Math 285
Midterm 1 Practice

Full Name: ______________________________
Net ID: ______________________________
UIN: ______________________________

Circle your class section: B1 C1 F1

- You must not communicate with other students during this test.
- No written materials of any kind allowed.
- No phones, calculators, iPods or electronic devices of any kind allowed for ANY reason, including checking the time (you may use a simple wristwatch).
- Do not turn this page until instructed to do so.
- There are many different versions of this exam.
- Violations of academic integrity (in other words, cheating) will be taken extremely seriously.

Fill in the following answers on the scantron form now:

93. A
94. A
95. D
96. C

Multiple Choice Questions. Mark answers to questions 1 to 8 on your scantron form

1. (3 points) The second order differential equation

$$\frac{d^2y}{dt^2} = \frac{dy}{dt}\sin y$$

is equivalent to which pair of first order differential equations:

(A) $\frac{dw}{dt} = w\sin y,\ \ \frac{dy}{dt} = w(y)$

(B) ★ $w\frac{dw}{dy} = w\sin y,\ \ \frac{dy}{dt} = w(y)$

(C) $\frac{dw}{dy} = w\sin y,\ \ \frac{dy}{dt} = w(y)$

(D) $w\frac{dw}{dt} = w\sin y,\ \ \frac{dy}{dt} = w(y)$

(E) $\frac{dy}{dt} = w\sin y,\ \ \frac{dw}{dy} = y(w)$

Solution. Set $w = dy/dt$ and use $d^2y/dt^2 = d/dt(dy/dt) = dy/dt \times d/dy(dy/dt) = wdw/dy$

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2. (3 points) The Wronskian of the two functions

$$y_1(t) = e^{3t^2+8} \quad \text{and} \quad y_2(t) = e^{3t^2-4}$$

is:

(A) None of the above.

(B) $12te^{3t^2+4}$

(C) ★ 0

(D) $12te^{6t^2+4}$

(E) $12te^{9t^4+12t^2-32}$

Solution.

$$\begin{aligned} W &= e^{3t^2+8}\left(e^{3t^2-4}\right)' - e^{3t^2-4}\left(e^{3t^2+8}\right)' \\ &= 6te^{(3t^2+8)+(3t^2-4)} - 6te^{(3t^2+8)+(3t^2-4)} \\ &= 6te^{6t^2+4} - 6te^{6t^2+4} = 0 \end{aligned}$$

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3. (3 points) The existence and uniqueness theorem for linear differential equations ensures that the solution of

$$(t-2)(t+3)y'' + t(t-2)y' + t^2 y = \frac{1}{t+5}, y(-6) = 2, y'(-6) = 5$$

exists for all t in the region defined by:

(A) $(-\infty, -5) \cup (-5, -3) \cup (-3, 2) \cup (2, \infty)$

(B) $(2, \infty)$

(C) ★ $(-\infty, -5)$

(D) $(-3, \infty)$

(E) $(-5, -3)$

Solution. The points of discontinuity are at $t = -5, -3, 2$ and $t_0 = -6$, so the region is the interval $(-\infty, -5)$.

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4. (3 points) The differential equation

$$\frac{dy}{dt} = \frac{t+4y}{5t-y}$$

can be transformed by a substitution into the separable differential equation:

(A) $(5t-y)dy = (t+4y)dt$

(B) ★ $t\frac{dv}{dt} + v = \frac{1+4v}{5-v}$

(C) $\frac{dv}{dt} = \frac{1+4v}{5-v}$

(D) $t\frac{dv}{dt} + v = \frac{t+4y}{5t-y}$

(E) $t\frac{dv}{dt} - v = t^2\frac{1+4v}{5-v}$

Solution. Make the substitution $v = y/t$

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5. (3 points) The differential equation

$$y' + (\sin t)\, y = (\cos t)\, y^5$$

can be transformed by a substitution into the linear equation:

(A) $v' + 4(\sin t)v = 4\cos t$

(B) $v' - 2(\sin t)v = -2\cos t$

(C) ★ $v' - 4(\sin t)v = -4\cos t$

(D) $v' + 2(\sin t)v = 2\cos t$

(E) $v' - (\sin t)v = -\cos t$

Solution. This is a Bernoulli equation. Make the substitution $v = y^{-4} = \frac{1}{y^4}$

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6. (3 points) The order of the differential equation below is (higher derivatives are denoted by a Roman numeral, i.e. $y^{(iv)}$ is d^4y/dt^4)

$$(y')^5 + 3(y')^4 + 2(y')^3 - (y')^2 + 10y' = 0$$

(A) 2

(B) 5

(C) ★ 1

(D) 4

(E) 3

Solution. 1 because y' is the highest derivative that appears

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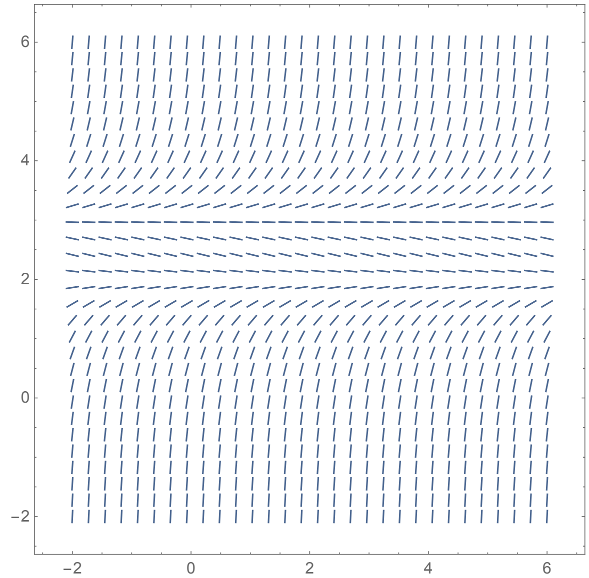
7. (3 points) Identify which differential equation below corresponds to the following direction field:



(A) $y' = 2 - y$

(B) ★ $y' = (y-3)(y-2)$

(C) $y' = -y(y-2)$

(D) $y' = y - 3$

(E) $y' = y(y-3)$

Solution. $y' = (y-3)(y-2)$ because there are equilibria at $y = 2$ and $y = 3$ and $y = 2$ is stable (check the signs of $(y-3)(y-2)$).

8. (3 points) For the linear differential equation

$$t^2 y' - y t^2 \sin t = t \cos t$$

an integrating factor is:

(A) $e^{-\sin t + t\cos t}$

(B) $\frac{1}{\cos t}$

(C) ★ $e^{\cos t}$

(D) $e^{-\cos t}$

(E) $\cos t$

Solution. $\mu = e^{-\int \sin t dt} = e^{\cos t}$

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9. (10 points) Consider the initial value problem

$$\frac{dy}{dt} = \left(\frac{3y+4}{4t+5}\right)^2, y(-1) = -1$$

a) Find the general solution of the differential equation (you do not need to solve for y).

b) Find the unique solution satisfying the initial condition (you do need to solve for y this time).

c) On what interval does the solution exist?

Solution. a) This problem is separable, so we get

$$\int \frac{dy}{(3y+4)^2} = \int \frac{dt}{(4t+5)^2} + C \Rightarrow -\frac{1}{3}\frac{1}{3y+4} = -\frac{1}{4}\frac{1}{4t+5} + C$$

b) Applying the initial condition gives us

$$
\begin{aligned}
-\frac{1}{3}\frac{1}{-3+4} &= -\frac{1}{4}\frac{1}{-4+5} + C \Rightarrow C = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12} \\
-\frac{1}{3}\frac{1}{3y+4} &= -\frac{1}{4}\frac{1}{4t+5} - \frac{1}{12} \Rightarrow \frac{4}{3y+4} = \frac{3}{4t+5} + 1 = \frac{4t+8}{4t+5} \\
y &= \frac{4}{3}\left(-1 + \frac{4t+5}{4t+8}\right)
\end{aligned}
$$

c) We cannot permit $t = -5/4$ since this is a point of discontinuity of the right side of the differential equation. For any t larger than this the solution exists and the initial value at $t = -1$ is included. So the domain is $(-5/4, \infty)$.

10. (10 points) Consider the initial value problem

$$y'' - 4y' + 4y = 0, y(0) = a, y'(0) = b$$

a) Find a fundamental set of solutions for the differential equation.

b) What is the general solution of the differential equation?

c) What is the solution of the initial value problem?

Solution. a) The characteristic equation and its solution are

$$r^2 - 4r + 4 = 0 = (r-2)^2 \Rightarrow r = 2, 2$$

Because of the repeated root, a fundamental set of solutions is

$$y_1 = e^{2t}, y_2 = te^{2t}$$

b) The general solution is then

$$y = c_1 e^{2t} + c_2 te^{2t}$$

with c_1 and c_2 arbitrary constants.

c) First compute: $y' = 2c_1 e^{2t} + 2c_2 te^{2t} + c_2 e^{2t}$. Then the initial conditions give us

$$c_1 = a, 2c_1 + c_2 = b \Rightarrow y = ae^{2t} + (b - 2a)te^{2t}$$

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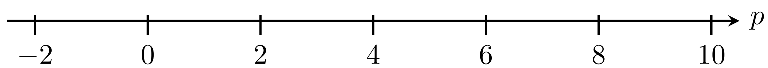
11. (10 points)

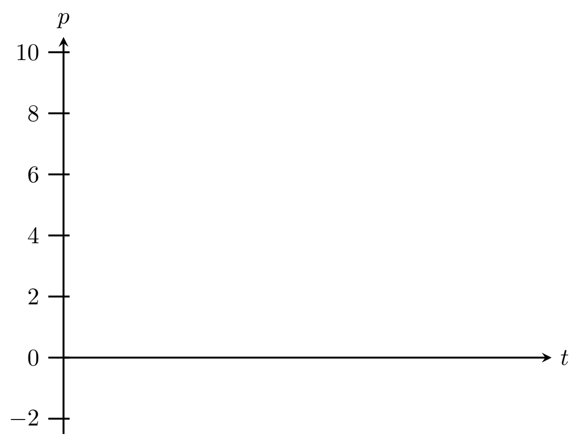
a) A certain species has population level $p(t)$ modelled by the initial value problem

$$\frac{dp}{dt} = -3p(2-p)(8-p), p(0) = p_0 > 0$$

On the phase line below label all critical values/equilibrium solutions for this model and the direction of flow between them.

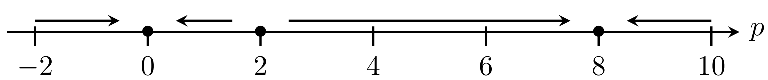


b) On the axes below draw some representative solution curves, including the equilibrium solutions and some solution curves above and below each of them.

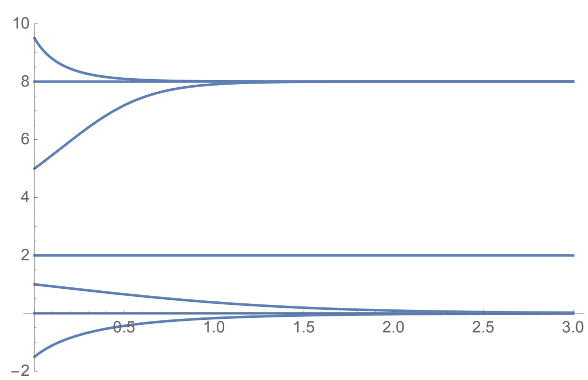


c) Under what conditions on p_0 will the population go extinct? When will extinction actually happen?

Solution. a)



b)



c) Extinction occurs when $p_0 < 2$. But it will take infinitely long to occur.