# Math 285 Lecture Note: Week 3

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## 1 Nonlinear differential equations

Last time, we have seen that if we have a separable equation $y' = F(x)G(y)$ or $M(x)dx + N(y)dy = 0$, then we can find a solution.

**Example 1.1.** Consider

$$y' = \frac{2x}{y + x^2 y} = \frac{2x}{1+x^2}y.$$

Then,

$$ydy = \frac{2x}{1+x^2}dx$$

and so $y^2 = 2\ln(1+x^2) + C$.

We have seen how to find a solution of an ODE that is separable. In this section, we discuss other cases where we can find a solution even though the ODE is not separable nor linear.

**Example 1.2** (Homogeneous equations)**.** We call an ODE $y' = F(x,y)$ is homogenous if $F(tx,ty) = F(x,y)$ for all $t \neq 0$. In this case, we can replace $F(x,y)$ with $F(1, y/x)$. Consider an ODE

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}.$$

This is not separable but we can make it separable by introducing a new variable. Let $v = y/x$, then the RHS can be written as

$$\frac{x^2 + xy + y^2}{x^2} = 1 + v + v^2.$$

On the other hands, we have $xv = y$ and so

$$v + x\frac{dv}{dx} = \frac{dy}{dx}.$$

Thus, we get

$$\begin{aligned}
x\frac{dv}{dx} &= 1 + v^2 \\
\frac{1}{1+v^2}dv &= \frac{1}{x}dx \\
\arctan(v) &= \ln|x| + C \\
v(x) &= \tan(\ln|x| + C) \\
y(x) &= x\tan(\ln|x| + C).
\end{aligned}$$

**Example 1.3** (Bernoulli equations)**.** Consider an ODE

$$y' + p(t)y = q(t)y^n.$$

If $n = 0, 1$, then it is linear so that we can solve it. Suppose $n \neq 0, 1$. First, $y(t) = 0$ is a trivial solution. Suppose $y(t) \neq 0$. Dividing $y^n$ of the both sides,

$$y^{-n}y' + p(t)y^{1-n} = q(t).$$

Let $v = y^{1-n}$, then $v' = (1-n)y^{-n}y'$ and so the ODE can be written as

$$\frac{1}{1-n}v' + p(t)v = q(t),$$

which is solvable. For example, let $y' + y = xy^2$, then for $v = y^{-1}$ we have

$$v' - v = -x.$$

Thus,

$$v = -e^t \int xe^{-t}\, dt = x + 1 + Ce^t.$$

Recall how we solve separable equations using differential forms. Suppose we have $M(x)dx + N(y)dy = 0$. Consider a function $F(x, y)$. The differential form of $F(x, y)$ is

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy.$$

Thus, if we can find a function $F(x, y)$ that satisfies

$$\frac{\partial F}{\partial x} = M(x), \qquad \frac{\partial F}{\partial y} = N(y),$$

then the equation can be written as

$$dF(x, y) = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = M(x)dx + N(y)dy = 0,$$

which implies $F(x, y) = C$. This formulation gives a generalization of the separation method: even though we have an ODE

$$M(x, y)dx + N(x, y)dy = 0$$

(not separable), we can solve the equation if we are able to find such a function $F(x, y)$ satisfying

$$\frac{\partial F}{\partial x} = M(x, y), \qquad \frac{\partial F}{\partial y} = N(x, y).$$

**Example 1.4.** Consider $(y - 3x^2)dx + (2y + x)dy = 0$. This is not separable. Let

$$\frac{\partial F}{\partial x}(x, y) = y - 3x^2, \qquad \frac{\partial F}{\partial y}(x, y) = 2y + x.$$

From the first equation, $F(x, y) = xy - x^3 + \phi(y)$ for some $\phi(y)$. If we plug this into the second equation,

$$\frac{\partial F}{\partial y}(x, y) = x + \phi'(y) = 2y + x,$$

which yields $\phi(y) = y^2$. Let $F(x, y) = y^2 + xy - x^3$, then the equation says $dF(x, y) = 0$. Thus, $y^2 + xy - x^3 = C$.

## 2 Existence and Uniqueness theorems

**Theorem 2.1** (Existence and Uniqueness: Linear case)**.** *Consider a first order ODE*

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0.$$

*Suppose $p(t)$ and $g(t)$ are continuous on an open interval $(\alpha, \beta)$ that contains $t_0$. Then, there exists a unique function $y = \phi(t)$ that satisfies the ODE with the initial condition.*

*Proof.* Let

$$\mu(t) = \exp\left(\int_{t_0}^{t} p(t)\, dt\right),$$

then $\mu(t)$ is well-defined on $(\alpha, \beta)$, $\mu(t_0) = 1$, and

$$\begin{aligned}
\mu(t)y' + \mu(t)p(t)y &= \frac{d}{dt}(\mu(t)y) = \mu(t)g(t),\\
\mu(t)y &= \int_{t_0}^{t} \mu(t)g(t)\, dt + C,\\
y &= \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(t)g(t)\, dt + C\right).
\end{aligned}$$

Since $y(t_0) = y_0$, we get $C = y_0$. Thus,

$$y = \frac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(t)g(t)\, dt + y_0\right)$$

is a unique solution of the linear ODE on $(\alpha, \beta)$. $\square$

**Example 2.2.** Consider a linear ODE

$$(t-5)y' + (\ln t)y = 2t, \qquad y(1) = 2.$$

Then, we can write

$$y' + \frac{\ln t}{t-5}y = \frac{2t}{t-5}$$

where $p(t) = \frac{\ln t}{t-5}$ and $g(t) = \frac{2t}{t-5}$. Note that $p(t)$ is continuous on $(0, 5) \cup (5, \infty)$ and $g(t)$ is continuous on $(-\infty, 5) \cup (5, \infty)$. Since the initial condition is given at $t = 1$, there exists a unique solution $\phi$ defined on the interval $(0, 5)$.

**Theorem 2.3** (Existence and Uniqueness: General case)**.** *Consider an ODE*

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0.$$

*Suppose $f$ and $\frac{\partial f}{\partial y}$ are continuous in some open rectangle containing $(t_0, y_0)$ (that is, $\alpha < t < \beta$ and $\gamma < y < \delta$). Then, for $t \in (t_0 - h, t_0 + h) \subset (\alpha, \beta)$, there exists a unique solution $y = \phi(t)$ that satisfies the ODE with the initial value condition on the interval $(t_0 - h, t_0 + h)$.*

**Example 2.4.** Consider a nonlinear ODE

$$\frac{dy}{dt} = \frac{1 + t^2}{2(1 - y)} = f(t, y), \qquad y(2) = 2.$$

Note that $f$ and

$$\frac{\partial f}{\partial y} = \frac{1+t^2}{2(1-y)^2}$$

are continuous if $y \neq 1$. Thus, the theorem says that there exists a unique solution $\phi(t)$ in $(2-h, 2+h)$ for some $h > 0$. Since the ODE is separable, we get

$$2y - y^2 = t + \frac{1}{3}t^3 + C$$

and so

$$y = 1 \pm \sqrt{C - t - \frac{1}{3}t^3}.$$

By the initial condition, we get

$$y(2) = 2 = 1 \pm \sqrt{C - 2 - \frac{8}{3}}$$

and so $C = 17/3$ and

$$y = 1 + \sqrt{\frac{17}{3} - t - \frac{1}{3}t^3}.$$

Note that the solution is defined on $(-\infty, 2.18576\ldots)$.

**Example 2.5** (Finite time blowup)**.** Consider a nonlinear ODE

$$y' = 2ty^2, \qquad y(0) = y_0.$$

It is easy to see that $f(t,y) = 2ty^2$ and

$$\frac{\partial f}{\partial y} = 4ty$$

are continuous for all $t$ and $y$. Thus, there exists a unique solution $\phi(t)$ defined on $(-h, h)$ for some $h > 0$. Note that the solution of the equation is obtained by the separation method: If $y_0 = 0$, then $y(t) = 0$ is the unique solution. Suppose $y_0 \neq 0$, then

$$\begin{aligned} -\frac{1}{y} &= t^2 + C \\ y &= -\frac{1}{t^2 + C} \\ y &= -\frac{1}{t^2 - \frac{1}{y_0}} \\ y &= \frac{y_0}{1 - y_0 t^2}. \end{aligned}$$

If $y_0 < 0$, then the solution is defined for all $t$. Even though $f$ and $\frac{\partial f}{\partial y}$ are continuous for all $t$ and $y$, the solution is only defined on $(-\frac{1}{\sqrt{y_0}}, \frac{1}{\sqrt{y_0}})$ if $y_0 > 0$. Note that $y(t) \to \infty$ if $t \to 1/\sqrt{y_0}$.

**Example 2.6** (When the uniqueness fails)**.** Let $y' = \sqrt{y}$ with $y(0) = 0$, then $\frac{\partial f}{\partial y}$ is not continuous when $y = 0$. Thus, the theorem does not apply. Indeed, there are infinitely many solutions with the initial condition.

# 3 Autonomous Equations and Population Dynamics, part 1

In this section, we consider an ODE of the form

$$\frac{dy}{dt} = f(y).$$

This is called autonomous. In particular, we investigate the equation by looking at $f(y)$. Typical examples of this type of differential equations are population model.

## 3.1 Exponential Growth Model

The easiest population model is the one where $f(y)$ is proportional to $y$. Consider $y' = ry$ with $y(0) = y_0$. The constant $r$ is called the rate of growth (if $r > 0$) or decline (if $r < 0$). Then, the solution is $y(t) = y_0 e^{rt}$.

## 3.2 Logistic Growth Model

In general, the rate of growth also depends on the population. As the population grows, the resources are getting short so that the rate of growth decreases according to the population. Consider $y' = h(y)y$ where

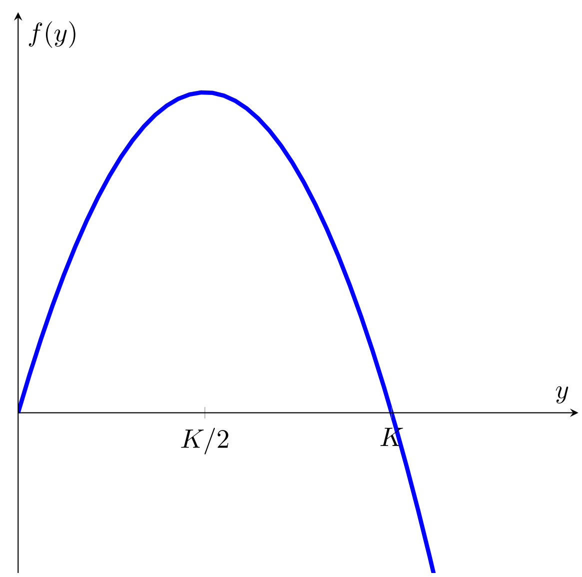
(i) $h(y) \cong r$ if $y$ is small,

(ii) $h(y)$ decreases as $y$ grows larger,

(iii) $h(y) < 0$ if $y$ is large enough.

Consider the case where $h(y) = r(1 - y/K)$, that is,

$$y' = r(1 - y/K)y = f(y), \qquad y(0) = y_0.$$




(i) Since $f(y) = 0$ if $y = 0, K$, $y(t) = 0$ and $y(t) = K$ are solutions for the equation. These are called *equilibrium solutions* because they are constant as time increases. The roots of $f(y)$ is called *critical points.*

(ii) If $0 < y < K$, then $f(y) > 0$. The equation says that $y' > 0$ and so $y(t)$ increases. On the other hand, if $y > K$, then $y' < 0$, which means $y(t)$ decreases. We can describe this on the phase line.

(iii) If $y > K$, the graph of the solution is concave up. Since $f(y)$ has its maximum at $y = K/2$, the graph of the solution is concave up when $0 < y < K/2$ and concave down when $K/2 < y < K$.

(iv) The existence and uniqueness theorem tells that the solution curves do not intersect each other. Thus, if the initial condition is $y(0) = y_0 \in (0, K)$, then $y(t) \in (0, K)$ for all $t$.

(v) By separation method, we can solve the equation:

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

(vi) If $y_0 > 0$, then $y(t) \to K$ as $t \to \infty$. The equilibrium solution $y(t) = K$ is called *asymptotically stable* and $y(t) = 0$ is called an *unstable equilibrium solution.*

## References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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