Exercise 4.2.3

Prove that if T is a measure-preserving transformation, then so are the induced transformations. (1)

Proof + reasoning:

Let $T : (X, \mathcal{A}, \mu) \to (X, \mathcal{A}, \mu)$ be a measure-preserving transformation. (2)

The primitive transformation is similar to a suspension with ceiling f. (3)

What is the natural measure on $(X_f, \mathcal{F})$? The only obvious one is the product of μ and the counting measure on $\mathbb{N}$. Although not explicitly stated, this is exactly the measure μ_f (4)

Let's start with proving that the derivative transformation is measure preserving. First, let's check that it is measurable. (5)

What is the natural σ-algebra on A? Clearly, it should be the 'subspace' sigma algebra, also referred to as the trace σ-algebra. (6)

Let $\mathcal{E}$ be the trace σ-algebra with respect to A and $A \in \mathcal{A}$. (7)

Let $B \in \mathcal{E}$. (8)

By definition, $B = C \cap A$, $C \in \mathcal{A}$. (9)

We want to show that $T_A^{-1}(B) \in \mathcal{E}$, i.e. that $T_A^{-1}(B) = B' \cap A$ for some $B' \in \mathcal{E}$. (10)

A standard way to show that a set such as $T_A^{-1}(B)$ is measurable is to equate it to a countable combination of measurable sets. (11)

$$T_A^{-1}(B) = \bigcup_{k \geq 1} (T^{-k} \cap A)? \tag{12}$$

Statement (12) seems false, in that $\bigcup_{k\geq 1}(T^{-k} \cap A)$ contains points that aren't in $T_A^{-1}(A)$, i.e. given $y \in A$, if $x \in T^{-1}(y)$, and $\exists z \in A$ s.t. $T(z) = x$, then $z \in \bigcup_{k=1}^{\infty} T^{-k}(y)$ but $z \notin T_A^{-1}(y)$. (13)

Let's try a different idea. Intuitively, the set $T^{-1}(B)$ can be partitioned into $T^{-1}(B) \cap A$, which clearly is a subset of $T_A^{-1}(B)$, and the 'remainder', $T^{-1}(B) \setminus A$. The remainder may still contain points x of which the inverse image 'eventually' intersects with A, meaning that there exists a $k \in \mathbf{N}$ with $T^{-k}(x) \in A$. By repeatedly taking the inverse image of the remainder and intersecting with A, we should obtain all points in $T_A^{-1}(B)$. Let's formalize this. (14)

Let (R_n) and (D_n) be sequences of sets defined inductively by letting

$$\begin{aligned} R_0 &= B, & D_0 &= \emptyset \\ R_1 &= T^{-1}(B) \setminus A, & D_1 &= T^{-1}(B) \cap A \\ R_{n+1} &= T^{-1}(R_n) \setminus A, & D_{n+1} &= T^{-1}(R_n) \cap A \quad \forall n \geq 2 \end{aligned} \tag{15}$$

Let $D := \bigcup_{n\geq 1} D_n$ (16)

Conjecture 1. $D = T_A^{-1}(B)$

Proof. Let $n \geq 2$

$$\begin{aligned}
R_n &= T^{-1}(R_{n-1}) \setminus A \\
&= T^{-1}(R_{n-1}) \cap A^c \\
&= T^{-1}(T^{-1}(R_{n-2}) \cap A^c) \cap A^c \\
&= (T^{-2}(R_{n-2}) \cap T^{-1}(A^c)) \cap A^c \\
&= (T^{-n}(B) \cap \cdots \cap T^{-1}(A^c)) \cap A^c \\
&= T^{-n}(B) \cap \left(\bigcap_{i=0}^{n-1} T^{-i}(A^c) \right).
\end{aligned}$$

This gives

$$D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A.$$

By definition, $T_A^{-1}(B)$ is the set of points $y \in A$ such that $T(y) \in B$ or such that there exists a $k \in \mathbb{N}$ with $k \geq 2$ such that $T^k(y) \in B$ and $T^i(y) \notin A$ for all $i \in \{1, \ldots, k-1\}$.

From the above, , it follows that $D = T_A^{-1}(B)$. $\square$

T is $\mathcal{A}$-measurable, so from $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A$ it follows that $D_n \in \mathcal{E}$. Since D is a countable union of such D_n, $D \in \mathcal{E}$, so by conjecture 1, $T_A^{-1}(B) \in \mathcal{E}$, so T_A is $\mathcal{E}$-measurable. (17)

Now we need to show that T_A is measure-preserving. I think the sets D_n are disjoint. Let's prove this using contradiction. (18)

Let $i, j \in \mathbb{N}$ with $i > j \geq 1$. (19)

Suppose $D_j \cap D_i \neq \emptyset$. (20)

By (20), there exists $x \in D_j \cap D_i$. By the fact that $D_n = T^{-n}(B) \cap \left(\bigcap_{i=1}^{n-1} T^{-i}(A^c) \right) \cap A$, it follows that $T^j(x) \in B$. (21)

Since $j < i$, $T^j(x) \in A^c$, but this contradicts (21), so D_j and D_i are disjoint. (22)

We haven't used any results from the given section. The only theorem mentioned in section 4.2 is the Poincaré recurrence theorem, which essentially states that almost all points in B eventually return to B. In our case, it seems to imply that $\mu(B)$ is a lower bound of $\mu(D)$. (23)

What can I say about the measure of the sequence D_n and R_n? Intuitively, at each step n, the measure of R_n is preserved, but divided among D_{n+1} and R_{n+1}. This, together with the disjointness of D_i, seems to imply that the measure of all D_i up to D_n together with the measure of R_n is constant over time, and equal to $\mu(R_0) = \mu(B)$. That could give us an upper bound for $\mu(D)$, which together with (23) might allow us to conclude. (24)

From (15), it is clear that $\forall n \in \mathbb{N}$, $D_n \cap R_n = \emptyset$ (25)

$$\mu(R_{n+1}) + \mu(D_{n+1}) = \mu(R_n) \quad \forall n \in \mathbb{N} \tag{26}$$

From (26), $\mu(R_n)$ is decreasing. (27)

By (22),
$$\mu(D) = \sum_{n \geq 1} \mu(D_n) \tag{28}$$

From (26),
$$\sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \mu(R_n) \quad \forall n \in \mathbb{N} \tag{29}$$

By (28), (29) and (27),
$$\mu(D) = \lim_{n \to \infty} \sum_{1 \leq i \leq n} \mu(D_n) = \mu(B) - \lim_{n \to \infty} \mu(R_n) \tag{30}$$

By the Poincaré recurrence theorem,
$$\mu(T_A^{-1}(B)) \geq \mu(B) \tag{31}$$

By conjecture 1 and (31), $\mu(D) \geq \mu(B)$. (32)

By (32) and (30),
$$\mu(B) - \lim_{n \to \infty} \mu(R_n) = \mu(D) \geq \mu(B) \tag{33}$$

From (33),
$$\lim_{n \to \infty} \mu(R_n) = 0 \tag{34}$$

From (34), (30) and conjecture 1,
$$\mu(T_A^{-1}(B)) = \mu(D) = \mu(B) \tag{35}$$

By (35), T_A is measure-preserving. (36)

Next, we want to show that the primitive transformation is measure-preserving. (37)

Let $T_f : X_f \to X_f$ be the primitive transformation, where $f : X \to \mathbb{N}$ is measurable. (38)

By basic measure theory, it suffices to show, to prove T_f is measure-preserving, that T_f preserves the measure of all elements of a generating set of the σ-algebra. (39)

Let $A \in \mathcal{A}$ and $k \in \mathbb{N}$. (40)

Note, $(A \times \{k\}) \cap X_f = (A \cap C_k) \times \{k\}$ where $C_k = f^{-1}(\{n \in \mathbb{N} : n \geq k\})$ (41)

$X_f = \{(x, k) : x \in X, 1 \leq k \leq f(x)\} \subseteq X \times \mathbb{N}$ (42)

Suppose $k > 1$. By (41):
$$T_f^{-1}((A \times \{k\}) \cap X_f) = T_f^{-1}((A \cap C_k) \times \{k\}) = (A \cap C_k) \times \{k - 1\} \tag{43}$$

Suppose $k = 1$ (44)
$$T_f^{-1}(A \times \{k\}) = \bigcup_{i \geq 1} (T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \tag{45}$$

From (43) and (45), $T_f^{-1}(A \times \{k\}) \cap X_f \in \mathcal{U}_f \quad \forall k$ (46)

Now we show that T_f preserves the measure: (47)

If $k > 1$, then by (43) and (41),

$$\begin{aligned}\mu_f(T_f^{-1}((A \times \{k\}) \cap X_f)) &= \mu_f((A \cap C_k) \times \{k-1\}) \\ &= \mu_f((A \cap C_k) \times \{k\} \cap X_f)\end{aligned} \tag{48}$$

If $k = 1$, then by (45) and T being measure-preserving,

$$\begin{aligned}\mu_f(T_f^{-1}(A \times \{1\})) &= \mu_f\left(\bigcup_{i\geq 1}(T^{-1}(f^{-1}(i) \cap A) \times \{i\})\right) \\ &= \sum_{i\geq 1}\mu_f(T^{-1}(f^{-1}(i) \cap A) \times \{i\}) \\ &= \sum_{i\geq 1}\mu(T^{-1}(f^{-1}(i) \cap A)) \\ &= \mu\left(\bigcup_{i\geq 1}T^{-1}(f^{-1}(i) \cap A)\right) \\ &= \mu(T^{-1}(f^{-1}(\mathbb{N}) \cap A)) \\ &= \mu(T^{-1}(A)) \\ &= \mu(A) \\ &= \mu_f(A \times \{1\})\end{aligned} \tag{49}$$

By (48) and (49), the primitive transformation is measure-preserving. (50)