Homework 7

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 1, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

   (a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row.

   (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ along the second row.

   (c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row.

2. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

3. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

4. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called nilpotent if $M^k = O$ for some integer k. Prove that if M is nilpotent, then $\det(M) = 0$.

5. Prove that if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar, then $\det(A) = \det(B)$.

6. Let $1 \le k < n$. Suppose that $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ can be written in the form
$$M = \begin{pmatrix} A & B \\ O & I_{n-k} \end{pmatrix}$$
where $A \in \mathcal{M}_{k\times k}(\mathbb{R})$ and $B \in \mathcal{M}_{k\times(n-k)}(\mathbb{R})$. Prove that $\det(M) = \det(A)$.

7. Let $1 \le k < n$. Suppose that $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ can be written in the form
$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$
where $A \in \mathcal{M}_{k\times k}(\mathbb{R})$, $B \in \mathcal{M}_{k\times(n-k)}(\mathbb{R})$, and $C \in \mathcal{M}_{(n-k)\times(n-k)}(\mathbb{R})$. Prove that $\det(M) = \det(A)\det(C)$.

8. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ have the form
$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Compute $\det(A + tI_n)$. (Note that if $n = 1$, $A = (a_0)$. If $n = 2$, then

$$A = \begin{pmatrix} 0 & a_0 \\ -1 & a_1 \end{pmatrix}.$$

If $n = 3$, then

$$A = \begin{pmatrix} 0 & 0 & a_0 \\ -1 & 0 & a_1 \\ 0 & -1 & a_2 \end{pmatrix}.$$

Compute $\det(A + tI_n)$ for $n = 2, 3$ first and guess the formula for general n. Use an induction on n.)