Parabolic focusing

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This is a technical document to ease calculations of beam parabolic focusing parameters. Let us start with Shröedinger equation

$$\mathbf{i}\frac{\partial E}{\partial z} + \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) = 0. \tag{1}$$

Solution can be derived in the following form:

$$E = A(z)\exp(-ar^2)\exp(\mathbf{i}br^2), \tag{2}$$

where $A \in \mathbb{C}$, $a, b \in \mathbb{R}$. By substituting (2) into (1):

$$\mathbf{i}A' - \mathbf{i}Aa'r^2 - Ab'r^2 - 4DA(a - \mathbf{i}b) + 4DA(a - \mathbf{i}b)^2 r^2 = 0, \tag{3}$$

we can derive the following relations:

$$a' = -8Dab \tag{4}$$

$$-b' + 4D(a^2 - b^2) = 0 \tag{5}$$

$$\mathbf{i}A' - 4DA(a - \mathbf{i}b) = 0 \tag{6}$$

From (4) and (5) we have:

$$-a''a + \frac{3}{2}a'^2 = 32D^2a^4 \tag{7}$$

$$b = -\frac{a'}{8Da} \tag{8}$$

The final solution is the following:

$$a = \frac{1}{\frac{16D^2}{R_0^2}z^2 + R_0^2} \tag{9}$$

$$b = \frac{4Dz}{R_0^2}\frac{1}{\frac{16D^2}{R_0^2}z^2 + R_0^2} \tag{10}$$

Inverse formulas read as follows:

$$R_0^2 = \frac{a}{a^2 + b^2} \tag{11}$$

$$z = \frac{1}{4D} \frac{b}{a^2 + b^2} \tag{12}$$

Let us also derive something to get R_0 from z_1 and initial radius R_1 $(1/a(z_1))$. From (9) we have:

$$R_0^2 = \frac{R_1^2}{2} - \sqrt{\frac{R_1^4}{4} - 16D^2 z_1^2}$$

Let us also rewrite for focal diameter:

$$D_0^2 = \frac{D_1^2}{2} - \sqrt{\frac{D_1^4}{4} - 256D^2 z_1^2}$$

Here R_0^2 is beam radius square $(1/a(z = 0))$ at focal point $(z = 0)$. Now let us find solution for complex anplitude A which can be rewritten in the following form: $A = |A| \exp(\mathbf{i}\phi)$. Then from (6) we have:

$$\mathbf{i}A'A * -4D|A|^2(a - \mathbf{i}b) = 0$$

$$-\mathbf{i}A *' A - 4D|A|^2(a + \mathbf{i}b) = 0$$

Then

$$\left(|A|^2\right)' - 8D|A|^2 b = 0$$

Using (4) we can derive the following:

$$\frac{\left(|A|^2\right)'}{|A|^2} + \frac{a'}{a} = 0$$

Then for $|A|^2$:

$$|A|^2 a = \text{const}$$

and for ϕ:

$$\phi' + 4Da = 0$$

1 FWHM

Let us also add convenience relations for beam width in FWHM notation. Let the field amplitude to be set in the form

$$A(r) \propto \exp\left(-\frac{r^2}{r_{A/e}^2}\right)$$

Then intensity would be

$$I(r) \propto A^2 \propto \exp\left(-2\frac{r^2}{r_{A/e}^2}\right)$$

here r_0 is field raduis at $1/e$. We want intesity radius at 1/2 level, so

$$-2\frac{r^2}{r_{A/e}^2} = -\ln(2)$$

$$r_{I/2} = \sqrt{\frac{\ln(2)}{2}}\ r_{A/e} \approx 0.588\ r_{A/e}$$

and vice versa

$$r_{A/e} \approx 1.7\ r_{I/2}$$