Section 1.1 : Systems of Linear Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Linear Equations

Matrices

Flat Geometry

Section 1.1 Systems of Linear Equations

Topics

We will cover these topics in this section.

1. Systems of Linear Equations
2. Matrix Notation
3. Elementary Row Operations
4. Questions of Existence and Uniqueness of Solutions

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of solutions, and whether the system is consistent or inconsistent.
2. Apply elementary row operations to solve linear systems of equations.
3. Express a set of linear equations as an augmented matrix.

A Single Linear Equation

Nonlinear

- $x_1^2 + 2 = 0$
- $x_1 \cdot x_2 + x_3 = 0$

A linear equation has the form

$$a_1x_1^1 + a_2x_2^1 + \cdots + a_nx_n^1 = b$$

$a_1, \ldots, a_n$ and b are the **coefficients**, $x_1, \ldots, x_n$ are the **variables** or **unknowns**, and n is the **dimension**, or number of variables.

For example,

- $2x_1 + 4x_2 = 4$ is a line in two dimensions
- $3x_1 + 2x_2 + 1x_3 = 6$ is a plane in three dimensions

2, 4, 4 : Coefficients

x_1, x_2 : Variables

dimension = 2

dimension = 3

Collection of Equations

Systems of Linear Equations = Linear System.

When we have more than one linear equation, we have a **linear system** of equations. For example, a linear system with two equations is

$$\begin{cases} x_1 + 1.5x_2 + \pi x_3 = 4 \\ 5x_1 + 0 \cdot x_2 + 7x_3 = 5 \end{cases}$$

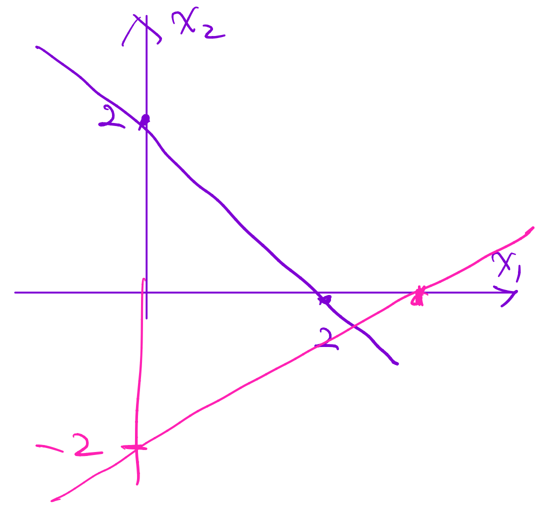
2 equations
3 variable.

Definition: Solution to a Linear System

The set of all possible values of $x_1, x_2, \ldots x_n$ that satisfy all equations is the **solution** to the system.

A system can have a unique solution, no solution, or an infinite number of solutions.

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 - 3x_2 = 6 \end{cases}$$



Solution of system

= "Intersection"

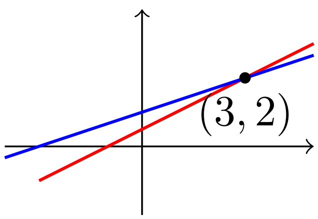
2 Eqns = Intersection of 2 lines

2 Variables = Straight line

Two Variables

Consider the following systems. How are they different from each other?

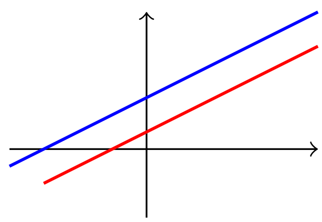
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$



non-parallel lines

1 solution

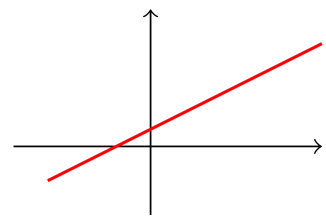
$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \end{aligned}$$



parallel lines

No Solution

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 1 \end{aligned}$$

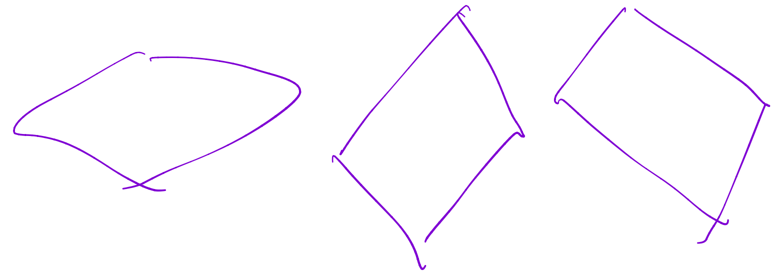


identical lines

Infinitely many Solutions

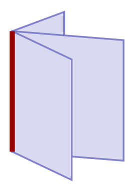
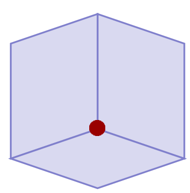
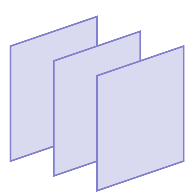
1 Eqn, 3 Variables $\Rightarrow$ plane

3 Eqn, 3 Variables

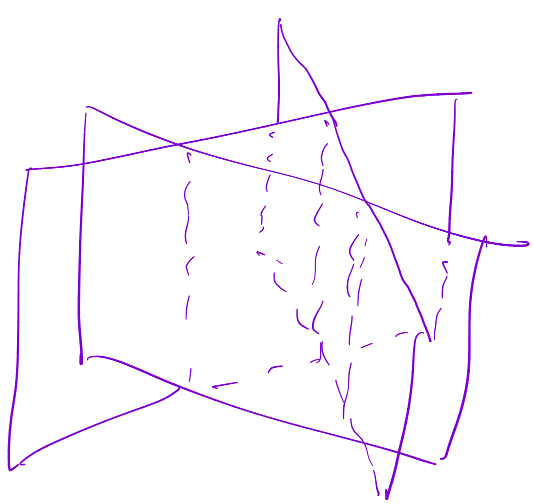


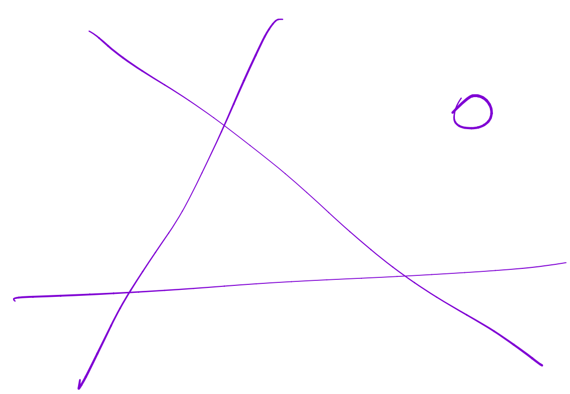
Three-Dimensional Case

An equation $a_1x_1 + a_2x_2 + a_3x_3 = b$ defines a plane in $\mathbb{R}^3$. The **solution** to a system of **three equations** is the set of intersections of the planes.

solution set	sketch	number of solutions
line		∞
point		1
empty		0

Question





Row Reduction by Elementary Row Operations

How can we find the solution set to a set of linear equations?
We can manipulate equations in a linear system using **row operations**.

1. (Replacement/Addition) Add a multiple of one row to another.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply a row by a non-zero scalar.

Let's use these operations to solve a system of equations.

$$\begin{cases} x_1 + x_2 = 2 \\ 2x_1 - 3x_2 = 6 \end{cases}$$

Using these

"Reduce # of variables"

Example 1

Identify the solution to the linear system.

$$\begin{array}{cccc} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

Augmented Matrices

It is redundant to write x_1, x_2, x_3 again and again, so we rewrite systems using matrices. For example,

$$\begin{array}{cccc} x_1 & -2x_2 & +x_3 & = 0 \\ & 2x_2 & -8x_3 & = 8 \\ 5x_1 & & -5x_3 & = 10 \end{array}$$

can be written as the **augmented matrix**,

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

The vertical line reminds us that the first three columns are the coefficients to our variables x_1, x_2, and x_3.

Consistent Systems and Row Equivalence

Definition (Consistent)

A linear system is **consistent** if it has at least one ___________.

Definition (Row Equivalence)

Two matrices are **row equivalent** if a sequence of ___________

______________________ transforms one matrix into the other.

Note: if the augmented matrices of two linear systems are row equivalent, then they have the same solution set.

Fundamental Questions

Two questions that we will revisit many times throughout our course.

1. Does a given linear system have a solution? In other words, is it consistent?
2. If it is consistent, is the solution unique?

Section 1.2 : Row Reduction and Echelon Forms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

Section 1.2 : Row Reductions and Echelon Forms

Topics
We will cover these topics in this section.

1. Row reduction algorithm
2. Pivots, and basic and free variables
3. Echelon forms, existence and uniqueness

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a linear system in terms of the number of leading entries, free variables, pivots, pivot columns, pivot positions.
2. Apply the row reduction algorithm to reduce a linear system to echelon form, or reduced echelon form.
3. Apply the row reduction algorithm to compute the coefficients of a polynomial.

Definition: Echelon Form and RREF

A rectangular matrix is in **echelon form** if

1. All zero rows (if any are present) are at the bottom.
2. The first non-zero entry (or **leading entry**) of a row is to the right of any leading entries in the row above it (if any).
3. All elements below a leading entry (if any) are zero.

A matrix in echelon form is in **reduced row echelon form** (RREF) if

1. All leading entries, if any, are equal to 1.
2. Leading entries are the only nonzero entry in their respective column.

Example of a Matrix in Echelon Form

$\blacksquare$ = non-zero number, $\quad * =$ any number

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1

Which of the following are in RREF?

$a)\quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$b)\quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$c)\quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$d)\quad \begin{bmatrix} 0 & 6 & 3 & 0 \end{bmatrix}$

$e)\quad \begin{bmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Definition: Pivot Position, Pivot Column

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A.

A **pivot column** is a column of A that contains a pivot position.

Example 2: Express the matrix in reduced row echelon form and identify the pivot columns.

$$\begin{bmatrix} 0 & -3 & -6 & 4 \\ -1 & -2 & -1 & 3 \\ -2 & -3 & 0 & 3 \end{bmatrix}$$

Row Reduction Algorithm

The algorithm we used in the previous example produces a matrix in RREF. Its steps can be stated as follows.

Step 1a Swap the 1st row with a lower one so the leftmost nonzero entry is in the 1st row

Step 1b Scale the 1st row so that its leading entry is equal to 1

Step 1c Use row replacement so all entries below this 1 are 0

Step 2a Swap the 2nd row with a lower one so that the leftmost nonzero entry below 1st row is in the 2nd row

etc. . . .

Now the matrix is in echelon form, with leading entries equal to 1.

Last step Use row replacement so all entries above each leading entry are 0, starting from the right.

Basic And Free Variables

Consider the augmented matrix

$$\left[A \mid \vec{b} \right] = \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 7 & 0 & 4 \\ 0 & 0 & 1 & 4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]$$

The leading one's are in first, third, and fifth columns. So:

- the pivot variables of the system $A\vec{x} = \vec{b}$ are x_1, x_3, and x_5.
- The free variables are x_2 and x_4. **Any choice** of the free variables leads to a solution of the system.

Note that A does not have basic variables or free variables. Systems have variables.

Existence and Uniqueness

Theorem

A linear system is consistent if and only if (exactly when) the last column of the **augmented** matrix does not have a pivot. This is the same as saying that the RREF of the augmented matrix does **not** have a row of the form

$$\left(\begin{array}{ccccc|c} 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

Moreover, if a linear system is consistent, then it has

1. a unique solution if and only if there are no free variables.
2. infinitely many solutions that are parameterized by free variables.

Section 1.3 : Vector Equations

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.3: Vector Equations

Topics

We will cover these topics in this section.

1. Vectors in $\mathbb{R}^n$, and their basic properties
2. Linear combinations of vectors

Objectives

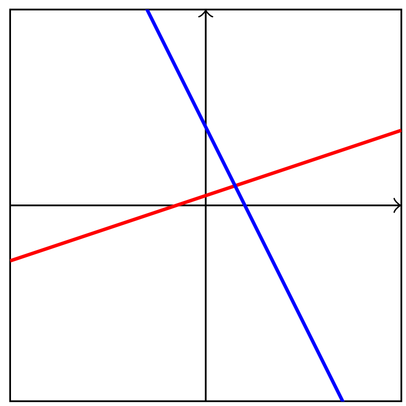
For the topics covered in this section, students are expected to be able to do the following.

1. Apply geometric and algebraic properties of vectors in $\mathbb{R}^n$ to compute vector additions and scalar multiplications.
2. Characterize a set of vectors in terms of **linear combinations**, their **span**, and how they are related to each other geometrically.

Motivation

We want to think about the **algebra** in linear algebra (systems of equations and their solution sets) in terms of **geometry** (points, lines, planes, etc).

$$\color{red}{x - 3y = -3}$$
$$\color{blue}{2x + y = 8}$$



- This will give us better insight into the properties of systems of equations and their solution sets.
- To do this, we need to introduce n-dimensional space $\mathbb{R}^n$, and **vectors** inside it.

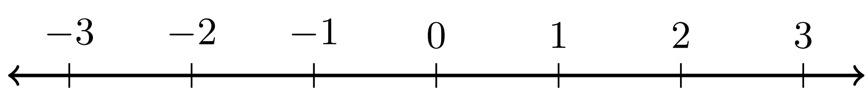
$\mathbb{R}^n$

Recall that $\mathbb{R}$ denotes the collection of all real numbers.

Let n be a positive whole number. We define

$$\mathbb{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

When $n = 1$, we get $\mathbb{R}$ back: $\mathbb{R}^1 = \mathbb{R}$. Geometrically, this is the **number line**.

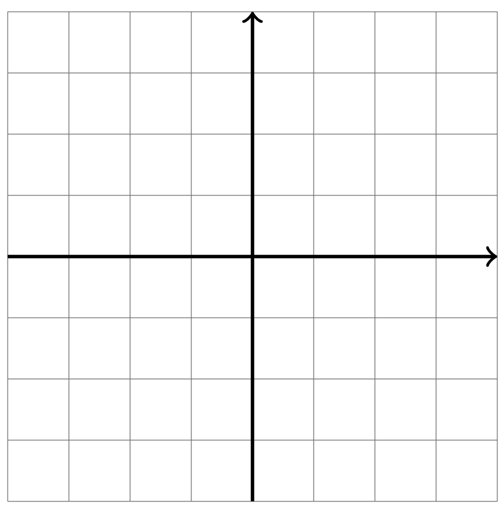


$\mathbb{R}^2$

Note that:

- when $n = 2$, we can think of $\mathbb{R}^2$ as a **plane**
- every point in this plane can be represented by an ordered pair of real numbers, its x- and y-coordinates

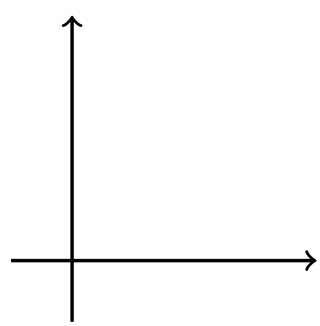
Example: Sketch the point $(3, 2)$ and the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



Vectors

In the previous slides, we were thinking of elements of $\mathbb{R}^n$ as **points**: in the line, plane, space, etc.

We can also think of them as **vectors**: arrows with a given length and direction.



For example, the vector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ points **horizontally** in the amount of its x-coordinate, and **vertically** in the amount of its y-coordinate.

Vector Algebra

When we think of an element of $\mathbb{R}^n$ as a vector, we write it as a matrix with n rows and one column:

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Suppose

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Vectors have the following properties.

1. **Scalar Multiple**:

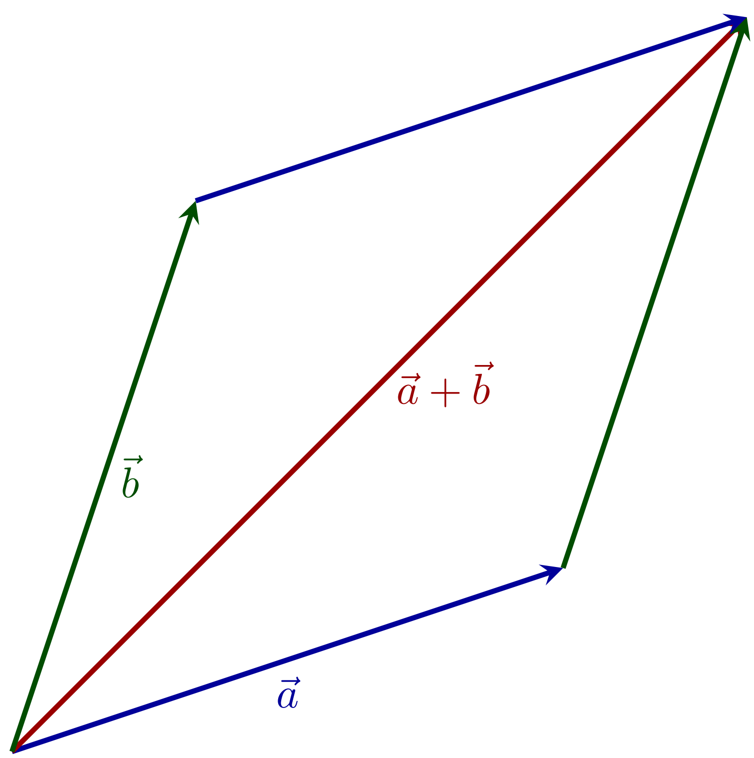
$$c\vec{u} =$$

2. **Vector Addition**:

$$\vec{u} + \vec{v} =$$

Note that vectors in higher dimensions have the same properties.

Parallelogram Rule for Vector Addition



Linear Combinations and Span

Definition

1. Given vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p \in \mathbb{R}^n$, and scalars $c_1, c_2, \dots, c_p$, the vector below

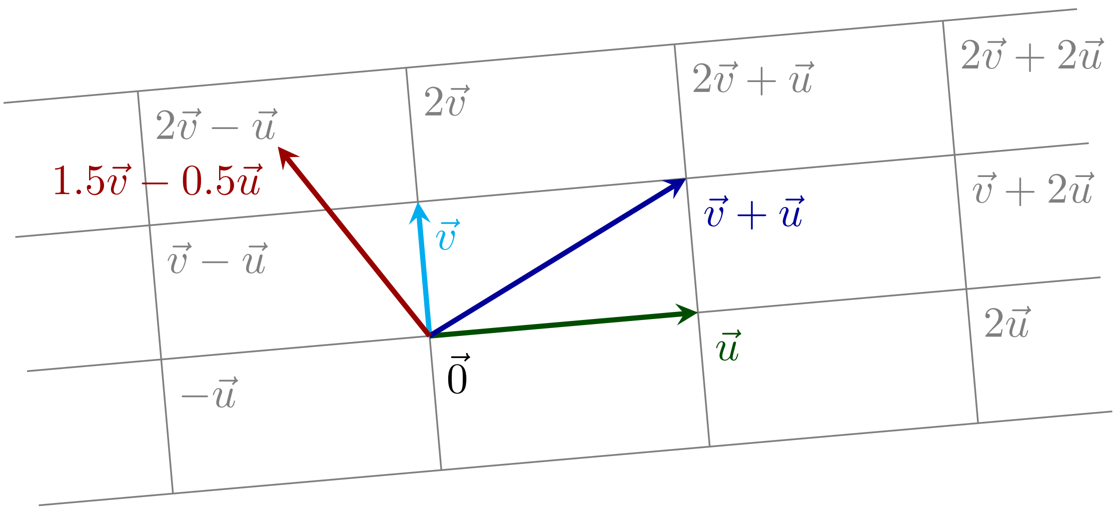
$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

   is called a **linear combination of** $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ **with weights** $c_1, c_2, \dots, c_p$**.**

2. The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$ is called the **Span** of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$.

Geometric Interpretation of Linear Combinations

Note that any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other, span $\mathbb{R}^2$. In other words, any vector in $\mathbb{R}^2$ can be represented as a linear combination of two vectors that are not multiples of each other.



Example

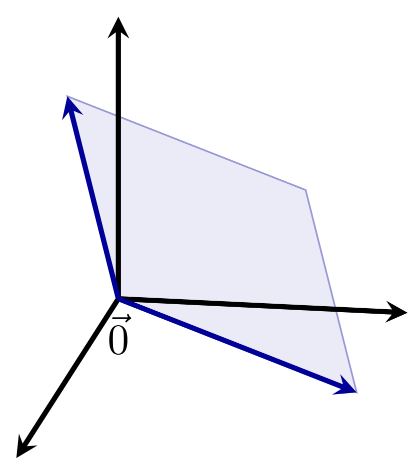
Is $\vec{y}$ in the span of vectors $\vec{v}_1$ and $\vec{v}_2$?

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \text{ and } \vec{y} = \begin{pmatrix} 7 \\ 4 \\ 15 \end{pmatrix}.$$

The Span of Two Vectors in $\mathbb{R}^3$

In the previous example, did we find that $\vec{y}$ is in the span of $\vec{v}_1$ and $\vec{v}_2$?

In general: Any two non-parallel vectors in $\mathbb{R}^3$ span a plane that passes through the origin. Any vector in that plane is also in the span of the two vectors.



Section 1.4 : The Matrix Equation

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

"Mathematics is the art of giving the same name to different things."
- H. Poincaré

In this section we introduce another way of expressing a linear system that we will use throughout this course.

1.4 : Matrix Equation $A\vec{x} = \vec{b}$

Topics

We will cover these topics in this section.

1. Matrix notation for systems of equations.
2. The matrix product $A\vec{x}$.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Compute matrix-vector products.
2. Express linear systems as vector equations and matrix equations.
3. Characterize linear systems and sets of vectors using the concepts of span, linear combinations, and pivots.

Notation

symbol	**meaning**
$\in$	belongs to
$\mathbb{R}^n$	the set of vectors with n real-valued elements
$\mathbb{R}^{m \times n}$	the set of real-valued matrices with m rows and n columns

Example: the notation $\vec{x} \in \mathbb{R}^5$ means that $\vec{x}$ is a vector with five real-valued elements.

Linear Combinations

Definition

A is a $m \times n$ matrix with columns $\vec{a}_1, \ldots, \vec{a}_n$ and $x \in \mathbb{R}^n$, then the **matrix vector product** $A\vec{x}$ is a linear combination of the columns of A:

$$A\vec{x} = \begin{bmatrix} | & | & \cdots & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & \cdots & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Note that $A\vec{x}$ is in the span of the columns of A.

Example

The following product can be written as a linear combination of vectors:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} =$$

Solution Sets

Theorem

If A is a $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$, and $x \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^m$, then the solutions to

$$A\vec{x} = \vec{b}$$

has the same set of solutions as the vector equation

$$x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$$

which as the same set of solutions as the set of linear equations with the augmented matrix

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Existence of Solutions

Theorem

The equation $A\vec{x} = \vec{b}$ has a solution if and only if $\vec{b}$ is a linear combination of the columns of A.

Example

For what vectors $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ does the equation have a solution?

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 8 & 4 \\ 0 & 1 & -2 \end{pmatrix} \vec{x} = \vec{b}$$

The Row Vector Rule for Computing $A\vec{x}$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad\qquad \\ \\ \end{bmatrix}$$

Summary

We now have four **equivalent** ways of expressing linear systems.

1. A system of equations:

$$\begin{aligned} 2x_1 + 3x_2 &= 7 \\ x_1 - x_2 &= 5 \end{aligned}$$

2. An augmented matrix:

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 5 \end{array}\right]$$

3. A vector equation:

$$x_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

4. As a matrix equation:

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Each representation gives us a different way to think about linear systems.

Section 1.5 : Solution Sets of Linear Systems

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.5 : Solution Sets of Linear Systems

Topics

We will cover these topics in this section.

1. Homogeneous systems
2. Parametric **vector** forms of solutions to linear systems

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Express the solution set of a linear system in parametric vector form.
2. Provide a geometric interpretation to the solution set of a linear system.
3. Characterize homogeneous linear systems using the concepts of free variables, span, pivots, linear combinations, and echelon forms.

Homogeneous Systems

Definition

Linear systems of the form __________________ are **homogeneous**.

Linear systems of the form __________________ are **inhomogeneous**.

Because homogeneous systems always have the **trivial solution**, $\vec{x} = \vec{0}$, the interesting question is whether they have __________________ solutions.

Observation

$A\vec{x} = \vec{0}$ has a nontrivial solution

$\iff$ there is a free variable

$\iff$ A has a column with no pivot.

Example: a Homogeneous System

Identify the free variables, and the solution set, of the system.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 - 5x_3 &= 0 \\ x_1 - 2x_3 &= 0 \end{aligned}$$

Parametric Forms, Homogeneous Case

In the example on the previous slide we expressed the solution to a system using a vector equation. This is a **parametric form** of the solution.

In general, suppose the free variables for $A\vec{x} = \vec{0}$ are $x_k, \ldots, x_n$. Then all solutions to $A\vec{x} = \vec{0}$ can be written as

$$\vec{x} = x_k\vec{v}_k + x_{k+1}\vec{v}_{k+1} + \cdots + x_n\vec{v}_n$$

for some $\vec{v}_k, \ldots, \vec{v}_n$. This is the **parametric form** of the solution.

Example 2 (non-homogeneous system)

Write the parametric vector form of the solution, and give a geometric interpretation of the solution.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ 2x_1 - x_2 - 5x_3 &= 11 \\ x_1 - 2x_3 &= 6 \end{aligned}$$

(Note that the left-hand side is the same as Example 1).

Section 1.7 : Linear Independence

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.7 : Linear Independence

Topics

We will cover these topics in this section.

- Linear independence
- Geometric interpretation of linearly independent vectors

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Characterize a set of vectors and linear systems using the concept of linear independence.
2. Construct dependence relations between linearly dependent vectors.

Motivating Question

What is the smallest number of vectors needed in a parametric solution to a linear system?

Linear Independence

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ in $\mathbb{R}^n$ are **linearly independent** if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

has only the **trivial** solution. It is **linearly dependent** otherwise.

In other words, $\{\vec{v}_1, \dots, \vec{v}_k\}$ are linearly dependent if there are real numbers $c_1, c_2, \dots, c_k$ **not all zero** so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

Consider the vectors:

$$\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$$

To determine whether the vectors are linearly independent, we can set the linear combination to the zero vector:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = V\vec{c} \stackrel{??}{=} \vec{0}$$

Linear independence: There is NO non-zero solution $\vec{c}$

Linear dependence: There is a non-zero solution $\vec{c}$.

Example 1

For what values of h are the vectors linearly independent?

$$\begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}, \begin{bmatrix} 1 \\ h \\ 1 \end{bmatrix}, \begin{bmatrix} h \\ 1 \\ 1 \end{bmatrix}$$

Example 2 (One Vector)

Suppose $\vec{v} \in \mathbb{R}^n$. When is the set $\{\vec{v}\}$ linearly dependent?

Example 3 (Two Vectors)

Suppose $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$. When is the set $\{\vec{v}_1, \vec{v}_2\}$ linearly dependent? Provide a geometric interpretation.

Two Theorems

Fact 1. Suppose $\vec{v}_1, \ldots, \vec{v}_k$ are vectors in $\mathbb{R}^n$. If $k > n$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

Fact 2. If any one or more of $\vec{v}_1, \ldots, \vec{v}_k$ is $\vec{0}$, then $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is linearly dependent.

Section 1.8 : An Introduction to Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra

1.8 : An Introduction to Linear Transforms

Topics

We will cover these topics in this section.

1. The definition of a linear transformation.
2. The interpretation of matrix multiplication as a linear transformation.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct and interpret linear transformations in $\mathbb{R}^n$ (for example, interpret a linear transform as a projection, or as a shear).
2. Characterize linear transforms using the concepts of
   - existence and uniqueness
   - domain, co-domain and range

From Matrices to Functions

Let A be an $m \times n$ matrix. We define a function

$$T : \mathbb{R}^n \to \mathbb{R}^m, \quad T(\vec{v}) = A\vec{v}$$

This is called a **matrix transformation**.

- The **domain** of T is $\mathbb{R}^n$.
- The **co-domain** or **target** of T is $\mathbb{R}^m$.
- The vector $T(\vec{x})$ is the **image** of $\vec{x}$ under T
- The set of all possible images $T(\vec{x})$ is the **range**.

This gives us **another** interpretation of $A\vec{x} = \vec{b}$:

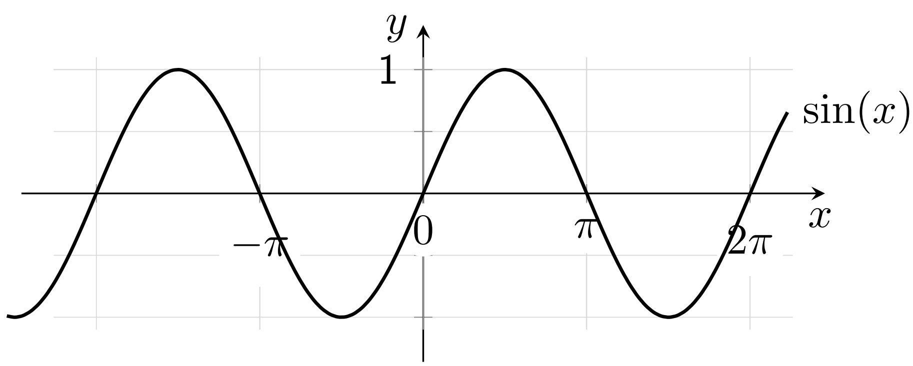
- set of equations
- augmented matrix
- matrix equation
- vector equation
- linear transformation equation

Functions from Calculus

Many of the functions we know have **domain** and **codomain** $\mathbb{R}$.We can express the **rule** that defines the function $\sin$ this way:

$$f\colon \mathbb{R} \to \mathbb{R} \qquad f(x) = \sin(x)$$

In calculus we often think of a function in terms of its graph, whose horizontal axis is the **domain**, and the vertical axis is the **codomain**.



This is ok when the domain and codomain are $\mathbb{R}$. It's hard to do when the domain is $\mathbb{R}^2$ and the codomain is $\mathbb{R}^3$. We would need five dimensions to draw that graph.

Example 1

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 7 \\ 5 \\ 7 \end{bmatrix}$.

a) Compute $T(\vec{u})$.

b) Calculate $\vec{v} \in \mathbb{R}^2$ so that $T(\vec{v}) = \vec{b}$

c) Give a $\vec{c} \in \mathbb{R}^3$ so there is no $\vec{v}$ with $T(\vec{v}) = \vec{c}$

or: Give a $\vec{c}$ that is not in the range of T.

or: Give a $\vec{c}$ that is not in the span of the columns of A.

Linear Transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.
- $T(c\vec{v}) = cT(\vec{v})$ for all $\vec{v} \in \mathbb{R}^n$, and c in $\mathbb{R}$.

So if T is linear, then

$$T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = c_1T(\vec{v}_1) + \cdots + c_kT(\vec{v}_k)$$

This is called the **principle of superposition**. The idea is that if we know $T(\vec{e}_1), \ldots, T(\vec{e}_n)$, then we know every $T(\vec{v})$.

Fact: Every matrix transformation T_A is linear.

Example 2

Suppose T is the linear transformation $T(\vec{x}) = A\vec{x}$. Give a short geometric interpretation of what $T(\vec{x})$ does to vectors in $\mathbb{R}^2$.

1) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

3) $A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for $k \in \mathbb{R}$

Example 3

What does T_A do to vectors in $\mathbb{R}^3$?

a) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 4

A linear transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ satisfies

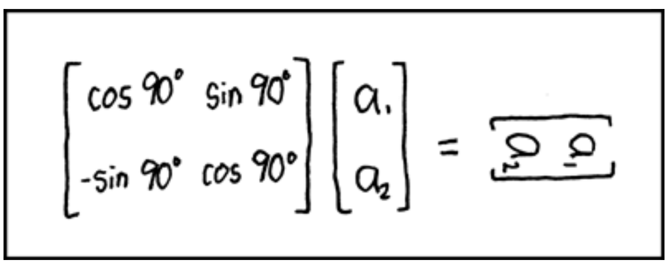
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}5\\-7\\2\end{bmatrix}, \qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-3\\8\\0\end{bmatrix}$$

What is the matrix that represents T?

Section 1.9 : Linear Transforms

Chapter 1 : Linear Equations

Math 1554 Linear Algebra



https://xkcd.com/184

1.9 : Matrix of a Linear Transformation

Topics

We will cover these topics in this section.

1. The **standard vectors** and the **standard matrix**.
2. Two and three dimensional transformations in more detail.
3. **Onto** and **one-to-one** transformations.

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Identify and construct linear transformations of a matrix.
2. Characterize linear transformations as onto and/or one-to-one.
3. Solve linear systems represented as linear transforms.
4. Express linear transforms in other forms, such as as matrix equations or as vector equations.

Definition: The Standard Vectors

The **standard vectors** in $\mathbb{R}^n$ are the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where:

$$\vec{e}_1 = \qquad \vec{e}_2 = \qquad \vec{e}_n =$$

For example, in $\mathbb{R}^3$,

$$\vec{e}_1 = \qquad \vec{e}_2 = \qquad \vec{e}_3 =$$

A Property of the Standard Vectors

Note: if A is an $m \times n$ matrix with columns $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, then

$$A\vec{e}_i = \vec{v}_i, \text{ for } i = 1, 2, \ldots, n$$

So multiplying a matrix by $\vec{e}_i$ gives column i of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \vec{e}_2 =$$

The Standard Matrix

Theorem

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there is a unique matrix A such that

$$T(\vec{x}) = A\vec{x}, \qquad \vec{x} \in \mathbb{R}^n.$$

In fact, A is a $m \times n$, and its j^{th} column is the vector $T(\vec{e}_j)$.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

The matrix A is the **standard matrix** for a linear transformation.

Rotations

Example 1

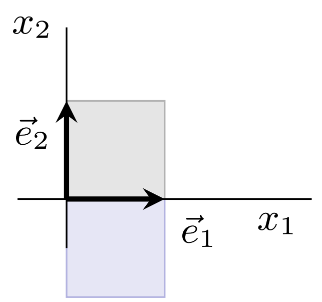
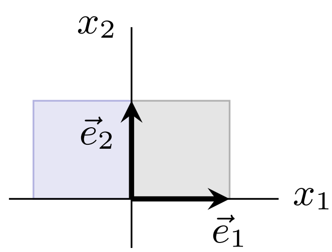
What is the linear transform $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\vec{x}) = \vec{x} \text{ rotated counterclockwise by angle } \theta?$$

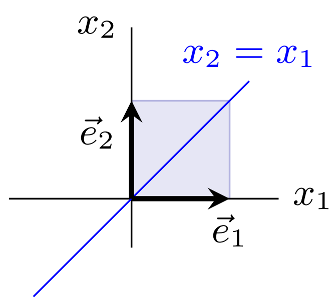
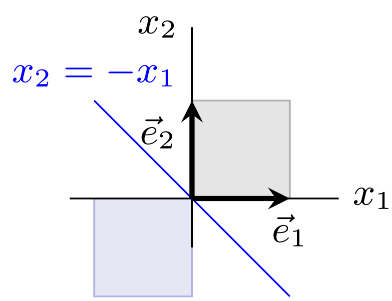
Standard Matrices in $\mathbb{R}^2$

- There is a long list of geometric transformations of $\mathbb{R}^2$ in our textbook, as well as on the next few slides (reflections, rotations, contractions and expansions, shears, projections, . . .)
- Please familiarize yourself with them: you are expected to memorize them (or be able to derive them)

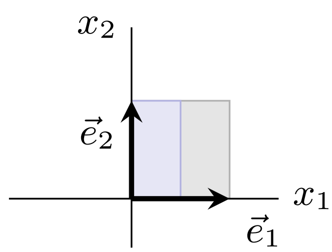
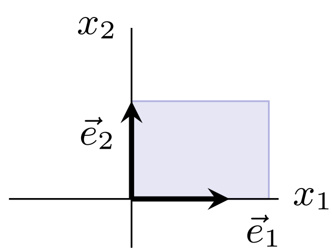
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
reflection through x_2-axis		$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

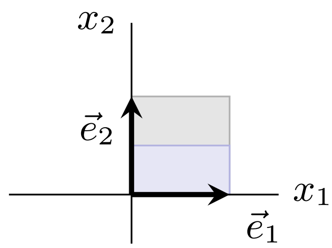
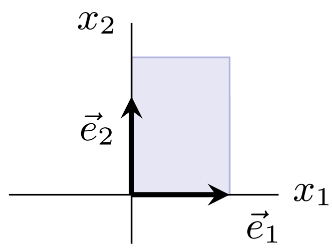
Two Dimensional Examples: Reflections

transformation	image of unit square	standard matrix
reflection through $x_2 = x_1$		$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
reflection through $x_2 = -x_1$		$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

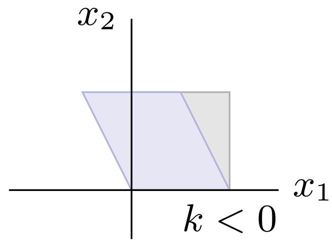
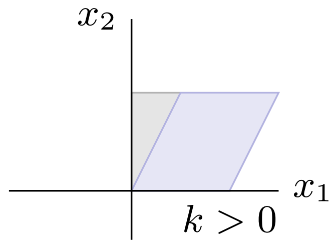
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Horizontal Contraction		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$. $\lvert k\rvert < 1$
Horizontal Expansion		$\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$, $k > 1$

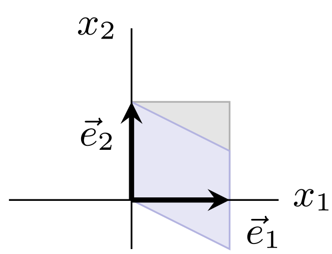
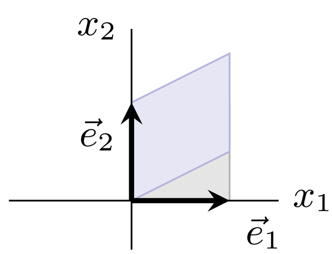
Two Dimensional Examples: Contractions and Expansions

transformation	**image of unit square**	**standard matrix**
Vertical Contraction		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $\lvert k \rvert < 1$
Vertical Expansion		$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k > 1$

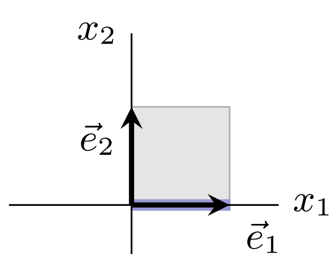
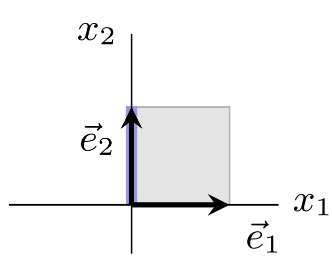
Two Dimensional Examples: Shears

transformation	**image of unit square**	**standard matrix**
Horizontal Shear(left)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k < 0$
Horizontal Shear(right)		$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Shears

transformation	image of unit square	standard matrix
Vertical Shear(down)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k < 0$
Vertical Shear(up)		$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, $k > 0$

Two Dimensional Examples: Projections

transformation	**image of unit square**	**standard matrix**
Projection onto the x_1-axis		$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$
Projection onto the x_2-axis		$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Onto

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto** if for all $\vec{b} \in \mathbb{R}^m$ there is a $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

Onto is an **existence property:** for any $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.

Examples

- A rotation on the plane is an onto linear transformation.
- A projection in the plane is not onto.

Useful Fact

T is onto if and only if its standard matrix has a pivot in every row.

One-to-One

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one** if for all $\vec{b} \in \mathbb{R}^m$ there is at most one (possibly no) $\vec{x} \in \mathbb{R}^n$ so that $T(\vec{x}) = \vec{b}$.

One-to-one is a uniqueness property, it does not assert existence for all $\vec{b}$.

Examples

- A rotation on the plane is a one-to-one linear transformation.
- A projection in the plane is not one-to-one.

Useful Facts

- T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the zero vector, $\vec{x} = \vec{0}$.
- T is one-to-one if and only if the standard matrix A of T has no free variables.

Example

Complete the matrices below by entering numbers into the missing entries so that the properties are satisfied. **If it isn't possible to do so, state why**.

a) A is a 2×3 standard matrix for a one-to-one linear transform.
$A = \begin{pmatrix} 1 & 0 & \\ 0 & & 1 \end{pmatrix}$

b) B is a 3×2 standard matrix for an onto linear transform.

$$B = \begin{pmatrix} 1 & \\ & \\ & \end{pmatrix}$$

c) C is a 3×3 standard matrix of a linear transform that is one-to-one and onto.

$$C = \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}$$

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is onto.
2. The matrix A has columns which span $\mathbb{R}^m$.
3. The matrix A has m pivotal columns.

Theorem

For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A these are equivalent statements.

1. T is one-to-one.
2. The unique solution to $T(\vec{x}) = \vec{0}$ is the trivial one.
3. The matrix A linearly independent columns.
4. Each column of A is pivotal.

Additional Examples

1. Construct a matrix $A \in \mathbb{R}^{2\times 2}$, such that $T(\vec{x}) = A\vec{x}$, where T is a linear transformation that rotates vectors in $\mathbb{R}^2$ counterclockwise by $\pi/2$ radians about the origin, then reflects them through the line $x_1 = x_2$.
2. Define a linear transformation by

$$T(x_1, x_2) = (3x_1 + x_2, 5x_1 + 7x_2, x_1 + 3x_2)$$

Is T one-to-one? Is T onto?