Sec 10.2: Fourier Series, part 2

Math 285 Spring 2020

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Recall

Recall that if a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right),$$

then

$$a_n = \frac{1}{L}(f, \cos \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx,$$

$$b_n = \frac{1}{L}(f, \sin \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.$$

Example

Example

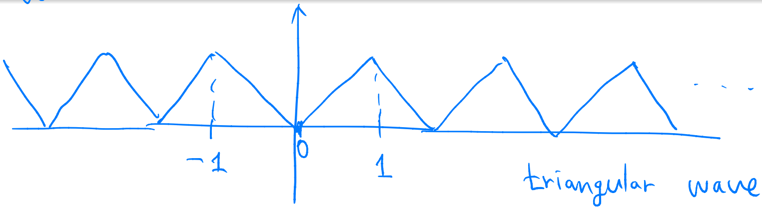
Consider a periodic function f defined by

Period $= 2L$

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

In this case, $L = 1$

and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.



triangular wave.

Assume $f(x) = \dfrac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(m\pi x) + b_m \sin(m\pi x)\right).$

Example

① $a_0 = \frac{1}{L}(f, 1) = \int_{-1}^{1} f(x)\,dx$

$= \int_0^1 x\,dx + \int_{-1}^{0} (-x)\,dx$

$= \frac{1}{2} + \frac{1}{2} = 1.$

② $a_n = \frac{1}{L}(f, \cos(n\pi x)) = \int_{-1}^{1} f(x)\cdot\cos(n\pi x)\,dx$

$= \int_0^1 x\cdot\cos(n\pi x)\,dx + \underbrace{\int_{-1}^{0} (-x)\cdot\cos(n\pi x)\,dx}_{= \int_0^1 x\cos(n\pi x)\,dx}$

by change of variable

$= 2\int_0^1 x\cdot\underbrace{\cos(n\pi x)}\,dx$

$= 2\left(\underbrace{\left[x\cdot\frac{\sin(n\pi x)}{n\pi}\right]_0^1}_{=0} - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx\right)$

$= \frac{2}{n^2\pi^2}(\cos(n\pi) - 1).$

Example

$a_0 = 1$

$$\underline{a_n} = \frac{2}{n^2\pi^2}(\cos(n\pi) - 1) \qquad (\cos(n\pi) = (-1)^n)$$

$$= \frac{2}{n^2\pi^2}((-1)^n - 1) \qquad n \geq 1$$

$$= \begin{cases} -\frac{4}{n^2\pi^2} & \text{if } n : \text{odd} \\ 0 & \text{if } n : \text{even} \end{cases}$$

③ $$b_n = \frac{1}{L}(f, \sin(n\pi x)) = \int_{-1}^{1} f(x) \sin(n\pi x)\,dx$$

$$= \underbrace{\int_0^1 x \sin(n\pi x)\,dx} + \underbrace{\int_{-1}^{0} (-x) \cdot \sin(n\pi x)\,dx}_{= -\int_0^1 x \sin(n\pi x)\,dx}$$

$= 0.$

for all $n \geq 1$.

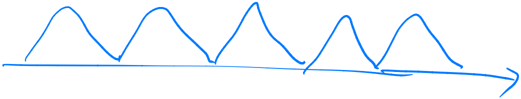
Example

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(m\pi x) + \cancelto{0}{b_m \sin(m\pi x)} \right)$$

$$= \frac{1}{2} + \sum_{\substack{m=1, \\ m:\text{odd}}}^{\infty} \left(-\frac{4}{m^2\pi^2}\right) \cos(m\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cdot \cos\left((2k-1)\pi x\right)$$

$$(\; m = 2k-1 \;) \qquad \blacksquare$$



Example

Example

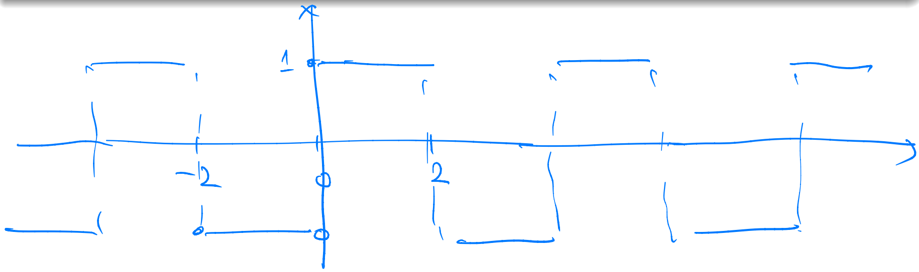
Consider a periodic function f defined by

$$f(x) = \begin{cases} 1, & 0 \leq x < 2, \\ -1, & -2 \leq x < 0, \end{cases}$$

and $f(x + 4) = f(x)$ for all $x \in \mathbb{R}$.

$4 = 2L$

$\boxed{L = 1}$



Example

① $a_0 = \frac{1}{L}(f, 1) = \frac{1}{2}\int_{-2}^{2} f(x)\, dx$

$= \frac{1}{2}\left(\int_0^2 1\, dx + \int_{-2}^{0} (-1)\, dx \right) = 0.$

② $a_n = \frac{1}{L}\left(f, \cos\left(\frac{n\pi}{L}x\right)\right) = \frac{1}{2}\int_{-2}^{2} f(x)\cdot \cos\left(\frac{n\pi}{2}x\right)dx$

$= \frac{1}{2}\left(\underbrace{\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx} + \underbrace{\int_{-2}^{0} (-1) \cos\left(\frac{n\pi x}{2}\right) dx}_{= -\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx} \right)$

$= 0$

Example

(3) $b_n = \frac{1}{L}\left(f, \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{1}{2}\left(\underbrace{\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx}\ + \underbrace{\int_{-2}^{0}(-1)\cdot\sin\left(\frac{n\pi x}{2}\right)dx}\right)$$

$$\underbrace{+\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx} \qquad x=-u$$

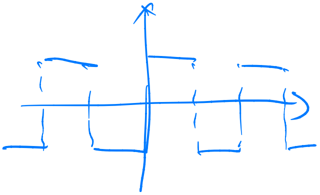
$$= \int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right)$$

$$= \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{4}{n\pi}, & n: \text{odd} \\ 0, & n: \text{even} \end{cases}$$

Example



$$a_n = 0 \quad \forall n = 0, 1, 2, \cdots$$

$$b_n = \begin{cases} \dfrac{4}{n\pi} & n: \text{odd} \\ 0 & n: \text{even} \end{cases}$$

$$f(x) = \cancel{\frac{a_0}{2}}^{\,0} + \sum_{m=1}^{\infty} \left(\cancel{a_m}^{\,0} \cos\left(\frac{m\pi x}{2}\right) + b_m \sin\left(\frac{m\pi x}{2}\right) \right)$$

$$= \sum_{\substack{m=1, \\ m:\text{odd}}}^{\infty} \frac{4}{m\pi} \sin\left(\frac{m\pi x}{2}\right)$$

$$(m = 2k-1)$$

$$= \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \cdot \sin\left(\frac{(2k-1)\pi}{2} x\right)$$