Math 285 Lecture Note: Week 12

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1 Fourier Series, part 2 (Sec 10.2)

Recall that if a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right),$$

then

$$\begin{aligned} a_n &= \frac{1}{L}(f, \cos \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \\ b_n &= \frac{1}{L}(f, \sin \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx. \end{aligned}$$

Example 1.1. Consider a periodic function f defined by

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. In this case $L = 1$. Suppose f can be written as a Fourier series. Let's find a_m and b_m. First,

$$a_0 = (f, 1) = \frac{1}{L} \int_{-1}^{1} f(x) \, dx = 1.$$

For $n = 1, 2, \cdots$, we have

$$\begin{aligned} a_n &= \frac{1}{L}(f, \cos(n\pi x)) = \int_{-1}^{1} f(x) \cos(n\pi x) \, dx \\ &= 2 \int_0^1 x \cos(n\pi x) \, dx \\ &= 2 \left(\left[\frac{x \sin(n\pi x)}{n\pi} \right]_0^1 - \frac{1}{n\pi} \int_0^1 \sin(n\pi x) \, dx \right) \\ &= \frac{2}{n^2\pi^2} (\cos(n\pi) - 1) \\ &= \begin{cases} -\frac{4}{n^2\pi^2}, & m \text{ is odd}, \\ 0, & m \text{ is even}, \end{cases} \end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{L}(f, \sin(n\pi x)) = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\
&= \int_0^1 x\sin(n\pi x)\,dx - \int_{-1}^{0} x\sin(n\pi x)\,dx \\
&= \int_0^1 x\sin(n\pi x)\,dx - \int_0^1 x\sin(n\pi x)\,dx \\
&= 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(x) &= \frac{1}{2} - \sum_{m=1, m \text{ is odd}}^{\infty} \frac{4}{m^2\pi^2}\cos(m\pi x) \\
&= \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.
\end{aligned}$$

Example 1.2. Consider a periodic function f defined by

$$f(x) = \begin{cases} 1, & 0 \le x < 2, \\ -1, & -2 \le x < 0, \end{cases}$$

and $f(x+4) = f(x)$ for all $x \in \mathbb{R}$. In this case $L = 2$. Suppose f can be written as a Fourier series. Let's find a_m and b_m. First,

$$a_0 = (f, 1) = \frac{1}{L}\int_{-2}^{2} f(x)\,dx = 0.$$

For $n = 1, 2, \cdots$, we have

$$\begin{aligned}
a_n &= \frac{1}{L}(f, \cos\left(\frac{n\pi x}{2}\right)) \\
&= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= \frac{1}{2}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)\,dx - \frac{1}{2}\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right)\,dx \\
&= 0.
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{L}(f, \sin\left(\frac{n\pi x}{2}\right)) \\
&= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\
&= \int_0^2 \sin\left(\frac{n\pi x}{2}\right)\,dx \\
&= -\frac{2}{n\pi}[\cos\left(\frac{n\pi x}{2}\right)]_0^2 \\
&= \frac{2}{n\pi}(1 - \cos(n\pi)) \\
&= \begin{cases} \frac{4}{n\pi}, & m \text{ is odd}, \\ 0, & m \text{ is even}. \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \sum_{m=1, m \text{ is odd}}^{\infty} \frac{4}{m\pi} \sin\left(\frac{m\pi x}{2}\right) \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)} \sin\left(\frac{(2k-1)\pi x}{2}\right). \end{aligned}$$

2 The Fourier Convergence Theorem (Sec 10.3)

Suppose a function f is given. If f is periodic with period $2L > 0$ and integrable on $[-L, L]$, then we can compute

$$\begin{aligned} a_n &= \frac{1}{L}(f, \cos\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx, \\ b_n &= \frac{1}{L}(f, \sin\frac{n\pi x}{L}) = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx. \end{aligned}$$

Define

$$S_N(x) = \frac{a_0}{2} + \sum_{m=1}^{N} \left(a_m \cos\frac{m\pi x}{L} + b_m \sin\frac{m\pi x}{L}\right)$$

for each $N = 1, 2, \cdots$.

Question 2.1.

(i) Does $S_N(x)$ converge as $N \to \infty$ for each x?

(ii) Suppose $S_N(x)$ converges to a function, say $S(x)$, as $N \to \infty$ for each x. Is the limit $S(x)$ equal to $f(x)$?

Definition 2.2. A function f is called piecewise continuous on an interval $[a, b]$ if there exists a partition of $[a, b]$, $a = x_0 < x_1 < \cdots < x_n = b$ such that

(i) f is continuous on an open subinterval (x_{i-1}, x_i) for each $i = 1, 2, \cdots, n$, and

(ii) the limits

$$\lim_{x \to x_{i-1}+} f(x), \qquad \lim_{x \to x_i-} f(x)$$

are finite for each $i = 1, 2, \cdots, n$.

Example 2.3. Let $f(x)$ be a periodic function with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$, then it is piecewise continuous.

Example 2.4. Let $f(x) = \frac{1}{x}$ for $x \neq 0$, then it is not piecewise continuous.

Theorem 2.5. *Suppose f and f' are piecewise continuous on $[-L, L]$. Assume that f is periodic with period $2L$, that is, $f(x + 2L) = f(x)$. Then, $S_N(x)$ converges to a function $S(x)$ as $N \to \infty$ for each x. Furthermore, $S(x) = f(x)$ if f is continuous at x and*

$$S(x) = \frac{1}{2}(f(x+) + f(x-))$$

otherwise.

Example 2.6. Consider a periodic function f with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$. Note that f is discontinuous at $x = 2k-1$, $k \in \mathbb{Z}$. In this case $L = 1$. Let's find a_m and b_m. First,

$$a_0 = (f, 1) = \frac{1}{L}\int_{-1}^{1} f(x)\,dx = 0.$$

For $n = 1, 2, \cdots$, we have

$$\begin{aligned} a_n &= \frac{1}{L}(f, \cos(n\pi x)) = \int_{-1}^{1} f(x)\cos(n\pi x)\,dx \\ &= \int_{-1}^{1} x\cos(n\pi x)\,dx \\ &= \left[\frac{x\sin(n\pi x)}{n\pi}\right]_{-1}^{1} - \frac{1}{n\pi}\int_{-1}^{1} \sin(n\pi x)\,dx \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{L}(f, \sin(n\pi x)) = \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \\ &= \int_{-1}^{1} x\sin(n\pi x)\,dx \\ &= \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1} \cos(n\pi x)\,dx \\ &= -\frac{2\cos(n\pi)}{n\pi} \\ &= -\frac{2(-1)^n}{n\pi} \end{aligned}$$

Therefore,

$$S_N(x) = -\frac{2}{\pi}\sum_{m=1}^{N} \frac{(-1)^m}{m}\sin(m\pi x).$$

Since f satisfies the assumptions of the Fourier convergence theorem, we see that $S_N(x)$ converges to $S(x)$ as $N \to \infty$ for each x and

$$f(x) = -\frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m}\sin(m\pi x)$$

for $x \neq 2k-1$, $k \in \mathbb{Z}$. Note that $S(2k-1) = 0$ and

$$\frac{1}{2}(f((2k-1)+) + f((2k-1)-)) = \frac{1}{2}(-1+1) = 0$$

for all $k \in \mathbb{Z}$.

Example 2.7. Consider a periodic function f defined by

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. We have seen that

$$S_N(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{N} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

Since f satisfies the assumptions of the Fourier convergence theorem, we have

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

In particular, if $x = 0$, then

$$f(0) = 0 = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

and so

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

3 Even and Odd Functions (Sec 10.4)

Definition 3.1. A function f is called even if $f(-x) = f(x)$ for all x in the domain. A function f is called odd if $f(-x) = -f(x)$ for all x in the domain.

Example 3.2 (Even functions)**.**

(i) $\cos(mx)$ for any m.

(ii) x^k for even integers k.

(iii) $f(x) + f(-x)$ for any function f.

Example 3.3 (Odd functions)**.**

(i) $\sin(mx)$ and $\tan(mx)$ for any m.

(ii) x^k for odd integers k.

(iii) $f(x) - f(-x)$ for any function f.

Proposition 3.4. *Let f, f_1, f_2 be even and g, g_1, g_2 be odd.*

(i) *$f_1 \pm f_2$, $f_1 f_2$, $g_1 g_2$, f_1/f_2, g_1/g_2 are even functions.*

(ii) *$g_1 \pm g_2$, fg, and f/g are odd functions.*

(iii) *If f and g are differentiable, then f' is odd and g' is even.*

(iv) $\displaystyle\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$ *and* $\displaystyle\int_{-L}^{L} g(x)\,dx = 0.$

3.1 Fourier cosine series

Suppose f and f' are piecewise continuous on $[-L, L]$. Assume that f is even and periodic with period $2L$. That is, $f(x) = f(-x)$ and $f(x + 2L) = f(x)$ for all x. Since $f(x)\cos(m\pi x/L)$ is even and $f(x)\sin(m\pi x/L)$ is odd, we have $b_m = 0$ for all $m = 1, 2, \cdots$. By the Fourier convergence theorem, we obtain

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} \right)$$

Such a series is called a Fourier cosine series.

Example 3.5. Consider a periodic function f defined by

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ -x, & -1 \le x < 0, \end{cases}$$

and $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. Since f is even, f has a Fourier cosine series. Indeed, we have seen that

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

3.2 Fourier sine series

Suppose f and f' are piecewise continuous on $[-L, L]$. Assume that f is odd and periodic with period $2L$. That is, $f(x) = f(-x)$ and $f(x+2L) = f(x)$ for all x. Since $f(x)\cos(m\pi x/L)$ is odd and $f(x)\sin(m\pi x/L)$ is even, we have $a_m = 0$ for all $m = 0, 1, 2, \cdots$. By the Fourier convergence theorem, we obtain

$$f(x) = \sum_{m=1}^{\infty} \left(b_m \sin \frac{m\pi x}{L} \right)$$

Such a series is called a Fourier sine series.

Example 3.6. Consider a periodic function f with period 2 defined by $f(x) = x$ on $[-1, 1)$ and $f(x+2) = f(x)$. Note that f is discontinuous at $x = 2k-1$, $k \in \mathbb{Z}$. Since f is odd, it has a Fourier sine series. Indeed, we have seen that

$$f(x) = -\frac{2}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin(m\pi x)$$

for $x \neq 2k-1$, $k \in \mathbb{Z}$.

3.3 Even and odd periodic extension

Suppose we are given a function f on $[0, L]$. We want to represent it as a Fourier series on $[0, L]$. To do this, we first extend f to be a periodic function. There are a lot of ways to do that. We assume that f is nice enough that the Fourier convergence theorem is applicable.

3.3.1 Extension to Cosine series

Define g by

$$g(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(-x), & -L \le x < 0, \end{cases}$$

and $g(x+2L) = g(x)$. Then, $g(x)$ is an even periodic function with period $2L$. Thus, it has a Fourier cosine series

$$g(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right)$$

where

$$a_m = \frac{1}{L} \int_{-L}^{L} g(x) \cos\left(\frac{m\pi x}{L}\right) dx = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{m\pi x}{L}\right) dx.$$

In particular, if $x \in [0, L]$ then $g(x) = f(x)$ and so

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right).$$

This is called a Fourier cosine series of f.

3.3.2 Extension to Sine series

Define h by

$$h(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ -f(-x), & -L \leq x < 0, \end{cases}$$

and $h(x+2L) = h(x)$. Then, $h(x)$ is an odd periodic function with period $2L$. Thus, it has a Fourier sine series

$$h(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)$$

where

$$b_m = \frac{1}{L}\int_{-L}^{L} h(x)\sin\left(\frac{m\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)\,dx.$$

In particular, if $x \in [0, L]$ then $h(x) = f(x)$ and so

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right).$$

This is called a Fourier sine series of f.

Example 3.7. Suppose $f(x) = x$ on $[0,1)$ and define $g(x)$ by $g(x+2) = g(x)$ and

$$g(x) = \begin{cases} x, & 0 \leq x < 1, \\ -x, & -1 \leq x < 0. \end{cases}$$

We have seen that

$$g(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2}.$$

If we extend f to be an odd periodic function h with period 2 as above, then

$$h(x) = -\frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m}\sin(m\pi x)$$

for $x \neq 2k-1$, $k \in \mathbb{Z}$. In particular, we have

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty} \frac{\cos((2k-1)\pi x)}{(2k-1)^2} = -\frac{2}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^m}{m}\sin(m\pi x)$$

for $x \in [0,1)$.

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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