Section 4.9 : Applications to Markov Chains

Chapter 4 : Vector Spaces

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. Markov chains
2. Steady-state vectors
3. Convergence

Objectives

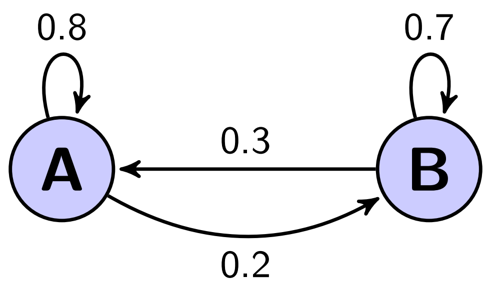
For the topics covered in this section, students are expected to be able to do the following.

1. Construct stochastic matrices and probability vectors.
2. Model and solve real-world problems using Markov chains (e.g. - find a steady-state vector for a Markov chain)
3. Determine whether a stochastic matrix is regular.

Example 1

- A small town has two libraries, A and B.
- After 1 month, among the books checked out of A,
  - 80% returned to A
  - 20% returned to B
- After 1 month, among the books checked out of B,
  - 30% returned to A
  - 70% returned to B

If both libraries have 1000 books today, how many books does each library have after 1 month? After one year? After n months? A place to simulate this is `http://setosa.io/markov/index.html`



	0	1	2	...
A	1000	800 + 300	$0.8 \cdot 1100 + 0.3 \cdot 900$	
B	1000	200 + 700	$0.2 \cdot 1100 + 0.7 \cdot 900$	

① Instead of counting # of books,
Consider the proportions.

At the beginning

$$\begin{bmatrix} 1000 \\ 1000 \end{bmatrix} \longrightarrow \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \vec{x}_0 \quad \text{Initial Status.}$$

② Describe the status after k months

$$\vec{x}_k = \begin{bmatrix} - \\ - \end{bmatrix}$$

from A, from B

$$\vec{x}_1 = \begin{bmatrix} 0.8 \cdot 0.5 + 0.3 \cdot 0.5 \\ 0.2 \cdot 0.5 + 0.7 \cdot 0.5 \end{bmatrix}$$

from A, from B

$$= \underbrace{\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}}_{\vec{x}_0} = A \vec{x}_0$$

$$\vec{x}_2 = A \cdot \vec{x}_1 = A(A\vec{x}_0) = A^2 \cdot \vec{x}_0$$

$$\vdots$$

$$\vec{x}_{1000} = A^{1000} \cdot \vec{x}_0$$

Thm If A is "Nice"

then $\vec{X}_k$ close to "A Certain

Vector" as k large

doesn't Not depend on $\vec{X}_0$

Example 1 Continued

The books are equally divided by between the two branches, denoted by $\vec{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$. What is the distribution after 1 month, call it $\vec{x}_1$? After two months?

After k months, the distribution is $\vec{x}_k$, which is what in terms of $\vec{x}_0$?

$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

Markov Chains

A few definitions:

- A **probability vector** is a vector, $\vec{x}$, with non-negative elements that sum to 1.
- A **stochastic matrix** is a square matrix, P, whose columns are probability vectors.
- A **Markov chain** is a sequence of probability vectors $\vec{x}_k$, and a stochastic matrix P, such that:

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

- A **steady-state vector** for P is a vector $\vec{q}$ such that $P\vec{q} = \vec{q}$.

Prob.(Current | All Past history)

= Prob(Current k^{th} | $(k-1)^{th}$ history)

If x_0 : steady-state

$x_1 = P \cdot x_0 = x_0$

$x_2 = x_0$

$x_k = x_0$

Example 2

Determine a steady-state vector for the stochastic matrix

$$\begin{pmatrix} .8 & .3 \\ .2 & .7 \end{pmatrix}$$

Convergence

We often want to know what happens to a process,

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

as $k \to \infty$.

Definition: a stochastic matrix P is **regular** if there is some k such that P^k only contains strictly positive entries.

Theorem

If P is a regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$, and $\vec{x}_{k+1} = P\vec{x}_k$ converges to $\vec{q}$ as $k \to \infty$.

Example 3

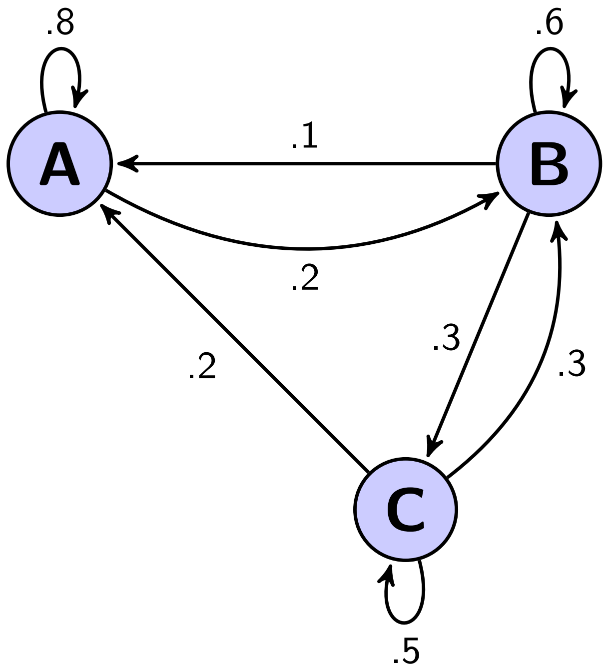
A car rental company has 3 rental locations, A, B, and C. Cars can be returned at any location. The table below gives the pattern of rental and returns for a given week.

		rented from		
		A	B	C
	A	.8	.1	.2
returned to	B	.2	.6	.3
	C	.0	.3	.5

There are 10 cars at each location today.

a) Construct a stochastic matrix, P, for this problem.

b) What happens to the distribution of cars after a long time? You may assume that P is regular.



$$P = \begin{bmatrix} .8 & .1 & .2 \\ .2 & .6 & .3 \\ .0 & .3 & .5 \end{bmatrix}$$

Section 5.1 : Eigenvectors and Eigenvalues

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics and Objectives

Topics
We will cover these topics in this section.

1. Eigenvectors, eigenvalues, eigenspaces
2. Eigenvalue theorems

Objectives
For the topics covered in this section, students are expected to be able to do the following.

1. Verify that a given vector is an eigenvector of a matrix.
2. Verify that a scalar is an eigenvalue of a matrix.
3. Construct an eigenspace for a matrix.
4. Apply theorems related to eigenvalues (for example, to characterize the invertibility of a matrix).

Eigenvectors and Eigenvalues

If $A \in \mathbb{R}^{n \times n}$, and there is a $\vec{v} \neq \vec{0}$ in $\mathbb{R}^n$, and

$$A\vec{v} = \lambda\vec{v}$$

then $\vec{v}$ is an **eigenvector** for A, and $\lambda \in \mathbb{C}$ is the corresponding **eigenvalue**.

Note that

- We will only consider square matrices.
- If $\lambda \in \mathbb{R}$, then
  - when $\lambda > 0$, $A\vec{v}$ and $\vec{v}$ point in the same direction
  - when $\lambda < 0$, $A\vec{v}$ and $\vec{v}$ point in opposite directions
- Even when all entries of A and $\vec{v}$ are real, λ can be complex (a rotation of the plane has no **real** eigenvalues.)
- We explore complex eigenvalues in Section 5.5.

Example 1

Which of the following are eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$? What are the corresponding eigenvalues?

a) $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

c) $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example 2

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

Eigenspace

Definition

Suppose $A \in \mathbb{R}^{n \times n}$. The eigenvectors for a given λ span a subspace of $\mathbb{R}^n$ called the λ-**eigenspace** of A.

Note: the λ-eigenspace for matrix A is $\mathrm{Nul}(A - \lambda I)$.

Example 3
Construct a basis for the eigenspaces for the matrix whose eigenvalues are given, and sketch the eigenvectors.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}, \quad \lambda = -1, 2$$

Theorems

Proofs for the most these theorems are in Section 5.1. If time permits, we will explain or prove all/most of these theorems in lecture.

1. The diagonal elements of a triangular matrix are its eigenvalues.

2. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A.

3. Stochastic matrices have an eigenvalue equal to 1.

4. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are eigenvectors that correspond to distinct eigenvalues, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are linearly independent.

Warning!

We can't determine the eigenvalues of a matrix from its reduced form.

Row reductions change the eigenvalues of a matrix.

Example: suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The eigenvalues are $\lambda = 2, 0$, because

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$
$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

- But the reduced echelon form of A is:
- The reduced echelon form is triangular, and its eigenvalues are:

Section 5.2 : The Characteristic Equation

Chapter 5 : Eigenvalues and Eigenvectors

Math 1554 Linear Algebra

Topics and Objectives

Topics

We will cover these topics in this section.

1. The characteristic polynomial of a matrix
2. Algebraic and geometric multiplicity of eigenvalues
3. Similar matrices

Objectives

For the topics covered in this section, students are expected to be able to do the following.

1. Construct the characteristic polynomial of a matrix and use it to identify eigenvalues and their multiplicities.
2. Characterize the long-term behaviour of dynamical systems using eigenvalue decompositions.

The Characteristic Polynomial

Recall:

λ is an eigenvalue of $A \Leftrightarrow (A - \lambda I)$ is not __________________

Therefore, to calculate the eigenvalues of A, we can solve

$$\det(A - \lambda I) =$$

The quantity $\det(A - \lambda I)$ is the **characteristic polynomial** of A.

The quantity $\det(A - \lambda I) = 0$ is the **characteristic equation** of A.

The roots of the characteristic polynomial are the ______________ of A.

Example

The characteristic polynomial of $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ is:

So the eigenvalues of A are:

Characteristic Polynomial of 2×2 Matrices

Express the characteristic equation of

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in terms of its determinant. What is the equation when M is singular?

Algebraic Multiplicity

Definition

The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Example

Compute the algebraic multiplicities of the eigenvalues for the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Geometric Multiplicity

Definition

The **geometric multiplicity** of an eigenvalue λ is the dimension of $\mathrm{Null}(A - \lambda I)$.

1. Geometric multiplicity is always at least 1. It can be smaller than algebraic multiplicity.
2. Here is the basic example:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

   $\lambda = 0$ is the only eigenvalue. Its algebraic multiplicity is 2, but the geometric multiplicity is 1.

Example

Give an example of a 4×4 matrix with $\lambda = 0$ the only eigenvalue, but the geometric multiplicity of $\lambda = 0$ is one.

Recall: Long-Term Behavior of Markov Chains

Recall:

- We often want to know what happens to a Markov Chain

$$\vec{x}_{k+1} = P\vec{x}_k, \quad k = 0, 1, 2, \ldots$$

  as $k \to \infty$.
- If P is regular, then there is a ________________________

Now lets ask:

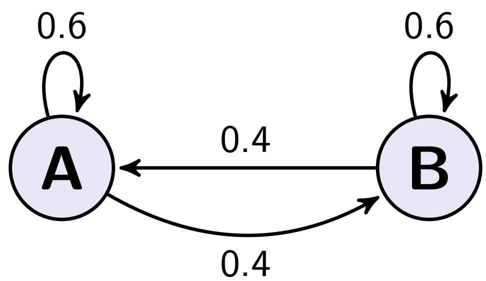
- If we don't know whether P is regular, what else might we do to describe the long-term behavior of the system?
- What can eigenvalues tell us about the behavior of these systems?

Example: Eigenvalues and Markov Chains

Consider the Markov Chain:

$$\vec{x}_{k+1} = P\vec{x}_k = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \vec{x}_k, \quad k = 0, 1, 2, 3, \ldots, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This system can be represented schematically with two nodes, A and B:



Goal: use eigenvalues to describe the long-term behavior of our system.

What are the eigenvalues of P?

What are the corresponding eigenvectors of P?

Use the eigenvalues and eigenvectors of P to analyze the long-term behaviour of the system. In other words, determine what $\vec{x}_k$ tends to as $k \to \infty$.

Similar Matrices

Definition

Two $n \times n$ matrices A and B are **similar** if there is a matrix P so that $A = PBP^{-1}$.

Theorem

If A and B similar, then they have the same characteristic polynomial.

If time permits, we will explain or prove this theorem in lecture. Note:

- Our textbook introduces similar matrices in Section 5.2, but doesn't have exercises on this concept until 5.3.
- Two matrices, A and B, do not need to be similar to have the same eigenvalues. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Additional Examples (if time permits)

1. True or false.
   a) If A is similar to the identity matrix, then A is equal to the identity matrix.
   b) A row replacement operation on a matrix does not change its eigenvalues.
2. For what values of k does the matrix have one real eigenvalue with algebraic multiplicity 2?

$$\begin{pmatrix} -3 & k \\ 2 & -6 \end{pmatrix}$$