Math 285 Lecture Note: Week 4

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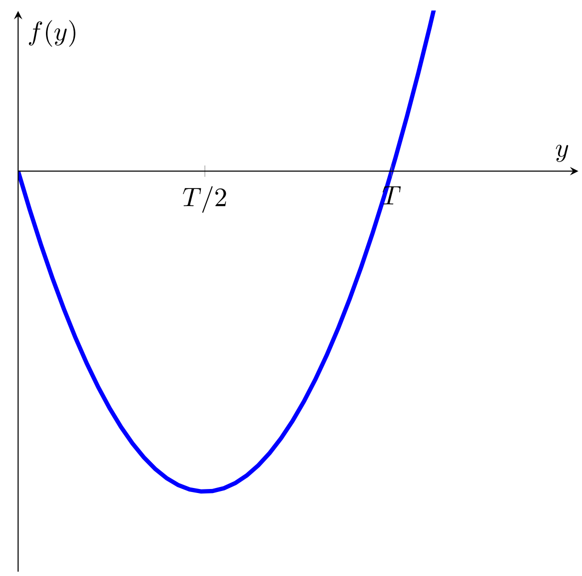
1 Autonomous Equations and Population Dynamics, part 2

1.1 A critical threshold

Consider

$$y' = -r(1 - \frac{y}{T})y.$$

We draw the graph of $f(y)$.



(i) $y(t) = 0$ and $y(t) = T$ are equilibrium solutions.

(ii) If $0 < y < T$, then y decreases. If $y > T$, then y increases. This T is called a critical threshold.

(iii) $y(t) = 0$ is asymptotically stable and $y(t) = T$ is unstable.

(iv) If $0 < y < T/2$, the graph is concave up. If $T/2 < y < T$, the graph is concave down. If $y > T$, the graph is concave up.

(v) By the separation method, the solution is

$$y(t) = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}.$$

(vi) If $y_0 \in (0, T)$, then $y(t) \to 0$ as $t \to \infty$. If $y_0 > T$, then the solution blows up in finite time

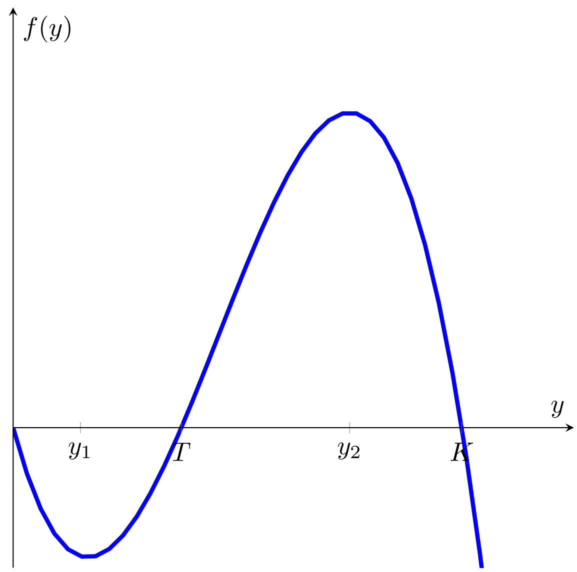
$$t_* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}.$$

1.2 Logistic growth with a threshold

Consider

$$y' = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y$$

where $r > 0$ and $0 < T < K$. We draw the graph of $f(y)$.



Suppose $f(y)$ has local minimum at $y = y_1$ and local maximum at $y = y_2$.

(i) $y(t) = 0, T, K$ are equilibrium solutions.

(ii) If $0 < y < T$, then y decreases. If $T < y < K$ then y increases. If $y > K$, then y decreases.

(iii) $y(t) = 0$ and $y(t) = K$ are asymptotically stable and $y(t) = T$ is unstable.

(iv) If $0 < y < y_1$, then the graph is concave up. If $y_1 < y < T$, then the graph is concave down. If $T < y < y_2$, then the graph is concave up. If $y_2 < y < K$, the graph is concave down. If $y > K$, the graph if concave up.

2 Homogeneous Equations with Constant Coefficients

A second order ODE has the form

$$y'' = F(t, y, y').$$

If F is linear, the equation can be written as

$$y'' + p(t)y' + q(t)y = g(t).$$

When we discuss initial value problems for first order ODEs, the initial condition is given at one point (t_0, y_0). However, for second order ODEs, the initial condition consists of $y(t_0) = y_0$ and $y'(t_0) = y'_0$. A second order ODE is called homogeneous if $g(t) = 0$. If not, it is called nonhomogeneous.

In this section, we discuss second order homogeneous linear ODEs with constant coefficients. We start with a simple example.

Example 2.1. Consider $y'' - 4y = 0$ with $y(0) = 2$ and $y'(0) = 8$. Suppose ϕ and ψ are solutions for the equation. Then, the linearity of the equation yields the following:

(i) For any $a, b \in \mathbb{R}$, $a\phi(t)$ and $b\psi(t)$ are also solutions. This is because

$$(a\phi(t))'' = (a\phi'(t))' = a\phi''(t) = 4a\phi(t).$$

(ii) The sum $\phi + \psi$ is also a solution. This is because

$$(\phi(t) + \psi(t))'' = (\phi'(t) + \psi'(t))' = \phi''(t) + \psi''(t) = 4\phi(t) + 4\psi(t).$$

Indeed, $\phi(t) = e^{2t}$ and $\psi(t) = e^{-2t}$ are solutions so that $y(t) = a\phi(t) + b\psi(t) = ae^{2t} + be^{-2t}$ is also a solution for $a, b \in \mathbb{R}$. The constants will be determined by the initial conditions. Since $y'(t) = 2ae^{2t} - 2be^{-2t}$, we have

$$\begin{aligned} y(0) &= a + b = 2, \\ y'(0) &= 2a - 2b = 8. \end{aligned}$$

Thus, $a = 3$ and $b = -1$. The solution is $y(t) = 3e^{2t} - e^{-2t}$.

Suppose we have $y'' + py' + qy = 0$ where $p, q \in \mathbb{R}$. Based on the previous example, we put $y(t) = e^{rt}$. Then,

$$y'' + py' + qy = (r^2 + pr + q)e^{rt} = 0.$$

So, $y(t) = e^{rt}$ is a solution to the equation if $r^2 + pr + q = 0$. The last equation is called the *characteristic equation.*

Example 2.2. Consider $y'' + 4y' + 3y = 0$ with $y(0) = 3$ and $y'(0) = -5$. If $y(t) = e^{rt}$ is a solution, then

$$y'' + 4y' + 3y = (r^2 + 4r + 3)e^{rt} = 0.$$

The characteristic equation is $r^2 + 4r + 3 = (r + 1)(r + 3) = 0$. This holds if $r = -1, -3$. Thus, $\phi(t) = e^{-t}$ and $\psi(t) = e^{-3t}$ are solutions to the equation. You can check that

$$y(t) = a\phi(t) + b\psi(t) = ae^{-t} + be^{-3t}$$

is also a solution as we have seen in the previous example. By the initial conditions, we get

$$\begin{aligned} y(0) &= a + b = 3, \\ y'(0) &= -a - 3b = -5, \end{aligned}$$

which yields $a = 2$ and $b = 1$. Thus, the solution is $y(t) = 2e^{-t} + e^{-3t}$.

3 Solutions of Linear Homogeneous Equations; the Wronskian

Theorem 3.1 (Existence and Uniqueness)**.** *Consider*

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \qquad y'(t_0) = y_0'$$

where $p(t), q(t), g(t)$ are continuous on $I = (\alpha, \beta)$ and $t_0 \in I$. Then, there exists a unique solution on I.

Proof. Beyond the scope of the course. □

Example 3.2. Let $t(t - 5)y'' + 3ty' + 4y = 2$ with $y(2) = 2$ and $y'(2) = 1$. By normalizing the equations, we get

$$y'' + \frac{3}{t-5}y' + \frac{4}{t(t-5)}y = \frac{2}{t(t-5)}.$$

Thus, the coefficients are continuous on $(-\infty, 0) \cup (0, 5) \cup (5, \infty)$. Since $2 \in (0, 5)$, the longest interval in which the initial value problem has a unique solution is $(0, 5)$.

We consider a second order homogeneous linear ODE of the form

$$y'' + p(t)y' + q(t)y = 0$$

where $p(t)$ and $q(t)$ are continuous on the interval $I = (\alpha, \beta)$.

We define the differential operator L (here, an operator is a map from a set of functions to another set of functions) by $\phi \mapsto L[\phi]$,

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t).$$

In this context, a solution of the equation can be thought of as a function ϕ such that $L[\phi] = 0$. That is, the set of solutions is the set of "roots" of the differential operator L.

Theorem 3.3 (Principle of Superposition)**.** *If y_1 and y_2 are solutions to*

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

then $c_1y_1 + c_2y_2$ is also a solution for $c_1, c_2 \in \mathbb{R}$. In other words, if $L[y_1] = L[y_2] = 0$, then $L[c_1y_1 + c_2y_2] = 0$ for all $c_1, c_2 \in \mathbb{R}$.

Proof. For any functions y_1, y_2 and $c_1, c_2 \in \mathbb{R}$, we have

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= (c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &= c_1L[y_1] + c_2L[y_2], \end{aligned}$$

which proves the theorem. □

Example 3.4. Note that the superposition property holds for any linear differential equations. Consider a nonlinear equation $2y'y = 1$. Then, $y_1(t) = \sqrt{t}$ and $y_2(t) = \sqrt{t+1}$ are solutions. But, one can see that $c_1y_1 + c_2y_2$ is not a solution for any $c_1, c_2 \neq 0$.

For a 2nd linear equation, we can find infinitely many solutions if we know two different solutions. A natural question is if having two solutions is enough. This is the case if the two solutions are "truely" different.

Definition 3.5. For two functions $y_1(t)$ and $y_2(t)$, the Wronskian of y_1 and y_2 is a function of t defined by

$$W[y_1, y_2](t) = \det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Example 3.6. If $y_1(t) = e^{-t}$ and $y_2(t) = e^{-3t}$, then

$$W[y_1, y_2](t) = \det \begin{pmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{pmatrix} = e^{-t}e^{-3t} - 3e^{-t}e^{-3t} = -2e^{-4t}.$$

Theorem 3.7. *Let y_1 and y_2 be solutions to*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where $p(t), q(t)$ are continuous on an open interval $I = (\alpha, \beta)$. Then, every solution to the equation has the form $\phi(t) = c_1y_1(t) + c_2y_2(t)$ for some $c_1, c_2 \in \mathbb{R}$ if and only if

$$W[y_1, y_2](t_0) = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0) \neq 0$$

for some $t_0 \in I$.

Definition 3.8. Let y_1 and y_2 be solutions to

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where $p(t), q(t)$ are continuous on an open interval $I = (\alpha, \beta)$. A set $\{y_1, y_2\}$ is called a fundamental set of solutions if $W[y_1, y_2](t_0) \neq 0$ for some $t_0 \in I$. In this case,

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is called the general solution.

Example 3.9. Consider $y'' + 4y' + 3y = 0$ with $y(0) = 3$ and $y'(0) = -5$. The characteristic equation is $r^2 + 4r + 3 = (r+1)(r+3) = 0$. Thus, $y_1(t) = e^{-t}$ and $y_2(t) = e^{-3t}$ are solutions to the equation. Since

$$W[y_1, y_2](t) = -2e^{-4t} \neq 0$$

for all $t \in \mathbb{R}$, $\{e^{-t}, e^{-3t}\}$ is a fundamental set of solutions and

$$y(t) = c_1 e^{-t} + c_2 e^{-3t}$$

is the general solution.

Theorem 3.10. *Let y_1 and y_2 be solutions to*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where $p(t), q(t)$ are continuous on an open interval $I = (\alpha, \beta)$. If

$$\begin{aligned} y_1(t_0) &= 1, \qquad y_1'(t_0) = 0 \\ y_2(t_0) &= 0, \qquad y_2'(t_0) = 1 \end{aligned}$$

for some $t_0 \in I$, then $\{y_1, y_2\}$ is a fundamental set of solutions.

Example 3.11. Consider $y'' - y = 0$. It is easy to see that e^t, e^{-t} are solutions. We also have seen that $y_1(t) = \cosh(t)$ and $y_2(t) = \sinh(t)$ are solutions. Since

$$\begin{aligned} y_1(t_0) &= 1, \qquad y_1'(t_0) = 0 \\ y_2(t_0) &= 0, \qquad y_2'(t_0) = 1, \end{aligned}$$

they form a fundamental set of solutions.

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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