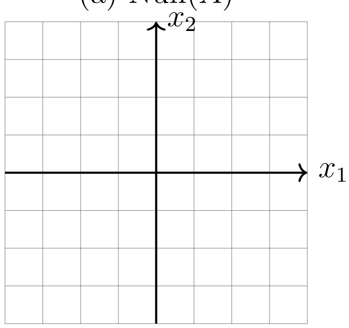
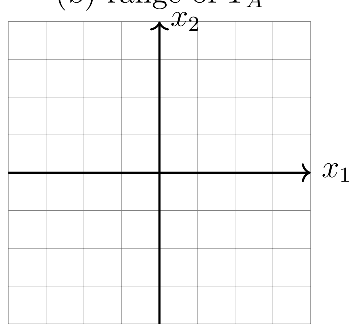
Midterm 2 Lecture Review Activity, Math 1554

1. (3 points) T_A is the linear transform $x \to Ax$, $A \in \mathbb{R}^{2\times 2}$, that projects points in $\mathbb{R}^2$ onto the x_2-axis. Sketch the nullspace of A, the range of the transform, and the column space of A. How are the range and column space related to each other?

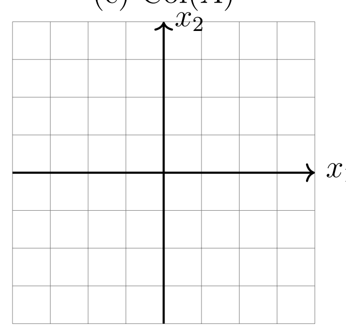
(a) Null(A)



(b) range of T_A



(c) Col(A)



2. Indicate **true** if the statement is true, otherwise, indicate **false**.

	true	false
a) $S = \{\vec{x} \in \mathbb{R}^3 \,\vert\, x_1 = a, x_2 = 4a, x_3 = x_1 x_2\}$ is a subspace for any $a \in \mathbb{R}$.	○	○
b) If A is square and non-zero, and $A\vec{x} = A\vec{y}$ for some $\vec{x} \neq \vec{y}$, then $\det(A) \neq 0$.	○	○

3. If possible, write down an example of a matrix or quantity with the given properties. If it is not possible to do so, write *not possible*.

   (a) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 1$. $A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

   (b) A is 2×2, Col$A$ is spanned by the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\dim(\text{Null}(A)) = 0$. $A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

   (c) A is in RREF and $T_A : \mathbb{R}^3 \to \mathbb{R}^3$. The vectors u and v are a basis for the range of T.

   $$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, A = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

4. Indicate whether the situations are possible or impossible by filling in the appropriate circle.

		possible	impossible
4.i)	Vectors $\vec{u}$ and $\vec{v}$ are eigenvectors of square matrix A, and $\vec{w} = \vec{u}+\vec{v}$ is also an eigenvector of A.	○	○
4.ii)	$T_A = A\vec{x}$ is one-to-one, $\dim(\text{Col}(A)) = 4$, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$.	○	○

5. (2 points) Fill in the blanks.

   (a) If A is a 6×4 matrix in RREF and $\text{rank}(A) = 4$, what is the rank of A^T? $\boxed{}$

   (b) $T_A = A\vec{x}$, where $A \in \mathbb{R}^{2\times 2}$, is a linear transform that first rotates vectors in $\mathbb{R}^2$ clockwise by π radians about the origin, then scales their x-component by a factor of 3, then projects them onto the x_1-axis. What is the value of $\det(A)$? $\boxed{}$

6. (3 points) A virus is spreading in a lake. Every week,

   - 20% of the healthy fish get sick with the virus, while the other healthy fish remain healthy but could get sick at a later time.
   - 10% of the sick fish recover and can no longer get sick from the virus, 80% of the sick fish remain sick, and 10% of the sick fish die.

   Initially there are exactly 1000 fish in the lake.

   a) What is the stochastic matrix, P, for this situation? Is P regular?

   b) Write down any steady-state vector for the corresponding Markov-chain.

Math 1554 Linear Algebra   Spring   2022

Midterm 2

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________   GTID Number: ___________________

Student GT Email Address: ________________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______   TA Name ________________________

Circle your instructor:

Prof Barone   Prof Shirani   Prof Simone   Prof Timko

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
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- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 8 pages of questions.

Midterm 2. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (10 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	If $k > n$ and $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ spans $\mathbb{R}^n$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a basis for $\mathbb{R}^n$.
◯	◯	If $A, B, C \in \mathbb{R}^{n\times n}$ and $AB = I_n = BC$, then $A = C$.
◯	◯	If $A \in \mathbb{R}^{n\times n}$ is invertible, then $A^T A$ is invertible.
◯	◯	If $A\vec{x} \neq A\vec{y}$ for all vectors $\vec{x} \neq \vec{y}$, then Null$(A) \neq \{\vec{0}\}$.
◯	◯	If LU is the LU factorization of a square matrix A, then A is invertible if and only if U is invertible.
◯	◯	If the rank of an $n \times n$ matrix A is equal to n, then all diagonal entries of a row echelon form of A are nonzero.
◯	◯	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
◯	◯	If P is the stochastic matrix of a Markov chain, then any probability vector in Null$(P - I)$ is a steady-state vector for the Markov chain.
◯	◯	If M is an $n \times n$ matrix and $\det(M^{2022}) = 1$, then M has linearly independent columns.
◯	◯	If $A, B \in \mathbb{R}^{n\times n}$, $\det A = 2$, and $\det B = -3$, then the product AB is invertible.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with $\dim(\text{Null}(A)) = 3$.
◯	◯	A is the standard matrix of a one-to-one linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^5$ with rank$(A) = 2$.
◯	◯	A is a matrix whose columns do not form a basis for Col(A).
◯	◯	A is a 5×3 matrix with rank$(A) = 2\dim(\text{Null}(A))$.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) The column space of a matrix A is spanned by the vector $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the null space of A has dimension 2. Which one of the following statements is **false**? *Choose only one.*

- ○ $\text{rank}(A) = 1$.
- ○ A is a 2×3 matrix.
- ○ If U is an echelon form of A, then $\{\vec{v}\}$ is a basis for $\text{Col}(U)$.
- ○ The linear system $A\vec{x} = c\vec{v}$ is consistent for all values of $c \in \mathbb{R}$.

2. (2 points) Suppose $A, B \in \mathbb{R}^{n\times n}$ with $AB = -BA$ and $A^2 = B^2$. Fill in the blanks in the following equation **using only numbers** to make it true.

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^2 = \begin{bmatrix} \underline{\qquad} A^2 & \underline{\qquad} I_n \\ \underline{\qquad} I_n & \underline{\qquad} A^2 \end{bmatrix}.$$

You do not need to justify your reasoning for questions on this page.

3. (2 points) Let $\mathcal{H}$ be a subspace of $\mathbb{R}^3$ that is composed of all vectors $\vec{x} = (x_1, x_2, x_3)$ that satisfy the following two equations:

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \end{aligned}$$

What is the dimension of $\mathcal{H}$?

$$\dim \mathcal{H} = \boxed{}$$

4. (2 points) Let $\mathcal{V}$ be a subspace of $\mathbb{R}^3$ that is spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} \right\}$$

What is the dimension of $\mathcal{V}$?

$$\dim \mathcal{V} = \boxed{}$$

You do not need to justify your reasoning for questions on this page.

5. (4 points) Find the LU factorization of A, where

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{bmatrix},$$

by filling in the blanks below.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \_\_\_ & 1 & 0 \\ \_\_\_ & \_\_\_ & 1 \end{bmatrix}, \quad U = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ 0 & \_\_\_ & \_\_\_ \\ 0 & 0 & \_\_\_ \end{bmatrix}.$$

6. (4 points) Find a basis for the $\lambda = -1$ eigenspace of the matrix A. *Hint: Check your answer.*

$$A = \begin{bmatrix} -7 & -6 & -6 \\ 6 & 5 & 6 \\ 3 & 3 & 2 \end{bmatrix}$$

You do not need to justify your reasoning for questions on this page.

7. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2) = (x_1 + 3x_2, x_1 + x_2).$$

Let R be the rectangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(1, 0)$, $(0, 3)$, $(1, 3)$.

(i) What is the standard matrix of T?

(ii) What is the area of the rectangle R?

(iii) Find the area of the image of R under the linear transformation T.

8. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Let

$$A = \begin{pmatrix} 1 & -1 & k \\ 1 & h & 2 \\ h & 1 & -2 \end{pmatrix}$$

(a) Find the value of h and the value of k such that $\dim(\text{Null}(A)) = 2$.

$h =$ ☐ $k =$ ☐

(b) Let $k = 4$. For what values of h is $\dim(\text{Col}(A)) = 2$.

$h =$ ☐

(c) Let $h = 0$ and $k = 0$. Is the vector $\vec{v} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ in the null space of A?

Note: Compute $A\vec{v}$ and use this calculation to clearly justify your answer in a few words using the space below for full credit.

○ yes ○ no

9. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Compute $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 2 & 5 \\ 0 & 1 & -2 \end{bmatrix}^{-1}$. *Hint: Check your answer!*

10. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Suppose

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Solve $LU\vec{x} = \vec{b}$ for $\vec{x}$.

$$\vec{x} = \boxed{}$$

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Math 1554 Linear Algebra Spring 2022

Midterm 2 Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ______________________ GTID Number: ___________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ______________________

Circle your instructor:

Prof Barone Prof Shirani Prof Simone Prof Timko

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Midterm 2 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

1. (a) (10 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	Any set of vectors in $\mathbb{R}^4$ which is linearly independent must be a basis for a subspace of $\mathbb{R}^4$.
◯	◯	If $A, B, C \in \mathbb{R}^{n\times n}$ and $AB = I_n = BC$, then $A = C^{-1}$.
◯	◯	If $A \in \mathbb{R}^{n\times n}$ is invertible, then $\det(A^T A) = 1$.
◯	◯	If $A\vec{x} = A\vec{y}$ for all vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$, then $\text{Null}(A) = \mathbb{R}^n$.
◯	◯	If LU is the LU factorization of a square matrix A, then $\det(A) = \det(U)$.
◯	◯	If the rank of an $n \times n$ matrix A is equal to $n - 1$, then all entries of some row of A are zero.
◯	◯	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
◯	◯	If P is the stochastic matrix of a Markov chain, then any probability vector in $\text{Null}(P - I)$ is a steady-state vector for the Markov chain.
◯	◯	If M is an $n \times n$ matrix and $\det(M^2) = 1$, then $\det(M) = 1$.
◯	◯	If $A, B \in \mathbb{R}^{n\times n}$, $\det A = 1$, and $\det B = -1$, then the product AB is row equivalent to the identity matrix.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$ with $\dim(\text{Null}(A)) = 3$.
◯	◯	A is the standard matrix of an onto linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with dim $\text{Nul}(A) = 1$.
◯	◯	A is a matrix whose columns do not form a basis for $\text{Col}(A)$.
◯	◯	A is a 5×4 matrix with $\text{rank}(A) = 2\dim(\text{Null}(A))$.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) The column space of a matrix A is spanned by the vectors $\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and the null space of A has dimension 1. Which one of the following statements is **false**? *Choose only one.*

- ◯ $\text{rank}(A) = 2$.
- ◯ A is a 3×2 matrix.
- ◯ $\{\vec{w}\}$ is a basis for $\text{Col}(A)$.
- ◯ The linear system $A\vec{x} = (\vec{v} + \vec{w})$ is consistent.

2. (2 points) Suppose $A, B \in \mathbb{R}^{n \times n}$ with $A^2 = B = -B^2$. Fill in the blanks in the following equation **using only numbers** to make it true.

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^2 = \begin{bmatrix} \_\_\_\_A^3 & \_\_\_\_A^3 \\ \_\_\_\_A^3 & \_\_\_\_A^3 \end{bmatrix}.$$

You do not need to justify your reasoning for questions on this page.

3. (2 points) Let $\mathcal{H}$ be a subspace of $\mathbb{R}^4$ that is composed of all vectors $\vec{x} = (x_1, x_2, x_3, x_4)$ that satisfy the following two equations:

$$\begin{aligned} x_1 + 3x_2 - 3x_3 + 2x_4 &= 0 \\ -2x_1 - 6x_2 + 6x_3 - 4x_4 &= 0 \end{aligned}$$

What is the dimension of $\mathcal{H}$?

$$\dim\mathcal{H} = \boxed{}$$

4. (2 points) Let $\mathcal{V}$ be a subspace of $\mathbb{R}^3$ that is spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ 0 \\ 0 \end{pmatrix} \right\}$$

What is the dimension of $\mathcal{V}$?

$$\dim\mathcal{V} = \boxed{}$$

Midterm 2 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

5. (4 points) Find the LU factorization of A, where

$$A = \begin{bmatrix} 3 & -6 & 3 \\ 6 & -7 & 2 \\ -1 & 2 & 0 \end{bmatrix},$$

by filling in the blanks below.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \underline{\quad\quad} & 1 & 0 \\ \underline{\quad\quad} & \underline{\quad\quad} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} \underline{\quad\quad} & \underline{\quad\quad} & \underline{\quad\quad} \\ 0 & \underline{\quad\quad} & \underline{\quad\quad} \\ 0 & 0 & \underline{\quad\quad} \end{bmatrix}.$$

6. (4 points) Find a basis for the $\lambda = -2$ eigenspace of the matrix A. *Hint: Check your answer.*

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

You do not need to justify your reasoning for questions on this page.

7. (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(x_1, x_2) = (2x_1 - 3x_2, -6x_1 + 9x_2).$$

Let R be the rectangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(2,0)$, $(0,5)$, $(2,5)$.

(i) What is the standard matrix of T?

(ii) What is the area of the rectangle R?

(iii) Find the area of the image of R under the linear transformation T.

8. (6 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Let

$$A = \begin{pmatrix} -1 & 2 & k \\ 1 & h & -1 \\ h & 4 & 2 \end{pmatrix}$$

(a) Find the value of h and the value of k such that $\dim(\text{Null}(A)) = 2$.

$h = \boxed{}$ $k = \boxed{}$

(b) Let $k = 2$. For what values of h is $\dim(\text{Col}(A)) = 2$.

$h = \boxed{}$

(c) Let $h = -2$ and $k = 0$. Is the vector $\vec{v} = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$ in the null space of A?

Note: Compute $A\vec{v}$ and use this calculation to clearly justify your answer in a few words using the space below for full credit.

○ yes ○ no

9. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Compute $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & -4 \\ 0 & \frac{1}{2} & 0 \end{bmatrix}^{-1}$. *Hint: Check your answer!*

10. (4 points) ***Show work*** *on this page with work under the problem, and* ***your answer in the box.***

Suppose

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}.$$

Solve $LU\vec{x} = \vec{b}$ for $\vec{x}$.

$$\vec{x} = \boxed{}$$

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Math 1554 Linear Algebra Fall 2022

Midterm 2

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ___________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler

Prof Blumenthal Prof Sun Prof Shirani

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You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	If A, B and C are $n \times n$ matrices, B is invertible and $AC = B$, then C is invertible.
◯	◯	If $A = LU$ is an LU-factorization of a square matrix A, then $\det(A) = \det(U)$
◯	◯	If $\vec{x}$ is a vector in $\mathbb{R}^3$ and B is a basis for $\mathbb{R}^3$, then $[\vec{x}]_B$ has 3 entries.
◯	◯	If A, B and C are $n \times n$ matrices, A is invertible and $AB = AC$, then $B = C$.
◯	◯	If $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, then the set of solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$.
◯	◯	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
◯	◯	If two matrices A, B share an eigenvector $\vec{v}$, with eigenvalue λ for matrix A and eigenvalue μ for the matrix B, then $\vec{v}$ is an eigenvector of the matrix $(A + 2B)$ with eigenvalue $\lambda + 2\mu$.
◯	◯	For any 2×2 real matrix A, we have $\det(-A) = -\det(A)$.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A matrix $A \in \mathbb{R}^{n\times n}$ such that A is invertible and A^T is singular.
◯	◯	A 3×3 matrix A with $\dim(\text{Null}(A)) = 0$ such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solution.
◯	◯	$T : \mathbb{R}^3 \to \mathbb{R}^3$ that is onto and its standard matrix has determinant equal to -1.
◯	◯	Two square matrices A, B with $\det(A)$ and $\det(B)$ both non-zero, and the matrix AB is singular.

You do not need to justify your reasoning for questions on this page.

(c) (3 points) If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible write `NP` in the space.

$$\begin{pmatrix} 1 & & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 1 & & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & \\ 1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

(d) (2 points) Let A be a 3×3 upper triangular matrix and assume that the volume of the parallelpiped determined by the columns of A is equal to 1. Which of the following statements is FALSE?

- ○ A is invertible.
- ○ The diagonal entries of A are either 1 or -1.
- ○ For every 3×3 matrix B we have $|\det(AB)| = |\det(B)|$.
- ○ If B is a matrix obtained by interchanging two rows of A, then the volume of the parallelepiped determined by the columns of B is equal to 1.

You do not need to justify your reasoning for questions on this page.

2. (2 points) Suppose A and B are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix

$$\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \underline{\quad\quad} & \underline{\quad\quad} \\ \underline{\quad\quad} & \underline{\quad\quad} \end{pmatrix}.$$

3. (2 points) Suppose A is a $m \times n$ matrix and B is $m \times 5$ matrix. Find the dimensions of the matrix C in the block matrix

$$\begin{pmatrix} A & B \\ I_n & C \end{pmatrix}.$$

C has $\boxed{\quad\quad}$ rows and $\boxed{\quad\quad}$ columns.

You do not need to justify your reasoning for questions on this page.

4. Fill in the blanks.

(a) (3 points) Give a matrix A whose column space is spanned by the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. If this is not possible, write `NP` in the box.

$$A = \boxed{}$$

(b) (3 points) Use the determinant to find all values of $\lambda \in \mathbb{R}$ such that the following matrix is singular.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 5 \\ \lambda & 2 & 3 \end{pmatrix}.$$

$$\lambda = \boxed{}$$

Midterm 2. Your initials: _______________

You do not need to justify your reasoning for questions on this page.

5. (3 points) Find the value of h such that the matrix

$$A = \begin{pmatrix} 5 & h \\ 1 & 3 \end{pmatrix}$$

has an eigenvalue with algebraic multiplicity 2.

$$h = \boxed{}$$

6. (3 points) Let $\mathcal{P}_B$ be a parallelogram that is determined by the columns of the matrix $B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$, and $\mathcal{P}_C$ be a parallelogram that is determined by the columns of the matrix $C = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. Suppose A is the standard matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $\mathcal{P}_B$ to $\mathcal{P}_C$. What is the value of $|\det(A)|$?

$$|\det(A)| = \boxed{}$$

7. (5 points) **Show all work for problems on this page.**
Given that 4 is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & -2 & 2 \\ 2 & 2 & -2 \\ 1 & -1 & 4 \end{pmatrix},$$

find an eigenvector $\vec{v}$ of A such that $A\vec{v} = 4\vec{v}$.

$\vec{v}$= $\boxed{}$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 8 \\ 2 & 8 & 6 \end{pmatrix}.$$

$L =$ $\boxed{}$

$U =$ $\boxed{}$

9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k, \ k = 0, 1, 2, \ldots$.
Suppose P has eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1/2$ and $\lambda_3 = 0$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
(i) If $\vec{x}_0 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_2$, then what is $\vec{x}_3$?

$\vec{x}_3$= $\boxed{}$

(ii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_1$?
Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$\vec{x}_1$= $\boxed{}$

(iii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$\lim\limits_{k\to\infty} \vec{x}_k$= $\boxed{}$

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Math 1554 Linear Algebra Fall 2022

Midterm 2 Make-up

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ____________________

Student GT Email Address: ______________________________@gatech.edu

Section Number (e.g. A3, G2, etc.) ________ TA Name __________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler

Prof Blumenthal Prof Sun Prof Shirani

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
- Calculators, notes, cell phones, books are not allowed.
- Use dark and clear writing: your exam will be scanned into a digital system.
- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	If A, B and C are $n \times n$ matrices, B is invertible and $AC = B$, then C is invertible.
○	○	If $A = LU$ is an LU-factorization of a square matrix A, then $\det(A) = \det(U)$
○	○	If $\vec{x}$ is a vector in $\mathbb{R}^3$ and B is a basis for $\mathbb{R}^3$, then $[\vec{x}]_B$ has 3 entries.
○	○	If A, B and C are $n \times n$ matrices, A is invertible and $AB = AC$, then $B = C$.
○	○	If $A \in \mathbb{R}^{m\times n}$ and $\vec{b} \in \mathbb{R}^m$, then the set of solutions $\vec{x}$ to the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^n$.
○	○	The set of all probability vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.
○	○	If two matrices A, B share an eigenvector $\vec{v}$, with eigenvalue λ for matrix A and eigenvalue μ for the matrix B, then $\vec{v}$ is an eigenvector of the matrix $(A + 2B)$ with eigenvalue $\lambda + 2\mu$.
○	○	For any 2×2 real matrix A, we have $\det(-A) = -\det(A)$.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A matrix $A \in \mathbb{R}^{n\times n}$ such that $\det(A) = 0$ and A^T is row equivalent to I_n.
○	○	A 3×3 matrix A with $\dim(\text{Null}(A)) = 1$ such that the system $A\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solution.
○	○	$T_A : \mathbb{R}^2 \to \mathbb{R}^3$ that is onto and $\text{Col}(A) = \mathbb{R}^3$.
○	○	Two square matrices A, B with $\det(A)$ and $\det(B)$ both non-zero, and the matrix AB is invertible.

You do not need to justify your reasoning for questions on this page.

(c) (3 points) If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are not invertible. If it is not possible write `NP` in the space.

$$\begin{pmatrix} 1 & & 1 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 0 & & 3 \\ 1 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & \\ 1 & 2 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

(d) (2 points) Let A be a 3×3 upper triangular matrix and assume that the volume of the parallelpiped determined by the columns of A is equal to 1. Which of the following statements is FALSE?

- ◯ A is invertible.
- ◯ The diagonal entries of A are either 1 or -1.
- ◯ For every 3×3 matrix B we have $|\det(AB)| = |\det(B)|$.
- ◯ If B is a matrix obtained by interchanging two rows of A, then the volume of the parallelepiped determined by the columns of B is equal to 1.

Midterm 2 Make-up. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

2. (2 points) Suppose A and B are invertible $n \times n$ matrices. Find the inverse of the partitioned matrix

$$\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{pmatrix}.$$

3. (2 points) Suppose A is a $m \times q$ matrix and B is $5 \times r$ matrix. Find the dimensions of the matrix C in the block matrix

$$\begin{pmatrix} A & B \\ I_q & C \end{pmatrix}.$$

C has $\boxed{\qquad}$ rows and $\boxed{\qquad}$ columns.

You do not need to justify your reasoning for questions on this page.

4. Fill in the blanks.

(a) (3 points) Give a matrix A whose column space is spanned by the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix}-2\\2\\1\end{pmatrix}$. If this is not possible, write `NP` in the box.

$$A=\boxed{}$$

(b) (3 points) Use the determinant to find all values of $\lambda \in \mathbb{R}$ such that the following matrix is singular.

$$\begin{pmatrix} 3 & -1 & \lambda \\ 5 & 1 & 4 \\ -9 & 3 & 2 \end{pmatrix}.$$

$$\lambda = \boxed{}$$

You do not need to justify your reasoning for questions on this page.

5. (3 points) Let H be the set of solutions to the system below, where $b_1 = b_2 = b_3 = 0$. Let K be the set of vectors $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system below is consistent. Find $\dim(H)$ and $\dim(K)$.

$$\begin{aligned} 2x_1 - x_2 + 2x_3 - 2x_4 &= b_1 \\ 4x_1 - 2x_2 + 4x_3 - 4x_4 &= b_2 \\ 8x_1 - 4x_2 + 8x_3 - 8x_4 &= b_3 \end{aligned}$$

$\dim(H) = \boxed{}$ $\qquad$ $\dim(K) = \boxed{}$

6. (3 points) Let $\mathcal{P}_B$ be a parallelogram that is determined by vectors $\vec{b}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{b}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Suppose $\mathcal{P}_B$ is mapped to another parallelogram $\mathcal{P}_C$ by a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. Let $A = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$ be the standard matrix of T. What is the area of $\mathcal{P}_C$?

$\text{area}(\mathcal{P}_C) = \boxed{}$

7. (5 points) **Show all work for problems on this page.**
   Given that 4 is an eigenvalue of the matrix

$$A = \begin{pmatrix} 6 & 2 & -8 \\ 2 & 6 & -8 \\ 2 & 2 & -4 \end{pmatrix},$$

   find a basis for the $\lambda = 4$ eigenspace.

$\vec{v}=$ $\boxed{}$

8. (6 points) Find the LU-factorization of

$$A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 6 \\ 1 & -1 & 8 \end{pmatrix}.$$

$L =$ $\boxed{}$

$U =$ $\boxed{}$

9. (6 points) **Show all work for problems on this page.**
Consider the Markov chain $\vec{x}_{k+1} = P\vec{x}_k,\ k = 0, 1, 2, \ldots$.
Suppose P has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1/2$ and $\lambda_3 = 1$. Let $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ be eigenvectors corresponding to λ_1, λ_2, and λ_3, respectively:

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Note: you may leave your answers as linear combinations of the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.
(i) If $\vec{x}_0 = \frac{1}{6}\vec{v}_1 + \frac{1}{3}\vec{v}_2 + \frac{1}{2}\vec{v}_3$, then what is $\vec{x}_2$?

$\vec{x}_2$= $\boxed{}$

(ii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_1$?
Hint: write $\vec{x}_0$ as a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$\vec{x}_1$= $\boxed{}$

(iii) If $\vec{x}_0 = \begin{pmatrix} 1/4 \\ 1/2 \\ 1/4 \end{pmatrix}$, then what is $\vec{x}_k$ as $k \to \infty$?

$\lim_{k\to\infty} \vec{x}_k$= $\boxed{}$

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