Math 416: Abstract Linear Algebra

Midterm 1, Fall 2019

Date: September 25, 2019

NAME:______________________________

READ THE FOLLOWING INFORMATION.

- This is a 50-minute exam.
- This exam contains 10 pages (including this cover page) and 6 questions. Total of points is 100.
- Books, notes, and other aids are not allowed. Collaboration is forbidden.
- Show all steps to earn full credit.
- Do not unstaple pages. Loose pages will be ignored.

Question	Points	Score
1	20	
2	20	
3	15	
4	10	
5	15	
6	20	
Total:	100	

1. (20 points) Circle True or False. Do not justify your answer.

(a) True False Let V be a vector space over $\mathbb{R}$, $a, b \in \mathbb{R}$, and $v \in V$. If $a \cdot v = b \cdot v$, then $a = b$.

(b) True False Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ and $W = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \operatorname{tr}(A) = 0\}$, then it is a subspace of V.

(c) True False Let V be a vector space over $\mathbb{R}$ and W_1, W_2 be subspaces of V, then the union $W_1 \cup W_2$ is a subspace of V.

(d) True False For each v in a vector space V over $\mathbb{R}$, $\{v\}$ is linearly independent.

(e) True False Let $v_1 = (1, 1, 0)$ and $v_2 = (0, 1, 2)$, then $(3, 1, -4) \in \operatorname{Span}(\{v_1, v_2\})$.

(f) True False The matrix $\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ is row-equivalent to $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix}$.

(g) True False The system of linear equations associated to an augmented matrix

$$(A, b) = \begin{pmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

has infinitely many solutions.

(h) True False The set of all solution to a system of linear equations with n variables is a subspace of $\mathbb{R}^n$.

(i) True False $V = \mathcal{P}_n(\mathbb{R})$ be the set of all real polynomials $p(x)$ with $\deg(p) \leq n$ (here, $\deg(p)$ denotes the degree of $p(x)$). Then the dimension of V is n.

(j) True False Let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of $\mathbb{R}^n$. If S is linearly independent, then S is a basis for $\mathbb{R}^n$.

2. Consider a system of linear equations

$$\begin{cases} x_1 + 2x_2 - 4x_3 - x_4 = 0 \\ x_1 + 3x_2 - 7x_3 = 0 \\ x_1 + 2x_3 - 2x_4 = 0. \end{cases}$$

(a) (5 points) Write down the augmented matrix A corresponding to the above system of linear equations.

(b) (5 points) Find a reduced row-echelon form of A (a matrix in reduced row-echelon form which is row-equivalent to A). Please label your individual row operations.

(c) (5 points) Find the solution set of the above system.

(d) (5 points) Find a basis of the solution set.

3. Let V be a vector space over $\mathbb{R}$.

   (a) (10 points) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{2u - v, u + v\}$ is linearly independent.

   (b) (5 points) Suppose $\{u, v\}$ is linearly independent. Is $\{au - v, u + v\}$ linearly independent for $a \in \mathbb{R}$? Prove it or give a counterexample.

4. (10 points) Let

$$W_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 + 3x_2 - x_3 - 9x_4 = 0, x_1 + 2x_2 + x_3 = 0\},$$
$$W_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 - 3x_3 - 4x_4 = 0\}.$$

Find the dimension of $W_1 \cap W_2$.

5. Let V be the set of all (2×2) matrices and W a subspace of V consisting of all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$.

   (a) (5 points) Find two (2×2) matrices $A, B \in V$ such that $A \in W$ and $B \notin W$.

   (b) (5 points) Find a basis β for W.

(c) (5 points) Find a basis γ for V such that $\beta \subset \gamma$.

6. Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ be the set of all $(n \times n)$ matrices with real entries. Define

$$U = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^t = A\}, \qquad W = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^t = -A\}.$$

(a) (10 points) Show that W is a subspace of V.

(b) (10 points) Show that every $A \in V$ can be uniquely written as $A = B + C$ for $B \in U$ and $C \in W$.