Math 285 Lecture Note: Week 10

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1 The Method of Undetermined Coefficients (Sec 4.3)

The method of undetermined coefficients works for higher order DEs. Consider

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t).$$

The general solution is the sum of the general solution to $L[y] = 0$ and a particular solution to $L[y] = g(t)$. We have seen how to find the general solution to $L[y] = 0$ using the characteristic equation. If $g(t)$ is a mixture of polynomials, exponential, sine or cosine functions, then a particular solution $Y(t)$ has one of the following form:

(i) If $g(t) = (t^n + \cdots)$, then $Y(t) = (A_n t^n + \cdots)$.

(ii) If $g(t) = (t^n + \cdots)e^{kt}$, then $Y(t) = (A_n t^n + \cdots)e^{kt}$.

(iii) If $g(t) = (t^n + \cdots)\sin(kt)$ (or $\cos(kt)$), then $Y(t) = (A_n t^n + \cdots)\cos(kt) + (B_n t^n + \cdots)\sin(kt)$.

(iv) Multiply $Y(t)$ by t until it does not contain a solution to the homogeneous equation.

If $g(t)$ is given by the sum of those functions, that is, $L[y] = g_1(t) + g_2(t)$, then we find particular solutions Y_1 and Y_2 to the equations $L[y] = g_1(t)$ and $L[y] = g_2(t)$ respectively. Then, $Y(t) = Y_1(t) + Y_2(t)$.

Example 1.1. Consider $L[y] = y^{(4)} + 2y''' - 2y' - y = 6e^{-t}$. First, we solve the corresponding homogeneous equation. The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 2\lambda - 1 = (\lambda + 1)^3(\lambda - 1) = 0.$$

Thus, $\{e^{-t}, te^{-t}, t^2e^{-t}, e^t\}$ is a fundamental set of solutions and the general solution is

$$y_c(t) = C_1 e^{-t} + C_2 te^{-t} + C_3 t^2 e^{-t} + C_4 e^t.$$

Since $g(t) = 6e^{-t}$ is a solution to $L[y] = 0$, the candidate for $Y(t)$ is Ate^{-t}. This is, however, again a solution. We repeat this until it is not a solution. Thus, $Y(t) = At^3e^{-t}$. We need to compute $L[Y]$ to determine A. For simplicity, let $f(t) = At^3$, then

$$\begin{aligned}
Y(t) &= fe^{-t} \\
Y'(t) &= (f' - f)e^{-t} \\
Y''(t) &= (f'' - 2f' + f)e^{-t} \\
Y'''(t) &= (f''' - 3f'' + 3f' - f)e^{-t} \\
Y''''(t) &= (f'''' - 4f''' + 6f'' - 4f' + f)e^{-t}.
\end{aligned}$$

So, we get $L[Y] = (f^{(4)} - 2f^{(3)})e^{-t} = -12Ae^{-t} = 6e^{-t}$ and so $A = -\frac{1}{2}$. Therefore, the general solution is

$$y(t) = y_c(t) + Y(t) = C_1 e^{-t} + C_2 te^{-t} + C_3 t^2 e^{-t} + C_4 e^t - \frac{1}{2} t^3 e^{-t}.$$

Remark 1.2. In the previous example, one can see that if $y(t) = fe^{-t}$ is a solution to $L[y] = 0$, then f satisfies $f^{(4)} - 2f^{(3)} = 0$. It is straightforward to see that the general solution to $f^{(4)} - 2f^{(3)} = 0$ is

$$f(t) = C_1 + C_2 t + C_3 t^2 + C_4 e^{2t}.$$

Note that $y(t) = f(t)e^{-t} = C_1 e^{-t} + C_2 te^{-t} + C_3 t^2 e^{-t} + C_4 e^t$ is indeed the general solution to $L[y] = y^{(4)} + 2y''' - 2y' - y = 0$. Can we find $f^{(4)} - 2f^{(3)} = 0$ without computing derivatives? Indeed, the characteristic equations of the new DE has a close relation with that of the original equation:

$$\lambda^4 + 2\lambda^3 - 2\lambda - 1 = (\lambda + 1)^4 - 2(\lambda + 1)^3.$$

Example 1.3. Consider $L[y] = y''' + y' = te^{-t} + \cos t$. The general solution to $L[y] = 0$ is

$$y_c(t) = C_1 + C_2 \cos t + C_3 \sin t.$$

Since $g(t) = te^{-t} + \cos t$, we find particular solutions to $L[y] = te^{-t}$ and $L[y] = \cos t$ separately. A particular solution to $L[y] = te^{-t}$ is $Y_1(t) = (At + B)e^{-t}$. Since

$$L[Y_1] = (-2At + (4A - 2B))e^{-t} = te^{-t},$$

$A = -1/2$ and $B = 1$. Thus, $Y_1(t) = (-\frac{1}{2}t + 1)e^{-t}$. For $L[y] = \cos t$, $Y_2(t) = At \cos t + Bt \sin t$ because $\cos t$ is a solution to $L[y] = 0$. Then,

$$L[Y_2] = -2A \cos t - 2B \sin t = \cos t$$

implies that $A = -\frac{1}{2}$ and $B = 0$. Therefore,

$$y(t) = y_c(t) + Y_1(t) + Y_2(t) = C_1 + C_2 \cos t + C_3 \sin t + (-\frac{1}{2}t + 1)e^{-t} - \frac{1}{2}t \cos t.$$

2 Two-Point Boundary Value Problems, part 1 (Sec 10.1)

We consider $y'' + p(t)y' + q(t)y = g(t)$. Previously, the initial value problem refers the DE with the condition of the form

$$y(t_0) = y_0, \qquad y'(t_0) = y_0'.$$

In this section, we will consider $y'' + p(x)y' + q(x)y = g(x)$ with

$$y(\alpha) = y_0, \qquad y(\beta) = y_1$$

for some $\alpha < \beta$. This is call a two-point boundary value problem. Our goal is to find solutions $y = \phi(x)$ that satisfies the DE in $x \in (\alpha, \beta)$ with the boundary condition.

Definition 2.1. A two-point boundary value problem is called *homogeneous* if $g(t) = y_0 = y_1 = 0$. Otherwise, we call it *nonhomogeneous*.

It is natural to ask if the Existence and Uniqueness theorem is available. In general, the answer is no. First, we consider nonhomogeneous case.

Example 2.2 (Nonhomogeneous with a unique solution)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\frac{\pi}{2}) = 2$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = 1$ and $C_2 = 2$. This shows that there exists a unique solution.

Example 2.3 (Nonhomogeneous with infinitely many solution)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = -1$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = 1$. Since there is no restriction on C_2, there are infinitely many solutions.

Example 2.4 (Nonhomogeneous with no solutions)**.** Consider $y'' + y = 0$ with $y(0) = 1$ and $y(\pi) = 2$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

there are no C_1 and C_2 that satisfy the boundary condition. This shows that the solution does not exist.

If the boundary problem is homogeneous, we always have a trivial solution, $y(x) = 0$.

Example 2.5 (Homogeneous with infinitely many solutions)**.** Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\pi) = 0$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

we have $C_1 = 0$. Since there is no restriction on C_2, there are infinitely many solutions.

Example 2.6 (Homogeneous with a unique solution)**.** Consider $y'' + y = 0$ with $y(0) = 0$ and $y(\frac{\pi}{2}) = 0$. Since the general solution is

$$y(x) = C_1 \cos x + C_2 \sin x,$$

one can see that $C_1 = C_2 = 0$. This shows that there exists a unique solution $y(x) = 0$.

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

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