Sec 11.1: The Occurrence of Two-Point Boundary Value Problems

Math 285 Spring 2020

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Recall: Heat Conduction Problem

Previously, we have seen the heat conduction equation $\alpha^2 u_{xx} = u_t$ with boundary conditions $u(0,t) = 0$ (or $u_x(0,t) = 0$) and $u(L,t) = 0$ (or $u_x(L,t) = 0$) and initial condition $u(x,0) = f(x)$.

We used the method of separation of variables to deduce two ODEs

$$\begin{aligned} &X'' + \lambda X = 0, \qquad X(0) = X(L) = 0, \\ &T' + \alpha^2 \lambda T = 0. \end{aligned}$$

It turned out that the ODE for X with the boundary conditions leads to eigenvalue problems. We have shown that for some λ_n, there exists nontrivial solutions for the boundary problem.

Then, we solved the ODE for T and used the superposition property to get the solution.

Generalization

$$r \cdot u_t = p \cdot u_{xx} + p' \cdot u_x - q \cdot u$$

We consider the partial differential equations of the form

$$r(x)u_t = \underbrace{(p(x)u_x)_x} - q(x)u.$$

with boundary conditions

$$= (p \cdot u_x)_x = \frac{\partial p}{\partial x} \cdot u_x + p \cdot u_{xx}$$

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0, \qquad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2$ with $\alpha_1^2 + \alpha_2^2 > 0$ and $\beta_1^2 + \beta_2^2 > 0$.

For example, the heat conduction problem is the case where
$\cancel{p(x) = 1 = r(x)}$ and $q(x) = 0$.

$$\underline{u_t = \alpha^2 \cdot u_{xx}}$$

$p(x) = \alpha^2 \qquad q(x) = 0, \qquad r(x) = 1$

Generalization

Let $u(x,t) = X(x)T(t)$, then

$$\overbrace{r(x)X(x)T'(t)}^{u_t} = \overbrace{(p(x)X'(x))'T(t)}^{(p\cdot u_x)_x} - \overbrace{q(x)X(x)T(t)}^{q\cdot u}$$

$$\frac{T'(t)}{T(t)} = \frac{(p(x)X'(x))'}{r(x)X(x)} - \frac{q(x)}{r(x)} = -\lambda.$$

Thus, we have $T' + \lambda T = 0$

$$(p(x)X')' - q(x)X + \lambda r(x)X = 0.$$

generalization of $X'' + \lambda X = 0.$

$\left(\begin{array}{c} p = r = 1 \\ q = 0. \end{array} \right)$

The boundary conditions read

$$\alpha_1 X(0) + \alpha_2 X'(0) = 0, \qquad \beta_1 X(L) + \beta_2 X'(L) = 0$$

Example

Consider the case where $p(x) = r(x) = 1$, $q(x) = 0$, $\alpha_2 = 0$, $\alpha_1 = \beta_1 = \beta_2 = 1$, and $L = \pi$. That is, $X'' + \lambda X = 0$ with $X(0) = 0$ and $X(\pi) + X'(\pi) = 0$.

Case 1 $\quad \lambda = -\mu^2 < 0$

$$X(x) = c_1 \cdot \cosh(\mu x) + c_2 \sinh(\mu x)$$

$$X(0) = c_1 = 0.$$

$$X'(x) = \mu c_2 \cosh(\mu x)$$

$$X(\pi) + X'(\pi) = c_2 \cdot (\sinh(\mu\pi) + \mu \cosh(\mu\pi))$$

$$= 0$$

Suppose $c_2 \neq 0$. then

$$\sinh \mu\pi + \mu \cosh \mu\pi = 0$$

$$0 < \mu = \underbrace{-\tanh \pi\mu}_{>0} < 0$$

Contradiction

No negative eigenvalues

Case 2 : $\lambda = 0$.

$$X'' = 0 \quad \Rightarrow \quad X(x) = C_1 + C_2 x$$

$$X(0) = C_1 = 0.$$

$$X(\pi) + X'(\pi) = C_2\pi + C_2 = 0$$

$$C_2 = 0$$

$$\therefore \quad \underline{X(x) = 0}$$

0 is Not an eigenvalue.

Case 3: $\lambda = \mu^2 > 0$

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$$

$$X(0) = c_1 = 0$$

$$X(\pi) + X'(\pi) = c_2 \underbrace{(\sin \mu\pi + \mu \cos \mu\pi)}_{=0} = 0$$

Suppose $c_2 \neq 0$.

$$\underline{\mu = -\tan(\mu\pi)}$$

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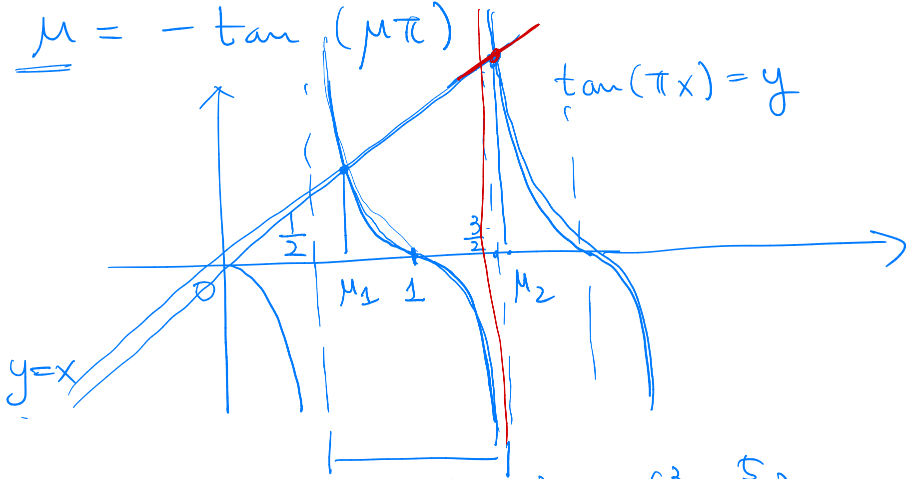

Every interval $(\frac{1}{2}, \frac{3}{2})$, $(\frac{3}{2}, \frac{5}{2})$, $\cdots$

, $(n-\frac{1}{2}, n+\frac{1}{2})$, $\exists\ \mu_n$ s.t

$\mu_n = -\tan(\mu_n \pi)$

$\underline{\lambda_n = \mu_n^2}$ $\exists$ X_n s.t

$$X_n'' + \lambda_n \cdot X_n = 0$$

$$\begin{cases} X_n(0) = 0 \\ X_n(\pi) + X_n'(\pi) = 0 \end{cases}$$

<u>Rmk</u>

1. $\lambda_1 < \lambda_2 < \lambda_2 < \cdots \cdots$

2. $\lim_{n\to\infty} \lambda_n = \infty$ $\left(\mu_n \in (n-\frac{1}{2}, n+\frac{1}{2}) \right)$

3. As $n \to \infty$ $\lambda_n \approx (n - \frac{1}{2})^2$