# Sec 10.6: Other Heat Conduction Problems

Math 285 Spring 2020

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# Recall

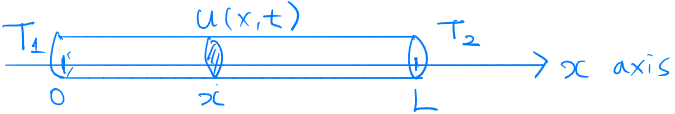
Heat equation: Let $u(x,t)$ be the temperature of a cross section at $x$ and time $t$. Then, $u$ is governed by the heat conduction equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \qquad t > 0.$$

The constant $\alpha^2$ is called the thermal diffusivity.

Initial condition: The initial temperature of the bar is given by $u(x,0) = f(x)$ for $0 \le x \le L$.

Boundary conditions: The ends of the bar are held at fixed temperatures $u(0,t) = T_1$ and $u(L,t) = T_2$ for all $t > 0$.



# Recall

Method of separation of variables: Use $u(x,t) = X(x)T(t)$ to derive two ordinary differential equations.

Solution: If $u(0,t) = u(L,t) = 0$, then

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{\alpha^2 m^2 \pi^2}{L^2} t} \sin\left(\frac{m\pi}{L} x\right)$$

where

$$C_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx.$$

① $u(0,t) = T_1$ , $u(L,t) = T_2$

② $u_x(0,t) = u_x(L,t) = 0$ .

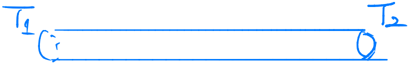
# Nonhomogeneous boundary conditions

Consider a heat conduction problem for a straight bar of length $L > 0$.

Suppose the ends of the bar are held at constant temperatures $T_1$ and $T_2$.

Then, the corresponding heat conduction equation with boundary conditions is

$$\begin{array}{rr} \text{Eqn}: & \alpha^2 u_{xx} = u_t, \\ \text{Boundary Cond.}\left\{\begin{array}{r} \\ \\ \end{array}\right. & \begin{array}{r} u(0,t) = \underline{T_1}, \\ u(L,t) = \underline{T_2}, \end{array} \\ \text{Initial Cond.}: & u(x,0) = f(x). \end{array}$$



# The steady state solution

Let $v(x) = \lim_{t\to\infty} u(x,t)$ be the steady state temperature distribution.

Then $v(x)$ satisfies $v'' = 0$ with $v(0) = T_1$ and $v(L) = T_2$.

$\alpha^2 u_{xx} = \underline{u_t}$

$u_t \to 0$ as $t \to \infty$

$v_{xx} = 0 = v''$

$v(x) = Ax + B$

$v(0) = B = T_1$

$v(L) = A \cdot L + T_1 = T_2$

So, $A = \frac{1}{L}(T_2 - T_1)$

$\boxed{v(x) = \frac{1}{L}(T_2 - T_1)x + T_1}$

Let $w(x,t) = u(x,t) - v(x)$.

$w_{xx} = u_{xx} - v_{xx}$

$= u_{xx}$

$w_t = u_t - v_t = u_t$

$\alpha^2 w_{xx} = w_t$.

$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$

$w(0,t) = u(0,t) - v(0) = 0$

$w(L,t) = \underbrace{u(L,t)}_{T_2} - \underbrace{v(L)}_{T_2} = 0$

# The steady state solution

Let $v(x) = \lim_{t\to\infty} u(x,t)$ be the steady state temperature distribution.

Then $v(x)$ satisfies $v'' = 0$ with $v(0) = T_1$ and $v(L) = T_2$.

$$v(x) = \frac{1}{L}(T_2 - T_1)x + T_1 \quad (*)$$

Therefore,

$$u(x,t) = w(x,t) + v(x)$$

Let $w(x,t) = u(x,t) - v(x)$.

$$\begin{cases} \alpha^2 w_{xx} = w_t \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = f(x) - v(x) \end{cases}$$

Solution :

$$w(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{\alpha^2 m^2 \pi^2}{L^2} t} \sin\left(\frac{m\pi}{L} x\right) \quad (**)$$

$$C_m = \frac{2}{L}\int_0^L (f(x) - v(x)) \cdot \sin\left(\frac{m\pi}{L} x\right) dx$$

# Example

$\alpha^2 = 5$ $L = 4$

## Example

Consider $5u_{xx} = u_t$ with $u(0,t) = \overset{T_1}{10}$, $u(\underline{4},t) = \overset{T_2}{2}$, and $u(x,0) = x$. $= f(x)$

$$v(x) = \frac{1}{L}(T_2 - T_1)\,x + T_1 = 10 - 2x \quad (**)$$

$$w(x,t) = u(x,t) - v(x)$$

$$= \sum_{m=1}^{\infty} C_m\, e^{-\frac{5m^2\pi^2}{4}t} \cdot \sin\left(\frac{m\pi}{4}x\right) \quad (*)$$

$$C_m = \frac{2}{4}\int_0^4 \Big(\underbrace{x - (10-2x)}_{f(x)\,-\,v(x)}\Big)\cdot \sin\left(\frac{m\pi}{4}x\right)dx$$

$$= \frac{1}{2}\int_0^4 (3x - 10)\sin\left(\frac{m\pi}{4}x\right)dx$$

$$u(x,t) = \underset{(*)}{w(x,t)} + \underset{(**)}{v(x)}$$

# Example

## Example

Consider $5u_{xx} = u_t$ with $u(0,t) = 10$, $u(4,t) = 2$, and $u(x,0) = x$.

# Bar with insulated ends

Suppose that the ends of the bar are perfectly insulated so that there is no passage of heat through them.

This model is governed by

$$\alpha^2 u_{xx} = u_t,$$
$$u_x(0,t) = u_x(L,t) = 0,$$
$$u(x,0) = f(x).$$

Let $u(x,t) = X(x) \cdot T(t)$, then

$$\begin{cases} X'' + \lambda X = 0 \\ T' + \lambda \alpha^2 T = 0. \end{cases}$$

$$u_x(0,t) = X'(0) \cdot T(t) = 0 = X'(L) \cdot T(t) = u_x(L,t)$$

$$\Rightarrow \quad X'(0) = X'(L) = 0.$$

# Bar with insulated ends

Solve $X'' + \lambda X = 0$ with $X'(0) = X'(L) = 0$.

Case 1 $\lambda = -\mu^2 < 0$.

$$X(x) = C_1 \cdot e^{\mu t} + C_2 e^{-\mu t}$$

Initial conditions $\Rightarrow$ $C_1 = C_2 = 0$.

Case 2 $\lambda = 0$.

$$X(x) = C_1 x + C_2$$

Initial condition $\Rightarrow$ $C_1 = 0$.

Let $\lambda_0 = 0$, $X_0(x) = 1$.

Case 3 $\lambda = \mu^2 > 0$

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

# Bar with insulated ends

Initial Condition $\Rightarrow$ $C_2 = 0$ & $C_1 \cdot \sin(\mu L) = 0$

$\Rightarrow$ $C_2 = 0$ & $\underline{\dfrac{\mu L}{\pi} \in \mathbb{N}}$

For each $n \in \mathbb{N}$,

$(\mu = \dfrac{n\pi}{L})$

$$\lambda_n = \frac{n^2 \pi^2}{L^2}$$

$$X_n(x) = \cos\left(\frac{n\pi}{L} x\right)$$

$$T' + \lambda_n \alpha^2 T = 0$$

$$T_n(t) = C \cdot e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

# Bar with insulated ends

$$u_n(x,t) = X_n(x) \cdot T_n(t)$$

$$= C_n \cdot e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cdot \cos\left(\frac{n\pi}{L} x\right)$$

$$u_0(x,t) = \underbrace{X_0(x)}_{=1} \cdot \underbrace{T_0(t)}_{=C_0} = C_0 .$$

$$u(x,t) = \frac{C_0}{2} + \sum_{m=1}^{\infty} C_m \cdot e^{-\frac{\alpha^2 m^2 \pi^2}{L^2} t} \cos\left(\frac{m\pi}{L} x\right)$$

$$u(x,0) = f(x)$$

$$= \frac{C_0}{2} + \sum_{m=1}^{\infty} C_m \cos\left(\frac{m\pi}{L} x\right).$$

The Fourier cosine series of $f$ gives

# Bar with insulated ends

$$c_m = \frac{2}{L}\int_0^L f(x)\cdot\cos\left(\frac{m\pi}{L}x\right)\,dx.$$