Exercise 2.8.3.

Prove the following generalization of Proposition 2.1.2. If a commutative group G acts by homeomorphisms on a compact metric space X, then there is a non-empty, closed G-invariant subset X' on which G acts minimally. (1)

Proof + reasoning:

Let's try to adapt the proof of Proposition 2.1.2 to the more general case. (2)

There are four theorems stated in section 2.8, but they don't seem applicable in this exercise. (3)

Let $\mathcal{C}$ be the collection of non-empty, closed G-invariant subsets of X, with the partial ordering given by inclusion. (4)

Since $X \in \mathcal{C}$, $\mathcal{C}$ is not empty. (5)

Suppose $\mathcal{K} \subseteq \mathcal{C}$ is a totally ordered subset. Then, any finite intersection of elements of $\mathcal{K}$ is nonempty, so by the finite intersection property for compact sets, $\bigcap_{K\in\mathcal{K}} K \neq \emptyset$. Thus, by Zorn's lemma, $\mathcal{C}$ contains a minimal element M. (6)

So far, we have followed the proof of 2.1.2 almost exactly. How can we conclude that G acts minimally on M? (7)

Intuitively speaking, we have very little constructive information about M, so proof by contradiction seems like a good strategy. (8)

Suppose that G does not act minimally on M. (9)

Then, there exists a point $b \in M$ and a nonempty open set $C \subseteq M$ such that $Gb \cap C = \emptyset$. (10)

To conclude the proof, we should find a contradiction, given (10) and (6). We want to find some set $Y \subseteq M$ that is closed, nonempty, and G-invariant. Let's start with some possible natural choices for Y, and work from there. (11)

Note C is not closed, $\mathrm{cl}(C)$ is not necessarily G-invariant, and $M \setminus C$ is also not necessarily G-invariant. On the other hand, $M \setminus GC$ seems like a good candidate. (12)

Since $GC = \bigcup_{g\in G} gC$, and each $g \in G$ is a homeomorphism, GC is open. (13)

So, since M is closed, $M \setminus GC$ is closed. (14)

Since $b \in M$ and $Gb \cap C = \emptyset$, $b \in M \setminus GC$, so $M \setminus GC$ is nonempty. (15)

If $m \in M \setminus GC$ and $g \in G$, then $m \neq g^{-1}c$, so $gm \neq C$, so since M is G-invariant, $gm \in M \setminus GC$, so $M \setminus GC$ is G-invariant. (16)

By (14)–(16), $M \setminus GC$ is a closed, nonempty, G-invariant proper subset of M, which contradicts (6), so (9) is false. Hence, G acts minimally on M. (17)