ELEC 221 Lecture 08
Introducing the Fourier transform

Tuesday 04 October 2022

Announcements

- Assignment 3 due Friday
- Assignment 4 available later this week (due after midterm)
- Quiz 4 today (beginning of class)

Today

Learning outcomes:

- Explain the concept of CT Fourier transform, and distinguish it from the CT Fourier series
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

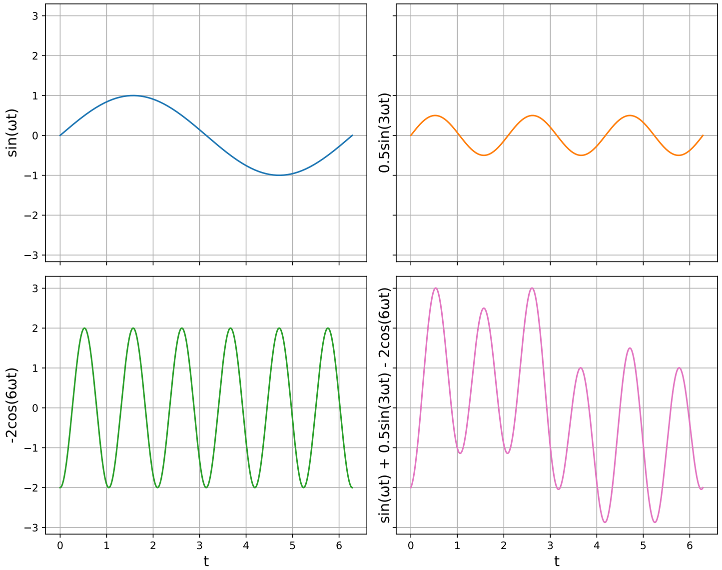
So far, we have been working with the Fourier series representation of **periodic** CT and DT signals:

CT synthesis equation:

CT analysis equation:

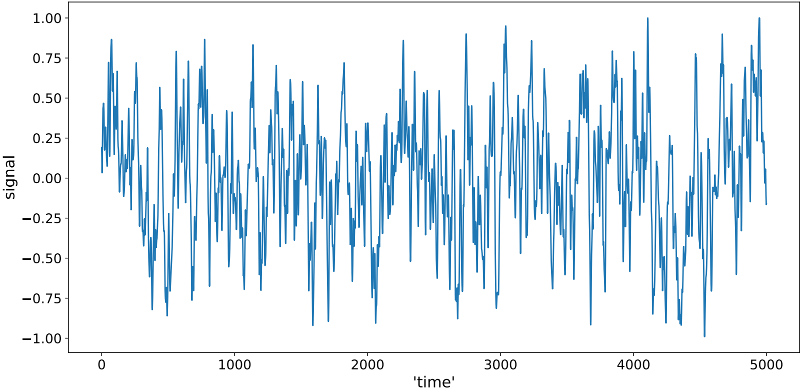
When the signal is periodic it can be represented using only the integer harmonics at the *same frequency* ω.

Recap: Fourier series



Towards the Fourier transform

On Thursday, we were working with audio signals:

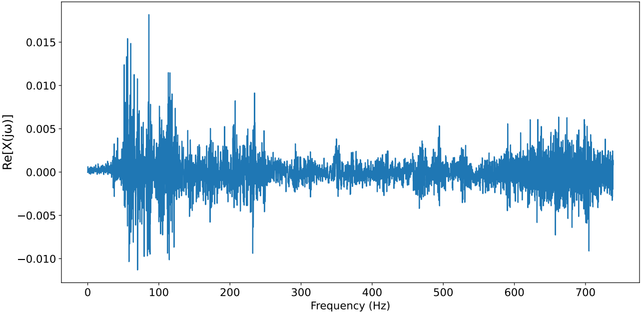


This is *not* a periodic signal.

But, we were still doing *something* with Fourier analysis to it:

```
fourier_coefficients = np.fft.rfft(audio)

frequencies = np.fft.rfftfreq(
    len(audio), 1 / sample_rate
)
```



The **Fourier transform** extends our Fourier series methods to **aperiodic signals**. It involves a **spectrum** of different frequencies.

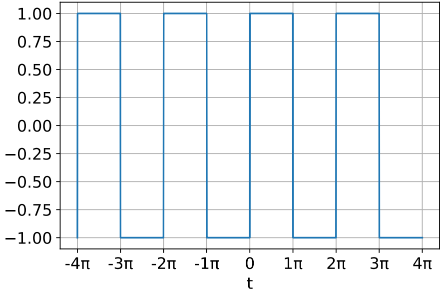
Fourier series:

Fourier transform:

How do we get here?

Towards the Fourier transform

Remember in lecture 4, we looked at a 2π-periodic square wave:



We derived its Fourier series representation

Let's generalize this a bit. Consider the following square wave:

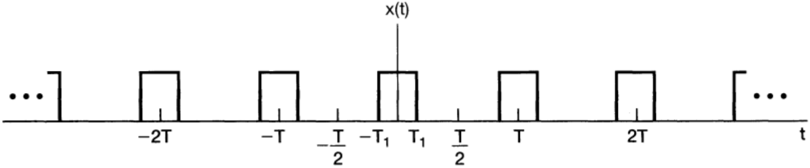


Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

Let's compute its Fourier coefficients.

$$c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega t}$$

Start with c_0:

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

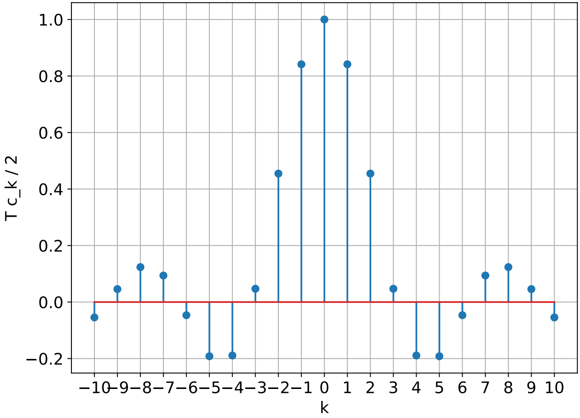
Now the c_k:

What does this function look like?

Let's rearrange a bit:

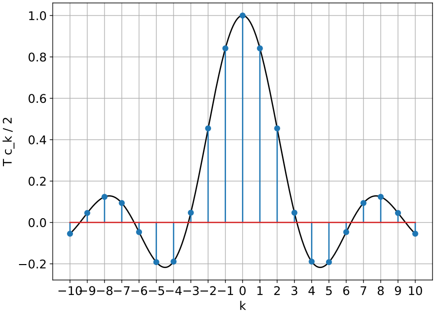
Let's plot the "important part" for different values of k.

Towards the Fourier transform



(Set $T_1 = \omega = 1$ to plot)

Towards the Fourier transform



These are **samples** of the function

at *integer values of* k.

Towards the Fourier transform

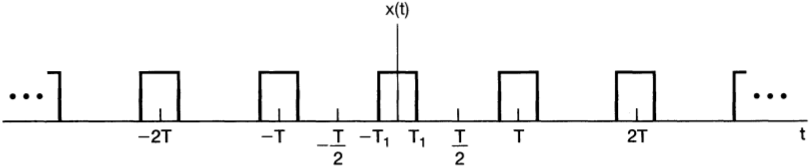
$$f(k) = \begin{cases} 1, & k = 0 \\ \frac{\sin(k\omega T_1)}{k\omega}, & k \neq 0 \end{cases}$$

Let's consider this differently, i.e., as a function of $\tilde{\omega}$:

The Fourier coefficients are samples of this function taken at *integer multiples* $k\omega$, where $\omega = 2\pi/T$.

Towards the Fourier transform

Suppose T grows (but T_1 stays the same)?



What happens to our samples from this function?

$$c_k \sim \frac{\sin(k\omega T_1)}{k\omega}$$

Image credit: Oppenheim chapter 4.1

Initially, we have some spacing of samples at integer values of $\omega = 2\pi/T$.

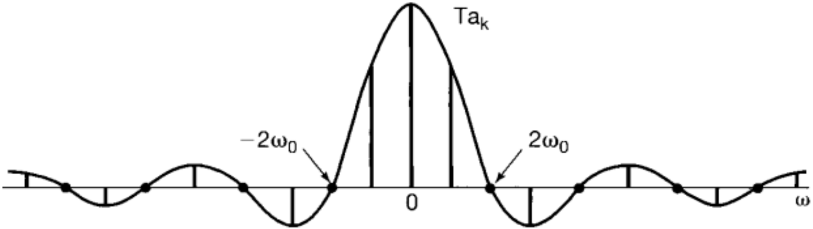
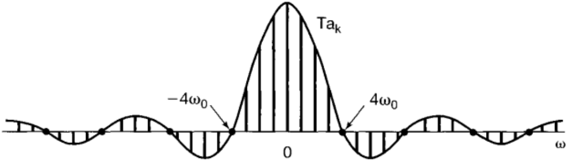


Image credit: Oppenheim chapter 4.1

As T grows, $\omega = 2\pi/T$ becomes smaller and smaller, so the integer multiples of it get closer and closer together.



(b)

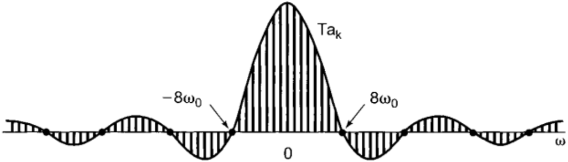


Image credit: Oppenheim chapter 4.1

Towards the Fourier transform

Eventually, ω becomes so small that instead of

we may as well just consider the sum over integer multiples as a continuous integral over all possible ω:

...but what does this have to do with non-periodic signals?

Given any aperiodic signal $x(t)$, we can always "pretend" it's periodic by constructing a **periodic extension**, $\tilde{x}(t)$ with period T.

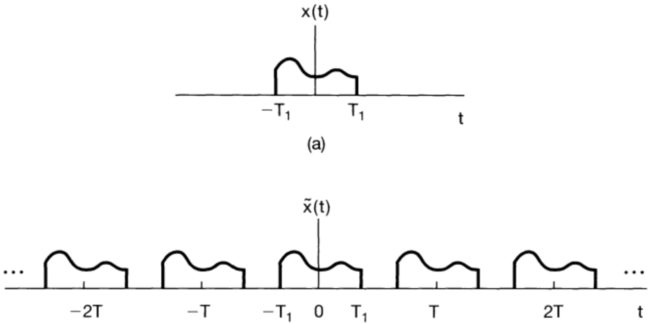


Image credit: Oppenheim chapter 4.1

Motivation: Fourier transform

Now that we made $\tilde{x}(t)$ look periodic, we can write it as a Fourier series (where $\omega = 2\pi/T$):

Motivation: Fourier transform

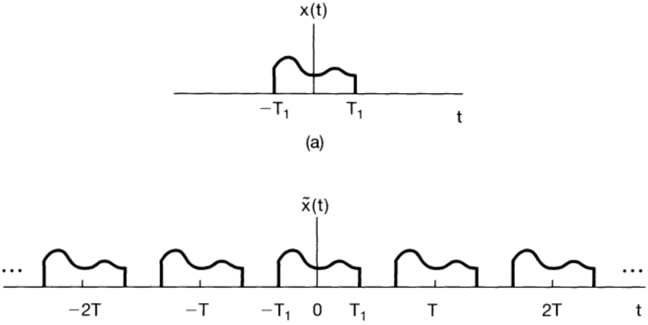


Image credit: Oppenheim chapter 4.1

Motivation: Fourier transform

What happens to the coefficients?

Let's define

so that

We can put this back in our Fourier series:

Motivation: Fourier transform

Now consider what happens when $T \to \infty$...

Two important things:

1. $\tilde{x}(t)$ will look just like $x(t)$ for large enough T
2. ω will get smaller and smaller

The Fourier transform

This is the **Fourier transform**.

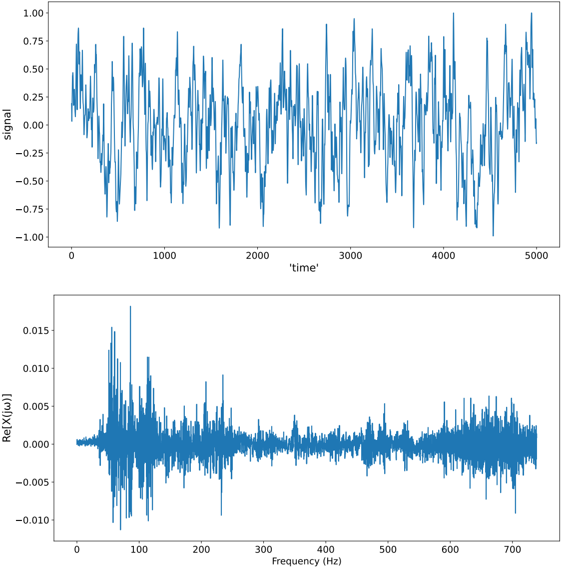
Inverse Fourier transform (synthesis equation):

Fourier transform (analysis equation, or Fourier *spectrum*):

Note: Sometimes the $1/2\pi$ prefactor appears on the spectrum, or sometimes both versions have $1/\sqrt{2\pi}$.

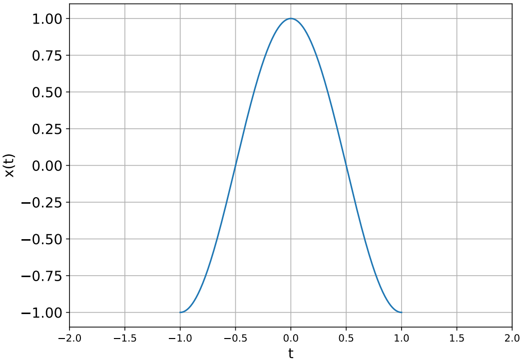
The Fourier transform

On Thursday, what we saw was a discretized version of this:



Example

Compute the Fourier spectrum of:



Example

$$x(t) = \begin{cases} \cos(\pi t), & |t| \leq 1 \\ 0, & |t| > 1 \end{cases}$$

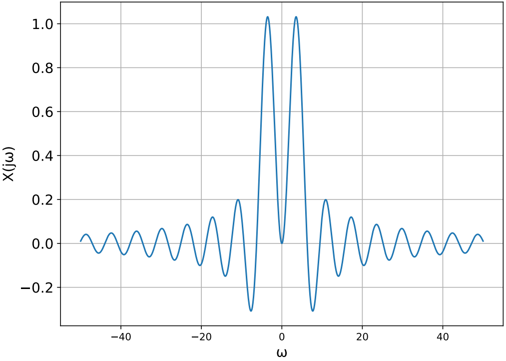
Start from the definition:

Example

$$X(j\omega) \qquad \frac{1}{2}\int_{-1}^{1} e^{j(\pi-\omega)t}dt + \frac{1}{2}\int_{-1}^{1} e^{-j(\pi+\omega)t}dt$$

Example

$$X(j\omega) = \frac{\sin(\omega)}{\pi - \omega} - \frac{\sin(\omega)}{\pi + \omega}$$



Fourier transform and impulse response

You've actually already (unknowingly) seen the Fourier transform when we discussed system functions and frequency response.

Recall the convolution integral representation of signals as a set of shifted, weighted impulses:

Put this in an LTI system with impulse response $h(t)$:

Fourier transform and impulse response

$$x(t) \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

We found that, when the signal in question is a complex exponential, that

Fourier transform and impulse response

The system function $H(j\omega)$, or frequency response

is the **Fourier transform of the impulse response**!

We can use the inverse Fourier transform to obtain the impulse response from the frequency response:

Fourier transform and impulse response

The same thing works in discrete time:

The impulse response can be obtained by computing the inverse discrete Fourier transform (recall we have only $\omega \in [0, 2\pi)$):

We will cover the DTFT in detail next week / after the midterm; but this should help you solve some A3 problems.

Today's learning outcomes were:

- Explain the concept of CT Fourier transform, and distinguish it from the CT Fourier series
- Compute the Fourier spectrum of a CT signal
- Describe how the Fourier transform relates impulse and frequency response of a system

What topics did you find unclear today?

For next time

Content:

- Properties of the CT Fourier *transform*
- Convolution properties of the Fourier transform and time/frequency duality

Action items:

1. Assignment 3 is due Friday
2. Assignment 4 released later this week
3. Midterm 1 next Thursday

Recommended reading:

- From today's class: Oppenheim 4.0-4.1
- For next class: Oppenheim 4.2-4.4