Homework 8

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: November 8, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$.
   (a) Find the characteristic polynomial of A.
   (b) Determine all the eigenvalues of A.
   (c) For each eigenvalue λ, find E_λ.
   (d) If possible, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

2. Let $A, D \in \mathcal{M}_{n\times n}(\mathbb{R})$ and D be a diagonal matrix.
   (a) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ for some $d_1, d_2, \cdots, d_n \in \mathbb{R}$. Show that $D^k = \operatorname{diag}(d_1^k, d_2^k, \cdots, d_n^k)$ for all integers $k \geq 1$.
   (b) Let $D = \operatorname{diag}(d_1, d_2, \cdots, d_n)$ for some $d_1, d_2, \cdots, d_n \in \mathbb{R} \setminus \{0\}$. Show that D is invertible and $D^{-1} = \operatorname{diag}(d_1^{-1}, d_2^{-1}, \cdots, d_n^{-1})$.
   (c) Suppose there exists an invertible matrix Q and $A = QDQ^{-1}$. Show that $A^k = QD^kQ^{-1}$ for all integers $k \geq 1$.
   (d) Suppose A is invertible and there exists an invertible matrix Q and $A = QDQ^{-1}$. Show that $A^{-1} = QD^{-1}Q^{-1}$.

3. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and λ_1, λ_2 be two distinct eigenvalues for A. Let $E_{\lambda_1}, E_{\lambda_2}$ be the eigenspaces of A corresponding to λ_1, λ_2 respectively. Prove that $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.

4. Let V be a finite dimensional vector space over $\mathbb{R}$ and $T : V \to V$ linear. Let β be a basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$.

5. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ and v be an eigenvector of A corresponding to an eigenvalue λ. Show that v is an eigenvector of A^k corresponding to an eigenvalue λ^k for all integers $k \geq 1$.

6. Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Find an expression for A^k for all integers $k \geq 1$.