Sec 11.2: Sturm–Liouville Boundary Value Problems

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Recall: Eigenvalues

We recall the definition of eigenvalues and eigenfunctions for a general differential operator.

$y'' + \lambda y = 0 \qquad y(0) = y(L) = 0.$

Definition

Let $L[y]$ be a differential operator. $\left(\text{Ex}: \; L[y] = -y'' \right)$

We say that λ is a real (or complex) eigenvalue of L with a homogeneous boundary conditions if $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) and the differential equation $L[y] = \lambda y$ with the given homogeneous boundary conditions has nontrivial solutions.

The nontrivial solutions are called eigenfunctions.

① $L[y] = -y'' = \lambda y \Rightarrow y'' + \lambda y = 0.$

② $L[y] = -x^2 y'' + 2y' = \lambda \cdot y \Rightarrow x^2 y'' - 2y' + \lambda y = 0.$

The Sturm–Liouville boundary problem

We study the Sturm–Liouville boundary problem, which consists of a differential equation of the form

$L[y] = -(py')' + qy$

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

$= \lambda r y$

$p \cdot y'' + p' \cdot y' - qy + \lambda r y = 0.$

on $[0, 1]$ with boundary conditions

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0, \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$$

with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$.

We further assume that p, p', q, r are continuous on $[0, 1]$ and $p(x), r(x) > 0$ for all $x \in [0, 1]$. In this case, the problem is called *regular*.

$u_t = (p \cdot u_x)_x - q(x) y$

Examples of the Sturm–Liouville boundary problem

Example

If $p(x) = r(x) = 1$ and $q(x) = 0$, then the problem is $y'' + \lambda y = 0$ with

$$\alpha_1 y(0) + \alpha_2 y'(0) = 0,$$
$$\beta_1 y(1) + \beta_2 y'(1) = 0.$$

$$(p \cdot y')' - q y + \lambda r y = 0$$

$P = r = 1$ & $q = 0$,

$$(y')' - 0 \cdot y + \lambda y$$

$$= y'' + \lambda y = 0.$$

Examples of the Sturm–Liouville boundary problem

Example

If $p(x) = x^k$, $q(x) = 0$, and $r(x) = x^{k-2}$, then

$$\begin{aligned}(p(x)y')' - q(x)y + \lambda r(x)y &= x^k y'' + kx^{k-1}y' + \lambda x^{k-2}y \\ &= x^{k-2}\underbrace{(x^2y'' + kxy' + \lambda y)}_{=0} \\ &= 0.\end{aligned}$$

Thus, the problem $x^2y'' + kxy' + \lambda y = 0$ with

$$\alpha_1 y(1) + \alpha_2 y'(1) = 0,$$
$$\beta_1 y(2) + \beta_2 y'(2) = 0.$$

also belongs to the class.

- $L[y] = -x^2y'' - kxy'$
$= \lambda y$

Lagrange's Identity

$(py')' - qy + \lambda r y = 0$

Let u and v be functions on $[0,1]$ with continuous second derivatives and L a differential operator defined by $L[y] = -(p(x)y')' + q(x)y$. Then,

$= \lambda r y$

$$\int_0^1 (L[u]v - uL[v])\,dx = -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]_0^1$$

$$= p(0)\left(u'(0)v(0) - u(0)v'(0)\right) - p(1)\left(u'(1)v(1) - u(1)v'(1)\right).$$

Integration by parts.

$= 0$

Furthermore, if u and v satisfies the boundary condition

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0, \\ \beta_1 y(1) + \beta_2 y'(1) = 0 \end{cases}$$

with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$, then

$$\int_0^1 (L[u]v - uL[v])\,dx = 0.$$

$L[y] = -(py')' + qy = \lambda r y$

Remark

$(L[u], v) = \int_0^1 L[u] \cdot v \, dx = \int_0^1 L[v] \cdot u \, dx = (u, L[v])$

The Lagrange's identity with the boundary condition can be written as

$$(L[u], v) = (u, L[v])$$

where the inner product on $[0, 1]$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)\, dx.$$

The Lagrange's identity also holds for complex-valued functions $\cancel{u}$ f and $\cancel{v}$ g with the complex inner product

conjugate bar

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx.$$

$\overline{(a + bi)} = a - bi$

Remark

The Lagrange's identity with the boundary condition can be written as

$$(L[u], v) = (u, L[v])$$

where the inner product on $[0, 1]$ is defined by

$$(f, g) = \int_0^1 f(x)g(x)\, dx.$$

The Lagrange's identity also holds for complex-valued functions u and v with the complex inner product

$L[u] = \underline{\lambda r u}$

$L[v] = \underline{\lambda r v}$

$$(f, g) = \int_0^1 f(x)\overline{g(x)}\, dx.$$

Rmk: If u, v are solutions to S-T bdry problem, then $(\underline{L[u]}, v) = (u, \underline{L[v]})$.

Theorem

Every eigenvalue of the Sturm–Liouville problem is real.

$$L[y] = -(py')' + qy = \lambda r y$$

with Bdry conditions.

<u>Proof</u> Suppose λ is an eigenvalue and ϕ is the corresponding eigenfunction.

① $L[\phi] = \lambda r \phi$

② $(L[\phi], \phi) = (\phi, L[\phi])$

Theorem

Every eigenvalue of the Sturm–Liouville problem is real.

$$(L[\phi], \phi) = (\phi, L[\phi])$$

$$\int_0^1 \underbrace{L[\phi]}_{= \lambda r \phi} \cdot \overline{\phi}\, dx = \int \phi \cdot \overline{\underbrace{L[\phi]}_{= \lambda r \phi}}\, dx$$

$$\int_0^1 \lambda r \phi \cdot \overline{\phi}\, dx = \int \phi \cdot \overline{\lambda}\, \overline{r}\, \overline{\phi}\, dx$$

$$r(x) > 0 \Rightarrow r(x) : \text{real} \Rightarrow \overline{r(x)} = r(x)$$

$$\lambda \int_0^1 (r \phi \cdot \overline{\phi})\, dx = \overline{\lambda} \int_0^1 (r \phi \cdot \overline{\phi})\, dx$$

Theorem

Every eigenvalue of the Sturm–Liouville problem is real.

$$(\lambda-\bar{\lambda}) \underbrace{\int_0^1 \underbrace{r(x)}_{>0} \underbrace{\phi(x)\overline{\phi(x)}}_{>0}\, dx}_{>0 \text{ (Not trivial)}} = 0$$

$\Rightarrow \quad (\lambda-\bar{\lambda})=0$

$\Rightarrow \quad \lambda=\bar{\lambda}$

$\Rightarrow \quad \lambda$: real.

$\boxed{\text{///}}$

Theorem

If ϕ_n and ϕ_m are two eigenfunctions of the Sturm–Liouville problem corresponding to distinct eigenvalues λ_n and λ_m, then

$$(r\phi_n, \phi_m) = \int_0^1 \phi_n(x)\phi_m(x)\, r(x) dx = 0.$$

We call ϕ_n and ϕ_m are orthogonal on $[0, 1]$ with the weight $r(x)$.

Proof ①

$$L[\phi_n] = \lambda_n r \phi_n$$
$$L[\phi_m] = \lambda_m r \phi_m$$

② $(L[\phi_n], \phi_m) = (\phi_n, L[\phi_m])$

$$(\lambda_n r\phi_n, \phi_m) = (\phi_n, \lambda_m r \phi_m)$$

$$0 \neq (\lambda_n - \lambda_m) \cdot (r\phi_n, \phi_m) = 0$$

Theorem

Every eigenvalue of the Sturm–Liouville problem is simple; that is, if ϕ_1 *and* ϕ_2 *are the corresponding eigenfunctions of the same eigenvalue* λ*, then they are linearly dependent. Furthermore, the eigenvalues form an infinite sequence* $\lambda_1 < \lambda_2 < \cdots$ *and* $\lim_{n\to\infty} \lambda_n = \infty$*.*

Definition

An eigenfunction ϕ is normalized with respect to the weight r if

$$(r\phi, \phi) = \int_0^1 \phi^2 \, r(x) dx = 1.$$

Remark

Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues and ϕ_n the corresponding normalized eigenfunctions. Then,

$$(r\phi_n, \phi_m) = \int_0^1 \phi_n \phi_m \, r(x) dx = \begin{cases} 1, & m = n \\ 0 & m \neq n. \end{cases}$$

In this case, we call the set $\{\phi_n : n = 1, 2, \cdots\}$ is orthonormal.

Suppose that we are given a function $f(x)$ and want to represent it in terms of the normalized eigenfunctions ϕ_n as we did in the heat conduction equation. To be specific, the question is to find C_n such that

$$f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x). \qquad = \frac{f(x+) + f(x-)}{2}$$

Using the orthogonality and the normalization, it is obvious to guess that

$$C_n = \int_0^1 f(x)\phi_n(x)\, r(x) dx.$$

$y'' + \lambda y = 0 \Rightarrow$ eigenfunctions are $\rightarrow f(x) = \sum$ sines cosines

sine or cosine

Theorem

Let $\phi_1, \phi_2, \cdots$ be the normalized eigenfunctions of the Sturm–Liouville problem. If $f(x)$ and $f'(x)$ are piecewise continuous on $[0,1]$, then

$$\sum_{n=1}^{\infty} C_n \phi_n(x) = \frac{1}{2}(f(x+) + f(x-))$$

with

$$C_n = \int_0^1 f(x)\phi_n(x)\, r(x) dx.$$