ELEC 221 Lecture 06
Introducing filters

Tuesday 27 September 2022

Announcements

- Assignment 2 due tomorrow
- Assignment 3 available tomorrow
- Quiz 3 today

We explored periodic DT complex exponential signals:

$$x[n] = e^{j\omega n} = e^{j\frac{2\pi}{N}n}$$

We found that these signals behave differently than CT signals...

Difference 1: we only need to consider ω in the range $[0, 2\pi)$.

$$\begin{aligned} x[n] &= e^{j(\omega+2\pi)n} \\ &= e^{j\omega n} \cdot e^{2\pi j n} \\ &= e^{j\omega n} \end{aligned}$$

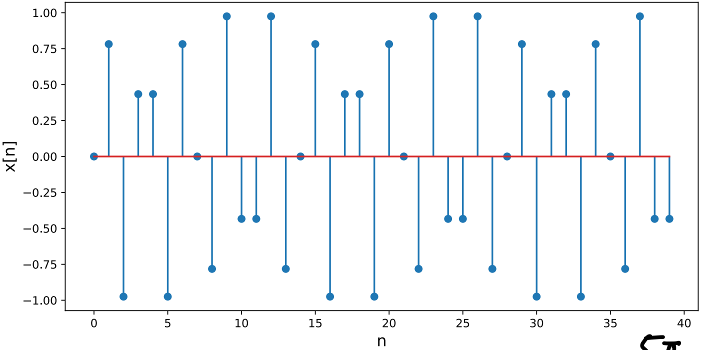
Difference 2: there are additional criteria for periodicity.

$$
\begin{aligned}
x[n+N] &= e^{j\omega(n+N)} \\
&= e^{j\omega n}e^{j\omega N} \\
&= e^{j\omega n} \quad \text{only if } \omega N = 2\pi m
\end{aligned}
$$

$$
\begin{aligned}
x[n+N] &= e^{jw(n+N)} \\
&= e^{jwn} \cdot e^{jwN} \\
&= e^{jwn} \qquad \omega N = 2\pi m
\end{aligned}
$$

Last time

Example: $x[n] = \sin(5\pi n/7)$ is periodic.



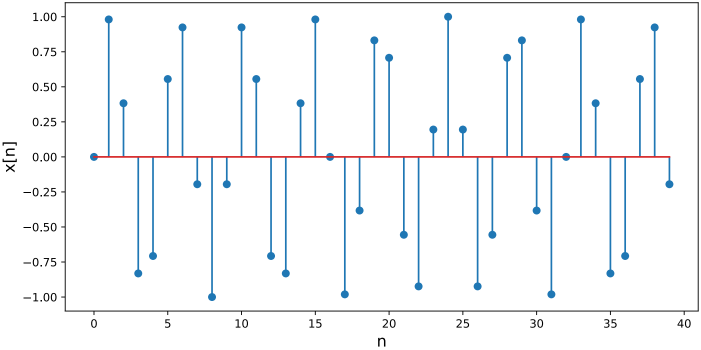
$\omega = \frac{5\pi}{7}$

In CT, period of $x(t) = \sin(5\pi t/7)$ is $T = 14/5$.
In DT, period of $x[n] = \sin(\underbrace{5\pi n/7})$ is $N = 14$.

$\frac{5\pi n}{7} = 2\pi m$

Last time

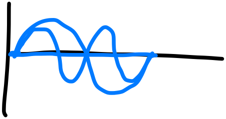
Example: $\underbrace{x[n] = \sin(5n/7)}$ is NOT periodic.



In CT, period of $x(t) = \sin(5t/7)$ is $\underbrace{T = 14\pi/5}$.

$e^{jk\omega n}$

Difference 3: there are only finitely many harmonics.



$$x_0[n] = 1$$

$$x_1[n] = e^{j\omega n} = e^{j\frac{2\pi}{N}n}$$

$$x_2[n] = e^{j2\cdot\frac{2\pi}{N}n}$$

$$\vdots$$

$$x_{N-1}[n] = e^{j(N-1)\cdot\frac{2\pi}{N}n}$$

$$x_N[n] = e^{jN\frac{2\pi}{N}n} = e^{2\pi jn} = 1$$

We expressed DT complex exponential signals with fundamental period N as Fourier series.

DT synthesis equation:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk \cdot \frac{2\pi}{N} n}$$

DT analysis equation:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk \frac{2\pi n}{N}}$$

We introduced the idea of **system functions** and **frequency response** of LTI systems.

$$x(t) \to y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

$$x[n] \to y[n] = \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}$$

$H(j\omega)$ in CT, and $H(e^{j\omega})$ in DT, are called the **frequency response** of the system.

Today

This week, we are going to look at our first major application of signals and systems: filters.

Learning outcomes:

- Define filters and identify a basic set of ideal frequency-selective filters
- Plot the frequency response of a filter in CT and DT
- Define and distinguish between finite-impulse-response (FIR) and infinite-impulse-response (IIR) filters in DT

In-class activity on Thursday: apply filters in the context of audio processing.

Filters are LTI systems that can be used to separate out, combine, or modify the components of a signal at specific frequencies.

Two key types:

- **Frequency-shaping**: change the amplitudes of parts of a signal at specified frequencies
- **Frequency-selective**: eliminate or attentuate parts of a signal at specified frequencies

CT frequency-selective filters

A convenient way to understand (and to design) filters is by looking at their **frequency response**.

Recall that the **system function** of a CT system $H(j\omega)$, is a measure of how much a system “modifies” a particular frequency:

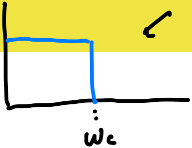
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k \underbrace{H(jk\omega)} e^{jk\omega t}$$

By selecting the frequency response at will, we can change the behaviour of a system.

Example. Consider a system function defined as follows.

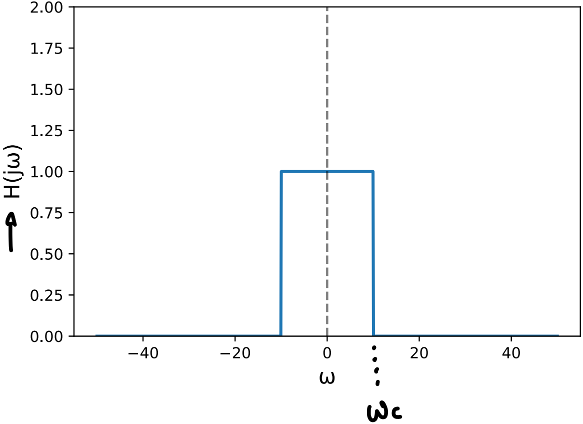
$$H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

What does this look like, and what does it do?



This is an **ideal lowpass filter**.

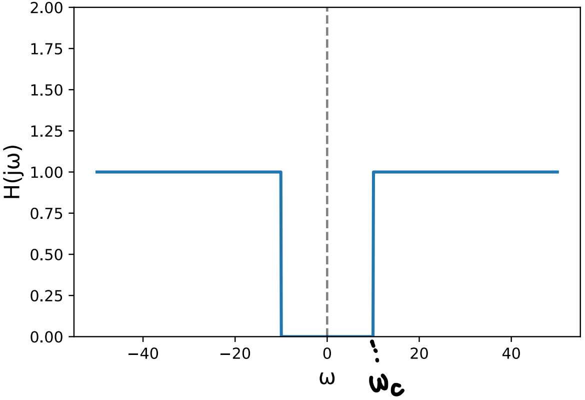
We often represent the frequency response graphically:



CT frequency-selective filters

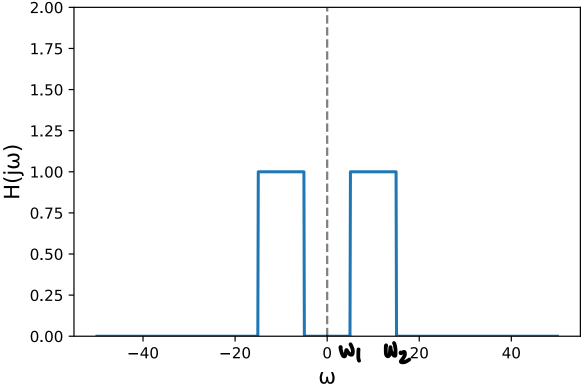
We can also consider an ideal **highpass filter**:

$$H(j\omega) = \begin{cases} 1 & |\omega| \geq \omega_c \\ 0 & |\omega| < \omega_c \end{cases}$$

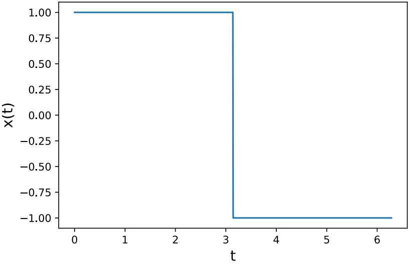


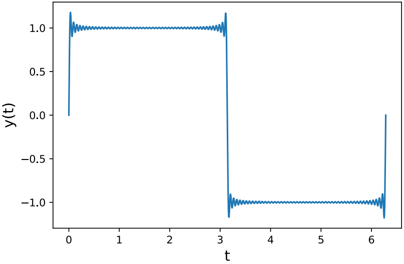
Or an ideal **bandpass filter**:

$$H(j\omega) = \begin{cases} 1 & \omega_1 \le |\omega| \le \omega_2 \\ 0 & \text{otherwise} \end{cases}$$



We have inadvertently already encountered an example of what can happen when we apply a lowpass filter: the Gibbs phenomenon.





We can actually reduce the effect of the Gibbs phenomenon using a different kind of frequency response called σ approximation.

Lowpass filters in practice

In reality some filters are made up of real, physical components, described by differential equations, and involve tradeoffs in their construction.

Example: We can construct a lowpass filter using a resistor and a capacitor.

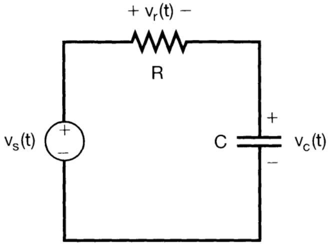
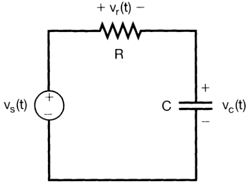


Image credit: Oppenheim chapter 3.10.

Suppose we input some complex exponential signal $v_s(t) = e^{j\omega t}$.



Can derive two expressions for the current, using the resistor and capacitor respectively:

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

Image credit: Oppenheim chapter 3.10.

Put these together to form a differential equation:

$$C \frac{dv_c(t)}{dt} = \frac{v_s(t) - v_c(t)}{R}$$

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

What is the voltage across the capacitor?

Can show that if $v_s(t) = e^{j\omega t}$, then $v_c(t) = H(j\omega) e^{j\omega t}$ for some scaling $H(j\omega)$.

Lowpass filters in practice

$$v_c(t) = H(j\omega)\, e^{j\omega t}$$

Plug this into the ODE:

$$RC \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

$$RC \cdot \frac{d}{dt}\left(H(j\omega) e^{j\omega t}\right) + H(j\omega)\, e^{j\omega t} = e^{j\omega t}$$

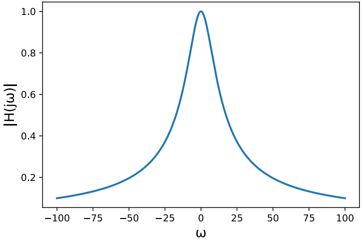
$$RC \cdot H(j\omega) \cdot j\omega\, e^{j\omega t} + H(j\omega)\, e^{j\omega t} = e^{j\omega t}$$

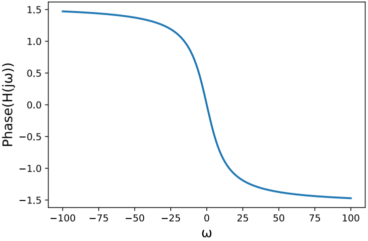
$$\left[RC \cdot H(j\omega) \cdot j\omega + H(j\omega)\right] \cdot \cancel{e^{j\omega t}} = \cancel{e^{j\omega t}}$$

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

Lowpass filters in practice

Results in the following frequency response (setting $RC = 0.1$):





Adjusting the value of RC controls the frequency response.
Increasing RC cuts off more frequencies.

However, there are tradeoffs involved in the design of such filters.

Example: we want the system to response *quickly* when we start giving it input. How does it respond to the unit step?

You can show that the unit impulse response of this filter is

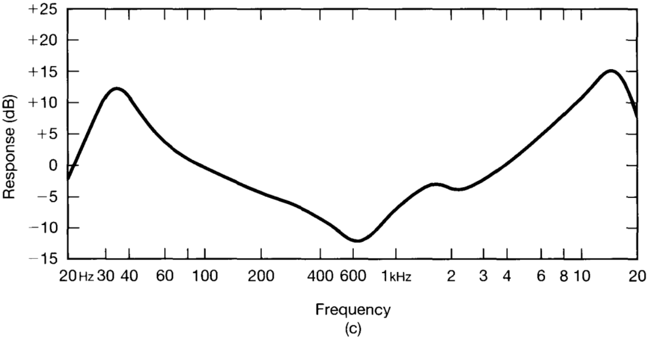
$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

and the unit step response is

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau = \left(1 - e^{-t/RC}\right) u(t)$$

High *RC* means tight frequency response, but slow step response.

Especially common in audio process: not eliminating any frequencies, just changing their strengths.



Decibels are units of $20\log_{10}|H(j\omega)|$; frequency in Hz is $\omega/2\pi$. (We will play with these on Thursday!)

Image credit: Oppenheim chapter 3.10.

DT filters

Recall that in DT, the frequency increases up until $\omega = \pi$, then decreases as it approaches 2π.

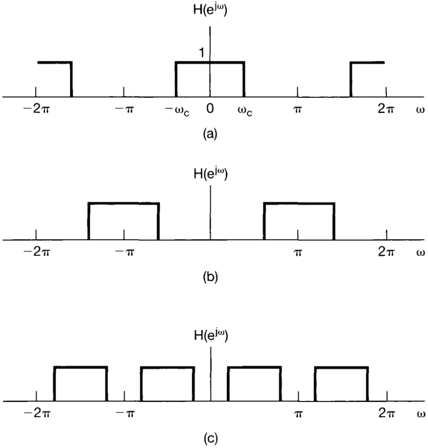


Image credit: Oppenheim chapter 3.9.

DT filters

There are two major categories of DT filters:

1. Infinite impulse response (IIR)
2. Finite impulse response (FIR)

DT filters (IIR)

$$y[n] = a \cdot y[n-1] + x[n]$$

Example: we can generate a low- or high-pass filter using a system described by a first-order difference equation.

$$y[n] - ay[n-1] = x[n]$$

$$y[n] - ay[n-1] = x[n]$$

We can use the same method as in the CT case to compute the frequency response: we know that complex exponential signals are eigenfuctions, so $y[n] = H(e^{j\omega})e^{j\omega n}$.

DT filters (IIR)

$$x[n] = e^{j\omega n} \qquad y[n] = H(e^{j\omega}) \cdot e^{j\omega n}$$

Work through the math:

$$y[n] - ay[n-1] = x[n]$$

$$H(e^{j\omega})e^{j\omega n} - a \cdot H(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}$$

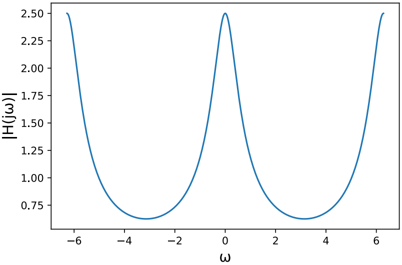
$$H(e^{j\omega})(1 - ae^{-j\omega})e^{j\omega n} = e^{j\omega n}$$

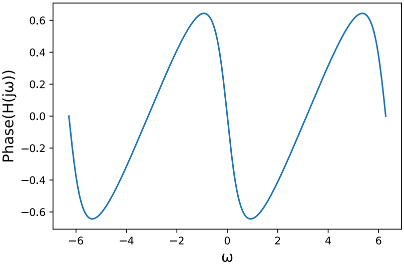
$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

Depending on the value of a, this could be either a highpass or lowpass filter.

DT filters (IIR)

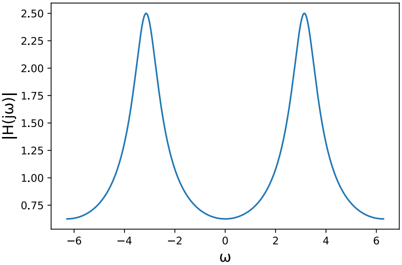
Case: $a > 0$

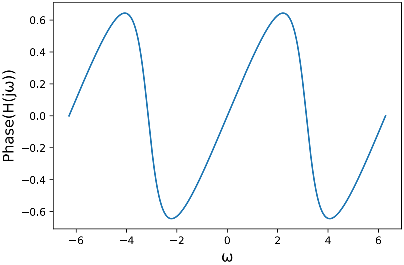




DT filters (IIR)

Case: $a < 0$





DT filters (IIR)

What happens to the impulse response of this filter? Need to solve the differential equation... consider a signal $x[n] = K\delta[n]$.

$$y[n] - ay[n-1] = x[n] = K \cdot \delta[n]$$

Assuming the system starts at rest ($y[-1] = 0$), we can solve recursively:

$$y[0] = x[0] = K \cdot \delta[0] = K$$

$$y[1] = ay[0] + x[1] = aK + \cancel{K \cdot \delta[1]} = aK$$

$$y[2] = ay[1] + x[2] = a^2K + \cancel{K \cdot \delta[2]} = a^2K$$

$$\vdots$$

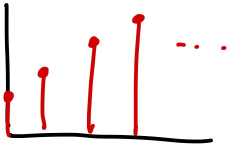
$$y[n] = ay[n-1] + x[n] = a^nK$$

DT filters (IIR)

If we put in the unit impulse ($K = 1$), we get

$$
\begin{aligned}
y[0] &= 1 \\
y[1] &= a \\
y[2] &= a^2 \\
&\vdots
\end{aligned}
$$

$$y[n] = a^n$$

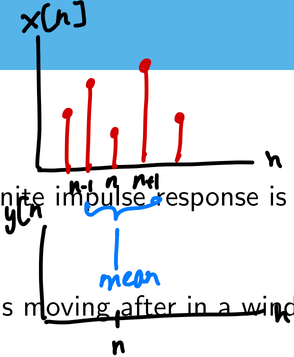


so the impulse response is

$$h[n] = a^n \cdot u[n]$$

This system has an *infinite impulse response*!

DT filters (FIR)



An example of a DT filter with a finite impulse response is the moving average.

Suppose the output of the system is moving after in a window of size M around each point:

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

What is its impulse response and frequency response?

Impulse response: set $x[n] = \delta[n]$

$$h[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} \delta[n-k]$$

$$= \frac{1}{2M+1}$$

This is clearly finite (as a result: these filters are stable systems).

$H(e^{j\omega})$

DT filters (FIR)

Frequency response: set $x[n] = e^{j\omega n}$,

$$
\begin{aligned}
y[n] &= \frac{1}{2M+1} \sum_{k=-M}^{M} e^{j\omega(n-k)} \\
&= \frac{1}{2M+1} \sum_{k=-M}^{M} e^{-j\omega k} \cdot e^{j\omega n} \\
&= H(e^{j\omega})e^{j\omega}
\end{aligned}
$$

So

$$H(e^{j\omega}) = \frac{1}{2M+1} \sum_{k=-M}^{M} e^{-j\omega k}$$

What does this look like? Let's plot it...

Recap

Today's learning outcomes were:

- Define filters and identify a basic set of ideal frequency-selective filters
- Plot the frequency response of a filter in CT and DT
- Define and distinguish between finite-impulse-response (FIR) and infinite-impulse-response (IIR) filters in DT

What topics did you find unclear today?

For next time

Content:

$$H(jk\omega) = \frac{jk\omega}{2+k\omega} \qquad H(2j\omega) =$$

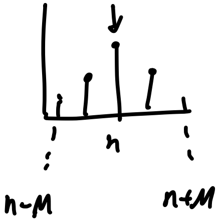
- Fun with filters! In-class activity with opportunity for some **extra credit**.

$$H(j\omega) = \frac{j\omega}{2+\omega} \rightarrow \frac{j2\omega}{2+2\omega}$$

- To prepare for class:
  - Produce a 10-15s clip of your favourite song in `.wav` format
  - Set up a Python environment (Google Colab works fine)
  - Loosely organize yourselves into groups of 3-4 (it will help for at least one person to have musical training)

Action items:

1. Assignment 2 is due tomorrow at 23:59
2. Assignment 3 is available tomorrow



Recommended reading:

- From today's class: Oppenheim 3.9-3.12
- For next class: Oppenheim 5.0-5.3