Exercise 1.8.3.

Let $\phi : \Sigma_2 = \{0,1\}^{\mathbb{Z}} \to H$ be the map that assigns to each infinite sequence $\omega = (\omega_i) \in \Sigma_2$ the unique point $\phi(\omega) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{\omega_i})$. (1)

Prove that ϕ is a bijection and that both ϕ and ϕ^{-1} are continuous. (2)

Proof + reasoning:

We will show ϕ is injective. (3)

Suppose $x, y \in \Sigma_2$ with $\phi(x) = \phi(y)$. Then

$$\bigcap_{-\infty}^{\infty} f^{-i}(R_{x_i}) = \phi(x) = \phi(y) = \bigcap_{-\infty}^{\infty} f^{-i}(R_{y_i}). \tag{4}$$

Is f injective? It seems so, since f is a map that stretches and bends the space D into a horseshoe, none of the regions of the horseshoe seem to overlap, and ϕ is a conjugacy. (5)

From the description of f, we see that f is injective. (6)

By definition,

$$R_0 = f(D_0) \cap R \quad \text{and} \quad R_1 = f(D_1) \cap R. \tag{7}$$

From (6), (7), and $D_0 \cap D_1 = \emptyset$, we get $R_0 \cap R_1 = \emptyset$. (8)

By (6) and (8), $f^{-i}(R_0) \cap f^{-i}(R_1) = \emptyset$ for all $i \in \mathbb{Z}$. (9)

From (4) and (9), $x_i = y_i$ for all $i \in \mathbb{Z}$, so $x = y$, so ϕ is injective. (10)

Let's show ϕ is surjective. (11)

Note $f(R) \cap R = R_1 \cup R_0$, and $f^{-1}(R) \subseteq R$. (12)

By (12), $f^{-i}(R) = f^{-i}(R_0) \cup f^{-i}(R_1)$ for all $i \geq 1$. (13)

Clearly, $R_0 \cap R_1 = \emptyset$. (14)

By (6) and (12), for all $i \geq 0$,

$$\big(f^{-i}(R_0) \cup f^{-i}(R_1)\big) \cap R = f^{-i}(R_0 \cup R_1) \cap R = f^{-i+1}(R) \cap f^{-i}(R) \cap R. \tag{15}$$

Let $x \in H$ and $j \in \mathbb{Z}$. By (13), (14) and (15), $x \in f^j(R_0)$ or $x \in f^j(R_1)$, but not both. (16)

By (16), we can define $x_j = 0$ if $x \in f^j(R_0)$ and $x_j = 1$ if $x \in f^j(R_1)$. Clearly, this gives a sequence $(x_j)_{j\in\mathbb{Z}} \in \Sigma_2$ such that $\phi((x_j)_{j\in\mathbb{Z}}) = x$. So ϕ is surjective. (17)

By (17) and (10), ϕ is bijective. (18)

Next, we show that ϕ is continuous. It seems that sets of the form $f^{-i}(R_{\omega_i}) \times R_{\omega_{i+1}}, \ldots, R_{\omega_n}$ are open, and even form a basis for the topology. Proving this will simplify the remaining parts of the exercise, since it suffices to prove that the inverse images of basis sets are open. (19)

Let $\omega \in \phi^{-1}(A \times B)$. For a sequence $\omega \in \{0,1\}^{\mathbb{Z}}$, define

$$R_{\omega_{-m},\ldots,\omega_m} = \bigcap_{i=-m}^{m} f^{-i}(R_{\omega_i}). \tag{20}$$

Define $\mathcal{C}_m = \{R_{\omega_{-m},\dots,\omega_m} \times R_{\omega_0,\dots,\omega_m}\}$, $\omega \in \{0,1\}^{\mathbb{Z}}$, $m \in \mathbb{N}$, and define $\mathcal{C} = \bigcup_{m\in\mathbb{N}} \{H \cap C : C \in \mathcal{C}_m\}$. (21)

Conjecture: $\mathcal{C}$ is a basis for the topology on H. (22)

Proof: Let $C \in \mathcal{C}$. Then $C = H \cap (R^- \times R^+)$ where $R^- \times R^+ \in \mathcal{C}_m$ for some $m \in \mathbb{N}$. (23)

Is C open? Intuitively, yes, since the sets in $\mathcal{C}_m$ are closed and bounded, we can contain them in open sets in $\mathbb{R}^2$ each containing no other points from H. (24)

Note $R^- = [x_1, x_2]$ and $R^+ = [y_1, y_2]$ for $x_1, x_2, y_1, y_2 \in \mathbb{R}$. (25)

By (14), for all $D \neq D' \in \mathcal{C}_m$, $D \cap D' = \emptyset$. (26)

By (25) and (26), there exist open intervals $I^-, I^+ \subset \mathbb{R}$ such that $R^- \times R^+ \subseteq I^- \times I^+$ and such that $(I^- \times I^+) \cap D = \emptyset$ for all $D \in \mathcal{C}_m$ with $D \neq R^- \times R^+$. (27)

Clearly, $\mathcal{C}_m$ covers H. (28)

By (27) and (28), $H \cap (R^- \times R^+) = H \cap (I^- \times I^+)$. (29)

By (29), C is open in H. (30)

Aren't we done at this point, since H is covered by $\mathcal{C}_m$ for all m? No, we still need to check that we can build each open U out of sets in $\mathcal{C}$ that are contained in U. Intuitively, this seems true because the rectangles in $\mathcal{C}_m$ get arbitrarily small as m increases. Let's prove this. (31)

Let A and B be open intervals in $\mathbb{R}$. (32)

Let $x \in H \cap (A \times B)$. (33)

Let $\varepsilon = \min\{d(x, y) : y \in A \times B\}$. (34)

Because $A \times B$ is open, $\varepsilon > 0$. (35)

Let $k = \min\{n \in \mathbb{N} : \mu^{-n} \leq \varepsilon, \lambda^n \leq \varepsilon\}$. (36)

By (35), and since $\lambda < 1/2$ and $\mu > 2$, $k > 0$. (37)

If $R^- \times R^+ \in \mathcal{C}_m$, then R^- has width equal to $\mu^{-k} \leq \varepsilon$ and R^+ has width equal to $\lambda^k \leq \varepsilon$. (38)

By (38), (34) and (28), there exists an $R^- \times R^+ \in \mathcal{C}_k$ such that $x \in R^- \times R^+$ and $R^- \times R^+ \subseteq A \times B$. (39)

By (30), $H \cap R^- \times R^+$ is open, so $\mathcal{C}$ is a basis for the topology on H. (40)

We can use (22) to prove ϕ is continuous, since we just need to prove $\phi^{-1}(C)$ is open for $C \in \mathcal{C}$. (41)

Let $C \in \mathcal{C}$. $C = H \cap (R_{\omega_{-m},\dots,\omega_m} \times R_{\omega_0,\dots,\omega_m})$ for $\omega \in \{0,1\}^{\mathbb{Z}}$. (42)

Let's show $\phi^{-1}(C) = B(\omega, 2^{-m})$. (43)

Let $j \in \{-m, -m+1, \dots, m\}$. (44)

Suppose $z \in \phi^{-1}(C)$. By definition of ϕ,

$$\phi(z) = \bigcap_{i\in\mathbb{Z}} f^{-i}(R_{z_i}) \subseteq f^j(R_{z_j}). \tag{45}$$

Since $\phi(z) \in R^+ \times R^-$, $\phi(z) \in f^j(R_{\omega_j})$. (46)

By (14), $f^j(R_1) \cap f^j(R_0) = \emptyset$, so $z_j = \omega_j$, so $z \in B(\omega, 2^{-m})$. (47)

Clearly $B(\omega, 2^{-m}) \subseteq \phi^{-1}(C)$. (48)

By (47) and (48), $\phi^{-1}(C) = B(\omega, 2^{-m})$, so $\phi^{-1}(C)$ is open, so, by (22), ϕ is continuous. (49)

Now let's show ϕ^{-1} is continuous. (50)

Let $B(\gamma, 2^{-n})$ be an open ball in Σ_2. By the same argument as for (49),

$$B(\gamma, 2^{-n}) = \phi^{-1}(H \cap (R_{\gamma_{-n},\ldots,\gamma_n} \times R_{\gamma_0,\ldots,\gamma_n})). \tag{51}$$

So $\phi(B(\gamma, 2^{-n})) = H \cap (R_{\gamma_{-n},\ldots,\gamma_n} \times R_{\gamma_0,\ldots,\gamma_n})$. (52)

By (52) and (22), $\phi(B(\gamma, 2^{-n}))$ is open, so ϕ^{-1} is continuous. (53)