Homework 11

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: December 11, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $V = \mathbb{C}^3$ be equipped with the standard inner product and $\varphi : V \to \mathbb{C}$ be a linear transformation defined by $\varphi(x_1, x_2, x_3) = x_1 - (2+i)x_2 + 4ix_3$. Find $y \in V$ such that $\varphi(x) = \langle x, y\rangle$ for all $x \in V$.

Solution: Let $y = (1, 2+i, -4i)$, then

$$\langle x, y\rangle = y^* x = \begin{pmatrix} 1 & -(2+i) & 4i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 - (2+i)x_2 + 4ix_3 = \varphi(x)$$

for all $x = (x_1, x_2, x_3) \in \mathbb{C}^3$.

2. Let $V = \mathbb{R}^3$, $W = \mathrm{Span}(\{(1, -2, 0), (1, 0, 1)\})$, and β be the standard basis for V. Compute $[T]_\beta$ where $T = \mathrm{proj}_W$ is the orthogonal projection onto W. (Hint: Let

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix},$$

then $W = \mathcal{R}(A)$.)

Solution: Note that A has full rank (that is, $\operatorname{rank}(A) = 2$). It follows that A^*A is invertible and

$$\begin{aligned}
[T]_\beta &= A(A^*A)^{-1}A^* \\
&= \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{9} \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{9} \begin{pmatrix} 1 & 4 \\ -4 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\
&= \frac{1}{9} \begin{pmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{pmatrix}.
\end{aligned}$$

3. Let V be a finite dimensional inner product space over F and $T : V \to V$ normal. Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$. (Hint: show $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$ and use it.)

Solution: Let $x \in \mathcal{N}(T)$, then

$$\|T^*(x)\|^2 = \langle T^*(x), T^*(x)\rangle = \langle TT^*(x), x\rangle = \langle T^*T(x), x\rangle = \langle T(x), T(x)\rangle = 0,$$

which shows that $x \in \mathcal{N}(T^*)$. By symmetry, we conclude that $\mathcal{N}(T) = \mathcal{N}(T^*)$.
Let $x \in \mathcal{R}(T)$, then there exists $y \in V$ such that $x = T(y)$. In the class, we have shown that $\mathcal{N}(T^*) = \mathcal{R}(T)^\perp$ and so $\mathcal{N}(T) = \mathcal{R}(T^*)^\perp$. Since $\mathcal{R}(T^*)$ is finite dimensional, we have

$$\mathcal{N}(T)^\perp = (\mathcal{R}(T^*)^\perp)^\perp = \mathcal{R}(T^*).$$

Thus, it suffices to show that

$$\langle x, z\rangle = \langle T(y), z\rangle = 0$$

for all $z \in \mathcal{N}(T)$. It follows from $\mathcal{N}(T) = \mathcal{N}(T^*)$ that

$$\langle x, z\rangle = \langle T(y), z\rangle = \langle y, T^*(z)\rangle = 0$$

for all $z \in \mathcal{N}(T)$.

4. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $f(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0$ be the characteristic polynomial of A over $\mathbb{C}$ (that is, $a_0, a_1, \cdots, a_n \in \mathbb{C}$). Use Schur's theorem to show that

$$\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}.$$

Solution: Since every polynomial splits over $\mathbb{C}$, we have

$$f(t) = (\lambda_1 - t)\cdots(\lambda_n - t)$$

and there exists a unitary matrix Q such that

$$Q^*AQ = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Thus,

$$\operatorname{tr}(A) = \operatorname{tr}(Q^*AQ) = \sum_{i=1}^{n} \lambda_i.$$

Since

$$f(t) = (\lambda_1 - t)\cdots(\lambda_n - t) = (-1)^n t^n + (-1)^{n-1}(\lambda_1 + \cdots + \lambda_n)t^{n-1} + \cdots,$$

we conclude that $\operatorname{tr}(A) = (-1)^{n-1}a_{n-1}$.

5. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $V = \mathbb{C}^n$.
   (a) Show that AA^* is positive semidefinite.
   (b) Suppose A is self-adjoint. Show that A^2 is positive semidefinite.

Solution:

(a) First, note that AA^* is self-adjoint because $(AA^*)^* = AA^*$. If $x \in V$, then

$$\langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle \geq 0.$$

Thus, AA^* is positive semidefinite.

(b) It follows that for $x \in V$

$$\left\langle A^2x, x\right\rangle = \langle Ax, A^*x\rangle = \langle Ax, Ax\rangle \geq 0.$$