CÁLCULO DIFERENCIAL E INTEGRAL I

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EXERCÍCIOS

RESOLVIDOS

EXERCÍCIO RESOLVIDO

Resolva as integrais:

$$grau\ P(x) \geq grau\ G(x)$$

1) $$\int \frac{x^3 - 3x + 2}{x + 2}\,dx = \int (x^2 - 2x + 1)\,dx = \frac{x^3}{3} - 2\frac{x^2}{2} + x + C = \frac{x^3}{3} - x^2 + x + C$$

$$\begin{array}{rrr|l}
x^3 & & -3x + 2 & \; x + 2 \\ \cline{4-4}
\underline{-x^3 - 2x^2} & & & x^2 - 2x + 1 \\
& -2x^2 & -3x & \\
& \underline{+2x^2} & \underline{+4x} & \\
& & x + 2 & \\
& & \underline{-x - 2} & \\
& & 0 &
\end{array}$$

2) $$\int \frac{x^3 + x}{x - 1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x - 1}\right)dx = \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2ln|x - 1| + C$$

$$\begin{array}{rrr|l}
x^3 & & +x & \; x - 1 \\ \cline{4-4}
\underline{-x^3 + x^2} & & & x^2 + x + 2 \\
& x^2 & +x & \\
& \underline{-x^2} & \underline{+x} & \\
& & 2x & \\
& & \underline{-2x + 2} & \\
& & 2 &
\end{array}$$

3) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx$

Neste caso o $grau\ P(x) < grau\ G(x),$ então vamos fatorar o denominador

$Fatorando\ \ o\ denominador, obtemos: 2x^3+3x^2-2x = x(2x-1)(x+2)$

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}dx = \int \frac{A}{x}dx + \int \frac{B}{2x-1}dx + \int \frac{C}{x+2}dx$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2} = \frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{x(2x-1)(x+2)}$$

Eliminando os denominadores temos:$x^2+2x-1 = A(2x^2+3x-2)+B(x^2+2x)+C(2x^2-x)$

3.

$$x^2 + 2x - 1 = (2A + B + 2C)x^2 + (3A + 2B - C)x - 2A$$

Obtemos o sistema $\begin{cases} 2A + B + 2C = 1 \\ 3A + 2B - C = 2 \\ -2A = -1 \end{cases} \rightarrow A = \frac{1}{2}, B = \frac{1}{5}, C = -\frac{1}{10}$

$$logo. \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx = \int \frac{A}{x} dx + \int \frac{B}{2x - 1} dx + \int \frac{C}{x + 2} dx$$

$$= \frac{1}{2} \int \frac{dx}{x} + \frac{1}{5} \int \frac{dx}{2x - 1} - \frac{1}{10} \int \frac{dx}{x + 2}$$

$$nlnA = lnA^n$$
$$lnA + lnB = \ln(A.B)$$
$$lnA - lnB = ln\frac{A}{B}$$

$$= \frac{1}{2} \ln(x) + \frac{1}{5} \ln(2x - 1) - \frac{1}{10} \ln(x + 2) + \mathrm{C}$$

4) $\displaystyle\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx$

Dividindo $\frac{P(x)}{G(x)}$, obtemos: $$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1}$$

$$Fatorando\ \ o\ denominador, obtemos: x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1) = (x-1)^2(x+1)$$

Decompondo em frações parciais

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\frac{4x}{x^3 - x^2 - x + 1} = \frac{A(x-1)(x+1) + B(x+1) + C(x-1)^2}{(x-1)^2(x+1)} = \frac{(A+C)x^2 + (B-2C)x + (-A+B+C)}{(x-1)^2(x+1)}$$

Eliminando os denominadores e igualando os polinômios

$$4x = (A+C)x^2 + (B-2C)x + (-A+B+C)$$

4.

Obtemos o sistema
$$\begin{cases} A + C = 0 \\ B - 2C = 4 \\ -A + B + C = 0 \end{cases} \rightarrow A = 1, B = 2, C = -1$$

e podemos resolver a integral

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1} = x + 1 + \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

$$\frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \frac{4x}{x^3 - x^2 - x + 1} = x + 1 + \frac{1}{(x-1)} + \frac{2}{(x-1)^2} + \frac{-1}{(x+1)}$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} dx = \int x dx + \int dx + \int \frac{1}{(x-1)} dx + \int \frac{2}{(x-1)^2} dx - \int \frac{1}{(x+1)} dx$$

$$= \frac{x^2}{2} + x + \ln(x-1) - \frac{2}{x-1} - \ln(x+1) + C$$

5) $\int \frac{2x^2 - x + 4}{x^3 + 4x} dx$

Como $x^3 + 4x = x(x^2 + 4)$ não pode mais ser fatorado, escrevemos

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$2x^2 - x + 4 = A(x^2 + 4) + (Bx + C)x = (A + B)x^2 + Cx + 4A$$

$$\begin{cases} A + B = 2 \\ C = -1 \\ 4A = 4 \end{cases} \rightarrow A = 1, B = 1, C = -1$$

$$\int \frac{2x^2 - x + 4}{x(x^2 + 4)} dx = \int \frac{dx}{x} + \int \frac{xdx}{x^2 + 4} - \int \frac{dx}{x^2 + 4}$$

$$= \ln(x) + \frac{1}{2}\ln(x^2 + 4) - \frac{1}{2} arctg\left(\frac{x}{2}\right) + C$$

6) $\int \frac{x^3 + 3x - 1}{x^4 - 4x^2} dx$

$Fatorando\ \ o\ denominador, obtemos\ x = 2, x = -2, x = 0\ com\ multiplicidade\ 2$

$$\frac{x^3 + 3x - 1}{x^4 - 4x^2} = \frac{x^3 + 3x - 1}{(x-2)(x+2)x^2}$$

Decompondo em frações parciais

$$\frac{x^3 + 3x - 1}{(x-2)(x+2)x^2} = \frac{A_1}{x-2} + \frac{A_2}{x+2} + \frac{B_1}{x^2} + \frac{B_2}{x}$$

Eliminando os denominadores e igualando os polinômios

$$x^3 + 3x - 1 = (x+2)x^2 A_1 + (x-2)x^2 A_2 + (x-2)(x+2)B_1 + (x-2)(x+2)xB_2$$

$$= A_1(x^3 + 2x^2) + A_2(x^3 - 2x^2) + B_1(x^2 - 4) + B_2(x^3 - 4x)$$

6.

$$= A_1(x^3 + 2x^2) + A_2(x^3 - 2x^2) + B_1(x^2 - 4) + B_2(x^3 - 4x)$$

$$x^3 + 3x - 1 = (A_1 + A_2 + B_2)x^3 + (2A_1 - 2A_2 + B_1)x^2 - 4B_2x - 4B_1$$

Obtemos o sistema

$$\begin{cases} A_1 + A_2 + B_2 = 1 \\ 2A_1 - 2A_2 + B_1 = 0 \\ -4B_2 = 3 \\ -4B_1 = -1 \end{cases} \to A_1 = \frac{13}{16}, A_2 = \frac{15}{16}, B_1 = \frac{1}{4}, B_2 = -\frac{3}{4}$$

$$\int \frac{x^3 + 3x - 1}{x^4 - 4x^2} dx = \frac{13}{16}\int \frac{dx}{x-2} + \frac{15}{16}\int \frac{dx}{x+2} + \frac{1}{4}\int \frac{dx}{x^2} - \frac{3}{4}\int \frac{dx}{x}$$

$$= \frac{13}{16} ln|x - 2| + \frac{15}{16} ln|x + 2| - \frac{1}{4x} - \frac{3}{4} ln|x| + C$$

$$7.\int \frac{x-8}{x^3-4x^2+4x}dx = \frac{3}{x-2} + ln\left[\frac{(x-2)^2}{x^2}\right] + C$$

$$raízes\ de\ \ x^3 - 4x^2 + 4x \rightarrow x = 2(raiz\ \ dupla), x = 0(raiz\ simples)$$

$$nlnA = lnA^n$$
$$lnA + lnB = \ln(A.B)$$
$$lnA - lnB = ln\frac{A}{B}$$

$$x = 2 \xrightarrow{fatores} \frac{A}{x-2} + \frac{B}{(x-2)^2}\ x = 0 \xrightarrow{fatores} \frac{C}{x}$$

$$\frac{x-8}{x^3-4x^2+4x} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x}$$

$$\frac{x-8}{x^3-4x^2+4x} = \frac{Ax^2 - 2Ax + Bx + Cx^2 - 4Cx + 4C}{(x-2)^2x} = \frac{(A+C)x^2 + (-2A+B-4C)x + 4C}{(x-2)^2x}$$

$$\begin{cases} A + C = 0 \\ -2A + B - 4C = 1 \rightarrow A = 2, B = -3, C = -2 \\ 4C = -8 \end{cases}$$

$$\int \frac{x-8}{x^3-4x^2+4x}dx = \int \frac{A}{x-2}dx + \int \frac{B}{(x-2)^2}dx + \int \frac{C}{x}dx = \int \frac{2}{x-2}dx + \int \frac{-3}{(x-2)^2}dx + \int \frac{-2}{x}dx =$$

$$= 2\ln(x-2) + \frac{3}{x-2} - 2\ln(x) + C = \frac{3}{x-2} + ln\left[\frac{(x-2)^2}{x^2}\right] + C$$

8. $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}$

O integrando contém $x^{1/2}\ e\ \ x^{1/3}$, portanto fazemos a substituição $y = x^{1/6}, \quad x = y^6, \quad dx = 6y^5dy$

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} = \int \frac{(y^6)^{\frac{1}{2}}}{1+(y^6)^{\frac{1}{3}}}(6y^5)dy = 6\int \frac{y^8}{1+y^2}dy$$

$$\frac{y^8}{1+y^2} \rightarrow como\ o\ grau\ do\ numerador\ \text{é}\ maior\ que\ o\ grau\ do\ denominador\ fazemos\ primeiro\ a\ divisão$$

$$\frac{y^8}{1+y^2} = y^6 - y^4 + y^2 - 1 + \frac{1}{1+y^2}$$

$$\int \frac{\sqrt{x}}{1+\sqrt[3]{x}} = 6\int \left(y^6 - y^4 + y^2 - 1 + \frac{1}{1+y^2}\right)dy$$

$$= \frac{6}{7}y^7 - \frac{6}{5}y^5 + 2y^3 - 6y + 6\text{arctg}(y) + C$$

Como $y = x^{1/6}$

$$= \frac{6}{7}x^{7/6} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6arctg(x^{\frac{1}{6}}) + C$$

9. $\int \frac{dx}{\sqrt{x^3} - 3\sqrt{x} + 2} = \int \frac{dx}{x^{3/2} - 3x^{\frac{1}{2}} + 2}$

fazemos a seguinte substituição

$x = y^n, onde\ n = m.m.c\ (2,2) = 2 \Rightarrow x = y^2, portanto\ dx = 2ydy$

$$\int \frac{dx}{x^{3/2} - 3x^{\frac{1}{2}} + 2} = \int \frac{2ydy}{(y^2)^{2/3} - 3(y^2)^{\frac{1}{2}} + 2} = 2\int \frac{ydy}{y^3 - 3y + 2} = 2\int \frac{ydy}{(y-1)^2 + (y+2)} =$$

$$\frac{y}{(y-1)^2(y+2)} = \frac{A}{(y-1)^2} + \frac{B}{y-1} + \frac{C}{+2} \Rightarrow \frac{y}{(y-1)^2(y+2)} = \frac{A(y+2) + B(y-1)(y+2) + C(y-1)^2}{(y-1)^2(y+2)}$$

$$y = Ay + 2A + By^2 + By - 2B + Cy^2 - 2Cy + C$$

$$y = (B + C)y^2 + (A + B - 2C)y + 2A - 2B + C \Rightarrow \begin{cases} B + C = 0 \\ A + B - 2C = 1 \\ 2A - 2B + C = 0 \end{cases}$$

$$A = \frac{1}{3},\ B = \frac{2}{9},\ C = -\frac{2}{9}.$$

9:

$$\int \frac{dx}{\sqrt{x^3} - 3\sqrt{x} + 2} = 2\int \frac{ydy}{(y-1)^2(y+2)} = 2\left[\frac{1}{3}\int \frac{dy}{(y-1)^2} + \frac{2}{9}\int \frac{dy}{y-1} - \frac{2}{9}\int \frac{dy}{y+2}\right]$$

$$= \frac{2}{3}\int (y-1)^2 dy + \frac{4}{9}\int \frac{dy}{y-1} - \frac{4}{9}\int \frac{dy}{y+2}$$

$$= \frac{2}{3} . \frac{(y-1)^{-1}}{(-1)} + \frac{4}{9}\ln(y-1) - \frac{4}{9}\ln(y+2) + C$$

$$= -\frac{2}{3(y-1)} + \frac{4}{9}\ln\left(\frac{y-1}{y+2}\right) + C \qquad lnA - lnB = ln\frac{A}{B}$$

Considerando que $y = \sqrt{x}$, obtemos:

$$\int \frac{dx}{\sqrt{x^3} - 3\sqrt{x} + 2} = -\frac{2}{3(\sqrt{x}-1)} + \frac{4}{9}\ln\left(\frac{\sqrt{x}-1}{\sqrt{x}+2}\right) + C$$

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EXERCÍCIOS PROPOSTOS

Resolva as seguintes integrais:

1. $\int \frac{2x^3}{x^2+x}dx = x^2 - 2x + 2\ln(x+1) + C$

2. $\int \frac{x^2+5x+4}{x^2-2x+1}dx = x + 7\ln(x-1) - \frac{10}{x-1} + C$

3. $\int \frac{x^3+2x^2+4}{2x^2+2}dx = \frac{x^2}{4} + x - \frac{1}{4}\ln(x^2+1) + arctgx + C$

4. $\int \frac{dx}{x^3+9x} = \frac{1}{9}\left(\ln(x) - \frac{1}{2}ln(x^2+9)\right) + C$

5. $\int \frac{dx}{x^3-4x^2} = \frac{1}{16}ln\left(\frac{x-4}{x}\right) + \frac{1}{4x} + C$

6. $\int \frac{x^4-x^3-3x^2-2x+2}{x^3+x^2-2x}dx = \frac{x^2}{2} - 2x - ln(x) + 3ln(x+2) - ln(x-1) + c$

7. $$\int \frac{\left(\sqrt{x^3} + \sqrt{x} - 2x\right)}{\sqrt{x} - 1} dx = \frac{x^2}{2} - \frac{2}{3} \sqrt{x^3} + C$$

8. $$\int \frac{\sqrt{x}\, dx}{\sqrt{x^3} + x} = 2\ ln\ (\sqrt{x} + 1) + \mathrm{c}$$

9. $$\int \frac{dx}{x\sqrt{1 + x}} = ln \left(\frac{\sqrt{1 + x} - 1}{\sqrt{1 + x} + 1}\right) + \mathrm{c}$$

10. $$\int \frac{\sqrt{x - 1}}{x} dx = 2\sqrt{x - 1} - 2\mathrm{arctg}\left(\sqrt{x - 1}\right) + C$$