Lecture 7 - 27/03

→ 7.1 Limits of functions

example:

$$g(x) = (1+x^2)^{1/x^2} \quad , \quad \text{at } x = 0$$

$\text{dom}(g) = \mathbb{R} \setminus \{0\}$ $\quad \rightarrow (1+0^2)^{1/0}$ ↯

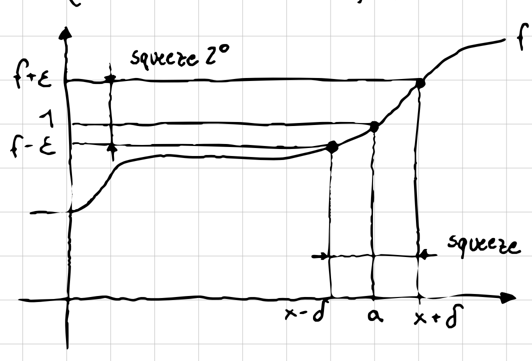
What happens when x approaches 0?

$$\lim_{x \to 0} g(x) = e$$

- definition of limit

$f: A \to \mathbb{R}$ $\quad a \in \mathbb{R}$, we say $\lim_{x \to 0} f(x) = 1$

if $(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \text{dom}(f) \setminus \{a\})\ (|x-a| < \delta \Rightarrow |f(x) - 1| < \varepsilon)$

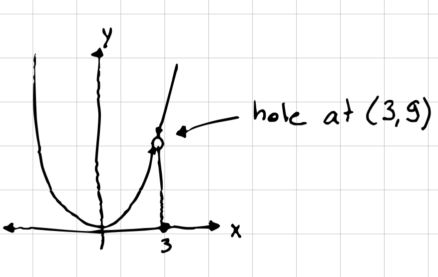


squeeze 1° : $x-\delta \to a \leftarrow x+\delta$

squeeze 1° ⇒ squeeze 2°

ex.

$$f(x) = \begin{cases} x^2 , & x \in \mathbb{R} \setminus \{3\} \\ 0 , & x = 3 \end{cases}$$



What is $\lim_{x \to 3} f(x) = 9$?

Proofe in Notes!

Theorem 1:

Let f and g be real functions, and $a \in \mathbb{R}$. Suppose the following:

$$\lim_{x \to a} f(x) = L \quad \text{and} \quad \lim_{x \to a} g(x) = M$$

(i) $\lim_{x \to a} |f(x)| = |L|$ $\qquad$ (ii) $\lim_{x \to a} (f \pm g)(x) = L \pm M$

(iii) $\lim_{x \to a} (f \cdot g)(x) = LM$ $\qquad$ (iv) $\lim_{x \to a} (f/g)(x) = L/M$

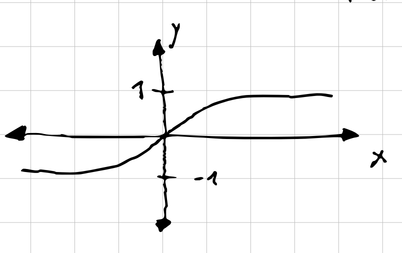
(v) $\lim_{x \to a} (k \cdot f)(x) = k \cdot L$ $\qquad$ (vi) $\lim_{x \to a} (f(x))^{m/n} = L^{m/n}, \quad L > 0$

→ 7.2 Limits of Infinity

example:

$$f(x) = \frac{x}{\sqrt{x^2+1}} \qquad \frac{2}{\sqrt{x^2}} \qquad \frac{2}{x}$$

$$\lim_{x \to +\infty} f(x) = 1 \qquad \lim_{x \to -\infty} f(x) = -1$$

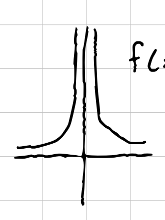


- definition of a limit $\pm\infty$: $\lim_{x \to \pm\infty} f(x) = L$, if

$(\forall \varepsilon > 0)(\exists M > 0)$ s.t. $(x > M \Rightarrow |f(x) - L| < \varepsilon)$
$x < -M$

similarly: $\lim_{x \to a} f(x) = +\infty$ if. $(\forall M > 0)(\exists \delta > 0)\ (|x-a| < \delta \Rightarrow f(x) > M)$
$< -M$

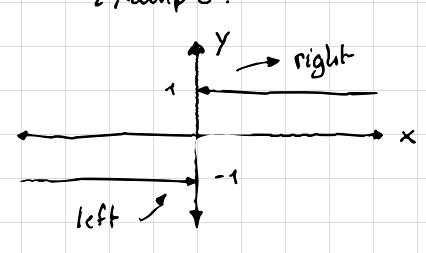
examples: $\lim_{x \to \pm\infty} 1/x^2 = 0 \qquad \lim_{x \to 0} 1/x^2 = +\infty$



$f(x) = 1/x^2$

→ 7.3 Left and right limit

example: $\quad f(x) = \text{sgn}(x) = x/|x| \quad , \quad \text{dom}(f) = \mathbb{R} \setminus \{0\}$



$$f(x) = \begin{cases} 1 , & x > 0 \\ -1 , & x < 0 \\ \text{und.} , & x = 0 \end{cases}$$

→ ~~definition~~ left limit : (from right side) $\lim_{x \to a+} f(x) = L$

→ right limit : (from left side) $\lim_{x \to a-} f(x) = L$

Theorem 2:

$$\lim_{x \to a} f(x) = L \quad \text{IFF} \quad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L$$

→ 7.4 Squeeze Theorem

Theorem 3 (Squeeze Theorem):

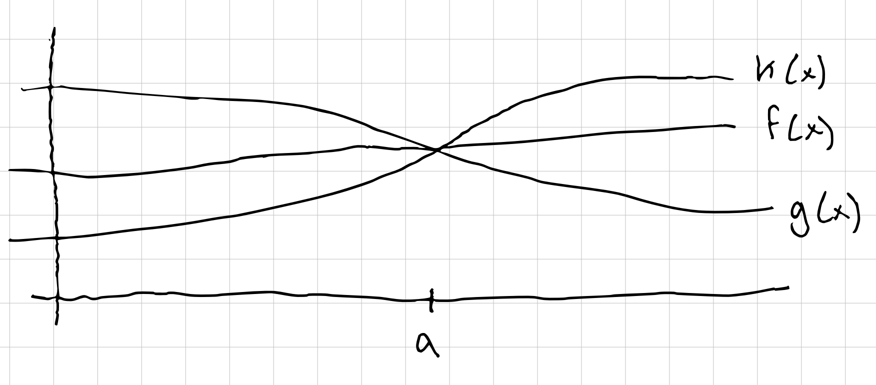
Let $f, g,$ be. real. functions defined on an open iterated ..
I that contains a but f, g, h do not have, to, be defined. Then, if

$$f(x) \leq g(x) \leq h(x)$$

$$\forall x \in I \setminus \{a\}$$

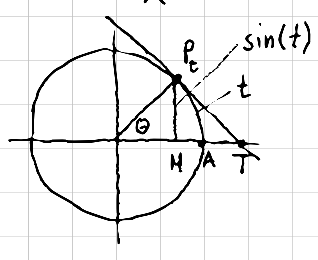
and if $\lim f(x) \leq \lim h(x) = L \qquad \lim g(x) = L$

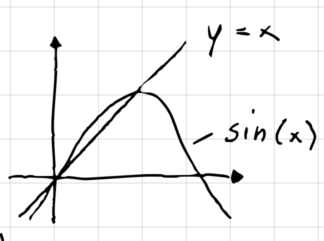
graphically



ex.

$\lim_{x \to 0} \frac{\sin(x)}{x} \quad \rightarrow \quad \text{dom}(f(x)) = \mathbb{R} \setminus \{0\}$





P_tT : tangent of the circle

$$P_tH < \text{arc}(\widehat{P_tA}) < P_tT$$

$\sin(t) < t < \tan(t) \left(= \frac{\sin(t)}{\cos(t)}\right) \qquad / : \sin(t)$

$1 < t/\sin(t) < \frac{1}{\cos(t)} \qquad /()^{-1}$

Squeeze Theorem ← $\quad 1 > \frac{\sin(t)}{t} > \cos(t)$

$$\lim_{t \to 0} 1 > \lim_{t \to 0} \frac{\sin(t)}{t} > \lim_{t \to 0} \cos(t)$$

goes to 0 $\qquad$ also goes to 0 $\qquad$ goes to 0

→ 7.5 Continuity

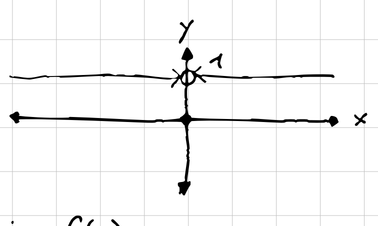
def: $\quad f: \mathbb{R} \to \mathbb{R}$ is continuous at

$c \in \mathbb{R}$, $\underline{c \in \text{dom}(f)}$ $\quad \lim_{x \to c} f(x) = f(c)$

Otherwise we say $f(x)$ is discontinuous

example:

$$f(x) = \begin{cases} 1 , & x \neq 0 \\ 0 , & x = 0 \end{cases}$$



$\lim_{x \to 0} f(x) = 1$ $\qquad$ is $f(x)$ CONT at $x = 0$?

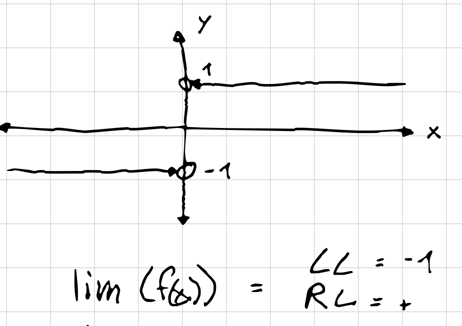
$f(0) = 0$ $\qquad$ → not CONT.!

example from exam:

$$f(x) = \begin{cases} \frac{x}{|x|} , & x \neq 0 \\ c , & x = 0 \end{cases}$$

1) Find c's that make f CONT.

$$f(x) = \begin{cases} -1 , & x < 0 \\ 1 , & x > 0 \\ c , & x = 0 \end{cases}$$



$\lim_{x \to 0} (f(x)) = \begin{matrix} LL = -1 \\ RL = + \end{matrix}$ $\qquad LL \neq RL$ ↯

no CL