Last update: 4:30pm, Dec 5.

Math 570: Mathematical Logic HOMEWORK 11 **Due: Dec 7–Dec 8**

1. Let T be a σ-theory, where σ is a finite signature.

(a) Prove that if T is decidable, then it has a recursive completion.

HINT: Use the Deduction theorem:

$$T \cup \{\varphi_0, \dots, \varphi_{n-1}\} \vdash \theta \text{ if and only if } T \vdash \bigwedge_{i<n} \varphi_i \to \theta.$$

(b) Deduce **Church's theorem**: Any σ-theory that recursively interprets PA is undecidable.

2. Prove that the following are equivalent for each σ_{arthm}-sentence θ.

(1) $\text{PA} \models \theta$.

(2) $\boldsymbol{N} \models \mathbf{Provable}_{\text{PA}}([\theta])$.

(3) $\text{PA} \models \mathbf{Provable}_{\text{PA}}([\theta])$.

3. Let φ and ψ denote a σ_{arthm}-sentence. Which implications hold between the following statements? For each implication, prove it or give an example of a pair φ, ψ for which it fails.

(1) $\text{PA} \vdash \varphi \implies \text{PA} \vdash \psi$;

(2) $\text{PA} \vdash \varphi \to \psi$.

4. For each of the following σ_{arthm}-sentences, either prove that PA satisfies it for every σ_{arthm}-sentence θ or provide an example of θ for which PA does not satisfy it.

(a) $\mathbf{Provable}_{\text{PA}\cup\{\neg\theta\}}([\theta]) \to \mathbf{Provable}_{\text{PA}}([\theta])$

(b) $\mathbf{Provable}_{\text{PA}}([\theta]) \to \neg\mathbf{Provable}_{\text{PA}}([\neg\theta])$

(c) $\mathbf{Provable}_{\text{PA}}(\mathbf{Provable}_{\text{PA}}([\theta])) \to \mathbf{Provable}_{\text{PA}}([\theta])$

5. (a) Show that the set of Σ_n^0 relations is closed under finite unions/intersections, projections, and recursive preimages, i.e. under the operations $\vee, \wedge, \exists^{\mathbb{N}}$, and taking a preimage under a recursive function.

(b) Conclude that the set of Π_n^0 relations is closed under finite unions/intersections, co-projections, and recursive preimages, i.e. under the operations $\vee, \wedge, \forall^{\mathbb{N}}$, and taking a preimage under a recursive function.

(c) (Optional for $n \geq 2$, mandatory for $n = 1$) Prove that Σ_n^0 is closed under recursive images.

Hint: For $n \geq 2$, to make the induction on n work, first figure out what the corresponding statement is for Π^0_n. To do so, look at what happens with projections (they are examples of recursive images).