Sec 10.2: Fourier Series, part 1

Math 285 Spring 2020

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Periodic functions

Definition

A function f is periodic with period $T > 0$ if

(i) $x + T$ belongs to the domain of f whenever x does, and

(ii) $f(x+T) = f(x)$ for all x.

The smallest period $T > 0$ is called the fundamental period of f.

Ex ① $\cos t$, $\sin t$: $2\pi, 4\pi, 6\pi, \cdots$

② $\tan t$:  $\pi, 2\pi, 3\pi, \cdots$

③ $|\sin t|$:  $\pi, 2\pi, 3\pi, \cdots$

Note If $T > 0$ is a period, then

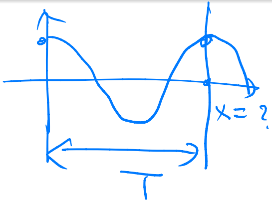
So are $2T, 3T, 4T, \cdots$

Periodic functions

Example

It is easy to see that $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$ are periodic with the same period $2L/m$.

$$\cos\left(\frac{m\pi}{L}x\right)$$



$$\frac{m\pi}{L}x = 2\pi$$

$$x = 2\pi \times \frac{L}{m\pi}$$

$$= \frac{2L}{m}$$

Periods: $\frac{2L}{m}, \frac{2L}{m}\times 2, \frac{2L}{m}\times 3, \cdots, 2L, \cdots$

Periodic functions

Proposition

If f and g are periodic functions with common period T, then so is $c_1 f + c_2 g$ for any $c_1, c_2 \in \mathbb{R}$.

Proof We know $f(x+T) = f(x)$, $g(x+T) = g(x)$

$$(c_1 f + c_2 g)(x+T) = c_1 f(x+T) + c_2 g(x+T)$$
$$= c_1 f(x) + c_2 g(x)$$
$$= (c_1 f + c_2 g)(x) \qquad \square$$

Ex What is a period of $\sin 2t + \cos 3t$?

$\pi, 2\pi, 3\pi, 4\pi, \cdots \;/\; \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, \cdots$

$\therefore \quad T = 2\pi, 4\pi, 6\pi, \cdots$

Inner product and Orthogonality

Definition

For functions f and g on $[\alpha, \beta]$, we define the standard inner product of f and g by

$$(f, g) = \int_{\alpha}^{\beta} f(x)g(x)\, dx.$$

Ex $f(x) = 1$, $g(x) = x$ on $[0, 1]$

$$(f, g) = \int_0^1 f \cdot g \, dx$$

$$= \int_0^1 1 \cdot x \, dx$$

$$= \frac{1}{2}.$$

Inner product and Orthogonality

Remark

The inner product has the following properties:

(i) *(Linearity)* $(cf+g,h) = c(f,h) + (g,h)$;

(ii) *(Symmetry)* $(f,g) = (g,f)$;

(iii) *(Positive-definite)* $(f,f) \geq 0$ *and* $(f,f) = 0$ *if and only if* $f = 0$.

Indeed, if a relation $(\cdot,\cdot)$ *satisfies these three assumptions, we call it an inner product. An elementary example of inner product is dot product.*

In $\mathbb{R}^2$, $(1,2)\cdot(3,4) = 1\times 3 + 2\times 4 = 11$

Two vectors are perpendicular (orthogonal) if

(dot product) $= 0$.

Inner product and Orthogonality

Definition

We say that functions f and g are orthogonal on $[\alpha, \beta]$ if $(f, g) = 0$. We say that a set of functions are mutually orthogonal if any two functions in the set are orthogonal.

Ex ① $f = x$, $g = x^2$ on $[-1, 1]$

$$(f, g) = \int_{-1}^{1} x \cdot x^2 \, dx = 0 \quad \text{orthogonal.}$$

② $f = 1$, $g = x$ on $[-1, 1]$

$$(f, g) = \int_{-1}^{1} 1 \cdot x \, dx = 0 \quad \text{orthogonal.}$$

Inner product and Orthogonality

Definition

We say that functions f and g are orthogonal on $[\alpha, \beta]$ if $(f, g) = 0$. We say that a set of functions are mutually orthogonal if any two functions in the set are orthogonal.

③ $\{1, x, x^2\}$ is not mutually orthogonal on $[-1, 1]$ because

$$(1, x^2) = \int_{-1}^{1} 1 \cdot x^2 \, dx = \frac{2}{3} \neq 0$$

eventhough $\{1, x\}$, $\{x, x^2\}$ are orthogonal.

Inner product and Orthogonality

Example

$\mathbb{N} \cup \{0\}$

The set $\{\sin \frac{m\pi x}{L}, \cos \frac{m\pi x}{L} : m \in \mathbb{N}\}$ is mutually orthogonal. on $[-L, L]$

$$= \{0, \sin\left(\frac{\pi}{L}x\right), \sin\left(\frac{2\pi}{L}x\right), \cdots$$
$$1, \cos\left(\frac{\pi}{L}x\right), \cos\left(\frac{2\pi}{L}x\right), \cdots \}$$

① $\left(\sin\left(\frac{m\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right)\right)$

$$= \int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \cdot \cos\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{1}{2}\int_{-L}^{L}\left(\underline{\sin\left(\frac{(m+n)\pi}{L}x\right)} + \underline{\sin\left(\frac{(m-n)\pi}{L}x\right)}\right) dx$$

$$= 0.$$ for any m, n.

Inner product and Orthogonality

Example

The set $\{\sin \frac{m\pi x}{L}, \cos \frac{m\pi x}{L} : m \in \mathbb{Z}\}$ is mutually orthogonal.

② $\left(\cos\left(\frac{m\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right) \right)$

$$= \int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cdot \cos\left(\frac{n\pi}{L}x\right) dx$$

$$= \frac{1}{2} \int_{-L}^{L} \left(\cos\left(\frac{(m-n)\pi}{L}x\right) + \cos\left(\frac{(m+n)\pi}{L}x\right) \right) dx$$

$$= \begin{cases} \text{If } m=n, & \frac{1}{2}\int_{-L}^{L} \left(1 + \cos\left(\frac{2m\pi}{L}x\right)\right) dx = L \\ \text{If } m \neq n, & 0 \end{cases}$$

$$\boxed{\begin{aligned} \left(\cos\left(\tfrac{m\pi}{L}x\right), \cos\left(\tfrac{n\pi}{L}x\right) \right) &= \\ \left(\sin\left(\tfrac{m\pi}{L}x\right), \sin\left(\tfrac{n\pi}{L}x\right) \right) &= \end{aligned} \begin{cases} L, & m=n \\ 0, & m \neq n \end{cases}}$$

Fourier series

Suppose that a function f can be written as

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).$$

Assume that the infinite sum in the RHS converges for each $x \in [-L, L]$.

Note that f is periodic with period $2L$. Our goal is to relate f with the coefficients a_m, b_m.

Fourier series

To this end, we compute

$$(f, \cos\frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx,$$
$$(f, \sin\frac{n\pi x}{L}) = \int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,$$

for each $n = 0, 1, 2, \cdots$.

① $n=0$. $\left(f, \sin\left(\frac{n\pi}{L}x\right)\right) = (f, 0) = 0$

$$\left(f, \cos\left(\frac{n\pi}{L}x\right)\right) = (f, 1) = \int_{-L}^{L} f\,dx$$

$$= \int_{-L}^{L} \frac{a_0}{2}\,dx + \underbrace{\sum_{m=1}^{\infty}\int_{-L}^{L}\left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right)dx}_{=0}$$

$$= \frac{a_0}{2}\cdot 2L = a_0 L$$

Fourier series

② $n \neq 0$.

$$\left(f, \cos\left(\frac{n\pi}{L}x\right)\right) = \underbrace{\int_{-L}^{L} \frac{a_0}{2} \cos\left(\frac{n\pi}{L}x\right) dx}_{=0}$$

$$+ \sum_{m=1}^{\infty} a_m \underbrace{\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cdot \cos\left(\frac{n\pi}{L}x\right) dx}_{= \left(\cos\left(\frac{m\pi}{L}x\right), \cos\left(\frac{n\pi}{L}x\right)\right) = \begin{cases} L, & m=n \\ 0, & m \neq n \end{cases}}$$

$$+ \sum_{m=1}^{\infty} b_m \underbrace{\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right) \cdot \cos\left(\frac{n\pi}{L}x\right) dx}_{=0}$$

$$= a_n \cdot L$$

$$\left(f, \sin\left(\frac{n\pi}{L}x\right)\right) = b_n \cdot L \qquad \text{for all } n = 1, 2, \cdots$$

Fourier series

Summary If

$$f(x) = \overbrace{\underbrace{\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right)}}^{\text{Fourier series.}}$$

then

$$(*)\begin{cases} a_n = \dfrac{1}{L}\left(f, \cos\left(\dfrac{n\pi}{L}x\right)\right) \\ b_n = \dfrac{1}{L}\left(f, \sin\left(\dfrac{n\pi}{L}x\right)\right). \end{cases}$$

Next Question

Suppose we are given a "nice" function f.
Using $(*)$, we can find the Fourier series of f. Q: f = Fourier series?