Homework 9

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 15, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $A = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$. Find $\lim_{m\to\infty} A^m$ if it exists.

Solution: The characteristic polynomial is

$$f(t) = (0.4 - t)(0.3 - t) - 0.42 = t^2 - 0.7t - 0.3 = (t-1)(t+0.3).$$

The corresponding eigenspaces are

$$E_1 = \mathrm{Span}(\{(7,6)\}), \qquad E_{-0.3} = \mathrm{Span}(\{(1,-1)\}).$$

Thus,

$$A = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} -0.3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ -1 & 6 \end{pmatrix}^{-1} = \frac{1}{13} \begin{pmatrix} 1 & 7 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} -0.3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & 1 \end{pmatrix}$$

and so

$$\begin{aligned} A^m &= \frac{1}{13} \begin{pmatrix} 1 & 7 \\ -1 & 6 \end{pmatrix} \begin{pmatrix} (-0.3)^m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & -7 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{13} \begin{pmatrix} 6(-0.3)^m + 7 & -7(-0.3)^m + 7 \\ -6(-0.3)^m + 6 & 7(-0.3)^m + 6 \end{pmatrix}. \end{aligned}$$

Therefore, $\lim_{m\to\infty} A^m$ exists and

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} \frac{1}{13} \begin{pmatrix} 6(-0.3)^m + 7 & -7(-0.3)^m + 7 \\ -6(-0.3)^m + 6 & 7(-0.3)^m + 6 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 7 & 7 \\ 6 & 6 \end{pmatrix}.$$

2. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $u = (1, 1, \cdots, 1)$.
   (a) Show that A is a transition matrix if and only if $A_{ij} \geq 0$ and $uA = u$.
   (b) Show that if A and B are transition matrices then AB is also a transition matrix.
   (c) From Part (b), deduce that if A is a transition matrix then A^m is a transition matrix for all integers $m \geq 1$.

Solution:

(a) Suppose that A is a transition matrix, then $A_{ij} \geq 0$ and

$$uA = \left(\sum_{i=1}^{n} A_{i1}, \sum_{i=1}^{n} A_{i2}, \cdots, \sum_{i=1}^{n} A_{in}\right) = u.$$

Suppose $A_{ij} \geq 0$ and $uA = u$, then the identity above implies that the sum of the entries in each column of A equals to 1. Thus, A is a transition matrix.

(b) Suppose A and B are transition matrices, then for each $j = 1, 2, \cdots, n$,

$$\sum_{i=1}^{n} (AB)_{ij} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n}\sum_{i=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n} B_{kj} = 1.$$

Since

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} \geq 0$$

for each $i, j = 1, 2, \cdots, n$, we conclude that AB is also a transition matrix.

(c) If $m = 1$, then it is trivial. Suppose that $m \geq 2$ and the claim is true for $m - 1$. Since $A^m = AA^{m-1}$ and A, A^{m-1} are transition matrices by the induction hypothesis, Part (b) yields that A^m is a transition matrix.

3. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^t.

Solution: It suffices to show that the characteristic polynomial of A is the same as that of A^t. Indeed, it follows that

$$f_A(x) = \det(A - xI_n) = \det((A - xI_n)^t) = \det(A^t - xI_n) = f_{A^t}(x).$$

4. Let V be an inner product space over F.
   (a) Show that $\|cx\| = |c|\|x\|$ for all $x \in V$ and $c \in F$.
   (b) Show that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.

Solution:

(a) By definition,

$$\|cx\|^2 = \langle cx, cx\rangle = c\bar{c}\,\langle x, x\rangle = |c|^2\|x\|^2.$$

(b) It follows from the definition of the norm that

$$\|x+y\|^2 + \|x-y\|^2 = (\|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2) + (\|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2) = 2\|x\|^2 + 2\|y\|^2.$$

5. Let $V = \mathbb{C}^2$ and define $\langle x, y\rangle = xAy^*$ for $x, y \in V$, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

That is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\langle x, y\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix}.$$

Show that $\langle \cdot, \cdot \rangle$ is an inner product.

Solution: Let $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2) \in V$ and $c \in \mathbb{C}$.

(a) $\langle x + z, y\rangle = (x + z)Ay^* = xAy^* + zAy^* = \langle x, y\rangle + \langle z, y\rangle$.

(b) $\langle cx, y\rangle = (cx)Ay^* = c(xAy^*) = c\,\langle x, y\rangle$.

(c) Note that

$$A^* = \overline{A^t} = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} = A.$$

Since $\langle x, y\rangle \in \mathbb{C}$, we have $\langle x, y\rangle = \langle x, y\rangle^t$, which implies that

$$\langle x, y\rangle = \langle x, y\rangle^t = (xAy^*)^t = \overline{(xAy^*)^*} = \overline{yA^*x^*} = \overline{yAx^*} = \overline{\langle y, x\rangle}.$$

(d) It follows that

$$\begin{aligned}
\langle x, x\rangle &= \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} \\
&= \begin{pmatrix} x_1 - ix_2 & ix_1 + 2x_2 \end{pmatrix} \begin{pmatrix} \overline{x_1} \\ \overline{x_2} \end{pmatrix} \\
&= (x_1 - ix_2)\overline{x_1} + (ix_1 + 2x_2)\overline{x_2} \\
&= |x_1|^2 - ix_2\overline{x_1} + ix_1\overline{x_2} + 2|x_2|^2 \\
&= (x_1 - ix_2)\overline{(x_1 - ix_2)} + |x_2|^2 \\
&= |x_1 - ix_2|^2 + |x_2|^2 \geq 0.
\end{aligned}$$

If $\langle x, x\rangle = 0$, then $x_2 = 0$ and $x_1 - ix_2 = 0$. Thus, $x = 0$ and we conclude that $\langle x, x\rangle > 0$ if $x \neq 0$.

6. Let $V = \mathbb{C}^n$ and $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Let $\langle x, y\rangle = \sum_{i=1}^n x_i\overline{y_i}$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.
   (a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in V$.
   (b) Suppose that there exists $B \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$. Show that $B = A^*$.

Solution:

(a) Let $x, y \in \mathbb{C}^n$, then

$$\begin{aligned}
\langle x, Ay \rangle &= \sum_{i=1}^{n} x_i \overline{(Ay)_i} \\
&= \sum_{i=1}^{n} x_i \sum_{j=1}^{n} \overline{A_{ij}}\, \overline{y_j} \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} x_i A^*_{ji} \right) \overline{y_j} \\
&= \sum_{j=1}^{n} (A^* x)_j \overline{y_j} \\
&= \langle A^* x, y \rangle .
\end{aligned}$$

(b) Suppose that there exists $B \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in V$. Then, Part (a) implies that

$$\langle A^* x, y \rangle = \langle Bx, y \rangle$$

for all $x, y \in V$. For each $i, j = 1, \cdots, n$, let $x = e_j$ and $y = e_i$ where $\{e_1, \cdots, e_n\}$ is the standard basis for $\mathbb{C}^n$, then

$$\langle A^* x, y \rangle = (A^*)_{ij} = B_{ij} = \langle Bx, y \rangle ,$$

which shows that $B = A^*$.