Homework 2 Solution

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 13, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.
   (a) $(-2, 0, 3)$, $(1, 3, 0)$, $(2, 4, -1)$
   (b) $(3, 4, 1)$, $(1, -2, 1)$, $(-2, -1, 1)$
   (c) $(5, 1, -5)$, $(1, -2, -3)$, $(-2, 3, -4)$

Solution:

(a) $(-2, 0, 3) = 4(1, 3, 0) - 3(2, 4, -1)$

(b) Let $(3, 4, 1) = a(1, -2, 1) + b(-2, -1, 1)$, then we have

$$\begin{cases} a - 2b = 3 \\ -2a - b = 4 \\ a + b = 1. \end{cases}$$

From the last two equations, $a = -5$ and $b = 6$. But, $(-5, 6)$ is not a solution to the first equation.

(c) Let $(5, 1, -5) = a(1, -2, -3) + b(-2, 3, -4)$, then

$$\begin{cases} a - 2b = 5 \\ -2a + 3b = 1 \\ -3a - 4b = -5. \end{cases}$$

From the first two equations, we see $a = -17$ and $b = -11$. But $(-17, -11)$ is not a solution to the third equation.

2. Let V be the set of all $A \in M_{2\times 2}(\mathbb{R})$ such that $A_{ij} = 0$ if $i > j$. Note that V is a subspace of $M_{2\times 2}(\mathbb{R})$. Let

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Prove that S generates V.

Solution: Recall that Span(S) is the smallest subspace containing S. Since V is a subspace of $M_{2\times 2}(\mathbb{R})$ containing S, Span(S) $\subseteq V$. Note that every matrix A in V can be written as

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$. Since

$$A = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$A \in \text{Span}(S)$. Thus, $\text{Span}(S) = V$.

3. Let V be a vector space and S_1, S_2 subsets of V such that $S_1 \subseteq S_2$.
   (a) Show that $\text{Span}(S_1) \subseteq \text{Span}(S_2)$.
   (b) Show that if S_1 generates V, then S_2 also generates V.

Solution:

(a) Let $x \in \text{Span}(S_1)$, then

$$x = a_1 v_1 + \cdots + a_n v_n$$

for some $a_i \in \mathbb{R}$ and $v_i \in S_1$. Since $S_1 \subseteq S_2$, $v_i \in S_2$. Thus, $x \in \text{Span}(S_2)$.

(b) By Theorem 1.5, $\text{Span}(S_2) \leq V$. Thus, $V = \text{Span}(S_1) \subseteq \text{Span}(S_2) \subset V$ and $\text{Span}(S_2) = V$.

4. Let V be a vector space and S_1, S_2 subsets of V. (Note that if $S = \varnothing$, then we define $\text{Span}(S) = \{0\}$.)
   (a) Prove that $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1) \cap \text{Span}(S_2)$.
   (b) Give an example in which $\text{Span}(S_1 \cap S_2) = \text{Span}(S_1) \cap \text{Span}(S_2)$.
   (c) Give an example in which $\text{Span}(S_1 \cap S_2) \neq \text{Span}(S_1) \cap \text{Span}(S_2)$.

Solution:

(a) Note that $S_1 \cap S_2 \subseteq S_1$ and $S_1 \cap S_2 \subseteq S_2$. By Problem 3 part (a), we have $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_1)$ and $\text{Span}(S_1 \cap S_2) \subseteq \text{Span}(S_2)$.

(b) Let $V = \mathbb{R}$ and $S_1 = S_2 = \{0\}$. Then, $\text{Span}(S_1) = \text{Span}(S_2) = \text{Span}(S_1 \cap S_2) = \{0\}$.

(c) Let $V = \mathbb{R}$, $S_1 = \{0, 1\}$, and $S_2 = \{0, 2\}$. Then, $\text{Span}(S_1) = \text{Span}(S_2) = \text{Span}(S_1) \cap \text{Span}(S_2) = \mathbb{R}$ and $\text{Span}(S_1 \cap S_2) = \{0\}$

5. Consider a system of linear equations

$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

where $a_{ij}, b_i \in \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, n\}$. Suppose that there are two distinct solutions $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$. Show that there are infinitely many solutions to $(*)$.

Solution: Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ be two distinct solutions to $(*)$. Let $c \in \mathbb{R}$ and

$$z = (z_1, \cdots, z_n) = (1-c)x + cy = (1-c)(x_1, \cdots, x_n) + c(y_1, \cdots, y_n).$$

For any $i \in \{1, 2, \cdots, m\}$, we have

$$\begin{aligned} a_{i1}z_1 + \cdots a_{in}z_n &= a_{i1}((1-c)x_1 + cy_1) + \cdots a_{in}((1-c)x_n + cy_n) \\ &= (1-c)(a_{i1}x_1 + \cdots a_{in}x_n) + c(a_{i1}y_1 + \cdots a_{in}y_n) \\ &= (1-c)b_i + cb_i \\ &= b_i, \end{aligned}$$

which implies that z is a solution to $(*)$ for each $c \in \mathbb{R}$. Suppose $c_1 \neq c_2$, $c_1, c_2 \in \mathbb{R}$, and $(1-c_1)x + c_1 y = (1-c_2)x + c_2 y$, then $(c_1 - c_2)x = (c_1 - c_2)y$ and $x = y$. This is a contradiction. Therefore, there are infinitely many solutions.

6. Let $M, N, L \in M_{m\times n}(\mathbb{R})$. If M is row-equivalent to N, then we denote by $M \sim N$. Prove the following.
   (a) $M \sim M$.
   (b) If $M \sim N$, then $N \sim M$.
   (c) If $M \sim N$ and $N \sim L$, then $M \sim L$.

Solution:

(a) By multiplying the first row by 1, one gets M from M.

(b) Since $M \sim N$, there exists a sequence of row operations $P_1, \cdots, P_k$ for some $k \in \mathbb{N}$. For each $i = 1, 2, \cdots, k$, define $\overline{P_i}$ by the undoing process of P_i. (If P_i is swapping two rows, then $\overline{P_i} = P_i$. If P_i is multiplying a row by $c \in \mathbb{R} \setminus \{0\}$, then $\overline{P_i}$ is to multiply the same row by $\frac{1}{c}$. If P_i is to add one row R_s to another row R_t, then $\overline{P_i}$ is to subtract R_s from R_t.) Note that $\overline{P_i}$ is also a row operation for each $i = 1, 2, \cdots, k$. Then, a sequence of row operations $\overline{P_k}, \overline{P_{k-1}}, \cdots, \overline{P_1}$ turns N to M. Thus, $N \sim M$.

(c) If $M \sim N$ and $N \sim L$, then there are two row operations $P_1, P_2, \cdots, P_k$ and $Q_1, Q_2, \cdots, Q_l$ that turn M and N to N and L respectively. Then, a sequence of row operations

$$P_1, P_2, \cdots, P_k, Q_1, Q_2, \cdots, Q_l$$

transforms M to L, which implies that $M \sim L$.

7. Find reduced row-echelon forms of the following matrices.
   (a) $\begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & -3 & 6 \\ 2 & 3 & -1 & 4 & 8 \end{pmatrix}$
   (b) $\begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{pmatrix}$
   (c) $\begin{pmatrix} 1 & 2 & 6 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & -1 & 15 \\ 1 & 3 & 10 & -5 \end{pmatrix}$

Solution:

(a)

$$\begin{pmatrix}1&2&-1&1&5\\1&4&-3&-3&6\\2&3&-1&4&8\end{pmatrix}\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-R_1}\begin{pmatrix}1&2&-1&1&5\\0&2&-2&-4&1\\0&-1&1&2&-2\end{pmatrix}$$

$$\xrightarrow[R_1\to R_1-R_2]{R_3\to R_3+\frac{1}{2}R_2}\begin{pmatrix}1&0&1&5&4\\0&2&-2&-4&1\\0&0&0&0&-\frac{3}{2}\end{pmatrix}$$

$$\xrightarrow[R_2\to\frac{1}{2}R_2]{R_3\to-\frac{2}{3}R_3}\begin{pmatrix}1&0&1&5&4\\0&1&-1&-2&\frac{1}{2}\\0&0&0&0&1\end{pmatrix}\to\begin{pmatrix}1&0&1&5&0\\0&1&-1&-2&0\\0&0&0&0&1\end{pmatrix}.$$

(b)

$$\begin{pmatrix}1&2&2&0&2\\1&0&8&5&-6\\1&1&5&5&3\end{pmatrix}\xrightarrow[R_2\to R_2-R_1]{R_3\to R_3-R_1}\begin{pmatrix}1&2&2&0&2\\0&-2&6&5&-8\\0&-1&3&5&1\end{pmatrix}$$

$$\xrightarrow[R_2\to-R_2]{R_2\leftrightarrow R_3}\begin{pmatrix}1&2&2&0&2\\0&1&-3&-5&-1\\0&-2&6&5&-8\end{pmatrix}$$

$$\xrightarrow[R_3\to R_3+2R_2]{R_1\to R_1-2R_2}\begin{pmatrix}1&0&8&10&4\\0&1&-3&-5&-1\\0&0&0&-5&-10\end{pmatrix}$$

$$\xrightarrow{R_3\to-\frac{1}{5}R_3}\begin{pmatrix}1&0&8&10&4\\0&1&-3&-5&-1\\0&0&0&1&2\end{pmatrix}\to\begin{pmatrix}1&0&8&0&-16\\0&1&-3&0&9\\0&0&0&1&2\end{pmatrix}$$

(c)

$$\begin{pmatrix}1&2&6&-1\\2&1&1&8\\3&1&-1&15\\1&3&10&-5\end{pmatrix}\xrightarrow[R_4\to R_4-R_1]{R_2\to R_2-2R_1,R_3\to R_3-3R_1}\begin{pmatrix}1&2&6&-1\\0&-3&-11&10\\0&-5&-19&18\\0&1&4&-4\end{pmatrix}$$

$$\xrightarrow[R_3\leftrightarrow R_4]{R_2\leftrightarrow R_4}\begin{pmatrix}1&2&6&-1\\0&1&4&-4\\0&-3&-11&10\\0&-5&-19&18\end{pmatrix}$$

$$\xrightarrow[R_4\to R_4+5R_1]{R_1\to R_1-2R_2,R_3\to R_3+3R_2}\begin{pmatrix}1&0&-2&7\\0&1&4&-4\\0&0&1&-2\\0&0&1&-2\end{pmatrix}$$

$$\xrightarrow{R_4\to R_4-R_3}\begin{pmatrix}1&0&-2&7\\0&1&4&-4\\0&0&1&-2\\0&0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&0&3\\0&1&0&4\\0&0&1&-2\\0&0&0&0\end{pmatrix}.$$

8. Consider a system of linear equations

$$(*)\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

where $a_{ij} \in \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, n\}$. Let V be the set of all solutions $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to $(*)$. Show that V is a subspace of $\mathbb{R}^n$.

Solution: First, $(0, 0, \cdots, 0) \in V$ because it is a solution to the system of equations. Let $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in V$, then for each $i = 1, 2, \cdots, m$,

$$a_{i1}(x_1 + y_1) + \cdots a_{in}(x_n + y_n) = (a_{i1}x_1 + \cdots a_{in}x_n) + (a_{i1}y_1 + \cdots a_{in}y_n) = 0.$$

Thus $x + y \in V$. Let $x = (x_1, \cdots, x_n) \in V$ and $c \in \mathbb{R}$, then

$$a_{i1}(cx_1) + \cdots a_{in}(cx_n) = c(a_{i1}x_1 + \cdots a_{in}x_n) = 0.$$

Thus $cx \in V$. Therefore, V is a subspace.