Math 285 Final Practice

1. (5 points) What is the correct integrating factor to solve the following ODE for $y(t)$?

$$3t^2y' + t^3y + t\sin t = 0$$

   A. $\mu(t) = t^3$
   B. $\mu(t) = \frac{1}{t^3}$
   C. $\mu(t) = e^{\frac{1}{6}t^2}$
   D. $\mu(t) = 1$
   E. None of these

Solution: The integrating factor is

$$\mu(t) = \exp\left(\int \frac{t^3}{3t^2}\,dt\right) = e^{\frac{1}{6}t^2}.$$

2. (5 points) Which one of the following is a solution to $ty' = 2y + 1$?
   A. $y = t^2 - 1$
   B. $y = \frac{1}{2}(t-1)$
   C. $y = 0$
   D. $y = \frac{1}{2}(t^2 - 1)$
   E. $y = 2t - \frac{1}{2}$

Solution: By separation,

$$\begin{aligned} \frac{1}{2y+1}\,dy &= \frac{1}{t}\,dt. \\ \frac{1}{2}\ln(2y+1) &= \ln t + C \\ 2y + 1 &= Ct^2 \\ y &= Ct^2 - \frac{1}{2}. \end{aligned}$$

3. (5 points) The function $P(t) = P_0$ is a stable solution to the following equation

$$\frac{dP}{dt} = (P-1)(P+3)P^2$$

if

A. $P_0 = 1$

B. $P_0 = 0$

C. $P_0 = 1$ and $P_0 = -3$

D. $P_0 = 1$ and $P_0 = 0$

E. $P_0 = -3$

Solution: The equilibrium solutions are $P(t) = -3$, $P(t) = 0$, and $P(t) = 1$. Since the sign of $(P-1)(P+3)P^2$ does not change near 0, $P(t) = 0$ is not stable. Since the sign of $(P-1)(P+3)P^2$ changes from negative to positive near 1, $P(t) = 1$ is not stable. Since the sign of $(P-1)(P+3)P^2$ changes from positive to negative near -3, $P(t) = 1$ is stable.

4. (5 points) The following nonlinear equation for $y(x)$ can be transformed with a substitution into which separable equation for $v(x)$?

$$y' = \frac{x^2 - xy + 2y^2}{x^2}$$

A. $xv' = 1 - 2v + 2v^2$

B. $xv' = 1 - v + 2v^2$

C. $v' = 1 - v + 2v^2$

D. $v' = x^2 - 2v + 2v^2$

E. None of these

Solution: Let $v = y/x$, then $xv = y$. So, we have $xv' + v = y'$ and

$$y' = xv' + v = 1 - v + 2v^2 = \frac{x^2 - xy + 2y^2}{x^2}.$$

5. (5 points) Consider

$$y''' - 5y'' + 8y' - 4y = 0$$

Which one of the following is NOT a solution?

A. $y = e^t$

B. $y = te^t + 3e^{2t}$

C. $y = 2e^{2t} - e^t$

D. $y = -te^{2t}$

E. None of these

Solution: Since the characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2 = 0,$$

the fundamental solutions are e^t, e^{2t}, te^{2t}.

6. (5 points) Identify the correct form of a particular solution for the following differential equation.

$$y'' + 2y' = 4te^{-2t} + 1.$$

A. $Y(t) = (At + B)e^{-2t} + Ct$
B. $Y(t) = (At^2 + Bt)e^{-2t} + C$
C. $Y(t) = (At^2 + Bt)e^{-2t} + Ct$
D. $Y(t) = (At + B)e^{-2t} + C$
E. $Y(t) = (At + B)e^{-2t}$

Solution: Since the characteristic equation is

$$\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0,$$

the fundamental solutions are $1, e^{-2t}$. Thus, a particular solution is of the form

$$Y(t) = t(At + B)e^{-2t} + Ct.$$

7. (5 points) The motion of a certain spring–mass system is governed by

$$u'' + \gamma u' + ku = 0$$

for some constants $\gamma, k > 0$. The motion is overdamped if

A. $\gamma = 1$ and $k = 2$
B. $\gamma = 3$ and $k = 3$
C. $\gamma = 2$ and $k = \frac{3}{2}$
D. $\gamma = 4$ **and** $k = 3$
E. None of these

Solution: The motion is overdamped if

$$\gamma^2 - 4k > 0.$$

8. (5 points) The existence and uniqueness theorem guarantees that there exists a unique solution of the initial value problem $(t-1)(t+2)y'' + y'\sqrt{5-t} + e^t y = 1$ with $y(2) = 1$ and $y'(2) = 3$ on an open interval

   A. $(-\infty, 1)$
   B. $(-2, 1)$
   C. $(1, \infty)$
   D. None of these
   E. $(1, 5)$

Solution: Dividing $(t-1)(t+2)$, the coefficients are continuous if $t \neq 1, -2$ and $t \leq 5$. Since the initial conditions are given at $t = 2 \in (1, 5)$, the answer is $(1, 5)$.

9. (5 points) Consider the following boundary value problem for a variant of the wave equation:

$$\begin{aligned} &u_{tt} = u_{xx} + u, \qquad \text{for } 0 < x < 1, \quad t > 0, \\ &u(0,t) = u_x(1,t) = 0 \qquad \text{for } t \geq 0, \\ &u(x,0) = f(x), \qquad u_t(x,0) = 0 \qquad \text{for } 0 \leq x \leq 1. \end{aligned}$$

   Then separated solutions must satisfy which of the following sets of equations?

   A. $X'' + \lambda X = 0$ with $X(0) = X'(1) = 0$, and $T'' + (\lambda - 1)T = 0$ with $T'(0) = 0$.
   B. $X'' + \lambda X = 0$ with $X(0) = X(1) = 0$, and $T'' + \lambda T = 0$ with $T'(0) = 0$.
   C. $X'' + (\lambda - 1)X = 0$ with $X(0) = X'(1) = 0$, and $T'' + \lambda T = 0$ with $T'(0) = 0$.
   D. $X'' + (\lambda - 1)X = 0$ with $X(0) = X(1) = 0$, and $T'' + (\lambda + 1)T = 0$ with $T'(0) = 0$.
   E. None of these

Solution: Let $u(x,t) = X(x)T(t)$, then the equation is

$$XT'' = X''T + XT$$
$$\frac{T''}{T} - 1 = \frac{X''}{X} = -\lambda.$$

So, we have $X'' + \lambda X = 0$ and $T'' + (\lambda - 1)T = 0$. The boundary conditions $u(0,t) = u_x(1,t) = 0$ imply $X(0) = X'(1) = 0$, and the initial condition $u_t(x,0) = 0$ yields with $T'(0) = 0$.

10. (5 points) Let $X(x) = e^{10x} + e^{-10x}$. Find a nonzero function $Y(y)$ such that the product $u(x, y) = X(x)Y(y)$ satisfies Laplace's equation $u_{xx} + u_{yy} = 0$.

    A. $Y(y) = \sinh(10y)$

    B. $Y(y) = \cos(10y)$

    C. $Y(y) = \sin(5y)$

    D. $Y(y) = e^{10y} - e^{-10y}$

    E. None of these

Solution: From $u(x, y) = X(x)Y(y)$, we have

$$100 = \frac{X''}{X} = -\frac{Y''}{Y},$$

that is, $Y'' + 100Y = 0$. The general solution for Y is $Y(y) = C_1 \cos(10y) + C_2 \sin(10y)$.

11. (5 points) If λ_1 is the smallest eigenvalue of $y'' + \lambda y = 0$ with $y'(0) = y(\pi) = 0$, what is the corresponding eigenfunction?

    A. $\cos(\frac{t}{4})$

    B. $\sin(\frac{t}{2})$

    C. $\cos(\frac{t}{2})$

    D. $\sin(\frac{t}{4})$

    E. None of these

Solution: If $\lambda = -\mu^2 < 0$, then $y(t) = c_1 \cosh(\mu t) + c_2 \sinh(\mu t)$. Since $y'(0) = c_2 = 0$ and $y(\pi) = c_1 \cosh(\mu\pi) = 0$, $y(t) = 0$.

If $\lambda = 0$, then $y(t) = c_1 + c_2 t$. Since $y'(0) = c_2 = 0$ and $y(\pi) = c_1 = 0$, $y(t) = 0$.

If $\lambda = \mu^2 > 0$, then $y(t) = c_1 \cos \mu t + c_2 \sin \mu t$. Since $y'(0) = c_2 = 0$ and $y(\pi) = c_1 \cos \mu\pi = 0$, if $c_1 \neq 0$ then $\mu = n - \frac{1}{2}$ for each $n \in \mathbb{N}$.

Thus, the smallest eigenvalue is $\frac{1}{4}$ and the corresponding eigenfunction is $y_1(t) = C \cos(\frac{t}{2})$.

12. (5 points) The equation

$$y'' - 2xy' + \lambda y = 0$$

can be transformed into the form $(p(x)y')' + q(x)y = 0$ with

    A. $p(x) = e^{-2x}$ and $q(x) = \lambda$

B. $p(x) = -x^2$ and $q(x) = -\lambda x^2$

C. $p(x) = 2x$ and $q(x) = \lambda$

D. $p(x) = e^{-x^2}$ **and** $q(x) = \lambda e^{-x^2}$

E. None of these

Solution: It suffices to find $\mu(x)$ such that

$$\mu(x)y'' - 2x\mu(x)y' + \lambda\mu(x)y = (p(x)y')' + q(x)y = p(x)y'' + p'(x)y' + q(x)y.$$

That is, $p(x) = \mu(x)$, $\lambda\mu(x) = q(x)$, and

$$-2x\mu(x) = p'(x) = \mu'(x)$$

Solving the equation for μ, we have

$$\mu(x) = Ce^{-x^2}.$$

So, $p(x) = Ce^{-x^2}$ and $q(x) = C\lambda e^{-x^2}$