Chapter 5. Distributions of Functions of Random Variables

Math 3215 Summer 2023

Georgia Institute of Technology

Section 1.
Functions of One Random Variable

Functions of One Random Variable

Let X be a random variable.

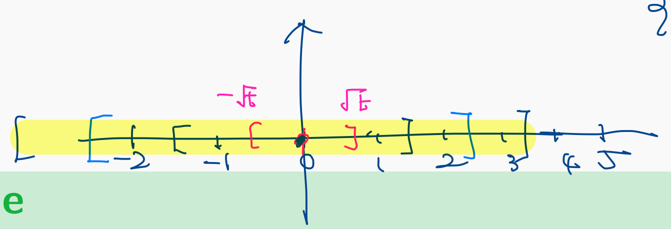
Define $Y = u(X)$ for some function u.

We discuss how to find the distribution of Y from that of X.

look at CDF

Functions of One Random Variable

$$\text{pmf of } X = f_X(x) = \begin{cases} \frac{1}{8}, & x = -2, -1, 0, 1, 2, 3, 4, 5 \quad \underline{\underline{8}} \\ 0, & \text{o.w.} \end{cases}$$



Example

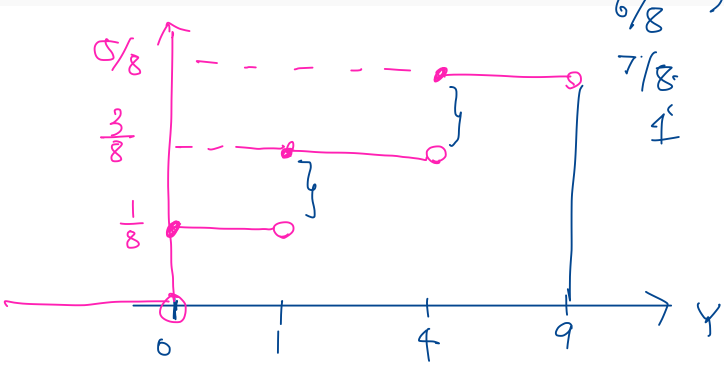
Let X have a discrete uniform distribution on the integers from $\underline{-2}$ to 5.

Find the distribution of $Y = X^2$. ≥ 0

$$P(\underline{Y} \leq t) = P(\underline{X^2 \leq t}) = P(-\sqrt{t} \leq X \leq \sqrt{t})$$

$t \geq 0$

$$= \begin{cases} \frac{1}{8}, & 0 \leq t < 1 \\ \frac{3}{8}, & 1 \leq \sqrt{t} < 2 \rightarrow 1 \leq t < 4 \\ 5/8, & 2 \leq \sqrt{t} < 3 \rightarrow 4 \leq t < 9 \\ 6/8, & 3 \leq \sqrt{t} < 4 \rightarrow 9 \leq t < 16 \\ 7/8, & 4 \leq \sqrt{t} < 5 \rightarrow 16 \leq t < 25 \\ 1, & 5 \leq \sqrt{t} \rightarrow t \geq 25 \end{cases}$$



$$P(\underline{Y} = \underline{k}) = P(X = \underline{\sqrt{k}} \text{ or } \underline{-\sqrt{k}})$$

$k = 0, 1, 4, 9, 16, 25$

$$f_Y(16) = f_Y(9) = f_Y(0) = \frac{1}{8}, \quad f_Y(1) = \frac{2}{8} = f_Y(4)$$

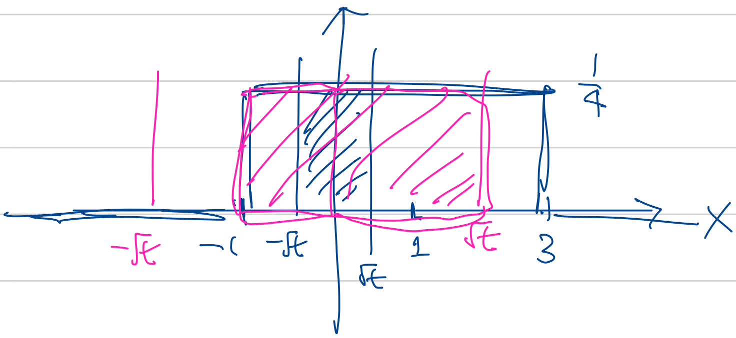
$$f_Y(16) = f_Y(25)$$

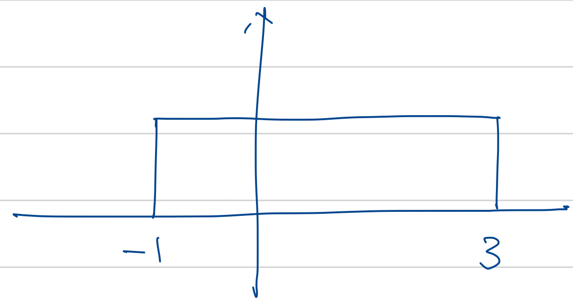
$X \sim \text{Unif}(-1,3)$ $\qquad Y = X^2 \geq 0$

$P(Y \le t) = 0 \quad$ if $\quad t < 0$

If $\quad 0 < t < 1$

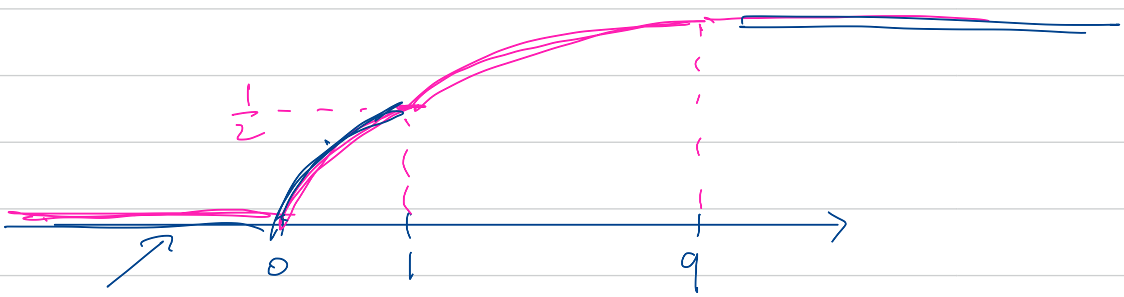
$P(Y \le t) = P(-\sqrt{t} \le X \le \sqrt{t}) = 2 \cdot \sqrt{t} \cdot \frac{1}{4} = \frac{\sqrt{t}}{2}$





If $\quad 1 < \sqrt{t} < 3, \quad P(Y \le t) = \frac{1}{4} + \frac{1}{4}\sqrt{t}$

If $\quad \sqrt{t} \ge 3 \quad P(Y \le t) = 1$



$F_Y(t)$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \begin{cases} 0 & t \le 0 \text{ or } t \ge 9 \\ \frac{1}{4\sqrt{t}} & 0 < t < 1 \\ \frac{1}{8\sqrt{t}} & 1 \le t \le 9 \end{cases}$$

CDF Technique

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad , \quad \Gamma(n) = (n-1)! \quad \Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

Example

Let X have a gamma distribution with pdf

$$f_X(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}.$$

$(\log = \ln = \log_e)$

Find the distribution of $Y = e^X$.

$$F_Y(t) = P(Y \le t) = P(e^X \le t) = \underbrace{P(X \le \log t)} = F_X(\log t)$$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \frac{d}{dt}\left(F_X(\log t)\right)$$

$$\underset{\substack{\uparrow \\ \text{Chain Rule}}}{=} F_X'(\log t) \cdot (\log t)'$$

$$= f_X(\log t) \cdot \frac{1}{t}$$

$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} (\log t)^{\alpha-1} \cdot e^{-\frac{1}{\theta}\log t} \cdot \frac{1}{t}.$$

$$e^{-\frac{1}{\theta}\log t} = e^{\log\left(t^{-\frac{1}{\theta}}\right)} = t^{-\frac{1}{\theta}}$$

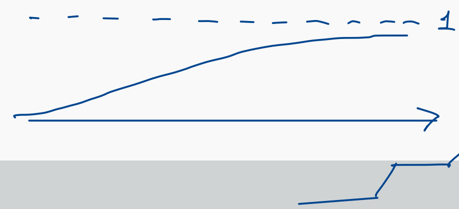
$$= \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot (\log t)^{\alpha-1} \cdot t^{-\frac{1}{\theta}-1} \quad //$$

CDF Technique

CDF : ① non-decreasing

② $\lim_{x \to -\infty} F(x) = 0$

③ $\lim_{x \to \infty} F(x) = 1$



Theorem

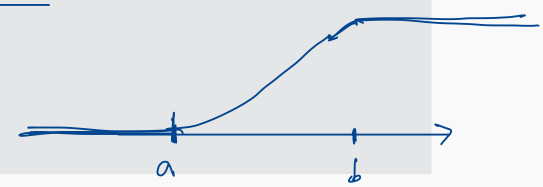
Let X be a random variable with cdf F_X

Suppose F is strictly increasing, $F(a) = 0$, $F(b) = 1$.

no flat in (a, b)

Let $Y \sim U(0,1)$.

Then, $X = F^{-1}(Y)$.



Proof

Let $Z = F^{-1}(Y)$

$F(F^{-1}(t)) = t, \ 0 \le t \le 1$

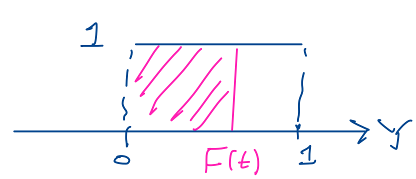
$F^{-1}(F(s)) = s, \ a < s < b$

$F_Z(t) = P(Z \le t)$

$= P(F_X^{-1}(Y) \le t)$ $\quad \because F$ is increasing

$= P(\underbrace{F_X(F_X^{-1}(Y))}_{Y} \le F_X(t))$

$= P(\underbrace{Y}_{\sim \mathrm{Unif}(0,1)} \le F_X(t))$, $F_X(t) \in [0,1]$



$= F_X(t)$ $\quad \Rightarrow \quad Z = X$ in distribution

$\quad \quad \quad \quad \quad \quad \quad \ \ \| $

$\quad \quad \quad \quad \quad \quad F_X^{-1}(Y)$.

▨

Change of Variables

Example

Let X have the pdf $f(x) = 3(1-x)^2$ for $0 < x < 1$.

Find the distribution of $Y = (1-X)^3$.

Exercise

Let X have the pdf $f(x) = 4x^3$, $0 < x < 1$.

Find the pdf of $Y = X^2$. > 0 $\quad t \geq 0$ $\quad$ otherwise $F_Y = 0$.

$$F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(X^2 \leq t)$$

$$= \mathbb{P}(-\sqrt{t} \leq X \leq \sqrt{t}) \leftarrow$$

$$= \mathbb{P}(X \leq \sqrt{t}) - \mathbb{P}(X < -\sqrt{t})$$

$$= \mathbb{P}(X \leq \sqrt{t}) = F_X(\sqrt{t})$$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \frac{d}{dt} F_X(\sqrt{t})$$

$$= F_X'(\sqrt{t}) \cdot (\sqrt{t})'$$

$0 < \sqrt{t} < 1$

$$= f_X(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \cancel{4}^{2} \cdot (\sqrt{t})^{\cancel{3}\,2} \cdot \frac{1}{\cancel{2}\cancel{\sqrt{t}}} = 2t .$$

0 otherwise .

$$f_Y(t) = \begin{cases} 2t , & 0 < t < 1 \\ 0 , & \text{o.w.} \end{cases}$$

$Y = u(X)$, u is strictly increasing.

$$F_Y(t) = \mathbb{P}(u(X) \leq t) \qquad (v = u^{-1})$$
$$= \mathbb{P}(X \leq v(t))$$
$$= \underline{F_X(v(t))}$$

$$f_Y(t) = f_X(v(t)) \cdot v'(t)$$

Section 2.
Transformations of Two Random Variables

Transformations of Two Random Variables

If X_1 and X_2 are two continuous-type random variables with joint pdf $f(x_1, x_2)$.

$x_1 x_2$

ex) $\begin{cases} Y_1 = X_1 + X_2 = u_1(X_1, X_2) \\ Y_2 = X_1 \cdot X_2 = u_2(X_1, X_2) \end{cases}$

Let $Y_1 = u_1(X_1, X_2)$, $Y_2 = u_2(X_1, X_2)$.

If $X_1 = v_1(Y_1, Y_2)$, $X_2 = v_2(Y_1, Y_2)$, then the joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2} = |J| f_{X_1,X_2}(v_1(y_1, y_2), v_2(y_1, y_2))$$

(Change of Variables)

where J is the Jacobian given by

$$J := \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}.$$

determinant.

Transformations of Two Random Variables

Example

Let X_1 and X_2 have the joint pdf

$$f(x_1, x_2) = 2, \qquad 0 < x_1 < x_2 < 1.$$

Find the joint pdf of $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = X_2$.

$$\begin{cases} Y_1 = u_1(X_1, X_2) = \dfrac{X_1}{X_2} \\ \underline{\underline{Y_2}} = u_2(X_1, X_2) = X_2 \end{cases} \Rightarrow \quad X_1 = \underline{\underline{X_2}} \cdot Y_1 = Y_1 \cdot Y_2$$

$$\Rightarrow \begin{cases} X_1 = \underline{Y_1 \cdot Y_2} = v_1(Y_1, Y_2) \\ X_2 = \underline{Y_2} = v_2(Y_1, Y_2) \end{cases}$$

$$|J| = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ 0 & 1 \end{vmatrix} = |\underline{y_2}|$$

$$f_{Y_1, Y_2}(y_1, y_2) = \underbrace{f_{X_1, X_2}(y_1 \cdot y_2, y_2)} \cdot |y_2| = \begin{cases} 2y_2, & 0 < y_1 y_2 < y_2 < 1 \\ 0, & \text{o.w.} \end{cases}$$

Exercise

Let X_1 and X_2 be independent random variables, each with pdf

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Find the joint pdf of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$.

Section 3.
Several Independent Random Variables

Independent random variables

Recall that X_1 and X_2 are independent if

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A)\mathbb{P}(X_2 \in B)$$

for all A, B.

In particular, if X_1 and X_2 have joint pdfs, then $f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$.

Independent random variables

Definition

In general, we say $X_1, X_2, \cdots, X_n$ are independent if $\{X_1 \in A_1\}, \{X_2 \in A_2\}, \cdots, \{X_n \in A_n\}$ are mutually independent, for any choice of $A_1, A_2, \cdots, A_n$.

In particular, if $X_1, X_2, \cdots, X_n$ has joint pdfs, then the joint pdf is the product. of marginals

If $X_1, X_2, \cdots, X_n$ are independent and have the same distribution, we say they are i.i.d. or a random sample of size n from that common distribution.

Independent, Identically distributed

For $\{X_{i_1} \in A_{i_1}\}, \cdots, \{X_{i_k} \in A_{i_k}\}$

$$P(X_{i_1} \in A_{i_1}, \cdots, X_{i_k} \in A_{i_k}) = P(X_{i_1} \in A_{i_1}) \cdots P(X_{i_k} \in A_{i_k})$$

Independent random variables

$Y \sim Exp(\lambda)$ $\qquad$ $\mathbb{P}(Y \underset{(=)}{>} t) = e^{-\lambda t}$

$X_1, X_2, X_3 \sim Exp(1)$ i.i.d.

Example

Let X_1, X_2, X_3 be a Indep. random sample from a distribution with pdf

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Exp (1)

Find $\mathbb{P}(0 < X_1 < 1, 2 < X_2 < 4, 3 < X_3 < 7)$.

$= \mathbb{P}(0 < X_1 < 1) \cdot \mathbb{P}(2 < X_2 < 4)\, \mathbb{P}(3 < X_3 < 7)$

$= \left(\mathbb{P}(X_1 > 0) - \mathbb{P}(X_1 \geq 1)\right)\left(\mathbb{P}(X_2 > 2) - \mathbb{P}(X_2 \geq 4)\right)\left(\mathbb{P}(X_3 > 3) - \mathbb{P}(X_3 \geq 7)\right)$

$= (1 - e^{-1})(e^{-2} - e^{-4})(e^{-3} - e^{-7})$.

$= e^{-5} \cdot (1 - e^{-1})(1 - e^{-2})(1 - e^{-4})$.

$$Var(X) = \mathbb{E}[(X - \mathbb{E}X)^2]$$

$$Cov(X,Y) = \mathbb{E}[\underbrace{(X - \mathbb{E}X)}\cdot(\underbrace{Y - \mathbb{E}[Y]})]$$

$$= \mathbb{E}[\bar{X} \cdot \bar{Y}]$$

Expectation and Variance

without indep.

Theorem

Let $X_1, X_2, \cdots, X_n$ be a sequence of random variables. Then,

$$\mathbb{E}[\underbrace{X_1 + X_2 + \cdots + X_n}] = \underbrace{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots} + \mathbb{E}[X_n].$$

If they are independent, then

$$\mathbb{E}[X_1 X_2 \cdots X_n] = \mathbb{E}[X_1]\mathbb{E}[X_2] \cdots \mathbb{E}[X_n]$$

and

$$\mathrm{Var}[X_1 + X_2 + \cdots + X_n] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + \cdots + \mathrm{Var}[X_n].$$

$\bar{X} = X - \mathbb{E}X$, $\bar{Y} = Y - \mathbb{E}Y$

Note

$$Var(X+Y) = Var(\bar{X} + \bar{Y}) \qquad \mathbb{E}\bar{X} = \mathbb{E}\bar{Y} = 0$$

$$\mathbb{E}[(\overbrace{X+Y}^{\parallel} - \underbrace{\mathbb{E}[X+Y]})^2] = \mathbb{E}[(\bar{X} + \bar{Y})^2]$$

$$= \mathbb{E}[\bar{X}^2 + 2\bar{X}\cdot\bar{Y} + \bar{Y}^2]$$

$$= \underbrace{\mathbb{E}[\bar{X}^2]} + 2\underbrace{\mathbb{E}[\bar{X}\bar{Y}]}_{?} + \underbrace{\mathbb{E}[\bar{Y}^2]}$$

$$= Var(X) + Var(Y) + 2\,Cov(X,Y)$$

Exercise

Let X_1, X_2, X_3 be i.i.d. Geometric with $p = \frac{3}{4}$.

Let Y be the minimum of X_1, X_2, X_3.

Find $\mathbb{P}(Y > 4)$.

Section 4.
The Moment-Generating Function Technique

The Moment-Generating Function

Theorem

If $X_1, X_2, \cdots, X_n$ are independent and have the mgfs $M_{X_i}(t)$, then the mgf of $Y = a_1X_1 + \cdots a_nX_n$ is $M_Y(t) = M_{X_1}(a_1t)\cdots M_{X_n}(a_nt)$.

Theorem

If $X_1, X_2, \cdots, X_n$ are i.i.d., then the mgf of $Y = X_1 + \cdots + X_n$ is $M_Y(t) = M_X(t)^n$. If $\overline{X} = \frac{X_1+\cdots+X_n}{n}$, then the mgf is $M_{\overline{X}}(t) = M_X(\frac{t}{n})^n$.

The Moment-Generating Function

Example

Let $X_1, X_2, \cdots, X_n$ be i.i.d. Bernoulli with p.

Let $Y = X_1 + \cdots + X_n$.

Find the mgf of Y.

The Moment-Generating Function

Example

Let $X_1, X_2, \cdots, X_n$ be i.i.d. exponential with θ.

Let $Y = X_1 + \cdots + X_n$.

Find the mgf of Y.

Exercise

Let X_1, X_2, X_3 be independent Poisson with means $2, 1, 4$.

Find the mgf of $Y = X_1 + X_2 + X_3$.

Section 6.
The Central Limit Theorem

The Central Limit Theorem

Let $X_1, X_2, \cdots, X_n$ be i.i.d. with common distribution X.

Let $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2$.

Let $\overline{X} = \frac{X_1 + \cdots + X_n}{n}$, then $\mathbb{E}[\overline{X}] = \mu$ and $\text{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

Let $W = \frac{\overline{X} - \mu}{\sigma\sqrt{n}}$, then

$\mathbb{E}[W] =$

$\text{Var}(W) =$

The Central Limit Theorem

Theorem

If μ and σ^2 are finite, then the distribution of W converges to that of the standard normal distribution as $n \to \infty$.

The convergence is in the following sense: If n is large, for the standard normal Z,

$$\mathbb{P}(W \leq x) \approx= \mathbb{P}(Z \leq x) =: \Phi(x) = \int_{\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{|y|^2}{2}} \, dy.$$

The Central Limit Theorem

Example

Let $\overline{X}$ be the mean of a random sample of $n = 25$ currents (in milliamperes) in a strip of wire in which each measurement has a mean of 15 and a variance of 4.

Find the approximate probability $\mathbb{P}(14.4 < \overline{X} < 15.6)$.

The Central Limit Theorem

Example

Let $\overline{X}$ denote the mean of a random sample of size 25 from the distribution whosepdf is $f(x) = \frac{x^3}{4}$, $0 < x < 2$.

Find the approximate probability $\mathbb{P}(1.5 \leq \overline{X} \leq 1.65)$.

Exercise

Let X equal the maximal oxygen intake of a human on a treadmill, where the measurements are in milliliters of oxygen per minute per kilogram of weight.

Assume that, for a particular population, the mean of X is $\mu = 54.030$ and the standard deviation is $\sigma = 5.8$.

Let $\overline{X}$ be the sample mean of a random sample of size $n = 47$.

Find $P(52.761 \leq \overline{X} \leq 54.453)$, approximately.

Section 8.
Chebyshev's Inequality and Convergence in Probability

Chebyshev's Inequality

Theorem

If the random variable X has a mean μ and variance σ^2, then for every $k \geq 1$,

$$\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

In particular $\varepsilon = k\sigma$, then

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

Chebyshev's Inequality

Example

Suppose X has a mean of 25 and a variance of 16.

Find the lower bound of $\mathbb{P}(17 < X < 33)$.

The Law of Large Numbers

Definition

We say a sequence of random variables X_n converges to a random variable X in probability if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

The Law of Large Numbers

Theorem

Let $X_1, X_2, \cdots, X_n$ be i.i.d. with common distribution X.

Let $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

Then, $\overline{X}$ converges to μ in probability.

Exercise

If X is a random variable with mean 3 and variance 16, use Chebyshev's inequality to find

1. A lower bound for $\mathbb{P}(23 < X < 43)$.
2. An upper bound for $\mathbb{P}(|X - 31| \geq 14)$.