Math 285 Final Conflict Exam

1. (5 points) What is the correct integrating factor to solve the following ODE for $y(t)$?

$$(e^{2t}+1)y' - e^{2t}y + \tan t = 0$$

   A. None of these
   B. $\mu(t) = \sqrt{e^{2t}+1}$
   C. $\mu(t) = \dfrac{1}{\sqrt{e^{2t}+1}}$
   D. $\mu(t) = e^{2t}+1$
   E. $\mu(t) = \dfrac{1}{e^{2t}+1}$

Solution: An integrating factor is

$$\mu(t) = \exp\left(-\int \frac{e^{2t}}{e^{2t}+1}\,dt\right) = \exp\left(-\frac{1}{2}\ln(e^{2t}+1)\right) = \frac{1}{\sqrt{e^{2t}+1}}.$$

2. (5 points) Which one of the following is a solution to $ty' = y^2+1$?
   A. $y^2 = 4t-1$
   B. $y = \tan(\ln(t+3))$
   C. $y^2+1 = 2\tan t$
   D. $y = \tan(\ln(2t))$
   E. None of these

Solution: Since the equation is separable, we have

$$\begin{aligned} ty' &= y^2+1 \\ \frac{y'}{y^2+1} &= \frac{1}{t} \\ \arctan(y) &= \ln t + C \\ y &= \tan(\ln t + C). \end{aligned}$$

3. (5 points) Let $y(t)$ be a solution to

$$y' = y^2(y-4)(y-K)$$

with $y(0) = y_0$. For which value of y_0 and K, do we have

$$\lim_{t\to\infty} y(t) = 4?$$

A. $y_0 = 3$ **and** $K = 6$

B. $y_0 = 6$ and $K = 5$

C. $y_0 = 2$ and $K = 4$

D. None of these

E. $y_0 = 3$ and $K = 2$

> **Solution:** If $K \leq 4$, then $y(t) = 4$ is not stable. If $K > 4$, then $\lim_{t\to\infty} y(t) = 4$ if $0 < y_0 < K$.

4. (5 points) The differential equation

$$\frac{dy}{dt} = \frac{t}{t+2y}$$

can be transformed by a substitution into the separable differential equation:

A. $tv' = \frac{1-v-2v^2}{1+2v}$

B. $v' = \frac{v}{v+2}$

C. None of these

D. $tv' = \frac{1}{1+2v}$

E. $v' = -\frac{v^2+v}{v+2}$

> **Solution:** Let $v = y/t$, then $tv' + v = y'$. Thus,
>
> $$tv' + v = \frac{dy}{dt} = \frac{1}{1+2v} = \frac{1}{1+2(y/t)} = \frac{t}{t+2y}.$$

5. (5 points) Which one of the following is a solution to $y''' + py'' + qy' + ry = 0$ if its characteristic equation is $(\lambda - 2)(\lambda^2 + 4\lambda + 5) = 0$?

A. $y(t) = 3e^{-2t}$

B. $y(t) = e^{2t} - e^{-2t}\cos t$

C. $y(t) = e^{-t}\cos t + 2e^{-t}\sin t$

D. $y(t) = 5 + e^{-2t}\cos t - 3e^{-2t}\sin t$

E. None of these

Solution: Since the roots for the characteristic equation are $\lambda = 2, -2 \pm i$, the fundamental solutions are $e^{2t}, e^{-2t}\cos t, e^{-2t}\sin t$. Thus, the general solution is

$$y(t) = C_1 e^{2t} + C_2 e^{-2t}\cos t + C_3 e^{-2t}\sin t$$

6. (5 points) Identify the correct form of a particular solution for the following differential equation.

$$y^{(4)} + 4y'' = t\sin 2t + \cos t.$$

A. $Y(t) = (At + B)\cos 2t + (Ct^2 + Dt)\sin 2t + E\cos t$

B. None of these

C. $Y(t) = (At^2 + Bt)\sin 2t + C\cos t$

D. $Y(t) = (At + B)\cos 2t + (Ct + D)\sin 2t + Et\cos t + Ft\sin t$

E. $Y(t) = (At^2 + Bt)\cos 2t + (Ct^2 + Dt)\sin 2t + E\cos t + F\sin t$

Solution: The fundamental solutions for the homogeneous equation are

$$1, t, \cos 2t, \sin 2t.$$

Thus, a particular solution is of the form

$$Y(t) = t(At + B)\cos 2t + t(Ct + D)\sin 2t + E\cos t + F\sin t.$$

7. (5 points) Consider

$$mu'' + \gamma u' + ku = F_0\cos(\omega t)$$

where $m, k, \omega > 0$ and $\gamma, F_0 \geq 0$. Which one of the following is correct for the vibration system?

A. The solution to the vibration system is periodic if $F_0 = 0$.

B. The natural frequency is determined by k and m only.

C. The oscillator is overdamped if $F_0 = 0$ and $\gamma^2 < 4mk$.

D. The oscillator is in resonance if $\omega = \sqrt{\frac{k}{m}}$.

E. None of these.

Solution: The solution to the vibration system is periodic in time if $F_0 = 0$ and $\gamma = 0$. The natural frequency is $\omega_0 = \sqrt{k/m}$. The oscillator is overdamped if $F_0 = 0$ and $\gamma^2 > 4mk$. The oscillator is in resonance if $\omega = \sqrt{\frac{k}{m}}$, $\gamma = 0$, and $F_0 \neq 0$.

8. (5 points) The existence and uniqueness theorem guarantees that there exists a unique solution of the initial value problem

$$\sin^2(\frac{\pi}{4}t)y'' + \sqrt{(t+2)(t-3)}y' + (t^2+1)y = 2$$

with $y(5) = -3$ and $y'(5) = 1$ on an open interval

A. $(4, 8)$

B. $(3, 6)$

C. $(-2, 8)$

D. $(4, \infty)$

E. None of these

Solution: First, we divide by $\sin^2(\frac{\pi}{4}t)$. Then the coefficients are continuous if $t \neq 4k$ for all integers $k \in \mathbb{Z}$, $t > 3$, and $t < -2$.

9. (5 points) Let $X(x) = 2\sin(2x)$, then $u(x,t) = X(x)T(t)$ is a solution to the variant of the wave equation $u_{tt} = u_{xx} + u$ if

A. None of these

B. $T(t) = \sin(2t)$

C. $T(t) = \cosh(2t)$

D. $T(t) = \cos(\sqrt{3}t)$

E. $T(t) = \sinh(\sqrt{3}t)$

Solution: Since $u(x,t) = X(x)T(t)$ and $X(x) = 2\sin(2x)$, then the wave equation yields

$$XT'' = X''T + XT$$
$$\frac{T''}{T} = \frac{X''}{X} + 1 = -3.$$

So, we have $T'' + 3T = 0$ and so $T(t) = C_1\cos(\sqrt{3}t) + C_2\sin(\sqrt{3}t)$.

10. (5 points) If you were to solve Laplace's equation $u_{xx} + u_{yy} = 0$ in a rectangle $\mathcal{R} = \{(x, y) : 0 < x < 3, 0 < y < 5\}$ with

$$u(0, y) = 0, \qquad u(3, y) = 0,$$
$$u(x, 0) = x(3 - x), \qquad u(x, 5) = 0,$$

using the method of separation of variables, what would be the correct form of $X_n(x)$?

A. $X_n(x) = \cos(\frac{n^2\pi^2}{9}x)$
B. $X_n(x) = \sinh(\frac{n\pi}{5}x)$
C. None of these
D. $X_n(x) = \tan(n\pi x)$
E. $X_n(x) = \sin(\frac{n\pi}{3}x)$

Solution: By the method of separation of variables, we get $X'' + \lambda X = 0$ with $X(0) = X(3) = 0$. Thus, the eigenvalues and the eigenfunctions are

$$\lambda_n = \frac{n^2\pi^2}{9}, \qquad X_n = \sin(\frac{n\pi}{3}x).$$

11. (5 points) Which one of the following is correct for the eigenvalue problem $y'' + \lambda y = 0$ with $y(0) - y'(0) = 0$ and $y(1) = 0$?

A. There exists a nontrivial solution if $\lambda = \pi^2$.
B. If λ is a positive eigenvalue, then it satisfies $\sqrt{\lambda} + \tan\sqrt{\lambda} = 0$.
C. 0 is an eigenvalue.
D. There are finitely many eigenvalues.
E. None of these

Solution: If $\lambda = 0$, then $y(x) = c_1 + c_2 x$. By the boundary conditions, we get

$$y(0) - y'(0) = c_1 - c_2 = 0$$
$$y(1) = c_1 + c_2 = 0,$$

which implies that $c_1 = c_2 = 0$. Thus, 0 is not an eigenvalue.

If $\lambda = \pi^2$, then $y(x) = c_1\cos(\pi x) + c_2\sin(\pi x)$. The boundary conditions yield $c_1 + c_2\pi = 0$ and $c_1 = 0$. Thus, $y(x) = 0$ is the only solution.

If $\lambda = \mu^2 > 0$ for some $\mu > 0$, then $y(x) = c_1\cos(\mu x) + c_2\sin(\mu x)$. By the boundary conditions, we get

$$y(0) - y'(0) = c_1 - \mu c_2 = 0$$
$$y(1) = c_1\cos(\mu) + c_2\sin(\mu) = 0.$$

Thus, we obtain $\mu\cos\mu + \sin\mu = 0$, or $\mu = -\tan\mu$. Since there are infinitely many $\mu > 0$ that satisfies $\mu = -\tan\mu$, there exist infinitely many eigenvalues.

12. (5 points) The equation

$$x^2y'' + xy' + (x^2-4)y = 0$$

can be transform into the form $(p(x)y')' + q(x)y = 0$ with

A. $p(x) = x^3$ and $q(x) = x^3 - 4x$

B. $p(x) = e^{\frac{1}{2}x^2}$ and $q(x) = e^{\frac{1}{2}x^2}(1 - \frac{4}{x^2})$

C. $p(x) = x$ **and** $q(x) = x - \frac{4}{x}$

D. None of these

E. $p(x) = x^2$ and $q(x) = x^2 - 4$

Solution: Multiplying $\mu(x)$ of the both sides, we have

$$x^2\mu(x)y'' + x\mu(x)y' + (x^2-4)\mu(x)y = 0.$$

Let $(x^2\mu)' = x\mu$, then $\mu(x) = \frac{1}{x}$. Therefore, $p(x) = x$ and $q(x) = \frac{x^2-4}{x}$.