Math 416 Lecture Note

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1 Linear transformations

In this section, we study a map between two vector spaces. In particular, we are interested in a map that preserves the structure of vector spaces.

Definition 1.1. Let V and W be vector spaces over $\mathbb{R}$. A map $T : V \to W$ is a linear transformation from V to W if

(i) $T(x + y) = T(x) + T(y)$ for all $x, y \in V$,

(ii) $T(cx) = cT(x)$ for all $x \in V$ and $c \in \mathbb{R}$.

Remark 1.2. Let $T : V \to W$ be a map. One can show that T is linear if and only if $T(0) = 0$ and $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in \mathbb{R}$.

Example 1.3. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (2x - y, x + y)$. First, $T(0, 0) = 0$. For $v_1 = (x_1, y_2)$, $v_2 = (x_2, y_2)$, and $c \in \mathbb{R}$, we have

$$\begin{aligned} T(cv_1 + v_2) &= T(cx_1 + x_2, cy_1 + y_2) \\ &= (2(cx_1 + x_2) - (cy_1 + y_2), (cx_1 + x_2) + (cy_1 + y_2)) \\ &= c(2x_1 - y_1, x_1 + y_1) + (2x_2 - y_2, x_2 + y_2) \\ &= cT(v_1) + T(v_2). \end{aligned}$$

Example 1.4 (Reflection)**.** Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x, -y)$. It is easy to see that T is linear. The map T is indeed a reflection about x-axis.

Example 1.5 (The transpose of matrices)**.** Define $T : \mathcal{M}_{m\times n}(\mathbb{R}) \to \mathcal{M}_{n\times m}(\mathbb{R})$ by $T(A) = A^t$, then T is linear.

Example 1.6 (Integration)**.** Let V be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define $T : V \to \mathbb{R}$ by

$$T(f) = \int_0^1 f(x)\, dx,$$

then T is linear.

Definition 1.7. Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. The null space (or kernel) $\mathcal{N}(T)$ of T is the set of all $v \in V$ such that $T(v) = 0$. The range (or image) $\mathcal{R}(T)$ of T is the set of all $T(v) \in W$ for $v \in V$. That is,

$$\mathcal{N}(T) = \{v \in V : T(v) = 0\}, \qquad \mathcal{R}(T) = \{T(v) \in W : v \in V\}.$$

Example 1.8 (The trace of matrices)**.** Define $T : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = \mathrm{tr}(A)$, then T is linear. The null space of T is

$$\mathcal{N}(T) = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}.$$

Also, the range of T is $\mathcal{R}(T) = \mathbb{R}$. (Why?) Note that $\mathcal{N}(T) \leq \mathcal{M}_{n\times n}(\mathbb{R})$ and $\mathcal{R}(T) \leq \mathbb{R}$. The next theorem tells that this holds in general.

Theorem 1.9. *Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. Then, $\mathcal{N}(T) \leq V$ and $\mathcal{R}(T) \leq W$.*

Proof. It is easy to see that $T(0_V) = 0_W$. This implies $0_V \in \mathcal{N}(T)$. Let $x, y \in \mathcal{N}(T)$ and $c \in \mathbb{R}$, then it suffices to show that $cx + y \in \mathcal{N}(T)$ (see Exercise 18 in [FIS, p. 21]). Indeed, we have

$$T(cx + y) = cT(x) + T(y) = 0.$$

Thus, $\mathcal{N}(T)$ is a subspace of V.

It follows from $T(0_V) = 0_W$ that $0_W \in \mathcal{R}(T)$. Let $x, y \in \mathcal{R}(T)$ and $c \in \mathbb{R}$, then there exist $v, w \in V$ such that $T(v) = x$ and $T(w) = y$. Then,

$$cx + y = cT(v) + T(w) = T(cv + w) \in \mathcal{R}(T).$$

Thus, $\mathcal{R}(T)$ is a subspace of W. □

Theorem 1.10. *Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. If $\beta = \{v_1, \cdots, v_n\}$ is a basis for V, then*

$$\mathcal{R}(T) = \mathrm{Span}(T(\beta)) = \mathrm{Span}(\{T(v_1), \cdots, T(v_n)\}).$$

Proof. Since $T(v_i) \in \mathcal{R}(T)$ for all i, we have $\mathrm{Span}(T(\beta)) \subseteq \mathcal{R}(T)$. Let $w \in \mathcal{R}(T)$, then there exists $v \in V$ such that $T(v) = w$. Since β is a basis for V, $v = a_1v_1 + \cdots + a_nv_n$ for some $a_1, \cdots, a_n \in \mathbb{R}$. Then,

$$w = T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n) \in \mathcal{R}(T).$$

□

Question 1.11. The result implies $\dim(\mathcal{R}(T)) \leq \dim(V)$. Then, what can we say about the difference $\dim(V) - \dim(\mathcal{R}(T))$?

Example 1.12 (Projection)**.** Let $n, m \in \mathbb{N}$ and $m < n$. Define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_m, 0, \cdots, 0)$. Then, $\dim(\mathbb{R}^n) = n$ and $\dim(\mathcal{R}(T)) = m$. Note that

$$\mathcal{N}(T) = \{(0, \cdots, 0, x_{m+1}, x_{m+2}, \cdots, x_n)\}$$

and $\dim(\mathcal{N}(T)) = n - m = \dim(\mathbb{R}^n) - \dim(\mathcal{R}(T))$. Next time, we will see that this is true in general.

2 The dimension theorem

Definition 2.1. Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. If $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are finite dimensional, we define the nullity of T by the dimension of $\mathcal{N}(T)$, and the rank of T by the dimension of $\mathcal{R}(T)$.

Theorem 2.2 (Dimension Theorem)**.** *Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. If V is finite dimensional, then*

$$\dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(V).$$

Proof. Since V is finite dimensional and $\mathcal{N}(T) \leq V$, there exists a basis $\beta = \{v_1, \cdots, v_m\}$ for $\mathcal{N}(T)$. By Basis extension theorem, we have a basis $\widetilde{\beta} = \{v_1, \cdots, v_m, \cdots, v_n\}$ for V. We claim that $\gamma = \{T(v_{m+1}), \cdots, T(v_n)\}$ is a basis for $\mathcal{R}(T)$. By the previous theorem, $T(\widetilde{\beta})$ spans $\mathcal{R}(T)$. Since $T(v_i) = 0$ for $i = 1, 2, \cdots, m$, we see that γ spans $\mathcal{R}(T)$. Let

$$a_{m+1}T(v_{m+1}) + \cdots + a_nT(v_n) = 0,$$

then

$$a_{m+1}T(v_{m+1}) + \cdots + a_nT(v_n) = T(a_{m+1}v_{m+1} + \cdots + a_nv_n) = 0.$$

Let $v = a_{m+1}v_{m+1} + \cdots + a_nv_n$, then $v \in \mathcal{N}(T)$ so

$$v = b_1v_1 + \cdots + b_mv_m = a_{m+1}v_{m+1} + \cdots + a_nv_n.$$

Since $\widetilde{\beta}$ is linearly independent, all a_i and b_i are zero. Thus, $T(v_{m+1}), \cdots, T(v_n)$ is linearly independent. ☐

Example 2.3 (The trace of matrices)**.** Define $T : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathbb{R}$ by $T(A) = \mathrm{tr}(A)$. Since $\mathcal{R}(T) = \mathbb{R}$ and $\dim(\mathcal{M}_{n\times n}(\mathbb{R})) = n^2$, Dimension Theorem yields that $\dim(\mathcal{N}(T)) = n^2 - 1$ where

$$\mathcal{N}(T) = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}.$$

Theorem 2.4. *Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. Then, T is one-to-one if and only if $\mathcal{N}(T) = \{0\}$.*

Proof. Suppose T is one-to-one. If $T(v) = 0 = T(0)$, then $v = 0$. Thus $\mathcal{N}(T) = \{0\}$.

Suppose $\mathcal{N}(T) = \{0\}$. If $T(v) = T(w)$, then $T(v) - T(w) = T(v - w) = 0$, which means $v - w = 0$. ☐

Example 2.5 (The transpose of matrices)**.** Define $T : \mathcal{M}_{m\times n}(\mathbb{R}) \to \mathcal{M}_{n\times m}(\mathbb{R})$ by $T(A) = A^t$. It is easy to see that T is linear and $\mathcal{N}(T) = \{0\}$. By the theorem, we see that T is one-to-one.

Theorem 2.6. *Let V and W be vector spaces over $\mathbb{R}$ with $\dim(V) = \dim(W) < \infty$ and $T : V \to W$ linear. Then, the following are equivalent:*

(i) *T is one-to-one.*

(ii) $\dim(\mathcal{R}(T)) = \dim(V) = \dim(W)$.

(iii) *T is onto.*

Proof. (i) $\Rightarrow$ (ii): By the previous theorem, $\mathcal{N}(T) = \{0\}$ so $\dim(\mathcal{N}(T)) = 0$. The result follows from Dimension Theorem.

(ii) $\Rightarrow$ (iii): Since $\mathcal{R}(T) \leq W$ and $\dim(\mathcal{R}(T)) = \dim(W)$, $\mathcal{R}(T) = W$.

(iii) $\Rightarrow$ (i): Note that $\dim(\mathcal{R}(T)) = \dim(W) = \dim(V)$. By Dimension Theorem, $\dim(\mathcal{N}(T)) = 0$ which implies $\mathcal{N}(T) = \{0\}$ and so $\mathcal{N}(T) = \{0\}$ and T is 1–1. ☐

Theorem 2.7. *Let V and W be vector spaces over $\mathbb{R}$ and $\beta = \{v_1, v_2, \cdots, v_n\}$ a basis for V. Let $w_1, w_2, \cdots, w_n \in W$, there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for each $i = 1, 2, \cdots, n$.*

Proof. For any $v \in V$, there exists a unique expression $v = a_1v_1 + \cdots + a_nv_n$ because β is a basis for V. Thus, we define

$$T(v) = T(a_1v_1 + \cdots + a_nv_n) = a_1T(v_1) + \cdots + a_nT(v_n) = a_1w_1 + \cdots + a_nw_n.$$

Note that T is well-defined. (If there are two different linear combinations of v, then T may not be well-defined.)

Since $0 = 0v_1 + \cdots + 0v_n$, $T(0) = 0$. Let $x, y \in V$ and $c \in \mathbb{R}$, then

$$x = a_1v_1 + \cdots a_nv_n, \qquad y = b_1v_1 + \cdots b_nv_n.$$

Thus,

$$\begin{aligned}
T(cx + y) &= T(c\sum_{i=1}^{n} a_iv_i + \sum_{i=1}^{n} b_iv_i) \\
&= T(\sum_{i=1}^{n}(ca_i + b_i)v_i) \\
&= \sum_{i=1}^{n}(ca_i + b_i)w_i \\
&= cT(x) + T(y).
\end{aligned}$$

This means that T is linear.

Suppose there are two linear transformations U, T such that $U(v_i) = T(v_i) = w_i$. Then for any $v = \sum_{i=1}^{n} a_iv_i$,

$$U(v) = U(\sum_{i=1}^{n} a_iv_i) = \sum_{i=1}^{n} a_iU(v_i) = \sum_{i=1}^{n} a_iw_i = \sum_{i=1}^{n} a_iT(v_i) = T(v).$$

Thus, $U = T$ as a map. □

Corollary 2.8. *Let V and W be vector spaces over $\mathbb{R}$ and $\beta = \{v_1, v_2, \cdots, v_n\}$ a basis for V. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \cdots, n$, then $U = T$.*

Example 2.9 (Rotation)**.** Let $\theta \in [0, 2\pi]$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, then it is straightforward to see that T is linear. Consider a map $U : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates points in $\mathbb{R}^2$ counterclockwise through an angle θ about the origin. In particular, we have $U(1, 0) = (\cos\theta, \sin\theta)$ and $U(0, 1) = (-\sin\theta, \cos\theta)$. Since $\{(1, 0), (0, 1)\}$ is a basis for $\mathbb{R}^2$, $T(1, 0) = U(1, 0)$, and $T(0, 1) = U(0, 1)$. Thus, we have $T = U$.

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

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