Math 416 Lecture Note: Week 7

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1 Invertibility and isomorphisms

Notation 1.1. Let V be a vector space over $\mathbb{R}$. The map $I_V : V \to V$ is defined by $I_V(x) = x$ for all $x \in V$.

Definition 1.2. Let V and W be vector spaces over $\mathbb{R}$, and $T : V \to W$ be linear.

(i) A function $U : W \to V$ is an inverse of T if $TU = I_W$ and $UT = I_V$.

(ii) We call T invertible if T has an inverse.

Remark 1.3. Let V and W be vector spaces over $\mathbb{R}$, and $T : V \to W$ be linear.

(i) If T has an inverse, it is unique. We denote by $T^{-1} : W \to V$.

(ii) T is invertible if and only if T is one-to-one and onto.

These hold for a general function $f : A \to B$, see Appendix B of [FIS].

Theorem 1.4. *Let V and W be vector spaces over $\mathbb{R}$, and $T : V \to W$ be linear. If T is invertible, then T^{-1} is linear.*

Proof. Note that $0 = T^{-1}T(0) = T^{-1}(0)$. Let $x, y \in W$ and $c \in \mathbb{R}$, then there exist $v, w \in V$ such that $T(v) = x$ and $T(w) = y$ because T is onto. It then follows that

$$\begin{aligned} T^{-1}(cx + y) &= T^{-1}(cT(v) + T(w)) \\ &= T^{-1}(T(cv + w)) \\ &= cv + w \\ &= cT^{-1}(x) + T^{-1}(y). \end{aligned}$$

□

Example 1.5. Let $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, -x)$ and $U(x, y) = (-y, x)$. We have seen that T and U are rotations by angle $\pi/2$ clockwise and counterclockwise. Noth that $UT(x, y) = U(y, -x) = (x, y)$ and $TU(x, y) = (x, y)$. Thus U is the inverse of T.

Theorem 1.6. *Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ be linear. Let β be a basis for V. If T is invertible, then $\gamma = \{T(v) : v \in \beta\}$ is a basis for W.*

Proof. Previously, we have seen that γ spans $\mathcal{R}(T)$. Since T is onto, γ spans W. To see γ is linearly independent, let

$$a_1T(v_1) + \cdots + a_nT(v_n) = T(a_1v_1 + \cdots + a_nv_n) = 0$$

where $a_i \in \mathbb{R}$ and $v_i \in \beta$. Since T is one-to-one, we have $a_1v_1 + \cdots + a_nv_n = 0$. Since β is linearly independent, $a_1 = \cdots = a_n = 0$, which implies that γ is a basis for W. □

Corollary 1.7. *Let V and W be vector spaces over $\mathbb{R}$, and $T : V \to W$ be linear. If T is invertible, then V is finite-dimensional if and only if W is finite-dimensional. In this case,* $\dim(V) = \dim(W)$.

Definition 1.8. Let V and W be vector spaces over $\mathbb{R}$ We say V is isomorphic to W if there exists an invertible linear map $T : V \to W$. Such a map T is called an isomorphism. We denote by $V \cong W$.

Remark 1.9. Let V, W, and Z be vector spaces over $\mathbb{R}$. One can see that

(i) $V \cong V$ for all vector spaces V.

(ii) If $V \cong W$, then $W \cong V$.

(iii) If $V \cong W$ and $W \cong Z$, then $V \cong Z$.

That is, the relation $\cong$ is an equivalence relation.

Example 1.10. Let $T : \mathbb{R}^3 \to \mathcal{P}_2(\mathbb{R})$ be defined by $T(a, b, c) = a + bx + cx^2$, then T is an isomorphism.

Theorem 1.11. *Let V and W be vector spaces over $\mathbb{R}$ and $\dim(V) < \infty$. Then, $V \cong W$ if and only if $\dim(V) = \dim(W) < \infty$.*

Proof. Suppose $\dim(V) = \dim(W) = n$. Let $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{w_1, \cdots, w_n\}$ be bases for V and W. Define $T : V \to W$ by $T(v_i) = w_i$ for each $i = 1, 2, \cdots, n$. (This map is well-defined and unique.) It is easy to see that T is onto. By Dimension theorem, $\dim(\mathcal{N}(T)) = 0$, which implies that T is one-to-one. Thus, T is an isomorphism. $\square$

Remark 1.12. Let V be a vector space over $\mathbb{R}$, then $V \cong \mathbb{R}^n$ if and only if $\dim(V) = n$. In this case, we have an explicit isomorphism.

Theorem 1.13. *Let V be an n-dimensional vector space over $\mathbb{R}$ with a basis β. Then a map $\phi_\beta : V \to \mathbb{R}^n$ defined by $\phi_\beta(v) = [v]_\beta$ is an isomorphism. We call ϕ_β the standard representation of V with respect to β.*

Proof. Homework. $\square$

Theorem 1.14. *Let V and W be finite dimensional vector spaces over $\mathbb{R}$ with $\dim(V) = n$ and $\dim(W) = m$. Then, $\mathcal{L}(V, W) \cong \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathcal{M}_{m\times n}(\mathbb{R})$.*

Proof. Define maps $\Theta : \mathcal{L}(V, W) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ and $\overline{\Theta} : \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \to \mathcal{L}(V, W)$ by $\Theta(T) = \phi_\gamma \circ T \circ (\phi_\beta)^{-1}$ and $\overline{\Theta}(S) = (\phi_\gamma)^{-1} \circ T \circ \phi_\beta$.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\Theta(T)"] \arrow[d, "(\phi_\beta)^{-1}"'] & \mathbb{R}^m \\ V \arrow[r, "T"] & W \arrow[u, "\phi_\gamma"'] \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\overline{\Theta}(S)"] \arrow[d, "\phi_\beta"'] & W \\ \mathbb{R}^n \arrow[r, "S"] & \mathbb{R}^m \arrow[u, "(\phi_\gamma)^{-1}"'] \end{tikzcd}$$

Then, it is easy to see that they are linear and $\Theta\overline{\Theta} = I_{\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)}$ and $= \overline{\Theta}\Theta = I_{\mathcal{L}(V,W)}$. $\square$

2 Matrices: invertibility and rank

Definition 2.1. A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is invertible if there exists $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $AB = BA = I_n$.

Example 2.2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Remark 2.3. The inverse is unique. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Suppose that there exist $B, C \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $AB = BA = AC = CA = I_n$. Then, we have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

Remark 2.4. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$, then $AB = I_n$ implies $BA = I_n$. (Homework)

Theorem 2.5. *Let V and W be finite dimensional vector spaces over $\mathbb{R}$ with bases β and γ. Then, $T : V \to W$ is an isomorphism if and only if $[T]_\beta^\gamma$ is invertible and*

$$([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta.$$

Proof. Suppose that T is an isomorphism. Then, we have

$$[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = [TT^{-1}]_\gamma^\gamma = [I_W]_\gamma^\gamma = I$$

and

$$[T^{-1}]_\gamma^\beta [T]_\beta^\gamma = [T^{-1}T]_\beta^\beta = [I_V]_\beta^\beta = I.$$

Suppose $A = [T]_\beta^\gamma$ is invertible. Let $\beta = \{v_1, \cdots, v_n\}$ and $\gamma = \{w_1, \cdots, w_n\}$ be bases for V and W. Let $B = (B_{ij})$ be the inverse of A, that is, $AB = BA = I$. Define $U : W \to V$ by

$$U(w_j) = \sum_{i=1}^{n} B_{ij} v_i.$$

It is straightforward to see that $UT(v_i) = v_i$ and $TU(w_i) = w_i$ for all $i = 1, 2, \cdots, n$. Thus U is the inverse of T and T is an isomorphism. ☐

Example 2.6. Let $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (y, -x)$ and $U(x, y) = (-y, x)$. We have seen that T and U are rotations by angle $\pi/2$ clockwise and counterclockwise and inverse each other. Let β be the standard basis for $\mathbb{R}^2$, then

$$[T]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad [U]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$[UT]_\beta = [TU]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [T]_\beta [U]_\beta = [U]_\beta [T]_\beta.$$

Remark 2.7. In particular, if $T = I_V$, then the matrix $[I_V]_\beta^\gamma$ is invertible and $([I_V]_\beta^\gamma)^{-1} = [I_V]_\gamma^\beta$.

Remark 2.8. We recall that the null space of A is the set of all solutions of the linear system $LS(A, 0)$. One can see that the system of linear equations can be written as $x_1[A]_1 + \cdots + x_n[A]_n = 0$. That is,

$$\mathcal{N}(A) = \{x : Ax = 0\} = \{x : L_A(x) = 0\} = \mathcal{N}(L_A).$$

Theorem 2.9. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Then, the following are equivalent.*

(i) *A is invertible.*

(ii) *L_A is an isomorphism and* $(L_A)^{-1} = L_{A^{-1}}$.

(iii) *The null space of A is* $\{0\}$.

(iv) *The set of the columns of A is linearly independent.*

(v) *For any $b \in \mathbb{R}^n$, the linear system $LS(A, b)$ has a unique solution.*

Proof. (i)$\Rightarrow$(ii): Homework.

(ii)$\Rightarrow$(iii): Since L_A is one-to-one, the null space of L_A (so A) is $\{0\}$.

(iii)$\Rightarrow$(iv): Let $x_1[A]_1 + \cdots + x_n[A]_n = 0$, then $(x_1, \cdots, x_n)$ belongs the null space of A. Thus, $x_1 = \cdots = x_n = 0$ and so the columns of A are linearly independent.

(iv)$\Rightarrow$(v): Since the columns of A are linearly independent, they spans $\mathbb{R}^n$ and are a basis for $\mathbb{R}^n$. For b, there exist $x_1, \cdots, x_n$ such that $x_1[A]_1 + \cdots + x_n[A]_n = b$. In other words, we get $Ax = b$ where $x = (x_1, \cdots, x_n)$.

(v)$\Rightarrow$(i): Let B be such that $[B]_i$ is the solution of $LS(A, e_i)$, then we have $A[B]_i = e_i$ for all i. Thus, we get $AB = I_n$. By Homework, we conclude A, B are invertible and so $B = A^{-1}$. $\square$

Remark 2.10. The proof of (v)$\Rightarrow$(i) gives how to compute the inverse of a matrix. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. We want to find the inverse of A. Let B be the inverse of A, then $AB = I_n$. In particular, for each $i = 1, 2, \cdots, n$,

$$A[B]_i = e_i.$$

This means that $[B]_i$ is the solution of the linear system $LS(A, e_i)$. Thus, it is enough to solve n linear systems. In practice, we find a reduced row echelon form of a $(n \times 2n)$ matrix (A, I_n). If A is invertible, then the RREF will be (I_n, B). Here B is the inverse of A. In practice, we find a reduced row echelon form of a $(n \times 2n)$ matrix (A, I_n). If A is invertible, then the RREF will be (I_n, B). Here B is the inverse of A

Example 2.11. Let $A = \begin{pmatrix} 0 & 3 & 2 \\ -1 & 4 & 2 \\ 3 & -4 & -1 \end{pmatrix}$. To find the inverse of A, we consider

$$(A, I_3) = \begin{pmatrix} 0 & 3 & 2 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 & 1 & 0 \\ 3 & -4 & -1 & 0 & 0 & 1 \end{pmatrix}.$$

By row operations, we can see that

$$(A, I_3) \sim \begin{pmatrix} 1 & 0 & 0 & -4 & 5 & 2 \\ 0 & 1 & 0 & -5 & 6 & 2 \\ 0 & 0 & 1 & 8 & -9 & -3 \end{pmatrix}.$$

Thus, the inverse of A is

$$A^{-1} = \begin{pmatrix} -4 & 5 & 2 \\ -5 & 6 & 2 \\ 8 & -9 & -3 \end{pmatrix}.$$

Definition 2.12. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$. We define

$$\mathrm{Col}(A) = \mathrm{Span}(\text{the columns of } A),$$
$$\mathrm{Row}(A) = \mathrm{Span}(\text{the rows of } A).$$

Theorem 2.13. *Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, then* $\dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A))$. *We call it the rank of A and denote by* $\mathrm{rank}(A)$.

Proof. Note that $\mathrm{Col}(A) = \mathcal{R}(L_A)$. By Dimension theorem, we have

$$n = \dim(\mathbb{R}^n) = \dim(\mathcal{N}(L_A)) + \dim(\mathcal{R}(L_A)) = \dim(\mathcal{N}(A)) + \dim(\mathrm{Col}(A)).$$

Also, we have seen that the dimension of $\mathrm{Row}(A)$ is the same as the number of non-zero rows of a RREF of A, which is the same as the number of pivot columns. Since $\dim(\mathcal{N}(L_A))$ is the number of non-pivot columns, we have

$$n = \dim(\mathcal{N}(A)) + \dim(\mathrm{Row}(A)).$$

This completes the proof. $\square$

3 The change of coordinate matrix

Example 3.1. Let $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ and consider $L_A : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $\beta' = \{v_1 = (1,1), v_2 = (-1,1)\}$. Since $Av_1 = 8v_1$ and $Av_2 = 2v_2$, we have

$$[L_A]_\beta = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}, \qquad [L_A]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$$

Question 3.2. In general, if we have $T : V \to W$ with bases β, β' for V and γ, γ' for W, can we find a link between $[T]_\beta^\gamma$ and $[T]_{\beta'}^{\gamma'}$?

Question 3.3. In the example, $[L_A]_{\beta'}$ is a diagonal matrix so that it is nicer than $[L_A]_\beta$. Can we find a basis β' so that the matrix representation is nicer?

To this end, we consider the identity map I_V. Let V be a vector space over $\mathbb{R}$ with bases β, β'. Recall that $[I_V]_\beta^{\beta'}$ is invertible and $([I_V]_\beta^{\beta'})^{-1} = [I_V]_{\beta'}^\beta$.

Theorem 3.4. *Let V and W be vector spaces over $\mathbb{R}$, β, β' bases for V, and γ, γ' bases for W. Let $T : V \to W$.*

(i) *For any $v \in V$, $[v]_{\beta'} = [I_V]_\beta^{\beta'}[v]_\beta$.*

(ii) *$[T]_{\beta'}^{\gamma'} = P[T]_\beta^\gamma Q$ where $P = [I_W]_\gamma^{\gamma'}$ and $Q = [I_V]_{\beta'}^\beta$.*

(iii) *If $V = W$, then $[T]_{\beta'} = Q^{-1}[T]_\beta Q$ where and $Q = [I_V]_{\beta'}^\beta$.*

Proof. (i) Recall that $[T(v)]_\gamma = [T]_\beta^\gamma[v]_\beta$. It follows that $[I_V]_\beta^{\beta'}[v]_\beta = [I_V(x)]_{\beta'} = [x]_{\beta'}$.

(ii) We recall the following: If $T : V \to W$ and $U : W \to Z$ are linear, then $[UT]_\alpha^\gamma = [U]_\beta^\gamma[T]_\alpha^\beta$. It follows that

$$[T]_{\beta'}^{\gamma'} = [I_W \circ T \circ I_V]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta$$

(iii) This follows from $([I_V]_\beta^{\beta'})^{-1} = [I_V]_{\beta'}^\beta$.

□

Example 3.5. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $\beta' = \{v_1 = (1,1), v_2 = (-1,1)\}$. Then, we have

$$e_1 = \frac{1}{2}(v_1 - v_2), \qquad e_2 = \frac{1}{2}(v_1 + v_2)$$

and so

$$[I_{\mathbb{R}^2}]_\beta^{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Note that $(8,6) = 7v_1 - v_2$ so that $[(8,6)]_{\beta'} = (7,-1)^t$. Then, we have

$$[(8,6)]_{\beta'} = [I_{\mathbb{R}^2}]_\beta^{\beta'}[(8,6)]_\beta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

Example 3.6. Let $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ and consider $L_A : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\beta = \{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and $\beta' = \{v_1 = (1,1), v_2 = (-1,1)\}$. We have seen that

$$[L_A]_\beta = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}, \qquad [L_A]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}.$$

One can check that

$$[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} =: Q, \qquad [I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = ([I_{\mathbb{R}^2}]_{\beta'}^{\beta})^{-1}$$

and

$$[L_A]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix} = [I_{\mathbb{R}^2}]_{\beta}^{\beta'}[L_A]_{\beta}[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Definition 3.7. We say $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar when there exists an invertible matrix $Q \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $B = Q^{-1}AQ$.

Remark 3.8. Note that this is an equivalence relation. Indeed, $A = I_n^{-1}AI_n$. If $B = Q^{-1}AQ$, then, $A = (Q^{-1})^{-1}BQ^{-1}$. Suppose $B = Q^{-1}AQ$ and $C = R^{-1}BR$, then

$$C = R^{-1}BR = R^{-1}(Q^{-1}AQ)R = (R^{-1}Q^{-1})A(QR) = (QR)^{-1}A(QR).$$

Example 3.9. Let P be a plain in $\mathbb{R}^3$ given by $x + y + z = 0$. Our goal is to construct a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ which is an orthogonal projecto onto P. That is, for each $x \in \mathbb{R}^3$, $T(x)$ is the closest point on P to x. Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$. First, we note that $\{v_1 = (1,-1,0), v_2 = (0,1,-1)\}$ is a basis for P. Since $v_3 = (1,1,1) \notin P$, we see that $\beta' = \{v_1, v_2, v_3\}$ is another basis for $\mathbb{R}^3$. Observe that $T(v_1) = v_1$ and $T(v_2) = v_2$ because $v_1, v_2 \in P$. Since v_3 is normal to P, we have $T(v_3) = 0$. Thus, we have

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$[T]_{\beta} = [I_{\mathbb{R}^3}]_{\beta'}^{\beta}[T]_{\beta'}[I_{\mathbb{R}^3}]_{\beta}^{\beta'} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

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