# Practice for Final Exam

Math 416, Abstract linear algebra, Fall 2019
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1. Let $W = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 1\}$.
   (a) Determine whether $W$ is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$.
   (b) Find $\mathrm{Span}(W)$.

2. Let $V$ be a vector space. We say a subspace $W$ is nontrivial if $W \neq \{0\}$ and $W \neq V$.
   (a) Find an example of nontrivial subspaces $W_1$ and $W_2$ such that $W_1 \cup W_2$ is not a subspace of $V$.
   (b) Find an example of nontrivial subspaces $W_1$ and $W_2$ such that $W_1 \cup W_2$ is a subspace of $V$.

3. Let $V$ be a vector space and $W$ a subspace of $V$ with $\dim(V) = 5$ and $\dim(W) = 4$. Show that if $v \notin W$, then $V = \mathrm{Span}(\{v\} \cup W)$.

4. Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.
   (a) Find the rank of $A$.
   (b) Find the solution set of a system of linear equations
   $$\begin{cases} 2y + z = 0 \\ x + 3y + z = 0 \\ -x + y + z = 0. \end{cases}$$
   (c) Find $a, b, c \in \mathbb{R}$ such that a system of linear equations
   $$\begin{cases} 2y + z = a \\ x + 3y + z = b \\ -x + y + z = c \end{cases}$$
   has no solution.

5. Let $V$ be a vector space over $\mathbb{R}$ of dimension 3 and $\beta = \{v_1, v_2, v_3\}$ be a basis for $V$. Let $T : V \to V$ be a linear transformation such that $T(v_1) = v_3$, $T(v_2) = v_1$, and $T(v_3) = v_2$.
   (a) Compute the matrix $[T]_\beta$.
   (b) Prove that $T \circ T \circ T = I_V$.
   (c) Let $\gamma = \{u_1, u_2, u_3\}$ where
   $$u_1 = v_1, \qquad u_2 = v_1 + v_2, \qquad u_3 = v_1 + v_2 + v_3.$$
   Compute the matrix $[I_V]_\beta^\gamma$.

6. Let $A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$ with $\det(A) = 3$.

(a) Compute the determinant of $A^{-1}$.

(b) Compute the determinant of $2A^{-1}$.

(c) Compute the determinant of $(2A)^{-1}$.

(d) Compute the determinant of $A^t$.

(e) Compute the determinant of $B = \begin{pmatrix} a_1+b_1 & c_1 & b_1 \\ a_2+b_2 & c_2 & b_2 \\ a_3+b_3 & c_3 & b_3 \end{pmatrix}$.

7. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ be invertible. Show that if $A+B$ is invertible then $A^{-1}+B^{-1}$ is also invertible.

8. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of $A$.

(b) Find all the eigenspaces of $A$.

(c) Find a diagonal matrix $D$ and an orthogonal matrix $Q$ such that $D = Q^*AQ$.

9. Find an example of a nonzero matrix $A \in \mathcal{M}_{3\times 3}(\mathbb{R})$ such that $A$ is not diagonalizable and the characteristic polynomial $f(t)$ splits.

10. Let $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$ be the transition matrix of a Markov chain.

(a) We say that a transition matrix is *regular* if there exists a positive integer $k \geq 1$ such that all entries of $A^k$ are positive. Show that $A$ is regular.

(b) Find the eigenspace of $A$ corresponding to $\lambda = 1$.

(c) Is $A^t$ also a transition matrix?

11. Let $V = \mathbb{R}^2$ with the nonstandard inner product $\langle x, y\rangle = 4x_1y_1 + x_2y_2$.

(a) Prove directly from the axioms that the above formula defines an inner product on $V$.

(b) Find an orthonormal basis $\beta = \{v_1, v_2\}$ of $V$ with respect this nonstandard inner product.

(c) Let $A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Prove that the adjoint of $L_A$ is $L_B$.

12. Let $V$ be an inner product space over $\mathbb{C}$ and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y\rangle\, z$.

(a) Prove that $T$ is linear.

(b) Find an explicit expression for $T^*$.

13. Let $V$ be an inner product space over $\mathbb{R}$ and $S = \{v_1, v_2, v_3\}$ be an orthonormal basis of $V$.

(a) Let

$$w_1 = v_1, \qquad w_2 = \frac{1}{2}(\sqrt{3}v_2 + v_3), \qquad w_3 = \frac{1}{2}(v_2 - \sqrt{3}v_3).$$

Prove that $S' = \{w_1, w_2, w_3\}$ is also an orthonormal basis of $V$.

(b) Find a basis of $\{w_3\}^\perp$.

14. Let $V$ be an inner product space over $\mathbb{C}$ and $v, w \in V$. Define

$$A = \begin{pmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{pmatrix}.$$

(a) Show that $A$ is self-adjoint.

(b) Show that

$$xAx^* = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} \geq 0$$

for all $x = (x_1, x_2) \in \mathbb{C}^2$.

15. Let $V$ be an inner product space over $\mathbb{C}$ and $T : V \to V$ be a self-adjoint linear transformation. Show that $\mathcal{N}(T) \cap \mathcal{R}(T) = \{0\}$.

16. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Determine whether $A$ is normal or self-adjoint.