# Homework 10

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: December 4, 2019

**Textbooks**: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $V = \mathbb{R}^3$ be equipped with the standard inner product. Apply the Gram–Schmidt process to a basis $\beta = \{v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (1,3,3)\}$ for $V$ to obtain an orthonormal basis for $V$.

2. Let $V$ be an inner product space over $F$ and $W$ a finite dimensional subspace of $V$. Let $\beta$ be a basis for $W$.
   (a) Show that $W^\perp$ is a subspace of $V$.
   (b) Show that $W \cap W^\perp = \{0\}$.
   (c) Show that $z \in W^\perp$ if and only if $\langle z, x\rangle = 0$ for all $x \in \beta$.

3. Let $V$ be an inner product space over $F$, $S_1, S_2$ be subsets of $V$, and $W$ be a finite dimensional subspace of $V$.
   (a) Show that if $S_1 \subseteq S_2$, then $S_2^\perp \subseteq S_1^\perp$.
   (b) Show that $\mathrm{Span}(S_1) \leq (S_1^\perp)^\perp$.
   (c) Show that $W = (W^\perp)^\perp$.

4. A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called unitary if $Q$ is invertible and $Q^{-1} = Q^*$. Show that $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is unitary if and only if the set of the columns of $A$ is orthonormal.

5. Let $V$ be an inner product space over $F$, $T : V \to V$ linear, and $y \in V$. Let $\varphi(x) : V \to F$ be defined by $\varphi(x) = \langle T(x), y\rangle$. Show that $\varphi$ is linear.

6. Let $V$ be an inner product space over $F$, $c \in F$, and $S, T : V \to V$ linear.
   (a) Show that $(cS + T)^* = \overline{c}S^* + T^*$.
   (b) Show that $(ST)^* = T^*S^*$.
   (c) Show that $(T^*)^* = T$ and $I^* = I$.

7. Let $V$ be an inner product space over $F$.
   (a) (Parseval's identity) Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for $V$. Show that
   $$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$$
   for all $x, y \in V$.
   (b) (Bessel's indequality) Let $S = \{v_1, \cdots, v_n\}$ be an orthonormal subset for $V$. Show that
   $$\sum_{i=1}^{n} |\langle x, v_i\rangle|^2 \leq \|x\|^2$$
   for all $x \in V$.