

## Midterm 3 Lecture Review Activity, Math 1554

1. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) If $S$ is a two-dimensional subspace of $\mathbb{R}^{50}$ , then the dimension of $S^\perp$ is 48.	<input type="radio"/>	<input type="radio"/>
b) An eigenspace is a subspace spanned by a single eigenvector.	<input type="radio"/>	<input type="radio"/>
c) The $n \times n$ zero matrix can be diagonalized.	<input type="radio"/>	<input type="radio"/>
d) A least-squares line that best fits the data points $(0, y_1), (1, y_2), (2, y_3)$ is unique for any values $y_1, y_2, y_3$ .	<input type="radio"/>	<input type="radio"/>

2. If possible, give an example of the following.

2.1) A matrix,  $A$ , that is in echelon form, and  $\dim((\text{Row } A)^\perp) = 2$ ,  $\dim((\text{Col } A)^\perp) = 1$

2.2) A singular  $2 \times 2$  matrix whose eigenspace corresponding to eigenvalue  $\lambda = 2$  is the line  $x_1 = 2x_2$ . The other eigenspace of the matrix is the  $x_2$  axis.

2.3) A subspace  $S$ , of  $\mathbb{R}^4$ , that satisfies  $\dim(S) = \dim(S^\perp) = 3$ .

2.4) A  $2 \times 3$  matrix,  $A$ , that is in RREF.  $(\text{Row } A)^\perp$  is spanned by  $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ .

3. Circle **possible** if the set of conditions are create a situation that is possible, otherwise, circle **impossible**. For the situations that are possible give an example.

3.1)  $A$  is  $n \times n$ ,  $A\vec{x} = A\vec{y}$  for a particular  $\vec{x} \neq \vec{y}$ ,  $\vec{x}$  and  $\vec{y}$  are in  $\mathbb{R}^n$ , and  $\dim((\text{Row } A)^\perp) \neq 0$ .

**possible**

**impossible**

3.2)  $A$  is  $n \times n$ ,  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A$ , and  $\dim((\text{Col}(A - \lambda I))^\perp) = 0$ .

**possible**

**impossible**

3.3)  $\text{proj}_{\vec{v}}\vec{u} = \text{proj}_{\vec{u}}\vec{v}$ ,  $\vec{v} \neq \vec{u}$ , and  $\vec{u} \neq \vec{0}$ ,  $\vec{v} \neq \vec{0}$ .

**possible**

**impossible**

4. Consider the matrix  $A$ .

$$A = \begin{pmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Construct a basis for the following subspaces and state the dimension of each space.

4.1)  $(\text{Row } A)^\perp$

4.2)  $\text{Col } A$

4.3)  $(\text{Col } A)^\perp$

## Midterm 3

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

Name: \_\_\_\_\_ GTID Number: \_\_\_\_\_

Student GT Email Address: \_\_\_\_\_@gatech.edu

Section Number (e.g. A3, G2, etc.) \_\_\_\_\_ TA Name \_\_\_\_\_

Circle your instructor:

Prof Vilaca Da Rocha    Prof Kafer    Prof Barone    Prof Wheeler  
Prof Blumenthal    Prof Sun    Prof Shirani

### Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

Midterm 3. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose  $A$  is an  $m \times n$  matrix and  $\vec{b} \in \mathbb{R}^m$  unless otherwise stated. Select **true** if the statement is true for all choices of  $A$  and  $\vec{b}$ . Otherwise, select **false**.

true    false

- 
- |                       |                       |  |
|-----------------------|-----------------------|--|
| <input type="radio"/> | <input type="radio"/> | A matrix $A \in \mathbb{R}^{n \times n}$ and its transpose $A^T$ have the same eigenvectors.   |
| <input type="radio"/> | <input type="radio"/> | An invertible matrix $A$ is diagonalizable if and only if its inverse $A^{-1}$ is diagonalizable.  |
| <input type="radio"/> | <input type="radio"/> | If $\vec{u} \cdot \vec{v} = \vec{u} \cdot \vec{w}$ , then $\vec{u}$ is orthogonal to $(\vec{w} - \vec{v})$ .   |
| <input type="radio"/> | <input type="radio"/> | If the vectors $\vec{u}$ and $\vec{v}$ are orthogonal then $\ \vec{u} + \vec{v}\  = \ \vec{u}\  + \ \vec{v}\ $ .   |
| <input type="radio"/> | <input type="radio"/> | If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and $W$ is a subspace of $\mathbb{R}^n$ , then $\ \text{proj}_W(\vec{y})\ $ is the shortest distance between $W$ and $\vec{y}$ . |
| <input type="radio"/> | <input type="radio"/> | If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and $W$ is a subspace of $\mathbb{R}^n$ , then $\vec{y} - \text{proj}_W(\vec{y})$ is in $W^\perp$ .                              |
| <input type="radio"/> | <input type="radio"/> | If $W$ is a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ such that $\vec{y} \cdot \vec{w} = 0$ for some vector $\vec{w} \in W$ , then $\vec{y} \in W^\perp$ .         |
| <input type="radio"/> | <input type="radio"/> | The line of best fit $y = \beta_0 + \beta_1 x$ for the points $(1, 2)$ , $(1, 3)$ , and $(1, 4)$ is unique.  |
- 

- (b) (4 points) Indicate whether the following situations are possible or impossible.

possible    impossible

- 
- |                       |                       |   |
|-----------------------|-----------------------|---|
| <input type="radio"/> | <input type="radio"/> | A $5 \times 5$ real matrix $A$ such that $A$ has no real eigenvalues.   |
| <input type="radio"/> | <input type="radio"/> | An $m \times n$ matrix $U$ where $U^T U = I_n$ and $n > m$ .  |
| <input type="radio"/> | <input type="radio"/> | A 2-dimensional subspace $W$ of $\mathbb{R}^3$ and a vector $\vec{y} \in W$ such that $\ \vec{v}_1 - \vec{y}\  = \ \vec{v}_2 - \vec{y}\ $ where $\vec{v}_1, \vec{v}_2 \in W^\perp$ and $\vec{v}_1 \neq \vec{v}_2$ . |
| <input type="radio"/> | <input type="radio"/> | A matrix $A \in \mathbb{R}^{3 \times 4}$ such that the linear system $A\vec{x} = \vec{b}$ has a unique least-squares solution.  |
-

Midterm 3. Your initials: \_\_\_\_\_

*You do not need to justify your reasoning for questions on this page.*

(c) (2 points) An  $m \times n$  matrix  $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$  has non-zero orthogonal columns and  $A^T A = 2I_n$ . Which of the following statements is FALSE?

- ☐  $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$  for every  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ .
- ☐  $n \leq m$ .
- ☐ If we apply the Gram-Schmidt process to  $\{\vec{a}_1, \dots, \vec{a}_n\}$  we obtain the same set  $\{\vec{a}_1, \dots, \vec{a}_n\}$ .
- ☐ If  $A = QR$  is the QR factorization of  $A$ , then  $R$  is a diagonal matrix.

2. (3 points) Find  $a, b, c$  so that the set of vectors  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  is an orthogonal set.

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} -2 \\ 0 \\ a \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} -1 \\ b \\ c \end{pmatrix}$$

$a =$	<input type="text"/>
$b =$	<input type="text"/>
$c =$	<input type="text"/>

Midterm 3. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

3. (8 points) Fill in the blanks.

(a) Suppose  $\vec{u}$  and  $\vec{v}$  are orthogonal vectors in  $\mathbb{R}^n$  and that  $\vec{v}$  is a unit vector.

If  $(2\vec{u} + \vec{v}) \cdot (\vec{u} + 5\vec{v}) = 13$ , determine the length of  $\vec{u}$ .

$$\|\vec{u}\| = \boxed{\phantom{000}}$$

(b) The normal equations for the least-squares solution to  $A\vec{x} = \vec{b}$  are given by:

(c) Compute the length (magnitude) of the vector  $\vec{y}$ .

$$\vec{y} = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\|\vec{y}\| = \boxed{\phantom{000}}$$

(d) Let  $A = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$ . The vector  $\vec{v} = \begin{pmatrix} -1 - i \\ -1 + i \end{pmatrix}$  is an eigenvector of  $A$ . Find the associated eigenvalue  $\lambda$  for the eigenvector  $\vec{v}$  of  $A$ .

$$\lambda = \boxed{\phantom{000}}$$

Midterm 3. Your initials: \_\_\_\_\_

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4. (4 points) Fill in the blanks.

- (a) Let  $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$  and let  $\vec{y} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ . Calculate the projection of  $\vec{y}$  onto the subspace  $W$ , and find the distance from  $\vec{y}$  to  $W$ .

$\text{proj}_W(\vec{y}) =$

$\text{dist}(\vec{y}, W) =$

- (b) Let  $W = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ ,  $L = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ . Find **all** vectors  $\vec{u} \in L$  such that the distance from  $\vec{u}$  to  $W$  is equal to 2.

Midterm 3. Your initials: \_\_\_\_\_

5. (6 points) **Show all work for problems on this page.**

One of the eigenvalues of the matrix  $A$  is  $\lambda = 1$ . Diagonalize the matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$$

Diagram illustrating two stacked rectangles. The top rectangle is labeled  $P =$  and the bottom rectangle is labeled  $D =$ .

6. (3 points) Find a matrix  $A \in \mathbb{R}^{2 \times 2}$  such that  $\vec{v}_1$  is an eigenvector of  $A$  with eigenvalue  $\lambda_1 = 4$ , and  $\vec{v}_2$  is an eigenvector of  $A$  with eigenvalue  $\lambda_2 = -3$ .

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$A =$$



Midterm 3. Your initials: \_\_\_\_\_

7. (4 points) **Show all work for problems on this page.**

Let  $\mathcal{B} = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$  be a basis for a subspace  $W$  of  $\mathbb{R}^4$ , where

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \\ -1 \end{pmatrix}.$$

- (a) Apply the Gram-Schmidt process to the set of vectors  $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$  to find an orthogonal basis  $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  for  $W$ . Clearly show all steps of the Gram-Schmidt process.

$\mathcal{H} =$

- (b) In the space below, **check** that the vectors in the basis  $\mathcal{H}$  form an orthogonal set.

Midterm 3. Your initials: \_\_\_\_\_

8. (4 points) **Show all work for problems on this page.**

If  $A$  has the following QR factorization

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 0 & 3 \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix},$$

compute the least-square solution to the equation  $A\vec{x} = \vec{b}$ .

$\hat{x} =$

9. (4 points) Compute the least squares line  $y = c_1 + c_2x$  that best fits the data

$$\begin{array}{c|ccc} x & -1 & 0 & 1 \\ \hline y & 4 & 2 & 5 \end{array}$$

## Midterm 3 Make-up

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Section Number (e.g. A3, G2, etc.) \_\_\_\_\_ TA Name \_\_\_\_\_

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1. (a) (8 points) Suppose  $A$  is an  $m \times n$  matrix and  $\vec{b} \in \mathbb{R}^m$  unless otherwise stated. Select **true** if the statement is true for all choices of  $A$  and  $\vec{b}$ . Otherwise, select **false**.

true    false

- 
- |                       |                       |  |
|-----------------------|-----------------------|--|
| <input type="radio"/> | <input type="radio"/> | A matrix $A \in \mathbb{R}^{n \times n}$ and its transpose $A^T$ have the same eigenvectors.   |
| <input type="radio"/> | <input type="radio"/> | The line of best fit $y = \beta_0 + \beta_1 x$ for the points $(1, 1)$ , $(2, 1)$ , and $(3, 1)$ is unique.  |
| <input type="radio"/> | <input type="radio"/> | If the vectors $\vec{u}$ and $\vec{v}$ are orthogonal then $\ \vec{u} + \vec{v}\  = \ \vec{u}\  + \ \vec{v}\ $ .   |
| <input type="radio"/> | <input type="radio"/> | A triangular matrix $A$ is diagonalizable if and only if $A$ is invertible.  |
| <input type="radio"/> | <input type="radio"/> | If $A = PDP^{-1}$ where $D$ is a diagonal matrix, then $D$ and $A$ have the same eigenvectors.   |
| <input type="radio"/> | <input type="radio"/> | If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and $W$ is a subspace of $\mathbb{R}^n$ , then $\text{proj}_W(\vec{y})$ is in $W$ .  |
| <input type="radio"/> | <input type="radio"/> | If $\vec{y} \in \mathbb{R}^n$ is a nonzero vector and $W$ is a subspace of $\mathbb{R}^n$ , then $\ \vec{y} - \text{proj}_W(\vec{y})\ $ is the shortest distance between $W$ and $\vec{y}$ . |
| <input type="radio"/> | <input type="radio"/> | If $W$ is a subspace of $\mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^n$ such that $\vec{y} \cdot \vec{w} = 0$ for some vector $\vec{w} \in W$ , then $\vec{y} \in W^\perp$ .                   |
- 

- (b) (4 points) Indicate whether the following situations are possible or impossible.

possible    impossible

- 
- |                       |                       |   |
|-----------------------|-----------------------|---|
| <input type="radio"/> | <input type="radio"/> | A $3 \times 3$ real matrix $A$ such that $A$ has eigenvalues $2, 3, 2i + 3$ .   |
| <input type="radio"/> | <input type="radio"/> | An $m \times n$ matrix $U$ where $U^T U = I_n$ and $n > m$ .  |
| <input type="radio"/> | <input type="radio"/> | A 2-dimensional subspace $W$ of $\mathbb{R}^3$ and a vector $\vec{y} \in W$ such that $\ \vec{v}_1 - \vec{y}\  = \ \vec{v}_2 - \vec{y}\ $ where $\vec{v}_1, \vec{v}_2 \in W^\perp$ and $\vec{v}_1 \neq \vec{v}_2$ . |
| <input type="radio"/> | <input type="radio"/> | A vector $\vec{y} \in \mathbb{R}^3$ and a subspace $W$ in $\mathbb{R}^3$ such that $\vec{y} = \vec{w} + \vec{z}$ where $\vec{w}$ is in $W$ , but $\vec{z}$ is not in $W^\perp$ .                                    |
-

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

(c) (2 points) An  $m \times n$  matrix  $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$  has non-zero orthogonal columns and  $A^T A = 2I_n$ . Which of the following statements is FALSE?

- ☐  $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$  for every  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^n$ .
- ☐  $n \leq m$ .
- ☐ If we apply the Gram-Schmidt process to  $\{\vec{a}_1, \dots, \vec{a}_n\}$  we obtain the same set  $\{\vec{a}_1, \dots, \vec{a}_n\}$ .
- ☐ If  $A = QR$  is the QR factorization of  $A$ , then  $R$  is a diagonal matrix.

2. (3 points) Using **only** 0's and 1's in your answers, give an example of  $2 \times 2$  matrices that satisfy:  $A$  is invertible but not diagonalizable, and  $B$  is diagonalizable but not invertible.

$$A = \begin{pmatrix} & \\ & \end{pmatrix} \qquad B = \begin{pmatrix} & \\ & \end{pmatrix}$$

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

3. (8 points) Fill in the blanks.

(a) Suppose  $\vec{u}$  and  $\vec{v}$  are orthogonal vectors in  $\mathbb{R}^n$  and that  $\vec{v}$  is a unit vector.

If  $(3\vec{u} + \vec{v}) \cdot (\vec{u} + 4\vec{v}) = 31$ , determine the length of  $\vec{u}$ .

$$\|\vec{u}\| = \boxed{\phantom{000}}$$

(b) The normal equations for the least-squares solution to  $A\vec{x} = \vec{b}$  are given by:

(c) Compute the length (magnitude) of the vector  $\vec{y}$ .

$$\vec{y} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\|\vec{y}\| = \boxed{\phantom{000}}$$

(d) Let  $A = \begin{bmatrix} 3 & 3 \\ -3 & 3 \end{bmatrix}$ . The vector  $\vec{v} = \begin{pmatrix} 1+i \\ 1-i \end{pmatrix}$  is an eigenvector of  $A$ . Find the associated eigenvalue  $\lambda$  for the eigenvector  $\vec{v}$  of  $A$ .

$$\lambda = \boxed{\phantom{000}}$$

Midterm 3 Make-up. Your initials: \_\_\_\_\_

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4. (4 points) Fill in the blanks.

- (a) Let  $W = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$  and let  $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ . Calculate the projection of  $\vec{y}$  onto the subspace  $W$ , and find the distance from  $\vec{y}$  to  $W$ .

$\text{proj}_W(\vec{y}) =$

$\text{dist}(\vec{y}, W) =$

- (b) Let  $W = \text{span} \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$ ,  $L = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ . Find **all** vectors  $\vec{u} \in L$  such that the distance from  $\vec{u}$  to  $W$  is equal to 2.

5. (6 points) **Show all work for problems on this page.**

$$A = \begin{pmatrix} -2 & -3 & -2 \\ 2 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P = \begin{bmatrix} & \\ & \\ & \\ & \\ & \end{bmatrix}$$

$$D = \begin{bmatrix} & \\ & \\ & \\ & \\ & \end{bmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
$$A =$$



Midterm 3 Make-up. Your initials: \_\_\_\_\_

7. (4 points) **Show all work for problems on this page.**

Let  $\mathcal{B} = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$  be a basis for a subspace  $W$  of  $\mathbb{R}^4$ , where

$$\vec{x}_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

(a) Apply the Gram-Schmidt process to the set of vectors  $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$  to find an orthogonal basis  $\mathcal{H} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  for  $W$ . Clearly show all steps of the Gram-Schmidt process.

$\mathcal{H} =$

(b) In the space below, **check** that the vectors in the basis  $\mathcal{H}$  form an orthogonal set.

Midterm 3 Make-up. Your initials: \_\_\_\_\_

8. (4 points) **Show all work for problems on this page.**

If  $A$  has the following QR factorization

$$A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}, \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix},$$

compute the least-square solution to the equation  $A\vec{x} = \vec{b}$ .

$\hat{x} =$

9. (4 points) Compute the least squares line  $y = c_1x + c_2$  that best fits the data

$$\begin{array}{c|ccc} x & -1 & 0 & 1 \\ \hline y & 1 & -2 & 2 \end{array}$$

## Midterm 3

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Section Number (e.g. A3, G2, etc.) \_\_\_\_\_ TA Name \_\_\_\_\_

Circle your instructor:

Prof Barone

Prof Shirani

Prof Simone

Prof Timko

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1. (a) (8 points) Suppose  $A$  is an  $m \times n$  matrix and  $\vec{b} \in \mathbb{R}^m$  unless otherwise stated. Select **true** if the statement is true for all choices of  $A$  and  $\vec{b}$ . Otherwise, select **false**.

true	false	
<input type="radio"/>	<input type="radio"/>	The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.
<input type="radio"/>	<input type="radio"/>	If $W$ is the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ , then $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is a vector in $W^\perp$ .
<input type="radio"/>	<input type="radio"/>	If $U$ is a $3 \times 2$ matrix with orthonormal columns, then for every $\vec{y} \in \text{Col}(U)$ we have $\vec{y} = UU^T\vec{y}$ .
<input type="radio"/>	<input type="radio"/>	If the matrix $A$ has orthogonal columns, then $A^T A$ is a diagonal matrix.
<input type="radio"/>	<input type="radio"/>	Assume $n \neq m$ . If $A = QR$ is the QR factorization of $A \in \mathbb{R}^{n \times m}$ , then $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{n \times m}$ .
<input type="radio"/>	<input type="radio"/>	If $A = QR$ is a QR factorization of $A$ , then $A^T A = R^T R$ .
<input type="radio"/>	<input type="radio"/>	If $\hat{x}$ and $\hat{y}$ are least-squares solutions of $A\vec{x} = \vec{b}$ , then $\hat{x} - \hat{y} \in \text{Nul}(A)$ .
<input type="radio"/>	<input type="radio"/>	Suppose $A$ is such that $T_A$ is not one-to-one, and $\vec{b}$ is not in the range of $T$ . Then $A\vec{x} = \vec{b}$ has a unique least-squares solution.

- (b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
<input type="radio"/>	<input type="radio"/>	$A$ is a $7 \times 7$ diagonalizable matrix with exactly three distinct eigenvalues whose geometric multiplicities are 1, 2, and 3, respectively.
<input type="radio"/>	<input type="radio"/>	$\vec{u}$ and $\vec{v}$ are nonzero vectors such that $\ \vec{u} + \vec{v}\ ^2 = \ \vec{u}\ ^2 + \ \vec{v}\ ^2$ .
<input type="radio"/>	<input type="radio"/>	The distance between a vector $\vec{b} \in \mathbb{R}^m$ and the column space of a matrix $A \in \mathbb{R}^{m \times n}$ is zero, and the linear system $A\vec{x} = \vec{b}$ is inconsistent.
<input type="radio"/>	<input type="radio"/>	$\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$ , and there exists a linearly independent set of vectors $\{\vec{x}, \vec{y}\}$ in $\mathbb{R}^3$ such that $\text{Proj}_{\mathcal{W}}\vec{x} = \text{Proj}_{\mathcal{W}}\vec{y}$ .

Midterm 3. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

- (c) (2 points) The standard matrix of a linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  has orthonormal columns. Which one of the following statements is **false**?

Choose only one.

- ☐  $\|T(\vec{x})\| = \|\vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^3$ .
- ☐ If two non-zero vectors  $\vec{x}$  and  $\vec{y}$  in  $\mathbb{R}^3$  are scalar multiples of each other, then  $\|T(\vec{x} + \vec{y})\|^2 = \|T(\vec{x})\|^2 + \|T(\vec{y})\|^2$ .
- ☐ If  $\mathcal{P}$  is a parallelepiped in  $\mathbb{R}^3$ , then the volume of  $T(\mathcal{P})$  is equal to the volume of  $\mathcal{P}$ .
- ☐  $T$  is one-to-one.

2. (2 points) Suppose that, in the QR factorization of  $A$ , we have  $Q$  as given below. Find  $R$ .

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \quad Q = \frac{1}{2} \begin{bmatrix} 1 & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \\ 1 & -\sqrt{3} \\ 1 & 1/\sqrt{3} \end{bmatrix}$$

Note: Please fill in the blanks and do not place values in front of the matrix for this problem.

$$R = \begin{bmatrix} \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \end{bmatrix}$$

Midterm 3. Your initials: \_\_\_\_\_

*You do not need to justify your reasoning for questions on this page.*

3. (2 points) Using **only** 0's and 1's in your answer, give an example of a  $2 \times 2$  matrix that is invertible but not diagonalizable.

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

4. (6 points) Fill in the blanks.

- (a) Let  $\vec{u}, \vec{v} \in \mathbb{R}^n$  be orthogonal vectors each with length 2. Determine the length of the vector  $2\vec{u} + \vec{v}$ .
- (b) Suppose  $A$  is a  $7 \times 5$  matrix such that  $\dim(\text{Row } A)^\perp = 4$ . Determine the dimension of the column space of  $A$ .
- (c) Suppose  $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$  is an orthogonal basis for a subspace  $\mathcal{W}$  of  $\mathbb{R}^n$ , and  $\vec{x}$  belongs to the subspace  $\mathcal{W}$ . Suppose also that

$$\vec{v}_1 \cdot \vec{v}_1 = 2, \vec{v}_2 \cdot \vec{v}_2 = 4, \vec{v}_1 \cdot \vec{x} = 6, \text{ and } \vec{v}_2 \cdot \vec{x} = -4.$$

Find  $[\vec{x}]_{\mathcal{B}}$  the coordinates of  $\vec{x}$  in the basis  $\mathcal{B}$ .

Midterm 3. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

5. (6 points) Let  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  be an orthogonal basis for  $\mathbb{R}^3$ , where

$$\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Let  $\mathcal{W}$  be a subspace of  $\mathbb{R}^3$  that is spanned by  $\vec{u}_1$  and  $\vec{u}_2$ .

(a) Give an orthonormal basis for  $\mathcal{W}^\perp$ .

(b) What is the closest vector in  $\mathcal{W}$  to the vector  $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ ?

(c) Give a vector  $\vec{x} \in \mathbb{R}^3$  that satisfies all of the following conditions:

- $\vec{x}$  is orthogonal to  $\vec{u}_3$ .
- $\text{Proj}_{\mathcal{L}_1} \vec{x} = 2\vec{u}_1$ , where  $\mathcal{L}_1 = \text{span}\{\vec{u}_1\}$ .
- $\text{Proj}_{\mathcal{L}_2} \vec{x} = 3\vec{u}_2$ , where  $\mathcal{L}_2 = \text{span}\{\vec{u}_2\}$ .

Midterm 3. Your initials: \_\_\_\_\_

6. (6 points) **Show work** on this page with work under the problem, and **your answer in the box**.

Let  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ .

(i) List the eigenvalues of  $A$ .

(ii) Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ . If this is not possible, write NP.

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$



Midterm 3. Your initials: \_\_\_\_\_

7. (6 points) **Show work** on this page with work under the problem, and **your answer in the box**.

Find an orthogonal basis of the subspace spanned by the set of vectors shown below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

*Note: In order to receive full credit you must clearly show all steps of the Gram-Schmidt process applied to the vectors.*



Midterm 3. Your initials: \_\_\_\_\_

8. (8 points) **Show work** on this page with work under the problem, and **your answer in the box**.

In this problem, you will use the least-squares method to find the values  $\alpha$  and  $\beta$  which best fit the curve

$$y = \alpha \cdot \frac{1}{1+x^2} + \beta$$

to the data points  $(-1, 1)$ ,  $(0, -1)$ ,  $(1, 0)$  using the parameters  $\alpha$  and  $\beta$ .

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?

(ii) What is the augmented matrix for the normal equations for this system.

(iii) Find a least-squares solution to the linear system from (i) to determine the parameters  $\alpha$  and  $\beta$  of the best fitting curve.

$$\alpha = \boxed{\phantom{000}} \quad \beta = \boxed{\phantom{000}}$$

Math 1554 Linear Algebra Spring 2022

## Midterm 3 Make-up

**PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS**

Name: \_\_\_\_\_ GTID Number: \_\_\_\_\_

Student GT Email Address: \_\_\_\_\_@gatech.edu

Section Number (e.g. A3, G2, etc.) \_\_\_\_\_ TA Name \_\_\_\_\_

Circle your instructor:

Prof Barone

Prof Shirani

Prof Simone

Prof Timko

### Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose  $A$  is an  $m \times n$  matrix and  $\vec{b} \in \mathbb{R}^m$  unless otherwise stated. Select **true** if the statement is true for all choices of  $A$  and  $\vec{b}$ . Otherwise, select **false**.

true	false	
<input type="radio"/>	<input type="radio"/>	The matrix $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.
<input type="radio"/>	<input type="radio"/>	If $W$ is the subspace of $\mathbb{R}^3$ spanned by $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ , then $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is a vector in $W^\perp$ .
<input type="radio"/>	<input type="radio"/>	If $U$ is a $3 \times 2$ matrix with orthonormal columns, then $(UU^T)^2 = UU^T$ .
<input type="radio"/>	<input type="radio"/>	If $A^T A = 3I$ , then $A$ has orthogonal columns.
<input type="radio"/>	<input type="radio"/>	Assume $n \neq m$ . If $A = QR$ is the QR factorization of $A \in \mathbb{R}^{m \times n}$ , then $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$ .
<input type="radio"/>	<input type="radio"/>	If $A = QR$ is a QR factorization of $A$ , then $A^T A = R^T R$ .
<input type="radio"/>	<input type="radio"/>	If $\hat{x}$ and $\hat{y}$ are least-squares solutions of $A\vec{x} = \vec{b}$ , then $\hat{x} - \hat{y} \in \text{Nul}(A)$ .
<input type="radio"/>	<input type="radio"/>	Suppose $A$ is such that for every $\vec{b}$ the system $A\vec{x} = \vec{b}$ has a unique least-squares solution. Then $\det(A^T A) \neq 0$ .

- (b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
<input type="radio"/>	<input type="radio"/>	$A$ is a real $5 \times 5$ diagonalizable matrix with exactly three distinct real eigenvalues whose geometric multiplicities are 1, 1, and 2, respectively.
<input type="radio"/>	<input type="radio"/>	$\vec{u}$ and $\vec{v}$ are nonzero orthogonal vectors such that $\ \vec{u} + \vec{v}\ ^2 < \ \vec{u}\ ^2 + \ \vec{v}\ ^2$ .
<input type="radio"/>	<input type="radio"/>	The distance between a vector $\vec{b} \in \mathbb{R}^m$ and the column space of a matrix $A \in \mathbb{R}^{m \times n}$ is nonzero, and the linear system $A\vec{x} = \vec{b}$ has a unique least squares solution.
<input type="radio"/>	<input type="radio"/>	$\mathcal{W}$ is a 2-dimensional subspace of $\mathbb{R}^3$ , and there exists a linearly independent set of vectors $\{\vec{x}, \vec{y}\}$ in $\mathbb{R}^3$ such that $\text{Proj}_{\mathcal{W}} \vec{x} = 0$ and $\text{Proj}_{\mathcal{W}} \vec{y} = 0$ .

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

- (c) (2 points) The standard matrix of a linear transformation  $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  has orthonormal columns. Which one of the following statements is **false**?

Choose only one.

- ☐ If  $T(\vec{x}) = A\vec{x}$  then  $\det(A) = 1$ .
- ☐  $T(\vec{x}) \cdot T(\vec{y}) = T(\vec{x} \cdot \vec{y})$  for all  $\vec{x}, \vec{y}$  in  $\mathbb{R}^3$
- ☐  $\|T(\vec{x})\| = \|\vec{x}\|$  for all  $\vec{x}$  in  $\mathbb{R}^3$ .
- ☐  $T$  is onto.

2. (2 points) Suppose that, in the QR factorization of  $A$ , we have  $Q$  as given below. Find  $R$ .

$$A = \begin{bmatrix} -1 & 2 \\ 1 & 2 \\ 1 & 2 \\ -1 & -2 \end{bmatrix} \quad Q = \frac{1}{2} \begin{bmatrix} -1 & \sqrt{3} \\ 1 & 1/\sqrt{3} \\ 1 & 1/\sqrt{3} \\ -1 & -1/\sqrt{3} \end{bmatrix}$$

Note: Please fill in the blanks and do not place values in front of the matrix for this problem.

$$R = \begin{bmatrix} \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \\ \underline{\hspace{1cm}} & \underline{\hspace{1cm}} \end{bmatrix}$$

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

3. (2 points) Using **only** 0's and 1's in your answer, give an example of a  $2 \times 2$  matrix that is not invertible and not diagonalizable. If not possible, write NP.

$$\begin{pmatrix} & \\ & \end{pmatrix}$$

4. (6 points) Fill in the blanks.

- (a) Let  $\vec{u}, \vec{v} \in \mathbb{R}^n$  be vectors each with length 2 and suppose  $u \cdot v = -1$ . Determine the length of the vector  $3\vec{u} + \vec{v}$ .
- (b) Suppose  $A$  is a  $5 \times 7$  matrix such that  $\dim(\text{Nul } A)^\perp = 4$ . Determine the dimension of the orthogonal complement of the column space of  $A$ .
- (c) Suppose  $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$  is an orthogonal basis for a subspace  $\mathcal{W}$  of  $\mathbb{R}^n$ , and  $\vec{x}$  belongs to the subspace  $\mathcal{W}$ . Suppose also that

$$\vec{v}_1 \cdot \vec{x} = 4, \vec{v}_2 \cdot \vec{x} = -3, \vec{v}_1 \cdot \vec{v}_1 = 2, \text{ and } \vec{v}_2 \cdot \vec{v}_2 = 6.$$

Find  $[\vec{x}]_{\mathcal{B}}$  the coordinates of  $\vec{x}$  in the basis  $\mathcal{B}$ .

Midterm 3 Make-up. Your initials: \_\_\_\_\_

You do not need to justify your reasoning for questions on this page.

5. (6 points) Let  $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$  be an orthogonal basis for  $\mathbb{R}^3$ , where

$$\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Let  $\mathcal{W}$  be a subspace of  $\mathbb{R}^3$  that is spanned by  $\vec{u}_1$  and  $\vec{u}_2$ .

(a) Give an orthonormal basis for  $\mathcal{W}^\perp$ .

(b) What is the closest vector in  $\mathcal{W}$  to the vector  $\vec{y} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ ?

(c) Give a vector  $\vec{x} \in \mathbb{R}^3$  that satisfies all of the following conditions:

- $\vec{x}$  is orthogonal to  $\vec{u}_3$ .
- $\text{Proj}_{\mathcal{L}_1} \vec{x} = 2\vec{u}_1$ , where  $\mathcal{L}_1 = \text{span}\{\vec{u}_1\}$ .
- $\text{Proj}_{\mathcal{L}_2} \vec{x} = 3\vec{u}_2$ , where  $\mathcal{L}_2 = \text{span}\{\vec{u}_2\}$ .

Midterm 3 Make-up. Your initials: \_\_\_\_\_

6. (6 points) **Show work** on this page with work under the problem, and **your answer in the box**.

Let  $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ .

(i) List the eigenvalues of  $A$ .

(ii) Find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $A = PDP^{-1}$ . If this is not possible, write NP.

$$P = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \quad D = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$



Midterm 3 Make-up. Your initials: \_\_\_\_\_

7. (6 points) **Show work** on this page with work under the problem, and **your answer in the box**.

Find an orthogonal basis of the subspace spanned by the set of vectors shown below.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

*Note: In order to receive full credit you must clearly show all steps of the Gram-Schmidt process applied to the vectors.*



Midterm 3 Make-up. Your initials: \_\_\_\_\_

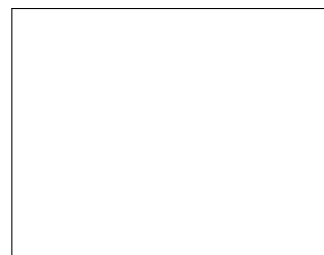
8. (8 points) **Show work** on this page with work under the problem, and **your answer in the box**.

In this problem, you will use the least-squares method to find the values  $\alpha$  and  $\beta$  which best fit the curve

$$y = \alpha \cdot 2^{-x^2} + \beta x$$

to the data points  $(-1, 2)$ ,  $(0, 0)$ ,  $(1, 1)$  using the parameters  $\alpha$  and  $\beta$ .

(i) What is the augmented matrix for the linear system of equations associated to this least squares problem?



(ii) What is the augmented matrix for the normal equations for this system.



(iii) Find a least-squares solution to the linear system from (i) to determine the parameters  $\alpha$  and  $\beta$  of the best fitting curve.

$$\alpha = \boxed{\phantom{000}} \quad \beta = \boxed{\phantom{000}}$$