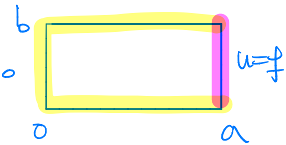
Sec 10.8: Laplace's Equation, part 2

Math 285 Spring 2020

Instructor: Daesung Kim

Recall: Dirichlet problem for a rectangle



Consider

$$\Delta u = u_{xx} + u_{yy} = 0$$

in the rectangle $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$ with

$$u(x,0) = 0, \qquad u(x,b) = 0 \quad \text{for } 0 < x < a,$$
$$u(0,y) = 0, \qquad u(a,y) = f(y) \quad \text{for } 0 \le y \le b,$$

where f is a function on $0 \le y \le b$.

The solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}x\right) \sin\left(\frac{n\pi}{b}y\right)$$

where

$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y) \sin\left(\frac{n\pi}{b}y\right)\, dy.$$

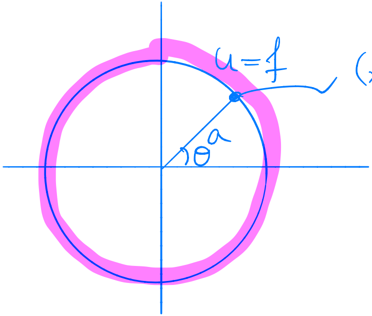
Dirichlet problem for a disk

Consider the 2-dimensional Laplace's equation $u_{xx} + u_{yy} = 0$ in the disk

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}.$$

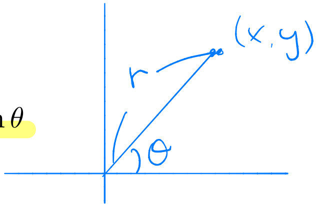
with the boundary condition $u(a\cos\theta, a\sin\theta) = f(\theta)$ for $0 \le \theta < 2\pi$.

For a disk, it is convenient to use polar coordinates.



$(x,y) = (a\cos\theta, a\sin\theta)$

Polar coordinates



Recall that polar coordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

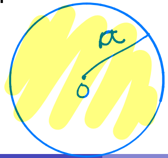
for $r > 0$ and $0 \le \theta < 2\pi$.

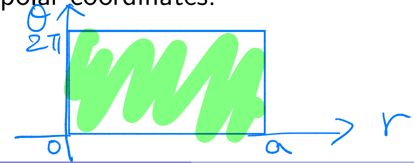
Using this, the disk can be written as

$$\mathcal{D} = \{(r,\theta) : 0 \le r < a, 0 \le \theta < 2\pi\}.$$

We abuse the notation $u(x,y) = u(r,\theta)$.

The boundary condition can be written as $u(a,\theta) = f(\theta)$. We translate the Laplace's equation in terms of polar coordinates.





Polar coordinates

$$u_{xx} + u_{yy} = 0$$

Interpret this in terms of r & θ.

Use the Chain Rule.

$$u_r = \frac{\partial u}{\partial r} = \underbrace{\frac{\partial u}{\partial x}}_{= u_x} \cdot \frac{\partial x}{\partial r} + \overbrace{\frac{\partial u}{\partial y}}^{= u_y} \cdot \frac{\partial y}{\partial r}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{cases} \dfrac{\partial x}{\partial r} = \cos\theta \\ \dfrac{\partial y}{\partial r} = \sin\theta \end{cases}$$

$$\boxed{u_r = \cos\theta \cdot u_x + \sin\theta \cdot u_y}$$

Polar coordinates

$$u_\theta = u_x \cdot \frac{\partial x}{\partial \theta} + u_y \cdot \frac{\partial y}{\partial \theta}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad (\because x = r\cos\theta)$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta \qquad (\because y = r\sin\theta)$$

$$\boxed{u_\theta = r\left(-\sin\theta \cdot u_x + \cos\theta \cdot u_y\right)}$$

Polar coordinates

$$
\begin{aligned}
u_{rr} &= \frac{\partial}{\partial r}(u_r) \\
&= \frac{\partial}{\partial r}\left(\underline{\cos\theta} \cdot u_x + \underline{\sin\theta} \cdot u_y \right) \\
&= \cos\theta \cdot \left(\underline{\frac{\partial}{\partial r} u_x} \right) + \sin\theta \cdot \left(\underline{\frac{\partial}{\partial r} u_y} \right) \\
&= \cos\theta \cdot \left(u_{xx} \cdot \frac{\partial x}{\partial r} + u_{xy} \frac{\partial y}{\partial r} \right) \\
&\quad + \sin\theta \left(u_{xy} \frac{\partial x}{\partial r} + u_{yy} \frac{\partial y}{\partial r} \right) \\
&= \underline{\cos^2\theta \cdot u_{xx} + 2\cos\theta \cdot \sin\theta\, u_{xy}} \\
&\quad + \underline{\sin^2\theta \cdot u_{yy}}.
\end{aligned}
$$

Polar coordinates

$$u_{\theta\theta} = \frac{\partial}{\partial\theta}(u_\theta)$$

$$= \frac{\partial}{\partial\theta}\left(r\left(\overbrace{-\sin\theta\, u_x + \cos\theta \cdot u_y}\right)\right)$$

$$= r \cdot \Big[-\underbrace{(\cos\theta\, u_x + \sin\theta \cdot u_y)}_{=u_r}$$

$$\begin{cases} -\sin\theta \cdot \left(u_{xx} \cdot \frac{\partial x}{\partial\theta} + u_{xy} \cdot \frac{\partial y}{\partial\theta} \right) \\ +\cos\theta \cdot \left(u_{xy} \frac{\partial x}{\partial\theta} + u_{yy} \frac{\partial y}{\partial\theta} \right) \Big] \end{cases}$$

$$= -r\, u_r + r^2 \cdot \left(\sin^2\theta\, u_{xx} - 2\sin\theta \cdot \cos\theta\, u_{xy} + \cos^2\theta\, u_{yy} \right)$$

Polar coordinates

$$u_{\theta\theta} = -r\,u_r + r^2\Big[\underbrace{\sin^2\theta}_{(1-\cos^2\theta)}\,u_{xx} - 2\sin\theta\cdot\cos\theta\,u_{xy} + \underbrace{\cos^2\theta}_{(1-\sin^2\theta)}\,u_{yy}\Big]$$

$$= -r\,u_r + r^2\big[u_{xx}+u_{yy}\big]$$

$$\boxed{\Delta u = u_{xx}+u_{yy}}$$

$$- r^2\Big[\underbrace{\cos^2\theta\,u_{xx} + 2\cos\theta\cdot\sin\theta\,u_{xy} + \sin^2\theta\,u_{yy}}_{u_{rr}}\Big]$$

$$= r^2[\Delta u] - r^2 u_{rr} - r u_r$$

$$\boxed{r^2 u_{rr} + r u_r + u_{\theta\theta}} = r^2\cdot\Delta u = 0$$

Separation of variables in Polar coordinates

So, if $u_{xx} + u_{yy} = 0$, then

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $u(r,\theta) = R(r)\Theta(\theta)$.

$$u(a,\theta) = f(\theta)$$

$$R(a)\cdot \underbrace{\Theta(\theta)}_{\text{periodic}} = f(\theta)$$

$$r^2 R''\Theta + rR'\Theta + R\cdot\Theta'' = 0$$

$$r^2 \frac{R''}{R} + r\cdot\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

$$\begin{cases} r^2 R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0. \quad (\Theta(t) = \Theta(t+2\pi)) \end{cases}$$

$$\Theta'' + \lambda \Theta = 0$$

<u>Case 1</u> $\lambda = -\mu^2 < 0$.

$$\Theta(t) = C_1 e^{\mu t} + C_2 e^{-\mu t}$$

$$\Theta(t) = \Theta(t+2\pi) \quad \text{for every } t.$$

$$\Rightarrow \quad C_1 = C_2 = 0.$$

$t = -\pi$.

$$\begin{cases} \Theta(-\pi) = C_1 e^{-\pi\mu} + C_2 e^{\pi\mu} = 0 \\ \Theta(\pi) = C_1 e^{\pi\mu} + C_2 e^{-\pi\mu} = 0 \end{cases}$$

Case 2 $\lambda = 0$. $\Rightarrow$ $u_0 = \text{const.}$

$$\Theta'' = 0 \qquad \Theta(t) = \underline{c_1} + c_2 t$$

$$\Theta(t) = \Theta(t + 2\pi)$$

$t = 0$. $\quad \Theta(0) = c_1 = \Theta(2\pi) = c_1 + c_2 \cdot 2\pi$

$$\Rightarrow \quad c_2 = 0.$$

So, $\underline{\Theta \text{ is a}}$ constant function.

$r^2 R'' + rR' = 0$, $\quad rR'' + R' = 0$

$$\underline{R(r)} = \underline{c_1} + c_2 \ln r \qquad \underline{R : \text{const.}}$$

$0 < r < a$, if $r \to 0$, $\ln r \to -\infty$, $\underline{c_2 = 0}$.

Case 3 $\quad \lambda = \mu^2 > 0$

$$\Theta(t) = C_1 \cos \mu t + C_2 \sin \mu t$$

$$\Theta(t) = \Theta(t + 2\pi)$$

$\Rightarrow \mu \in \mathbb{N}$. $\quad \mu = n$. $\quad \lambda = n^2$

$$r^2 R'' + r R' - n^2 R = 0.$$

① $r = e^t \Rightarrow \dfrac{d^2R}{dt^2} - n^2 R = 0$.

② $R(r) = r^k$

$$r^2 R'' + rR' - n^2 R = (k^2 - n^2) \cdot r^k = 0$$

$k = n, -n.$

$$R_n(r) = c_1 r^n + c_2 \cdot r^{-n}$$

If $r \downarrow 0$, $\underline{r^{-n} \uparrow \infty} \Rightarrow c_2 = 0$

$$R_n(r) = c \cdot r^n$$

$$\Theta_\theta(\theta) = c_1 \cos n\theta + c_2 \sin n\theta$$

$$u_n(r, \theta) = c \cdot r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n \left(c_n \cos n\theta + d_n \sin n\theta \right)$$

$$u(a,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} \underbrace{a^n \left(c_n \right.}\cos n\theta + d_n \sin n\theta \left.\right)$$

$$= f(\theta)$$

$$c_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta$$

$$d_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cdot \sin n\theta \, d\theta .$$