ELEC 221 Lecture 06
Introducing filters

Tuesday 27 September 2022

Announcements

- Assignment 2 due tomorrow
- Assignment 3 available tomorrow
- Quiz 3 today

Last time

We explored periodic DT complex exponential signals:

$$x[n] = e^{j\omega n} = e^{j\frac{2\pi}{N}n}$$

We found that these signals behave differently than CT signals...

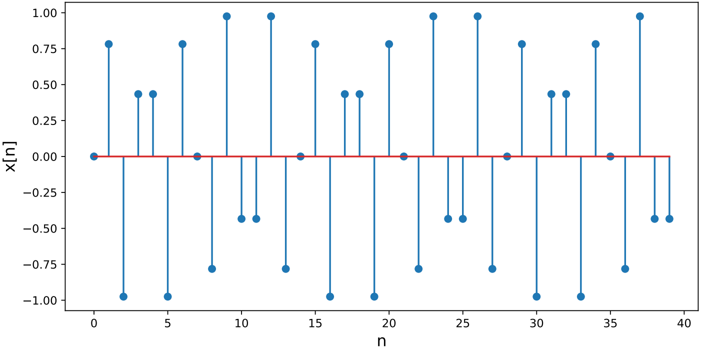
Difference 1: we only need to consider ω in the range $[0, 2\pi)$.

$$\begin{aligned} x[n] &= e^{j(\omega+2\pi)n} \\ &= e^{j\omega n}e^{j2\pi n} \\ &= e^{j\omega n} \end{aligned}$$

Difference 2: there are additional criteria for periodicity.

$$\begin{aligned} x[n+N] &= e^{j\omega(n+N)} \\ &= e^{j\omega n} e^{j\omega N} \\ &= e^{j\omega n} \quad \text{only if } \omega N = 2\pi m \end{aligned}$$

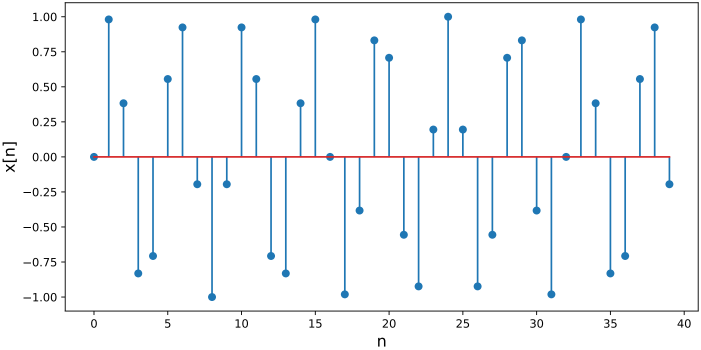
Example: $x[n] = \sin(5\pi n/7)$ is periodic.



In CT, period of $x(t) = \sin(5\pi t/7)$ is $T = 14/5$.
In DT, period of $x[n] = \sin(5\pi n/7)$ is $N = 14$.

Last time

Example: $x[n] = \sin(5n/7)$ is NOT periodic.



In CT, period of $x(t) = \sin(5t/7)$ is $T = 14\pi/5$.

Difference 3: there are only finitely many harmonics.

$$
\begin{aligned}
x_0[n] &= 1 \\
x_1[n] &= e^{j\frac{2\pi}{N}n} \\
x_2[n] &= e^{j2\frac{2\pi}{N}n} \\
\vdots\quad &\vdots\quad \vdots \\
x_{N-1}[n] &= e^{j(N-1)\frac{2\pi}{N}n} \\
x_N[n] &= e^{jN\frac{2\pi}{N}n} = e^{j2\pi n} = 1
\end{aligned}
$$

Last time

We expressed DT complex exponential signals with fundamental period N as Fourier series.

DT synthesis equation:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{jk\frac{2\pi n}{N}}$$

DT analysis equation:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi n}{N}}$$

We introduced the idea of **system functions** and **frequency response** of LTI systems.

$$
\begin{aligned}
x(t) \to y(t) &= \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t} \\
x[n] \to y[n] &= \sum_{k=0}^{N-1} c_k H(e^{jk\omega}) e^{jk\omega n}
\end{aligned}
$$

$H(j\omega)$ in CT, and $H(e^{j\omega})$ in DT, are called the **frequency response** of the system.

This week, we are going to look at our first major application of signals and systems: filters.

Learning outcomes:

- Define filters and identify a basic set of ideal frequency-selective filters
- Plot the frequency response of a filter in CT and DT
- Define and distinguish between finite-impulse-response (FIR) and infinite-impulse-response (IIR) filters in DT

In-class activity on Thursday: apply filters in the context of audio processing.

Filters are LTI systems that can be used to separate out, combine, or modify the components of a signal at specific frequencies.

Two key types:

- **Frequency-shaping**: change the amplitudes of parts of a signal at specified frequencies
- **Frequency-selective**: eliminate or attentuate parts of a signal at specified frequencies

A convenient way to understand (and to design) filters is by looking at their **frequency response**.

Recall that the **system function** of a CT system $H(j\omega)$, is a measure of how much a system "modifies" a particular frequency:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega) e^{jk\omega t}$$

By selecting the frequency response at will, we can change the behaviour of a system.

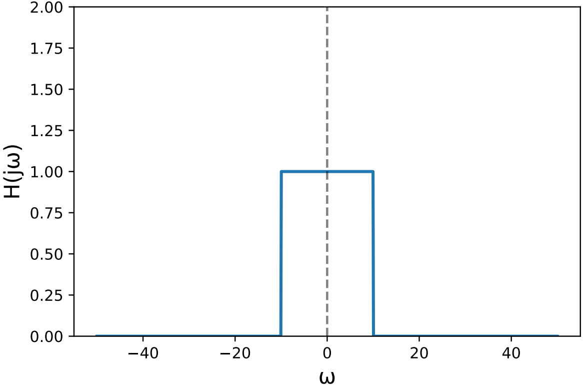
Example. Consider a system function defined as follows.

$$H(j\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

What does this look like, and what does it do?

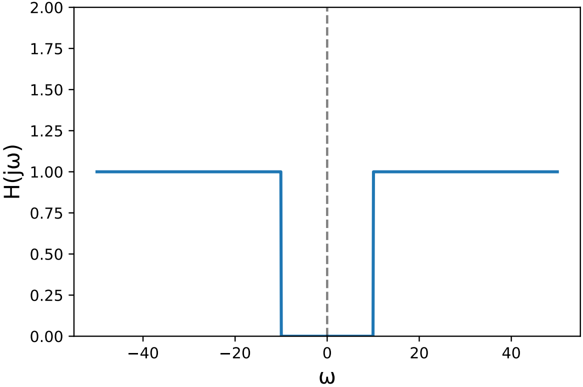
This is an **ideal lowpass filter**.

We often represent the frequency response graphically:



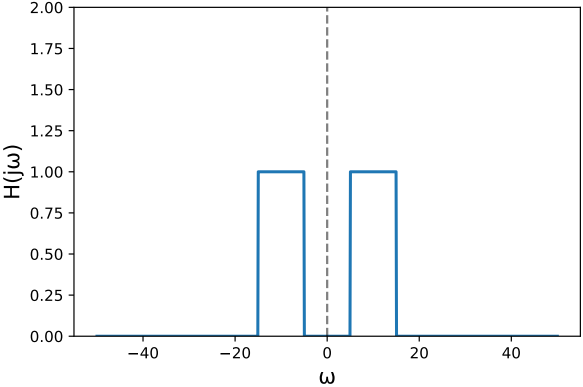
We can also consider an ideal **highpass filter**:

$$H(j\omega) = \begin{cases} 1, & |\omega| > \omega_c \\ 0, & |\omega| \leq \omega_c \end{cases}$$



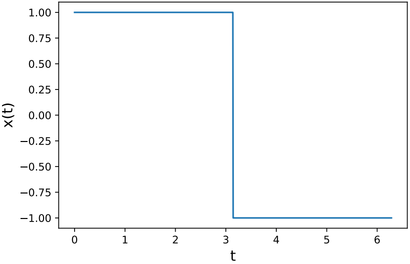
Or an ideal **bandpass filter**:

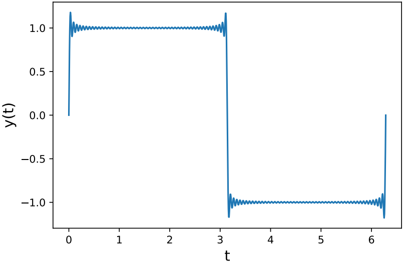
$$H(j\omega) = \begin{cases} 1, & \omega_1 \leq |\omega| \leq \omega_2 \\ 0, & \text{otherwise} \end{cases}$$



Filters and ringing

We have inadvertently already encountered an example of what can happen when we apply a lowpass filter: the Gibbs phenomenon.





We can actually reduce the effect of the Gibbs phenomenon using a different kind of frequency response called σ approximation.

In reality some filters are made up of real, physical components, described by differential equations, and involve tradeoffs in their construction.

Example: We can construct a lowpass filter using a resistor and a capacitor.

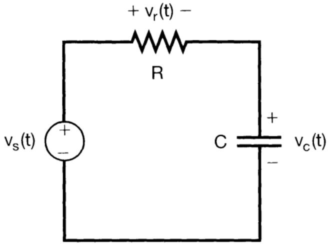
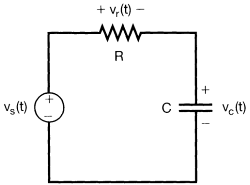


Image credit: Oppenheim chapter 3.10.

Lowpass filters in practice

Suppose we input some complex exponential signal $v_s(t) = e^{j\omega t}$.



Can derive two expressions for the current, using the resistor and capacitor respectively:

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$
$$i(t) = C\frac{dv_c(t)}{dt}$$

Image credit: Oppenheim chapter 3.10.

Put these together to form a differential equation:

$$\begin{aligned} C\frac{dv_c(t)}{dt} &= \frac{v_s(t) - v_c(t)}{R} \\ RC\frac{dv_c(t)}{dt} + v_c(t) &= v_s(t) \end{aligned}$$

What is the voltage across the capacitor?

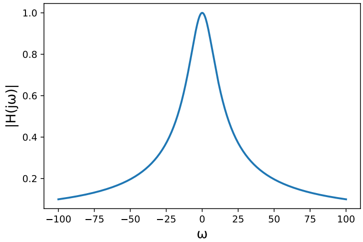
Can show that if $v_s(t) = e^{j\omega t}$, then $v_c(t) = H(j\omega)e^{j\omega t}$ for some scaling $H(j\omega)$.

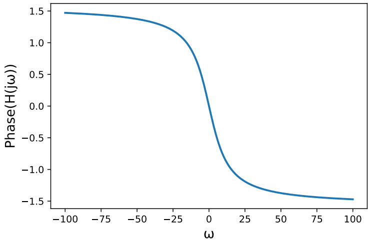
Lowpass filters in practice

Plug this into the ODE:

$$
\begin{aligned}
RC\frac{dv_c(t)}{dt} + v_c(t) &= v_s(t) \\
RC\frac{d}{dt}(H(j\omega)e^{j\omega t}) + H(j\omega)e^{j\omega t} &= e^{j\omega t} \\
RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} &= e^{j\omega t} \\
(RCj\omega + 1)H(j\omega)e^{j\omega t} &= e^{j\omega t} \\
H(j\omega) &= \frac{1}{1 + j\omega RC}
\end{aligned}
$$

Results in the following frequency response (setting $RC = 0.1$):





Adjusting the value of RC controls the frequency response.
Increasing RC cuts off more frequencies.

However, there are tradeoffs involved in the design of such filters.

Example: we want the system to response *quickly* when we start giving it input. How does it respond to the unit step?

You can show that the unit impulse response of this filter is

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

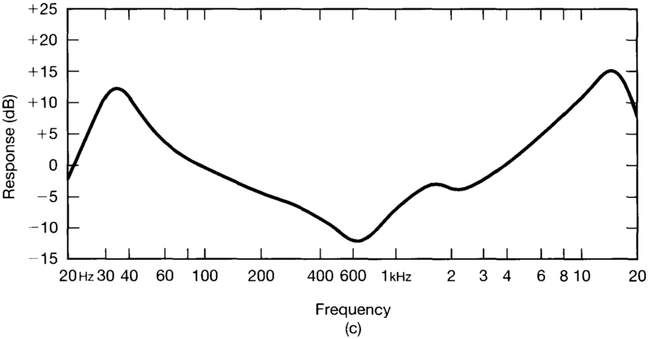
and the unit step response is

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau = (1 - e^{-t/RC}) u(t)$$

High RC means tight frequency response, but slow step response.

Frequency shaping filters

Especially common in audio process: not eliminating any frequencies, just changing their strengths.



Decibels are units of $20 \log_{10} |H(j\omega)|$; frequency in Hz is $\omega/2\pi$. (We will play with these on Thursday!)

Image credit: Oppenheim chapter 3.10.

DT filters

Recall that in DT, the frequency increases up until $\omega = \pi$, then decreases as it approaches 2π.

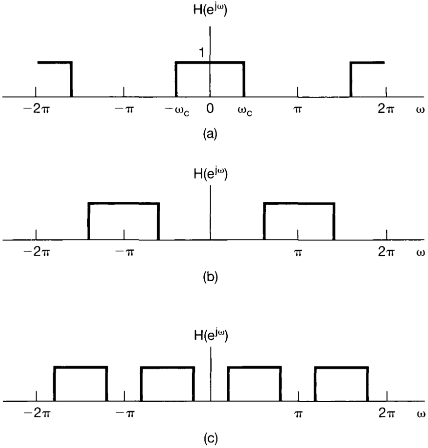


Image credit: Oppenheim chapter 3.9.

DT filters

There are two major categories of DT filters:

1. Infinite impulse response (IIR)
2. Finite impulse response (FIR)

DT filters (IIR)

Example: we can generate a low- or high-pass filter using a system described by a first-order difference equation.

$$y[n] - ay[n-1] = x[n]$$

We can use the same method as in the CT case to compute the frequency response: we know that complex exponential signals are eigenfuctions, so $y[n] = H(e^{j\omega})e^{j\omega n}$.

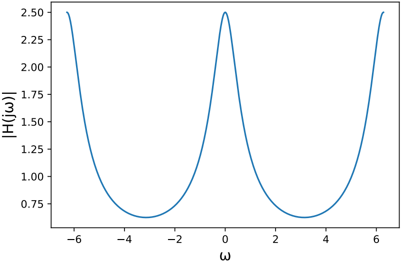
DT filters (IIR)

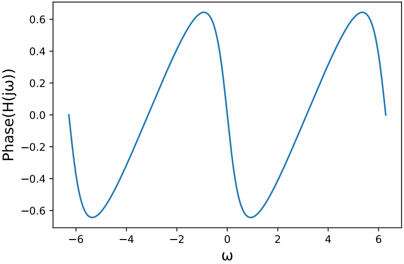
Work through the math:

$$\begin{aligned}
y[n] - ay[n-1] &= x[n] \\
H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} &= e^{j\omega n} \\
H(e^{j\omega})(1 - ae^{-j\omega})e^{j\omega n} &= e^{j\omega n} \\
H(e^{j\omega}) &= \frac{1}{1 - ae^{-j\omega}}
\end{aligned}$$

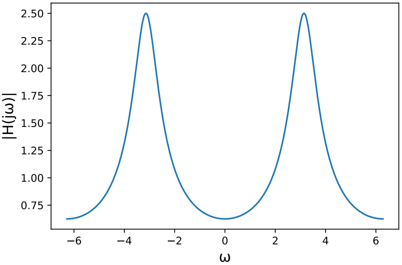
Depending on the value of a, this could be either a highpass or lowpass filter.

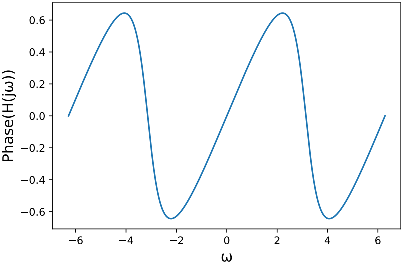
Case: $a > 0$


2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
|H(jω)|
−6
−4
−2
0
2
4
6
ω


0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
Phase(H(jω))
−6
−4
−2
0
2
4
6
ω

Case: $a < 0$


2.50
2.25
2.00
1.75
1.50
1.25
1.00
0.75
|H(jω)|
−6
−4
−2
0
2
4
6
ω


0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
Phase(H(jω))
−6
−4
−2
0
2
4
6
ω

What happens to the impulse response of this filter? Need to solve the differential equation... consider a signal $x[n] = K\delta[n]$.

$$y[n] - ay[n-1] = x[n] = K\delta[n]$$

Assuming the system starts at rest ($y[-1] = 0$), we can solve recursively:

$$\begin{aligned}
y[0] &= x[0] = K\delta[0] = K \\
y[1] &= ay[0] + x[1] = aK + K\delta[1] = aK \\
y[2] &= ay[1] + x[2] = a(aK) + K\delta[2] = a^2 K \\
\vdots & \quad \vdots \quad \vdots \\
y[n] &= ay[n-1] + x[n] = a^n K
\end{aligned}$$

DT filters (IIR)

If we put in the unit impulse ($K = 1$), we get

$$\begin{aligned}
y[0] &= 1 \\
y[1] &= a \\
y[2] &= a^2 \\
\vdots \;&\;\vdots\; \vdots \\
y[n] &= a^n
\end{aligned}$$

so the impulse response is

$$h[n] = a^n u[n]$$

This system has an *infinite impulse response*!

DT filters (FIR)

An example of a DT filter with a finite impulse response is the moving average.

Suppose the output of the system is moving after in a window of size M around each point:

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$$

What is its impulse response and frequency response?

DT filters (FIR)

Impulse response: set $x[n] = \delta[n]$

$$\begin{aligned} h[n] &= \frac{1}{2M+1} \sum_{k=-M}^{M} \delta[n-k] \\ &= \frac{1}{2M+1} \end{aligned}$$

This is clearly finite (as a result: these filters are stable systems).

DT filters (FIR)

Frequency response: set $x[n] = e^{j\omega n}$,

$$
\begin{aligned}
y[n] &= \frac{1}{2M+1} \sum_{k=-M}^{M} e^{j\omega(n-k)} \\
&= \frac{1}{2M+1} \sum_{k=-M}^{M} e^{-j\omega k} \cdot e^{j\omega n} \\
&= H(e^{j\omega}) e^{j\omega}
\end{aligned}
$$

So

$$
H(e^{j\omega}) = \frac{1}{2M+1} \sum_{k=-M}^{M} e^{-j\omega k}
$$

What does this look like? Let's plot it...

Recap

Today's learning outcomes were:

- Define filters and identify a basic set of ideal frequency-selective filters
- Plot the frequency response of a filter in CT and DT
- Define and distinguish between finite-impulse-response (FIR) and infinite-impulse-response (IIR) filters in DT

What topics did you find unclear today?

For next time

Content:

- Fun with filters! In-class activity with opportunity for some **extra credit**.
- To prepare for class:
  - Produce a 10-15s clip of your favourite song in `.wav` format
  - Set up a Python environment (Google Colab works fine)
  - Loosely organize yourselves into groups of 3-4 (it will help for at least one person to have musical training)

Action items:

1. Assignment 2 is due tomorrow at 23:59
2. Assignment 3 is available tomorrow

Recommended reading:

- From today's class: Oppenheim 3.9-3.12
- For next class: Oppenheim 5.0-5.3