Sec 10.5: Separation of Variables; Heat Conduction in a Rod

Math 285 Spring 2020

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Heat Equation

Consider a heat conduction problem for a straight bar of length $L > 0$. Suppose it has uniform cross section and homogeneous material.

Let the x-axis lie along the axis of the bar. Assume that the sides of the bar are perfectly insulated and each cross section has uniform temperature.

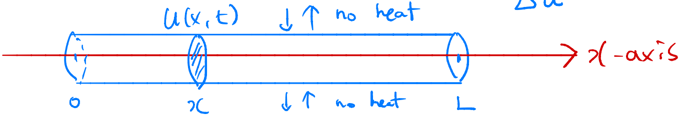
Heat equation: Let $u(x,t)$ be the temperature of a cross section at x and time t. Then, u is governed by the heat conduction equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \qquad t > 0.$$

$\alpha^2 (u_{xx} + u_{yy}) = u_t$

$\alpha^2 \underbrace{(u_{xx} + u_{yy} + u_{zz})}_{= \Delta u} = u_t$

The constant α^2 is called the thermal diffusivity.

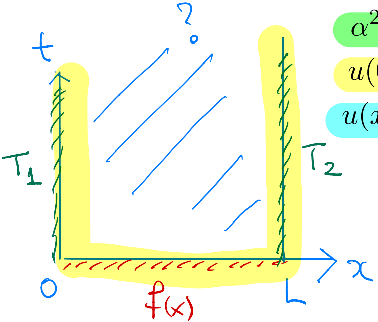


Initial and Boundary conditions

Initial condition: The initial temperature of the bar is given by $u(x,0) = f(x)$ for $0 \le x \le L$.

Boundary conditions: The ends of the bar are held at fixed temperatures $u(0,t) = T_1$ and $u(L,t) = T_2$ for all $t > 0$.

In this section, we focus on the case $T_1 = T_2 = 0$ find solutions to



$$\alpha^2 u_{xx} = u_t,$$
$$u(0,t) = u(L,t) = 0,$$
$$u(x,0) = f(x).$$

Separation of variables

superposition:

Consider the boundary problem

homogeneous

$$\alpha^2 u_{xx} = u_t,$$
$$u(0,t) = u(L,t) = 0.$$

If u_1, u_2 solu.
then $c_1u_1 + c_2u_2$ is also a solu.

for $0 < x < L$ and $t > 0$. (We drop the initial condition for a moment.)

Main Idea: Let $u(x,t) = X(x)T(t)$. This is called the method of separation of variables.

Boundary condition

$u(0,t) = X(0) \cdot T(t) = 0$ for all t
$u(L,t) = X(L) \cdot T(t) = 0$ for all t

If $X(0) \neq 0$, $T(t) = 0 \ \forall t$, $u = 0 \ \forall t$.

$\Rightarrow$ $X(0) = 0$ & $X(L) = 0$.

Separation of variables

Equation

$$u(x,t) = X(x) \cdot T(t)$$

$$\alpha^2 u_{xx} = u_t$$

$$\alpha^2 u_{xx} = \underbrace{\alpha^2 X''(x) \cdot T(t) = X(x) \cdot T'(t)} = u_t$$

$$\frac{X''}{X} = \frac{1}{\alpha^2} \cdot \frac{T'}{T} = \text{Constant} = -\lambda$$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + \alpha^2 \lambda T = 0 \end{cases} \quad \Bigg| \quad X(0) = X(L) = 0.$$

Solutions for $X(x)$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0. \end{cases}$$

WANT TO FIND NONTRIVIAL SOL.

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$$

for $n = 1, 2, 3, \cdots$

$$T' + \alpha^2 \lambda T = 0$$

Solutions for $T(t)$

$$T' + \alpha^2 \frac{n^2\pi^2}{L^2} T = 0$$

$$T_n(t) = C_1 \exp\left(-\frac{\alpha^2 n^2 \pi^2}{L^2} \cdot t\right) = C e^{-\frac{\alpha^2 n^2\pi^2}{L^2} t}$$

for each $n = 1, 2, 3, \cdots$

$$u_n(x,t) = X_n(x) \cdot T_n(t)$$
$$= C_n \cdot \exp\left(-\frac{\alpha^2 n^2 \pi^2}{L^2} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right)$$

is a solution to

$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u(0,t) = u(L,t) = 0 \end{cases}$$

Initial condition and Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \cdot e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} \cdot t} \cdot \sin\left(\frac{n\pi}{L} x\right).$$

Is a solution to $\begin{cases} \alpha^2 u_{xx} = u_t \\ u(0,t) = u(L,t) = 0. \end{cases}$

Assume $u(x,0) = f(x)$

Fourier sine series.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \cdot \sin\left(\frac{n\pi}{L} x\right) = f(x)$$

determine this to make this sense.

Recall Fourier Sine series of $f(x)$

on $[0, L]$

$$\begin{cases} h(x) = \begin{cases} f(x) & \text{on } \underline{[0, L]} \\ -f(-x) & \text{on } [-L, 0] \end{cases} \\ h(x+2L) = h(x) \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)$$

$$C_n = \frac{1}{L}\int_{-L}^{L} \underbrace{h(x)}_{\text{odd}} \underbrace{\sin\left(\frac{n\pi}{L}x\right)}_{\text{odd}} dx$$

$$= \underline{\frac{2}{L}\int_{0}^{L} f(x) \cdot \sin\left(\frac{n\pi}{L}x\right) dx}.$$

Finally,

$$u(x,t) = \sum_{n=1}^{\infty} c_n \cdot e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L} x\right)$$

$$\underline{c_n} = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

is a solution to

$$\begin{cases} \alpha^2 u_{xx} = u_t \\ u(x,0) = f(x) \\ u(0,t) = u(L,t) = 0. \end{cases}$$