

# Chapter 5. Distributions of Functions of Random Variables

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Math 3215 Summer 2023

Georgia Institute of Technology

## **Section 1.**

# **Functions of One Random Variable**

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## Functions of One Random Variable

Let  $X$  be a random variable.

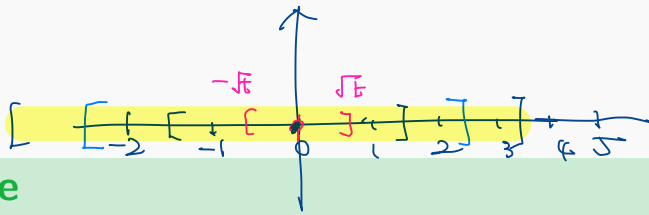
Define  $Y = u(X)$  for some function  $u$ .

We discuss how to find the distribution of  $Y$  from that of  $X$ .

↓  
look at CDF

## Functions of One Random Variable

pmf of  $X = f_X(x) = \begin{cases} \frac{1}{8} & , x = -2, -1, 0, 1, 2, 3, 4, 5 \\ 0 & , \text{o.w.} \end{cases}$



### Example

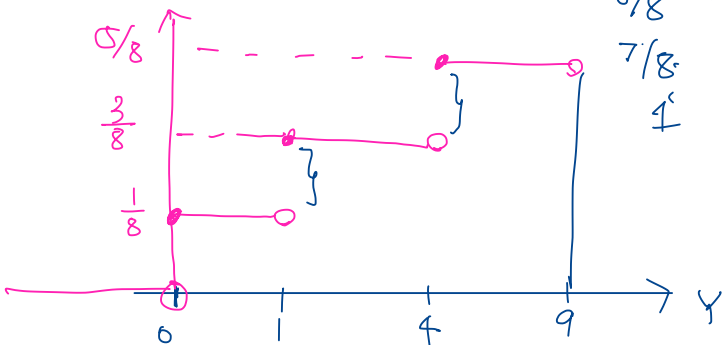
Let  $X$  have a discrete uniform distribution on the integers from -2 to 5.

Find the distribution of  $Y = X^2$ .  $\geq 0$

$$P(\underline{Y} \leq t) = P(\underline{X^2} \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t})$$

$t \geq 0$

$$= \begin{cases} \frac{1}{8} & , & 0 \leq t < 1 \\ \frac{3}{8} & , & 1 \leq \sqrt{t} < 2 \rightarrow 1 \leq t < 4 \\ \frac{5}{8} & , & 2 \leq \sqrt{t} < 3 \rightarrow 4 \leq t < 9 \\ \frac{6}{8} & , & 3 \leq \sqrt{t} < 4 \rightarrow 9 \leq t < 16 \\ \frac{7}{8} & , & 4 \leq \sqrt{t} < 5 \rightarrow 16 \leq t < 25 \\ 1 & , & 5 \leq \sqrt{t} \rightarrow t \geq 25 \end{cases}$$



$$P(\underline{Y} = \underline{k}) = P(X = \sqrt{k} \text{ or } -\sqrt{k})$$

$k = 1, 4, 9, 16, 25$

$$f_Y(16) = f_Y(9) = f_Y(0) = \frac{1}{8} \quad , \quad f_Y(1) = \frac{2}{8} = f_Y(4)$$

$f_Y(25)$

$$X \sim \text{Unif}(-1, 3)$$

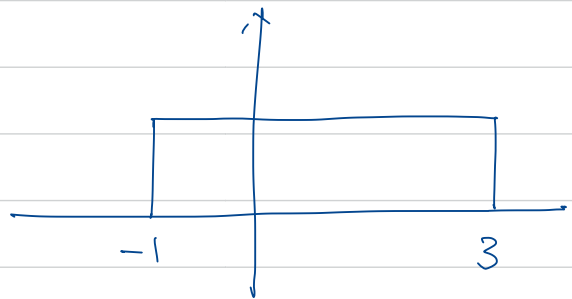
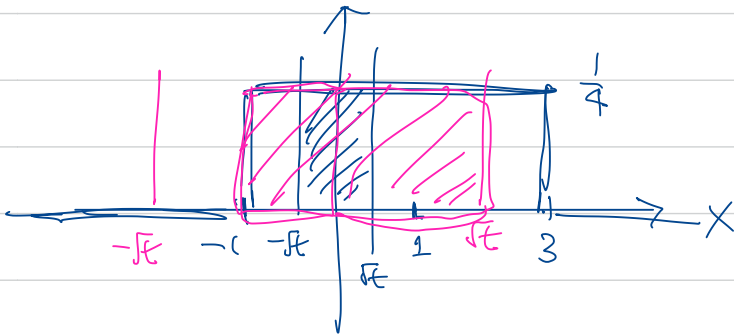
$$Y = X^2 \geq 0$$

$$P(Y \leq t) = 0 \quad \text{if } t \leq 0$$

If

$$0 < t < 1$$

$$P(Y \leq t) = P(-\sqrt{t} \leq X \leq \sqrt{t}) = 2 \cdot \sqrt{t} \cdot \frac{1}{4} = \frac{\sqrt{t}}{2}$$



If

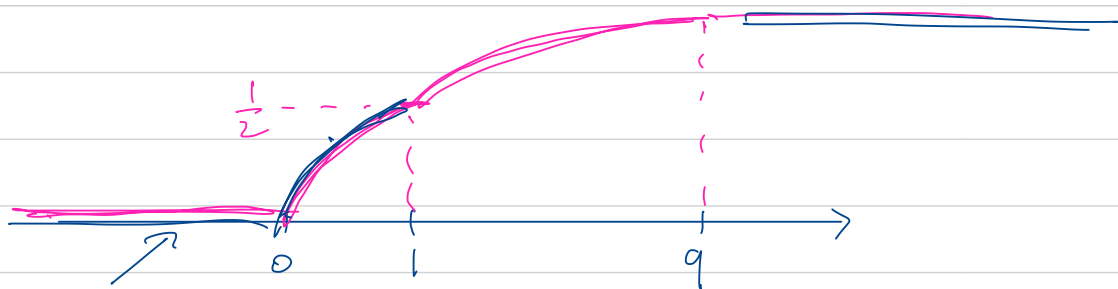
$$1 \leq \sqrt{t} < 3$$

$$P(Y \leq t) = \frac{1}{4} + \frac{1}{4}\sqrt{t}$$

If

$$\sqrt{t} \geq 3$$

$$P(Y \leq t) = 1$$



$F_Y(t)$

$$f_Y(t) = \frac{d}{dt} F_Y(t) = \begin{cases} 0 & t \leq 0 \text{ or } t \geq 9 \\ \frac{1}{4\sqrt{t}} & 0 < t < 1 \\ \frac{1}{8\sqrt{t}} & 1 \leq t \leq 9 \end{cases}$$

## CDF Technique

### Example

Let  $X$  have a gamma distribution with pdf

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}.$$

Find the distribution of  $Y = e^X$ .

## CDF Technique

### Theorem

Let  $X$  be a random variable with cdf  $F$ .

Suppose  $F$  is strictly increasing,  $F(a) = 0$ ,  $F(b) = 1$ .

Let  $Y \sim U(0, 1)$ .

Then,  $X = F^{-1}(Y)$ .

## Change of Variables

### Example

Let  $X$  have the pdf  $f(x) = 3(1 - x)^2$  for  $0 < x < 1$ .

Find the distribution of  $Y = (1 - X)^3$ .



## Exercise

Let  $X$  have the pdf  $f(x) = 4x^3$ ,  $0 < x < 1$ .

Find the pdf of  $Y = X^2$ .

## **Section 2.**

# **Transformations of Two Random Variables**

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## Transformations of Two Random Variables

If  $X_1$  and  $X_2$  are two continuous-type random variables with joint pdf  $f(x_1, x_2)$ .

Let  $Y_1 = u_1(X_1, X_2)$ ,  $Y_2 = u_2(X_1, X_2)$ .

If  $X_1 = v_1(Y_1, Y_2)$ ,  $X_2 = v_2(Y_1, Y_2)$ , then the joint pdf of  $Y_1$  and  $Y_2$  is

$$f_{Y_1, Y_2} = |J| f_{X_1, X_2}(v_1(y_1, y_2), v_2(y_1, y_2))$$

where  $J$  is the Jacobian given by

$$J := \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}.$$

## Transformations of Two Random Variables

### Example

Let  $X_1$  and  $X_2$  have the joint pdf

$$f(x_1, x_2) = 2, \quad 0 < x_1 < x_2 < 1.$$

Find the joint pdf of  $Y_1 = \frac{X_1}{X_2}$  and  $Y_2 = X_2$ .

## Exercise

Let  $X_1$  and  $X_2$  be independent random variables, each with pdf

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Find the joint pdf of  $Y_1 = X_1 - X_2$  and  $Y_2 = X_1 + X_2$ .

## **Section 3.**

# **Several Independent Random Variables**

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## Independent random variables

Recall that  $X_1$  and  $X_2$  are independent if

$$\mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A)\mathbb{P}(X_2 \in B)$$

for all  $A, B$ .

In particular, if  $X_1$  and  $X_2$  have pdfs, then  $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$ .

## Independent random variables

### Definition

In general, we say  $X_1, X_2, \dots, X_n$  are independent if  $\{X_1 \in A_1\}, \{X_2 \in A_2\}, \dots, \{X_n \in A_n\}$  are mutually independent, for any choice of  $A_1, A_2, \dots, A_n$ .

In particular, if  $X_1, X_2, \dots, X_n$  has pdfs, then the joint pdf is the product.

If  $X_1, X_2, \dots, X_n$  are independent and have the same distribution, we say they are i.i.d. or a random sample of size  $n$  from that common distribution.



## Independent random variables

### Example

Let  $X_1, X_2, X_3$  be a random sample from a distribution with pdf

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Find  $\mathbb{P}(0 < X_1 < 1, 2 < X_2 < 4, 3 < X_3 < 7)$ .

## Expectation and Variance

### Theorem

Let  $X_1, X_2, \dots, X_n$  be a sequence of random variables. Then,

$$\mathbb{E}[X_1 + X_2 + \dots + X_n] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n].$$

If they are independent, then

$$\mathbb{E}[X_1 X_2 \dots X_n] = \mathbb{E}[X_1] \mathbb{E}[X_2] \dots \mathbb{E}[X_n]$$

and

$$\text{Var}[X_1 + X_2 + \dots + X_n] = \text{Var}[X_1] + \text{Var}[X_2] + \dots + \text{Var}[X_n].$$

## Exercise

Let  $X_1, X_2, X_3$  be i.i.d. Geometric with  $p = \frac{3}{4}$ .

Let  $Y$  be the minimum of  $X_1, X_2, X_3$ .

Find  $\mathbb{P}(Y > 4)$ .

## **Section 4.**

# **The Moment-Generating Function Technique**

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## The Moment-Generating Function

### Theorem

If  $X_1, X_2, \dots, X_n$  are independent and have the mgfs  $M_{X_i}(t)$ , then the mgf of  $Y = a_1X_1 + \dots + a_nX_n$  is  $M_Y(t) = M_{X_1}(a_1t) \cdots M_{X_n}(a_nt)$ .

### Theorem

If  $X_1, X_2, \dots, X_n$  are i.i.d., then the mgf of  $Y = X_1 + \dots + X_n$  is  $M_Y(t) = M_X(t)^n$ . If  $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ , then the mgf is  $M_{\bar{X}}(t) = M_X(\frac{t}{n})^n$ .

## The Moment-Generating Function

### Example

Let  $X_1, X_2, \dots, X_n$  be i.i.d. Bernoulli with  $p$ .

Let  $Y = X_1 + \dots + X_n$ .

Find the mgf of  $Y$ .

## The Moment-Generating Function

### Example

Let  $X_1, X_2, \dots, X_n$  be i.i.d. exponential with  $\theta$ .

Let  $Y = X_1 + \dots + X_n$ .

Find the mgf of  $Y$ .

## Exercise

Let  $X_1, X_2, X_3$  be independent Poisson with means 2, 1, 4.

Find the mgf of  $Y = X_1 + X_2 + X_3$ .



## **Section 6.**

# **The Central Limit Theorem**

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## The Central Limit Theorem

Let  $X_1, X_2, \dots, X_n$  be i.i.d. with common distribution  $X$ .

Let  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ .

Let  $\bar{X} = \frac{X_1 + \dots + X_n}{n}$ , then  $\mathbb{E}[\bar{X}] = \mu$  and  $\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$ .

Let  $W = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ , then

$\mathbb{E}[W] =$

$\text{Var}(W) =$

## The Central Limit Theorem

### Theorem

If  $\mu$  and  $\sigma^2$  are finite, then the distribution of  $W$  converges to that of the standard normal distribution as  $n \rightarrow \infty$ .

The convergence is in the following sense: If  $n$  is large, for the standard normal  $Z$ ,

$$\mathbb{P}(W \leq x) \approx \mathbb{P}(Z \leq x) =: \Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{|y|^2}{2}} dy.$$

## The Central Limit Theorem

### Example

Let  $\bar{X}$  be the mean of a random sample of  $n = 25$  currents (in milliamperes) in a strip of wire in which each measurement has a mean of 15 and a variance of 4.

Find the approximate probability  $\mathbb{P}(14.4 < \bar{X} < 15.6)$ .

## The Central Limit Theorem

### Example

Let  $\bar{X}$  denote the mean of a random sample of size 25 from the distribution whose pdf is  $f(x) = \frac{x^3}{4}$ ,  $0 < x < 2$ .

Find the approximate probability  $\mathbb{P}(1.5 \leq \bar{X} \leq 1.65)$ .

## Exercise

Let  $X$  equal the maximal oxygen intake of a human on a treadmill, where the measurements are in milliliters of oxygen per minute per kilogram of weight.

Assume that, for a particular population, the mean of  $X$  is  $\mu = 54.030$  and the standard deviation is  $\sigma = 5.8$ .

Let  $\bar{X}$  be the sample mean of a random sample of size  $n = 47$ .

Find  $P(52.761 \leq \bar{X} \leq 54.453)$ , approximately.

## **Section 8.**

# **Chebyshev's Inequality and Convergence in Probability**

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## Chebyshev's Inequality

### Theorem

If the random variable  $X$  has a mean  $\mu$  and variance  $\sigma^2$ , then for every  $k \geq 1$ ,

$$\mathbb{P}(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

In particular  $\varepsilon = k\sigma$ , then

$$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$



## Chebyshev's Inequality

### Example

Suppose  $X$  has a mean of 25 and a variance of 16.

Find the lower bound of  $\mathbb{P}(17 < X < 33)$ .

## The Law of Large Numbers

### Definition

We say a sequence of random variables  $X_n$  converges to a random variable  $X$  in probability if for every  $\varepsilon > 0$ ,

$$\lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \varepsilon) = 0.$$

## The Law of Large Numbers

### Theorem

Let  $X_1, X_2, \dots, X_n$  be i.i.d. with common distribution  $X$ .

Let  $\mathbb{E}[X] = \mu$  and  $\text{Var}(X) = \sigma^2$ .

Then,  $\bar{X}$  converges to  $\mu$  in probability.

## Exercise

If  $X$  is a random variable with mean 3 and variance 16, use Chebyshev's inequality to find

1. A lower bound for  $\mathbb{P}(23 < X < 43)$ .
2. An upper bound for  $\mathbb{P}(|X - 31| \geq 14)$ .