Wave equation

Daniil A. Fadeev

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This is a technical documentation for numerical algorithm

1 wave equation

Maxwell equations:

$$\mathrm{rot} H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c} j, \tag{1}$$

$$\mathrm{rot} E = -\frac{1}{c}\frac{\partial H}{\partial t}. \tag{2}$$

Equation for current:

$$\frac{\partial j}{\partial t} = \frac{e^2}{m} n E. \tag{3}$$

From Maxwell equations we have:

$$\mathrm{rot}\,\mathrm{rot} E + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial j}{\partial t} = 0$$

Substituting equation for current we obtain wave equation:

$$-\Delta_\perp E - \frac{\partial^2 E}{\partial z^2} + \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \frac{4\pi e^2}{mc^2} n E = 0 \tag{4}$$

If $rot = 0$ we have plasma oscillations:

$$\frac{\partial^2 E}{\partial t^2} + \frac{4\pi e^2}{m} n E = 0 \tag{5}$$

Okay, $4\pi e^2 n/m$ is a square of plasma frequency. Let us continue. Passing to new variables:

$$z = z, \tag{6}$$

$$\tau = t - \frac{z}{c}. \tag{7}$$

Using rules

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial \tau}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}$$

We obtain the following equation:

$$-\Delta_\perp E + \frac{2}{c}\frac{\partial^2 E}{\partial z \partial \tau} + \frac{4\pi e^2}{mc^2} nE = 0, \tag{8}$$

$\partial/\partial z^2$ is neglected here. Plasma density is calculated as follows:

$$\frac{\partial n}{\partial \tau} = N_0 \mathrm{w}_0 \left(\frac{|E|}{E_0}\right)^{-0.625} \exp\left(-\frac{E_0}{|E|}\right), \tag{9}$$

where N_0 is O_2 neutral density $n_0 = 10^{19}\,\mathrm{cm}^{-3}$ (O_2 ionization potential is 14.01 ev while N_2 ionization potential is 15.51 ev, air consists of 20% O_2 and 80% N_2)

2 Dimensionless units:

We uze:

$$z_0 = \frac{2}{k_0} = 0.248\,\mu\mathrm{m},\ k0 = \frac{2\pi}{\lambda} = 80553.65779\,\mathrm{cm}^{-1}$$

$$\tau_0 = \frac{T}{2\pi} = 0.41\,\mathrm{fs},\ T = \frac{\lambda}{c} - \text{optic period at } 780\,\mathrm{nm}$$

equation is written as

$$-\frac{1}{x_0^2}\Delta_\perp E + \frac{2}{c\tau_0 z_0}\frac{\partial^2 E}{\partial z \partial \tau} + n_0 \frac{4\pi e^2}{mc^2} nE = 0, \tag{10}$$

where al operators and variables without ineds are dimensionless. In calculations we use the following form of equation:

$$-2\Delta_\perp E + \frac{\partial^2 E}{\partial z \partial \tau} + nE = 0.$$

Thus

$$\frac{1}{x_0^2} = 2 \times \frac{2}{c\tau_0 z_0} = 2k_0^2$$

$$n_0 \frac{4\pi e^2}{mc^2} = k_0^2$$

So

$$x_0 = \frac{1}{\sqrt{2}k_0} = 0.087\,\mu\mathrm{m} \tag{11}$$

$$n_0 = \frac{\omega^2 m}{4\pi e^2} = 2 \times 10^{21}\,\mathrm{cm}^{-3} \tag{12}$$

In numerical scheme equation for plasma density reads as follows:

$$\frac{\partial n}{\partial \tau} = \left(\frac{|E|}{E_0}\right)^{-0.625} \exp\left(-\frac{E_0}{|E|}\right),$$

Thus

$$\frac{n_0}{\tau_0} = N_0 \mathrm{w}_0$$

So

$$\mathrm{w}_0 = \frac{n_0}{\tau_0 N_0} = 2.4\,\mathrm{fs}^{-1}$$