Homework 4

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: October 4, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Prove that T is a linear transformation, find bases for $\mathcal{N}(T)$ and $\mathcal{R}(T)$, and compute $\dim(\mathcal{N}(T))$ and $\dim(\mathcal{R}(T))$.
   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x - y, 2z)$.
   (b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 0, 2x - y)$.
   (c) $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$. (Here, $\mathcal{P}_n(\mathbb{R})$ is the set of all polynomials $p(x)$ with real coefficients with $\deg(p(x)) \le n$.)

Solution:

(a) We have $T(0, 0, 0) = (0, 0)$. Let $c \in \mathbb{R}$ and $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$, then

$$\begin{aligned} T(c(x_1, y_1, z_1) + (x_2, y_2, z_2)) &= ((cx_1 + x_2) - (cy_1 + y_2), 2(cz_1 + z_2)) \\ &= c(x_1 - y_1, 2z_1) + (x_2 - y_2, 2z_2) \\ &= cT(x_1, y_1, z_1) + T(x_2, y_2, z_2). \end{aligned}$$

Thus, T is linear. If $T(x, y, z) = (x - y, 2z) = (0, 0)$, then $x = y$ and $z = 0$. Thus,

$$\mathcal{N}(T) = \{(t, t, 0) : t \in \mathbb{R}\}.$$

Let $\beta = \{(1, 1, 0)\}$, then β spans $\mathcal{N}(T)$ and β is linearly independent (because β consists of a nonzero vector). So, β is a basis for $\mathcal{N}(T)$ and $\dim(\mathcal{N}(T)) = 1$. By Dimension theorem $\dim(\mathcal{R}(T)) = 2 = \dim(\mathbb{R}^2)$. Since $\mathcal{R}(T) \le \mathbb{R}^2$, we have $\mathcal{R}(T) = \mathbb{R}^2$. The standard basis for $\mathbb{R}^2$ is also a basis for $\mathcal{R}(T)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x + y, 0, 2x - y)$.
We have $T(0, 0) = (0, 0, 0)$. Let $c \in \mathbb{R}$ and $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, then

$$\begin{aligned} T(c(x_1, y_1) + (x_2, y_2)) &= ((cx_1 + x_2) + (cy_1 + y_2), 0, 2(cx_1 + x_2) - (cy_1 + y_2)) \\ &= c(x_1 + y_1, 0, 2x_1 - y_1) + (x_2 + y_2, 0, 2x_2 - y_2) \\ &= cT(x_1, y_1) + T(x_2, y_2). \end{aligned}$$

Thus, T is linear. If $T(x, y) = (x + y, 0, 2x - y) = (0, 0, 0)$, then $x + y = 0$ and $2x - y = 0$. Thus, $x = y = 0$ and

$$\mathcal{N}(T) = \{(0, 0)\}.$$

So, $\beta = \varnothing$ is a basis for $\mathcal{N}(T)$ and $\dim(\mathcal{N}(T)) = 0$. We claim that

$$\mathcal{R}(T) = \{(t, 0, s) : t, s \in \mathbb{R}\}.$$

It is obvious that $\mathcal{R}(T) \subseteq \{(t, 0, s) : t, s \in \mathbb{R}\}$. Let $t, s \in \mathbb{R}$. Define $x = \frac{1}{3}(t + s)$ and $y = \frac{1}{3}(2t - s)$, then

$$(t, 0, s) = (\frac{1}{3}(t+s) + \frac{1}{3}(2t-s), 0, \frac{2}{3}(t+s) - \frac{1}{3}(2t-s)) = (x + y, 0, 2x - y) \in \mathcal{R}(T).$$

Thus we obtain $\mathcal{R}(T) = \{(t, 0, s) : t, s \in \mathbb{R}\}$. Let $\gamma = \{(1, 0, 0), (0, 0, 1)\}$, then γ is linearly independent because it is a subset of the standard basis for $\mathbb{R}^3$. Since $\mathcal{R}(T) = \{(t, 0, s) : t, s \in \mathbb{R}\}$, it is obvious that γ spans $\mathcal{R}(T)$. Thus, γ is a basis for $\mathcal{R}(T)$ and $\dim(\mathcal{R}(T)) = 2$.

(c) We have $T(0) = 0$. Let $c \in \mathbb{R}$ and $p(x), q(x) \in \mathcal{P}_2(\mathbb{R})$. Then,

$$\begin{aligned} T(cp(x) + q(x)) &= x(cp(x) + q(x)) + (cp(x) + q(x))' \\ &= c(xp(x) + p'(x)) + (xq(x) + q'(x)) \\ &= cT(p(x)) + T(q(x)). \end{aligned}$$

Thus, T is linear. Let $p(x) = ax^2 + bx + c$ and $T(p) = 0$, then

$$T(p) = xp(x) + p'(x) = ax^3 + bx^2 + cx + 2ax + b = ax^3 + bx^2 + (2a + c)x + b = 0.$$

So, we have $a = b = c = 0$, that is, $\mathcal{N}(T) = \{0\}$. Thus, $\varnothing$ is a basis for $\mathcal{N}(T)$ and $\dim(\mathcal{N}(T)) = 0$.

Let $p(x) = ax^2 + bx + c$, then

$$\begin{aligned} T(p) &= xp(x) + p'(x) \\ &= ax^3 + bx^2 + cx + 2ax + b \\ &= a(x^3 + 2x) + b(x^2 + 1) + cx. \end{aligned}$$

Let $\beta = \{x, x^2 + 1, x^3 + 2x\}$, then β spans $\mathcal{R}(T)$. By Dimension Theorem, $\dim(\mathcal{R}(T)) = \dim(\mathcal{P}_2(\mathbb{R})) = 3$. Since $|\beta| = 3$, we conclude that β is a basis for $\mathcal{R}(T)$.

2. Let V and W be vector spaces over $\mathbb{R}$ and $T : V \to W$ linear.
   (a) Let $\{w_1, \cdots, w_k\}$ be a linearly independent subset of $\mathcal{R}(T)$. Prove that if $S = \{v_1, \cdots, v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, \cdots, k$, then S is linearly independent.
   (b) Prove that T is one-to-one if and only if T carries linearly independent subsets of V onto linearly independent subsets of W.

Solution:

(a) Let $a_1v_1 + \cdots + a_kv_k = 0$, then

$$T(a_1v_1 + \cdots + a_kv_k) = a_1T(v_1) + \cdots + a_kT(v_k) = a_1w_1 + \cdots + a_kw_k = 0,$$

which implies that $a_1 = \cdots = a_k = 0$. Thus, S is linearly independent.

(b) Suppose T is one-to-one. Let $S = \{v_1, \cdots, v_k\}$ be a linearly independent subset of V. Let $a_1T(v_1) + \cdots + a_kT(v_k) = 0$, then

$$a_1T(v_1) + \cdots + a_kT(v_k) = T(a_1v_1 + \cdots + a_kv_k) = 0.$$

Since $\mathcal{N}(T) = \{0\}$, we have $a_1 v_1 + \cdots + a_k v_k = 0$. Since S is linearly independent, we conclude that $a_1 = \cdots = a_k = 0$ as desired.

Suppose that T carries linearly independent subsets of V onto linearly independent subsets of W and T is not one-to-one. Then, there exist $x \in V$ such that $T(x) = 0$ and $x \neq 0$. Since $\{x\}$ is linearly independent and $\{T(x)\}$ is not, this is a contradiction. Thus, T is one-to-one.

3. Let V and W be finite dimensional vector spaces over $\mathbb{R}$ and $T : V \to W$ linear.
   (a) Prove that if $\dim(V) < \dim(W)$, then T cannot be onto.
   (b) Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one.

Solution:

(a) Suppose $\dim(V) < \dim(W)$ and T is onto. By Dimension Theorem, $\dim(W) = \dim(\mathcal{R}(T)) = \dim(V) - \dim(\mathcal{N}(T)) \leq \dim(V)$, which is a contradiction.

(b) Suppose $\dim(V) > \dim(W)$ and T is one-to-one. Since $\mathcal{N}(T) = \{0\}$ and $\mathcal{R}(T) \leq W$, it follows from Dimension Theorem that $\dim(W) \geq \dim(\mathcal{R}(T)) = \dim(V)$, which is a contradiction.

4. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathcal{N}(T) = \mathcal{R}(T)$.
   (b) Give an example of two distinct linear transformations U and T such that $\mathcal{N}(U) = \mathcal{N}(T)$ and $\mathcal{R}(U) = \mathcal{R}(T)$.

Solution:

(a) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (0, x)$, then

$$\mathcal{N}(T) = \mathcal{R}(T) = \{(0, t) : t \in \mathbb{R}\}.$$

(b) Define $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (0, x)$ and $U(x, y) = (0, 2x)$. Then,

$$\mathcal{N}(T) = \mathcal{R}(T) = \mathcal{N}(U) = \mathcal{R}(U) = \{(0, t) : t \in \mathbb{R}\}$$

but $T \neq U$.

5. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform. Show that there exist $a, b, c, d \in \mathbb{R}$ such that $T(x, y) = (ax + by, cx + dy)$.

Solution: Choose $a, b, c, d \in \mathbb{R}$ by $T(1, 0) = (a, c)$ and $T(0, 1) = (b, d)$. Then, for any $(x, y) \in \mathbb{R}^2$, we have

$$T(x, y) = xT(1, 0) + yT(0, 1) = x(a, c) + y(b, d) = (ax + by, cx + dy).$$

6. Let V be a vector space and $T : V \to V$ linear. A subspace $W \leq V$ is called T-invariant if $T(x) \subseteq W$ for all $x \in W$.
   (a) Show that the subspaces $\{0\}, V, \mathcal{N}(T), \mathcal{R}(T)$ are T-invariant.
   (b) Suppose that V is finite dimensional and $W \leq V$ is T-invariant. Show that if $V = \mathcal{R}(T) \oplus W$ then $W = \mathcal{N}(T)$.

Solution:

(a) Since $T(0) = 0$, $\{0\}$ is T-invariant. Since $T(x) \in V$ for all $x \in V$, V is T-invariant. Since $T(x) = 0 \in \mathcal{N}(T)$ for all $x \in \mathcal{N}(T)$, $\mathcal{N}(T)$ is T-invariant. Since $T(x) \in \mathcal{R}(T)$ for all $x \in V$ (so for $x \in \mathcal{R}(T)$), $\mathcal{R}(T)$ is T-invariant.

(b) Since W is T-invariant and $\mathcal{R}(T) \cap W = \{0\}$, if $y \in W$, then $T(y) \in W \cap \mathcal{R}(T)$. This implies $T(y) = 0$ and $y \in \mathcal{N}(T)$. That is, $W \subseteq \mathcal{N}(T)$. Note that $V = \mathcal{R}(T) \oplus W$ implies $\dim(V) = \dim(\mathcal{R}(T)) + \dim(W)$. By Dimension theorem, we have

$$\dim(\mathcal{N}(T)) = \dim(V) - \dim(\mathcal{R}(T)) = \dim(W),$$

which yields $W = \mathcal{N}(T)$.