Math 416 Lecture Note: Week 8

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1 Introduction to determinants

We define the determinant of a square matrix $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ by the "(signed) volume" of the parallelogram spanned by the row vectors of A. For $n = 1$, it is natural to define $\det(A) = a$ where $A = (a)$. In this section, we focus on the case $n = 2$.

Let $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Let v, w be the row vectors of A, that is,

$$A = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $v = (a, b)$ and $w = (c, d)$. We consider the determinant of A as a function from $\mathcal{M}_{2\times 2}(\mathbb{R})$ to $\mathbb{R}$. If the determinant represents the area of the parallelogram spanned by v, w, then it should satisfy the following:

(i) $\det(I_2) = 1$.

(ii) (multilinear) For any $u, v, w \in \mathbb{R}^2$ and $c \in \mathbb{R}$,

$$\det\begin{pmatrix} v \\ cu + w \end{pmatrix} = c\begin{pmatrix} v \\ u \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}, \qquad \det\begin{pmatrix} cu + v \\ w \end{pmatrix} = c\begin{pmatrix} u \\ w \end{pmatrix} + \begin{pmatrix} v \\ w \end{pmatrix}.$$

(iii) (alternating) For any $v \in \mathbb{R}^2$,

$$\det\begin{pmatrix} v \\ v \end{pmatrix} = 0.$$

For each (z, w), the map

$$(x, y) \in \mathbb{R}^2 \mapsto \det\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathbb{R}$$

is linear, so that there exist constants $T(z, w), S(z, w)$ depending on z, w such that

$$\det\begin{pmatrix} x & y \\ z & w \end{pmatrix} = T(z, w)x + S(z, w)y.$$

On the other hand, if $x = 1$ and $y = 0$, then the map

$$(z, w) \mapsto \det\begin{pmatrix} 1 & 0 \\ z & w \end{pmatrix} = T(z, w)$$

is linear. Similarly, $S(z, w)$ is also linear. Thus, there exist $p, q, r, s \in \mathbb{R}$ such that $T(z, w) = pz + qw$ and $S(z, w) = rz + sw$, which leads to

$$\det\begin{pmatrix} x & y \\ z & w \end{pmatrix} = (pz + qw)x + (rz + sw)y.$$

We determine the constants p, q, r, s using the property (i) and (iii). Indeed, we have

$$\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = q = 1, \qquad \det\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = p = 0$$
$$\det\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix} = s = 0, \qquad \det\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = q + r = 0.$$

Thus, if the determinant satisfy the above properties, it should be $\det(A) = ad - bc$.

Definition 1.1. For $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$, the determinant of A is defined by $\det(A) = ad - bc$.

Remark 1.2. One can check that $\det(A) = ad - bc$ indeed satisfy the property (i), (ii), and (iii). That is, the determinant is the unique alternating multilinear map from $\mathcal{M}_{2\times 2}(\mathbb{R})$ to $\mathbb{R}$ with $\det(I_2) = 1$. For $n \geq 3$, the determinant can be defined in the same way. In general, there exists a unique alternating multilinear map from $\mathcal{M}_{n\times n}(\mathbb{R})$ to $\mathbb{R}$ which maps I_n to 1. We call the map the determinant. We will not go over the uniqueness in this course. Next time, we construct the determinant for $n \geq 3$ and show that it is actually alternating and multilinear.

Theorem 1.3. *Let $A, B \in \mathcal{M}_{2\times 2}(\mathbb{R})$.*

(i) $\det(AB) = \det(A)\det(B)$.

(ii) *A is invertible if and only if* $\det(A) \neq 0$. *In this case,* $\det(A^{-1}) = (\det(A))^{-1}$.

Proof. Exercise. □

We will check that the determinant really measures the area of the parallelogram spanned by the row vectors.

Theorem 1.4. *Let $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$, then $|\det(A)|$ is the area of the parallelogram spanned by the row vectors of A.*

Proof.

$$A = \begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

It is easy to see that

$$T_\theta\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

rotates a vector by angle $\theta \in [0, 2\pi)$ counterclockwise. Let

$$\begin{pmatrix}p\\ q\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}, \qquad \begin{pmatrix}r\\ s\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}c\\ d\end{pmatrix}.$$

Then, we have (using the transpose)

$$B = \begin{pmatrix}p & q\\ r & s\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Thus, we get

$$\det(B) = \det\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\det(A) = \det(A).$$

By rotation, it is okay to assume that one of the row vectors is parallel to the x-axis. We assume that

$$B = \begin{pmatrix}p & 0\\ r & s\end{pmatrix},$$

then $\det(A) = \det(B) = ps$. In this case the base of the parallelogram is $|p|$ and the height is $|s|$, which completes the proof. □

Remark 1.5. The sign of the determinant represents the orientation of the row vectors of a matrix. We say that $\{v, w\}$ is positively oriented if v can be rotated counterclockwise through $\theta \in (0, \pi)$ to coincide with w, and negatively oriented otherwise. Then, $\{v, w\}$ is positively oriented if and only if $\det(A) > 0$ where

$$A = \begin{pmatrix} v \\ w \end{pmatrix}.$$

2 Definition of determinants

Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Recall that $\det(A) = A_{11}$ for $n = 1$, and $\det(A) = A_{11}A_{22} - A_{12}A_{21}$ for $n = 2$. In this section, we define the determinant for $n \geq 3$. This definition also agrees with the previous ones for $n = 1, 2$.

Definition 2.1. For $A \in \mathcal{M}_{n\times n}(\mathbb{R})$, $\widetilde{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained by deleting the i-th row and the j-th column of A.

Example 2.2. Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$. Then,

$$\widetilde{A}_{12} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \widetilde{A}_{23} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \widetilde{A}_{31} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}.$$

Definition 2.3. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. The determinant of A is defined by

$$\det(A) = |A| = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}).$$

The cofactor C_{ij} of A in row i and column j is defined by

$$C_{ij} = (-1)^{1+j} \det(\widetilde{A}_{ij}).$$

That is, the determinant can be written as

$$\det(A) = A_{11}C_{11} + A_{12}C_{12} + \cdots + A_{1n}C_{1n}.$$

This formula is called the cofactor expansion along the first row of A.

Example 2.4. $\det(I_n) = 1$.

Example 2.5. Consider $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix}$. Then,

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{pmatrix} = 1(-1)^{1+1} \det \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} + 2(-1)^{1+2} \det \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} + 3(-1)^{1+3} \det \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 7.$$

Remark 2.6. We will see that this determinant is alternating multilinear, as we have seen for $n = 2$ last time. It is well-known that such a map from $\mathcal{M}_{n\times n}(\mathbb{R})$ to $\mathbb{R}$ is unique.

Theorem 2.7. *Let $A, B, C \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $k \in \mathbb{R}$. Let $r \in \{1, 2, \cdots, n\}$ and u, v, w the r-th rows of A, B, C respectively. If A, B, C are the same except in row r where $u = kv + w$, then*

$$\det(A) = k\det(B) + \det(C).$$

Proof. Use an induction on n. If $n = 1$, it is trivial. Assume that $n > 1$ and the theorem holds for $n - 1$. If $r = 1$, then $A_{1j} = kB_{1j} + C_{1j}$ and $\widetilde{A}_{1j} = \widetilde{B}_{1j} = \widetilde{C}_{1j}$ for all $j = 1, 2, \cdots, n$. Thus,

$$\begin{aligned} \det(A) &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\widetilde{A}_{1j}) \\ &= k \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \det(\widetilde{B}_{1j}) + \sum_{j=1}^{n} (-1)^{1+j} C_{1j} \det(\widetilde{C}_{1j}) \\ &= k\det(B) + \det(C). \end{aligned}$$

Suppose $r > 1$, then $\widetilde{A}_{1j}, \widetilde{B}_{1j}, \widetilde{C}_{1j}$ are the same except in row $(r-1)$ where $u' = kv' + w'$. Here, u', v', w' are the $(r-1)$-th rows of $\widetilde{A}_{1j}, \widetilde{B}_{1j}, \widetilde{C}_{1j}$. By the induction hypothesis, we have

$$\det(\widetilde{A}_{1j}) = k\det(\widetilde{B}_{1j}) + \det(\widetilde{C}_{1j}).$$

Since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n}(-1)^{1+j}A_{1j}\det(\widetilde{A}_{1j}) \\
&= k\sum_{j=1}^{n}(-1)^{1+j}B_{1j}\det(\widetilde{B}_{1j}) + \sum_{j=1}^{n}(-1)^{1+j}C_{1j}\det(\widetilde{C}_{1j}) \\
&= k\det(B) + \det(C).
\end{aligned}$$

□

Corollary 2.8. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. If A has a row consisting of of all zeros, then* $\det(A) = 0$.

Theorem 2.9. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $i \in \{1, 2, \cdots, n\}$, then*

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}A_{ij}\det(\widetilde{A}_{ij}).$$

Proof. This follow from the multi-linearity and [FIS, Lemma in p. 213]. See [FIS, Theorem 4.4 in p. 215]. □

Corollary 2.10. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $n \geq 2$. If A has two identical rows, then* $\det(A) = 0$.

Proof. Use induction on n. If $n = 2$, we have seen this the last time. Suppose $n \geq 3$ and the result holds for $n-1$. Let rows r and s of A be identical for $r \neq s$. Let i be an integer such that $1 \leq i \leq n$, $i \neq r, s$. Then $\widetilde{A}_{ij}$ is an $(n-1)\times(n-1)$ matrix with two identical rows. By the induction hypothesis, $\det(\widetilde{A}_{ij}) = 0$ for all $j = 1, 2, \cdots, n$. Thus,

$$\det(A) = \sum_{j=1}^{n}(-1)^{i+j}A_{ij}\det(\widetilde{A}_{ij}) = 0.$$

□

Theorem 2.11. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. If A is not invertible (or, equivalently* $\operatorname{rank}(A) < n$*), then* $\det(A) = 0$.

Proof. If A is not invertible, then the set of the row vectors of A is linearly dependent. This means that one of the row vectors can be written as a linear combination of the others. Suppose

$$v_i = a_1v_1 + \cdots + a_{i-1}v_{i-1} + a_{i+1}v_{i+1} + \cdots + a_nv_n$$

for some I, where $v_1, \cdots, v_n$ are the row vectors of A. Then, A can be written as

$$A = a_1B_1 + \cdots + a_{i-1}B_{i-1} + a_{i+1}B_{i+1} + \cdots + a_nB_n$$

where B_i are $n \times n$ matrices having two identical rows. Also A and B_i differ only in one row. Thus, by the multilinearity and the alternating property, we have

$$\det(A) = a_1\det(B_1) + \cdots + a_{i-1}\det(B_{i-1}) + a_{i+1}\det(B_{i+1}) + \cdots + a_n\det(B_n) = 0.$$

□

3 Determinants and row operations

Recall that there are three types of the row operations:

(1) Swap two rows of M. ($R_i \leftrightarrow R_j$)

(2) Multiply one row by a nonzero constant $c \in \mathbb{R}$. ($R_i \to cR_i$)

(3) Add one row to another. ($R_i \to R_i + R_j$).

We study how these operation affect on the determinant.

Theorem 3.1. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. If $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ is obtained by $R_i \leftrightarrow R_j$, then* $\det(B) = -\det(A)$.

Proof. Let R_i and R_j be the i-th and j-th row vectors of A, then

$$A = \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{pmatrix}, \qquad B = \begin{pmatrix} \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \end{pmatrix}.$$

By the linearity and the alternating property, we get

$$\begin{aligned} 0 = \det \begin{pmatrix} \vdots \\ R_i + R_j \\ \vdots \\ R_i + R_j \\ \vdots \end{pmatrix} &= \det \begin{pmatrix} \vdots \\ R_i + R_j \\ \vdots \\ R_i \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ R_i + R_j \\ \vdots \\ R_j \\ \vdots \end{pmatrix} \\ &= \det \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_i \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ R_j \\ \vdots \\ R_i \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ R_j \\ \vdots \\ R_j \\ \vdots \end{pmatrix} = \det(A) + \det(B), \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $c \in \mathbb{R}$. If $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ is obtained by $R_i \to cR_i$, then* $\det(B) = c\det(A)$.

Proof. This follows from the linearity. □

Theorem 3.3. *Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $c \in \mathbb{R}$. If $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ is obtained by $R_i \to R_i + cR_j$, then* $\det(B) = \det(A)$.

Proof. Let R_i and R_j be the i-th and j-th row vectors of A, then

$$A = \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{pmatrix}, \qquad B = \begin{pmatrix} \vdots \\ R_i + cR_j \\ \vdots \\ R_j \\ \vdots \end{pmatrix}.$$

By the linearity and the alternating property, we get

$$\det(B) = \det \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{pmatrix} + c \det \begin{pmatrix} \vdots \\ R_j \\ \vdots \\ R_j \\ \vdots \end{pmatrix} = \det \begin{pmatrix} \vdots \\ R_i \\ \vdots \\ R_j \\ \vdots \end{pmatrix} = \det(A).$$

□

These observations tell us how to compute the determinant using the row operations.

Theorem 3.4. *If A is upper or lower triangular, then $\det(A)$ is the product of diagonal entries of A.*

Proof. Suppose A is lower triangular. (The same argument works for the other case.) We use an induction on n. If $n = 1$, it is trivial. Suppose $n \geq 2$ and the result holds for $n - 1$. Let

$$A = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ A_{21} & A_{22} & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ A_{n1} & A_{n2} & A_{n3} & \cdots & A_{nn} \end{pmatrix}.$$

By the cofactor expansion of A along the first row, we get

$$\det(A) = A_{11} \det(\widetilde{A}_{11}).$$

Since $\widetilde{A}_{11}$ is lower triangular, the result follows from the induction hypothesis. □

Example 3.5.

$$A = \begin{pmatrix} 3 & -7 & 4 \\ 1 & -2 & 1 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -2 & 1 \\ 3 & -7 & 4 \\ 2 & -1 & -2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 2 & -1 & -2 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3 - 2R_1} \begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 3 & -4 \end{pmatrix} \xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -4 \end{pmatrix}$$
$$\xrightarrow{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix} = B.$$

Thus, $\det(A) = \det(B) = -1$.

References

[FIS] Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.

[Bee] Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

[Hol] W. H. Holzmann, http://www.cs.uleth.ca/ holzmann/notes/reduceduniq.pdf

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