Lecture 19 - 06/06

19.1 Exponential Functions

Four categories:

→ 1. Exponentiation

$f(x) = a^x \qquad x \in \mathbb{Q}\ ,\quad a > 0$

→ 2. Addition - Multiplication Homomorphism

$f(x+y) = f(x) \cdot f(y)$

ex. $3^8 = 3^{5+3} = 3^5 \cdot 3^3$

→ 3. Growth proportional to the value

$\begin{cases} f'(x) = d \cdot f(x) \quad , \quad d \notin 0 \\ f(0) = 1 \end{cases}$

ex $f(x) = e^{\alpha x} \ ,\quad f'(x) = \alpha \cdot e^{\alpha x}$

→ 4. Integration of $1/x$

define that: $\ln(y) = \int_1^y 1/x \, dx$

We say that e^y is an Inverse of $\ln(y)$

s.t. $e^{\ln(y)} = y \ ,\quad \ln(e^y) = y$

- We will define (1) and show (2) as property
- We will define (4) and show (1), (2), (3) as property

Exp. Function via (1)

def. An ex. function is a function of the form $f(x) = a^x$, $a > 0$, a is called base and x is exponent.

We define a^x as follows

$a^0 = 1$

$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n} \ ,\quad n \in \mathbb{N}$

$a^{-n} = \dfrac{1}{a^n}$

$a^{m/n} = \sqrt[n]{a^m} \ ,\quad \begin{matrix} n \in \mathbb{N} \\ m \in \mathbb{Z} \setminus \{0\} \end{matrix}$

Here we assume that we can always find some b, s.t $b^n = a^1 (= a^m)$

What if $x \notin \mathbb{Q}$, ex. $x = \pi$

$a^x = \lim\limits_{\substack{r \in \mathbb{Q} \\ r \to x}} a^r$

Property 1 Let $a, b > 0$, $x, y \in \mathbb{R}$

$a^0 = 1$

$a^{x+y} = a^x \cdot a^y$

$a^{-x} = \frac{1}{a^x}$

$a^{x-y} = \frac{a^x}{a^y}$

$(a^x)^y = a^{x \cdot y}$

$(a \cdot b)^x = a^x \cdot b^x$

$\boxed{\text{Proof}}$ of $a^{x+y} = a^x \cdot a^y$

$x, y \in \mathbb{N}$

$a^{x+y} = \underbrace{a \cdot a \cdot \ldots \cdot a}_{x} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{y} = a^x \cdot a^y$

$x \in \mathbb{Z}\ ,\quad x < 0$

$x' = -x \qquad x' \in \mathbb{N}\ (x = -x')$, for simplicity $y < 0$

$a^x = \frac{1}{a^{x'}}$

$x, y \in \mathbb{Q}$

$x = p/q \ ,\quad y = r/t \qquad$ for $p, q, r, t \in \mathbb{Z}$

$a^{p/q + r/t} = a^{\frac{pt + rq}{qt}} = \sqrt[qt]{a^{pt + rq}}$

$= \sqrt[qt]{a^{pt}} \cdot \sqrt[qt]{a^{rq}} = a^{p/q} \cdot a^{r/t}$

→ 19.2 Logarithmic Functions

The function $f(x) = a^x$;