Lecture 4 - 15/03

→ 4.1 Series

- definition: Let (a_n) be a sequence. Then we define anoter (s_n) like:

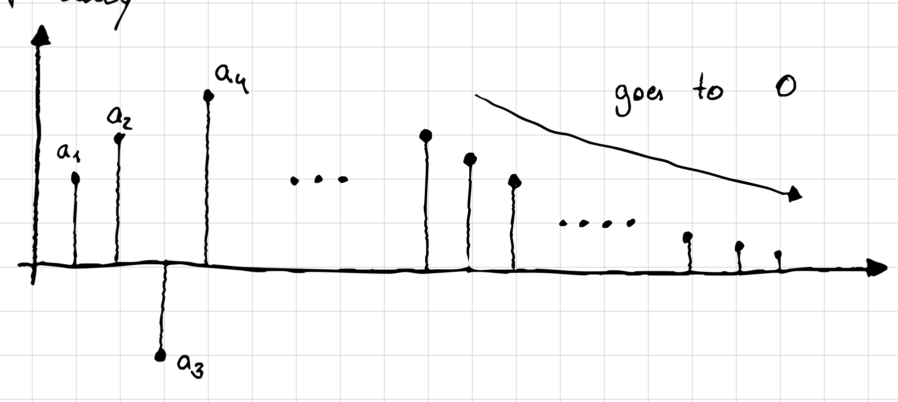
$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_n = a_1 + a_2 \dots + a_n = \sum_{i=1}^{n} a_1$$

- Then, we define series as: $\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} s_n$
- $\sum_{n=1}^{\infty} a_i$ converges IFF (s_n) converges, and if $\lim_{n \to \infty} s_n = s$ then $\sum_{n \to \infty} a_n = s$

Theorem 1: If a series $\sum_{n \to \infty}^{\infty} a_n$ converges, then $\lim_{n \to \infty} a_n = 0$

→ In other words, if $\sum_{n \to \infty}^{\infty} a_n$ converges, the terms of series are getting smaller and smaller (but not vice-versa)

Graphically:



→ 4.2 Harmonic Series

- Important series: $a_n = 1/n$, $\sum_{n=1}^{\infty} 1/n$ → Harmonic Series

$\lim_{n \to \infty} a_n = 0$ $\quad \sum_{n=1}^{\infty} a_n = +\infty$

Theorem 2:

$$\sum_{n=1}^{\infty} 1/n \text{ is divergent } \otimes$$

Proof.

$1/n$ replaced by $1/2^k$ s.t it's larger than n

$$\otimes \geq 1 + \tfrac{1}{2} + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + (1/16 \dots + 1/16) + \dots$$

$$\otimes \geq 1 + 1/2 + 1/2 + 1/2 + \dots = +\infty$$

so $\sum_{n=1}^{\infty} 1/n = +\infty$

→ 4.3 Geometric Series

$$\sum_{n=1}^{\infty} a \cdot r^{n-1} \qquad \begin{array}{l} a \in \mathbb{R} \\ r \in \mathbb{R} \setminus \{1\} \end{array}$$

- partial sum s_n (Series): $s_n = a + ar + ar^2 + \dots a \cdot r^{n-1} \quad \Big| \cdot \frac{1-r}{1-r}$

we transform by multiplying with $\frac{(1-r)}{(1-r)}$

$$s_n = \frac{(a - \cancel{ar}) + (\cancel{ar} - \cancel{ar^2}) + (\cancel{ar^2} - \cancel{ar^3}) \dots (\cancel{ar^{n-1}} - ar^n)}{1-r}$$

$$= \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r} = a \cdot \frac{r^n - 1}{r-1}$$

We observe that $\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left[\frac{a(r^n - 1)}{r-1} \right]$

$$= \frac{a}{r-1} \lim_{n \to \infty} (r^n - 1) = \frac{a}{r-1} \left[\lim_{n \to \infty} r^n - 1 \right] \begin{array}{ll} \to +\infty & |r| > 1 \\ \to \frac{a}{1-r} & |r| < 1 \end{array}$$

Theorem 3:

Let $\sum_{n=1}^{\infty} a r^{n-1}$ be a geometric series, then:

- if $|r| < 1$ then it converges to $\frac{a}{1-r}$
- if $|r| \geq 1$ then it diverges

ex. $\sum_{n=1}^{\infty} \left(\tfrac{1}{2}\right)^n \to \tfrac{1}{2} \sum_{n=1}^{\infty} 1 \cdot \left(\tfrac{1}{2}\right)^{n-1}$

ex $\sum_{n=3}^{\infty} \frac{2^{n+3}}{e^{n-3}} = 2^6 \sum_{n=3}^{\infty} \frac{2^{n-3}}{e^{n-3}} = 2^6 \sum_{n=3}^{\infty} (2/e)^{n-3} = 2^6 \sum_{n=0}^{\infty} (2/e)^n \quad \left|\tfrac{e}{2}\right| < 1$

→ 4.4 Partial Fraction Extension (PFE)

- Recall: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$
- $Q(x) = (x - a_1) \cdot (x - a_2) \dots (x - a_n)$

  ex. $E(x) = x^2 + 2x + 1 = (x+1)(x+1)$, but we need $a_i \neq a_j \quad i \neq j$
- Degree of polynomial Q = highest power
- Observe: $\frac{P(x)}{Q(x)}$, where $\deg(P) < \deg(Q)$

we transform: $\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \dots + \frac{A_n}{x - a_n}$, $A_n \in \mathbb{R}$

How to find $A_1 \dots A_n$? Examples in Notes!