Hardy–Stein identity and Fourier multipliers

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Fourier multiplier operators

Definition

Let $m : \mathbb{R}^n \to \mathbb{C}$ be a function in L^∞ and $1 \leq p \leq \infty$.

An operator T_m is defined by $\widehat{T_m f}(\xi) = m(\xi)\widehat{f}(\xi)$ for $f \in L^2 \cap L^p$.

If $\|T_m f\|_p \lesssim \|f\|_p$ for all $f \in L^2 \cap L^p$, then T_m can be extended to L^p.

The operator $T_m : L^p \to L^p$ is called the Fourier multiplier operator with symbol m.

Examples

(1) Riesz transforms $R_j : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ with $\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|}\widehat{f}(\xi)$.

(2) Beurling–Ahlfors operator $B : L^p(\mathbb{C}) \to L^p(\mathbb{C})$ with $\widehat{Bf}(\xi) = \frac{\overline{\xi}}{\xi}\widehat{f}(\xi)$.

Question

For a given symbol m, can one define the Fourier multiplier T_m on L^p?

Fourier multiplier operators

Hörmander multiplier theorem

If $k > \frac{n}{2}$ and $m : \mathbb{R}^n \to \mathbb{C}$ is C^k away from the origin with a differentiability condition

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}, \qquad \forall \xi \in \mathbb{R}^n$$

for $|\alpha| \leq k$, then $\|T_m\|_{p \to p} \lesssim 1$.

Theorem (Bañuelos, Bogdan 2007)

If ν is a symmetric Lévy measure and $\phi \in L^\infty$ with $\phi(z) = \phi(-z)$, then T_m with

$$m(\xi) = \frac{\int (\cos(\xi \cdot z) - 1)\phi(z)\,\nu(dz)}{\int (\cos(\xi \cdot z) - 1)\,\nu(dz)}$$

is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$.

Littlewood–Paley theory

One way of deriving L^p boundedness of Fourier multipliers is Littlewood–Paley theory.

Example: Second-order Riesz transforms

Consider $T_m = R_i R_j$ with $m(\xi) = -\frac{\xi_i \xi_j}{|\xi|^2}$.

For $f \in L^2 \cap L^p$ and $h \in L^2 \cap L^q$ with $\|h\|_q = 1$,

$$\begin{aligned}
\int R_i R_j f(x) h(x)\, dx &= \frac{1}{(2\pi)^n} \int \frac{\xi_i \xi_j}{|\xi|^2} \widehat{f}(\xi) \overline{\widehat{h}(\xi)}\, d\xi \\
&= \frac{1}{(2\pi)^n} \int \int_0^\infty (\xi_i e^{-\frac{t}{2}|\xi|^2} \widehat{f}) \overline{(\xi_j e^{-\frac{t}{2}|\xi|^2} \widehat{h})}\, dt d\xi \\
&= \int \int_0^\infty \partial_i P_t f(x) \partial_j P_t h(x)\, dt dx \\
&\le \int \int_0^\infty |\nabla P_t f(x)| |\nabla P_t h(x)|\, dt dx \\
&\le \|G(f)\|_p \|G(h)\|_q
\end{aligned}$$

where $P_t f(x) = \mathbb{E}_x[f(B_t)]$ and $G(f)(x) = \left(\int_0^\infty |\nabla P_t f(x)|^2\, dt\right)^{\frac{1}{2}}$.

Littlewood–Paley theory

The L^p boundedness of the Second-order Riesz transforms $R_i R_j$ follows from that of the Littlewood–Paley square function $G(f)$.

There are two approaches to obtain (two-sided) L^p boundedness of $G(f)$:

(1) Analytic way: Calderon–Zygmund theory.

(2) Probabilistic way: Martingale transforms and Burkholder–Davis–Gundy inequality.

Hardy–Stein identity

An alternative analytic way to prove L^p-boundedness for $1 < p < 2$ is to use Hardy–Stein identity:

$$\int_{\mathbb{R}^n} |f|^p \, dx = p(p-1) \int_{\mathbb{R}^n} \int_0^\infty y|u|^{p-2} |\nabla u|^2 \, dy dx$$

where $u = u_f$ is the harmonic extension of f.

This was used to bound the squre function

$$g(f)(x)^2 = \int_0^\infty y|\nabla u|^2 \, dy.$$

This was proved by Stein (1970) using chain rule

$$\Delta(u^p) = p(p-1)|u|^{p-2}|\nabla u|^2 + p|u|^{p-1}\Delta u$$

and Green's theorem.

This proof can be adapted to Markovian semigroups whose generators satisfy the chain rule.

It is known that such a chain rule requires the process to have continuous trajectories.

Hardy–Stein identity

The identity can also be derived using the so-called background radiation process by Gundy and Varopoulos.

Consider the $(n+1)$ dimensional Brownian motion $Z_t = (X_t, Y_t)$ starting at (x, s) $(s > 0)$

Let $W_t = u_f(Z_t)$, then it follows from Itô's formula that

$$W_t = W_0 + \int_0^t \nabla u_f(Z_s) \cdot dZ_s.$$

Applying Itô's formula for $\varphi(x) = |x|^p$,

$$\mathbb{E}_{(x,s)}[|u_f(Z_\tau)|^p] - \mathbb{E}_{(x,s)}[|u_f(Z_0)|^p] = \frac{p(p-1)}{2}\mathbb{E}_{(x,s)}[\int_0^\tau |u_f(Z_t)|^{p-2}|\nabla u_f(Z_t)|^2\, dt]$$

where $\tau = \inf\{t \geq 0 : Y_t = 0\}$. Integrating over $\mathbb{R}^n$ in x and taking $s \to \infty$,

$$\int_{\mathbb{R}^n} |f|^p\, dx = p(p-1)\int_{\mathbb{R}^n}\int_0^\infty y|u|^{p-2}|\nabla u|^2\, dydx.$$

Hardy–Stein identity

The same argument works for the space-time Brownian motion.

Let $T > 0$ be fixed $Z_t = P_{T-t} f(B_t)$ where B_t is the n-dimensional Brownian motion and P_t is the heat semigroup, then

$$Z_t = Z_0 + \int_0^t \nabla P_{T-s} f(B_s) \cdot dB_s.$$

By Itô's formula as before,

$$\mathbb{E}_x[|Z_T|^p] - \mathbb{E}_x[|Z_0|^p] = \frac{p(p-1)}{2}\mathbb{E}_x[\int_0^T |P_{T-t} f(B_t)|^{p-2} |\nabla P_{T-t} f(B_t)|^2 \, dt].$$

Integrating over $\mathbb{R}^n$ in x and taking $T \to \infty$,

$$\int |f|^p \, dx = \frac{p(p-1)}{2} \int_{\mathbb{R}^n} \int_0^\infty |P_t f(x)|^{p-2} |\nabla P_t f(x)|^2 \, dy dx.$$

With the help of the maximal ergodic theorem, this Hardy–Stein identity provides $\|G(f)\|_p \lesssim \|f\|_p$ for $1 < p < 2$ where

$$G(f)(x) = \left(\int_0^\infty |\nabla P_t f(x)|^2 \, dt \right)^{\frac{1}{2}}.$$

Hardy–Stein identity for Lévy processes

Bañuelos, Bogdan, Luks (2016) extended the Hardy–Stein identity to a pure jump Lévy process and proved the L^p-boundedness of the corresponding square functions and the Fourier multipliers as an application.

Let X_t be a pure jump Lévy process with a symmetric Lévy measure ν.

Assume that the characteristic exponent $\psi(\xi)$ satisfies the Hartman–Wintner condition

$$\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{\log|\xi|} = \infty.$$

Let $P_t f(x) = \mathbb{E}_x[f(X_t)]$, then

Theorem (Bañuelos, Bogdan, Luks 2016)

$$\int |f|^p \, dx = \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(P_t f(x), P_t f(x+y)) \, \nu(dy) dx dt$$

where $F(a,b) = |b|^p - |a|^p - pa|a|^{p-2}(b-a)$.

Hardy–Stein identity for Lévy processes

In particular, if $p = 2$ then $F(a,b) = (a-b)^2$ and

$$\|f\|_2^2 = \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (P_t f(x) - P_t f(x+y))^2 \, \nu(dy) dx dt$$

Since $F(a,b) \approx (b-a)^2 (|b| \vee |a|)^{p-2}$,

$$\|f\|_p^p \approx \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (P_t f(x+y) - P_t f(x))^2 (|P_t f(x)| \vee |P_t f(x+y)|)^{p-2} \, \nu(dy) dx dt.$$

Theorem (Bañuelos, Bogdan, Luks 2016) For a square function $\widetilde{G}(f)$ defined by

$$\widetilde{G}(f)(x) = \left(\int_0^\infty \int_{\{|P_t f(x)| > |P_t f(x+y)|\}} (P_t f(x+y) - P_t f(x))^2 \, \nu(dy) dt \right)^{\frac{1}{2}},$$

we have $\|\widetilde{G}(f)\|_p \approx \|f\|_p$ for $1 < p < \infty$.

Hardy–Stein identity for Lévy processes

The proof of the Hardy–Stein identity is based on the semigroup analysis.

They considered $\xi(t) = |P_{T-t}f|^p$ along the path $t \mapsto P_t$, that is,

$$P_T|f|^p(x) - |P_Tf(x)|^p = P_T(\xi(T))(x) - P_0(\xi(0))(x) = \int_0^T \frac{d}{dt} P_t(\xi(t))(x)\, dt.$$

The proof relies on the symmetry of the Lévy measure ν, which assures that the infinitesimal generator can be written as

$$Lf(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} (f(x+y) - f(x))\, \nu(dy).$$

Main result

We give a new proof of the Hardy–Stein identity for a pure jump Lévy process using Itô's formula for jump processes, which works for non-symmetric Lévy measure and a certain class of martingales.

Theorem (Bañuelos, K. 2019)

For a pure jump Lévy process with a (possible non-symmetric) Lévy measure ν,

$$\int |f|^p \, dx = \int_0^\infty \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} F(P_t f(x), P_t f(x+y)) \, \nu(dy) dx dt.$$

Lévy processes

Definition

A stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a Lévy process if

(i) Independent increments: for $0 \leq t_0 < t_1 < \cdots < t_n < \infty$, $\{X_{t_k} - X_{t_{k-1}}\}_{k \geq 1}$ are independent.

(ii) Stationary increments: for $0 < s < t < \infty$ and a Borel set $A \in \mathbb{R}^n$, $\mathbb{P}(X_t - X_s \in A) = \mathbb{P}(X_{t-s} \in A)$.

(iii) Stochastically continuous: for all $\varepsilon > 0$ and $s \geq 0$, $\lim_{t \to s} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0$.

Lévy-Khintchine formula

For $\xi \in \mathbb{R}^n$, $\mathbb{E}[e^{i\xi X_t}] = e^{-t\psi(\xi)}$ with

$$\psi(\xi) = ib \cdot \xi + \frac{1}{2}\xi \cdot A\xi + \int_{\mathbb{R}^n} (1 - e^{i\xi y} + i\xi \cdot y 1_{B_1(0)}(y))\nu(dy)$$

where $b \in \mathbb{R}^n$, A a $n \times n$ positive semi-definite symmetric matrix, and

$$\int_{\mathbb{R}^n \setminus \{0\}} \min\{1, y^2\}\nu(dy) < \infty.$$

The converse also holds.

Lévy processes

Decomposition

The jump of X_t at time s is denoted by $\Delta X_s = X_s - X_{s-}$.

For $t \geq 0$, a Borel subset $A \subseteq \mathbb{R}^n$, we define the jump measure of for X_t by

$$\begin{aligned} N(t, A) &= \text{the number of jumps during time } [0, t] \text{ of size in } A \\ &= |\{s \in [0, t] : \Delta X_s \in A\}|. \end{aligned}$$

Note that this is a Poisson random measure with intensity $dt \otimes \nu$.

One can decompose X_t into continuous part and jump part:

$$\begin{aligned} X_t &= X_t^c + \sum_{s:0 \leq s \leq t} \Delta X_s \\ &= bt + G(t) + \int_{|x| \geq 1} x N(t, dx) + \int_{|x|<1} x \widetilde{N}(t, dx) \\ &= \text{(Gaussian with drift)} + \text{(Pure jump Lévy)} \end{aligned}$$

where $\widetilde{N}(t, A) := N(t, A) - t\nu(A)$ is the compensated jump measure.

Lévy processes

Semigroup and generator

Let X_t be a Lévy process with (b, A, ν).

The semigroup $P_t f(x) := \mathbb{E}^x[f(X_t)]$ and the infinitesimal generator

$$\begin{aligned}\mathcal{L}f(x) &= \lim_{t\downarrow 0} \frac{P_t f(x) - f(x)}{t} \\ &= -b \cdot \nabla f(x) + \frac{1}{2} \sum_{i,j=1}^{n} A_{ij} \partial_{ij} f(x) \\ &\qquad + \int_{\mathbb{R}^n} (f(x+y) - f(y) - y \nabla f(x) \cdot 1_{B_1(0)}(y))\, \nu(dy).\end{aligned}$$

We focus on a pure jump Lévy process (the case $b = 0, A = 0$).

Itô's formula

Setting

Let $(Z_t)_{t\geq 0}$ be a stochastic process defined by

$$Z_t = Z_0 + M_t + A_t + \int_0^t \int_{\mathbb{R}^n} G(s,x)\, N(ds,dx) + \int_0^t \int_{\mathbb{R}^n} H(s,x)\, \widetilde{N}(ds,dx)$$

where

(1) M_t: a continuous square integrable local martingale

(2) A_t: a continuous adapted process of bounded variation with $A_0 = 0$

(3) N_t: a jump measure of a Lévy process X_t with ν.

(4) $G(t,x) = (G_1(t,x), \cdots, G_n)$ and $H(t,x) = (H_1(t,x), \cdots, H_n(t,x))$: n-dimensional predictable processes such that $G_i(t,x)H_j(t,x) = 0$,

$$\int_0^t \int_{\mathbb{R}^n} |G_i(t,x)|\, N(ds,dx) < \infty \qquad \text{a.s.},$$

$$\mathbb{E}[\int_0^t \int_{\mathbb{R}^n} |H_i(s\wedge\tau_n,x)|^2\, \nu(dx)ds] < \infty,$$

for all $t > 0$, $i,j = 1,2,\cdots,n$, where $\tau_n \to \infty$.

Itô's formula

Statement

If φ is $C^2(\mathbb{R}^n)$ and $H(t,x)$ satisfies

$$\sup_{0\leq t\leq T}\sup_{x\in\mathbb{R}^n}|H(t,x)|<\infty,$$

then we have

$$\begin{aligned}
\varphi(Z_t) = \varphi(Z_0) &+ \int_0^t \nabla\varphi(Z_s)\cdot dM_s \\
&+ \int_0^t \nabla\varphi(Z_s)\cdot dA_s + \frac{1}{2}\sum_{i,j=1}^n \int_0^t \partial_{ij}\varphi(Z_s)d\langle M_i, M_j\rangle_s \\
&+ \int_0^t\int_{\mathbb{R}^n} \left(\varphi(Z_{s-}+G)-\varphi(Z_{s-})\right) N(ds,dx) \\
&+ \int_0^t\int_{\mathbb{R}^n} \left(\varphi(Z_{s-}+H)-\varphi(Z_{s-})\right) \widetilde{N}(ds,dx) \\
&+ \int_0^t\int_{\mathbb{R}^n} \left(\varphi(Z_{s-}+H)-\varphi(Z_{s-})-H\cdot\nabla\varphi(Z_{s-})\right)\nu(dx)ds.
\end{aligned}$$

Key step: Application of Itô's formula

Let X_t be a pure jump Lévy process and P_t the corresponding semigroup.

Let $Y_t = P_{T-t}f(X_t)$, then Itô's formula yields

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}^n} H(s,y)\, \widetilde{N}(ds,dy)$$

where $H(t,x) = P_{T-t}f(X_{t-}+x) - P_{T-t}f(X_{t-})$.

(Note: Since $P_{T-t}f(x)$ is time-dependent, it requires additional work to apply Itô's formula.)

Again, we apply Itô's formula for $\varphi(x) = |x|^p$ to obtain

$$\begin{aligned}|Y_t|^p - |Y_0|^p = &\int_0^t \int_{\mathbb{R}^n} (|Y_{s-} + H(s,y)|^p - |Y_{s-}|^p)\, \widetilde{N}(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}^n} (|Y_{s-} + H(s,y)|^p - |Y_{s-}|^p - p|Y_{s-}|^{p-2} H(s,y) \cdot Y_{s-})\, \nu(dy) ds.\end{aligned}$$

Taking $\int_{\mathbb{R}^n} \mathbb{E}_x[\cdot]\, dx$ and letting $T \to \infty$, we obtain the result.

Hardy–Stein type identity for a martingale

As we have seen in the proof, once we have

$$Y_t = Y_0 + \int_0^t \int_{\mathbb{R}^n} H(s,y)\, \widetilde{N}(ds,dy),$$

we can apply Itô to obtain the identity. This leads to the following.

Theorem (Bañuelos, K. 2019)

Let $1 < p < \infty$, ν a Lévy measure, and $H(t,x)$ be a n-dimensional predictable process satisfying

$$\mathbb{E}[\int_0^t \int_{\mathbb{R}^n} |H_i(s \wedge \tau_n, x)|^2\, \nu(dx)ds] < \infty, \qquad \sup_{0 \le t \le T} \sup_{x \in \mathbb{R}^n} |H(t,x)| < \infty.$$

If

$$M_t = M_0 + \int_0^t \int_{\mathbb{R}^n} H(s,y)\, \widetilde{N}(ds,dy),$$

is uniformly integrable in $L^2 \cap L^p$, then

$$\mathbb{E}|M_\infty|^p - \mathbb{E}|M_0|^p = \int_0^\infty \int_{\mathbb{R}^n} \mathbb{E}[F(M_{s-}, M_{s-} + H(s,y))]\, \nu(dy)ds.$$

Where is Hartman–Wintner condition used?

Recall that Hartman–Wintner condition is

$$\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{\log|\xi|} = \infty$$

where $\psi(\xi) = \int_{\mathbb{R}^n} (1 - e^{i\xi\cdot y} + i\xi \cdot y 1_{B_1}(y))\,\nu(dy)$.

It follows that $P_{T-t}f(x) \in C^2$ (we need this for Itô's formula).

Another consequence is the ultracontractivity of P_t, that is,

$$\|P_t f\|_\infty \le C_t^{\frac{1}{p}} \|f\|_p$$

with $\lim_{t\to\infty} C_t = 0$. This provides

(1) $|Y_0|^p \to 0$ as $T \to 0$ in the proof;

(2) $H(t,x)$ is uniformly bounded, which is required in the Itô's formula.

Application to Square functions and Fourier multiplier

For square functions, it turns out that the symmetry assumption on ν is necessary.

The reason is because

$$\widetilde{G}(f)(x) = \left(\int_0^\infty \int_{\{|P_t f(x)| > |P_t f(x+y)|\}} (P_t f(x+y) - P_t f(x))^2 \, \nu(dy) dt \right)^{\frac{1}{2}}$$

and $\{|P_t f(x)| > |P_t f(x+y)|\}$ does not work well with non-symmetric ν.

In order to bypass this difficulty, we used the symmetrization argument motivated by [Bañuelos, Bielaszewski, Bogdan 2011].

Consider $\widetilde{X}_t = X_{\frac{t}{2}} + \widehat{X}_{\frac{t}{2}}$ where $\widehat{X}_t$ is the dual process and $\widetilde{\nu}(B) = \frac{1}{2}(\nu(B) + \nu(-B))$.

Then, we have square function bounds with respect to $\widetilde{X}$ and $\widetilde{\nu}$ by [Bañuelos, Bogdan, Luks 2016].

The L^p-boundedness of the Fourier multiplier with respect to ν can be derived using the symmetrized square function.

Main result

Theorem (Bañuelos, K. 2019)

For a bounded function ϕ, let

$$\Lambda_\phi(f,g)(x) = \iiint (P_t f(x+y) - P_t f(x))(P_t g(x+y) - P_t g(x))\phi(t,y)\,\nu(dy)dtdx.$$

Then,

(i) $\Lambda_\phi(f,g)$ is absolutely convergent for $f \in L^2 \cap L^p$ and $g \in L^2 \cap L^q$;

(ii) there is a unique linear operator S_ϕ on L^p such that $\Lambda_\phi(f,g) = \langle S_\phi(f), g\rangle$ and $S_\phi = T_m$ where

$$m(\xi) = \int_0^\infty \int_{\mathbb{R}^n} |e^{i\xi\cdot y} - 1|^2 e^{-2tRe(\psi(\xi))}\phi(t,y)\,\nu(dy)dt.$$

Thank you!