

# Chapter 4. Bivariate Distributions

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Math 3215 Spring 2024

Georgia Institute of Technology

**Section 1.**  
**Bivariate Distributions of the**  
**Discrete Type**

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## Motivation

Suppose that we observe the maximum daily temperature,  $X$ , and maximum relative humidity,  $Y$ , on summer days at a particular weather station.

We want to determine a relationship between these two variables. <sup>Random.</sup>

For instance, there may be some pattern between temperature and humidity that can be described by an appropriate curve  $Y = u(X)$ .

PMF ( Prob. Mass Function ) :  $f(x) = P(X=x)$

## Joint distribution

discrete

Let  $X$  and  $Y$  be two random variables defined on a discrete sample space.

Let  $S$  denote the corresponding two-dimensional space of  $X$  and  $Y$ , the two random variables of the discrete type.

### Definition

The function  $f(x, y) = P(X = x, Y = y)$  is called the joint probability mass function (joint PMF) of  $X$  and  $Y$ .

$$f(x, y) = P(\{X = x\} \cap \{Y = y\})$$

## Joint distribution

Joint PMF = "prob."

=  $\mathbb{P}(\text{---})$

Note that

- $0 \leq f(x, y) \leq 1$   $\mathbb{P}(\mathcal{S})$
- $\sum_{(x, y) \in \mathcal{S}} f(x, y) = 1$
- $\mathbb{P}((X, Y) \in A) = \sum_{(x, y) \in A} f(x, y)$

## Joint distribution

### Example (4 faced)

Roll a pair of fair dice.

Let  $X$  denote the smaller and  $Y$  the larger outcome on the dice.

Find the joint PMF of  $(X, Y)$ .

$$f(x, y) = \begin{cases} \frac{1}{16} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{16} \\ \vdots \\ \vdots \end{cases}$$

$$(x, y) = \begin{matrix} (1, 1) \\ (1, 2) \\ (1, 3) \\ (1, 4) \\ \underline{(2, 2)} \\ (2, 3) \\ (2, 4) \\ (3, 3) \\ (3, 4) \\ (4, 4) \end{matrix}$$

## Marginal distribution

### Definition

Let  $X$  and  $Y$  have the joint probability mass function  $f(x, y)$ .

The probability mass function of  $X$ , which is called the marginal probability mass function of  $X$ , is defined by

$$f_X(x) = \sum_y f(x, y) = \mathbb{P}(X = x).$$

$$\begin{aligned} f_X(x) &= \mathbb{P}(X = x) \\ &= \sum_{\substack{\text{all possible} \\ y}} \mathbb{P}(\underbrace{\{X = x\} \cap \{Y = y\}}_y) \\ &= \sum_y f(x, y) \end{aligned}$$

$$f_Y(y) = \sum_x f(x, y)$$

Def  $X, Y$  RVs  
General  $\uparrow$  Indep. if  $P(X \in A \text{ and } Y \in B) = P(X \in A) P(Y \in B)$   
 for all possible  $A, B$ .

## Marginal distribution

**Definition** ( $X, Y$  : discrete)

We say  $X$  and  $Y$  are independent if

$$\text{Joint PMF} = P(X = x, Y = y) = P(X = x)P(Y = y) = \text{Product of Marginal PMFs}$$

for all  $(x, y) \in S$ .

Equivalently,  $f(x, y) = f_X(x)f_Y(y)$  for all  $x, y$ .

Otherwise, we say  $X$  and  $Y$  are dependent.



## Marginal distribution

### Example

Let the joint PMF of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{x+y}{21}$$

for  $x = 1, 2, 3$  and  $y = 1, 2$ .

Find the marginal PMFs of  $X$  and  $Y$ .

Determine whether they are independent.

$$\begin{aligned} f_X(x) &= P(X=x) = \sum_y f(x, y) = f(x, 1) + f(x, 2) \\ &= \frac{1}{21}(x+1) + \frac{1}{21}(x+2) = \frac{1}{21}(2x+3) \end{aligned}$$

$$\begin{aligned} f_Y(y) &= P(Y=y) = \sum_x f(x, y) = f(1, y) + f(2, y) + f(3, y) \\ &= \frac{1}{21} \cdot ((y+1) + (y+2) + (y+3)) = \frac{1}{21}(3y+6) = \frac{1}{7}(y+2) \end{aligned}$$

$$f(x, y) = f_X(x) \cdot f_Y(y)$$

$$\frac{1}{21}(x+y) \neq \frac{1}{21}(2x+3) \cdot \frac{1}{7}(y+2)$$

why?

dependent.

## Marginal distribution

### Example

Let the joint PMF of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{xy^2}{30} = \left( \text{a function of } x \right) \cdot \left( \text{function of } y \right)$$

for  $x = 1, 2, 3$  and  $y = 1, 2$ .

Find the marginal PMFs of  $X$  and  $Y$ .

Determine whether they are independent.

$$\begin{aligned} f_X(x) &= f(x, 1) + f(x, 2) = \frac{x \cdot 1^2}{30} + \frac{x \cdot 2^2}{30} \\ &= \frac{x}{30} \cdot (5) = \frac{x}{6} \end{aligned}$$

$$\begin{aligned} f_Y(y) &= f(1, y) + f(2, y) + f(3, y) \\ &= \frac{1 \cdot y^2}{30} + \frac{2 \cdot y^2}{30} + \frac{3 \cdot y^2}{30} = \frac{6}{30} \cdot y^2 = \frac{y^2}{5} \end{aligned}$$

$$f(x, y) \stackrel{?}{=} f_X(x) \cdot f_Y(y)$$

$$\frac{xy^2}{30} \quad \Rightarrow \quad \frac{x}{6} \cdot \frac{y^2}{5}$$

## Expectations

### Definition

Let  $X_1$  and  $X_2$  be random variables of the discrete type with the joint PMF  $f(x_1, x_2)$  on the space  $S$ . If  $u(X_1, X_2)$  is a function of these two random variables, then

$$\underline{\mathbb{E}[u(X_1, X_2)]} = \sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2).$$

In particular, if  $u(x_1, x_2) = x_1$ , then

$$\mathbb{E}[u(X_1, X_2)] = \mathbb{E}[X_1] = \sum_{(x_1, x_2) \in S} x_1 f(x_1, x_2) = \sum_{x_1} x_1 f_{X_1}(x_1).$$

Ex

$$\begin{aligned} u(X_1, X_2) &= X_1 \rightarrow \mathbb{E}[X_1] = \sum_{x_1, x_2} x_1 f(x_1, x_2) \\ " &= X_2 \rightarrow \mathbb{E}[X_2] = \sum_{x_1, x_2} x_2 f(x_1, x_2) \\ " &= X_1 + X_2 \rightarrow \mathbb{E}[X_1 + X_2] \\ " &= X_1 \cdot X_2 \rightarrow \mathbb{E}[X_1 \cdot X_2] = \sum_{x_1, x_2} (x_1 + x_2) f(x_1, x_2) \\ &\quad \downarrow \\ &\quad \mathbb{E}[X_1 \cdot X_2] = \sum_{x_1, x_2} x_1 \cdot x_2 f(x_1, x_2) \end{aligned}$$

## Expectations

### Example

There are eight similar chips in a bowl: three marked  $(0,0)$ , two marked  $(1,0)$ , two marked  $(0,1)$ , and one marked  $(1,1)$ .

A player selects a chip at random.

Let  $X_1$  and  $X_2$  represent those two coordinates.

Find the joint PMF.

Compute  $\mathbb{E}[X_1 + X_2]$ .

$$f(x_1, x_2) = \begin{cases} 3/8 & , (x_1, x_2) = (0, 0) \\ 2/8 & , (1, 0) \\ 2/8 & , (0, 1) \\ 1/8 & , (1, 1) \end{cases}$$

$$\begin{aligned} \mathbb{E}[X_1 + X_2] &= \sum_i (x_1 + x_2) f(x_1, x_2) \\ &= (0+0) \cdot \frac{3}{8} + (1+0) \cdot \frac{2}{8} + (0+1) \cdot \frac{2}{8} + (1+1) \cdot \frac{1}{8} \\ &= 0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4} . \end{aligned}$$

## Trinomial distribution

Consider an experiment with three outcomes, say perfect, seconds, and defective.

Let  $p_1, p_2, p_3$  be the corresponding probabilities.

Repeat the experiment  $n$  times and let  $X, Y$  be the numbers of perfect and seconds.

We say  $(X, Y)$  has **the trinomial distribution**.

## Trinomial distribution

### Example

In manufacturing a certain item, it is found that in normal production about 95% of the items are good ones, 4% are "seconds," and 1% are defective.

A company has a program of quality control by statistical methods, and each hour an online inspector observes 20 items selected at random, counting the number  $X$  of seconds and the number  $Y$  of defectives.

Suppose that the production is normal.

Find the probability that, in this sample of size  $n = 20$ , at least two seconds or at least two defective items are discovered.

## Exercise

Roll a pair of four-sided dice, one red and one black.

Let  $X$  equal the outcome of the red die and let  $Y$  equal the sum of the two dice.

Find the joint PMF.

Are they independent?

## **Section 2.**

### **The Correlation Coefficient**

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## Covariance and Correlation coefficient

### Definition

The covariance of  $X$  and  $Y$  is

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)].$$

The correlation coefficient of  $X$  and  $Y$  is

$$\rho = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

## Covariance and Correlation coefficient

### Properties

1. If  $X$  and  $Y$  are independent, then  $\text{Cov}(X, Y) = 0$ .
2.  $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$ .
3.  $-1 \leq \rho \leq 1$ .

## Covariance and Correlation coefficient

### Example

Let the joint PMF of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{x + 2y}{18}$$

for  $x = 1, 2$  and  $y = 1, 2$ .

Compute  $\text{Cov}(X, Y)$  and  $\rho$ .

## The Least Squares Regression Line

Suppose we are trying to see if there is a pattern or a certain relation between two random variables  $X$  and  $Y$ .

One of natural ways is to consider a linear relation between  $X$  and  $Y$ , that is, to figure out the best possible slope  $b$  such that  $Y - \mu_Y = b(X - \mu_X)$  has small errors.

We measure the error by  $\mathbb{E}[((Y - \mu_Y) - b(X - \mu_X))^2]$ .

## The Least Squares Regression Line

One can see by some calculus that the error is minimized when

$$b = \rho \frac{\sigma_Y}{\sigma_X}$$

and the minimum error is  $\sigma_Y^2(1 - \rho^2)$ .

The line  $Y - \mu_Y = \rho \frac{\sigma_Y}{\sigma_X}(X - \mu_X)$  is called **the line of best fit**, or **the least squares regression line**.

## The Least Squares Regression Line

### Example

Let  $X$  equal the number of ones and  $Y$  the number of twos and threes when a pair of fair four-sided dice is rolled.

Then  $X$  and  $Y$  have a trinomial distribution.

Find the least squares regression line.

## Uncorrelated

We say  $X, Y$  are uncorrelated if  $\rho = 0$ .

If  $X, Y$  are independent then they are uncorrelated.

However, the converse is not true.

## Uncorrelated

### Example

Let  $X$  and  $Y$  have the joint pmf  $f(x, y) = \frac{1}{3}$  for  $(x, y) = (0, 1), (1, 0), (2, 1)$ .



## Exercise

The joint pmf of  $X$  and  $Y$  is  $f(x, y) = \frac{1}{6}$ ,  $0 < x + y < 2$ , where  $x$  and  $y$  are nonnegative integers.

Find the covariance and the correlation coefficient.

## **Section 3.**

### **Conditional Distributions**

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## Conditional distribution

### Definition

**The conditional probability mass function** of  $X$ , given that  $Y = y$ , is defined by

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)}.$$

## Conditional distribution

### Example

Let the joint pmf of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{x + y}{21}$$

for  $x = 1, 2, 3$  and  $y = 1, 2$ . We have shown that

$$f_X(x) = \frac{2x + 3}{21}, \quad f_Y(y) = \frac{3y + 6}{21}.$$

Find the conditional PMFs.

## Conditional distribution

### Definition

**The conditional expectation** of  $Y$  given  $X = x$  is defined by

$$\mathbb{E}[Y|X = x] = \sum_y y f_{Y|X}(y|x).$$

The conditional variance of  $Y$  given  $X = x$  is defined by

$$\begin{aligned}\text{Var}(Y|X = x) &= \mathbb{E}[(Y - \mathbb{E}[Y|X = x])^2|X = x] \\ &= \mathbb{E}[Y^2|X = x] - (\mathbb{E}[Y|X = x])^2.\end{aligned}$$

## Conditional distribution

### Example

Let the joint PMF of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{x + y}{21}$$

for  $x = 1, 2, 3$  and  $y = 1, 2$ .

Find  $\mathbb{E}[Y|X = 3]$  and  $\text{Var}(Y|X = 3)$ .

## Conditional expectation as a function and a random variable

One can consider  $\mathbb{E}[Y|X = x]$  as a function of  $x$ .

Say  $h(x) = \mathbb{E}[Y|X = x]$

We define a random variable  $\mathbb{E}[Y|X] = h(X)$ .

## Contional expectation as a function and a random variable

### Example

Let the joint pmf of  $X$  and  $Y$  be defined by

$$f(x, y) = \frac{x + y}{21}$$

for  $x = 1, 2, 3$  and  $y = 1, 2$ . One can see that  $\mathbb{E}[Y|X = 1] = \frac{8}{5}$   $\mathbb{E}[Y|X = 2] = \frac{11}{7}$   
 $\mathbb{E}[Y|X = 3] = \frac{14}{9}$

Find the PMF of  $\mathbb{E}[Y|X]$  and  $\mathbb{E}[\mathbb{E}[Y|X]]$ .



## Conditional expectation as a function and a random variable

### Theorem

1.  $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$
2.  $\text{Var}(Y) = \mathbb{E}[\text{Var}(Y|X)] + \text{Var}(\mathbb{E}[Y|X])$

## Conditional expectation as a function and a random variable

### Example

Let  $X$  have a Poisson distribution with mean 4, and let  $Y$  be a random variable whose conditional distribution, given that  $X = x$ , is binomial with sample size  $n = x + 1$  and probability of success  $p$ .

Find  $\mathbb{E}[Y]$  and  $\text{Var}(Y)$ .

## Linear case

Suppose  $\mathbb{E}[Y|X = x]$  is linear in  $x$ , that is,  $\mathbb{E}[Y|X = x] = a + bx$ .

Then we have  $\mu_Y = a + b\mu_X$  and  $\mathbb{E}[XY] = a\mu_X + b\mathbb{E}[X^2]$ .

Solving for  $a$ , we have

$$a = \mu_Y - \rho \frac{\sigma_Y}{\sigma_X} \mu_X, \quad b = \rho \frac{\sigma_Y}{\sigma_X}.$$

Thus,

$$\mathbb{E}[Y|X = x] = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X} (x - \mu_X).$$

## Linear case

### Example

Let  $X$  and  $Y$  have the trinomial distribution with parameters  $n, p_X, p_Y$ , that is, the joint pmf is given by

$$f(x, y) = \binom{n}{x, y} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}.$$

Find  $\mathbb{E}[Y|X = x]$ .

## Exercise

A miner is trapped in a mine containing 3 doors.

The first door leads to a tunnel that will take him to safety after 3 hours of travel.

The second door leads to a tunnel that will return him to the mine after 5 hours of travel.

The third door leads to a tunnel that will return him to the mine after 7 hours.

If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?

## **Section 4.**

### **Bivariate Distributions of the Continuous Type**

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## Joint PDF

### Definition

An integrable function  $f(x, y)$  is **the joint probability density function** of two random variables  $X, Y$  if

- $f(x, y) \geq 0$
- $\iint f(x, y) \, dx dy = 1$
- $\mathbb{P}((X, Y) \in A) = \iint_A f(x, y) \, dx dy$

**The marginal density functions** for  $X, Y$  are

$$f_X(x) = \int f(x, y) \, dy, \quad f_Y(y) = \int f(x, y) \, dx.$$

## Joint PDF

### Example

Let  $X$  and  $Y$  have the joint PDF

$$f(x, y) = \frac{4}{3}(1 - xy)$$

for  $0 < x, y < 1$ . Find  $f_X$ ,  $f_Y$ , and  $\mathbb{P}(Y \leq \frac{X}{2})$ .



## Joint PDF

### Example

Let  $X$  and  $Y$  have the joint PDF

$$f(x, y) = \frac{3}{2}x^2(1 - |y|)$$

for  $-1 < x, y < 1$ .

Find  $\mathbb{E}[X]$  and  $\mathbb{E}[Y]$ .

## Independent random variables

### Definition

Two random variables  $X, Y$  with joint pdf are independent if and only if  $f(x, y) = f_X(x)f_Y(y)$ .

## Independent random variables

### Example

Let  $X$  and  $Y$  have the joint pdf  $f(x, y) = 2$  for  $0 < x < y < 1$ .

Compute  $\mathbb{P}(0 < X, Y < \frac{1}{2})$ .

Are they independent?

## Conditional densities and Conditional Expectation

### Definition

**The conditional density** of  $Y$  given  $X = x$  is defined by

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}.$$

As in the discrete case, the conditional expectation and the conditional variance are defined by

$$\begin{aligned}\mathbb{E}[Y|X = x] &= \int y f_{Y|X}(y|x) dy, \\ \text{Var}(Y|X = x) &= \mathbb{E}[(Y - \mathbb{E}[Y|X = x])^2 | X = x].\end{aligned}$$

## Conditional densities and Conditional Expectation

### Example

Let  $X$  and  $Y$  have the joint PDF  $f(x, y) = 2$  for  $0 < x < y < 1$ .

Then,  $f_X(x) = 2(1 - x)$  for  $0 < x < 1$  and  $f_Y(y) = 2y$  for  $0 < y < 1$ .

Find  $\mathbb{E}[X|Y = y]$  and  $\mathbb{E}[Y|X = x]$ .

## Conditional densities and Conditional Expectation

### Example

Let  $X$  be  $U(0, 1)$ , and let the conditional distribution of  $Y$ , given  $X = x$  be  $U(x, 2x)$ . Find  $\mathbb{E}[Y]$  and  $\text{Var}(Y)$ .

## Exercise

Let  $f(x, y) = 2e^{-x-y}$ ,  $0 < x \leq y < \infty$ , be the joint pdf of  $X$  and  $Y$ .

Find  $f_X(x)$  and  $f_Y(y)$ . Are  $X$  and  $Y$  independent?

## **Section 5.**

### **The Bivariate Normal Distribution**

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## Motivation

Let  $X$  be a random variable.

We construct a random variable  $Y$  in the following way:

The conditional distribution of  $Y$  given  $X = x$  satisfies

1. it is normal for each  $x$
2.  $\mathbb{E}[Y|X = x]$  is linear in  $x$
3.  $\text{Var}(Y|X = x)$  is constant in  $x$

## Motivation

Then,  $Y|X = x$  is normal with mean  $\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$  and variance  $\sigma_Y^2(1 - \rho^2)$ .

The conditional density is

$$f_{Y|X}(y|x) = \frac{1}{\sigma_Y \sqrt{2\pi} \sqrt{1 - \rho^2}} \exp \left( -\frac{(y - (\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)))^2}{2\sigma_Y^2(1 - \rho^2)} \right)$$

## Bivariate normal distribution

If  $X$  itself has normal distribution,  $(X, Y)$  is called **a bivariate normal random variables**.

## Bivariate normal distribution

### Definition

We say  $(X, Y)$  has a bivariate normal distribution with mean vector  $\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$  and covariance matrix  $\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$  if its joint pdf is given by

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{\bar{x}^2}{\sigma_X^2} - 2\frac{\rho\bar{x}\bar{y}}{\sigma_X\sigma_Y} + \frac{\bar{y}^2}{\sigma_Y^2}\right)\right)$$

where  $\bar{x} = x - \mu_X$  and  $\bar{y} = y - \mu_Y$ .

## Bivariate normal distribution

### Example

Let us assume that in a certain population of college students, the respective grade point averages, say  $X$  and  $Y$ , in high school and the first year of college have a bivariate normal distribution with parameters  $\mu_X = 2.9$ ,  $\mu_Y = 2.4$ ,  $\sigma_X = 0.4$ ,  $\sigma_Y = 0.5$ , and  $\rho = 0.6$ .

Find  $\mathbb{P}(2.1 < Y < 3.3 | X = 3.2)$ .

## Bivariate normal distribution

### Theorem

If  $X$  and  $Y$  have a bivariate normal distribution with correlation coefficient  $\rho$ , then  $X$  and  $Y$  are independent if and only if  $\rho = 0$ .

## Exercise

For a female freshman in a health fitness program, let  $X$  equal her percentage of body fat at the beginning of the program and  $Y$  equal the change in her percentage of body fat measured at the end of the program.

Assume that  $X$  and  $Y$  have a bivariate normal distribution with

$\mu_X = 24.5$ ,  $\mu_Y = -0.2$ ,  $\sigma_X = 4.8$ ,  $\sigma_Y = 3$ , and  $\rho = -0.32$ .

Find  $\mathbb{P}(1.3 < Y < 5.8)$ ,  $\mathbb{E}[Y|X = x]$ , and  $\text{Var}(Y|X = x)$ .

