Sec 10.8: Laplace's Equation, part 1

Math 285 Spring 2020

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Laplace's equation

Consider a 2-dimensional heat equation $\alpha^2(\overset{\Delta u}{\overset{\shortparallel}{\underbrace{u_{xx} + u_{yy}}_{\text{Laplacian}}}}) = u_t$.

Suppose that there exists a steady state temperature distribution, then $\lim_{t\to\infty} u_t(x, y, t) = 0$.

Let $v(x, y) = \lim_{t\to\infty} u(x, y, t)$, then it satisfies

$$\Delta u = v_{xx} + v_{yy} = 0.$$

In 1 dim

$u'' = 0$

$u(x)$: straight line.

This is called Laplace's equation.

Since there is no time dependence, we do not have initial condition.

In particular, we are interested in the Laplace's equation in a region. in $\mathbb{R}^2$.

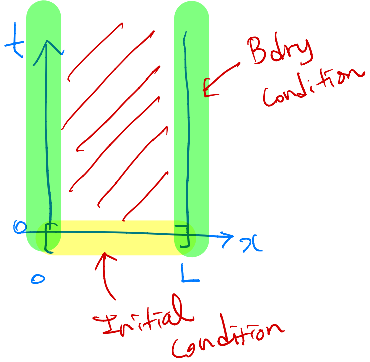
Dirichlet and Neumann Problems

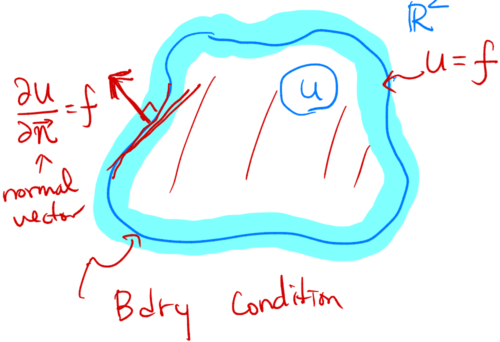
$\Delta u = u_{xx} + u_{yy} = 0$

The problem of finding a solution of Laplace's equation with prescribed function values on the boundary is called a Dirichlet problem.

The problem of finding a solution of Laplace's equation with prescribed normal derivatives on the boundary is called a Neumann problem.

In this lecture, we focus on a Dirichlet problem for a rectangle.





Dirichlet problem for a rectangle

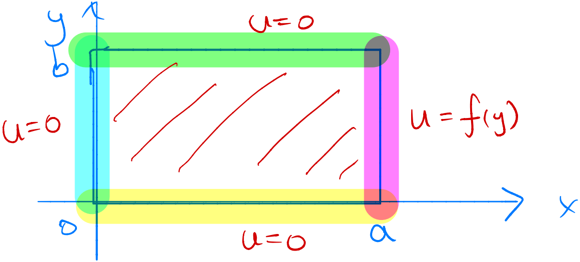
Consider

$$u_{xx} + u_{yy} = 0$$

in the rectangle $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$ with

$$u(x,0) = 0, \qquad u(x,b) = 0 \quad \text{for } 0 < x < a,$$
$$u(0,y) = 0, \qquad u(a,y) = f(y) \quad \text{for } 0 \le y \le b,$$

where f is a function on $0 \le y \le b$.



IDEA

Separation of variables.

Dirichlet problem for a rectangle

Let $u(x,y) = X(x) \cdot Y(y)$

$u_{xx} = X'' \cdot Y \qquad u_{yy} = X \cdot Y''$

$u_{xx} + u_{yy} = 0$

$X''Y + X \cdot Y'' = 0$

$\dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$

$\dfrac{X''}{X} = -\dfrac{Y''}{Y} = \lambda$

$X'' - \lambda X = 0$

$X(0) = 0$

$Y'' + \lambda Y = 0$

$u(x,0) = X(x) \cdot Y(0) = 0$

$u(x,b) = X(x) \cdot Y(b) = 0$

$\Rightarrow \quad Y(0) = Y(b) = 0$

$u(0,y) = X(0) \cdot Y(y) = 0$

Dirichlet problem for a rectangle

① $Y'' + \lambda Y = 0$, $Y(0) = Y(b) = 0$.

$$\lambda_n = \frac{n^2\pi^2}{b^2} \quad \text{for each } n \in \mathbb{N}.$$

$$Y_n(y) = \sin\left(\frac{n\pi}{b} y\right)$$

② $X'' - \lambda_n X = 0$, $X(0) = 0$

$$X_n(x) = c_1 \cosh\left(\frac{n\pi}{b} x\right) + c_2 \sinh\left(\frac{n\pi}{b} x\right)$$

$$X_n(0) = 0 \Rightarrow c_1 = 0$$

$$X_n(x) = c \cdot \sinh\left(\frac{n\pi}{b} x\right)$$

Dirichlet problem for a rectangle

$$u(x,y) = \sum_{n=1}^{\infty} X_n(x) \cdot Y_n(y)$$

$$= \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b} x\right) \cdot \sin\left(\frac{n\pi}{b} y\right)$$

$$u(a,y) = f(y)$$

$$= \sum_{n=1}^{\infty} \overbrace{\underbrace{C_n \cdot \sinh\left(\frac{n\pi}{b} a\right)}_{\text{coefficient}}}^{\text{" } C_n'} \cdot \sin\left(\frac{n\pi}{b} y\right)$$

$$C_n' = \frac{2}{L} \int_0^L f(y) \cdot \sin\left(\frac{n\pi}{b} y\right) dy$$

$$L = b.$$

Dirichlet problem for a rectangle

$$C_n' = C_n \cdot \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

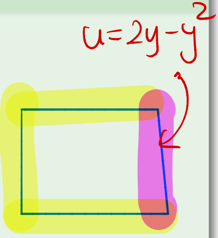
$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)dy$$

$$u(x,y) = \sum_{n=1}^{\infty} C_n \cdot \sinh\left(\frac{n\pi}{b}x\right)\cdot\sin\left(\frac{n\pi}{b}y\right)$$

Example

Consider $u_{xx} + u_{yy} = 0$ in the rectangle

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2\}$$



$u = 0$

with

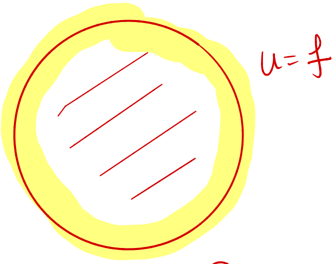
$$u(x,0) = 0, \qquad u(x,2) = 0 \quad \text{for } 0 < x < 2,$$
$$u(0,y) = 0, \qquad u(2,y) = 2y - y^2 \quad \text{for } 0 \le y \le 2.$$

Then, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{2}x\right) \sin\left(\frac{n\pi}{2}y\right)$$

where $a = b = 2$ $\quad \dfrac{\cancel{2}}{\cancel{b}\sinh\left(\frac{n\pi a}{b}\right)} = \dfrac{1}{\sinh(n\pi)}$

$$C_n = \frac{1}{\sinh(n\pi)} \int_0^2 (2y - y^2) \sin\left(\frac{n\pi}{2}y\right)\, dy = \frac{16(1-(-1)^n)}{\pi^3 n^3 \sinh(n\pi)}.$$



↓ Polar coordinates