Math 416 Lecture Note: Week 12

Daesung Kim

1 Orthogonality

Definition 1.1. Let V be an inner product space over F, $x, y \in V$, and S a subset of V.

(i) We say x and y are orthogonal if $\langle x, y\rangle = 0$.

(ii) We say S is orthogonal if $\langle z, w\rangle = 0$ for all $z, w \in S$, $z \neq w$.

(iii) We say x is a unit vector if $\|x\| = 1$.

(iv) We say S is orthonormal if S is orthogonal and consists of unit vectors.

Example 1.2. Let $V = \mathbb{R}^2$ be equipped with the standard inner product (dot product), then $S_1 = \{(1,0),(0,1)\}$ is orthonormal, $S_2 = \{(1,1),(1,-1)\}$ is orthogonal, and $S_3 = \{(1,0),(1,1)\}$ is not orthogonal.

Definition 1.3. Let V be an inner product space over F. A orthonormal basis for V is a basis which is orthonormal.

Theorem 1.4. *Let V be an inner product space over F and $S = \{v_1, \cdots, v_k\}$ be an orthogonal subset of V. If $y \in \mathrm{Span}(S)$, then*

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i\rangle}{\|v_i\|^2} v_i.$$

Proof. Since $y \in \mathrm{Span}(S)$, there exist $a_1, \cdots, a_k$ such that

$$y = a_1 v_1 + \cdots + a_k v_k.$$

Then, for each $i = 1, 2, \cdots, k$,

$$\langle y, v_i\rangle = \left\langle \sum_{j=1}^{k} a_j v_j, v_i \right\rangle = \sum_{j=1}^{k} a_j \langle v_j, v_i\rangle = a_i \|v_i\|^2$$

because S is orthogonal. This finishes the proof. $\square$

Corollary 1.5. *Let V be an inner product space over F and $S = \{v_1, \cdots, v_k\}$ be an orthonormal subset of V. If $y \in \mathrm{Span}(S)$, then*

$$y = \sum_{i=1}^{k} \langle y, v_i\rangle v_i.$$

Corollary 1.6. *Let V be an inner product space over F and S be an orthogonal subset of V consisting of nonzero vectors. Then, S is linearly independent.*

Proof. Let $v_1, \cdots, v_k \in S$ and $\sum_{i=1}^k a_i v_i = 0$. We apply the previous theorem with $y = 0$, then

$$a_i = \frac{\langle 0, v_i \rangle}{\|v_i\|^2} = 0$$

for all $i = 1, 2, \cdots, k$. Thus, S is linearly independent. $\square$

Example 1.7. Let $V = \mathbb{R}^3$ be equipped with dot product and $v = 2e_1 + 3e_2 + 4e_3$. Then,

$$\langle v, e_1 \rangle = 2, \qquad \langle v, e_2 \rangle = 3, \qquad \langle v, e_3 \rangle = 4.$$

Definition 1.8. A matrix $A \in \mathcal{M}_{n\times n}(F)$ is called unitary when $F = \mathbb{C}$ (and orthogonal when $F = \mathbb{R}$) if Q is invertible and $Q^{-1} = Q^*$.

We simply use the terminology "orthogonal" for both cases $F = \mathbb{R}$ and $F = \mathbb{C}$.

Theorem 1.9. *A matrix $A \in \mathcal{M}_{n\times n}(F)$ is orthogonal if and only if the set of the columns of A is orthonormal.*

Proof. HW. $\square$

Remark 1.10. Let $V = F^n$, β be an orthonormal basis, and γ the standard basis. Then, $Q = [I]_\beta^\gamma$ is orthogonal.

2 Gram–Schmidt orthogonalization process

Theorem 2.1. *Every finite dimensional inner product space has an orthonormal basis.*

Proof. It suffices to show that there exists a basis which is orthogonal. Let V be an inner product space over F and β is a basis. If $n = 1$, then it is trivial. For $n \geq 2$, we will construct an orthogonal basis β' from β. This method is called the Gram–Schmidt orthogonalization process. To illustrate this process, suppose $\beta = \{v_1, v_2\}$ is a basis for V. We want to find another basis $\beta' = \{w_1, w_2\}$ that is orthonormal. First, let $w_1 = v_1/\|v_1\|$. The idea of defining w_2 is to decompose v_2 as the sum of the orthogonal and non-orthogonal part with respect to w_1. That is,

$$v_2 = \overbrace{\langle v_2, w_1\rangle w_1}^{\text{Non-orthogonal to } w_1} + \underbrace{(v_2 - \langle v_2, w_1\rangle w_1)}_{\text{Orthogonal to } w_1}.$$

Thus, we define $w_2' = v_2 - \langle v_2, w_1\rangle w_1$ and $w_2 = w_2'/\|w_2'\|$. In general, we can construct an orthonormal basis by the induction. We complete this by the next theorem. $\square$

Theorem 2.2 (the Gram–Schmidt orthogonalization process)**.** *Let V be an inner product space over F and $S = \{v_1, \cdots, v_n\}$ be a linearly independent subset of V. We define $S' = \{w_1, \cdots, w_n\}$ by the following inductive way: Let $w_1 = v_1$. For $k = 2, \cdots, n$,*

$$w_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, w_j\rangle}{\|w_j\|^2} w_j.$$

Then, S' is orthogonal and $\mathrm{Span}(S) = \mathrm{Span}(S')$.

Proof. Use an induction on n. If $n = 1$, then $S = S'$ so that it is done. Suppose $n \geq 2$ and the theorem is true for $n - 1$. That is, if we let $T = \{v_1, \cdots, v_{n-1}\}$ and $T' = \{w_1, \cdots, w_{n-1}\}$, then T' is orthogonal and $\mathrm{Span}(T) = \mathrm{Span}(T')$ by the induction hypothesis. For $k = 1, 2, \cdots, n-1$, we have

$$\begin{aligned}
\langle w_n, w_k\rangle &= \left\langle v_n - \sum_{j=1}^{n-1} \frac{\langle v_k, w_j\rangle}{\|w_j\|^2} w_j, w_k \right\rangle \\
&= \langle v_n, w_k\rangle - \sum_{j=1}^{n-1} \frac{\langle v_k, w_j\rangle}{\|w_j\|^2} \langle w_j, w_k\rangle \\
&= \langle v_n, w_k\rangle - \frac{\langle v_k, w_k\rangle}{\|w_k\|^2} \langle w_k, w_k\rangle \\
&= 0.
\end{aligned}$$

Thus, $S' = T \cup \{w_n\}$ is orthogonal.

Note that $\mathrm{Span}(T) = \mathrm{Span}(T')$, $S = T \cup \{v_n\}$, and $S' = T' \cup \{w_n\}$. To show $\mathrm{Span}(S) = \mathrm{Span}(S')$, it suffices to show that $v_n \in \mathrm{Span}(S')$ and $w_n \in \mathrm{Span}(S)$. This follows from

$$v_n = w_n + \sum_{j=1}^{n-1} \frac{\langle v_k, w_j\rangle}{\|w_j\|^2} w_j \in \mathrm{Span}(S')$$

and

$$w_n = v_n - \sum_{j=1}^{n-1} \frac{\langle v_k, w_j\rangle}{\|w_j\|^2} w_j \in \mathrm{Span}(S)$$

because $\{w_1, \cdots, w_{n-1}\} \subset \mathrm{Span}(S)$. $\square$

Example 2.3. Let $S = \{v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (1,3,3)\}$ be a linearly independent subset of $\mathbb{R}^3$. By the Gram–Schmidt process, we get $w_1 = v_1$,

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\|w_1\|^2} w_1 = (0,1,1) - \frac{1}{2}(1,0,1) = (-\frac{1}{2}, 1, \frac{1}{2}) = \frac{1}{2}(-1,2,1),$$

and

$$\begin{aligned} w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2\rangle}{\|w_2\|^2} w_2 \\ &= (1,3,3) - 2(1,0,1) - \frac{4}{3}(-1,2,1) \\ &= \frac{1}{3}(1,1,-1). \end{aligned}$$

Then, $\beta' = \{w_1, w_2, w_3\}$ is an orthogonal basis for $\mathbb{R}^3$.

Example 2.4. Let $V = \mathcal{P}(\mathbb{R})$ be the space of all polynomials with real coefficients and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be defined by

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(x)g(x)\,dx.$$

Then, $\langle \cdot, \cdot \rangle$ is an inner product. Let $e_n(x) = x^n$ for each integer $n \geq 0$ and $\beta = \{e_n(x) : n \geq 0\}$. We apply the Gram-Schmidt process to β. Let $f_0(x) = 1$,

$$f_1(x) = e_1(x) - \frac{\langle e_1, f_0\rangle}{\|f_0\|^2} f_0(x) = x - \frac{1}{2}\int_{-1}^{1} x\,dx = x,$$

and

$$f_2(x) = e_2(x) - \frac{\langle e_2, f_0\rangle}{\|f_0\|^2} f_0(x) - \frac{\langle e_2, f_1\rangle}{\|f_1\|^2} f_1(x) = x^2 - \frac{1}{3}.$$

We can continue this procedure and get $\beta' = \{f_n : n \geq 0\}$. We call $f_n(x)$ the Legendre polynomial.

Definition 2.5. Let V be an inner product space over F and β is an orthonormal subset of V. We define the Fourier coefficients of x relative to β by $c_y = \langle x, y\rangle$ for each $y \in \beta$.

Example 2.6. Let $V = C([-1,1])$ be the space of all continuous functions $f : [-1,1] \to \mathbb{R}$ and $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ be defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Let

$$S = \{\frac{1}{\sqrt{2}}\} \cup \{\sin(\pi n x) : n = 1, 2, \cdots\} \cup \{\cos(\pi n x) : n = 1, 2, \cdots\}.$$

Note that S is an orthonormal subset. Let $h(x) = |x| \in V$. To see the Fourier coefficients of f relative to S, we compute that

$$\left\langle h, \frac{1}{\sqrt{2}} \right\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} |x|\,dx = \sqrt{2}\int_{0}^{1} x\,dx = \frac{1}{\sqrt{2}},$$

$\langle h, \sin(\pi n x)\rangle = 0$ by symmetry, and

$$\langle h, \cos(\pi n x)\rangle = 2\int_0^1 x\cos(\pi n x)\,dx = -\frac{2}{\pi^2 n^2}\int_0^{\pi n} \sin x\,dx = \begin{cases} 0, & \text{if } n \text{ is even,} \\ -\frac{4}{\pi^2 n^2} & \text{if } n \text{ is odd.} \end{cases}$$

It is well-known (Parseval's identity) that

$$f(x) = \left\langle f, \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} (\langle f, \sin(\pi n x)\rangle \sin(\pi n x) + \langle f, \cos(\pi n x)\rangle \cos(\pi n x)).$$

This yields that

$$|x| = \frac{1}{2} - \sum_{n\colon \text{ odd}} \frac{4\cos(\pi n x)}{\pi^2 n^2}$$

for $x \in [-1, 1]$. In particular, if $x = 0$, then

$$\sum_{n\colon \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8},$$

which implies in turn that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3 Orthogonal complements and projections

Definition 3.1. Let V be an inner product space over F and S a nonempty subset of V. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in V that are orthogonal to S.

Example 3.2. Note that $\{0\}^{\perp} = V$ and $V^{\perp} = \{0\}$.

Example 3.3. Let $V = \mathbb{R}^2$ and $S = \{e_1\}$, then $S^{\perp} = \mathrm{Span}(\{e_2\})$. If $S = \{v\}$, $v \neq 0$, and $\langle v, w\rangle = 0$, then $S = \mathrm{Span}(\{w\})$.

Example 3.4. Let $V = \mathbb{R}^3$ and $S = \{e_1\}$, then $S^{\perp} = \mathrm{Span}(\{e_2, e_3\})$. Note that S is indeed the yz-plane. If $S = \{e_1, e_2\}$, then $S^{\perp}$ is the z-axis.

Theorem 3.5. *Let V be an inner product space over F and W a finite dimensional subspace of V. Then, every vector y in V can be written uniquely as $y = w + z$ where $w \in W$ and $z \in W^{\perp}$. Moreover, if $\{u_1, \cdots, u_n\}$ is an orthonormal basis for W, then*

$$w = \sum_{i=1}^{n} \langle y, u_i\rangle\, u_i.$$

Proof. Let $y \in V$, then we can decompose it into the orthogonal part and non-orthogonal part as we did in the Gram-Schmidt process. That is,

$$y = \left(\sum_{i=1}^{n} \langle y, u_i\rangle\, u_i\right) + \left(y - \sum_{i=1}^{n} \langle y, u_i\rangle\, u_i\right) = w + z.$$

One can see that $w \in W$ and for each u_j,

$$\langle z, u_i\rangle = \left\langle y - \sum_{i=1}^{n} \langle y, u_i\rangle\, u_i, u_j \right\rangle = \langle y, u_j\rangle - \sum_{i=1}^{n} \langle y, u_i\rangle \langle u_i, u_j\rangle = 0,$$

which implies that $z \in W^{\perp}$ (HW). So, it suffices to show that this decomposition is unique. This follows from the fact that $W \cap W^{\perp} = \{0\}$ (HW). Suppose $y = w + z = w' + z'$ where $w, w' \in W$ and $z, z' \in W^{\perp}$. Then,

$$w - w' = z' - z \in W \cap W^{\perp} = \{0\}$$

and so $w = w'$ and $z = z'$. ☐

Definition 3.6. Let V be an inner product space over F and W a finite dimensional subspace of V. The orthogonal projection of y onto W, denoted by $\mathrm{proj}_W(y)$, is a vector w in W such that $y = w + z$ and $z \in W^{\perp}$. The theorem ensures that this definition is well-defined.

Remark 3.7. Let V be an inner product space over F and W a finite dimensional subspace of V. We define $\mathrm{proj}_W : V \to W$ by $y \to \mathrm{proj}_W(y)$. One can see that this is indeed a linear transformation.

Example 3.8. Let W be a plain in $\mathbb{R}^3$ given by $x + y + z = 0$. Then, W is a subspace of $\mathbb{R}^3$, $W = \mathrm{Span}(\{v_1, v_2\})$, and $W^{\perp} = \mathrm{Span}(\{v_3\})$ where $v_1 = (1, -1, 0)$, $v_2 = (0, 1, -1)$, and $v_3 = (1, 1, 1)$.

For each $u \in \mathbb{R}^3$, we define $T(u)$ by the closest vector in W from u. We define a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $u \to T(u)$. Previously, we have seen that

$$T(x, y, z) = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2x - y - z \\ -x + 2y - z \\ -x - y + 2z \end{pmatrix}.$$

Let $A = [T]_{\gamma}$ and γ be the standard basis for $\mathbb{R}^3$, then

$$I - A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} - & v_3 & - \\ - & v_3 & - \\ - & v_3 & - \end{pmatrix}.$$

In particular, $u - T(u) \in W^{\perp}$ for all $u \in \mathbb{R}^3$.

This example suggests that $T(u) = \mathrm{proj}_W(u)$, that is, T is the orthogonal projection onto W. In other words, we can interpret $\mathrm{proj}_W(y)$ as the "closest" vector in W from y. The next result asserts that this is the case.

Corollary 3.9. *Let V be an inner product space over F and W a finite dimensional subspace of V. If $y \in V$ and $w = \mathrm{proj}_W(y)$, then*

$$\|y - x\| \geq \|y - w\|$$

for all $x \in W$. Moreover, equality holds if and only if $x = w$.

Proof. Let $y = w + z$ where $w \in W$ and $z \in W^\perp$. Since $w - x \in W$, we have

$$\begin{aligned}
\|y - x\|^2 &= \|(w - x) + z\|^2 \\
&= \|w - x\|^2 + \langle w - x, z\rangle + \langle z, w - x\rangle + \|z\|^2 \\
&= \|w - x\|^2 + \|z\|^2 \\
&\geq \|z\|^2 \\
&= \|y - w\|^2.
\end{aligned}$$

□

Theorem 3.10. *Let V be a finite dimensional inner product space over F and W a subspace of V, then*

$$\dim(V) = \dim(W) + \dim(W^\perp).$$

Proof. Let $\dim(V) = n$, $\dim(W) = k$, and $1 \leq k \leq n$. Suppose $\{w_1, \cdots, w_k\}$ be a basis for W. By the Gram–Schmidt process, we get an orthonormal basis $\beta = \{v_1, \cdots, v_k\}$ for W. By the basis extension theorem, we have a basis $\{v_1, \cdots, v_k, w_{k+1}, \cdots, w_n\}$ for V. By the Gram–Schmidt process again, we get an orthonormal basis $\gamma = \{v_1, \cdots, v_k, v_{k+1}, \cdots, v_n\}$ for V. Note that along the process the vectors in β does not change because β is already orthonormal. Let $\beta' = \{v_{k+1}, \cdots, v_n\}$. We claim that β' is a basis for $W^\perp$. First, β' is linearly independent because it is a subset of β which is linearly independent. And $\beta' \subset W^\perp$ because γ is orthonormal and so β' is perpendicular to W. It suffices to show that β' spans $W^\perp$. Suppose $w \in W^\perp$, then

$$w = \sum_{i=1}^{n} a_i v_i$$

for some $a_i \in F$. Since $w \in W^\perp$ and γ is orthonormal, we get

$$\langle w, v_j\rangle = \left\langle \sum_{i=1}^{n} a_i v_i, v_j \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, v_j\rangle = a_j = 0$$

for all $j = 1, 2, \cdots, k$. Thus,

$$w = \sum_{i=1}^{n} a_i v_i = \sum_{i=k+1}^{n} a_i v_i,$$

which implies that β' spans $W^\perp$ as desired. □

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu