Exercise 1.9.3.

Let $\mathbb{T}$ denote the set of sequences $(\phi_i)_{i=0}^{\infty}$ where $\phi_i \in S^1$ and $\phi_i = 2\phi_{i+1} \mod 1$ for all i. Let $\alpha : \mathbb{T} \to \mathbb{T}$ be defined by

$$(\phi_0, \phi_1, \dots) \mapsto (2\phi_1, \phi_1, \phi_2, \dots). \tag{1}$$

Show that $\mathbb{T}$ is a topological group. (2)

Show that α is an automorphism (3)

Proof + reasoning:

Let's first formulate basic lemmas to use throughout the exercise. (4)

Lemma 1. *$\forall x \in \mathbb{R}, \forall y \in \mathbb{R},\ a, b \in \mathbb{N} : x \equiv_a y \Rightarrow bx \equiv_a by$.*

Proof. $\exists p \in \mathbb{Z} : x = y + pa$, so $bx = by + bpa$, so $bx \equiv_a by$. □

Lemma 2. *$\forall x \in \mathbb{R}, a, b \in \mathbb{N},\ a(x \mod b) \equiv_b ax$.*

Proof. Clearly, $x \mod b \equiv_b x$. By lemma 1, $a(x \mod b) \equiv_b ax$. □

Lemma 3. *$\forall x, y \in \mathbb{R},\ \forall k \in \mathbb{N},\ (x \mod k) + (y \mod k) \equiv_k x + y$.*

Proof. $x + y = (x \mod k) + (y \mod k) + pk + qk = (x \mod k) + (y \mod k) + (p + q)k$. So, $x + y \equiv_k (x \mod k) + (y \mod k)$. □

Given ψ and ϕ in $\mathbb{T}$, define $(\psi + \phi)_i = (\psi_i + \phi_i) \mod 1$. (5)

Let ψ and ϕ be elements of $\mathbb{T}$. It suffices to show $\psi_i + \phi_i \equiv_1 2(\psi + \phi)_{i+1}$. (6)

By (1), lemma 3 and lemma 2,

$$\begin{aligned}
\psi_i + \phi_i &= (2\psi_{i+1} \mod 1) + (2\phi_{i+1} \mod 1) \\
&\equiv_1 2\psi_{i+1} + 2\phi_{i+1} \\
&= 2(\psi_{i+1} + \phi_{i+1}) \\
&\equiv_1 2((\psi + \phi)_{i+1} \mod 1) \\
&\equiv_1 2(\psi + \phi)_{i+1}
\end{aligned} \tag{7}$$

From (7) and (6), $\mathbb{T}$ is closed under addition. (8)

For $\phi \in \mathbb{T}$, define $(-\phi)_i := -\phi_i$. (9)

Clearly, this is the inverse of ϕ. (10)

Now, let's check continuity. (11)

Suppose that $\phi^n \to \phi$ and $\psi^n \to \psi$ in $(S^1)^{\mathbb{N}_0}$. Let $n \geq 0$. By assumption, $\exists K_n$ s.t. $\forall j \leq n,\ \phi^i_j = \phi_j,\ \psi^i_j = \psi_j$, hence $(\phi^i - \psi^i)_j = \phi_j - \psi_j$. (12)

By (12), $\forall \varepsilon > 0, \exists k$ s.t. $d(\phi^i - \psi^i, \phi - \psi) \leq \varepsilon\ \forall i \geq k$, so $\lim_{n\to\infty} d(\phi^n - \psi^n, \phi - \psi) = 0$, hence $(\phi, \psi) \mapsto \phi - \psi$ is continuous. (13)

Now, we check α is a group automorphism. Is it possible to take a shortcut? I think that if α preserves the product and identity and is bijective, then it follows that α preserves inverses and that α^{-1} is a group homomorphism. (14)

Let $\phi, \psi \in \mathbb{T}$. Show $\alpha(\phi) + \alpha(\psi) = \alpha(\phi + \psi)$. By lemma 3 and lemma 2,

$$\begin{aligned}(\alpha(\phi) + \alpha(\psi))_i &= ((2\phi_i) \bmod 1 + (2\psi_i) \bmod 1) \bmod 1 \\ &\equiv_1 2(\phi_i + \psi_i) \\ &\equiv_1 2\left((\phi + \psi)_i \bmod 1\right) \\ &= \alpha(\phi + \psi)_i. \end{aligned} \tag{15}$$

Clearly, α preserves the identity. (16)

Now, let's show α is bijective. (17)

Suppose $\alpha(\phi) = \alpha(\psi)$. Then $\phi_{i-1} = \alpha(\phi)_i = \alpha(\psi)_i = \psi_{i-1}\ \forall i \geq 1$, so $\phi = \psi$, hence α is injective. (18)

Let $\phi \in \mathbb{T}$. Let $(\phi')_i := \phi_{i+1}\ \forall i \geq 1$. (19)

$\alpha(\phi') = \phi$, so α is surjective. (20)

By (18) and (20), α is bijective. By (15), (16), and (18), α is a group automorphism. (21)

Now we need to show α is a homeomorphism. Note that α is a map between product topologies. This should simplify our proofs. (22)

For product topologies the 1-d cylinders form a subbasis. So, to show that α is continuous it suffices to show that $\forall i \in \mathbb{N}, \pi_i \circ \alpha$ is continuous. (23)

Let $i \in \mathbb{N}$. Note $\pi_i \circ \alpha : (\phi_0, \phi_1, \dots) \mapsto \begin{cases} \phi_{i-1} & \text{if } i \geq 1 \\ 2\phi_0 & \text{if } i = 0 \end{cases}$. (24)

The map $r : S^1 \to S^1 : s \mapsto 2s \mod 1$ is clearly continuous. (25)

By (25), if $A \in \mathcal{T}(S^1)$, then

$$(\pi_i \circ \alpha)^{-1}(A) = \begin{cases} (S^1)^{i-1} \times A \times S^1 \times \cdots & \text{if } i \geq 1 \\ r^{-1}(A) \times S^1 \times \cdots & \text{if } i = 0 \end{cases} \tag{26}$$

By (26) and (25), $\pi_i \circ \alpha$ is continuous, so by (23), α is continuous. (27)

Note $\pi_i \circ \alpha^{-1} : (\phi_0, \phi_1, \dots) \mapsto \phi_{i+1}$. This is clearly continuous, so by (23), α^{-1} is continuous, so by (27) and (21), α is a homeomorphism. (28)