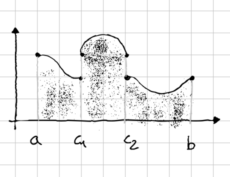
Lecture 16 - 24/05

→ 16.1 Piecewise continious Functions



f is not CONT on $[a,b]$, but it is CONT on $(a, c_1), (c_1, c_2), (c_2, b)$

definition:

let $c_0 < c_1 < c_2 \dots < c_n$

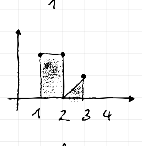
a function f is defined on $[c_0, c_n]$ except some points $c_0 \dots c_i$ is called piecewise continious on $[c_0, c_n]$, if for each $i = 1 \dots n$ exists a function g_i that is CONT on $[c_{i-1}, c_i]$ and

$$f(x) = g_i(x) \qquad x \in (c_{i-1}, c_i)$$

We define:

$$\int_{c_0}^{c_n} f(x)\, dx = \sum_{i=1}^{n} \int_{c_{i-1}}^{c_i} g_i(x)\, dx$$

Ex $\int_1^3 f(x)\, dx \qquad f(x) = \begin{cases} 2 & , 1 < x \leq 2 \\ x-2 & , 2 < x \leq 3 \end{cases}$



f is CONT except on 2

$(\lim_{x \to 2^-} f(x) = 2 \neq \lim_{x \to 2^+} f(x) = 0)$

But f is piecewise CONT, so

$$\int_1^3 f(x) = \int_1^2 f(x) + \int_2^3 f(x) = \int_1^2 2\,dx + \int_2^3 (x-2)\, dx$$

$$2x \Big|_1^2 + \frac{(x-2)^2}{2}\Big|_2^3 = 4 - 2 + \frac{1^2}{2} - \frac{0^2}{2} = \frac{5}{2} \quad \Delta$$

→ 16.2 Indefinite Integral

We observe that we can omit bounds when defining anti-derivative. So we study the problem of finding F, s.t. $F(x) = \int f(x)\, dx$ → indefinite Integral

for example

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} \qquad n \neq -1$$

or $\int \sin(x)\, dx = -\cos(x)$

$\int \cos(x)\, dx = \sin(x)$

<u>Motivation for C:</u>

$$\int (x+1)^2\, dx \to (x+1)^3 \tfrac{1}{3} = \frac{x^3}{3} + x^2 + x + \tfrac{1}{3} = \otimes_1$$

$$\neq$$

$$\to (x^2 + 2x + 1)\, dx = \int x^2 dx + 2\int x\, dx + \int 1 dx = \frac{x^3}{3} + x^2 + x = \otimes_2$$

Both $\otimes_1$, $\otimes_2$ are correct and they are identical up to $\tfrac{1}{3}$

For this reason, for indefinite Integrals, it is a convention to write

$$\int f(x)\, dx = F(x) + C \quad , \quad C \in \mathbb{R}$$

→ 16.4 Method of Integration by Parts

We know that

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Now if we integrate both sides:

$$f(x) \cdot g(x) = \int f'(x)\, g(x)\, dx + \int f(x) \cdot g'(x)\, dx$$

$$\boxed{\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x) \cdot dx}$$

Ex. $\int x e^x\, dx$

$g(x) = x \quad , \quad g'(x) = 1 \quad , \quad f'(x) = e^x \quad , \quad f(x) = e^x$

$$\Rightarrow \quad x \cdot e^x - \int 1 \cdot e^x\, dx = x \cdot e^x - e^x + C$$

↳ anti-derivative $\qquad = (x-1)e^x + C$

→ 16.3 Methods of Substitution

From the Chain-rule we have that:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

If we integrate both sides, we get:

$$f(g(x)) \cdot g'(x) = \int f'(g(x)) \cdot g'(x) \cdot dx$$

Ex $\int \underset{g'(x)}{x^2} \underset{f'(g(x))}{\sqrt{1+x^3}}\, dx$

$$\int \tfrac{1}{3} \underset{\overset{\shortparallel}{g(x)}}{(3x^2)} \cdot \tfrac{2}{3} \underbrace{\left(\tfrac{3}{2} \cdot \sqrt{\underset{\overset{\shortparallel}{g(x)}}{(1+x^3)}}\right)}_{f'(g(x))} dx \qquad \begin{aligned} f(x) &= 1 + x^3 \\ g(x) &= x^3 \end{aligned}$$

$$= \tfrac{2}{9} \int 3x^2 \cdot \left(\tfrac{3}{2} \sqrt{(1+x^3)}\right) dx \overset{\text{Subs}}{=} \tfrac{2}{9} (1+x^3)^{3/2} \quad \Delta$$

Ex $\int \frac{x}{x^2+1}\, dx \qquad$ Hint $\int \frac{1}{x}\, dx = \ln|x| + C$

$(\ln|x|)' = \frac{1}{x}$

$$\frac{1}{2} \int \underset{\substack{\shortparallel \\ g'(x) \\ \shortparallel \\ g(x) = x^2}}{2x} \cdot \underbrace{\frac{1}{x^2+1}}_{\overset{\shortparallel}{f'(g(x))}}\, dx$$

$$f'(g(x)) = \tfrac{1}{2} \ln|x^2 + 1| + C$$

Theorem 1

Let g be DIFF on $[a,b]$ and let $g(a) = A$ and $g(b) = B$. Also assume that f is CONT on Range(g). Then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_A^B f(t)\, dt$$

Ex $\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}$

$g(x) = \sqrt{x+1} \to g'(x) = \tfrac{1}{2} \frac{1}{\sqrt{x+1}}$

$f'(x) = \cos(x) \to f(t) = \sin(t)$

$g(0) = \sqrt{x+1} = 1$

$g(8) = \sqrt{9} = 3 \qquad$ so

$$\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}} = 2\int_0^8 \frac{\cos\sqrt{x+1}}{2\sqrt{x+1}} = 2\int_0^8 \cos(t)\, dt$$

$$= 2\sin(t)\Big|_1^3 = 2\sin(3) - 2\sin(1) \quad \Delta$$