Lecture 8 - 29/03

→ 8.1 Continuity

Theorem 1:

g, f are CONT, then following are CONT.

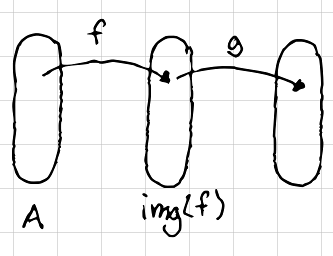
(i) $f+g$ (iii) $k \cdot f$ (v) $f^{1/n}$

(ii) $(f \cdot g)$ (iv) f/g

Theorem 2:

f, g are functions, f, g are CONT, $img(f) \subseteq dom(g)$

and g is continious on A.



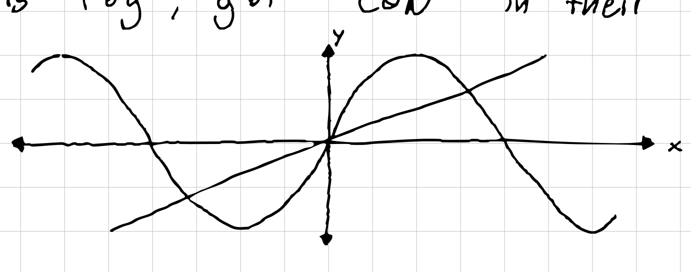
Then $g \circ f = g(f(x))$ is CONT in A

Remark: f is CONT in intervall A if it is CONT for all $a \in A$

example:

$f(x) = \sin(x)$, $g(x) = \sqrt{x}$

Is $f \circ g$, $g \circ f$ CON in their domain ?



$f(x) = \sin(x) \Rightarrow dom(f) = (-\infty, +\infty)$, $img(f) = [-1, +1]$

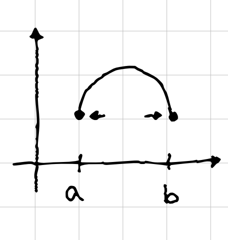
$g(x) = \sqrt{x} \Rightarrow dom(g) = [0, +\infty)$, $img(g) = [0, +\infty)$

$f \circ g = \sin(\sqrt{x})$, cond. $img(g) \subseteq dom(f)$, $[0, +\infty) \subseteq (-\infty, +\infty)$ ✓

$g \circ f = \sqrt{\sin(x)}$, cond. $img(f) \subseteq dom(g)$, $[-1, 1] \subseteq [0, +\infty)$ X

f is CONT on <u>closed</u> intervall $[a, b]$

(i) CONT (a, b) (ii) $\lim_{x \to a^+} f(x) = f(a)$ (iii) $\lim_{x \to b^-} f(x) = f(b)$



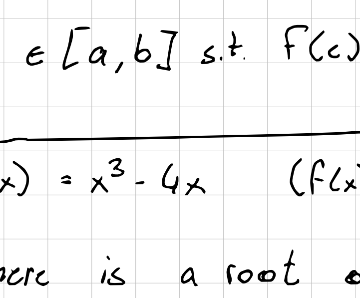
Theorem 3 (MIN - MAX):

let f be CONT on $[a, b]$. Then exists p, q s.t.

$m = f(p) \leq f(x) \leq f(q) = M$

m = absolute min of f at $[a, b]$

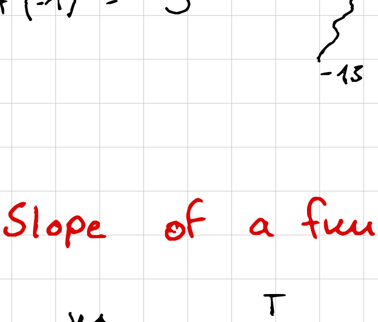
M = absolute max of f at $[a, b]$



Theorem 4 (Intermediate - Value Theorem) aka Bolzano - Theorem:

Let f be CONT. $[a, b]$, assume $f(a) \leq f(b)$, then for every $s \in [f(a), f(b)]$ exists $c \in [a, b]$ s.t. $f(c) = s$

$f(c) = s$

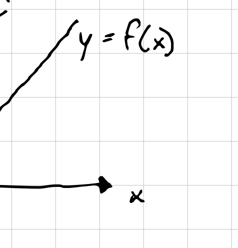


example: $f(x) = x^3 - 4x$ $(f(x) = 0)$

Show that there is a root of f between $-3, -1$

$f(-3) = -15$

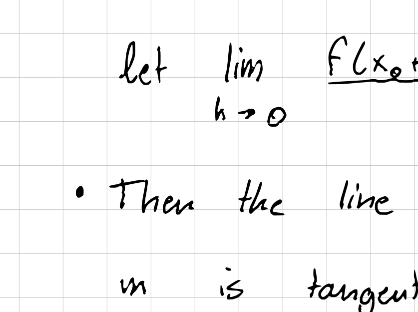
$f(-1) = 3$



From Inter- Theorem, we now there must be some number s.t $f(x) = 0$

$x(x^2 - 4) = x(x-2)(x+2)$ $x \begin{cases} 0 \\ 2 \\ -2 \end{cases}$

→ 8.2 Slope of a function



T = tangent

formally def:

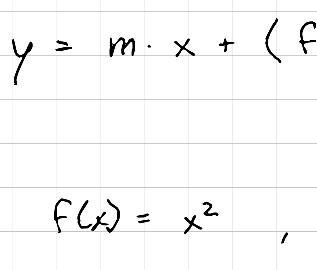
f CONT at $x = x_0$:

let $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = m$

- Then the line that paues $(x_0, f(x_0))$ with slope m is tangent for f at $(x_0, f(x_0))$

- Linear functions recap:

→ Function of the form $f(x) = y = k \cdot x + n$



k = slope

n = displacement of height

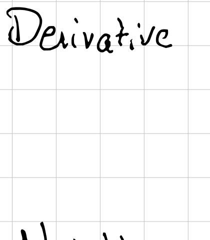
- Tangent

· at $(x_0, f(x_0))$ is a line that

$y = m \cdot x + (f(x_0) - m x_0)$

example:

$f(x) = x^2$, $x_0 = 1$



$$m(\text{tangent}) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(\cancel{1}+h)^2 - \cancel{1}}{h} = \frac{2h + h^2}{h} = 2$$

→ 8.3 Derivatives

Derivative of a function f is the function f'

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Notation</u>:

- if $f'(x)$ exists we say that is Differentiable (DIFF) at x.
- We read $f'(x)$ as "f-prime"
- For $f(x)$ we also write $\frac{d}{dx} f(x)$, $\frac{df}{dx}$, $\frac{\delta f}{\delta x}$

example:

a) $f(x) = x^n$, $n \in \mathbb{N}$

$$f(x)' = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + \binom{n}{1} \cdot x^{n-1} \cdot \cancel{h} + \binom{n}{2} \cdot x^{n-2} \cdot h^{\cancel{2}} + \ldots + h^n - \cancel{x^n}}{\cancel{h}}$$

0

$= n \cdot x^{n-1}$ → $f(x)$ is DIFF at $\forall x \in \mathbb{R}$

and $f(x)' = n \cdot x^{n-1}$

b) $f(x) = |x|$ $\Rightarrow f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

- is f CONT in $\mathbb{R}$?
- is f DIFF in $\mathbb{R}$?

$\Updownarrow$ $\lim_{x \to c} f(x) = f(c)$ or $LL = RL$

We observe that $x_0 = 0$ → $LL = RL = f(c) = 0$, check if:

$LL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0$

$RL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

- <u>See</u>: $(x^n)' = n \cdot x^{n-1}$

$\Rightarrow x > 0 \Rightarrow 1 \cdot x^0 = 1$

$\Rightarrow -x < 0 \Rightarrow (-x)' = -(x)' = -1$

For $x = 0$, we need to check:

$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist, since:

RL $\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h - 0}{h} = 1$

LL $\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h - 0}{h} = -1$

$LL \neq RL \Rightarrow$ CL does not exist

f is not DIFF at $x = 0$

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ \text{undef.}, & x = 0 \end{cases}$$