Math 416 Lecture Note: Week 16

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1 Self-adjoint operators

Definition 1.1. Let V be an inner product space over F and $T : V \to V$ linear. We say that T is self-adjoint when $F = \mathbb{R}$ (or Hermitian when $F = \mathbb{C}$) if $T = T^*$. A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ (or $A \in \mathcal{M}_{n\times n}(\mathbb{C})$) is self-adjoint (or Hermitian) if $A = A^*$.

We simply use the terminology "self-adjoint" for both cases $F = \mathbb{R}$ and $F = \mathbb{C}$.

Remark 1.2. If T is self-adjoint, then T is normal.

Example 1.3. Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ and $T = L_A$, then T is self-adjoint.

Theorem 1.4. *Let V be an inner product space over F and $T : V \to V$ linear. If T is self-adjoint, then every eigenvalue of T is real.*

Proof. Let $T(x) = \lambda x$ for $\lambda \in F$ and $x \in V$ with $x \neq 0$. Then,

$$\lambda \langle x, x\rangle = \langle T(x), x\rangle = \langle x, T^*(x)\rangle = \langle x, T(x)\rangle = \langle x, \lambda x\rangle = \overline{\lambda} \langle x, x\rangle .$$

Thus, we have $(\lambda - \overline{\lambda}) \langle x, x\rangle = 0$. Since $x \neq 0$, we have $\langle x, x\rangle \neq 0$ and so $\lambda = \overline{\lambda}$, which implies that $\lambda \in \mathbb{R}$. $\square$

Lemma 1.5. *Let V be an inner product space over $\mathbb{R}$ and $T : V \to V$ linear. If T is self-adjoint, then the characteristic polynomial $f(t)$ of T splits over $\mathbb{R}$. As a consequence, there exists at least one eigenvector v for T with a real eigenvalue λ.*

Proof. Let β be an orthonormal basis for V and $A = [T]_\beta$. Consider $L_A : \mathbb{C} \to \mathbb{C}$. Then the characteristic polynomial of L_A is the same as that of T, say $f(t)$. By the fundamental theorem of algebra, $f(t)$ splits over $\mathbb{C}$. Since the roots of $f(t)$ should be real by the previous theorem, $f(t)$ splits over $\mathbb{R}$. $\square$

Example 1.6. Let $A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$ and $T = L_A$. The characteristic polynomial is

$$f(t) = (1 - t)(2 - t) - 1 = t^2 - 3t + 1 = (t - \frac{3 + \sqrt{5}}{2})(t - \frac{3 - \sqrt{5}}{2}).$$

Thus, the eigenvalues are real. In fact, there are two distinct eigenvalues so that A is diagonalizable.

An interesting example of self-adjoint operators is the projection operator.

Proposition 1.7. *Let V be a finite dimensional inner product space over F and $W \leq V$. Then $T = \mathrm{proj}_W : V \to V$ is self-adjoint.*

Proof. If $x \in V$, then it can be written as $x = \mathrm{proj}_W(x) + z$ uniquely, where $z \in W^\perp$. In particular,

$$\langle x, w\rangle = \langle \mathrm{proj}_W(x), w\rangle$$

for all $w \in W$. Thus, for all $x, y \in V$,

$$\langle \mathrm{proj}_W(x), y\rangle = \langle \mathrm{proj}_W(x), \mathrm{proj}_W(y)\rangle = \langle x, \mathrm{proj}_W(y)\rangle .$$

$\square$

Theorem 1.8. *Let V be an inner product space over $\mathbb{R}$ and $T : V \to V$ linear. Then, T is self-adjoint if and only if there exists an orthonormal basis β for V consisting of eigenvectors of T.*

Proof. Suppose that there exists an orthonormal basis β for V consisting of eigenvectors of T. Then, $[T]_\beta$ is diagonal and $[T^*]_\beta = ([T]_\beta)^*$ is also diagonal. Since every diagonal entry is an eigenvalue and so they are real, we get $[T^*]_\beta = ([T]_\beta)^* = [T]_\beta$. Thus, T is self-adjoint.

Suppose that T is self-adjoint. By Schur's theorem, there exists an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$ such that $[T]_\beta$ is upper triangular. Since $([T]_\beta)^* = [T^*]_\beta = [T]_\beta$, we conclude that $[T]_\beta$ is diagonal. □

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

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