Review for Midterm 2

Math 285 Spring 2020

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Homogeneous equations with constant coefficients

Idea: Use the characteristic equations.

If $y'' + py' + qy = 0$ and p, q are real numbers, the characteristic equation is $\lambda^2 + p\lambda + q = 0$.

r_1, r_2

(i) Real root: $e^{r_1 t}$ and $e^{r_2 t}$.

(ii) Complex roots $r \pm i\mu$: $e^{rt}\cos(\mu t)$ and $e^{rt}\sin(\mu t)$.

(iii) Repeat roots: e^{rt} and te^{rt}.

r $\times t$

This works for higher order DEs.

Practice Exam

Problem 5

Find the general solution of $y'' + 4y' + 5y = 0$.

$$\lambda^2 + 4\lambda + 5 = 0 \longrightarrow \lambda = -2 \pm i$$

quadratic formula

$$\rightarrow e^{-2t} \cdot \cos t, \; e^{-2t} \sin t$$

$$y(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t.$$

Practice Exam

Problem 8

Find the general solution of the following differential equation.

$$y''' + 8y = 0.$$

$\lambda^3 + 8 = 0$ $\quad \lambda = -2$ is a root.

$(\lambda+2)(\lambda^2 - 2\lambda + 4) = 0$ $\quad \therefore \lambda = -2, \ 1 \pm \sqrt{3}\, i$

$2\lambda^2$

$-2\lambda^2$

e^{-2t}

$e^{t} \cos \sqrt{3} t$

$e^{t} \sin \sqrt{3} t$

$\therefore y(t) = C_1 e^{-2t} + C_2 e^{t} \cos \sqrt{3} t + C_3 e^{t} \sin \sqrt{3} t.$

Nonhomogeneous Equation: Undetermined Coefficients

Consider $\underbrace{y'' + p(t)y' + q(t)y}_{L[y]=} = \boxed{g(t)}$. The general solution is

$$y(t) = (\text{Gen. Sol. of Homogeneous Eqn}) + (\text{A Particular Solution}).$$

of Nonhom.

How to find a particular solution?

coefficients determined by Equation

(i) $g(t) = (t^n + \cdots)$: $Y(t) = (A_n t^n + \cdots)$.

(ii) $g(t) = (t^n + \cdots)e^{kt}$: $Y(t) = (A_n t^n + \cdots)e^{kt}$.

(iii) $g(t) = (t^n + \cdots)\sin(kt)$ (or $\cos(kt)$):
$Y(t) = (A_n t^n + \cdots)\cos(kt) + (B_n t^n + \cdots)\sin(kt)$.

(iv) If $Y(t)$ contains a solution to the homogeneous equation, multiply it by t.

$L[y] = g_1(t) + g_2(t)$

$g_1(t) \to Y_1$, $g_2(t) \to Y_2$

$Y(t) = Y_1 + Y_2$.

Practice Exam

Problem 4

Identify the correct form of a particular solution for the following differential equation.

$$y'' - y = t^2 e^{-t} + 10t^3.$$

① $y'' - y = 0$

$\lambda^2 - 1 = 0 \quad \therefore \lambda = \pm 1 \rightarrow e^{t}, e^{-t}$

② $t^2 e^{-t} \rightsquigarrow Y_1(t) = t(A_0 t^2 + A_1 t + A_2) e^{-t}$

③ $10 t^3 \rightsquigarrow Y_2(t) = (B_0 t^3 + B_1 t^2 + B_2 t + B_3)$

$$Y(t) = Y_1(t) + Y_2(t)$$

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Practice Exam

Problem 14

Find the solution to the following initial value problem

$$y'' - 2y' - 3y = 3te^{2t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

① $y'' - 2y' - 3y = 0$

$\lambda^2 - 2\lambda - 3 = 0$ $\quad \therefore \lambda = 3, -1$

e^{3t}, e^{-t}

② $3t e^{2t} \longrightarrow (At + B)e^{2t} = Y(t)$

$Y'' - 2Y' - 3Y = (2A - 3B)e^{2t} - 3Ate^{2t}$

$= 3te^{2t}$

$$\begin{cases} 2A - 3B = 0 \\ -3A = 3 \end{cases} \qquad \therefore \quad A = -1, \quad B = -\frac{2}{3}.$$

$$Y(t) = -\left(t + \frac{2}{3}\right)e^{2t}$$

$$y(t) = \underline{C_1} e^{-t} + \underline{C_2} e^{3t} - \left(t + \frac{2}{3}\right)e^{2t}.$$

using $\quad y(0) = 1, \quad y'(0) = 0.$

$$\Rightarrow \quad y(t) = \frac{2}{3}e^{-t} + e^{3t} - \left(t + \frac{2}{3}\right)e^{2t}.$$

$\blacksquare$

Reduction of order

If $y_1(t)$ is a solution, then another solution y_2 can be obtained by letting $y_2 = vy_1$ for some $v(t)$.

$\longrightarrow \dfrac{\text{Condition for } v}{\text{DE.}}$

Practice Exam

Problem 10

The equation $t^2y'' - 4ty' + 6y = 0$ has a solution $y_1(t) = t^2$. If $y_2(t) = v(t)y_1(t)$ is another solution to the equation, then what equation does $v(t)$ satisfy?

$$t^2 y_2'' - 4t y_2' + 6y_2$$

$$= t^2 \cdot \left(v'' \underbrace{y_1}_{t^2} + 2v' \underbrace{y_1'}_{2t} + \boxed{v y_1''} \right) - 4t \left(v' \underbrace{y_1}_{t^2} + \boxed{v y_1'} \right) + \boxed{6 v y_1}$$

$$= v \underbrace{\left(t^2 y_1'' - 4t y_1' + 6y_1 \right)}_{=0 \ (\because y_1 \text{ is a solution})} + \left(t^4 v'' + \cancel{4t^3 v'} - \cancel{4t^3 v'} \right)$$

$$= t^4 v'' = 0.$$

$$\therefore \ v'' = 0.$$

Practice Exam

Problem 13

Consider $t^2y'' - t(t+2)y' + (t+2)y = 0$ for $t > 0$.

(i) Verify that $y_1(t) = t$ is a solution to the equation.

(ii) Find the general solution to the equation using the method of reduction of order.

(i) $t^2 \cdot \underbrace{(t)''}_{=0} - t\cdot(t+2)\cdot\underbrace{(t)'}_{=1} + (t+2)\cdot t = 0.$

(ii) $y(t) = v(t)\cdot y_1(t) = t\cdot v$

$t^2\cdot(v''y_1 + 2v'y_1' + \boxed{v y_1''}) - t(t+2)(v'y_1 + \boxed{v y_1'})$
$+ \boxed{(t+2) v y_1}$

$= t^3 v'' + 2t^2 v' - t^2(t+2)v' = t^3\cdot(v''-v') = 0$

$$v'' - v' = 0$$

$$\lambda^2 - \lambda = 0 \rightarrow \lambda = 0, 1 \rightarrow 1 \;\&\; e^t$$

$$v(t) = c_1 + c_2 e^t$$

$$y(t) = t \cdot v(t) = \boxed{c_1 t + c_2 t e^t}$$

$$t^2 e^t$$

General Solution? Yes because

$$W[t, te^t] = \det\begin{pmatrix} t & te^t \\ 1 & (t+1)e^t \end{pmatrix} \neq 0 \quad \text{if } t > 0$$

Practice Exam

Spring - Mass System.

Problem 6

Suppose $m, k, \omega, F_0 > 0$ and $r \geq 0$. What can you say for the vibration system?

$$ms'' + rs' + ks = F_0 \cos(\omega t)$$

(A) When $F_0 = 0$ and $\boxed{r^2 \geq 4mk}$, the general solution eventually becomes monotone and decays to 0.

True.

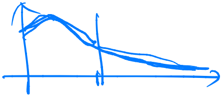
$$ms'' + rs' + ks = 0$$

$$m\lambda^2 + r\lambda + k = 0, \quad \lambda = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m}$$

If $r^2 \geq 4mk$, then $(r^2 - 4mk) \geq 0$

Real roots (possibly repeated).

$$y(t) = e^{-\frac{r}{2m}t} \cdot \left(\begin{cases} c_1 + c_2 t \\ c_1 e^{\sqrt{D}t} + c_2 e^{-\sqrt{D}t} \end{cases} \right)$$



Practice Exam

Problem 6

Suppose $m, k, \omega, F_0 > 0$ and $r \geq 0$. What can you say for the vibration system?

$$ms'' + rs' + ks = F_0 \cos(\omega t)$$

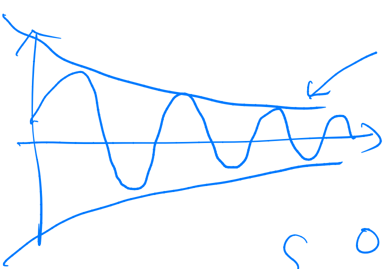
(B) When $F_0 = 0$ and $0 < r^2 < 4mk$, the general solution oscillates and decays to 0.

True

$ms'' + rs' + ks = 0$

$$\lambda = \frac{-r \pm \sqrt{\underbrace{r^2 - 4mk}_{<0}}}{2m}$$

$$y(t) = e^{-\frac{r}{2m}t} \cdot \left(C_1 \cos \mu t + C_2 \sin \mu t\right)$$



$$e^{-\frac{\gamma}{2m}t} = 1$$

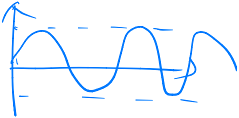
oscillates

decays to 0

as $t \to \infty$.

Note If $\underline{\gamma = 0}$ & $\gamma^2 - 4mk < 0$

then oscillates



but **Not** decay to 0.

Practice Exam

Problem 6

Suppose $m, k, \omega, F_0 > 0$ and $r \geq 0$. What can you say for the vibration system?

$$ms'' + rs' + ks = F_0 \cos(\underline{\omega t})$$

$\omega_0 =$

(C) $\boxed{\text{Resonance}}$ happens when $r = 0$ and $\sqrt{\frac{k}{m}} = \omega$. True.

Forced without damping & $\omega = \omega_0 = \sqrt{\frac{k}{m}}$

Practice Exam

Problem 6

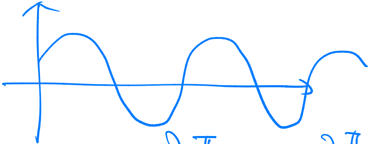
Suppose $m, k, \omega, F_0 > 0$ and $r \geq 0$. What can you say for the vibration system?

$$ms'' + rs' + ks = F_0 \cos(\omega t)$$

(D) When $F_0 = 0$ and $r = 0$, the general solution is periodic function.

$$ms'' + ks = 0$$

Free vibration w.o. damping



Period $T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{m}{k}}$.

Practice Exam

Problem 9

The motion of a certain spring–mass system is governed by

$$u'' + 4u = 0$$

with $u(0) = 1$ and $u'(0) = 1$. What are the amplitude and the natural frequency?

$\lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$

$u(t) = C_1 \cos 2t + C_2 \sin 2t$

$u(0) = C_1 = 1$

$u'(t) = -2C_1 \sin 2t + 2C_2 \cos 2t$

$u'(0) = 2C_2 = 1 \qquad \therefore \quad C_2 = \frac{1}{2}$

$$u(t) = \cos t + \frac{1}{2}\sin t$$

$$R = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2.$$

$\blacksquare$.

Linearly Independent

Definition

Let $f_1, \cdots, f_n$ be functions on I. We say $f_1, \cdots, f_n$ are linearly dependent on I if there exist constants $k_1, \cdots, k_n$ not all zero such that

$$k_1 f_1(t) + \cdots + k_n f_n(t) = 0$$

for all $t \in I$. If not, we say $f_1, \cdots, f_n$ are linearly independent on I

Note If

$k_1 f_1 + \cdots + k_n f_n = 0$ implies $k_1 = \cdots = k_n = 0$,

then $\{f_1, \cdots, f_n\}$ is linearly independent

Practice Exam

Problem 7

Which of these is NOT a set of linearly independent solutions?

(A) $\cos x, \sin x, \cos 2x$ — lin. indep.

(B) $1, x, x^2$

(C) $x, x^2 + x, 2x^2$

(D) e^x, e^{2x}, e^{3x}

linearly indep.

(E) None of these

(A): $\cancel{A \cdot \cos x} + \cancel{B \sin x} + C \cdot \sin 2x = 0$

$x = 0. \rightarrow A = 0.$

$x = \frac{\pi}{2} \rightarrow B \cdot \sin\frac{\pi}{2} + C \underbrace{\sin \pi}_{0} = B = 0$

$x = \frac{\pi}{4} \rightarrow C = 0 \Rightarrow A = B = C = 0$

$\triangle x$ $\boxed{x^2+x}$ $2x^2$

$$A x + B(x^2+x) + C\cdot(2x^2) = 0 \quad -(\ast)$$

Let $A = 1$, then $B = -1$

So, $x - (x^2+x) + C(2x^2) = 0$

$$\underbrace{(2C-1)}_{=0}\cdot x^2 = 0$$

$\therefore C = \frac{1}{2}$

$(\ast)$ holds $A = 1$, $B = -1$, $C = \frac{1}{2}$

$\{x, x^2+x, 2x^2\}$ lin. dep. $\blacksquare$