Conditional distribution

Definition

The conditional expectation of Y given $X = x$ is defined by

$$\mathbb{E}[Y|X = x] = \sum_y y f_{Y|X}(y|x).$$

The conditional variance of Y given $X = x$ is defined by

$$\begin{aligned} \mathrm{Var}(Y|X = x) &= \mathbb{E}[(Y - \mathbb{E}[Y|X = x])^2|X = x] \\ &= \mathbb{E}[Y^2|X = x] - (\mathbb{E}[Y|X = x])^2. \end{aligned}$$

Contional expectation as a function and a random variable

One can consider $\mathbb{E}[Y|X = x]$ as a function of x.

Say $h(x) = \mathbb{E}[Y|X = x]$

We define a random variable $\mathbb{E}[Y|X] = h(X)$.

Contional expectation as a function and a random variable

Theorem

1. $\mathbb{E}[\mathbb{E}[Y|X]] = \mathbb{E}[Y]$
2. $\mathsf{Var}(Y) = \mathbb{E}[\mathsf{Var}(Y|X)] + \mathsf{Var}(\mathbb{E}[Y|X])$

Exercise

A miner is trapped in a mine containing 3 doors.

The first door leads to a tunnel that will take him to safety after 3 hours of travel.

The second door leads to a tunnel that will return him to the mine after 5 hours of travel.

The third door leads to a tunnel that will return him to the mine after 7 hours.

If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?