Math 285 Midterm 3 Free Response Questions

Due: 4/29 (Wed) at 8 pm

1. Consider the two point boundary problem

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0 \tag{A}$$

with $y(1) = -1$ and $y(2) = 1$.

(a) (5 points) Let $x = e^t$. Show that the equation can be written as

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0.$$

(b) (5 points) Find the general solution to the equation (A).

(c) (5 points) Find the solution to the two point boundary problem.

Solution:

(a) Let $x = e^t$. Note that $\frac{dx}{dt} = x$. By the Chain rule,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = x\frac{dy}{dx}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(x\frac{dy}{dx}) = \frac{dx}{dt}\frac{dy}{dx} + x\frac{d}{dt}(\frac{dy}{dx}) = x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.$$

(b) Since the characteristic equation of $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$ is $\lambda^2 - 3\lambda + 2 = 0$ and its roots are $\lambda = 1, 2$, the general solution is

$$y(x) = y(e^t) = c_1e^t + c_2e^{2t} = c_1x + c_2x^2.$$

(c) By the boundary conditions, we have $y(1) = c_1 + c_2 = -1$ and $y(2) = 2c_1 + 4c_2 = 1$. Solving for c_1 and c_2, we get $c_1 = -5/2$ and $c_2 = 3/2$.

2. Consider the heat conduction equation $4u_{xx} = u_t$ with $u(0,t) = u_x(\pi, t) = 0$ and $u(x, 0) = x$.

(a) (3 points) Find a pair of two ordinary differential equations using the method of separation of variables $u(x,t) = X(x)T(t)$.

(b) (3 points) Find the boundary condition for $X(x)$.

(c) (9 points) Show that

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-(2m-1)^2 t} \sin\left(\frac{(2m-1)x}{2}\right)$$

is a solution for some C_m.

Solution:

(a) Let $u(x,t) = X(x)T(t)$, then $4X''T = XT' = 0$. Dividing by XT, we have

$$\frac{X''}{X} = \frac{1}{4}\frac{T'}{T} = -\lambda.$$

Thus, $X'' + \lambda X = 0$ and $T' + 4\lambda T = 0$.

(b) Since $u(0,t) = u_x(\pi,t) = 0$, we have $X(0)T(t) = X'(\pi)T(t) = 0$. This leads to $X(0) = X'(\pi) = 0$.

(c) Suppose $\lambda = -\mu^2 < 0$, then $X(x) = c_1 \cosh(\mu x) + c_2 \sinh(\mu x)$. By the boundary conditions, $c_1 = c_2 = 0$. (2 points)

If $\lambda = 0$, then $X(x) = c_1 + c_2 x$. By the boundary conditions, $c_1 = c_2 = 0$. (2 points)

Assume that $\lambda = \mu^2 > 0$, then $X(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$. By the boundary conditions, $c_1 = 0$ and $c_2 \mu \cos(\mu x) = 0$. If $c_2 \neq 0$, then $\cos(\mu\pi) = 0$ which is equivalent to

$$\mu\pi = (n - \frac{1}{2})\pi.$$

Therefore, for each $n \in \mathbb{N}$, $\lambda_n = (n - \frac{1}{2})^2$ and

$$X_n(x) = C \sin\left(\frac{(2n-1)x}{2}\right).$$

(2 points)

For each n, the general solution of $T' + 4\lambda_n T = 0$ is $T_n(t) = Ce^{-(2n-1)^2 t}$. (2 points)

By the superposition property,

$$u(x,t) = \sum_{m=1}^{\infty} X_m(x)T_m(t) = \sum_{m=1}^{\infty} C_m e^{-(2m-1)^2 t} \sin\left(\frac{(2m-1)x}{2}\right)$$

is a solution for some C_m. (1 points)