Math 285 Lecture Note: Week 13 and 14

Daesung Kim

1 Separation of Variables; Heat Conduction in a Rod (Sec 10.5)

Consider a heat conduction problem for a straight bar of length $L > 0$. Suppose it has uniform cross section and homogeneous material. Let the x-axis lie along the axis of the bar. Assume that the sides of the bar are perfectly insulated and each cross section has uniform temperature. Let $u(x,t)$ be the temperature of a cross section at x and time t. Then, u is governed by the heat conduction equation

$$\alpha^2 u_{xx} = u_t, \qquad 0 < x < L, \qquad t > 0.$$

The constant α^2 is called the thermal diffusivity.

We further assume that the initial temperature of the bar is given by $u(x,0) = f(x)$ for $0 \le x \le L$ and the ends of the bar are held at fixed temperatures $u(0,t) = T_1$ and $u(L,t) = T_2$ for all $t > 0$. In this section, we focus on the case $T_1 = T_2 = 0$ find solutions to

$$\alpha^2 u_{xx} = u_t, \qquad u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x). \tag{1.1}$$

First, we consider the boundary problem

$$\alpha^2 u_{xx} = u_t, \qquad u(0,t) = u(L,t) = 0. \tag{1.2}$$

for $0 < x < L$ and $t > 0$. (In other words, we drop the initial temperature distribution for a moment.) Note that this boundary problem has a trivial solution $u(x,t) = 0$ for all x and t. However, this may not satisfy $u(x,0) = f(x)$ except when $f = 0$. Thus, we want to find nontrivial solutions to (1.2).

The idea is to consider the case where $u(x,t)$ is a product of two functions $X(x)$ and $T(t)$. Let $u(x,t) = X(x)T(t)$, then $\alpha^2 u_{xx} = u_t$ implies

$$\begin{aligned} \alpha^2 X''(x)T(t) &= X(x)T'(t) \\ \frac{X''(x)}{X(x)} &= \frac{1}{\alpha^2}\frac{T'(t)}{T(t)} \end{aligned}$$

Note that the boundary conditions read $X(0)T(t) = X(L)T(t) = 0$ for all $t > 0$. If $X(0) \neq 0$ or $X(L) \neq 0$, then $T(t) = 0$ for all $t > 0$. Since we are looking for nontrivial solutions, it is reasonable to assume that $X(0) = X(L) = 0$.

Suppose $\frac{X''(x)}{X(x)}$ is a constant for all x and t, that is, $X''(x) + \lambda X(x) = 0$. If $u(x,t) = X(x)T(t)$ is nontrivial, the constant λ should be an eigenvalue and $X(x)$ is an eigenfunction. Thus, we get

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = C\sin\left(\frac{n\pi}{L}x\right),$$

for all $n \in \mathbb{N}$. Plugging this λ to the equation for $T(t)$ and solving it, we have

$$\begin{aligned} T'(t) &= -\frac{n^2\pi^2\alpha^2}{L^2}T(t), \\ T_n(t) &= C\exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right) \end{aligned}$$

for all $n \in \mathbb{N}$. Thus,

$$u_n(x,t) = C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

for all $n \in \mathbb{N}$.

Since the boundary problem (1.2) is homogeneous, if u_1 and u_2 are solutions then so is a linear combination $c_1u_1 + c_2u_2$. (This is called superposition.) So, we have

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi}{L}x\right).$$

The last step is to impose the initial temperature distribution $u(x,0) = f(x)$. From the above solution, we have

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) = f(x).$$

We now use a Fourier series representation for f to determine C_n for all $n \in \mathbb{N}$. To this end, we extend f to an odd function on $[-L, L]$ (i.e. define $f(x)$ for $x \in [-L, 0)$ by $-f(-x)$, and $f(x+2L) = f(x)$ for all x) and assume that the Fourier convergence theorem is applicable (i.e. f and f' are piecewise continuous). If we choose

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx,$$

then

$$u(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L^2}t\right)\sin\left(\frac{n\pi}{L}x\right).$$

is a solution to the boundary problem (1.1).

2 Other Heat Conduction Problems (Sec 10.6)

Nonhomogeneous boundary conditions

Consider a heat conduction problem for a straight bar of length $L > 0$. Suppose the ends of the bar are held at constant temperatures T_1 and T_2. Then, the corresponding heat conduction equation with boundary conditions is $\alpha^2 u_{xx} = u_t$ with

$$u(0,t) = T_1, \qquad u(L,t) = T_2, \qquad u(x,0) = f(x). \tag{2.1}$$

Let $v(x) = \lim_{t\to\infty} u(x,t)$ be the steady state temperature distribution, then it will satisfy $v'' = 0$ with $v(0) = T_1$ and $v(L) = T_2$. Solving the boundary problem, we get

$$v(x) = T_1(1 - \frac{x}{L}) + T_2\frac{x}{L} = T_1 + \left(\frac{T_2 - T_1}{L}\right)x.$$

Let $w(x,t) = u(x,t) - v(x)$, then we have $\alpha^2 w_{xx} = w_t$ with

$$w(0,t) = w(L,t) = 0, \qquad w(x,0) = f(x) - v(x) = f(x) - T_1 - \left(\frac{T_2 - T_1}{L}\right)x.$$

We have seen in the previous section that

$$w(x,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

where

$$C_n = \frac{2}{L}\int_0^L (f(x) - v(x)) \sin\left(\frac{n\pi}{L}x\right)\, dx,$$

Therefore, the solution is

$$u(x,t) = v(x) + w(x,t) = T_1 + \left(\frac{T_2 - T_1}{L}\right)x + \sum_{n=1}^{\infty} C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L}t\right)\sin\left(\frac{n\pi}{L}x\right).$$

Bar with insulated ends

Suppose that the ends of the bar are perfectly insulated so that there is no passage of heat through them. This model is governed by $\alpha^2 u_{xx} = u_t$ with

$$u_x(0,t) = u_x(L,t) = 0, \qquad u(x,0) = f(x). \tag{2.2}$$

We use the method of separation of variables. Let $u(x,t) = X(x)T(t)$, then

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda$$

as before. For $X(x)$, we have $X'' + \lambda X = 0$ with $X'(0) = X'(L) = 0$. By considering $Z(x) := X'(x)$, we get

$$\lambda_n = \frac{n^2\pi^2}{L},$$
$$X_n(x) = \cos\left(\frac{n\pi}{L}x\right)$$

for all $n = 0, 1, 2, \cdots$. For each λ_n, a solution to $T' = -\lambda_n \alpha^2 T$ is

$$T_n(t) = \exp\left(-\frac{n^2\pi^2\alpha^2}{L}t\right).$$

Thus, we have

$$u(x,t) = \frac{C_0}{2} + \sum_{m=1}^{\infty} C_n \exp\left(-\frac{n^2\pi^2\alpha^2}{L}t\right)\cos\left(\frac{n\pi}{L}x\right).$$

Since the initial temperature distribution is

$$u(x,0) = \frac{C_0}{2} + \sum_{m=1}^{\infty} C_n \cos\left(\frac{n\pi}{L}x\right) = f(x),$$

we use the Fourier cosine series for f to determine C_n. That is,

$$C_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\, dx$$

for all $n = 0, 1, 2, \cdots$.

3 The Wave Equation: Vibrations of an Elastic String (Sec 10.7)

3.1 Model

Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level. Let the x-axis lie along the string. Let $u(x,t)$ be the vertical displacement by the string at the point x at time t. Then, $u(x,t)$ satisfies the PDE

$$a^2 u_{xx} = u_{tt} \tag{3.1}$$

for $0 < x < L$ and $t > 0$. The equation is called the 1-dimensional wave equation. Since the ends are fixed, we have the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0 \tag{3.2}$$

for all $t \geq 0$. We prescribe two initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{3.3}$$

for all $0 \leq x \leq L$. We note that the wave equation (3.1) can be generalized to higher dimensions:

$$a^2(u_{xx} + u_{yy}) = u_{tt}, \qquad a^2(u_{xx} + u_{yy} + u_{zz}) = u_{tt}, \qquad \cdots .$$

3.2 Nonzero initial displacement

We consider the wave equation (3.1) with boundary condition (3.2) and initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = 0. \tag{3.4}$$

As we did for the heat equation, we use the method of separation of variables. Let $u(x,t) = X(x)T(t)$, then

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = -\lambda$$

so that

$$X'' + \lambda X = 0, \qquad T'' + a^2\lambda T = 0.$$

The boundary conditions (3.2) read

$$X(0) = X(L) = 0.$$

Therefore, for each $n \in \mathbb{N}$, we have

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right).$$

For these λ_n, the general solution to $T'' + a^2\lambda T = 0$ is

$$T_n(t) = k_1 \cos\left(\frac{n\pi a}{L}t\right) + k_2 \sin\left(\frac{n\pi a}{L}t\right).$$

Using the initial condition $u_t(x,0) = 0$, we have $T'(0) = 0$ so that $k_2 = 0$. Thus,

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

is a solution to (3.1) with (3.2) and $u_t(x,0) = 0$. Since this boundary problem is homogeneous, the superposition property yields that

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right)$$

is also a solution. Finally, we consider the initial condition $u(x,0) = f(x)$. Since

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right),$$

we use the Fourier sine series of f to determine C_n

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

Remark 3.1. Note that for each $n \in \mathbb{N}$, $u_n(x,t)$ is periodic in time t and position x. The quantity $n\pi a/L$ for $n \in \mathbb{N}$ are called the *natural frequencies* of the string. The factor $\sin(n\pi x/L)$ represents the displacement pattern, which is called a natural mode of vibration. The period of position $2L/n$ is called the *wavelength* of the mode.

Example 3.2. We consider $4u_{xx} = u_{tt}$ with $u(0,t) = u(2,t) = 0$, $u(x,0) = f(x)$, and $u_t(x,0) = 0$ where

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{2}x\right)\cos\left(n\pi t\right)$$

with

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx \\ &= \int_0^1 x\sin\left(\frac{n\pi}{2}x\right)\,dx + \int_1^2 (2-x)\sin\left(\frac{n\pi}{2}x\right)\,dx \\ &= \frac{8}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right). \end{aligned}$$

3.3 Nonzero initial velocity

We consider the wave equation (3.1) with the boundary condition (3.2) and the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = g(x). \tag{3.5}$$

Using the same argument, we get

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sin\left(\frac{n\pi}{L}x\right),$$

and $T'' + a^2\lambda T = 0$ with $T(0) = 0$. Since the general solution for T is

$$T_n(t) = k_1\cos\left(\frac{n\pi a}{L}t\right) + k_2\sin\left(\frac{n\pi a}{L}t\right),$$

the initial condition $T(0) = 0$ implies $k_1 = 0$. Thus,

$$u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$$

is a solution to (3.1) with (3.2) and $u(x,0) = 0$. Since this boundary problem is homogeneous, the superposition property yields that

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$$

is also a solution. Finally, the initial condition $u_t(x,0) = g(x)$ yields

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} D_n \frac{n\pi a}{L}\sin\left(\frac{n\pi}{L}x\right).$$

We use the Fourier sine series of g to determine C_n

$$D_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx.$$

3.4 General case

We are ready to find a solution $u(x,t)$ to (3.1) with boundary conditions (3.2) and initial conditions (3.3). To this end, we find two solutions $v(x,t)$ and $w(x,t)$ such that $v(x,t)$ is a solution with initial condition (3.4) and $w(x,t)$ with (3.5). That is,

$$\begin{aligned} &a^2 v_{xx} = v_{tt}, \qquad v(0,t) = v(L,t) = 0, \qquad v(x,0) = f(x), \qquad v_t(x,0) = 0, \\ &a^2 w_{xx} = w_{tt}, \qquad w(0,t) = w(L,t) = 0, \qquad w(x,0) = 0, \qquad w_t(x,0) = g(x), \end{aligned}$$

Define $u(x,t) = v(x,t) + w(x,t)$, then

$$\begin{aligned} a^2 u_{xx} &= a^2(v_{xx} + w_{xx}) = a^2 v_{xx} + a^2 w_{xx} = v_{tt} + w_{tt} = u_{tt}, \\ u(0,t) &= v(0,t) + w(0,t) = 0, \\ u(L,t) &= v(L,t) + w(L,t) = 0, \\ u(x,0) &= v(x,0) + w(x,0) = f(x) + 0 = f(x), \\ u_t(x,0) &= v_t(x,0) + w_t(x,0) = 0 + g(x) = g(x). \end{aligned}$$

Thus,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi a}{L}t\right) + \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi a}{L}t\right)$$

with

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)\,dx, \\ D_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)\,dx. \end{aligned}$$

References

[BD] Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 10th Edition, Wiley

Department of Mathematics, University of Illinois at Urbana-Champaign
E-mail address: daesungk@illinois.edu