Homework 8

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 8, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$.

   (a) Find the characteristic polynomial of A.

   (b) Determine all the eigenvalues of A.

   (c) For each eigenvalue λ, find E_λ.

   (d) If possible, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

Solution:

(a)

$$\begin{aligned}
f(t) &= \det(A - tI_n) \\
&= \det \begin{pmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{pmatrix} \\
&= -t \det \begin{pmatrix} 1-t & -1 \\ 2 & 5-t \end{pmatrix} + 2 \det \begin{pmatrix} -1 & -1 \\ 2 & 5-t \end{pmatrix} - 3 \det \begin{pmatrix} -1 & 1-t \\ 2 & 2 \end{pmatrix} \\
&= -t((1-t)(5-t)+2) + 2(t-5+2) - 3(-2+2(t-1)) \\
&= -t(t^2-6t+7) + 2t - 6 - 6t + 12 \\
&= -t^3 + 6t^2 - 11t + 6 \\
&= -(t^3 - 6t^2 + 11t - 6).
\end{aligned}$$

(b) To solve the equation $f(t) = 0$,

$$f(t) = -t^3 + 6t^2 - 11t + 6 = -(t-1)(t^2-5t+6) = -(t-1)(t-2)(t-3).$$

Thus, the eigenvalues of A are $1, 2, 3$.

(c) For $\lambda = 1$, we have

$$A - I_n = \begin{pmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$E_1 = \mathcal{N}(A - I_n) = \{t(1, 1, -1) : t \in \mathbb{R}\}.$$

For $\lambda = 2$, we have

$$A - 2I_n = \begin{pmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$E_2 = \mathcal{N}(A - 2I_n) = \{t(1, -1, 0) : t \in \mathbb{R}\}.$$

For $\lambda = 3$, we have

$$A - 3I_n = \begin{pmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$E_3 = \mathcal{N}(A - 3I_n) = \{t(1, 0, -1) : t \in \mathbb{R}\}.$$

(d) Let $\gamma = \{v_1 = (1, 1, -1), v_2 = (1, -1, 0), v_3 = (1, 0, -1)\}$ and $T = L_A$, then $T(v_1) = v_1$, $T(v_2) = 2v_2$, and $T(v_3) = 3v_3$. Since A has three distinct eigenvalues, γ is a basis for $\mathbb{R}^3$ and A is diagonalizable. In fact, $[L_A]_\gamma = \operatorname{diag}(1, 2, 3)$ and

$$A = [L_A]_\beta = [I_{\mathbb{R}^3}]_\gamma^\beta [L_A]_\gamma [I_{\mathbb{R}^3}]_\beta^\gamma.$$

Thus, $D = [L_A]_\gamma = \operatorname{diag}(1, 2, 3)$ and

$$Q = [I_{\mathbb{R}^3}]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix}.$$

2. Let $A, D \in \mathcal{M}_{n\times n}(\mathbb{R})$ and D be a diagonal matrix.
   (a) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ for some $d_1, d_2, \cdots, d_n \in \mathbb{R}$. Show that $D^k = \operatorname{diag}(d_1^k, d_2^k, \cdots, d_n^k)$ for all integers $k \geq 1$.
   (b) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ for some $d_1, d_2, \cdots, d_n \in \mathbb{R} \setminus \{0\}$. Show that D is invertible and $D^{-1} = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \cdots, d_n^{-1})$.
   (c) Suppose there exists an invertible matrix Q and $A = QDQ^{-1}$. Show that $A^k = QD^kQ^{-1}$ for all integers $k \geq 1$.
   (d) Suppose A is invertible and there exists an invertible matrix Q and $A = QDQ^{-1}$. Show that $A^{-1} = QD^{-1}Q^{-1}$.

Solution:

(a) Use an induction on k. If $k = 1$, then it is trivial. Suppose that $k \geq 2$ and the result holds for

$k-1$. Then,

$$\begin{aligned}(D^k)_{ij} &= (DD^{k-1})_{ij} \\ &= \sum_{k=1}^{n} (D)_{ik}(D^{k-1})_{kj} \\ &= (D)_{ii}(D^{k-1})_{ij} + (D)_{ij}(D^{k-1})_{jj}\end{aligned}$$

Since D and D^{k-1} are diagonal, if $i \neq j$ then $(D^k)_{ij} = 0$. If $i = j$, then

$$(D^k)_{ii} = (D)_{ii}(D^{k-1})_{ii} = d_i^k$$

by the induction hypothesis.

(b) Let $C = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \cdots, d_n^{-1})$, then

$$DC = \operatorname{diag}(d_1, d_2, \cdots, d_n) \operatorname{diag}(d_1^{-1}, d_2^{-1}, \cdots, d_n^{-1}) = I.$$

Thus, D is invertible and $C = D^{-1}$.

(c) Use an induction on k. If $k = 1$, then it is trivial. Suppose that $k \geq 2$ and the result holds for $k-1$. Then,

$$A^k = AA^{k-1} = (QDQ^{-1})(QD^{k-1}Q^{-1}) = QD(Q^{-1}Q)D^{k-1}Q^{-1} = QD^kQ^{-1}.$$

(d) Let $B = QD^{-1}Q^{-1}$. It follows that

$$AB = (QDQ^{-1})(QD^{-1}Q^{-1}) = QD(Q^{-1}Q)D^{-1}Q^{-1} = Q(DD^{-1})Q^{-1} = QQ^{-1} = I.$$

3. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and λ_1, λ_2 be two distinct eigenvalues for A. Let $E_{\lambda_1}, E_{\lambda_2}$ be the eigenspaces of A corresponding to λ_1, λ_2 respectively. Prove that $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

Solution: It is obvious that $\{0\} \subseteq E_{\lambda_1} \cap E_{\lambda_2}$. Suppose $v \in E_{\lambda_1} \cap E_{\lambda_2}$. Then, $Av = \lambda_1 v = \lambda_2 v$ and so $(\lambda_1 - \lambda_2)v = 0$. Since $\lambda_1 - \lambda_2 \neq 0$, we conclude that $v = 0$. Thus, $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

4. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ linear. Let β be a basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

Solution: Suppose that λ is an eigenvalue of T. Then, there exists a nonzero vector v such that $T(v) = \lambda v$. Thus,

$$[T]_\beta [v]_\beta = [T(v)]_\beta = \lambda [v]_\beta.$$

Since $[v]_\beta \neq 0$, λ is an eigenvalue of $[T]_\beta$.

Suppose λ is an eigenvalue of $[T]_\beta$ for some basis β for V. Then, there exists a nonzero vector $w = (a_1, \cdots, a_n)$ such that $[T]_\beta w = \lambda w$. Define $v = a_1 v_1 + \cdots + a_n v_n$ where $\beta = \{v_1, v_2, \cdots, v_n\}$. Then, $[T]_\beta w = \lambda w$ can be written as

$$T(v) = \lambda v,$$

which implies that λ is an eigenvalue of T.

5. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and v be an eigenvector of A corresponding to an eigenvalue λ. Show that v is an eigenvector of A^k corresponding to an eigenvalue λ^k for all integers $k \geq 1$.

Solution: Use an induction on k. If $k = 1$, then it is trivial. Suppose that $k \geq 2$ and the result holds for $k-1$. Then,

$$A^k v = A(A^{k-1}v) = A(\lambda^{k-1}v) = \lambda^{k-1}Av = \lambda^k v.$$

6. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find an expression for A^k for all integers $k \geq 1$.

Solution: The characteristic polynomial is

$$f(t) = \det(A - tI_2) = \det \begin{pmatrix} 1-t & 4 \\ 2 & 3-t \end{pmatrix} = (t-1)(t-3) - 8 = t^2 - 4t - 5 = (t+1)(t-5).$$

Thus -1 and 5 are the eigenvalues of A. Since

$$A + I_2 = \begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

and

$$A - 5I_2 = \begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

the eigenspaces are $E_{-1} = \{t(2,-1) : t \in \mathbb{R}\}$ and $E_5 = \{t(1,1) : t \in \mathbb{R}\}$. Let $\gamma = \{(2,-1),(1,1)\}$, then γ is a basis for $\mathbb{R}^2$ and

$$A = [L_A]_\beta = [I]_\gamma^\beta [L_A]_\gamma ([I]_\gamma^\beta)^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1}$$

where β is the standard basis for $\mathbb{R}^2$. By Problem 2, we have

$$\begin{aligned} A^k &= \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}^k \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \\ &= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & 5^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2(-1)^k & 5^k \\ -(-1)^k & 5^k \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 5^k + 2(-1)^k & 2(5^k) - 2(-1)^k \\ 5^k - (-1)^k & 2(5^k) + (-1)^k \end{pmatrix}. \end{aligned}$$