Math 285 Lecture Note: Week 8

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1 Forced Vibrations with damping

We study forced vibrations with damping effect. Suppose there is an external force given by $F_0 \cos(\omega t)$, then we have

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos(\omega t)$$

where $\gamma \neq 0$ and $F_0 > 0$. Then the general solution is

$$u(t) = C_1 y_1(t) + C_2 y_2(t) + A\cos(\omega t) + B\sin(\omega t)$$

where $\{y_1, y_2\}$ is a fundamental set of solutions to the homogeneous equation $mu''(t) + \gamma u'(t) + ku(t) = 0$. Let $u_c(t) = C_1 y_1(t) + C_2 y_2(t)$ and $U(t) = A\cos(\omega t) + B\sin(\omega t)$. Note that $u_c(t)$ is the general solution to the homogeneous equation and $U(t)$ is a particular solution to the nonhomogeneous equation.

Recall that C_1, C_2 are determined by the initial conditions and A, B are determined by the equation. We have seen that $u_c(t)$ has three different forms depending on the sign of $D = \frac{\gamma^2 - 4mk}{4m^2}$:

$$u_c(t) = \begin{cases} e^{-\frac{\gamma}{2m}t}(C_1 e^{\sqrt{D}t} + C_2 e^{-\sqrt{D}t}), & \text{if } D > 0 \text{ or } \Gamma > 4, \\ e^{-\frac{\gamma}{2m}t}(C_1 + C_2 t), & \text{if } D = 0 \text{ or } \Gamma = 4, \\ e^{-\frac{\gamma}{2m}t}(C_1 \cos\mu t + C_2 \sin\mu t), & \text{if } D > 0 \text{ or } \Gamma < 4, \end{cases}$$

where $\mu = \sqrt{-D} = \sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}$ is the quasi frequency and $\Gamma = \frac{\gamma^2}{mk}$. Note that

$$D = \frac{k}{4m}\left(\frac{\gamma^2}{mk} - 4\right) = \omega_0^2 \left(\frac{\Gamma}{4} - 1\right)$$

In particular, $u_c(t) \to 0$ as $t \to \infty$. For this reason, $u_c(t)$ is called the transient solution. Since $u_c(t)$ dies out as t increases, the solution $u(t)$ tends to be close to $U(t)$ as t goes. In this sense, $U(t)$ is called the steady state solution or the forced response.

Let's find A and B. By replacing $u(t)$ with $U(t)$ in the equation, we get

$$\begin{aligned} mU''(t) + \gamma U'(t) + kU(t) &= -m\omega^2(A\cos(\omega t) + B\sin(\omega t)) + \gamma\omega(-A\sin(\omega t) + B\cos(\omega t)) + k(A\cos(\omega t) + B\sin(\omega t)) \\ &= (-m\omega^2 A + \gamma\omega B + kA)\cos(\omega t) + (-m\omega^2 B - \gamma\omega A + kB)\sin(\omega t) \\ &= F_0 \cos(\omega t), \end{aligned}$$

which yields

$$\begin{aligned} (k - m\omega^2)A + \gamma\omega B &= F_0, \\ -\gamma\omega A + (k - m\omega^2)B &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} A &= \frac{(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2} F_0 = \frac{m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} F_0, \\ B &= \frac{\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2} F_0 = \frac{\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} F_0, \end{aligned}$$

where $\omega_0 = \sqrt{k/m}$. To simplify this, we introduce

$$\alpha = m(\omega_0^2 - \omega^2), \qquad \beta = \gamma\omega, \qquad \Delta = \sqrt{\alpha^2 + \beta^2},$$

then

$$A = \frac{\alpha}{\Delta^2} F_0, \qquad B = \frac{\beta}{\Delta^2} F_0.$$

As before, $U(t)$ can be written as

$$U(t) = A\cos\omega t + B\sin\omega t = R\cos(\omega t - \delta)$$

where $R = \sqrt{A^2 + B^2} = F_0/\Delta$ and $\delta \in [0, 2\pi)$ satisfying

$$\cos\delta = \frac{A}{R} = \frac{\alpha}{\Delta}, \qquad \sin\delta = \frac{B}{R} = \frac{\beta}{\Delta}.$$

Let's focus on the behavior of R according to ω. Indeed, it suffices to consider Δ. Note that

$$\begin{aligned}
\Delta^2 &= \alpha^2 + \beta^2 \\
&= m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2 \\
&= m^2\omega_0^4 \left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \frac{\gamma^2}{m^2\omega_0^2}\frac{\omega^2}{\omega_0^2} \right] \\
&= k^2 \left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\frac{\omega^2}{\omega_0^2} \right] \\
&= k^2 \left[\left(\frac{\omega^2}{\omega_0^2} - (1 - \frac{\Gamma}{2})\right)^2 + \frac{1}{4}\Gamma(4 - \Gamma) \right]
\end{aligned}$$

where $\Gamma = \frac{\gamma^2}{mk}$. One can see that $R \to F_0/k$ as $\omega \to 0$ and $R \to 0$ as $\omega \to \infty$. If $0 < \Gamma < 2$, then R has its maximum (or Δ has its minimum) when

$$\omega^2 = \omega_0^2(1 - \frac{\Gamma}{2}).$$

In this case, the maximum of R is

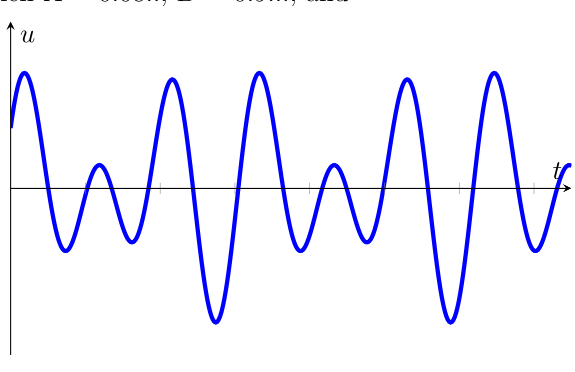
$$R_{\max} = \frac{2F_0}{k\sqrt{\Gamma(4-\Gamma)}}.$$

In particular, if γ is small, then Γ is also small. Thus, the amplitude of the steady state solution R attains its maximum if ω is close to the natural frequency ω_0. If $\Gamma \geq 2$, then R has its maximum (or Δ has its minimum) when $\omega = 0$. In this case, the maximum of R is F_0/k.

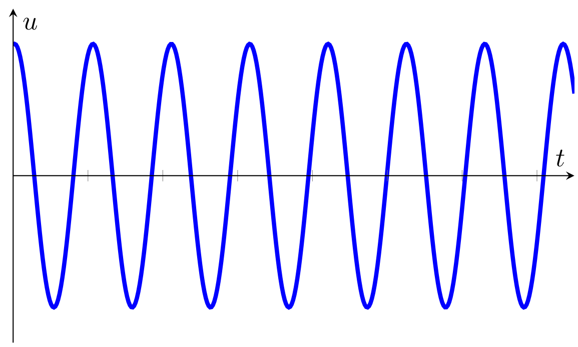
Example 1.1. We consider the case where γ is small. Let $u'' + 0.2u' + 9.01u = 5\cos(\omega t)$. The general solution is

$$u(t) = e^{-0.1t}(C_1\cos(3t) + C_2\sin(3t)) + A\cos(\omega t) + B\sin(\omega t).$$

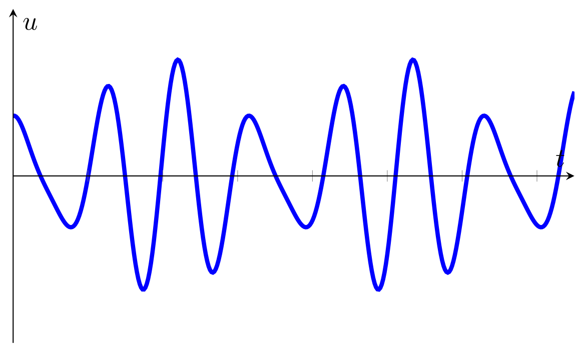
If $\omega = 2$ and $C_1 = C_2 = 1$, then $A = 0.08..$, $B = 0.9...$, and



If $\omega = 3$ and $C_1 = C_2 = 1$, then $A = 8.4...$, $B = 0.14...$, and



If $\omega = 4$ and $C_1 = C_2 = 1$, then $A = 0.08...$, $B = -0.7...$, and



2 General Theory of Higher Order Linear Equations

An n-th order linear differential equation is of the form

$$\frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t).$$

We assume that $p_1(t), \cdots, p_n(t), g(t)$ are continuous on $I = (\alpha, \beta)$. Let

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y,$$

then we simply write $L[y] = g(t)$.

Theorem 2.1 (Existence and Uniqueness)**.** *If $p_1(t), \cdots, p_n(t), g(t)$ are continuous on $I = (\alpha, \beta)$, then there exists a unique solution to the equation*

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = g(t)$$

on I.

Definition 2.2. The Wronskian of $y_1, \cdots, y_n$ is defined by

$$W[y_1, \cdots, y_n](t) = \det \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}.$$

Theorem 2.3. *Suppose $p_1(t), \cdots, p_n(t), g(t)$ are continuous on $I = (\alpha, \beta)$ and $y_1, \cdots, y_n$ are solutions to*

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0.$$

If $W[y_1, \cdots, y_n](t_0) \neq 0$ for some $t_0 \in I$, then every solution to $L[y] = 0$ can be written as

$$C_1 y_1 + \cdots + C_n y_n.$$

In this case $\{y_1, \cdots, y_n\}$ is called a fundamental set of solutions and $y(t) = C_1 y_1 + \cdots + C_n y_n$ is called the general solution.

Definition 2.4. Let $f_1, \cdots, f_n$ be functions on I. We say $f_1, \cdots, f_n$ are linearly dependent on I if there exist constants $k_1, \cdots, k_n$ not all zero such that

$$k_1 f_1(t) + \cdots + k_n f_n(t) = 0$$

for all $t \in I$. If not, we say $f_1, \cdots, f_n$ are linearly independent on I

Example 2.5. Let $f_1 = 2t - 3$, $f_2 = t^2 + 1$, and $f_3 = 2t^2 - t$. Consider

$$\begin{aligned} k_1 f_1 + k_2 f_2 + k_3 f_3 &= k_1(2t-3) + k_2(t^2+1) + k_3(2t^2 - t) \\ &= (k_2 + 2k_3)t^2 + (2k_1 - k_3)t + (k_2 - 3k_1). \end{aligned}$$

Therefore,

$$\begin{cases} k_2 + 2k_3 = 0 \\ 2k_1 - k_3 = 0 \\ k_2 - 3k_1 = 0 \end{cases}$$

(why?). Thus, $k_1 = k_2 = k_3 = 0$, which means that f_1, f_2, f_3 are linearly independent.

Theorem 2.6. *Let $y_1, \cdots, y_n$ be solutions to*

$$L[y] = \frac{d^n y}{dt^n} + p_1(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + p_{n-1}(t)\frac{dy}{dt} + p_n(t)y = 0.$$

A set $\{y_1, \cdots, y_n\}$ is a fundamental set of solutions if and only if $\{y_1, \cdots, y_n\}$ is linearly independent on I.

Example 2.7. Consider $y''' + y' = 0$. Let $v = y'$, then the given equation is $v'' + v = 0$. Thus, $v = y' = A\cos t + B\sin t$ and so

$$y = C_1 \cos t + C_2 \sin t + C_3.$$

It is natural to guess that $\{1, \cos t, \sin t\}$ is a fundamental set of solutions. There are two ways to verify that. One can directly show that the Wronskian is not zero. That is,

$$\begin{aligned} W[1, \cos t, \sin t](t) &= \det \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix} \\ &= \det \begin{pmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{pmatrix} \\ &= 1. \end{aligned}$$

The other way is to check whether $\{1, \cos t, \sin t\}$ is linearly independent. Suppose there exist constants k_1, k_2, k_3 such that

$$k_1 + k_2 \cos t + k_3 \sin t = 0$$

for all t. Taking derivative, we have $-k_2 \sin t + k_3 \cos t = 0$. Putting $t = 0$, we get $k_3 = 0$. Then, it follows that $k_1 = k_3 = 0$, as desired.

3 Homogeneous Equations with Constant Coefficients

Consider the homogeneous equation with constant coefficients

$$L[y] = a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = 0$$

where $a_0, \cdots, a_n$ are real and $a_0 \neq 0$. If $y(t) = e^{\lambda t}$ is a solution, then

$$Z(\lambda) = a_0 \lambda^n + \cdots + a_1 \lambda + a_0 = 0,$$

which is called the characteristic equation. It is well-known that $Z(\lambda)$ has n complex roots (including repeated roots) and

$$Z(\lambda) = a_0(\lambda - r_1) \cdots (\lambda - r_n).$$

If $r_1, \cdots, r_n$ are n distinct real roots, then $\{e^{r_1 t}, \cdots, e^{r_n t}\}$ is a fundamental set of solutions and the general solution is

$$y(t) = C_1 e^{r_1 t} + \cdots + C_n e^{r_n t}.$$

Example 3.1. Consider $y''' + 2y'' - y' - 2y = 0$, then the characteristic equation is

$$\begin{aligned} Z(\lambda) &= \lambda^3 + 2\lambda^2 - \lambda - 2 \\ &= \lambda^2(\lambda + 2) - (\lambda + 2) \\ &= (\lambda^2 - 1)(\lambda + 2) \\ &= (\lambda - 1)(\lambda + 1)(\lambda + 2). \end{aligned}$$

Thus, the equation has three distinct real roots $\lambda = 1, -1, -2$ and $\{e^t, e^{-t}, e^{-2t}\}$ is a fundamental set of solutions. The general solution is

$$y(t) = C_1 e^t + C_2 e^{-t} + C_3 e^{-2t}.$$

Suppose $Z(\lambda)$ has complex roots, say $\lambda = r + i\mu$. Since the coefficients are real, the conjugate $r - i\mu$ is also a root. Thus, this complex roots correspond to

$$e^{rt} \cos \mu t, \qquad e^{rt} \sin \mu t.$$

Example 3.2. Consider $y''' - y'' + y' - y = 0$, then the characteristic equation is

$$\begin{aligned} Z(\lambda) &= \lambda^3 - \lambda^2 + \lambda - 1 \\ &= \lambda^2(\lambda - 1) + (\lambda - 1) \\ &= (\lambda^2 + 1)(\lambda - 1). \end{aligned}$$

Thus, the equation has one real root and two complex roots $\lambda = 1, i, -i$ and $\{e^t, \cos t, \sin t\}$ is a fundamental set of solutions. The general solution is

$$y(t) = C_1 e^t + C_2 \cos t + C_3 \sin t.$$

Suppose $Z(\lambda)$ has repeated roots. To be specific, suppose $Z(\lambda)$ has a factor $(\lambda - r)^s$ where s is the maximum power. Then, s is called the multiplicity of the root r. In this case, the corresponding solutions are

$$e^{rt}, te^{rt}, \cdots, t^{s-1} e^{rt}.$$

If a complex root, say $r + i\mu$, has multiplicity s, then the conjugate $r - i\mu$ has the same multiplicity. In this case the corresponding solutions are

$$e^{rt} \cos \mu t, \qquad e^{rt} \sin \mu t, \qquad te^{rt} \cos \mu t, \qquad te^{rt} \sin \mu t, \qquad \cdots \qquad t^{s-1} e^{rt} \cos \mu t, \qquad t^{s-1} e^{rt} \sin \mu t.$$

Example 3.3. Consider $y^{(6)} + 2y^{(4)} + y'' = 0$, then the characteristic equation is

$$\begin{aligned} Z(\lambda) &= \lambda^6 + 2\lambda^4 + \lambda^2 \\ &= \lambda^2(\lambda^2 + 1)^2. \end{aligned}$$

Thus, the equation has roots $\lambda = 0, i, -i$ with multiplicity 2. Thus, $\{1, t, \cos t, \sin t, t\cos t, t\sin t\}$ is a fundamental set of solutions and the general solution is

$$y(t) = C_1 + C_2 t + C_3 \cos t + C_4 \sin t + C_5 t \cos t + C_6 t \sin t.$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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