Math 285: Differential Equations

Midterm Exam 1, Spring 2020

Date: February 19, 2020

READ THE FOLLOWING INFORMATION.

- This is a 90-minute exam.
- This exam contains 10 pages (including this cover page), 10 multiple choice questions, and 2 free response questions. Total of points is 100.
- No books, notes, calculators, or electronic devices allowed.
- You must not communicate with other students during this test.
- There are several different versions of this exam.
- Do not turn this page until instructed to.

1. Fill in your information:

Full Name: ____

UIN (Student Number): ____

NetID: ____

Circle your section: E1 (1 pm) F1 (2 pm)

2. Fill in the following answers on the Scantron form:

91. A
92. A
93. A
94. A
95. D
96. C

Multiple Choice Questions: Mark answers to questions 1 to 10 on your scantron form.

1. (5 points) For which values of r and K, is the constant function $y(t) = -2$ an asymptotic stable equilibrium solution to the following equation?

$$y' = r(y + 2)(y - K).$$

(A) $r = -2$ and $K = 1$

(B) ★ $r = 1$ and $K = 0$

(C) $r = -1$ and $K = -2$

(D) $r = 2$ and $K = -3$

(E) None of these

Solution. In order for $y(t) = -2$ to be asymptotic stable, $f(y) = r(y+2)(y-K)$ is negative if y is close to -2 from right and positive if y is close to -2 from left. Thus, this holds if $r > 0$ and $K > -2$ or if $r < 0$ and $K < -2$.

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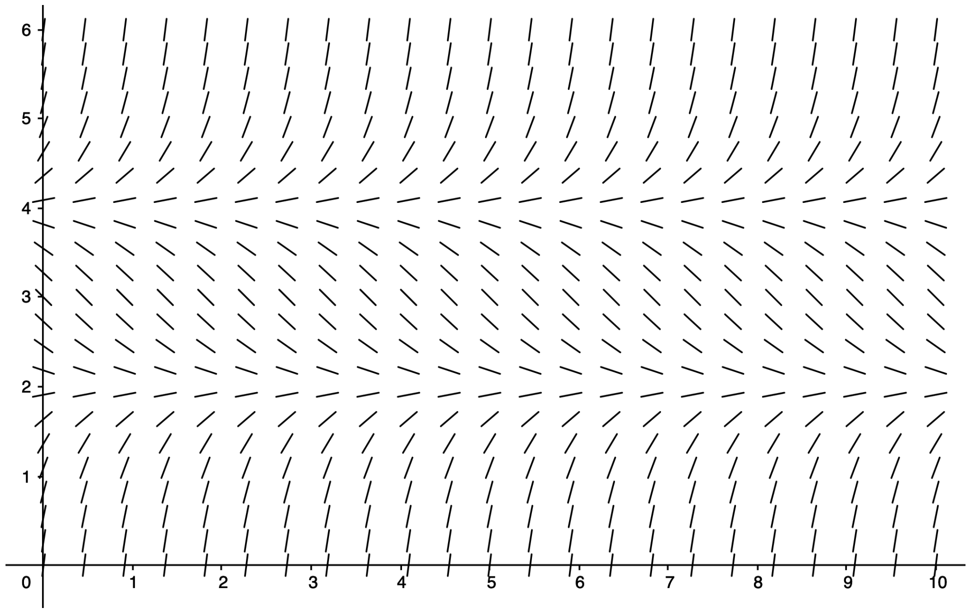
2. (5 points) Identify which differential equation below corresponds to the following direction field.



(A) None of these

(B) $y' = y(y-4)$

(C) ★ $y' = (y-2)(y-4)$

(D) $y' = (2-y)(y-4)$

(E) $y' = y-2$

Solution. The slopes are zero if $y = 2, 4$. If $y > 4$ or $y < 2$, then the slope is positive. If $2 < y < 4$, then the slope is negative. Thus, $y' = (y-2)(y-4)$.

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3. (5 points) The following nonlinear equation for $y(x)$ can be transformed with a substitution into which separable equation for $v(x)$?

$$y' = \frac{-x^2 + 2xy + y^2}{x^2} + \sec\left(\frac{y}{x}\right)$$

(A) None of these

(B) ★ $xv' = -1 + v + v^2 + \sec v$

(C) $v' = 1 - 2v + v^2$

(D) $v' = -1 + 2v + v^2 + \sec v$

(E) $xv' = -x^2 + v + v^2 + \sec v$

Solution. Let $v = y/x$, then $v + xv' = y'$ and the RHS is $-1 + 2v + v^2 + \sec v$.

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4. (5 points) Solve the following initial value problem.

$$y' - 4y = e^{2t}, \qquad y(1) = 0.$$

(A) $y(t) = e^{6t} - e^{10t-4}$

(B) ★ $y(t) = \frac{1}{2}(e^{4t-2} - e^{2t})$

(C) $y(t) = e^{2} - e^{2t}$

(D) None of these

(E) $y(t) = \frac{1}{2}(e^{-2} - e^{-2t})$

Solution. Since the integrating factor is $\mu(t) = e^{-4t}$, we have

$$y(t) = e^{4t} \int e^{-2t} dt = e^{4t}(-\frac{1}{2}e^{-2t} + C).$$

Since $y(1) = 0$, the solution is

$$y(t) = \frac{1}{2}(e^{4t-2} - e^{2t}).$$

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5. (5 points) What is the correct integrating factor to solve the following ODE for $y(t)$?

$$t^4 y' + 3t^3 y + 1 = 0.$$

(A) $\mu(t) = 3t$

(B) ★ $\mu(t) = t^3$

(C) $\mu(t) = \frac{1}{t^3}$

(D) None of these

(E) $\mu(t) = \frac{3}{t}$

Solution. Dividing the equation by t^4, we get $p(t) = 3/t$ and

$$\mu(t) = \exp(3 \int \frac{1}{t}\, dt) = t^3.$$

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6. (5 points) How would you classify the following equation for $y(t)$?

$$(1+t^2)(y''')^2 + e^t \ln y = \frac{y}{t}.$$

(A) A third order linear differential equation.

(B) None of these

(C) ★ A third order nonlinear differential equation.

(D) A sixth order linear differential equation.

(E) A sixth order nonlinear differential equation.

Solution. The highest derivative is y''' and there are nonlinear terms $(y''')^2$ and $\ln y$.

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7. (5 points) Determine the values of r for which the given differential equation has solutions of the form $y(t) = e^{rt}$.

$$y'' + y' - 6y = 0.$$

(A) $r = 1, -3$

(B) ★ $r = 2, -3$

(C) $r = 2, 3$

(D) None of these

(E) $r = -2, 3$

Solution. Plugging $y(t) = e^{rt}$ into the equation, we get $(r^2 + r - 6)e^{rt} = 0$. Since $e^{rt} \neq 0$, we have $r = 2, -3$.

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8. (5 points) Consider the following nonlinear ODE for $y(x)$. Which initial condition would guarantee a unique solution to the initial value problem?

$$yy' + 2xy' = \frac{\ln x}{x-3}$$

(A) $y(3) = -6$

(B) $y(3) = 2$

(C) ★ $y(-1) = 1$

(D) None of these

(E) $y(-1) = 2$

Solution. Since $yy' + 2xy' = y'(y+2x)$, we have

$$y' = \frac{\ln x}{(y+2x)(x-3)} = f(x,y).$$

Then, f and $\frac{\partial f}{\partial y}$ are continuous if $y \neq -2x$ and $x \neq 3$.

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9. (5 points) Find the solution of

$$\frac{dy}{dt} = 8 - 3y.$$

(A) ★ $y(t) = Ce^{-3t} + \frac{8}{3}$ where C is an arbitrary constant.

(B) None of these

(C) $y(t) = Ce^{-t} + \frac{8}{3}$ where C is an arbitrary constant.

(D) $y(t) = e^{-3t} + C$ where C is an arbitrary constant.

(E) $y(t) = e^{-t} + C$ where C is an arbitrary constant.

Solution. Since it is separable,

$$\frac{1}{3y-8} dy = -dt$$

and so $\frac{1}{3} \ln |3y - 8| = -t + C$. Thus, $y(t) = Ce^{-3t} + \frac{8}{3}$.

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10. (5 points) The existence and uniqueness theorem for linear differential equations ensures that the solution of

$$(t-4)(t+1)y' + \sqrt{1-t}y = 1 - 2t^2, \qquad y(0) = 3$$

exists for all t in the region defined by:

(A) ★ $(-1, 1)$

(B) None of these

(C) $(-\infty, -1)$

(D) $(-1, 4)$

(E) $(4, \infty)$

Solution. Dividing the equation by $(t-4)(t+1)$, we have

$$y' + \frac{\sqrt{1-t}}{(t-4)(t+1)}y = \frac{1-2t^2}{(t-4)(t+1)}.$$

The first coefficient is continuous on $(-\infty, -1) \cup (-1, 1)$ and the second on $(-\infty, -1) \cup (-1, 4) \cup (4, \infty)$. Since the initial condition $0 \in (-1, 1)$, the interval is $(-1, 1)$.

Free Response Questions: Make sure you show your work. You may get partial credit.

11. (10 points) Consider the following differential equation.

$$y' = \frac{x^4 + 3y}{x}, \qquad x > 0.$$

(i) Determine whether the equation is linear. Give a reason why.

(ii) Solve the equation with the initial condition $y(2) = 4$.

Solution.

(i) It is linear because it can be written as $y' - \frac{3}{x}y = x^3$.

(ii) The integrating factor is $\mu(x) = x^{-3}$. Thus,

$$y(x) = \frac{1}{\mu(t)} \int \mu(x)x^3 \, dx = x^3(x + C) = x^4 + Cx^3.$$

Since $y(2) = 4$, $C = -\frac{3}{2}$ and so

$$y(x) = x^4 - \frac{3}{2}x^3.$$

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12. (10 points) Find the solution to the following initial value problem.

$$(x^4+1)y' = 2x^3y^2, \qquad y(0) = \frac{3}{2}$$

and specify the interval where the solution is defined.

Solution. The equation is separable and

$$\frac{1}{y^2}dy = \frac{2x^3}{x^4+1}dx.$$

Taking integration of the both sides, we have

$$-\frac{1}{y} = \frac{1}{2}\ln(x^4+1) + C.$$

Using the initial condition, $C = -2/3$. Thus,

$$y(x) = \frac{6}{4 - 3\ln(x^4+1)}.$$

Since the denominator is zero if

$$x = \pm(e^{4/3}-1)^{\frac{1}{4}},$$

the solution is defined on the interval

$$(-(e^{4/3}-1)^{\frac{1}{4}}, (e^{4/3}-1)^{\frac{1}{4}}).$$