Homework 4

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: October 4, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Prove that T is a linear transformation, find bases for $\mathcal{N}(T)$ and $\mathcal{R}(T)$, and compute $\dim(\mathcal{N}(T))$ and $\dim(\mathcal{R}(T))$.
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x - y, 2z)$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 0, 2x - y)$.
   (c) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$. (Here, $\mathcal{P}_n(\mathbb{R})$ is the set of all polynomials $p(x)$ with real coefficients with $\deg(p(x)) \leq n$.)
2. Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear.
   (a) Let $\{w_1, \cdots, w_k\}$ be a linearly independent subset of $\mathcal{R}(T)$. Prove that if $S = \{v_1, \cdots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent.
   (b) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.
3. Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and $T : V \to W$ linear.
   (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
   (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.
4. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{N}(T) = \mathcal{R}(T)$.
   (b) Give an example of two distinct linear transformations U and T such that $\mathcal{N}(U) = \mathcal{N}(T)$ and $\mathcal{R}(U) = \mathcal{R}(T)$.
5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform. Show that there exist $a, b, c, d \in \mathbb{R}$ such that $T(x, y) = (ax + by, cx + dy)$.
6. Let V be a vector space and $T : V \to V$ linear. A subspace $W \leq V$ is called T-invariant if $T(x) \subseteq W$ for all $x \in W$.
   (a) Show that the subspaces $\{0\}, V, \mathcal{N}(T), \mathcal{R}(T)$ are T-invariant.
   (b) Suppose that V is finite dimensional and $W \leq V$ is T-invariant. Show that if $V = \mathcal{R}(T) \oplus W$ then $W = \mathcal{N}(T)$.