Homework 6

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: October 18, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be invertible matrices.
   (a) Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
   (b) Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Solution:

(a) By the associativity, we have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

and

$$(B^{-1}A^{-1})(AB) = B^{-1}(AA^{-1})B = B^{-1}I_nB = B^{-1}B = I_n,$$

which leads the result.

(b) It follows from the fact that $(AB)^t = B^tA^t$ that

$$A^t(A^{-1})^t = (A^{-1}A)^t = (I_n)^t = I_n$$

and

$$(A^{-1})^tA^t = (AA^{-1})^t = (I_n)^t = I_n,$$

which implies that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

2. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Prove that A is invertible if and only if L_A is invertible and $(L_A)^{-1} = L_{A^{-1}}$.

Solution: Suppose A is invertible. Then, there exists a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$. Since

$$L_AL_{A^{-1}} = L_{AA^{-1}} = L_{I_n} = I_{\mathbb{R}^n}$$

and

$$L_{A^{-1}}L_A = L_{A^{-1}A} = L_{I_n} = I_{\mathbb{R}^n},$$

we conclude that L_A is invertible and $(L_A)^{-1} = L_{A^{-1}}$.

Suppose L_A is invertible. Let β be the standard basis for $\mathbb{R}^n$. Define $B = [(L_A)^{-1}]_\beta$. Note that $[L_A]_\beta = A$. It follows that

$$AB = [L_A]_\beta[(L_A)^{-1}]_\beta = [L_A(L_A)^{-1}]_\beta = [I_{\mathbb{R}^n}]_\beta = I_n$$

and

$$BA = [(L_A)^{-1}]_\beta[L_A]_\beta = [(L_A)^{-1}L_A]_\beta = [I_{\mathbb{R}^n}]_\beta = I_n,$$

which completes the proof.

3. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$.
   (a) Prove that if AB is invertible, then A and B are invertible.
   (b) Prove that if $AB = I_n$, then $A = B^{-1}$.

Solution:

(a) Since AB is invertible, L_{AB} is invertible. Since $L_{AB} = L_A L_B$, we see that L_A is onto and L_B is one-to-one by Problem 3 of Homework 5. By Dimension theorem,

$$\dim(\mathbb{R}^n) = \dim(\mathcal{N}(L_A)) + \dim(\mathcal{R}(L_A)) = \dim(\mathcal{N}(L_A)) + \dim(\mathbb{R}^n),$$

which leads to $\dim(\mathcal{N}(L_A)) = 0$. Thus, $\mathcal{N}(L_A) = \{0\}$ and L_A is one-to-one. By Dimension theorem again,

$$\dim(\mathbb{R}^n) = \dim(\mathcal{N}(L_B)) + \dim(\mathcal{R}(L_B)) = \dim(\mathcal{R}(L_B)),$$

which leads to $\mathcal{R}(L_B) = \mathbb{R}^n$. Thus, L_B is onto. Thus L_A and L_B are invertible. Thus, A and B are invertible.

(b) Since $AB = I_n$, in particular AB is invertible. By the part (a), there exist A^{-1} and B^{-1}. Thus, we have

$$A = A(BB^{-1}) = (AB)B^{-1} = I_nB^{-1} = B^{-1}$$

as desired.

4. Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Solution: If A and B are similar, then there exists an invertible matrix Q such that $A = Q^{-1}BQ$. It then follows from the fact $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (see Problem 4 of HW5) that

$$\mathrm{tr}(A) = \mathrm{tr}((Q^{-1}B)Q) = \mathrm{tr}(Q(Q^{-1}B)) = \mathrm{tr}((QQ^{-1})B) = \mathrm{tr}(I_nB) = \mathrm{tr}(B).$$

5. Let V be a finite-dimensional vector space over $\mathbb{R}$ with basis β and $\dim(V) = n$. Define $\phi_\beta : V \to \mathbb{R}^n$ by $\phi_\beta(v) = [v]_\beta$. Show that ϕ_β is an isomorphism.

Solution: We already know that ϕ_β is a linear transformation. Since $\dim(V) = n = \dim(\mathbb{R}^n)$, by Dimension theorem, it suffices to show that ϕ_β is one-to-one. Suppose $\phi_\beta(v) = [v]_\beta = 0$. This means that if we write

$$v = a_1 v_1 + \cdots + a_n v_n$$

where $a_i \in \mathbb{R}$ and $v_i \in \beta$, then $a_1 = \cdots = a_n$. Thus we conclude that $v = 0$. Since $\mathcal{N}(\phi_\beta) = \{0\}$, the map is one-to-one. By Dimension theorem,

$$n = \dim(V) = \dim(\mathcal{N}(\phi_\beta)) + \dim(\mathcal{R}(\phi_\beta)) = \dim(\mathcal{R}(\phi_\beta)) \leq \dim(\mathbb{R}^n)$$

and so $\dim(\mathcal{R}(\phi_\beta)) = n$. Thus, ϕ_β is an isomorphism.

6. Let T be the linear map on $\mathbb{R}^2$ defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}.$$

Let

$$\beta = \left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad \beta' = \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Find $[T]_\beta, [I_{\mathbb{R}^2}]_{\beta'}^{\beta}, [I_{\mathbb{R}^2}]_{\beta}^{\beta'}$, and $[T]_{\beta'}$.

Solution: Since $T(e_1) = 2e_1 + e_2$ and $T(e_2) = e_1 - 3e_2$,

$$[T]_\beta = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}.$$

Since $I_{\mathbb{R}^2}(v_1) = e_1 + e_2$ and $I_{\mathbb{R}^2}(v_2) = e_1 + 2e_2$,

$$[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Since $[I_{\mathbb{R}^2}]_{\beta}^{\beta'} = ([I_{\mathbb{R}^2}]_{\beta'}^{\beta})^{-1}$, we have

$$[I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

(Also, one can directly find the matrix using the fact that $I_{\mathbb{R}^2}(e_1) = 2v_1 - v_2$ and $I_{\mathbb{R}^2}(e_2) = -v_1 + v_2$.) Finally, we have

$$\begin{aligned} [T]_{\beta'} &= [I_{\mathbb{R}^2}]_{\beta}^{\beta'} [T]_\beta [I_{\mathbb{R}^2}]_{\beta'}^{\beta} \\ &= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}. \end{aligned}$$

7. In $\mathbb{R}^2$, let L be the line $y = mx$ where $m \neq 0$. Find expressions for the following linear transformations $T(x, y)$.
   (a) T is the reflection of $\mathbb{R}^2$ about L.
   (b) T is the projection on L along the line perpendicular to L. (That is, for each $(x, y) \in \mathbb{R}^2$, $T(x, y)$ is the closest point on L to (x, y).)

Solution:

(a) Let $\beta = \{e_1 = (1, 0), e_2 = (0, 1)\}$ and $\beta' = \{v_1 = (1, m), v_2 = (m, -1)\}$. Then, β' is a basis for $\mathbb{R}^2$, $T(v_1) = v_1$, and $T(v_2) = -v_2$. Thus,

$$\begin{aligned}
[T]_\beta &= [I_{\mathbb{R}^2}]^\beta_{\beta'}[T]_{\beta'}[I_{\mathbb{R}^2}]^{\beta'}_\beta \\
&= [I_{\mathbb{R}^2}]^\beta_{\beta'}[T]_{\beta'}([I_{\mathbb{R}^2}]^\beta_{\beta'})^{-1} \\
&= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}^{-1} \\
&= \frac{1}{m^2+1}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \\
&= \frac{1}{m^2+1}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}.
\end{aligned}$$

Thus, we obtain

$$T(x, y) = \frac{1}{m^2+1}((1-m^2)x + 2my, 2mx + (m^2-1)y).$$

(b) Let $\beta = \{e_1 = (1, 0), e_2 = (0, 1)\}$ and $\beta' = \{v_1 = (1, m), v_2 = (m, -1)\}$. Then, β' is a basis for $\mathbb{R}^2$, $T(v_1) = v_1$, and $T(v_2) = 0$. Thus,

$$\begin{aligned}
[T]_\beta &= [I_{\mathbb{R}^2}]^\beta_{\beta'}[T]_{\beta'}[I_{\mathbb{R}^2}]^{\beta'}_\beta \\
&= [I_{\mathbb{R}^2}]^\beta_{\beta'}[T]_{\beta'}([I_{\mathbb{R}^2}]^\beta_{\beta'})^{-1} \\
&= \begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix}^{-1} \\
&= \frac{1}{m^2+1}\begin{pmatrix} 1 & 0 \\ m & 0 \end{pmatrix}\begin{pmatrix} 1 & m \\ m & -1 \end{pmatrix} \\
&= \frac{1}{m^2+1}\begin{pmatrix} 1 & m \\ m & m^2 \end{pmatrix}.
\end{aligned}$$

Thus, we obtain

$$T(x, y) = \frac{1}{m^2+1}(x + my, mx + m^2y).$$