Lecture 3 - 13/03

→ 3.1 Sequences

$(1, 2, 3, 4 \ldots)$

→ Elements can repeat
→ Order matters

→ Denotation: $(a_n)_{n \in \mathbb{N}}$ or simply (a_n)

ex $a_n = 2n \Rightarrow (2, 4, 6, 8 \ldots)$

→ Convergence

- definition: A sequence is converging to some $\alpha \in \mathbb{R}$ if

$$\text{if } (\forall \varepsilon > 0)\ (\exists N_\varepsilon \in \mathbb{N})\ (\forall n > N_\varepsilon)\ |a_n - \alpha| < \varepsilon$$

choose a small number as difference

there will be a number where the distance will be smaller from there on

example

$(a_n) = \frac{n}{n+1}$

$$|a_n - \alpha| < \varepsilon$$

$$\left|\frac{n}{n+1} - 1\right| = \left|-\frac{1}{n+1}\right| = \frac{1}{n+1} < \varepsilon$$

$$1 < \varepsilon(n+1)$$

$$1 < \varepsilon n + \varepsilon$$

$$\frac{1-\varepsilon}{\varepsilon} < n$$

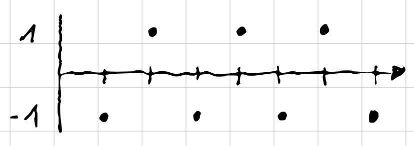
- definition 2:

if a sequence a_n converges to $\alpha \in \mathbb{R}$

$$\boxed{\lim_{n \to \infty} a_n = \alpha}$$

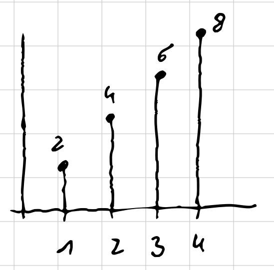
otherwise a_n is divergent.

ex. $a_n = (-1)^n$



DIVERGENT

ex $a_n = 2^n$
power sequence



$\lim_{n \to \infty} a_n = +\infty$

DIVERGENCE OF INFINITY

$$\lim_{n \to \infty} a_n = +\infty$$

$$\text{if } (\forall M > 0)\ (\exists N_M \in \mathbb{N})\ (\forall n > N_M)\ a_n > M$$

- Property I:

$$\lim_{n \to \infty} a_n = +\infty \quad \text{IFF} \quad \lim 1/a_n = 0$$

- Property II

$b \in \mathbb{R}$ and $a_n = b^n$

if $b = 1$ then $\lim a_n = 1$

if $b \in (-1, 1)$ then $\lim a_n = 0$

if $b \notin [-1, 1)$ then a_n diverges

→ 3.2 Properties about convergence

a) a_n is bounded from above if

$(\exists M \in \mathbb{R})\ (\forall n)\ a_n \leq M$

b) below

$(\exists M \in \mathbb{R})\ (\forall n)\ a_n \geq M$

c) bounded IFF above and below

- Every convergent sequence is bounded

Theorem 4: a_n and b_n sequence s.t. $\lim a_n \to \alpha$, $\lim b_n \to \beta$

them:

(i) $\lim(-a_n) = -\alpha$ (ii) $\lim(a_n + b_n) = \alpha + \beta$

(iii) $\lim(a_n - b_n) = \alpha - \beta$ (iiii) $\lim(a_n \cdot b_n) = \alpha \cdot \beta$

(iiiii) $\lim(k \cdot a_n) = k \cdot \alpha$ (iiiiii) $\lim(1/b_n) = 1/\beta$

(iiiiiii) $\lim(a_n/b_n) = \alpha/\beta \to b_n \neq 0, \beta \neq 0$

→ 3.3 Monotonic Sequences

A sequence is

a) monotone increasing if $a_{n+1} \geq a_n$, $n \in \mathbb{N}$

b) monotone decreasing if $a_{n+1} \leq a_n$, $n \in \mathbb{N}$

c) ultimatly monotone increasing if $a_{n+1} \geq a_n$, $n \geq N$

d) ultimatly monotone decreasing if $a_{n+1} \leq a_n$, $n \geq N$

Theorem 5:

a) ultimatly monotone increasing + bounded above = convergent

b) ultimatly monotone decreasing + bounded below = convergent

→ 3.4 Squeeze Theorem for Sequences

aka "Carabinieri Theorem"

$$\lim b_n = \lim c_n = L$$

$$b_n \leq a_n \leq c_n$$

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