Homework 3

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 20, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Find the solution sets to the following linear systems.

   (a) $\begin{cases} 2x_1 - 3x_2 + x_3 + 7x_4 = 14 \\ 2x_1 + 8x_2 - 4x_3 + 5x_4 = -1 \\ x_1 + 3x_2 - 3x_3 = 4 \\ -5x_1 + 2x_2 + 3x_3 + 4x_4 = -19 \end{cases}$

   (b) $\begin{cases} 2x_1 + 4x_2 + 5x_3 + 7x_4 = -26 \\ x_1 + 2x_2 + x_3 - x_4 = -4 \\ -2x_1 - 4x_2 + x_3 + 11x_4 = -10 \end{cases}$

   (c) $\begin{cases} 2x_1 + x_2 + 7x_3 - 2x_4 = 4 \\ 3x_1 - 2x_2 + 11x_4 = 13 \\ x_1 + x_2 + 5x_3 - 3x_4 = 1 \end{cases}$

2. Determine whether the two matrices are row-equivalent.

   (a) $\begin{pmatrix} 1 & 4 & 3 & -1 & 5 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 1 & 6 & 5 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

   (b) $\begin{pmatrix} 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ 1 & -2 & -2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$

3. Let V be a vector space over $\mathbb{R}$ and S a subset of V. Show that S is linearly independent if and only if for any $v_1, \cdots, v_n \in S$,

$$x_1 v_1 + \cdots + x_n v_n = 0 \quad \Rightarrow \quad x_1 = \cdots = x_n = 0.$$

4. Let $n \geq 2$ and $V = \mathbb{R}^n$. Define $v_1, v_2, \cdots, v_n \in V$ by $v_n = (1, 0, \cdots, 0)$ and for $i = 2, \cdots, n$, the i-th entry of v_i are 1 and j-th entry is zero for each $j > i$. That is,

$$\begin{aligned} v_1 &= (1, 0, \cdots, 0), \\ v_2 &= (a_{12}, 1, 0, \cdots, 0), \\ &\vdots \\ v_n &= (a_{1n}, \cdots, a_{n-1,n}, 1). \end{aligned}$$

(a) Show that $S = \{v_1, \cdots, v_n\}$ is linearly independent.

(b) Show that $S = \{v_1, \cdots, v_n\}$ generates V.

5. Let $n \geq 1$ and $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Let $S = \{\sin(2^k x) : k = 1, 2, \cdots, n\}$ be a subset of V. Show that S is linearly independent.

6. Let V be a vector space over $\mathbb{R}$ and $u, v \in V$ with $u \neq v$. Show that $\{u, v\}$ is linearly dependent if and only if $u = c_1 v$ or $v = c_2 u$ for some $c_1, c_2 \in \mathbb{R}$.

7. Let V be a vector space over $\mathbb{R}$.

   (a) Let $u, v \in V$ and $u \neq v$. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.

   (b) Let $n \in \mathbb{N}$. Let S be the set of n distinct elements $v_1, \cdots, v_n$ in V (that is, $v_i \neq v_j$ for all $i \neq j$). Let $A = (A_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R})$ and define $T = \{w_1, \cdots, w_n\}$ where

   $$w_j = A_{1j} v_1 + \cdots + A_{nj} v_n$$

   for each $j = 1, 2, \cdots, n$. Suppose S is linearly independent. Show that T is linearly independent if and only if the linear system associated to $(A, 0)$ has exactly one (trivial) solution.

8. Let W be the set of all (2×2) matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$. Find a basis for W.

9. Find bases for the following subspaces of $\mathbb{R}^5$.

   (a) $W_1 = \{(x_1, x_2, x_3, x_4, x_5) : x_1 - x_3 - x_4 = 0\}$

   (b) $W_2 = \{(x_1, x_2, x_3, x_4, x_5) : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$

10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $W_1, W_2 \leq V$. Show that if $W_1 \cap W_2 = \{0\}$, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. (Recall that $W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$ is a subspace of V.)