Math 285 Lecture Note: Week 2

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1 Classification of Differential Equations

There are several types of differential equations. Suppose that we consider a differential equation with unknown function f. This function f can be a function of one variable (time for example) or several variables (like space and time). Let

$$f = f(t, x, \cdots).$$

Definition 1.1. An ordinary differential equation (ODE) is an equation with $f = f(t)$ and the derivatives in one variable. A partial differential equation (PDE) is an equation with $f = f(t, x, \cdots)$ and the derivatives in several variables.

Example 1.2.

$$\frac{d^2 f}{dx^2} + f = 0 \qquad \text{(Harmonic oscillator)}$$
$$\frac{d^2 f}{dx^2} + \sin(f) = 0 \qquad \text{(Motion of pendulum)}$$
$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \qquad \text{(Laplace equation)}$$
$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} \qquad \text{(Heat equation)}$$
$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} \qquad \text{(Wave equation)}$$

Notation 1.3. We use the notations

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}, \qquad \frac{\partial^2 f}{\partial x \partial t} = f_{xt}, \cdots$$

Definition 1.4. A system of differential equations is a collection of differential equations with several unknown functions. For example,

$$\begin{cases} \dfrac{df}{dx} = 3f(x) - g(x), \\ \dfrac{dg}{dx} = 2f(x) + g(x). \end{cases}$$

Definition 1.5. The order of a differential equation is the order of the highest derivative that appears in the equation. For example, the heat equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}$$

is of order 2.

Another way to categorize differential equations is linearity. In general, a differential equation can be written as

$$F(t, x, \cdots, f, f', f'', \cdots) = 0$$

where $t, x, \cdots$ are variables and f, F are functions.

Definition 1.6. We consider an ODE given by

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

where y is a function of t and $y^{(n)}$ denotes the n-th derivative. We say the ODE is linear if F is linear, that is,

$$F(t, y, y', \cdots, y^{(n)}) = a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y + g(t) = 0.$$

If not, we say it is nonlinear.

Example 1.7. Note that

$$y'' + 2y' - 3y = \tan(t)$$

is a second-order linear differential equation and

$$y'' + \sin(y) = 0, \qquad yy'' = y^2$$

are nonlinear equations.

Definition 1.8. A solution of the ODE given by

$$F(t, y, y', \cdots, y^{(n)}) = 0$$

on the interval (α, β) is a function $\phi(t)$ such that

$$F(t, \phi, \phi', \cdots, \phi^{(n)}) = 0$$

for every $t \in (\alpha, \beta)$.

Example 1.9. Note that $y = \phi(t) = e^t$ is a solution to the second order linear ODE

$$y'' - y = 0$$

for all $t \in \mathbb{R}$. To verify this, we substitute y with e^t, i.e.,

$$\phi''(t) - \phi(t) = e^t - e^t = 0.$$

Note also that e^t is not the only solution. $\psi(t) = e^{-t}$ is also a solution. Furthermore, any linear combination of $\phi(t)$ and $\psi(t)$ is also a solution. That is,

$$f(t) = a\phi(t) + b\psi(t)$$

is a solution for $a, b \in \mathbb{R}$. A particular example of this linear combination is $\sinh(t)$ and $\cosh(t)$.

2 Linear equations; method of integrating factors

In this section, we focus on how to find an explicit solution to the first order linear ODE of the form

$$F(t, y, y') = 0$$

where F is linear. In other words, we consider the following form

$$P(t)\frac{dy}{dt} + Q(t)y = R(t)$$

where $P(t), Q(t), R(t)$ are given functions. For example,

$$t\frac{dy}{dt} - y = t^2 e^{-t}.$$

In this case $P(t) = t$, $Q(t) = -1$, and $R(t) = t^2 e^{-t}$. The idea of solving this type of ODEs is to use the product rule:

$$\frac{d}{dt}(P(t)y) = P(t)\frac{dy}{dt} + P'(t)y.$$

If we have $P'(t) = Q(t)$, then

$$\begin{aligned} P(t)\frac{dy}{dt} + Q(t)y &= \frac{d}{dt}(P(t)y) = R(t) \\ P(t)y &= \int R(t)\,dt \\ y &= \frac{1}{P(t)}\int R(t)\,dt. \end{aligned}$$

Example 2.1. Consider an ODE

$$(t^3+1)\frac{dy}{dt} + 3t^2 y = \sin t.$$

Since $P(t) = (t^3+1)$ and $Q(t) = 3t^2 = P'(t)$, it follows from the previous argument that

$$y = \frac{1}{t^3+1}\int \sin t\,dt = \frac{-\cos t + C}{t^3+1}$$

is a solution to the ODE.

In general, $Q(t)$ may not be the derivative of $P(t)$. Before dealing with general cases, we consider the case where $P(t)$ and $Q(t)$ are constants.

Example 2.2. Consider an ODE

$$\frac{dy}{dt} + 2y = t.$$

The idea is to multiply a new function $\mu(t)$

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = t\mu(t).$$

If we have $\mu'(t) = 2\mu(t)$, then we can apply the previous technique. To find such a function μ, we solve the ODE

$$\begin{aligned} \frac{1}{\mu}\frac{d\mu}{dt} &= 2 \\ \ln|\mu(t)| &= 2t + C \\ \mu(t) &= Ce^{2t}. \end{aligned}$$

Let $\mu(t) = e^{2t}$, then the original ODE can be written as

$$\begin{aligned} e^{2t}\frac{dy}{dt} + 2e^{2t}y = \frac{d}{dt}(e^{2t}y) &= te^{2t} \\ e^{2t}y &= \int te^{2t}\,dt \\ &= \frac{1}{2}(te^{2t} - \int e^{2t}\,dt) \\ &= \frac{1}{4}(2te^{2t} - e^{2t} + C) \end{aligned}$$

and so

$$y = \frac{1}{4}(2t - 1 + Ce^{-2t}).$$

Example 2.3. Consider an ODE

$$y' - 3y = \cos t, \qquad y(0) = 0.$$

Solving the auxiliary ODE

$$\frac{d\mu}{dt} = -3\mu,$$

we let $\mu(t) = e^{-3t}$. Then the original ODE gives

$$\begin{aligned} \mu(t)y' - 3\mu(t)y = \frac{d}{dt}(\mu(t)y) &= \mu(t)\cos t \\ e^{-3t}y &= \int e^{-3t}\cos t\,dt \\ &= \frac{1}{10}e^{-3t}(\sin t - 3\cos t) + C \\ y(t) &= \frac{1}{10}(\sin t - 3\cos t) + Ce^{3t}. \end{aligned}$$

Since

$$y(0) = -\frac{3}{10} + C = 0,$$

we get

$$y(t) = \frac{1}{10}(\sin t - 3\cos t + 3e^{3t}).$$

We are ready to discuss how to solve a first order linear ODE

$$P(t)\frac{dy}{dt} + Q(t)y = R(t).$$

By dividing $P(t)$ of both sides, we consider a first order linear ODE of the standard form

$$\frac{dy}{dt} + p(t)y = r(t)$$

where $p(t), r(t)$ are given. We introduce a new function $\mu(t)$ and multiply by $\mu(t)$

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)r(t).$$

We want to find $\mu(t)$ such that

$$\frac{d}{dt}\mu(t) = \mu(t)p(t).$$

Indeed, we have

$$\frac{1}{\mu(t)}\frac{d}{dt}\mu(t) = \frac{d}{dt}(\ln|\mu(t)|) = p(t)$$

and

$$\ln|\mu(t)| = \int p(t)\,dt.$$

Let $\mu(t) = \exp(\int p(t)\,dt)$, then $\mu'(t) = \mu(t)p(t)$. Thus,

$$\mu(t)\frac{dy}{dt} + \mu(t)p(t)y = \mu(t)\frac{dy}{dt} + \mu'(t)y = (\mu(t)y)' = \mu(t)r(t).$$

Therefore, we get

$$y = \frac{1}{\mu(t)}\int \mu(t)r(t)\,dt.$$

Example 2.4. Consider

$$t\frac{dy}{dt} - y = t^2e^{-t}.$$

Dividing by t of both sides, we get

$$\frac{dy}{dt} - \frac{1}{t}y = te^{-t}$$

and so $p(t) = -\frac{1}{t}$ and $r(t) = te^{-t}$. The previous argument yields

$$\begin{aligned}\ln|\mu(t)| &= \int p(t)\,dt = -\int \frac{1}{t}\,dt = -\ln|t| + C,\\ \mu(t) &= \frac{C}{t}\end{aligned}$$

where C is an arbitrary constant. Thus, solutions of the equation are

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\int \mu(t)r(t)\,dt\\ &= \frac{1}{C}t\int \frac{C}{t}te^{-t}\,dt\\ &= t\int e^{-t}\,dt\\ &= t(-e^{-t} + C).\end{aligned}$$

Example 2.5. Consider an ODE

$$(\cos t)\frac{dy}{dt} + (\sin t)y = \cos^3 t, \qquad y(0) = 2$$

for $t \in (-\pi/2, \pi/2)$. As before, we consider

$$\mu(t)\frac{dy}{dt} + \mu(t)\tan t y = \mu(t)\cos^2 t.$$

Then, the auxiliary ODE is

$$\begin{aligned}\frac{d\mu}{dt} &= \mu(t)\tan t \\ \ln|\mu(t)| &= \int \tan t\, dt \\ &= \ln|\sec t| + C \\ \mu(t) &= C\sec t.\end{aligned}$$

Simply, we put $mu(t) = \sec t$ then

$$\begin{aligned}\mu(t)\frac{dy}{dt} + \mu(t)\tan ty = \frac{d}{dt}(\mu(t)y) &= \sec t\cos^2 t = \cos t \\ y &= \cos t\int \cos t\, dt = \cos t(\sin t + C).\end{aligned}$$

Since $y(0) = C = 2$, we obtain

$$y = (\sin t + 2)\cos t.$$

3 Nonlinear differential equations; separable equations

In this section, we discuss how to solve nonlinear first order ODEs. Previously, we have seen an ODE of the form

$$\frac{dy}{dt} = F(y).$$

The idea was to bring $F(y)$ to the other side and apply the Chain rule, which leads to

$$\frac{d}{dt}(G(y)) = \frac{1}{F(y)}\frac{dy}{dt} = 1$$

and so $G(y) = t + C$. This method indeed works for a more general ODE. Consider a first order ODE of the form

$$\frac{dy}{dt} = F(t,y).$$

where $F(t,y)$ is a product of functions $F_1(t)$ and $F_2(y)$. Then,

$$\frac{1}{F_2(y)}\frac{dy}{dt} = F_1(t).$$

If we find a function G such that $G'(y) = \frac{1}{F_2(y)}$, then

$$\begin{aligned}\frac{d}{dt}(G(y)) &= F_1(t), \\ G(y) &= \int F_1(t)\, dt.\end{aligned}$$

Example 3.1. Consider an ODE

$$y' = \frac{x^2 y}{1+x^3}.$$

Then,

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2}{1+x^3}.$$

To apply the chain rule, we find a function $G(y)$ such that

$$G'(y) = \frac{1}{y}.$$

By integrating of the both sides, we get

$$G(y) = \ln|y| + C.$$

Let $C = 0$, then

$$\ln|y| = \int \frac{x^2}{1+x^3}\,dx = \frac{1}{3}\ln|1+x^3| + C = \ln(e^C|1+x^3|^{\frac{1}{3}}).$$

Thus, the solution is

$$y = C|1+x^3|^{\frac{1}{3}}$$

This method can be understood in terms of differential forms. We can rewrite the previous form of ODEs as

$$\begin{aligned}
\frac{dy}{dx} &= F_1(x)F_2(y) \\
\frac{1}{F_2(y)}\,dy &= F_1(x)\,dx \\
-F_1(x)\,dx + \frac{1}{F_2(y)}\,dy &= 0.
\end{aligned}$$

So, we simply consider an ODE of the form

$$M(x)\,dx = N(y)\,dy.$$

In this case, we take integration of both sides with respect to x and y respectively, which yields

$$\int M(x)\,dx = \int N(y)\,dy.$$

Such an equation is said to be separable.

Example 3.2. Consider an ODE

$$xdx + ye^{-x}dy = 0, \qquad y(0) = 1.$$

Then,

$$\begin{aligned}
xe^x dx &= -ydy \\
\int xe^x dx &= -\int ydy \\
(x-1)e^x &= -\frac{1}{2}y^2 + C \\
y^2 &= 2(1-x)e^x + C \\
y &= \pm\sqrt{2(1-x)e^x + C}.
\end{aligned}$$

Since $y(0) = 1$, the sign is plus and we get

$$y(0) = 1 = \sqrt{2+C},$$

which yields $C = -1$. Therefore, the solution is

$$y = \sqrt{2(1-x)e^x - 1}.$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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