Math 285 Lecture Note: Week 11

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1 Two-Point Boundary Value Problems, part 2 (Sec 10.1)

In this section, we focus on a homogenous boundary value problem $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ for some $L > 0$.

Recall that a boundary value problem $y'' + p(x)y' + q(x)y = g(x)$ with $y(\alpha) = y_0$ and $y(\beta) = y_1$ is called *homogeneous* if $g(x) = 0$ and $y_0 = y_1 = 0$. Otherwise, it is called *nonhomogeneous*.

If a boundary value problem is nonhomogeneous, it has (i) a unique solution, (ii) infinitely many solutions, or (iii) no solutions. If it is homogeneous, the problem always has a trivial solution $y = 0$. So, it has (i) a unique solution or (ii) infinitely many solutions.

Definition 1.1. Let $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ for $L > 0$. We call λ is an *eigenvalue* of the boundary value problem if it has nontrivial solutions. The solutions are called the corresponding *eigenfunctions*.

Our goal is to find all eigenvalues and eigenfunctions of $y'' + \lambda y = 0$ with $y(0) = 0$ and $y(L) = 0$ where $L > 0$.

Case 1: $\lambda > 0$.

For notational simplicity, let $\lambda = \mu^2$ for $\mu \in \mathbb{R}$. The general solution to $y'' + \mu^2 y = 0$ is

$$y(x) = C_1 \cos \mu x + C_2 \sin \mu x.$$

The boundary conditions yield $C_1 = 0$ and

$$y(L) = C_2 \sin \mu L = 0.$$

If $C_2 \neq 0$, then $\mu L = n\pi$ for $n \in \mathbb{N}$. Thus, if $\lambda \neq n^2(\pi/L)^2$ then the boundary value problem has a unique trivial solution. The eigenvalues are $\lambda = n^2(\pi/L)^2$ for all $n \in \mathbb{N}$ and the corresponding eigenfunctions are $C \sin(n\pi x/L)$.

Case 2: $\lambda < 0$.

Let $\lambda = -\mu^2$ for $\mu \in \mathbb{R}$. The general solution to $y'' - \mu^2 y = 0$ is

$$y(x) = C_1 \cosh \mu x + C_2 \sinh \mu x$$

where $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. The boundary conditions yield $C_1 = 0$ and

$$y(L) = C_2 \sinh \mu L = 0.$$

Since $L \neq 0$, $C_2 = 0$. Thus, the equation does not have nontrivial solutions. There is no negative eigenvalue.

Case 3: $\lambda = 0$.

The general solution to $y'' = 0$ is $y(x) = C_1x + C_2$. The boundary conditions yield $y(0) = C_2 = 0$ and $y(L) = C_1L = 0$. Thus, the equation does not have nontrivial solutions and so 0 is not an eigenvalue.

Remark 1.2. In general, let $L[y]$ be a differential operator. For example, $L[y] = -y''$ or $L[y] = -x^2y''+2xy'$. Suppose boundary conditions are given by $y(\alpha) = y_0$ (or $y'(\alpha) = y_0$) and $y(\beta) = y_1$ (or $y'(\beta) = y_1$). Then, λ is an eigenvalue of $L[y]$ with the boundary conditions if $L[y] = \lambda y$ with the boundary conditions has nontrivial solutions.

2 Fourier Series, part 1 (Sec 10.2)

2.1 Periodic functions

Definition 2.1. A function f is periodic with period $T > 0$ if

(i) $x + T$ belongs to the domain of f if x does, and

(ii) $f(x + T) = f(x)$ for all x.

The smallest period $T > 0$ is called the fundamental period of f.

Example 2.2. It is easy to see that $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$ are periodic with the same period $2L/m$.

Proposition 2.3. *If f and g are periodic functions with common period T, then so is $c_1f + c_2g$ for any $c_1, c_2 \in \mathbb{R}$.*

2.2 Inner product and Orthogonality

Definition 2.4. For functions f and g on $[\alpha, \beta]$, we define the standard inner product of f and g by

$$(f, g) = \int_\alpha^\beta f(x)g(x)\,dx.$$

Remark 2.5. The inner product has the following properties:

(i) (Linearity) $(cf + g, h) = c(f, h) + (g, h)$;

(ii) (Symmetry) $(f, g) = (g, f)$;

(iii) (Positive-definite) $(f, f) \geq 0$ and $(f, f) = 0$ if and only if $f = 0$.

Indeed, if a relation $(\cdot, \cdot)$ satisfies these three assumptions, we call it an inner product. An elementary example of inner product is dot product.

Definition 2.6. We say that functions f and g are orthogonal on $[\alpha, \beta]$ if $(f, g) = 0$. We say that a set of functions are mutually orthogonal if any two functions in the set are orthogonal.

Example 2.7. One can see that

$$\begin{aligned}
&\left(\sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\right) = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0, & m \neq n, \\ L, & m = n, \end{cases} \\
&\left(\sin\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\right) = 0, \\
&\left(\cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\right) = \begin{cases} 0, & m \neq n, \\ L, & m = n. \end{cases}
\end{aligned}$$

Thus, the set $\{\sin\frac{m\pi x}{L}, \cos\frac{m\pi x}{L} : m \in \mathbb{Z}\}$ is mutually orthogonal.

2.3 Fourier series

Suppose that a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

Assume that the infinite sum in the RHS converges for each $x \in [-L, L]$. Note that f is periodic with period $2L$. Our goal is to relate f with the coefficients a_m, b_m. To this end, we compute

$$\begin{aligned} (f, \cos \frac{n\pi x}{L}) &= \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \\ (f, \sin \frac{n\pi x}{L}) &= \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx, \end{aligned}$$

for each $n = 0, 1, 2, \cdots$. In fact, we have

$$\begin{aligned} (f, \cos \frac{n\pi x}{L}) &= \frac{a_0}{2} \int_{-L}^{L} \cos \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L}\, dx \\ &= a_n L \end{aligned}$$

by orthogonality for $n \in \mathbb{N}$. (We note that the above computation is not rigorous. To be precise, one needs to justify whether the infinite sum and integrals are interchangeable, and if the sum converges. This is beyond the scope of the course.) Similarly,

$$\begin{aligned} (f, \sin \frac{n\pi x}{L}) &= \frac{a_0}{2} \int_{-L}^{L} \sin \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx + \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L}\, dx \\ &= b_n L \end{aligned}$$

for $n \in \mathbb{N}$. If $n = 0$, then

$$(f, \cos \frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\, dx = a_0 L.$$

Therefore, we conclude that

$$\begin{aligned} a_n &= \frac{1}{L} (f, \cos \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx, \\ b_n &= \frac{1}{L} (f, \sin \frac{n\pi x}{L}) = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx. \end{aligned}$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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