Sec 10.7: The Wave Equation: Vibrations of an Elastic String

Math 285 Spring 2020

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Recall: Nonzero initial displacement

We consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

with boundary condition

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

① $u(x,0) = 0$
$u_t(x,0) = g(x)$

② $u(x,0) = f(x)$
$u_t(x,0) = g(x)$

Then, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

with

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

Nonzero initial velocity

We consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

with boundary condition

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = g(x).$$

Let $u(x,t) = X(x) \cdot T(t)$

$u_{xx} = X'' \cdot T$

$u_{tt} = X \cdot T''$ $\Rightarrow$ $\dfrac{X''}{X} = \dfrac{1}{a^2} \dfrac{T''}{T} = -\lambda$

Nonzero initial velocity

$$\begin{cases} X'' + \lambda X = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$$

$$u(0,t) = u(L,t) = 0$$

$$\underline{X(0)} \cdot T(t) = \underline{X(L)} \cdot T(t) = 0 \qquad \forall\, t > 0.$$

$$X(0) = X(L) = 0.$$

$$\Rightarrow \quad \lambda_n = \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, \cdots$$

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right)$$

Nonzero initial velocity

$$T'' + a^2 \lambda_n T = 0.$$

$$u(x,0) = 0$$

$$X(x) \cdot T(0) = 0 \quad \text{for all } x.$$

$$T(0) = 0$$

$$T_n(t) = A \cdot \cos\left(\frac{an\pi}{L} t\right) + B \sin\left(\frac{an\pi}{L} t\right)$$

$$T_n(0) = A = 0.$$

$$T_n(t) = C_n' \sin\left(\frac{an\pi}{L} t\right)$$

Nonzero initial velocity

$$X_n = \sin\left(\frac{n\pi}{L}x\right), \quad T_n = C_n \cdot \sin\left(\frac{an\pi}{L}t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) \cdot T_n(t)$$

$$= \sum_{n=1}^{\infty} C_n \cdot \sin\left(\frac{an\pi}{L}t\right) \cdot \sin\left(\frac{n\pi}{L}x\right)$$

$$u_t(x,0) = g(x)$$

$$u_t(x,t) = \frac{\partial}{\partial t}\left(\sum_{n=1}^{\infty} C_n \sin\left(\frac{an\pi}{L}t\right) \cdot \sin\left(\frac{n\pi}{L}x\right)\right)$$

$$= \sum_{n=1}^{\infty} \frac{\partial}{\partial t}\left(C_n \sin\left(\frac{an\pi}{L}t\right) \cdot \sin\left(\frac{n\pi}{L}x\right)\right)$$

Nonzero initial velocity

$$u_t(x,t) = \sum_{n=1}^{\infty} C_n \cdot \frac{an\pi}{L} \cos\left(\frac{an\pi}{L} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} C_n \frac{an\pi}{L} \sin\left(\frac{n\pi}{L} x\right)$$

$$= g(x)$$

$$\Rightarrow \quad C_n \cdot \frac{an\pi}{L} = \frac{2}{L} \int_0^L g(x) \cdot \sin\left(\frac{n\pi}{L} x\right) dx$$

$$C_n = \frac{2}{an\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx .$$

General case

We consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

with boundary condition

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

General case

① $v(x,t)$ is the solution to

$$a^2 v_{xx} = v_{tt}$$

$$v(0,t) = v(L,t) = 0$$

$$v(x,0) = f(x), \quad v_t(x,0) = g(x).$$

$$v(x,t) = \sum_{n=1}^{\infty} C_n \cdot \cos\left(\frac{an\pi}{L} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right)$$

$$C_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi}{L} x\right) dx$$

General case

② w is the solution to

$$a^2 w_{xx} = w_{tt}$$

$$w(0,t) = w(L,t) = 0$$

$$w(x,0) = 0 \quad , \quad w_t(x,0) = g(x)$$

$$w(x,t) = \sum_{n=1}^{\infty} D_n \cdot \sin\left(\frac{an\pi}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

$$D_n = \frac{2}{an\pi} \int_0^L g(x) \cdot \sin\left(\frac{n\pi}{L} x\right) dx$$

General case

Let $u(x,t) = v(x,t) + w(x,t)$

(i) $a^2 u_{xx} = a^2(v_{xx} + w_{xx})$

$= v_{tt} + w_{tt}$

$= u_{tt}$

(ii) $u(0,t) = v(0,t) + w(0,t)$

$= 0$

$u(L,t) = v(L,t) + w(L,t)$

$= 0$.

General case

③ $u(x,0) = \underbrace{v(x,0)}_{= f(x)} + \underbrace{w(x,0)}_{= 0} = f(x)$

$u_t(x,0) = \underbrace{v_t(x,0)}_{= 0} + \underbrace{w_t(x,0)}_{= g(x)} = g(x)$.

Thus, $u(x,t)$ satisfies

$$a^2 u_{xx} = u_{tt}$$

$$u(0,t) = u(L,t) = 0$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x)$$