Quantitative isoperimetric inequalities arising from stochastic processes *

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Abstract

It is well known that isoperimetric type inequalities hold for a large class of quantities arising from Brownian motion. Banuelos and Mendez-Hernandez showed that such inequalities can be extended to a wide class of Levy processes. A stability question is if the inequality will be about to achieving the equality when a given domain is close to being a ball. This question can be answered by quantitative improvement of such inequalities in terms of the asymmetry. In this talk, we discuss the quantitative isoperimetric inequalities for the expected lifetime of Brownian motion and α-stable processes, and some related open problems.

1 Isoperimetric type inequalities

The classical isoperimetric inequality says that if D is a set of finite perimeter with $|D| = 1$, the perimeter is minimized when D is a ball. This inequality has been of great interest for a long time. Isoperimetric inequality can be generalized as follows. Instead of the perimeter, we can think any nonnegative functional for sets. If the functional is minimized (or maximized) under fixed volume when a set is a ball (up to translation), we call it a generalized isoperimetric inequality for the functional. There are many examples for such functional that satisfy isoperimetric inequality.

Consider the first eigenvalue for the Laplacian in a set D, which is defined by

$$\lambda_1(D) = \inf_{u\in W_0^{1,2}(D)\setminus\{0\}} \frac{\int_D |\nabla u|^2\,dx}{\int_D |u|^2\,dx}.$$

If $|D| = 1$, then we have

$$\lambda_1(D) \geq \lambda_1(B)$$

where B is a centered ball with $|B| = 1$. This inequality is called Faber–Krahn inequality.

To explain how Faber–Krahn is related to the classical isoperimetric, we recall the symmetric decreasing rearrangement.

For a set D in $\mathbb{R}^d$ with $|D| < \infty$, its symmetric rearrangement D^* is a centered ball with the same volume. For a nonnegative function, how do we define such rearrangement? The idea is to use layer cake representation

$$f(x) = \int_0^\infty 1_{\{f(x)>t\}}\,dt.$$

For a nonnegative function f with $|\{x \in \mathbb{R}^d : f(x) > t\}| < \infty$ for all $t > 0$, the symmetric decreasing rearrangement f^* is defined by

$$f^*(x) = \int_0^\infty 1_{\{f(x)>t\}^*}\,dt.$$

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Using the classical isoperimetric inequality and the coarea formula, we can derive the Faber–Krahn as follows: For $u \in W_0^{1,2}(D)$, we have

$$\begin{aligned}
\int_D |\nabla u|^2\,dx &= \int_0^\infty \int_{\{u=t\}} |\nabla u|\,d\mathcal{H}^{d-1}dt \\
&\geq \int_0^\infty \frac{P(\{u>t\})^2}{\int_{\{u=t\}} |\nabla u|^{-1}\,d\mathcal{H}^{d-1}}\,dt \\
&\geq \int_0^\infty \frac{P(D_t^*)}{-\mu'(t)}\,dt \\
&= \int_{D^*} |\nabla u^*|^2\,dx
\end{aligned}$$

where $D_t = \{u > t\}$ and $\mu(t) = |D_t|$. Since u and its rearrangement u^* have the same L^2 norm, Faber–Krahn follows. Note that the inequality for the L^2 norm of ∇u is called Polya–Szego inequality.

We also have isoperimetric inequality for torsional rigidity of a set D, defined by

$$T(D) = \max_u \frac{\left(\int_D u\,dx\right)^2}{\int_D |\nabla u|^2\,dx}.$$

Saint-Venant inequality says that $T(D^*) \geq T(D)$. Torsional rigidity has a probablistic interpretation:

$$T(D) = \int_D \mathbb{E}_x[\tau_D]\,dx$$

where τ_D is the exit time of Brownian motion from D.

There are other quantities from probability that satisfy isoperimetric type inequalities. For example, if D is a bounded domain in $\mathbb{R}^d$, then the survival probability for Brownian motion $\mathbb{P}_x(\tau_D > t)$ is maximized when $D = D^*$. To see this, we recall the rearrangement inequality by Brascamp–Lieb–Luttinger: for nonnegative functions $f_1, \cdots, f_m$, we have

$$\int \cdots \int \prod_{i=1}^m f_i(\sum_{j=1}^k b_{ij}x_j)\,dx_1 \cdots dx_k \leq \int \cdots \int \prod_{i=1}^m f_i^*(\sum_{j=1}^k b_{ij}x_j)\,dx_1 \cdots dx_k.$$

Let X_t be Brownian motion and D a "nice" domain with finite volume. For $m \geq 1$, we have

$$\begin{aligned}
&\mathbb{P}_z(X_{\frac{t}{m}} \in D, \cdots, X_{\frac{mt}{m}} \in D) \\
&= \int \cdots \int p(\frac{t}{m}, z, x_1)p(\frac{2t}{m}, x_1, x_2) \cdots p(\frac{mt}{m}, x_{m-1}, x_m) 1_D(x_1) \cdots 1_D(x_m)\,dx_1 \cdots dx_m \\
&\leq \int \cdots \int p(\frac{t}{m}, z, x_1)^* p(\frac{2t}{m}, x_1 - x_2, 0)^* \cdots p(\frac{mt}{m}, x_{m-1} - x_m, 0)^* 1_{D^*}(x_1) \cdots 1_{D^*}(x_m)\,dx_1 \cdots dx_m \\
&= \int \cdots \int p(\frac{t}{m}, 0, x_1)p(\frac{2t}{m}, x_1, x_2) \cdots p(\frac{mt}{m}, x_{m-1}, x_m) 1_{D^*}(x_1) \cdots 1_{D^*}(x_m)\,dx_1 \cdots dx_m \\
&= \mathbb{P}_0(X_{\frac{t}{m}} \in D^*, \cdots, X_{\frac{mt}{m}} \in D^*).
\end{aligned}$$

Since X_t has continuous path, we obtain $\mathbb{P}_z(\tau_D > t) \leq \mathbb{P}_0(\tau_{D^*} > t)$ by letting $m \to \infty$. Not only this, we also have

$$\mathbb{E}_z[(\tau_D)^p] \leq \mathbb{E}_0[(\tau_{D^*})^p], \qquad Q_D(t) = \int_D \mathbb{P}_z(\tau_D > t)\,dz \leq Q_{D^*}(t).$$

Banuelos and Mendez-Hernandez extended these to a wide class of Levy processes. The idea is to symmetrize Levy process. The main result is

$$\mathbb{E}_z[f(X_t)e^{-\int_0^t V(X_s)\,ds}; \tau_D^X > t] \leq \mathbb{E}_0[f^*(X_t^*)e^{-\int_0^t V^*(X_s^*)\,ds}; \tau_{D^*}^{X^*} > t].$$

In particular, if X_t is symmetric α-stable process, then $X_t = X_t^*$.

2 Stability questions

In general, isoperimetric inequality can be written as $\mathcal{F}(D) \geq \mathcal{F}(D^*)$ (or the opposite) under $|D| = 1$ (this is for simplicity). We define the deficit

$$\delta(D) = \mathcal{F}(D) - \mathcal{F}(D^*) \geq 0.$$

A stablity quations is if D is close to $D*$ when $\delta(D) \to 0$. This question can be answered by a stronger argument

$$\delta(D) \geq C\mathcal{A}(D)$$

where $\mathcal{A}(D)$ is the Fraenkel asymmetry given by

$$\mathcal{A}(D) = \inf_{x \in \mathbb{R}^d} \frac{|D \triangle (D^* + x)|}{|D|}.$$

2.1 Existing results

(i) Classical Isoperimetric

- [Hal92]: Non-sharp. $\delta(D) \geq C_d \mathcal{A}(D)^4$.
- [FMP08]: Sharp. $\delta(D) \geq C_d \mathcal{A}(D)^2$

(ii) Fractional perimeter

- [FMM11]: $\delta(D) \geq C_{d,\alpha} \mathcal{A}(D)^{\frac{8}{\alpha}}$ for $0 < \alpha < 2$. Seems non-sharp. It is conjectured that the sharp exponent is 2.

(iii) Faber–Krahn (the first eigenvalue) and torsional rigidity

- [Mel92], [HN94]
- [BDPV15]: Sharp. $\delta(D) \geq C_d \mathcal{A}(D)^2$

(iv) Fractional Faber–Krahn and fractional torsional rigidity

- [BCV20]: $\delta(D) \geq C_{d,\alpha} \mathcal{A}(D)^{\frac{6}{\alpha}}$

2.2 Main results

Let $u_D(x) = \mathbb{E}_x[\tau_D]$, then it was shown in [Kim20] that

(i) $1 - \dfrac{\|u_D\|_\infty}{\|u_{D^*}\|_\infty} \geq C_d \mathcal{A}(D)^3$

(ii) $1 - \dfrac{\|u_D\|_p^p}{\|u_{D^*}\|_p^p} \geq C_{d,p} \mathcal{A}(D)^{2+p}$ for $1 \leq p < \infty$

(iii) For fractional torsional rigidity,

$$\delta(D) \geq C_{d,\alpha} \mathcal{A}(D)^{2+\frac{2}{\alpha}}.$$

The main idea is to use Hansen–Nadirashvili's asymmetry of level sets estimates, which leads to a quantitative inequality for $u_D(x)$.

2.3 Open questions

(i) For Brownian motion, quantitative inequalities for many quantities are open, for example, survival probability, heat contents, the p-th moments of the exit time, etc.

(ii) Sharp quantitative inequalities for fractional perimeter, fractional first eigenvalue, and fractional torsional rigidity are open.

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