Sec 10.7: The Wave Equation: Vibrations of an Elastic String

Math 285 Spring 2020

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Model

Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level.

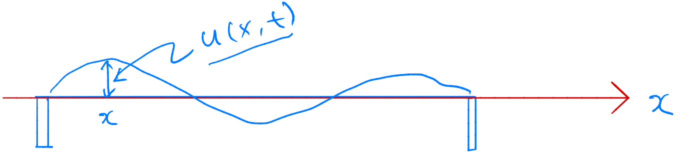
Let the x-axis lie along the string. Let $u(x,t)$ be the vertical displacement by the string at the point x at time t.

For higher dim, replace u_{xx} with $\Delta u = (u_{xx} + u_{yy})$ or $(u_{xx} + u_{yy} + u_{zz})$

Then, $u(x,t)$ satisfies the PDE

$$a^2 u_{xx} = u_{tt}$$

for $0 < x < L$ and $t > 0$. The equation is called the 1-dimensional wave equation.



Model

Since the ends are fixed, we have the boundary conditions

$$u(0,t)=0, \qquad u(L,t)=0$$

for all $t \geq 0$.

We prescribe two initial conditions

$$u(x,0)=\underbrace{f(x)}_{\text{initial displacement}}, \qquad u_t(x,0)=\underbrace{g(x)}_{\text{initial velocity}}$$

for all $0 \leq x \leq L$.

Heat eqn : $\alpha^2 u_{xx} = u_t$ → 1 initial condition

Wave eqn : $a^2 u_{xx} = u_{tt}$ → 2 initial conditions

Nonzero initial displacement

We consider the wave equation

$$a^2 u_{xx} = u_{tt}$$

with boundary condition

$$u(0,t) = 0, \qquad u(L,t) = 0$$

and initial conditions

$f(x)$ $\qquad$ $g(x)$

0 $\qquad$ $g(x)$ } Next lecture.

$$u(x,0) = f(x), \qquad u_t(x,0) = 0.$$

Use the method of separation of variables.

Let $u(x,t) = X(x) \cdot T(t)$.

Nonzero initial displacement

$u(x,t) = X(x) \cdot T(t)$

Equation $\quad a^2 u_{xx} = a^2 X''(x) \cdot T(t) = X(x) \cdot T''(t) = u_{tt}$

Divide by $a^2 X \cdot T$, then

$$\frac{X''}{X} = \frac{1}{a^2} \frac{T''}{T} = -\lambda$$

$$\Rightarrow \quad \begin{cases} X'' + \lambda X = 0 \\ T'' + a^2 \lambda T = 0 \end{cases}$$

Boundary Conditions : $\quad u(0,t) = u(L,t) = 0$

$$X(0) \cdot T(t) = 0 = X(L) \cdot T(t)$$

$$\Rightarrow \quad X(0) = X(L) = 0 .$$

Nonzero initial displacement

Initial Conditions

$$\begin{cases} u(x,0) = f(x) \quad \text{Fourier series} \\ u_t(x,0) = X(x) \cdot T'(0) = 0 \end{cases}$$

$$\Rightarrow \quad T'(0) = 0.$$

$$u(x,t) = X(x) \cdot T(t)$$

$$\Rightarrow \quad ① \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases} \quad \& \quad ② \begin{cases} T'' + \lambda a^2 T = 0 \\ T'(0) = 0 \end{cases}$$

$$\& \quad ③ \; u(x,0) = f(x)$$

Nonzero initial displacement

① $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$

$\lambda_n = \frac{n^2\pi^2}{L^2}$, $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$.

for all $n \in \mathbb{N}$.

② $\begin{cases} T'' + \lambda_n a^2 T = 0, \quad T'' + \frac{a^2 n^2 \pi^2}{L^2} T = 0 \\ T'(0) = 0 \end{cases}$

$T_n(t) = k_1 \cos\left(\frac{an\pi}{L}t\right) + k_2 \cdot \sin\left(\frac{an\pi}{L}t\right)$

$T'(0) = 0 \Rightarrow k_2 = 0 \quad \therefore T_n(t) = \cdot \cos\left(\frac{an\pi}{L}t\right)$

Nonzero initial displacement

$$X_n = \sin\left(\frac{n\pi}{L}x\right), \quad T_n = \cos\left(\frac{an\pi}{L}t\right)$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} c_n \cdot \cos\left(\frac{an\pi}{L}t\right) \cdot \sin\left(\frac{n\pi}{L}x\right)$$

is a solution (by Superposition) to

$$\begin{cases} \underline{a^2 u_{xx} = u_{tt}}, & u(0,t) = u(L,t) = \underline{\underline{0}} \\ , & u_t(x,0) = \underline{\underline{0}}. \end{cases}$$

$$u(x,0) = f(x)$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}x\right)$$

Nonzero initial displacement

where $\quad c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$

Remark

Note that for each $n \in \mathbb{N}$, $u_n(x,t)$ is periodic in time t and position x.

The quantity $n\pi a/L$ for $n \in \mathbb{N}$ are called the *natural frequencies* of the string.

The factor $\sin(n\pi x/L)$ represents the displacement pattern, which is called a natural mode of vibration.

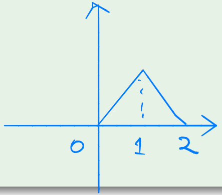
The period of position $2L/n$ is called the *wavelength* of the mode.

$$u_n(x,t) = \cos\left(\frac{an\pi}{L} t\right) \cdot \sin\left(\frac{n\pi}{L} x\right)$$

Example

We consider $4u_{xx} = u_{tt}$ with $u(0,t) = u(2,t) = 0$, $u(x,0) = f(x)$, and $u_t(x,0) = 0$ where

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$



$a = 2$, $L = 2$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cdot \cos(n\pi t) \cdot \sin\left(\frac{n\pi}{2} x\right)$$

$$C_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{2} x\right) dx$$

$$= \int_0^2 f(x) \sin\left(\frac{n\pi}{2} x\right) dx$$

$$C_n = \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

$$= \int_0^1 x \sin\left(\frac{n\pi}{2}x\right) dx + \int_1^2 (2-x) \sin\left(\frac{n\pi}{2}x\right) dx$$

Integration by Parts

$$= \left(\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \cancel{\cos}\left(\frac{n\pi}{2}\right)\right)$$

$$+ \left(\frac{4}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi} \cancel{\cos}\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{8}{n^2\pi^2} \cdot \sin\left(\frac{n\pi}{2}\right).$$