Math 416: Abstract Linear Algebra

Midterm 2 Solution, Fall 2019

Date: October 23, 2019

1. Let $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be a linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2.$$

Let $\beta = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$ be bases for $\mathcal{M}_{2\times 2}(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ respectively.

(a) (4 points) Determine $[T]_\beta^\gamma$.

(b) (3 points) Find a basis for the null space $\mathcal{N}(T)$.

(c) (3 points) Find the dimension of the range $\mathcal{R}(T)$ using the Dimension theorem.

Solution:

(a) Since $T(e_1) = 1$, $T(e_2) = 2x$, $T(e_3) = x^2$, and $T(e_4) = x^2$, the matrix representation is

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

(b) Suppose $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0$, then $a = b = c + d = 0$. Thus, we have

$$\begin{aligned} \mathcal{N}(T) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a = b = c + d = 0 \right\} \\ &= \left\{ \begin{pmatrix} 0 & 0 \\ t & -t \end{pmatrix} : t \in \mathbb{R} \right\} \\ &= \operatorname{Span}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}. \end{aligned}$$

Since $\left\{ \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$ is linearly independent (because it is not zero), it is a basis for $\mathcal{N}(T)$.

(c) By Dimension theorem, we have

$$\dim(\mathcal{M}_{2\times 2}(\mathbb{R})) = \dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)).$$

Since $\dim(\mathcal{M}_{2\times 2}(\mathbb{R})) = 4$ and $\dim(\mathcal{N}(T)) = 1$ by Part (b), we obtain $\dim(\mathcal{R}(T)) = 4 - 1 = 3$

2. Let $A = \begin{pmatrix} 3 & 7 & -2 \\ 1 & 2 & 4 \\ 1 & 2 & -1 \end{pmatrix}$.

   (a) Compute $\det(A)$ by cofactor expansion along the second row.

   (b) Compute $\det(A)$ by a different method that involves row operations.

Solution:

(a) We have

$$\begin{aligned}
\det(A) &= \det \begin{pmatrix} 3 & 7 & -2 \\ 1 & 2 & 4 \\ 1 & 2 & -1 \end{pmatrix} \\
&= (-1)^{2+1} \det \begin{pmatrix} 7 & -2 \\ 2 & -1 \end{pmatrix} + 2(-1)^{2+2} \det \begin{pmatrix} 3 & -2 \\ 1 & -1 \end{pmatrix} + 4(-1)^{2+3} \det \begin{pmatrix} 3 & 7 \\ 1 & 2 \end{pmatrix} \\
&= (-1)(-7+4) + 2(-3+2) - 4(6-7) \\
&= 3 - 2 + 4 = 5.
\end{aligned}$$

(b) We have

$$A = \begin{pmatrix} 3 & 7 & -2 \\ 1 & 2 & 4 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 4 \\ 3 & 7 & -2 \\ 1 & 2 & -1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & -14 \\ 0 & 0 & -5 \end{pmatrix} = B.$$

Since B is upper triangular, $\det(B)$ is the product of all diagonal entries, which is -5. Since the row operations only change the sign of the determinant, we conclude that $\det(A) = 5$.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ with $A \neq O$ and $A^2 = O$.
   (a) Show that A is not invertible.
   (b) Show that $\dim(\mathcal{N}(A)) = 1$.
   (c) By (b), there exists $v \in \mathbb{R}^2$ such that $v \notin \mathcal{N}(A)$. Let $\beta = \{v, Av\}$. Show that β is a basis for $\mathbb{R}^2$.
   (d) Find the matrix representation $[L_A]_\beta$.

Solution:

(a) Suppose A is invertible, then there exists A^{-1} and

$$A^{-1}A^2 = (A^{-1}A)A = A = O,$$

which contradicts the assumption.

(b) Consider $L_A : \mathbb{R}^2 \to \mathbb{R}^2$. First note that $\mathcal{N}(A) = \mathcal{N}(L_A)$ and $0 \leq \dim(\mathcal{N}(L_A)) \leq \dim(\mathbb{R}^2) = 2$ because $\mathcal{N}(L_A) \leq \mathbb{R}^2$. Since L_A is not invertible, L_A is not one-to-one. (If it is one-to-one, then Dimension theorem implies that L_A is invertible.) Thus, L_A is not one-to-one so that $\dim(\mathcal{N}(L_A)) \neq 0$. Suppose $\dim(\mathcal{N}(L_A)) = 2$, then $L_A(v) = Av = 0$ for all $v \in \mathbb{R}^2$. This implies that $A = 0$, which contradicts the assumption. Thus, we conclude that $\dim(\mathcal{N}(L_A)) = 1$.

(c) Since $v \notin \mathcal{N}(A)$ and $0 \in \mathcal{N}(A)$, we have $v \neq 0$. Since $v \notin \mathcal{N}(A)$, we have $Av \neq 0$. Let $av + bAv = 0$. Multiplying A of both sides, we get

$$A(av + bAv) = aAv + bA^2v = aAv = 0$$

and so $a = 0$. This also implies that $bAv = 0$ and so $b = 0$. Thus, it is linearly independent and so a basis for $\mathbb{R}^2$ (because the dimension of $\mathbb{R}^2$ is 2).

(d) Since $L_A(v) = Av$ and $L_A(Av) = A^2v = 0$, we have

$$[L_A]_\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

4. Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and

$$\gamma = \{v_1 = (1, -1, 0), v_2 = (0, -1, 1), v_3 = (1, 1, 1)\}$$

be another basis for $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(v_1) = v_1$, $T(v_2) = v_2$, and $T(v_3) = 0$.

(a) Write down $[T]_\beta$ in terms of $[I_{\mathbb{R}^3}]_\gamma^\beta$ and $[T]_\gamma$.

(b) Determine $[I_{\mathbb{R}^3}]_\gamma^\beta$ and $[T]_\gamma$.

(c) Show that $T^2 = T$.

Solution:

(a)

$$[T]_\beta = [I_{\mathbb{R}^3}]_\gamma^\beta [T]_\gamma [I_{\mathbb{R}^3}]_\beta^\gamma = [I_{\mathbb{R}^3}]_\gamma^\beta [T]_\gamma ([I_{\mathbb{R}^3}]_\gamma^\beta)^{-1}.$$

(b)

$$[I_{\mathbb{R}^3}]_\gamma^\beta = \begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad [T]_\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Note that $T^2(v_1) = T(T(v_1)) = T(v_1)$, $T^2(v_2) = T(T(v_2)) = T(v_2)$, and $T^2(v_3) = 0 = T(v_3)$. Since γ is a basis, the uniqueness of such linear transformations yields $T^2 = T$.

5. (10 points) Circle True or False. Do not justify your answer.

(a) **TRUE** False Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. Then, T is one-to-one if and only if $\dim(\mathcal{R}(T)) = \dim(V)$.

(b) True **FALSE** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, then the dimension of the set of all linear transformations $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is $m + n$.

(c) True **FALSE** If $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are invertible, then AB is also invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(d) **TRUE** False The vector spaces $\mathcal{M}_{2\times 3}(\mathbb{R})$ and $\mathcal{P}_5(\mathbb{R})$ are isomorphic.

(e) **TRUE** False If T and S are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^4$ such that $T(1, 0) = S(1, 0)$ and $T(2, 3) = S(2, 3)$, then $T = S$.