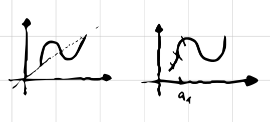
# Lecture 13 - 08/05

## → 13.01 Taylor Polynomial

Given a function f, we want to find a polynomial of n degree that approximates f.

$$f(x) \sim a_n x^n + a_{n-1} x^{n-1} + \dots a_1 x^1 + a_0$$

$n = 1$ $\qquad f(x) = a_1 x^1 + a_0 = P_1$



Look at Point $x = a$

- $P_1(a) = f(a)$
- $P_1'(a) = f'(a)$

$P_1 = \underline{\underline{c}} + \underline{\underline{d}}(x - \underline{\underline{a}})$ // $\underbrace{c - da}_{a_0} + \underbrace{dx}_{a_1}$

$P_1'(x) = d = f'(x)$

$P_1(x) = c + f'(a)(x-a)$

$P_1(a) = c + f'(a)(a-a) = c = f(a)$

All together: $P_1(x) = f(a) + f'(a)(x-a)$

↳ extend this to $n = 2$?

↳ We can repeat this idea with $P_2(x) = c + d(x-a) + l(x-a)^2$

we want to find $c, d, l$ by repeating the same steps like $n=1$ and get that

$$P_2 = f(a) + f'(a)(x-a) + \underbrace{f''(a)}_{l}(x-a)^2 \dots\dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We have that $P_n^{(n)}(a) = f^{(n)}(a)$

→ definition:

Polynomial $P_n$ is called Taylor Polynomial of f at Point a of size n. If $a = 0$ it is called

MAC-LAURIN

ex. Find Mac-Laurin Polynomial of $f(x) = e^x$

$f'(0) = e^0 = 1$

$f''(0) = e^0 = 1$

$$\Rightarrow P_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} \dots \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \dots \frac{1}{n!}x^n$$

$$P_n = \sum_{k=0}^{n} \frac{x^k}{k!}$$

ex. Find Mac-Laurin Poly of $f(x) = \sin(x)$

(0) $f(0) = \sin(x) = 0$

(1) $f'(0) = \cos(x) = 1$

(2) $f''(0) = -\sin(x) = 0$

(3) $f'''(0) = -\cos(x) = -1$

(4) $f''''(0) = \sin(x) = 0$

→ pattern of Poly repeats after 4 steps

$$f^{(n)}(0) = \begin{cases} 1 & n = 1, 5, 9, 13 \\ 0 & n = 3, 7, 11, 15 \end{cases}$$

$$\sin(x) \sim P_n(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

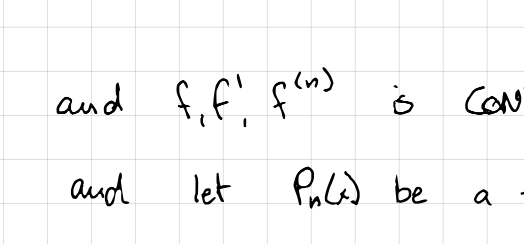
$$P_{2k-1}(x) = P_{2k}(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 \dots + \frac{(-1)\, x^{2k-1}}{(2k-1)!}$$

$(-1)^{g(k)} = 1 \quad (k = 1, 5, 9, 13 \dots)$

$(-1)^{g(k)} = -1 \quad (k = 3, 7, 11, 15 \dots)$

$g(k) = k - 1$ → ex. $n = 5 = 2k - 1 \to k = 3 \to k - 1 = 2$

## → 13.2 Taylor Theorem and Lagrange remainder




→ How big is the error?

→ Can we lower the error?

**Theorem 1**

Let f be a function that is $(n+1)$ DIFF on $(a, c)$ and $f, f', f^{(n)}$ is CONT on $[c, d]$. Let $a, x \in (c, d)$ and let $P_n(x)$ be a taylor for f at a.

Than the error $E_n(x) = f(x) - P_n(x)$ is given by

$$E_n(x) = \frac{f^{(n+1)}(s)}{(n+1)!}(x-a)^{n+1}$$ whe s is between x and a

→ Error is called Langrange Remainder and formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{n+1}(s)}{(n+1)!}(x-a)^n$$

- Some Mac-Laurin Formulas:

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \frac{x^n}{n!} + E_n(x)$

b) $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \frac{(-1)^n x^{2n}}{(2n)!} + E_{2n}(x)$

c) $\sin(x) = 1 - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \frac{(-1)^n x^{2n+1}}{(2n)!} + E_{2n}(x)$

d) $\frac{1}{1-x} = 1 + x + x^2 + \dots + x^n + E_n(x)$

e) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots (-1)^n \cdot \frac{x^n}{n!} + E_n(x)$

ex. Given $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Find Mac-Laurin of f of order 2n

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \dots + \frac{x^{2n}}{n!}$$

$$e^{(-x)} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} \dots \frac{(-x)^{2n}}{n!}$$

$$e^x - e^{(-x)} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{2!} \dots + \frac{x^{2n}}{n!}\right)$$

$$\frac{e^x - e^{(-x)}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{2!} \dots + \frac{x^{2n}}{n!}$$

ex. $f(x) = \cos^2(x)$. Find Mac-Laurin of f of order n.

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4} + \dots (-1)^n \frac{x^n}{(2n)!}$$

$f(x) = \cos(x) \cdot \cos(x)$

Recall: $\cos(2x) = \cos^2(x) - \sin^2(x)$

$1 = \sin^2(x) + \cos^2(x) \Rightarrow 1 - \cos^2(x) = \cos^2(x)$

$\cos^2(x) - 1 - \cos^2(x) = \cos(2x) - 1$

$\Rightarrow \cos^2(x) = \frac{1 + \cos(2x)}{2}$

$$\cos(2x) = 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4} \dots (-1)^n \frac{(2x)^n}{(2n)!}$$

$$1 + \cos(2x) = 2 - \frac{(2x)^2}{2} + \dots\dots (-1)^n \frac{(2x)^2}{(2n)!}$$

$$\frac{1 + \cos(2x)}{2} = 1 - \frac{1}{2} \cdot \frac{(2x)^2}{2} + \frac{1}{2} \cdot \frac{(2x)^4}{4} \dots (-1)^n \cdot \frac{1}{2} \cdot \frac{(2x)^2}{(2n)!}$$

## Generalize Taylor for $n = \infty$

Recall: $\sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad \left(+ E_r(f)\right)$

- Analytical functions:

f is analytical if Taylor for $n = +\infty$ "CONV" to $f(x)$ at some "small" interval containing c.