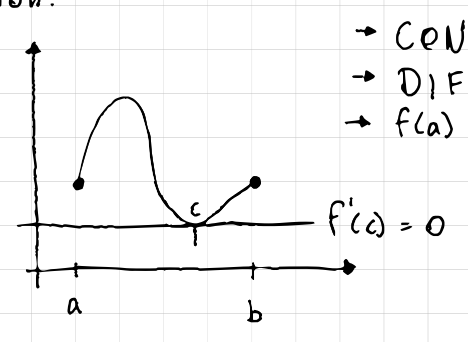
Lecture 10 - 12/04/2023

→ 10.1 Rolle's Theorem

Let f be a real function continuous on $[a, b]$ and DIFF on (a, b), and suppose $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$

Intuition:



→ CONT $[a, b]$
→ DIFF (a, b)
→ $f(a) = f(b)$

→ Lema 1:

Let f be a function, CONT $[a, b]$, DIFF (a, b).
Assume that f has a MAX or MIN value at some c, and that $f'(c)$ exists, then $f'(c) = 0$

• Proof:

Suppose c is MAX of f, then $f(x) - f(c) \leq 0$

Since $f'(c)$ exists. CL for RL = LL

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{\overbrace{f(x) - f(c)}^{\leq 0}}{\underbrace{x - c}_{\geq 0}} = \lim_{x \to c^-} \frac{\overbrace{f(x) - f(c)}^{\leq 0}}{\underbrace{x - c}_{< 0}}$$

$$\left.\begin{array}{l} RL \text{ is always } \leq 0 \\ LL \text{ is always } \geq 0 \end{array}\right\} \Rightarrow RL = LL = 0 \; - \; f'(c) = 0$$

→ Proof Rolle's Theorem

We have 2 possible cases:

- f is a constant function on $[a, b]$, then $f'(c) = 0$ for any $c \in [a,b]$
- f is not a constant function on $[a, b]$, then it must exist $d \in (a,b)$ s.t.

$(*)$ $f(d) > f(a) = f(b)$ or

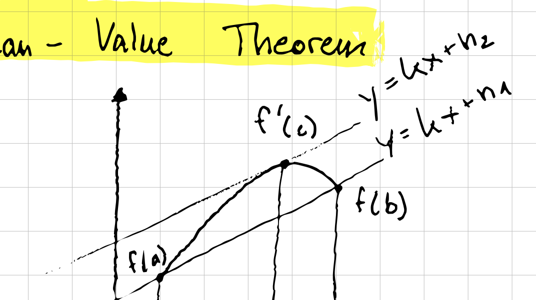
$(**)$ $f(d) < f(a) = f(b)$

Assume $(*)$, case $(**)$ can be shown similarly.
From MIN-MAX Theorem (Lec 8), we know that there must exist some c s.t.

$$f(c) \geq f(d) > f(a) = f(b)$$

Since $f(c)$ is MAX from Lemma 1, we know that $f'(c) = 0$.

Mean-Value Theorem



$$k = \frac{y_2 - y_1}{x_2 - x_1}$$

- f is CONT $[a, b]$
- f is DIFF (a,b)

Then there exists a $c \in (a,b)$ s.t. $\boxed{f'(c) = \frac{f(b) - f(a)}{b - a}}$

Proof:

Consider a function g as follows:

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

f CONT $[a, b]$ $\Rightarrow$ g CONT $[a, b]$

f DIFF (a, b) $\Rightarrow$ g DIFF (a,b)

$$g(a) = \cancel{f(a)} - \cancel{f(a)} - \frac{f(b) - f(a)}{b - a} \cdot \overbrace{(a - a)}^{= 0} = 0$$

$$g(b) = \cancel{f(b)} - \left[\cancel{f(a)} - \cancel{\frac{f(b) - f(a)}{b - a}} \cdot \cancel{(b - a)}\right] = 0$$

From Rolle's Theorem we know that there must be a point c s.t. $g'(c) = 0$

$$g'(x) = f'(x) - \left[0 + \frac{f(b) - f(a)}{b - a} \cdot 1\right]$$

So $f'(c) = 0$

$$g'(c) = 0 = f'(c) - \frac{f(b) - f(a)}{b - a} \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

→ 10.3 Increasing and Decreasing Functions

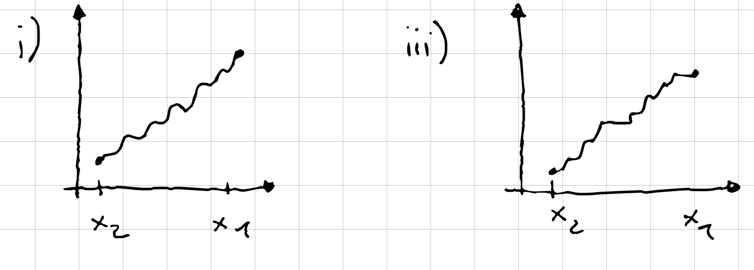
Let f be a function on (a, b) and for any two points $x_1, x_2 \in (a, b)$

(i) $f(x_1) > f(x_2)$ when $x_1 > x_2$ → f is INC on (a, b)

(ii) $f(x_1) < f(x_2)$ when $x_1 > x_2$ → f is DEC on (a, b)

(iii) $f(x_1) \geq f(x_2)$ when $x_1 > x_2$ → f is NOT DEC on (a,b)

(iiii) $f(x_1) \leq f(x_2)$ when $x_1 > x_2$ → f is NOT INC on (a, b)



Theorem 3:

(i) If $f'(x) > 0$ $\forall x \in (a,b)$ → INC

(ii) If $f'(x) < 0$ $\forall x \in (a, b)$ → DEC

(iii) If $f'(x) \geq 0$ $\forall x \in (a, b)$ → NON DEC

(iiii) If $f'(x) \leq 0$ $\forall x \in (a, b)$ → NON INC

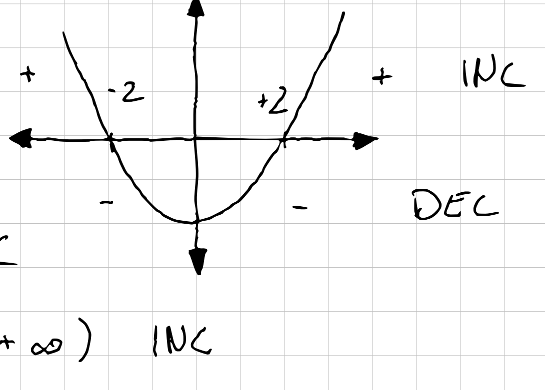
ex 1)

$$f(x) = x^3 - 12x + 11$$

• Find intervals on which f is increasing and decreasing (sketch it)

f is CONT and DIFF on $\mathbb{R}$

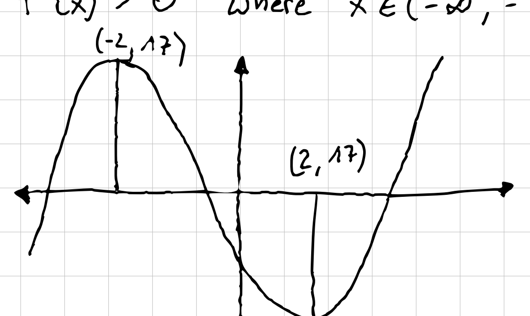
$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$



at $x = \pm 2$ $f'(x) = 0$

$f'(x) < 0$ where $x \in (-2, 2)$ DEC

$f'(x) > 0$ where $x \in (-\infty, -2) \cup (2, +\infty)$ INC



	$-\infty$	-2	$+2$	$+\infty$
f	↗	↘	↗	
f'	+	−	+	

$f(-2) = 17$

$f(2) = -15$

ex 2)

$f(x) = |x|$

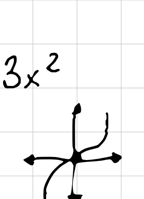
$f'(x) = 1$ $x > 0$

$f'(x) = -1$ $x < 0$

ex 3)

$f(x) = x^3$ $f(x)' = 3x^2$

$f'(0) = 0$



$(0, 0)$ is not MIN/MAX