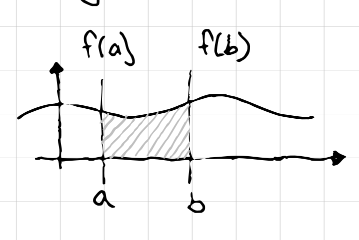
Lecture 14 - 10.05

14.1 Integrals

Two ways to describe Integral:

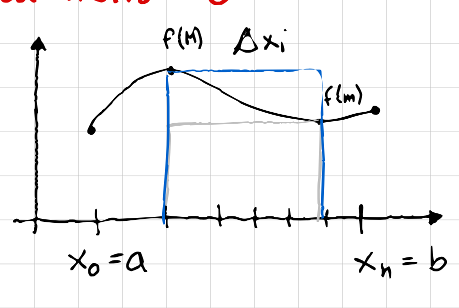
a) Integral is the area under the curve of a function



Area $(a, b, f(b), f(a))$ = Integral of f

b) Integral of f is another function F, s.t. f is antiderivative of F. $\rightarrow F'(x) = f(x)$

14.2 Partitions & sums



$P = \{x_0, x_1 \dots x_n\}$

also store results:

$\Delta x_i = x_i - x_{i-1}$

- Let f be CONT on $[a, b]$

  since f is bounded (from CONT) we have MIN and MAX

  $f(m) \leq f(x) \leq f(M)$

  $f(m) \cdot \Delta x_i \leq A_i \leq f(M) \cdot \Delta x_i$

def:

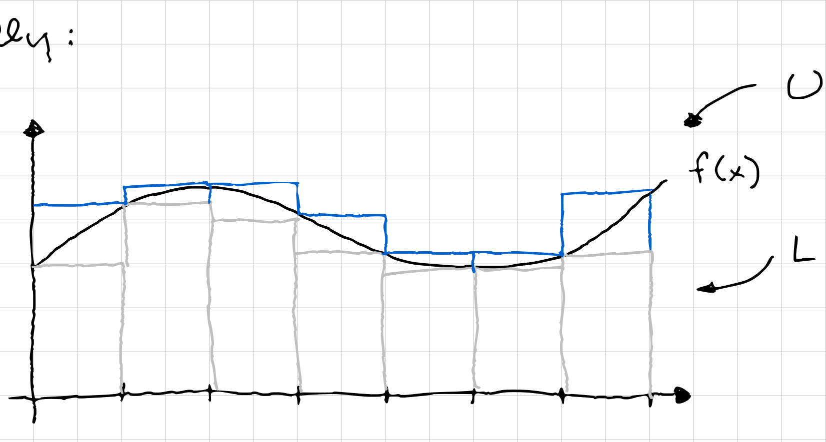
Lower Riemann sum

$$L(f, P) = f(m) \cdot \Delta x_1 + \dots f(m) \Delta x_n = \sum_{i=1}^{n} f(m) \cdot \Delta x_i$$

Upper Riemann sum

$$U(f, P) = f(M) \cdot \Delta x_1 + \dots f(M) \Delta x_n = \sum_{i=1}^{n} f(M) \cdot \Delta x_i$$

Graphically:



example:

$f(x) = x^2$, $[0, a]$, $a \in \mathbb{R}$

first we fix some partitions P_n

$x_i - x_{i-1}$ is same length, $\Delta x_i = \Delta x = a/n$, $X = a/n \cdot i$

$$L(f, P) = \sum_{i=1}^{n} f(m) \cdot x_i = \sum_{i=1}^{n} f(x_{i-1}) \cdot a/n$$

$$= \sum_{i=1}^{n} f\left(\frac{a}{n} \cdot i-1\right) \cdot a/n = \sum_{i=1}^{n} (a/n \cdot i-1)^2 \cdot a/n$$

$$= a^2/n^2 \cdot a/n \sum_{i=1}^{n} (i-1)^2 = \frac{a^3}{n^3} \sum_{i=1}^{n} (i-1)^2$$

Recall: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{a^3}{n^3} \cdot \frac{(n-1)\, n\, (2(n-1)+1)}{6}$$

$$\underline{L(f, P)} = \frac{(n-1) \cdot (2n-1) \cdot a^3}{6n^2}$$

$$U(f, P) = \sum_{n=1}^{\infty} f(M) \Delta x_i = \sum_{n=1}^{\infty} (x_i)^2 \Delta x_i = \sum_{n=1}^{\infty} \left(\frac{i}{n} \cdot a\right)^2 a/n$$

$$\underline{U(f, P)} = \frac{(n+1) \cdot (2n+1) \cdot a^3}{6n^2}$$

→ 14.3 The definite Integral

We want $U(f, P_n)$ get closer to $L(f, P_n)$.

P^b is refinement of P^a, by adding more partitions.

$$L(f, P^a) \leq L(f, P^b) \leq U(f, P^b) \leq U(f, P^a)$$

SUP/INF: $A = \{1/n \mid n \in \mathbb{N}\} = \{1, 1/2, 1/3, 1/4, 1/5 \dots 1/n\}$

sup A: 1 (can also not belong to set)

inf A: 0

Let:

$L = \sup\{L(f, P_n) \mid P \text{ is partitioning } [a, b]\}$

$U = \inf\{U(f, P_n) \mid P \text{ is partitioning } [a, b]\}$

def 2:

Let f be CONT $[a, b]$ / bounded on $[a, b]$

If exists number $I = L = U$, then f is Riemann Integrable on $[a, b]$ and we write

$$I = \int_a^b f(x)\, dx$$

(showing bounded later)

Observation 1:

To show that f is RI it is sufficient to find a sequence of refinements $P_1 \subseteq P_2 \dots P_n$

s.t. $\lim_{n \to \infty} L(f, P_n) = \lim_{n \to \infty} U(f, P_n)$

$\sup\{L(f, P_n)\} = \inf\{U(f, P_n)\}$

$\int_b^a$ bounds of the integral

ex 2: $f(x) = x^2$ $[0, a]$

$\Delta x_i = a/n$ $X_i = a/n \cdot i$

We showed: $U(f, P_n) = \frac{(n+1)(2n+1) \cdot a^3}{6n^2}$

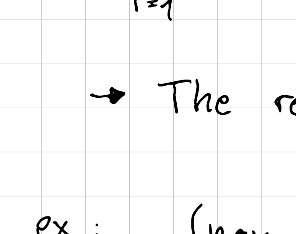
$L(f, P_n) = \frac{(n+1)(2n-1) \cdot a^3}{6n^2}$

$\lim_{n=\infty} L(f, P_n) = \frac{1}{3} a^3$

$\lim_{n=\infty} U(f, P_n) = \frac{1}{3} a^3$

$$\int_b^a x^2\, dx = \frac{a^3}{3}$$

→ Defining Integral for bounded functions



$P_n = x_0 = a, x_1, x_2 \dots x_n = b$

$\Delta x = x_i - x_{i-1}$

$m_i = \inf_{x \in (x_{i-1}, x_i)} f(x)$

$M_i = \sup_{x \in (x_{i-1}, x_i)} f(x)$

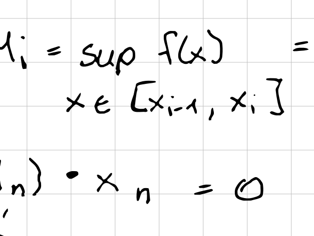
$$L(f, P_n) = \sum_{i=1}^{n} m_i \cdot \Delta x_i$$

$$U(f, P_n) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$

→ The rest of the theory is the same

ex: (non integrable function)

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



Let P be any partitioning of $[a, b]$ of size n s.t.

$$a = x_0 < x_1 < x_2 \dots x_n = b$$

$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = 0$ $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = 1$

$$L(f, P_n) = \underset{=0}{f(l_1)} \cdot \Delta x_i + \dots + \underset{=0}{f(l_n)} \cdot x_n = 0$$

$$U(f, P_n) = \overset{=1}{f(u_1)} \cdot \Delta x_i + \dots + \overset{=1}{f(u_n)} \cdot x_n = a-b$$

Since we picked arbitrary partition P of $[a, b]$.

$U = \inf\{U(f, P) \mid P \text{ on } [a, b]\} = b-a$

$L = \sup\{L(f, P) \mid P \text{ on } [a, b]\} = 0$

$U \neq L$ so f is not Riemann integrable Δ