathbb{N}, n \geq 3$ implies $(n^2 - 4)^{1/2} \notin \mathbb{N}$ (21)

Proof of (21): Suppose $(n^2 - 4)^{1/2} = k$ where $k \in \mathbb{N}$ (22)

Then $n^2 - 4 = k^2$ (23)

So $n^2 - k^2 = 4$ (24)

Clearly, $n > k$ (25)

Then $n^2 - k^2 \geq n^2 - (n-1)^2 = n^2 - n^2 + 2n - 1 = 2n - 1$ (26)

So $n^2 - k^2 > 4$ (27)

(27) contradicts (24), so $\square_{c34}$ (28)

Now we can conclude. By (16) and (21), $((a_{11} + a_{22})^2 - 4)^{1/2} \notin \mathbb{N}$ (29)

So, by (29), (15) and (**??**), $((a_{11} + a_{22})^2 - 4)^{1/2}$ is irrational (30)

So, by (30) and (11), λ is irrational (31)

Let $x \in \mathbb{T}^2$. Without loss of generality, $\lambda > 1 > \lambda^{-1}$, where λ and λ^{-1} are the eigenvalues of A. The stable manifold $W^u(x)$ is the line through x parallel to v where v is the eigenvector corresponding to λ. (32)

Denote $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. A line parallel to v has slope equal to $\frac{v_2}{v_1}$ (33)

$$Av = \lambda v \tag{34}$$

$$\begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix} \tag{35}$$

$$a_{11}v_1 + a_{12}v_2 = \lambda v_1 \tag{36}$$

$$a_{21}v_1 + a_{22}v_2 = \lambda v_2 \tag{37}$$

$$v_2(\lambda - a_{22}) = a_{21}v_1 \tag{38}$$

$$v_1(\lambda - a_{11}) = a_{12}v_2 \tag{39}$$

Since v is an eigenvector, $v_1 \neq 0$ or $v_2 \neq 0$ (40)

By (31), $(\lambda - a_{22})$ and $(\lambda - a_{11})$ are irrational. (41)

If $v_1 \neq 0$, then by (41) and (39), $v_2 \neq 0$ (42)

If $v_2 \neq 0$, then by (41) and (38), $v_1 \neq 0$ (43)

So, $v_1 \neq 0$ and $v_2 \neq 0$ (44)

By (44) and (38), $v_2 v_1^{-1} = (\lambda - a_{22})^{-1} a_{21} \neq 0$ (45)

By (45) and (41), $v_2 v_1^{-1}$ is irrational (46)

Denote $x = (x_1, x_2)$, let $t \in \mathbb{R}^+$ (47)

Let $\phi^t_{\frac{v_1}{v_2}}(x) := (x_1 + \frac{v_1}{v_2}t, x_2 + t) \mod 1$ (48)

Then $\bigcup_{t \in \mathbb{R}^+} \phi^t_{\frac{v_1}{v_2}}(x) \subseteq W^u(x)$ (49)

By exercise 1.11.1, the orbit of $\phi_{\frac{v_1}{v_2}}$ is dense, so $W^u(x)$ is dense. For the stable manifold, the proof is similar. (50)

Exercise 1.8.3.

Let $\phi : \Sigma_2 = \{0,1\}^{\mathbb{Z}} \to H$ be the map that assigns to each infinite sequence $\omega = (\omega_i) \in \Sigma_2$ the unique point $\phi(\omega) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{\omega_i})$. (1)

Prove that ϕ is a bijection and that both ϕ and ϕ^{-1} are continuous. (2)

Proof + reasoning:

Suppose $x, y \in \Sigma_2$ with $\phi(x) = \phi(y)$. Then

$$\bigcap_{-\infty}^{\infty} f^{-i}(R_{x_i}) = \phi(x) = \phi(y) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{y_i}). \tag{3}$$

From the description of f, we see that f is injective. (4)

By definition,

$$R_0 = f(D_0) \cap R \quad \text{and} \quad R_1 = f(D_1) \cap R. \tag{5}$$

From (4), (5), and $D_0 \cap D_1 = \emptyset$, we get $R_0 \cap R_1 = \emptyset$. (6)

By (4) and (6), $f^{-i}(R_0) \cap f^{-i}(R_1) = \emptyset$ for all $i \in \mathbb{Z}$. (7)

From (3) and (7), $x_i = y_i$ for all $i \in \mathbb{Z}$, so $x = y$, so ϕ is injective. (8)

Note $f(R) \cap R = R_1 \cup R_0$, and $f^{-1}(R) \subseteq R$. (9)

By (9), $f^{-i}(R) = f^{-i}(R_0) \cup f^{-i}(R_1)$ for all $i \geq 1$. (10)

Clearly, $R_0 \cap R_1 = \emptyset$. (11)

By (4) and (9), for all $i \geq 0$,

$$\big(f^{-i}(R_0) \cup f^{-i}(R_1)\big) \cap R = f^{-i}(R_0 \cup R_1) \cap R = f^{-i+1}(R) \cap f^{-i}(R) \cap R. \tag{12}$$

Let $x \in H$ and $j \in \mathbb{Z}$. By (10), (11) and (12), $x \in f^j(R_0)$ or $x \in f^j(R_1)$, but not both. (13)

By (13), we can define $x_j = 0$ if $x \in f^j(R_0)$ and $x_j = 1$ if $x \in f^j(R_1)$. Clearly, this gives a sequence $(x_j)_{j\in\mathbb{Z}} \in \Sigma_2$ such that $\phi((x_j)_{j\in\mathbb{Z}}) = x$. So ϕ is surjective. (14)

By (14) and (8), ϕ is bijective. (15)

Let $\omega \in \phi^{-1}(A \times B)$. For a sequence $\omega \in \{0,1\}^{\mathbb{Z}}$, define

$$R_{\omega_{-m},\ldots,\omega_m} = \bigcap_{i=-m}^{m} f^{-i}(R_{\omega_i}). \tag{16}$$

Define $\mathcal{C}_m = \{R_{\omega_{-m},\ldots,\omega_m} \times R_{\omega_0,\ldots,\omega_m}\}$, $\omega \in \{0,1\}^{\mathbb{Z}}$, $m \in \mathbb{N}$, and define $\mathcal{C} = \bigcup_{m\in\mathbb{N}}\{H \cap C : C \in \mathcal{C}_m\}$. (17)

Conjecture 2. *$\mathcal{C}$ is a basis for the topology on H.*

Proof: Let $C \in \mathcal{C}$. Then $C = H \cap (R^- \times R^+)$ where $R^- \times R^+ \in \mathcal{C}_m$ for some $m \in \mathbb{N}$. (18)

Note $R^- = [x_1, x_2]$ and $R^+ = [y_1, y_2]$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. (19)

By (11), for all $D \neq D' \in \mathcal{C}_m$, $D \cap D' = \emptyset$. (20)

By (19) and (20), there exist open intervals $I^-, I^+ \subset \mathbb{R}$ such that $R^- \times R^+ \subseteq I^- \times I^+$ and such that $(I^- \times I^+) \cap D = \emptyset$ for all $D \in \mathcal{C}_m$ with $D \neq R^- \times R^+$. (21)

Clearly, $\mathcal{C}_m$ covers H. (22)

By (21) and (22), $H \cap (R^- \times R^+) = H \cap (I^- \times I^+)$. (23)

By (23), C is open in H. (24)

Let A and B be open intervals in $\mathbb{R}$. (25)

Let $x \in H \cap (A \times B)$. (26)

Let $\varepsilon = \min\{d(x, y) : y \in A \times B\}$. (27)

Because $A \times B$ is open, $\varepsilon > 0$. (28)

Let $k = \min \{n \in \mathbb{N} : \mu^{-n} \leq \varepsilon, \lambda^n \leq \varepsilon\}$. (29)

By (28), and since $\lambda < 1/2$ and $\mu > 2$, $k > 0$. (30)

If $R^- \times R^+ \in \mathcal{C}_m$, then R^- has width equal to $\mu^{-k} \leq \varepsilon$ and R^+ has width equal to $\lambda^k \leq \varepsilon$. (31)

By (31), (27) and (22), there exists an $R^- \times R^+ \in \mathcal{C}_k$ such that $x \in R^- \times R^+$ and $R^- \times R^+ \subseteq A \times B$. (32)

By (24), $H \cap R^- \times R^+$ is open, so $\mathcal{C}$ is a basis for the topology on H. (33)

Let $C \in \mathcal{C}$. $C = H \cap (R_{\omega_{-m},\ldots,\omega_m} \times R_{\omega_0,\ldots,\omega_m})$ for $\omega \in \{0,1\}^{\mathbb{Z}}$. (34)

Let $j \in \{-m, -m+1, \ldots, m\}$. (35)

Suppose $z \in \phi^{-1}(C)$. By definition of ϕ,

$$\phi(z) = \bigcap_{i \in \mathbb{Z}} f^{-i}(R_{z_i}) \subseteq f^j(R_{z_j}). \tag{36}$$

Since $\phi(z) \in R^+ \times R^-$, $\phi(z) \in f^j(R_{\omega_j})$. (37)

By (11), $f^j(R_1) \cap f^j(R_0) = \emptyset$, so $z_j = \omega_j$, so $z \in B(\omega, 2^{-m})$. (38)

Clearly $B(\omega, 2^{-m}) \subseteq \phi^{-1}(C)$. (39)

By (38) and (39), $\phi^{-1}(C) = B(\omega, 2^{-m})$, so $\phi^{-1}(C)$ is open, so, by 2, ϕ is continuous. (40)

Let $B(\gamma, 2^{-n})$ be an open ball in Σ_2. By the same argument as for (40),

$$B(\gamma, 2^{-n}) = \phi^{-1}(H \cap (R_{\gamma_{-n},\ldots,\gamma_n} \times R_{\gamma_0,\ldots,\gamma_n})). \tag{41}$$

So $\phi(B(\gamma, 2^{-n})) = H \cap (R_{\gamma_{-n},\ldots,\gamma_n} \times R_{\gamma_0,\ldots,\gamma_n})$. (42)

By (42) and 2, $\phi(B(\gamma, 2^{-n}))$ is open, so ϕ^{-1} is continuous. (43)

Exercise 1.9.3.

Let $\mathbb{T}$ denote the set of sequences $(\phi_i)_{i=0}^{\infty}$ where $\phi_i \in S^1$ and $\phi_i = 2\phi_{i+1} \mod 1$ for all i. Let $\alpha : \mathbb{T} \to \mathbb{T}$ be defined by

$$(\phi_0, \phi_1, \dots) \mapsto (2\phi_1, \phi_1, \phi_2, \dots). \tag{1}$$

Show that $\mathbb{T}$ is a topological group. (2)

Show that α is an automorphism (3)

Proof + reasoning:

Lemma 1. *$\forall x \in \mathbb{R}, \forall y \in \mathbb{R},\ a, b \in \mathbb{N} : x \equiv_a y \Rightarrow bx \equiv_a by.$*

Proof. $\exists p \in \mathbb{Z} : x = y + pa$, so $bx = by + bpa$, so $bx \equiv_a by$. □

Lemma 2. *$\forall x \in \mathbb{R}, a, b \in \mathbb{N},\ a(x \mod b) \equiv_b ax.$*

Proof. Clearly, $x \mod b \equiv_b x$. By lemma 1, $a(x \mod b) \equiv_b ax$. □

Lemma 3. *$\forall x, y \in \mathbb{R},\ \forall k \in \mathbb{N},\ (x \mod k) + (y \mod k) \equiv_k x + y.$*

Proof. $x + y = (x \mod k) + (y \mod k) + pk + qk = (x \mod k) + (y \mod k) + (p + q)k$. So, $x + y \equiv_k (x \mod k) + (y \mod k)$. □

Given ψ and ϕ in $\mathbb{T}$, define $(\psi + \phi)_i = (\psi_i + \phi_i) \mod 1$. (4)

Let ψ and ϕ be elements of $\mathbb{T}$. It suffices to show $\psi_i + \phi_i \equiv_1 2(\psi + \phi)_{i+1}$. (5)

By (1), lemma 3 and lemma 2,

$$\begin{aligned}
\psi_i + \phi_i &= (2\psi_{i+1} \mod 1) + (2\phi_{i+1} \mod 1) \\
&\equiv_1 2\psi_{i+1} + 2\phi_{i+1} \\
&= 2(\psi_{i+1} + \phi_{i+1}) \\
&\equiv_1 2((\psi + \phi)_{i+1} \mod 1) \\
&\equiv_1 2(\psi + \phi)_{i+1}
\end{aligned} \tag{6}$$

From (6) and (5), $\mathbb{T}$ is closed under addition. (7)

For $\phi \in \mathbb{T}$, define $(-\phi)_i := -\phi_i$. (8)

Clearly, this is the inverse of ϕ. (9)

Suppose that $\phi^n \to \phi$ and $\psi^n \to \psi$ in $(S^1)^{\mathbb{N}_0}$. Let $n \geq 0$. By assumption, $\exists K_n$ s.t. $\forall j \leq n,\ \phi^i_j = \phi_j,\ \psi^i_j = \psi_j$, hence $(\phi^i - \psi^i)_j = \phi_j - \psi_j$. (10)

By (10), $\forall \varepsilon > 0, \exists k$ s.t. $d(\phi^i - \psi^i, \phi - \psi) \leq \varepsilon\ \forall i \geq k$, so $\lim_{n\to\infty} d(\phi^n - \psi^n, \phi - \psi) = 0$, hence $(\phi, \psi) \mapsto \phi - \psi$ is continuous. (11)

Let $\phi, \psi \in \mathbb{T}$. Show $\alpha(\phi) + \alpha(\psi) = \alpha(\phi + \psi)$. By lemma 3 and lemma 2,

$$\begin{aligned}
(\alpha(\phi) + \alpha(\psi))_i &= ((2\phi_i) \bmod 1 + (2\psi_i) \bmod 1) \bmod 1 \\
&\equiv_1 2(\phi_i + \psi_i) \\
&\equiv_1 2\left((\phi + \psi)_i \bmod 1\right) \\
&= \alpha(\phi + \psi)_i.
\end{aligned} \tag{12}$$

Clearly, α preserves the identity. (13)

Suppose $\alpha(\phi) = \alpha(\psi)$. Then $\phi_{i-1} = \alpha(\phi)_i = \alpha(\psi)_i = \psi_{i-1}\ \forall i \geq 1$, so $\phi = \psi$, hence α is injective. (14)

Let $\phi \in \mathbb{T}$. Let $(\phi')_i := \phi_{i+1}\ \forall i \geq 1$. (15)

$\alpha(\phi') = \phi$, so α is surjective. (16)

By (14) and (16), α is bijective. By (12), (13), and (14), α is a group automorphism. (17)

For product topologies the 1-d cylinders form a subbasis. So, to show that α is continuous it suffices to show that $\forall i \in \mathbb{N}, \pi_i \circ \alpha$ is continuous. (18)

Let $i \in \mathbb{N}$. Note $\pi_i \circ \alpha : (\phi_0, \phi_1, \dots) \mapsto \begin{cases} \phi_{i-1} & \text{if } i \geq 1 \\ 2\phi_0 & \text{if } i = 0 \end{cases}$. (19)

The map $r : S^1 \to S^1 : s \mapsto 2s \mod 1$ is clearly continuous. (20)

By (20), if $A \in \mathcal{T}(S^1)$, then

$$(\pi_i \circ \alpha)^{-1}(A) = \begin{cases} (S^1)^{i-1} \times A \times S^1 \times \cdots & \text{if } i \geq 1 \\ r^{-1}(A) \times S^1 \times \cdots & \text{if } i = 0 \end{cases} \tag{21}$$

By (21) and (20), $\pi_i \circ \alpha$ is continuous, so by (18), α is continuous. (22)

Note $\pi_i \circ \alpha^{-1} : (\phi_0, \phi_1, \dots) \mapsto \phi_{i+1}$. This is clearly continuous, so by (18), α^{-1} is continuous, so by (22) and (17), α is a homeomorphism. (23)

Exercise 1.10.3.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. (1)

Show that $-f$ is a Lyapunov function for the gradient flow. (2)

Show that the trajectories of the gradient flow are orthogonal to the level sets of f. (3)

Proof + reasoning:

The gradient flow is the flow of the differential equation $\dot{x} = \nabla f(x)$. (4)

Let $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$. Note $(f \circ g_x)(0) = f(x)$ and $(f \circ g_x)(t) = f(g^t(x))$. (5)

By (5), if $(f \circ g_x)'(s) \geq 0$ for all $s \in \mathbb{R}^+$ then $-f$ is Lyapunov. (6)

By (4), $g_x'(t) = \nabla f(g_x(t))$. (7)

By the multivariate chain rule and (7),

$$(f \circ g_x)'(t) = \langle \nabla f(g_x(t)), g_x'(t) \rangle = \langle g_x'(t), g_x'(t) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^n$. (8)

By definition of inner products, $\langle g_x'(t), g_x'(t) \rangle \geq 0$. (9)

By (8), (9) and (6), $-f$ is Lyapunov. (10)

Let $x \in \mathbb{R}^n$. (11)

Define the level set $C := f^{-1}(f(x))$. (12)

Let $T_x = \{\dot{\gamma}(0) : \exists \varepsilon > 0 \text{ s.t. } \gamma : (-\varepsilon, \varepsilon) \to C \text{ is smooth and } \gamma(0) = x\}$. (13)

By (7), $(f \circ g_x)'(0) = \nabla f(g_x(0)) = \nabla f(x) = g_x'(0)$. (14)

Let $V \in T_x$, with corresponding path $\gamma : (-\varepsilon, \varepsilon) \to C$. (15)

Since $\gamma(t) \in C$ for all $t \in (-\varepsilon, \varepsilon)$, $f(\gamma(t)) = f(\gamma(0))$ for all $t \in (-\varepsilon, \varepsilon)$. (16)

By the multivariate chain rule,

$$\begin{aligned}
\langle \nabla f(x), V \rangle &= \sum_{k=1}^{n} V_k \frac{\partial f}{\partial y_k}(x) \\
&= \sum_{k=1}^{n} V_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\
&= \sum_{k=1}^{n} \dot{\gamma}(0)_k \frac{\partial f}{\partial y_k}(\gamma(0)) \\
&= (f \circ \gamma)'(0).
\end{aligned} \tag{17}$$

By (16), $(f \circ \gamma)'(0) = 0$, so by (17), $\langle \nabla f(x), V \rangle = 0$, so by (14), the trajectories of the gradient flow are orthogonal to the level sets of f. (18)

Exercise 1.11.3.

Suppose $1, s$ and αs are real numbers that are linearly independent over $\mathbb{Q}$. (1)

Show that every orbit of the time-s map ϕ_α^s is dense in $\mathbb{T}^2$. (2)

Proof + reasoning:

Let $x \in \mathbb{T}^2$, $y \in \mathbb{T}^2$, $\varepsilon' > 0$ and $\varepsilon = \frac{\varepsilon'}{2\sqrt{2}}$. (3)

Let $\mathcal{P}_\varepsilon$ be a partition of $\mathbb{T}^2$ into finitely many squares of the form $[a, b)^2$, where $\frac{\varepsilon}{2} < |a - b| < \varepsilon$. (4)

By the pigeonhole principle, there exists a $P \in \mathcal{P}_\varepsilon$ and $k > m$ in $\mathbb{Z}$ such that $\phi_\alpha^{ks}(x)$ and $\phi_\alpha^{ms}(x)$ are in P. (5)

By (5), $d(z, \phi_\alpha^{(k-m)s}(z)) < \sqrt{2}\varepsilon$ for all $z \in \mathbb{T}^2$, where d is the metric on $\mathbb{T}^2$. (6)

Conjecture 3. *There exists a $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $y \in \mathbb{T}^2$*

$$\frac{(\phi_\alpha^{(k-m)s}(y))_2 - y_2}{(\phi_\alpha^{(k-m)s}(y))_1 - y_1} = \beta.$$

Proof. Suppose for contradiction that $s = 0$. Then for $p = 1, q = 1, r = 0$ we have $p\alpha s + qs + r = 0$, a contradiction, so $s \neq 0$. Similarly, $\alpha s \neq 0$. Suppose for contradiction that $\alpha s \in \mathbb{Q}$. Let $p = 1, q = -\alpha s, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\alpha s \notin \mathbb{Q}$. Suppose for contradiction that $\frac{1}{\alpha} \in \mathbb{Q}$. Then s is irrational. Let $p = \frac{1}{\alpha}, q = -1, r = 0$. Then $p\alpha s + qs + r = 0$, a contradiction, so $\frac{1}{\alpha}$ is irrational. Let $y \in \mathbb{T}^2$. Then

$$\frac{(\phi_\alpha^{(k+m)s}(y))_2 - y_2}{(\phi_\alpha^{(k+m)s}(y))_1 - y_1} = \frac{(k-m)s}{(k-m)\alpha s} = \frac{1}{\alpha}$$

So, with $\beta = \frac{1}{\alpha}$, the statement follows. $\square$

Let γ be the line in $\mathbb{T}^2$ starting from x in the direction of $x - \phi_\alpha^{m-k}(x)$. (7)

Let β be the slope of γ, which is finite and in $\mathbb{R} \setminus \mathbb{Q}$ by 3. (8)

By (8), considering γ as a subset of $\mathbb{T}^2$, we have

$$\begin{aligned} \gamma \cap (y_1 \times \mathbb{T}) &= \bigcup_{n \geq 0} \{(y_1, (x_2 + \beta(y_1 - x_1) + \beta n) \bmod 1)\} \\ &= \bigcup_{n \geq 0} \left\{ \left(y_1, R_\beta^n(x_2 + \beta(y_1 - x_1))\right) \right\}. \end{aligned} \tag{9}$$

By (8), R_β has dense semiorbits. (10)

By (10) and (9), there exists a $z \in \gamma \cap (y_1 \times (y_2 - \varepsilon, y_2 + \varepsilon))$. (11)

By (6) and Conjecture 3, there exists a $p \in \mathbb{N}$ such that

$$d(\phi_\alpha^{p(k-m)s}(x), z) < \sqrt{2}\varepsilon \tag{12}$$

By (12) and (11),

$$\begin{aligned} d(\phi_\alpha^{p(k-m)s}(x), y) &\leq d(\phi_\alpha^{p(k-m)s}(x), z) + d(z, y) \\ &\leq \sqrt{2}\varepsilon + \varepsilon \\ &\leq 2\sqrt{2}\varepsilon \\ &\leq \varepsilon'. \end{aligned} \tag{13}$$

By (13), every orbit of ϕ_α^s is dense in $\mathbb{T}^2$. (14)

Exercise 1.12.3.

Compute the Lyapunov exponents for the solenoid. (1)

Proof + reasoning:

Let $F\colon S^1 \times D^2 \to S^1 \times D^2$ be the solenoid. Let $x, y \in \mathbb{R}$ and let $\lambda \in (0, \frac{1}{2})$. (2)

Note $F(\phi, x, y) = (2\phi, \lambda x + \frac{1}{2}\cos(2\pi\phi), \lambda y + \sin(2\pi\phi))$. (3)

By writing out the composition, we see that:

$$\begin{aligned}
F^n(\phi, x, y)_1 &= 2^n\phi \\
F^n(\phi, x, y)_2 &= \lambda^n x + \frac{1}{2}\lambda^{n-1}\cos(2\pi\phi) + \cdots + \frac{1}{2}\lambda^0\cos(2^{n-1}\pi\phi) \\
&= \lambda^n x + \frac{1}{2}\sum_{i=0}^{n-1}\lambda^i\cos(2^{n-1-i}\pi\phi) \\
F^n(\phi, x, y)_3 &= \lambda^n y + \frac{1}{2}\sum_{i=0}^{n-1}\lambda^i\sin(2^{n-1-i}\pi\phi)
\end{aligned} \tag{4}$$

By (4), denoting $\delta_{ij} := \frac{\partial F_i}{\partial z_j}(\phi, x, y)$, we can express $dF^n(\phi, x, y)$ as follows:

$$\begin{aligned}
\delta_{11} &= 2^n \\
\delta_{21} &= -\frac{1}{2}\sum_{i=0}^{n-1}\lambda^i 2^{n-1-i}\pi\sin(2^{n-1-i}\pi\phi) \\
&= -\frac{\pi}{2}\sum_{i=0}^{n-1}\lambda^i 2^{n-1-i}\sin(2^{n-1-i}\pi\phi) \\
\delta_{31} &= \frac{\pi}{2}\sum_{i=0}^{n-1}\lambda^i 2^{n-1-i}\cos(2^{n-1-i}\pi\phi) \\
\delta_{22} &= \lambda^n \\
\delta_{33} &= \lambda^n \\
\delta_{ij} &= 0 \quad \text{otherwise}
\end{aligned} \tag{5}$$

The Lyapunov exponent is defined as

$$\chi(\phi, x, y, v) = \lim_{n\to\infty}\frac{1}{n}\log\|dF^n(\phi, x, y)v\| \tag{6}$$

By (5),

$$dF^n(\phi, x, y)v = v_1(2^n + \delta_{21} + \delta_{31}) + \lambda^n(v_2 + v_3) \tag{7}$$

Note, since $\lambda \in (0, \frac{1}{2})$, that $|\delta_{21}| \le \pi \cdot n \cdot 2^n$ and $|\delta_{31}| \le \pi \cdot n \cdot 2^n$. (8)

Suppose $v_1 \neq 0$. By (8) and (7), for n sufficiently large,

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{2}\cdot\frac{1}{n}\log\left(\|dF^n(\phi,x,y)v\|^2\right)\\
&= \frac{1}{2n}\log\left(v^2(2^n+\delta_{21}+\delta_{31})^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\leq \frac{1}{2n}\log\left((v_1\cdot 3\pi\cdot n\cdot 2^n)^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\leq \frac{1}{2n}\log\left((v_1\cdot 4\pi\cdot n\cdot 2^n)^2\right)\\
&= \frac{1}{n}\log(v_1\cdot 4\pi\cdot n\cdot 2^n)\\
&= \frac{1}{n}\left(\log(v_1\cdot 4\pi\cdot n)+n\log(2)\right)\\
&\xrightarrow{n\to\infty}\log(2)
\end{aligned} \tag{9}$$

By (9), $\chi(\phi,x,y,v)\leq\log(2)$. (10)

For the lower bound, by (11) and (9),

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{2n}\log\left(v_1^2(2^n+\delta_{21}+\delta_{31})^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\geq \frac{1}{2n}\log\left(v_1^2\cdot 2^{2n}\right)\\
&= \frac{1}{n}\log(v_1\cdot 2^n)\\
&= \log(2)+\frac{1}{n}\log(v_1)\xrightarrow{n\to\infty}\log(2)
\end{aligned} \tag{11}$$

By (10) and (11), $\chi(\phi,x,y,v)=\log(2)$. (12)

Suppose $v_1=0$. By (13) and (7),

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{n}\log(\lambda^n(v_2+v_3))\\
&= \log(\lambda)+\frac{1}{n}\log(v_2+v_3)\\
&\xrightarrow{n\to\infty}\log(\lambda)
\end{aligned} \tag{13}$$

By (13), $\chi(\phi,x,y,v)=\log(\lambda)$. (14)

By (14) and (12), the Lyapunov exponents are $\log(2)$ and $\log(\lambda)$. □ (15)

Exercise 2.1.3.

Let $f : X \to X$ be a topological dynamical system. (1)

Show that $\mathcal{R}(f) \subseteq \mathrm{NW}(f)$. (2)

Proof + reasoning:

Let $x \in \mathcal{R}(f)$. (3)

Let U be a neighborhood of x, and V an open set such that $V \subseteq U$ and $x \in V$. (4)

By (3) and (4), there exists a recurrent point z in V. (5)

By (5), there exists an increasing sequence (m_k) such that

$$f^{m_k}(z) \to z \quad \text{and} \quad m_k \to \infty. \tag{6}$$

Since V is a neighborhood of z, by (6) there exists an $M \geq 1$ such that $\forall i \geq M$, $f^{m_i}(z) \in V$, so $f^{m_M}(z) \in U$, hence $f^{m_M}(U) \cap U \neq \emptyset$. (7)

By (7), $\mathcal{R}(f) \subseteq \mathrm{NW}(f)$. (8)

Exercise 2.2.3.

Is the product of two topologically transitive systems topologically transitive? (1)

Is a factor of a topologically transitive system topologically transitive? (2)

Proof + reasoning:

Let R_α be the circle translation, where α is irrational. (3)

It is known that R_α is topologically transitive. (4)

Let $(a, b) \in S^1 \times S^1$. (5)

If $a \geq b$, then the orbit of (a, b) under $R_\alpha \times R_\alpha$ is contained in $l_1 \cup l_2$ where

$$\begin{aligned} l_1 &= \{t(a-b, 0) + (1-t)(1, b-a+1) : t \in [0,1]\}, \\ l_2 &= \{t(1, b-a+1) + (1-t)(1, b-a+1) : t \in [0,1]\}. \end{aligned} \tag{6}$$

If $b \geq a$, then the same holds with

$$\begin{aligned} l_1 &= \{t(0, a-b+1) + (1-t)(b-a, 1) : t \in [0,1]\}, \\ l_2 &= \{t(b-a, 0) + (1-t)(1, a-b+1) : t \in [0,1]\}. \end{aligned} \tag{7}$$

In both cases, l_1 and l_2 are lines contained in $[0, 1) \times [0, 1)$. Since these lines are clearly not dense in $S^1 \times S^1$, the forward orbit of (a, b) is not dense in $S^1 \times S^1$. Hence, $R_\alpha \times R_\alpha$ is not topologically transitive. (8)

Suppose $f : X \to X$ and $g : Y \to Y$ are topological dynamical systems, that π is a topological semiconjugacy from f to g, and that f is topologically transitive with point $x \in X$ with dense forward orbit. (9)

Let $U \subseteq Y$ be open. Since π is continuous, $\pi^{-1}(U)$ is open, so by (9), there exists a $k \in \mathbb{N}$ such that $f^k(x) \in \pi^{-1}(U)$. (10)

By (9), $\pi \circ f^k(x) = g^k(\pi(x))$. (11)

By (10) and (11), $g^k(\pi(x)) \in U$, so $\pi(x)$ is dense, hence a factor of a topologically transitive system is topologically transitive. (12)

Exercise 2.3.3.

Show that a factor of a topologically mixing system is also topologically mixing. (1)

Proof + reasoning:

Let $f : X \to X$ and $g : Y \to Y$ be topological dynamical systems, and π a topological semiconjugacy from f to g. (2)

Let U and V be nonempty open sets in Y. (3)

Since π is surjective and continuous, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are nonempty and open. (4)

By (2) and (4), there exists an $N \in \mathbf{N}$ such that for all $n \geq N$, $f^n(\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \varnothing$. (5)

By (2),

$$\begin{aligned}\pi\left(f^n(\pi^{-1}(U)) \cap \pi^{-1}(V)\right) &\subseteq \pi(f^n(\pi^{-1}(U))) \cap \pi(\pi^{-1}(V)) \\ &= g^n(\pi(\pi^{-1}(U))) \cap \pi(\pi^{-1}(V)) \\ &= g^n(U) \cap V.\end{aligned} \tag{6}$$

By (6) and (5), $g^n(U) \cap V \neq \varnothing$, so g is topologically mixing, hence a factor of a topologically mixing system is topologically mixing. (7)

Exercise 2.5.3.

Let $\{a_n\}$ be a subadditive sequence of non-negative real numbers, i.e. (1)

$0 \leq a_{m+n} \leq a_m + a_n$ for all $m, n \geq 0$. (2)

Show that $\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 0} \frac{a_n}{n}$. (3)

Proof + reasoning:

Let $k \in \mathbb{N}_+$. (4)

Let $n \geq k$. (5)

By (5), $n = mk + m'$, where $m \in \mathbb{N}$ and $m' < k$. (6)

By (6), and the subadditivity of (a_n),

$$\begin{aligned}
\frac{a_n}{n} - \frac{a_k}{k} &= \frac{a_{mk+m'}}{n} - \frac{a_k}{k} \\
&\leq \frac{a_{mk} + a_{m'}}{n} - \frac{a_k}{k} \\
&\leq \frac{ma_k}{mk + m'} + \frac{ka_1}{n} - \frac{a_k}{k} \\
&\xrightarrow{n\to\infty} 0
\end{aligned} \tag{7}$$

Hence, $\lim_{n\to\infty} \frac{a_n}{n}$ is a lower bound for $\left\{\frac{a_n}{n} : n \geq 1\right\}$. (8)

Additionally, if $C \leq \frac{a_m}{m}$ for all $m \in \mathbb{N}$, then clearly $C \leq \lim_{n\to\infty} \frac{a_n}{n}$, so by (8),

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 0} \frac{a_n}{n}. \tag{9}$$

Exercise 2.7.3.

Give a non-trivial example of a homeomorphism f of a compact metric space (X, d) such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for every pair $x, y \in X$. (1)

Proof + reasoning:

Define $f : S^1 \to S^1$ by

$$f(x) = \begin{cases} \frac{1}{2}x & \text{if } x \in [0, \frac{1}{2}) \\ \frac{3}{2}x - \frac{1}{2} & \text{if } x \in [\frac{1}{2}, 1) \end{cases} \tag{2}$$

Clearly, f is a homeomorphism such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$ for all pairs x, y. (3)

Exercise 2.8.3.

Prove the following generalization of Proposition 2.1.2. If a commutative group G acts by homeomorphisms on a compact metric space X, then there is a non-empty, closed G-invariant subset X' on which G acts minimally. (1)

Proof + reasoning:

Let $\mathcal{C}$ be the collection of non-empty, closed G-invariant subsets of X, with the partial ordering given by inclusion. (2)

Since $X \in \mathcal{C}$, $\mathcal{C}$ is not empty. (3)

Suppose $\mathcal{K} \subseteq \mathcal{C}$ is a totally ordered subset. Then, any finite intersection of elements of $\mathcal{K}$ is nonempty, so by the finite intersection property for compact sets, $\bigcap_{K \in \mathcal{K}} K \neq \emptyset$. Thus, by Zorn's lemma, $\mathcal{C}$ contains a minimal element M. (4)

Suppose that G does not act minimally on M. (5)

Then, there exists a point $b \in M$ and a nonempty open set $C \subseteq M$ such that $Gb \cap C = \emptyset$. (6)

Since $GC = \bigcup_{g \in G} gC$, and each $g \in G$ is a homeomorphism, GC is open. (7)

So, since M is closed, $M \setminus GC$ is closed. (8)

Since $b \in M$ and $Gb \cap C = \emptyset$, $b \in M \setminus GC$, so $M \setminus GC$ is nonempty. (9)

If $m \in M \setminus GC$ and $g \in G$, then $m \neq g^{-1}c$, so $gm \neq C$, so since M is G-invariant, $gm \in M \setminus GC$, so $M \setminus GC$ is G-invariant. (10)

By (8)–(10), $M \setminus GC$ is a closed, nonempty, G-invariant proper subset of M, which contradicts (4), so (5) is false. Hence, G acts minimally on M. (11)

Exercise 3.1.3.

Use a higher block presentation to prove that for any block code $c : X \to Y$ there is a subshift Z and an isomorphism $f : Z \to X$ such that $c \circ f : Z \to Y$ is a $(0,0)$-block code. (1)

Proof + reasoning:

Let $c : X \to Y$ be a block code, with corresponding function $\alpha : W_{a+b+1} \to \mathcal{A}_m$. (2)

Letting $k = a + b + 1$ and $l = b$, the higher block presentation d of X can be written as

$$d(x)_i = x_{i-a} \dots x_{i+b}, \quad i \in \mathbb{Z} \tag{3}$$

Since $\mathrm{im}(d) \subseteq \Sigma_{W_{a+b+1}(X)}$ we have $W_1(\mathrm{im}(d)) \subseteq W_{a+b+1}(X)$. (4)

If $\omega \in W_{a+b+1}(X)$, then for some sequence $x \in X$ and $i \in \mathbb{Z}$,

$$\omega = x_{i-a} \dots x_{i+b}, \quad \text{so} \quad d(x)_i = \omega,$$

so $\omega \in W_1(\mathrm{im}(d))$. (5)

By (5) and (4), $W_1(\mathrm{im}(d)) = W_{a+b+1}(X)$. (6)

By Exercise 3.1.2, d is an isomorphism onto its image. Let $d^{-1} : \mathrm{im}(d) \to X$ be its inverse. (7)

If $z \in \mathrm{im}(d)$ and $i \in \mathbb{Z}$, then there exists a unique x such that $d(x) = z$, so

$$\begin{aligned}(c \circ d^{-1})(z)_i &= c(d^{-1}(z))_i \\ &= c(x)_i \\ &= \alpha(x_{i-a} \dots x_{i+b}) \\ &= \alpha(z_i)\end{aligned} \tag{8}$$

By (8) and (5), $c \circ d^{-1} : \mathrm{im}(d) \to Y$ is a $(0,0)$-block code and, by (7), d^{-1} is an isomorphism. (9)

Exercise 3.2.3.

Show that every edge shift is an SFT. (1)

If Σ_B^e is an edge shift with graph Γ_B, then Σ_B^e is precisely the set of sequences that do not contain the words $e'e$ of length 2 in which the target of e is not equal to the source of e'. (2)

Since this collection of words is finite, Σ_B^e is an SFT. (3)

Exercise 4.2.3

Prove that if T is a measure-preserving transformation, then so are the induced transformations. (1)

Proof + reasoning:

Let $T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ be a measure-preserving transformation. (2)

Let's start with proving that the derivative transformation is measure preserving. First, let's check that it is measurable. (3)

Let $\mathcal{E}$ be the trace σ-algebra with respect to $A \in \mathcal{A}$. (4)

Let $B \in \mathcal{E}$. (5)

By definition, $B = C \cap A$, $C \in \mathcal{A}$. (6)

Let (R_n) and (D_n) be sequences of sets defined inductively by letting

$$\begin{aligned} R_0 &= B, & D_0 &= \emptyset \\ R_1 &= T^{-1}(B) \setminus A, & D_1 &= T^{-1}(B) \cap A \\ R_{n+1} &= T^{-1}(R_n) \setminus A, & D_{n+1} &= T^{-1}(R_n) \cap A \quad \forall n \geq 2 \end{aligned} \tag{7}$$

Let $D := \bigcup_{n \geq 1} D_n$ (8)

Conjecture 4. $D = T_A^{-1}(B)$

Proof. Let $n \geq 2$

$$\begin{aligned} R_n &= T^{-1}(R_{n-1}) \setminus A \\ &= T^{-1}(R_{n-1}) \cap A^c \\ &= T^{-1}(T^{-1}(R_{n-2}) \cap A^c) \cap A^c \\ &= (T^{-2}(R_{n-2}) \cap T^{-1}(A^c)) \cap A^c \\ &= (T^{-n}(B) \cap \cdots \cap T^{-1}(A^c)) \cap A^c \\ &= T^{-n}(B) \cap \left(\bigcap_{i=0}^{n-1} T^{-i}(A^c) \right). \end{aligned}$$

This gives

$$D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A.$$

By definition, $T_A^{-1}(B)$ is the set of points $y \in A$ such that $T(y) \in B$ or such that there exists a $k \in \mathbb{N}$ with $k \geq 2$ such that $T^k(y) \in B$ and $T^i(y) \notin A$ for all $i \in \{1, \ldots, k-1\}$.

From the above, , it follows that $D = T_A^{-1}(B)$. $\square$

T is $\mathcal{A}$-measurable, so from $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A$ it follows that $D_n \in \mathcal{E}$. Since D is a countable union of such D_n, $D \in \mathcal{E}$, so by conjecture 4, $T_A^{-1}(B) \in \mathcal{E}$, so T_A is $\mathcal{E}$-measurable. (9)

Let $i, j \in \mathbb{N}$ with $i > j \geq 1$. (10)

Suppose $D_j \cap D_i \neq \emptyset$. (11)

By (11), there exists $x \in D_j \cap D_i$. By the fact that $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c)\right) \cap A$, it follows that $T^j(x) \in B$. (12)

Since $j < i$, $T^j(x) \in A^c$, but this contradicts (12), so D_j and D_i are disjoint. (13)

From (7), it is clear that $\forall n \in \mathbb{N}$, $D_n \cap R_n = \emptyset$ (14)

$$\mu(R_{n+1}) + \mu(D_{n+1}) = \mu(R_n) \quad \forall n \in \mathbb{N} \tag{15}$$

From (15), $\mu(R_n)$ is decreasing. (16)

By (13),

$$\mu(D) = \sum_{n \geq 1} \mu(D_n) \tag{17}$$

From (15),

$$\sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \mu(R_n) \quad \forall n \in \mathbb{N} \tag{18}$$

By (17), (18) and (16),

$$\mu(D) = \lim_{n \to \infty} \sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \lim_{n \to \infty} \mu(R_n) \tag{19}$$

By the Poincaré recurrence theorem,

$$\mu(T_A^{-1}(B)) \geq \mu(B) \tag{20}$$

By conjecture 4 and (20), $\mu(D) \geq \mu(B)$. (21)

By (21) and (19),

$$\mu(B) - \lim_{n \to \infty} \mu(R_n) = \mu(D) \geq \mu(B) \tag{22}$$

From (22),

$$\lim_{n \to \infty} \mu(R_n) = 0 \tag{23}$$

From (23), (19) and conjecture 4,

$$\mu(T_A^{-1}(B)) = \mu(D) = \mu(B) \tag{24}$$

By (24), T_A is measure-preserving. (25)

Let $T_f : X_f \to X_f$ be the primitive transformation, where $f : X \to \mathbb{N}$ is measurable. (26)

Let $A \in \mathcal{A}$ and $k \in \mathbb{N}$. (27)

Note, $(A \times \{k\}) \cap X_f = (A \cap C_k) \times \{k\}$ where $C_k = f^{-1}(\{n \in \mathbb{N} : n \geq k\})$ (28)

$X_f = \{(x, k) : x \in X, 1 \leq k \leq f(x)\} \subseteq X \times \mathbb{N}$ (29)

Suppose $k > 1$. By (28):

$$T_f^{-1}((A \times \{k\}) \cap X_f) = T_f^{-1}((A \cap C_k) \times \{k\}) = (A \cap C_k) \times \{k - 1\} \tag{30}$$

Suppose $k = 1$ (31)

$$T_f^{-1}(A \times \{k\}) = \bigcup_{i \geq 1} (T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \tag{32}$$

From (30) and (32), $T_f^{-1}(A \times \{k\}) \cap X_f \in \mathcal{U}_f \quad \forall k$ (33)

If $k > 1$, then by (30) and (28),

$$\begin{aligned}\mu_f(T_f^{-1}((A \times \{k\}) \cap X_f)) &= \mu_f((A \cap C_k) \times \{k-1\}) \\ &= \mu_f((A \cap C_k) \times \{k\} \cap X_f)\end{aligned} \tag{34}$$

If $k = 1$, then by (32) and T being measure-preserving,

$$\begin{aligned}\mu_f(T_f^{-1}(A \times \{1\})) &= \mu_f\left(\bigcup_{i\geq 1}(T^{-1}(f^{-1}(i) \cap A) \times \{i\}\right) \\ &= \sum_{i\geq 1} \mu_f(T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \\ &= \sum_{i\geq 1} \mu(T^{-1}(f^{-1}(i) \cap A)) \\ &= \mu\left(\bigcup_{i\geq 1} T^{-1}(f^{-1}(i) \cap A)\right) \\ &= \mu(T^{-1}(f^{-1}(\mathbb{N}) \cap A)) \\ &= \mu(T^{-1}(A)) \\ &= \mu(A) \\ &= \mu_f(A \times \{1\})\end{aligned} \tag{35}$$

By (34) and (35), the primitive transformation is measure-preserving. (36)

Exercise 4.3.3.

A measure-preserving transformation or flow T of a probability space $(X, \mathcal{U}, \mu)$ is called *(strong) mixing* if

$$\lim_{t\to\infty} \mu(T^t(A) \cap B) = \mu(A) \cdot \mu(B)$$

for any two measurable sets $A, B \in \mathcal{U}$. (1)

Equivalently, T is mixing if

$$\lim_{t\to\infty} \int_X f(T^t(x))g(x)\, d\mu = \int_X f\, d\mu \int_X g\, d\mu$$

for any two bounded measurable functions. (2)

Transformation T is called *weak mixing* if $\forall A, B \in \mathcal{U}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left|\mu(T^{-i}(A) \cap B) - \mu(A)\mu(B)\right| = 0. \tag{3}$$

Equivalently, T is weak mixing if for all bounded measurable functions,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left|\int_X f(T^i(x))g(x)\, d\mu - \int_X f\, d\mu \int_X g\, d\mu\right| = 0. \tag{4}$$

Flow T is called weak mixing if $\forall A, B \in \mathcal{U}$,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left|\mu(T^{-s}(A) \cap B) - \mu(A)\mu(B)\right|\, ds = 0. \tag{5}$$

Equivalently, T is weak mixing if for all bounded measurable functions,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \left|\int_X f(T^s(x))g(x)\, d\mu - \int_X f\, d\mu \int_X g\, d\mu\right| ds = 0. \tag{6}$$

Show that the two definitions of strong and weak mixing given in terms of sets and bounded measurable functions are equivalent. (7)

Proof + reasoning:

Let T be a measure-preserving flow on $(X, \mathcal{U}, \mu)$. (8)

Assume (5) holds. (9)

Suppose f and g are simple, with

$$f = \sum_{i\le n} \mathbf{1}_{A_i} a_i, \quad g = \sum_{j\le n} \mathbf{1}_{A_j} b_j. \tag{10}$$

Define

$$M := \lim_{t\to\infty} \frac{1}{t} \int_0^t \left|\int_X f(T^s(x))g(x)\, d\mu - \int_X f\, d\mu \int_X g\, d\mu\right| ds. \tag{11}$$

Then

$$M=\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\int_X f(T^s(x))g(x)\,d\mu-\sum_{i,j\leq n}\mu(A_i)\mu(A_j)a_ib_j\right|ds. \tag{12}$$

Let $h:\mathbb{R}\to\mathbb{R}$ be an arbitrary function. For all $x\in\mathbb{R}$, we have that $f(h(x))=a_i$ if $h(x)\in A_i$. Hence, for all $x\in\mathbb{R}$,

$$f(h(x))=\sum_{i\leq n}a_i\mathbf{1}_{h_{-1}(A_i)}(x). \tag{13}$$

By (13),

$$\begin{aligned}\int_X f(T^s(x))g(x)\,d\mu&=\int_X\left(\sum_{i\leq n}\mathbf{1}_{T^{-s}(A_i)}a_i\right)\left(\sum_{j\leq n}\mathbf{1}_{A_j}b_j\right)d\mu\\&=\sum_{i,j\leq n}\mu(T^{-s}(A_i)\cap A_j)a_ib_j.\end{aligned} \tag{14}$$

Clearly,

$$\int_X f\,d\mu\int_X g\,d\mu=\left(\sum_{i\leq n}\mu(A_i)a_i\right)\left(\sum_{j\leq n}\mu(A_j)b_j\right)=\sum_{i,j\leq n}\mu(A_i)\mu(A_j)a_ib_j. \tag{15}$$

By (14) and (15),

$$M=\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\sum_{i,j\leq n}a_ib_j\left(\mu(T^{-s}(A_i)\cap A_j)-\mu(A_i)\mu(A_j)\right)\right|ds. \tag{16}$$

By (16),

$$\begin{aligned}M&\leq\lim_{t\to\infty}\frac{1}{t}\int_0^t\sum_{i,j\leq n}|a_ib_j|\left|\mu(T^{-s}(A_i)\cap A_j)-\mu(A_i)\mu(A_j)\right|ds\\&=\sum_{i,j\leq n}|a_ib_j|\lim_{t\to\infty}\frac{1}{t}\int_0^t\left|\mu(T^{-s}(A_i)\cap A_j)-\mu(A_i)\mu(A_j)\right|ds\\&=0.\end{aligned} \tag{17}$$

Assume that f and g are measurable and bounded by some $C>0$. (18)

By (36), f and g are the uniform limits of sequences (f_n) and (g_n) respectively, where f_n and g_n are simple functions that are bounded by C. (19)

By the dominated convergence theorem,

$$\begin{aligned}M&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\Big|\int_X\lim_{n\to\infty}f_n(T^s(x))\lim_{n\to\infty}g_n(x)\,d\mu-\int_X\lim f_n\,d\mu\int_X\lim g_n\,d\mu\Big|\,ds\\&=\lim_{t\to\infty}\frac{1}{t}\int_0^t\lim_{n\to\infty}\Big|\int_X f_n(T^s(x))g_n(x)\,d\mu-\int_X f_n\,d\mu\int_X g_n\,d\mu\Big|\,ds\end{aligned} \tag{20}$$

Note,
$$\left|\int_X f_n(T^s(x))g_n(x)\,d\mu - \int_X f_n\,d\mu \int_X g_n\,d\mu\right| \le 2C\mu(X). \tag{21}$$
By (20), by the fact that the absolute value is continuous, and by (21) together with the dominated convergence theorem,

$$\begin{aligned} M &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \lim_{n\to\infty}\left|\int_X f_n(T^s(x))g_n(x)\,d\mu - \int_X f_n\,d\mu \int_X g_n\,d\mu\right| ds \\ &= \lim_{t\to\infty}\lim_{n\to\infty} \frac{1}{t}\int_0^t \left|\int_X f_n(T^s(x))g_n(x)\,d\mu - \int_X f_n\,d\mu \int_X g_n\,d\mu\right| ds \end{aligned} \tag{22}$$

Let $\mathcal{T} = (0,\delta)$, where $\delta > 0$. (23)

Let $\Delta_1 > 0$. (24)

There exists a $k \in \mathbb{N}$ such that for all $n \ge k$ and all $x \in X$, $f_n(T^s(x))g_n(x) \in B(f(T^s(x))g(x), \Delta_1)$, where $B(x,\epsilon)$ denotes a ball around x of radius ϵ, so

$$\int_X f_n(T^s(x))g_n(x)\,d\mu \in B\left(\int_X f(T^s(x))g(x)\,d\mu, \Delta_1\mu(X)\right) \tag{25}$$

Let $\Delta_2 \ge 0$. Then there exists an m such that for $n \ge m$,

$$\int_X f_n\,d\mu \in B\left(\int_X f\,d\mu, \Delta_2\mu(X)\right) \quad \text{and} \quad \int_X g_n\,d\mu \in B\left(\int_X g\,d\mu, \Delta_2\mu(X)\right). \tag{26}$$

By (25) and (26), for all $n \ge \max(m,k)$,

$$\begin{aligned} &\frac{1}{t}\int_0^t \left|\int_X f_n(T^s(x))g_n(x)\,d\mu - \int_X f\,d\mu \int_X g\,d\mu\right| ds \\ &\in B\Big(\frac{1}{t}\int_0^t \left|\int_X f(T^s(x))g(x)\,d\mu - \int_X f\,d\mu \int_X g\,d\mu\right| ds, \Delta_1\mu(X) + 2\Delta_2\mu(X)\Big) \end{aligned} \tag{27}$$

Since Δ_1 and Δ_2 were arbitrary, and (27) does not depend on t,

$$\|h_n(t) - h(t)\|_{\mathcal{T}} \to 0. \tag{28}$$

Therefore, by the Moore–Osgood theorem,

$$M = \lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t}\int_0^t \left|\int_X f_n(T^s(x))g_n(x)\,d\mu - \int_X f_n\,d\mu \int_X g_n\,d\mu\right| ds = 0. \tag{29}$$

Clearly, (6) implies (5), so (5) and (6) are equivalent. (30)

We will skip the proof of (3) ⇔ (4), since it is likely very similar to the one for (5) ⇔ (6). (31)

Next, suppose
$$\lim_{t\to\infty} \mu(T^{-t}(A)\cap B) = \mu(A)\cdot\mu(B)$$
for any two measurable sets $A, B \in \mathcal{U}$. (32)

By (32) and by dominated convergence,

$$\begin{aligned}
\lim_{t\to\infty}\int_X f_n(T^{-t}(x))g_n(x)\,d\mu &= \lim_{t\to\infty}\int_X \left(\sum_{i=1}^n \mathbf{1}_{T^{-t}(A_i)}a_i^n\right)\left(\sum_{j=1}^n \mathbf{1}_{A_j}b_j^n\right)\,d\mu \\
&= \lim_{t\to\infty}\int_X \left(\sum_{i,j\le n} \mathbf{1}_{T^{-t}(A_i)\cap A_j}(x)a_i^n b_j^n\right)\,d\mu \\
&= \sum_{i,j\le n}\lim_{t\to\infty}\int_X \mathbf{1}_{T^{-t}(A_i)\cap A_j}(x)a_i^n b_j^n\,d\mu \\
&= \sum_{i,j\le n}\lim_{t\to\infty}\mu(T^{-t}(A_i)\cap A_j)a_i^n b_j^n \\
&= \sum_{i,j\le n}\mu(A_i)\mu(A_j)a_i^n b_j^n.
\end{aligned} \tag{33}$$

So,

$$\int_X f\,d\mu\int_X g\,d\mu. \tag{34}$$

Using (19), (34) and dominated convergence, it follows from an argument similar to the one used to derive (29) that

$$\begin{aligned}
M_s(f,g) := \lim_{t\to\infty}\int_X f(T^t(x))g(x)\,d\mu &\\
= \lim_{t\to\infty}\int_X \lim_{n\to\infty} f_n(T^t(x))\lim_{n\to\infty} g_n(x)\,d\mu &\\
= \lim_{t\to\infty}\lim_{n\to\infty}\int_X f_n(T^t(x))g_n(x)\,d\mu &\\
= \lim_{n\to\infty}\lim_{t\to\infty}\int_X f_n(T^t(x))g_n(x)\,d\mu &\\
= \lim_{n\to\infty}\int_X f_n\,d\mu\int_X g_n\,d\mu &\\
= \int_X f\,d\mu\int_X g\,d\mu. &
\end{aligned} \tag{35}$$

In other words, statement (1) implies statement (2). Since the converse is trivial, (1) and (2) are equivalent. (36)