Sec 4.3: The Method of Undetermined Coefficients

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Instructor: Daesung Kim

Method

The method of undetermined coefficients works for higher order DEs.

Consider

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t).$$

If $g(t)$ is a mixture of polynomials, exponential, sine or cosine functions, then a particular solution $Y(t)$ has one of the following form:

(i) If $g(t) = (t^n + \cdots)$, then $Y(t) = (A_n t^n + \cdots)$.

(ii) If $g(t) = (t^n + \cdots)e^{kt}$, then $Y(t) = (A_n t^n + \cdots)e^{kt}$.

(iii) If $g(t) = (t^n + \cdots)\sin(kt)$ (or $\cos(kt)$), then $Y(t) = (A_n t^n + \cdots)\cos(kt) + (B_n t^n + \cdots)\sin(kt)$.

(iv) Multiply $Y(t)$ by t until it does not contain a solution to the homogeneous equation.

Method

Suppose $g(t)$ is given by the sum of those functions, that is, $L[y] = g_1(t) + g_2(t)$.

Find particular solutions Y_1 and Y_2 to the equations $L[y] = g_1(t)$ and $L[y] = g_2(t)$ respectively, then $Y(t) = Y_1(t) + Y_2(t)$.

Example

Example

Consider $L[y] = y^{(4)} + 2y''' - 2y' - y = 6e^{-t}$.

General solution to $L[y] = 0$:

$$\lambda^4 + 2\lambda^3 - 2\lambda - 1 = (\lambda - 1)(\lambda + 1)^3 = 0$$

$$\lambda = \underline{1}, \underline{-1} \text{ multiplicity} = 3$$

$$e^t \qquad \underbrace{e^{-t}, te^{-t}, t^2e^{-t}}_{3 \text{ functions.}}$$

$$\therefore \quad y_c(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t} + c_4 e^t.$$

Example

Example

Consider $L[y] = y^{(4)} + 2y''' - 2y' - y = 6e^{-t}$.

A particular solution to $L[y] = 6e^{-t}$:

$g(t) = 6e^{-t} \longrightarrow \underline{Ae^{-t}} \xrightarrow{\times t} \underline{Ate^{-t}}$

$\xrightarrow{\times t} \underline{At^2e^{-t}}$

$\xrightarrow{\times t} At^3e^{-t} = Y(t)$

Determine A : $\underbrace{L[Y]}_{\text{compute this!}} = 6e^{-t}$

$f(t) = At^3$, $Y(t) = f \cdot e^{-t}$. (for simplicity)

Example

$$Y = f \cdot e^{-t}$$

$$Y' = f' \cdot e^{-t} + f \cdot (-e^{-t}) = (f' - f) e^{-t}$$

$$Y'' = (f'' - 2f' + f) e^{-t}$$

$$Y''' = (f''' - 3f'' + 3f' - f) e^{-t}$$

$$Y^{(4)} = (f^{(4)} - 4f''' + 6f'' - 4f' + f) e^{-t}$$

$$L[Y] = Y^{(4)} + 2Y''' - 2Y' - Y = (f^{(4)} - 2f^{(3)}) \cdot e^{-t}$$

$$= -12A e^{-t} = 6 e^{-t} \qquad \therefore \quad A = -\frac{1}{2}$$

$$Y(t) = -\frac{1}{2} t^3 e^{-t}$$

$$\therefore \quad y(t) = y_c(t) + Y(t) = C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} + C_4 e^{t}$$

$$- \frac{1}{2} t^3 e^{-t}$$

Remark

Suppose $y(t) = f(t) \cdot e^{-t}$ is a solution to $L[y]=0$

$$\Rightarrow \quad L[y] = \left(f^{(4)} - 2f^{(3)}\right) \cdot e^{-t} = 0$$

$$\Rightarrow \quad \underbrace{f^{(4)} - 2f^{(3)}}\, = 0$$

$$\Rightarrow \quad f(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{2t}$$

$$\Rightarrow \quad y(t) = f(t) \cdot e^{-t}$$

$$= C_1 e^{-t} + C_2 t e^{-t} + C_3 t^2 e^{-t} + C_4 e^{t}$$

general solution to $L[y]=0$.

$$\boxed{\begin{aligned} &\lambda^4 + 2\lambda^3 - 2\lambda - 1 \\ &= \underbrace{(\lambda+1)^4 - 2(\lambda+1)^3} \end{aligned}}$$

Notice : $f^{(4)} - 2f^{(3)} = 0$

Can we find this without computing derivatives?

Example

Example

Consider $L[y] = y''' + y' = te^{-t} + \cos t$.

General solution to $L[y] = 0$:

$$\lambda^3 + \lambda = 0$$

$$\lambda(\lambda^2 + 1) = 0$$

$$\lambda = 0 \quad , \pm i$$

$$\downarrow \qquad \qquad \downarrow$$

$$1 \qquad \cos t, \ \sin t$$

$$y_c(t) = C_1 + C_2 \cos t + C_3 \sin t.$$

Example

Example

Consider $L[y] = y''' + y' = te^{-t} + \cos t$.

A particular solution to $L[y] = \underline{te^{-t}}$:

Not a solution to $L[y]=0$.

$$Y_1(t) = (At+B)e^{-t}$$

$$L[Y_1] = (-2At + (4A-2B))e^{-t} = te^{-t}$$

$$\begin{cases} -2A = 1 \\ 4A-2B=0 \end{cases} \quad \therefore \quad A = -\frac{1}{2}, \quad B=-1$$

$$Y_1(t) = -\frac{1}{2}(t+2)e^{-t}.$$

Example

Example

Example

Consider $L[y] = y''' + y' = te^{-t} + \cos t$.

A particular solution to $L[y] = \cos t$:

A solution to $L[y]=0$.

$Y_2(t) = t(A\cos t + B\sin t)$

$L[Y_2] = -2A\cos t - 2B\sin t = \cos t$

$A = -\frac{1}{2}$, $B=0$.

$Y_2(t) = -\frac{1}{2}t\cos t$.

$\therefore\ y(t) = y_c + Y_1 + Y_2$

$= c_1 + c_2\cos t + c_3\sin t - \frac{1}{2}(t+2)e^{-t} - \frac{1}{2}t\cos t$.

Example