Analysis Theorems

Theorem 2.1:

let $A = ax^2 + bx + c = 0$, then
(i) $A \geq 0$ if $a > 0$ and $b^2 + 4ac \leq 0$
(ii) A has roots if $b^2 + 4ac \geq 0$
(iii) x_1, x_2 roots of A, then if $a > 0$, then $\{x \mid A \geq 0, y = [x_1, x_2]\}$

Theorem 2.2: Binomial Theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} \cdot b^r$$
$$\downarrow\ a^n + n \cdot a^{n-1} \cdot b + \binom{n}{r} a^{n-r} \cdot b^r + \dots + nab^{n-r} + b^n$$

Theorem 3.1:

$a_n > 0,\ \forall n \geq 1$
$\lim_{n \to \infty} a_n = \infty$ IFF $\lim_{n \to \infty} 1/a_n = 0$

Theorem 3.2:

$a_n = b^n,\ b \in \mathbb{R}$

a) if $b = 1$, then $\lim_{n \to \infty} a_n = 1$
b) for $b \in (-1, 1)$, then $\lim_{n \to \infty} a_n = 0$
c) for $b \in (-1, 1]$, then a_n DIV

Theorem 3.3:

Every convergent sequence is bounded

Theorem 3.4:

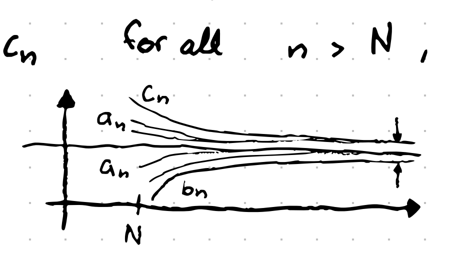
a_n and b_n are sequences, s.t. $\lim_{n \to \infty} a_n = \alpha$, $\lim_{n = \infty} b_n = \beta$, then
(i) $\lim (-a_n) = -\alpha$
(ii) $\lim (a_n + b_n) = \alpha + \beta$ (v) $\lim (k \cdot a_n) = k \cdot \alpha$
(iii) $\lim (a_n - b_n) = \alpha - \beta$ (vi) $\lim (1/b_n) = 1/\beta$
(iv) $\lim (a_n \cdot b_n) = \alpha \cdot \beta$ (vii) $\lim (a_n/b_n) = \alpha/\beta$

Theorem 3.5:

a) a_n is ultimatly monotonic increasing + bounded above = CONV.
b) b_n is ultimally monotonic decreasing + bounded below = CONV.

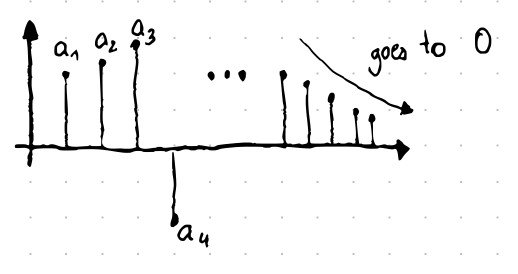
Theorem 3.6: Squeeze Theorem

$\lim b_n = \lim c_n = L$, then if exists an $N \in \mathbb{N}$ s.t.
$b_n \leq a_n \leq c_n$ for all $n > N$, then $\lim a_n = L$



Theorem 4.1:

if $\sum_{n=1}^{\infty} a_n$ CONV., then $\lim a_n = 0$



Theorem 4.2:

Harmonic Series $\sum_{n=1}^{\infty} 1/n$ is DIV. ⊛

Proof:

→ $1/n$ replaced by $1/2^k$ s.t. it's larger than n
⊛ $\geq 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8) + \dots$
⊛ $\geq 1 + 1/2 + 1/2 + 1/2 + \dots = +\infty$
→ $\sum_{n=1}^{\infty} 1/n = +\infty$

Theorem 4.2

Let $\sum_{n=1}^{\infty} ar^{n-1}$ be a geometric series, then

- if $|r| < 1$ then it converges to $\frac{a}{1-r}$
- if $|r| \geq 1$ then it diverges

ex $\sum_{n=3}^{\infty} \frac{2^{n+3}}{e^{n-3}} = 2^6 \sum_{n=3}^{\infty} \frac{2^{n-3}}{e^{n-3}} = 2^6 \sum_{n=3}^{\infty} (2/e)^{n-3}$ $|2/e| < 1$

Theorem 5.1 Comparison Test:

Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} a_n$ be two positiv series, then

- if $x_n \leq a_n$ for $n \in \mathbb{N} \Rightarrow \sum_{n=1}^{\infty} a_n$ CONV $\Rightarrow \sum_{n=1}^{\infty} x_n$ CONV
- if $x_n \geq a_n$ for $n \in \mathbb{N} \Rightarrow \sum_{n=1}^{\infty} a_n$ DIV $\Rightarrow \sum_{n=1}^{\infty} x_n$ DIV

Theorem 5.2:

Let $\alpha \in \mathbb{R}$

- if $\alpha > 1$, then series $\sum_{n=1}^{\infty} 1/n^{\alpha}$ CONV
- if $\alpha \leq 1$, then series $\sum_{n=1}^{\infty} 1/n^{\alpha}$ DIV

Theorem 5.3 Asymptotic Test:

Let $\sum a_n$ and $\sum x_n$ be a series of positive terms, assume that $a_n \sim x_n$, then

- if $\sum a_n$ CON $\to \sum x_n$ CON ex $\sqrt{n^2+1} \sim n$
- if $\sum x_n$ DIV $\to \sum a_n$ DIV

Theorem 5.4 Ratio Test:

Let $\sum_{n=1}^{\infty} a_n$ be a positive series, then

- if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$, then series CONV
- if $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} > 1$, then series DIV

Theorem 7.1:

Let f and g be real functions and $a \in \mathbb{R}$. Suppose

$\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

(i) $\lim_{x \to a} |f(x)| = |L|$ (ii) $\lim (f \pm g)(x) = L \pm M$

Theorem 7.2:

(iii) $\lim (f \cdot g)(x) = LM$ (iv) $\lim (f/g)(x) = L/M$
(v) $\lim (k \cdot f)(x) = k \cdot L$ (vi) $\lim (f(x))^{m/n} = L^{m/n}$
$\lim_{x \to a^+} f(x) = \lim_{\lim \to a^-} f(x) = L$

Theorem 7.3 $\lim_{x \to} f(x) = L$ IFF **Squeeze Theorem:**

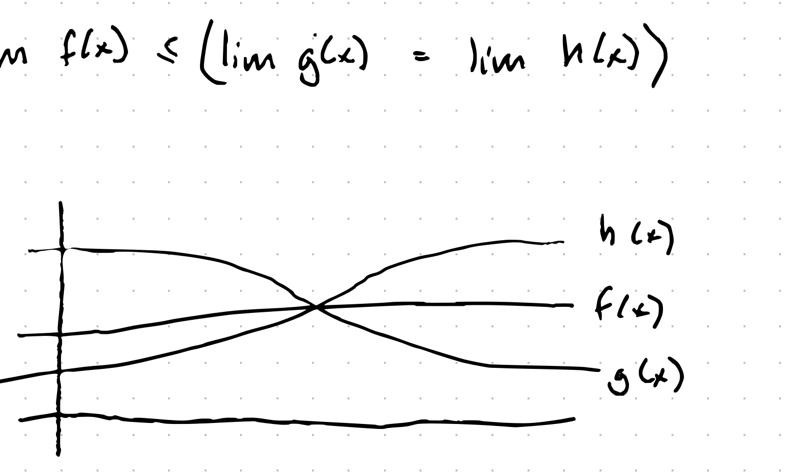
Let f, g and h be real functions defined on an open iterated I that contains a, but f, g, h do not have to be defined. Then, if

$$f(x) \leq g(x) \leq h(x), \quad \forall x \in I \setminus \{a\}$$

and if

$$\lim f(x) \leq (\lim g(x) = \lim h(x))$$

graphically:



ex: $\lim_{x \to 0} \frac{\sin(x)}{x} \Rightarrow 1 > \frac{\sin(x)}{x} > \cos(x)$

$\lim(1) > \lim \frac{\sin(x)}{x} > \lim(\cos(x))$
$\downarrow$ $\downarrow$
$0 \longrightarrow$ goal $\longleftarrow 0$

Theorem 8.1:

g and f are CONT, then the following are CONT

(i) $f+g$ (iii) $k \cdot f$ (v) $f^{1/n}$
(ii) $(f \cdot g)$ (iv) f/g