Practice for Exam 1

Math 3215 Summer 2023

Georgia Institute of Technology

Practice Problems

Consider a thick coin with three possible outcomes of a toss (Heads, Tails, and Edge) for which Heads and Tails are equally likely, but Heads is five times as likely than Edge. What is the probability of Heads?

$$\begin{cases} P(H) = P(T) = 5P(E) \\ P(H) + P(T) + P(E) = 1 \end{cases} \qquad \therefore \; P(H) = \frac{5}{11}$$

Practice Problems

Suppose A, B and C are events with $\mathbb{P}(A) = 0.43, \mathbb{P}(B) = 0.40, \mathbb{P}(C) = 0.32, \mathbb{P}(A \cap B) = 0.29, \mathbb{P}(A \cap C) = 0.22, \mathbb{P}(B \cap C) = 0.20$ and $\mathbb{P}(A \cap B \cap C) = 0.15$. Find $\mathbb{P}(A^c \cap B^c \cap C^c)$.

$$\mathbb{P}(A^c \cap B^c \cap C^c) = 1 - \mathbb{P}(A \cup B \cup C)$$

$$= 1 - \big(\mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) + \mathbb{P}(A \cap B \cap C) \big)$$

$$= 1 - (0.43 + 0.4 + 0.32 - 0.29 - 0.22 - 0.2 + 0.15)$$

$$= 0.41$$

Practice Problems

If $\mathbb{P}(A) = 0.4, \mathbb{P}(B) = 0.5, \mathbb{P}(B \mid A) = 0.75$, find $\mathbb{P}(A \mid B)$ and $\mathbb{P}(A \mid B^c)$.

$$\mathbb{P}(A|B) = \mathbb{P}(B|A) \cdot \frac{\mathbb{P}(A)}{\mathbb{P}(B)} = 0.75 \cdot \frac{0.4}{0.5} = 0.6$$

$$\mathbb{P}(B^c|A) + \mathbb{P}(B|A) = 1$$

$$\mathbb{P}(A|B^c) = \mathbb{P}(B^c|A) \cdot \frac{\mathbb{P}(A)}{\mathbb{P}(B^c)} = (1 - \mathbb{P}(B|A)) \cdot \frac{\mathbb{P}(A)}{(1 - \mathbb{P}(B))}$$

$$= 0.25 \cdot \frac{0.4}{0.5} = 0.2$$

Practice Problems

Seventy percent of the light aircraft that disappear while in flight in Neverland are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 90% of the aircraft not discovered do not have such a locator. Suppose a light aircraft that has just disappeared has an emergency locator. What is the probability that it will not be discovered?

$P(D) = 0.7$

$P(E \mid D) = 0.6 \Rightarrow P(D \cap E) = 0.42$

$P(E^c \mid D^c) = 0.9 \Rightarrow P(D^c \cap E^c) = 0.27$

$\Downarrow$

$P(E) = 0.45$

$$P(D^c \mid E) = \frac{P(E \cap D^c)}{P(E)}$$

$$= \frac{P(E) - P(E \cap D)}{P(E)} = \frac{0.03}{0.45} = \frac{1}{15}.$$

Practice Problems

In a particular sports event, it is known that 2% of the athletes consume a certain kind of drug. A blood test for detecting this drug is 96% effective, meaning that if the drug is present in the blood of the athlete, the test will return a positive result 96% of the time. However, the test also yields a false positive result (it gives a positive result when the drug is not present) for 1% of the drug-free persons tested.

1. What is the probability that for a randomly selected sthlete, the test is positive?
2. What is the probability that a randomly seleted athlete consumed the drug, given that the test is positive?

$\mathbb{P}(D) = 0.02$, $\mathbb{P}(P \mid D) = 0.96$, $\mathbb{P}(P \mid D^c) = 0.01$

Practice Problems

1. $$\mathbb{P}(P) = \mathbb{P}(P \mid D)\cdot\mathbb{P}(D) + \mathbb{P}(P \mid D^c)\cdot\mathbb{P}(D^c)$$

$$= 0.96 \cdot 0.02 + 0.01 \cdot 0.98$$

$$= 0.0192 + 0.0098 = 0.029$$

2. $$\mathbb{P}(D \mid P) = \frac{\mathbb{P}(P \mid D)\cdot\mathbb{P}(D)}{\mathbb{P}(P)} = \frac{0.0192}{0.029} \approx 0.66$$

Practice Problems

During the first week of the semester, 80% of customers at a local convenience store bought either beer or potato chips (or both). 60% bought potato chips. 30% of the customers bought both beer and potato chips. Are events a randomly selected customer bought potato chips and a randomly selected customer bought beer independent?

$\mathbb{P}(P \cup B) = 0.8$ $\quad$ $\mathbb{P}(P) = 0.6$ $\quad$ $\mathbb{P}(P \cap B) = 0.3$

$$\Rightarrow \quad \mathbb{P}(B) = \mathbb{P}(P \cup B) - \mathbb{P}(P) + \mathbb{P}(P \cap B) = 0.5$$

$$\mathbb{P}(P \cap B) = 0.3 = \mathbb{P}(P) \cdot \mathbb{P}(B)$$

$\Rightarrow$ Independent.

Practice Problems

Let X be an integer-valued random variable with probability mass function given by $f(x) = \frac{C}{3^x}$ for $x = 2, 3, 4, \cdots$.

1. Find C.
2. FInd the probability that X takes an odd value.

$$\sum_{x=2}^{\infty} \frac{C}{3^x} = \frac{C/3^2}{1 - 1/3} = \frac{C}{6} = 1 \quad \therefore C = 6.$$

$$P(X \text{ is odd}) = \sum_{k=1}^{\infty} f(2k+1) = \frac{6/3^3}{1 - \frac{1}{3^2}} = \frac{6}{27-3}$$

$$= \frac{1}{4}$$

Practice Problems

A certain basketball player knows that on average he will successfully make 78% of his free throw attempts. Assuming all throw attempts are independent. What are the expectation and the variance of the number of successful throws in 1020 attempts.

X = # of successful throws $\sim \text{Bin}(1020, 0.78)$

$E[X] = 1020 \cdot 0.78 = 795.6$

$\text{Var}(X) = 1020 \cdot 0.78 \cdot 0.22 = 175.032$

Practice Problems

An airline overbooks a flight, selling more tickets for the flight than there are seats on the plane (figuring that it's likely that some people won't show up). The plane has 100 seats, and 110 people have booked the flight. Each person will show up for the flight with probability 0.9, independently.

1. Let X be the number of people showing up. Is X a binomial random variable? Justify your answer. If so, find the parameters.
2. Find the approximate probability that exactly 100 passengers are shown up.

1. Showing up a person is indep. with the same prob. 0.9

$\Rightarrow X \sim Bin(110, 0.9)$

Practice Problems

2. $Y = 110 - X \sim \mathrm{Bin}(110, 0.1) \approx \mathrm{Pois}(11)$

$$\mathbb{P}(X = 100) = \mathbb{P}(Y = 10) \approx e^{-11} \frac{11^{10}}{10!}.$$

Practice Problems

A particular brand of candy-coated chocolate comes in six different colors. Suppose 30% of all pieces are brown, 20% are blue, 15% are red, 15% are yellow, 10% are green, and 10% are orange. Thirty pieces are selected at random. What is the probability that 10 are brown, 8 are blue, 7 are red, 3 are yellow, 2 are green, and none are orange?

$$\frac{30!}{10!\,8!\,7!\,3!\,2!}(0.3)^{10}\cdot(0.2)^{8}(0.15)^{7}\cdot(0.15)^{3}(0.1)^{2}\cdot$$

Practice Problems

Suppose you roll two 5 faced dice, with faces labeled 1,2,3,4,5, and each equally likely to appear on top. Let X denote the smaller of the two numbers that appear. If both dice show the same number, then X is equal to that common number.

1. Find the PMF of X.
2. Compute $\mathbb{P}(X \leq 3 | X > 1)$ and $\mathbb{E}[5X - 2]$.

k	$f(k)$
1	9/25
2	7/25
3	5/25
4	3/25
5	1/25

$$\mathbb{P}(X \leq 3 \mid X > 1) = \frac{\mathbb{P}(X = 2, 3)}{\mathbb{P}(X > 1)} = \frac{7+5}{7+5+3+1}$$

$$= \frac{3}{4}$$

$$\mathbb{E}[5X-2] = 5\,\mathbb{E}[X] - 2 = 9$$

$$\mathbb{E}[X] = \frac{1}{25}(1\cdot 9 + 2\cdot 7 + 3\cdot 5 + 4\cdot 3 + 5\cdot 1)$$

$$= \frac{55}{25} = \frac{11}{5}$$