Lecture 15 - 17/05

15.1 Classes of Integrable Functions

We write $f \in \mathcal{R}[a,b]$ if f is Riemann Integrable on $[a,b]$.

Theorem 1:

If f is MONOTONIC (INC or DEC) on $[a,b]$ then $f \in \mathcal{R}[a,b]$.

Theorem 2:

If f is CONT on $[a,b]$ then $f \in \mathcal{R}[a,b]$.

LEMMA:

if $f \in \mathcal{R}[a,b]$ IFF $(\forall \varepsilon > 0)(\exists P \text{ of } [a,b])$

$$U(f,P) - L(f,P) < \varepsilon$$

15.2 Properties of Integrals

Theorem 3: Let $f, g \in \mathcal{R}[a,b]$ and $\mathcal{R}[b,c]$, then

a) $\int_a^a f(x)\,dx = 0$

b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

c) $\int_a^b (A f(x) + B g(x))\,dx = A\int_a^b f(x) + B\int_a^b g(x)$; $A, B \in \mathbb{R}$

d) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

e) if $f(x) \leq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \leq \int_a^b f(x)\,dx$$

f) $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$ "triangle equality"

g) let f be an odd function (i.e. $f(-x) = -f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 0$$

h) let f be an even function (i.e. $f(-x) = f(x)$), then

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$$

Intuitions:

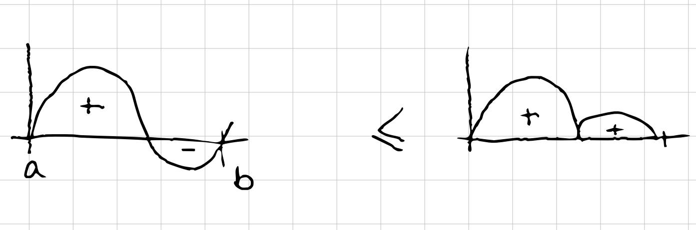
a) Since $\Delta x_i = 0$

b) extend def. of Δx_i for $\int_b^a$ s.t. $\Delta x_i < 0$

c) Integral is just an infinite sum $\Delta x \to 0$, so they can be manipulated just as sums

d) idea that Integral is just an area under the curve

f)



15.3 The fundamental Theorem of Calculus

→ 3 Blue 1 Brown : "Essance of calculus"

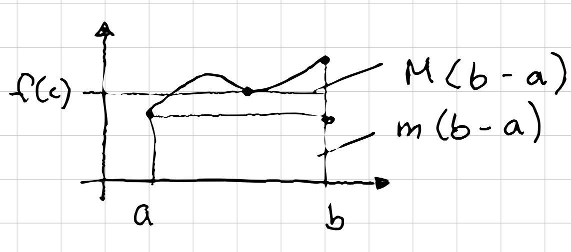
Let f be CONT on $[a,b]$, then

- m min $[a,b]$
- M Max $[a,b]$

$$m \leq f(x) \leq M$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Graphically:



$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$$

Theorem 4 (Mean-Value for Integrals): (Proof later)

If f is CONT on $[a,b]$ then exists $c \in [a,b]$ s.t.

$$\int_a^b f(x)\,dx = (b-a)\cdot f(c)$$

def: Let f and F be two function, If $F'(x) = f(x)$, then F is <u>antiderivative</u> of f.

Theorem 5 (Fundamental TH of Calculus) Part A:

Let f be CONT on $[a,b]$. For each $x \in [a,b]$ we define:

$$F(x) = \int_a^x f(t)\,dt$$

then

- F is DIFF on $[a,b]$, and
- $F'(x) = f(x)$

$\boxed{\text{Proof:}}$

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right)$$

$$\underset{\text{TH 4}}{=} \lim_{h\to 0}\frac{1}{h}\left(\int_x^{x+h} f(t)\,dt\right) \underset{\substack{\text{TH 4}\\ c\in[x,x+h]}}{=} \lim_{h\to 0}\frac{1}{\cancel{h}}\Big(\underbrace{((x+h)-x)}_{\cancel{h}}\, f(x+\Theta h)\Big)$$

$$= \lim_{h\to 0} f(x+\Theta h) = \underline{\underline{f(x)}} \quad \Delta$$

$F'(x)$ exists and $F'(x) = f(x)$

Theorem 6 Part B:

Let f be CONT on $[a,b]$, and let G be an anti-derivative of f i.e. $G'(x) = f(x)$, then

$$\int_a^b f(t)\,dt = G(b) - G(a)$$

$\boxed{\text{Proof:}}$

- Let $f(x) = \int_a^x f(t)\,dt$ from TH 5, then
  $F'(x) = f(x) = G'(x)$ ie. $F'(x) = G'(x)$, $x \in [a,b]$
- Then must be: $(C)' = 0$

  $F(x) = G(x) + C$ for some constant $C \in \mathbb{R}$

  $\int_a^x f(t)\,dt = f(x) = G(x) + C$

Let $x = a$, then $\int_a^a f(t)\,dt = 0$, so $G(a) + C = 0$

$\Rightarrow C = -G(a)$

So $\int_a^b f(t)\,dt = G(b) - G(a)$ □

Observations:

- TH 6 tells us that to compute integral we need to know antiderivative
- not all functions f would have antiderivative, but one can still compute the Integral

Ex. $\int_0^a x^2\,dx$

find antiderivative of f s.t $F'(x) = f$

$F(x) = \frac{1}{3}x^3 = \frac{x^3}{3}$ $\qquad F'(x) = \frac{1}{\cancel{3}}\cancel{3}x^2 = x^2$ ✓

Apply TH 6:

$$\int_0^a x^2\,dx = F(a) - F(0) = \frac{a^3}{3} - \overset{0}{\overbrace{\frac{0^3}{3}}} = \frac{a^3}{3}$$

Ex 2: $\int_{-1}^{2}(x^2 - 3x + 2)\,dx$ ⊗

$$⊗ = \int_{-1}^2 x^2\,dx - \int_{-1}^2 3x\,dx + \int_{-1}^2 2\,dx =$$

$$= \int_{-1}^2 x^2\,dx - 3\int_{-1}^2 x\,dx + 2\int_{-1}^2 dx =$$

We search the anti-derivative of each Integral

Recall $F(x)\Big|_a^b = F(b) - F(a)$

$$= \frac{x^3}{3}\Big|_{-1}^2 - 3\cdot\left(\frac{x^2}{2}\Big|_{-1}^2\right) + 2\left(x\Big|_{-1}^2\right)$$

$$= \frac{1}{3}(8-(-1)) - \frac{3}{2}(4-1) + 2(2-(-1))$$

$$= 3 - \frac{9}{2} + 6 = 9 - \frac{9}{2} = \frac{18-9}{2} = \frac{9}{2}$$