# Math 285 Lecture Note: Week 15 and 16

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## 1 Laplace's Equation (Sec 10.8)

Consider a 2-dimensional heat equation $\alpha^2(u_{xx}+u_{yy}) = u_t$. If a steady state temperature distribution exists, $u$ is a function of $x$ and $y$ and satisfies

$$u_{xx} + u_{yy} = 0.$$

This is called Laplace's equation. Since there is no time dependence, we do not have initial condition.

In 1-dimension, boundary conditions refer prescribed function values or derivatives at the ends of a given interval. In higher dimension, information at two points is not sufficient. In general, boundary conditions are conditions at all points of the boundary.

The problem of finding a solution of Laplace's equation with prescribed function values on the boundary is called a Dirichlet problem. The problem of finding a solution of Laplace's equation with prescribed normal derivatives on the boundary is called a Neumann problem.

In this section, we focus on 2-dimensional Dirichlet problems for a rectangle and a disk.

### 1.1 Dirichlet problem for a rectangle

Consider

$$u_{xx} + u_{yy} = 0 \tag{1.1}$$

in the rectangle $\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : 0 < x < a, 0 < y < b\}$ with

$$\begin{aligned} &u(x, 0) = 0, \qquad u(x, b) = 0 \quad \text{for } 0 < x < a, \\ &u(0, y) = 0, \qquad u(a, y) = f(y) \quad \text{for } 0 \le y \le b, \end{aligned} \tag{1.2}$$

where $f$ is a function on $0 \le y \le b$. By the method of separation, we let $u(x, y) = X(x)Y(y)$ and have

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The boundary conditions (1.3) read

$$Y(0) = Y(b) = 0, \qquad X(0) = 0$$

Solving $Y'' + \lambda Y = 0$ with $Y(0) = Y(b) = 0$, we get

$$\lambda_n = \frac{n^2\pi^2}{b^2}$$
$$Y_n(y) = \sin\left(\frac{n\pi}{b}y\right).$$

For each $\lambda_n$, we solve $X'' - \lambda_n X = 0$ with $X(0) = 0$ to obtain

$$X_n(x) = \sinh\left(\frac{n\pi}{b}x\right)$$

and so

$$u(x,y) = \sum_{n=1}^{\infty} C_n u_n(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{b}x\right)\sin\left(\frac{n\pi}{b}y\right).$$

Now, the condition $u(a,y) = f(y)$ implies

$$u(a,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi}{b}y\right) = f(y).$$

Using the Fourier sine series of $f$, the constants $C_n$ are determined by

$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b f(y)\sin\left(\frac{n\pi}{b}y\right)\,dy.$$

**Example 1.1.** Consider $u_{xx} + u_{yy} = 0$ in the rectangle $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2\}$ with

$$\begin{aligned} u(x,0) &= 0, \qquad u(x,2) = 0 \quad \text{for } 0 < x < 2, \\ u(0,y) &= 0, \qquad u(2,y) = f(y) \quad \text{for } 0 \le y \le 2, \end{aligned} \tag{1.3}$$

where $f(y) = 2y - y^2$. Then, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{2}x\right)\sin\left(\frac{n\pi}{2}y\right)$$

where

$$C_n = \frac{1}{\sinh(n\pi)} \int_0^2 (2y - y^2)\sin\left(\frac{n\pi}{2}y\right)\,dy = \frac{16(1-(-1)^n)}{\pi^3 n^3 \sinh(n\pi)}.$$

## 1.2 Dirichlet problem for a disk

Consider the 2-dimensional Laplace's equation in the disk

$$\mathcal{D} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}.$$

with the boundary condition $u(a\cos\theta, a\sin\theta) = f(\theta)$ for $0 \le \theta < 2\pi$. For a disk, it is convenient to use polar coordinates. Recall that polar coordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta$$

for $r > 0$ and $0 \le \theta < 2\pi$. Using this, the disk can be written as $\mathcal{D} = \{(r,\theta) : 0 \le r < a, 0 \le \theta < 2\pi\}$. We abuse the notation $u(x,y) = u(r,\theta)$, then the boundary condition can be written as $u(a,\theta) = f(\theta)$. We translate the Laplace's equation in terms of polar coordinates. By chain rule, we have

$$\begin{aligned} u_r &= \frac{\partial x}{\partial r}u_x + \frac{\partial y}{\partial r}u_y = (\cos\theta)u_x + (\sin\theta)u_y, \\ u_\theta &= \frac{\partial x}{\partial \theta}u_x + \frac{\partial y}{\partial \theta}u_y = (-r\sin\theta)u_x + (r\cos\theta)u_y, \\ u_{rr} &= \frac{\partial x}{\partial r}(u_r)_x + \frac{\partial y}{\partial r}(u_r)_y = (\cos^2\theta)u_{xx} + (2\cos\theta\sin\theta)u_{xy} + (\sin^2\theta)u_{yy}, \\ u_{\theta\theta} &= \frac{\partial x}{\partial \theta}(u_\theta)_x + \frac{\partial y}{\partial \theta}(u_\theta)_y \\ &= (r^2\sin^2\theta)u_{xx} + (-2r^2\sin\theta\cos\theta)u_{xy} + (r^2\cos^2\theta)u_{yy} + (-r\cos\theta)u_x + (-r\sin\theta)u_y \\ &= r^2(u_{xx} + u_{yy}) - r^2u_{rr} - ru_r. \end{aligned}$$

So, if $u_{xx} + u_{yy} = 0$, then

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0.$$

Let $u(r, \theta) = R(r)\Theta(\theta)$, then

$$\begin{aligned} r^2 R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0 \\ r^2 \frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda. \end{aligned}$$

Thus, we get

$$r^2 R'' + rR' - \lambda R = 0, \qquad \Theta'' + \lambda\Theta = 0.$$

We first solve the equation for $\Theta$. Note that we do not have any boundary conditions for $\Theta$. Instead, $\Theta$ is periodic with period $2\pi$.

Suppose $\lambda = -\mu^2 < 0$, then

$$\Theta(\theta) = c_1 e^{\mu\theta} + c_2 e^{-\mu\theta}.$$

Since $\Theta$ is periodic, $c_1$ and $c_2$ should be zero. That is, $\Theta = 0$.

Suppose $\lambda = 0$, then $\Theta(\theta) = c_1 + c_2\theta$. Due to the periodicity, $c_2 = 0$, that is, $\Theta$ is a constant. In this case, we have $rR'' + R' = 0$ and so the general solution is

$$R(r) = c_1 + c_2 \ln r.$$

If $r$ tends to 0, $\ln r$ diverges. Since we are interested in the case where $u$ is bounded in the disk $\mathcal{D}$, $c_2$ should be 0 and $R$ is also a constant. Thus, the fundamental solution corresponding to $\lambda = 0$ is $u_0(r, \theta) = 1$.

Suppose $\lambda = \mu^2 > 0$, then

$$\Theta(\theta) = c_1 \cos(\mu\theta) + c_2 \sin(\mu\theta).$$

Since $\Theta$ is periodic with period $2\pi$, $\mu$ should be a positive integer. For each $n \in \mathbb{N}$, we want to solve $r^2 R'' + rR' - n^2 R = 0$. Let $R(r) = r^k$, then

$$r^2 R'' + rR' - n^2 R = (k(k-1) + k - n^2) r^k = 0,$$

which means $k = n, -n$. Thus, the general solution is

$$R(r) = c_1 r^n + c_2 r^{-n}.$$

As $r \to 0$, $r^{-n}$ does not converge. So, $c_2 = 0$ and the fundament solution corresponding to $\lambda = n^2$ is

$$u_n(r, \theta) = r^n (c_1 \cos(n\theta) + c_2 \sin(n\theta)).$$

Therefore, a solution to $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$ is

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos n\theta + d_n \sin n\theta).$$

Finally, the boundary condition $u(a, \theta) = f(\theta)$ yields

$$u(a, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} a^n (c_n \cos n\theta + d_n \sin n\theta) = f(\theta).$$

Using the Fourier sine series of $f$, the coefficients $c_n$ and $d_n$ are determined by

$$\begin{aligned} c_n &= \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \cos n\theta \, d\theta, \\ d_n &= \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) \sin n\theta \, d\theta. \end{aligned}$$

## 2 The Occurrence of Two-Point Boundary Value Problems (Sec 11.1)

Previously, we have seen the heat conduction equation $\alpha^2 u_{xx} = u_t$ with boundary conditions $u(0,t) = 0$ (or $u_x(0,t) = 0$) and $u(L,t) = 0$ (or $u_x(L,t) = 0$) and initial condition $u(x,0) = f(x)$. We used the method of separation of variables to deduce two ODEs

$$\begin{aligned} &X'' + \lambda X = 0, \qquad X(0) = X(L) = 0, \\ &T' + \alpha^2 \lambda T = 0. \end{aligned}$$

It turned out that the ODE for $X$ with the boundary conditions leads to eigenvalue problems. We have shown that for some $\lambda_n$, there exists nontrivial solutions for the boundary problem. Then, we solved the ODE for $T$ and used the superposition property to get the solution.

Our goal of this and the next lecture is to generalize the heat conduction problem. We consider the partial differential equations of the form

$$r(x)u_t = (p(x)u_x)_x - q(x)u$$

with boundary conditions

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0, \qquad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_1\alpha_2 \neq 0$ and $\beta_1\beta_2 \neq 0$. For example, the heat conduction problem is the case where $p(x) = 1 = r(x)$ and $q(x) = 0$.

Let $u(x,t) = X(x)T(t)$, then

$$\begin{aligned} r(x)X(x)T'(t) &= (p(x)X'(x))'T(t) - q(x)X(x)T(t) \\ \frac{T'(t)}{T(t)} &= \frac{(p(x)X'(x))'}{r(x)X(x)} - \frac{q(x)}{r(x)} = -\lambda. \end{aligned}$$

Thus, we have $T' + \lambda T = 0$

$$(p(x)X')' - q(x)X + \lambda r(x)X = 0.$$

The boundary conditions read

$$\alpha_1 X(0) + \alpha_2 X'(0) = 0, \qquad \beta_1 X(L) + \beta_2 X'(L) = 0$$

To solve the PDE $r(x)u_t = (p(x)u_x)_x - q(x)u$, it suffices to understand the eigenvalue problem $(p(x)X')' - q(x)X + \lambda r(x)X = 0$ with the boundary conditions. This is called Sturm–Liouville theory.

**Example 2.1.** Consider the case where $p(x) = r(x) = 1$, $q(x) = 0$, $\alpha_2 = 0$, $\alpha_1 = \beta_1 = \beta_2 = 1$, and $L = \pi$. That is, $X'' + \lambda X = 0$ with $X(0) = 0$ and $X(\pi) + X'(\pi) = 0$.

Suppose $\lambda = -\mu^2 < 0$, then

$$X(x) = c_1 \cosh \mu x + c_2 \sinh \mu x.$$

Note that $X(0) = c_1 = 0$ and

$$X(\pi) + X'(\pi) = c_2(\sinh \mu\pi + \mu \cosh \mu\pi) = 0.$$

If $c_2 \neq 0$, then $\mu$ satisfies $\sinh \mu\pi + \mu \cosh \mu\pi = 0$ and so

$$\mu = -\tanh \mu\pi.$$

Since $-\tanh \mu\pi < 0$ for $\mu > 0$, there is no such $\mu$. That is, there is no negative eigenvalue.

Suppose $\lambda = 0$, then $X(x) = c_1 + c_2 x$. By the boundary conditions, $c_1 = 0$ and $X(\pi) + X'(\pi) = c_2(\pi + 1) = 0$. Thus, $X(x) = 0$, which means that 0 is not an eigenvalue.

Suppose $\lambda = \mu^2 > 0$, then

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$

Note that $X(0) = c_1 = 0$ and

$$X(\pi) + X'(\pi) = c_2(\sin \mu\pi + \mu \cos \mu\pi) = 0.$$

If $c_2 \neq 0$, then $\mu$ satisfies $\sin \mu\pi + \mu \cos \mu\pi = 0$ and so

$$\mu = -\tan \mu\pi.$$

For each $n \in \mathbb{N}$, there exists $\mu_n \in (n - \frac{1}{2}, n + \frac{1}{2})$ such that $\mu_n = -\tan \mu_n \pi$. For each eigenvalue $\lambda_n = \mu_n^2$, the corresponding eigenfunction is

$$\phi_n(x) = k_n \sin \sqrt{\lambda_n} x$$

for arbitrary constant $k_n$. Note that as $n \to \infty$ $\lambda_n = \mu_n^2 \approx (n - \frac{1}{2})^2$.

# 3 Sturm–Liouville Boundary Value Problems (Sec 11.2)

We recall the definition of eigenvalues and eigenfunctions for a general differential operator.

**Definition 3.1.** Let $L[y]$ be a differential operator. We say that $\lambda$ is a real (or complex) eigenvalue of $L$ with a homogeneous boundary conditions if $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) and the differential equation $L[y] = \lambda y$ with the given homogeneous boundary conditions has nontrivial solutions. The nontrivial solutions are called eigenfunctions.

In this section, we study the Sturm–Liouville boundary problem, which consists of a differential equation of the form

$$(p(x)y')' - q(x)y + \lambda r(x)y = 0$$

on $[0, 1]$ with boundary conditions

$$\begin{aligned} \alpha_1 y(0) + \alpha_2 y'(0) &= 0, \\ \beta_1 y(1) + \beta_2 y'(1) &= 0 \end{aligned}$$

with $\alpha_1\alpha_2 \neq 0$ and $\beta_1\beta_2 \neq 0$. We further assume that $p, p', q, r$ are continuous on $[0, 1]$ and $p(x), r(x) > 0$ for all $x \in [0, 1]$. In this case, the problem is called *regular*.

**Example 3.2.** If $p(x) = r(x) = 1$ and $q(x) = 0$, then the problem is $y'' + \lambda y = 0$ with

$$\begin{aligned} \alpha_1 y(0) + \alpha_2 y'(0) &= 0, \\ \beta_1 y(1) + \beta_2 y'(1) &= 0. \end{aligned}$$

**Example 3.3.** If $p(x) = x^k$, $q(x) = 0$, and $r(x) = x^{k-2}$, then

$$\begin{aligned} (p(x)y')' - q(x)y + \lambda r(x)y &= x^k y'' + kx^{k-1}y' + \lambda x^{k-2} y \\ &= x^{k-2}(x^2 y'' + kxy' + \lambda y) \\ &= 0. \end{aligned}$$

Thus, the problem $x^2y'' + kxy' + \lambda y = 0$ with

$$\begin{aligned} \alpha_1 y(1) + \alpha_2 y'(1) &= 0, \\ \beta_1 y(2) + \beta_2 y'(2) &= 0. \end{aligned}$$

also belongs to the class.

**Proposition 3.4** (Lagrange's identity). *Let $u$ and $v$ be functions on $[0,1]$ with continuous second derivatives and $L$ a differential operator defined by $L[y] = -(p(x)y')' + q(x)y$. Then,*

$$\begin{aligned}\int_0^1 (L[u]v - uL[v])\,dx &= -p(x)\left[u'(x)v(x) - u(x)v'(x)\right]_0^1 \\ &= p(0)\left(u'(0)v(0) - u(0)v'(0)\right) - p(1)\left(u'(1)v(1) - u(1)v'(1)\right).\end{aligned}$$

*Furthermore, if $u$ and $v$ satisfies the boundary condition*

$$\begin{aligned}\alpha_1 y(0) + \alpha_2 y'(0) = 0,\\ \beta_1 y(1) + \beta_2 y'(1) = 0\end{aligned}$$

*with $\alpha_1\alpha_2 \neq 0$ and $\beta_1\beta_2 \neq 0$, then*

$$\int_0^1 (L[u]v - uL[v])\,dx = 0.$$

**Remark 3.5.** The Lagrange's identity with the boundary condition can be written as

$$(L[u], v) = (u, L[v])$$

where the inner product on $[0,1]$ is defined by

$$(f,g) = \int_0^1 f(x)g(x)\,dx.$$

The Lagrange's identity also holds for complex-valued functions $u$ and $v$ with the complex inner product

$$(f,g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

**Theorem 3.6.** *Every eigenvalue of the Sturm–Liouville problem is real.*

*Proof.* Let $\lambda$ be an eigenvalue and $\phi$ the corresponding eigenfunction. Since $\phi$ is a solution to the Sturm–Liouville problem , we have

$$\int_0^1 L[\phi]\overline{\phi}\,dx = \int_0^1 \phi\overline{L[\phi]}\,dx$$

and $L[\phi] = \lambda r(x)\phi$. Since $r(x) > 0$ and $\phi$ is nontrivial, we get

$$\int_0^1 L[\phi]\overline{\phi}\,dx = \lambda\int_0^1 r(x)\phi\overline{\phi}\,dx = \overline{\lambda}\int_0^1 r(x)\phi\overline{\phi}\,dx = \int_0^1 \phi\overline{L[\phi]}\,dx$$

and so $\lambda = \overline{\lambda}$. □

**Theorem 3.7.** *If $\phi_n$ and $\phi_m$ are two eigenfunctions of the Sturm–Liouville problem corresponding to distinct eigenvalues $\lambda_n$ and $\lambda_m$, then*

$$\int_0^1 \phi_n(x)\phi_m(x)\,r(x)dx = 0.$$

*We call $\phi_n$ and $\phi_m$ are orthogonal on $[0,1]$ with the weight $r(x)$.*

*Proof.* We have $L[\phi_n] = \lambda_n r(x)\phi_n$ and $L[\phi_m] = \lambda_m r(x)\phi_m$. It follows from the Lagrange's identity that

$$\begin{aligned}\int_0^1 L[\phi_n]\phi_m\,dx - \int_0^1 \phi_n L[\phi_m]\,dx &= \lambda_n\int_0^1 \phi_n(x)\phi_m(x)\,r(x)dx - \lambda_m\int_0^1 \phi_n(x)\phi_m(x)\,r(x)dx \\ &= (\lambda_n - \lambda_m)\int_0^1 \phi_n(x)\phi_m(x)\,r(x)dx \\ &= 0.\end{aligned}$$

Since $\lambda_n - \lambda_m \neq 0$, we obtain the desired result. □

**Theorem 3.8.** *Every eigenvalue of the Sturm–Liouville problem is simple; that is, if $\phi_1$ and $\phi_2$ are the corresponding eigenfunctions of the same eigenvalue $\lambda$, then they are linearly dependent. Furthermore, the eigenvalues form an infinite sequence $\lambda_1 < \lambda_2 < \cdots$ and $\lim_{n\to\infty} \lambda_n = \infty$.*

**Definition 3.9.** An eigenfunction $\phi$ is normalized with respect to the weight $r$ if

$$\int_0^1 \phi^2 \, r(x)dx = 1.$$

**Remark 3.10.** Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues and $\phi_n$ the corresponding normalized eigenfunctions. Then,

$$\int_0^1 \phi_n \phi_m \, r(x)dx = \begin{cases} 1, & m = n \\ 0 & m \neq n. \end{cases}$$

In this case, we call the set $\{\phi_n : n = 1, 2, \cdots\}$ is orthonormal.

Suppose that we are given a function $f(x)$ and want to represent it in terms of the normalized eigenfunctions $\phi_n$ as we did in the heat conduction equation. To be specific, the question is to find $C_n$ such that

$$f(x) = \sum_{n=1}^{\infty} C_n \phi_n(x).$$

Using the orthogonality and the normalization, it is obvious to guess that

$$C_n = \int_0^1 f(x)\phi_n(x) \, r(x)dx.$$

**Theorem 3.11.** *Let $\phi_1, \phi_2, \cdots$ be the normalized eigenfunctions of the Sturm–Liouville problem. If $f(x)$ and $f'(x)$ are piecewise continuous on $[0, 1]$, then*

$$\sum_{n=1}^{\infty} C_n \phi_n(x) = \frac{1}{2}(f(x+) + f(x-))$$

*with*

$$C_n = \int_0^1 f(x)\phi_n(x) \, r(x)dx.$$

# References


[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley



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