Homework 1

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 6, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Prove Corollary 1 in section 1.2 of [FIS] (page 11).
2. Prove Corollary 2 in section 1.2 of [FIS] (page 12).
3. Let $V = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be vectors in V, and $c \in \mathbb{R}$. Define $x + y = (x_1 + y_1, x_2 + y_2)$ and $cx = (cx_1, c^2 x_2)$. Is V a vector space over $\mathbb{R}$? Justify your answer.
4. Let V, W be vector spaces over $\mathbb{R}$. Define the product of $V \times W$ by
$$V \times W = \{(v, w) : v \in V, w \in W\}.$$
For $(v_1, w_1), (v_2, w_2) \in V \times W$ and $c \in \mathbb{R}$, define
$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \qquad c(v_1, w_1) = (cv_1, cw_1).$$
Show that $V \times W$ is a vector space over $\mathbb{R}$.
5. Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries. Prove the following.
   (a) $(aA + bB)^t = aA^t + bB^t$ for any $a, b \in \mathbb{R}$ and $A, B \in M_{m\times n}(\mathbb{R})$, where $m, n \in \mathbb{N}$.
   (b) $\mathrm{tr}(aA + bB) = a\,\mathrm{tr}(A) + b\,\mathrm{tr}(B)$ for any $a, b \in \mathbb{R}$ and $A, B \in M_{n\times n}(\mathbb{R})$, where $n \in \mathbb{N}$.
6. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operation of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify you answer.
   (a) $W_1 = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y - z = 0\}$.
   (b) $W_2 = \{(x, y, z) \in \mathbb{R}^3 : x = y - 3z + 1\}$.
   (c) $W_3 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z\}$.
   (d) $W_4 = \{(x, y, z) \in \mathbb{R}^3 : x = 2y, y = -z\}$.
7. Let $F_0(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$. Define addition and scalar multiplication by $(f + g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for any $f, g \in F_0(\mathbb{R})$, $x, c \in \mathbb{R}$. Show that $F_0(\mathbb{R})$ is a vector space over $\mathbb{R}$.
8. Let W_1, W_2 be subspaces of a vector space V over $\mathbb{R}$. Show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.
9. Let W_1, W_2 be subspaces of a vector space V over $\mathbb{R}$. Define
$$W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}.$$

(a) Show that $W_1 + W_2$ is a subspace of V.

(b) Let U be a subspace of V and $W_1, W_2 \subseteq U$. Show that $W_1 + W_2 \leq U$. (This implies that $W_1 + W_2$ is the smallest subspace of V containing W_1 and W_2.)

10. Let V be a vector space over $\mathbb{R}$. We say that V is the direct sum of W_1 and W_2 if $W_1, W_2 \leq V$, $W_1 \cap W_2 = \{0\}$, and $W_1 + W_2 = V$. We denote by $V = W_1 \oplus W_2$. Let $W_1, W_2 \leq V$. Show that $V = W_1 \oplus W_2$ if and only if every $x \in V$ can be uniquely written as $x = x_1 + x_2$ for $x_1 \in W_1$ and $x_2 \in W_2$.