In-Class Midterm 1 Review, Math 1554

1. Consider the matrix A and vectors $\vec{b}_1$ and $\vec{b}_2$.

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}, \quad \vec{b}_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

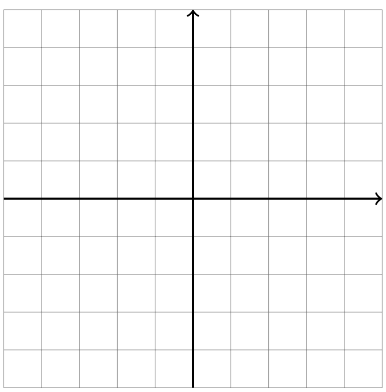
If possible, on the grids below, draw

(i) the two vectors and the span of the columns of A,

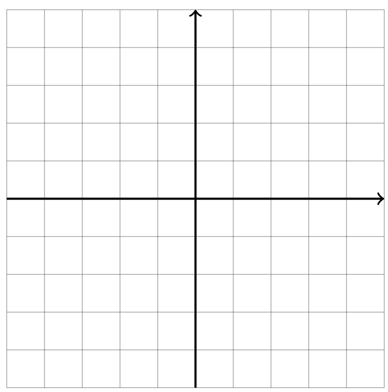
(ii) the solution set of $A\vec{x} = \vec{b}_1$.

(iii) the solution set of $A\vec{x} = \vec{b}_2$.

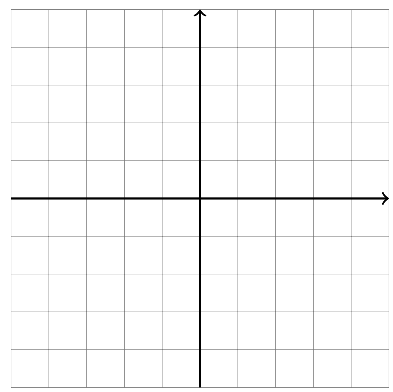
(i) $\vec{b}_1$, $\vec{b}_2$, column span



ii) solution set $Ax = \vec{b}_1$



iii) solution set $Ax = \vec{b}_2$



2. Indicate **true** if the statement is true, otherwise, indicate **false**. For the statements that are false, give a counterexample.

	true	false	counterexample
a) If $A \in \mathbb{R}^{M\times N}$ has linearly dependent columns, then the columns of A cannot span $\mathbb{R}^M$.	○	○	
b) If there are some vectors $\vec{b} \in \mathbb{R}^M$ that are not in the range of $T(\vec{x}) = A\vec{x}$, then there cannot be a pivot in every row of A.	○	○	
c) If the transform $\vec{x} \to A\vec{x}$ projects points in $\mathbb{R}^2$ onto a line that passes through the origin, then the transform cannot be one-to-one.	○	○	

3. If possible, write down an example of a matrix with the following properties. If it is not possible to do so, write *not possible.*

   (a) A linear system that is homogeneous and has no solutions.

   (b) A standard matrix A associated to a linear transform, T. Matrix A is in RREF, and $T_A : \mathbb{R}^3 \to \mathbb{R}^4$ is one-to-one.

   (c) A 3×7 matrix A, in RREF, with exactly 2 pivot columns, such that $A\vec{x} = \vec{b}$ has exactly 5 free variables.

4. Consider the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & 7 & 0 & -5 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

   (a) Express the augmented matrix $(\, A \,|\, \vec{b}\,)$ in RREF.

   (b) Write the set of solutions to $A\vec{x} = \vec{b}$ in parametric vector form. Your answer must be expressed as a vector equation.

Math 1554 Linear Algebra Fall 2022

Midterm 1

PLEASE PRINT YOUR NAME CLEARLY IN ALL CAPITAL LETTERS

Name: ________________________ GTID Number: ____________________

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Section Number (e.g. A3, G2, etc.) ________ TA Name ________________________

Circle your instructor:

Prof Vilaca Da Rocha Prof Kafer Prof Barone Prof Wheeler
Prof Blumenthal Prof Sun Prof Shirani

Student Instructions

- **Show your work** and justify your answers for all questions unless stated otherwise.
- **Organize your work** in a reasonably neat and coherent way.
- **Simplify your answers** unless explicitly stated otherwise.
- **Fill in circles** completely. Do not use check marks, X's, or any other marks.
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- Exam pages are double sided. Be sure to complete both sides.
- Leave a 1 inch border around the edges of exams.
- The last page is for scratch work. Please use it if you need extra space.
- This exam has 7 pages of questions.

You do not need to justify your reasoning for questions on this page.

1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
◯	◯	If A has a pivot in every column then the system $A\vec{x} = \vec{b}$ has a unique solution.
◯	◯	Suppose A is a 6×4 matrix with 4 pivots, then there is $\vec{b}$ such that $A\vec{x} = \vec{b}$ has no solution.
◯	◯	The sets $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_1 + \vec{v}_2, -\vec{v}_1 - \vec{v}_2\}$ have the same span.
◯	◯	If A and B are square $n \times n$ matrices, then $A^2 - B^2 = (A - B)(A + B)$.
◯	◯	The matrix equation $A\vec{x} = \vec{0}$ is always consistent.
◯	◯	Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are nonzero vectors in $\mathbb{R}^n$ and the sets $\{\vec{v}_1, \vec{v}_2\}$, $\{\vec{v}_1, \vec{v}_3\}$, and $\{\vec{v}_2, \vec{v}_3\}$ are all linearly independent. Then, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.
◯	◯	If $A\vec{v} = 0$, $A\vec{u} = 0$ and $\vec{w} = 3\vec{v} - 2\vec{u}$, then $A\vec{w} = 0$.
◯	◯	Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\vec{x}) = \vec{b}$ has a solution for every $\vec{b} \in \mathbb{R}^m$. Then T is one-to-one.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
◯	◯	A 7×5 matrix A with linearly independent columns.
◯	◯	A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that is not onto and its standard matrix has linearly independent columns.
◯	◯	$T : \mathbb{R}^3 \to \mathbb{R}^2$ that is onto and its standard matrix has exactly one non-pivotal column.
◯	◯	Two non-zero matrices A, B of size 2×2 with $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

You do not need to justify your reasoning for questions on this page.

(c) (2 points) Let

$$\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 3h & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

be an augmented matrix of a system of linear equations. For which values of h does the system have a free variable? *Choose the best option.*

- ○ 0 only
- ○ $\frac{1}{3}$ only
- ○ 1 only
- ○ for all values of h
- ○ for no values of h

(d) (2 points) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^1$ maps each of the standard unit vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ to 1. Which of the following statements is TRUE? *Select only one.*

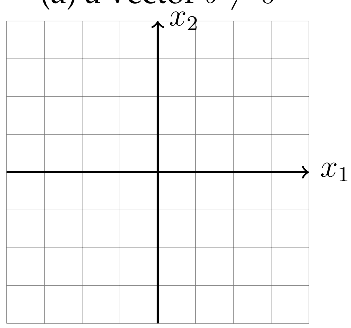
- ○ T is one-to-one.
- ○ T is not onto.
- ○ The solution set of $T(\vec{x}) = \vec{0}$ spans a plane in $\mathbb{R}^3$.
- ○ The range of T is $\{1\}$.

Midterm 1. Your initials: _______________

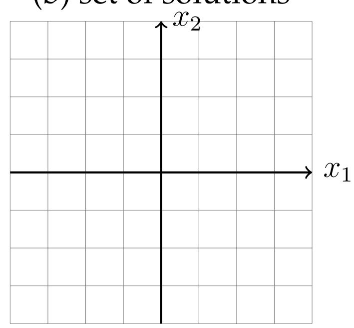
You do not need to justify your reasoning for questions on this page.

2. (4 points) Suppose $A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and sketch (a) a non-zero vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent, and (b) the set of solutions to $A\vec{x} = \vec{0}$.

(a) a vector $\vec{b} \neq \vec{0}$



(b) set of solutions



3. (2 points) Consider the linear system in variables x_1, x_2, x_3 with unknown constants below.

$$\begin{aligned} a_1x_1 + a_2x_2 + a_3x_3 &= b_1 \\ c_1x_1 + c_2x_2 + c_3x_3 &= b_2 \end{aligned}$$

Which of the following statements about the solution set of this system are possible?
Select all that apply.

- ◯ The solution set is empty.
- ◯ The solution set is a single point.
- ◯ The solution set is a line.
- ◯ The solution set is a plane.

You do not need to justify your reasoning for questions on this page.

4. Fill in the blanks.

   (a) (3 points) Let A be a coefficient matrix of size 2×2 and B be a coefficient matrix of size 3×2. Construct an example of two augmented matrices $\left[A|\vec{b}\right]$ and $\left[B|\vec{d}\right]$ which are both in RREF and such that the systems $A\vec{x} = \vec{b}$ and $B\vec{x} = \vec{d}$ each have the exact same unique solution $x_1 = 3$ and $x_2 = 6$. If this is not possible write NP in each box.

   $\left[A|\vec{b}\right] = \boxed{}$ $\qquad$ $\left[B|\vec{d}\right] = \boxed{}$

   (b) (2 points) Let $\vec{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find c_1, c_2 such that $\vec{b} = c_1\vec{u}_1 + c_2\vec{u}_2$.

   $c_1 = \boxed{}$ $\qquad$ $c_2 = \boxed{}$

Midterm 1. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

5. (8 points) Let T be a linear transformation that maps $\vec{v}_1$ to $T(\vec{v}_1)$ and $\vec{v}_2$ to $T(\vec{v}_2)$, where

$$\vec{v}_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad T(\vec{v}_1) = \begin{pmatrix} 1 \\ 3 \\ 0 \\ 1 \end{pmatrix}, \quad T(\vec{v}_2) = \begin{pmatrix} 3 \\ -1 \\ -2 \\ 1 \end{pmatrix}.$$

(i) What is domain and codomain of T?

domain is $\boxed{}$

codomain is $\boxed{}$

(ii) Is it true that $\mathbb{R}^2 = \text{span}\{\vec{v}_1, \vec{v}_2\}$? ◯ yes ◯ no

(iii) Write $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as linear combinations of $\vec{v}_1$ and $\vec{v}_2$.

$\vec{e}_1 = \boxed{}$

$\vec{e}_2 = \boxed{}$

(iv) What is the standard matrix of T? $\boxed{}$

(v) Is T one-to-one? ◯ yes ◯ no

6. **Show all work for problems on this page.**

   (a) (3 points) For what value of k will matrix A have exactly two pivots?

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & k \end{pmatrix}$$

$$k = \boxed{}$$

   (b) (4 points) Find b and c such that $AB = BA$.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & b \\ c & 0 \end{pmatrix}$$

$$b = \boxed{} \qquad c = \boxed{}$$

7. (4 points) **Show your work for problems on this page.**
   Write down the parametric vector form for solutions to the homogeneous equation $A\vec{x} = \vec{0}$.

$$A = \begin{bmatrix} 1 & -1 & -2 & -3 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ -1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

8. (4 points) Determine whether the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.
   Justify your answer in the space below.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -2 \\ 2 \\ -9 \end{bmatrix}$$

○ linearly independent ○ linearly dependent

Math 1554 Linear Algebra   Spring   2022

Midterm 1

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Section Number (e.g. A3, G2, etc.) _______ TA Name ________________________

Circle your instructor:

Prof Barone   Prof Shirani   Prof Simone   Prof Timko

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1. (a) (8 points) Suppose A is an $m \times n$ matrix and $\vec{b} \in \mathbb{R}^m$ unless otherwise stated. Select **true** if the statement is true for all choices of A and $\vec{b}$. Otherwise, select **false**.

true	false	
○	○	The span of two non-zero vectors in $\mathbb{R}^3$ is necessarily a plane.
○	○	If an echelon form of A has a row of zeros, then the system $A\vec{x} = \vec{b}$ has a free variable.
○	○	If $A\vec{v} = \vec{b}$, and $[A \mid \vec{b}]$ is row equivalent to $[C \mid \vec{d}]$, then $C\vec{v} = \vec{d}$.
○	○	If the columns of A span $\mathbb{R}^m$, then $A\vec{x} = \vec{b}$ is consistent for any $\vec{b} \in \mathbb{R}^m$.
○	○	If $A\vec{x} = \vec{0}$ has a non-trivial solution, then the columns of A are linearly dependent.
○	○	If $\vec{v}$ and $\vec{u}$ are solutions of a homogeneous system of linear equations, then $\vec{v} + \vec{u}$ is also a solution of that system.
○	○	If the columns of a matrix A are linearly dependent, then the system $A\vec{x} = \vec{b}$ can not have a unique solution.
○	○	If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(\vec{x}) = \vec{b}$ has no solution for some $\vec{b} \in \mathbb{R}^n$, then T is not one-to-one.

(b) (4 points) Indicate whether the following situations are possible or impossible.

possible	impossible	
○	○	A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that is not onto.
○	○	A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ that is onto and its standard matrix has two non-pivotal columns.
○	○	A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$ that is onto and its standard matrix has linearly dependent columns.
○	○	Three non-zero matrices A, B, C of size 2×2 with $AC = BC$ and $A \neq B$.

Midterm 1. Your initials: ______________

You do not need to justify your reasoning for questions on this page.

(c) (2 points) Let

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & h^2 & 0 \end{array}\right]$$

be a row echelon form of an augmented matrix of a system of linear equations. For which values of h is the system consistent? *Choose the best option.*

◯ for all values of h
◯ 0 only
◯ for no values of h
◯ 1 and -1 only

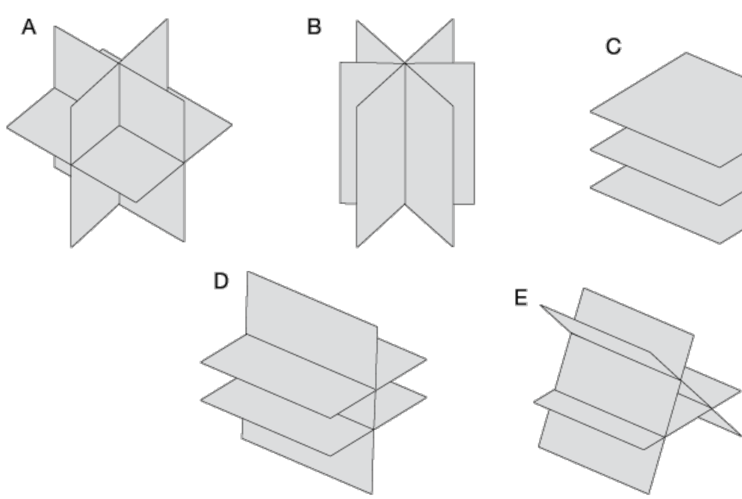
(d) (2 points) Let

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & \pi & 2 & -1 \\ 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

be a row echelon form of an augmented matrix of a system of equations. How many solutions does the system have? *Choose the best option.*

◯ 0
◯ 1
◯ infinitely many

(e) (2 points) Suppose $A\vec{x} = \vec{b}$ is a system of three linear equations in three variables. If the system $A\vec{x} = \vec{b}$ is consistent, which of the following could be the graphs in $\mathbb{R}^3$ of the three equations represented by the rows of $[A \mid b]$? *Circle all pictures that apply.*



Midterm 1. Your initials: ______________

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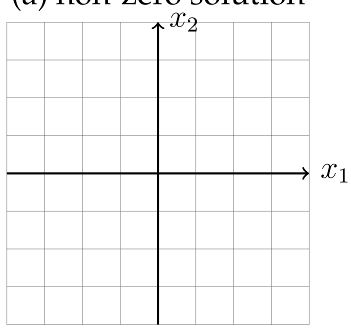
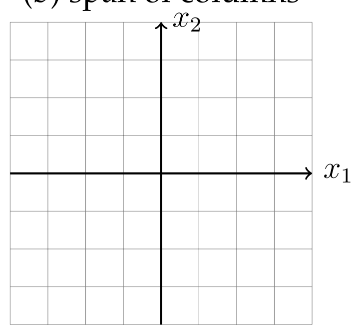
2. (4 points) Suppose $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ and sketch a) a non-zero solution to $A\vec{x} = \vec{0}$, and b) the span of the columns of A.

(a) non-zero solution



(b) span of columns



3. (5 points) Let $A \in \mathbb{R}^{3\times5}, B \in \mathbb{R}^{4\times3}$ and $\vec{a} \in \mathbb{R}^3, \vec{b} \in \mathbb{R}^4, \vec{c} \in \mathbb{R}^5$. Which of the following are defined? *Choose all the expressions which are defined.*

   ○ $B\vec{b}$

   ○ $A\vec{c}$

   ○ $A(B\vec{a})$

   ○ $B(A\vec{c})$

   ○ $B(\vec{a} + \vec{b})$

4. (3 points) In each of the following cases, indicate whether $A\vec{x} = \vec{b}$ has no solutions, a unique solution, infinitely many solutions, or if this can not be determined with the given information.

no solution	unique solution	infinitely many solutions	can't be determined	
○	○	○	○	$A \in \mathbb{R}^{3\times4}$, $\vec{b} = \vec{0}$, and A has 2 pivots
○	○	○	○	$A \in \mathbb{R}^{5\times2}$, $\vec{b} = \vec{0}$, and A has 2 pivots
○	○	○	○	$A \in \mathbb{R}^{3\times5}$ and A has 3 pivots

Midterm 1. Your initials: ______________

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5. Fill in the blanks.

   (a) (2 points) If the augmented matrix $[A \mid \vec{b}]$ of a system of equations is 3×6 and the system has two pivot (basic) variables, then how many free variables does it have?

   $\boxed{}$

   (b) (2 points) For what value(s) of h is the following set of vectors linearly dependent?

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}, \begin{pmatrix} 1 \\ h \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ h \end{pmatrix} \right\}$$

$$h = \boxed{}$$

Midterm 1. Your initials: _______________

6. **Show all work for problems on this page.**

   (a) (1 point) Let $\vec{b} = \begin{bmatrix} 3 \\ -4 \\ -6 \\ 1 \end{bmatrix}$, $\vec{a_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{a_3} = \begin{bmatrix} 3 \\ -3 \\ -5 \\ 2 \end{bmatrix}$. Is $\vec{b}$ in the span of $\vec{a_1}, \vec{a_2}$, and $\vec{a_3}$?

   ○ Yes

   ○ No

   (b) (2 points) If you answered yes to part (a), write $\vec{b}$ as a linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$. If you answered no, give an echelon form of the augmented matrix $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3 \mid \vec{b}]$.

7. (3 points) **Show your work for problems on this page.**

   Suppose that we have

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 4 & 0 & -1 & 3 \\ 0 & 0 & 1 & 5 & 2 \end{array}\right]$$

   Find the parametric vector form for the solutions of $A\vec{x} = \vec{b}$.

8. (2 points) Suppose $A\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $A\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Compute $A(2\vec{v} - \vec{u})$.

9. (8 points) **Show all work for problems on this page.** Consider the linear transformation defined by $T(x_1, x_2) = (x_1 + x_2, x_1, x_1 - x_2)$ with domain $\mathbb{R}^2$.

(i) What is the codomain of T?

(ii) What is the standard matrix of T?

(iii) Is T onto? ◯ yes ◯ no

(iv) Write an equation using the variables b_1, b_2, and b_3 which is satisfied exactly when $T(x_1, x_2) = (b_1, b_2, b_3)$ has a solution for x_1, x_2.

(v) What is the range of T?