Practice for Midterm 3

Math 416, Abstract linear algebra, Fall 2019
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1. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.
   (a) Find all the eigenvalues of A.
   (b) Find all the eigenspaces.
   (c) Find a diagonal matrix D and an invertible matrix Q such that $D = Q^{-1}AQ$.

2. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ with $A^2 = A$.
   (a) Show that the only possible eigenvalues for A are 0 and 1.
   (b) Show that A has at least one eigenvalue.
   (c) Show that $\mathcal{R}(L_A) = \mathcal{N}(A - I)$.
   (d) Show that A is diagonalizable. (Hint: use the Dimension theorem.)

3. Let $A = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$, then the characteristic polynomial is $f(t) = (3-t)^2(5-t)$. Thus, 3 and 5 are the eigenvalues for A.
   (a) Find the algebraic multiplicities of $\lambda = 3$ and $\lambda = 5$.
   (b) Find the geometric multiplicities of $\lambda = 3$ and $\lambda = 5$.
   (c) Determine whether A is diagonalizable or not. Justify your answer.

4. Let u and v be column vectors in $\mathbb{R}^n$ and $A = uv^t \in \mathcal{M}_{n\times n}(\mathbb{R})$. Suppose that $\langle u, v\rangle \neq 0$ where $\langle \cdot, \cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n$.
   (a) Show that A has rank 1.
   (b) Show that u is an eigenvector for A. Find the corresponding eigenvalue.
   (c) Let $W = \{v\}^\perp$. Show that every vector in W is an eigenvector for A. Find the corresponding eigenvalue.
   (d) Conclude that A is diagonalizable.

5. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be the characteristic polynomial. Show that $a_0 = \det(A)$.

6. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ with $A^k = O$ for some $k \geq 1$. Show that the only possible eigenvalue for A is 0.

7. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be a transition matrix such that $A_{ij} > 0$ for all $i, j = 1, 2, \cdots, n$, and $\sum_{j=1}^n A_{ij} = 1$ for all $i = 1, 2, \cdots, n$. Show that

$$\lim_{m\to\infty} A^m = \frac{1}{n}\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

8. Let $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ be a transition matrix associated to a Markov chain. Suppose that $\frac{1}{2}$ and $\frac{1}{3}$ are eigenvalues of A.

(a) Show that A is diagonalizable.

(b) Suppose $\mathcal{N}(A-I) = \{t(1,3,4) : t \in \mathbb{R}\}$. Find the limit $\lim_{m\to\infty} A^m$.

9. Let $V = \mathbb{R}^3$ and define $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1y_1 + x_2y_2$. Show that this is not an inner product.

10. Let $V = \mathcal{M}_{2\times 2}(\mathbb{R})$ and define an inner product $\langle A, B\rangle = \operatorname{tr}(B^tA)$ for $A, B \in V$. Show that $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ 3 & -4 \end{pmatrix}$ are orthogonal.

11. Let V be an inner product space over $\mathbb{R}$ and $S = \{v_1, \cdots, v_k\}$ an orthonormal subset of V. Show that

$$\|a_1v_1 + a_2v_2 + \cdots + a_kv_k\|^2 = a_1^2 + a_2^2 + \cdots + a_k^2$$

for any $a_1, \cdots, a_k \in \mathbb{R}$.

12. Let V be an inner product space over $\mathbb{R}$ and S a subset of V. Show that if $x, y \in \operatorname{Span}(S)$ and $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in S$ then $x = y$.