Math 416: Abstract Linear Algebra

Midterm 2, Fall 2019

Date: October 23, 2019

NAME:______________________________

READ THE FOLLOWING INFORMATION.

- This is a 50-minute exam.
- This exam contains 7 pages (including this cover page) and 5 questions. Total of points is 50.
- Books, notes, and other aids are not allowed except for one page of cheat sheet. Collaboration is forbidden.
- Show all steps to earn full credit.
- Do not unstaple pages. Loose pages will be ignored.

Question	Points	Score
1	10	
2	8	
3	14	
4	8	
5	10	
Total:	50	

1. Let $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ be a linear transformation defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2.$$

Let $\beta = \left\{ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$ be bases for $\mathcal{M}_{2\times 2}(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$ respectively.

(a) (3 points) Determine $[T]_\beta^\gamma$.

(b) (4 points) Find a basis for the null space $\mathcal{N}(T)$.

(c) (3 points) Find the dimension of the range $\mathcal{R}(T)$ using the Dimension theorem.

2. Let $A = \begin{pmatrix} 3 & 7 & -2 \\ 1 & 2 & 4 \\ 1 & 2 & -1 \end{pmatrix}$.

   (a) (4 points) Compute $\det(A)$ by cofactor expansion along the second row.

   (b) (4 points) Compute $\det(A)$ by a different method that involves row operations.

3. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$ with $A \neq O$ and $A^2 = O$.

   (a) (2 points) Show that A is not invertible.

   (b) (5 points) Show that $\dim(\mathcal{N}(A)) = 1$.

(c) (5 points) Note that there exists $v \in \mathbb{R}^2$ such that $v \notin \mathcal{N}(A)$ by Part (b). Let $\beta = \{v, Av\}$. Show that β is a basis for $\mathbb{R}^2$.

(d) (2 points) Find the matrix representation $[L_A]_\beta$.

4. Let $\beta = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$ and

$$\gamma = \{v_1 = (1, -1, 0), v_2 = (0, -1, 1), v_3 = (1, 1, 1)\}$$

be another basis for $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(v_1) = v_1$, $T(v_2) = v_2$, and $T(v_3) = 0$.

(a) (3 points) Write down $[T]_\beta$ in terms of $[I_{\mathbb{R}^3}]_\gamma^\beta$ and $[T]_\gamma$.

(b) (2 points) Determine $[I_{\mathbb{R}^3}]_\gamma^\beta$ and $[T]_\gamma$.

(c) (3 points) Show that $T^2 = T$.

5. (10 points) Circle True or False. Do not justify your answer.

(a) True False Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and $T : V \to W$ linear. Then, T is one-to-one if and only if $\dim(\mathcal{R}(T)) = \dim(V)$.

(b) True False Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, then the dimension of the set of all linear transformations $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is $m + n$.

(c) True False If $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are invertible, then AB is also invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(d) True False The vector spaces $\mathcal{M}_{2\times 3}(\mathbb{R})$ and $\mathcal{P}_5(\mathbb{R})$ are isomorphic.

(e) True False If T and S are linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^4$ such that $T(1, 0) = S(1, 0)$ and $T(2, 3) = S(2, 3)$, then $T = S$.