Math 285 Lecture Note: Week 1

Daesung Kim

1 Some Basic Mathematical Models; Direction Fields

A differential equation is an equation consisting of functions and their derivatives. Why are we interested in such equations? Suppose we want to understand phenomena like

(i) the motion of a falling object, fluids,

(ii) the flow of current in electric circuits,

(iii) the trend of populations, etc.

One way is to consider the rate of change. This rate of change can be described as a derivative. Suppose a function $y(t)$ represents the quantity that we want to understand. Then, the rate of change is a derivative of y in time t, which denotes

$$\frac{dy}{dt} = y' = y_t.$$

Using some knowledge (like physics law), observation (experiments), or statistical inference, we can build a relation between the rate of change and other quantities like

$$\frac{dy}{dt} = F(t, y, \cdots).$$

This procedure is called a mathematical modeling. We end up obtaining a differential equation. This is a motivation of the study of differential equations.

Example 1.1. We are interested in understanding a falling object with air resistance. Using Newton's second law, one can build a mathematical model like

$$\frac{dv}{dt} = 5 - v.$$

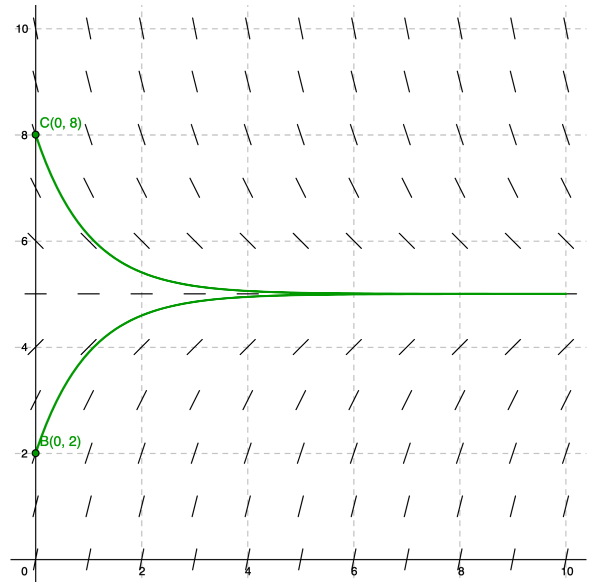
A following question is how to find the function v that satisfies the equation. Such v is called a solution. We first observe that

(i) the RHS represents the slope of v;

(ii) the RHS depends only on v, not on time t;

(iii) if $v > 5$, then $v' < 0$; if $v < 5$, then $v' > 0$; if $v = 5$, then $v' = 0$.

For further investigation, we draw a picture on $t - v$ graph.



This graph is called the direction field. The directional field suggests the following observations:

(i) There are many solutions.

(ii) If we can specify one velocity (like at time 0), then there will be one solution.

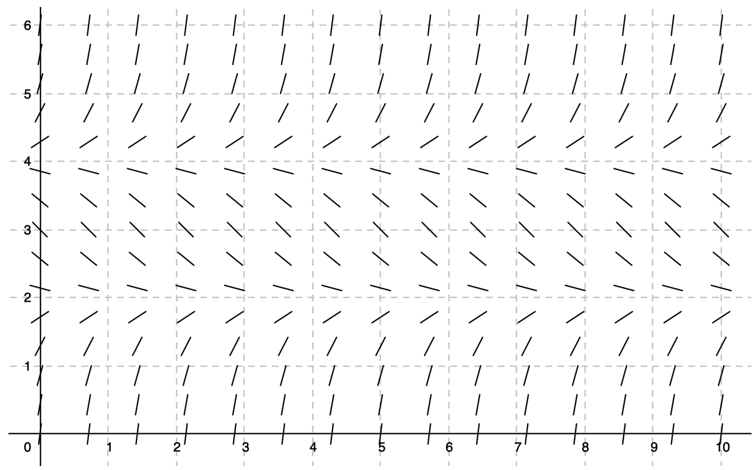
(iii) It is easy to see that $y(t) = 5$ is a solution. This is called an equilibrium solution.

(iv) The trend of velocity will be (dramatically) changed according to the choice of the initial value.

Example 1.2. Consider a differential equation

$$\frac{dp}{dt} = (p-2)(p-4).$$

The directional fields is



2 Solutions of Some Differential Equations

In this section, we study differential equations of the form

$$\frac{dy}{dt} = F(t, y).$$

In particular, we focus on easy cases where the function F depends only on t or on y.

Example 2.1. Consider a differential equation

$$\frac{dy}{dt} = t^2.$$

If the RHS is a function of t, then finding a solution $y = y(t)$ is just a matter of integration. Indeed,

$$f(t) = \frac{1}{3}t^3 + C$$

for some constant C.

Now, we focus on the case where $F(t, y)$ is a function of y.

Example 2.2. Consider a differential equation

$$\frac{dv}{dt} = 5 - v.$$

Then, we have

$$\frac{1}{5-v}\frac{dv}{dt} = 1$$

The idea is to exploit the chain rule. Recall that

$$\frac{d}{dt}G(v(t)) = G'(v)\frac{dv}{dt}.$$

We want to find G that satisfies

$$G'(v) = \frac{1}{5-v}.$$

Once finding such function G, the equation can be written as

$$\frac{1}{5-v}\frac{dv}{dt} = G'(v)\frac{dv}{dt} = \frac{d}{dt}G(v(t)) = 1.$$

Now the RHS does not depend on v. By integrating of both sides, we get

$$G(v(t)) = t + C.$$

How do we find such G? Taking integration of both sides, we get

$$G(v) = \int \frac{1}{5-v}\, dv = -\ln|5-v| + C.$$

We can just pick one function G. So, let $G(v) = -\ln|v-5|$, then

$$\begin{aligned} -\ln|v-5| &= t + C \\ |v-5| &= e^{-t+C} \\ v &= 5 + Ce^{-t} \end{aligned}$$

where C is an arbitrary constant. (Note that the constants C may be different from line to line.)

The key idea is to replace the original equation to easier one. This principle is one of the main theme in this course.

Example 2.3. Consider a differential equation

$$\frac{dy}{dt} = 1 + y^2.$$

Similarly, we let $G'(y) = \frac{1}{1+y^2}$ and solve this auxiliary differential equation. By taking integration, we have

$$G(y) = \arctan(y) = t + C$$

and so $y(t) = \tan(t + C)$.

Example 2.4. Consider a differential equation

$$\frac{dp}{dt} = (p-2)(p-4).$$

Similarly,

$$\frac{1}{(p-2)(p-4)}\frac{dp}{dt} = 1.$$

Let

$$G'(p) = \frac{1}{2}\left(\frac{1}{p-4} - \frac{1}{p-2}\right),$$

then

$$\begin{aligned} G(p) &= \frac{1}{2}\int\left(\frac{1}{p-4} - \frac{1}{p-2}\right)\,dp \\ &= \frac{1}{2}\left(\ln|p-4| - \ln|p-2|\right) \\ &= \frac{1}{2}\ln\left|\frac{p-4}{p-2}\right| \\ &= t + C. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{p-4}{p-2} = 1 - \frac{2}{p-2} &= Ce^{2t}, \\ \frac{2}{p-2} &= 1 + Ce^{2t}, \\ p(t) &= 2 + \frac{2}{1+Ce^{2t}}. \end{aligned}$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu