Sec 4.2: Homogeneous Equations with Constant Coefficients

Math 285 Spring 2020

Daesung Kim

Goal

Consider the homogeneous higher order differential equation with constant coefficients

$$L[y] = a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = 0$$

where $a_0, \cdots, a_n$ are real and $a_0 \neq 0$.

Our goal is to find the general solution using $\boxed{\text{the characteristic equation}}$ as we did for the second order differential equations.

Notation

$$\frac{d^n y}{dt^n} = y^{(n)} \neq y^n = \overbrace{y \cdot y \cdots y}^{n \text{ times}}$$

Characteristic Equations

Definition

Consider $\underbrace{a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0}_{}$ $\overset{L[y]}{=}$. The characteristic equation is defined by

$$Z(\lambda) = a_0\lambda^n + \cdots + a_{n-1}\lambda + a_n = 0.$$

Let $y(t) = e^{\lambda t}$, then $y^{(n)} = \lambda^n e^{\lambda t}$.

$$L[y] = e^{\lambda t} \cdot \underbrace{(a_0\lambda^n + \cdots + a_n)}_{Z(\lambda)} = 0$$

Will find the roots of $Z(\lambda)$ and relate with solution of DE.

Characteristic Equations

Definition

Suppose $r_1, r_2, \cdots, r_k$ are the distinct roots for $Z(\lambda)$. One can also write

$$Z(\lambda) = a_0(\lambda - r_1)^{m_1} \cdots (\lambda - r_k)^{m_k}$$

where m_i are positive integers. We call m_i the multiplicity of r_i

Fact Every polynomial can be factored

$$Z(\lambda) = a_0(\lambda - r_1) \cdots (\lambda - r_n)$$

(Fundamental Thm of Algebra)

Ex $Z(\lambda) = (\lambda - 1)^2 (\lambda - 2)^1$

The multiplicity of $1 = 2$

" of $2 = 1$

Case 1: Real Roots

Suppose that $r_1, \cdots, r_n$ are n distinct real roots.

$$Z(\lambda) = a_0 \underbrace{(\lambda - r_1)}_{e^{r_1 t}} \underbrace{(\lambda - r_2)}_{e^{r_2 t}} \cdots \underbrace{(\lambda - r_n)}_{e^{r_n t}}$$

$\{e^{r_1 t}, \cdots, e^{r_n t}\}$: Fundamental set of solutions

(because $W[e^{r_1 t}, \cdots, e^{r_n t}] \neq 0$.)

$$y(t) = C_1 e^{r_1 t} + \cdots + C_n e^{r_n t}.$$

Case 1: Real Roots

Example

Consider $y''' + 2y'' - y' - 2y = 0$.

Characteristic Eqn: $\underbrace{\lambda^3 + 2\lambda^2} - \underbrace{\lambda - 2} = 0$

$$\lambda^2 \underbrace{(\lambda+2)} - \underbrace{(\lambda+2)} = 0$$

$$(\lambda+2)(\lambda^2-1) = 0$$

$$(\lambda+2)(\lambda-1)(\lambda+1) = 0 \qquad \therefore \lambda = -2, -1, 1$$

$$e^{-2t} \qquad e^{-t} \qquad e^{t}$$

$$y(t) = C_1 e^{-2t} + C_2 e^{-t} + C_3 e^{t}.$$

Case 2: Complex Roots

Suppose $Z(\lambda) = 0$ has complex roots. Let $Z(\underline{r + i\mu}) = 0$.

Fact If $r+i\mu$ is a root, then the conjugate $\overline{(r+i\mu)} = r - i\mu$ is also a root. (This is because $a_0, \cdots, a_n$ of $Z(\lambda)$ are real.)

Ex $Z(\lambda) = \lambda - i = 0$. $\underline{i}$: root but $\bar{i} = -i$: not a root.

$$\begin{cases} r + i\mu \\ r - i\mu \end{cases} \Longrightarrow e^{rt}\cos\mu t \ \& \ e^{rt}\sin\mu t.$$

Case 2: Complex Roots

Example

Consider $y''' - y'' + y' - y = 0$.

$$\lambda^3 - \lambda^2 + \lambda - 1 = \lambda^2(\lambda - 1) + (\lambda - 1)$$

$$= (\lambda - 1)(\lambda^2 + 1)$$

$$= (\lambda - 1)(\lambda + i)(\lambda - i) = 0$$

e^t

$\lambda = i, \ -i = \bar{i}$

$e^0 \cdot \cos t, \ e^0 \cdot \sin t$

$= \cos t, \ = \sin t$

$W \neq 0$ (Exercise)

$$y(t) = C_1 e^t + C_2 \cos t + C_3 \sin t.$$

Case 3: Repeated Roots

Suppose $Z(\lambda)$ has repeated roots.

Case 3: Repeated Roots

Suppose $Z(\lambda)$ has repeated roots.

Assume $Z(\lambda)$ has a factor $\boxed{(\lambda - r)^s}$ where r is real and $s > 1$ is the multiplicity of r.

$$Z(\lambda) = a_0 \cdot \underbrace{(\lambda - r)^s}_{\deg = s} (\cdots)$$

$\deg = s \rightsquigarrow$ need s solutions

Recall

$$Z(\lambda) = \underbrace{(\lambda - 1)^2}$$

$$\to e^t, te^t$$

$$(\lambda - r)^s \longrightarrow \underbrace{e^t, te^t, t^2e^t, \cdots, t^{s-1}e^t}_{s \text{ solutions}}.$$

Case 3: Repeated Roots

Suppose $r + i\mu$ is a complex root with multiplicity $s > 1$.

$$Z(\lambda) = a_0 (\lambda - (r + i\mu))^s (\cdots\cdots)$$

$$= a_0 \underbrace{(\lambda - (r+i\mu))^s (\lambda - (r - i\mu))^s}_{\deg = 2s} (\cdots\cdots)$$

$a_0, \cdots, a_n$: real

$\deg = 2s$

$\rightsquigarrow$ Need $2s$ solutions

For $\lambda = r \pm i\mu$, $e^{rt}\cos\mu t$, $e^{rt}\sin\mu t$

$\times t$

$te^{rt}\cos\mu t$, $te^{rt}\sin\mu t$

$\vdots$

$t^{s-1}e^{rt}\cos\mu t$, $t^{s-1}e^{rt}\sin\mu t$.

Case 3: Repeated Roots

Example

Consider $y^{(6)} + 2y^{(4)} + y'' = 0$.

$$\lambda^6 + 2\lambda^4 + \lambda^2 = \lambda^2(\lambda^4 + 2\lambda^2 + 1)$$

$$= \lambda^2(\lambda^2+1)^2$$

$$= \underbrace{\lambda^2}_{\boxed{1,\ t}} \underbrace{(\lambda+i)^{2}(\lambda-i)^{2}}_{\boxed{\cos t,\ \sin t,\ t\cos t,\ t\sin t}} = 0$$

$\times t$

$$y(t) = C_1 + C_2 t + C_3 \cos t + C_4 \sin t + C_5 t\cos t + C_6 t\sin t.$$