Homework 1 Solution

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 6, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Prove Corollary 1 in section 1.2 of [FIS] (page 11).

Solution: Suppose there are two zero vectors, say 0 and $\overline{0}$. Then the definition of zero vector implies

$$0 = 0 + \overline{0} = \overline{0}.$$

Thus, it is unique.

2. Prove Corollary 2 in section 1.2 of [FIS] (page 12).

Solution: Let $x \in V$. Suppose there are $y, z \in V$ such that $x + y = x + z = 0$. It then follows from Theorem 1.1 that $y = z$, which implies that the inverse of x is unique.

3. Let $V = \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\}$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be vectors in V, and $c \in \mathbb{R}$. Define $x + y = (x_1 + y_1, x_2 + y_2)$ and $cx = (cx_1, c^2 x_2)$. Is V a vector space over $\mathbb{R}$? Justify your answer.

Solution: It is not a vector space because

$$(1 + 1) \cdot (1, 1) = (2, 4) \neq (1, 1) + (1, 1).$$

4. Let V, W be vector spaces over $\mathbb{R}$. Define the product of $V \times W$ by

$$V \times W = \{(v, w) : v \in V, w \in W\}.$$

For $(v_1, w_1), (v_2, w_2) \in V \times W$ and $c \in \mathbb{R}$, define

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \qquad c(v_1, w_1) = (cv_1, cw_1).$$

Show that $V \times W$ is a vector space over $\mathbb{R}$.

Solution: Let $(v_1, w_1), (v_2, w_2) \in V \times W$ and $c \in \mathbb{R}$. Since $v_1 + v_2 \in V$, $w_1 + w_2 \in W$, $cv_1 \in V$, and $cw_1 \in W$, we have $(v_1, w_1) + (v_2, w_2) \in V \times W$ and $c(v_1, w_1) \in V \times W$.

(1) $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) = (v_2 + v_1, w_2 + w_1) = (v_2, w_2) + (v_1, w_1)$ for any $(v_1, w_1), (v_2, w_2) \in V \times W$.

(2) For any $(v_1, w_1), (v_2, w_2), (v_3, w_3) \in V \times W$,

$$\begin{aligned}((v_1, w_1) + (v_2, w_2)) + (v_3 + w_3) &= ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3) \\ &= (v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) \\ &= (v_1, w_1) + ((v_2, w_2) + (v_3 + w_3)).\end{aligned}$$

(3) Let 0_V and 0_W be the zero vectors for V and W respectively. Let $0_{V\times W} = (0_V, 0_W)$, then $(v, w) + 0_{V\times W} = (v + 0_V, w + 0_W) = (v, w)$ for all $(v, w) \in V \times W$.

(4) For $(v, w) \in V \times W$, $(v, w) + (-v, -w) = (v + (-v), w + (-w)) = (0_V, 0_W) = 0_{V\times W}$.

(5) $1(v, w) = (1v, 1w) = (v, w)$ for all $(v, w) \in V \times W$.

(6) $(ab)(v, w) = ((ab)v, (ab)w) = (a(bv), a(bw)) = a(bv, bw) = a(b(v, w))$ for all $(v, w) \in V \times W$ and $a, b \in \mathbb{R}$.

(7) For all $a \in \mathbb{R}$ and $(v_1, w_1), (v_2, w_2) \in V \times W$,

$$\begin{aligned}a((v_1, w_1) + (v_2, w_2)) &= a(v_1 + v_2, w_1 + w_2) \\ &= (a(v_1 + v_2), a(w_1 + w_2)) \\ &= (av_1 + av_2, aw_1 + aw_2) \\ &= (av_1, aw_1) + (av_2, aw_2) \\ &= a(v_1, w_1) + a(v_2, w_2).\end{aligned}$$

(8) For all $a, b \in \mathbb{R}$ and $(v, w) \in V \times W$, $(a + b)(v, w) = ((a + b)v, (a + b)w) = (av + bv, aw + bw) = (av, aw) + (bv, bw) = a(v, w) + b(v, w)$.

5. Let $M_{m\times n}(\mathbb{R})$ be the set of all $m \times n$ matrices with real entries. Prove the following.
   (a) $(aA + bB)^t = aA^t + bB^t$ for any $a, b \in \mathbb{R}$ and $A, B \in M_{m\times n}(\mathbb{R})$, where $m, n \in \mathbb{N}$.
   (b) $\operatorname{tr}(aA + bB) = a \operatorname{tr}(A) + b \operatorname{tr}(B)$ for any $a, b \in \mathbb{R}$ and $A, B \in M_{n\times n}(\mathbb{R})$, where $n \in \mathbb{N}$.

Solution: Let $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, n\}$. The (i, j)-th entries of a matrix $M \in M_{m\times n}(\mathbb{R})$ is denoted by M_{ij}.

(a) For each $i \in \{1, 2, \cdots, m\}$ and $j \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned}((aA + bB)^t)_{ji} &= (aA + bB)_{ij} \\ &= aA_{ij} + bB_{ij} \\ &= a(A^t)_{ji} + b(B^t)_{ji} \\ &= (aA^t + bB^t)_{ji}.\end{aligned}$$

(b) We have

$$\begin{aligned}\operatorname{tr}(aA+bB) &= \sum_{i=1}^{n}(aA+bB)_{ii}\\ &= \sum_{i=1}^{n}(aA_{ii}+bB_{ii})\\ &= a\sum_{i=1}^{n}A_{ii}+b\sum_{i=1}^{n}B_{ii}\\ &= a\operatorname{tr}(A)+b\operatorname{tr}(B).\end{aligned}$$

6. Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operation of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify you answer.
   (a) $W_1 = \{(x,y,z) \in \mathbb{R}^3 : 2x+3y-z=0\}$.
   (b) $W_2 = \{(x,y,z) \in \mathbb{R}^3 : x=y-3z+1\}$.
   (c) $W_3 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2=z\}$.
   (d) $W_4 = \{(x,y,z) \in \mathbb{R}^3 : x=2y, y=-z\}$.

Solution:

(a) First, we have $(0,0,0) \in W_1$. Let $(x_1,y_1,z_1),(x_2,y_2,z_2) \in W_1$, then

$$2(x_1+x_2)+3(y_1+y_2)-(z_1+z_2) = 2x_1+3y_1-z_1+2x_2+3y_2-z_2 = 0$$

and $(x_1,y_1,z_1)+(x_2,y_2,z_2) \in W_1$. Let $c \in \mathbb{R}$ and $(x,y,z) \in W_1$, then

$$2(cx)+3(cy)-(cz) = c(2x+3y-z) = 0,$$

which yields $c(x,y,z) \in W_1$. By Theorem 1.3, W_1 is a subspace of $\mathbb{R}^3$.

(b) W_2 is not a subspace because $(0,0,0) \notin W_2$.

(c) $(1,1,1) \in W_3$ but $(2,2,2) \notin W_3$. Thus, W_3 is not a subspace.

(d) First, we have $(0,0,0) \in W_4$. Let $(x_1,y_1,z_1),(x_2,y_2,z_2) \in W_4$, then $(x_1+x_2) = 2(y_1+y_2)$ and $(y_1+y_2) = -(z_1+z_2)$. So, $(x_1,y_1,z_1)+(x_2,y_2,z_2) \in W_4$. Let $c \in \mathbb{R}$ and $(x,y,z) \in W_1$, then $cx = 2(cy)$ and $cy = -cz$, which yields $c(x,y,z) \in W_4$. By Theorem 1.3, W_4 is a subspace of $\mathbb{R}^3$.

7. Let $F_0(\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$. Define addition and scalar multiplication by $(f+g)(x) = f(x)+g(x)$ and $(cf)(x) = cf(x)$ for any $f,g \in F_0(\mathbb{R})$, $x,c \in R$. Show that $F_0(\mathbb{R})$ is a vector space over $\mathbb{R}$.

Solution: Let $\mathcal{F}(\mathbb{R},\mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. In the class, we have seen that $\mathcal{F}(\mathbb{R},\mathbb{R})$ is a vector space. Since $F_0(\mathbb{R})$ is a subset of $\mathcal{F}(\mathbb{R},\mathbb{R})$, it suffices to show that $F_0(\mathbb{R})$ is a subspace. Let $f,g \in F_0(\mathbb{R})$ and $c \in \mathbb{R}$, then $(f+g)(0) = f(0)+g(0) = 0$ and $(cf)(0) = cf(0) = 0$ and so $f+g \in F_0(\mathbb{R})$ and $cf \in F_0(\mathbb{R})$. The zero function 0 also satisfies $0(0) = 0$ so $0 \in F_0(\mathbb{R})$. Therefore, $F_0(\mathbb{R})$ is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$ and by Theorem 1.3, it is a vector space.

8. Let W_1, W_2 be subspaces of a vector space V over $\mathbb{R}$. Show that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Solution: ($\Rightarrow$): Assume that $W_1 \cup W_2$ is a subspace of V. Suppose $W_1 \not\subseteq W_2$ and $W_2 \not\subseteq W_1$. Let $x \in W_1 \setminus W_2$ and $y \in W_2 \setminus W_1$. Since $x, y \in W_1 \cup W_2$ and $W_1 \cup W_2$ is closed under addition, we have $x + y \in W_1 \cup W_2$. If $x + y \in W_1$, then $(x + y) + (-x) = y \in W_1$, which is a contradiction. If $x + y \in W_2$, then $(x + y) + (-y) = x \in W_2$, which is a contradiction. Thus, $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

($\Leftarrow$): If $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_2$ or $W_1 \cup W_2 = W_1$, respectively. In both cases, $W_1 \cup W_2$ is a subspace.

9. Let W_1, W_2 be subspaces of a vector space V over $\mathbb{R}$. Define

$$W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}.$$

(a) Show that $W_1 + W_2$ is a subspace of V.

(b) Let U be a subspace of V and $W_1, W_2 \subseteq U$. Show that $W_1 + W_2 \leq U$. (This implies that $W_1 + W_2$ is the smallest subspace of V containing W_1 and W_2.)

Solution:

(a) Since V is a vector space, $W_1 + W_2$ is a subset of V. Since $0 = 0 + 0$, we have $0 \in W_1 + W_2$. Let $c \in \mathbb{R}$, $w_1 \in W_1$, and $w_2 \in W_2$. Then, there exist $x, z \in W_1$, and $y, w \in W_2$ such that $w_1 = x + y$ and $w_2 = z + w$. Since $x + z \in W_1$ and $y + w \in W_2$, we have

$$w_1 + w_2 = (x + y) + (z + w) = (x + z) + (y + w) \in W_1 + W_2.$$

Since $cx \in W_1$ and $cy \in W_2$,

$$c(x + y) = cx + cy \in W_1 + W_2.$$

Thus, $W_1 + W_2$ is a subspace of V.

(b) It suffices to show that $W_1 + W_2 \subseteq U$. Let $z \in W_1 + W_2$, then there exist $x \in W_1$ and $y \in W_2$. Since $x, y \in U$ and U is a subspace of V, $x + y = z \in U$, which finishes the solution.

10. Let V be a vector space over $\mathbb{R}$. We say that V is the direct sum of W_1 and W_2 if $W_1, W_2 \leq V$, $W_1 \cap W_2 = \{0\}$, and $W_1 + W_2 = V$. We denote by $V = W_1 \oplus W_2$. Let $W_1, W_2 \leq V$. Show that $V = W_1 \oplus W_2$ if and only if every $x \in V$ can be uniquely written as $x = x_1 + x_2$ for $x_1 \in W_1$ and $x_2 \in W_2$.

Solution: ($\Rightarrow$): Let $x \in V$. Since $V = W_1 \oplus W_2$, there exist $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$. Suppose $x = x_1 + x_2 = \widetilde{x}_1 + \widetilde{x}_2$, then $x_1 + (-\widetilde{x}_1) \in W_1$ and

$$\begin{aligned} x_1 + (-\widetilde{x}_1) &= (x_1 + x_2) + ((-\widetilde{x}_1) + (-x_2)) \\ &= (\widetilde{x}_1 + \widetilde{x}_2) + ((-\widetilde{x}_1) + (-x_2)) \\ &= \widetilde{x}_2 + (-x_2) \in W_2. \end{aligned}$$

Since $W_1 \cap W_2 = \{0\}$, $x_1 + (-\widetilde{x}_1) = \widetilde{x}_2 + (-x_2) = 0$, which yields $x_1 = \widetilde{x}_1$ and $x_2 = \widetilde{x}_2$.

($\Leftarrow$): By the hypothesis, we have $V = W_1 + W_2$. Suppose there exists a nonzero element $x \in W_1 \cap W_2$. Since $W_1 \cap W_2$ is a vector space, $cx \in W_1 \cap W_2$ for all $c \in \mathbb{R}$. Then, $4x = 3x + x = 2x + 2x$. Note that $3x \neq 2x$ and $x \neq 2x$ because $x \neq 0$. Thus, the decomposition is not unique, which is a contradiction. So, $W_1 \cap W_2 = \{0\}$ and $V = W_1 \oplus W_2$.