Homework 11

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

This will not be corrected.

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $V = \mathbb{C}^3$ be equipped with the standard inner product and $\varphi : V \to \mathbb{C}$ be a linear transformation defined by $\varphi(x_1, x_2, x_3) = x_1 - (2+i)x_2 + 4ix_3$. Find $y \in V$ such that $\varphi(x) = \langle x, y\rangle$ for all $x \in V$.

2. Let $V = \mathbb{R}^3$, $W = \mathrm{Span}(\{(1,-2,0),(1,0,1)\})$, and β be the standard basis for V. Compute $[T]_\beta$ where $T = \mathrm{proj}_W$ is the orthogonal projection onto W. (Hint: Let

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 0 & 1 \end{pmatrix},$$

then $W = \mathcal{R}(A)$.)

3. Let V be a finite dimensional inner product space over F and $T : V \to V$ normal. Prove that $\mathcal{N}(T) = \mathcal{N}(T^*)$ and $\mathcal{R}(T) = \mathcal{R}(T^*)$. (Hint: show $\mathcal{R}(T^*) = \mathcal{N}(T)^\perp$ and use it.)

4. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of A over $\mathbb{C}$ (that is, $a_0, a_1, \cdots, a_n \in \mathbb{C}$). Use Schur's theorem to show that

$$\mathrm{tr}(A) = (-1)^{n-1} a_{n-1}.$$

5. Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $V = \mathbb{C}^n$.
   (a) Show that AA^* is positive semidefinite.
   (b) Suppose A is self-adjoint. Show that A^2 is positive semidefinite.