Math 285: Differential Equations

Practice for midterm 2, Spring 2020

READ THE FOLLOWING INFORMATION.

- This is a 90-minute exam.
- No books, notes, calculators, or electronic devices allowed.
- You must not communicate with other students during this test.
- There are several different versions of this exam.
- Do not turn this page until instructed to.

1. Fill in your information:

Full Name: ______________________

UIN (Student Number): ______________________

NetID: ______________________

Circle your section: B1 (9 am) C1 (10 am) E1 (1 pm) F1 (2 pm)

2. Fill in the following answers on the Scantron form:

95. D
96. C

Multiple Choice Questions: Mark answers to questions 1 to 10 on your scantron form.

1. (5 points) If $y_1 = e^{-t} + e^{3t}$ and $y_2 = 2e^{-t} + 3e^{3t}$, then the Wronskian $W[y_1, y_2](t)$ is

(A) $9e^{6t}$

(B) None of these

(C) $e^{2t} - e^{-2t}$

(D) $2e^{2t} + 3e^{6t}$

(E) ★ $4e^{2t}$

Solution. By definition, we have

$$\begin{aligned}
W[y_1, y_2](t) &= y_1 y_2' - y_1' y_2 \\
&= (e^{-t} + e^{3t})(-2e^{-t} + 9e^{3t}) - (-e^{-t} + 3e^{3t})(2e^{-t} + 3e^{3t}) \\
&= -2e^{-2t} - 2e^{2t} + 9e^{2t} + 9e^{6t} + 2e^{-2t} - 6e^{2t} + 3e^{2t} - 9e^{6t} \\
&= 4e^{2t}.
\end{aligned}$$

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2. (5 points) The motion of a certain spring–mass system is governed by

$$u'' + \gamma u + 9 = 0$$

for some constant $\gamma > 0$. The motion is underdamped if

(A) ★ $\gamma \in (0, 6)$

(B) $\gamma \in (6, \infty)$

(C) None of these

(D) $\gamma \in (-6, 6)$

(E) $\gamma = 6$

Solution. The characteristic equation $\lambda^2 + \gamma\lambda + 9 = 0$ has complex roots if $\gamma^2 - 36 < 0$, which implies $|\gamma| < 6$.

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3. (5 points) What can you say for the method of undetermined coefficients?

$$y'' + p(x)y' + q(x)y = g(x) \qquad (1)$$

(A) ★ None of these

(B) It applies to system (1) with non-constant coefficients p and q

(C) It is more general than the method of variation of parameter.

(D) It helps to find the general solution to the homogeneous equation corresponding to (1)

(E) It provides us a formula for finding a specific solution to (1) provided that we know the general solution of the corresponding homogeneous equation

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Solution. (i) The method of undetermined coefficients only applies to system (1) with constant coefficients p and q. While the method of variation of parameters works for non-constant system (1). (ii) Neither the method of undetermined coefficients nor the method of variation of parameter helps with finding the general solution to the homogeneous equation corresponding to (1). (iii) is a statement for the method of variation of parameters. (iv)The method of variation of parameter is a more general method than the method of undetermined coefficients.

Remark: Given y_1 is a solution of the corresponding homogeneous equation of (1). Using the method of reduction of order, we can ALWAYS find another independent solution y_2 satisfying $W(y_1, y_2) \neq 0$.

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4. (5 points) Identify the correct form of a particular solution for the following differential equation.

$$y'' - y = t^2 e^{-t} + 10t^3.$$

(A) $Y(t) = (A_0 + A_1 t + A_2 t^2)e^{-t} + (B_0 + B_1 t + B_2 t^2 + B_3 t^3)$

(B) ★ $Y(t) = t(A_0 + A_1 t + A_2 t^2)e^{-t} + (B_0 + B_1 t + B_2 t^2 + B_3 t^3)$

(C) $Y(t) = t^2(A_0 + A_1 t + A_2 t^2)e^{-t} + (B_0 + B_1 t + B_2 t^2 + B_3 t^3)$

(D) None of these

(E) $Y(t) = (A_0 + A_1 t + A_2 t^2 + A_3 t^3)e^{-t}$

Solution. Since the homogeneous equation $y'' - y = 0$ has solutions e^t, e^{-t}, a particular solution is of the form

$$Y(t) = t(A_0 + A_1 t + A_2 t^2)e^{-t} + (B_0 + B_1 t + B_2 t^2 + B_3 t^3).$$

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5. (5 points) The general solution of $y'' + 4y' + 5y = 0$ is

(A) None of these

(B) ★ $y(t) = e^{-2t}(C_1 \cos t + C_2 \sin t)$

(C) $y(t) = e^{2t}(C_1 + C_2 t)$

(D) $y(t) = C_1 e^{-5t} + C_2 e^{t}$

(E) $y(t) = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$

Solution. Since the roots for the characteristic equation are $\lambda = -2 \pm i$, the general solution is

$$y(t) = e^{-2t}(C_1 \cos t + C_2 \sin t).$$

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6. (5 points) What can you say for the vibration system?

$$ms'' + rs' + ks = F_0 \cos(\omega t) \qquad (1)$$

(A) When $F_0 = 0$ $r^2 \geq 4mk$, the general solution eventually becomes monotone and decays to 0.

(B) When $F_0 = 0$ $r^2 < 4mk$, the general solution oscillates and decays to 0.

(C) Resonance happens when $r = 0$ and $\sqrt{\frac{k}{m}} = \omega$.

(D) When $F_0 = 0$ $r = 0$, the general solution is periodic function.

(E) ★ All of these.

Solution. (i) Resonane happens when the natural frequency $\sqrt{\frac{k}{m}}$ is equal to the driven force frequency. (ii)-(iv) are free vibration systems. And we study the general solution in class and these statements are all correct.

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7. (5 points) Which of these is NOT a set of linearly independent solutions?

(A) $\cos x, \sin x, \cos 2x$

(B) $1, x, x^2$

(C) ★ $x, x^2 + x, 2x^2$

(D) e^x, e^{2x}, e^{3x}

(E) None of these

Solution. The set $\{x, x^2 + x, 2x^2\}$ is linearly dependent because

$$k_1 x + k_2(x^2 + x) + k_3(2x^2) = 0$$

holds for all x if $k_1 = 2$, $k_2 = -2$, and $k_3 = 1$.

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8. (5 points) Find the general solution of the following differential equation.

$$y''' + 8y = 0.$$

(A) ★ $C_1e^{-2t} + C_2e^t \cos\sqrt{3}t + C_3e^t \sin\sqrt{3}t$

(B) $C_1e^{2t} + C_2e^t \cos\sqrt{3}t + C_3e^t \sin\sqrt{3}t$

(C) $C_1e^{2t} + C_2e^{-t} \cos\sqrt{3}t + C_3e^{-t} \sin\sqrt{3}t$

(D) $C_1e^{-2t} + C_2e^{-t} \cos\sqrt{3}t + C_3e^{-t} \sin\sqrt{3}t$

(E) None of these

Solution. The characteristic equation is $\lambda^3 + 8 = 0$. Since it can be factored

$$\lambda^3 + 8 = (\lambda + 2)(\lambda^2 - 2\lambda + 4) = 0,$$

the roots are $\lambda = -2, 1 \pm \sqrt{3}i$. Thus, the general solution is

$$y(t) = C_1e^{-2t} + C_2e^t \cos\sqrt{3}t + C_3e^t \sin\sqrt{3}t.$$

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9. (5 points) The motion of a certain spring–mass system is governed by

$$u'' + 4u = 0$$

with $u(0) = 1$ and $u'(0) = 1$. What are the amplitude and the natural frequency?

(A) $R = \sqrt{2}$ and $\omega_0 = 1$

(B) $R = \sqrt{5}/2$ and $\omega_0 = 1$

(C) ★ $R = \sqrt{5}/2$ and $\omega_0 = 2$

(D) None of these

(E) $R = \sqrt{2}$ and $\omega_0 = 2$

Solution. The general solution is $u(t) = A \cos 2t + B \sin 2t$. By the initial conditions, $A = 1$ and $B = \frac{1}{2}$. Thus, $R = \sqrt{A^2 + B^2} = \sqrt{5}/2$ and $\omega_0 = 2$

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10. (5 points) The equation $t^2y'' - 4ty' + 6y = 0$ has a solution $y_1(t) = t^2$. If $y_2(t) = v(t)y_1(t)$ is another solution to the equation, then the function $v(t)$ satisfies

(A) None of these

(B) ★ $v'' = 0$

(C) $t^2v'' + 2t(t-2)v' = 0$

(D) $t^4v'' + 2t^3(t-2)v' = 0$

(E) $tv'' - 4v' = 0$

Solution. We have

$$\begin{aligned} t^2y_2'' - 4ty_2' + 6y_2 &= t^2(v''y_1 + 2v'y_1' + vy_1'') - 4t(v'y_1 + vy_1') + 6(vy_1) \\ &= t^4v'' + 4v't^3 - 4t^3v' + v(t^2y_1'' - 4ty_1' + 6y_1) \\ &= t^4v'' = 0, \end{aligned}$$

which implies $v'' = 0$.

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11. (5 points) The existence and uniqueness theorem guarantees that there exists a unique solution of the initial value problem $(t-1)(t+3)y'' + (t-4)y' + \sin t y = e^{-3t}$ with $y(-1) = 2$ and $y'(-1) = 0$ on an open interval

(A) $(1, 4)$

(B) ★ $(-3, 1)$

(C) None of these

(D) $(-\infty, -3)$

(E) $(4, \infty)$

Solution. After dividing by $(t-1)(t+3)$, the coefficients are continuous if $t \neq 1, -3$. Since the initial conditions are given at $t = -1$, the interval is $(-3, 1)$.

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12. (5 points) The function $u(t) = -\cos 2t + \sin 2t$ can be written as $u(t) = R\cos(\omega_0 t - \delta)$ where

(A) $R = \sqrt{3}$, $\omega_0 = 2$, $\delta = 3\pi/4$

(B) ★ $R = \sqrt{2}$, $\omega_0 = 2$, $\delta = 3\pi/4$

(C) $R = \sqrt{2}$, $\omega_0 = 2$, $\delta = \pi/4$

(D) None of these

(E) $R = \sqrt{3}$, $\omega_0 = 1$, $\delta = 5\pi/4$

Solution. It is easy to check that $R = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ and $\omega_0 = 2$. If $\cos(\delta) = -1/\sqrt{2}$ and $\sin(\delta) = 1/\sqrt{2}$, then $\delta = 3\pi/4$.

Free Response Questions: Make sure you show your work. You may get partial credit.

13. (10 points) Consider $t^2y'' - t(t+2)y' + (t+2)y = 0$ for $t > 0$.

(i) Verify that $y_1(t) = t$ is a solution to the equation.

(ii) Find the general solution to the equation using the method of reduction of order.

Solution.

(i) We have $t^2y_1'' - t(t+2)y_1' + (t+2)y_1 = -t(t+2) + t(t+2) = 0$.

(ii) Let $y_2 = vy_1$. By the equation,

$$\begin{aligned} t^2y_2'' - t(t+2)y_2' + (t+2)y_2 &= t^2(v''y_1 + 2v'y_1' + vy_1'') - t(t+2)(v'y_1 + vy_1') + (t+2)vy_1 \\ &= t^2(tv'' + 2v') - t^2(t+2)v' \\ &= t^3(v'' - v') \\ &= 0. \end{aligned}$$

Thus, $v'' - v' = 0$. By solving the equation, we get $v(t) = C_1e^t + C_2$. Since $W[t, te^t] \neq 0$, the general solution is $y(t) = C_1te^t + C_2t$.

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14. (10 points) Find the solution to the following initial value problem

$$y'' - 2y' - 3y = 3te^{2t}, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution. The general solution of $y'' - 2y' - 3y = 0$ is $C_1e^{-t} + C_2e^{3t}$. By the method of undetermined coefficients, a particular solution has a form

$$Y(t) = (At + B)e^{2t}.$$

We then have

$$\begin{aligned} Y'' - 2Y' - 3Y &= (4Ae^{2t} + 4(At + B)e^{2t}) - 2(Ae^{2t} + 2(At + B)e^{2t}) - 3(At + B)e^{2t} \\ &= (2A - 3B)e^{2t} - 3Ate^{2t} \\ &= 3te^{2t}, \end{aligned}$$

which means $2A - 3B = 0$ and $-3A = 3$. Thus, $A = -1$, $B = -2/3$, and the solution is

$$y(t) = C_1e^{-t} + C_2e^{3t} - (t + \frac{2}{3})e^{2t}.$$

By the initial conditions,

$$y(t) = \frac{2}{3}e^{-t} + te^{3t} - (t + \frac{2}{3})e^{2t}.$$