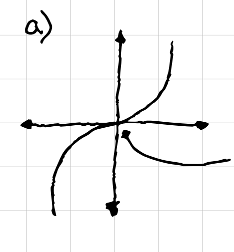
Lecture 12 - 19/04/23

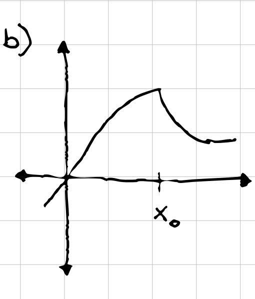
→ 12.1 Inflection Points

Let f be a function a point $(x_0, f(x_0))$ is called INFLECTION point if

a) f has a tangent on $(x_0, f(x_0))$

b) f has opposite concavities on each side of x_0

a)  $(0,0)$ is INF point

b) 

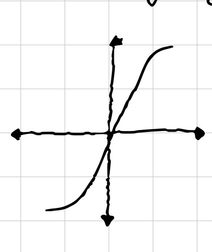
$\lim_{x \to x_0^-} f'(x) \neq \lim_{x \to x_0^+} f'(x)$

- Observation:

Point A can be satisfied, either

1) f' exists at $(x_0, f(x_0))$

2) f has a vertical line tangent at $(x_0, f(x_0))$



$f(x) = x^{1/3}$ $\quad f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} = \frac{1}{3} \frac{1}{x^{2/3}}$

$f'(x)$ is not defined at $\underline{\underline{0}}$

→ tangent is y axis

Theorem 1:

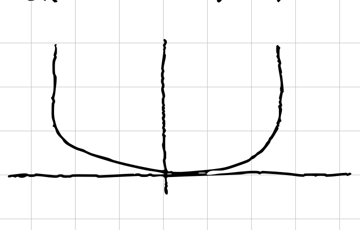
Let f be a function on (a, b), then:

(i) if $f''(x) > 0$ on (a,b) then f is CONV on (a,b)

(ii) if $f''(x) < 0$ on (a,b) then f is CONC on (a,b)

(iii) if f has an INFL point in $(x_0, f(x_0))$ and $f''(x_0)$ exists, then $f''(x_0) = 0$

ex. $f(x) = x^4$



$f'(x) = 4x^3$
$f''(x) = 12x^2$

$f''(0) = 0$, however $(0,0)$ is not INFL

ex. Study concavity of $f(x) = x^6 - 10x^4$

$f'(x) = 6x^5 - 40x^3$

$f''(x) = 30x^4 - 120x^2$

$f''(x) > 0 \to$ increasing

$30x^4 - 120x^2 > 0$

$30x^2(x^2 - 4) > 0 \begin{cases} -2 \\ 2 \end{cases}$ INF

$f'' < 0$, $x \in (-\infty, -2) \cup (2, +\infty)$ CONV

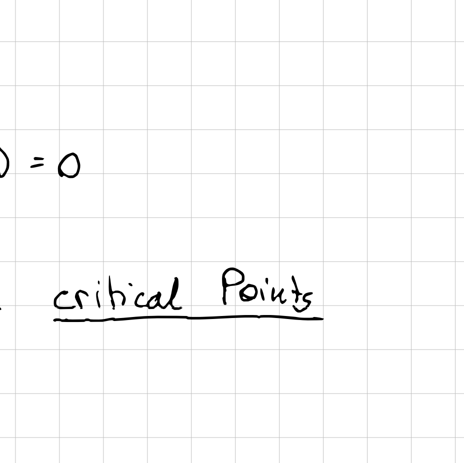
$f'' > 0$, $x \in (-2, 2)$ CONC

$f'(x) = 2x^3(3x^2 - 20)$

$0 = 2x^3(3x^2 - 20) \begin{cases} \sqrt{\frac{20}{3}} \\ -\sqrt{\frac{20}{3}} \\ 0 \end{cases}$

	$-\infty$		$-\sqrt{\frac{20}{3}}$		-2		0		2		$\sqrt{\frac{20}{3}}$		$+\infty$
f			MIN				MAX				MIN		
f'		$-$		$+$		$+$		$-$		$-$		$+$	
f''		$+$		$+$		$-$		$-$		$+$		$+$	
					INFL				INFL				

Plotting



Local MIN/MAX

Recall f has MIN/MAX x_0 and $f'(x_0)$ exists $\Rightarrow f'(x_0) = 0$

Points where $f(x)$ are called <u>critical Points</u>

Theorem 2

a) if $f'(x) = 0$ and $f''(x_0) < 0$ then f has local MAX at $(x_0, f(x_0))$

b) If $f'(x) = 0$ and $f''(x_0) > 0$ the f ha local MIN at $(x_0, f(x_0))$

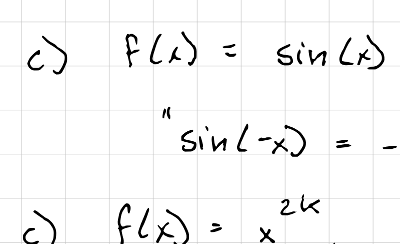
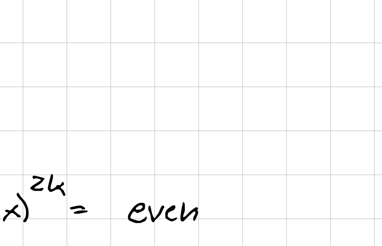
c) if $f'(x_0) = 0$ and $f''(x_0) = 0 \Rightarrow$ no conclusion.

→ 12.2 Even and Odd Function

Suppose for f, $-x \in dom(f)$ where $x \in dom(f)$

1) if f is even, $f(-x) = f(x)$

2) if f is odd, $f(-x) = -f(x)$

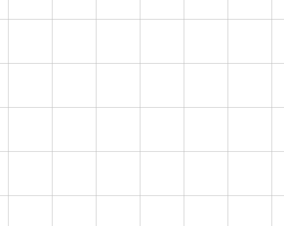
1)  b) 

ex.

a) $f(x) = x$ - odd

b) $f(x) = \cos(x)$ - even

"$\cos(-x) = -\cos(x)$"



c) $f(x) = \sin(x)$ - odd

"$\sin(-x) = -\sin(x)$"

c) $f(x) = x^{2k}$, $f(-x) = (-x)^{2k} =$ even

d) $f(x) = x^{2k+1}$, $f(-x) = (-x)^{2k+1} =$ odd

→ 12.3 Asymptotes

→ Vertical Asymptote (VA)

f has V.A in $X = a$ if either

a) $\lim_{x \to a^+} f(x) = \pm\infty$

b) $\lim_{x \to a^-} f(x) = \pm\infty$

ex $f(x) = \frac{1}{x^2 - x}$ $\quad dom(f) = \mathbb{R} \setminus \{0, 1\}$

VA 1 a) $\lim_{x \to 0^+} \frac{1}{x^2 - x} = -\infty$

VA 2 b) $\lim_{x \to 0^-} \frac{1}{x^2 - x} = +\infty$

VA 3 a) $\lim_{x \to 1^+} \frac{1}{x^2 - x} = -\infty$

VA 4 b) $\lim_{x \to 1^-} \frac{1}{x^2 - x} = +\infty$

→ Horizontal Asymptote (H.A)

f has H.A $y = L$ $(L \in \mathbb{R})$ if either

a) $\lim_{x \to +\infty} f(x) = L$

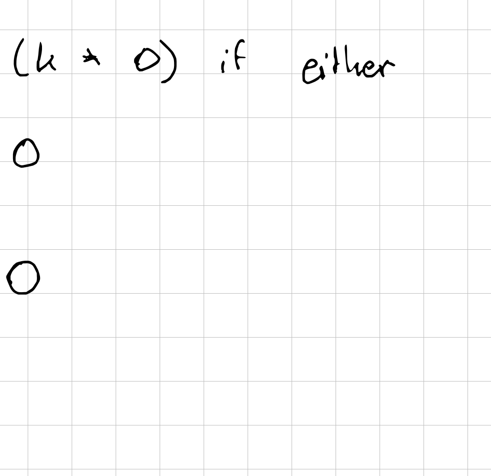
b) $\lim_{x \to -\infty} f(x) = L$

ex

$f(x) = \frac{1}{x^2 - x}$

a) $\lim_{x \to +\infty} \frac{1}{x^2 - x} = 0^+$

b) $\lim_{x \to -\infty} \frac{1}{x^2 - x} = 0^+$



→ Oblique Asymptote

f has an O.A $y = kx + n$ $(k \neq 0)$ if either

a) $\lim_{x \to +\infty} [f(x) - (kx + n)] = 0$

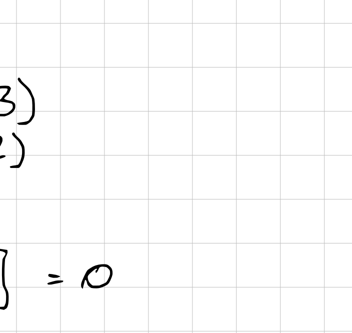
b) $\lim_{x \to -\infty} [f(x) - (kx + n)] = 0$

ex $f(x) = \frac{x^2 + 1}{x}$

We observe that $x > 0$, $f(x) = \frac{x^2+1}{x} = \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x}$ $\to 0$

$\lim_{x \to \infty} f(x) - x = (\cancel{x} + \underbrace{\frac{1}{x}}_{\to 0} - \cancel{x}) = 0$

$y = x$



→ More generally, rational functions $f(x) = \frac{P(x)}{Q(x)}$ where $\deg(P) = \deg(Q) + 1$ are having O.A.

ex

$f(x) = \frac{x^3}{x^2 + x + 1}$ $\quad \begin{matrix} \deg(3) \\ \deg(2) \end{matrix}$

$\lim_{x \to +\infty} \left[\frac{x^3}{x^2 + x + 1} - (kx + n)\right] = 0$

$\lim_{x \to +\infty} \left[\frac{x^3}{x^2 + x + 1} - \frac{(kx + n) \cdot (x^2 + x + 1)}{x^2 + x + 1}\right] = 0$

$\lim_{x \to +\infty} \left[\frac{x^3 - kx^3 - kx^2 - kx - nx^2 - nx - n}{x^2 + x + 1}\right] = 0$

group same power

$\lim_{x \to +\infty} \left[\frac{\underbrace{(1-k)}_{0} x^3 + \underbrace{(-k-n)}_{0} x^2 + (-k-n)x - n}{x^2 + x + 1}\right] = 0$

$\left.\begin{matrix} k = 1 \\ n = -1 \end{matrix}\right\} y = x - 1 = \text{O.A}$