Homework 10

Math 416, Abstract linear algebra, Fall 2019
Instructor: Daesung Kim

Due date: December 4, 2019

Textbooks: In the assignment, the two texts are abbreviated as follows:

- [FIS]: Freidberg, Insel, and Spence, *Linear Algebra*, 4th edition, 2002.
- [Bee]: Beezer, *A First Course in Linear Algebra*, Version 3.5, 2015.

1. Let $V = \mathbb{R}^3$ be equipped with the standard inner product. Apply the Gram–Schmidt process to a basis $\beta = \{v_1 = (1,0,1), v_2 = (0,1,1), v_3 = (1,3,3)\}$ for V to obtain an orthonormal basis for V.

Solution: Let $w_1 = v_1/\|v_1\| = \frac{1}{\sqrt{2}}(1,0,1)$. By the Gram–Schmidt process, we have

$$w_2' = v_2 - \langle v_2, w_1\rangle w_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \frac{1}{2}(-1,2,1).$$

Let $w_2 = w_2'/\|w_2'\| = \frac{1}{\sqrt{6}}(-1,2,1)$. Similarly,

$$\begin{aligned} w_3' &= v_3 - \langle v_3, w_1\rangle w_1 - \langle v_3, w_2\rangle w_2 \\ &= (1,3,3) - 2(1,0,1) - \frac{4}{3}(-1,2,1) \\ &= \frac{1}{3}(1,1,-1) \end{aligned}$$

and $w_3 = w_3'/\|w_3'\| = \frac{1}{\sqrt{3}}(1,1,-1)$. Thus, $\{w_1, w_2, w_3\}$ is an orthonormal basis for V.

2. Let V be an inner product space over F and W a finite dimensional subspace of V. Let β be a basis for W.
   (a) Show that $W^\perp$ is a subspace of V.
   (b) Show that $W \cap W^\perp = \{0\}$.
   (c) Show that $z \in W^\perp$ if and only if $\langle z, x\rangle = 0$ for all $x \in \beta$.

Solution:

(a) Note that $0 \in W^\perp$ because $\langle 0, x\rangle = 0$ for all $x \in W$. Let $x, y \in W^\perp$ and $c \in F$, then for $w \in W$,

$$\langle cx + y, w\rangle = c\langle x, w\rangle + \langle y, w\rangle = 0,$$

which means that $cx + y \in W^\perp$. Thus, $W^\perp \leq V$.

(b) Let $x \in W \cap W^\perp$, then $\langle x, x\rangle = 0$. Thus, $x = 0$ and so $W \cap W^\perp = \{0\}$.

(c) Suppose $z \in W^{\perp}$, then it is trivial that $\langle z, x\rangle = 0$ for all $x \in \beta$. Suppose that $z \in V$ satisfies $\langle z, x\rangle = 0$ for all $x \in \beta$. For $w \in W$, w can be written as $w = \sum_{i=1}^{k} a_i v_i$ where $a_i \in F$ and $v_i \in \beta$. Then,

$$\langle z, w\rangle = \sum_{i=1}^{k} \overline{a_i} \langle z, v_i\rangle = 0.$$

Thus, $z \in W^{\perp}$.

3. Let V be an inner product space over F, S_1, S_2 be subsets of V, and W be a finite dimensional subspace of V.
   (a) Show that if $S_1 \subseteq S_2$, then $S_2^{\perp} \subseteq S_1^{\perp}$.
   (b) Show that $\mathrm{Span}(S_1) \leq (S_1^{\perp})^{\perp}$.
   (c) Show that $W = (W^{\perp})^{\perp}$.

Solution:

(a) Suppose $S_1 \subseteq S_2$ and $x \in S_2^{\perp}$. Then, $\langle x, s\rangle = 0$ for all $s \in S_2$. This holds for all $s \in S_1$ and so $x \in S_1^{\perp}$.

(b) Since $(S_1^{\perp})^{\perp}$ is a subspace of V, it suffices to show that $S_1 \subseteq (S_1^{\perp})^{\perp}$. Let $x \in S_1$, then

$$\langle x, s\rangle = 0$$

for all $s \in S_1^{\perp}$ by definition. This yields that $x \in (S_1^{\perp})^{\perp}$.

(c) By Part (b), we have $W \leq (W^{\perp})^{\perp}$. Let $x \in (W^{\perp})^{\perp}$, then $\langle x, y\rangle = 0$ for all $y \in W^{\perp}$. Note that x can be uniquely written as $x = w + z$ where $w \in W$ and $z \in W^{\perp}$. Then,

$$0 = \langle x, y\rangle = \langle z, y\rangle$$

for all $y \in W^{\perp}$. In particular, if we choose $y = z$ for each z, then we get $x = w = \mathrm{proj}_W(x)$ and so $x \in W$. Thus, we conclude that $W = (W^{\perp})^{\perp}$.

4. A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is called unitary if Q is invertible and $Q^{-1} = Q^*$. A matrix $A \in \mathcal{M}_{n\times n}(\mathbb{C})$ is unitary if and only if the set of the columns of A is orthonormal.

Solution: Suppose that A is unitary, then $A^*A = AA^* = I$. Let $\{v_1, \cdots, v_n\}$ be the columns of A, then

$$I_{ij} = \delta_{ij} = (A^*A)_{ij} = (v_i)^* v_j = \langle v_i, v_j\rangle$$

for all $i, j = 1, 2, \cdots, n$. Thus, $\{v_1, \cdots, v_n\}$ is orthonormal.

Suppose $\{v_1, \cdots, v_n\}$ is the set of columns of A which is orthonormal. Since

$$\delta_{ij} = \langle v_i, v_j\rangle = (v_i)^* v_j = (A^*A)_{ij},$$

for all i, j, we get $A^*A = I$. Thus, A is invertible and $A^{-1} = A^*$.

5. Let V be an inner product space over F, $T : V \to V$ linear, and $y \in V$. Let $\varphi(x) : V \to F$ be defined by $\varphi(x) = \langle T(x), y\rangle$. Show that φ is linear.

Solution: Note that $\varphi(0) = \langle T(0), y\rangle = \langle 0, y\rangle = 0$. Let $v, w \in V$ and $c \in F$, then

$$\varphi(cv + w) = \langle T(cv + w), y\rangle = \langle cT(v) + T(w), y\rangle = c\,\langle T(v), y\rangle + \langle T(w), y\rangle = c\varphi(v) + \varphi(w).$$

6. Let V be an inner product space over F, $c \in F$, and $S, T : V \to V$ linear.
   (a) Show that $(cS + T)^* = \overline{c}S^* + T^*$.
   (b) Show that $(ST)^* = T^*S^*$.
   (c) Show that $(T^*)^* = T$ and $I^* = I$.

Solution:

(a) For $x, y \in V$,

$$\begin{aligned}\langle x, (cS + T)^*(y)\rangle &= \langle (cS + T)(x), y\rangle \\ &= c\,\langle S(x), y\rangle + \langle T(x), y\rangle \\ &= c\,\langle x, S^*(y)\rangle + \langle x, T^*(y)\rangle \\ &= \langle x, \overline{c}S^*(y) + T^*(y)\rangle\,.\end{aligned}$$

Since this holds for all $x, y \in V$, we conclude that $(cS + T)^* = \overline{c}S^* + T^*$.

(b) For $x, y \in V$,

$$\begin{aligned}\langle x, (ST)^*(y)\rangle &= \langle (ST)(x), y\rangle \\ &= \langle T(x), S^*(y)\rangle \\ &= \langle x, (T^*S^*)(y)\rangle\,.\end{aligned}$$

Thus, $(ST)^* = T^*S^*$.

(c) For $x, y \in V$,

$$\begin{aligned}\langle x, (T^*)^*(y)\rangle &= \langle (T^*)(x), y\rangle \\ &= \langle x, T(y)\rangle\end{aligned}$$

and

$$\begin{aligned}\langle x, I^*(y)\rangle &= \langle I(x), y\rangle \\ &= \langle x, y\rangle \\ &= \langle x, I(y)\rangle\,.\end{aligned}$$

7. Let V be an inner product space over F.
   (a) (Parseval's identity) Let $\beta = \{v_1, \cdots, v_n\}$ be an orthonormal basis for V. Show that

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle\,\overline{\langle y, v_i\rangle}$$

   for all $x, y \in V$.

(b) (Bessel's indequality) Let $S = \{v_1, \cdots, v_n\}$ be an orthonormal subset for V. Show that

$$\sum_{i=1}^{n} |\langle x, v_i \rangle|^2 \leq \|x\|^2$$

for all $x \in V$.

Solution:

(a) Since

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i, \qquad y = \sum_{i=1}^{n} \langle y, v_i \rangle v_i,$$

we have

$$\begin{aligned}
\langle x, y \rangle &= \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i, \sum_{j=1}^{n} \langle y, v_j \rangle v_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_j \rangle} \langle v_i, v_j \rangle \\
&= \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.
\end{aligned}$$

(b) Let $W = \mathrm{Span}(S)$ and $x \in V$, then S is an orthonormal basis for W and

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i + z$$

where $z \in W^{\perp}$. Then,

$$\begin{aligned}
\|x\|^2 &= \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i + z, \sum_{j=1}^{n} \langle x, v_j \rangle v_j + z \right\rangle \\
&= \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i, \sum_{j=1}^{n} \langle x, v_j \rangle v_j \right\rangle + \left\langle \sum_{i=1}^{n} \langle x, v_i \rangle v_i, z \right\rangle + \left\langle z, \sum_{j=1}^{n} \langle x, v_j \rangle v_j \right\rangle + \|z\|^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x, v_i \rangle \overline{\langle x, v_j \rangle} \langle v_i, v_j \rangle + \|z\|^2 \\
&\geq \sum_{i=1}^{n} |\langle x, v_i \rangle|^2.
\end{aligned}$$