Sec 10.1: Two-Point Boundary Value Problems, part 1

Math 285 Spring 2020

Instructor: Daesung Kim

Overview

Partial DE ① Heat Eqn : $\alpha^2 u_{xx} = u_t$ (10.6)

② Wave Eqn : $\alpha^2 u_{xx} = u_{tt}$ (10.7)

③ Laplace Eqn : $u_{xx} + u_{yy} = 0$ (10.8)

One Eqn with Two variables

$\downarrow$ Separation (10.5) $\leftarrow$ Fourier Series (10.2–4)

Two ordinary DE (10.1)

Eigenvalues & Eigenfunctions

The initial value problem

We consider $y'' + p(t)y' + q(t)y = g(t)$. Previously, the initial value problem refers the DE with the condition of the form

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

the same time input.

Two-point boundary value problem

We will consider $y'' + p(x)y' + q(x)y = g(x)$ with ← the same as before.

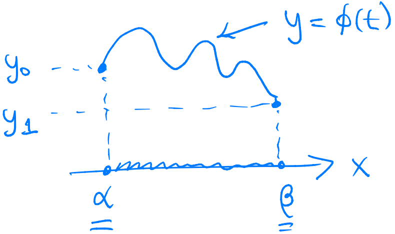
$$y(\alpha) = y_0, \qquad y(\beta) = y_1$$: Diff. conditions

different inputs

for some $\alpha < \beta$.

This is call a two-point boundary value problem.

Our goal is to find solutions $y = \phi(x)$ that satisfies the DE in $x \in (\alpha, \beta)$ with the boundary condition.



Some time

Replace $y(\alpha)$ with $y'(\alpha)$

$y(\beta)$ with $y'(\beta)$

Two-point boundary value problem

Definition

A two-point boundary value problem is called *homogeneous* if $g(t) = y_0 = y_1 = 0$. Otherwise, we call it *nonhomogeneous*.

BVP is homogeneous if DE is homogeneous $(g(x) = 0)$

and $y(\alpha) = y(\beta) = 0$.

Ex

$$\begin{cases} y'' - 2y' + 3y = 0 \quad (\text{hom.}) \\ y(0) = 0, \ \underline{y(1) = 1}. \end{cases} \quad \text{Nonhom.}$$

$$\begin{cases} y'' - 2y' = e^t \\ y(0) = y(1) = 0 \end{cases} \quad \text{Nonhom.}$$

Nonhomogeneous case

Example (Nonhomogeneous with a unique solution)

Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\frac{\pi}{2}) = 2$.

Gen. Solution: $\lambda^2 + 1 = 0$ $\quad \lambda = \pm i$

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = C_1 \cdot \cos(0) = C_1 = 1$$

$$y\left(\frac{\pi}{2}\right) = C_2 \sin\left(\frac{\pi}{2}\right) = C_2 = 2$$

$$\therefore \quad y(x) = \underline{\cos x + 2 \sin x}$$

a unique solution.

Nonhomogeneous case

Example (Nonhomogeneous with infinitely many solution)

Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = -1$.

$$y(x) = c_1 \cos x + c_2 \sin x$$

$$y(0) = c_1 = 1$$

$$y(\pi) = c_1 \underbrace{\cos(\pi)}_{= -1} + c_2 \underbrace{\sin(\pi)}_{= 0}$$

$$= -c_1 = -1$$

Choose any number

$$y(x) = \cos x + c_2 \sin x$$

Infinitely many solutions.

Nonhomogeneous case

Example (Nonhomogeneous with no solutions)

Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = 2.$ -1

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = C_1 = 1$$

$$y(\pi) = \underbrace{\cos \pi}_{=-1} + C_2 \underbrace{\sin \pi}_{=0}$$

$$= -1 \neq 2$$

No Solutions

Nonhom. BVP

(i) No Solution

(ii) Unique Sol

(iii) Infinitely many Solution

Homogeneous case

Example (~~No~~nhomogeneous with infinitely many solutions)

Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\pi) = 0$.

$$y(x) = C_1 \cos x + C_2 \sin x$$

$$y(0) = C_1 = 0$$

$$y(\pi) = -C_1 = 0$$

$$y(x) = C_2 \cdot \sin x$$

No conditions on C_2

Infinitely many solutions.

Homogeneous case

Example (~~Non~~homogeneous with a unique solution)

Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\frac{\pi}{2}) = 0$.

$y(x) = C_1 \cos x + C_2 \sin x$

$y(0) = C_1 = 0$

$y(\frac{\pi}{2}) = C_2 = 0$

$y(x) = \underline{\underline{0}}$

unique solution

Hom. BVP

$\Rightarrow$ Trivial Solution $y=0$

(i) Unique solution ($y=0$)

(ii) Nontrivial Solution.

- Nonhomogeneous BVP
  - No solutions
  - a unique solution
  - Infinitely many solution

☆

- Homogeneous BVP
  - Trivial Solution ($y=0$)
  - Nontrivial Solution. $y \to cy$ is also a solution
    - ↳ Infinitely many solutions.