Lecture 20 - 07/06

→ 20.1 Exponential Functions via Integration

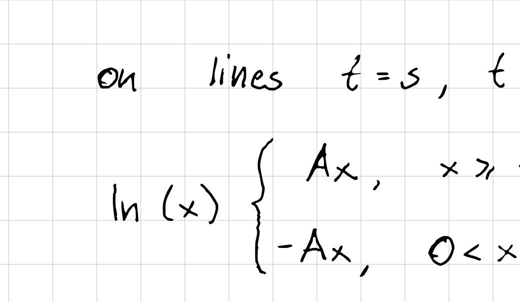
$x' = 1$

$(x^{-1})' = -1 x^{-1} \prec \frac{1}{x}$

definition:

For $x > 0$, let A_x be the area under the curve $y(t) = 1/t$

⊗



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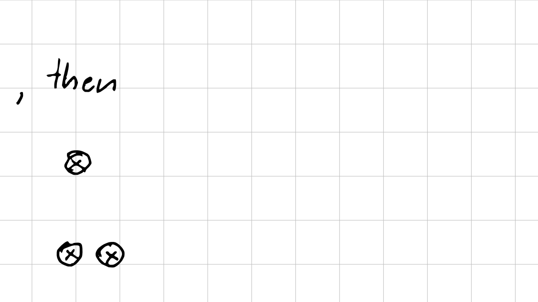


on lines $t = s$, $t = x$, then

$$\ln(x) \begin{cases} A_x, & x \geq 1 \quad ⊗ \\ -A_x, & 0 < x < 1 \quad ⊗⊗ \end{cases}$$

- by definition

$\ln(1) = 0$

$\ln(x) \geq 0, \quad x \geq 1$

$\ln(x) < 0, \quad 0 < x < 1$

• ln is monotonic

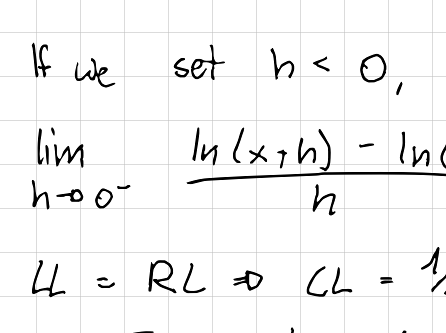
$\ln(x') > \ln(x), \quad x' > x$

↳ ln is 1-1

Theorem 1:

$\ln(x)' = 1/x$

Proof:



• $A = A_{x+h} - A_x$

• $h \cdot \frac{1}{x+h} < A < h \, 1/x \quad /h$

$\frac{1}{x+h} < A/h < 1/x$

$$\frac{1}{x+h} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$$

→ let $h \to 0$ by squeeze $\frac{1}{x} < \frac{\ln(x+h) - \ln(x)}{h} < \frac{1}{x}$

$\quad\quad \overset{\shortparallel}{\frac{1}{x}}$ (RL)

If we set $h < 0$, we can see that

$$\lim_{h \to 0^-} \frac{\ln(x+h) - \ln(x)}{h} = \frac{1}{x} \quad (LL)$$

$LL = RL \Rightarrow CL = 1/x$

$\Rightarrow [\ln(x)]' = 1/x$, for $x > 0$ □

Theorem 2:

(i) $\ln(x \cdot y) = \ln(x) + \ln(y)$

(ii) $\ln(1/x) = -\ln(x)$

(iii) $\ln(x/y) = \ln(x) - \ln(y)$

(iv) $\ln(x^r) = r \ln(x)$

• Proof of $\ln(x \cdot y) = \ln(x) + \ln(y)$

$$\ln(x \cdot y) = \int_1^{xy} 1/t \, dt = \int_1^x 1/t \, dt + \int_x^{xy} 1/t \, dt$$

$xy \geq 1$

$$= \ln(x) + \int_x^{xy} 1/t = \begin{bmatrix} u(t) = 1/x \cdot t \\ du = 1/x \, dt \\ dt = x \cdot du \end{bmatrix} \begin{bmatrix} \text{when } t = x, \; u = 1 \\ t = xy, \; u = y \\ t = ux \end{bmatrix}$$

$$= \ln(u) + \int_1^y \frac{x \cdot du}{u \cdot x} = \ln(x) + \int_1^y 1/u \, du = \ln(x) + \ln(y) \; □$$

Graphical properties for $\ln(x)$

$$\lim_{x \to +\infty} \ln(x) = +\infty$$

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

Property 1: for $x \in \mathbb{R} \setminus \{0\}$

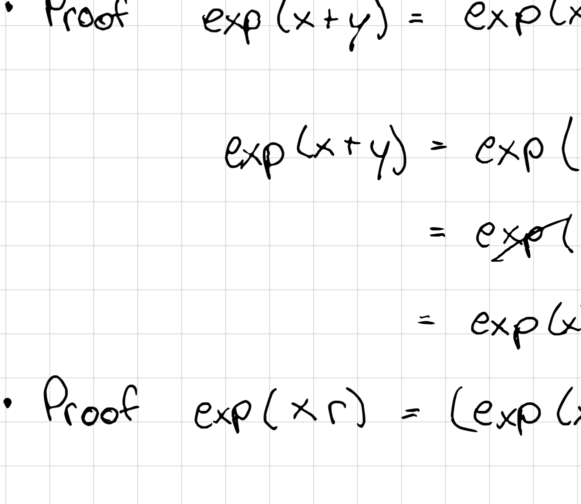
$(\ln|x|)' = 1/x$ and $\int 1/x \, dx = \ln|x| + C$

→ 20.2 Exponential Functions

$\ln(x)$ is 1-1 on $(0, +\infty)$

then we define the inverse $\ln^{-1} = \exp$ s.t.

$y = \exp(x)$ IFF $x = \ln(y)$



Theorem 3:

(i) $(\exp(x))^r = \exp(x r)$

(ii) $\exp(x + y) = \exp(x) \cdot \exp(y)$

(iii) $\exp(-x) = 1/\exp(x)$

(iv) $\exp(x - y) = \exp(x)/\exp(y)$

• Proof $\exp(x+y) = \exp(x) \cdot \exp(y)$

$$\exp(x+y) = \exp(\ln(\exp(x)) + \ln(\exp(y)))$$

$$= \cancel{\exp}(\cancel{\ln}(\exp(x) \cdot \exp(y)))$$

$$= \exp(x) \cdot \exp(y) \quad □$$

• Proof $\exp(x r) = (\exp(x))^r$

$\exp(xr) = y \iff xr = \ln(y)$

$x = \frac{1}{r} \ln(y) = x = \ln(y^{\frac{1}{r}})$

$\exp(x) = y^{1/r}$

$\exp(x)^r = y = \exp(xr)$ □

• definition 2:

let $\exp(1) = e$

From TH3: $\exp(r) = \exp(r \cdot 1) = [\exp(1)]^r = e^r$

Why does e exist?

$\exp(1) = e \iff 1 = \int_1^e 1/t \, dt \quad (1 = \ln|e|)$

ln is CONT on $(1, +\infty)$ s.t.

$(1, +\infty) \longmapsto (0, +\infty)$ so it must exist some number from $(1, +\infty)$ that is mapped to $1 \in (0, +\infty)$. $\ln(1) = e$

Calculate e^x

$y(x) = e^x \iff \ln(y(x)) = x$

$1 = y(x)' \cdot \frac{1}{y(x)} \iff y(x) = y'(x)$

$(e^x)' = e^x \Rightarrow \int e^x \, dx = e^x + C$

→ 20.3 Properties of log and exponential functions

• definition 3:

$$a^x = e^{\ln(a^x)} = e^{x \ln(a)}, \quad a > 0, x \in \mathbb{R}$$

ex. find 2^π

$$2^\pi = e^{\ln(2^\pi)} = e^{\pi \ln(2)}$$

• Property 2:

$(a^x)' = a^x \ln(a)$

• Proof Prop. 2

$$(a^x)' = (e^{\ln(a^x)})' = (\ln a^x)' \, e^{\ln(a^x)}$$

$$= (x \cdot \ln(a))' \cdot a^x = \ln(a) \cdot a^x \quad □$$

We defined e^x. Based on that we define $\log_a x$ as an inverse function of e^x, since e^x is 1-1.

• Prop. 3:

$$(\log_a x)' = \frac{1}{\ln(a) \cdot x}$$

• Proop of Prop. 3:

$y(x) = \log_a x \iff a^{y(x)} = x$

$\ln(a) \cdot a^{y(x)} \cdot y'(x) = 1$

$$\Rightarrow y'(x) = \frac{1}{\ln(a) \cdot a^{y(x)}} = \frac{1}{\ln(a) \cdot a^{\log_a x}}$$

$$y(x) = \log_a x = \frac{1}{\ln(a) \cdot x}$$

→ 20.4 Limit of $(1 + \frac{x}{n})^n$

Theorem 5:

$$\lim_{n \to \infty} (1 + x/n)^n = e^x$$

¡ Proof in Notes !