Practice Exam

1 (a) W is NOT a subspace because $0 \notin W$.

(b) Claim: ~~$\cancel{W}$~~ $V = \mathrm{Span}(W)$

For any $A \in V$,

if $tr(A) \neq 0$, then ~~$cA \in W$~~

$$A = tr(A) \cdot \left(\frac{1}{tr(A)} A\right)$$

$$= c \cdot B \quad \text{where} \quad B \in W$$

if $tr(A) = 0$, then

$$A = \frac{1}{2}\left(A + \frac{1}{n} I_n\right) - \frac{1}{2}\left(\cancel{A} - A + \frac{1}{n} I_n\right)$$

$$= \frac{1}{2} B - \frac{1}{2} C$$

where $B, C \in W$.

2 (a) $V = \mathbb{R}^2$, $W_1 = \{(x,0)\}$, $W_2 = \{(0,y)\}$

then $W_1 \cup W_2$ is not a subspace.

(b) For any V and a nontrivial subspace $W \leq V$,

let $W = W_1 = W_2$ then $W_1 \cup W_2 = W \leq V$.

3 Let $\beta = \{v_1, \cdots, v_4\}$ be a basis for W.

<u>Claim</u>: $\gamma = \{v_1, \cdots, v_4, v\}$ lin. indep.

Let $a_1 v_1 + \cdots + a_4 v_4 + a_5 v = 0$.

If $a_5 = 0$, then $a_1 = \cdots = a_4$ because β is lin. indep.

If $a_5 \neq 0$ then $v = -\frac{a_1}{a_5} v_1 - \cdots - \frac{a_4}{a_5} v_4$

$\in$ ~~Span(W)~~ W. Contradiction!

$\therefore \gamma$ is a basis for V.

4

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

(a) $\text{rank}(A) = \dim(\text{Col}(A))$
$= \dim(\text{Row}(A))$
$= 2.$

(b) Solution set $= \mathcal{N}(A)$
$= \{ (t, -t, 2t) \ ; \ t \in \mathbb{R} \}$

(c)
$$\begin{pmatrix} 0 & 2 & 1 & a \\ 1 & 3 & 1 & b \\ -1 & 1 & 1 & c \end{pmatrix} \to \begin{pmatrix} 1 & 3 & 1 & b \\ 0 & 2 & 1 & a \\ -1 & 1 & 1 & c \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 3 & 1 & b \\ 0 & 2 & 1 & a \\ 0 & \underset{0}{\cancel{4}} & \underset{0}{\cancel{2}} & \underset{-2a}{b+c} \end{pmatrix}$$

No solution if $\underline{b+c \neq 2a}$.

5 (a)

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) $T^3(v_1) = v_1$, $T^3(v_2) = v_2$, $T^3(v_3) = v_3$

$\Rightarrow \quad T^3 = I_V$

(c)

$$[I_V]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} v_1 = u_1 \\ v_2 = u_2 - u_1 \\ v_3 = u_3 - u_2 \end{pmatrix}$$

6 (a) $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{3}$

(b) $\det(2 \cdot A^{-1}) = 2^3 \cdot \det(A^{-1}) = \dfrac{8}{3}$

(c) $\det((2A)^{-1}) = \dfrac{1}{\det(2A)} = \dfrac{1}{8 \cdot \det A} = \dfrac{1}{24}$.

(d) $\det A^t = \det A = 3$

(e) $\det B = \det \begin{pmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{pmatrix} = -\det A = -3$.

7 Let $x \in N(A^{-1}+B^{-1})$, then

$A^{-1}x = -B^{-1}x$

$A(A^{-1}x) = x = -BA^{-1}x$

$\therefore (A+B)(A^{-1}x) = 0$ i.e. $A^{-1}x \in N(A+B)$

Since $A+B$ is invertible, $A^{-1}x = 0$ $\therefore x = 0$

$\therefore N(A^{-1}+B^{-1}) = \{0\} \Rightarrow A^{-1}+B^{-1}$ invertible.

8 (a) $f(t) = \det\begin{pmatrix} 2-t & 1 \\ 1 & 2-t \end{pmatrix} = t^2 - 4t + 3$

$= (t-1)(t-3)$

(b) $A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \therefore E_1 = \{(t, -t)\}$

$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \quad \therefore E_3 = \{(t, t)\}$

(c) $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \beta = \{v_1, v_2\}$

$[L_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$= [I]_\gamma^\beta \, [L_A]_\gamma \, [I]_\beta^\gamma$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$= Q^* A Q$

$\therefore D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$

9 $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$f(t) = \det\begin{pmatrix} -t & 1 & 1 \\ 0 & -t & 1 \\ 0 & 0 & -t \end{pmatrix} = -t^3$

$E_0 = \{(t, 0, 0)\}, \quad \dim(E_0) = 1 \neq 3$

$\therefore$ A is not diagonalizable.

10 (a) $\cancel{A^2 = \left(\frac{3}{8} \quad \frac{1}{2} \quad \frac{3}{8}\right.}$

$$A = \frac{1}{4}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

$$A^2 = \frac{1}{16}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{pmatrix} = \frac{1}{16}\begin{pmatrix} 6 & 8 & 6 \\ 2 & 4 & 4 \\ 8 & 4 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{3}{8} \end{pmatrix} \quad \therefore \quad A \text{ is regular.}$$

(b) $$A - I = \frac{1}{4}\begin{pmatrix} -2 & 2 & 1 \\ 0 & -2 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$N(A-I) = E_1 = \{(t, \underset{\frac{1}{2}t}{\cancel{2t}}, t) : t \in \mathbb{R}\}$$

(c) No. because

$$A^t = \frac{1}{4}\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and}$$

$$\frac{1}{4}(2+2+1) \neq 1.$$

11 (a)

- $\langle x+z, y\rangle = 4(x_1+z_1)y_1 + (x_2+z_2)y_2$
  $= (4x_1y_1 + x_2y_2) + (4z_1y_1 + z_2y_2)$
  $= \langle x, y\rangle + \langle z, y\rangle$
- $\langle cx, y\rangle = 4(cx_1)y_1 + (cx_2)\cdot y_2$
  $= c(4x_1y_1 + x_2y_2)$
  $= c\langle x, y\rangle$
- $\langle x, y\rangle = 4x_1y_1 + x_2y_2$
  $= 4y_1x_1 + y_2\cdot x_2$
  $= \langle y, x\rangle$
- $\langle x, x\rangle = 4x_1^2 + x_2^2 \geq 0$ and

$=$ only if $x_1 = 0$ & $x_2 = 0$ i.e. $x = 0$

$\therefore \langle , \rangle$ is an inner product.

(b) Let $\beta = \{e_1, e_2\} = \{(1,0), (0,1)\}$

$v_1 = \frac{1}{\|e_1\|} e_1 = \frac{1}{2}(1,0)$

$\langle e_1, e_1\rangle = 4 = \|e_1\|^2$

$v_2' = e_2 - \langle v_1, e_2\rangle v_1$

$= (0,1) - 0$

$v_2 = \frac{1}{\|v_2'\|} v_2' = (0,1)$

$\therefore \{\frac{1}{2}(1,0), (0,1)\}$: ON.

(c) $(L_A)^* = L_B$

$\iff \langle L_A(x), y\rangle = \langle x, L_B(y)\rangle \quad \forall x, y \in \mathbb{R}^2$

$$\begin{aligned}
RHS &= \langle x, L_B(y)\rangle \\
&= \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ 2y_1 + y_2 \end{pmatrix} \right\rangle \\
&= 4x_1y_1 + 2x_2y_1 + x_2y_2
\end{aligned}$$

$$\begin{aligned}
LHS &= \langle L_A(x), y\rangle \\
&= \left\langle \begin{pmatrix} x_1 + \frac{1}{2}x_2 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \\
&= 4(x_1 + \tfrac{1}{2}x_2)y_1 + x_2y_2 \\
&= 4x_1y_1 + 2x_2y_1 + x_2y_2
\end{aligned}$$

12 (a)

$$\begin{aligned}
T(cx_1 + x_2) &= \langle cx_1 + x_2, y\rangle z \\
&= (c\langle x_1, y\rangle + \langle x_2, y\rangle) z \\
&= c\langle x_1, y\rangle z + \langle x_2, y\rangle z \\
&= cT(x_1) + T(x_2)
\end{aligned}$$

(b)

$$\begin{aligned}
&\langle T(x_1), x_2\rangle \\
&= \langle \langle x_1, y\rangle z, x_2\rangle \\
&= \langle x_1, y\rangle \langle z, x_2\rangle \\
&= \langle x_1, \overline{\langle z, x_2\rangle}\, y\rangle \\
&= \langle x_1, \underline{\langle x_2, z\rangle y}\rangle
\end{aligned}$$

$\therefore$ $\underline{T^*(x) = \langle x, z\rangle y}$

13 (a) $\langle w_1, w_1\rangle = \langle v_1, v_1\rangle = 1$

$\langle w_1, w_2\rangle = \langle v_1, \frac{\sqrt{3}}{2}v_2 + \frac{1}{2}v_3\rangle$

$= \frac{\sqrt{3}}{2}\langle v_1, v_2\rangle + \frac{1}{2}\langle v_1, v_3\rangle = 0$

$\langle w_1, w_3\rangle = \langle v_1, \frac{1}{2}v_2 - \frac{\sqrt{3}}{2}v_3\rangle$

$= \frac{1}{2}\langle v_1, v_2\rangle - \frac{\sqrt{3}}{2}\langle v_1, v_3\rangle = 0$

$\langle w_2, w_2\rangle = \langle \frac{\sqrt{3}}{2}v_2 + \frac{1}{2}v_3, \frac{\sqrt{3}}{2}v_2 + \frac{1}{2}v_3\rangle$

$= \frac{3}{4}\langle v_2, v_2\rangle + \sqrt{3}\langle v_2, v_3\rangle + \frac{1}{4}\langle v_3, v_3\rangle$

$= 1$

$\langle w_2, w_3\rangle = \langle \frac{\sqrt{3}}{2}v_2 + \frac{1}{2}v_3, \frac{1}{2}v_2 - \frac{\sqrt{3}}{2}v_3\rangle$

$= \frac{\sqrt{3}}{4}\langle v_2, v_2\rangle + \left(\frac{1}{4} - \frac{3}{4}\right)\langle v_2, v_3\rangle - \frac{\sqrt{3}}{4}\langle v_3, v_3\rangle$

$= 0$

$\langle w_3, w_3\rangle = 1.$

(b) $\dim(\{w_3\}^\perp) = 2$

$w_1, w_2 \in \{w_3\}^\perp$

$\{w_1, w_2\}$ lin. indep.

$\therefore \{w_1, w_2\}$ is a basis of $\{w_3\}^\perp$.

14 (a) $A^* = \begin{pmatrix} \overline{\langle v, v \rangle} , & \overline{\langle w, v \rangle} \\ \overline{\langle v, w \rangle} , & \overline{\langle w, w \rangle} \end{pmatrix}$

$= \begin{pmatrix} \langle v, v \rangle , & \langle v, w \rangle \\ \langle w, v \rangle , & \langle w, w \rangle \end{pmatrix}$

$= A$

(b) $x A x^* = \left(x_1 \langle v, v \rangle + x_2 \langle w, v \rangle , \; x_1 \langle v, w \rangle + x_2 \langle w, w \rangle \right) \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix}$

$= x_1 \langle v, v \rangle \overline{x_1} + x_2 \langle w, v \rangle \overline{x_1} + x_1 \langle v, w \rangle \overline{x_2} + x_2 \langle w, w \rangle \overline{x_2}$

$= \langle x_1 v , x_1 v \rangle + \langle x_2 w , x_1 v \rangle + \langle x_1 v , x_2 w \rangle + \langle x_2 w , x_2 w \rangle$

$= \langle x_1 v + x_2 w , x_1 v + x_2 w \rangle$

≥ 0.

15 Let $x \in N(T) \cap R(T)$

then $\exists y$ s.t. $x = Ty$

$\langle x, x \rangle = \langle x, Ty \rangle = \langle \underset{0}{\underset{\shortparallel}{Tx}}, y \rangle = 0$

$\therefore x = 0$.

16 $A^* = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \neq A \quad \therefore$ Not self-adjoint.

$$A^*A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$AA^* = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\therefore$ A is <u>normal</u>.