Math 416 Lecture Note: Week 9

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1 Properties of determinants

The goal of this section is to show the following: if $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$, then $\det(AB) = \det(A)\det(B)$.

To this end, we use the row operations.

Definition 1.1. An $n \times n$ elementary matrix is the result of doing a single row operation to I_n.

Example 1.2.

$$I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad I_3 \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$I_3 \xrightarrow{R_3 \to 5R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Theorem 1.3. *Let E be an elementary matrix obtained by a row operation $\mathcal{R}$ from I_n. If $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, then a matrix obtained by $\mathcal{R}$ from A is EA.*

Proof. Execise. $\square$

Example 1.4.

$$\begin{pmatrix} 3 & -7 & 4 \\ 1 & -2 & 1 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -2 & 1 \\ 3 & -7 & 4 \\ 2 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -7 & 4 \\ 1 & -2 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -2 & 1 \\ 3 & -7 & 4 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 3 & -7 & 4 \\ 2 & -1 & -2 \end{pmatrix}$$

Theorem 1.5. *Every elementary matrix is invertible.*

Proof. Suppose E is obtained by $\mathcal{R}$ from I_n. Let $\mathcal{R}'$ be the row operation that reverses $\mathcal{R}$, that is,

$$A \xrightarrow{\mathcal{R}} B \xrightarrow{\mathcal{R}'} A$$

for all matrices A. Let E' be the elementary matrix corresponding to $\mathcal{R}'$, then

$$EE' = I_n = E'E.$$

$\square$

Theorem 1.6. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, then A is invertible if and only if A is the product of elementary matrices.*

Proof. Suppose that A is the product of elementary matrices. Let

$$A = E_1 E_2 \cdots E_k$$

for some elementary matrices E_i. Since they are invertible, A is also invertible and

$$A^{-1} = E_k^{-1} \cdots E_2^{-1} E_1^{-1}.$$

Suppose that A is invertible, then there exists a sequence of row operations $\mathcal{R}$ such that

$$(A|I_n) \xrightarrow{\mathcal{R}} (I_n|A^{-1}).$$

This implies that there exists a sequence of elementary matrices $E_1, \cdots, E_k$ such that

$$E_1 \cdots E_k A = I_n.$$

Multiplying A^{-1} from right, we get

$$A^{-1} = E_1 \cdots E_k,$$

which leads to $A = E_k^{-1} \cdots E_1^{-1}$. Since E_i^{-1} is also elementary, the proof is complete. $\square$

Theorem 1.7. *Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$, then* $\det(AB) = \det(A)\det(B)$.

Proof. In the HW, we have seen that AB is invertible if and only if A and B are invertible. If AB is not invertible, then one of the matrices is not invertible. If a matrix is not invertible, then the determinant is zero. Thus, the result holds.

We claim that if A is an elementary matrix, then $\det(AB) = \det(A)\det(B)$. If A is obtained by $R_i \leftrightarrow R_j$ from I_n, then AB is obtained by the same operation from B. In this case, we have $\det(A) = -1\det(I_n) = -1$ and

$$\det(AB) = -\det(B) = \det(A)\det(B).$$

This also works for the other row operations.

Suppose AB is invertible. Then, A and B are also invertible. Thus, $\det(AB) \neq 0$, $\det(A) \neq 0$, and $\det(B) \neq 0$. There exist elementary matrices $E_1, \cdots, E_k$ such that

$$A = E_1 \cdots E_k.$$

Using the claim, we get

$$\det(AB) = \det(E_1)\det(E_2 \cdots E_k B) = \cdots = \det(E_1)\det(E_2)\cdots\det(E_k)\det(B) = \det(A)\det(B).$$

$\square$

Corollary 1.8. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, then A is invertible (or, equivalently* $\operatorname{rank}(A) < n$*) if and only if* $\det(A) \neq 0$. *Furthermore, if A is invertible, then* $\det(A^{-1}) = \det(A)^{-1}$.

Proof. ($\Rightarrow$): If A is invertible, then there exists A^{-1} such that $AA^{-1} = A^{-1}A = I_n$. Thus,

$$\det(AA^{-1}) = \det(A)\det(A^{-1}) = 1.$$

Thus, $\det(A) \neq 0$.

($\Leftarrow$): If A is not invertible, then we have seen that $\det(A) = 0$. $\square$

Corollary 1.9. *Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$.*

(i) $\det(AB) = \det(BA)$.

(ii) *If A is similar to B, then* $\det(A) = \det(B)$.

Proof. Homework. $\square$

Theorem 1.10. *Let* $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, *then* $\det(A) = \det(A^t)$.

Proof. If $\det(A) = 0$, then the set of columns of A is linearly dependent. Then we have

$$n > \dim(\mathrm{Col}(A)) = \dim(\mathrm{Row}(A)) = \dim(\mathrm{Col}(A^t)).$$

Thus, the set of columns of A^t is linearly dependent and so $\det(A^t) = 0$.

Suppose $\det(A) \neq 0$, then A is invertible. Thus, A^t is also invertible and $\det(A^t) \neq 0$ by the previous HW. There exist elementary matrices $E_1, E_2, \cdots, E_k$ such that

$$A = E_1 E_2 \cdots E_k.$$

By the HW, we have

$$A^t = E_k^t \cdot E_2^t E_1^t.$$

Since $\det(E_i) = \det(E_i^t)$ by HW, the proof is complete. $\square$

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

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