Homework 3

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: September 20, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Find the solution sets to the following linear systems.

(a) $\begin{cases} 2x_1 - 3x_2 + x_3 + 7x_4 = 14 \\ 2x_1 + 8x_2 - 4x_3 + 5x_4 = -1 \\ x_1 + 3x_2 - 3x_3 = 4 \\ -5x_1 + 2x_2 + 3x_3 + 4x_4 = -19 \end{cases}$

(b) $\begin{cases} 2x_1 + 4x_2 + 5x_3 + 7x_4 = -26 \\ x_1 + 2x_2 + x_3 - x_4 = -4 \\ -2x_1 - 4x_2 + x_3 + 11x_4 = -10 \end{cases}$

(c) $\begin{cases} 2x_1 + x_2 + 7x_3 - 2x_4 = 4 \\ 3x_1 - 2x_2 + 11x_4 = 13 \\ x_1 + x_2 + 5x_3 - 3x_4 = 1 \end{cases}$

Solution:

(a) The augmented matrix is row-reduces to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Thus the solution set is

$$S = \{(1, -3, -4, 1)\}.$$

(b) The augmented matrix is row-reduces to

$$\begin{pmatrix} 1 & 2 & 0 & -4 & 2 \\ 0 & 0 & 1 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the solution set is

$$S = \{(2 - 2s + 4t, s, -6 - 3t, t) : s, t \in \mathbb{R}\}.$$

(c) The augmented matrix is row-reduces to

$$\begin{pmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & 3 & -4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus the solution set is

$$S = \{(3-2s-t, -2-3s+4t, s, t) : s, t \in \mathbb{R}\}.$$

2. Determine whether the two matrices are row-equivalent.

(a) $\begin{pmatrix} 1 & 4 & 3 & -1 & 5 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 1 & 6 & 5 & 9 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 5 & 7 & 0 \\ 0 & 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$

(b) $\begin{pmatrix} 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ 1 & -2 & -2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$

Solution:

(a) Since

$$\begin{pmatrix} 1 & 4 & 3 & -1 & 5 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 1 & 6 & 5 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 7/5 & 7/5 & 0 \\ 0 & 1 & 2/5 & -3/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and a RREF is unique, they are not row-equivalent.

(b) The second matrix is a RREF of the first. They are row-equivalent.

3. Let V be a vector space over $\mathbb{R}$ and S a subset of V. Show that S is linearly independent if and only if for any $v_1, \cdots, v_n \in S$,

$$x_1 v_1 + \cdots + x_n v_n = 0 \quad \Rightarrow \quad x_1 = \cdots = x_n = 0.$$

Solution: ($\Rightarrow$): Assume S is linearly independent. Suppose that there exist $v_1, \cdots, v_n \in S$ such that $x_1 v_1 + \cdots + x_n v_n = 0$ and $x_j \neq 0$ for some $j \in \{1, \cdots, n\}$. Dividing by x_j, we see that v_j is a linear combination of other vectors. This is a contradiction.

($\Leftarrow$): Suppose that S is linearly dependent. Then there exists $v \in S$ such that

$$v = x_1 v_1 + \cdots + x_n v_n$$

for some $v_1, \cdots, v_n \in S$ and $x_1, \cdots, x_n \in \mathbb{R}$. We then have

$$v + x_1 v_1 + \cdots + x_n v_n = 0$$

and $v, v_1, \cdots, v_n \in S$. By the assumption, we get $1 = x_1 = \cdots = x_n = 0$, which is a contradiction.

4. Let $n \geq 2$ and $V = \mathbb{R}^n$. Define $v_1, v_2, \cdots, v_n \in V$ by $v_n = (1, 0, \cdots, 0)$ and for $i = 2, \cdots, n$, the i-th entry of v_i are 1 and j-th entry is zero for each $j > i$. That is,

$$\begin{aligned} v_1 &= (1, 0, \cdots, 0), \\ v_2 &= (a_{12}, 1, 0, \cdots, 0), \\ &\vdots \\ v_n &= (a_{1n}, \cdots, a_{n-1,n}, 1). \end{aligned}$$

(a) Show that $S = \{v_1, \cdots, v_n\}$ is linearly independent.

(b) Show that $S = \{v_1, \cdots, v_n\}$ generates V.

Solution:

(a) Let $x_1 v_1 + \cdots + x_n v_n = 0$. This gives rise to a system of linear equations

$$\begin{cases} x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0 \\ \vdots \\ x_n = 0. \end{cases}$$

Then, the corresponding augmented matrix A is row-equivalent to

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} & 0 \\ 0 & 1 & \cdots & a_{2n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix}.$$

This tells us that the solution set of A is $\{(0, \cdots, 0)\}$. By Problem 3, S is linearly independent.

(b) Let $v \in \mathbb{R}^n$. We need to show that there exist $x_1, \cdots, x_n$ not all zero such that $x_1 v_1 + \cdots + x_n v_n = 0$. In other words, a system of linear equations

$$\begin{cases} x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = v_1 \\ x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = v_2 \\ \vdots \\ x_n = v_n \end{cases}$$

is consistent. The corresponding augmented matrix A is row-equivalent to

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} & v_1 \\ 0 & 1 & \cdots & a_{2n} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & v_n \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \cdots & 0 & w_1 \\ 0 & 1 & \cdots & 0 & w_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 1 & w_n \end{pmatrix}.$$

Since the last column is not pivot, the linear system has a solution (indeed, exactly one solution). Thus, S spans V.

5. Let $n \geq 1$ and $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Let $S = \{\sin(2^k x) : k = 1, 2, \cdots, n\}$ be a subset of V. Show that S is linearly independent.

Solution: Let $a_1, a_2, \cdots, a_n \in \mathbb{R}$. Suppose

$$\sum_{k=1}^{n} a_k \sin(2^k x) = a_1 \sin(2x) + a_2 \sin(4x) + \cdots + a_n \sin(2^n x) = 0$$

for all $x \in \mathbb{R}$. Put $x = \frac{\pi}{4}$, then

$$\begin{aligned} 0 &= \sum_{k=1}^{n} a_k \sin(2^{k-2}\pi) \\ &= a_1 \sin(\frac{\pi}{2}) + a_2 \sin(\pi) + a_3 \sin(2\pi) + \cdots + a_n \sin(2^{n-2}\pi) \\ &= a_1. \end{aligned}$$

Letting $x = \frac{\pi}{8}$, we get

$$\begin{aligned} 0 &= \sum_{k=2}^{n} a_k \sin(2^{k-2}\pi) \\ &= a_2 \sin(\frac{\pi}{2}) + a_3 \sin(\pi) + a_3 \sin(2\pi) + \cdots + a_n \sin(2^{n-3}\pi) \\ &= a_2. \end{aligned}$$

By repeating this procedure, we conclude that $a_1 = a_2 = \cdots = a_n = 0$, which implies that S is linearly independent.

6. Let V be a vector space over $\mathbb{R}$ and $u, v \in V$ with $u \neq v$. Show that $\{u, v\}$ is linearly dependent if and only if $u = c_1 v$ or $v = c_2 u$ for some $c_1, c_2 \in \mathbb{R}$.

Solution: ($\Rightarrow$): Suppose that $\{u, v\}$ is linearly dependent, then there exists $a, b \in \mathbb{R}$, not both zero, such that $au + bv = 0$. If $a = 0$, then $v = 0$ because $b \neq 0$. In this case, we have $v = 0 \cdot u$. Suppose $a \neq 0$, then $u = cv$ where $c = -b/a$.

($\Leftarrow$): By definition!

7. Let V be a vector space over $\mathbb{R}$.
   (a) Let $u, v \in V$ and $u \neq v$. Prove that $\{u, v\}$ is linearly independent if and only if $\{u + v, u - v\}$ is linearly independent.
   (b) Let $n \in \mathbb{N}$. Let S be the set of n distinct elements $v_1, \cdots, v_n$ in V (that is, $v_i \neq v_j$ for all $i \neq j$). Let $A = (A_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R})$ and define $T = \{w_1, \cdots, w_n\}$ where

$$w_j = A_{1j} v_1 + \cdots + A_{nj} v_n$$

   for each $j = 1, 2, \cdots, n$. Suppose S is linearly independent. Show that T is linearly independent if and only if the linear system associated to $(A, 0)$ has exactly one (trivial) solution.

Solution:

(a) ($\Rightarrow$): Let $a(u + v) + b(u - v) = 0$, then

$$(a + b)u + (a - b)v = 0.$$

Since $\{u, v\}$ is linearly independent, $a + b = 0$ and $a - b = 0$. Soving the linear system for a and b, we get $a = b = 0$. Thus, $\{u + v, u - v\}$ is linearly independent.

($\Leftarrow$): Let $au + bv = 0$, then

$$2au + 2bv = (a + b)(u + v) + (a - b)(u - v) = 0.$$

Since $\{u + v, u - v\}$ is linearly independent, we have $a + b = 0$ and $a - b = 0$. Thus, $a = b = 0$ and so $\{u, v\}$ is linearly independent.

(b) Since S is linearly independent and

$$\begin{aligned} x_1 w_1 + \cdots + x_n w_n &= \sum_{j=1}^{n} x_j w_j \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} A_{ij} v_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} x_j A_{ij} \right) v_i, \end{aligned}$$

$x_1 w_1 + \cdots + x_n w_n = 0$ if and only if $\sum_{j=1}^{n} x_j A_{ij} = 0$ for each $i = 1, 2, \cdots, n$. That is, $x_1 w_1 + \cdots + x_n w_n = 0$ if and only if $(x_1, \cdots, x_n)$ is a solution to the linear system $LS(A, 0)$.

($\Rightarrow$): Suppose T is linearly independent. By Problem 3, if $x_1 w_1 + \cdots + x_n w_n = 0$ then $a_1 = \cdots = a_n = 0$. This implies that the linear system $LS(A, 0)$ has the only solution $(x_1, \cdots, x_n) = (0, \cdots, 0)$.

($\Leftarrow$): Suppose the linear system associated to $(A, 0)$ has exactly one (trivial) solution. Then $x_1 w_1 + \cdots + x_n w_n = 0$ implies $(x_1, \cdots, x_n) = (0, \cdots, 0)$. Thus T is linearly independent.

8. Let W be the set of all (2×2) matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$. Find a basis for W.

Solution: Let

$$\beta = \left\{ A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

For any $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in W$, we have

$$A = aA_1 + bA_2 + cA_3.$$

Thus β spans W. Let

$$aA_1 + bA_2 + cA_3 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = O,$$

then $a = b = c = 0$, which implies β is linearly independent.

9. Find bases for the following subspaces of $\mathbb{R}^5$.

(a) $W_1 = \{(x_1, x_2, x_3, x_4, x_5) : x_1 - x_3 - x_4 = 0\}$

(b) $W_2 = \{(x_1, x_2, x_3, x_4, x_5) : x_2 = x_3 = x_4, x_1 + x_5 = 0\}$

Solution:

(a) Since

$$\begin{aligned} W_1 &= \{(x_1, x_2, x_3, x_4, x_5) : x_1 - x_3 - x_4 = 0\} \\ &= \{(x_3 + x_4, x_2, x_3, x_4, x_5) : x_2, x_3, x_4, x_5 \in \mathbb{R}\} \\ &= \{x_2(0,1,0,0,0) + x_3(1,0,1,0,0) + x_4(1,0,0,1,0) + x_5(0,0,0,0,1) : x_2, x_3, x_4, x_5 \in \mathbb{R}\}, \end{aligned}$$

it is natural to consider

$$\beta = \{(0,1,0,0,0), (1,0,1,0,0), (1,0,0,1,0), (0,0,0,0,1)\}.$$

The above observation explains that β spans W_1. To see β is linearly independent, let

$$x_2(0,1,0,0,0) + x_3(1,0,1,0,0) + x_4(1,0,0,1,0) + x_5(0,0,0,0,1) = 0.$$

Then, $x_2 = \cdots = x_5 = 0$.

(b) Since

$$\begin{aligned} W_2 &= \{(x_1, x_2, x_3, x_4, x_5) : x_2 = x_3 = x_4, x_1 + x_5 = 0\} \\ &= \{(x_1, x_2, x_2, x_2, -x_1) : x_1, x_2 \in \mathbb{R}\} \\ &= \{x_1(1,0,0,0,-1) + x_2(0,1,1,1,0) : x_1, x_2 \in \mathbb{R}\}, \end{aligned}$$

$\gamma = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ is a basis for W_2 as before.

10. Let V be a finite dimensional vector space over $\mathbb{R}$ and $W_1, W_2 \leq V$. Show that if $W_1 \cap W_2 = \{0\}$, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$. (Recall that $W_1 + W_2 = \{x + y : x \in W_1, y \in W_2\}$ is a subspace of V.)

Solution: Let $\dim(W_1) = m$ and $\dim(W_2) = n$. Let $\beta = \{v_1, \cdots, v_m\}$ and $\gamma = \{w_1, \cdots, w_n\}$ be bases for W_1 and W_2. It is enough to show that

$$\alpha = \{v_1, \cdots, v_m, w_1, \cdots, w_n\}$$

is a basis for $W_1 + W_2$ and $|\alpha| = m + n$. Suppose

$$a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots + b_n w_n = 0$$

and the coefficients are not all zero. Then, there exist $a_i \neq 0$ and $b_j \neq 0$ because β and γ are linearly independent. Then,

$$a_1 v_1 + \cdots + a_m v_m = -b_1 w_1 - \cdots - b_n w_n \in W_1 \cap W_2 = \{0\},$$

which contradicts to the assumption. Thus α is linearly independent. For any $x \in W_1 + W_2$, there exist $v \in W_1$ and $w \in W_2$ such that $x = v + w$. Since β and γ are bases for W_1 and W_2, we have

$$x = v + w = a_1 v_1 + \cdots + a_m v_m + b_1 w_1 + \cdots + b_n w_n.$$

Thus α spans $W_1 + W_2$. Therefore, α is a basis for $W_1 + W_2$.