Section 5.3 : Diagonalization

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Motivation: it can be useful to take large powers of matrices, for example A^k, for large k.

But: multiplying two $n \times n$ matrices requires roughly n^3 computations. Is there a more efficient way to compute A^k?

Topics and Objectives

Topics

1. Diagonal, similar, and diagonalizable matrices
2. Diagonalizing matrices

Learning Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Determine whether a matrix can be diagonalized, and if possible diagonalize a square matrix.
2. Apply diagonalization to compute matrix powers.

Diagonal Matrices

A matrix is **diagonal** if the only non-zero elements, if any, are on the main diagonal.

The following are all diagonal matrices.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \end{bmatrix}, \quad I_n, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We'll only be working with diagonal square matrices in this course.

Powers of Diagonal Matrices

If A is diagonal, then A^k is easy to compute. For example,

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 0.5 \end{pmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (0.5)^2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 3^k & 0 \\ 0 & (0.5)^k \end{bmatrix}$$

But what if A is not diagonal?

Goal

$$A = P \cdot D \cdot P^{-1}$$

$$= P \cdot \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix} P^{-1}$$

$$A^2 = P \cdot \underbrace{D \boxed{P^{-1} \cdot P} \cdot D} \cdot P^{-1} = P \cdot D^2 \cdot P^{-1}$$

$$A^k = P \cdot D^k \cdot P^{-1}$$

Diagonalization

Suppose $A \in \mathbb{R}^{n\times n}$. We say that A is **diagonalizable** if it is similar to a diagonal matrix, D. That is, we can write

$$\boxed{A = PDP^{-1}}$$

$AP = PD.$

$$\underline{A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & d_n \end{bmatrix}$$

$$= \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix} = \begin{bmatrix} d_1 \cdot v_1 & d_2 \cdot v_2 & \cdots & d_n \cdot v_n \end{bmatrix}$$

$$\begin{cases} Av_1 = d_1 v_1 \\ Av_2 = d_2 v_2 \\ \vdots \\ Av_n = d_n v_n \end{cases}$$

Always

$AP = PD$

$$P = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & d_n \end{bmatrix}$$

Diagonalization

Theorem

If A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors.

Note: the symbol $\Leftrightarrow$ means " if and only if ".

Also note that $A = PDP^{-1}$ if and only if

$$A = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n] \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]^{-1}$$

where $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent eigenvectors, and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (**in order**).

Example 1

Diagonalize if possible.

$$A = \begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix} = P \cdot D\, P^{-1}$$

$\lambda_1 = 2$ $\qquad \lambda_2 = -1$ $\qquad \longrightarrow D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$\underline{\underline{v_1 =}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad v_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\qquad P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\text{Null}\,(A - 2I) = \text{Null} \begin{bmatrix} 0 & 6 \\ 0 & -3 \end{bmatrix} = \text{Null} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\text{Null}\,(A + I) = \text{Null} \begin{pmatrix} 3 & 6 \\ 0 & 0 \end{pmatrix} = \text{Null} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

Example 2

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Distinct Eigenvalues

Theorem

If A is $n \times n$ and has n distinct eigenvalues, then A is diagonalizable.

Why does this theorem hold?

Is it necessary for an $n \times n$ matrix to have n distinct eigenvalues for it to be diagonalizable?

Non-Distinct Eigenvalues

Theorem. Suppose

- A is $n \times n$
- A has distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, $k \leq n$
- $a_i =$ algebraic multiplicity of λ_i
- $d_i =$ dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n \Leftrightarrow d_i = a_i$ for all i
3. A is diagonalizable $\Leftrightarrow$ the eigenvectors, for all eigenvalues, together form a basis for $\mathbb{R}^n$.

Example 3

The eigenvalues of A are $\lambda = 3, 1$. If possible, construct P and D such that $AP = PD$.

$$A = \begin{pmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{pmatrix}$$

Additional Example (if time permits)

Note that

$$\vec{x}_k = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x}_{k-1}, \quad \vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad k = 1, 2, 3, \ldots$$

generates a well-known sequence of numbers.

Use a diagonalization to find a matrix equation that gives the n^{th} number in this sequence.

Basis of Eigenvectors

Express the vector $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and find the coordinates of $\vec{x}_0$ in the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_0\}$.

$[\vec{x}_0]_{\mathcal{B}} =$

Let $P = [\vec{v}_1 \ \vec{v}_2]$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and find $[A^k \vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$[A^k \vec{x}_0]_{\mathcal{B}} =$

Basis of Eigenvectors - part 2

Let $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as before.

Again define $P = [\vec{v}_1 \ \vec{v}_2]$ but this time let $D = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$, and now find $[A^k\vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$[A^k\vec{x}_0]_{\mathcal{B}} =$

Basis of Eigenvectors - part 3

Let $\vec{x}_0 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as before.

Again define $P = [\vec{v}_1 \ \vec{v}_2]$ but this time let $D = \begin{bmatrix} 2 & 0 \\ 0 & 3/2 \end{bmatrix}$, and now find $[A^k \vec{x}_0]_{\mathcal{B}}$ where $A = PDP^{-1}$, for $k = 1, 2, \ldots$.

$[A^k \vec{x}_0]_{\mathcal{B}} =$

Chapter 5 : Eigenvalues and Eigenvectors

5.5 : Complex Eigenvalues

Topics and Objectives

Topics

1. Complex numbers: addition, multiplication, complex conjugate
2. Complex eigenvalues and eigenvectors.
3. Eigenvalue theorems

Learning Objectives

1. Use eigenvalues to determine identify the rotation and dilation of a linear transform.
2. Rotation dilation matrices.
3. Find complex eigenvalues and eigenvectors of a real matrix.
4. Apply theorems to characterize matrices with complex eigenvalues.

Motivating Question
What are the eigenvalues of a rotation matrix?

Imaginary Numbers

Recall: When calculating roots of polynomials, we can encounter square roots of negative numbers. For example:

$$x^2 + 1 = 0$$

The roots of this equation are:

We usually write $\sqrt{-1}$ as i (for "imaginary").

Addition and Multiplication

The imaginary (or complex) numbers are denoted by $\mathbb{C}$, where

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify $\mathbb{C}$ with $\mathbb{R}^2$: $\quad a + bi \leftrightarrow (a, b)$

We can add and multiply complex numbers as follows:

$(2 - 3i) + (-1 + i) =$

$(2 - 3i)(-1 + i) =$

Complex Conjugate, Absolute Value, Polar Form

We can **conjugate** complex numbers: $\overline{a+bi} =$ ____________

The **absolute value** of a complex number: $|a+bi| =$ ____________

We can write complex numbers in **polar form**: $a + ib = r(\cos\phi + i\,\sin\phi)$

Complex Conjugate Properties

If x and y are complex numbers, $\vec{v} \in \mathbb{C}^n$, it can be shown that:

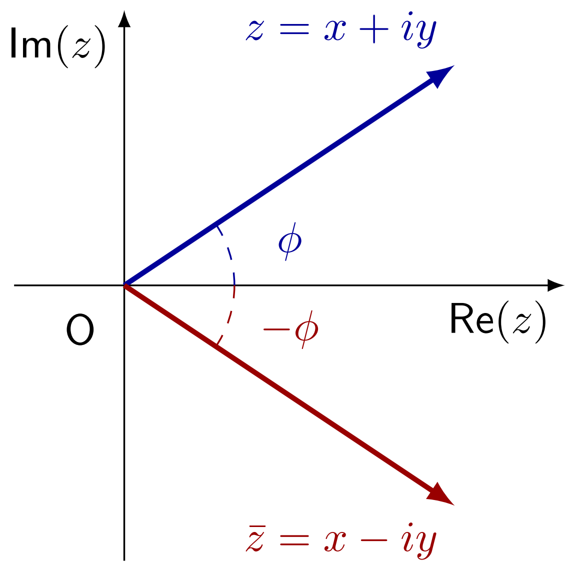
- $\overline{(x+y)} = \overline{x} + \overline{y}$
- $\overline{A\vec{v}} = A\overline{\vec{v}}$
- $\text{Im}(x\overline{x}) = 0$.

Example True or false: if x and y are complex numbers, then

$$\overline{(xy)} = \overline{x}\ \overline{y}$$

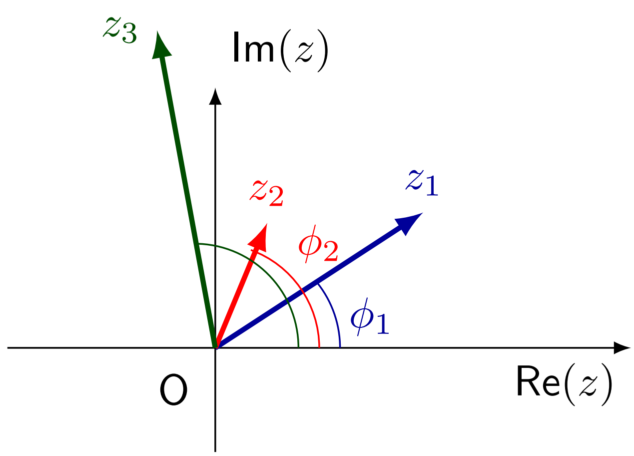
Polar Form and the Complex Conjugate

Conjugation reflects points across the real axis.



Euler's Formula

Suppose z_1 has angle ϕ_1, and z_2 has angle ϕ_2.



The product $z_1 z_2$ has angle $\phi_1 + \phi_2$ and modulus $|z|\,|w|$. Easy to remember using Euler's formula.

$$\boxed{z = |z|\,\mathrm{e}^{i\phi}}$$

The product $z_1 z_2$ is:

$$z_3 = z_1 z_2 = (|z_1|\,\mathrm{e}^{i\phi_1})(|z_2|e^{i\phi_2}) = |z_1|\,|z_2|\,\mathrm{e}^{i(\phi_1+\phi_2)}$$

Complex Numbers and Polynomials

Theorem: Fundamental Theorem of Algebra

Every polynomial of degree n has exactly n complex roots, counting multiplicity.

Theorem

1. If $\lambda \in \mathbb{C}$ is a root of a real polynomial $p(x)$, then the conjugate $\overline{\lambda}$ is also a root of $p(x)$.
2. If λ is an eigenvalue of real matrix A with eigenvector $\vec{v}$, then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{\vec{v}}$.

Example

Four of the eigenvalues of a 7×7 matrix are $-2, 4+i, -4-i$, and i. What are the other eigenvalues?

Example

The matrix that rotates vectors by $\phi = \pi/4$ radians about the origin, and then scales (or dilates) vectors by $r = \sqrt{2}$, is

$$A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

What are the eigenvalues of A? Find an eigenvector for each eigenvalue.

Example

The matrix in the previous example is a special case of this matrix:

$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Calculate the eigenvalues of C and express them in polar form.

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$$C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Calculate the eigenvalues of C and express them in polar form.

Example

Find the complex eigenvalues and an associated complex eigenvector for each eigenvalue for the matrix.

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$