Sec 10.4: Even and Odd Functions

Math 285 Spring 2020

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Even and Odd Functions

Definition

A function f is called *even* if $f(-x) = f(x)$ for all x in the domain.
A function f is called *odd* if $f(-x) = -f(x)$ for all x in the domain.

Examples

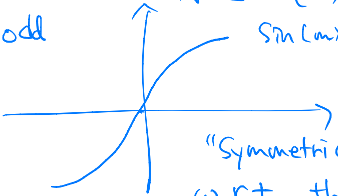
Even : $\cos(m \cdot x) = \cos(-mx)$

$x^2 = (-x)^2, \quad x^4 = (-x)^4, \quad \cdots, \quad x^k, \quad k : \text{even}$

Odd : $\sin(m \cdot x) = -\sin(-mx)$

$x = -(-x), \quad x^3 = -(-x)^3, \quad \cdots, \quad x^k, \quad k : \text{odd}$

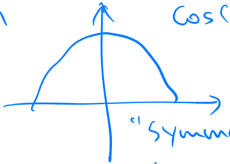
odd



$\sin(mx)$

"Symmetric w.r.t the origin"

even



$\cos(mx)$

"symmetric about y-axis"

proposition

Let f, f_1, f_2 be even and g, g_1, g_2 be odd.

(i) $f_1 \pm f_2$, $f_1 f_2$, $g_1 g_2$, f_1/f_2, g_1/g_2 are even functions.

(ii) $g_1 \pm g_2$ are odd functions.

(iii) If f and g are differentiable, then f' is odd and g' is even.

(iv) $\displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$ and $\displaystyle\int_{-L}^{L} \overset{\text{odd}}{g(x)}\,dx = 0.$

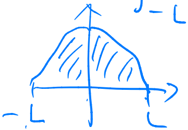
(iv) $$\int_{-L}^{L} \underbrace{f(x)}_{\text{even}}\,dx = \int_0^L f(x)\,dx + \underbrace{\int_{-L}^{0} f(x)\,dx} = 2\int_0^L f\,dx,$$

$$\left(\begin{aligned} x &= -u \\ dx &= -du \end{aligned}\right)$$

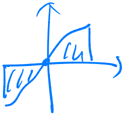
$$= -\int_L^0 f(-u)\,du$$

$$= \int_0^L \underbrace{f(-u)}_{= f(u)}\,du$$

$$= \int_0^L f(x)\,dx$$



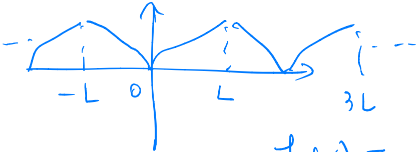
$$\int_{-L}^{L} \underbrace{g(x)}_{\text{odd}}\,dx = 0.$$



Fourier cosine series

Fourier convergence thm.

Suppose f and f' are piecewise continuous on $[-L, L]$. Assume that f is even and periodic with period $2L$. That is, $f(x) = f(-x)$ and $f(x + 2L) = f(x)$ for all x.



$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right)$$

$$a_m = \frac{1}{L}\int_{-L}^{L} \overbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\cos\left(\frac{m\pi}{L}x\right)}_{\text{even}}}^{\text{even}} dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)dx$$

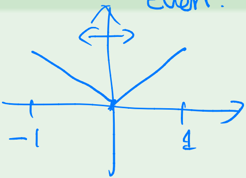
$$b_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \cdot \underbrace{\sin\left(\frac{m\pi}{L}x\right)}_{\text{odd}}}_{\text{odd}} dx = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi}{L}x\right)$$: Fourier cosine series of f.

Fourier cosine series

Example

even.



Consider a periodic function f defined by

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

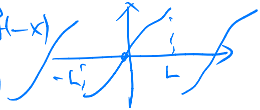
Since f is even, f has a Fourier cosine series. Indeed, we have seen that

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)\pi x).$$

Fourier sine series

Suppose f and f' are piecewise continuous on $[-L, L]$. Assume that f is odd and periodic with period $2L$. That is, $f(x) = -f(-x)$ and $f(x + 2L) = f(x)$ for all x.

$f(x) = -f(-x)$



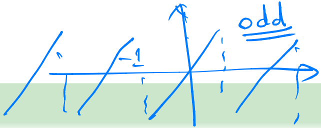
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right)$$

$$a_m = \frac{1}{L}\int_{-L}^{L} \overbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\cos\left(\frac{m\pi}{L}x\right)}_{\text{even}}}^{\text{odd}}\, dx = 0$$

$$b_m = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}}\ \underbrace{\sin\left(\frac{m\pi}{L}x\right)}_{\text{odd}}}_{\text{even}}\, dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L}x\right)$$: Fourier sine series of f.

Fourier sine series



Example

Consider a periodic function f with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$. Note that f is discontinuous at $x = 2k-1$, $k \in \mathbb{Z}$.

$f(x) = -f(-x)$

Since f is odd, it has a Fourier sine series. Indeed, we have seen that

$$f(x) = -\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\sin(m\pi x)$$

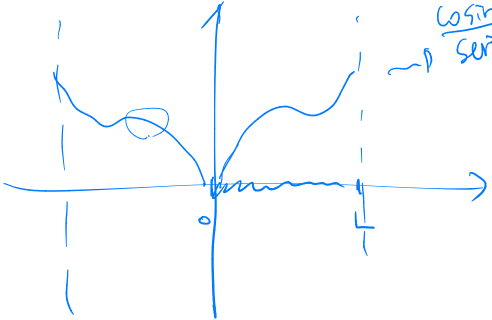
for $x \neq 2k-1$, $k \in \mathbb{Z}$.

Even and odd periodic extension

Suppose we are given a function f on $[0, L]$. We assume that f is nice enough that the Fourier convergence theorem is applicable.

We want to represent it as a Fourier series on $[0, L]$. To do this, we first extend f to be a periodic function.

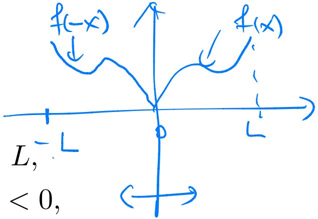
There are a lot of ways to do that.



$\rightsquigarrow$ Cosine series

$\downarrow$ Sine series

Extension to Cosine series



Define g by

$$g(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(-x), & -L \le x < 0, \end{cases}$$

and $g(x + \underline{2L}) = g(x)$.

$g(x) = g(-x)$ even.

$$g(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi}{L} x\right) \quad \text{all } x$$

$$a_m = \frac{2}{L} \int_0^L \underline{\underline{f(x)}} \cdot \cos\left(\frac{m\pi}{L} x\right) dx$$

On $[0, L]$, $g(x) = f(x)$

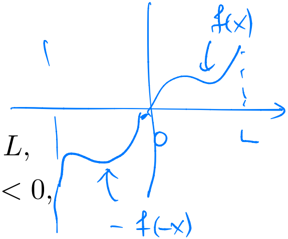
$$\underline{f(x)} = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi}{L} x\right) \quad x \in [0, L]$$

Cosine series of f

Extension to Sine series

Define h by

$$h(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(-x), & -L \le x < 0, \end{cases}$$



and $h(x + 2L) = h(x)$. Odd periodic

$$h(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L} x\right) \quad \text{all } x$$

$$b_m = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{m\pi}{L} x\right) dx$$

On $[0, L]$, $h = f$,

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi}{L} x\right) \quad \text{for } x \in [0, L].$$

Sine series of f.

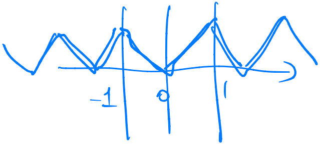
Example

Example

Suppose $f(x) = x$ on $[0, 1]$. Find its cosine and sine series.

① Even : $g(x) = \begin{cases} f(x) = x , & [0, 1] \\ f(-x) = -x , & [-1, 0] \end{cases}$.

$g(x+2) = g(x)$



$x = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}$

on $[0,1]$

② Odd : $h(x) = \begin{cases} f(x) = x & \text{on } [0, 1] \\ -f(-x) = x & \text{on } [-1, 0] \end{cases}$, $h(x+2) = h(x)$

$x = -\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin(m\pi x)$ on $[0, 1$

Example

Example

Suppose $f(x) = x$ on $[0, 1]$. Find its cosine and sine series.