Tutorial 1

Some notes about the tutorials

- Attendance is not mandatory, but encouraged
- Feel free to make topic suggestions on Canvas
- TAs will make a post on Fridays asking for suggestions on topics to revisit
- If there's time left from the tutorial session after covering the suggested topics, it will be used as office hours

Frequency Response

From Lecture 5 slides

And we know that these complex exponential signals are eigenfunctions of LTI systems:

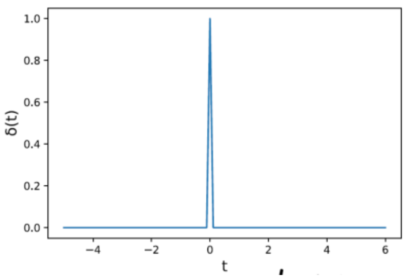
$$
\begin{aligned}
x(t) \rightarrow y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \\
x[n] \rightarrow y[n] &= \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}
\end{aligned}
$$

In general, $H(s)$ is called the **system function**; when we are limiting to $H(j\omega)$ we term this the **frequency response**.

In other words, when we input a complex exponential signal at a particular frequency, how does the system respond?

to know how a system will behave to any given signal

unit impulse



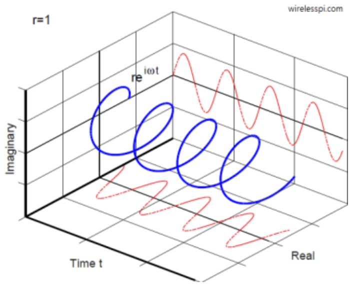
System

$h(t)$

impulse response

to know how a system will **behave to** any given **frequency**

complex exponential



System

$\mathrm{H}(j\omega)$

frequency response

Frequency Response

Three steps:

1. Determine the Fourier coefficients of the signal
2. Determine the frequency response
3. Use 1 and 2 to obtain Fourier coefficients of the output signal

1. Determine Fourier coefficients of input signal

Using an example from Quiz 2

Quiz 2 Fourier coefficients

The continuous-time Fourier series of a periodic function is defined as $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t}$.

Consider the function below: $x(t) = 2 + 2\sin(\omega t) + 2\cos(2\omega t)$

What are the real and imaginary parts of the first 3 Fourier coefficients? (Hint: $1/j = -j$.)

Helpful to remember:

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$
$$\frac{1}{j} = -j$$

- Expand to complex exponentials

$$x(t) = 2 + 2\frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) + 2\frac{1}{2}\left(e^{2j\omega t} + e^{-2j\omega t}\right)$$

When dealing with periodic signals such as these, calculating C_k is a straight forward process

$$x(t) = 2 - j\left(e^{j\omega t} - e^{-j\omega t}\right) + \left(e^{2j\omega t} + e^{-2j\omega t}\right)$$

$$C_0 = 2$$
$$C_1 = \frac{1}{j} = -j;\ \ C_{-1} = C_1^* = j$$
$$C_2 = C_{-2} = 1$$

2. Determine frequency response

Consider the impulse response function

$$h(t) = e^{-t}u(t)$$

We obtain the frequency response of the system as follows

$$H(j\omega) = \int_0^\infty e^{-\tau}e^{-j\omega\tau}d\tau = \int_0^\infty e^{-\tau(1+j\omega)}d\tau$$

$$= \frac{-1}{1+j\omega}e^{-\tau(1+j\omega)}\Big|_0^\infty$$

For $\tau = \infty$, $e^{-\tau(1+j\omega)} \to 0$, whereas for $\tau = 0$, $e^{-\tau(1+j\omega)} = 1$

So, it follows that the frequency response is

$$H(j\omega) = \frac{1}{1+j\omega}$$

Helpful to remember:
Since $u(t)$ will be 0 for any $t < 0$, the lower limit of the integral can be modified to reflect that, i.e., making the lower limit 0 instead of $-\infty$.

From Oppenheim, Example 3.16

3. Calculate the new coefficients for the signal

Original coefficients

$$C_0 = 2$$

$$C_1 = \frac{1}{j} = -j; \;\; C_{-1} = C_1^* = j$$

$$C_2 = C_{-2} = 1$$

Frequency response

$$H(j\omega) = \frac{1}{1 + j\omega}$$

New coefficients can be calculated using

$$y(t) = \sum_{k=-\infty}^{\infty} C_k H(jk\omega) e^{jk\omega t}$$

Where

$$\tilde{C}_k = C_k H(jk\omega)$$

New coefficients

$$\tilde{C}_0 = 2\left(\frac{1}{1+j\omega(0)}\right) = 2$$

$$\tilde{C}_1 = \tilde{C}_{-1}^* = -j\left(\frac{1}{1+j\omega(1)}\right) = \frac{-j}{1+j\omega}$$

$$\tilde{C}_2 = \tilde{C}_{-2} = 1\left(\frac{1}{1+j\omega(2)}\right) = \frac{1}{1+2j\omega}$$

The Real and Imaginary parts of C_k

In order to separate the real and imaginary parts of the coefficients, we multiply and divide by the conjugate of the denominator.
E.g.

$$\tilde{C}_1 = \tilde{C}^*_{-1} = \frac{-j}{(1+j\omega)}\frac{(1-j\omega)}{(1-j\omega)} = \frac{-\omega-j}{1+\omega^2}$$

$$\Re(\tilde{C}_1) = \frac{-\omega}{1+\omega^2},\ \Im(\tilde{C}_1) = \frac{-j}{1+\omega^2}$$

and

$$\tilde{C}_2 = \tilde{C}_{-2} = \frac{1}{(1+2j\omega)}\frac{(1-2j\omega)}{(1-2j\omega)} = \frac{1-2j\omega}{1+4\omega^2}$$

$$\Re(\tilde{C}_2) = \frac{1}{1+4\omega^2},\ \Im(\tilde{C}_2) = \frac{-2j\omega}{1+4\omega^2}$$

Another example

1. Coefficients of $x(t)$

$$x(t) = 1 + \cos(\omega t) + 3\cos(3\omega t)$$

$$x(t) = 1 + \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) + \frac{3}{2}\left(e^{3j\omega t} + e^{-3j\omega t}\right)$$

$$C_0 = 1;\;\; C_1 = C_{-1} = \frac{1}{2};\;\; C_3 = C_{-3} = \frac{3}{2}$$

2. Frequency response of $h(t) = \frac{1}{2}e^{-\frac{t}{6}}u(t)$

$$H(j\omega) = \int_0^{-\infty} \frac{1}{2} e^{-\tau/6} e^{-j\omega\tau} d\tau = \frac{-1}{2\left(\frac{1}{6}+j\omega\right)} e^{-\tau\left(\frac{1}{6}+j\omega\right)}\Bigg|_0^{\infty}$$

$$H(j\omega) = \frac{1}{2\left(\frac{1}{6}+j\omega\right)}$$

Another example (continuation)

3. New coefficients $\tilde{C}_k$

$$\tilde{C}_0 = 3$$

$$\tilde{C}_1 = \tilde{C}_{-1} = \frac{1}{\frac{2}{3}+4j\omega} = \frac{\frac{2}{3}}{\frac{4}{3}+16\omega^2} - \frac{4j\omega}{\frac{4}{3}+16\omega^2}$$

$$\tilde{C}_3 = \tilde{C}_{-3} = \frac{3}{\frac{2}{3}+12j\omega} = \frac{2}{\frac{4}{9}+144\omega^2} - \frac{36j\omega}{\frac{4}{9}+144\omega^2}$$