Math 416: Abstract Linear Algebra

Midterm 1 Solution, Fall 2019

Date: September 25, 2019

1. (20 points) Circle True or False. Do not justify your answer.

(a) True **FALSE** Let V be a vector space over $\mathbb{R}$, $a, b \in \mathbb{R}$, and $v \in V$. If $a \cdot v = b \cdot v$, then $a = b$.
If $v = 0$, then $2v = 3v$ but $2 \neq 3$.

(b) **TRUE** False Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ and $W = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : \mathrm{tr}(A) = 0\}$, then it is a subspace of V.

(c) True **FALSE** Let V be a vector space over $\mathbb{R}$ and W_1, W_2 be subspaces of V, then the union $W_1 \cup W_2$ is a subspace of V.

(d) True **FALSE** For each v in a vector space V over $\mathbb{R}$, $\{v\}$ is linearly independent.
If $v = 0$, then $\{0\}$ is linearly dependent.

(e) **TRUE** False Let $v_1 = (1, 1, 0)$ and $v_2 = (0, 1, 2)$, then $(3, 1, -4) \in \mathrm{Span}(\{v_1, v_2\})$.

(f) **TRUE** False The matrix $\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 2 \end{pmatrix}$ is row-equivalent to $\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \end{pmatrix}$.

(g) True **FALSE** The system of linear equations associated to an augmented matrix

$$(A, b) = \begin{pmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

has infinitely many solutions.
Since the last column is pivot, there is no solution.

(h) True **FALSE** The set of all solution to a system of linear equations with n variables is a subspace of $\mathbb{R}^n$.
If the system is not homogeneous, the zero is not in the solution set.

(i) True **FALSE** $V = \mathcal{P}_n(\mathbb{R})$ be the set of all real polynomials $p(x)$ with $\deg(p) \leq n$ (here, $\deg(p)$ denotes the degree of $p(x)$). Then the dimension of V is n.
The dimension is $n + 1$.

(j) **TRUE** False Let $S = \{v_1, v_2, \cdots, v_n\}$ be a subset of $\mathbb{R}^n$. If S is linearly independent, then S is a basis for $\mathbb{R}^n$.

2. Consider a system of linear equations

$$\begin{cases} x_1 + 2x_2 - 4x_3 - x_4 = 0 \\ x_1 + 3x_2 - 7x_3 = 0 \\ x_1 + 2x_3 - 2x_4 = 0. \end{cases}$$

(a) (5 points) Write down the augmented matrix A corresponding to the above system of linear equations.

Solution: $A = \begin{pmatrix} 1 & 2 & -4 & -1 & 0 \\ 1 & 3 & -7 & 0 & 0 \\ 1 & 0 & 2 & -2 & 0 \end{pmatrix}$

(b) (5 points) Find a reduced row-echelon form of A (a matrix in reduced row-echelon form which is row-equivalent to A). Please label your individual row operations.

Solution:

$$\begin{aligned} A &\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 & -4 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & -2 & 6 & -1 & 0 \end{pmatrix} \\ &\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & 2 & -3 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ &\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 + 3R_3} \begin{pmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} =: R. \end{aligned}$$

(c) (5 points) Find the solution set of the above system.

Solution: Since the solution set of the linear system is equal to that of $LS(R)$, we have

$$\begin{cases} x_1 = -2x_3 \\ x_2 = 3x_3 \\ x_3 = x_3 \\ x_4 = 0. \end{cases}$$

and $\mathcal{N}(A) = \{(-2t, 3t, t, 0) : t \in \mathbb{R}\}$.

(d) (5 points) Find a basis of the solution set.

Solution: Let $\beta = \{(-2, 3, 1, 0)\}$, then β is linearly independent because $(-2, 3, 1, 0) \neq 0$. Also, β spans $\mathcal{N}(A)$ because every vector in $\mathcal{N}(A)$ can be written as

$$(-2t, 3t, t, 0) = t(-2, 3, 1, 0) \in \mathcal{N}(A).$$

3. Let V be a vector space over $\mathbb{R}$.

(a) (10 points) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{2u - v, u + v\}$ is linearly independent.

Solution: Suppose that $\{u, v\}$ is linearly independent. Let

$$a(2u - v) + b(u + v) = 0.$$

Then,

$$a(2u - v) + b(u + v) = (2a + b)u + (-a + b)v = 0,$$

which yields $2a + b = 0$ and $-a + b = 0$. Thus, $a = b = 0$ and so $\{2u - v, u + v\}$ is linearly independent.

Suppose that $\{2u - v, u + v\}$ is linearly independent. Let $au + bv = 0$, then

$$0 = 3au + 3bv = (a - b)(2u - v) + (a + 2b)(u + v),$$

which implies $a - b = 0$ and $a + 2b = 0$. Thus, $a = b = 0$ and so $\{u, v\}$ is linearly independent.

Common Mistake: Suppose that $\{2u-v, u+v\}$ is linearly independent. Then, $a(2u-v)+b(u+v) = 0$ implies $a = b = 0$. Since

$$a(2u - v) + b(u + v) = (2a + b)u + (-a + b)v = 0$$

and $a = b = 0$ implies $2a + b = -a + b = 0$, $\{u, v\}$ is linearly independent.

This proof has a serious problem. To show $\{u, v\}$ is linearly independent, you need to start from $cu + dv = 0$ and show that $c = d = 0$. The key step of this problem is to show that there exist a, b such that $c = 2a + b$ and $d = -a + b$ for any $c, d \in \mathbb{R}$. Once proving this, you can conclude that $a = b = 0$ from the fact that $\{2u - v, u + v\}$ is linearly independent and in turn that $c = d = 0$.

(b) (5 points) Suppose $\{u, v\}$ is linearly independent. Is $\{au - v, u + v\}$ linearly independent? Prove it or give a counterexample.

Solution: If $a = -1$, then $au - v = -u - v = -(u + v)$. Thus, $\{au - v, u + v\}$ is linearly dependent.

4. (10 points) Let

$$W_1 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 + 3x_2 - x_3 - 9x_4 = 0, x_1 + 2x_2 + x_3 = 0\},$$
$$W_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 - 3x_3 - 4x_4 = 0\}.$$

Find the dimension of $W_1 \cap W_2$.

Solution: Note that

$$W_1 \cap W_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : 2x_1 + 3x_2 - x_3 - 9x_4 = x_1 + 2x_2 + x_3 = x_1 - 2x_2 - 3x_3 - 4x_4 = 0\}.$$

That is, $W_1 \cap W_2$ is the solution set of the linear system

$$\begin{cases} x_1 - 2x_2 - 3x_3 - 4x_4 = 0, \\ 2x_1 + 3x_2 - x_3 - 9x_4 = 0, \\ x_1 + 2x_2 + x_3 = 0. \end{cases}$$

The corresponding coefficient matrix is

$$\begin{aligned} A = &\begin{pmatrix} 1 & -2 & -3 & -4 \\ 2 & 3 & -1 & -9 \\ 1 & 2 & 1 & 0 \end{pmatrix} \\ \to &\begin{pmatrix} 1 & -2 & -3 & -4 \\ 0 & 7 & 5 & -1 \\ 0 & 4 & 4 & 4 \end{pmatrix} \\ \to &\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -8 \end{pmatrix} \\ \to &\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{pmatrix}. \end{aligned}$$

Since the null space of A is the same as the solution of the homogeneous system of linear equations (by definition) and the dimension of the null space of A is the number of non-pivot columns of a RREF of A, the dimension of $W_1 \cap W_2$ is 1.

5. Let V be the set of all (2×2) matrices and W a subspace of V consisting of all matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$.

(a) (5 points) Find two (2×2) matrices $A, B \in V$ such that $A \in W$ and $B \notin W$.

Solution: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then we have $\operatorname{tr}(A) = 1 - 1 = 0$ and $\operatorname{tr}(B) = 1 + 1 = 2$. Thus, $A \in W$ and $B \notin W$.

(b) (5 points) Find a basis β for W.

Solution: Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

Let $A \in W$, then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a + d = 0$. Thus,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \operatorname{Span}(\beta).$$

So, $\operatorname{Span}(\beta) = W$.
Suppose

$$a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, we get $a = b = c = 0$ so that β is linearly independent. We conclude that β is a basis for W.

(c) (5 points) Find a basis γ for V such that $\beta \subset \gamma$.

Solution: Define $\gamma = \beta \cup \{B\}$ where $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the part (a), we have seen that $B \notin W = \operatorname{Span}(\beta)$. Thus, γ is linearly independent. Since $\dim(V) = 4 = |\gamma|$, we conclude that γ is a basis for V.

6. Let $V = \mathcal{M}_{n\times n}(\mathbb{R})$ be the set of all $(n \times n)$ matrices with real entries. Define

$$U = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^t = A\}, \qquad W = \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) : A^t = -A\}.$$

(a) (10 points) Show that W is a subspace of V.

Solution: Since $O^t = O = -O$, $O \in W$. Let $A, B \in W$ and $c \in \mathbb{R}$, then

$$(cA + B)^t = cA^t + B^t = -cA - B = -(cA + B).$$

Thus, $cA + B \in W$. Therefore, W is a subspace of V.

(b) (10 points) Show that every $A \in V$ can be uniquely written as $A = B + C$ for $B \in U$ and $C \in W$.

Solution: Let $A \in V$ and define $B = \frac{1}{2}(A + A^t)$, $C = \frac{1}{2}(A - A^t)$. Then, one can see that $A = B + C$,

$$B^t = \left(\frac{1}{2}(A + A^t)\right)^t = \frac{1}{2}(A^t + (A^t)^t) = \frac{1}{2}(A + A^t) = B,$$

and

$$C^t = \left(\frac{1}{2}(A - A^t)\right)^t = \frac{1}{2}(A^t - (A^t)^t) = -\frac{1}{2}(A - A^t) = -C.$$

Thus, $B \in U$ and $C \in W$.
Suppose that $A = B_1 + C_1 = B_2 + C_2$ for $B_1, B_2 \in U$ and $C_1, C_2 \in W$. Then,

$$B_1 - B_2 = C_2 - C_1 \in U \cap W.$$

If $D \in U \cap W$, then $D^t = D = -D$ and so $D = 0$. This yields that $B_1 = B_2$ and $C_1 = C_2$. Thus the decomposition is unique.