Lecture 9 - 03/04/2023

Theorem 1 (DIFF <=> CONT):

If f is DIFF at $c \in \mathbb{R}$, then f is CONT.

- Proof:

Assumption: $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$ // alt. to $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

h

c x

Note: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)}$

We want to show: $\lim_{x \to c} f(x) = f(c) = \lim_{x \to c} (f(x) - f(c)) = 0$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) \iff \lim_{x \to c} f(x) - f(c) = (\lim_{x \to c} x - c) \cdot f'(x)$$
$$= 0 \cdot f'(x)$$

- CONT does not imply DIFF: CONT $\not\Rightarrow$ DIFF

Theorem 2:

Let f and g be functions that are DIFF at point c

(i) $(k \cdot f)' \cdot (c) = k \cdot f'(c)$

(ii) $(f \pm g)'(c) = f'(c) \pm g'(c)$

(iii) $(f \cdot g)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$

(iv) $(1/g)(c)$ exists if $g(c) \neq 0$ and $(1/g)(c) = \frac{-g'(c)}{[g(c)]^2}$

COROLLARY:

If f and g are DIFF at c and $g(c) = 0$, then

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c) \cdot g(c) - f(c) \cdot g'(c)}{[g(c)]^2}$$

(Proof in Notes)

$$\left(\frac{f}{g}\right)'(c) = \left(f \cdot \frac{1}{g}\right)'(c)$$

example:

$f(x) = 1 - x^2$

$g(x) = 1 + x^2$

Compute $(f/g)' = \frac{f'g - g'f}{g^2}$

$f'(x) = (1 - x^2)' = (1)' - (x^2)' = 0 - 2x = -2x$

$g'(x) = (1 + x^2)' = (1)' + (x^2)' = 0 + 2x = 2x$

$$\left(\frac{(1 - x^2)}{(1 + x^2)}\right)' = \frac{(-2x) \cdot (1 + x^2) - (2x) \cdot (1 - x^2)}{(1 + x^2)^2}$$

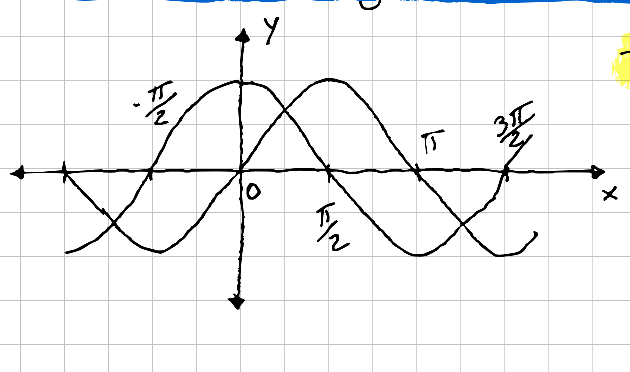
$$\frac{-2x - \cancel{2x^3} - 2x + \cancel{2x^3}}{(1 + x^2)^2} = \frac{-4x}{(1 + x^2)^2}$$

Theorem 3 (Chain - Rule):

f, and g are function, s.t. f is DIFF at c, and g is DIFF at $f(c)$. $g \circ f$ is DIFF at c:

$$(g \circ f)(c) = g(f(c)) = (g' \circ f)(c) \cdot f'(c) = g'(f(c)) \cdot f'(c)$$

Derivatives of trigonometric functions



Theorem 4:

a) $(\sin(x))' = \cos(x)$

b) $(\cos(x))' = -\sin(x)$

$$\sin(\alpha + \beta) = \sin(\alpha) \cdot \cos(\beta) + \cos(\alpha) \cdot \sin(\beta)$$

$$(\sin(x))' = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x) \overbrace{\cos(h)}^{\to 1} + \cos(x) \sin(h) - \overbrace{\sin(x)}^{\to 0}}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x) \overbrace{(\cos(h) - 1)}^{\to 0} + \cos(x) \cdot \sin(h)}{h}$$

$$= \lim_{h \to 0} \cos(x) \cdot \frac{\sin(h)}{h} \to 1 = \cos(x)$$