Math 285 Final Free Response Questions: Solution

Due: 5/12 (Tue) at 8 pm

1. (a) (5 points) Let $F, G : \mathbb{R} \to \mathbb{R}$ be twice differentiable functions. Show that $u(x,t) = F(x-2t) + G(x+2t)$ is a solution to the wave equation $4u_{xx} = u_{tt}$.

   (b) (4 points) Show that $u(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t))$ is a solution to the wave equation $4u_{xx} = u_{tt}$ with $u(x,0) = f(x)$ and $u_t(x,0) = 0$, where f is a twice differentiable function on $\mathbb{R}$.

   (c) (6 points) Let $u(x,t) = \frac{1}{2}(f(x-2t) + f(x+2t))$ and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 2, & x \in [-2, 2) \\ 0, & \text{otherwise.} \end{cases}$$

   Draw the graphs of $u(x, \frac{1}{2})$, $u(x,1)$, and $u(x,2)$ in the $x-u$ plane.

Solution:

(a) We have

$$u_{xx}(x,t) = F''(x-2t) + G''(x-2t)$$

(2 points) and

$$u_{tt} = \frac{\partial}{\partial t}(-2F'(x-2t) + 2G'(x+2t)) = 4(F''(x-2t) + G''(x+2t))$$

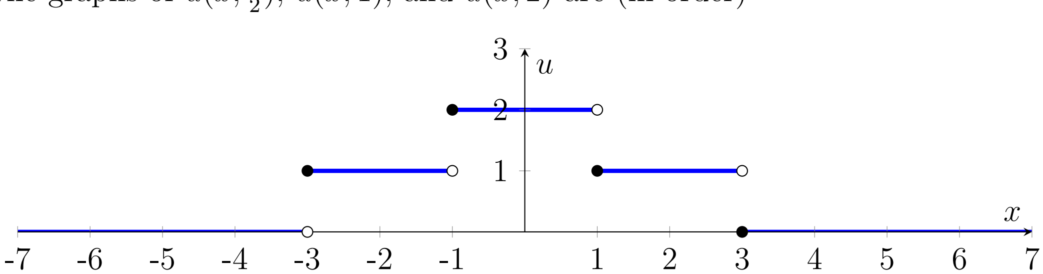
(2 points). Thus, $4u_{xx} = u_{tt}$ (1 point).

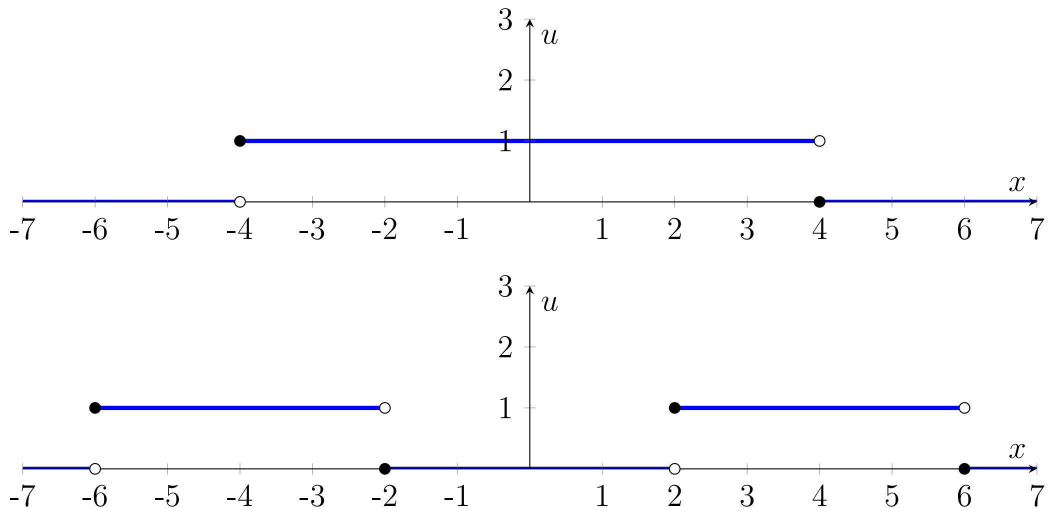
(b) By the part (a), $u(x,t)$ satisfies $4u_{xx} = u_{tt}$ (1 point). If $t = 0$, then $u(x,0) = \frac{1}{2}(f(x) + f(x)) = f(x)$ (1 point). Since

$$u_t(x,t) = \frac{1}{2}(-2f'(x-2t) + 2f'(x+2t)),$$

we have $u_t(x,0) = \frac{1}{2}(-2f'(x) + 2f'(x)) = 0$ (2 points).

(c) The graphs of $u(x, \frac{1}{2})$, $u(x,1)$, and $u(x,2)$ are (in order)





(2 points each).

2. Let $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2\}$. A function $u(x,y)$ defined on $\mathcal{R}$ is called *harmonic* if $u_{xx} + u_{yy} = 0$.

   (a) (10 points) Find a harmonic function $u(x,y)$ defined on $\mathcal{R}$ with the boundary conditions

$$\begin{aligned} u(x,0) &= f(x), \qquad u(x,2) = 0 \quad \text{for } 0 < x < 2, \\ u(0,y) &= 0, \qquad u(2,y) = 0 \quad \text{for } 0 < y < 2, \end{aligned}$$

   where $f(x)$ is given by

$$f(x) = \begin{cases} x, & 0 < x \le 1, \\ 2 - x, & 1 < x < 2. \end{cases}$$

   (Guide: First, use the method of separation of variables. In particular, let $u(x,y) = X(x)Y(y)$. Then, find two ordinary differential equations for $X(x)$ and $Y(y)$ with appropriate boundary conditions. Solve those ODEs to find fundamental solutions. Finally, find a solution using the Fourier series. Show your work in detail.)

   (b) (5 points) Find an example of a harmonic function $u(x,y)$ defined on $\mathcal{R}$ such that $u(x,y)$ attains its maximum at $(1,1)$. Justify your answer. (That is, find a function $u(x,y)$ such that $u_{xx} + u_{yy} = 0$ and $u(1,1) \ge u(x,y)$ for all $(x,y) \in \mathcal{R}$.)

Solution:

(a) Let $u(x,y) = X(x)Y(y)$, then the method of separation of variables yields

$$X'' + \lambda X = 0, \qquad X(0) = X(2) = 0$$

and

$$Y'' - \lambda Y = 0, \qquad Y(2) = 0$$

(2 points). Solving the eigenvalue problem for X, we get

$$\lambda_n = \frac{n^2\pi^2}{4}, \qquad X(x) = \sin(\frac{n\pi}{2}x)$$

(2 points). For each λ_n, we solve the ODE for $Y(y)$ with $Y(2) = 0$ to get

$$Y_n(y) = \sinh(\frac{n\pi}{2}y) - \tanh(n\pi)\cosh(\frac{n\pi}{2}y)$$

(2 points). Therefore,

$$\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} C_n X_n(x) Y_n(y) \\ &= \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{2}x)(\sinh(\frac{n\pi}{2}y) - \tanh(n\pi)\cosh(\frac{n\pi}{2}y)) \end{aligned}$$

(2 points). Since

$$u(x,0) = -\sum_{n=1}^{\infty} C_n \tanh(n\pi)\sin(\frac{n\pi}{2}x) = f(x),$$

we have

$$-C_n \tanh(n\pi) = \int_0^2 f(x)\sin(\frac{n\pi}{2}x)\,dx = \frac{8\sin(n\pi/2)}{n^2\pi^2}$$

(2 points). Therefore, the solution is

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{2}x)(\sinh(\frac{n\pi}{2}y) - \tanh(n\pi)\cosh(\frac{n\pi}{2}y))$$

with

$$C_n = -\frac{8\sin(n\pi/2)\coth(n\pi)}{n^2\pi^2}.$$

(b) The constant function $u(x,y) = 1$ is harmonic and attains its maximum at $(1,1)$.

(i) Any attempts to find an example: 3 points.

(ii) Any attempts to justify it is harmonic: 1 point.

(iii) Any attempts to justify it attains its maximum at (1,1): 1 point.

Remark on Problem 2-(b):

Coming up with constant functions is natural for the following reasons.

It is good to start with an easier case first. Consider the one dimension case. Suppose u is a function defined on the interval $[0, 2]$ such that $u'' = 0$ (that is, harmonic). Then, $u(x) = ax + b$. If it attains its maximum at $x = 1$ (in the middle of the interval which is analogous to $(1, 1)$ in $\mathcal{R}$), then a should be zero and so u is a constant function. In fact, this holds for higher dimensions.

Laplace's equation, as explained in the lecture, can be thought of as a limit of the heat equation as $t \to \infty$. In other words, solutions to Laplace's equation (i.e., harmonic functions) represents steady state (equilibrium) temperature distributions. Keeping this in mind, if a solution attains the maximum at the middle of the given region, this means that the heat has been equally distributed. So, solutions should be constant intuitively.

In general, harmonic functions have very nice properties, which are called the mean value property and maximum principle. The first one says that if u is harmonic, then $u(x_0)$ equals to the average of $u(x)$ in a neighborhood of x_0. This leads to the maximum principle, which states that every harmonic function attains its maximum at the boundary, and if it attain maximum inside the given region, then it should be constant.

The maximum principle says that constant functions are the only possible answers for the part 2-(b). If you obtain some functions that is not constant for this part, then probably (1) it is not harmonic (for example, it could not be twice differentiable), (2) or it is not defined on the whole region $\mathcal{R}$ (some students only defined the function on a part of $\mathcal{R}$), (3) or it actually does not attain its maximum at $(1, 1)$.