Review: Final

Math 285 Spring 2020

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Problem 1

What is the correct integrating factor to solve the following ODE for $y(t)$?

$$3t^2y' + t^3y + t\sin t = 0$$

$$y' + p(t)\cdot y = q(t)$$

$$\underbrace{\mu(t)}\cdot y' + \underbrace{\mu(t)\,p(t)}\,y = q(t)\cdot\underbrace{\mu(t)}_{\text{Integrating factor}}$$

$$\underline{\mu' = \mu\cdot p} \Rightarrow \mu = e^{\int p(t)\,dt}$$

$$(\mu\cdot y)' = \mu\cdot q$$

$$y = \frac{1}{\mu}\int \mu\cdot q$$

Problem 1

What is the correct integrating factor to solve the following ODE for $y(t)$?

$$3t^2 y' + t^3 y + t\sin t = 0$$

Divide by $3t^2$,

$$y' + \underbrace{\frac{1}{3}t}_{=p(t)}\, y = \underbrace{-\frac{1}{3t}\sin t}_{=q(t)}$$

$$\mu(t) = e^{\int p(t)\,dt}$$

$$= C_1 \cdot e^{\frac{1}{6}t^2}$$

ANS : C_1.

Problem 2

Which one of the following is a solution to $ty' = 2y + 1$?

$$t\frac{dy}{dt} = (2y+1)$$

$$\underbrace{\frac{dy}{2y+1}}_{} = \frac{dt}{t}$$

$$\frac{1}{2}\ln(2y+1) = \ln t + C'$$

$$\ln(2y+1) = 2\ln t + C' = \ln(C' \cdot t^2)$$

$$2y+1 = C't^2$$

$$y = C't^2 - \frac{1}{2}$$

ANS : D

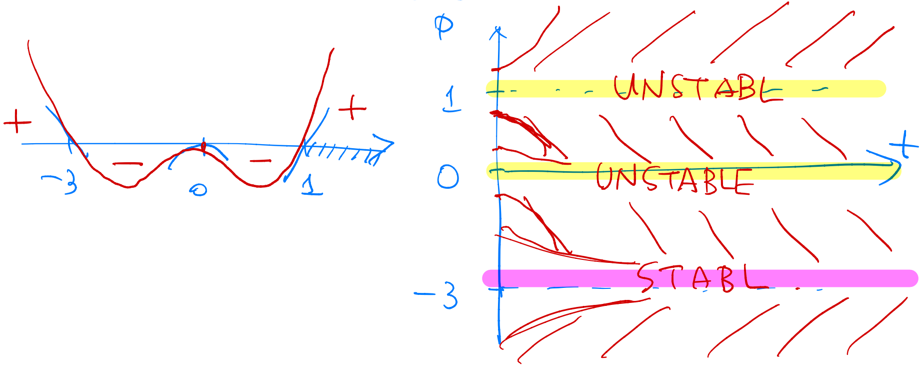
Problem 3

For what value of P_0, is the function $P(t) = P_0$ a stable solution to the following equation

ANS : $P_0 = 3$ (E).

$$\frac{dP}{dt} = (\underline{P-1})(P+3)P^2?$$

$$= \underbrace{f(P)}$$ Autonomous Eq.



Problem 4

The following nonlinear equation for $y(x)$ can be transformed with a substitution into which separable equation for $v(x)$?

ANS : A

$$y' = \underbrace{\frac{x^2 - xy + 2y^2}{x^2}}_{\text{Homogeneous.}}$$

$$= \frac{x^2}{x^2} - \frac{xy}{x^2} + 2 \cdot \frac{y^2}{x^2}$$

$$= 1 - \left(\frac{y}{x}\right) + 2\left(\frac{y}{x}\right)^2$$

$$\begin{cases} v = \dfrac{y}{x} \\ x \cdot v = y \\ y' = xv' + v \end{cases}$$

$$y' = xv' + v = 1 - v + 2v^2$$

$$xv' = 1 - 2v + 2v^2$$

Problem 5

Consider

$$y''' - 5y'' + 8y' - 4y = 0$$

ANS: B.

Which one of the following is NOT a solution?

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4)$

$\lambda = 1, \quad 1 - 5 + 8 - 4 = 0. \quad (\lambda - 1)$ is a factor

$= (\lambda - 1)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2$

Fund. solutions: $e^t, \; e^{2t}, \; te^{2t}$

A: e^t ✓ C: $2e^{2t} - e^t$ ✓, D: $-te^{2t}$ ✓, B $\cancel{te^t}$ $+3e^{2t}$

Problem 6

Identify the correct form of a particular solution for the following differential equation.

ANS : C.

$$\underbrace{y'' + 2y'}_{=0} = \underbrace{4te^{-2t} + 1}.$$

① $\lambda^2 + 2\lambda = 0$

$\lambda = 0, -2$. Fund. Sol. : $1, e^{-2t}$

② $4te^{-2t} \rightsquigarrow t(At+B)e^{-2t}$

$1 \longrightarrow Ct$

$Y(t) = t(At+B)e^{-2t} + Ct$

Problem 7

The motion of a certain spring–mass system is governed by

$$u'' + \gamma u' + ku = 0$$

for some constants $\gamma, k > 0$. Under what γ and k, is the motion overdamped?

Overdamped if two real roots, $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Critically damped if repeated root. $C_1 e^{rt} + C_2 t e^{rt}$

Underdamped if complex roots, Sine & Cosines

Problem 7

The motion of a certain spring–mass system is governed by

$$u'' + \gamma u' + ku = 0$$

ANS : D.

for some constants $\gamma, k > 0$. Under what γ and k, is the motion overdamped?

$\lambda^2 + \gamma\lambda + k = 0$ two distinct real roots

iff $\gamma^2 - 4k > 0$.

D: $\gamma = 4$, $k = 3$

$4^2 - 4 \cdot 3 > 0$. $\Rightarrow$ Overdamped.

Problem 8

Find an open interval on which there exists a unique solution of the initial value problem $(t-1)(t+2)y'' + y'\sqrt{5-t} + e^t y = 1$ with $y(2) = 1$ and $y'(2) = 3$.

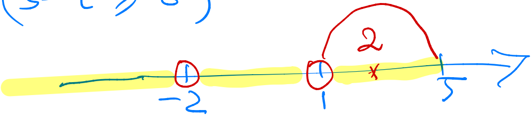
ANS: $(1,5)$ — E

Divide by $(t-1)(t+2)$

$$y'' + \underbrace{\frac{\sqrt{5-t}}{(t-1)(t+2)}} y' + \underbrace{\frac{e^t}{(t-1)(t+2)}} y = \underbrace{\frac{1}{(t-1)(t+2)}}$$

Coefficients are continuous if

$t \neq 1, -2, (5-t \geq 0)$



Problem 9

Consider the following boundary value problem for a variant of the wave equation:

$$
\begin{aligned}
&u_{tt} = u_{xx} + u, \qquad \text{for } 0 < x < 1, \quad t > 0,\\
&u(0,t) = u_x(1,t) = 0 \qquad \text{for } t \geq 0,\\
&u(x,0) = f(x), \qquad u_t(x,0) = 0 \qquad \text{for } 0 \leq x \leq 1.
\end{aligned}
$$

Then separated solutions must satisfy which of the following sets of equations?

$$u(x,t) = X(x) \cdot T(t)$$

$$X \cdot T'' = X'' \cdot T + XT$$

$$\frac{T''}{T} = \frac{X''}{X} + 1, \quad \frac{T''}{T} - 1 = \frac{X''}{X} = -\lambda$$

Problem 9

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$$
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&u(x,0) = f(x), \qquad u_t(x,0) = 0 \qquad \text{for } 0 \leq x \leq 1.
\end{aligned}
$$

Then separated solutions must satisfy which of the following sets of equations?

$T'' - T = -\lambda T$, $T'' + (\lambda - 1)T = 0$

$X'' + \lambda X = 0$.

Problem 9

Consider the following boundary value problem for a variant of the wave equation:

ANS : A

$$u_{tt} = u_{xx} + u, \qquad \text{for } 0 < x < 1, \quad t > 0,$$
$$u(0,t) = u_x(1,t) = 0 \qquad \text{for } t \geq 0,$$
$$u(x,0) = f(x), \qquad u_t(x,0) = 0 \qquad \text{for } 0 \leq x \leq 1.$$

Then separated solutions must satisfy which of the following sets of equations?

$X(0) \cdot T(t) = 0$, $X'(0) \cdot T(t) = 0$

$\Rightarrow \quad X(0) = X'(0) = 0$.

$X(x) \cdot T(0) = f(x)$, $X(x) \cdot T'(0) = 0$

$\Rightarrow \quad T'(0) = 0$

Problem 10

Let $X(x) = e^{10x} + e^{-10x}$. Find a nonzero function $Y(y)$ such that the product $u(x,y) = X(x)Y(y)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

$$u_{xx} + u_{yy} = X'' \cdot Y + X \cdot Y'' = 0.$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = 100$$

$$X = e^{10x} + e^{-10x}$$

$$X' = 10 \cdot \left(e^{10x} - e^{-10x} \right)$$

$$X'' = 100 \left(e^{10x} + e^{-10x} \right)$$

$$= 100 \cdot X$$

$$\therefore \quad Y'' + 100Y = 0.$$

Problem 10

Let $X(x) = e^{10x} + e^{-10x}$. Find a nonzero function $Y(y)$ such that the product $u(x,y) = X(x)Y(y)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

$Y(y) = \underline{C_1 \cdot \cos(10y) + C_2 \cdot \sin(10y)}$

$X \cdot Y$ is a solution to $u_{xx} + u_{yy} = 0$.

B. $Y(y) = \cos(10y)$. is a solution.

Problem 11

If λ_1 is the smallest eigenvalue of $y'' + \lambda y = 0$ with $y'(0) = y(\pi) = 0$, what is the corresponding eigenfunction?

① $\lambda = -\mu^2 < 0$

$$y(x) = C_1 \cdot \cosh(\mu x) + C_2 \cdot \sinh(\mu x)$$

$$y'(x) = \mu\left(C_1 \sinh(\mu x) + C_2 \cdot \cosh(\mu x)\right)$$

$$y'(0) = C_2 \cdot \mu = 0. \qquad \therefore C_2 = 0.$$

$$y(\pi) = C_1 \cdot \underbrace{\cosh(\mu\pi)}_{\neq 0} = 0. \qquad \therefore C_1 = 0$$

$\therefore$ No negative eigenvalues.

Problem 11

If λ_1 is the smallest eigenvalue of $y'' + \lambda y = 0$ with $y'(0) = y(\pi) = 0$, what is the corresponding eigenfunction?

2 $\quad \lambda = 0. \qquad y(x) = c_1 + c_2 x$

$y'(x) = c_2$

$y'(0) = c_2 = 0.$

$y(\pi) = c_1 = 0.$

0 is Not an eigenvalue.

Problem 11

If λ_1 is the smallest eigenvalue of $y'' + \lambda y = 0$ with $y'(0) = y(\pi) = 0$, what is the corresponding eigenfunction?

③ $\lambda = \mu^2 > 0.$

$$y(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x)$$

$$y'(x) = -C_1 \mu \sin(\mu x) + C_2 \mu \cos(\mu x)$$

$$y'(0) = C_2 \mu = 0 \qquad \therefore \ C_2 = 0.$$

$$y(\pi) = C_1 \cdot \underbrace{\cos(\mu \pi)}_{=0} = 0.$$

Suppose $C_1 \neq 0$, $\quad \mu \pi = (n - \frac{1}{2})\pi.$

Problem 11

If λ_1 is the smallest eigenvalue of $y'' + \lambda y = 0$ with $y'(0) = y(\pi) = 0$, what is the corresponding eigenfunction?

$$\mu_n = \left(n - \frac{1}{2}\right), \quad \lambda_n = \left(n - \frac{1}{2}\right)^2, \ n \in \mathbb{N}.$$

$$y_n(x) = C \cdot \cos\left(\left(n - \frac{1}{2}\right)x\right).$$

The smallest eigenvalue is $\lambda_1 = \frac{1}{4}$.

$$y_1(x) = C \cdot \underbrace{\cos\left(\frac{x}{2}\right)}$$

ANS : (C).

Problem 12

Find $p(x)$ and $q(x)$ with which the equation

$$y'' - 2xy' + \lambda y = 0$$

can be transform~~ed~~ed into the form $(p(x)y')' + q(x)y = 0$.

$$\underline{y''} \; \underline{-2xy' + \lambda y = 0}$$

$$\Downarrow$$

$$\underline{(p \cdot y')'} + q \cdot y = \underline{p \cdot y'' + p' y + q y = 0}.$$

$$\mu(x) y'' - 2x \mu y' + \lambda \mu y = 0.$$

$$\Rightarrow \quad \underline{\mu' = -2x\mu}$$

Problem 12

Find $p(x)$ and $q(x)$ with which the equation

$$y'' - 2xy' + \lambda y = 0$$

can be transform into the form $(p(x)y')' + q(x)y = 0$.

ANS : D

$$\mu' = -2x\mu$$

$$\frac{\mu'}{\mu} = -2x$$

$$\ln \mu = -x^2 + C$$

$$\mu(x) = C \cdot e^{-x^2}$$

$$(e^{-x^2} \cdot y')' + \lambda e^{-x^2} y = 0$$

$$p(x) = e^{-x^2}$$

$$q(x) = \lambda e^{-x^2}$$

$$e^{-x^2} y'' - 2x e^{-x^2} y' + \lambda e^{-x^2} y = 0$$