Math 416 Lecture Note: Week 13

Daesung Kim

1 The adjoint of a linear operator

Theorem 1.1. *Let V be a finite dimensional inner product space over F, and $T : V \to V$ linear. Then, there exists a unique linear transform $T^* : V \to V$ such that $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.*

Definition 1.2. Let V be a finite dimensional inner product space over F, and $T : V \to V$ linear. The adjoint of T is the unique linear transformation $T^* : V \to V$ satisfying $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.

Lemma 1.3. *Let V be a finite dimensional inner product space over F and $\varphi : V \to F$ be linear. Then, there exists a unique vector $y \in V$ such that $\varphi(x) = \langle x, y\rangle$.*

Proof. Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V and

$$y = \sum_{i=1}^{n} a_i v_i$$

for some $a_i \in F$ which will be determined later. If $\varphi(x) = \langle x, y\rangle$ holds for all $x \in V$, then in particular, we have

$$\varphi(v_j) = \langle v_j, y\rangle = \sum_{i=1}^{n} \overline{a_i} \langle v_j, v_i\rangle = \overline{a_j}.$$

Let $a_j = \overline{\varphi(v_j)}$. For any $x = \sum_{i=1}^{n} b_i v_i$, it then follows that

$$\langle x, y\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} b_i \varphi(v_j) \langle v_i, v_j\rangle = \sum_{i=1}^{n} b_i \varphi(v_i) = \varphi(x).$$

Suppose there are two vectors $y, z \in V$ such that $\langle x, y\rangle = \varphi(x) = \langle x, z\rangle$ for all $x \in V$. Then,

$$\langle x, y - z\rangle = 0$$

for all $x \in V$. In particular, if $x = y - z$, then $\|y - z\| = 0$ which implies that $y = z$. $\square$

Proof of Theorem 1.1. (i) For fixed $y \in V$, we define a map $\varphi(x) : V \to F$ by $\varphi(x) = \langle T(x), y\rangle$. One can see that this is linear (HW).

(ii) By the previous lemma, for each $y \in V$, there exists a unique vector $y' \in V$ such that $\varphi(x) = \langle T(x), y\rangle = \langle x, y'\rangle$. This enables us to define a map $T^* : V \to V$ by $T^*(y) = y'$. This map is well-defined because such a vector y' is unique for each $y \in V$.

(iii) To show T^* is linear, let $x, y \in V$ and $c \in F$. Then,

$$\begin{aligned}
\langle z, T^*(cx + y)\rangle &= \langle T(z), cx + y\rangle \\
&= \overline{c} \langle T(z), x\rangle + \langle T(z), y\rangle \\
&= \overline{c} \langle z, T^*(x)\rangle + \langle z, T^*(y)\rangle \\
&= \langle z, cT^*(x) + T^*(y)\rangle
\end{aligned}$$

for all $z \in V$. Thus, $T^*(cx + y) = cT^*(x) + T^*(y)$.

(iv) Let $S : V \to V$ be a linear transformation such that $\langle T(x), y\rangle = \langle x, S(y)\rangle$ for all $x, y \in V$. Then, $\langle T(x), y\rangle = \langle x, S(y)\rangle = \langle x, T^*(y)\rangle$ and so

$$\langle x, S(y) - T^*(y)\rangle = 0$$

for all x. Thus, $S(y) = T^*(y)$ for all $y \in V$ as desired.

□

Proposition 1.4. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ and $T = L_A$, then $T^* = L_{A^*}$.*

Proof. It suffices to show that

$$\langle Ax, y\rangle = \langle x, A^* y\rangle$$

for all $x, y \in \mathbb{C}^n$ where $\langle \cdot, \cdot\rangle$ is the dot product. (HW) □

Proposition 1.5. *Let V be a finite dimensional inner product space over F, $T : V \to V$ linear, and β an orthonormal basis for V. Then,*

$$[T^*]_\beta = ([T]_\beta)^*.$$

Proof. Let $\beta = \{v_1, \cdots, v_n\}$. Let $[T]_\beta = A = (A_{ij})$ and $[T^*]_\beta = B = (B_{ij})$. Then, $\langle T(v_j), v_i\rangle = \langle v_j, T^*(v_i)\rangle$,

$$\langle T(v_j), v_i\rangle = \left\langle \sum_{k=1}^n A_{kj} v_k, v_i \right\rangle = A_{ij},$$

and

$$\langle v_j, T^*(v_i)\rangle = \left\langle v_j, \sum_{k=1}^n B_{kj} v_k \right\rangle = \overline{B_{ji}} = (B^*)_{ij},$$

which yields $A = B^*$. □

Remark 1.6. Let V be an inner product space over F and $S, T : V \to V$ linear. One can see the following properties (HW).

(i) $(cS + T)^* = \overline{c} S^* + T^*$.

(ii) $(ST)^* = T^* S^*$.

(iii) $(T^*)^* = T$ and $I^* = I$.

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu