HW10 Solutions in Math 55 Fall 2016

with Professor Stankova

Problems from: Rosen, *Discrete Mathematics and Its Applications*, 7th edition. ISBN 978–0–07–338309–5.

Section 7.2

Exercise 2.

Let a denote the probability of rolling a number aside from 3. Then $p(3) = 2a$, and we have in particular that

$$p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = a + a + 2a + a + a + a = 7a = 1$$

so that $a = 1/7$. Thus the probability of rolling any number aside from 3 is $a = 1/7$, and the probability of rolling a 3 is $2a = 2/7$.

Exercise 6.

It is probably easiest to just list out the permutations and note which permutations have which of the desired properties:

Permutation	1 prec. 3	3 prec. 1	3 prec. 1 and 3 prec. 2
123	X		
132	X		
213	X		
231		X	
312		X	X
321		X	X
Count	3	3	2

Since there are 6 permutations in total and each permutation is equally likely, the probability that 1 precedes 3 is $3/6 = 1/2$, and similarly (by symmetry, in fact) the probability that 3 precedes 1 is $3/6 = 1/2$. The probability that 3 precedes both 1 and 2 is $2/6 = 1/3$.

Exercise 8.

Now we need to be a bit more clever in counting permutations that satisfy the desired properties. For parts a) and b), note that in any permutation, either 1 precedes 2 or 2 precedes 1, these possibilities are mutually exclusive, and by symmetry there are an equal number of permutations of each type. Thus exactly half of the $n!$ permutations are of the first type, and exactly half of them are of the second type, so the probability of each outcome is exactly $1/2$.

For part c), to count the number of permutations in which 1 immediately precedes 2, we can consider them as a block that moves together, and then count permutations on $n-1$ objects, the numbers 3 through n and the "number" 01 which we treat as a single object. Thus there are $(n-1)!$ such permutations in total, and the probability of 1 immediately preceding 2 is just $(n-1)!/n! = 1/n$.

For part d), note that since $n \geq 4$, the numbers n and $n-1$ are distinct from the numbers 1 and 2. Now we can count the number of permutations where n precedes 1 and $n-1$ precedes 2 by choosing independently the positions in the string representation of the permutation where n and 1 and where $n-1$ and 2 will be located, and the choosing a permutation on the remaining $n-4$ numbers. Thus we have

$$\binom{n}{2}\binom{n-2}{2}(n-4)! = \frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}(n-4)! = \frac{n!}{4}$$

different possibilities. Dividing by the total number of permutations gives a probability $1/4$.

Finally, for part e), notice that we can independently choose the locations of 1, 2 and n, the permutation of these three numbers, and the permutation of the remaining $n-3$ numbers. There are $C(n,3)$ ways to choose the locations of 1, 2 and n in the permutations, 2 permutations of these three numbers in which n precedes both 1 and 2, and $(n-3)!$ ways to choose a permutation on the remaining numbers. This gives

$$2\binom{n}{3}(n-3)! = \frac{2n(n-1)(n-2)(n-3)!}{6} = \frac{n!}{3}$$

Thus by dividing by the total number of permutations, we see that this event has probability $1/3$.

Exercise 12.

Note that for any E and F, we have $E \cup F = E \cup (F \setminus E) = F \cup (E \setminus F)$, where both of the latter two expressions are disjoint unions. In particular, recall that if U and V are disjoint events, then the probability of their union is the sum of their probabilities. Thus

$$p(E \cup F) = p(E \cup (F \setminus E)) = p(E) + p(F \setminus E) \geq p(E)$$

and

$$p(E \cup F) = p(F \cup (E \setminus F)) = p(F) + p(E \setminus F) \geq p(F)$$

Thus $p(E \cup F) \geq \max(p(E), p(F))$. This proves the first part, since in our case, $\max(p(E), p(F)) = 0.8$.

For the second part, note that

$$p(E \cap F) = p(E) + p(F) - p(E \cup F) \geq p(E) + p(F) - 1$$

since $-p(E \cup F) \geq -1$. In particular, in our case, $p(E) + p(F) - 1 = 0.4$, so this proves the desired bound.

Exercise 13.

The second part of exercise 12 proves this fact to obtain the desired bounds on $p(E \cap F)$.

Exercise 15.

We prove Boole's inequality by induction on the number of events in the union. For just one event, the inequality is obvious, since

$$p(E_1) \leq p(E_1)$$

This proves the base case for our induction. For the inductive step, suppose as the inductive hypothesis that the inequality holds when taking a union of k events from the sample space, where $k \geq 1$. Then we have

$$\begin{aligned}
&p\big(E_1 \cup E_2 \cup \cdots \cup E_k \cup E_{k+1}\big) \\
&= p\big((E_1 \cup E_2 \cup \cdots \cup E_k) \cup E_{k+1}\big) \\
&= p(E_1 \cup E_2 \cup \cdots \cup E_k) + p(E_{k+1}) - p((E_1 \cup E_2 \cup \cdots \cup E_k) \cap E_{k+1}) \\
&\leq p(E_1 \cup E_2 \cup \cdots \cup E_k) + p(E_{k+1}) \\
&\overset{\text{IH}}{\leq} p(E_1) + p(E_2) + \cdots + p(E_k) + p(E_{k+1})
\end{aligned}$$

this completes the inductive step, since it proves Boole's inequality for $k+1$ events under the assumption that the inequality holds for k events. By mathematical induction, we conclude that Boole's inequality holds for n events for any $n \geq 1$.

Exercise 16.

By taking complements and using the independence of E and F, we see that

$$\begin{aligned}
&p(\overline{E} \cap \overline{F}) \\
&= 1 - p(\overline{\overline{E} \cap \overline{F}}) \\
&= 1 - p(E \cup F) \\
&= 1 - (p(E) + p(F) - p(E \cap F)) \\
&= 1 - p(E) - p(F) + p(E \cap F) \\
&= 1 - p(E) - p(F) + p(E)p(F) \\
&= (1 - p(E))(1 - p(F)) \\
&= p(\overline{E})p(\overline{F})
\end{aligned}$$

This proves that $\overline{E}$ and $\overline{F}$ are independent.

Exercise 18.

To answer the problem, it is helpful to answer the related question: Given n people chosen at random, what is the probability p_n that none of them were born on the same day of the week? At least two people being born on the same day of the week out of a group of n is the negation of this event, so the probability we want will be $1 - p_n$.

To determine the probability that n people chosen at random were all born on different days of the week, we go through the people in sequence and consider the possible days of the week which haven't already been taken by a previous person. The first person clearly can't conflict with a previous person's day, so for $n = 1$, the probability is $p_1 = 1$. For $n = 2$, the second person can't have the first person's day of the week, so there's a 6 in 7 chance of no conflict, i.e. $p_2 = p_1 \cdot (6/7) = 6/7$. Similarly for the third person, since two days of the week have already been taken, there's now a 5 in 7 chance of no conflict with the previous two people, so the new probability is the probability that the first two people don't have a conflict times the probability that the last person's day of the week doesn't conflict with the previously taken days of the week. Thus $p_3 = p_2 \cdot (5/7) = (7 \cdot 6 \cdot 5)/7^3$. Continuing this pattern, we see that

$$p_n = \frac{7 \cdot 6 \cdots (7 - (n - 1))}{7^n}$$

for $n = 1, 2, \ldots, 7$. In fact, the formula also works for $n \geq 8$, since it will always give a value of 0, which is the correct probability since by the pigeonhole principle there will always be two people born on the same day of the week.

In particular, for this problem, the probability that two people chosen at random were born on the same day of the week is $1 - p_1 = 1 - 6/7 = 1/7$, the probability that n people chosen at random were born on the same day of the week is $1 - p_n$, and by a quick computation we can see that $1 - p_3$ is slightly less than 0.39, but $1 - p_4$ is more than 0.65, so 4 people need to be chosen at random to make the probability that two were born on the same day of the week greater than 1/2.

Exercise 20.

Similarly to number 18, the probability p_n of no two people out of a group of n randomly chosen people having the same birthday is given by

$$p_n = \frac{366 \cdot 365 \cdots (366 - (n - 1))}{366^n}$$

So we're looking for the smallest integer n such that $1 - p_n > 1/2$. We can find this directly using a computer. The following Python code is a simple brute-force method of finding the desired value:

```
def p(bins, n):
    prob = 1.0
    for i in range(0,n):
        prob = prob * float(bins - i) / bins
    return prob

n = 1
while (1 - p(366, n)) <= 0.5:
    n = n + 1

print "Smallest n:"
print n
```

This reveals that the minimum number of people needed in order for the probability that two of them have the same birthday to be greater than 1/2 is $n = 23$ people. The probability in this case is 0.506.

Exercise 24.

The two events in question are E, the event that exactly four heads appear when a fair coin is flipped five times, and F, the event that the first coin flip came up tails. The conditional probability we want is $p(E \mid F) = p(E \cap F)/p(F)$. We can see that $p(E \cap F) = 1/32$, since there is exactly one outcome where exactly four heads came up and the first flip was also tails, and there are 32 equally likely possibilities in total.

Likewise, we can compute $p(F) = 16/32$ by counting 16 possible outcomes for the remaining coin flips after the first comes up tails. Thus the conditional probability is just

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/32}{16/32} = \frac{1}{16}$$

Exercise 26.

There are four bit strings of length three with an odd number of 1s, four bit strings of length three which start with a 1, and two bit strings of length three which both start with a 1 and have an odd number of 1s. Thus $p(E) = 4/8 = 1/2$, $p(F) = 4/8 = 1/2$, and $p(E \cap F) = 2/8 = 1/4 = 1/2 \cdot 1/2 = p(E)p(F)$. This shows that the two events are in fact independent.

Exercise 28.

We will liberally make use of Theorem 2, which gives the probability for a given number successes in a sequence of independent Bernoulli trials. For part a), we can compute the probability directly by

$$p_a = \binom{5}{3}(0.51)^3(0.49)^2 \approx 0.318$$

For part b), notice that at least one boy is the complement of no boys, so we compute the probability that the family consists of all girls:

$$p_b^{(1)} = \binom{5}{0}(0.51)^0(0.49)^5 \approx 0.028$$

Then the probability we want is just $p_b = 1 - p_b^{(1)} \approx 0.972$.

For part c), the same strategy works, but switching the roles of the girls and the boys:

$$p_c^{(1)} = \binom{5}{5}(0.51)^5(0.49)^0 \approx 0.035$$

Then similarly, the probability we want is $p_c = 1 - p_c^{(1)} \approx 0.965$.

Finally, for part d) we've already computed the necessary parts in parts b) and c), namely $p_b^{(1)}$ computes the probability of having all girls in the family, and $p_c^{(1)}$ computes the probability of having all boys. Thus the probability of having all children the same sex is just $p_d = p_b^{(1)} + p_c^{(1)} \approx 0.063$.

Exercise 30.

Since the bits are independent, the probability that a string contains no 0s is given by the product of probabilities that each bit is a 1. That is, if b_i representes the ith bit, then the probability we want is

$$p = \prod_{i=1}^{10} p(b_i = 1)$$

For part a), if $p(b_i = 1) = 1/2$ for each i, then $p = 1/2^{10} = 1/1024$.

For part b), if $p(b_i = 1) = 0.6$ for each i, then $p = (0.6)^{10} \approx 0.00605$.

For part c), if $p(b_i = 1) = 1/2^i$ for each i, then we have

$$p = \prod_{i=1}^{10} p(b_i = 1) = \prod_{i=1}^{10} 2^{-i} = 2^{-\sum_{i=1}^{10} i} = 2^{-45} \approx 3.5 \cdot 10^{-13}$$

Exercise 34.

As in number 28, the results here make liberal use of Theorem 2. For part a) the probability is given directly by

$$\binom{n}{0} p^0 (1-p)^n = (1-p)^n$$

The event for part b) is the complement of that for part a), so the probability is just

$$1 - (1-p)^n$$

For part c), the event includes both the outcomes with exactly 0 successes, and the outcomes with exactly 1 success. This can be computed by

$$\binom{n}{0} p^0 (1-p)^n + \binom{n}{1} p^1 (1-p)^{n-1} = (1-p)^n + np(1-p)^{n-1}$$

Finally, for part d), the event of outcomes with at least two successes is the complement of the event in part c), so the probability is the complement of the one in part c), or

$$1 - (1-p)^n - np(1-p)^{n-1}$$

Exercise 14.

We use mathematical induction on the number n of events in the intersection. Clearly if $n = 1$ the inequality holds because $p(E_1) \geq p(E_1) - (1 - 1)$. Now for the inductive step, let $k \geq 1$, and suppose as an inductive hypothesis that the inequality holds for an intersection of any k events. Then we can use the two-event version of Bonferroni's inequality from problem number 13, along with the inductive hypothesis, to see that

$$\begin{aligned}
&p\big(E_1 \cap E_2 \cap \cdots \cap E_k \cap E_{k+1}\big) \\
&= p\big((E_1 \cap E_2 \cap \cdots \cap E_k) \cap E_{k+1}\big) \\
&\geq p(E_1 \cap E_2 \cap \cdots \cap E_k) + p(E_{k+1}) - 1 \\
&\overset{\text{IH}}{\geq} p(E_1) + p(E_2) + \cdots + p(E_k) - (k-1) + p(E_{k+1}) - 1 \\
&= p(E_1) + p(E_2) + \cdots + p(E_k) + p(E_{k+1}) - ((k+1) - 1)
\end{aligned}$$

This proves the inequality for an intersection of $k+1$ events, and completes the inductive step. By mathematical induction, we can conclude that this generalization of Bonferroni's inequality holds for an intersection of any finite collection of events.

Exercise 32.

For all of the parts of this problem, it is convenient to restate the event being considered in terms of two events:

$$E = \{\text{bit strings starting with a 1}\}, \quad F = \{\text{bit strings ending with a 00}\}$$

The event we want to find probabilities for is $E \cup F$. These events are not disjoint, however, so we will rewrite the union as a disjoint union as follows

$$E \cup F = E \cup (F \setminus E)$$

Then E represents all of the bit strings which start with a 1, and F represents all of the bit strings which end with a 00, but also do not start with a 1, so in particular also start with a 0. Now since the above union is now disjoint, the probability can be computed by adding the probabilities of the two disjoint events. If b_i denotes the value of the ith bit in the bitstrings being considered, then we can write

$$\begin{aligned}
&p(E \cup (F \setminus E)) \\
&= p(E) + p(F \setminus E) \\
&= p(b_1 = 1) + p(b_1 = 0) \cdot p(b_9 = 0) \cdot p(b_{10} = 0)
\end{aligned}$$

From this we can compute the probabilities for the different conditions directly.

For part a), if $p(b_i = 1) = p(b_i = 0) = 1/2$, we get $p = 1/2 + (1/2)^3 = 5/8$.

For part b), if $p(b_i = 1) = 0.6$ and $p(b_i = 0) = 0.4$, we get $p = 0.6 + (0.4)^3 = 0.664$.

For part c), if $p(b_i = 1) = 1/2^i$ and $p(b_i = 0) = 1 - 1/2^i$, we get

$$\begin{aligned}
p &= p(b_1 = 1) + p(b_1 = 0) \cdot p(b_9 = 0) \cdot p(b_{10} = 0) \\
&= \frac{1}{2} + \frac{1}{2} \cdot (1 - 2^{-9}) \cdot (1 - 2^{-10}) \approx 0.9985
\end{aligned}$$

Exercise 38.

The information the honest observer gives allows you to consider conditional probabilities, conditioned on what the observer said about the event. For part a), we want to compute the conditional probability of the event E of dice rolls adding up to seven, conditioned on the event F that at least one of the dice is a 6. To do this, we need to compute the probabilities $p(E \cap F)$ and $p(F)$. Namely, there are 11 distinct dice rolls which have at least one 6, and out of those 11, exactly 2 have a sum of 7. Thus

$$p = p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{2/36}{11/36} = \frac{2}{11}$$

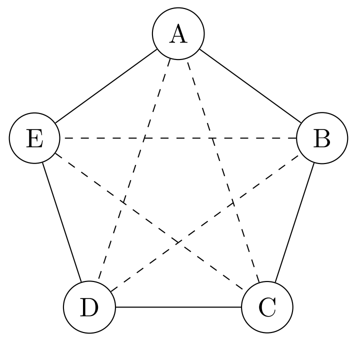
For part b), the computation is similar. However, in this case we instead let F be the event that at least one of the dice is 5. As before, there are 11 distinct dice rolls which have at least one 5, and also as before, exactly 2 of these 11 have a sum of 7. Thus the computation of the conditional probability is the same, and the probability is again p = 2/11.

Another potential interpretation of this problem is that the honest observer now additionally requires that one of the dice came up 5, as well as at least one of them coming up 6. In this case, since the only two possibilities in the conditioning event F are $(6, 5)$ and $(5, 6)$, none of these possibilities have a sum of 7, so the conditional probability would be zero.

Section 6.2

Exercise 26.

The following diagram of the relations between five people gives an example of how a group of five people may not have a group of three mutual friends or a group of three mutual enemies. Solid lines between people indicate friends, and dashed lines indicate enemies (or vice-versa since the relative roles of enemies and friends is symmetric).



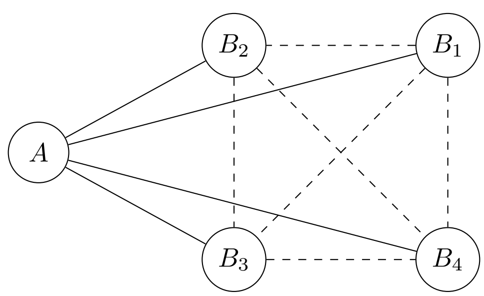
Exercise 27.

We will prove that there are either three mutual friends or four mutual enemies; the second part is proven analogously since the role of friends and enemies in this problem is symmetric.

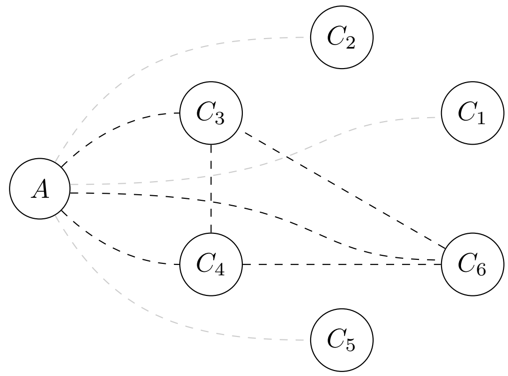
If there are three mutual friends among the ten people, then we are done. So suppose that no group of three people has everyone being mutual friends. Single out one person, A. By the (generalized) pigeonhole principle, it must be the case that A is either friends with four people, or enemies with six people (since he has a friend or an enemy relationship with the 9 other people in the group).

Suppose that A is friends with four people, B_1, B_2, B_3, and B_4. If it were the case that B_i was friends with B_j for any $i \neq j$, then we would have that A, B_i and B_j would be a group of three mutual friends. Since we've assumed that no such group exists, it must be the case that B_1, B_2, B_3, and B_4 are a group of four mutual enemies, so we see that this case works.

(As in the previous problem, solid lines indicate friends, and dashed lines indicate enemies.)



Now suppose that instead, A is enemies with six people, $C_1, C_2, \ldots, C_6$. We have seen that any group of six people either has a group of three mutual friends, or a group of three mutual enemies. Since we are assuming that there are not three mutual friends, there must be three mutual enemies, C_i, C_j, and C_k. Then since A is enemies with all of the C_i, the group of A, C_i, C_j, and C_k is a group of four mutual enemies, so we see that this case works as well.



This covers all of the cases, so we see that in general, in a group of ten people who are each friends or enemies, there must either be a group of three mutual friends, or a group of four mutual enemies.

Exercise 28.

As in Exercise 27, pick a particular person A. Now by the generalized pigeonhole principle, it is either the case that A has 10 friends in the group of 20, or A has 10 enemies. If A has 10 friends, $B_1, B_2, \ldots, B_{10}$, then by the result of Exercise 27, the ten friends have among them either four mutual enemies, or three mutual friends. If they have four mutual enemies, then we are done. If instead they have three mutual friends, say B_i, B_j, and B_k, then the group of A, B_i, B_j, and B_k forms a group of four mutual friends since A is by assumption friends with each of the others. Thus in this case we are also done.

However, if A has 10 enemies, an analogous argument also works, since the result of Exercise 27 is proven both for four mutual friends/three mutual enemies and for three mutual friends/four mutual enemies. Thus in all possible cases, we see that there must either be a group of four mutual friends or a group of four mutual enemies, and this proves the result.

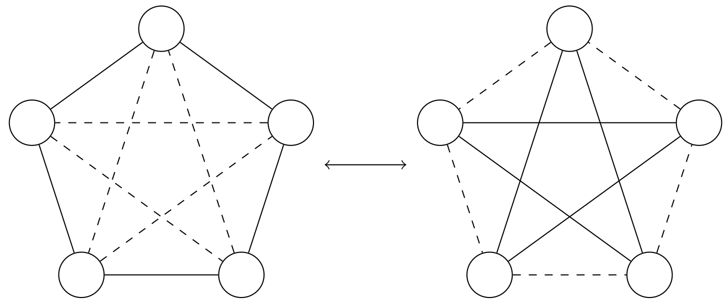
Exercise 29.

$R(2, n)$ denotes the minimum number of people at a party such that there are either 2 mutual friends or n mutual enemies, assuming that every pair of people at the party are friends or enemies. But this is very easy to compute with these numbers. $n-1$ mutual enemies at a party (what a party!) means that there are not 2 mutual friends or n mutual enemies. This counterexample shows that $R(2, n) \geq n$.

However, if n people are at a party, then either 2 of them are friends, or each pair of them are enemies, meaning that the whole group consists of n mutual enemies. This proves that any group of n people at a party must either have 2 mutual friends or n mutual enemies. This shows that $R(2, n) \leq n$. Together with the previous example, we see that $R(2, n)$ must be equal to n.

Exercise 30.

Call a party with r people such that no group of m people are mutual friends and no group of n people are mutual enemies an (m, n)-counterexample of size r. Ramsey numbers are symmetric with respect to the two parameters ($R(m, n) = R(n, m)$) because given any (m, n)-counterexample, we can form an (n, m)-counterexample by changing all the friend relations into enemy relations, and all the enemy relations into friend relations.



In particular, suppose by way of contradiction that $R(m,n) > R(n,m)$. Then in particular, since $R(m,n)$ is the smallest integer r such that there is no (m,n)-counterexample of size r, there must be an (m,n)-counterexample of size $R(m,n)-1$, and this can be restricted to give an (m,n)-counterexample of size $R(n,m)$. But then by the above construction, this (m,n)-counterexample of size $R(n,m)$ can be transformed into an (n,m)-counterexample of size $R(n,m)$. This is a contradiction, since $R(n,m)$ is by definition the smallest positive integer such that no such counterexample exists. Thus we must have $R(m,n) \leq R(n,m)$, and by a similar argument, we see that $R(n,m) \leq R(m,n)$. This proves equality of the two Ramsey numbers.

Section 7.3

Exercise 2.

A common version of Bayes' Theorem states that

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E)}$$

This version is reached halfway through the proof of Theorem 1, and in general the formulas for the denominator of Bayes' Theorem all are different representations of $p(()E)$.

In particular, for our problem, we can apply this version of Bayes' Theorem (here with the roles of E and F reversed) to get

$$p(E \mid F) = \frac{p(F \mid E)p(E)}{p(F)} = \frac{(5/8)(2/3)}{3/4} = \frac{5}{9}$$

Exercise 4.

Here we will use Bayes' Theorem as stated in Theorem 1. E is the event that Ann picked an orange ball, F_1 is the event that Ann picked from the first box, and F_2 is the event that Ann picked from the second box. Note that $F_1 = \overline{F_2}$. Thus

$$p(F_2 \mid E) = \frac{p(E \mid F_2)p(F_2)}{p(E \mid F_2)p(F_2) + p(E \mid F_1)p(F_1)} = \frac{(5/11)(1/2)}{(5/11)(1/2) + (3/7)(1/2)} = \frac{5/22}{34/77} = \frac{35}{68}$$

Exercise 6.

Here, we let F be the event that a player takes steroids, and let E be the event that a player tests positive. Then we have

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})} = \frac{(0.98)(0.05)}{(0.98)(0.05) + (0.12)(0.95)} = \frac{0.049}{0.163} \approx 0.3006$$

Thus the probability that a player testing positive actually has taken steroids is only about 30%!

Exercise 8.

For this problem, we will let F be the event that someone has the rare genetic disease, and E be the event that someone tests positive for the disease using our excellent test. The probabilities given can be interpreted as $p(F) = 1/10000$, $p(E \mid F) = 999/1000$, and $p(E \mid \overline{F}) = 2/10000$.

For part a), we have

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})} = \frac{\frac{999}{1000} \cdot \frac{1}{10000}}{\frac{999}{1000} \cdot \frac{1}{10000} + \frac{2}{10000} \cdot \frac{9999}{10000}} = \frac{999/10^7}{29988/10^8} = \frac{555}{1666} \approx 0.3331$$

Thus the probability that somebody has the genetic disease given a positive test result is only about 1 in 3.

For part b), we instead want to compute $p(\overline{F} \mid \overline{E})$, noting that conditional probability in general satisfies the property that $p(\overline{A} \mid B) = 1 - p(A \mid B)$. Then we have

$$p(\overline{F} \mid \overline{E}) = \frac{p(\overline{E} \mid \overline{F})p(\overline{F})}{p(\overline{E} \mid \overline{F})p(\overline{F}) + p(\overline{E} \mid F)p(F)} = \frac{\frac{9998}{10000} \cdot \frac{9999}{10000}}{\frac{9998}{10000} \cdot \frac{9999}{10000} + \frac{1}{1000} \cdot \frac{1}{10000}} = \frac{(9998 \cdot 9999)}{(9998 \cdot 9999) + 10} \approx 1 - 10^{-8}$$

Thus the probability of a negative test result being incorrect is around one in ten million; a negative test result will in general be highly accurate.

Exercise 10.

As usual, we will apply Bayes' theorem. Here, we let F be the event that a patient is infected with avian influenza, and E be the event that a patient tests positive for avian influenza.

For part a), we compute

$$p(F \mid E) = \frac{p(E \mid F)p(F)}{p(E \mid F)p(F) + p(E \mid \overline{F})p(\overline{F})} = \frac{0.97 \cdot 0.04}{0.97 \cdot 0.04 + 0.02 \cdot 0.96} = \frac{0.0388}{0.058} \approx 0.669$$

Thus the probability of a patient testing positive for avian influenza actually being infected is just slightly over 2/3.

For part b), notice that $p(\overline{F} \mid E) = 1 - p(F \mid E)$, so the probability of not having avian influenza given a positive test is just under 1/3.

For part c), we see that

$$p(F \mid \overline{E}) = \frac{p(\overline{E} \mid F)p(F)}{p(\overline{E} \mid F)p(F) + p(\overline{E} \mid \overline{F})p(\overline{F})} = \frac{0.03 \cdot 0.04}{0.03 \cdot 0.04 + 0.98 \cdot 0.96} = \frac{0.0012}{0.942} \approx 0.00127$$

If a patient tests negative, the probability that they are infected falls from 4% to around 0.13%.

Part d) is again looking for the complementary probability, so the probability of not being infected given a negative test is 1 minus the probability from the above, or around 99.87%.

Exercise 14.

This is a straightforward application of the generalized version of Bayes' Theorem.

$$\begin{aligned} p(F_2 \mid E) &= \frac{p(E \mid F_2)p(F_2)}{p(E \mid F_1)p(F_1) + p(E \mid F_2)p(F_2) + p(E \mid F_3)p(F_3)} \\ &= \frac{(3/8)(1/2)}{(2/7)(1/6) + (3/8)(1/2) + (1/2)(1/3)} = \frac{3/16}{45/112} = \frac{7}{15} \end{aligned}$$

Exercise 18.

Letting S denote the event that an email message is spam, and letting E denote the event that an email message contains the word "exciting", we estimate the probability $p(S \mid E)$ using Bayes' Theorem, along with the estimates for conditional probabilities given by the training sample of emails and the naive assumption that spam messages are as likely as non-spam messages.

$$p(S \mid E) = \frac{p(E \mid S)p(S)}{p(E \mid S)p(S) + p(E \mid \overline{S})p(\overline{S})} \approx \frac{\frac{40}{500} \cdot \frac{1}{2}}{\frac{40}{500} \cdot \frac{1}{2} + \frac{25}{200} \cdot \frac{1}{2}} = \frac{16}{41} < \frac{9}{10}$$

Thus if the threshold for reject spam is 0.9, we would not reject a message containing the word "exciting".

Exercise 20.

Let S denote the event that an email message is spam, U the event that an email message contains the word "undervalued", and T the event that an email message contains the word "stock". For part a), again using the assumption that spam and non-spam email is equally likely (hopefully pessimistic?), we see

$$p(S \mid U) = \frac{p(U \mid S)p(S)}{p(U \mid S)p(S) + p(U \mid \overline{S})p(\overline{S})} \approx \frac{\frac{200}{2000} \cdot \frac{1}{2}}{\frac{200}{2000} \cdot \frac{1}{2} + \frac{25}{1000} \cdot \frac{1}{2}} = 4/5 < 9/10$$

Thus we find that the word "undervalued" by itself is not suspect enough to reject at the threshold of 0.9.

For part b), we calculate

$$p(S \mid T) = \frac{p(T \mid S)p(S)}{p(T \mid S)p(S) + p(T \mid \overline{S})p(\overline{S})} \approx \frac{\frac{400}{2000} \cdot \frac{1}{2}}{\frac{400}{2000} \cdot \frac{1}{2} + \frac{60}{1000} \cdot \frac{1}{2}} = 10/13 < 9/10$$

Again we find that the single word, this time "stock", is by itself not suspect enough to reject at the threshold of 0.9.

Exercise 21.

Let S denote the event that an email message is spam, E the event that an email message contains the word "enhancement", and H the event that an email message contains the word "herbal". We make the same assumptions as in Example 4 about independence of the events, conditional independence, and equal likelihood of spam versus non-spam. Then we estimate

$$p(S \mid E \cap H) = \frac{p(E \mid S)p(H \mid S)}{p(E \mid S)p(H \mid S) + p(E \mid \overline{S})p(H \mid \overline{S})} \approx \frac{\frac{1500}{10000} \cdot \frac{800}{10000}}{\frac{1500}{10000} \cdot \frac{800}{10000} + \frac{20}{5000} \cdot \frac{200}{5000}} = 75/76 > 9/10$$

Here, we find that the message will be rejected as spam at a threshold of 0.9.

Exercise 23.

Your author struggled with this problem for over an hour before realizing that the result is in fact *not true* as stated.

Before we give a counterexample, let us be more clear about what the problem is specifying. First, we require that E_1 and E_2 be independent events. Second, we require that $E_1 \mid S$ and $E_2 \mid S$ are independent, or in other words, that

$$p(E_1 \cap E_2 \mid S) = p(E_1 \mid S)p(E_2 \mid S)$$

Finally, the statement that "we have no prior knowledge regarding whether or not the message is spam" can be interpreted probabilistically as: $p(S) = p(\overline{S}) = 1/2$.

The following table describes a set of probabilities which provides a counterexample to the proposed identity.

Set	Probability
$E_1 \cap E_2 \cap S$	$2/36$
$\overline{E_1} \cap E_2 \cap S$	$4/36$
$E_1 \cap \overline{E_2} \cap S$	$4/36$
$\overline{E_1} \cap \overline{E_2} \cap S$	$8/36$
$E_1 \cap E_2 \cap \overline{S}$	$7/36$
$\overline{E_1} \cap E_2 \cap \overline{S}$	$5/36$
$E_1 \cap \overline{E_2} \cap \overline{S}$	$5/36$
$\overline{E_1} \cap \overline{E_2} \cap \overline{S}$	$1/36$

You can check that this is a probability distribution (the disjoint probabilities add up to 1) such that

a. $p(E_1) = p(E_2) = p(S) = 1/2$

b. $p(E_1 \cap E_2) = 1/4 = p(E_1)p(E_2)$

c. $p(E_1 \mid S) = p(E_2 \mid S) = 1/3$

d. $p(E_1 \cap E_2 \mid S) = 1/9 = p(E_1 \mid S)p(E_2 \mid S)$

Thus the distribution satisfies all of the necessary conditions in the problem statement. However, we can compute

$$p(S \mid E_1 \cap E_2) = \frac{p(E_1 \cap E_2 \cap S)}{p(E_1 \cap E_2)} = \frac{1/18}{1/4} = \frac{2}{9}$$

Also,

$$\frac{p(E_1 \mid S)p(E_2 \mid S)}{p(E_1 \mid S)p(E_2 \mid S) + p(E_1 \mid \overline{S})p(E_2 \mid \overline{S})} = \frac{(1/3)(1/3)}{(1/3)(1/3) + (2/3)(2/3)} = \frac{1/9}{5/9} = \frac{1}{5}$$

Thus the two sides of the desired identity are not equal in this case, so the proposed result is not true in general.

Note that the reason the book solution to this odd-numbered exercise is an invalid argument is because of the final sentence: "Because of the assumed independence of E_1, E_2, and S, we have $p(E_1 \cap E_2 \mid S) = p(E_1 \mid S) \cdot p(E_2 \mid S)$, and similarly for $\overline{S}$." The statement that $E_1 \mid S$ and $E_2 \mid S$ are independent does not imply that $E_1 \mid \overline{S}$ and $E_2 \mid \overline{S}$ are! This is the case even with the other restrictions of the problem, as the given counterexample demonstrates. There, we have $p(E_1 \mid \overline{S}) = p(E_2 \mid \overline{S}) = 2/3$, but $p(E_1 \cap E_2 \mid \overline{S}) = 7/18 \neq 4/9$.

If we make the additional assumption that E_1 and E_2 are independent when conditioned on $\overline{S}$, then this fixes the issue, and the proof can be conducted as discussed in the book solution by applying Bayes' theorem, and then using the conditional independence.