Math 285 Midterm 3 Practice Exam Solution

1. (5 points) Find a solution to $y^{(4)} + y'' = 24t$.

   A. $y = 4t^3 + 1$

   B. $y = 6t^3 + 5t^2$

   C. $y = 4t^2 + t$

   D. $y = 4t^3 + t^2$

   E. None of these

Solution: The characteristic equation is $\lambda^4 + \lambda^2 = \lambda^2(\lambda^2+1) = 0$. Thus, the roots are $\lambda = 0, i, -i$ and fundamental solutions to the homogeneous equation are $1, t, \cos t, \sin t$. By the method of undetermined coefficients, a particular solution is $Y = t^2(At+B)$. Since $Y^{(4)} + Y'' = 6At + 2B = 24t$, we have $A = 4$ and $B = 0$. Therefore, $Y = 4t^3$ is a particular solution.

2. (5 points) Find the smallest number λ such that $y'' + \lambda y = 0$ with $y'(0) = y'(\pi) = 0$ has a nontrivial solution.

   A. 1

   B. 0

   C. $\frac{1}{2}$

   D. π^2

   E. None of these

Solution: If $\lambda = -\mu^2 < 0$, then $y(t) = c_1 e^{\mu t} + c_2 e^{-\mu t}$. By the boundary conditions, we get $c_1 = c_2 = 0$. Thus, if $\lambda < 0$, there is no nontrivial solution. If $\lambda = 0$, then $y(t) = 1$ is a solution. Thus 0 is the smallest number.

3. (5 points) Suppose that a function $f(t)$ which is periodic of period 2π has the Fourier series

$$f(t) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m^2 + 3m} \cos mt.$$

Evaluate the integral

$$\int_{-\pi}^{\pi} f(t) \cos 6t \, dt.$$

A. $\frac{1}{54}$
B. $\frac{1}{27}$
C. $\frac{\pi}{54}$
D. $\frac{\pi}{27}$
E. None of these

Solution: This follows from the fact that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos mt\,dt = \frac{(-1)^m}{m^2+3m}.$$

4. (5 points) Consider the function $f(t)$ defined on $\mathbb{R}$ such that $f(t) = f(t+2\pi)$ and

$$f(t) = \begin{cases} 3, & -\pi \le t < 0, \\ e^{\pi^2}, & t = 0, \\ -1, & 0 < t < \pi. \end{cases}$$

Let $S(t)$ be the Fourier series of $f(t)$. What is $S(0)$?

A. e^{π^2}
B. 2
C. 0
D. 1
E. None of these

Solution: This follows from the fact that $f(0-) = 3$, $f(0+) = -1$, and

$$S(x) = \frac{1}{2}(f(x-) + f(x+)).$$

5. (5 points) Let $f(t)$ be a function on $[0, 2]$ given by $f(t) = 2t$. Find the Fourier sine series for $f(t)$ of period 4.

A. $\displaystyle\sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{2}t\right)$ where $a_m = \displaystyle\frac{1}{2}\int_0^2 t\sin\left(\frac{m\pi}{2}t\right)\,dt.$
B. $\displaystyle\sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{2}t\right)$ where $a_m = \displaystyle\frac{1}{2}\int_0^2 t\sin\left(\frac{m\pi}{2}t\right)\,dt.$

C. $\displaystyle\sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{4}t\right)$ where $a_m = \displaystyle\int_0^2 t\sin\left(\frac{m\pi}{4}t\right)\,dt.$

D. $\displaystyle\sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{2}t\right)$ **where** $a_m = 2\displaystyle\int_0^2 t\sin\left(\frac{m\pi}{2}t\right)\,dt.$

E. $\displaystyle\sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{4}t\right)$ where $a_m = \displaystyle\int_{-2}^2 t\sin\left(\frac{m\pi}{4}t\right)\,dt.$

Solution: In this case, $L = 2$. The Fourier sine series of f is

$$S(x) = \sum_{m=1}^{\infty} a_m \sin\left(\frac{m\pi}{2}t\right)$$

where

$$a_m = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{m\pi}{2}t\right)\,dt = 2\int_0^2 t\sin\left(\frac{m\pi}{2}t\right)\,dt.$$

6. (5 points) Let f and g be functions defined on $\mathbb{R}$. Which one of the followings is NOT correct?

A. If f is even, then f' is odd.

B. The function $\sin 3t + \cos 2t$ is periodic with period 2π.

C. If f is even and g is odd, then $f(x) + g(x)$ is even.

D. If f is even and g is odd, then $\displaystyle\int_{-4}^4 f(x)g(x)\,dx = 0.$

E. If f is periodic with period 4 and $f(x) = x$ for $0 < x < 2$, then $f(x) = x - 4$ for $4 < x < 6$.

Solution:

A. If f is even, then $f(x) = f(-x)$. Taking the derivative of both sides, we get $f'(x) = -f'(-x)$, which means f' is odd.

B. Let $h(t) = \sin 3t + \cos 2t$, then $h(t + 2\pi) = \sin(3t + 6\pi) + \cos(2t + 4\pi) = \sin 3t + \cos 2t = h(t)$.

C. If $f = 1$ and $g = x$, then f is even and g is odd but $f(x)+g(x)$ is not even nor odd.

D. If f is even and g is odd, then $f(x)g(x)$ is odd and so $\displaystyle\int_{-4}^4 f(x)g(x)\,dx = 0.$

E. If $4 < x < 6$, then $0 < x - 4 < 2$. Since f is periodic with period 4, we have $f(x) = f(x-4) = x - 4$ for $4 < x < 6$.

7. (5 points) Find a pair of ordinary differential equations from the partial differential equation $xu_{xx} + u_t = 0$ using the method of separation of variables.
   - A. $X''(x) + \lambda X(x) = 0$ and $T'(t) + \lambda x T(t) = 0$
   - **B.** $xX''(x) + \lambda X(x) = 0$ **and** $T'(t) - \lambda T(t) = 0$
   - C. $X''(x) + \lambda x X(x) = 0$ and $\lambda T'(t) - T(t) = 0$
   - D. $X''(x) - \lambda x X(x) = 0$ and $T'(t) - \lambda T(t) = 0$
   - E. None of these

Solution: Let $u(x,t) = X(x)T(t)$, then $xu_{xx} + u_t = 0$ can be written as

$$x\frac{X''}{X} = -\frac{T'}{T} = -\lambda.$$

Thus, $xX''(x) + \lambda X(x) = 0$ and $T'(t) - \lambda T(t) = 0$.

8. (5 points) Consider the heat conduction problem

$$5u_{xx} = u_t, \quad 0 < x < 3,$$
$$u(0,t) = u(3,t) = 0, \quad u(x,0) = f(x)$$

for some function f defined on $[0,3]$. Which one of the followings is correct?
   - **A. If** $f(x) = \sin \pi x$**, then the solution is** $u(x,t) = e^{-5\pi^2 t} \sin \pi x$.
   - B. If $u(x,t)$ and $v(x,t)$ are solutions, then $u(x,t) + v(x,t)$ is also a solution.
   - C. The thermal diffusivity is 3.
   - D. The solution is

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{5m^2\pi^2}{3}t} \sin\left(\frac{m\pi}{3}x\right)$$

   for some C_m.
   - E. None of these.

Solution: Note that the solution is

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{5m^2\pi^2}{9}t} \sin\left(\frac{m\pi}{3}x\right)$$

where

$$C_m = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{m\pi}{3}x\right)\,dx.$$

A. If $f(x) = \sin \pi x$, then

$$\begin{aligned} C_m &= \frac{2}{3}\int_0^3 \sin\pi x \sin\left(\frac{m\pi}{3}x\right)\,dx = \frac{1}{3}\int_{-3}^3 \sin\pi x\sin\left(\frac{m\pi}{3}x\right)\,dx \\ &= \begin{cases} 1, & m = 3, \\ 0, & m \neq 3. \end{cases} \end{aligned}$$

B. If $u(x,t)$ and $v(x,t)$ are solutions and $f(x) = 1$, then $u(x,0) + v(x,0) = 2 \neq f(x)$.

C. The thermal diffusivity is 5.

D. The solution is

$$u(x,t) = \sum_{m=1}^{\infty} C_m e^{-\frac{5m^2\pi^2}{9}t} \sin\left(\frac{m\pi}{3}x\right)$$

for some C_m.

9. (5 points) What is the steady state solution $v(x)$ for the following problem?

$$\begin{aligned} 5u_{xx} &= u_t, \qquad 0 < x < 6, \quad t \geq 0, \\ u(0,t) &= 10, \qquad u(6,t) = 2. \end{aligned}$$

A. $v(x) = \frac{5}{2}x - 1$

B. $v(x) = 0$

C. $v(x) = x + 5$

D. $v(x) = x - 10$

E. $v(x) = 10 - \frac{4}{3}x$

Solution: The steady state solution $v(x)$ satisfies

$$v_{xx} = 0, \qquad v(0) = 10, \qquad v(6) = 2.$$

Thus, $v(x) = c_1 x + c_2$ and the boundary conditions yield $c_1 = -\frac{4}{3}$ and $c_2 = 10$.