Math 285 Midterm 2 Conflict Exam

Multiple choice questions

1. (5 points) The general solution of the third order differential equation

$$y''' + py'' + qy' + ry = 0$$

is $C_1 + C_2e^{-2t}\cos t + C_3e^{-2t}\sin t$. What are p, q, r?

A. $p = 1$, $q = 2$, $r = 3$

B. $p = 1$, $q = -4$, $r = 5$

C. $p = 0$, $q = 4$, $r = 5$

D. $p = 4$, $q = -5$, $r = 0$

E. $p = 4$**,** $q = 5$**,** $r = 0$

Solution: Since the general solution is $C_1 + C_2e^{-2t}\cos t + C_3e^{-2t}\sin t$, the characteristic equation has roots $\lambda = 0, -2 \pm i$. Thus,

$$\lambda^3 + p\lambda^2 + q\lambda + r = \lambda(\lambda + 2 - i)(\lambda + 2 + i) = \lambda^3 + 4\lambda^2 + 5\lambda.$$

2. (5 points) Identify the correct form of a particular solution for the following differential equation.

$$y'' + 2y' = 2te^{-2t} - t^3.$$

A. $Y(t) = Ate^{-2t} + Be^{-2t} + C + Dt + Et^2 + Ft^3$

B. $Y(t) = Ate^{-2t} + Be^{-2t} + Ct + Dt^2 + Et^3 + Ft^4$

C. $Y(t) = At^2e^{-2t} + Bte^{-2t} + Ct + Dt^2 + Et^3 + Ft^4$

D. None of these

E. $Y(t) = At^2e^{-2t} + Bte^{-2t} + C + Dt + Et^2 + Ft^3$

Solution: The general solution to $y'' + 2y' = 0$ os $C_1 + C_2e^{-2t}$. A particular solution to $y'' - 2y' = 2te^{-2t}$ is $Y_1 = t(At + B)e^{-2t}$. A particular solution to $y'' - 2y' = -t^3$ is $Y_2 = t(C + Dt + Et^2 + Ft^3)$. Thus, the answer is $Y = Y_1 + Y_2 = At^2e^{-2t} + Bte^{-2t} + C + Dt + Et^2 + Ft^3$.

3. (5 points) Which one of the followings is correct?

A. $W[f, g] = W[g, f]$ for any functions f and g.

B. If f and g are linearly independent, then $W[f,g](t) \neq 0$ for all $t \in \mathbb{R}$.

C. $W[\cos t, 2\cos t] \neq 0$ for all $t \in \mathbb{R}$.

D. $W[2f - 3g, f + g] = 5W[f,g]$ **for any functions** f **and** g**.**

E. None of these

Solution:

A. $W[f,g] = -W[g,f]$ for any functions f and g.

B. If $f = t$ and $g = t^2$, then they are linearly independent but $W[f,g](t) = t^2 = 0$ if $t = 0$.

C. $W[\cos t, 2\cos t] = 0$ for all $t \in \mathbb{R}$.

4. (5 points) Which of these is NOT a set of linearly independent solutions?

A. $e^x, 2e^x - e^{2x}, e^x + 3e^{2x}$

B. $\cos x, \sin x, \cos 2x$

C. $2, x, x\ln x$

D. None of these

E. $x, x^2 + x, 2x^3$

Solution: We have

$$-7e^x + 3(2e^x - e^{2x}) + (e^x + 3e^{2x}) = 0.$$

5. (5 points) The existence and uniqueness theorem guarantees that there exists a unique solution of the initial value problem

$$(t+1)(t+3)y''' - 3y' + (t+3)y = \tan\left(\frac{\pi t}{4}\right)$$

with $y(1) = 2$, $y'(1) = 1$, and $y''(1) = 0$ on an open interval

A. $(-1, \infty)$

B. $(-1, 2)$

C. $(-3, -1)$

D. $(-1, 4)$

E. None of these

Solution: Dividing by $(t+1)(t+3)$, one sees that the coefficients are continuous except $t \neq -3, -1, 2(2k+1)$ for all $k \in \mathbb{Z}$. Thus, the interval is $(-1, 2)$.

6. (5 points) What conclusion can you draw for the solution of the following initial value problem?

$$u'' + u' + u = 0, \quad u(0) = 2, \quad u'(0) = 0.$$

   A. The solution oscillates with a constant amplitude.

   B. The solution oscillates with decaying amplitude.

   C. The solution oscillates with growing amplitude.

   D. The solution eventually becomes monotone and decays to zero.

   E. None of these

Solution: This is a damped free vibration system. The corresponding characteristic equation has a pair of conjugate complex roots with negative real part. The solution oscillates with decaying amplitude.

7. (5 points) If $y(t) = v(t)e^t$ is a solution to $y'' - 2y' + y = 0$, then $v(t)$ satisfies

   A. $v'' - v' = 0$

   B. $v'' + v' + v = 0$

   C. $v'' + v' = 0$

   D. $v'' = 0$

   E. None of these

Solution: We have

$$y'' - 2y' + y = (v''e^t + 2v'e^t + ve^t) - 2(v'e^t + ve^t) + ve^t = v''e^t = 0.$$

8. (5 points) For which frequency ω of the driven force is the following undamped forced vibration in resonance?

$$u'' + 4\,u = 5\,\cos(\omega t)$$

   A. $\omega = 1$

   B. $\omega = 2$

   C. $\omega = 3$

   D. $\omega = 4$

E. None of these

Solution: An undamped forced oscillator is in resonance when the driven force frequency is equal to the natural frequency $\sqrt{\frac{k}{m}}$. In this case when $\omega = \sqrt{4} = 2$.

9. (5 points) If $\cos\delta = \frac{5}{13}$ and $\sin\delta = \frac{12}{13}$, then $5\cos 2t - 12\sin 2t$ is

   A. $13\cos(2t - \delta)$

   B. $\cos(2t - \delta)$

   C. $13\cos(2t + \delta)$

   D. $\cos(2t + \delta)$

   E. None of these

Solution: We have

$$5\cos 2t - 12\sin 2t = 13(\cos\delta\cos 2t - \sin\delta\sin 2t) = 13\cos(2t + \delta).$$

10. (5 points) The motion of a certain spring-mass system is governed by

$$y'' + 4y' + 4y = 0$$

with $y(0) = 1$ and $y'(0) = 1$. Which one of the following equations is correct?

   A. This is overdamped.

   B. $\lim_{t\to\infty} y(t) = 0$**.**

   C. There exists $t_0 > 0$ such that $y(t_0) = 0$.

   D. The solution is $y(t) = e^{-2t}(t + 1)$.

   E. None of these

Solution: The motion is critically damped and the solution is $y(t) = e^{-2t}(3t + 1)$. Since $y(t) > 0$ for all $t > 0$, there does not exist $t_0 > 0$ such that $y(t_0) = 0$. We have $\lim_{t\to\infty} y(t) = 0$.

Free response questions

11. (10 points) Find the solution to the following initial value problem

$$y'' + 4y = 4\sin(2t), \qquad y(0) = 2, \qquad y'(0) = -1.$$

Solution: The general solution to $y''+4y = 0$ is $C_1\cos(2t)+C_2\sin(2t)$. A particular solution is of the form $Y(t) = t(A\cos(2t) + B\sin(2t))$. Putting this back to the original equation, we have

$$Y'' + 4Y = -4(A\sin(2t) - B\cos(2t)) = 4\sin(2t).$$

Thus, $Y(t) = -t\cos(2t)$ and the general solution is

$$y(t) = C_1\cos(2t) + C_2\sin(2t) - t\cos(2t).$$

By the initial conditions, $C_1 = 2$ and $C_2 = 0$. Thus,

$$y(t) = 2\cos(2t) - t\cos(2t).$$

12. Consider

$$2t^2y'' + ty' - 3y = 0, \qquad t > 0.$$

(a) (2 points) Verify that $y_1(t) = \frac{1}{t}$ is a solution.

(b) (8 points) Find another solution $y_2(t)$ such that $W[y_1, y_2](t) \neq 0$ for all $t > 0$.

Solution:

(a) We have

$$2t^2y_1'' + ty_1' - 3y_1 = 2t^2(\frac{2}{t^3}) + t(-\frac{1}{t^2}) - 3\frac{1}{t} = 0.$$

(b) Let $y(t) = v(t)y_1(t)$ be a solution, then

$$\begin{aligned} 2t^2y'' + ty' - 3y &= 2t^2(v''y_1 + 2v'y_1' + vy_1'') + t(v'y_1 + vy_1') - 3vy_1 \\ &= 2tv'' - 3v' \\ &= 0 \end{aligned}$$

(2 points). Let $w = v'$, then $2tw' = 3w$. By the separation method, $w = Ct^{3/2}$ and so $v = C_1 + C_2t^{5/2}$ (4 points). Therefore,

$$y(t) = C_1t^{-1} + C_2t^{3/2}.$$

Let $y_2(t) = t^{3/2}$, then it is a solution and

$$W[y_1, y_2](t) = \frac{5}{2}t^{-\frac{1}{2}} \neq 0$$

for all $t > 0$ (2 points).