Lecture 5 - 20/03

→ 5.1 Comparison test for positive series

$$\sum_{n=1}^{\infty} a_n \begin{cases} \text{sum } \alpha \text{ (conv.)} \\ \text{sum } \pm\infty \text{ (div.)} \end{cases} \qquad a_n \geq 0,\ n \in \mathbb{N}$$

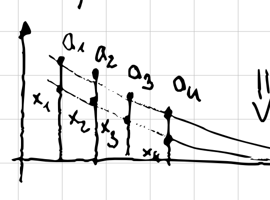
Theorem 1 (comparison test):

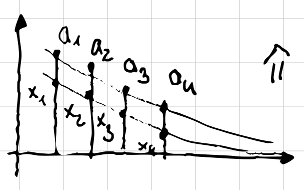
Let $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} a_n$ be two positive series, then

(i) if $x_n \leq a_n$ for $n \in \mathbb{N}$ $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is convergent $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ is convergent

(ii) if $x_n \geq a_n$ for $n \in \mathbb{N}$ $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ is divergent $\Rightarrow$ $\sum_{n=1}^{\infty} x_n$ is divergent

graphically

(i)  "push" from above

(ii)  "push" from below

(Proof in Notes!)

→ example 1

- Show that $\sum_{n=1}^{\infty} 1/n^2$ is convergent.

Recall $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$ → $\frac{1}{n^2} < \frac{1}{n(n+1)} \cdot 2 \quad \Big|\cdot n^2(n+1)$

$$n+1 < 2n$$

$$1 < n \quad \text{true for } n > 1$$

so: $\frac{1}{n^2} < \frac{2}{n(n+1)}$, $\forall n > 1$

Since $\sum \frac{2}{n(n+1)} = 2 \Rightarrow 2\sum\frac{1}{n(n+1)} = 2\cdot 1 = 1$,

so $\sum 1/n^2$ is convergent as well!

Question: does $\sum_{n=1}^{\infty} 1/n^\alpha$ $\alpha \in \mathbb{R}$ converge?

Theorem 2: Let $\alpha \in \mathbb{R}$, then

(i) if $\alpha > 1$, then series $\sum_{n=1}^{\infty} 1/n^\alpha$ converges

(ii) if $\alpha \leq 1$, then series $\sum_{n=1}^{\infty} 1/n^\alpha$ diverges

example

$\sum_{n=1}^{\infty} 1/\sqrt{n} = 1/n^{1/2}$ → DIV.

$\sum_{n=1}^{\infty} 1/n\sqrt{n} = 1/n^{3/2}$ → CONV.

$\sum_{n=1}^{\infty} 1/n^{1,000\ldots1}$ → CONV.

→ 5.2 Asymptotic Test

another example:

$\sum_{n=1}^{\infty} \frac{n+1}{n^3-n-1}$ → $\frac{n+1}{n^3-n-1}$ behaves like $\left.\begin{matrix} \frac{n}{n^3} & n \to +\infty \\ \text{''} & n \to +\infty \end{matrix}\right\} \sim \frac{n}{n^3} = \frac{1}{n^2}$

increasing / decreasing

$$\frac{n+1}{n^3-n-1} < \frac{2n}{n^3-n-1} < \frac{2n}{n^3-\frac{1}{2}n^3} = \frac{2n}{\frac{1}{2}n^3} = \frac{4}{n^2}$$

$\sum_{n=1}^{\infty} 4/n^2$ is convergent, since $\sum_{n=1}^{\infty} 1/n^2$ is convergent. By comparison test $\sum_{n=1}^{\infty} \frac{n+1}{n^3-n-1}$ is also convergent

another example: $\sum_{n=1}^{\infty} a_n$

$$a_n = \frac{n+1}{n^2+2n+4} \sim \frac{n}{n^2} = \frac{1}{n} \quad \text{DIV.}$$

decrease / increase

$$\frac{n+1}{n^2+2n+4} > \frac{n}{n^2+2n+4} > \frac{n}{3n^2} = \frac{1}{3}\cdot\frac{1}{n}$$

$$\text{(with } n^2 \leq n^2\text{)}$$

$\sum_{n=1}^{\infty} \frac{1}{3}\frac{1}{n} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$, from comparison test $\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n+4} \to +\infty$

Δ

Let a_n and x_n be two positive sequences, we say that x_n is behaving (asymptotically) similar to a_n

$$x_n \sim a_n$$

if $\lim \frac{x_n}{a_n} = k \quad k \in (a, +\infty)$

ex. 5 $\sqrt{n^2+1} \sim n$, obv. $n^2+1 \not\sim n$

Theorem 3 (Asymptotic Theorem):

Let $\sum a_n$ and $\sum x_n$ be a series of positive terms, assume that $x_n \sim a_n$, then:

(i) if $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} x_n$ converges

(ii) if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} x_n$ diverges

→ 5.3 Ratio Test

Theorem 4 (Ratio Test):

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series, then,

(i) if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, then the series CONV.

(ii) if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$, then the series DIV.

ex.

$\sum_{n=1}^{\infty} 1/n \qquad \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} = 1 \qquad$ DIV.

$\sum_{n=1}^{\infty} 1/n^2 \qquad \lim_{n\to\infty} \frac{1/n^2+1}{\frac{1}{n^2}} = \frac{n^2}{n+1} = 1 \qquad$ DIV.

$\sum_{n=1}^{\infty} \frac{x^n}{n!} \qquad \lim_{n\to\infty} \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1} \to 0 \qquad$ CONV.