Review: Midterm 3

Math 285 Spring 2020

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Problem 1

Find a solution to $y^{(4)} + y'' = 24t.$ ANS : $4t^3+1$. (A)

Characteristic Eqn : $\lambda^4 + \lambda^2 = \lambda^2 \cdot (\lambda^2+1) = 0$

$\lambda = 0 \; , \; i \; , \; -i$

$2 + 1 + 1 = 4.$

Fundamental Solutions : $1, t, \cos t, \sin t$

$Y(t) = t^2(At+B)$: a particular Solution

$Y'''(t) = 0$, $Y''(t) = 6At + 2B$

$Y^{(4)} + Y'' = 6At + 2B = 24t \quad \therefore A = 4, \; B = 0.$

$Y(t) = 4t^3$

$y(t) = C_1 \cdot 1 + C_2 \cdot t + C_3 \cdot \cos t + C_4 \sin t + 4t^3$ ▨

Problem 2

real eigenvalue

Find the smallest number λ such that $y'' + \lambda y = 0$ with $y'(0) = y'(\pi) = 0$ has a nontrivial solution.

eigenfunction.

① Suppose $\lambda = -\mu^2 < 0$

$$y'' - \mu^2 y = 0$$

$$y(x) = C_1 \cdot e^{\mu x} + C_2 e^{-\mu x}$$

$$y'(x) = C_1 \mu e^{\mu x} + C_2 (-\mu) e^{-\mu x}$$

$$y'(0) = C_1 \mu - C_2 \mu = (C_1 - C_2)\mu^{\neq 0} = 0$$

$$y'(\pi) = C_1 \mu e^{\mu\pi} - C_2 \mu e^{-\mu\pi}$$

$$= (C_1 - C_2 e^{-2\mu\pi}) \cdot \mu e^{\mu\pi \, \neq 0} = 0$$

Problem 2

Find the smallest number λ such that $y'' + \lambda y = 0$ with $y'(0) = y'(\pi) = 0$ has a nontrivial solution. ANS : 0 , (B) .

$$\begin{cases} C_1 - C_2 = 0 \\ C_1 - C_2 e^{-2\mu\pi} = 0 \end{cases}$$

$e^{-2\mu\pi} \neq 1$ because $\mu \neq 0$ $\Rightarrow$ $\begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$

$y(x) = 0$: trivial solution .

There is no negative such λ eigenvalue.

the smallest one .

② $\lambda = 0$

$y'' = 0$

$y(x) = C_1 x + C_2$

$y'(x) = C_1$

$y'(0) = y'(\pi) = C_1 = 0$

$y(x) = C_2$ (Constant)

Problem 3

Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series $f(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2+3m} \cos mt$. Evaluate the integral $\int_{-\pi}^{\pi} f(t) \cos 6t \, dt$.

If $f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi}{L} t\right) + b_m \sin\left(\frac{m\pi}{L} t\right) \right)$

$a_m = \frac{1}{L}\left(f, \cos\left(\frac{m\pi}{L} t\right)\right) = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{m\pi}{L} t\right) dt$

$b_m = \frac{1}{L}\int_{-L}^{L} f(t) \sin\left(\frac{m\pi}{L} t\right) dt$.

$L = \pi$. $b_m = 0$, $a_0 = 0$, $a_m = \frac{(-1)^m}{m^2+3m}$

Problem 3

Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series $f(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2+3m} \cos mt$. Evaluate the integral $\int_{-\pi}^{\pi} f(t) \cos 6t \, dt$. $= \frac{\pi}{54}$. (c)

$L = \pi$

$$a_m = \frac{(-1)^m}{m^2+3m} = \frac{1}{L}\int_{-L}^{L} f(t) \cos\left(\frac{m\pi}{L} t\right) dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos(mt)\, dt$$

$m = 6$

$$a_6 = \frac{1}{6^2+3\cdot 6} = \frac{1}{54} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos(6t)\, dt$$

Problem 4

from left

$L = \pi.$

Consider the function $f(t)$ defined on $\mathbb{R}$ such that $f(t) = f(t + 2\pi)$ and

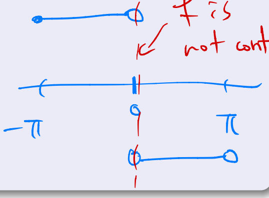
$f(t-) = \lim_{s \to t-} f(s)$

$f(t+) = \lim_{s \to t+} f(s)$

$$f(t) = \begin{cases} 3, & -\pi \le t < 0, \\ e^{\pi^2}, & t = 0, \\ -1, & 0 < t < \pi. \end{cases}$$

from right

f is not conti.



Let $S(t)$ be the Fourier series of $f(t)$. What is $S(0)$?

$$S(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mt) + b_m \sin(mt) \right)$$

Fourier convergence thm : if f, f' are piecewise continuous

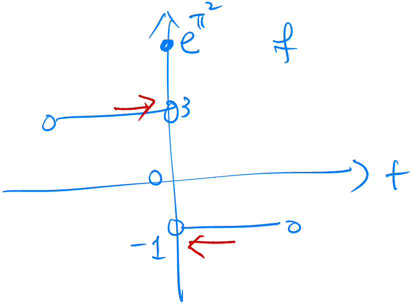
$$= \begin{cases} f(t) & : \text{ if } f \text{ is continuous at } t \\ \frac{1}{2}\left(f(t-) + f(t+)\right) & : \text{ if not} \end{cases}$$

Problem 4

Consider the function $f(t)$ defined on $\mathbb{R}$ such that $f(t) = f(t + 2\pi)$ and

$$f(t) = \begin{cases} 3, & -\pi \le t < 0, \\ e^{\pi^2}, & t = 0, \\ -1, & 0 < t < \pi. \end{cases}$$

Let $S(t)$ be the Fourier series of $f(t)$. What is $S(0)$? ANS : 1 (D)



$$S(0) = \frac{1}{2}\Big(\underbrace{f(0-)}_{=3} + \underbrace{f(0+)}_{=(-1)}\Big)$$

$$= \frac{1}{2}(3 - 1)$$

$$= 1$$

$$f(0) = e^{\pi^2}$$

Problem 5

Let $f(t)$ be a function on $[0, 2]$ given by $f(t) = 2t$. Find the Fourier sine series for $f(t)$ of period 4. $L=2$ ANS : D

$$h(t) = \begin{cases} f(t), & 0 \le t \le 2 \\ -f(-t), & -2 < t < 0 \end{cases} \qquad h(t) = h(t+4)$$

$$h(t) = \sum_{m=1}^{\infty} C_m \sin\left(\frac{m\pi}{2} t\right) = f(t) \text{ on } [0,2]$$

$$C_m = \frac{2}{L} \int_0^L \underline{f(t)} \sin\left(\frac{m\pi}{2} t\right) dt$$

$$= \int_0^2 2t \sin\left(\frac{m\pi}{2} t\right) dt$$

Problem 6

Let f and g be functions defined on $\mathbb{R}$. Which one of the followings is NOT correct?

A If f is even, then f' is odd.

B The function $\sin 3t + \cos 2t$ is periodic with period 2π.

C If f is even and g is odd, then $f(x) + g(x)$ is even.

D If f is even and g is odd, then $\int_{-4}^{4} f(x)g(x)\,dx = 0$.

E If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

A. f : even

$f(x) = f(-x)$

$f'(x) = f'(-x)\cdot(-x)'$

$= -f'(-x) \Rightarrow f'$: odd.

Problem 6

Let f and g be functions defined on $\mathbb{R}$. Which one of the followings is NOT correct?

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C If f is even and g is odd, then $f(x) + g(x)$ is even.

D If f is even and g is odd, then $\int_{-4}^{4} f(x)g(x)\,dx = 0$.

E If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

$\sin 3t$: $\frac{2\pi}{3}$, $\frac{4\pi}{6}$, $\frac{6\pi}{6}$, $\frac{8\pi}{6}$, $\cdots$, 2π.

$\cos 2t$: π, 2π, 3π, $\cdots$

Problem 6

Let f and g be functions defined on $\mathbb{R}$. Which one of the followings is NOT correct?

A If f is even, then f' is odd.

B The function $\sin 3t + \cos 2t$ is periodic with period 2π.

C If f is even and g is odd, then $f(x) + g(x)$ is even. False

D If f is even and g is odd, then $\displaystyle\int_{-4}^{4} f(x)g(x)\,dx = 0$.

E If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

C : For example. $f = 1$ (even)

$g = x$ (odd)

$(f+g)(x) = x + 1 \neq (f+g)(-x) = -x + 1$:

Not even

Problem 6

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C If f is even and g is odd, then $f(x) + g(x)$ is even.

D If f is even and g is odd, then $\displaystyle\int_{-4}^{4} f(x)g(x)\,dx = 0$. : True

E If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

D : f : even g : odd $\Rightarrow$ $f \cdot g$: odd

$$\int_{-L}^{L} (\text{even}) = 2\int_{0}^{L} (\text{even})$$

$$\int_{-L}^{L} (\text{odd}) = 0$$

Problem 6

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B The function $\sin 3t + \cos 2t$ is periodic with period 2π.

C If f is even and g is odd, then $f(x) + g(x)$ is even.

D If f is even and g is odd, then $\int_{-4}^{4} f(x)g(x)\,dx = 0$.

E If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

True.

Let $x \in (4, 6)$ then $x-4 \in (0,2)$

$f(\underline{x-4}) = x-4 = f(x)$

because f is periodic w/ period 4

$\cdots = f(x-4) = f(x) = f(x+4) = \cdots$

Problem 7

Find a pair of ordinary differential equations from the partial differential equation $xu_{xx} + u_t = 0$ using the method of separation of variables.

Start with $u(x,t) = X(x) \cdot T(t)$

ANS: B.

$x \cdot u_{xx} = x \cdot X''(x) \cdot T(t)$

$u_t = X(x) \cdot T'(t)$

$x \cdot u_{xx} + u_t = x \cdot X'' \cdot T + X \cdot T' = 0$

Divide by $X \cdot T$ = u

(or λ)

$$x \cdot \frac{X''}{X} = -\frac{T'}{T} = -\lambda$$

$x \cdot X'' + \lambda X = 0$ & $T' - \lambda T = 0$.

Problem 8

Consider the heat conduction problem

$L = 3$

$\alpha^2 = 5$ Thermal diffusivity

$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

A If $f(x) = \sin \pi x$, then the solution is $u(x,t) = e^{-5\pi^2 t} \sin \pi x$.

Recall

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{\alpha^2 m^2 \pi^2}{L^2} t} \cdot \sin\left(\frac{m\pi}{L} x\right)$$
$$= \sum_{m=1}^{\infty} C_m e^{-\frac{5 m^2 \pi^2}{9} t} \sin\left(\frac{m\pi}{3} x\right)$$

$$C_m = \frac{2}{3} \int_0^3 f(x) \sin\left(\frac{m\pi}{3} x\right) dx$$

Problem 8

Consider the heat conduction problem

$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

A If $f(x) = \sin \pi x$, then the solution is $u(x,t) = e^{-5\pi^2 t} \sin \pi x$.

$$C_m = \frac{2}{3}\int_0^3 \overbrace{\underbrace{\sin(\pi x)}_{\text{odd}} \cdot \underbrace{\sin\left(\frac{m\pi}{3}x\right)}_{\text{odd}}}^{\text{even}} dx$$

$$= \frac{1}{3}\int_{-3}^3 \sin(\pi x) \cdot \sin\left(\frac{m\pi}{3}x\right) dx$$

$$= \begin{cases} 1 & \text{if } m = 3 \\ 0 & \text{otherwise} \end{cases}$$

(orthogonality of sine functions

Problem 8

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$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

A If $f(x) = \sin \pi x$, then the solution is $u(x,t) = e^{-5\pi^2 t} \sin \pi x$. True.

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{5n^2\pi^2}{9}t} \sin\left(\frac{n\pi}{3}x\right)$$

$$= e^{-\frac{5}{9}\pi^2 t} \sin(\pi x)$$

Problem 8

Consider the heat conduction problem

$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

B If $u(x,t)$ and $v(x,t)$ are solutions, then $u(x,t) + v(x,t)$ is also a solution.

False

If $f(x) = 1$

$u(x,0) = 1 = v(x,0)$

$u(x,0) + v(x,0) = 2 \neq 1 = f(x)$

$\Rightarrow$ $u + v$ is NOT a solution

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$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

C The thermal diffusivity is $\cancel{3}$. 5 $3 = L$

Problem 8

Consider the heat conduction problem

$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?

$L=3$

D The solution is

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{5m^2\pi^2}{3}t} \sin\left(\frac{m\pi}{3}x\right)$$

$\times 2$

for some C_m.

False.

Problem 9

What is the steady state solution $v(x)$ for the following problem?

$$5u_{xx} = u_t, \qquad 0 < x < 6, \quad t \geq 0,$$

$$u(0,t) = 10, \qquad u(6,t) = 2.$$

ANS: $v(x) = -\frac{4}{3}x + 10.$ (E)

Recall $v(x) = \lim_{t \to \infty} u(x,t)$

$5u_{xx} = u_t$ as $t \to \infty$

$\downarrow \qquad \downarrow$

$5v'' = 0 \Rightarrow v'' = 0$

$\Rightarrow v(x) = Ax + B$

$v(0) = 10, \quad v(6) = 2$

$v(0) = A \cdot 0 + B = B = 10, \quad v(6) = 6A + 10 = 2, \quad A = -\frac{4}{3}$