Homework 7

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 1, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

   (a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row.

   (b) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ along the second row.

   (c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row.

Solution:

(a)

$$\begin{aligned} \det \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix} &= (-1)^{1+2} \det \begin{pmatrix} -1 & -3 \\ 2 & 0 \end{pmatrix} + 2(-1)^{1+3} \det \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \\ &= -6 - 6 = -12 \end{aligned}$$

(b)

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} &= 4(-1)^{2+1} \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 5(-1)^{2+2} \det \begin{pmatrix} 1 & 3 \\ 7 & 9 \end{pmatrix} + 6(-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} \\ &= -4(18-24) + 5(9-21) - 6(8-14) \\ &= 24 - 60 + 36 = 0 \end{aligned}$$

(c)

$$\begin{aligned} \det \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix} &= (-1)(-1)^{3+1} \det \begin{pmatrix} 0 & 2 \\ 1 & 5 \end{pmatrix} + 3(-1)^{3+2} \det \begin{pmatrix} 1 & 2 \\ 0 & 5 \end{pmatrix} \\ &= 2 - 15 = -13 \end{aligned}$$

2. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

Solution: Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ be an upper triangular matrix. Use an induction on n. If $n = 1$, it is trivial. Suppose $n \geq 2$ and the result holds for $n-1$. By the cofactor expansion along the last row, we have

$$\det(A) = (-1)^{n+n} A_{nn} \det(\widetilde{A}_{nn}) = A_{nn} \det(\widetilde{A}_{nn}).$$

Note that for each $1 \leq i, j \leq n-1$, the (i, j) entry of A is the same as that of $\widetilde{A}_{nn}$. This implies that $\widetilde{A}_{nn}$ is upper triangular. By the induction hypothesis, we get

$$\begin{aligned} \det(A) &= A_{nn} \det(\widetilde{A}_{nn}) \\ &= A_{nn} (\widetilde{A}_{nn})_{11} \cdots (\widetilde{A}_{nn})_{n-1,n-1} \\ &= A_{11} A_{22} \cdots A_{nn}. \end{aligned}$$

3. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

Solution: Suppose that E is obtained by $R_i \leftrightarrow R_j$ from I_n, then $E_{kk} = 1$ for $k \neq i, j$, $E_{ij} = E_{ji} = 1$, and others are zero. Thus, $E^t = E$ and so $\det(E^t) = \det(E)$.

Suppose that E is obtained by $R_i \to cR_i$ from I_n for some nonzero constant c. Then, E is diagonal so that $E^t = E$ and $\det(E^t) = \det(E)$.

Suppose that E is obtained by $R_i \to R_i + cR_j$ from I_n for some nonzero constant c. Then, E^t is an elementary matrix obtained by $R_j \to R_j + cR_i$ from I_n. Thus, $\det(E^t) = 1 = \det(E)$.

4. A matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ is called nilpotent if $M^k = O$ for some integer k. Prove that if M is nilpotent, then $\det(M) = 0$.

Solution: We claim that $\det(M^k) = (\det(M))^k$. If $k = 1$, it is trivial. Suppose $k \geq 2$ and the claim is true for $k - 1$. Then, $M^k = MM^{k-1}$ and so

$$\det(M^k) = \det(MM^{k-1}) = \det(M)\det(M^{k-1}) = \det(M)\det(M)^{k-1} = \det(M)^k.$$

Since $\det(M^k) = 0$, we conclude that $\det(M) = 0$.

5. Prove that if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar, then $\det(A) = \det(B)$.

Solution: If $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are similar, then there exists an invertible Q such that $A = Q^{-1}BQ$. Thus,

$$\begin{aligned} \det(A) &= \det(Q^{-1}BQ) \\ &= \det(Q^{-1})\det(B)\det(Q) \\ &= \det(Q)^{-1}\det(B)\det(Q) \\ &= \det(B). \end{aligned}$$

6. Let $1 \le k < n$. Suppose that $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I_{n-k} \end{pmatrix}$$

where $A \in \mathcal{M}_{k\times k}(\mathbb{R})$ and $B \in \mathcal{M}_{k\times(n-k)}(\mathbb{R})$. Prove that $\det(M) = \det(A)$.

Solution: By the cofactor expansion along the bottom row, we get

$$\begin{aligned} \det \begin{pmatrix} A & B \\ O & I_{n-k} \end{pmatrix} &= (-1)^{n+n} \det \begin{pmatrix} A & B' \\ O & I_{n-k-1} \end{pmatrix} \\ &= \det \begin{pmatrix} A & B' \\ O & I_{n-k-1} \end{pmatrix} \end{aligned}$$

where $B' \in \mathcal{M}_{k\times(n-k-1)}(\mathbb{R})$. Repeating this procedure, we see

$$\begin{aligned} \det \begin{pmatrix} A & B \\ O & I_{n-k} \end{pmatrix} &= \det \begin{pmatrix} A & B' \\ O & I_{n-k-1} \end{pmatrix} \\ &= (-1)^{(n-1)+(n-1)} \det \begin{pmatrix} A & B'' \\ O & I_{n-k-2} \end{pmatrix} \\ &= \cdots \\ &= \det \begin{pmatrix} A & b \\ O & 1 \end{pmatrix} \\ &= \det(A). \end{aligned}$$

7. Let $1 \le k < n$. Suppose that $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$$

where $A \in \mathcal{M}_{k\times k}(\mathbb{R})$, $B \in \mathcal{M}_{k\times(n-k)}(\mathbb{R})$, and $C \in \mathcal{M}_{(n-k)\times(n-k)}(\mathbb{R})$. Prove that $\det(M) = \det(A)\det(C)$.

Solution: Suppose that C is not invertible, then $\det(C) = 0$ and the rows of C are linearly dependent. This implies that the rows of M are also linearly dependent. Thus, $\det(M) = 0 = \det(A)\det(C)$.

Suppose that C is invertible. Then there exists a sequence of row operations that transforms C to I_{n-k}. Say

$$C \xrightarrow{\mathcal{R}} I_{n-k}.$$

Note that along this sequence of row operations, the determinant is multiplied by $\det(C)$. By applying the sequence of row operations $\mathcal{R}$ to M, one can see that

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \xrightarrow{\mathcal{R}} \begin{pmatrix} A & \widetilde{B} \\ O & I_{n-k} \end{pmatrix} =: N$$

where $\widetilde{B} \in \mathcal{M}_{k\times(n-k)}(\mathbb{R})$, and $\det(M) = \det(C)\det(N)$. (Note that there is no change on A because all entries below A are zero, whereas B has been changed to $\widetilde{B}$ along $\mathcal{R}$). By Problem 6, we have $\det(N) = \det(A)$. Thus we conclude that $\det(M) = \det(A)\det(C)$ as desired.

8. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ have the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Compute $\det(A + tI_n)$.

Solution: We observe that if $n = 2$ then

$$\det(A + tI_2) = \det \begin{pmatrix} t & a_0 \\ -1 & t + a_1 \end{pmatrix} = t(t + a_1) + a_0 = t^2 + a_1 t + a_0$$

and if $n = 3$ then

$$\begin{aligned} \det(A + tI_2) &= \det \begin{pmatrix} t & 0 & a_0 \\ -1 & t & a_1 \\ 0 & -1 & t + a_2 \end{pmatrix} \\ &= t \det \begin{pmatrix} t & a_1 \\ -1 & t + a_2 \end{pmatrix} + a_0 \det \begin{pmatrix} -1 & t \\ 0 & -1 \end{pmatrix} \\ &= t(t(t + a_2) + a_1) + a_0 \\ &= t^3 + a_2 t^2 + a_1 t + a_0. \end{aligned}$$

From these observation, we claim that $\det(A + tI_n) = t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0$. We use an induction on n. We have seen that the claim is true for $n = 1, 2, 3$. Suppose $n \geq 4$ and it is true for $n - 1$. By the cofactor expansion along the first row, we get

$$\begin{aligned} \det(A + tI_n) &= \det \begin{pmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & t + a_{n-1} \end{pmatrix} \\ &= t \det \begin{pmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -1 & t + a_{n-1} \end{pmatrix} + (-1)^{1+n} a_0 \det \begin{pmatrix} -1 & t & 0 & \cdots & 0 \\ 0 & -1 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \\ &= t(t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1) + (-1)^{n+1} a_0 \det(B). \end{aligned}$$

where

$$B = \begin{pmatrix} -1 & t & 0 & \cdots & 0 \\ 0 & -1 & t & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & t \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \in \mathcal{M}_{(n-1)\times(n-1)}(\mathbb{R}).$$

Since B is upper triangular, it follows from Problem 2 that $\det(B) = (-1)^{n-1}$. Therefore, we obtain

$$\begin{aligned} f(t) &= t(t^{n-1} + a_{n-1}t^{n-2} + \cdots + a_2 t + a_1) + (-1)^{n+1} a_0 \det(B) \\ &= t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + (-1)^{2n} a_0 \\ &= t^n + a_{n-1}t^{n-1} + \cdots + a_2 t^2 + a_1 t + a_0. \end{aligned}$$