Math 416: Abstract Linear Algebra

Midterm 3 Solution, Fall 2019

Date: November 20, 2019

1. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ with $A^2 = A$.
   (a) (4 points) Show that the only possible eigenvalues for A are 0 and 1.
   (b) (4 points) Show that $\mathcal{R}(L_A) = \mathcal{N}(A - I_n)$.
   (c) (4 points) Show that A is diagonalizable.

Solution:

(a) Let $\lambda \in \mathbb{R}$ be an eigenvalue of A and v the corresponding eigenvector, then

$$\lambda v = Av = A^2 v = A(\lambda v) = \lambda(Av) = \lambda^2 v.$$

Since $v \neq 0$, we get $\lambda^2 - \lambda = 0$, which implies that $\lambda = 0, 1$.

(b) Suppose $v \in \mathcal{R}(L_A)$, then there exists $w \in \mathbb{R}^n$ such that $v = Aw$. Then,

$$(A - I_n)v = (A - I_n)Aw = (A^2 - A)w = 0.$$

Thus, $v \in \mathcal{N}(A - I)$.
Suppose $v \in \mathcal{N}(A - I)$. Then, $(A - I_n)v = 0$ and so $v = Av \in \mathcal{R}(L_A)$.

(c) By the Dimension theorem and Part (b), we have

$$n = \dim(\mathcal{N}(L_A)) + \dim(\mathcal{R}(L_A)) = \dim(\mathcal{N}(A)) + \dim(\mathcal{N}(A - I_n)).$$

Since $E_0 = \mathcal{N}(A)$ and $E_1 = \mathcal{N}(A - I_n)$, we can find n linearly independent vectors from E_0 and E_1. Thus, A is diagonalizable.

2. Let $A = \begin{pmatrix} 0 & 4 & -1 & 1 \\ -2 & 6 & -1 & 1 \\ -2 & 8 & -1 & -1 \\ -2 & 8 & -3 & 1 \end{pmatrix}$, then the characteristic polynomial is $f(t) = (t-2)^2(t+2)(t-4)$. Thus, $2, -2$, and 4 are the eigenvalues for A.

   (a) (3 points) Find the algebraic multiplicities $\mathrm{m_{alg}}(\lambda)$ for $\lambda = 2, -2, 4$.

   (b) (3 points) Find the geometric multiplicities $\mathrm{m_{geo}}(\lambda)$ for $\lambda = 2, -2, 4$.

   (c) (4 points) Determine whether A is diagonalizable or not. Justify your answer.

Solution:

(a) Since the characteristic polynomial is $f(t) = (t-2)^2(t+2)(t-4)$, we have

$$\mathrm{m_{alg}}(2) = 2, \qquad \mathrm{m_{alg}}(-2) = 1, \qquad \mathrm{m_{alg}}(4) = 1.$$

(b) Since $1 \leq \mathrm{m_{geo}}(\lambda) \leq \mathrm{m_{alg}}(\lambda)$, we have

$$\mathrm{m_{geo}}(-2) = 1, \qquad \mathrm{m_{geo}}(4) = 1.$$

To see the geometric multiplicity for $\lambda = 2$, we use row operations to get

$$\begin{aligned} A - 2I &= \begin{pmatrix} -2 & 4 & -1 & 1 \\ -2 & 4 & -1 & 1 \\ -2 & 8 & -3 & -1 \\ -2 & 8 & -3 & -1 \end{pmatrix} \\ &\to \begin{pmatrix} -2 & 4 & -1 & 1 \\ -2 & 8 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\to \begin{pmatrix} 2 & 0 & -1 & -3 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Since the dimension of the null space is the same as the number of non-pivot columns, we see that $\dim(A - 2I) = 2$. Therefore, $\mathrm{m_{geo}}(2) = 2$.

(c) Since the characteristic polynomial splits over $\mathbb{R}$ and $\mathrm{m_{alg}}(\lambda) = \mathrm{m_{geo}}(\lambda)$ for all $\lambda = 2, -2, 4$, A is diagonalizable.

3. Let $A = \frac{1}{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$.

   (a) (3 points) Write down the definition of a transition matrix and show that A is a transition matrix.

   (b) (4 points) Find the eigenspace E_1 corresponding to $\lambda = 1$.

   (c) (3 points) Compute $\lim_{m\to\infty} A^m$.

Solution:

(a) A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ is a transition matrix if $A_{ij} \geq 0$ and $\sum_{i=1}^n A_{ij} = 1$ for each $j = 1, 2, \cdots, n$.

Since the sum of each column is

$$\frac{1}{10}(1+2+3+4) = 1$$

and every entry of A is positive, the given matrix A is a transition matrix.

(b) Let $u = (1, 1, 1, 1)^t$, then

$$Au = \frac{1}{10}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus, u is an eigenvector of A corresponding to 1. Since every entry of A is strictly positive, E_1 is 1-dimensional. Thus,

$$E_1 = \operatorname{Span}(\{u\}) = \{t(1, 1, 1, 1) : t \in \mathbb{R}\}.$$

(c) Since every entry of A is strictly positive, we have learned that the limit exists and

$$L = \lim_{m\to\infty} A^m = \begin{pmatrix} v & v & v & v \end{pmatrix}$$

where v is the unique probability vector in E_1 as a column vector. Thus,

$$\lim_{m\to\infty} A^m = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

4. (8 points) Let V be an inner product space over $\mathbb{R}$ and S a subset of V. Show that if $x, y \in \operatorname{Span}(S)$ and $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in S$ then $x = y$.

Solution: Note that $\langle x, z\rangle = \langle y, z\rangle$ implies $\langle x - y, z\rangle = 0$ for all $z \in S$. Since $x - y \in \operatorname{Span}(S)$, there exist $a_1, \cdots, a_n \in \mathbb{R}$ and $z_1, \cdots, z_n \in S$ such that

$$x - y = a_1 z_1 + \cdots + a_n z_n.$$

Thus,

$$\langle x - y, x - y\rangle = \langle x - y, a_1 z_1 + \cdots + a_n z_n\rangle = \sum_{i=1}^{n} a_i \langle x - y, z_i\rangle = 0.$$

Since $\langle \cdot, \cdot\rangle$ is an inner product, $\langle x - y, x - y\rangle = 0$ implies $x = y$.

5. (10 points) Circle True or False. Do not justify your answer.

(a) **TRUE** False If $A, B \in \mathcal{M}_{3\times 3}(\mathbb{R})$ have the same eigenvalues $\lambda = 1, 6, 7$, then A and B are similar.

> **Solution:** There exist invertible matrices Q_1 and Q_2 such that
>
> $$\mathrm{diag}(1,6,7) = Q_1^{-1} A Q_1 = Q_2^{-1} B Q_2.$$
>
> Thus,
>
> $$A = Q_1 Q_2^{-1} B Q_2 Q_1^{-1} = (Q_2 Q_1^{-1})^{-1} B (Q_2 Q_1^{-1}).$$

(b) True **FALSE** Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. If $A - \lambda I_n$ is onto, then λ is an eigenvalue of A.

> **Solution:** If $A - \lambda I_n$ is onto, then it is invertible. So, $\det(A - \lambda I_n) \neq 0$ and λ is not an eigenvalue of A.

(c) **TRUE** False If $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ are transition matrices, then AB is also a transition matrix.

> **Solution:** By Homework.

(d) True **FALSE** Every finite dimensional inner product space has a unique orthonormal basis.

> **Solution:** In $\mathbb{R}^2$, $\{(1,0),(0,1)\}$ and $\{(\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{\sqrt{3}}{2}, \frac{1}{2})\}$ are two different orthonormal bases.

(e) True **FALSE** Let $V = \mathbb{R}^3$ and define

$$\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = x_1 y_1 + x_2 y_2$$

for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in V$. Then, $\langle \cdot, \cdot \rangle$ is an inner product.

> **Solution:** If $x = (0,0,1)$, then $\langle x, x \rangle = 0$ but $x \neq 0$.