Homework 5

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: October 11, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

   Compute $A(3B + 2C), (AB)D, A(BD)$.

2. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ and $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations defined by

$$T(f(x)) = xf'(x) + 2f(x), \qquad U(a + bx + cx^2) = (a + b, c, a - b).$$

   Let $\beta = \{1, x, x^2\}$ and $\gamma = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$. Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$.

3. Let V, W, and Z be vector spaces. Let $T : V \to W$ and $U : W \to Z$ be linear.
   (a) Prove that if UT is one-to-one, then T is one-to-one.
   (b) Prove that if UT is onto, then U is onto.
   (c) Prove that U and T are one-to-one and onto, then so is UT.

4. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$.
   (a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(A) = \mathrm{tr}(A^t)$, and $(AB)^t = B^tA^t$.
   (b) Are there exist $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $AB - BA = I_n$? Justify your answer.

5. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$. Define $\langle A, B\rangle = \mathrm{tr}(AB^t)$.
   (a) Show that $\langle A, B\rangle = \langle B, A\rangle$.
   (b) Show that $\langle A, A\rangle \geq 0$ and equality holds if and only if $A = O$.

6. Determine whether T is invertible and justify your answer.
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (3x - y, y, 4x)$.
   (b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x - 2z, y, 3x + 4y)$.

7. Let V and W finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.
   (a) Prove that $T(V_0)$ is a subspace of W
   (b) Prove that $\dim(V_0) = \dim(T(V_0))$.