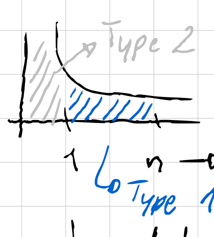
→ 17.1 Improper Integral TYPE 1

What if f is not bounded in $[a, b]$ or if , $a, b = \pm\infty$

i) Either $a = -\infty$ and/or $b = -\infty$

ex $f = 1/x^2$



ii) although f is not bouded at a or b eg $f = 1/x^2$ is not bouded at $[0, 1]$

→ Improper Integral of Type 2

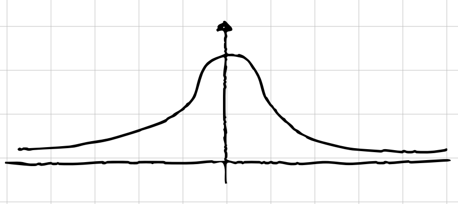
def Type 1:

Let f be CONT on $[a, +\infty]$, we define Type 1 Improper Integral as

$$\int_a^{+\infty} f(x)\,dx = \lim_{R \to +\infty} \int_a^R f(x)\,dx$$

If limit exists, we say that Integrals CONV, otherwise DIV

Integral $\int_{-\infty}^{+\infty} f(x)\,dx$



is defined if for some $a \in \mathbb{R}$ both $\int_{-\infty}^{a} f(x)dx$ and $\int_a^{+\infty} f(x)\,dx$ exists.

Observation

$\int_{-\infty}^{a} f(x)\,dx$ and $\int_a^{+\infty} f(x)\,dx$ is a stronger proof than $\lim_{R \to +\infty} \int_{-R}^{R} f(x)dx$

IMPORTANT EXAMPLE

Show that Integral $\int_a^{+\infty} 1/x^n\,dx$, $a > 0$, $n \in \mathbb{Z}$

$n \neq 1$

$$\int_a^{+\infty} 1/x^n\,dx = \lim_{k \to \infty} \int_a^k 1/x^n\,dx = \lim_{k \to \infty} \left[\frac{-x^{n+1}}{-n+1}\right]\Big|_a^k$$

$$= \lim_{k \to \infty} \frac{1}{-n+1}\left[k^{-n+1} - a^{-n+1}\right] = \lim_{k \to \infty} \frac{1}{n+1}\left[a^{-n+1} - k^{-n+1}\right]$$

$\boxed{n > 1}$:

eg. $n = 5$ $\quad k^{-5+1} = 1/k^4 \to k^{-n+1} \to 0$

$\to \frac{1}{n+1} \cdot a^{1-n}$ CONV

$\boxed{n < 1}$:

eg. $n = -5$ $\quad k^{5+1} = k^6 \to +\infty$ DIV

$\boxed{n = 1}$ $\quad \lim_{k \to \infty} \int 1/x\,dx = \lim_{k \to \infty} \ln(x)\Big|_a^{\infty}$

$$= \lim_{k \to \infty} \left[\ln|k| - \ln|a|\right] = +\infty \quad \text{DIV}$$

→ 17.2 Improper Integral of Type 2

definition Type 2:

f CONT on some $[a, b]$ and possibly unbounded near point a, we define the improper integral as follows.

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$$

Similarly if f is CONT on $[a, b]$ and possibly unbounded near b we define:

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

Type 2 Integrals can CONV, DIV to $\pm\infty$ or just DIV.

IMPORTANT EXAMPLE

$\int_0^a 1/x^p$ for which P des this CONV/DIV ?

$$\int_0^a 1/x^p = \lim_{c \to 0^+} \int_c^a x^{-p}\,dx$$

$$= \lim_{c \to 0^+} \frac{x^{-p+1}}{-p+1}\Big|_a^c$$

$$= \lim_{c \to 0^+} \frac{1}{-p+1}\left[a^{-p+1} - c^{-p+1}\right] = \lim_{c \to 0^+} \frac{1}{p-1}\left[c^{-p-1} - a^{-p+1}\right] = \otimes$$

$\boxed{p > 1}$ eg. $p = 5$ $\quad c^{-5-1} = 1/c^6$

$c^{-p-1} \xrightarrow[c \to 0^+]{} +\infty$, $\otimes = +\infty$ → DIV

$\boxed{p < 1}$ eg. $p = -5$ $\quad c^{5-1} = c^4$

$c \to 0$, $\otimes = \frac{1}{p-1}(-a^{-p+1}) = \frac{a^{1-p}}{1-p} \to$ CONV

Similar to ex 1, we can show case $p = 1$ Δ

17.3 Comparison Test For improper Integrals

Sometimes we can test CONV of an Integral, even if we cant compute the actual Integral

Theorem 1:

Let a, b be either two numbers in $\mathbb{R}$ s.t. $a < b$ or $a = -\infty$ or $b = +\infty$.

Let, f, g be two CONT functions on a, b s.t. $0 \leq f(x) < g(x)$ $x \in (a, b)$. Then, if $\int_a^b g(x)\,dx$ CONV, so does $\int_a^b f(x)\,dx$ CONV, and

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

Equivalently if $\int_a^b f(x)$ DIV → $\int_a^b g(x)\,dx$ DIV

$\boxed{\text{Example}}$: show that $\int_0^{+\infty} e^{-x^2}\,dx$ CONV and find some upper-bound

- We observe $\frac{1}{e^{x^2}} < \frac{1}{e^x}$ for $x \geq 1$
- for $0 \leq x \leq 1$ → $1/e^{x^2} \leq 1$

$$\int_0^{+\infty} e^{-x^2} = \int_0^1 e^{-x^2} + \int_1^{\infty} e^{-x^2} \leq \int_0^1 1\,dx + \int_1^{\infty} e^{-x}\,dx =$$

$$= 1 + \lim_{R \to \infty} \int_1^R e^{-x}\,dx$$

$$= 1 + \lim_{R \to \infty} \left| e^{-x} \right|_1^R$$

$$= 1 + \lim_{R \to \infty} (-\overset{\to 0}{e^{-R}} + e^{-1})$$

$$= 1 + e^{-1} = \underline{\underline{1 + 1/e}}$$

ex 5:

$\int_0^{+\infty} \frac{dx}{\sqrt{x + x^3}}$ → not defined in $x = 0$ and $x = +\infty$
→ it is a combination of Type 1 & 2

$$= \underbrace{\int_0^1 \frac{dx}{\sqrt{x + x^3}}}_{I_1} + \underbrace{\int_1^{\infty} \frac{dx}{\sqrt{x + x^3}}}_{I_2}$$

I_2:

$$I_2 \leq \int_1^{\infty} x^{-3/2}\,dx = \lim_{R \to \infty} -2x^{-1/2}\Big|_1^R$$

$$\lim_{R \to \infty} (-2\overset{\to 0}{R^{1/2}} + 2) = 2$$

$$\frac{1}{\sqrt{x + x^3}} \nleq \frac{1}{x^{3/2}} \to \frac{1}{\sqrt{x + x^3}} \leq \frac{1}{\sqrt{x}} \qquad x \in [0, 1]$$

I_1:

$$I_1 \leq \int_0^1 1/\sqrt{x}\,dx = \lim_{c \to 0^+} \int_c^1 x^{-1/2}\,dx$$

$$= \left[(\,)' = x^{1/2}\right] \quad 2x^{1/2} = 2\tfrac{1}{2} \cdot x^{1/2 - 1}$$

$$= \lim_{c \to 0^+} 2\left[x^{1/2}\right]\Big|_c^1 = 2 - 0 = 2$$

All together, $\int_0^{+\infty} \frac{1}{\sqrt{x + x^3}}\,dx \leq 2 + 2 = 4$