Lecture 11 - 17/04/23

→ 11.1 Derivatives

If a function $f'(x)$, which is the derivative of $f(x)$ is also DIFF, we can get $f''(x)$ which is $(f(x)')'$. We can continue as long as $f^n(x)$ is DIFF.

→ 11.2 L'Hopital Rule and Indeterminate Forms

Indeterminate Forms

TYPE	EXAMPLE
$[\frac{0}{0}]$	$\lim\limits_{x \to 0} \frac{\sin(x)}{x}$
$[\frac{\infty}{\infty}]$	$\lim\limits_{x \to \infty} x^2/e^x$
$[0 \infty]$	$\lim\limits_{x \to +0} \ln(1/x) \cdot x$
$[\infty - \infty]$	$\lim\limits_{x \to +\infty} e^x - x^x$
$[0^0]$	$\lim\limits_{x \to 0^+} x^x$
$[\infty^0]$	$\lim\limits_{x \to \frac{\pi}{2}^-} \tan(x)^{\cos(x)}$
$[1^\infty]$	$\lim\limits_{x \to \infty} (1 + 1/x)^x$

Theorem 1 First L'Hopital Rule

Let g, f be DIFF in (a,b) and $c \in (a,b)$, $g(x)' \neq 0$, for $x \in (a,b) \setminus \{c\}$. Then if

* $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$

* $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$ $\quad L \in \mathbb{R}$ or $\pm\infty$

then $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$

- The Theorem also holds, when $x \to c^-$ or $x \to c^+$

a) f, g should be DIFF on (a,c) or (c,b) respectivly

b) Also holds when $c = \pm\infty$

- Multiple Applications of L'Hopital rule

In some cases $\lim \frac{f'(x)}{g'(x)}$ is also $[\frac{0}{0}]$. However we can try to apply L'Hopital $\lim \frac{f''(x)}{g''(x)}$, and if exists an L, $\lim \frac{f'(x)}{g'(x)} = L$ and $\lim \frac{f(x)}{g(x)} = L$. We can apply the rule arbitrary many times.

ex.

$$\lim_{x \to 0} \frac{2\sin(x) - \sin(2x)}{2e^x - 2 - 2x - x^2} \quad [\tfrac{0}{0}]$$

$$\lim_{x \to 0} \frac{2\cos(x) - 2\cos(2x)}{2e^x - 2x} \quad [\tfrac{0}{0}]$$

$$\lim_{x \to 0} \frac{-2\sin(x) + 4\sin(2x)}{2e^x - 2} \quad [\tfrac{0}{0}]$$

$$\lim_{x \to 0} \frac{-2\sin(x) + 8\cos(2x)}{2e^x} = \frac{-2+8}{2} = \underline{\underline{3}}$$

Observation:

- if $\frac{f(x)}{g(x)} \neq [\frac{0}{0}]$ we cannot apply L'Hopital

  $\lim\limits_{x \to 1^+} \frac{x}{\ln(x)} \quad [\frac{1}{0}]$

- If $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)}$ does not exist, it does not mean that there is no limit!

Theorem 2 Second L'Hopital Rule:

f, g are DIFF on (a,b) and $c \in (a,b)$, $g'(x) \neq 0$ on $(a,b) \setminus \{c\}$. Then if:

- $\lim\limits_{x \to c} g'(x) = \pm\infty$
- $\lim\limits_{x \to c} \frac{f'(x)}{g'(x)} = L$ $\quad L \in \mathbb{R}$ or $\pm\infty$

then: $\lim\limits_{x \to c} \frac{f(x)}{g(x)} = L$

ex.

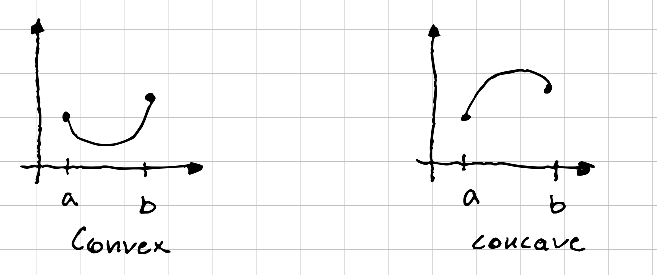
$$\lim_{x \to +\infty} \frac{x^2}{e^x} \quad [\tfrac{\infty}{\infty}]$$

$$\lim_{x \to +\infty} \frac{2x}{e^x} \quad [\tfrac{\infty}{\infty}]$$

$$\lim_{x \to +\infty} 2/e^x = 0 \quad \Rightarrow \quad \lim_{x \to +\infty} x^2/e^x = 0$$

→ 11.3 Convex and Concave Functions

Graphical:



A function is

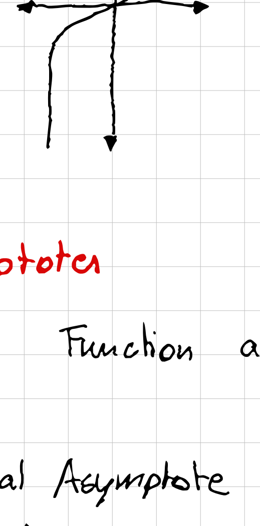
... convex on (a,b) if f' exists on (a,b) and is increasing.

... concave on (a,b) if f' exists on (a,b) and is decreasing.

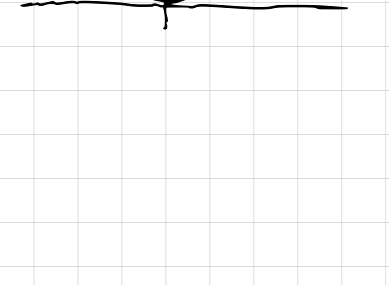
ex. $f(x) = x^3$ $\quad$ CONV $(-\infty, 0)$ ↘

$f'(x) = 3x^2$ $\quad$ CONC $(0, +\infty)$ ↗

$f(x)$



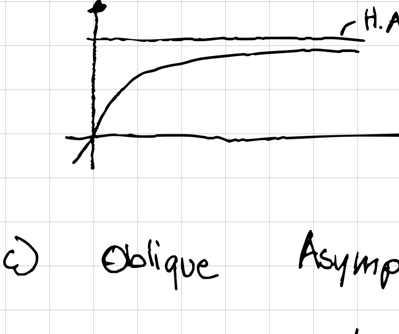
$f'(x)$



→ 11.3 Asymptotes

Idea: Function approaches a line at point or $\pm\infty$

a) Vertical Asymptote



$$\lim_{x \to a^+} f(x) = \pm\infty$$

$$\lim_{x \to 0^+} 1/x = +\infty$$

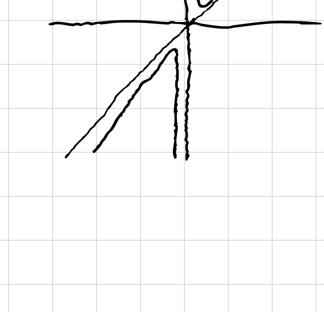
$$\lim_{x \to 0^-} 1/x = -\infty$$

b) Horizontal Asymptote

$$\lim_{x \to \infty^-} f(x) = L_1$$

$$\lim_{x \to \infty^-} f(x) = L_2$$

c) Oblique Asymptote



$$\lim_{x \to \pm\infty} [f(x) - (kx+n)] = 0$$

ex.

$$f(x) = \frac{x^2+1}{x}$$

$$\lim_{x \to \infty} f(x) - x$$

$$\lim_{x \to \infty} \frac{x^2+1}{x} - x = 0$$

$$\lim_{x \to \infty} \cancel{x} + \frac{1}{x} - \cancel{x} = 0 \quad ✓$$