Homework 9

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 15, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $A = \begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$. Find $\lim_{m\to\infty} A^m$ if it exists.

2. Let $A, B \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $u = (1, 1, \cdots, 1)$.
   (a) Show that A is a transition matrix if and only if $A_{ij} \geq 0$ and $uA = u$.
   (b) Show that if A and B are transition matrices then AB is also a transition matrix.
   (c) From Part (b), deduce that if A is a transition matrix then A^m is a transition matrix for all integers $m \geq 1$.

3. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^t.

4. Let V be an inner product space over F.
   (a) Show that $\|cx\| = |c|\|x\|$ for all $x \in V$ and $c \in F$.
   (b) Show that $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$.

5. Let $V = \mathbb{C}^2$ and define $\langle x, y\rangle = xAy^*$ for $x, y \in V$, where
$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$
That is, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,
$$\langle x, y\rangle = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix}.$$
Show that $\langle\cdot,\cdot\rangle$ is an inner product.

6. Let $V = \mathbb{C}^n$ and $A \in \mathcal{M}_{n\times n}(\mathbb{C})$. Let $\langle x, y\rangle = \sum_{i=1}^n x_i\overline{y_i}$ for $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$.
   (a) Prove that $\langle x, Ay\rangle = \langle A^*x, y\rangle$ for all $x, y \in V$.
   (b) Suppose that there exists $B \in \mathcal{M}_{n\times n}(\mathbb{C})$ such that $\langle x, Ay\rangle = \langle Bx, y\rangle$ for all $x, y \in V$. Show that $B = A^*$.