Homework 2

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 13, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.
   (a) $(-2, 0, 3)$, $(1, 3, 0)$, $(2, 4, -1)$
   (b) $(3, 4, 1)$, $(1, -2, 1)$, $(-2, -1, 1)$
   (c) $(5, 1, -5)$, $(1, -2, -3)$, $(-2, 3, -4)$

2. Let V be the set of all $A \in M_{2\times 2}(\mathbb{R})$ such that $A_{ij} = 0$ if $i > j$. Note that V is a subspace of $M_{2\times 2}(\mathbb{R})$. Let
$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$
   Prove that S generates V.

3. Let V be a vector space and S_1, S_2 subsets of V such that $S_1 \subseteq S_2$.
   (a) Show that $\mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$.
   (b) Show that if S_1 generates V, then S_2 also generates V.

4. Let V be a vector space and S_1, S_2 subsets of V. (Note that if $S = \varnothing$, then we define $\mathrm{Span}(S) = \{0\}$.)
   (a) Prove that $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.
   (b) Give an example in which $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.
   (c) Give an example in which $\mathrm{Span}(S_1 \cap S_2) \neq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

5. Consider a system of linear equations
$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$
   where $a_{ij}, b_i \in \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, n\}$. Suppose that there are two distinct solutions $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_n)$. Show that there are infinitely many solutions to $(*)$.

6. Let $M, N, L \in M_{m\times n}(\mathbb{R})$. If M is row-equivalent to N, then we denote by $M \sim N$. Prove the following.
   (a) $M \sim M$.
   (b) If $M \sim N$, then $N \sim M$.

(c) If $M \sim N$ and $N \sim L$, then $M \sim L$.

7. Find reduced row-echelon forms of the following matrices.

(a) $\begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & -3 & 6 \\ 2 & 3 & -1 & 4 & 8 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 & 2 & 0 & 2 \\ 1 & 0 & 8 & 5 & -6 \\ 1 & 1 & 5 & 5 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & 6 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & -1 & 15 \\ 1 & 3 & 10 & -5 \end{pmatrix}$

8. Consider a system of linear equations

$$(*) \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

where $a_{ij} \in \mathbb{R}$ for $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, n\}$. Let V be the set of all solutions $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ to $(*)$. Show that V is a subspace of $\mathbb{R}^n$.