Sec 10.1: Two-Point Boundary Value Problems, part 2

Math 285 Spring 2020

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Recall

Consider a boundary value problem

$$y'' + p(x)y' + q(x)y = g(x)$$

with $y(\alpha) = y_0$ and $y(\beta) = y_1$, $\alpha < \beta$.

It is called *homogeneous* if $g(x) = 0$ and $y_0 = y_1 = 0$. Otherwise, it is called *nonhomogeneous*.

If a boundary value problem is nonhomogeneous, it has either (i) a unique solution, (ii) infinitely many solutions, or (iii) no solutions.

If it is homogeneous, the problem always has a trivial solution $\underline{y = 0.}$ So, it has either (i) a unique (trivial) solution or (ii) infinitely many (nontrivial) solutions.

$$\begin{cases} y'' + p(x)y' + q(x)y = 0 \\ y(\alpha) = y(\beta) = 0 \end{cases}$$

Eigenvalues and Eigenfunctions

Consider $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ for $L > 0$

Definition

We call λ is an *eigenvalue* of the boundary value problem if it has nontrivial solutions. The solutions are called the corresponding *eigenfunctions*.

Our goal is to find all eigenvalues and eigenfunctions.

Case 1: $\lambda > 0$

Let $\lambda = \mu^2 > 0$.

$y'' + \mu^2 y = 0$.

$r^2 + \mu^2 = 0$

$r = \pm \mu i$

$\Rightarrow y(t) = \underline{C_1 \cos \mu t + C_2 \sin \mu t}$

$y(0) = C_1 = 0$

$y(L) = \underbrace{C_2}_{\neq 0} \underbrace{\sin(\mu L)}_{=0} = 0$

<u>Goal</u>: Find λ (or μ) s.t. $\begin{cases} y'' + \mu^2 y = 0 \\ y(0) = y(L) = 0 \end{cases}$

has <u>nontrivial</u> solutions

$\Rightarrow C_2 \neq 0$

$\Rightarrow \sin(\mu L) = 0$

Case 1: $\lambda > 0$

$\sin(\mu L) = 0 \Rightarrow \mu L$ = a multiple of π

$= m \cdot \pi$

where m is an integer.

$(m \in \mathbb{Z})$

$$\mu = \frac{m\pi}{L}$$

Thus $\lambda = \mu^2 = \dfrac{m^2 \pi^2}{L^2}$ are eigenvalues

and the corresponding eigenfunctions are

$$y(t) = C \cdot \sin(\mu x) = C \cdot \sin\left(\frac{m\pi}{L} x\right).$$

Case 2: $\lambda < 0$

Let $\lambda = -\mu^2 < 0$.

$y'' - \mu^2 y = 0$

$r^2 - \mu^2 = 0$

$r = \pm\mu$

$y(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$

$= \tilde{C}_1 \cosh(\mu x) + \tilde{C}_2 \sinh(\mu x)$

where $\begin{cases} \cosh(\mu x) = \frac{1}{2}(e^{\mu x} + e^{-\mu x}) \\ \sinh(\mu x) = \frac{1}{2}(e^{\mu x} - e^{-\mu x}) \end{cases}$.

Note

$\{\cosh(\mu x), \sinh(\mu x)\}$ is also a fundamental set of solutions.

$\sinh(0) = 0$

$\cosh(0) = 1$.

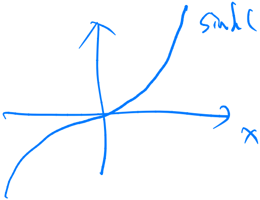
Case 2: $\lambda < 0$

$y(x) = C_1 \cosh(\mu x) + C_2 \sinh(\mu x)$

$y(0) = C_1 = 0$

$y(L) = C_2 \sinh(\mu L) = 0$

Nontrivial Solution $\Rightarrow$ $C_2 \neq 0$ $\Rightarrow$ $\sinh(\mu L) = 0.$



$\Rightarrow$ $\underbrace{\mu}_{>0} \underbrace{L}_{>0} = 0$ Contradiction.

therefore $\begin{cases} y'' - \mu^2 y = 0 \\ y(0) = y(L) = 0 \end{cases}$ has a unique

solution. $\Rightarrow$ No Eigenvalues for $\lambda < 0$.

Case 3: $\lambda = 0$

$$y'' = 0$$

$$y(x) = C_1 + C_2 t$$

$$y(0) = C_1 = 0$$

$$y(L) = C_2 L = 0$$

Nontrivial Solution $\Rightarrow$ $C_2 \neq 0 \Rightarrow \overset{>0}{L} = 0$

Contradiction.

$$\begin{cases} y'' = 0 \\ y(0) = y(L) = 0 \end{cases}$$ has a unique solution (trivial)

$\lambda = 0$ is NOT an eigenvalue.

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(L) = 0 \end{cases}$$

Eigenvalues : $\dfrac{m^2\pi^2}{L^2}$ $(m = 1, 2, 3, \cdots)$

Eigenfunctions : $C_1 \sin\left(\dfrac{m\pi}{L} x\right)$.

Remark

In general, let $L[y]$ be a differential operator.

For example, $L[y] = -y''$ or $L[y] = -x^2y'' + 2xy'$.

Suppose boundary conditions are given by $y(\alpha) = y_0$ (or $y'(\alpha) = y_0$) and $y(\beta) = y_1$ (or $y'(\beta) = y_1$). $\alpha < \beta$

Then, λ is an eigenvalue of $L[y]$ with the boundary conditions if $\underline{L[y] = \lambda y}$ with the boundary conditions has <u>nontrivial solutions</u>.

<u>Ex</u> $L[y] = -y''$, $L[y] = \lambda y$

$$-y'' = \lambda y$$

$$y'' + \lambda y = 0.$$

$\underline{L[y]} = \underline{-x^2y'' + 2xy' = \lambda} y$