Math 416 Lecture Note: Week 3

Daesung Kim

1 Existence and Uniqueness of RREF

Recall 1.1. (i) There is a correspondence between a system of linear equations and a matrix (an augmented matrix).

(ii) There are three types of row operations; (i) Swap two rows, (ii) multiply on by c, (iii) add one to another.

(iii) The solution set of a linear system is preserved under row operations.

(iv) Once transforming a matrix to a RREF, it is easy to find a solution the the associated linear system.

Question 1.2. For each matrix, can we always obtain a reduced row-echelon matrix by row operations?

Definition 1.3. Let $M \in \mathcal{M}_{m\times n}(\mathbb{R})$. A reduced row-echelon form (RREF) of M is a matrix that is in reduced row-echelon form and row-equivalent to M.

Definition 1.4. A *pivot column* of a matrix in RREF is a column containing a leading 1.

Theorem 1.5. *For each $M \in \mathcal{M}_{m\times n}(\mathbb{R})$, there exists a unique reduced row-echelon form of M.*

Proof of the Existence. Use an induction on the number of rows, m. Let $M \in \mathcal{M}_{m\times n}(\mathbb{R})$. Assume $m = 1$. Let M_{1j} be the leftmost nonzero entry of M. (If $M = O$, M is in RREF.) By applying the row operation $R_1 \to \frac{1}{M_{1j}} R_1$, we get a RREF.

Let $m \geq 2$ and $M \in \mathcal{M}_{m\times n}(\mathbb{R})$. Assume that every matrix with the number of rows less than m is row-equivalent to a RREF. The i-th row and the j-th column are denoted by R_i and C_j. Let C_j be the leftmost nonzero column vector of M and M_{ij} a nonzero entry in C_j and R_i. First, we swap R_1 and R_j. Then, multiply R_1 by $\frac{1}{M_{ij}}$ so as to make the first entry of C_j is equal to 1. Let $C_j = (c_1, c_2, \cdots, c_m)^t$ and apply the row operation $R_k \to R_k - c_k R_1$ for each $k = 2, \cdots, m$. Then we get

$$\widetilde{M} = \left(\begin{array}{ccc|c|cccc} 0 & \cdots & 0 & 1 & d_1 & \cdots & \cdots & d_t \\ \hline & & & 0 & A_{11} & A_{12} & \cdots & A_{1t} \\ & O & & \vdots & \vdots & \ddots & \vdots & A_{1t} \\ & & & 0 & A_{s1} & A_{s2} & \cdots & A_{st} \end{array}\right).$$

Note that $M \sim \widetilde{M}$ and $s = m - 1$. By the hypothesis, there exists a sequence of row operations that turns

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1t} \\ \vdots & \ddots & \vdots & A_{1t} \\ A_{s1} & A_{s2} & \cdots & A_{st} \end{pmatrix}$$

to a reduced row-echelon form $\widetilde{A}$. If we apply the sequence of row operations to $\widetilde{M}$, we get

$$\widetilde{M} = \left(\begin{array}{ccc|c|ccc} 0 & \cdots & 0 & 1 & d_1 & \cdots & d_t \\ \hline & & & 0 & & & \\ & O & & \vdots & & \widetilde{A} & \\ & & & 0 & & & \end{array}\right).$$

Suppose $\widetilde{A}_{ij}$ is a leading 1 of $\widetilde{A}$. Then, apply a row operation to make $d_j = 0$. Finally, we obtain a RREF matrix that is row-equivalent to the original matrix M. $\square$

Lemma 1.6. *Let $M, N \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $M \sim N$. Let $M = (C_1, \cdots, C_n)$ and $N = (D_1, \cdots, D_n)$ where C_j and D_j are column vectors for M and N, respectively. Let $k \le n$ and $\{i(1), \cdots, i(k)\} \subset \{1, 2, \cdots, n\}$. Let*

$$\begin{aligned} M' &= (C_{i(1)}, C_{i(2)}, \cdots, C_{i(k)}), \\ N' &= (D_{i(1)}, D_{i(2)}, \cdots, D_{i(k)}), \end{aligned}$$

then $M' \sim N'$.

Proof. Exercise. $\square$

Proof of the Uniqueness. (Due to Holzmann [Hol].) Suppose M has two distinct RREFs $R = (R_1, \cdots, R_n)$ and $S = (S_1, \cdots, S_n)$ where R_i and S_i are column vectors of R and S. Let $k \in \{1, \cdots, n\}$ be the smallest number for which $R_k \neq S_k$ and $R_j = S_j$ for all $j < k$. Let $I = \{i(1), \cdots, i(t-1)\} \subset \{1, 2, \cdots, k-1\}$ be such that $R_{i(j)} = S_{i(j)}$ are pivot for all $j = 1, 2, \cdots, t-1$. Note that $I \neq \varnothing$. (Suppose $I = \varnothing$. This means that R_k is pivot and S_k is zero or vice versa. Since a pivot column is not row-equivalent to the zero column, this is a contradiction.) Let $i(t) = k$. Define new matrices $\widetilde{R}$ and $\widetilde{S}$ by

$$\begin{aligned} \widetilde{R} &= (R_{i(1)}, R_{i(2)}, \cdots, R_{i(t)}) \\ \widetilde{S} &= (S_{i(1)}, S_{i(2)}, \cdots, S_{i(t)}). \end{aligned}$$

There are three possibility for $\widetilde{R}$ and $\widetilde{S}$; (i) R_k is non-pivot and S_k is pivot, (ii) R_k is pivot and S_k is non-pivot, (ii) both R_k and S_k are non-pivot. The first case is

$$\widetilde{R} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & r_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & r_{t-1} \\ \hline & & & & 0 \\ & & O & & \vdots \\ & & & & 0 \end{array}\right), \quad \widetilde{S} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hline & & & & 1 \\ & & O & & 0 \\ & & & & \vdots \end{array}\right).$$

Consider the corresponding linear systems of these two matrices. The solutions for the first system will be $r_1, \cdots, r_{t-1}$, whereas there is no solution to the second system. Thus, this case will not happen. For the same reason,

$$\widetilde{R} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \hline & & & & 1 \\ & & O & & 0 \\ & & & & \vdots \end{array}\right), \quad \widetilde{S} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_{t-1} \\ \hline & & & & 0 \\ & & O & & \vdots \\ & & & & 0 \end{array}\right),$$

is not possible. The third possibility is

$$\widetilde{R} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & r_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & r_{t-1} \\ \hline & & & & 1 \\ & & O & & 0 \\ & & & & \vdots \end{array}\right), \qquad \widetilde{S} = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & s_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & s_{t-1} \\ \hline & & & & 0 \\ & & O & & \vdots \\ & & & & 0 \end{array}\right).$$

If we compare the solution sets of the corresponding systems, we get $r_i = s_i$ for all i, which is a contradiction. Therefore, a RREF of a matrix is unique. $\square$

Example 1.7. Suppose we have two matrices with the same size

$$A = \begin{pmatrix} 1 & 2 & 2 & 0 \\ 1 & 0 & 8 & 5 \\ 1 & 1 & 5 & 5 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Can we determine whether A is row-equivalent to B? Note that B is in RREF and

$$A \sim \begin{pmatrix} 1 & 0 & 8 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the two matrices have diffrent RREFs, by the uniqueness of RREF, we conclude that A is not row-equivalent to B.

2 Solution spaces to linear systems

Example 2.1. Consider (3×5) matrix

$$\begin{pmatrix} 1 & 0 & 8 & 0 & -16 \\ 0 & 1 & -3 & 0 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

and its corresponding linear system

$$\begin{cases} x_1 + 8x_3 = -16 \\ x_2 - 3x_3 = 9 \\ x_4 = 2. \end{cases}$$

Whenever we choose a value for x_3, we obtain one solution. Thus, x_3 acts as a "free variable" and there are infinitely many solutions. Indeed, the solution set is

$$\{(-8t - 16, 3t + 9, t, 2) : t \in \mathbb{R}\}.$$

Example 2.2. Consider (3×5) matrix

$$\begin{pmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and its corresponding linear system

$$\begin{cases} x_1 + x_3 + 5x_4 = 0 \\ x_2 - x_3 - 2x_4 = 0 \\ 0 = 1. \end{cases}$$

There is no solution to the system. Note that the last column of the RREF has a leading 1.

Example 2.3. Consider (3×4) matrix

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

and its corresponding linear system

$$\begin{cases} x_1 = 3 \\ x_2 = 4 \\ x_3 = -2. \end{cases}$$

In this case, the solution is unique.

Definition 2.4. A linear system is *consistent* if it has at least one solution. If there is no solution, it is called *inconsistent*.

Recall that a column vector of a matrix in RREF is pivot if it has a leading 1.

Theorem 2.5. *Let a matrix $M \in \mathcal{M}_{m\times n}(\mathbb{R})$ be in RREF. We use the notation $M = (C_1, C_2, \cdots, C_n)$ where C_i is the i-th column vector of M. Then, the corresponding linear system is inconsistent if and only if C_n is a pivot column.*

Proof. If C_n is a pivot column, then one of the equations is $0 = 1$ as we have seen in the previous example. So, the system is inconsistent.

Suppose that the system is inconsistent and C_n is not a pivot column. Suppose that $1 \leq i(1) \leq i(2) \leq \cdots \leq i(k) \leq n$, $C_{i(1)}, C_{i(2)}, \cdots, C_{i(k)}$ are pivot columns, and $C_n = (c_1, c_2, \cdots, c_m)^t$. Define

$$\begin{cases} x_{i(j)} = c_j, & j = 1, 2, \cdots, k \\ x_j = 0, & \text{otherwise}. \end{cases}$$

We claim that $(x_1, x_2, \cdots, x_n)$ is a solution to the linear system. Fix $j = 1, 2, \cdots, k$. The equation that corresponds to the row in the matrix containing a leading 1 of $C_{i(j)}$ can be written as

$$x_{i(j)} + (\text{a linear combinations of } x_p\text{'s such that } p > i(j) \text{ and } p \neq i(l) \text{ for all } l) = x_{i(j)} = c_j.$$

$\square$

Theorem 2.6. *Every linear system has either no solution, exactly one solution, or infinitely many solutions.*

Proof. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ be a coefficient matrix, b a constant vector, and $M = (A, b)$ an augmented matrix in $\mathcal{M}_{m\times(n+1)}(\mathbb{R})$. Assume that M has no zero rows.

If the $(n+1)$-th column is pivot, then the system is inconsistent by the previous theorem.

Suppose the $(n+1)$-th column is not pivot. Since there is no zero row, every row has a leading zero, which means that $m \leq n$. If $m = n$, then a RREF of M is

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_m \end{array}\right)$$

and so that the system has exactly one solution. If $m < n$, there exists at least one non pivot column. Since a non-pivot variable is "free", we can find infinitely many solution by choosing different values for the non-pivot variable. $\square$

3 Linear dependent and independent

Question 3.1. Let V be a vector space over $\mathbb{R}$ and S a nonempty subset of V. We say S generates V if $\mathrm{Span}(S) = V$. Note that for any $x \in V$, $\mathrm{Span}(S \cup \{x\}) = V$. It is natural to think the opposite way. That is, it is interesting to see whether the generation property can be preserved if we remove some elements from S. Furthermore, can we find "the smallest subset" that also generates V?

Definition 3.2. Let V be a vector space over $\mathbb{R}$. We say that $v_1, \cdots, v_n \in V$ are linearly dependent if there exist $a_1, \cdots, a_n \in \mathbb{R}$, not all zero, such that $a_1 v_1 + \cdots + a_n v_n = 0$. A subset S of V is called linearly dependent if there exist linearly dependent vectors $v_1, \cdots, v_n$ in S.

Example 3.3. Let $V = \mathbb{R}^3$, $v_1 = (1, 1, 1)$, $v_2 = (2, 0, 1)$, and $v_3 = (0, 2, 1)$. Are they linearly dependent? To see this, let $av_1 + bv_2 + cv_3 = 0$, which gives rise to a linear system

$$\begin{cases} a + 2b = 0 \\ a + 2c = 0 \\ a + b + c = 0. \end{cases}$$

The corresponding augment matrix is

$$(v_1, v_2, v_3) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $a = -2c$ and $b = c$. Let $c = 1$, then $-2v_1 + v_2 + v_3 = 0$ and they are linearly dependent.

Definition 3.4. Let V be a vector space over $\mathbb{R}$. A subset S of V is linearly independent if it is not linearly dependent.

Remark 3.5. Let V be a vector space over $\mathbb{R}$, then we have the following.

(i) $\varnothing$ is linearly independent.

(ii) $\{0\}$ is linearly dependent.

(iii) For each $v \in V \setminus \{0\}$, $\{v\}$ is linearly independent.

(iv) (Homework) A set S is linearly independent if and only if for any $v_1, \cdots, v_n$,

$$a_1 v_1 + \cdots + a_n v_n = 0 \quad \Rightarrow \quad a_1 = \cdots = a_n = 0.$$

Theorem 3.6. *Let V be a vector space over $\mathbb{R}$, S a subset of V, and $v \in V$. If $\mathrm{Span}(S \cup \{v\}) = V$ and $v \in \mathrm{Span}(S)$, then $\mathrm{Span}(S) = V$.*

Proof. It suffices to show that $\mathrm{Span}(S \cup \{v\}) \subset \mathrm{Span}(S)$. Note that $S \subset \mathrm{Span}(S)$ and $v \in \mathrm{Span}(S)$. Since $\mathrm{Span}(S \cup \{v\})$ is the smallest subspace that contains S and v and $\mathrm{Span}(S)$ is a subspace of V, the proof is complete. □

Example 3.7. Let $V = \mathbb{R}^3$, $v_1 = (1, 1, 1)$, $v_2 = (2, 0, 1)$, $v_3 = (0, 2, 1)$, and $S = \{v_1, v_2, v_3\}$. What is $\mathrm{Span}(S)$? As we have seen above, one vector is a linear combination of the others: $v_1 = \frac{1}{2}v_2 + \frac{1}{2}v_3$, $v_2 = 2v_1 - v_3$, $v_3 = 2v_1 - v_2$. Thus, $\mathrm{Span}(S) = \mathrm{Span}(\{v_1, v_2\}) = \mathrm{Span}(\{v_2, v_3\}) = \mathrm{Span}(\{v_3, v_1\})$. It then follows that

$$\mathrm{Span}(S) = \mathrm{Span}(\{v_1, v_2\}) = \{av_1 + bv_2 : a, b \in \mathbb{R}\} = \{(a + 2b, a, a + b) : a, b \in \mathbb{R}\}$$

and an equation for $\mathrm{Span}(S)$ is $x + y - 2z = 0$.

Theorem 3.8. *Let V be a vector space over $\mathbb{R}$ and $S_1 \subset S_2 \subset V$.*

(i) *If S_1 is linearly dependent, then so is S_2.*

(ii) *If S_2 is linearly independent, then so is S_1.*

Proof. Exercise. $\square$

Theorem 3.9. *Let V be a vector space over $\mathbb{R}$ and S linearly independent. Let $v \in V \setminus S$. Then, $S \cup \{v\}$ is linearly dependent if and only if $v \in \mathrm{Span}(S)$.*

Proof. ($\Rightarrow$): There exist $v_1, \cdots, v_n \in S$ and $a_1, \cdots, a_n, a_{n+1} \in \mathbb{R}$, not all zero, such that

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n + a_{n+1}v = 0.$$

If $a_{n+1} = 0$, then $v_1, \cdots, v_n$ is linearly dependent, which is a contradiction. Thus, $a_{n+1} \neq 0$. By dividing the equation by $-a_{n+1}$, we get

$$v = -\frac{a_1}{a_{n+1}}v_1 - \cdots - \frac{a_n}{a_{n+1}}v_n.$$

So, $v \in \mathrm{Span}(S)$.

($\Leftarrow$): There exists $v_1, \cdots, v_n \in S$ and $a_1, \cdots, a_n \in \mathbb{R}$ such that $v = a_1v_1 + \cdots + a_nv_n$. Thus,

$$a_1v_1 + \cdots + a_nv_n + a_{n+1}v = 0$$

where $a_{n+1} = -1$ and $a_1, \cdots, a_{n+1}$ are not all zero. So, $S \cup \{v\}$ is linearly dependent. $\square$

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

[Hol] W. H. Holzmann, http://www.cs.uleth.ca/ holzmann/notes/reduceduniq.pdf

Department of Mathematics, University of Illinois at Urbana-Champaign
*E-mail address:*daesungk@illinois.edu