Homework 5

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: October 11, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute $A(3B + 2C), (AB)D, A(BD)$.

Solution:

$$\begin{aligned} A(3B+2C) &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \left(3 \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 5 & 2 & -1 \\ 10 & -1 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 35 & -1 & 17 \\ 0 & 5 & -8 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} (AB)D &= \left(\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} \right) \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 3 & 3 \\ -2 & -1 & -8 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ -26 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} A(BD) &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -7 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 29 \\ -26 \end{pmatrix}. \end{aligned}$$

2. Let $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ and $U : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations defined by

$$T(f(x)) = xf'(x) + 2f(x), \qquad U(a + bx + cx^2) = (a + b, c, a - b).$$

Let $\beta = \{1, x, x^2\}$ and $\gamma = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$. Compute $[U]_\beta^\gamma$, $[T]_\beta$, and $[UT]_\beta^\gamma$.

Solution: Since $U(1) = e_1 + e_3$, $U(x) = e_1 - e_3$, and $U(x^2) = e_2$, we have

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Since $T(1) = 2$, $T(x) = x + 2x = 3x$, and $T(x^2) = 2x^2 + 2x^2 = 4x^2$, we have

$$[T]_\beta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

It follows that

$$[UT]_\beta^\gamma = [U]_\beta^\gamma [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 4 \\ 2 & -3 & 0 \end{pmatrix}.$$

3. Let V, W, and Z be vector spaces. Let $T : V \to W$ and $U : W \to Z$ be linear.
   (a) Prove that if UT is one-to-one, then T is one-to-one.
   (b) Prove that if UT is onto, then U is onto.
   (c) Prove that U and T are one-to-one and onto, then so is UT.

Solution:

(a) We claim that $\mathcal{N}(T) \subseteq \mathcal{N}(UT)$. Indeed, if $x \in \mathcal{N}(T)$, then $T(x) = 0$. Since U is linear, $UT(x) = U(T(x)) = U(0) = 0$ and so $x \in \mathcal{N}(UT)$. Since UT is one-to-one, $\mathcal{N}(UT) = \{0\}$ and so $\mathcal{N}(T) = \{0\}$. This implies that T is one-to-one.

(b) It suffices to show that $\mathcal{R}(UT) \subseteq \mathcal{R}(U)$. Suppose $v \in \mathcal{R}(UT)$, then there exists $x \in V$ such that $v = UT(x) = U(T(x))$. Thus, $v \in \mathcal{R}(U)$. Since $\mathcal{R}(UT) = Z$, we get $\mathcal{R}(U) = Z$ and U is onto.

(c) Suppose $UT(x) = 0$, then $T(x) = 0$ because U is one-to-one. Since T is one-to-one, $x = 0$ and UT is one-to-one. For any $z \in Z$, there exists $w \in W$ such that $U(w) = z$ because U is onto. Since T is onto, there exists $v \in V$ such that $T(v) = w$. Thus, $z = UT(v)$ and UT is onto.

4. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$.
   (a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(A) = \mathrm{tr}(A^t)$, and $(AB)^t = B^t A^t$.
   (b) Are there exist $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $AB - BA = I_n$? Justify your answer.

Solution:

(a) By definition,

$$\begin{aligned}\operatorname{tr}(AB) &= \sum_{i=1}^{n}(AB)_{ii} \\ &= \sum_{i=1}^{n}\left(\sum_{j=1}^{n}A_{ij}B_{ji}\right) \\ &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n}B_{ji}A_{ij}\right) \\ &= \sum_{j=1}^{n}(BA)_{jj} \\ &= \operatorname{tr}(BA).\end{aligned}$$

Since $A_{ii} = (A^t)_{ii}$ for all $i = 1, 2, \cdots, n$, we have

$$\operatorname{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n}(A^t)_{ii} = \operatorname{tr}(A^t).$$

It follows that for each $i, j = 1, 2, \cdots, n$,

$$((AB)^t)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n}(B^t)_{ik}(A^t)_{kj} = (B^tA^t)_{ij}.$$

(b) Suppose that there exist such matrices A, B, then the part (a) implies that

$$\operatorname{tr}(AB - BA) = \operatorname{tr}(AB) - \operatorname{tr}(BA) = 0.$$

However, we have $\operatorname{tr}(I_n) = n$ so that it is a contradiction. Thus, there are no such matrices.

5. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$. Define $\langle A, B\rangle = \operatorname{tr}(AB^t)$.
   (a) Show that $\langle A, B\rangle = \langle B, A\rangle$.
   (b) Show that $\langle A, A\rangle \geq 0$ and equality holds if and only if $A = O$.

Solution:

(a) By the part (a) of Problem 4, we have

$$\langle A, B\rangle = \operatorname{tr}(AB^t) = \operatorname{tr}((AB^t)^t) = \operatorname{tr}(BA^t) = \langle B, A\rangle.$$

(b) It follows that

$$\langle A, A\rangle = \operatorname{tr}(AA^t) = \sum_{i=1}^{n}(AA^t)_{ii} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}A_{ij}(A^t)_{ji}\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}(A_{ij})^2 \geq 0.$$

Suppose $\langle A, A\rangle = 0$, then $A_{ij} = 0$ for all $i, j = 1, 2, \cdots, n$ by the above observation. This implies that $A = O$. If $A = O$, then it is trivial that $\langle A, A\rangle = 0$

6. Determine whether T is invertible and justify your answer.

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (3x - y, y, 4x)$.

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x - 2z, y, 3x + 4y)$.

Solution:

(a) Since $\dim(\mathbb{R}^2) = 2 \neq 3 = \dim(\mathbb{R}^3)$, T cannot be an isomorphism.

(b) Suppose $T(x, y, z) = (3x - 2z, y, 3x + 4y) = 0$. Then, $y = 0$, $x = -4y/3 = 0$, and $z = 3x/2 = 0$. Thus, T is one-to-one. By Dimension theorem, we have $3 = \dim(\mathbb{R}^3) = \dim(\mathcal{N}(T)) + \dim(\mathcal{R}(T)) = \dim(\mathcal{R}(T))$, which implies that T is onto. Therefore, T is an isomorphism.

7. Let V and W finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $T(V_0)$ is a subspace of W

(b) Prove that $\dim(V_0) = \dim(T(V_0))$.

Solution:

(a) It is obvious that $T(V_0) \subseteq W$ and $0 \in T(V_0)$. Suppose that $x, y \in T(V_0)$ and $c \in \mathbb{R}$. Then, there exist $v, w \in V_0$ such that $T(v) = x$ and $T(w) = y$. Note that $cv + w \in V_0$ because it is a subspace of V. Thus we get

$$cx + y = cT(v) + T(w) = T(cv + w) \in T(V_0).$$

(b) Define $T|_{V_0} : V_0 \to T(V_0)$ by $T|_{V_0}(v) = T(v)$ for all $v \in V_0$. Note that $V_0, T(V_0)$ are vector spaces, $T|_{V_0}$ is linear. Furthermore, the map is onto by definition and one-to-one because T is one-to-one. Therefore the map is an isomorphism and as a consequence, $\dim(V_0) = \dim(T(V_0))$.