Exercise 1.5.3.

Suppose p is an attracting fixed point for f. Show that there is a neighborhood U of p such that the forward orbit of every point in U converges to p. (1)

Proof + reasoning:

Let's work out the meanings of the assumptions and of the required result. (2)

What is an attracting fixed point for f? A fixed point p is attracting if there exists a neighborhood U of p such that $\overline{U}$ is compact, $f(\overline{U}) \subseteq U$, and $\bigcap_{n\geq 0} f^n(U) = \{p\}$. (3)

From the definition of p, we are already given a candidate neighborhood U of p. Intuitively, it seems that $f^n(U)$ form a sequence of sets in which each is contained in the previous, with a single point p in the intersection, so orbits should converge to p or to a point in $\bigcap_{n\geq 0} f^n(\overline{U})$. But $f(\overline{U}) \subseteq U$ makes it likely that $\bigcap_{n\geq 0} f^n(\overline{U}) = \bigcap_{n\geq 0} f^n(U)$. Let's prove those intuitions. (4)

Is $f^{n+1}(U) \subseteq f^n(U)$ for all $n \in \mathbb{N}$? (5)

By assumption, there exists a neighborhood U of p such that $\overline{U}$ is compact, $f(\overline{U}) \subseteq U$, and $\bigcap_{n\geq 0} f^n(\overline{U}) = \{p\}$. (6)

Clearly,

$$U \subset \overline{U}. \tag{7}$$

From (7) and (6)

$$f(U) \subseteq f(\overline{U}) \subseteq U. \tag{8}$$

Therefore,

$$f^{n+1}(U) \subseteq f^n(U) \text{ for all } n \in \mathbb{N} \tag{9}$$

Is it true that $\bigcap_{n\geq 0} f^n(\overline{U}) = \bigcap_{n\geq 0} f^n(U)$? (10)

Clearly,

$$\bigcap_{n\geq 0} f^n(U) \subseteq \bigcap_{n\geq 0} f^n(\overline{U}) \tag{11}$$

Conversely,

$$\bigcap_{n\geq 0} f^n(\overline{U}) \subseteq \bigcap_{n\geq 1} f^n(\overline{U}) = \bigcap_{n\geq 0} f^{n+1}(\overline{U}) \subseteq \bigcap_{n\geq 0} f^n(U) \tag{12}$$

So, from (12)

$$\bigcap_{n\geq 0} f^n(U) = \bigcap_{n\geq 0} f^n(\overline{U}) \tag{13}$$

The answer to (10) is yes. Now, let's try to prove convergence of forward orbits. (14)

Let $x \in U$. Define $(x_n)_{n\in\mathbb{N}} = (f^n(x))_{n\in\mathbb{N}}$. (15)

Does (x_n) converge to p? (16)

Let's try proof by contradiction. (17)

Assume (x_n) does not converge. Then $\exists \varepsilon' > 0$ such that $\forall n : \exists k \geq n : d(f^k(x), p) > \varepsilon'$. (18)

I'm stuck. I know there are infinitely many points in $U \setminus B(p, \varepsilon')$. Intuitively, this should contradict the fact that $\bigcap_{n \geq 0} f^n(\overline{U})$ only contains p. (19)

Let me check what given assumptions I haven't used in my proof yet. I haven't used that p is a fixed point, that f is continuous, or the compactness of $\overline{U}$. (20)

Let me think of properties that we can use, or look up general consequences of these facts. (21)

In a metric space, the intersection of compact sets is compact. (22)

If some $y \in U$ is not in $f^n(U)$ for some $n \in \mathbb{N}$, it won't be in any $f^k(U)$ for $k \geq n$. (23)

In a compact metric space, all sequences have convergent subsequences. (24)

I think I can apply (24) to show a contradiction: (25)

From (18), there exists a sequence $f^{m_n}(x)$ such that $d(f^{m_n}(x), p) \geq \varepsilon$ for all $n \geq 0$. By compactness of $\overline{U}$, this sequence has a convergent subsequence $(f^{z_n}(x))_{n \geq 0}$ with $f^{z_n}(x) \to z \in \overline{U}$ and $z_n \to \infty$. (26)

Since f is continuous and $\overline{U}$ compact, $f^n(\overline{U})$ is compact for all $n \geq 0$. (27)

$\forall n \geq 0$ there exists K s.t. $f^{z_K}(x) \in f^n(\overline{U})$, hence $\forall m \geq K$, $f^{z_m}(x) \in f^n(\overline{U})$. (28)

From (28), the limit point z must be in $f^n(\overline{U})$ for all $n \geq 0$. (29)

Therefore, $z \in \bigcap_{n \geq 0} f^n(\overline{U}) = \{p\}$. (30)

So, $z = p$, which contradicts (18), so assumption (17) is false. Hence (x_n) converges. (31)

We still need to show that (x_n) converges to p, but the proof will likely be almost the same as above. (32)

Suppose $x_n \to q$ and $q \neq p$. (33)

Let $n \geq 0$. $(x_i)_{i \geq n}$ is contained in $f^n(\overline{U})$, which is compact, so $q \in f^n(\overline{U})$. (34)

By (34), $q \in \bigcap_{n \geq 0} f^n(\overline{U})$. (35)

By (13), $q \in \bigcap_{n \geq 0} f^n(U) = \{p\}$, (36)

which is a contradiction, so (33) is false. (37)

From (37) and (31), we get $x_n \to p$. Therefore, the forward orbit of any point in U converges to p. (38)