Sec 10.3: The Fourier Convergence Theorem

Math 285 Spring 2020

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Questions

Suppose a function f is given. If f is periodic with period $2L > 0$ and integrable on $[-L, L]$, then we can compute

$$a_n = \frac{1}{L}(f, \cos\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$b_n = \frac{1}{L}(f, \sin\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Define

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N}\left(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\right)$$

for each $N = 1, 2, \cdots$.

Questions

A periodic function f with period $2L > 0$ is integrable on $[-L, L]$ and $S_N(x)$ is defined by

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

for each $N = 1, 2, \cdots$.

Question

(i) Does $S_N(x)$ converge as $N \to \infty$ for each x?

(ii) Suppose $S_N(x)$ converges to a function, say $S(x)$, as $N \to \infty$ for each x. Is the limit $S(x)$ equal to $f(x)$?

Piecewise continuous functions

Definition

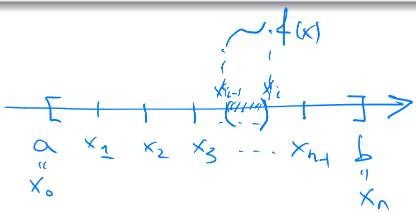
A function f is called piecewise continuous on an interval $[a, b]$ if there exists a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$ such that

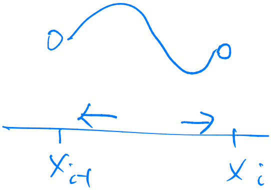
(i) f is continuous on an open subinterval (x_{i-1}, x_i) for each $i = 1, 2, \cdots, n$, and

(ii) the limits

$$\underbrace{\lim_{x \to x_{i-1}+} f(x)}_{\text{from right}}, \qquad \underbrace{\lim_{x \to x_i-} f(x)}_{\text{from left}}$$

are finite for each $i = 1, 2, \cdots, n$.

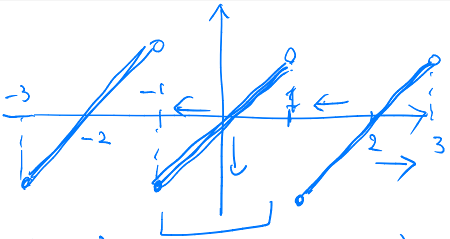




Example

Example

Let $f(x)$ be a periodic function with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$, then it is piecewise continuous.



$(-3, 1), \quad (-1, 1), \quad (1, 3), \quad (3, 5), \quad \cdots$

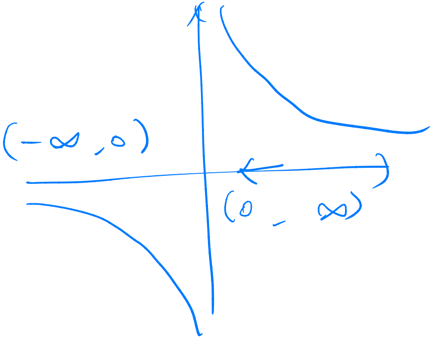
$\lim_{x \to -1+} f(x) = -1, \quad \lim_{x \to 1-} f(x) = 1$

are finite

Example

Example

Let $f(x) = \frac{1}{x}$ for $x \neq 0$, then it is not piecewise continuous.



f is conti. on $(0,\infty)$, $(-\infty,0)$

But, $\lim_{x\to 0+} f(x) = \infty$

& $\lim_{x \to 0-} f(x) = -\infty$

are not finite

Fourier Convergence Theorem

Theorem

① *Suppose f and f' are piecewise continuous on $[-L, L]$.*

Assume that f is periodic with period $2L$, that is, $f(x+2L) = f(x)$.

Then, $S_N(x)$ converges to a function $\cancel{S(N)}$ $S_N(x)$ as $N \to \infty$ for each x.

② *Furthermore, $S(x) = f(x)$ if f is continuous at x and*

$$S(x) = \frac{1}{2}(f(x+) + f(x-))$$

otherwise.

$$S(x) = \begin{cases} f(x) & , \\ & , \end{cases}$$

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N} \left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right) \right)$$

(Partial sum of Fourier series)

$\to S_N(x)$ as $N \to \infty$

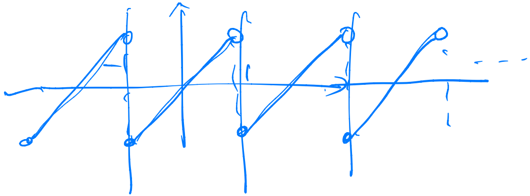
Example

Example

Consider a periodic function f with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$.

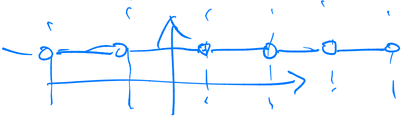
① f is piecewise conti.

Discontinuity occurs at $x = 2k-1 \quad \forall k \in \mathbb{Z}$.



② Period $= 2 = 2L \quad \therefore \quad L = \underline{1}$

③ f' piecewise continuous



Example

Find a_m, b_m

① $a_0 = \frac{1}{L}(f, 1) = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = 0$

② $a_m = \frac{1}{L}\left(f, \cos\left(\frac{m\pi}{L}x\right)\right) = \int_{-1}^{1} x \cos(m\pi x)\,dx$

$= \left[\frac{x \sin(m\pi x)}{m\pi}\right]_{-1}^{1} - \frac{1}{m\pi}\int_{-1}^{1} \sin(m\pi x)\,dx$

$= 0$

③ $b_m = \frac{1}{L}\left(f, \sin\left(\frac{m\pi}{L}x\right)\right) = \int_{-1}^{1} x \sin(m\pi x)\,dx$

$= \left[-\frac{x \cos(m\pi x)}{m\pi}\right]_{-1}^{1} + \frac{1}{m\pi}\underbrace{\int_{-1}^{1} \cos(m\pi x)\,dx}_{=0}$

$= -\frac{\cos(m\pi)}{m\pi} + \frac{(-1)\cos(-m\pi)}{m\pi} = -\frac{2\cos m\pi}{m\pi}$

Example

Note $\cos(n\pi) = (-1)^n$

$\cos(0\pi) = 1$, $\cos(\pi) = -1$, $\cos(2\pi) = 1$

$$b_m = -\frac{2\cos m\pi}{m\pi} = -\frac{2(-1)^m}{m\pi}$$

Therefore, $S_N(x) = -\frac{2}{\pi}\sum_{m=1}^{N} \frac{(-1)^m}{m} \sin(m\pi x)$

By Fourier Convergence thm,

$$S_N(x) \longrightarrow S(x) = -\frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin(m\pi x)$$

$$= f(x)$$

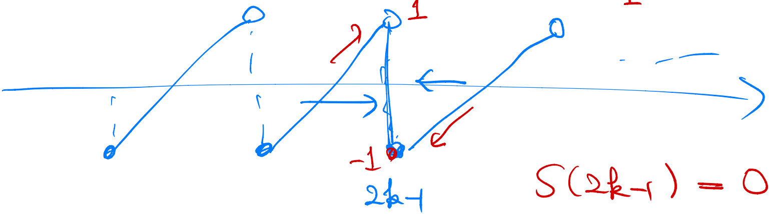
if $f(x)$ is continuous at x.

Example

f is conti. at $x \neq 2k-1$

$$S(x) = \begin{cases} f(x), & x \neq 2k-1 \text{ for } k \in \mathbb{Z} \\ \text{if } x = 2k-1, \text{ then} & \end{cases}$$

$$S(x) = \frac{1}{2}\Big(\underbrace{f((2k-1)+)}_{=-1} + \underbrace{f((2k-1)-)}_{=1} \Big)$$



$$S(2k-1) = 0$$

Example

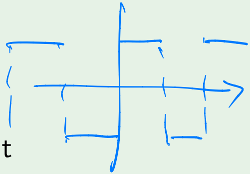
Example

① f is continuous

② f' is piecewise conti.

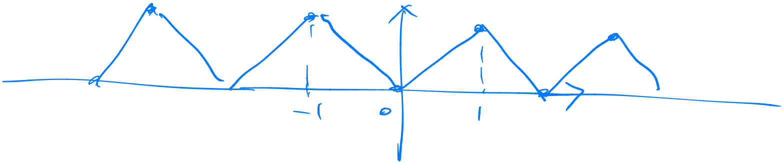
Consider a periodic function f defined by

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

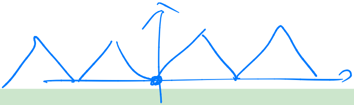


and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. We have seen that

$$S_N(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{N} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$



Example



Example

Since f satisfies the assumptions of the Fourier convergence theorem, we have

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

for all x

In particular, if $x = 0$, then

$$f(0) = 0 = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$$

and so

$$\boxed{\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$$