Exercise 1.12.3.

Compute the Lyapunov exponents for the solenoid. (1)

Proof + reasoning:

We need to calculate the matrix corresponding to the total derivative of F^n. (2)

Note the total derivative of $f\colon \mathbb{R}^n \to \mathbb{R}^n$ corresponds to the matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \tag{3}$$

We need to compute $dF^n(x)v$. Is there a shortcut that we can take? Since we are calculating the derivative of a composition of functions, we might use the chain rule. (4)

The chain rule for total derivatives states $d(f \circ g)(a) = df(g(a)) \circ dg(a)$. (5)

In our case, $d(f^n(x)) = df(f^{n-1}(a)) \circ df^{n-1}(a) = \cdots$ (6)

I'm not sure this will help us. Instead, let's calculate $f^n(x)$ directly, and from that the total derivative. (7)

Let $F\colon S^1 \times D^2 \to S^1 \times D^2$ be the solenoid. Let $x, y \in \mathbb{R}$ and let $\lambda \in (0, \frac{1}{2})$. (8)

Note $F(\phi, x, y) = (2\phi, \lambda x + \frac{1}{2}\cos(2\pi\phi), \lambda y + \sin(2\pi\phi))$. (9)

By writing out the composition, we see that:

$$\begin{aligned} F^n(\phi, x, y)_1 &= 2^n\phi \\ F^n(\phi, x, y)_2 &= \lambda^n x + \frac{1}{2}\lambda^{n-1}\cos(2\pi\phi) + \cdots + \frac{1}{2}\lambda^0\cos(2^{n-1}\pi\phi) \\ &= \lambda^n x + \frac{1}{2}\sum_{i=0}^{n-1}\lambda^i\cos(2^{n-1-i}\pi\phi) \\ F^n(\phi, x, y)_3 &= \lambda^n y + \frac{1}{2}\sum_{i=0}^{n-1}\lambda^i\sin(2^{n-1-i}\pi\phi) \end{aligned} \tag{10}$$

By (10), denoting $\delta_{ij} := \frac{\partial F_i}{\partial z_j}(\phi, x, y)$, we can express $dF^n(\phi, x, y)$ as follows:

$$\begin{aligned}
\delta_{11} &= 2^n \\
\delta_{21} &= -\frac{1}{2}\sum_{i=0}^{n-1} \lambda^i 2^{n-1-i}\pi \sin(2^{n-1-i}\pi\phi) \\
&= -\frac{\pi}{2}\sum_{i=0}^{n-1} \lambda^i 2^{n-1-i} \sin(2^{n-1-i}\pi\phi) \\
\delta_{31} &= \frac{\pi}{2}\sum_{i=0}^{n-1} \lambda^i 2^{n-1-i} \cos(2^{n-1-i}\pi\phi) \\
\delta_{22} &= \lambda^n \\
\delta_{33} &= \lambda^n \\
\delta_{ij} &= 0 \quad \text{otherwise}
\end{aligned} \tag{11}$$

The Lyapunov exponent is defined as

$$\chi(\phi, x, y, v) = \lim_{n\to\infty} \frac{1}{n} \log \|dF^n(\phi, x, y)v\| \tag{12}$$

Note the lim inf in the above definition. (13)

By (11),

$$dF^n(\phi, x, y)v = v_1(2^n + \delta_{21} + \delta_{31}) + \lambda^n(v_2 + v_3) \tag{14}$$

Intuitively, if $v_1 = 0$, then the Lyapunov exponent seems to be $\log(\lambda)$, but if $v_1 \neq 0$, then the first term of $dF^n(\phi, x, y)v$ dominates and the exponent seems to be $\log(2)$, given that δ_{21} and δ_{31} (which depend on n and ϕ) are small enough. (15)

So, we need to argue about the limiting behaviour of these terms. My idea is to bound $\chi(\phi, x, y, v)$ from above and below. We can find an upper bound using the fact that (co)sines are bounded by 1 and the fact that λ is less than $\frac{1}{2}$. (16)

Note, since $\lambda \in (0, \frac{1}{2})$, that $|\delta_{21}| \leq \pi \cdot n \cdot 2^n$ and $|\delta_{31}| \leq \pi \cdot n \cdot 2^n$. (17)

Suppose $v_1 \neq 0$. By (17) and (14), for n sufficiently large,

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{2}\cdot\frac{1}{n}\log\left(\|dF^n(\phi,x,y)v\|^2\right)\\
&= \frac{1}{2n}\log\left(v^2(2^n+\delta_{21}+\delta_{31})^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\leq \frac{1}{2n}\log\left((v_1\cdot 3\pi\cdot n\cdot 2^n)^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\leq \frac{1}{2n}\log\left((v_1\cdot 4\pi\cdot n\cdot 2^n)^2\right)\\
&= \frac{1}{n}\log(v_1\cdot 4\pi\cdot n\cdot 2^n)\\
&= \frac{1}{n}\left(\log(v_1\cdot 4\pi\cdot n)+n\log(2)\right)\\
&\xrightarrow{n\to\infty}\log(2)
\end{aligned} \tag{18}$$

By (18), $\chi(\phi,x,y,v)\leq\log(2)$. (19)

For the lower bound, by (20) and (18),

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{2n}\log\left(v_1^2(2^n+\delta_{21}+\delta_{31})^2+\lambda^{2n}(v_2+v_3)^2\right)\\
&\geq \frac{1}{2n}\log\left(v_1^2\cdot 2^{2n}\right)\\
&= \frac{1}{n}\log(v_1\cdot 2^n)\\
&= \log(2)+\frac{1}{n}\log(v_1)\xrightarrow{n\to\infty}\log(2)
\end{aligned} \tag{20}$$

By (19) and (20), $\chi(\phi,x,y,v)=\log(2)$. (21)

Suppose $v_1=0$. By (22) and (14),

$$\begin{aligned}
\frac{1}{n}\log\|dF^n(\phi,x,y)v\| &= \frac{1}{n}\log(\lambda^n(v_2+v_3))\\
&= \log(\lambda)+\frac{1}{n}\log(v_2+v_3)\\
&\xrightarrow{n\to\infty}\log(\lambda)
\end{aligned} \tag{22}$$

By (22), $\chi(\phi,x,y,v)=\log(\lambda)$. (23)

By (23) and (21), the Lyapunov exponents are $\log(2)$ and $\log(\lambda)$. $\square$ (24)