Financial Numerical Recipes in C++.

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June 2014

Contents

This book is a a discussion of the calculation of specific formulas in finance. The field of finance has seen a rapid development in recent years, with increasing mathematical sophistication. While the formalization of the field can be traced back to the work of Markowitz (1952) on investors mean-variance decisions and Modigliani and Miller (1958) on the capital structure problem, it was the solution for the price of a call option by Black and Scholes (1973); Merton (1973) which really was the starting point for the mathematicalization of finance. The fields of derivatives and fixed income have since then been the main fields where complicated formulas are used. This book is intended to be of use for people who want to both understand and use these formulas, which explains why most of the algorithms presented later are derivatives prices.

This project started when I was teaching a course in derivatives at the University of British Columbia, in the course of which I sat down and wrote code for calculating the formulas I was teaching. I have always found that implementation helps understanding these things. For teaching such complicated material it is often useful to actually look at the implementation of how the calculation is done in practice. The purpose of the book is therefore primarily pedagogical, although I believe all the routines presented are correct and reasonably efficient, and I know they are also used by people to price real options.

To implement the algorithms in a computer language I choose C++. My students keep asking why anybody would want to use such a backwoods computer language, they think a spreadsheet can solve all the worlds problems. I have some experience with alternative systems for computing, and no matter what, in the end you end up being frustrated with higher end "languages", such as *Matlab* og *R* (Not to mention the straitjacket which is is a spreadsheet.) and going back to implementation in a standard language. In my experience with empirical finance I have come to realize that nothing beats knowledge a **real** computer language. This used to be `FORTRAN`, then `C`, and now it is **C++**. All example algorithms are therefore coded in C++. I do acknowledge that matrix tools like `Matlab` are very good for rapid prototyping and compact calculations, and will in addition to C++ in places also illustrate the use of `Matlab`, as well as other (public domain) tools.

The manuscript has been sitting on the internet a few of years, during which it has been visited by a large number of people, to judge by the number of mails I have received about the routines. The present (2014) version mainly expands on the background discussion of the routines, it is more extensive, (but it does not replace a real textbook). I have also added some introductory material on how to program in C++, since a number of questions make it obvious this manuscript is used by a number of people who know finance but not C++. All the routines have been made to confirm to the new ISO/ANSI C++ standard, using such concepts as namespaces and the standard template library. The latest (2011) C++ standard introduced a few useful simplifications, which is incorporated in places.

The current manscript therefore has various intented audiences. Primarily it is for students of finance who desires to see a complete discussion and implementation of some formula. But the manuscript is also useful for students of finance who wants to learn C++, and for computer scientists who want to understand about the finance algorithms they are asked to implent and embed into their programs.

In doing the implementation I have tried to be as generic as possible in terms of the C++ used, but I have taken advantage of a some of the possibilities the language provides in terms of abstraction and modularization. This will also serve as a lesson in why a *real* computer language is useful. For example I have encapsulated the term structure of interest rate as an example of the use of `classes`.

This is not a textbook in the underlying theory, for that there are many good alternatives. For much of the material the best textbooks to refer to are Hull (2011) and McDonald (2013), which I have used as references. The notation of the present manuscipt is also similar to these books.

Chapter 1

On C++ and programming.

Contents

In this chapter I introduce C++ and discuss how to run programs written in C++. This is by no means a complete reference to programming in C++, it is designed to give enough information to understand the rest of the book. This chapter also only discusses a subset of C++, it concentrates on the parts of the language used in the remainder of this book. For really learning C++ a textbook is necessary. I have found Lippman and Lajoie (1998) an excellent introduction to the language.[1] The authorative source on the language is Stroustrup (1997b).

1.1 Compiling and linking

To program in C++ one has to first write a separate file with the program, which is then *compiled* into low-level instructions (machine language) and *linked* with libraries to make a complete executable program. The mechanics of doing the compiling and linking varies from system to system, and we leave these details as an exercise to the reader.

[1] I learned C++ from the previous edition of the book, Lippman (1992). From what I can tell the present editions still seems like a good way of learning the language, but C++ has changed a lot in recent years.

1.2 The structure of a C++ program

The first thing to realize about C++ is that it is a strongly typed language. Everything must be declared before it is used, both variables and functions. C++ has a few basic building blocks, which can be grouped into types, operations and functions.

1.2.1 Types

The types we will work with in this book are `bool`, `int`, `long`, `double` and `string`.

Here are some example definitions

```
bool this_is_true=true;
int i = 0;
long j = 123456789;
double pi = 3.14159265358979323846264643;
string s("this is a string");
```

The most important part of C++ comes from the fact that these basic types can be expanded by use of *classes*, of which more later.

1.2.2 Operations

To these basic types the common mathematical operations can be applied, such as addition, subtraction, multiplication and division:

```
int i = 100 + 50;
int j = 100 - 50;
int n = 100 * 2;
int m = 100 / 2;
```

These operations are defined for all the common datatypes, with exception of the `string` type. Such operations can be defined by the programmer for other datatypes as well.

Increment and decrement In addition to these basic operations there are some additional operations with their own shorthand. An example we will be using often is incrementing and decrementing a variable. When we want to increase the value of one item by one, in most languages this is written:

```
int i=0;
i = i+1;
i = i-1;
```

In `C++` this operation has its own shorthand

```
int i=0;
i++;
i--;
```

While this does not seem intuitive, and it is excusable to think that this operation is not really necessary, it does come in handy for more abstract data constructs. For example, as we will see later, if one defines a `date` class with the necessary operations, to get the next date will simply be a matter of

```
date d(1,1,1995);
d++;
```

These two statements will result in the date in `d` being 2jan95.

1.2.3 Functions and libraries

In addition to the basic mathematical operations there is a large number of additional operations that can be performed on any type. However, these are not parts of the core language, they are implemented as standalone functions (most of which are actually written in `C` or C++). These functions are included in the large *library* that comes with any C++ installation. Since they are not part of the core language they must be defined to the compiler before they can be used. Such definitions are performed by means of the *include* statement.

For example, the mathematical operations of taking powers and performing exponentiation are defined in the mathematical library `cmath`. In the C++ program one will write

```
#include <cmath>
```

`cmath` is actually a file with a large number of function defintions, among which one finds `pow(x,n)` which calculates x^n, and `exp(r)` which calculates e^r. The following programming stub calculates $a = 2^2$ and $b = e^1$.

```
#include <cmath>
double a = pow(2,2);
double b = exp(1);
```

which will give the variables `a` and `b` values of 4 and 2.718281828..., respectively.

1.2.4 Templates and libraries

The use of libraries is not only limited to functions. Also included in the standard library is generic data structures, which can be used on any data type. The example we will be considering the most is the `vector<>`, which defines an array, or vector of variables.

```
#include <vector>
vector<double> M(2);
M[0]=1.0;
M[1]=2.0;
M.push_back(3);
```

This example defines an array with three elements of type double

$$M = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Note some pecularities here. When first defining the vector with the statement

```
vector<double> M(2);
```

we defined an array of 2 elements of type `double`, which we then proceeded to fill with the values 1 and 2. When filling the array we addressed each element directly. Note that in the statement

```
M[0]=1.0;
```

lies one of the prime traps for programmers coming to `C` or C++ from another language. Indexing of arrays starts at zero, not at one. `M[0]` really means the *first* element of the array.

The last statement,

```
M.push_back(3);
```

shows the ability of the programmer of changing the size of the array after it has been defined. `push_back` is a standard operation on arrays which "pushes" the element onto the back of the array, extending the size of the array by one element. Most programming languages do not allow the programmer to specify variable-sized arrays "on the fly." In `FORTRAN` or `Pascal` we would usually have to set a maximum length for each array, and hope that we would not need to exceed that length. The `vector<>` template of C++ gets rid of the programmers need for "bookkeeping" in such array manipulations.

1.2.5 Flow control

To repeat statements several times one will use on of the possibilities for flow control, such as the `for` or `while` constucts. For example, to repeat an operation `n` times one can use the following `for` loop:

```
for (int i=0; i<n; i++) {
   some_operation(i);
};
```

The `for` statement has tree parts. The first part gives the initial condition (`i=0`). The next part the terminal condition (`i<n`), which says to stop when `i<n` is not fulfilled, which is at the `n`'th iteration. The last part is the increment statement (`i++`), saying what to do in each iteration. In this case the value of `i` is increased by one in each iteration. This is the typical `for` statement. One of the causes of `C`'s reputation for terseness is the possibility of elaborate `for` constructs, which end up being almost impossible to read. In the algorithms presented in this book we will try to avoid any obfuscated `for` statements, and stick to the basic cases.

1.2.6 Input Output

For any program to do anything useful it needs to be able to output its results. Input and output operations is defined in a couple of libraries, `iostream` and `fstream`. The first covers in/output to standard terminals and the second in/output to files.

To write to standard output `cout` (the terminal), one will do as follows:

```
#include <iostream>
cout << "This is a test" << endl;
```

To write to a file `"test.out"`, one will do as follows:

```
#include <fstream>
ofstream outf;
outf.open("test.out");
outf << "This is a test" << endl;
outf.clear();
outf.close();
```

1.2.7 Splitting up a program

Any nontrivial program in C++ is split into several pieces. Usually each piece is written as a function which returns a value of a given type. To illustrate we provide a complete example program, shown in **C++ Code 1.1**.

```
#include <iostream> // input output operations
#include <cmath>   // mathematics library
using namespace std;  // the above is part of the standard namespace

double power(double x, double n){
    // define a simple power function
    double p = exp(n*log(x));
    return p;
};

int main(){
    for (int n=1;n<6;n++){
        cout << " 2^" << n << " = " << power(2,n) << endl;
    };
};
```

C++ Code 1.1: A complete program

The program defines a function performing the mathematical power operation, `power(x,n)` which calculates x^n through the identity $x^n = e^{n\ln(x)}$. This function is then used to calculate and print the first 5 powers of 2.

When compiled, linked and run, the program will provide the following output

```
 2^1 = 2
 2^2 = 4
 2^3 = 8
 2^4 = 16
 2^5 = 32
```

1.2.8 Namespaces

To help in building large programs, the concept of a `namespace` was introduced. Namespaces are a means of keeping the variables and functions defined local to the context in which they are used. For now it is necessary to know that any function in the standard C++ library lies in its own namespace, called the standard namespace. To actually access these library functons it is necessary to explicitly specify that one wants to access the standard namespace, by the statement

```
using namespace std;
```

Instead of such a general approach, one can also specify the namespace on an element by element basis, but this is more a topic for specialized C++ texts, for the current purposes we will allow all routines access to the whole standard namespace.

1.3 Extending the language, the `class` concept.

One of the major advances of C++ relative to other programming languages is the programmers ability to extend the language by creating new data types and defining standard operations on these data types. This ability is why C++ is called an object oriented programming language, since much of the work in programming is done by creating *objects*. An object is best though of as a data structure with operations on it defined. How one uses an object is best shown by an example.

1.3.1 `date`, an example class

Consider the abstract concept of a date. A date can be specified in any number of ways. Let us limit ourselves to the Gregorian calendar. 12 august 2003 is a common way of specifying a date. However, it can also be represented by the strings: "2003/8/12", "12/8/2003" and so on, or by the number of years since 1 january 1900, the number of months since January, and the day of the month (which is how a UNIX programmer will think of it).

However, for most people writing programs the representation of a date is not relevant, they want to be able to enter dates in some abstract way, and then are conserned with such questions as:

- Are two dates equal?
- Is one date earlier than another?
- How many days is it between two dates?

A C++ programmer will proceed to use a *class* that embodies these uses of the concept of a date. Typically one will look around for an extant class which has already implemented this, but we will show a trivial such date class as an example of how one can create a class.

```
class date {
protected:
    int year_;
    int month_;
    int day_;
public:
    date();
    date(const int& d, const int& m, const int& y);

    bool valid() const;

    int day()   const;
    int month() const;
    int year()  const;

    void set_day   (const int& day   );
    void set_month (const int& month );
    void set_year  (const int& year );

    date operator ++(); // prefix
    date operator ++(int); // postfix
    date operator --(); // prefix
    date operator --(int); // postfix
};

bool operator == (const date&, const date&); // comparison operators
bool operator != (const date&, const date&);
bool operator < (const date&, const date&);
bool operator > (const date&, const date&);
bool operator <= (const date&, const date&);
bool operator >= (const date&, const date&);
```

Header file 1.1: Defining a `date` class

A class is defined in a header file, as shown in **Header file 1.1**. As internal representation of the date is chosen the three integers `day_`, `month_` and `year_`. This is the data structure which is then manipulated by the various functions defined below.

The functions are used to

- Create a date variable: `date(const int& d, const int& m, const int& y);`
- Functions outputting the date by the three integer functions `day()`, `month()` and `year()`.
- Functions setting the date `set_day(int)`, `set_month(int)` and `set_year(int)`, which are used by providing an integer as arguments to the function.
- Increment and decrement functions ++ and -
- Comparison functions <, <=, >, >=, == and `!`-.

After including this header file, programmers using such a class will then treat an object of type `date` just like any other.

For exmple,

```
date d(1,1,2001);
++d;
```

would result in the `date` object `d` containing the date 2 january 2001.

Any C++ programmer who want to use this `date` object will only need to look at the header file to know what are the possible functions one can use with a `date` object, and be happy about not needing to know anything about how these functions are implemented. This is the encapsulation part of object oriented programming, all relevant information about the `date` object is specified by the header file. This is the only point of interaction, all details about implementation of the class objects and its functions is not used in code using this object. In fact, the user of the class can safely ignore the class' privates, which is only good manners, anyway.

Let us look at the implementation of this.

C++ Code 1.2 defines the basic operations, initialization, setting the date, and checking whether a date is valid.

```
#include "date.h"

date::date(){ year_ = 0; month_ = 0; day_ = 0;};

date::date(const int& day, const int& month, const int& year){
    day_ = day;
    month_ = month;
    year_ = year;
};

int date::day() const { return day_;  };
int date::month() const { return month_; };
int date::year()  const { return year_;  };

void date::set_day (const int& day)    { date::day_ = day; };
void date::set_month(const int& month) { date::month_ = month; };
void date::set_year (const int& year)  { date::year_ = year; };

bool date::valid() const {
// This function will check the given date is valid or not.
// If the date is not valid then it will return the value false.
//  Need some more checks on the year, though
    if (year_ <0)                 return false;
    if (month_>12 || month_<1) return false;
    if (day_>31 || day_<1)    return false;
    if ((day_==31 && ( month_==2 || month_==4 || month_==6 || month_==9 || month_==11) ) )
        return false;
    if ( day_==30 && month_==2) return false;
    // should also check for leap years, but for now allow for feb 29 in any year
    return true;
};
```

C++ Code 1.2: Basic operations for the date class

For many abstract types it can be possible to define an ordering. For dates there is the natural ordering. **C++ Code 1.3** shows how such comparison operations is defined.

```
#include "date.h"

bool operator == (const date& d1,const date& d2){ // check for equality
    if (! (d1.valid() && (d2.valid())) ) { return false; }; /* if dates not valid, not clear what to do.
                                                              alternative: throw exception */
    return ((d1.day()==d2.day()) && (d1.month()==d2.month()) && (d1.year()==d2.year()));
};

bool operator < (const date& d1, const date& d2){
    if (! (d1.valid() && (d2.valid())) ) { return false; }; // see above remark
    if (d1.year()==d2.year()) { // same year
        if (d1.month()==d2.month()) { // same month
            return (d1.day()<d2.day());
        }
        else {
            return (d1.month()<d2.month());
        };
    }
    else { // different year
        return (d1.year()<d2.year());
    };
};

// remaining operators defined in terms of the above

bool operator <=(const date& d1, const date& d2){
  if (d1==d2) { return true; }
  return (d1<d2);
}

bool operator >=(const date& d1, const date& d2) {
    if (d1==d2) { return true;};
    return (d1>d2);
};

bool operator > (const date& d1, const date& d2) { return !(d1<=d2);};

bool operator !=(const date& d1, const date& d2){ return !(d1==d2);}
```

C++ Code 1.3: Comparison operators for the date class

C++ Code 1.4 shows operations for finding previous and next date, called an *iteration* operator.

```
#include "date.h"

date next_date(const date& d){
    if (!d.valid()) { return date(); }; //
    date ndat=date((d.day()+1),d.month(),d.year()); // first try adding a day
    if (ndat.valid()) return ndat;
    ndat=date(1,(d.month()+1),d.year()); // then try adding a month
    if (ndat.valid()) return ndat;
    ndat = date(1,1,(d.year()+1));       // must be next year
    return ndat;
}

date previous_date(const date& d){
    if (!d.valid()) { return date(); }; // return the default date
    date pdat = date((d.day()-1),d.month(),d.year()); if (pdat.valid()) return pdat; // try same month
    pdat = date(31,(d.month()-1),d.year()); if (pdat.valid()) return pdat; // try previous month
    pdat = date(30,(d.month()-1),d.year()); if (pdat.valid()) return pdat;
    pdat = date(29,(d.month()-1),d.year()); if (pdat.valid()) return pdat;
    pdat = date(28,(d.month()-1),d.year()); if (pdat.valid()) return pdat;
    pdat = date(31,12,(d.year()-1));                  // try previous year
    return pdat;
};

date date::operator ++(int){ // postfix operator
   date d = *this;
   *this = next_date(d);
   return d;
}

date date::operator ++(){ // prefix operator
   *this = next_date(*this);
   return *this;
}

date date::operator --(int){ // postfix operator, return current value
   date d = *this;
   *this = previous_date(*this);
   return d;
}

date date::operator --(){ // prefix operator, return new value
   *this = previous_date(*this);
   return *this;
};
```

C++ Code 1.4: Iterative operators for the date class

Exercise 1.1.

The function `valid()` in the date class accepts february 29'th in every year, but this should ideally only happen for leap years. Modify the function to return a `false` if the year is not a leap year.

Exercise 1.2.

A typical operating system has functions for dealing with dates, which your typical C++ implementation can call. Find the relevant functions in your implementation, and

1. Implement a function querying the operating system for the current date, and return this date.
2. Implement a function querying the operating system for the weekday of a given date, and return a representation of the weekday as a member of the set:

   ```
   {"mon","tue","wed","thu","fri","sat","sun"}
   ```

3. Reimplement the `valid()` function using a system call.

Exercise 1.3.

Once the date class is available, a number of obvious functions begs to be implemented. How would you

1. Add a given number of days to a date?
2. Go to the end or beginning of a month?
3. Find the distance betwen two dates (in days or in years)?
4. Extract a date from a string? (Here one need to make some assumptions about the format)

Exercise 1.4.

Take a look at how dates are dealt with in various computing environments, such as the operating system (`Unix`, `Windows`), applications (Spreadsheets), programming languages, etc. At what level of abstraction is the interface? Do you need to know how dates are implemented? For those with access to both `Matlab` and `Windows`, why would you say that `Matlab` has an "off-by-one" problem relative to `Windows`?

1.4 `Const` references

Let us now discuss a concept of more technical nature. Consider two alternative calls to a function, defined by function calls:

```
some_function(double r);
some_function(const double& r);
```

They both are called by an argument which is a double, and that argument is guaranteed to not be changed in the calling function, but they work differently. In the first case a copy of the variable referenced to in the argument is created for use in the function, but in the second case one uses the same variable, the argument is a *reference* to the location of the variable. The latter is more efficient, in particular when the argument is a large class. However, one worries that the variable referred to is changed in the function, which in most cases one do not want. Therefore the `const` qualifier, it says that the function can not modify its argument. The compiler will warn the programmer if an attempt is made to modify such a variable.

For efficiency, in most of the following routines arguments are therefore given as as constant references.

1.5 Other C++ concepts

A number of other C++ concepts, such as function prototypes and templates, will be introduced later in particular contexts. They only appear in a few places and is better introduced where they are used.

Chapter 2

Matrix Tools

Being computer literate entails being aware of a number of computer tools and being able to choose the most suitable tool for the problem at hand. Way to many people turns this around, and want to fit any problem to the computer tool they know. The tool that very often is *the* tool for business school students is a spreadsheet like *Excel*. Of course, a spreadsheet is very useful for very many business applications. However, it is not the best tool for more computationally intensive tasks.

While the bulk of the present book concerns itself with C++, in many applications in finance a very handy tool is a language for manipulating vectors and matrices using linear algebra. There are a lot of different possible programs that behaves very similarly, with a syntax taken from the mathematical formulation of linear algebra. An early tool of this sort was *matlab*, with a large number of programs copying much of the syntax of this program. As a result of this there is a proliferation of programs with similar syntax to `Matlab` doing similar analysis. General tools include the commercial package `Matlab` sold by Mathworks, the public domain programs `octave` and `scilab`. Tools that are similar, but more geared towards econometrics, include Gauss, Ox and *S* with its public domain "clone" `R`. As for what program to install, there is no right answer. For the basic learning of how these tools work, any of the mentioned packages will do the job. For students on a limited budget the public domain tools `octave`, `scilab` and R are obvious candidates. All of them perform the basic operations done by the commercial `Matlab` package, and good for learning the basics of such a matrix tool.

All of these tools are programs that lets the user manipulate vectors and matrices using very compact notation. While compact notation is always prone to tense, making programs using it unreadable, this is not such a large problem in `Matlab`, the notation tends to be so close how a mathematician would write them that programs can be relatively easy to follow. There are some pitfalls for the unwary user, in particular it is easy to "miss" the difference between a command operating on a whole matrix and the corresponding element by element operation. For example, consider the following short example, where the operator `^` means that the matrix `A` is taken to the power 2 (multiplied with itself), and the operator `.^` means that each element of the matrix `A` is taken to the power 2. The two commands give very different answers.

```
>> A = [1 1 ; 1 1]
A =
  1  1
  1  1
>> A^2
ans =
  2  2
  2  2
>> A.^2
ans =
```

```
  1   1
  1   1
```

The rest of this chapter gives an introduction to a tool like this.

2.1 The first screen

How you start the particular tool you are using depend both on which program and which operating system you are working on. The details of how to start it is left as an exercise to the reader.

The tools are interactive, they present you with a prompt, and expect you to start writing commands. We will

```
>>
```

as the prompt, which means that the program is ready to receive commands.

In the text output of the matrix tool will be shown typewritten as:

```
>>  A = [1, 2, 3;  4, 5, 6]
```

This particular command defines a matrix `A`, the matrix tool will respond to this command by printing the matrix that was just defined:

```
A =
  1 2 3
  4 5 6
```

2.2 Linear algebra

To use such a tool you need some knowledge of linear algebra. We assume the reader have this basic knowledge, if not a quick perusal of a standard mathematical text on linear algebra is called for.

2.2.1 Basic matrix operations

In matrix algebra a set of mathematical rules are given for operating on the basic elements real numbers, vectors, and matrices. In `Matlab` the type of each variable is determined when you first define it.

```
>> a=1
a = 1
>> b=2
b = 2
>> c=3
c = 3
>> y=[1;2;3]
y =
  1
  2
  3
>> x=[1,2,3]
x =
  1  2  3
```

```
>> A=[1 2 3;4 5 6]
A =
  1  2  3
  4  5  6
```

Observe that when filling in a vector or a matrix a space or a comma means a new number, a semicolon a new row. To suppress the printing of what you just defined, end the line with a semicolon:

```
>> A=[1,2,3,4];
>> A=[1,2,3,4]
A =
  1  2  3  4
>>
```

You can also use defined variables to define new variables, as long as the dimensions make sense. For example, given the above definitions:

```
>>B=[c x]
B =
  3  1  2  3
>> C = [A;x]
C =
  1  2  3
  4  5  6
  1  2  3
>> D = [A y]
error: number of rows must match
error: evaluating assignment expression near line 22, column 3
```

If the dimensioning is wrong, you get an error message, and the variable is not defined.

To see what is in a variable, tell `Matlab` to print the value by giving the name:

```
>> a
a = 1
>> A
A =
  1  2  3
  4  5  6
```

Note that `Matlab` is case-sensitive, both `A` and `a` are defined.

2.2.2 Arithmetic Matrix Operations.

We now get to the important parts of `Matlab`, namely, its built–in matrix arithmetic. Given the definitions above, let us add and subract a few elements according to the rules of matrix algebra. We first show how to manipulate numbers and vectors:

```
>> a=1
a =  1
>> b=2
b =  2
>> a+b
ans =  3
```

```
>> x=[1 2 3 4]
x =
   1   2   3   4
>> y=[4 3 2 1]
y =
   4   3   2   1
>> x+y
ans =
   5   5   5   5
>> y-x
ans =
   3   1  -1  -3
>> a*x+b*y
ans =
   9   8   7   6
```

similarly, for matrices:

```
A=[1 2 3; 4 5 6]
A =
   1   2   3
   4   5   6
>> B=[6 5 4; 3 2 1]
B =
   6   5   4
   3   2   1
>> A+B
ans =
   7   7   7
   7   7   7
>>A-B
ans =
  -5  -3  -1
   1   3   5
>> a*A+b*B
ans =
   13   12   11
   10    9    8
> A*B'
ans =
   28   10
   73   28
```

In linear algebra, you need to be aware that matrix multiplication is not element by element multiplication, it is a much more complex operation, where all possible vector combinations are multiplied with each other. When multiplying matrices, you need to be more careful about dimensions, but in terms of notation it is just like vector multiplication.

```
>> A=[1 2 3;4 5 6]
A =
  1  2  3
  4  5  6
>> B = [1 2;3 4; 5 6]
```

```
B =
  1  2
  3  4
  5  6
>> A*B
ans =
  22  28
  49  64
>> B*A
ans =
   9  12  15
  19  26  33
  29  40  51
```

For these matrices, both $\mathbf{AB}$ and $\mathbf{BA}$ are defined operations, but note that the results are different, in fact, even the dimension of the product is different.

If we let $\mathbf{B}$ be a 2×2 matrix, then multiplying $\mathbf{AB}$ is an error.

```
>> B=[1 2;3 4]
B =
  1  2
  3  4
>> A*B
error: nonconformant matrices (op1 is 2x3, op2 is 2x2)
>> B*A
ans =
   9  12  15
  19  26  33
```

Let us now discuss some standard matrix concepts.

The transpose of a matrix `A` is found in `Matlab` as `A'`:

```
>> A
A =
  1  2  3
  4  5  6
>> A'
ans =
  1  4
  2  5
  3  6
```

Two special matrices are the `null` and `identity` matrices:

```
>> null = zeros(3,3)
null =
  0  0  0
  0  0  0
  0  0  0
>> ident = eye(3,3)
ident =
  1  0  0
  0  1  0
```

```
  0  0  1
```

The *rank* of a matrix is is the number of independent rows or columns in the matrix, and calculated as

```
>> Ahi
A =
  1  2  3
  4  5  6
>> rank(A)
ans = 2
```

The *inverse* of a square matrix `A` is the matrix `inv(A)` such that *A*inv(A)* equals the identity matrix, or in mathematical notation $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.

```
>> D=[1 2;1 4]
D =
   1   2
   1   4
>> inv(D)
ans =
   2.00000  -1.00000
  -0.50000   0.50000
>> D^-1
ans =
   2.00000  -1.00000
  -0.50000   0.50000
```

To make sure that this *is* the inverse, multiply `D` and `inv(D)`:

```
>> D * inv(D)
ans =
  1  0
  0  1
```

Determinant

```
>> B
B =
  1  2
  3  4
>> det(B)
ans = -2
```

2.3 Solving linear equations

Consider the basic linear equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

This equation has a defined solution if the rank of **A** equals the rank of $[\mathbf{A}|\mathbf{b}]$. If **A** is nonsingular, we solve the linear equation by finding the unique solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Consider the linear equation

$$3x_1 + 4x_2 = 5$$

$$4x_1 + 6x_2 = 8$$

Write this in matrix form by defining

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

Let us first check that this system is solvable

```
>> A = [3 4;4 6]
A =
  3  4
  4  6
>> b=[5;8]
b =

  5
  8
>> rank(A)
ans = 2
>> rank ([A b])
ans = 2
```

Note how to create the augmented matrix $[A|b]$ by `[A b]`. The rank of the two is the same. Since **A** is square, we can calculate the solution as

```
>> inverse(A)
ans =
   3.0000  -2.0000
  -2.0000   1.5000
>> x = inverse(A) * b
x =
  -1
   2
```

The solution to the system of equations is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

In this case we calculated the solution by finding the inverse. But you should be aware that solving the system of equations by calculation of the inverse is not the numerically most stable way of doing the calculation. `Matlab` has built in a direct linear matrix solver, which is invoked by the *left division* operator

```
>> x = A\b
x =
  -1
   2
```

This solves the system of equations directly, and it is usually the preferred way to do this operation, unless the inverse is needed for other purposes.

2.4 Element by element operations

When a command is prefixed with a period, it means the command applies to each element of a vector or matrix, not the vector or matrix.

For example, with the two vectors below, consider the difference in multiplying the two and doing an element by element multiplication:

```
> x=[1 2 3 ]
x =
   1   2   3
> t=[1 2 3]
t =
   1   2   3
>> x*t'
ans =  14
> x.*t
ans =
   1   4   9
```

Similarly, when taking the exponent of a matrix

```
> A=[1 1;1 1]
A =
   1   1
   1   1

>> A^10
ans =
   512   512
   512   512
>> A.^10
ans =
   1   1
   1   1
```

2.5 Function definitions

2.6 m files

2.7 Flow control

2.8 Plotting

A very important use for a tool like `Matlab` is its ability to produce plots graphing mathematical relationships. For illustrating mathematical relations a two or three dimensional picture *can* be better than the thousands words of the old adage.

2.9 Libraries

2.10 References

You need the manual for your chosen package.

Chapter 3

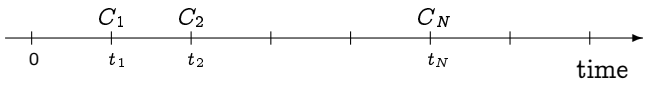
The value of time

Contents

Finance as a field of study is sometimes somewhat flippantly said to deal with the value of two things: *time* and *risk*. While this is not the whole story, there is a deal of truth in it. These are the two issues which is always present. We start our discussion by ignoring risk and only considering the implications of the fact that anybody prefers to get something earlier rather than later, or the value of time.

3.1 Present value

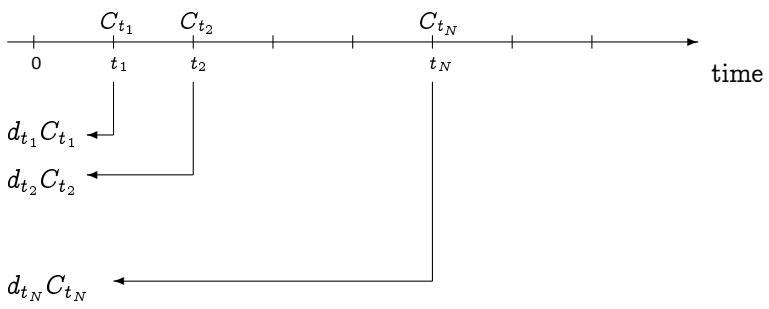
The present value is the current value of a stream of future payments. Let C_t be the cash flow at time t. Suppose we have N future cash flows that occur at times $t_1, t_2, \cdots, t_N$.



To find the *present* value of these future cash flows one need a set of prices of future cash flows. Suppose d_t is the price one would pay today for the right to receive one dollar at a future date t. Such a price is also called a *discount factor*. To complicate matters further such prices will differ depending on the riskiness of the future cash flows. For now we concentrate on one particular set of prices, the prices of *riskless* future cash flows. We will return to how one would adjust the prices for risky cash flows.

If one knows the set of prices for future claims of one dollar, $d_1, d_2, \ldots,$, one would calculate the present value as the sum of the present values of the different elements.

$$PV = \sum_{i=1}^{N} d_{t_i} C_{t_i}$$



However, knowing this set of current prices for cash flows at all future dates is not always feasible, and some way has to be found to simplify the data need inherent in such general present value calculations.

3.2 One interest rate with annual compounding

The best known way to simplify the present value calculation is to rewrite the discount factors in terms of interest rates, or yields, through the relationship:

$$d_t = \frac{1}{(1+r_t)^t}$$

where r_t is the interest rate (usually termed the spot rate) relevant for a t-period investment. To further simplify this calculation one can impose that this interest rate r is constant for all periods. This is termed a *flat* term structure. We will in the next chapter relax this simplifying assumption. The prices for valuing the future payments d_t is calculated from this interest rate:

$$d_t = \frac{1}{(1+r)^t},$$

In this case one would calculate the present value of a stream of cash flows paid at discrete dates $t = 1, 2, \ldots N$ as

$$PV = \sum_{t=1}^{N} \frac{C_t}{(1+r)^t}.$$

The implementation of this calculation is shown in **C++ Code 3.1**.

```
#include <cmath>
#include <vector>
using namespace std;
#include <iostream>
double cash_flow_pv_discrete(const vector<double>& cflow_times,
                             const vector<double>& cflow_amounts,
                             const double& r){
    double PV=0.0;
    for (int t=0; t<cflow_times.size();t++) {
        PV += cflow_amounts[t]/pow(1.0+r,cflow_times[t]);
    };
    return PV;
};
```

C++ Code 3.1: Present value with discrete compounding

Example

An investment project has an investment cost of 100 today, and produces cash flows of 75 each of the next two years. What is the Net Present Value of the project?

Matlab program:

```
C=[-100 75 75]
t=[0 1 2]
r = 0.1
d=(1/(1+r)).^t
NPV=C*d'
```

Output from Matlab program:

```
C =
 -100 75 75
t =
 0 1 2
r = 0.1000
d =
 1.0000 0.9091 0.8264
NPV = 30.165
```

C++ program:

```
vector<double> cflows; cflows.push_back(-100.0); cflows.push_back(75); cflows.push_back(75);
vector<double> times; times.push_back(0.0); times.push_back(1); times.push_back(2);
double r=0.1;
cout << " Present value, 10 percent discretely compounded interest = "
     << cash_flow_pv_discrete(times, cflows, r) << endl;
```

Output from C++ program:

```
 Present value, 10 percent discretely compounded interest = 30.1653
```

Given the assumption of a discrete, annual interest rate, there are a number of useful special cases of cash flows where one can calculate the present value in a simplified manner. Some of these are shown in the following exercises.

Exercise 3.1.

A perpetuity is a promise of a payment of a fixed amount X each period for the indefinite future. Suppose there is a fixed interest rate r.

1. Show that the present value of this sequence of cash flows is calculated simply as

$$PV = \sum_{t=1}^{\infty} \frac{X}{1+r} = \frac{X}{r}$$

Exercise 3.2.

A *growing perpetuity* is again an infinite sequence of cashflows, where the payment the first year is X and each consequent payment grows by a constant rate g, i.e, the time 2 payment is $X(1+g)$, the time 3 payment is $X(1+g)^2$, and so on.

1. Show that the present value of this perpetuity simplifies to

$$PV = \sum_{t=1}^{\infty} \frac{X(1+g)^{t-1}}{(1+r)^t} = \frac{X_1}{r-g}$$

Exercise 3.3.

An *annuity* is a sequence of cashflows for a given number of years, say T periods into the future. Consider an annuity paying a fixed amount X each period. The interest rate is r.

1. Show that the present value of this sequence of cash flows can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X}{(1+r)^t} = X\left[\frac{1}{r} - \frac{1}{r}\frac{1}{(1+r)^T}\right]$$

Exercise 3.4.

An *growing annuity* is a sequence of cashflows for a given number of years, say T periods into the future, where each payment grows by a given factor each year. Consider a T-period annuity that pays X the first period. After that, the payments grows at a rate of g per year, i.e. the second year the cash flow is $X(1+g)$, the third $X(1+g)^2$, and so on.

1. Show that the present value of this growing annuity can be simplified as

$$PV = \sum_{t=1}^{T} \frac{X(1+g)^{(t-1)}}{(1+r)^t} = X\left[\frac{1}{r-g} - \left(\frac{1+g}{1+r}\right)^T \frac{1}{r-g}\right]$$

Exercise 3.5.

Rank the following cash flows in terms of present value. Use an interest rate of 5%.

1. A perpetuity with an annual payment of \$100.
2. A growing perpetuity, where the first payment is \$75, and each subsequent payment grows by 2%.
3. A 10-year annuity with an annual payment of \$90.
4. A 10 year growing annuity, where the first payment is \$85, and each subsequent payment grows by 5%.

3.2.1 Internal rate of return.

In addition to its role in simplifying present value calculations, the interest rate has some further use. The percentage return on an investment is a summary measure of the investment's profitability. Saying that an investment earns 10% per year is a good way of summarizing the cash flows in a way that does not depend on the amount of the initial investment. The return is thus a relative measure of profitability. To estimate a return for a set of cash flows we calculate the *internal rate of return.* The internal rate of return for a set of cash flows is the interest rate that makes the present value of the cash flows equal to zero. When there is a uniquely defined internal rate of return we get a relative measure of the profitability of a set of cash flows, measured as a return, typically expressed as a percentage. Note some of the implicit assumptions made here. We assume that the same interest rate applies at all future dates (i.e. a flat term structure). The IRR method also assumes intermediate cash flows are reinvested at the internal rate of return.

Suppose the cash flows are $C_0, C_1, C_2, \ldots C_T$. Finding an internal rate of return is finding a solution y of the equation

$$\sum_{t=1}^{T} \frac{C_t}{(1+y)^t} - C_0 = 0$$

Note that this is a polynomial equation, and as T becomes large, there in no way to find an explicit solution to the equation. It therefore needs to be solved numerically. For well behaved cash flows, where we know that there is one IRR, the method implemented in **C++ Code 3.2** is suitable, it is an iterative process called bisection. It is an adaption of the bracketing approach discussed in (Press, Teukolsky, Vetterling, and Flannery, 1992, Chapter9),

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;
#include "fin_recipes.h"

const double ERROR=-1e30;

double cash_flow_irr_discrete(const vector<double>& cflow_times,
                              const vector<double>& cflow_amounts) {
    // simple minded irr function. Will find one root (if it exists.)
    // adapted from routine in Numerical Recipes in C.
    if (cflow_times.size()!=cflow_amounts.size()) return ERROR;
    const double ACCURACY = 1.0e-5;
    const int MAX_ITERATIONS = 50;
    double x1 = 0.0;
    double x2 = 0.2;

    // create an initial bracket, with a root somewhere between bot,top
    double f1 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x1);
    double f2 = cash_flow_pv_discrete(cflow_times, cflow_amounts, x2);
    int i;
    for (i=0;i<MAX_ITERATIONS;i++) {
        if ( (f1*f2) < 0.0) { break; }; //
        if (fabs(f1)<fabs(f2)) {
            f1 = cash_flow_pv_discrete(cflow_times,cflow_amounts, x1+=1.6*(x1-x2));
        }
        else {
            f2 = cash_flow_pv_discrete(cflow_times,cflow_amounts, x2+=1.6*(x2-x1));
        };
    };
    if (f2*f1>0.0) { return ERROR; };
    double f = cash_flow_pv_discrete(cflow_times,cflow_amounts, x1);
    double rtb;
    double dx=0;
    if (f<0.0) {
        rtb = x1;
        dx=x2-x1;
    }
    else {
        rtb = x2;
        dx = x1-x2;
    };
    for (i=0;i<MAX_ITERATIONS;i++){
        dx *= 0.5;
        double x_mid = rtb+dx;
        double f_mid = cash_flow_pv_discrete(cflow_times,cflow_amounts, x_mid);
        if (f_mid<=0.0) { rtb = x_mid; }
        if ( (fabs(f_mid)<ACCURACY) || (fabs(dx)<ACCURACY) ) return x_mid;
    };
    return ERROR; // error.
};
```

C++ Code 3.2: Estimation of the internal rate of return

Example

We are considering an investment with the following cash flows at dates 0, 1 and 2:

$$C_0 = -100, \quad C_1 = 10, \quad C_2 = 110$$

1. The current interest rate (with discrete, annual compounding) is 5%. Determine the present value of the cash flows.
2. Find the internal rate of return of this sequence of cash flows.

Matlab program:

```
C=[-100 10 110]
t=[0 1 2]
r = 0.05
d=(1/(1+r)).^t
NPV=C*d'
IRR = irr(C)
```

Output from Matlab program:

```
C =
 -100  10  110
t =
 0  1  2
r = 0.050000
d =
 1.0000  0.9524  0.9070
NPV = 9.2971
IRR = 0.100000
```

C++ program:

```
vector<double> cflows; cflows.push_back(-100.0); cflows.push_back(10.0); cflows.push_back(110.0);
vector<double> times; times.push_back(0.0); times.push_back(1); times.push_back(2);
double r=0.05;
cout << " present value, 5 percent discretely compounded interest = " ;
cout << cash_flow_pv_discrete(times, cflows, r) << endl;
cout << " internal rate of return, discrete compounding = ";
cout << cash_flow_irr_discrete(times, cflows) << endl;
```

Output from C++ program:

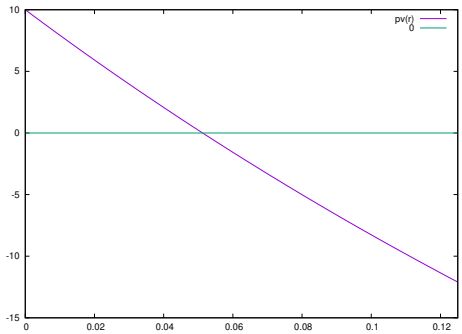
```
present value, 5 percent discretely compounded interest = 9.29705
internal rate of return, discrete compounding = 0.1
```

In addition to the above economic qualifications to interpretations of the internal rate of return, we also have to deal with technical problem stemming from the fact that any polynomial equation has potentially several solutions, some of which may be imaginary. By imaginary here we mean that we move away from the real line to the set of complex numbers. In economics we prefer the real solutions, complex interest rates are not something we have much intuition about... To see whether we are likely to have problems in identifying a single meaningful IRR, the code shown in code 3.3 implements a simple check. It is only a necessary condition for a unique IRR, not sufficient, so you may still have a well-defined IRR even if this returns false. The first test is just to count the number of sign changes in the cash flow. From Descartes rule we know that the number of real roots is one if there is only one sign change. If there is more than one change in the sign of cash flows, we can go further and check the *aggregated* cash flows for sign changes (See Norstrom (1972)).

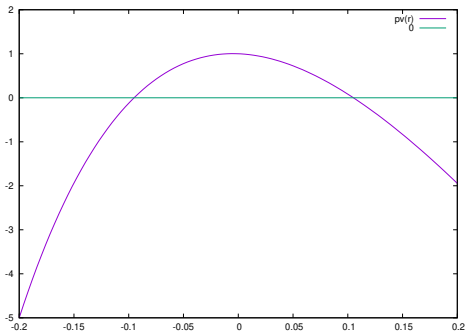
```
#include <cmath>
#include <vector>
using namespace std;

inline int sgn(const double& r){ if (r>=0) {return 1;} else {return -1;}; };

bool cash_flow_unique_irr(const vector<double>& cflow_times,
                          const vector<double>& cflow_amounts) {
    int sign_changes=0;   // first check Descartes rule
    for (int t=1;t<cflow_times.size();++t){
        if (sgn(cflow_amounts[t-1]) !=sgn(cflow_amounts[t])) sign_changes++;
    };
    if (sign_changes==0) return false; // can not find any irr
    if (sign_changes==1) return true;

    double A = cflow_amounts[0]; // check the aggregate cash flows, due to Norstrom
    sign_changes=0;
    for (int t=1;t<cflow_times.size();++t){
        if (sgn(A) != sgn(A+=cflow_amounts[t])) sign_changes++;
    };
    if (sign_changes<=1) return true;
    return false;
}
```

C++ Code 3.3: Test for uniqueness of IRR

A better way to gain an understanding for the relationship between the interest rate and the present value is simply to plot the present value as a function of the interest rate. The following picture illustrates the method for two different cash flows. Note that the set of cash flows on the right has two possble interest rates that sets the present value equal to zero.



$C_0 = -100, C_1 = 10, C_2 = 100$



$C_0 = -100, C_1 = 201, C_2 = -100$

Exercise 3.6.

An alternative way of estimating the IRR is to use an external subroutine that finds the root of a polynomial equation. Search for a suitable general subroutine for root finding and replace the IRR estimation with a call to this subroutine.

3.3 Continously compounded interest

Such discrete compounding as we have just discussed is not the only alternative way to approximate the discount factor. The discretely compounded case assumes that interest is added at discrete points in time (hence the name). However, an alternative assumption is to assume that interest is added continously. If compounding is continous, and r is the interest rate, one would calculate the current price d_t of reciving one dollar at a future date t as

$$d_t = e^{-rt},$$

Formula 3.1 summarizes some rules for translating between continously compounded and discretly compounded interest rates.

$$r = n \ln\left(1 + \frac{r_n}{n}\right)$$

$$r_n = n\left(e^{\frac{r}{n}} - 1\right)$$

$$\text{Future value } = e^{rt}$$

$$\text{Present value } = e^{-rt}$$

Notation: r_n: interest rate with discrete compounding, n: compounding periods per year. r: interest rate with continuous compounding, t: time to maturity.

Formula 3.1: Translating between discrete and continous compounding

Example

1. Given a 15% interest rate with monthly compounding, calculate the equivalent interest rate with continuous compounding.
2. Given a 12% interest rate with continuous compounding, find the equivalent interest rate with quarterly compounding.

Carrying out the calculations:

1. $r_{12} = 15\%$, $r = 12 \ln\left(1 + \frac{0.15}{12}\right) = 14.91\%$
2. $r = 12\%$, $r_4 = n\left(e^{\frac{r}{n}} - 1\right) = 4\left(e^{\frac{0.12}{4}} - 1\right) = 12.18\%$

Using `Matlab` to do the calculations:

`Matlab` program:

```
r  = 12 * log( 1+0.15/12)
r4 = 4 * ( exp(0.12/4)-1 )
```

Output from `Matlab` program:

```
r = 0.1491
r4 = 0.1218
```

3.3.1 Present value

Applying this to a set of cash flows at future dates $t_1, t_2, \ldots, t_n$, we get the following present value calculation:

$$PV = \sum_{i=1}^{n} e^{-rt_i} C_{t_i}$$

This calculation is implemented in **C++ Code 3.4**.

```
#include <cmath>
#include <vector>
using namespace std;

double cash_flow_pv( const vector<double>& cflow_times,
                     const vector<double>& cflow_amounts,
                     const double& r){
    double PV=0.0;
    for (int t=0; t<cflow_times.size();t++) {
        PV += cflow_amounts[t] * exp(-r*cflow_times[t]);
    };
    return PV;
};
```

C++ Code 3.4: Present value calculation with continously compounded interest

In much of what follows we will work with the case of continously compounded interest. There is a number of reasons why, but a prime reason is actually that it is easier to use continously compounded interest than discretely compounded, because it is easier to deal with uneven time periods. Discretely compounded interest is easy to use with evenly spaced cash flows (such as annual cash flows), but harder otherwise.

3.4 Further readings

The material in this chapter is covered in most standard textbooks of corporate finance (e.g. Brealey, Myers, and Allen (2020) or Ross, Westerfield, Jaffe, and Jordan (2019)) and investments (e.g. Bodie, Kane, and Marcus (2021), Haugen (2001) or Sharpe, Alexander, and Bailey (1999)).

Chapter 4

Bond Pricing with a flat term structure

Contents

In this section we use the present value framework of the previous chapter to price bonds and other fixed income securities. What distinguishes bonds is that the future payments are set when the security is issued. The simplest, and most typical bond, is a fixed interest, constant maturity bond with no default risk. There is however a large number of alternative contractual features of bonds. The bond could for example ba an annuity bond, paying a fixed amount each period. For such a bond the principal amount outstanding is paid gradually during the life of the bond. The interest rate the bond pays need not be fixed, it could be a floating rate, the interest rate paid could be a function of some market rate. Many bonds are issued by corporations, and in such cases there is a risk that the company issued the bond defaults, and the bond does not pay the complete promised amount. Another thing that makes bond pricing difficult in practice, is that interest rates tend to change over time.

We start by assuming that all the promised payments are certain.

Then the bond current price B_0 is found as the present value of these payments. The first step of pricing is to use the terms of the bond to find the promised payments. We start by considering a fixed interest bond with no default risk. Such bonds are typically bonds issued by governments. The bond is a promise to pay a face value F at the maturity date T periods from now. Each period the bond pays a fixed percentage amount of the face value as coupon C. The cash flows from the bond thus look as follows.

$t =$	0	1	2	3	$\cdots$	T
Coupon		C	C	C	$\cdots$	C
Face value						F
Total cash flows		$C_1 = C$	$C_2 = C$		$\cdots$	$C_T = C + F$

In general a bond price is found as the present value

$$B_0 = d_1 C_1 + d_2 C_2 + \cdots + d_T C_T = \sum_{t=1}^{T} d_t C_t$$

where d_t is the discount factor, or the time 0 price of a payment of 1 at time t. To fully specify the

problem it is necessary to find all discount factors d_t. In this chapter we will work with a specially simple specifiction of the term structure, namely that it is flat, and specified by the interest rate r.

4.1 Flat term structure with discrete, annual compounding

This is the simplest possible specification of a term structure,

$$d_t = \left(\frac{1}{1+r}\right)^t = \frac{1}{(1+r)^t}$$

4.1.1 Bond Price

The current bond price (B_0) is the present value of the cash flows from the bond

$$B_0 = \sum_{t=1}^{T} \left(\frac{1}{(1+r)^t}\right) C_t = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} \tag{4.1}$$

If we continue with the example of a standard fixed interest bond, where $C_t = C$ when $t < T$ and $C_T = C + F$, and show how the bond price will be calculated using `Matlab`.

Example

A 3 year bond with a face value of \$100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Determine the current bond price.

The current bond price: $B_0 = \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531$.

Here are the calculations:

`Matlab` program:

```
C =[10,10,110]
t = 1:3
r = 0.09
d=(1./(1+r).^t)
B= d * C'
```

Output from `Matlab` program:

```
C =
 10 10 110
t =
 1 2 3
r = 0.090000
d =
 0.9174 0.8417 0.7722
B = 102.53
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
cout << " bonds price = " << bonds_price_discrete(times, cflows, r) << endl;
```

Output from C++ program:

```
 bonds price     = 102.531
```

The general code in C++ for calculating the bond price with discrete annual compounding is shown in **C++ Code 4.1**.

```
#include <cmath>
#include <vector>
using namespace std;

double bonds_price_discrete(const vector<double>& times,
                            const vector<double>& cashflows,
                            const double& r) {
    double p=0;
    for (int i=0;i<times.size();i++) {
        p += cashflows[i]/(pow((1+r),times[i]));
    };
    return p;
};
```

C++ Code 4.1: Bond price calculation with discrete, annual compounding.

4.1.2 Yield to maturity

Since bonds are issued in terms of interest rate, it is also useful to find an interest rate number that summarizes the terms of the bond. The obvious way of doing that is asking the question: What is the internal rate of return on the investment of buying the bond now and keeping the bond to maturity? The answer to that question is the yield to maturity of a bond. The yield to maturity is the interest rate that makes the present value of the future coupon payments equal to the current bond price, that is, for a known price B_0, the yield is the solution y to the equation

$$B_0 = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} \tag{4.2}$$

This calculation therefore has the same qualifications as discussed earlier calculating IRR, it supposes reinvestment of coupon at the bond yield (the IRR).

There is much less likelihood we'll have technical problems with multiple solutions when doing this yield estimation for bonds, since the structure of cash flows is such that there exist only one solution to the equation. The algorithm for finding a bonds yield to maturity shown in **C++ Code 4.2** is thus simple bisection. We know that the bond yield is above zero and set zero as a lower bound on the bond yield. We then find an upper bound on the yield by increasing the interest rate until the bond price with this interest rate is negative. We then bisect the interval between the upper and lower until we are "close enough." **C++ Code 4.2** implements this idea.

```
#include <cmath>
using namespace std;

#include "fin_recipes.h"

double bonds_yield_to_maturity_discrete( const vector<double>& times,
                                         const vector<double>& cashflows,
                                         const double& bondprice) {
    const double ACCURACY = 1e-5;
    const int MAX_ITERATIONS = 200;
    double bot=0, top=1.0;
    while (bonds_price_discrete(times, cashflows, top) > bondprice) { top = top*2; };
    double r = 0.5 * (top+bot);
    for (int i=0;i<MAX_ITERATIONS;i++){
        double diff = bonds_price_discrete(times, cashflows,r) - bondprice;
        if (fabs(diff)<ACCURACY) return r;
        if (diff>0.0)  { bot=r;}
        else           { top=r; };
        r = 0.5 * (top+bot);
    };
    return r;
};
```

C++ Code 4.2: Bond yield calculation with discrete, annual compounding

Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Find the bond's current price.
2. Find the bond's yield to maturity.

Matlab program:

```
C = [ 10 10 110 ];
t = 1:3;
r=0.09;
B = C * (1./((1+r).^t))'
y = irr([-B C ])
```

Output from Matlab program:

```
B = 102.53
y = 0.090000
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " Bond price, 9 percent discretely compounded interest = " << B << endl;
cout << " bond yield to maturity = " << bonds_yield_to_maturity_discrete(times, cflows, B) << endl;
```

Output from C++ program:

```
Bond price, 9 percent discretely compounded interest = 102.531
bond yield to maturity = 0.09
```

4.1.3 Duration

When holding a bond one would like to know how sensitive the value of the bond is to changes in economic environment. The most relevent piece of the economic environment is the current interest rate. An important component of such calculation is the *duration* of a bond. The duration of a bond should be interpreted as the weighted average maturity of the bond, and is calculated as

$$\text{Duration} = \frac{\sum_t t \frac{C_t}{(1+r)^t}}{\text{Bond Price}}, \tag{4.3}$$

where C_t is the cash flow in period t, and r the interest rate. Using the bond price calculated in equation 4.1 we calculate duration as

$$D = \frac{\sum_t \frac{tC_t}{(1+r)^t}}{\sum_t \frac{C_t}{(1+r)^t}} \tag{4.4}$$

which is shown in **C++ Code 4.3**.

```
#include <cmath>
#include <vector>
using namespace std;

double bonds_duration_discrete(const vector<double>& times,
                               const vector<double>& cashflows,
                               const double& r) {
    double B=0;
    double D=0;
    for (int i=0;i<times.size();i++){
        D += times[i] * cashflows[i] / pow(1+r,times[i]);
        B += cashflows[i] / pow(1+r,times[i]);
    };
    return D/B;
};
```

C++ Code 4.3: Bond duration using discrete, annual compounding and a flat term structure

An alternative approach to calculating duration is calculate the yield to maturity y for the bond, and use that in estimating the bond price. This is called *Macaulay Duration*. First one calculates y, the yield to maturity, from

$$\text{Bond price} = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t}$$

and then use this y in the duration calculation:

$$\text{Macaulay duration} = \frac{\sum_t \frac{tC_t}{(1+y)^t}}{\sum_t \frac{C_t}{(1+y)^t}} \tag{4.5}$$

C++ Code 4.4 implements this calculation.

Note though, that in the present case, with a flat term structure, these should produce the same number. If the bond is priced correctly, the yield to maturity must equal the current interest rate. If $r = y$ the two calculations in equations (4.4) and (4.5) obviously produces the same number.

Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

```
#include "fin_recipes.h"

double bonds_duration_macaulay_discrete(const vector<double>& times,
                                        const vector<double>& cashflows,
                                        const double& bond_price) {
  double y = bonds_yield_to_maturity_discrete(times, cashflows, bond_price);
  return bonds_duration_discrete(times, cashflows, y); // use YTM in duration calculation
};
```

C++ Code 4.4: Calculating the Macaulay duration of a bond

1. Determine the current bond price.
2. Calculate the duration using the current interest rate.
3. Calculate the duration using the MaCaulay definition.

Need to calculate the following:
The current bond price: $B_0 = \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531.$
The bond's duration: $D = \frac{1}{102.531}\left(\frac{1 \cdot 10}{1.09} + \frac{2 \cdot 10}{1.09^2} + \frac{3 \cdot 110}{1.09^3}\right) = 2.74$

Matlab program:

```
C =[10,10,110];
t = 1:3;
r = 0.09;
B= C * (1./(1+r).^t)'
D= (1/B)*t.* C * (1./(1+r).^t)'
y = irr([-B C ])
DM= (1/B)*t.* C * (1./(1+y).^t)'
```

Output from Matlab program:

```
B = 102.53
D = 2.7390
y = 0.090000
DM = 2.7390
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration_discrete(times, cflows, r) << endl;
cout << " bond macaulay = " << bonds_duration_macaulay_discrete(times, cflows, B) << endl;
```

Output from C++ program:

```
bonds price    = 102.531
bond duration  = 2.73895
bond macaulay  = 2.73895
```

4.1.4 Measuring bond sensitivity to interest rate changes

Now, the reason for why we say that we can measure the sensitivity of a bond price using duration. To a first approximation, ΔB_0, the change in the bond price for a small change in the interest rate Δr, can be calculated

$$\frac{\Delta B_0}{B_0} \approx -\frac{D}{1+r}\Delta r$$

where D is the bond's duration. For simplicity one often calculates the term in front of the Δy in the above, $\frac{D}{1+y}$ directly and terms it the bond's *modified duration*.

$$\text{Modified Duration} = D^* = \frac{D}{1+r}$$

The sensitivity calculation is then

$$\frac{\Delta B_0}{B_0} \approx -D^*\Delta r$$

The modified duration is also written in term's of the bond's yield to maturity y, and is then

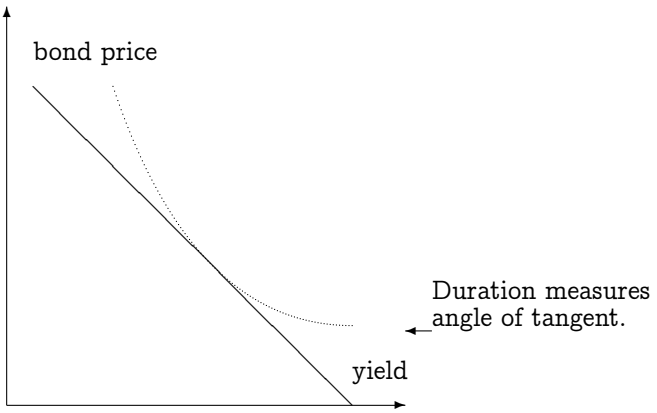
$$D^* = \frac{D}{1+y}$$

C++ Code 4.5 shows this calculation.

```
#include <vector>
using namespace std;
#include "fin_recipes.h"

double bonds_duration_modified_discrete (const vector<double>& times,
                                         const vector<double>& cashflows,
                                         const double& bond_price){
    double y = bonds_yield_to_maturity_discrete(times, cashflows, bond_price);
    double D = bonds_duration_discrete(times, cashflows, y);
    return D/(1+y);
};
```

C++ Code 4.5: Modified duration

Approximating bond price changes using duration is illustrated in the following figure.



The modified duration measures the angle of the tangent at the current bond yield. Approximating the change in bond price with duration is thus only a first order approximation. To improve on this approximation we also need to account for the curvature in the relationship between bond price and interest rate. To quantify this curvature we calculate the *convexity* of a bond.

$$\text{Convexity} = Cx = \frac{1}{B_0}\frac{1}{(1+r)^2}\sum_{t=1}^{T}(t+t^2)\frac{C_t}{(1+r)^t} \tag{4.6}$$

This calculation is implemented in **C++ Code 4.6**. To improve on the estimate of how the bond price

```
#include <cmath>
#include "fin_recipes.h"

double bonds_convexity_discrete(const vector<double>& times,
                                const vector<double>& cashflows,
                                const double& r) {
    double Cx=0;
    for (int i=0;i<times.size();i++){
        Cx+= cashflows[i]*times[i]*(times[i]+1)/(pow((1+r),times[i]));
    };
    double B=bonds_price_discrete(times, cashflows, r);
    return (Cx/(pow(1+r,2)))/B;
};
```

C++ Code 4.6: Bond convexity with a flat term structure and annual compounding

change when the interest rates changes you will then calculate

$$\frac{\Delta B_0}{B_0} \approx -D^*\Delta y + \frac{1}{2}Cx\left(\Delta y\right)^2$$

Formula 4.1 summarizes the above calculations.

Bond Price (B_0)

$$B_0 = \sum_{t=1}^{T}\frac{C_t}{(1+r)^t}$$

Yield to maturity y solves

$$B_0 = \sum_{t=1}^{T}\frac{C_t}{(1+y)^t}$$

Duration (D)

$$D = \frac{1}{B_0}\sum_{t=1}^{T}\frac{tC_t}{(1+r)^t}$$

Macaulay duration

$$D = \frac{1}{B_0}\sum_{t=1}^{T}\frac{tC_t}{(1+y)^t}$$

Modified duration

$$D^* = \frac{D}{1+y}$$

Convexity (Cx)

$$Cx = \frac{1}{B_0}\frac{1}{(1+r)^2}\sum_{t=1}^{T}(t+t^2)\frac{C_t}{(1+r)^t}$$

Approximating bond price changes

$$\frac{\Delta B_0}{B_0} \approx -D^*\Delta y$$

$$\frac{\Delta B_0}{B_0} \approx -D^*\Delta y + \frac{1}{2}\times Cx\times(\Delta y)^2$$

C_t: Cash flow at time t, r: interest rate, y: bond yield to maturity, B_0: current bond price. Bond pays coupon at evenly spaced dates $t = 1, 2, 3\ldots, T$.

Formula 4.1: Bond pricing formulas with a flat term structure and discrete, annual compounding

Example

A 3 year bond with a face value of \$100 makes annual coupon payments of 10%. The current interest rate (with annual compounding) is 9%.

1. Determine the current bond price.
2. Suppose the interest rate changes to 10%, determine the new price of the bond by direct calculation.
3. Use duration to estimate the new price and compare it to the correct price.
4. Use convexity to improve on the estimate using duration only.

Need to calculate the following:

The current bond price: $B_0 = \frac{10}{(1+0.09)^1} + \frac{10}{(1+0.09)^2} + \frac{110}{(1+0.09)^3} = 102.531.$

The bond's duration: $D = \frac{1}{102.531}\left(\frac{1\cdot 10}{1.09} + \frac{2\cdot 10}{1.09^2} + \frac{3\cdot 110}{1.09^3}\right) = 2.74$

The modified duration: $D^* = \frac{D}{1+r} = \frac{2.74}{1.09} = 2.51.$

The convexity: $Cx = \frac{1}{(1+0.09)^2}\frac{1}{102.531}\left(\frac{(1+1)\cdot 10}{1.09} + \frac{(2^2+2)\cdot 10}{1.09^2} + \frac{(3+3^2)\cdot 110}{1.09^3}\right) = 8.93.$

Here are the calculations:

Matlab program:

```
C =[10,10,110];
t = 1:3;
r = 0.09;
B= C * (1./(1+r).^t)'
D= (1/B)*t.* C * (1./(1+r).^t)'
Cx= (1/(1+r)^2) * (1/B)*t.^2.*C * (1./(1+r).^t)'
newB=C* (1./(1+0.1).^t)'
```

Output from Matlab program:

```
B = 102.53
D = 2.7390
Cx = 6.6272
newB = 100.000
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
double B = bonds_price_discrete(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration_discrete(times, cflows, r) << endl;
cout << " bond duration modified = " << bonds_duration_modified_discrete(times, cflows, B) << endl;
cout << " bond convexity =" << bonds_convexity_discrete(times, cflows, r) << endl;
cout << " new bond price = " << bonds_price_discrete(times, cflows, 0.1);
```

Output from C++ program:

```
bonds price    = 102.531
bond duration  = 2.73895
bond duration modified = 2.5128
bond convexity =8.93248
new bond price = 100
```

Using these numbers to answer the questions, let us see what happens when the interest rate increases to 10%.

This means the bond will be selling at par, equal to 100, which can be confirmed with direct computation:

$$B_0 = \frac{10}{(1+0.1)^1} + \frac{10}{(1+0.1)^2} + \frac{110}{(1+0.1)^3} = 100$$

Using duration to estimate the change in the bond price for a unit change in the interest rate:

$$\frac{\Delta B_0}{B_0} = -D^* \Delta y = -2.51 \cdot 0.01 = -0.0251$$

Using this duration based number to estimate the new bond price.

$$\hat{B} = B_0 + \left(\frac{\Delta B_0}{B_0}\right) B_0 = 102.531 - 0.0251 \cdot 102.531 = 99.957$$

Additionally using convexity in estimating the change in the bond price:

$$\frac{\Delta B_0}{B_0} = -D^* \Delta y + \frac{1}{2} C_x y^2 = -2.51 \cdot 0.01 + \frac{1}{2} 8.93 (0.01)^2 = -0.0251 + 0.00044 = -0.02465$$

$$\hat{B} = 102.531 \left(1 + \left(\frac{\Delta B_0}{B_0}\right) B_0\right) = 102.531(1 - 0.02465) = 100.0036$$

Exercise 4.1.

Perpetual duration [4]

The term structure is flat with annual compounding. Consider the pricing of a perpetual bond. Let C be the per period cash flow

$$B_0 = \sum_{t=1}^{\infty} \frac{C}{(1+r)^t} = \frac{C}{r}$$

1. Determine the first derivative of the price with respect to the interest rate.
2. Find the duration of the bond.

Exercise 4.2.

[5]

Consider an equally weighted portfolio of two bonds, A and B. Bond A is a zero coupon bond with 1 year to maturity. Bond B is a zero coupon bond with 3 years to maturity. Both bonds have face values of 100. The current interest rate is 5%.

1. Determine the bond prices.
2. Your portfolio is currently worth 2000. Find the number of each bond invested.
3. Determine the duration of the portfolio.
4. Determine the convexity of your position.

4.2 Continously compounded interest

We will go over the same concepts as covered in the previous section on bond pricing. There are certain subtle differences in the calculations. Formula 4.2 corresponds to the earlier summary in formula 4.1.

Bond Price B_0:

$$B_0 = \sum_i e^{-rt_i)} C_{t_i}$$

Yield to maturity y solves:

$$B_0 = \sum_i C_{t_i} e^{-yt_i}$$

Duration D:

$$D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-rt_i}$$

Macaulay duration

$$D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-yt_i}$$

Convexity Cx:

$$Cx = \frac{1}{B_0} \sum_i C_{t_i} t_i^2 e^{-rt_i}$$

Approximating bond price changes

$$\frac{\Delta B_0}{B_0} \approx -D\Delta y$$

$$\frac{\Delta B_0}{B_0} \approx -D\Delta y + \frac{1}{2} \times Cx \times (\Delta y)^2$$

Bond paying cash flows $C_{t_1}, C_{t_2}, \ldots$ at times $t_1, t_2, \ldots$. Notation: B_0: current bond price. e: natural exponent.

Formula 4.2: Bond pricing formulas with continously compounded interest and a flat term structure

Some important differences is worth pointing out. When using continously compounded interest, one does not need the concept of *modified duration*. In the continously compounded case one uses the calculated duration directly to approximate bond changes, as seen in the formulas describing the approximation of bond price changes. Note also the difference in the *convexity* calculation, one does not divide by $(1+y)^2$ in the continously compounded formula, as was done in the discrete case.

C++ Code 4.7, **C++ Code 4.8**, **C++ Code 4.9** and **C++ Code 4.10** show continously compounded analogs of the earlier codes for the discretely compounded case.

```
#include <cmath>
#include <vector>
using namespace std;

double bonds_price(const vector<double>& cashflow_times,
                   const vector<double>& cashflows,
                   const double& r) {
    double p=0;
    for (int i=0;i<cashflow_times.size();i++) {
        p += exp(-r*cashflow_times[i])*cashflows[i];
    };
    return p;
};
```

C++ Code 4.7: Bond price calculation with continously compounded interest and a flat term structure

```
#include <cmath>
#include <vector>
using namespace std;

double bonds_duration(const vector<double>& cashflow_times,
                      const vector<double>& cashflows,
                      const double& r) {
    double S=0;
    double D1=0;
    for (int i=0;i<cashflow_times.size();i++){
        S  +=                    cashflows[i] * exp(-r*cashflow_times[i]);
        D1 += cashflow_times[i] * cashflows[i] * exp(-r*cashflow_times[i]);
    };
    return D1 / S;
};
```

C++ Code 4.8: Bond duration calculation with continously compounded interest and a flat term structure

```
#include "fin_recipes.h"

double bonds_duration_macaulay(const vector<double>& cashflow_times,
                               const vector<double>& cashflows,
                               const double& bond_price) {
  double y = bonds_yield_to_maturity(cashflow_times, cashflows, bond_price);
  return bonds_duration(cashflow_times, cashflows, y); // use YTM in duration
};
```

C++ Code 4.9: Calculating the Macaulay duration of a bond with continously compounded interest and a flat term structure

```
#include <cmath>
#include "fin_recipes.h"

double bonds_convexity(const vector<double>& times,
                       const vector<double>& cashflows,
                       const double& r ) {
    double C=0;
    for (int i=0;i<times.size();i++){
        C += cashflows[i] * pow(times[i],2) * exp(-r*times[i]);
    };
    double B=bonds_price(times, cashflows,r);
    return C/B;
};
```

C++ Code 4.10: Bond convexity calculation with continously compounded interest and a flat term structure

Example

A 3 year bond with a face value of $100 makes annual coupon payments of 10%. The current interest rate (with continous compounding) is 9%.

1. Calculate the bond's price, yield to maturity, duration and convexity.
2. Suppose the interest rate falls to 8%. Estimate the new bond price using duration, and compare with the actual bond price using the correct interest rate.

Calculations:

Matlab program:

```
C =[10,10,110]
t = 1:3
r = 0.09
d=(1./(1+r).^t)
B= C * d'
D=1/B*C.*t*d'
Dadj=D/(1+r)
r=0.1
actualB = C*(1./(1+r).^t)'
```

Output from Matlab program:

```
C =
 10 10 110
t =
 1 2 3
r = 0.090000
d =
 0.9174 0.8417 0.7722
B = 102.53
D = 2.7390
Dadj = 2.5128
r = 0.1000
actualB = 100.000
```

C++ program:

```
vector<double> cflows; cflows.push_back(10); cflows.push_back(10); cflows.push_back(110);
vector<double> times; times.push_back(1); times.push_back(2); times.push_back(3);
double r=0.09;
double B = bonds_price(times, cflows, r);
cout << " bonds price = " << B << endl;
cout << " bond duration = " << bonds_duration(times, cflows, r) << endl;
cout << " bond convexity =" << bonds_convexity(times, cflows, r) << endl;
cout << " new bond price = " << bonds_price(times, cflows, 0.08);
```

Output from C++ program:

```
bonds price    = 101.464
bond duration  = 2.73753
bond convexity =7.86779
new bond price = 104.282
```

Exercise 4.3.

The term structure is flat, and compounding is continous. Consider two definitions of duration, the "usual" definition $D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-rt_i}$ and the Macaulay defintion: $D = \frac{1}{B_0} \sum_i t_i C_{t_i} e^{-yt_i}$, where B_0 is the current bond price, C_{t_i} is the coupon payment at date t_i, r is the current interest rate and y is the bond's yield to maturity.

1. Show that these two definitions will produce the same number if the bond is correctly priced.

4.3 Further readings

The material in this chapter is covered in most standard textbooks on investments (e.g. Bodie et al. (2021), Haugen (2001) or Sharpe et al. (1999)). More details can be found in textbooks on fixed income, such as Sundaresan (2001).

Chapter 5

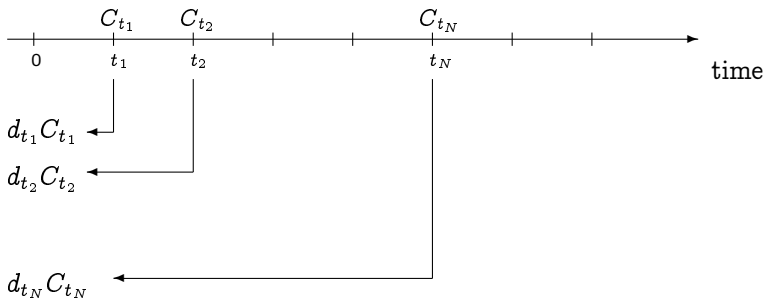
The term structure of interest rates and an object lesson

Contents

In this chapter we expand on the analysis of the previous chapter by relaxing the "one interest rate" assumption used there and allow the spot rates to change as you change the time you are discounting over.

Recall that we said that the present value of a set of cash flows is calculated as

$$PV = \sum_{i=1}^{N} d_{t_i} C_{t_i}$$



To make this applicable to cash flows received at any future date t we potentially need an infinite number of discount factors d_t. This is not feasible, so some lower dimensional way needs to be found to approximate d_t, but with more flexibility than the extremely strong assumption that there is one fixed interest rate r, and that the discount factor for any time t is calculated as either $d_t = 1/(1+r)^t$ (discrete compounding), or $d_t = e^{-rt}$ (continuous compounding), which we used in the previous chapter.

In this chapter we first show that this approximation of the discount factors can be done in either terms of discount factors directly, interest rates, or forward rates. Either of these are useful ways of formulating a term structure, and either of them can be used, since there are one to one transformations between either of these three. We then go on to demonstrate how a feature of C++, the ability to create an abstract datatype as an object, or class, is very useful for the particular application of defining and using a term structure. It is in fact this particular application, to create a term structure class, which really illustrates the power of C++, and why you want to use an object oriented language instead of classical langues like `FORTRAN` and *C*, or matrix languages like `Gauss` or `Matlab` for many financial calculations.

5.1 The interchangeability of discount factors, spot interest rates and forward interest rates

The term structure can be specified in terms of either discount factors, spot interest rates or forward interest rates. A discount factor is the current price for a future (time t) payment of one dollar. To find the current value PV of a cash flow C_t, we calculate $PV = d_t C_t$. This discount factor can also be specified in terms of interest rates, where we let r_t be the relevant interest rate (spot rate) for discounting a t-period cashflow. Then we know that the present value $PV = e^{-r_t t} C_t$. Since these two methods of calculating the present value must be consistent,

$$PV = d_t C_t = e^{-r_t t} C_t$$

and hence

$$d_t = e^{-r_t t}$$

Note that this equation calculates d_t given r_t. Rearranging this equation we find the spot rate r_t in terms of discount factors

$$r_t = \frac{-\ln(d_t)}{t}$$

An alternative concept that is very useful is a forward interest rate, the yield on borrowing at some future date t_1 and repaying it at a later date t_2. Let f_{t_1,t_2} be this interest rate. If we invest one dollar today, at the current spot rate spot rate till period t_1 and the forward rate for the period from t_1 to t_2 (which is what you would have to do to make an actual investment), you would get the following future value

$$FV = e^{r_{t_1} t_1} e^{f_{t_1,t_2}(t_2 - t_1)}$$

The present value of this forward value using the time t_2 discount factor has to equal one:

$$d_{t_2} FV = 1$$

These considerations are enough to calculate the relevant transforms. The forward rate for borrowing at time t_1 for delivery at time t_2 is calculated as

$$f_{t_1,t_2} = \frac{-\ln\left(\frac{d_{t_2}}{d_{t_1}}\right)}{t_2 - t_1} = \frac{\ln\left(\frac{d_{t_1}}{d_{t_2}}\right)}{t_2 - t_1}$$

The forward rate can also be calculated directly from yields as

$$f_{t_1,t_2} = r_{t_2} \frac{t_2}{t_2 - t_1} - r_{t_1} \frac{t_1}{t_2 - t_1}$$

$$d_t = e^{-r_t t}$$

$$r_t = \frac{-\ln(d_t)}{t}$$

$$f_{t_1,t_2} = \frac{\ln\left(\frac{d_{t_1}}{d_{t_2}}\right)}{t_2 - t_1}$$

$$f_{t_1,t_2} = r_{t_2}\frac{t_2}{t_2 - t_1} - r_{t_1}\frac{t_1}{t_2 - t_1}$$

Notation: d_t discount factor for payment at time t, r_t: spot rate applying to cash flows at time t. f_{t_1,t_2} forward rate between time t_1 and t_2, i.e. the interest rate you would agree on today on the future transactions.

```
#include <cmath>
using namespace std;

double term_structure_yield_from_discount_factor(const double& d_t, const double& t) {
    return (-log(d_t)/t);
}

double term_structure_discount_factor_from_yield(const double& r, const double& t) {
    return exp(-r*t);
};

double term_structure_forward_rate_from_discount_factors(const double& d_t1, const double& d_t2,
                                                         const double& time) {
    return (log (d_t1/d_t2))/time;
};

double term_structure_forward_rate_from_yields(const double& r_t1, const double& r_t2,
                                               const double& t1, const double& t2) {
    return r_t2*t2/(t2-t1)-r_t1*t1/(t2-t1);
};
```

C++ Code 5.1: Term structure transformations

C++ Code 5.1 shows the implementation of these transformations.

Example

You are given the one period spot rate $r_1 = 5\%$ and the two period discount factor $d_2 = 0.9$. Calculate the two period spot rate and the forward rate from 1 to 2.

C++ program:

```
double t1=1; double r_t1=0.05; double d_t1=term_structure_discount_factor_from_yield(r_t1,t1);
cout << " a " << t1 << " period spot rate of " << r_t1
    << " corresponds to a discount factor of " << d_t1 << endl;
double t2=2; double d_t2 = 0.9;
double r_t2 = term_structure_yield_from_discount_factor(d_t2,t2);
cout << " a " << t2 << " period discount factor of " << d_t2
    << " corresponds to a spot rate of " << r_t2 << endl;
cout << " the forward rate between " << t1 << " and " << t2
    << " is " << term_structure_forward_rate_from_discount_factors(d_t1,d_t2,t2-t1)
    << " using discount factors " << endl;
cout << " and is " << term_structure_forward_rate_from_yields(r_t1,r_t2,t1,t2)
    << " using yields " << endl;
```

Output from C++ program:

```
a 1 period spot rate of 0.05 corresponds to a discount factor of 0.951229
a 2 period discount factor of 0.9 corresponds to a spot rate of 0.0526803
the forward rate between 1 and 2 is 0.0553605 using discount factors
 and is 0.0553605 using yields
```

5.2 The term structure as an object

From the previous we see that the term structure can be described in terms of discount factors, spot rates or forward rates, but that does not help us in getting round the dimensionality problem. If we think in terms of discount factors, for a complete specification of the current term structure one needs an infinite number of discount factors $\{d_t\}_{t\in\mathbb{R}^+}$. It is perhaps easier to think about this set of discount factors as a *function* $d(t)$, that, given a nonnegative time t, returns the discount factor. Since we have established that there are three equivalent ways of defining a term structure, discount factors, spot rates and forward rates, we can therefore describe a term structure as a collection of three different functions that offer different views of the same underlying object.

A term structure is an abstract object that to the user should provide

- discount factors d (prices of zero coupon bonds).
- spot rates r (yields of zero coupon bonds).
- forward rates f

for any future maturity t. The user of a term structure will not need to know how the term structure is implemented, all that is needed is an interface that specifies the above three functions.

This is tailor made for being implemented as a C++ *class*. A class in C++ terms is a collection of data structures and functions that operate on these data structures. In the present context it is a way of specifying the three functions.

r(t)

d(t)

f(t)

5.2.1 Base class

Header File 5.1 shows how we describe the generic term structure as a C++ class.

```
#ifndef _TERM_STRUCTURE_CLASS_H_
#define _TERM_STRUCTURE_CLASS_H_

class term_structure_class {
public:
    virtual double r(const double& t) const; // yield on zero coupon bond
    virtual double d(const double& t) const; // discount factor/price of zero coupon bond
    virtual double f(const double& t1, const double& t2) const; // forward rate
    virtual ~term_structure_class();
};

#endif
```

Header file 5.1: Header file describing the `term_structure` base class

The code for these functions uses algorithms that are described earlier in this chapter for transforming between various views of the term structure. The term structure class merely provide a convenient interface to these algorithms. The code is shown in **C++ Code 5.2**

Note that the definitions of calculations are circular. Any given *specific* type of term structure has to over-ride at least one of the functions `r` (yield), `d` (discount factor) or `f` (forward rate).

We next consider two examples of *specific* term structures.

```
#include "fin_recipes.h"

term_structure_class::~term_structure_class(){};

double term_structure_class::f(const double& t1, const double& t2) const{
   double d1 = d(t1);
   double d2 = d(t2);
   return term_structure_forward_rate_from_discount_factors(d1,d2,t2-t1);
};

double term_structure_class::r(const double& t) const{
    return term_structure_yield_from_discount_factor(d(t),t);
};

double term_structure_class::d(const double& t) const {
    return term_structure_discount_factor_from_yield(r(t),t);
};
```

C++ Code 5.2: Default code for transformations between discount factors, spot rates and forward rates in a term structure class

5.2.2 Flat term structure.

The flat term structure overrides the `yield` member function of the base class.

The only piece of data this type of term structure needs is an interest rate.

```
#ifndef _TERM_STRUCTURE_CLASS_FLAT_
#define _TERM_STRUCTURE_CLASS_FLAT_

#include "term_structure_class.h"

class term_structure_class_flat : public term_structure_class {
private:
    double R_;                          // interest rate
public:
    term_structure_class_flat(const double& r);
    virtual ~term_structure_class_flat();
    virtual double r(const double& t) const;
    void set_int_rate(const double& r);
};

#endif
```

Header file 5.2: Header file for term structure class using a flat term structure

```
#include "fin_recipes.h"

term_structure_class_flat::term_structure_class_flat(const double& r){ R_ = r; };

term_structure_class_flat::~term_structure_class_flat(){};

double term_structure_class_flat::r(const double& T) const { if (T>=0) return R_; return 0; };

void term_structure_class_flat::set_int_rate(const double& r) { R_ = r; };
```

C++ Code 5.3: Implementing term structure class using a flat term structure

Example

The term structure is flat with $r = 5\%$. Determine the discount factors for years 1 and 2 and the forward rate between 1 and 2.

C++ program:

```
term_structure_class_flat ts(0.05);
double t1=1;
cout << "discount factor t1 = " << t1 << ":" << ts.d(t1) << endl;
double t2=2;
cout << "discount factor t2 = " << t2 << ":" << ts.d(t2) << endl;
cout << "spot rate t = " << t1 << ":" << ts.r(t1) << endl;
cout << "spot rate t = " << t2 << ":" << ts.r(t2) << endl;
cout << "forward rate from t1= " << t1 << " to t2= " << t2 << ":"
     << ts.f(t1,t2) << endl;
```

Output from C++ program:

```
discount factor t1 = 1:0.951229
discount factor t2 = 2:0.904837
spot rate t = 1:0.05
spot rate t = 2:0.05
forward rate from t1= 1 to t2= 2:0.05
```

5.3 Using the currently observed term structure.

A first step to a general term structure is to use market data on bond prices and interest rates to infer discount factors. The simplest way of presenting this problem is in terms of linear algebra. Suppose we have three bonds with cash flows at times 1, 2 and 3.

$$\begin{array}{rcl} B_1 & = & d_1C_{11} + d_2C_{12} + d_2C_{13} \\ B_2 & = & d_1C_{21} + d_2C_{22} + d_2C_{23} \\ B_1 & = & d_1C_{31} + d_2C_{32} + d_2C_{33} \end{array}$$

Writing this in terms of matrices

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Solving for the discount factors

$$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

or

$$\mathbf{d} = \mathbf{C}^{-1}\mathbf{B}$$

using the obvious matrix definitions

$$\mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Example

The following set of bond and bond prices is observed:

		Time (in years)						
	Bond	0	0.5	1	1.5	2	3	4
Bond	Price							
1	98		100					
2	96			100				
2	92				100			
3	118		10	10	10	110		
4	109		8	8	8	8	108	
5	112		9	9	9	9	9	109

The bonds are treasury secrurities, which can be viewed as nominally riskless.

1. For what maturities is it possible to infer discount factors?
2. Determine the implicit discount factors in the market.
3. What is the interest rates implied in this set of discount rates?

Here we can find the discount factors for maturities of 0.5,1,1.5,2,3 and 4.

Matlab program:

```
C=[100 0 0 0 0 0;0 100 0 0 0 0;0 0 100 0 0 0;10 10 10 110 0 0;8 8 8 8 108 0;9 9 9 9 9 109]
B=[96 94 92 118 109 112]
d=B*inv(C)'
t=[0.5 1 1.5 2 3 4]
r=d.^(-1./t)-1
```

Output from Matlab program:

```
C =
 100  0  0  0  0  0
 0  100  0  0  0  0
 0  0  100  0  0  0
 10  10  10  110  0  0
 8  8  8  8  108  0
 9  9  9  9  9  109
B =
 96  94  92  118  109  112
d =
 0.9600 0.9400 0.9200 0.8164 0.7399 0.6662
t =
 0.5000 1.0000 1.5000 2.0000 3.0000 4.0000
r =
 0.085069 0.063830 0.057162 0.106772 0.105628 0.106884
```

To just use todays term structure, we need to take the observations of discount factors d_t observed in the market and use these to generate a term structure. The simplest possible way of doing this is to linearly interpolate the currently observable discount factors.

5.3.1 Linear Interpolation.

If we are given a set of discount factors (d_t) for various maturities, the simplest way to construct a term structure is by straightforward linear interpolation between the observations we have to find an intermediate time. For many purposes this is "good enough." This interpolation can be on either zero coupon yields, discount factors or forward rates, we illustrate the case of linear interpolation of spot (zero coupon) rates.

Computer algorithm, linear interpolation of yields. Note that the algorithm assumes the yields are ordered in increasing order of time to maturity.

Example

You observe the following term structure of spot rates

Time	r
0.1	0.1
0.5	0.2
1	0.3
5	0.4
10	0.5

Interpolate spot rates (zero rates) at times 0.1, 0.5, 1, 3, 5 and 10.

```
#include <vector>
using namespace std;
#include "fin_recipes.h"

double term_structure_yield_linearly_interpolated(const double& time,
                                                  const vector<double>& obs_times,
                                                  const vector<double>& obs_yields) {
    // assume the yields are in increasing time to maturity order.
    int no_obs = int(obs_times.size());
    if (no_obs<1) return 0;
    double t_min = obs_times[0];
    if (time <= t_min) return obs_yields[0]; // earlier than lowest obs.

    double t_max = obs_times[no_obs-1];
    if (time >= t_max) return obs_yields[no_obs-1]; // later than latest obs

    int t=1;  // find which two observations we are between
    while ( (t<no_obs) && (time>obs_times[t]))  { ++t; };
    double lambda = (obs_times[t]-time)/(obs_times[t]-obs_times[t-1]);
    // by ordering assumption, time is between t-1,t
    double r = obs_yields[t-1] * lambda + obs_yields[t] * (1.0-lambda);
    return r;
};
```

C++ Code 5.4: Interpolated term structure from spot rates

C++ program:

```
vector<double> times;
vector<double> yields;
times.push_back(0.1); times.push_back(0.5); times.push_back(1);
yields.push_back(0.1); yields.push_back(0.2); yields.push_back(0.3);
times.push_back(5); times.push_back(10);
yields.push_back(0.4); yields.push_back(0.5);
cout << " yields at times: " << endl;
cout << " t=0.1 " << term_structure_yield_linearly_interpolated(0.1,times,yields) << endl;
cout << " t=0.5 " << term_structure_yield_linearly_interpolated(0.5,times,yields) << endl;
cout << " t=1 " << term_structure_yield_linearly_interpolated(1, times,yields) << endl;
cout << " t=3 " << term_structure_yield_linearly_interpolated(3, times,yields) << endl;
cout << " t=5 " << term_structure_yield_linearly_interpolated(5, times,yields) << endl;
cout << " t=10 " << term_structure_yield_linearly_interpolated(10, times,yields) << endl;
```

Output from C++ program:

```
yields at times:
t=0.1 0.1
t=0.5 0.2
t=1   0.3
t=3   0.35
t=5   0.4
t=10  0.5
```

5.3.2 Interpolated term structure class.

The interpolated term structure implemented here uses a set of observations of yields as a basis, and for observations in between observations will interpolate between the two closest. The following only provides implementations of calculation of the yield, for the other two rely on the base class code.

As shown in **Header File 5.3** and **C++ Code 5.5**, there is some more book-keeping involved here, need to have code that stores observations of times and yields.

```
#ifndef _TERM_STRUCTURE_CLASS_INTERPOLATED_
#define _TERM_STRUCTURE_CLASS_INTERPOLATED_

#include "term_structure_class.h"
#include <vector>
using namespace std;

class term_structure_class_interpolated : public term_structure_class {
private:
    vector<double> times_; // use to keep a list of yields
    vector<double> yields_;
    void clear();
public:
    term_structure_class_interpolated();
    term_structure_class_interpolated(const vector<double>& times, const vector<double>& yields);
    virtual ~term_structure_class_interpolated();
    term_structure_class_interpolated(const term_structure_class_interpolated&);
    term_structure_class_interpolated operator= (const term_structure_class_interpolated&);

    int no_observations() const { return times_.size(); };
    virtual double r(const double& T) const;
    void set_interpolated_observations(vector<double>& times, vector<double>& yields);
};

#endif
```

Header file 5.3: Header file describing a term structure class using linear interpolation between spot rates

```
#include "fin_recipes.h"

void term_structure_class_interpolated::clear(){
    times_.erase(times_.begin(), times_.end());
    yields_.erase(yields_.begin(), yields_.end());
};

term_structure_class_interpolated::term_structure_class_interpolated():term_structure_class(){clear();};

term_structure_class_interpolated::term_structure_class_interpolated(const vector<double>& in_times,
                                                                     const vector<double>& in_yields) {
    clear();
    if (in_times.size()!=in_yields.size()) return;
    times_= vector<double>(in_times.size());
    yields_= vector<double>(in_yields.size());
    for (int i=0;i<in_times.size();i++) {
        times_[i]=in_times[i];
        yields_[i]=in_yields[i];
    };
};

term_structure_class_interpolated::~term_structure_class_interpolated(){ clear();};

term_structure_class_interpolated::term_structure_class_interpolated(const term_structure_class_interpolated& term) {
    times_          = vector<double> (term.no_observations());
    yields_         = vector<double> (term.no_observations());
    for (int i=0;i<term.no_observations();i++){
        times_[i]   = term.times_[i];
        yields_[i]  = term.yields_[i];
    };
};

term_structure_class_interpolated
term_structure_class_interpolated::operator= (const term_structure_class_interpolated& term) {
    times_          = vector<double> (term.no_observations());
    yields_         = vector<double> (term.no_observations());
    for (int i=0;i<term.no_observations();i++){
        times_[i]   = term.times_[i];
        yields_[i]  = term.yields_[i];
    };
    return (*this);
};

double term_structure_class_interpolated::r(const double& T) const {
    return term_structure_yield_linearly_interpolated(T, times_, yields_);
};

void
term_structure_class_interpolated::set_interpolated_observations(vector<double>& in_times,
                                                                 vector<double>& in_yields) {
    clear();
    if (in_times.size()!=in_yields.size()) return;
    times_= vector<double>(in_times.size());
    yields_= vector<double>(in_yields.size());
    for (int i=0;i<in_times.size();i++) {
        times_[i]=in_times[i];
        yields_[i]=in_yields[i];
    };
};
```

C++ Code 5.5: Term structure class using linear interpolation between spot rates

Example

Time	r
0.1	0.05
1	0.07
5	0.08

Determine discount factors and spot rates at times 1 and 2, and forward rate between 1 and 2.

C++ program:

```
vector<double> times;  times.push_back(0.1);
vector<double> spotrates; spotrates.push_back(0.05);
times.push_back(1);     times.push_back(5);
spotrates.push_back(0.07);spotrates.push_back(0.08);
term_structure_class_interpolated ts(times,spotrates);
double t1=1;
cout << "discount factor t1 = " << t1 << ":" << ts.d(t1) << endl;
double t2=2;
cout << "discount factor t2 = " << t2 << ":" << ts.d(t2) << endl;
cout << "spot rate t = " << t1 << ":" << ts.r(t1) << endl;
cout << "spot rate t = " << t2 << ":" << ts.r(t2) << endl;
cout << "forward rate from t1= " << t1 << " to t2= " << t2 << ":"
    << ts.f(t1,t2) << endl;
```

Output from C++ program:

```
discount factor t1 = 1:0.932394
discount factor t2 = 2:0.865022
spot rate t = 1:0.07
spot rate t = 2:0.0725
forward rate from t1= 1 to t2= 2:0.075
```

5.4 Bond calculations with a general term structure and continous compounding

Coupon bond paying coupons at dates $t_1, t_2, \ldots$:
Bond Price B_0:

$$B_0 = \sum_i d_{t_i} C_{t_i} = \sum_i e^{-r_{t_i} t_i} C_{t_i}$$

Duration D:

$$D = \frac{1}{B_0} \sum_i t_i d_{t_i} C_{t_i}$$

$$D = \frac{1}{B_0} \sum_i t_i e^{-r_{t_i} t_i} C_{t_i}$$

$$D = \frac{1}{B_0} \sum_i t_i e^{-y t_i} C_{t_i}$$

Yield to maturity y solves:

$$B_0 = \sum_i C_{t_i} e^{-y t_i}$$

Convexity Cx:

$$Cx = \frac{1}{B_0} \sum_i t_i^2 d_{t_i} C_{t_i}$$

$$Cx = \frac{1}{B_0} \sum_i t_i^2 e^{-r_{t_i} t_i} C_{t_i}$$

$$Cx = \frac{1}{B_0} \sum_i t_i^2 e^{-y t_i} C_{t_i}$$

Formula 5.1: Bond pricing with a continously compounded term structure

C++ Code 5.6 and **C++ Code 5.7** illustrates how one would calculate bond prices and duration if one has a term structure class.

```
#include "fin_recipes.h"

double bonds_price(const vector<double>& cashflow_times,
                   const vector<double>& cashflows,
                   const term_structure_class& d) {
    double p = 0;
    for (unsigned i=0;i<cashflow_times.size();i++) {
        p += d.d(cashflow_times[i])*cashflows[i];
    };
    return p;
};
```

C++ Code 5.6: Pricing a bond with a term structure class

```
#include "fin_recipes.h"

double bonds_duration(const vector<double>& cashflow_times,
                      const vector<double>& cashflow_amounts,
                      const term_structure_class& d ) {
    double S=0;
    double D1=0;
    for (unsigned i=0;i<cashflow_times.size();i++){
        S += cashflow_amounts[i] * d.d(cashflow_times[i]);
        D1 += cashflow_times[i] * cashflow_amounts[i] * d.d(cashflow_times[i]);
    };
    return D1/S;
};
```

C++ Code 5.7: Calculating a bonds duration with a term structure class

```
#include "fin_recipes.h"
#include <cmath>

double bonds_convexity(const vector<double>& cashflow_times,
                       const vector<double>& cashflow_amounts,
                       const term_structure_class& d ) {
    double B=0;
    double Cx=0;
    for (unsigned i=0;i<cashflow_times.size();i++){
        B += cashflow_amounts[i] * d.d(cashflow_times[i]);
        Cx += pow(cashflow_times[i],2) * cashflow_amounts[i] * d.d(cashflow_times[i]);
    };
    return Cx/B;
};
```

C++ Code 5.8: Calculating a bonds convexity with a term structure class

Example

The term structure is flat with $r = 10\%$ continously compunded interest. Calculate price, duration, and convexity of a 10%, 2 year bond.

C++ program:

```
vector <double> times;  times.push_back(1);     times.push_back(2);
vector <double> cashflows; cashflows.push_back(10); cashflows.push_back(110);
term_structure_class_flat tsflat(0.1);
cout << " price = " << bonds_price (times, cashflows, tsflat) << endl;
cout << " duration = " << bonds_duration(times, cashflows, tsflat) << endl;
cout << " convexity = " << bonds_convexity(times, cashflows, tsflat) << endl;
```

Output from C++ program:

```
price = 99.1088
duration = 1.9087
convexity = 3.72611
```

References Shiller (1990) is a good reference on the use of the term structure.

Chapter 6

The Mean Variance Frontier

Contents

We now discuss a classical topic in finance, mean variance analysis. This leads to ways of accounting for the riskiness of cashflows.

Mean variance analysis concerns investors choices between portfolios of risky assets, and how an investor chooses portfolio weights. Let r_p be a portfolio return. We assume that investors preferences over portfolios p satisfy a mean variance utility representation, $u(p) = u(E[r_p], \sigma(r_p))$, with utility increasing in expected return ($\partial u / \partial E[r_p] > 0$) and decreasing in variance ($\partial u / \partial \text{var}(r_p) < 0$). In this part we consider the representation of the *portfolio opportunity set* of such decision makers. There are a number of useful properties of this opportunity set which follows purely from the mathematical formulation of the optimization problem. It is these properties we focus on here.

6.1 Setup

We assume there exists $n \geq 2$ risky securities, with expected returns $\mathbf{e}$

$$\mathbf{e} = \begin{bmatrix} E[r_1] \\ E[r_2] \\ \vdots \\ E[r_n] \end{bmatrix}$$

and covariance matrix $\mathbf{V}$:

$$\mathbf{V} = \begin{bmatrix} \sigma(r_1, r_1) & \sigma(r_1, r_2) & \dots & \\ \sigma(r_2, r_1) & \sigma(r_2, r_2) & \dots & \\ \vdots & & & \\ \sigma(r_n, r_1) & & \dots & \sigma(r_n, r_n) \end{bmatrix}$$

The covariance matrix $\mathbf{V}$ is assumed to be invertible.

A *portfolio* p is defined by a set of weights $\mathbf{w}$ invested in the risky assets.

$$\mathbf{w} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix},$$

where w_i is the fraction of the investors wealth invested in asset i. Note that the weights sum to one. The expected return on a portfolio is calculated as

$$E[r_p] = \mathbf{w}'\mathbf{e}$$

and the variance of the portfolio is

$$\sigma^2(r_p) = \mathbf{w}'\mathbf{V}\mathbf{w}$$

Example

An investor can invest in three assets with expected returns and variances as specified in the following table.

t Asset	$E[r]$	$\sigma^2(r)$
1	10%	0.20
2	11.5%	0.10
3	8%	0.15

The three assets are independent (uncorrelated).

1. Determine the expected return and standard deviation of an equally weighted portfolio of the three assets

Matlab program:

```
e=[0.1 0.11 0.08]
V=[ 0.2 0 0; 0 0.1 0 ; 0 0 0.15]
w=1/3*[1 1 1]
er= e*w'
sigma=sqrt(w*V*w')
```

Output from Matlab program:

```
e =
 0.100000 0.110000 0.080000
V =
 0.2000 0 0
 0 0.1000 0
 0 0 0.1500
w =
 0.3333 0.3333 0.3333
er = 0.096667
sigma = 0.2236
```

6.2 The minimum variance frontier

A portfolio is a *frontier* portfolio if it minimizes the variance for a given expected return, that is, a frontier portfolio p solves

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

subject to:

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

The set of all frontier portfolios is called the *minimum variance frontier*.

6.3 Calculation of frontier portfolios

Proposition 1 *If the matrix $\mathbf{V}$ is full rank, and there are no restrictions on shortsales, the weights $\mathbf{w}_p$ for a frontier portfolio p with mean $E[\tilde{r}_p]$ can be found as*

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}E[r_p]$$

where

$$\mathbf{g} = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1}$$

$$\mathbf{h} = \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}$$

$$A = \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

$$B = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

$$C = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$\mathbf{A} = \begin{bmatrix} B & A \\ A & C \end{bmatrix}$$

$$D = BC - A^2 = |\mathbf{A}|$$

Proof

Any minimum variance portfolio solves the program

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

subject to

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

Set up the Lagrangian corresponding to this problem

$$L(\mathbf{w}, \lambda, \gamma|\mathbf{e}, \mathbf{V}) = \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w} - \lambda\left(E[\tilde{r}_p] - \mathbf{w}'\mathbf{e}\right) - \gamma(1 - \mathbf{w}'\mathbf{1})$$

Differentiate

$$\frac{\partial L}{\partial \mathbf{w}} = \mathbf{w}'\mathbf{V} - \lambda \mathbf{e}' - \gamma \mathbf{1}' = 0$$

$$\frac{\partial L}{\partial \lambda} = E[r_p] - \mathbf{w}'\mathbf{e} = 0$$

$$\frac{\partial L}{\partial \gamma} = 1 - \mathbf{w}'\mathbf{1} = 0$$

Rewrite conditions above as (note that this requires the invertibility of $\mathbf{V}$.

$$\mathbf{w}' = \lambda \mathbf{e}' V^{-1} - \gamma \mathbf{1}' \mathbf{V}^{-1} \tag{6.1}$$

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p] \tag{6.2}$$

$$\mathbf{w}'\mathbf{1} = 1 \tag{6.3}$$

Post-multiply equation (6.1) with $\mathbf{e}$ and recognise the expression for $E[r_p]$ in the second equation

$$\mathbf{w}'\mathbf{e} = E[r_p] = \lambda \mathbf{e}'\mathbf{V}^{-1}\mathbf{e} + \gamma \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

Similarly post-multiply equation (6.1) with $\mathbf{1}$ and recognise the expression for 1 in the third equation

$$\mathbf{w}'\mathbf{1} = 1 = \lambda \mathbf{e}'\mathbf{V}^{-1}\mathbf{1} + \gamma \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

With the definitions of A, B, C and D above, this becomes the following system of equations

$$\left\{ \begin{array}{rcl} E[r_p] & = & \lambda B + \gamma A \\ 1 & = & \lambda A + \gamma C \end{array} \right\}$$

Solving for λ and γ, get

$$\gamma = \frac{B - AE[r_p]}{D}$$

$$\lambda = \frac{CE[r_p] - A}{D}$$

Plug in expressions for λ and γ into equation (6.1) above, and get

$$\mathbf{w}' = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1} + \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}E[r_p] = \mathbf{g} + \mathbf{h}E[r_p]$$

The portfolio defined by weights $\mathbf{g}$ is a portfolio with expected return 0. The portfolio defined by weights $(\mathbf{g}+\mathbf{h})$ is a portfolio with expected return 1. This implies the useful property that $\mathbf{g}\mathbf{1}' = 1$, and $\mathbf{h}\mathbf{1}' = 0$.

Example

An investor can invest in three assets with expected returns and variances as specified in the following table.

t Asset	$E[r]$	$\sigma^2(r)$
1	10%	0.20
2	11.5%	0.10
3	8%	0.15

The three assets are independent (uncorrelated).

1. What are the weights of the minimum variance portfolio with mean 9%?

Matlab program:

```
e=[0.1 0.11 0.08]'
V=[ 0.2 0 0; 0 0.1 0 ; 0 0 0.15]
r=0.09
n = length(e)
a = ones(1,n)*inv(V)*e
b = e'*inv(V)*e
c = ones(1,n)*inv(V)*ones(n,1)
A = [b a;a c]
d = det(A)
g = 1/d*(b*ones(1,n) - a*e')*inv(V)
h = h = 1/d*(c*e' - a*ones(1,n))*inv(V)
w=g+h*r
```

Output from Matlab program:

```
e =
 0.100000
 0.110000
 0.080000
V =
 0.2000 0 0
 0 0.1000 0
 0 0 0.1500
r = 0.090000
n = 3
a = 2.1333
b = 0.2137
c = 21.667
A =
 0.2137 2.1333
 2.1333 21.6667
d = 0.078333
g =
 0.021277 -2.680851 3.659574
h =
 2.1277 31.9149 -34.0426
w =
 0.2128 0.1915 0.5957
```

This calculation is put into a Matlab function in Matlab **Code 6.1**.

```
function w = min_variance_portfolio(e,V,r)
   n = length(e);
   a = ones(1,n)*inv(V)*e;
   b = e'*inv(V)*e;
   c = ones(1,n)*inv(V)*ones(n,1);
   A = [b a;a c];
   d = det(A);
   g = 1/d*(b*ones(1,n) - a*e')*inv(V);
   h = h = 1/d*(c*e' - a*ones(1,n))*inv(V);
   w=g+h*r;
end
```

Matlab **Code 6.1**: Calculation of minimum variance portfolio for given return

Calculating the weights of the minimum variance portfolio given an interest rate

$$\mathbf{w}_p = \mathbf{g} + \mathbf{h}E[r_p]$$

where

$$\mathbf{g} = \frac{1}{D}\left(B\mathbf{1}' - A\mathbf{e}'\right)\mathbf{V}^{-1}$$

$$\mathbf{h} = \frac{1}{D}\left(C\mathbf{e}' - A\mathbf{1}'\right)\mathbf{V}^{-1}$$

$$A = \mathbf{1}'\mathbf{V}^{-1}\mathbf{e}$$

$$B = \mathbf{e}'\mathbf{V}^{-1}\mathbf{e}$$

$$C = \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}$$

$$D = BC - A^2 =$$

Notation: r_p desired portfolio returns.$\mathbf{V}$: covariance matrix of asset returns. $\mathbf{e}$: vector of expected asset returns.

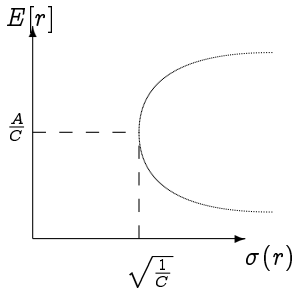
6.4 The global minimum variance portfolio

The portfolio that minimizes variance regardless of expected return is called the *global minimum variance portfolio*. Let mvp be the global minimum variance portfolio.

Proposition 2 (Global Minimum Variance Portfolio) *The global minimum variance portfolio has weights*

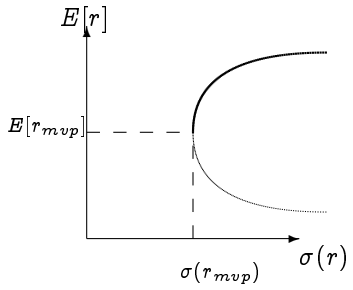
$$\mathbf{w}'_{mvp} = \left(\mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\right)^{-1}\mathbf{1}'\mathbf{V}^{-1} = \frac{1}{C}\mathbf{1}'\mathbf{V}^{-1},$$

expected return $E[r_{mvp}] = \frac{A}{C}$ *and variance* $var(r_{mvp}) = \frac{1}{C}$.



6.5 Efficient portfolios

Portfolios on the minimum variance frontier with expected returns higher than or equal to $E[r_{mvp}]$ are called *efficient* portfolios.



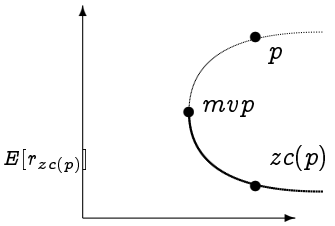
6.6 The zero beta portfolio

Proposition 3 *For any portfolio p on the frontier, there is a frontier portfolio $zc(p)$ satisfying*

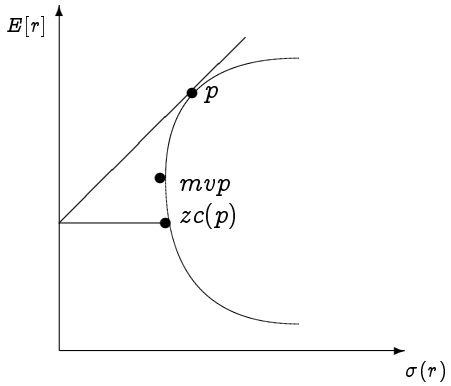
$$cov(r_{zc(p)}, r_p) = 0.$$

This portfolio is called the zero beta portfolio relative to p. The zero beta portfolio $zb(p)$ has return

$$E[r_{zc(p)}] = \frac{A}{C} - \frac{\frac{D}{C^2}}{E[r_p] - \frac{A}{C}}$$



Note that if p is an efficient portfolio on the mean variance frontier then $zc(p)$ is inefficient. Conversely, if p is inefficient $zc(p)$ is efficient.



6.7 Allowing for a riskless asset.

Suppose have N risky assets with weights $\mathbf{w}$ and one riskless assets with return r_f.
Intuitively, the return on a portfolio with a mix of risky and risky assets can be written as

$$E[r_p] = \text{weight in risky} \times \text{ return risky } + \text{weight riskless } \times r_f$$

which in vector form is:

$$E[r_p] = \mathbf{w}'\mathbf{e} + (1 - \mathbf{w}'\mathbf{1})r_f$$

Proposition 4 *An efficient portfolio in the presence of a riskless asset has the weights*

$$\mathbf{w}_p = \mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)\frac{E[r_p] - r_f}{H}$$

where

$$H = (\mathbf{e} - \mathbf{1}r_f)'\mathbf{V}^{-1}(\mathbf{e} - \mathbf{1}r_f)$$

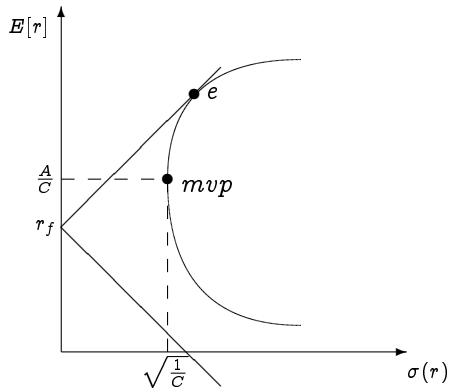
The variance of the efficient portfolio is

$$\sigma^2(r_p) = \frac{(E[r_p] - r_f)^2}{H}$$

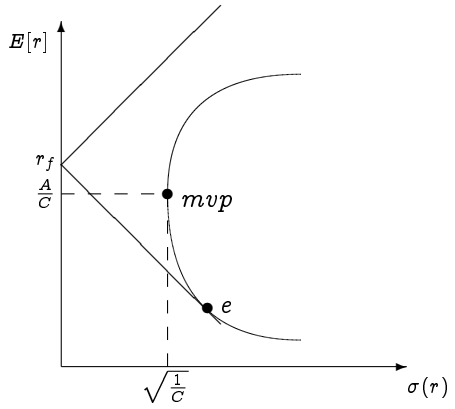
Note that standard deviation is a linear function of $E[r_p]$. The efficient set is a line in mean-standard deviation space.

6.8 Efficient sets with risk free assets.

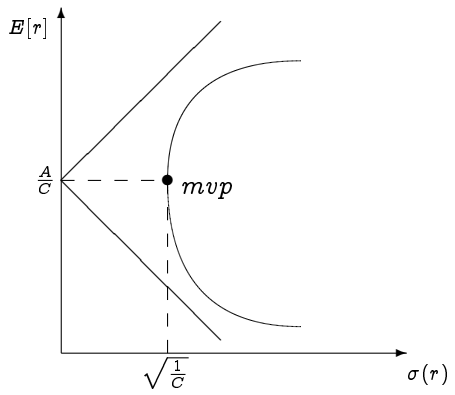
Suppose $r_f < \frac{A}{C}$. Then the efficient set is the line from $(0, r_f)$ through tangency on the efficient set of *risky* assets.



Suppose $r_f > \frac{A}{C}$. Then the efficient set is the two half-lines starting from $(0, r_f)$.



If $r_f = \frac{A}{C}$, the weight in the risk free asset is one. The risky portfolio is an zero investment portfolio. The efficient set consists of two asymptotes toward the efficient set of risky assets.



6.9 Short-sale constraints

So far the analysis has put no restrictions on the set of weights w_p that defines the minimum variance frontier. For practical applications, existence of negative weights is problematic, since this involves selling securities short.

This has led to the investigation of *restricted* mean variance frontiers, where the weights are constrained to be non-negative.

Definition 1 *A short sale resctricted minimum variance portfolio p solves*

$$\mathbf{w}_p = \arg\min_{\mathbf{w}} \frac{1}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$$

subject to

$$\mathbf{w}'\mathbf{e} = E[\tilde{r}_p]$$

$$\mathbf{w}'\mathbf{1} = 1$$

$$\mathbf{w}' \geq \mathbf{0}$$

Such short sale resctricted minimum variance portfolio portfolios are much harder to deal with analytically, since they do not admit a general solution, one rather has to investigate the Kuhn-Tucker conditions for corner solutions etc. To deal with this problem in practice one will use a subroutine for solving constrained optimization problems.

6.10 The Sharpe Ratio

The Sharpe ratio of a given portfolio p is defined as

$$S_p = \frac{E[r_p] - r_f}{\sigma(r_p)}$$

The Sharpe ratio S_p of a portfolio p is the slope of the line in mean-standard deviations space from the risk free rate through p. Note that in the case with a risk free asset, the tangency portfolio has the maximal Sharpe Ratio on the efficient frontier.

6.11 Equilibrium: CAPM

Under certain additional assumptions, an economy of mean variance optimizers will aggregate to an economy where the Capital Asset Pricing Model (CAPM) holds. Under the CAPM, any asset returns will satisfy.

$$E[r_i] = r_f + \beta_i(E[r_m] - r_f)$$

where r_i is the return on asset i, r_f the return on the risk free asset,

6.11.1 Treynor

$$T_p = \frac{r_p - r_f}{\beta_p}$$

6.11.2 Jensen

$$\alpha_p = r_p - (r_f + \beta_p(r_m - r_f)$$

6.12 Working with Mean Variance and CAPM

The computational problems in mean variance optimization and the CAPM are not major, except for the case of short sales constrained portfolios (quadratic programming). The issues of more concern is estimation of parameters such as covariances and betas.

```
function s= sharpe(r,rf)
    s=(mean(r)-mean(rf))/std(r-rf);
endfunction
```

Matlab **Code 6.2**: Sharpe Ratio

```
function t = treynor(r, rm, rf)
     beta = cov(r,rm)/var(rm);
     t = (mean(r-rf))/beta;
endfunction
```

Matlab **Code 6.3**: Treynor Ratio

```
function alpha = jensen(r, rm, rf)
   beta = cov(r,rm)/var(rm);
   alpha = mean(r) - (rf+beta*(mean(rm)-rf));
endfunction
```

Matlab **Code 6.4**: Jensens alpha

Readings and Sources The classical sources for this material are Merton (1972) and Roll (1977a). (Huang and Litzenberger, 1988, Ch 3) has a good textbook discussion of it.

6.13 Mean variance analysis using matrix libraries

Let us now consider how to implement mean variance analysis in C++. As shown using `Matlab`, the calculations are relatively simple matrix expressions. To implement the calculations in C++ the best way of doing it is to use a linear algebra class to deal with the calculations. For illustrative purposes we will show usage of two different matrix classes, `Newmat` and `IT++`. These classes are described in more detail in an appendix, with some references to how to obtain and install them.

In **C++ Code 6.1** and **C++ Code 6.2** we show how to do the basic mean variance calculations, calculating means and standard deviations, using the two classes. Note the similarity to using `Matlab` in the way the matrix and vector multiplications are carried out. Note also an important difference between the two classes. In `IT++` the indexing of elements start at zero, the usual C++ way. Hence, when adressing element `tmp(0,0)` in the `IT++` example we are pulling the first element. In **Newmat** the default indexing starts at one, the `Matlab` way. Therefore, in adressing `tmp(1,1)` in the `Newmat` example we are *also* pulling the first element. This serves as a warning to read the documentation carefully, the "off by one" error is a very common occurrence when mixing libraries like this.

```
#include <cmath>
using namespace std;

#include <itpp/itbase.h>
using namespace itpp;

double mv_calculate_mean(const vec& e, const vec& w){
    vec tmp = e.transpose()*w;
    return tmp(0);
};

double mv_calculate_variance(const mat& V, const vec& w){
    mat tmp = w.transpose()*V*w;
    return tmp(0,0);
};

double mv_calculate_st_dev(const mat& V, const vec& w){
    double var = mv_calculate_variance(V,w);
    return sqrt(var);
};
```

C++ Code 6.1: Mean variance calculations using `IT++`

Example

Mean variance calculations.

$$\mathbf{e} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Calculate mean, variance and stdev for portfolio

$$\mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

```
#include <cmath>
using namespace std;

#include "newmat.h"
using namespace NEWMAT;

double mv_calculate_mean(const Matrix& e, const Matrix& w){
    Matrix tmp = e.t()*w;
    return tmp(1,1);
};

double mv_calculate_variance(const Matrix& V, const Matrix& w){
    Matrix tmp = w.t()*V*w;
    return tmp(1,1);
};

double mv_calculate_st_dev(const Matrix& V, const Matrix& w){
    double var = mv_calculate_variance(V,w);
    return sqrt(var);
};
```

C++ Code 6.2: Mean variance calculations using `Newmat`

C++ program:

```
    cout << "Simple example of mean variance calculations " << endl;
    Matrix e(2,1);
    e(1,1)=0.05; e(2,1)=0.1;
    Matrix V(2,2);
    V(1,1)=1.0; V(2,1)=0.0;
    V(1,2)=0.0; V(2,2)=1.0;
    Matrix w(2,1);
    w(1,1)=0.5;
    w(2,1)=0.5;
    cout << " mean " << mv_calculate_mean(e,w) << endl;
    cout << " variance " << mv_calculate_variance(V,w) << endl;
    cout << " stdev " << mv_calculate_st_dev(V,w) << endl;
```

Output from C++ program:

```
Simple example of mean variance calculations
 mean 0.075
 variance 0.5
 stdev 0.707107
```

In **C++ Code 6.4** and **C++ Code 6.3** we show how to calculate the mean variance optimal portfolio for a given required return. This is the case where there are no constraints on the weight, and we use the analytical solution directly.

```
#include "newmat.h"
using namespace NEWMAT;

ReturnMatrix mv_calculate_portfolio_given_mean_unconstrained(const Matrix& e,
                                                             const Matrix& V,
                                                             const double& r){
    int no_assets=e.Nrows();
    Matrix ones = Matrix(no_assets,1); for (int i=0;i<no_assets;++i){ ones.element(i,0) = 1; };
    Matrix Vinv = V.i(); // inverse of V
    Matrix A = (ones.t()*Vinv*e); double a = A.element(0,0);
    Matrix B = e.t()*Vinv*e;   double b = B.element(0,0);
    Matrix C = ones.t()*Vinv*ones; double c = C.element(0,0);
    double d = b*c - a*a;
    Matrix Vinv1=Vinv*ones;
    Matrix Vinve=Vinv*e;
    Matrix g = (Vinv1*b - Vinve*a)*(1.0/d);
    Matrix h = (Vinve*c - Vinv1*a)*(1.0/d);
    Matrix w = g + h*r;
    w.Release();
    return w;
};
```

C++ Code 6.3: Calculating the unconstrained frontier portfolio given an expected return using `Newmat`

Example

Mean variance calculations.

$$\mathbf{e} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix}$$

Find the optmal minimum variance portfolio with return $r = 0.075$.

C++ program:

```
    cout << "Testing portfolio calculation " << endl;
    Matrix e(2,1);
    e(1,1)=0.05; e(2,1)=0.1;
    Matrix V(2,2);
    V(1,1)=1.0; V(2,1)=0.0;
    V(1,2)=0.0; V(2,2)=1.0;
    double r=0.075;
    Matrix w = mv_calculate_portfolio_given_mean_unconstrained(e,V,r);
    cout << " suggested portfolio: ";
    cout << " w1 = " << w(1,1) << " w2 = " << w(2,1) << endl;
```

Output from C++ program:

```
Testing portfolio calculation
 suggested portfolio:   w1 = 0.5 w2 = 0.5
```

```
#include <itpp/itbase.h>
using namespace itpp;

mat mv_calculate_portfolio_given_mean_unconstrained(const vec& e,
                                                    const mat& V,
                                                    const double& r){
    int no_assets=e.size();
    vec one = ones(no_assets);
    mat Vinv = inv(V);          // inverse of V
    mat A = one.transpose()*Vinv*e;
    double a = A(0,0);
    mat B = e.transpose()*Vinv*e;
    double b = B(0,0);
    mat C = one.transpose()*Vinv*one;
    double c = C(0,0);
    double d = b*c-a*a;
    mat Vinv1=Vinv*one;
    mat Vinve=Vinv*e;
    mat g = (Vinv1*b - Vinve*a)*(1.0/d);
    mat h = (Vinve*c - Vinv1*a)*(1.0/d);
    mat w = g + h*r;
    return w;
};
```

C++ Code 6.4: Calculating the unconstrained frontier portfolio given an expected return using IT++

Chapter 7

Futures algoritms.

Contents

In this we discuss algoritms used in valuing futures contracts.

7.1 Pricing of futures contract.

The futures price of an asset without payouts is the future value of the current price of the assset.

$$f_t = e^{r(T-t)} S_t$$

```
#include <cmath>
using namespace std;

double futures_price(const double& S,                    // current price of underlying asset
                     const double& r,                     // risk free interest rate
                     const double& time_to_maturity) {
    return exp(r*time_to_maturity)*S;
};
```

C++ Code 7.1: Futures price

Example

Let $S = 100$ and $r = 10\%$. What is the futures price for a contract with time to maturity of half a year?

C++ program:

```
    double S=100;   double r=0.10;    double time=0.5;
    cout << " futures price = " << futures_price(S,r, time) << endl;
```

Output from C++ program:

```
 futures price = 105.127
```

Chapter 8

Binomial option pricing

Contents

8.1 Options

Option and other derivative pricing is one of the prime "success stories" of modern finance. An option is a derivative security, the cash flows from the security is a function of the price of some *other* security, typically called the underlying security. A call option is a right, but not obligation, to buy a given quantity of the underlying security at a given price, called the exercise price K, within a certain time interval. A put option is the right, but not obligation, to *sell* a given quantity of the underlying security to an agreed excercise price within a given time interval. If an option can only be exercised (used) at a given date (the time interval is one day), the option is called an European Option. If the option can be used in a whole time period up to a given date, the option is called American.

An option will only be used if it is valuable to the option holder. In the case of a call option, this is when the exercise price K is lower than the price one alternatively could buy the underlying security for, which is the current price of the underlying security. Hence, options have never negative cash flows at maturity. Thus, for anybody to be willing to offer an option, they must have a cost when entered into. This cost, or price, is typically called an option *premium*. As notation, let C signify the price of a call option, P the price of a put option and S the price of the underlying security. All of these prices are indexed by time. We typically let 0 be "now" and T the final maturity date of the option. From the definition of the options, it is clear that at their last possible exercise date, the maturity date, they have cash flows.

$$C_T = \max(0, S_T - K)$$

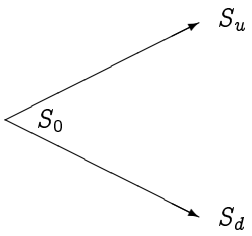
$$P_T = \max(0, K - S_T)$$

The challenge of option pricing is to determine the option premia C_0 (call price) and P_0 (put price).

8.2 Pricing

All pricing uses that the cashflows from the derivative is a direct function of the price of the underlying security. Pricing can therefore be done *relative* to the price of the underlying security. To price options

it is necessary to make assumptions about the probability distribution of movements of the underlying security. We start by considering this in a particularly simple framework, the binomial assumption. The price of the underlying is currently S_0. The price can next period only take on two values, S_u and S_d.



If one can find all possible future "states," an enumeration of all possibilities, one can value a security by constructing artificial "probabilities", called "state price probabilities," which one use to find an artificial expected value of the underlying security, which is then discounted at the risk free interest rate. The binomial framework is particularly simple, since there are only two possible states. If we find the "probability" q of one state, we also find the probability of the other as $(1-q)$. Equation 8.1 demonstrates this calculation for the underlying security.

$$S_0 = e^{-r}(qS_u + (1-q)S_d) \tag{8.1}$$

Now, any derivative security based on this underlying security can be priced using the same "probability" q. The contribution of binomial option pricing is in actually calculating the number q. To do valuation, start by introducing constants u and d implicitly defined by $S_u = uS_0$ and $S_d = dS_0$, and you get a price process as illustrated in figure 8.1.

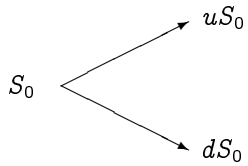


Figure 8.1: Binomial Tree

and calculate the artifical "probability" q as

$$q = \frac{e^r - d}{u - d}$$

The price of a one-period call option in a binomial framework is shown in **Formula 8.1** and implemented in **C++ Code 8.1**.

The "state price probability" q is found by an assumption of no arbitrage opportunities. If one has the possibility of trading in the underlying security and a risk free bond, it is possible to create a portfolio of these two assets that exactly duplicates the future payoffs of the derivative security. Since this portfolio has the same future payoff as the derivative, the price of the derivative has to equal the cost of the duplicating portfolio. Working out the algebra of this, one can find the expression for q as the function of the up and down movements u and d.

Exercise 8.1.

The price of the underlying security follows the binomial process

$$C_u = \max(0, S_u - K)$$

$$C_d = \max(0, S_d - K)$$

$$C_0 = e^{-r}\left(qC_u + (1-q)C_d\right)$$

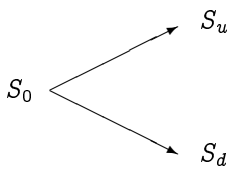
$$q = \frac{e^r - d}{u - d}$$

$S_u = uS_0$ and $S_d = dS_0$ are the possible values for the underlying security next period, u and d are constants, r is the (continously compounded) risk free interest rate and K is the call option exercise price.

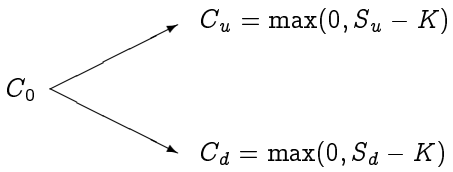
Formula 8.1: The single period binomal call option price

```
#include <cmath>          // standard mathematical library
#include <algorithm>          // defining the max() operator
using namespace std;

double option_price_call_european_binomial_single_period( const double& S, // spot price
                                                          const double& X,    // exercice price
                                                          const double& r,    // interest rate (per period)
                                                          const double& u,    // up movement
                                                          const double& d){  // down movement
   double p_up = (exp(r)-d)/(u-d);
   double p_down = 1.0-p_up;
   double c_u = max(0.0,(u*S-X));
   double c_d = max(0.0,(d*S-X));
   double call_price = exp(-r)*(p_up*c_u+p_down*c_d);
   return call_price;
};
```

C++ Code 8.1: Binomial European, one period



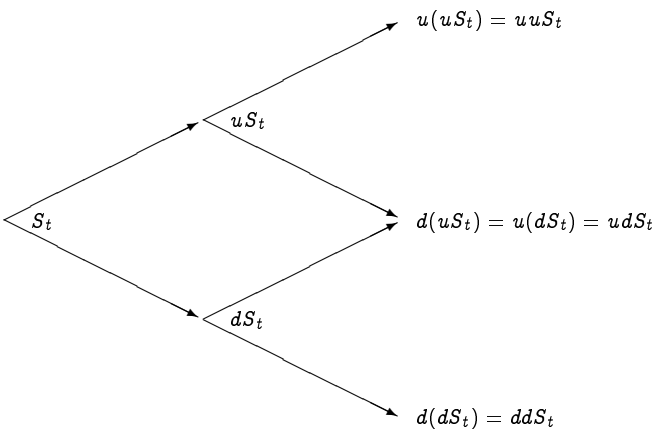
A one period call option has payoffs



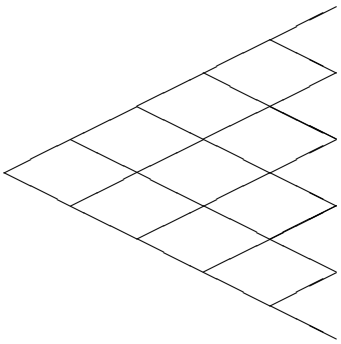
1. Show how one can combine a position in the underlying security with a position in risk free bonds to create a portfolio which exactly duplicates the payoffs from the call.
2. Use this result to show the one period pricing formula for a call option shown in formula 8.1.

8.3 Multiperiod binomial pricing

Of course, an assumption of only two possible future states next period is somewhat unrealistic, but if we iterate this assumption, and assume that every date, there are only two possible outcomes *next* date, but then, for each of these two outcomes, there is two new outcomes, as illustrated in the next figure:

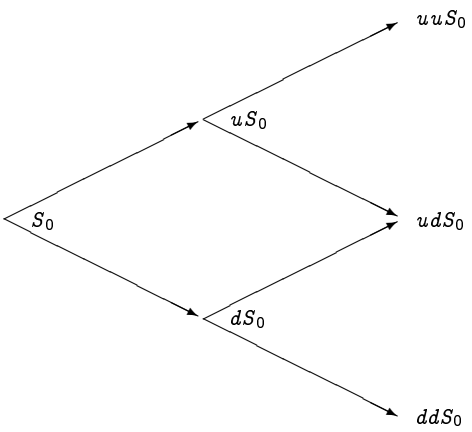


Iterating this idea a few times more, the number of different *terminal* states increases markedly, and we get closer to a realistic distribution of future prices of the underlying at the terminal date. Note that a crucial assumption to get a picture like this is that the factors u and d are the same on each date.

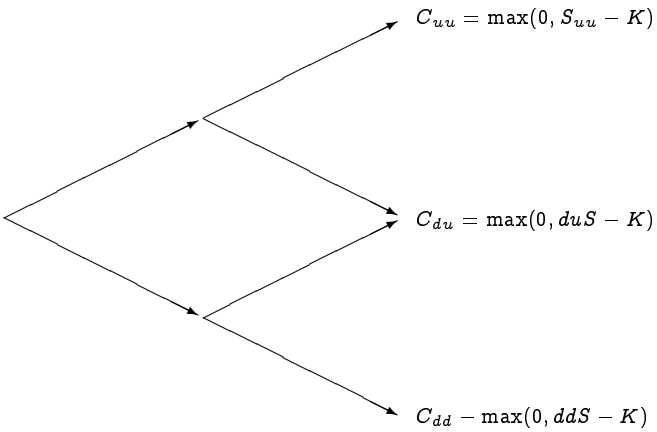


Pricing in a setting like this is done by working backwards, starting at the terminal date. Here we know all the possible values of the underlying security. For each of these, we calculate the payoffs from the derivative, and find what the set of possible derivative prices is *one period before*. Given these, we can find the option one period before this again, and so on. Working ones way down to the root of the tree, the option price is found as the derivative price in the first node.

For example, suppose we have two periods, and price a two period call option with exercise price K.



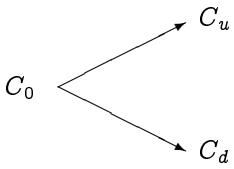
First step: Find terminal payoffs of derivative security:



Next step: Find the two possible call prices at time 1:

$$C_u = e^{-r}(qC_{uu} + (1-q)C_{ud})$$

$$C_d = e^{-r}(qC_{ud} + (1-q)C_{dd})$$



Final step: Using the two possible payoffs at time 1, C_u and C_d, find option value at time 0:

$$C_0 = e^{-r}(qC_u + (1-q)C_d)$$

Thus, binomial pricing really concerns "rolling backward" in a binomial tree, and programming therefore concerns an efficient way of traversing such a tree. The obvious data structure for describing such a tree is shown in **C++ Code 8.2**, where the value in each node is calculated from finding out the number of up and down steps are used to get to the particular node.

Exercise 8.2.

```
#include <vector>
#include <cmath>
using namespace std;

vector< vector<double> > binomial_tree(const double& S0,
                                       const double& u,
                                       const double& d,
                                       const int& no_steps){
    vector< vector<double> > tree;
    for (int i=1;i<=no_steps;++i){
        vector<double> S(i);
        for (int j=0;j<i;++j){
            S[j] = S0*pow(u,j)*pow(d,i-j-1);
        };
        tree.push_back(S);
    };
    return tree;
};
```

C++ Code 8.2: Building a binomial tree

In terms of computational efficiency the approcach of **C++ Code 8.2** will not be optimal, since it requires a lot of calls to the `pow()` functional call. More efficient would be to carry out the tree building by doing the multiplication from the previous node, for example the j'th vector is the $j-1$'th vector times u, and then one need to add one more node by multiplying the lowest element by d.

1. Implement such an alternative tree building procedure.

Basing the recursive calculation of a derivative price on a triangular array structure as shown in **C++ Code 8.2** is the most natural approach, but with some cleverness based on understanding the structure of the binomial tree, we can get away with the more efficienent algorithm that is shown in **C++ Code 8.3**. Note that here we only use one `vector<double>`, not a triangular array as built above.

Example

Let $S = 100.0$, $K = 100.0$, $r = 0.025$, $u = 1.05$ and $d = 1/u$.

1. Price one and two period European Call options.

C++ program:

```
double S = 100.0;  double K = 100.0;  double r = 0.025;
double u = 1.05;   double d = 1/u;
cout << " one period european call = "
     << option_price_call_european_binomial_single_period(S,K,r,u,d) << endl;
int no_periods = 2;
cout << " two period european call = "
     << option_price_call_european_binomial_multi_period_given_ud(S,K,r,u,d,no_periods) << endl;
```

Output from C++ program:

```
 one period european call =  3.64342
 two period european call =  5.44255
```

Exercise 8.3.

Implement pricing of single and multi period binomial put options.

Further reading The derivation of the single period binomial is shown in for example Bossaerts and Ødegaard (2001), Hull (2011) or McDonald (2013).

```
#include <cmath>          // standard mathematical library
#include <algorithm>      // defining the max() operator
#include <vector>         // STL vector templates
using namespace std;

double option_price_call_european_binomial_multi_period_given_ud(const double& S, // spot price
                                                                 const double& K,    // exercice price
                                                                 const double& r,    // interest rate (per period)
                                                                 const double& u,  // up movement
                                                                 const double& d,  // down movement
                                                                 const int& no_periods){ // no steps in binomial tree
    double Rinv = exp(-r);                // inverse of interest rate
    double uu = u*u;
    double p_up = (exp(r)-d)/(u-d);
    double p_down = 1.0-p_up;
    vector<double> prices(no_periods+1); // price of underlying
    prices[0] = S*pow(d, no_periods); // fill in the endnodes.
    for (int i=1; i<=no_periods; ++i) prices[i] = uu*prices[i-1];
    vector<double> call_values(no_periods+1); // value of corresponding call
    for (int i=0; i<=no_periods; ++i) { call_values[i] = max(0.0, (prices[i]-K));}; // call payoffs at maturity
    for (int step=no_periods-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
        };
    };
    return call_values[0];
};
```

C++ Code 8.3: Binomial multiperiod pricing of European call option

Chapter 9

Basic Option Pricing, the Black Scholes formula

Contents

The pricing of options and related instruments has been a major breakthrough for the use of financial theory in practical application. Since the original papers of Black and Scholes (1973) and Merton (1973), there has been a wealth of practical and theoretical applications. We will now consider the orginal Black Scholes formula for pricing options, how it is calculated and used. For the basic intuition about option pricing the reader should first read the discussion of the binomial model in the previous chapter, as that is a much better environment for understanding what is actually calculated.

An option is a *derivative security*, its value depends on the value, or price, of some other underlying security, called the *underlying* security.. Let S denote the value, or price, of this underlying security. We need to keep track of what time this price is observed at, so let S_t denote that the price is observed at time t. A call (put) option gives the holder the right, but not the obligation, to buy (sell) some underlying asset at a given price K, called the exercise price, on or before some given date T. If the option is a so called *European* option, it can only be used (exercised) at the maturity date. If the option is of the so called *American* type, it can be used (exercised) at any date up to and including the maturity date T. If exercised at time T, a call option provides payoff

$$C_T = \max(0, S_T - K)$$

and a put option provides payoff

$$P_T = \max(0, K - S_T)$$

The Black Scholes formulas provides analytical solutions for *European* put and call options, options which can only be exercised at the options maturity date. Black and Scholes showed that the additional

information needed to price the option is the (continously compounded) risk free interest rate r, the variability of the underlying asset, measured by the standard deviation σ of (log) price changes, and the time to maturity $(T-t)$ of the option, measured in years. The original formula was derived under the assumption that there are no payouts, such as stock dividends, coming from the underlying security during the life of the option. Such payouts will affection option values, as will become apparent later.

9.1 The formula

Formula 9.1 gives the exact formula for a call option, and the calculation of the same call option is shown in **C++ Code 9.1** and `Matlab` **Code 9.1**.

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Alternatively one can calculate d_1 and d_2 as

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}$$

S is the price of the underlying security, K the exercise price, r the (continously compounded) risk free interest rate, σ the standard deviation of the underlying asset, t the current date, T the maturity date, $T-t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution.

Formula 9.1: The Black Scholes formula

```
#include <cmath>          // mathematical C library
#include "normdist.h"     // the calculation of the cumularive normal distribution

double option_price_call_black_scholes(const double& S,     // spot (underlying) price
                                       const double& K,     // strike (exercise) price,
                                       const double& r,     // interest rate
                                       const double& sigma, // volatility
                                       const double& time) { // time to maturity
    double time_sqrt = sqrt(time);
    double d1 = (log(S/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
    double d2 = d1-(sigma*time_sqrt);
    return S*N(d1) - K*exp(-r*time)*N(d2);
};
```

C++ Code 9.1: Price of European call option using the Black Scholes formula

```
function c = black_scholes_call(S,K,r,sigma,time)
   time_sqrt = sqrt(time);
   d1 = (log(S/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
   d2 = d1-(sigma*time_sqrt);
   c = S * normcdf(d1) - K * exp(-r*time) * normcdf(d2);
endfunction
```

Matlab **Code 9.1**: Price of European call option using the Black Scholes formula

Example

Stock in company XYZ is currently trading at 50. Consider a call option on XYZ stock with an exercise price of $K = 50$ and time to maturity of 6 months. The volatility of XYZ stock has been estimated to be $\sigma = 30\%$. The current risk free interest rate (with continous compounding) for six month borrowing is 10%.

1. Determine the option price.

To calculate the price of this option we use the Black Scholes formula with inputs $S = 50$, $K = 50$, $r = 0.10$, $\sigma = 0.3$ and $(T - t) = 0.5$.

Matlab program:

```
S = 100;
K = 100;
r = 0.1;
sigma = 0.1;
time = 1;
c = black_scholes_call(S,K,r,sigma,time)
```

Output from Matlab program:

```
c = 10.308
```

C++ program:

```
double S = 50; double K = 50; double r = 0.10;
double sigma = 0.30; double time=0.50;
cout << " Black Scholes call price = ";
cout << option_price_call_black_scholes(S, K , r, sigma, time) << endl;
```

Output from C++ program:

```
Black Scholes call price = 5.45325
```

Exercise 9.1.

The Black Scholes price for a put option is:

$$p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where d_1 and d_2 are as for the call option:

$$d_1 = \frac{\ln\left(\frac{S}{X}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

S is the price of the underlying secrutity, K the exercise price, r the (continously compounded) risk free interest rate, σ the standard deviation of the underlying asset, $T - t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution.

1. Implement this formula.

9.2 Understanding the why's of the formula

To get some understanding of the Black Scholes formula and why it works will need to delve in some detail into the mathematics underlying its derivation. It does not help that there are a number of ways to prove the Black Scholes formula, depending on the setup. As it turns out, two of these ways are important to understand for computational purposes, the original Black Scholes continous time way, and the "limit of a binomial process" way of Cox, Ross, and Rubinstein (1979).

9.2.1 The original Black Scholes analysis

The primary assumption underlying the Black Scholes analyis concerns the stochastic process governing the price of the underlying asset. The price of the underlying asset, S, is assumed to follow a geometric Brownian Motion process, conveniently written in either of the shorthand forms

$$dS = \mu S dt + \sigma S dZ$$

or

$$\frac{dS}{S} = \mu dt + \sigma dZ$$

where μ and σ are constants, and Z is Brownian motion.

Using Ito's lemma, the assumption of no arbitrage, and the ability to trade continuously, Black and Scholes showed that the price of *any* contingent claim written on the underlying must solve the *partial differential equation* (9.1).

$$\frac{\partial f}{\partial S} rS + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \tag{9.1}$$

For any *particular* contingent claim, the terms of the claim will give a number of *boundary conditions* that determines the form of the pricing formula.

The pde given in equation (9.1), with the boundary condition $c_T = \max(0, S_T - K)$ was shown by Black and Scholes to have an analytical solution of functional form shown in the Black Scoles formula 9.1.

9.2.2 The limit of a binomial case

Another is to use the limit of a binomial process (Cox et al., 1979). The latter is particularly interesting, as it allows us to link the Black Scholes formula to the binomial, allowing the binomial framework to be used as an approximation.

9.2.3 The representative agent framework

A final way to show the BS formula to assume a representative agent and lognormality as was done in Rubinstein (1976).

9.3 Partial derivatives.

In trading of options, a number of partial derivatives of the option price formula is important.

9.3.1 Delta

The first derivative of the option price with respect to the price of the underlying security is called the *delta* of the option price. It is the derivative most people will run into, since it is important in hedging of options.

$$\frac{\partial c}{\partial S} = N(d_1)$$

C++ Code 9.2 shows the calculation of the delta for a call option.

```
#include <cmath>
#include "normdist.h"

double option_price_delta_call_black_scholes(const double& S, // spot price
                                             const double& K,     // Strike (exercise) price,
                                             const double& r,     // interest rate
                                             const double& sigma, // volatility
                                             const double& time){ // time to maturity
    double time_sqrt = sqrt(time);
    double d1 = (log(S/K)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;
    double delta = N(d1);
    return delta;
};
```

C++ Code 9.2: Calculating the delta of the Black Scholes call option price

9.3.2 Other Derivatives

The remaining derivatives are more seldom used, but all of them are relevant. All of them are listed in formula 9.3.2.

Delta (Δ)

$$\Delta = \frac{\partial c}{\partial S} = N(d_1)$$

Gamma (Γ)

$$\frac{\partial^2 c}{\partial S^2} = \frac{n(d_1)}{S\sigma\sqrt{T-t}}$$

Theta (Θ) (careful about which of these you want)

$$\frac{\partial c}{\partial (T-t)} = Sn(d_1)\frac{1}{2}\sigma\frac{1}{\sqrt{T-t}} + rKe^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -Sn(d_1)\frac{1}{2}\sigma\frac{1}{\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$$

Vega

$$\frac{\partial c}{\partial \sigma} = S\sqrt{T-t}n(d_1)$$

Rho (ρ)

$$\frac{\partial c}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2)$$

S is the price of the underlying security, K the exercise price, r the (continously compounded) risk free interest rate, σ the standard deviation of the underlying asset, t the current date, T the maturity date and $T-t$ the time to maturity for the option. $n(\cdot)$ is the normal distribution function $\left(n(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\right)$ and $N(\cdot)$ the cumulative normal distribution $\left(N(z) = \int_{-\infty}^{z} n(t)dt\right)$.

Formula 9.2: Partial derivatives of the Black Scholes call option formula

The calculation of all of these partial derivatives for a call option is shown in **C++ Code 9.3**.

Example

Consider again six month call options on XYZ stock. The option matures 6 months from now, at which time the holder of the option can recive one unit of the underlying security by paying the exercise price of $K = 50$. The current price of the underlying security is $S = 50$. The volatility of the underlying security is given as $\sigma = 30\%$. The current risk free interest rate (with continous compounding) for six month borrowing is 10%.

```
#include <cmath>
#include "normdist.h"
using namespace std;

void option_price_partials_call_black_scholes( const double& S, // spot price
                                               const double& K,    // Strike (exercise) price,
                                               const double& r,    // interest rate
                                               const double& sigma, // volatility
                                               const double& time, // time to maturity
                                               double& Delta, // partial wrt S
                                               double& Gamma, // second prt wrt S
                                               double& Theta, // partial wrt time
                                               double& Vega, // partial wrt sigma
                                               double& Rho){ // partial wrt r
  double time_sqrt = sqrt(time);
  double d1 = (log(S/K)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;
  double d2 = d1-(sigma*time_sqrt);
  Delta = N(d1);
  Gamma = n(d1)/(S*sigma*time_sqrt);
  Theta =- (S*sigma*n(d1))/(2*time_sqrt) - r*K*exp( -r*time)*N(d2);
  Vega = S * time_sqrt*n(d1);
  Rho  = K*time*exp(-r*time)*N(d2);
};
```

C++ Code 9.3: Calculating the partial derivatives of a Black Scholes call option

1. Calculate the partial derivatives (Delta, Gamma, Theta, Vega and Rho) for this option.

To calculate the partial derivatives we use inputs $S = 50$, $K = 50$, $r = 0.10$, $\sigma = 0.3$ and $(T - t) = 0.5$.

C++ program:

```
cout << " Black Scholes call partial derivatives " << endl;
double S = 50; double K = 50; double r = 0.10;
double sigma = 0.30; double time=0.50;
double Delta, Gamma, Theta, Vega, Rho;
option_price_partials_call_black_scholes(S,K,r,sigma, time, Delta, Gamma, Theta, Vega, Rho);
cout << " Delta = " << Delta << endl;
cout << " Gamma = " << Gamma << endl;
cout << " Theta = " << Theta << endl;
cout << " Vega = " << Vega << endl;
cout << " Rho = " << Rho << endl;
```

Output from C++ program:

```
Black Scholes call partial derivatives
 Delta = 0.633737
 Gamma = 0.0354789
 Theta = -6.61473
 Vega  = 13.3046
 Rho   = 13.1168
```

9.3.3 Implied Volatility.

In calculation of the option pricing formulas, in particular the Black Scholes formula, the only unknown is the standard deviation of the underlying stock. A common problem in option pricing is to find the implied volatility, given the observed price quoted in the market. For example, given c_0, the price of a call option, the following equation should be solved for the value of σ:

$$c_0 = c(S, K, r, \sigma, T - t)$$

Unfortunately, this equation has no closed form solution, which means the equation must be numerically solved to find σ. What is probably the algorithmic simplest way to solve this is to use a binomial search algorithm, which is implemented in **C++ Code 9.4**. We start by bracketing the sigma by finding a high sigma that makes the BS price higher than the observed price, and then, given the bracketing interval, we search for the volatility in a systematic way. shows such a calculation.

```
#include <cmath>
#include "fin_recipes.h"

double option_price_implied_volatility_call_black_scholes_bisections(const double& S,
                                                                     const double& K,
                                                                     const double& r,
                                                                     const double& time,
                                                                     const double& option_price){
    if (option_price<0.99*(S-K*exp(-time*r))) { // check for arbitrage violations.
        return 0.0;                             // Option price is too low if this happens
    };

    // simple binomial search for the implied volatility.
    // relies on the value of the option increasing in volatility
    const double ACCURACY = 1.0e-5; // make this smaller for higher accuracy
    const int MAX_ITERATIONS = 100;
    const double HIGH_VALUE = 1e10;
    const double ERROR = -1e40;

    // want to bracket sigma. first find a maximum sigma by finding a sigma
    // with a estimated price higher than the actual price.
    double sigma_low=1e-5;
    double sigma_high=0.3;
    double price = option_price_call_black_scholes(S,K,r,sigma_high,time);
    while (price < option_price) {
        sigma_high = 2.0 * sigma_high; // keep doubling.
        price = option_price_call_black_scholes(S,K,r,sigma_high,time);
        if (sigma_high>HIGH_VALUE) return ERROR; // panic, something wrong.
    };
    for (int i=0;i<MAX_ITERATIONS;i++){
        double sigma = (sigma_low+sigma_high)*0.5;
        price = option_price_call_black_scholes(S,K,r,sigma,time);
        double test = (price-option_price);
        if (fabs(test)<ACCURACY) { return sigma; };
        if (test < 0.0) { sigma_low = sigma; }
        else { sigma_high = sigma; }
    };
    return ERROR;
};
```

C++ Code 9.4: Calculation of implied volatility of Black Scholes using bisections

Instead of this simple bracketing, which is actually pretty fast, and will (almost) always find the solution, we can use the Newton–Raphson formula for finding the root of an equation in a single variable. The general description of this method starts with a function $f()$ for which we want to find a root.

$$f(x) = 0.$$

The function $f()$ needs to be differentiable. Given a first guess x_0, iterate by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

until

$$|f(x_i)| < \epsilon$$

where ϵ is the desired accuracy.[1]

In our case

$$f(x) = c_{obs} - c_{BS}(\sigma)$$

and, each new iteration will calculate

$$\sigma_{i+1} = \sigma_i + \frac{c_{obs} - c_{BS}(\sigma_i)}{-\frac{\partial c_{BS}()}{\partial \sigma}}$$

C++ Code 9.5 shows the calculation of implied volatility using Newton-Raphson.

```
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>
#include <iostream>
double option_price_implied_volatility_call_black_scholes_newton(const double& S,
                                                                 const double& K,
                                                                 const double& r,
                                                                 const double& time,
                                                                 const double& option_price) {
    if (option_price<0.99*(S-K*exp(-time*r))) { // check for arbitrage violations. Option price is too low if this happens
        return 0.0;
    };

    const int MAX_ITERATIONS = 100;
    const double ACCURACY = 1.0e-5;
    double t_sqrt = sqrt(time);

    double sigma = (option_price/S)/(0.398*t_sqrt); // find initial value
    for (int i=0;i<MAX_ITERATIONS;i++){
        double price = option_price_call_black_scholes(S,K,r,sigma,time);
        double diff = option_price -price;
        if (fabs(diff)<ACCURACY) return sigma;
        double d1 = (log(S/K)+r*time)/(sigma*t_sqrt) + 0.5*sigma*t_sqrt;
        double vega = S * t_sqrt * n(d1);
        sigma = sigma + diff/vega;
    };
    return -99e10; // something screwy happened, should throw exception
};
```

C++ Code 9.5: Calculation of implied volatility of Black Scholes using Newton-Raphson

Note that to use Newton-Raphson we need the derivative of the option price. For the Black-Scholes formula this is known, and we can use this. But for pricing formulas like the binomial, where the partial derivatives are not that easy to calculate, simple bisection is the preferred algorithm.

Example

Consider again six month call options on XYZ stock. The option matures 6 months from now, at which time the holder of the option can recive one unit of the underlying security by paying the exercise price of $K = 50$.

[1] For further discussion of the Newton-Raphson formula and bracketing, a good source is chapter 9 of Press et al. (1992)

The current price of the underlying security is $S = 50$. The current risk free interest rate (with continous compounding) for six month borrowing is 10%.

1. The current option price is $C = 2.5$. Determine the volatility implicit in this price.

The implied volatility is the σ which, input in the Black Scholes formula with these other inputs, will produce an option price of $C = 2.5$.
To calculate we use inputs $S = 50$, $K = 50$, $r = 0.10$ and $(T - t) = 0.5$.

C++ program:

```
double S = 50; double K = 50; double r = 0.10; double time=0.50;
double C=2.5;
cout << " Black Scholes implied volatility using Newton search = ";
cout << option_price_implied_volatility_call_black_scholes_newton(S,K,r,time,C) << endl;
cout << " Black Scholes implied volatility using bisections = ";
cout << option_price_implied_volatility_call_black_scholes_bisections(S,K,r,time,C) << endl;
```

Output from C++ program:

```
Black Scholes implied volatility using Newton search = 0.0500427
Black Scholes implied volatility using bisections = 0.0500419
```

9.4 References

Black and Scholes (1973) Merton (1973)
Gray and Gray (2001) has a simple proof of the formula.

Chapter 10

Warrants

Contents

A warrant is an option-like security on equity, but it is issued by the same company which has issued the equity, and when a warrant is exercised, a *new* stock is issued. This new stock is issued at a the warrant strike price, which is lower than the current stock price (If it wasn't the warrant would not be exercised.) Since the new stock is a a fractional right to all cashflows, this stock issue *waters out*, or *dilutes*, the equity in a company. The degree of dilution is a function of how many warrants are issued.

10.1 Warrant value in terms of assets

Let K be the strike price, n the number of shares outstanding and m the number of warrants issued. Assume each warrant is for 1 new share, and let A_t be the current asset value of firm. Suppose all warrants are exercised simultaneously. Then the assets of the firm increase by the number of warrants times the strike price of the warrant.

$$A_t + mK,$$

but this new asset value is spread over more shares, since each exercised warrant is now an equity. The assets of the firm is spread over all shares, hence each new share is worth:

$$\frac{A_t + mK}{m+n}$$

making each exercised warrant worth:

$$\frac{A_t + mK}{m+n} - K = \frac{n}{m+n}\left(\frac{A_t}{n} - K\right)$$

If we knew the current value of assets in the company, we could value the warrant in two steps:

1. Value the option using the Black Scholes formula and $\frac{A_t}{n}$ as the current stock price.
2. Multiply the resulting call price with $\frac{n}{m+n}$.

If we let W_t be the warrant value, the above arguments are summarized as:

$$W_t = \frac{n}{n+m} C_{BS}\left(\frac{A}{n}, K, \sigma, r, (T-t)\right),$$

where $C_{BS}(\cdot)$ is the Black Scholes formula.

10.2 Valuing warrants when observing the stock value

However, one does not necessarily observe the asset value of the firm. Typically one only observes the equity value of the firm. If we let S_t be the current stock price, the asset value is really:

$$A_t = nS_t + mW_t$$

Using the stock price, one would value the warrant as

$$W_t = \frac{n}{n+m} C_{BS}\left(\frac{nS_t + mW_t}{n}, K, \sigma, r, (T-t)\right)$$

or

$$W_t = \frac{n}{n+m} C_{BS}\left(S_t + \frac{m}{n}W_t, K, \sigma, r, (T-t)\right)$$

Note that this gives the value of W_t as a function of W_t. One need to solve this equation numerically to find W_t.

The numerical solution for W_t is done using the Newton-Rhapson method. Let

$$g(W_t) = W_t - \frac{n}{n+m} C_{BS}\left(S_t + \frac{m}{n}W_t, K, \sigma, r, (T-t)\right)$$

Starting with an initial guess for the warrant value W_t^o, the Newton-Rhapson method is that one iterates as follows

$$W_t^i = W_t^{i-1} - \frac{g(W_t^{i-1})}{g'(W_t^{i-1})},$$

where i signifies iteration i, until the criterion function $g(W_t^{i-1})$ is below some given accuracy ϵ. In this case

$$g'(W_t) = 1 - \frac{m}{m+n} N(d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{S_t + \frac{m}{n}W_t}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

An obvious starting value is to set calculate the Black Scholes value using the current stock price, and multiply it with $\frac{m}{m+n}$.

C++ **Code 10.1** implements this calculation.

Example

A stock is currently priced at $S = 48$. Consider warrants on the same company with exercise price $K = 40$ and time to maturity of six months. The company has $n = 10000$ shares outstanding, and has issued $m = 1000$ warrants. The current (continously compounded) risk free interest rate is 8%. Determine the current warrant price.

```
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>

const double EPSILON=0.00001;

double warrant_price_adjusted_black_scholes(const double& S,
                                            const double& K,
                                            const double& r,
                                            const double& sigma,
                                            const double& time,
                                            const double& m, // number of warrants outstanding
                                            const double& n){ // number of shares outstanding
    double time_sqrt = sqrt(time);
    double w = (n/(n+m))*option_price_call_black_scholes(S,K,r,sigma,time);
    double g = w-(n/(n+m))*option_price_call_black_scholes(S+(m/n)*w,K,r,sigma,time);
    while (fabs(g)>EPSILON) {
        double d1 = (log((S+(m/n))/K)+r*time)/(sigma*time_sqrt)+0.5*sigma*time_sqrt;
        double gprime = 1-(m/n)*N(d1);
        w=w-g/gprime;
        g = w-(n/(n+m))*option_price_call_black_scholes(S+(m/n)*w,K,r,sigma,time);
    };
    return w;
};
```

C++ Code 10.1: Adjusted Black Scholes value for a Warrant

C++ program:

```
    double S = 48; double K = 40; double r = 0.08; double sigma = 0.30;
    double time = 0.5; double m = 1000; double n = 10000;
    double w = warrant_price_adjusted_black_scholes(S,K,r,sigma, time, m, n);
    cout << " warrant price = " << w << endl;
```

Output from C++ program:

```
 warrant price = 10.142
```

Exercise 10.1.

The solution method assumes that all warrants are exercised simultanously. Can you see where this assumption is used?

10.3 Readings

McDonald (2013) and Hull (2011) are general references. A problem with warrants is that exercise of all warrants simultaneously is not necessarily optimal.

Press et al. (1992) discusses the Newton-Rhapson method for root finding.

Chapter 11

Extending the Black Scholes formula

Contents

11.1 Adjusting for payouts of the underlying.

For options on other financial instruments than stocks, we have to allow for the fact that the underlying may have payouts during the life of the option. For example, in working with commodity options, there is often some storage costs if one wanted to hedge the option by buying the underlying.

11.1.1 Continous Payouts from underlying.

The simplest case is when the payouts are done continuously. To value an European option, a simple adjustment to the Black Scholes formula is all that is needed. Let q be the *continuous payout* of the underlying commodity.

Call and put prices for European options are then given by formula 11.1, which are implemented in **C++ Code 11.1**.

Exercise 11.1.

The price of a put on an underlying security with a continous payout of q is:

$$p = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)$$

1. Implement this formula.

$$c = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

S is the price of the underlying secrutity, K the exercise price, r the risk free interest rate, q the (continous) payout and σ the standard deviation of the underlying asset, t the current date, T the maturity date, $T-t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution.

Formula 11.1: Analytical prices for European call option on underlying security having a payout of q

```
#include <cmath>          // mathematical library
#include "normdist.h"      // this defines the normal distribution
using namespace std;

double option_price_european_call_payout( const double& S, // spot price
                                          const double& X, // Strike (exercise) price,
                                          const double& r, // interest rate
                                          const double& q, // yield on underlying
                                          const double& sigma, // volatility
                                          const double& time) { // time to maturity
    double sigma_sqr = pow(sigma,2);
    double time_sqrt = sqrt(time);
    double d1 = (log(S/X) + (r-q + 0.5*sigma_sqr)*time)/(sigma*time_sqrt);
    double d2 = d1-(sigma*time_sqrt);
    double call_price = S * exp(-q*time)* N(d1) - X * exp(-r*time) * N(d2);
    return call_price;
};
```

C++ Code 11.1: Option price, continous payout from underlying

11.1.2 Dividends.

A special case of payouts from the underlying security is stock options when the stock pays dividends. When the stock pays dividends, the pricing formula is adjusted, because the dividend changes the value of the underlying.

The case of continuous dividends is easiest to deal with. It corresponds to the continuous payouts we have looked at previously. The problem is the fact that most dividends are paid at discrete dates.

European Options on dividend-paying stock.

To adjust the price of an European option for known dividends, we merely subtract the present value of the dividends from the current price of the underlying asset in calculating the Black Scholes value.

Example

Consider a stock option with the following data given:$S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time to maturity is one year. dividend yield = 5%. Dividend payments at times 0.25 and 0.75.

- Determine the option price

```
#include <cmath>           // mathematical library
#include <vector>
#include "fin_recipes.h"       // define the black scholes price

double option_price_european_call_dividends( const double& S,
                                             const double& K,
                                             const double& r,
                                             const double& sigma,
                                             const double& time_to_maturity,
                                             const vector<double>& dividend_times,
                                             const vector<double>& dividend_amounts ) {
    double adjusted_S = S;
    for (int i=0;i<dividend_times.size();i++) {
        if (dividend_times[i]<=time_to_maturity){
            adjusted_S -= dividend_amounts[i] * exp(-r*dividend_times[i]);
        };
    };
    return option_price_call_black_scholes(adjusted_S,K,r,sigma,time_to_maturity);
};
```

C++ Code 11.2: European option price, dividend paying stock

C++ program:

```
double S = 100.0;  double K = 100.0;
double r = 0.1;    double sigma = 0.25;
double time=1.0;
double dividend_yield=0.05;
vector<double> dividend_times; vector<double> dividend_amounts;
dividend_times.push_back(0.25); dividend_amounts.push_back(2.5);
dividend_times.push_back(0.75); dividend_amounts.push_back(2.5);
cout << " european stock call option with contininous dividend = "
     << option_price_european_call_payout(S,K,r,dividend_yield,sigma,time) << endl;
cout << " european stock call option with discrete dividend = "
     << option_price_european_call_dividends(S,K,r,sigma,time,dividend_times,dividend_amounts) << endl;
```

Output from C++ program:

```
european stock call option with contininous dividend = 11.7344
european stock call option with discrete dividend =  11.8094
```

11.2 American options

American options are much harder to deal with than European ones. The problem is that it may be optimal to use (exercise) the option before the final expiry date. This optimal exercise policy will affect the value of the option, and the exercise policy needs to be known when solving the pde. However, the exercise policy is not known. There is therefore no general analytical solutions for American call and put options. There are some special cases. For American call options on assets that do not have any payouts, the American call price is the same as the European one, since the optimal exercise policy is to not exercise. For American puts this this not the case, it may pay to exercise them early. When the underlying asset has payouts, it may also pay to exercise an American call option early. There is one known known analytical price for American call options, which is the case of a call on a stock that pays a known dividend *once* during the life of the option, which is discussed next. In all other cases the American price has to be approximated using one of the techniques discussed in later chapters: Binomial approximation, numerical solution of the partial differential equation, or another numerical approximation.

11.2.1 Exact american call formula when stock is paying one dividend.

When a stock pays dividend, a call option on the stock may be optimally exercised just before the stock goes ex-dividend. While the general dividend problem is usually approximated somehow, for the special case of one dividend payment during the life of an option an analytical solution is available, due to Roll–Geske–Whaley.

If we let S be the stock price, K the exercise price, D_1 the amount of dividend paid, t_1 the time of dividend payment, T the maturity date of option, we denote the time to dividend payment $\tau_1 = T - t_1$ and the time to maturity $\tau = T - t$.

A first check of early exercise is:

$$D_1 \leq K\left(1 - e^{-r(T-t_1)}\right)$$

If this inequality is fulfilled, early exercise is not optimal, and the value of the option is

$$c(S - e^{-r(t_1-t)}D_1, K, r, \sigma, (T-t))$$

where $c(\cdot)$ is the regular Black Scholes formula.

If the inequality is not fulfilled, one performs the calculation shown in **Formula 11.2** and implemented in **C++ Code 11.3**.

$$C = (S - D_1 e^{-r(t_1-t)})\left(N(b_1) + N(a_1, -b_1, \rho)\right) + Ke^{-r(T-t))}N(a_2, -b_2, \rho) - (K - D_1)e^{-r(t_1-t)}N(b_2)$$

where

$$\rho = -\sqrt{\frac{(t_1 - t}{T-t}}$$

$$a_1 = \frac{\ln\left(\frac{S - D_1 e^{-q(\tau_1}}{K}\right) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$a_2 = a_1 - \sigma\sqrt{T-t}$$

$$b_1 = \frac{\ln\left(\frac{S - D_1 e^{-r(t_1-t)}}{\bar{S}}\right) + (r + \frac{1}{2}\sigma^2)(t_1 - t)}{\sigma\sqrt{(t_1 - t)}}$$

$$b_2 = b_1 - \sigma\sqrt{T-t}$$

and $\bar{S}$ solves

$$c(\bar{S}, t_1) = \bar{S} + D_1 - K$$

S is the price of the underlying secrutity, K the exercise price, r the risk free interest rate, D_1 is the dividend amount and σ the standard deviation of the underlying asset, t the current date, T the maturity date, $T-t$ the time to maturity for the option and $N(\cdot)$ the cumulative normal distribution. $N()$ with one argument is the univariate normal cumulative distribution. $N()$ with three arguments is the bivariate normal distribution with the correlation between the two normals given as the third arguement.

Formula 11.2: Roll–Geske–Whaley price of american call option paying one fixed dividend

Example

Consider an option on a stock paying one dividend. The relevant data is $S = 100$, $K = 100$,r=0.1, $\sigma = 0.25$, $\tau = 1.0$, $\tau_1 = 0.5$ and $D_1 = 10$.

Price the option.

C++ program:

```
double S = 100.0;   double K = 100.0;
double r = 0.1;     double sigma = 0.25;
double tau = 1.0;   double tau1 = 0.5;
double D1 = 10.0;
cout << " american call price with one dividend = "
     << option_price_american_call_one_dividend(S,K,r,sigma,tau,D1, tau1)<< endl;
```

Output from C++ program:

```
 american call price with one dividend = 10.0166
```

Exercise 11.2.

The Black approximation to the price of an call option paying a fixed dividend is an approximation to the value of the call. Suppose the dividend is paid as some date t_1 before the maturity date of the option T. Blacks approximation calculates the value of two European options using the Black Scholes formula. One with expiry date equal to the ex dividend date of the options. Another with expiry date equal to the option expiry, but the current price of the underlying security is adjusted down by the amount of the dividend.

1. Implement Black's approximation.

```
#include <cmath>
#include "normdist.h" // define the normal distribution functions
#include "fin_recipes.h" // the regular black sholes formula

double option_price_american_call_one_dividend(const double& S,
                                               const double& K,
                                               const double& r,
                                               const double& sigma,
                                               const double& tau,
                                               const double& D1,
                                               const double& tau1){
   if (D1 <= K* (1.0-exp(-r*(tau-tau1)))) // check for no exercise
      return option_price_call_black_scholes(S-exp(-r*tau1)*D1,K,r,sigma,tau);
   const double ACCURACY = 1e-6;      // decrease this for more accuracy
   double sigma_sqr = sigma*sigma;
   double tau_sqrt = sqrt(tau);
   double tau1_sqrt = sqrt(tau1);
   double rho = - sqrt(tau1/tau);

   double S_bar = 0; // first find the S_bar that solves c=S_bar+D1-K
   double S_low=0;  // the simplest: binomial search
   double S_high=S; // start by finding a very high S above S_bar
   double c = option_price_call_black_scholes(S_high,K,r,sigma,tau-tau1);
   double test = c-S_high-D1+K;
   while ( (test>0.0) && (S_high<=1e10) ) {
      S_high *= 2.0;
      c = option_price_call_black_scholes(S_high,K,r,sigma,tau-tau1);
      test = c-S_high-D1+K;
   };
   if (S_high>1e10) { // early exercise never optimal, find BS value
      return option_price_call_black_scholes(S-D1*exp(-r*tau1),K,r,sigma,tau);
   };
   S_bar = 0.5 * S_high; // now find S_bar that solves c=S_bar-D+K
   c = option_price_call_black_scholes(S_bar,K,r,sigma,tau-tau1);
   test = c-S_bar-D1+K;
   while ( (fabs(test)>ACCURACY) && ((S_high-S_low)>ACCURACY) ) {
      if (test<0.0) { S_high = S_bar; }
      else { S_low = S_bar; };
      S_bar = 0.5 * (S_high + S_low);
      c = option_price_call_black_scholes(S_bar,K,r,sigma,tau-tau1);
      test = c-S_bar-D1+K;
   };
   double a1 = (log((S-D1*exp(-r*tau1))/K) +( r+0.5*sigma_sqr)*tau) / (sigma*tau_sqrt);
   double a2 = a1 - sigma*tau_sqrt;
   double b1 = (log((S-D1*exp(-r*tau1))/S_bar)+(r+0.5*sigma_sqr)*tau1)/(sigma*tau1_sqrt);
   double b2 = b1 - sigma * tau1_sqrt;
   double C = (S-D1*exp(-r*tau1)) * N(b1) + (S-D1*exp(-r*tau1)) * N(a1,-b1,rho)
      - (K*exp(-r*tau))*N(a2,-b2,rho) - (K-D1)*exp(-r*tau1)*N(b2);
   return C;
};
```

C++ Code 11.3: Option price, Roll–Geske–Whaley call formula for dividend paying stock

11.3 Options on futures

11.3.1 Black's model

For an European option written on a futures contract, we use an adjustment of the Black Scholes solution, which was developed in Black (1976). Essentially we replace S_0 with $e^{-r(T-t)r}F$ in the Black Scholes formula, and get the formula shown in 11.3 and implemented in **C++ Code 11.4**.

$$c = e^{-r(T-t)}\left(FN(d_1) - KN(d_2)\right)$$

where

$$d_1 = \frac{\ln\left(\frac{F}{K}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

F is the futures price, K is the exercise price, r the risk free interest rate, σ the volatility of the futures price, and $T-t$ is the time to maturity of the option (in years).

Formula 11.3: Black's formula for the price of an European Call option with a futures contract as the underlying security

```
#include <cmath>                                  // mathematics library
#include "normdist.h"                             // normal distribution
using namespace std;

double futures_option_price_call_european_black( const double& F, // futures price
                                                 const double& K,    // exercise price
                                                 const double& r,    // interest rate
                                                 const double& sigma, // volatility
                                                 const double& time){ // time to maturity
    double sigma_sqr = sigma*sigma;
    double time_sqrt = sqrt(time);
    double d1 = (log (F/K) + 0.5 * sigma_sqr * time) / (sigma * time_sqrt);
    double d2 = d1 - sigma * time_sqrt;
    return exp(-r*time)*(F * N(d1) - K * N(d2));
};
```

C++ Code 11.4: Price of European Call option on Futures contract

Example

Price a futures option in the Black setting. Information: $F = 50$, $K = 45$, $r = 8\%$, $\sigma = 0.2$, and time to maturity is a half year.

C++ program:

```
double F = 50.0;   double K = 45.0;
double r = 0.08;   double sigma = 0.2;
double time=0.5;
cout << " european futures call option = "
     << futures_option_price_put_european_black(F,K,r,sigma,time) << endl;
```

Output from C++ program:

```
 european futures call option = 0.851476
```

Exercise 11.3.

The Black formula for a put option on a futures contract is

$$p = e^{-r(T-t)}\left(KN(-d_2) - FN(-d_1)\right)$$

where the varibles are as defined for the call option.

1. Implement the put option price.

11.4 Foreign Currency Options

Another relatively simple adjustment of the Black Scholes formula occurs when the underlying security is a currency exchange rate (spot rate). In this case one adjusts the Black-Scholes equation for the interest-rate differential.

Let S be the spot exchange rate, and now let r be the domestic interest rate and r_f the foreign interest rate. σ is then the volatility of changes in the exchange rate. The calculation of the price of an European call option is then shown in formula 11.4 and implented in **C++ Code 11.5**.

$$c = Se^{-r_f(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

S is the spot exchange rate and K the exercise price. r is the domestic interest rate and r_f the foreign interest rate. σ is the volatility of changes in the exchange rate. $T-t$ is the time to maturity for the option.

Formula 11.4: European currency call

```
#include <cmath>
#include "normdist.h"   // define the normal distribution function

double currency_option_price_call_european( const double& S, // exchange_rate,
                                            const double& X,     // exercise,
                                            const double& r,     // r_domestic,
                                            const double& r_f,   // r_foreign,
                                            const double& sigma, // volatility,
                                            const double& time){ // time to maturity
    double sigma_sqr = sigma*sigma;
    double time_sqrt = sqrt(time);
    double d1 = (log(S/X) + (r-r_f+ (0.5*sigma_sqr)) * time)/(sigma*time_sqrt);
    double d2 = d1 - sigma * time_sqrt;
    return S * exp(-r_f*time) * N(d1) - X * exp(-r*time) * N(d2);
};
```

C++ Code 11.5: European Futures Call option on currency

Example

Price a European currency call given the following information $S = 50$, $K = 52$, $r = 8\%$, $r_f = 5\%$, $\sigma = 20\%$ and time to maturity = 0.5 years.

C++ program:

```
double S = 50.0;      double K = 52.0;
double r = 0.08;      double rf=0.05;
double sigma = 0.2; double time=0.5;
cout << " european currency call option = "
     << currency_option_price_call_european(S,K,r,rf,sigma,time) << endl;
```

Output from C++ program:

```
 european currency call option = 2.22556
```

Exercise 11.4.

The price for an european put for a currency option is

$$p = Ke^{-r(T-t)}N(-d_2) - Se^{-r_f(T-t)}N(-d_1)$$

1. Implement this formula.

11.5 Perpetual puts and calls

A *perpetal* option is one with no maturity date, it is inifinitely lived. Of course, only American perpetual options make any sense, European perpetual options would probably be hard to sell.[1] For both puts and calls analytical formulas has been developed. We consider the price of an American call, and discuss the put in an exercise. Formula 11.5 gives the analytical solution.

$$C^p = \frac{K}{h_1 - 1}\left(\frac{h_1 - 1}{h_1}\frac{S}{K}\right)^{h_1}$$

where

$$h_1 = \frac{1}{2} - \frac{r-q}{\sigma^2} + \sqrt{\left(\frac{r-q}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}}$$

S is the current price of the underlying security, K is the exercise price, r is the risk free interest rate, q is the dividend yield and σ is the volatility of the underlying asset.

Formula 11.5: Price for a perpetual call option

```
#include <cmath>
using namespace std;

double option_price_american_perpetual_call(const double& S,
                                            const double& K,
                                            const double& r,
                                            const double& q,
                                            const double& sigma){
    double sigma_sqr=pow(sigma,2);
    double h1 = 0.5 - ((r-q)/sigma_sqr);
    h1 += sqrt(pow(((r-q)/sigma_sqr-0.5),2)+2.0*r/sigma_sqr);
    double pric=(K/(h1-1.0))*pow(((h1-1.0)/h1)*(S/K),h1);
    return pric;
};
```

C++ Code 11.6: Price for an american perpetual call option

[1] Such options would be like the classical April fools present, a perpetual zero coupon bond...

Example

Price a perpetual call, given the following informtation $S = 50.0, K = 40.0, r = 0.05, q = 0.02, \sigma = 0.05$

C++ program:

```
double S=50.0;   double K=40.0;
double r=0.05;   double q=0.02;
double sigma=0.05;
double price = option_price_american_perpetual_call(S,K,r,q,sigma);
cout << " perpetual call price = " << price << endl;
```

Output from C++ program:

```
 perpetual call price = 19.4767
```

Exercise 11.5.

The price for a perpetual american put is

$$P^p = \frac{K}{1 - h_2} \left(\frac{h_2 - 1}{h_2} \frac{S}{K} \right)^{h_2}$$

where

$$h_2 = \frac{1}{2} - \frac{r - q}{\sigma^2} - \sqrt{\left(\frac{r - q}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}$$

1. Implement the calculation of this formula.

11.6 Readings

Hull (2011) and McDonald (2013) are general references. A first formulation of an analytical call price with dividends was in Roll (1977b). This had some errors, that were partially corrected in Geske (1979), before Whaley (1981) gave a final, correct formula. See Hull (2011) for a textbook summary. Black (1976) is the original development of the futures option. The original formulations of European foreign currency option prices are in Garman and Kohlhagen (1983) and Grabbe (1983). The price of a perpetual put was first shown in Merton (1973). For a perpetual call see McDonald and Siegel (1986). The notation for perpetual puts and calls follows the summary in (McDonald, 2013, pg. 393).

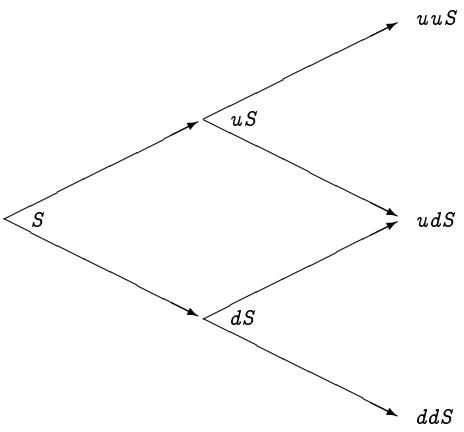
Chapter 12

Option pricing with binomial approximations

Contents

12.1 Introduction

We have shown binomial calculations given an up and down movement in chapter 8. However, binomial option pricing can also be viewed as an *approximation* to a continuous time distribution by judicious choice of the constants u and d. To do so one has to ask: Is it possible to find a parametrization (choice of u and d) of a binomial process



which has the same time series properties as a (continous time) process with the same mean and volatility? There is actually any number of ways of constructing this, hence one uses one degree of freedom on imposing that the nodes *reconnect*, by imposing $u = \frac{1}{d}$.

To value an option using this approach, we specify the number n of periods to split the time to maturity $(T - t)$ into, and then calculate the option using a binomial tree with that number of steps.

Given S, X, r, σ, T and the number of periods n, calculate

$$\Delta t = \frac{T - t}{n}$$

$$u = e^{\sigma\sqrt{\Delta t}}$$

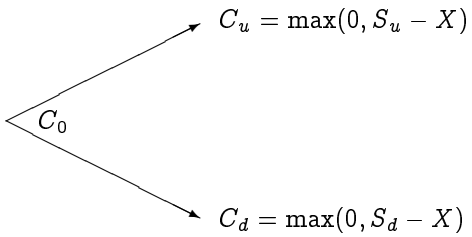
$$d = e^{-\sigma\sqrt{\Delta t}}$$

We also redefine the "risk neutral probabilities"

$$R = e^{r\Delta t}$$

$$q = \frac{R - d}{u - d}$$

To find the option price, will "roll backwards:" At node t, calculate the call price as a function of the two possible outcomes at time $t + 1$. For example, if there is one step,



find the call price at time 0 as

$$C_0 = e^{-r}(qC_u + (1 - q)C_d)$$

With more periods one will "roll backwards" as discussed in chapter 8

12.2 Pricing of options in the Black Scholes setting

Consider options on underlying securities not paying dividend.

12.2.1 European Options

For European options, binomial trees are not that much used, since the Black Scholes model will give the correct answer, but it is useful to see the construction of the binomial tree without the checks for early exercise, which is the American case.

The computer algorithm for a binomial in the following merits some comments. There is only one vector of call prices, and one may think one needs two, one at time t and another at time $t + 1$. (Try to write down the way you would solve it before looking at the algorithm below.) But by using the fact that the branches reconnect, it is possible to get away with the algorithm below, using one less array. You may want to check how this works. It is also a useful way to make sure one understands binomial option pricing.

```
#include <cmath>          // standard mathematical library
#include <algorithm>          // defining the max() operator
#include <vector>         // STL vector templates
using namespace std;

double option_price_call_european_binomial( const double& S, // spot price
                                            const double& K,    // exercice price
                                            const double& r,    // interest rate
                                            const double& sigma, // volatility
                                            const double& t,    // time to maturity
                                            const int& steps){ // no steps in binomial tree
   double R = exp(r*(t/steps));         // interest rate for each step
   double Rinv = 1.0/R;                 // inverse of interest rate
   double u = exp(sigma*sqrt(t/steps)); // up movement
   double uu = u*u;
   double d = 1.0/u;
   double p_up = (R-d)/(u-d);
   double p_down = 1.0-p_up;
   vector<double> prices(steps+1);   // price of underlying
   prices[0] = S*pow(d, steps); // fill in the endnodes.
   for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
   vector<double> call_values(steps+1);   // value of corresponding call
   for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity
   for (int step=steps-1; step>=0; --step) {
      for (int i=0; i<=step; ++i) {
         call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
      };
   };
   return call_values[0];
};
```

C++ Code 12.1: Option price for binomial european

12.2.2 American Options

An American option differs from an European option by the exercise possibility. An American option can be exercised at any time up to the maturity date, unlike the European option, which can only be exercised at maturity. In general, there is unfortunately no analytical solution to the American option problem, but in some cases it can be found. For example, for an American call option on non-dividend paying stock, the American price is the same as the European call.

It is in the case of American options, allowing for the possibility of early exercise, that binomial approximations are useful. At each node we calculate the value of the option as a function of the next periods prices, and then check for the value exercising of exercising the option now

C++ Code 12.2 illustrates the calculation of the price of an American call.

```
#include <cmath>
#include <vector>
using namespace std;

double option_price_call_american_binomial( const double& S,
                                            const double& K,
                                            const double& r,
                                            const double& sigma,
                                            const double& t,
                                            const int& steps) {
   double R = exp(r*(t/steps));
   double Rinv = 1.0/R;
   double u = exp(sigma*sqrt(t/steps));
   double d = 1.0/u;
   double p_up = (R-d)/(u-d);
   double p_down = 1.0-p_up;

   vector<double> prices(steps+1);   // price of underlying
   prices[0] = S*pow(d, steps); // fill in the endnodes.
   double uu = u*u;
   for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];

   vector<double> call_values(steps+1);   // value of corresponding call
   for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity

   for (int step=steps-1; step>=0; --step) {
      for (int i=0; i<=step; ++i) {
         call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
         prices[i] = d*prices[i+1];
         call_values[i] = max(call_values[i],prices[i]-K);     // check for exercise
      };
   };
   return call_values[0];
};
```

C++ Code 12.2: Price of American call option using a binomial approximation

Actually, for this particular case, the american price will equal the european.

12.2.3 `Matlab` implementation

To illustrate differences between `Matlab` and C++, consider the two implementations in `Matlab` **Code 12.1** and `Matlab` **Code 12.2**. The calculation of a put is more compact, with less loops, by some judicious use of array indexing.

```
function c = bin_am_call( S, K, r, sigma,t,steps)
    R = exp(r*(t/steps));
    Rinv = 1.0/R;
    u = exp(sigma*sqrt(t/steps));
    d = 1.0/u;
    p_up = (R-d)/(u-d);
    p_down = 1.0-p_up;
    prices = zeros(steps+1);

    prices(1) = S*(d^steps);
    uu = u*u;
    for i=2:steps
       prices(i) = uu*prices(i-1);
    end

   call_values = max(0.0, (prices-K));
   for step=steps:-1:1
      for i=1:step+1
         call_values(i) = (p_up*call_values(i+1)+p_down*call_values(i))*Rinv;
         prices(i) = d*prices(i+1);
         call_values(i) = max(call_values(i),prices(i)-K);
      end
   end
   c= call_values(1);
end
```

`Matlab` **Code 12.1**: Price of American call using a binomial approximation

```
function p = bin_am_put( S, K, r, sigma, t, steps)
    R = exp(r*(t/steps));
    Rinv = 1.0/R;
    u = exp(sigma*sqrt(t/steps));
    d = 1.0/u;
    p_up = (R-d)/(u-d);
    p_down = 1.0-p_up;

    prices = zeros(steps+1,1);
    prices(1) = S*(d^steps);
    uu = u*u;
    for i=2:steps+1
      prices(i) = uu*prices(i-1);
    end
    values = max(0.0, (K-prices));
    for step=steps:-1:1
       values = Rinv * ( p_up*values(2:step+1) + p_down*values(1:step) );
       prices = u*prices(1:step);
       values = max(values,K-prices);
   end
   p=values(1);
end
```

Matlab **Code 12.2**: Price of American put using a binomial approximation

Example

You are given the following information about an option: $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year. Price American calls and puts using binomial approximations with 100 steps.

C++ program:

```
double S = 100.0;   double K = 100.0;
double r = 0.1;     double sigma = 0.25;
double time=1.0;    int no_steps = 100;
cout << " european call= " << option_price_call_european_binomial(S,K,r,sigma,time,no_steps) << endl;
cout << " american call= " << option_price_call_american_binomial(S,K,r,sigma,time,no_steps) << endl;
cout << " american put = " << option_price_put_american_binomial(S,K,r,sigma,time,no_steps) << endl;
```

Output from C++ program:

```
european call= 14.9505
american call= 14.9505
american put = 6.54691
```

Matlab program:

```
S=100;
K=100;
r=0.1;
sigma=0.25;
time=1;
no_steps=100;
C = bin_am_call(S,K,r,sigma,time,no_steps)
P = bin_am_put(S,K,r,sigma,time,no_steps)
```

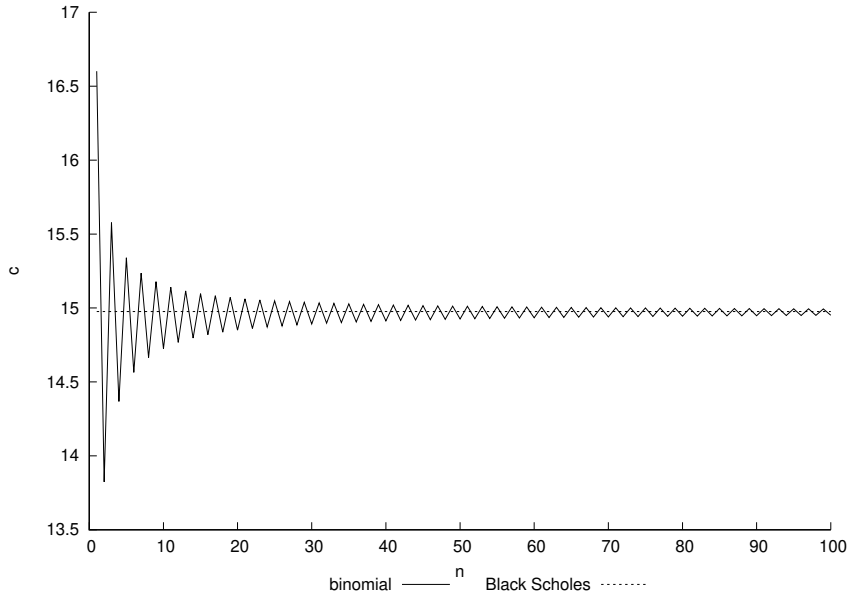
Output from Matlab program:

```
C = 14.951
P = 6.5469
```

12.3 How good is the binomial approximation?

To illustrate the behaviour of the binomial approximation figure 12.1 plots a comparison with the binomial approximation as a function of n, the number of steps in the binomial approximation, and the true (Black Scholes) value of the option. Note the "sawtooth" pattern, the binomial approximation jumps back and forth around the true value for small values of n, but rapidly moves towards the Black Scholes value as the n increases.

Figure 12.1: Illustrating convergence of binomial to Black Scholes



12.3.1 Estimating partials.

It is always necessary to calculate the partial derivatives as well as the option price. Let us start with Delta, the derivative of the option price with respect to the underlying. The binomial methods gives us ways to approximate these as well. How to find them in the binomial case are described in Hull (2011). The implementation in **C++ Code 12.3** is for the non–dividend case.

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double option_price_delta_american_call_binomial(const double& S,
                                                 const double& K,
                                                 const double& r,
                                                 const double& sigma,
                                                 const double& t,
                                                 const int& no_steps){ // steps in binomial

   double R = exp(r*(t/no_steps));
   double Rinv = 1.0/R;
   double u = exp(sigma*sqrt(t/no_steps));
   double d = 1.0/u;
   double uu= u*u;
   double pUp = (R-d)/(u-d);
   double pDown = 1.0 - pUp;

   vector<double> prices (no_steps+1);
   prices[0] = S*pow(d, no_steps);
   for (int i=1; i<=no_steps; ++i) prices[i] = uu*prices[i-1];

   vector<double> call_values (no_steps+1);
   for (int i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (prices[i]-K));

   for (int CurrStep=no_steps-1 ; CurrStep>=1; --CurrStep) {
      for (int i=0; i<=CurrStep; ++i) {
          prices[i] = d*prices[i+1];
          call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
          call_values[i] = max(call_values[i], prices[i]-K);      // check for exercise
      };
   };
   double delta = (call_values[1]-call_values[0])/(S*u-S*d);
   return delta;
};
```

C++ Code 12.3: Delta

Calculation of all the derivatives are shown in **C++ Code 12.4**

```
#include <cmath>
#include <algorithm>
#include "fin_recipes.h"

void option_price_partials_american_call_binomial(const double& S, // spot price
                                                  const double& K, // Exercise price,
                                                  const double& r,     // interest rate
                                                  const double& sigma, // volatility
                                                  const double& time, // time to maturity
                                                  const int& no_steps, // steps in binomial
                                                  double& delta, // partial wrt S
                                                  double& gamma, // second prt wrt S
                                                  double& theta, // partial wrt time
                                                  double& vega, // partial wrt sigma
                                                  double& rho){ // partial wrt r
   vector<double> prices(no_steps+1);
   vector<double> call_values(no_steps+1);
   double delta_t =(time/no_steps);
   double R = exp(r*delta_t);
   double Rinv = 1.0/R;
   double u = exp(sigma*sqrt(delta_t));
   double d = 1.0/u;
   double uu= u*u;
   double pUp = (R-d)/(u-d);
   double pDown = 1.0 - pUp;
   prices[0] = S*pow(d, no_steps);
   for (int i=1; i<=no_steps; ++i) prices[i] = uu*prices[i-1];
   for (int i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (prices[i]-K));
   for (int CurrStep=no_steps-1; CurrStep>=2; --CurrStep) {
      for (int i=0; i<=CurrStep; ++i) {
         prices[i] = d*prices[i+1];
         call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
         call_values[i] = max(call_values[i], prices[i]-K);      // check for exercise
      };
   };
   double f22 = call_values[2];
   double f21 = call_values[1];
   double f20 = call_values[0];
   for (int i=0;i<=1;i++) {
      prices[i] = d*prices[i+1];
      call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
      call_values[i] = max(call_values[i], prices[i]-K);      // check for exercise
   };
   double f11 = call_values[1];
   double f10 = call_values[0];
   prices[0] = d*prices[1];
   call_values[0] = (pDown*call_values[0]+pUp*call_values[1])*Rinv;
   call_values[0] = max(call_values[0], S-K);    // check for exercise on first date
   double f00 = call_values[0];
   delta = (f11-f10)/(S*u-S*d);
   double h = 0.5 * S * ( uu - d*d);
   gamma = ( (f22-f21)/(S*(uu-1)) - (f21-f20)/(S*(1-d*d)) ) / h;
   theta = (f21-f00) / (2*delta_t);
   double diff = 0.02;
   double tmp_sigma = sigma+diff;
   double tmp_prices = option_price_call_american_binomial(S,K,r,tmp_sigma,time,no_steps);
   vega = (tmp_prices-f00)/diff;
   diff = 0.05;
   double tmp_r = r+diff;
   tmp_prices = option_price_call_american_binomial(S,K,tmp_r,sigma,time,no_steps);
   rho = (tmp_prices-f00)/diff;
};
```

C++ Code 12.4: Hedge parameters

Example

Given the following information: $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year, Use 100 steps in the binomial approximation.

1. Estimate all the "greeks" for the option: delta(Δ), gamma, theta, vega and rho.

C++ program:

```
double S = 100.0;  double K = 100.0;
double r = 0.1;    double sigma = 0.25;
double time=1.0;   int no_steps = 100;
double delta, gamma, theta, vega, rho;
option_price_partials_american_call_binomial(S,K,r, sigma, time, no_steps,
                                             delta, gamma, theta, vega, rho);
cout << " Call price partials " << endl;
cout << " delta = " << delta << endl;
cout << " gamma = " << gamma << endl;
cout << " theta = " << theta << endl;
cout << " vega = " << vega << endl;
cout << " rho = " << rho << endl;
```

Output from C++ program:

```
 Call price partials
  delta = 0.699792
  gamma = 0.0140407
  theta = -9.89067
  vega  = 34.8536
  rho   = 56.9652
```

Exercise 12.1.

Consider an American call option on non-dividend paying stock, where $S = 100$, $K = 100$, $\sigma = 0.2$, $(T - t) = 1$ and $r = 0.1$.

1. Calculate the price of this option using Black Scholes
2. Calculate the price using a binomial approximation, using 10, 100 and 1000 steps in the approximation.
3. Discuss sources of differences in the estimated prices.

12.4 Adjusting for payouts for the underlying

The simplest case of a payout is the similar one to the one we saw in the Black Scholes case, a continous payout of y. This y needs to be brought into the binomial framework. Here we use it to adjust the probability:

$$\Delta t = \sqrt{(T-t)/n}$$

$$u = e^{\sigma \Delta t}$$

$$d = \frac{1}{u}$$

$$p_u = \frac{e^{(r-y)\Delta t} - d}{u - d}$$

$$p_d = 1 - p_u$$

For example, the last node

$$C_0 = e^{-r\Delta t}(p_u C_u + (1 - p_u)C_d)$$

```
#include <cmath>              // standard mathematical library
#include <algorithm>          // defines the max() operator
#include <vector>             // STL vector templates
using namespace std;

double option_price_call_american_binomial( const double& S, // spot price
                                            const double& K,    // exercice price
                                            const double& r,    // interest rate
                                            const double& y,    // continous payout
                                            const double& sigma, // volatility
                                            const double& t,    // time to maturity
                                            const int& steps) {   // no steps in binomial tree
   double R = exp(r*(t/steps));         // interest rate for each step
   double Rinv = 1.0/R;                 // inverse of interest rate
   double u = exp(sigma*sqrt(t/steps)); // up movement
   double uu = u*u;
   double d = 1.0/u;
   double p_up = (exp((r-y)*(t/steps))-d)/(u-d);
   double p_down = 1.0-p_up;
   vector<double> prices(steps+1);   // price of underlying
   prices[0] = S*pow(d, steps);
   for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1]; // fill in the endnodes.

   vector<double> call_values(steps+1);   // value of corresponding call
   for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-K)); // call payoffs at maturity

   for (int step=steps-1; step>=0; --step) {
      for (int i=0; i<=step; ++i) {
         call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
         prices[i] = d*prices[i+1];
         call_values[i] = max(call_values[i],prices[i]-K);     // check for exercise
      };
   };
   return call_values[0];
};
```

C++ Code 12.5: Binomial option price with continous payout

12.5 Pricing options on stocks paying dividends using a binomial approximation

12.5.1 Checking for early exercise in the binomial model.

If the underlying asset is a stock paying dividends during the maturity of the option, the terms of the option is not adjusted to reflect this cash payment, which means that the option value will reflect the dividend payments.

In the binomial model, the adjustment for dividends depend on whether the dividends are discrete or proportional.

12.5.2 Proportional dividends.

For proportional dividends, we simply multiply with an adjustment factor the stock prices at the ex–dividend date, the nodes in the binomial tree will "link up" again, and we can use the same "rolling back" procedure.

```
#include <cmath>
#include <algorithm>
#include <vector>
#include "fin_recipes.h"
#include <iostream>

double option_price_call_american_proportional_dividends_binomial(const double& S,
                                                                  const double& K,
                                                                  const double& r,
                                                                  const double& sigma,
                                                                  const double& time,
                                                                  const int& no_steps,
                                                                  const vector<double>& dividend_times,
                                                                  const vector<double>& dividend_yields) {
    // note that the last dividend date should be before the expiry date, problems if dividend at terminal node
    int no_dividends = int(dividend_times.size());
    if (no_dividends == 0) {
        return option_price_call_american_binomial(S,K,r,sigma,time,no_steps); // price w/o dividends
    };
    double delta_t = time/no_steps;
    double R = exp(r*delta_t);
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(delta_t));
    double uu= u*u;
    double d = 1.0/u;
    double pUp = (R-d)/(u-d);
    double pDown = 1.0 - pUp;
    vector<int> dividend_steps(no_dividends); // when dividends are paid
    for (int i=0; i<no_dividends; ++i) {
        dividend_steps[i] = (int)(dividend_times[i]/time*no_steps);
    };
    vector<double> prices(no_steps+1);
    vector<double> call_prices(no_steps+1);
    prices[0] = S*pow(d, no_steps); // adjust downward terminal prices by dividends
    for (int i=0; i<no_dividends; ++i) { prices[0]*=(1.0-dividend_yields[i]); };
    for (int i=1; i<=no_steps; ++i) {     prices[i] = uu*prices[i-1]; };
    for (int i=0; i<=no_steps; ++i) call_prices[i] = max(0.0, (prices[i]-K));

    for (int step=no_steps-1; step>=0; --step) {
        for (int i=0;i<no_dividends;++i) { // check whether dividend paid
            if (step==dividend_steps[i]) {
                for (int j=0;j<=(step+1);++j) {
                    prices[j]*=(1.0/(1.0-dividend_yields[i]));
                };
            };
        };
        for (int i=0; i<=step; ++i) {
            call_prices[i] = (pDown*call_prices[i]+pUp*call_prices[i+1])*Rinv;
            prices[i] = d*prices[i+1];
            call_prices[i] = max(call_prices[i], prices[i]-K);        // check for exercise
        };
    };
    return call_prices[0];
};
```

C++ Code 12.6: Binomial option price of stock option where stock pays proportional dividends

12.5.3 Discrete dividends

The problem is when the dividends are constant dollar amounts. In that case the nodes of the binomial tree do not "link up," and the number of branches increases dramatically, which means that the time to do the calculation is increased.

The algorithm presented in **C++ Code 12.7** implements this case, with no linkup, by constructing a binomial tree up to the ex-dividend date, and then, at the terminal nodes of that tree, call itself with one less dividend payment, and time to maturity the time remaining at the ex-dividend date. Doing that calculates the value of the option at the ex-dividend date, which is then compared to the value of exercising just before the ex-dividend date. It is a instructive example of using recursion in simplifying calculations, but as with most recursive solutions, it has a cost in computing time. For large binomial trees and several dividends this procedure is costly in computing time.

```
#include <cmath>
#include <vector>
#include "fin_recipes.h"
#include <iostream>
double option_price_call_american_discrete_dividends_binomial(const double& S,
                                                               const double& K,
                                                               const double& r,
                                                               const double& sigma,
                                                               const double& t,
                                                               const int& steps,
                                                               const vector<double>& dividend_times,
                                                               const vector<double>& dividend_amounts) {
    int no_dividends = int(dividend_times.size());
    if (no_dividends==0) return option_price_call_american_binomial(S,K,r,sigma,t,steps);// just do regular
    int steps_before_dividend = (int)(dividend_times[0]/t*steps);
    const double R = exp(r*(t/steps));
    const double Rinv = 1.0/R;
    const double u = exp(sigma*sqrt(t/steps));
    const double d = 1.0/u;
    const double pUp = (R-d)/(u-d);
    const double pDown = 1.0 - pUp;
    double dividend_amount = dividend_amounts[0];
    vector<double> tmp_dividend_times(no_dividends-1); // temporaries with
    vector<double> tmp_dividend_amounts(no_dividends-1); // one less dividend
    for (int i=0;i<(no_dividends-1);++i){
        tmp_dividend_amounts[i] = dividend_amounts[i+1];
        tmp_dividend_times[i] = dividend_times[i+1] - dividend_times[0];
    };
    vector<double> prices(steps_before_dividend+1);
    vector<double> call_values(steps_before_dividend+1);
    prices[0] = S*pow(d, steps_before_dividend);
    for (int i=1; i<=steps_before_dividend; ++i) prices[i] = u*u*prices[i-1];
    for (int i=0; i<=steps_before_dividend; ++i){
        double value_alive
            = option_price_call_american_discrete_dividends_binomial(prices[i]-dividend_amount,K, r, sigma,
                                                                      t-dividend_times[0],// time after first dividend
                                                                      steps-steps_before_dividend,
                                                                      tmp_dividend_times,
                                                                      tmp_dividend_amounts);
        call_values[i] = max(value_alive,(prices[i]-K)); // compare to exercising now
    };
    for (int step=steps_before_dividend-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            prices[i] = d*prices[i+1];
            call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
            call_values[i] = max(call_values[i], prices[i]-K);
        };
    };
    return call_values[0];
};
```

C++ Code 12.7: Binomial option price of stock option where stock pays discrete dividends

Example

Given the following information $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year, Calculate option price with two different assumptions about dividends:

1. Continuous payout $d = 0.02$.
2. Discrete payout, $d = 0.025$ at times 0.25 and 0.75.

C++ program:

```
double S = 100.0;  double K = 100.0;
double r = 0.10;     double sigma = 0.25;
double time=1.0;
int no_steps = 100;
double d=0.02;
cout << " call price with continuous dividend payout = "
     << option_price_call_american_binomial(S,K,r,d,sigma,time,no_steps) << endl;
vector<double> dividend_times; vector<double> dividend_yields;
dividend_times.push_back(0.25); dividend_yields.push_back(0.025);
dividend_times.push_back(0.75); dividend_yields.push_back(0.025);
cout << " call price with proportial dividend yields at discrete dates = "
     << option_price_call_american_proportional_dividends_binomial(S,K,r,sigma,time,no_steps,
                                                                   dividend_times, dividend_yields)
     << endl;
vector<double> dividend_amounts; dividend_amounts.push_back(2.5); dividend_amounts.push_back(2.5);
cout << " call price with proportial dividend amounts at discrete dates = "
     << option_price_call_american_discrete_dividends_binomial(S,K,r,sigma,time,no_steps,
                                                               dividend_times, dividend_amounts)
     << endl;
```

Output from C++ program:

```
call price with continuous dividend payout = 13.5926
call price with proportial dividend yields at discrete dates = 11.8604
call price with proportial dividend amounts at discrete dates = 12.0233
```

12.6 Option on futures

For American options, because of the feasibility of early exercise, the binomial model is used to approximate the option value for both puts and calls.

Example

$F = 50.0, K = 45.0, r = 0.08, sigma = 0.2$, time=0.5, no steps=100; Price the futures option

C++ program:

```
double F = 50.0;  double K = 45.0;
double r = 0.08;  double sigma = 0.2;
double time=0.5;
int no_steps=100;
cout << " european futures call option = "
     << futures_option_price_call_american_binomial(F,K,r,sigma,time,no_steps) << endl;
```

Output from C++ program:

```
european futures call option = 5.74254
```

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double futures_option_price_call_american_binomial(const double& F, // price futures contract
                                                   const double& K, // exercise price
                                                   const double& r, // interest rate
                                                   const double& sigma, // volatility
                                                   const double& time, // time to maturity
                                                   const int& no_steps) { // number of steps
   vector<double> futures_prices(no_steps+1);
   vector<double> call_values (no_steps+1);
   double t_delta= time/no_steps;
   double Rinv = exp(-r*(t_delta));
   double u = exp(sigma*sqrt(t_delta));
   double d = 1.0/u;
   double uu= u*u;
   double pUp = (1-d)/(u-d); // note how probability is calculated
   double pDown = 1.0 - pUp;
   futures_prices[0] = F*pow(d, no_steps);
   int i;
   for (i=1; i<=no_steps; ++i) futures_prices[i] = uu*futures_prices[i-1]; // terminal tree nodes
   for (i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (futures_prices[i]-K));
   for (int step=no_steps-1; step>=0; --step) {
      for (i=0; i<=step; ++i) {
         futures_prices[i] = d*futures_prices[i+1];
         call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
         call_values[i] = max(call_values[i], futures_prices[i]-K); // check for exercise
      };
   };
   return call_values[0];
};
```

C++ Code 12.8: Pricing an american call on an option on futures using a binomial approximation

12.7 Foreign Currency options

For American options, the usual method is approximation using binomial trees, checking for early exercise due to the interest rate differential.

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double currency_option_price_call_american_binomial(const double& S,
                                                    const double& K,
                                                    const double& r,
                                                    const double& r_f,
                                                    const double& sigma,
                                                    const double& time,
                                                    const int& no_steps) {
    vector<double> exchange_rates(no_steps+1);
    vector<double> call_values(no_steps+1);
    double t_delta= time/no_steps;
    double Rinv = exp(-r*(t_delta));
    double u = exp(sigma*sqrt(t_delta));
    double d = 1.0/u;
    double uu= u*u;
    double pUp = (exp((r-r_f)*t_delta)-d)/(u-d); // adjust for foreign int.rate
    double pDown = 1.0 - pUp;
    exchange_rates[0] = S*pow(d, no_steps);
    int i;
    for (i=1; i<=no_steps; ++i) {
        exchange_rates[i] = uu*exchange_rates[i-1]; // terminal tree nodes
    }
    for (i=0; i<=no_steps; ++i) call_values[i] = max(0.0, (exchange_rates[i]-K));
    for (int step=no_steps-1; step>=0; --step) {
        for (i=0; i<=step; ++i) {
            exchange_rates[i] = d*exchange_rates[i+1];
            call_values[i] = (pDown*call_values[i]+pUp*call_values[i+1])*Rinv;
            call_values[i] = max(call_values[i], exchange_rates[i]-K); // check for exercise
        };
    };
    return call_values[0];
};
```

C++ Code 12.9: Pricing an american call on an option on currency using a binomial approximation

Example

Price a futures currency option with the following information: $S = 50, K = 52, r = 0.08, r_f = 0.05, \sigma = 0.2$, time=0.5, number of steps = 100.

C++ program:

```
double S = 50.0;     double K = 52.0;
double r = 0.08;     double rf=0.05;
double sigma = 0.2; double time=0.5;
int no_steps = 100;
cout << " european currency option call = "
     << currency_option_price_call_american_binomial(S,K,r,rf,sigma,time,no_steps) << endl;
```

Output from C++ program:

```
 european currency option call = 2.23129
```

12.8 References

The original source for binomial option pricing was the paper by Cox et al. (1979). Textbook discussions are in Cox and Rubinstein (1985), Bossaerts and Ødegaard (2001) and Hull (2011).

Chapter 13

Finite Differences

Contents

13.1 Explicit Finite differences

The method of choice for any engineer given a differential equation to solve is to numerically approximate it using a finite difference scheme, which is to approximate the continous differential equation with a discrete *difference* equation, and solve this difference equation.

13.2 European Options.

For European options we do not need to use the finite difference scheme, but we show how one would find the european price for comparison purposes. We show the case of an explicit finite difference scheme in **C++ Code 13.1**. A problem with the explicit version is that it may not converge for certain combinations of inputs.

```
#include <cmath>
#include <vector>
using namespace std;

double option_price_put_european_finite_diff_explicit(const double& S,
                                                      const double& X,
                                                      const double& r,
                                                      const double& sigma,
                                                      const double& time,
                                                      const int& no_S_steps,
                                                      const int& no_t_steps) {
    double sigma_sqr = pow(sigma,2);
    int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; } // need no_S_steps to be even:
    double delta_S = 2.0*S/M;
    vector<double> S_values(M+1);
    for (unsigned m=0;m<=M;m++) { S_values[m] = m*delta_S; };
    int N=no_t_steps;
    double delta_t = time/N;

    vector<double> a(M);
    vector<double> b(M);
    vector<double> c(M);
    double r1=1.0/(1.0+r*delta_t);
    double r2=delta_t/(1.0+r*delta_t);
    for (unsigned int j=1;j<M;j++){
        a[j] = r2*0.5*j*(-r+sigma_sqr*j);
        b[j] = r1*(1.0-sigma_sqr*j*j*delta_t);
        c[j] = r2*0.5*j*(r+sigma_sqr*j);
    };
    vector<double> f_next(M+1);
    for (unsigned m=0;m<=M;++m) { f_next[m]=max(0.0,X-S_values[m]); };
    double f[M+1];
    for (int t=N-1;t>=0;--t) {
        f[0]=X;
        for (unsigned m=1;m<M;++m) {
            f[m]=a[m]*f_next[m-1]+b[m]*f_next[m]+c[m]*f_next[m+1];
        };
        f[M] = 0;
        for (unsigned m=0;m<=M;++m) { f_next[m] = f[m]; };
    };
    return f[M/2];
};
```

C++ Code 13.1: Explicit finite differences calculation of european put option

13.3 American Options.

We now compare the American versions of the same algoritms, the only difference being the check for exercise at each point. **C++ Code 13.2** shows the C++ code for an american put option and `Matlab` **Code 13.1** shows the same implemented in `Matlab`.

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double option_price_put_american_finite_diff_explicit( const double& S,
                                                       const double& K,
                                                       const double& r,
                                                       const double& sigma,
                                                       const double& time,
                                                       const int& no_S_steps,
                                                       const int& no_t_steps) {
    double sigma_sqr = sigma*sigma;
    int M=no_S_steps+(no_S_steps%2); // need no_S_steps to be even:
    double delta_S = 2.0*S/M;
    vector<double> S_values(M+1);
    for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; };
    int N=no_t_steps;
    double delta_t = time/N;

    vector<double> a(M);
    vector<double> b(M);
    vector<double> c(M);
    double r1=1.0/(1.0+r*delta_t);
    double r2=delta_t/(1.0+r*delta_t);
    for (int j=1;j<M;j++){
        a[j] = r2*0.5*j*(-r+sigma_sqr*j);
        b[j] = r1*(1.0-sigma_sqr*j*j*delta_t);
        c[j] = r2*0.5*j*(r+sigma_sqr*j);
    };
    vector<double> f_next(M+1);
    for (int m=0;m<=M;++m) { f_next[m]=max(0.0,K-S_values[m]); };
    vector<double> f(M+1);
    for (int t=N-1;t>=0;--t) {
        f[0]=K;
        for (int m=1;m<M;++m) {
            f[m]=a[m]*f_next[m-1]+b[m]*f_next[m]+c[m]*f_next[m+1];
            f[m] = max(f[m],K-S_values[m]); // check for exercise
        };
        f[M] = 0;
        for (int m=0;m<=M;++m) { f_next[m] = f[m]; };
    };
    return f[M/2];
};
```

C++ Code 13.2: Explicit finite differences calculation of american put option

```
function P = findiff_exp_am_put(S,K,r,sigma,time,no_S_steps,no_t_steps)
    sigma_sqr = sigma^2;
    M=no_S_steps + rem(no_S_steps,2); # need no_S_steps to be even:
    delta_S = 2*S/M;
    S_values = delta_S * (0:M)';
    N=no_t_steps;
    delta_t = time/N;

    r1=1/(1+r*delta_t);
    r2=delta_t/(1+r*delta_t);

    a=zeros(M-1,1);
    b=zeros(M-1,1);
    c=zeros(M-1,1);
    for j=1:M-1
       a(j) = r2*0.5*j*(-r+sigma_sqr*j);
       b(j) = r1*(1.0-sigma_sqr*j*j*delta_t);
       c(j) = r2*0.5*j*(r+sigma_sqr*j);
    endfor
    f_next = max(0,K-S_values);
    for  t=N-1:-1:0
       f = [ K; a.*f_next(1:M-1)+b.*f_next(2:M)+c.*f_next(3:M+1); 0];
       f = max(f,K-S_values);
       f_next=f;
    endfor
    P=f(1+M/2);
endfunction
```

Matlab **Code 13.1**: Explicit finite differences calculation of American put option

Example

Given the following parameters: $S = 50$, $K = 50$, $r = 10\%$ and $\sigma = 0.4$. The time to maturity is 0.4167.

Price European and American put option prices using finite differences with 20 steps in the S dimension and 11 steps in the time dimenstion.

C++ program:

```
double S = 50.0;
double K = 50.0;
double r = 0.1;
double sigma = 0.4;
double time=0.4167;
int no_S_steps=20;
int no_t_steps=11;
cout << " explicit finite differences, european put price = ";
cout << option_price_put_european_finite_diff_explicit(S,K,r,sigma,time,no_S_steps,no_t_steps)
     << endl;
cout << " explicit finite differences, american put price = ";
cout << option_price_put_american_finite_diff_explicit(S,K,r,sigma,time,no_S_steps,no_t_steps)
     << endl;
```

Output from C++ program:

```
explicit finite differences, european put price = 4.03667
explicit finite differences, american put price = 4.25085
```

Readings Brennan and Schwartz (1978) is one of the first finance applications of finite differences. Section 14.7 of Hull (1993) has a short introduction to finite differences. Wilmott, Dewynne, and Howison (1994) is an exhaustive source on option pricing from the perspective of solving partial differential equations.

13.4 Implicit finite differences

What really distinguishes C++ from standard C is the ability to *extend* the language by creating classes and collecting these classes into libraries. A library is a collection of classes and routines for one particular purpose. We have already seen this idea when creating the `date` and `term_structure` classes. However, one should not necessarily always go ahead and create such classes from scratch. It is just as well to use somebody else's class, as long as it is correct and well documented and fulfills a particular purpose.

13.5 An example matrix class

Use `Newmat` as an example matrix class.

13.6 Finite Differences

We use the case of implicit finite difference calculations to illustrate matrix calculations in action.

The method of choice for any engineer given a differential equation to solve is to numerically approximate it using a finite difference scheme, which is to approximate the continous differential equation with a discrete *difference* equation, and solve this difference equation.

In the following we implement *implicit finite differences*. Explicit finite differences was discussed earlier, we postponed the implicit case to now because it is much simplified by a matrix library.

13.7 American Options

Let us first look at how this pricing is implemented in `Matlab`. `Matlab` **Code 13.2** shows the implementation. Implementation of the same calculation in **C++ Code 13.3** using the `Newmat` library and **C++ Code 13.4** using IT++.

```
function P = findiff_imp_am_put(S,K,r,sigma,time,no_S_steps,no_t_steps)
     sigma_sqr = sigma^2;
     M=no_S_steps + rem(no_S_steps,2); # need no_S_steps to be even:
     delta_S = 2.0*S/double(M);
     S_values = delta_S* (1:M+1)';
     N=no_t_steps;
     delta_t = time/N;
     A = zeros(M+1,M+1);
     A(1,1)=1.0;
     for j=2:M
        A(j,j-1) = 0.5*j*delta_t*(r-sigma_sqr*j);
        A(j,j)   = 1.0 + delta_t*(r+sigma_sqr*j*j);
        A(j,j+1) = 0.5*j*delta_t*(-r-sigma_sqr*j);
    endfor
    A(M+1,M+1)=1.0;
    B = max(0,K-S_values);
    F = inv(A)*B;
    for  t=N-1:-1:1
        B = F;
        F = inv(A)*B;
        F=max(F,K-S_values);
    endfor
   P= F(M/2);
endfunction
```

`Matlab` **Code 13.2:** Calculation of price of American put using implicit finite differences

```
#include <cmath>
#include "newmat.h" // definitions for newmat matrix library
using namespace NEWMAT;

#include <vector>
#include <algorithm>
using namespace std;

double option_price_put_american_finite_diff_implicit(const double& S,
                                                       const double& K,
                                                       const double& r,
                                                       const double& sigma,
                                                       const double& time,
                                                       const int& no_S_steps,
                                                       const int& no_t_steps) {
    double sigma_sqr = sigma*sigma;
    int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:
    //    int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; }; // need no_S_steps to be even:
    double delta_S = 2.0*S/double(M);
    double S_values[M+1];
    for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; };
    int N=no_t_steps;
    double delta_t = time/N;

    BandMatrix A(M+1,1,1); A=0.0;
    A.element(0,0) = 1.0;
    for (int j=1;j<M;++j) {
        A.element(j,j-1) = 0.5*j*delta_t*(r-sigma_sqr*j); // a[j]
        A.element(j,j)  = 1.0 + delta_t*(r+sigma_sqr*j*j); // b[j];
        A.element(j,j+1) = 0.5*j*delta_t*(-r-sigma_sqr*j); // c[j];
    };
    A.element(M,M)=1.0;
    ColumnVector B(M+1);
    for (int m=0;m<=M;++m){ B.element(m) = max(0.0,K-S_values[m]); };
    ColumnVector F=A.i()*B;
    for(int t=N-1;t>0;--t) {
        B = F;
        F = A.i()*B;
        for (int m=1;m<M;++m) {  // now check for exercise
            F.element(m) = max(F.element(m), K-S_values[m]);
        };
    };
    return F.element(M/2);
};
```

C++ Code 13.3: Calculation of price of American put using implicit finite differences with the `Newmat` matrix library

```
#include <cmath>
//#include <vector>
//#include <algorithm>
using namespace std;

#include <itpp/base/vec.h>
#include <itpp/base/mat.h>
#include <itpp/base/matfunc.h>

using namespace itpp;

double option_price_put_american_finite_diff_implicit_itpp(const double& S,
                                                           const double& K,
                                                           const double& r,
                                                           const double& sigma,
                                                           const double& time,
                                                           const int& no_S_steps,
                                                           const int& no_t_steps) {
    double sigma_sqr = sigma*sigma;
    int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:

    double delta_S = 2.0*S/double(M);
    double S_values[M+1];
    for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; };
    int N=no_t_steps;
    double delta_t = time/N;

    mat A(M+1,M+1);
    A.zeros();
    A(0,0) = 1.0;
    for (int j=1;j<M;++j) {
        A(j,j-1) = 0.5*j*delta_t*(r-sigma_sqr*j); // a[j]
        A(j,j)   = 1.0 + delta_t*(r+sigma_sqr*j*j); // b[j];
        A(j,j+1) = 0.5*j*delta_t*(-r-sigma_sqr*j); // c[j];
    };
    A(M,M)=1.0;
    vec B(M+1);
    for (int m=0;m<=M;++m){ B(m) = max(0.0,K-S_values[m]); };
    mat InvA = inv(A);
    vec F=InvA*B;
    for(int t=N-1;t>0;--t) {
        B = F;
        F = InvA*B;
        for (int m=1;m<M;++m) {  // now check for exercise
            F(m) = max(F(m), K-S_values[m]);
        };
    };
    return F(M/2);
};
```

C++ Code 13.4: Calculation of price of American put using implicit finite differences with the IT++ matrix library

13.8 European Options

For European options we do not need to use the finite difference scheme, but for comparison purposes **C++ Code 13.5** show how one would find the European price.

```
#include <cmath>
#include "newmat.h" // definitions for newmat matrix library
using namespace NEWMAT;

#include <vector> // standard STL vector template
#include <algorithm>
using namespace std;

double option_price_put_european_finite_diff_implicit(const double& S,
                                                      const double& K,
                                                      const double& r,
                                                      const double& sigma,
                                                      const double& time,
                                                      const int& no_S_steps,
                                                      const int& no_t_steps) {
    double sigma_sqr = sigma*sigma;
    int M=no_S_steps + (no_S_steps%2); // need no_S_steps to be even:
    //   int M=no_S_steps; if ((no_S_steps%2)==1) { ++M; }; // need no_S_steps to be even:
    double delta_S = 2.0*S/M;
    vector<double> S_values(M+1);
    for (int m=0;m<=M;m++) { S_values[m] = m*delta_S; };
    int N=no_t_steps;
    double delta_t = time/N;

    BandMatrix A(M+1,1,1); A=0.0;
    A.element(0,0) = 1.0;
    for (int j=1;j<M;++j) {
        A.element(j,j-1) = 0.5*j*delta_t*(r-sigma_sqr*j); // a[j]
        A.element(j,j)  = 1.0 + delta_t*(r+sigma_sqr*j*j); // b[j];
        A.element(j,j+1) = 0.5*j*delta_t*(-r-sigma_sqr*j); // c[j];
    };
    A.element(M,M)=1.0;
    ColumnVector B(M+1);
    for (int m=0;m<=M;++m){ B.element(m) = max(0.0,K-S_values[m]); };
    ColumnVector F=A.i()*B;
    for(int t=N-1;t>0;--t) {
        B = F;
        F = A.i()*B;
    };
    return F.element(M/2);
};
```

C++ Code 13.5: Calculation of price of European put using implicit finite differences

Exercise 13.1.

The routines above uses direct multiplication with the inverse of $\mathbf{A}$. Are there alternative, numerically more stable ways of doing it?

Example

Given the parameters $S = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.4$, time=0.5, no S steps=200; no t steps=200;

1. Price European put options using both Black Scholes and implicit finite differences.
2. Price the American put option using implicit finite differences.

C++ program:

```
double S = 50.0;    double K = 50.0;
double r = 0.1;     double sigma = 0.4;   double time=0.5;
int no_S_steps=200; int no_t_steps=200;
cout << " black scholes put price = " << option_price_put_black_scholes(S,K,r,sigma,time)<< endl;
cout << " implicit Euro put price = ";
cout << option_price_put_european_finite_diff_implicit(S,K,r,sigma,time,no_S_steps,no_t_steps) << endl;
cout << " implicit American put price = ";
cout << option_price_put_american_finite_diff_implicit(S,K,r,sigma,time,no_S_steps,no_t_steps) << endl;
```

Output from C++ program:

```
black scholes put price = 4.35166
implicit Euro put price = 4.34731
implicit American put price = 4.60064
```

Matlab program:

```
S = 50.0;
K = 50.0;
r = 0.1;
sigma = 0.4;
time=0.5;
no_S_steps=200;
no_t_steps=200;
P= findiff_imp_am_put(S,K,r,sigma,time,no_S_steps,no_t_steps)
```

Output from Matlab program:

```
P = 4.6006
```

Exercise 13.2.

The Newmat library is only one of a large number of available matrix libraries, both public domain and commercial offerings. Look into alternative libraries and replace Newmat with one of these alternative libraries. What needs changing in the option price formulas?

13.9 References

Hull (2011).

Chapter 14

Option pricing by simulation

Contents

We now consider using Monte Carlo methods to estimate the price of an European option, and let us first consider the case of the "usual" European Call, which can priced by the Black Scholes equation. Since there is already a closed form solution for this case, it is not really necessary to use simulations, but we use the case of the standard call for illustrative purposes.

At maturity, a call option is worth

$$c_T = \max(0, S_T - X)$$

At an earlier date t, the option value will be the expected present value of this.

$$c_t = E[PV(\max(0, S_T - X)]$$

Now, an important simplifying feature of option pricing is the "risk neutral result," which implies that we can treat the (suitably transformed) problem as the decision of a risk neutral decision maker, if we also modify the expected return of the underlying asset such that this earns the risk free rate.

$$c_t = e^{-r(T-t)} E^*[\max(0, S_T - X)],$$

where $E^*[\cdot]$ is a transformation of the original expectation. One way to estimate the value of the call is to simulate a large number of sample values of S_T according to the assumed price process, and find the estimated call price as the average of the simulated values. By appealing to a law of large numbers, this average will converge to the actual call value, where the rate of convergence will depend on how many simulations we perform.

14.1 Simulating lognormally distributed random variables

Lognormal variables are simulated as follows. Let $\tilde{x}$ be normally distributed with mean zero and variance one. If S_t follows a lognormal distribution, then the one-period-later price S_{t+1} is simulated as

$$S_{t+1} = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)+\sigma\tilde{x}},$$

or more generally, if the current time is t and terminal date is T, with a time between t and T of $(T-t)$,

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\tilde{x}}$$

Simulation of lognormal random variables is illustrated by **C++ Code 14.1**.

```
#include <cmath>
using namespace std;
#include "normdist.h"

double simulate_lognormal_random_variable(const double& S, // current value of variable
                                          const double& r, // interest rate
                                          const double& sigma, // volatitily
                                          const double& time) { // time to final date
   double R = (r - 0.5 * pow(sigma,2) )*time;
   double SD = sigma * sqrt(time);
   return S * exp(R + SD * random_normal());
};
```

C++ Code 14.1: Simulating a lognormally distributed random variable

14.2 Pricing of European Call options

For the purposes of doing the Monte Carlo estimation of the price of an European call

$$c_t = e^{-r(T-t)}E[\max(0, S_T - X)],$$

note that here one merely need to simulate the terminal price of the underlying, S_T, the price of the underlying at any time between t and T is not relevant for pricing. We proceed by simulating lognormally distributed random variables, which gives us a set of observations of the terminal price S_T. If we let $S_{T,1}, S_{T,2}, S_{T,3}, \ldots S_{T,n}$ denote the n simulated values, we will estimate $E^*[\max(0, S_T - X)]$ as the average of option payoffs at maturity, discounted at the risk free rate.

$$\hat{c}_t = e^{-r(T-t)}\left(\sum_{i=1}^{n}\max\left(0, S_{T,i} - X\right)\right)$$

C++ Code 14.2 shows the implementation of a Monte Carlo estimation of an European call option.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1. Use 5000 simulations. Price put and call option using Black Scholes and simulation.

```
#include <cmath>  // standard mathematical functions
#include <algorithm>  // define the max() function
using namespace std;
#include "normdist.h" // definition of random number generator

double option_price_call_european_simulated( const double& S,
                                             const double& K,
                                             const double& r,
                                             const double& sigma,
                                             const double& time,
                                             const int& no_sims){
    double R = (r - 0.5 * pow(sigma,2))*time;
    double SD = sigma * sqrt(time);
    double sum_payoffs = 0.0;
    for (int n=1; n<=no_sims; n++) {
        double S_T = S* exp(R + SD * random_normal());
        sum_payoffs += max(0.0, S_T-K);
    };
    return exp(-r*time) * (sum_payoffs/double(no_sims));
};
```

C++ Code 14.2: European Call option priced by simulation

C++ program:

```
    double S=100.0; double K=100.0; double r=0.1; double sigma=0.25;
    double time=1.0; int no_sims=5000;
    cout << " call: black scholes price = " << option_price_call_black_scholes(S,K,r,sigma,time) << endl;
    cout << "     simulated price = "
         << option_price_call_european_simulated(S,K,r,sigma,time,no_sims) << endl;
    cout << " put: black scholes price = " << option_price_put_black_scholes(S,K,r,sigma,time) << endl;
    cout << "    simulated price = "
         << option_price_put_european_simulated(S,K,r,sigma,time, no_sims) << endl;
```

Output from C++ program:

```
 call:  black scholes price = 14.9758
        simulated price     = 14.8404
 put:  black scholes price = 5.45954
       simulated price     = 5.74588
```

14.3 Hedge parameters

It is of course, just as in the standard case, desirable to estimate hedge parameters as well as option prices. We will show how one can find an estimate of the option *delta*, the first derivative of the call price with respect to the underlying security: $\Delta = \frac{\partial c_t}{\partial S}$. To understand how one goes about estimating this, let us recall that the first derivative of a function f is defined as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Thinking of $f(S)$ as the option price formula $c_t = f\left(S; X, r, \sigma, (T-t)\right)$, we see that we can evaluate the option price at two different values of the underlying, S and $S+q$, where q is a small quantity, and estimate the option delta as

$$\widehat{\Delta} = \frac{f(S+q) - f(S)}{q}$$

In the case of Monte Carlo estimation, it is very important that this is done by using the same sequence

of random variables to estimate the two option prices with prices of the underlying S and $S + q$. **C++ Code 14.3** implements this estimation of the option delta.

```
#include <cmath>  // standard mathematical functions
#include <algorithm> // define the max() function
using namespace std;
#include "normdist.h" // definition of random number generator

double option_price_delta_call_european_simulated(const double& S,
                                                  const double& K,
                                                  const double& r,
                                                  const double& sigma,
                                                  const double& time,
                                                  const int& no_sims){
    double R = (r - 0.5 * pow(sigma,2))*time;
    double SD = sigma * sqrt(time);
    double sum_payoffs = 0.0;
    double sum_payoffs_q = 0.0;
    double q = S*0.01;
    for (int n=1; n<=no_sims; n++) {
        double Z = random_normal();
        double S_T = S* exp(R + SD * Z);
        sum_payoffs += max(0.0, S_T-K);
        double S_T_q = (S+q)* exp(R + SD * Z);
        sum_payoffs_q += max(0.0, S_T_q-K);
    };
    double c = exp(-r*time) * (sum_payoffs/no_sims);
    double c_q = exp(-r*time) * (sum_payoffs_q/no_sims);
    return (c_q-c)/q;
};
```

C++ Code 14.3: Estimate Delta of European Call option priced by Monte Carlo

One can estimate other hedge parameters in a simular way.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1. Use 5000 simulations. Calculate deltas of put and call option using Black Scholes and simulation.

C++ program:

```
    double S=100.0; double K=100.0; double r=0.1; double sigma=0.25;
    double time=1.0; int no_sims=5000;
    cout << " call: bs delta = " << option_price_delta_call_black_scholes(S,K,r,sigma,time)
        << "    sim delta = " << option_price_delta_call_european_simulated(S,K,r,sigma,time,no_sims)
        << endl;
    cout << " put: bs delta = " << option_price_delta_put_black_scholes(S,K,r,sigma,time)
        << "   sim delta = " << option_price_delta_put_european_simulated(S,K,r,sigma,time,no_sims)
        << endl;
```

Output from C++ program:

```
 call: bs delta  = 0.700208       sim delta = 0.701484
 put: bs delta  = -0.299792      sim delta = -0.307211
```

14.4 More general payoffs. Function prototypes

The above shows the case for a call option. If we want to price other types of options, with different payoffs we could write similar routines for every possible case. But this would be wasteful, instead a bit of thought allows us to write option valuations for any kind of option whose payoff depend on the value of the underlying at maturity, only. Let us now move toward a generic routine for pricing derivatives with Monte Carlo. This relies on the ability of C++ to write subroutines which one call with *function prototypes*, i.e. that in the call to to the subroutine/function one provides a function instead of a variable. Consider pricing of standard European put and call options. At maturity each option only depend on the value of the underlying S_T and the exercise price X through the relations

$$C_T = \max(S_T - X, 0)$$

$$P_T = \max(X - S_T, 0)$$

C++ Code 14.4 shows two C++ functions which calculates this.

```
#include <algorithm>
using namespace std;

double payoff_call(const double& S,
                   const double& K){
    return max(0.0,S-K);
};

double payoff_put (const double& S,
                   const double& K) {
    return max(0.0,K-S);
};
```

C++ Code 14.4: Payoff call and put options

The interesting part comes when one realises one can write a generic simulation routine to which one provide one of these functions, or some other function describing a payoff which only depends on the price of the underlying and some constant. **C++ Code 14.5** shows how this is done.

```
#include <cmath>
using namespace std;
#include "fin_recipes.h"

double derivative_price_simulate_european_option_generic(const double& S,
                                                         const double& K,
                                                         const double& r,
                                                         const double& sigma,
                                                         const double& time,
                                                         double payoff(const double& price, const double& X),
                                                         const int& no_sims) {
    double sum_payoffs=0;
    for (int n=0; n<no_sims; n++) {
       double S_T = simulate_lognormal_random_variable(S,r,sigma,time);
       sum_payoffs += payoff(S_T,K);
    };
    return exp(-r*time) * (sum_payoffs/no_sims);
};
```

C++ Code 14.5: Generic simulation pricing

Note the presence of the line

```
double payoff(const double& S, const double& K),
```

in the subroutine call. When this function is called, the calling program will need to provide a function to put there, such as the Black Scholes example above. The next example shows a complete example of how this is done.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1, no sims=5000.

C++ program:

```
double S = 100; double K = 100; double r = 0.1;
double sigma = 0.25; double time = 1.0; int no_sims = 50000;
cout << "Black Scholes call option price = " << option_price_call_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated call option price = "
     << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_call,no_sims)
       << endl;
cout << "Black Scholes put option price = " << option_price_put_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated put option price = "
       << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_put,no_sims)
       << endl;
```

Output from C++ program:

```
Black Scholes call option price = 14.9758
Simulated call option price     = 14.995
Black Scholes put option price  = 5.45954
Simulated put option price      = 5.5599
```

As we see, even with as many as 50,000 simuations, the option prices estimated using Monte Carlo still differs substantially from the "true" values.

14.5 Improving the efficiency in simulation

There are a number of ways of "improving" the implementation of Monte Carlo estimation such that the estimate is closer to the true value.

14.5.1 Control variates.

One is the method of control variates. The idea is simple. When one generates the set of terminal values of the underlying security, one can value several derivatives using the same set of terminal values. What if one of the derivatives we value using the terminal values is one which we have an analytical solution to? For example, suppose we calculate the value of an at the money European call option using both the (analytical) Black Scholes formula and Monte Carlo simulation. If it turns out that the Monte Carlo estimate overvalues the option price, we think that this will also be the case for other derivatives valued using the same set of simulated terminal values. We therefore move the estimate of the price of the derivative of interest downwards.

Thus, suppose we want to value an European put and we use the price of an at the money European call as the control variate. Using the same set of simulated terminal values $S_{T,i}$, we estimate the two options using Monte Carlo as:

$$\hat{p}_t = e^{-r(T-t)} \left(\sum_{i=1}^{n} \max\left(0, X - S_{T,i}\right) \right)$$

$$\hat{c}_t = e^{-r(T-t)} \left(\sum_{i=1}^{n} \max\left(0, S_{T,i} - X\right) \right)$$

We calculate the Black Scholes value of the call $\hat{c}_t^{bs}$, and calculate p_t^{cv}, the estimate of the put price with a control variate adjustment, as follows

$$\hat{p}_t^{cv} = \hat{p}_t + (c_t^{bs} - \hat{c}_t)$$

One can use other derivatives than the at-the-money call as the control variate, the only limitation being that it has a tractable analytical solution.

C++ Code 14.6 shows the implementation of a Monte Carlo estimation using an at-the-money European call as the control variate.

```
#include <cmath>
using namespace std;
#include "fin_recipes.h"

double
derivative_price_simulate_european_option_generic_with_control_variate(const double& S,
                                                                       const double& X,
                                                                       const double& r,
                                                                       const double& sigma,
                                                                       const double& time,
                                                                       double payoff(const double& S,
                                                                                     const double& X),
                                                                       const int& no_sims) {
    double c_bs = option_price_call_black_scholes(S,S,r,sigma,time);// price an at the money Black Scholes call
    double sum_payoffs=0;
    double sum_payoffs_bs=0;
    for (int n=0; n<no_sims; n++) {
        double S_T= simulate_lognormal_random_variable(S,r,sigma,time);
        sum_payoffs += payoff(S_T,X);
        sum_payoffs_bs += payoff_call(S_T,S); // simulate at the money Black Scholes price
    };
    double c_sim = exp(-r*time) * (sum_payoffs/no_sims);
    double c_bs_sim = exp(-r*time) * (sum_payoffs_bs/no_sims);
    c_sim += (c_bs-c_bs_sim);
    return c_sim;
};
```

C++ Code 14.6: Generic with control variate

14.5.2 Antithetic variates.

An alternative to using control variates is to consider the method of *antithetic* variates. The idea behind this is that Monte Carlo works best if the simulated variables are "spread" out as closely as possible to the true distribution. Here we are simulating unit normal random variables. One property of the normal is that it is symmetric around zero, and the median value is zero. Why don't we enforce this in the simulated terminal values? An easy way to do this is to first simulate a unit random normal variable Z, and then use both Z and $-Z$ to generate the lognormal random variables. **C++ Code 14.7** shows the implementation of this idea.

Boyle (1977) shows that the efficiency gain with antithetic variates is not particularly large. There are other ways of ensuring that the simulated values really span the whole sample space, sometimes called "pseudo Monte Carlo."

```
#include "fin_recipes.h"
#include "normdist.h"
#include <cmath>
using namespace std;

double
derivative_price_simulate_european_option_generic_with_antithetic_variate(const double& S,
                                                                          const double& K,
                                                                          const double& r,
                                                                          const double& sigma,
                                                                          const double& time,
                                                                          double payoff(const double& S,
                                                                                        const double& X),
                                                                          const int& no_sims) {
    double R = (r - 0.5 * pow(sigma,2) )*time;
    double SD = sigma * sqrt(time);
    double sum_payoffs=0;
    for (int n=0; n<no_sims; n++) {
        double x=random_normal();
        double S1 = S * exp(R + SD * x);
        sum_payoffs += payoff(S1,K);
        double S2 = S * exp(R + SD * (-x));
        sum_payoffs += payoff(S2,K);
    };
    return exp(-r*time) * (sum_payoffs/(2*no_sims));
};
```

C++ Code 14.7: Generic with antithetic variates

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1, no sims=50000. Price put and call option using Black Scholes and simulation. The simulation is to use both control variates and anthitetic methods.

C++ program:

```
double S = 100; double K = 100; double r = 0.1;
double sigma  = 0.25; double time = 1; int no_sims = 50000;
cout << "Black Scholes call option price = "
     << option_price_call_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated call option price = "
     << derivative_price_simulate_european_option_generic(S,K,r,sigma,time, payoff_call,no_sims)
     << endl;
cout << "Simulated call option price, CV = "
     << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time,
                                                                               payoff_call,no_sims)
     << endl;
cout << "Simulated call option price, AV = "
     << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time,
                                                                                  payoff_call,no_sims)
     << endl;
cout << "Black Scholes put option price = " << option_price_put_black_scholes(S,K,r,sigma,time) << endl;
cout << "Simulated put option price = "
     << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,payoff_put,no_sims) << endl;
cout << "Simulated put option price, CV = "
     << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time,
                                                                               payoff_put,no_sims)
     << endl;
cout << "Simulated put option price, AV = "
     << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time,
                                                                                  payoff_put,no_sims)
     << endl;
```

Output from C++ program:

```
Black Scholes call option price = 14.9758
Simulated call option price     = 14.995
Simulated call option price, CV = 14.9758
Simulated call option price, AV = 14.9919
Black Scholes put option price  = 5.45954
Simulated put option price      = 5.41861
Simulated put option price,  CV = 5.42541
Simulated put option price,  AV = 5.46043
```

14.6 More exotic options

These generic routines can also be used to price other options. Any European option that only depends on the terminal value of the price of the underlying security can be valued. Consider the *binary* options discussed by e.g. Hull (2011). An *cash or nothing call* pays a fixed amount Q if the price of the asset is above the exercise price at maturity, otherwise nothing. An *asset or nothing call* pays the price of the asset if the price is above the exercise price at maturity, otherwise nothing. Both of these options are easy to implement using the generic routines above, all that is necesary is to provide the payoff functions as shown in **C++ Code 14.8**.

```
double payoff_cash_or_nothing_call(const double& S,
                                   const double& K){
    if (S>=K) return 1;
    return 0;
};

double payoff_asset_or_nothing_call(const double& S,
                                    const double& K){
    if (S>=K) return S;
    return 0;
};
```

C++ Code 14.8: Payoff binary options

Now, many exotic options are not simply functions of the terminal price of the underlying security, but depend on the evolution of the price from "now" till the terminal date of the option. For example options that depend on the average of the price of the underlying (Asian options). For such cases one will have to simulate the whole path. We will return to these cases in the chapter on pricing of exotic options.

Example

Given $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$, time=1. Use 5000 simulations. Price cash or nothing option with $Q = 1$.

Price asset or nothing option.

In both cases compare results using control variate and anthitetic methods.

C++ program:

```
double S=100.0; double K=100.0; double r=0.1; double sigma=0.25;
double time=1.0; int no_sims=5000;
cout << " cash or nothing, Q=1: "
     << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
                                                          payoff_cash_or_nothing_call,
                                                          no_sims)
     << endl;
cout << " control variate "
     << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time,
                                                                                payoff_cash_or_nothing_call,
                                                                                no_sims)
     << endl;
cout << " antithetic variate "
     << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time,
                                                                                  payoff_cash_or_nothing_call,
                                                                                  no_sims)
     << endl;
cout << " asset or nothing: "
     << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
                                                          payoff_asset_or_nothing_call,
                                                          no_sims)
     << endl;
cout << " control variate "
     << derivative_price_simulate_european_option_generic_with_control_variate(S,K,r,sigma,time,
                                                                                payoff_asset_or_nothing_call,
                                                                                no_sims)
     << endl;
cout << " antithetic variate "
     << derivative_price_simulate_european_option_generic_with_antithetic_variate(S,K,r,sigma,time,
                                                                                  payoff_asset_or_nothing_call,
                                                                                  no_sims)
     << endl;
```

Output from C++ program:

```
cash or nothing, Q=1: 0.547427
control variate 1.02552
antithetic variate 0.549598
asset or nothing: 70.5292
control variate 69.8451
antithetic variate 70.2205
```

Exercise 14.1.

Consider the pricing of an European Call option as implemented in code 14.2, and the generic formula for pricing with Monte Carlo for European options that only depend on the terminal value of the underlying security, as implemented in code 14.5.

Note the difference in the implementation of the lognormal simulation of terminal values. Why can one argue that the first implementation is more efficient than the other?

14.7 References

Boyle (1977) is a good early source on the use of the Monte Carlo technique for pricing derivatives. Simulation is also covered in Hull (2011).

Chapter 15

Pricing American Options – Approximations

Contents

There has been developed some useful *approximations* to various specific options. It is of course American options that are approximated. We will look at a number of approximations, both for their intrinsic usefulness, but also as examples of different approaches to generating an approximated value. The first we will look at is the Johnson (1983) approximation to an American put. This is a purely empirical approach to estimating the functional forms, with parameters estimated using regressions. The Geske and Johnson (1984) approach is to view the American option as a the limit of a sequence of Bermudan options with increasing number of exercise dates. The American value is found by interpolating forward the Bermudan values. The Barone-Adesi and Whaley (1987) approximation decomposes the American option into two, the European value, and the early exercise premium. The early exercise premium is relatively small, and is easier to approximate. Finally, the Bjerksund and Stensland (1993) is an example of bounding the option value by finding a lower bound on its value. A typical approach to doing so is to specify some (suboptimal) exercise strategy. As soon as the exercise strategy is known, the option is easily valued.

Approximations are useful, but they should be used with caution. Typically, an approximation works well within a range of parameter values, but can go seriously wrong for some unfortunate combination of parameters. The user is well advised to to consider alternative approximations, and at a minimum use the price of the corresponding European option as a "sanity check" in the calculations. (In fact, in several of the following routines values are checked against Black Scholes values.)

15.1 The Johnson (1983) approximation

An early attempt at an analytical approximation to the price of an American put is provided by Johnson (1983). **Formula 15.1** provides the details, and **C++ Code 15.1** provides the implementation.

Example

Using the parameters $r = 0.125$, $S = 1.1$, $X = 1$, $\sigma = 0.5$ and time to maturity of one year, calculate the price of an american call using the Johnson (1983) approximation.

$$P(X) = \alpha p\left(Xe^{r(T-t)}\right) + (1-\alpha)p(X)$$

$$\alpha = \left(\frac{r(T-t)}{a_0 r(T-t) + a_1}\right)^l, \quad \text{where} \quad l = \frac{\ln(S/S_c)}{\ln(X/S_c)}$$

$$S_c = X\left(\frac{\gamma}{1+\gamma}\right)^m, \quad \text{where} \quad m = \frac{\sigma^2(T-t)}{b_0\sigma^2(T-t) + b_1} \quad \gamma = 2r/\sigma^2$$

The constants a_0, a_1, b_1 and b_2 are constants. Their parameter values are estimated as

$$a_0 = 3.9649 \quad a_1 = 0.032325$$

$$b_0 = 1.040803 \quad b_1 = 0.00963$$

S: Current value of underlying security. X: Exercise price of american put. σ: Volatility of underlying. r: risk freee interest rate. $(T-t)$: Time to maturity for option.

Formula 15.1: The functional form of the Johnson (1983) approximation to the value of an American put

C++ program:

```
double r=0.125;  double S=1.1;  double X=1;
double sigma=0.5;  double time = 1;
cout << " American put price using Johnson approximation = "
     << option_price_american_put_approximated_johnson(S, X, r, sigma, time)
     << endl;
```

Output from C++ program:

```
American put price using Johnson approximation = 0.11582
```

Barone-Adesi (2005) notes that this approximation is inaccurate outside of the parameter range considered in the paper.

```
#include <cmath>
#include "fin_recipes.h"

double option_price_american_put_approximated_johnson( const double& S,
                                                       const double& X,
                                                       const double& r,
                                                       const double& sigma,
                                                       const double& time ){
    double sigma_sqr=pow(sigma,2);
    double a0= 3.9649 ;
    double a1 = 0.032325;
    double b0 = 1.040803;
    double b1 = 0.00963;
    double gamma = 2*r/sigma_sqr;
    double m = (sigma_sqr*time)/(b0*sigma_sqr*time+b1);
    double Sc = X * pow (((gamma)/(1+gamma)),m);
    double l = (log(S/Sc))/(log(X/Sc) );
    double alpha = pow( ( (r*time)/(a0*r*time+a1) ), l );
    double P = alpha*option_price_put_black_scholes(S,X*exp(r*time),r,sigma,time)
        + (1-alpha)*option_price_put_black_scholes(S,X,r,sigma,time);
    double p=option_price_put_black_scholes(S,X,r,sigma,time); // for safety use the Black Scholes as lower bound
    return max(p,P);
};
```

C++ Code 15.1: The Johnson (1983) approximation to an american put price

15.2 An approximation to the American Put due to Geske and Johnson (1984)

Geske and Johnson (1984) develop an approximation for the American put problem. The solution technique is to view the American put as an sequence of Bermudan options, with the number of exercise dates increasing. The correct value is the limit of this sequence.

Define P_i to be the price of the put option with i dates of exercise left. P_1 is then the price of an european option, with the one exercise date the expiry date. P_2 is the price of an option that can be exercised twice, once halfway between now and expiry, the other at expiry. Geske-Johnson shows how these options may be priced, and then develops a sequence of approximate prices that converges to the correct price. An approximation involving 3 option evaluations is

$$\hat{P} = P_3 + \frac{7}{2}(P_3 - P_2) - \frac{1}{2}(P_2 - P_1)$$

To calculate these, first define

$$d_1(q,\tau) = \frac{\ln\left(\frac{S}{q}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2(q,\tau) = d_1 - \sigma\sqrt{\tau}$$

and

$$\rho_{12} = \frac{1}{\sqrt{2}}, \quad \rho_{13} = \frac{1}{\sqrt{3}}, \quad \rho_{23} = \sqrt{\frac{2}{3}}$$

In this notation, P_1 is the ordinary (european) Black Scholes value:

$$P_1 = Xe^{-rT}N_1(-d_2(X,T)) - SN_1(d_1(X,T))$$

To calculate P_2:

$$\begin{aligned} P_2 &= Xe^{-r\frac{T}{2}}N_1\left(-d_2\left(\overline{S}_{\frac{T}{2}}, \frac{T}{2}\right)\right) - SN_1\left(-d_1\left(\overline{S}_{\frac{T}{2}}, \frac{T}{2}\right)\right) \\ &+ Xe^{-rT}N_2\left(d_2\left(\overline{S}_{\frac{T}{2}}, \frac{T}{2}\right), -d_2(X,T); -\rho_{12}\right) - SN_2\left(d_1\left(\overline{S}_{\frac{T}{2}}, \frac{T}{2}\right), -d_1\left(X,T\right); -\rho_{12}\right) \end{aligned}$$

and $\overline{S}_{\frac{T}{2}}$ solves

$$S = X - p\left(S, X, \frac{T}{2}, r, \sigma\right) = \overline{S}_{\frac{T}{2}}$$

To calculate P_3:

$$\begin{aligned} P_3 &= Xe^{-r\frac{T}{3}}N_1\left(-d_2\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right)\right) - SN_1\left(-d_1\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right)\right) \\ &+ Xe^{-r\frac{2T}{3}}N_2\left(d_2\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right), -d_2\left(\overline{S}_{\frac{2T}{3}}, \frac{2T}{3}\right); -\rho_{12}\right) \\ &- SN_2\left(d_1\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right), -d_1\left(\overline{S}_{\frac{2T}{3}}, \frac{2T}{3}\right); -\rho_{12}\right) \\ &+ Xe^{-rT}N_3\left(d_1\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right), d_1\left(\overline{S}_{\frac{2T}{3}}, \frac{2T}{3}\right), -d_1(X,T); \rho_{12}, -\rho_{13}, -\rho_{23}\right) \\ &- SN_3\left(d_2\left(\overline{S}_{\frac{T}{3}}, \frac{T}{3}\right), d_2\left(\overline{S}_{\frac{2T}{3}}, \frac{2T}{3}\right), -d_2(X,T); \rho_{12}, -\rho_{13}, -\rho_{23}\right) \end{aligned}$$

where the critical stock prices solves

$$S = X - P_2\left(S, X, \frac{2T}{3}, r, \sigma\right) = \overline{S}_{T/3}$$

$$S = X - p\left(S, X, \frac{T}{3}, r, \sigma\right) = \overline{S}_{2T/3}$$

The main issue in implementation of this formula (besides keeping the notation straight) is the evalutation of P_3, since it involves the evaluation of a trivariate normal cumulative distribution. The evaluation of $N_3()$ is described later, since it involves calls to external routines for numerical intergration.

To solve for

$$S = X - p\left(S, X, \frac{T}{2}, r, \sigma\right) = \overline{S}_{\frac{T}{2}}$$

involves Newton steps. Define

$$g(S) = S - X + p\left(S, X, \frac{T}{2}, r, \sigma\right)$$

Iterate on S

$$S_i = S_i - \frac{g}{g'}$$

until g is equal to zero with some given tolerance.

```
#include <cmath>
#include "normdist.h"
#include "fin_recipes.h"
#include <iostream>

const double ACCURACY=1e-6;

inline double d1(const double& S,const double& X, const double& r, const double& sigma, const double& tau){
    return (log(S/X) + (r+0.5*pow(sigma,2))*tau)/(sigma*sqrt(tau));
};

inline double d2(const double& S, const double& X, const double& r, const double& sigma, const double& tau){
    return d1(S,X,r,sigma,tau)-sigma*sqrt(tau);
};

inline double calcP2(const double& S, const double& X, const double& r, const double& sigma, const double& time,
                     const double& t2, const double& S2_bar, const double& rho12){
    double P2 = X*exp(-r*t2)*N(-d2(S,S2_bar,r,sigma,t2));
    P2 -= S*N(-d1(S,S2_bar,r,sigma,t2));
    P2 += X*exp(-r*time)*N(d2(S,S2_bar,r,sigma,t2),-d2(S,X,r,sigma,time),-rho12);
    P2 -= S*N(d1(S,S2_bar,r,sigma,t2),-d1(S,X,r,sigma,time),-rho12);
    return P2;
};

double option_price_american_put_approximated_geske_johnson( const double& S, const double& X, const double& r,
                                                              const double& sigma, const double& time ){
    double P1 = option_price_put_black_scholes(S,X,r,sigma,time);
    double rho12=1.0/sqrt(2.0);   double rho13=1.0/sqrt(3.0);   double rho23=sqrt(2.0/3.0);
    double t2 = time/2.0;  double t23 = time*2.0/3.0; double t3 = time/3.0;
    double Si=S;     double S2_bar=S;
    double g=1;      double gprime=1;
    while (fabs(g)>ACCURACY){
        g=Si-X+option_price_put_black_scholes(Si,X,r,sigma,t2);
        gprime=1.0+option_price_delta_put_black_scholes(Si,X,r,sigma,t2);
        S2_bar=Si;
        Si=Si-g/gprime;
    };
    double P2 = calcP2(S,X,r,sigma,time,t2,S2_bar,rho12);
    P2=max(P1,P2); // for safety, use one less step as lower bound
    double S23_bar=S2_bar;
    g=1;
    while (fabs(g)>ACCURACY){
        g=Si-X+option_price_put_black_scholes(Si,X,r,sigma,t23);
        gprime=1.0+option_price_delta_put_black_scholes(Si,X,r,sigma,t23);
        S23_bar=Si;
        Si=Si-g/gprime;
    };
    double S3_bar=S23_bar;
    g=1;
    while (fabs(g)>ACCURACY){
        g=Si-X+option_price_put_black_scholes(Si,X,r,sigma,t3);
        gprime=1.0+option_price_delta_put_black_scholes(Si,X,r,sigma,t3);
        S3_bar=Si;
        Si=Si-g/gprime;
    };
    double P3 = X * exp(-r*t3) * N(-d2(S,S3_bar,r,sigma,t3));
    P3 -= S * N(-d1(S,S3_bar,r,sigma,t3));
    P3 += X*exp(-r*time)*N(d2(S,S3_bar,r,sigma,t3),-d2(S,S23_bar,r,sigma,t23),-rho12);
    P3 -= S*N(d1(S,S3_bar,r,sigma,t3),-d1(S,S23_bar,r,sigma,t23),-rho12);
    P3 += X*exp(-r*t23)*N3(d1(S,S3_bar,r,sigma,t3),d1(S,S23_bar,r,sigma,t23),-d1(S,X,r,sigma,time),rho12,-rho13,-rho23);
    P3 -= S*N3(d2(S,S3_bar,r,sigma,t3),d2(S,S23_bar,r,sigma,t23),-d2(S,X,r,sigma,time),rho12,-rho13,-rho23);
    P3=max(P2,P3); // for safety, use one less step as lower bound
    return P3+3.5*(P3-P2)-0.5*(P2-P1);
};
```

C++ Code 15.2: Geske Johnson approximation of American put

15.3 A quadratic approximation to American prices due to Barone–Adesi and Whaley.

We now discuss an approximation to the option price of an American option on a commodity, described in Barone-Adesi and Whaley (1987) (BAW).[1] The commodity is assumed to have a continuous payout b. The starting point for the approximation is the (Black-Scholes) stochastic differential equation valid for the value of any derivative with price V.

$$\frac{1}{2}\sigma^2 S^2 V_S S + bSV_S - rV + V_t = 0 \tag{15.1}$$

Here V is the (unknown) formula that determines the price of the contingent claim. For an European option the value of V has a known solution, the adjusted Black Scholes formula. For American options, which may be exercised early, there is no known analytical solution.

To do their approximation, BAW decomposes the American price into the European price and the early exercise premium

$$C(S,T) = c(S,T) + \varepsilon_C(S,T)$$

Here ε_C is the early exercise premium. The insight used by BAW is that ε_C must *also* satisfy the same partial differential equation. To come up with an approximation BAW transformed equation (15.1) into one where the terms involving V_t are neglible, removed these, and ended up with a standard linear homeogenous second order equation, which has a known solution.

The functional form of the approximation is shown in formula 15.2.

$$C(S,T) = \begin{cases} c(S,T) + A_2\left(\frac{S}{S^*}\right)^{q_2} & \text{if} \quad S < S^* \\ S - X & \text{if} \quad S \geq S^* \end{cases}$$

where

$$q_2 = \frac{1}{2}\left(-(N-1) + \sqrt{(N-1)^2 + \frac{4M}{K}}\right)$$

$$A_2 = \frac{S^*}{q_2}\left(1 - e^{(b-r)(T-t)}N\left(d_1(S^*)\right)\right)$$

$$M = \frac{2r}{\sigma^2},\ N = \frac{2b}{\sigma^2},\ K(T) = 1 - e^{-r(T-t)}$$

and S^* solves

$$S^* - X = c\left(S^*, T\right) + \frac{S^*}{q_2}\left(1 - e^{(b-r)(T-t)}N\left(d_1(S^*)\right)\right)$$

Notation: S Stock price. X: Exercise price.

Formula 15.2: The functional form of the Barone Adesi Whaley approximation to the value of an American call

In implementing this formula, the only problem is finding the critical value S^*. This is the classical problem of finding a root of the equation

$$g(S^*) = S^* - X - c(S^*) - \frac{S^*}{q_2}\left(1 - e^{(b-r)(T-t)}N\left(d_1(S^*)\right)\right) = 0$$

[1] The approximation is also discussed in Hull (2011).

This is solved using Newton's algorithm for finding the root. We start by finding a first "seed" value S_0. The next estimate of S_i is found by:

$$S_{i+1} = S_i - \frac{g()}{g'}$$

At each step we need to evaluate $g()$ and its derivative $g'()$.

$$g(S) = S - X - c(S) - \frac{1}{q_2} S \left(1 - e^{(b-r)(T-t)} N(d_1)\right)$$

$$g'(S) = (1 - \frac{1}{q_2}) \left(1 - e^{(b-r)(T-t)} N(d_1)\right) + \frac{1}{q_2}(e^{(b-r)(T-t)} n(d_1)) \frac{1}{\sigma\sqrt{T-t}}$$

where $c(S)$ is the Black Scholes value for commodities. Code 15.3 shows the implementation of this formula for the price of a call option.

Example

Consider the following set of parameters, used as an example in the Barone-Adesi and Whaley (1987) paper: $S = 100$, $X = 100$, $\sigma = 0.20$, $r = 0.08$, $b = -0.04$.

1. Price a call option with time to maturity of 3 months.

C++ program:

```
double S = 100; double X = 100;   double sigma = 0.20;
double r = 0.08; double b = -0.04; double time = 0.25;
cout << " Call price using Barone-Adesi Whaley approximation = "
     << option_price_american_call_approximated_baw(S,X,r,b,sigma,time) << endl;
```

Output from C++ program:

```
Call price using Barone-Adesi Whaley approximation = 5.74339
```

Exercise 15.1.

The Barone-Adesi – Whaley insight can also be used to value a put option, by approximating the value of the early exercise premium. For a put option the approximation is

$$P(S) = \begin{array}{ll} p(S,T) + A_1 \left(\frac{S}{S^{**}}\right)^{q_1} & \text{if} \quad S > S^{**} \\ X - S & \text{if} \quad S \leq S^{**} \end{array}$$

$$A_1 = -\frac{S^{**}}{q1}(1 - e^{(b-r)(T-t)} N(-d_1(S^{**}))$$

One again solves iteratively for S^{**}, for example by Newton's procedure, where now one would use

$$g(S) = X - S - p(S) + \frac{S}{q_1} \left(1 - e^{(b-r)(T-t)} N(-d_1)\right)$$

$$g'(S) = (\frac{1}{q_1} - 1) \left(1 - e^{(b-r)(T-t)} N(-d_1)\right) + \frac{1}{q_1} e^{(b-r)(T-t)} \frac{1}{\sigma\sqrt{T-t}} n(-d_1)$$

1. Implement the calculation of the price of an American put option using the BAW approach.

```
#include <cmath>
#include <algorithm>
using namespace std;
#include "normdist.h"            // normal distribution
#include "fin_recipes.h"      // define other option pricing formulas

const double ACCURACY=1.0e-6;

double option_price_american_call_approximated_baw( const double& S,
                                                    const double& X,
                                                    const double& r,
                                                    const double& b,
                                                    const double& sigma,
                                                    const double& time) {
    double sigma_sqr = sigma*sigma;
    double time_sqrt = sqrt(time);
    double nn = 2.0*b/sigma_sqr;
    double m = 2.0*r/sigma_sqr;
    double K = 1.0-exp(-r*time);
    double q2 = (-(nn-1)+sqrt(pow((nn-1),2.0)+(4*m/K)))*0.5;

    double q2_inf = 0.5 * ( -(nn-1) + sqrt(pow((nn-1),2.0)+4.0*m)); // seed value from paper
    double S_star_inf = X / (1.0 - 1.0/q2_inf);
    double h2 = -(b*time+2.0*sigma*time_sqrt)*(X/(S_star_inf-X));
    double S_seed = X + (S_star_inf-X)*(1.0-exp(h2));

    int no_iterations=0; // iterate on S to find S_star, using Newton steps
    double Si=S_seed;
    double g=1;
    double gprime=1.0;
    while ((fabs(g) > ACCURACY)
           && (fabs(gprime)>ACCURACY) // to avoid exploding Newton's
           && ( no_iterations++<500)
           && (Si>0.0)) {
        double c  = option_price_european_call_payout(Si,X,r,b,sigma,time);
        double d1 = (log(Si/X)+(b+0.5*sigma_sqr)*time)/(sigma*time_sqrt);
        g=(1.0-1.0/q2)*Si-X-c+(1.0/q2)*Si*exp((b-r)*time)*N(d1);
        gprime=( 1.0-1.0/q2)*(1.0-exp((b-r)*time)*N(d1))
            +(1.0/q2)*exp((b-r)*time)*n(d1)*(1.0/(sigma*time_sqrt));
        Si=Si-(g/gprime);
    };
    double S_star = 0;
    if (fabs(g)>ACCURACY) { S_star = S_seed; } // did not converge
    else { S_star = Si; };
    double C=0;
    double c  = option_price_european_call_payout(S,X,r,b,sigma,time);
    if (S>=S_star) {
        C=S-X;
    }
    else {
        double d1 = (log(S_star/X)+(b+0.5*sigma_sqr)*time)/(sigma*time_sqrt);
        double A2 = (1.0-exp((b-r)*time)*N(d1))* (S_star/q2);
        C=c+A2*pow((S/S_star),q2);
    };
    return max(C,c); // know value will never be less than BS value
};
```

C++ Code 15.3: Barone Adesi quadratic approximation to the price of a call option

15.4 An alternative approximation to american options due to Bjerksund and Stensland (1993)

Bjerksund and Stensland (1993) provides an alternative approximation to the price of American options. Their approximation is an example of calculating a lower bound for the option value. Their valuation relies on using an exercise strategy that is known, and although suboptimal, close enough to the (unknown) exercise strategy that the approximated value is

The corresponding put option can be estimated as

$$P(S, X, T, r, b, \sigma) = C(X, S, T, r - b, -b, \sigma)$$

The call option is found as

$$C(S,K,T,r,b,\sigma;X) = \alpha(X)S^{\beta} - \alpha(X)\varphi(S,T|\beta,X,X) + \varphi(S,T|1,X,X)$$

$$-\varphi(S,T|1,K,X) - X\varphi(S,T|0,X,X) + X\varphi(S,T|0,K,X)$$

where

$$\alpha(X) = (X-K)X^{-\beta}$$

$$\beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{b}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r}{\sigma^2}}$$

$$\varphi(S,T|\gamma,H,X) = e^{\lambda}S^{\gamma}\left[N(d_1) - \left(\frac{X}{S}\right)^{\kappa} N\left(d_2\right)\right]$$

$$\lambda = \left(-r + \gamma b + \frac{1}{2}\gamma(\gamma-1)\sigma^2\right)T$$

$$d_1 = -\frac{\ln(S/H) + \left(b + (\gamma - \frac{1}{2})\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_2 = -\frac{\ln(X^2/SH) + \left(b + (\gamma - \frac{1}{2})\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$\kappa = \frac{2b}{\sigma^2} + (2\gamma - 1)$$

$$X_T = B_0 + \left(B_{\infty} - B_0)(1 - e^{h(T)}\right)$$

$$h(T) = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{B_0}{B_{\infty} - B_0}\right)$$

or (suggested in a later paper Bjerksund and Stensland (2002))

$$h(T) = -\left(bT + 2\sigma\sqrt{T}\right)\left(\frac{K^2}{B_{\infty} - B_0}\right)$$

$$B_{\infty} = \frac{\beta}{\beta - 1}K$$

$$B_0 = \max\left(K, \left(\frac{r}{r-b}\right)K\right)$$

S: Price of underlying security. K: Exercise price.

Formula 15.3: The Bjerksund and Stensland (1993) lower bound approximation to the value of an American Call

```
#include "fin_recipes.h"
#include <cmath>
#include "normdist.h"

inline double phi(double S, double T, double gamma, double H, double X, double r, double b, double sigma){
    double sigma_sqr=pow(sigma,2);
    double kappa = 2.0*b/sigma_sqr + 2.0*gamma - 1.0;
    double lambda = (-r + gamma * b + 0.5*gamma*(gamma-1.0)*sigma_sqr)*T; // check this, says lambda in text
    double d1= - (log(S/H)+(b+(gamma-0.5)*sigma_sqr)*T)/(sigma*sqrt(T));
    double d2= - (log((X*X)/(S*H))+(b+(gamma-0.5)*sigma_sqr)*T)/(sigma*sqrt(T));
    double phi = exp(lambda) * pow(S,gamma) * (N(d1) - pow((X/S),kappa) * N(d2));
    return phi;
};

double option_price_american_call_approximated_bjerksund_stensland( const double& S,
                                                                    const double& K,
                                                                    const double& r,
                                                                    const double& b,
                                                                    const double& sigma,
                                                                    const double& T ){
    double sigma_sqr=pow(sigma,2);
    double B0=max(K,(r/(r-b)*K));
    double beta = (0.5 - b/sigma_sqr) + sqrt( pow((b/sigma_sqr-0.5),2) + 2.0 * r/sigma_sqr);
    double Binf = beta/(beta-1.0)*K;
    double hT= - (b*T + 2.0*sigma*sqrt(T))*((K*K)/(Binf-B0));
    double XT = B0+(Binf-B0)*(1.0-exp(hT));
    double alpha = (XT-K)*pow(XT,-beta);
    double C=alpha*pow(S,beta);
    C -= alpha*phi(S,T,beta,XT,XT,r,b,sigma);
    C += phi(S,T,1,XT,XT,r,b,sigma);
    C -= phi(S,T,1,K,XT,r,b,sigma) ;
    C -= K*phi(S,T,0,XT,XT,r,b,sigma);
    C += K*phi(S,T,0,K,XT,r,b,sigma);
    double c=option_price_european_call_payout(S,K,r,b,sigma,T); // for safety use the Black Scholes as lower bound
    return max(c,C);
};
```

C++ Code 15.4: Approximation of American Call due to Bjerksund and Stensland (1993)

```
#include "fin_recipes.h"

double option_price_american_put_approximated_bjerksund_stensland( const double& S,
                                                                   const double& X,
                                                                   const double& r,
                                                                   const double& q,
                                                                   const double& sigma,
                                                                   const double& T ){
    return option_price_american_call_approximated_bjerksund_stensland(X,S,r-(r-q),r-q,sigma,T);
};
```

C++ Code 15.5: Approximation of American put due to Bjerksund and Stensland (1993)

Exercise 15.2.

An option is "At the Money Forward" if the forward price F equals the exercise price.

1. Show that a reasonable approximation to the Black Scholes value of a call option on a non-dividend paying asset that is "At the Money Forward" is

   $$C = 0.4\sigma\sqrt{T-t}PV(F)$$

   where F is the forward price, $PV(\cdot)$ signifies the present value operator, $T-t$ is the time to maturity and σ is the volatility.

15.5 Readings

See Broadie and Detemple (1996) for some comparisions of various approximations. A survey of the pricing of options, including American options, is Broadie and Detemple (2004). Barone-Adesi (2005) summarizes the history of approximating the American put.

Chapter 16

Average, lookback and other exotic options

Contents

We now look at a type of options that has received a lot of attention in later years. The distinguishing factor of these options is that they depend on the *whole price path* of the underlying security between today and the option maturity.

16.1 Bermudan options

A Bermudan option is, as the name implies,[1] a mix of an European and American option. It is a standard put or call option which can only be exercised at discrete dates throughout the life of the option. The simplest way to do the pricing of this is again the binomial approximation, but now, instead of checking at every node whether it is optimal to exercise early, only check at the nodes corresponding to the potential exercise times. **C++ Code 16.1** shows the calculation of the Bermudan price using binomial approximations. The times as which exercise can happen is passed as a vector argument to the routine, and in the binomial a list of which nodes exercise can happen is calculated and checked at every step.

Example

Price a Bermudan put. The following informaton is given $S = 80$, $K = 100$ r $= 0.20$; time $= 1$; $\sigma = 0.25$, steps $= 500$, $q = 0.0$, Potential exercise times $= 0{,}25$, 0.5 and 0.75.

[1] Since Bermuda is somewhere between America and Europe...

C++ program:

```
double S=80;        double K=100;        double r = 0.20;
double time = 1.0; double sigma = 0.25;
int steps = 500;
double q=0.0;
vector<double> potential_exercise_times; potential_exercise_times.push_back(0.25);
potential_exercise_times.push_back(0.5); potential_exercise_times.push_back(0.75);
cout << " Bermudan put price = "
       << option_price_put_bermudan_binomial(S,K,r,q,sigma,time,potential_exercise_times,steps)
       << endl;
```

Output from C++ program:

```
 Bermudan put price = 15.8869
```

```
#include <cmath>              // standard C mathematical library
#include <algorithm>          // defines the max() operator
#include <vector>             // STL vector templates
using namespace std;

double option_price_put_bermudan_binomial( const double& S,
                                           const double& X,
                                           const double& r,
                                           const double& q,
                                           const double& sigma,
                                           const double& time,
                                           const vector<double>& potential_exercise_times,
                                           const int& steps) {
    double delta_t=time/steps;
    double R = exp(r*delta_t);
    double Rinv = 1.0/R;
    double u = exp(sigma*sqrt(delta_t));
    double uu = u*u;
    double d = 1.0/u;
    double p_up = (exp((r-q)*delta_t)-d)/(u-d);
    double p_down = 1.0-p_up;
    vector<double> prices(steps+1);
    vector<double> put_values(steps+1);

    vector<int> potential_exercise_steps; // create list of steps at which exercise may happen
    for (int i=0;i<potential_exercise_times.size();++i){
        double t = potential_exercise_times[i];
        if ( (t>0.0)&&(t<time) ) {
            potential_exercise_steps.push_back(int(t/delta_t));
        };
    };

    prices[0] = S*pow(d, steps); // fill in the endnodes.
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
    for (int i=0; i<=steps; ++i) put_values[i] = max(0.0, (X-prices[i])); // put payoffs at maturity
    for (int step=steps-1; step>=0; --step) {
        bool check_exercise_this_step=false;
        for (int j=0;j<potential_exercise_steps.size();++j){
            if (step==potential_exercise_steps[j]) { check_exercise_this_step=true; };
        };
        for (int i=0; i<=step; ++i) {
            put_values[i] = (p_up*put_values[i+1]+p_down*put_values[i])*Rinv;
            prices[i] = d*prices[i+1];
            if (check_exercise_this_step) put_values[i] = max(put_values[i],X-prices[i]);
        };
    };
    return put_values[0];
};
```

C++ Code 16.1: Binomial approximation to Bermudan put option

16.2 Asian options

The payoff depends on the average of the underlying price. An *average price call* has payoff

$$C_T = \max(0, \bar{S} - X),$$

where $\bar{S}$ is the average of the underlying in the period between t and T.

Another Asian is the *average strike call*

$$C_T = \max(0, S_T - \bar{S})$$

There are different types of Asians depending on how the average $\bar{S}$ is calculated. For the case of S being lognormal and the average $\bar{S}$ being a geometric average, there is an analytic formula due to Kemna and Vorst (1990). Hull (2011) also discusses this case. It turns out that one can calculate this option using the regular Black Scholes formula adjusting the volatility to $\sigma/\sqrt{3}$ and the dividend yield to

$$\frac{1}{2}\left(r + q + \frac{1}{6}\sigma^2\right)$$

in the case of continous sampling of the underlying price distribution.

C++ Code 16.2 shows the calculation of the analytical price of an Asian geometric average price call.

```
#include <cmath>
using namespace std;
#include "normdist.h" // normal distribution definitions

double
option_price_asian_geometric_average_price_call(const double& S,
                                                const double& K,
                                                const double& r,
                                                const double& q,
                                                const double& sigma,
                                                const double& time){
    double sigma_sqr = pow(sigma,2);
    double adj_div_yield=0.5*(r+q+sigma_sqr/6.0);
    double adj_sigma=sigma/sqrt(3.0);
    double adj_sigma_sqr = pow(adj_sigma,2);
    double time_sqrt = sqrt(time);
    double d1 = (log(S/K) + (r-adj_div_yield + 0.5*adj_sigma_sqr)*time)/(adj_sigma*time_sqrt);
    double d2 = d1-(adj_sigma*time_sqrt);
    double call_price = S * exp(-adj_div_yield*time)* N(d1) - K * exp(-r*time) * N(d2);
    return call_price;
};
```

C++ Code 16.2: Analytical price of an Asian geometric average price call

Example

Relevant parameters: $S = 100$, $K = 100$, $q = 0$, $r = 0.06$, $\sigma = 0.25$ and time to maturity is one year. Price an Asian geometric average price call.

C++ program:

```
double S=100; double K=100; double q=0;
double r=0.06; double sigma=0.25; double time=1.0;
cout << " Analytical geometric average = "
     << option_price_asian_geometric_average_price_call(S,K,r,q,sigma,time)
     << endl;
```

Output from C++ program:

```
 Analytical geometric average = 6.74876
```

16.3 Lookback options

The payoff from lookback options depend on the maximum or minimum of the underlying achieved through the period. The payoff from the lookback call is the terminal price of the undelying less the minimum value

$$C_T = \max(0, S_T - \min_{\tau \in [t,T]} S_\tau)$$

For this particular option an analytical solution has been found, due to Goldman, Sosin, and Gatto (1979), which is shown in **Formula 16.1** and implemented in **C++ Code 16.3**.

$$C = Se^{-q(T-t)}N(a_1) - Se^{-q(T-t)}\frac{\sigma^2}{2(r-q)}N(-a_1) - S_{min}e^{-r(T-t)}\left(N(a_2) - \frac{\sigma^2}{2(r-q)}e^{Y_1}N(-a3)\right)$$

$$a_1 = \frac{\ln\left(\frac{S}{S_{min}}\right) + (r - q + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$a_2 = a_1 - \sigma\sqrt{T-t}$$

$$a_3 = \frac{\ln\left(\frac{S}{S_{min}}\right) + \left(-r + q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$Y_1 = \frac{2\left(r - q - \frac{1}{2}\sigma^2\right)\ln\left(\frac{S}{S_{min}}\right)}{\sigma^2}$$

Formula 16.1: Analytical formula for a lookback call

```
#include <cmath>
using namespace std;
#include "normdist.h"

double option_price_european_lookback_call(const double& S,
                                           const double& Smin,
                                           const double& r,
                                           const double& q,
                                           const double& sigma,
                                           const double& time){
    if (r==q) return 0;
    double sigma_sqr=sigma*sigma;
    double time_sqrt = sqrt(time);
    double a1 = (log(S/Smin) + (r-q+sigma_sqr/2.0)*time)/(sigma*time_sqrt);
    double a2 = a1-sigma*time_sqrt;
    double a3 = (log(S/Smin) + (-r+q+sigma_sqr/2.0)*time)/(sigma*time_sqrt);
    double Y1 = 2.0 * (r-q-sigma_sqr/2.0)*log(S/Smin)/sigma_sqr;
    return S * exp(-q*time)*N(a1)- S*exp(-q*time)*(sigma_sqr/(2.0*(r-q)))*N(-a1)
        - Smin * exp(-r*time)*(N(a2)-(sigma_sqr/(2*(r-q)))*exp(Y1)*N(-a3));
};
```

C++ Code 16.3: Price of lookback call option

Example

Parameters $S = 100$, $Smin = S$ $q = 0$, $r = 0.06$, $\sigma = 0.346$, time $= 1.0$, Price an European lookback call.

C++ program:

```
double S=100; double Smin=S; double q = 0; double r = 0.06;
double sigma = 0.346; double time = 1.0;
cout << " Lookback call price = "
     << option_price_european_lookback_call(S,Smin,r,q,sigma,time) << endl;
```

Output from C++ program:

```
 Lookback call price = 27.0713
```

16.4 Monte Carlo Pricing of options whose payoff depend on the whole price path

Monte Carlo simulation can be used to price a lot of different options. The limitation is that the options should be European. American options can not be priced by simulation methods. There is (at least) two reasons for this. First, the optimal exercise policy would have to be known. But if the exercise policy was known there would be an analytical solution. Second, approximations using Monte Carlo relies on a law of large numbers. But if some of the sample paths were removed, the LLN would be invalidated.

In chapter 14 we looked at a general simulation case where we wrote a generic routine which we passed a payoff function to, and the payoff function was all that was necessary to define an option value. The payoff function in that case was a function of the *terminal* price of the underlying security. The only difference to the previous case is that we now have to generate a price *sequence* and write the terminal payoff of the derivative in terms of that, instead of just generating the terminal value of the underlying security from the lognormal assumption.

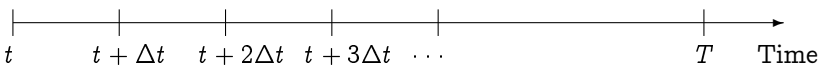
16.4.1 Generating a series of lognormally distributed variables

Recall that one will generate lognormally distributed variables as

$$S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}\tilde{x}}$$

where the current time is t and terminal date is T. To simulate a price sequence one splits this period into say N periods, each of length

$$\Delta t = \frac{T-t}{N}$$



Each step in the simulated price sequence is

$$S_{t+\Delta t} = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)\Delta+\sigma\sqrt{\Delta t}\tilde{x}}$$

C++ Code 16.4 shows how one would simulate a sequence of lognormally distributed variables.

This code is then used in the generic routine to do calculations, as shown in **C++ Code 16.5**.

To price an option we are then only in need of a definition of a payoff function. We consider a couple of examples. One is the case of an Asian option, shown in **C++ Code 16.6**.[2]

Another is the payoff for a lookback, shown in **C++ Code 16.7**.[3]

[2] Note the use of the `accumulate()` function, which is part of the C++ standard.

[3] Note the use of the `min_element()` and `max_element` functions, which are part of the C++ standard.

```
#include <cmath>
#include <vector>
using namespace std;
#include "normdist.h"

vector<double>
simulate_lognormally_distributed_sequence(const double& S,
                                          const double& r,
                                          const double& sigma,
                                          const double& time, // time to final date
                                          const int& no_steps){ // number of steps
    vector<double> prices(no_steps);
    double delta_t = time/no_steps;
    double R = (r-0.5*pow(sigma,2))*delta_t;
    double SD = sigma * sqrt(delta_t);
    double S_t = S;                    // initialize at current price
    for (int i=0;  i<no_steps; ++i) {
        S_t = S_t * exp(R + SD * random_normal());
        prices[i]=S_t;
    };
    return prices;
};
```

C++ Code 16.4: Simulating a sequence of lognormally distributed variables

```
#include <cmath>
using namespace std;
#include "fin_recipes.h"

double
derivative_price_simulate_european_option_generic(const double& S,
                                                  const double& K,
                                                  const double& r,
                                                  const double& sigma,
                                                  const double& time,
                                                  double payoff(const vector<double>& prices, const double& X),
                                                  const int& no_steps,
                                                  const int& no_sims) {
    double sum_payoffs=0;
    for (int n=0; n<no_sims; n++) {
        vector<double>prices = simulate_lognormally_distributed_sequence(S,r,sigma,time,no_steps);
        sum_payoffs += payoff(prices,K);
    };
    return exp(-r*time) * (sum_payoffs/no_sims);
};
```

C++ Code 16.5: Generic routine for pricing European options

```
#include <cmath>
#include <numeric>
#include <vector>
using namespace std;

double payoff_arithmetric_average_call(const vector<double>& prices, const double& K) {
    double sum=accumulate(prices.begin(), prices.end(),0.0);
    double avg = sum/double(prices.size());
    return max(0.0,avg-K);
};

double payoff_geometric_average_call(const vector<double>& prices, const double& K) {
    double logsum=log(prices[0]);
    for (unsigned i=1;i<prices.size();++i){ logsum+=log(prices[i]); };
    double avg = exp(logsum/double(prices.size()));
    return max(0.0,avg-K);
};
```

C++ Code 16.6: Payoff function for Asian call option

```
#include <vector>
#include <algorithm>
using namespace std;

double payoff_lookback_call(const vector<double>& prices, const double& unused_variable) {
   double m = *min_element(prices.begin(),prices.end());
   return prices.back()-m; // always positive or zero
};

double payoff_lookback_put(const vector<double>& prices, const double& unused_variable) {
    double m = *max_element(prices.begin(),prices.end());
    return m-prices.back(); // max is always larger or equal.
};
```

C++ Code 16.7: Payoff function for lookback option

16.5 Control variate

As discussed in chapter 14, a control variate is a price which we both have an analytical solution of and find the Monte Carlo price of. The differences between these two prices is a measure of the bias in the Monte Carlo estimate, and is used to adjust the Monte Carlo estimate of other derivatives priced using the same random sequence.

C++ Code 16.8 shows the Black Scholes price used as a control variate. An alternative could have been the analytical lookback

price, or the analytical solution for a geometric average price call shown earlier.

```
#include "fin_recipes.h"
#include <cmath>
using namespace std;

double
derivative_price_simulate_european_option_generic_with_control_variate(const double& S,
                                                                       const double& K,
                                                                       const double& r,
                                                                       const double& sigma,
                                                                       const double& time,
                                                                       double payoff(const vector<double>& prices,
                                                                                     const double& X),
                                                                       const int& no_steps,
                                                                       const int& no_sims) {
    double c_bs = option_price_call_black_scholes(S,S,r,sigma,time);// price an at the money Black Scholes call
    double sum_payoffs=0;
    double sum_payoffs_bs=0;
    for (int n=0; n<no_sims; n++) {
        vector<double> prices = simulate_lognormally_distributed_sequence(S,r,sigma,time, no_steps);
        double S1= prices.back();
        sum_payoffs += payoff(prices,K);
        sum_payoffs_bs += payoff_call(S1,S); // simulate at the money Black Scholes price
    };
    double c_sim = exp(-r*time) * (sum_payoffs/no_sims);
    double c_bs_sim = exp(-r*time) * (sum_payoffs_bs/no_sims);
    c_sim += (c_bs-c_bs_sim);
    return c_sim;
};
```

C++ Code 16.8: Control Variate

Example

Using the parameters $S = 100$, $K = 120$, $r = 0.10$, time $= 1.0$ $\sigma = 0.25$, no sims $= 10000$, no steps $= 250$, $q = 0$, price arithmetric and geometric average calls by generic simulation.

C++ program:

```
    cout << "Testing general simulation of European options " << endl;
    double S=100; double K=120; double r = 0.10;
    double time = 1.0; double sigma = 0.25; int no_sims = 10000; int no_steps = 250;
    double q=0;
    cout << " simulated arithmetric average "
        << " S= " << S << " r= " << r << " price="
        << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
                                                             payoff_arithmetric_average_call,
                                                             no_steps,no_sims)
        << endl;
    cout << " simulated geometric average = "
        << derivative_price_simulate_european_option_generic(S,K,r,sigma,time,
                                                             payoff_geometric_average_call,
                                                             no_steps,no_sims)
        << endl;
    cout << " analytical lookback put = "
        << option_price_european_lookback_put(S,S,r,q,sigma,time)
        << endl;
    cout << " simulated lookback put = "
        << derivative_price_simulate_european_option_generic(S,0,r,sigma,time,
                                                             payoff_lookback_put,
                                                             no_steps,no_sims)
        << endl;
    cout << " analytical lookback call = "
        << option_price_european_lookback_call(S,S,r,q,sigma,time)
        << endl;
    cout << " simulated lookback call = "
        << derivative_price_simulate_european_option_generic(S,0,r,sigma,time,
                                                             payoff_lookback_call,
                                                             no_steps,no_sims)
        << endl;
    cout << " simulated lookback call using control variates = "
        << derivative_price_simulate_european_option_generic_with_control_variate(S,0,r,sigma,time,
                                                                                   payoff_lookback_call,
                                                                                   no_steps,no_sims)
        << endl;
```

Output from C++ program:

```
Testing general simulation of European options
 simulated arithmetric average  S= 100 r= 0.1 price=1.49696
 simulated geometric average = 1.38017
 analytical lookback put = 16.2665
 simulated lookback put = 14.9846
 analytical lookback call = 22.8089
 simulated lookback call = 21.9336
 simulated lookback call using control variates = 22.0685
```

16.6 References

Exotic options are covered in Hull (2011). Rubinstein (1993) has an extensive discussion of analytical solutions to various exotic options. Gray and Gray (2001) also looks at some analytical solutions.

Chapter 17

Generic binomial pricing

Contents

17.1 Introduction

In earlier chapters we have seen a large number of different versions of the binomial pricing formula. In this chapter we see how we can build a framework for binomial pricing that lets us write a single generic routine that can be used for a binomial approximation of all sorts of derivatives. The important feature that lets us write such a generic routine is that the only place the terms of the derivative appears in the calculation is the calculation of the value at each node. Consider the binomial approximation of an American call in Chapter 12's C++ **Code 12.2**, repeated below as C++ **Code 17.1** for convenience.

The terms of the derivative only appears when calculating the value at the nodes, where the calculation `max(prices[]-K,0)` appears. The key to building a generic binomial routine lies in replacing this calculation with a generic routine. C++ **Code 17.2** shows how the generic ruotine is implemented.

```
#include <cmath>              // standard mathematical library
#include <algorithm>          // defines the max() operator
#include <vector>             // STL vector templates
using namespace std;

double option_price_call_american_binomial( const double& S, // spot price
                                            const double& X,    // exercice price
                                            const double& r,    // interest rate
                                            const double& sigma, // volatility
                                            const double& t,    // time to maturity
                                            const int& steps) { // no steps in binomial tree
    double R = exp(r*(t/steps));         // interest rate for each step
    double Rinv = 1.0/R;                 // inverse of interest rate
    double u = exp(sigma*sqrt(t/steps)); // up movement
    double d = 1.0/u;
    double p_up = (R-d)/(u-d);
    double p_down = 1.0-p_up;

    vector<double> prices(steps+1);   // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    double uu = u*u;
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];

    vector<double> call_values(steps+1);   // value of corresponding call
    for (int i=0; i<=steps; ++i) call_values[i] = max(0.0, (prices[i]-X)); // call payoffs at maturity

    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            call_values[i] = (p_up*call_values[i+1]+p_down*call_values[i])*Rinv;
            prices[i] = d*prices[i+1];
            call_values[i] = max(call_values[i],prices[i]-X);     // check for exercise
        };
    };
    return call_values[0];
};
```

C++ Code 17.1: Binomial price of American Call

```
#include <cmath>              // standard mathematical library
#include <algorithm>          // defines the max() operator
#include <vector>             // STL vector templates
using namespace std;

double option_price_generic_binomial( const double& S,
                                      const double& K,
                                      double generic_payoff(const double& S, const double& K),
                                      const double& r,
                                      const double& sigma,
                                      const double& t,
                                      const int& steps) {
    double R = exp(r*(t/steps));          // interest rate for each step
    double Rinv = 1.0/R;                  // inverse of interest rate
    double u = exp(sigma*sqrt(t/steps)); // up movement
    double d = 1.0/u;
    double p_up = (R-d)/(u-d);
    double p_down = 1.0-p_up;

    vector<double> prices(steps+1);   // price of underlying
    prices[0] = S*pow(d, steps); // fill in the endnodes.
    double uu = u*u;
    for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];

    vector<double> values(steps+1);   // value of corresponding call
    for (int i=0; i<=steps; ++i) values[i] = generic_payoff(prices[i],K); // payoffs at maturity

    for (int step=steps-1; step>=0; --step) {
        for (int i=0; i<=step; ++i) {
            values[i] = (p_up*values[i+1]+p_down*values[i])*Rinv; // value by not exercising
            prices[i] = d*prices[i+1];
            values[i] = max(values[i],generic_payoff(prices[i],K));     // check for exercise
        };
    };
    return values[0];
};
```

C++ Code 17.2: Generic binomial calculation

Using this routine is then merely a matter of providing a definition of the derivative payoff. **C++ Code 17.3** shows how the payoffs are defined for standard put and call options. Pricing American put and call options is then merely a matter of supplying these payoff definitions to the generic binomial routine.

```
#include <algorithm>
using namespace std;

double payoff_call(const double& S, const double& K){
    return max(0.0,S-K);
};

double payoff_put (const double& S, const double& K) {
    return max(0.0,K-S);
};
```

C++ Code 17.3: Payoff definitions for put and call options

Example

Consider American call and put option on non-dividend paying stock, where $S = 100$, $K = 100$, $\sigma = 0.2$, $(T - t) = 1$ and $r = 0.1$.

1. Price the options using binomial approximations with 100 steps.

C++ program:

```
double S = 100.0;
double K = 100.0;
double r = 0.1;
double sigma = 0.25;
double time_to_maturity=1.0;
int steps = 100;
cout << " american call price = "
     << option_price_generic_binomial(S,K,payoff_call, r, sigma, time_to_maturity, steps)
     << endl;
cout << " american put price = "
     << option_price_generic_binomial(S,K,payoff_put, r, sigma, time_to_maturity, steps)
     << endl;
```

Output from C++ program:

```
american call price = 14.9505
american put price = 6.54691
```

More exotic options can also be calculated using the same approach. For example, **C++ Code 17.4** shows payoff definitions for two binary options, options that pay off one dollar if the price of the underlying is above K (call) or below K (put). The typical such option is European, the check for whether the option is in the money happens at maturity, but one can also think of cases where the binary option pays off one dollar as soon as S, the price of the underlying, hits the "barrier" K.

Example

Consider binary options that pays one dollar if the price S of the underlying increases to K or above during some time period. Suppose the current price of the underlying is $S = 100$. Using $\sigma = 0.2$, $(T - t) = 1$ and $r = 0.1$, price such a binary option when $K = 120$.

```
double payoff_binary_call(const double& S, const double& K){
    if (S>=K) return 1;
    return 0;
};

double payoff_binary_put(const double& S, const double& K){
    if (S<=K) return 1;
    return 0;
};
```

C++ Code 17.4: Payoff definitions for binomial options

C++ program:

```
    double S = 100.0;
    double K = 120.0;
    double r = 0.1;
    double sigma = 0.25;
    double time_to_maturity=1.0;
    int steps = 100;
    cout << " binary option price = "
        << option_price_generic_binomial(S,K,payoff_binary_call, r, sigma, time_to_maturity, steps)
        << endl;
```

Output from C++ program:

```
 binary option price = 0.498858
```

17.2 Delta calculation

Deltas and other greeks are calculated using the same style of generic routine. **C++ Code 17.5** shows how to calculate delta using the same generic approach.

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double option_price_delta_generic_binomial(const double& S,
                                           const double& K,
                                           double generic_payoff(const double& S, const double& K),
                                           const double& r,
                                           const double& sigma,
                                           const double& t,
                                           const int& no_steps){
   double R = exp(r*(t/no_steps));
   double Rinv = 1.0/R;
   double u = exp(sigma*sqrt(t/no_steps));
   double d = 1.0/u;
   double uu= u*u;
   double pUp = (R-d)/(u-d);
   double pDown = 1.0 - pUp;

   vector<double> prices (no_steps+1);
   prices[0] = S*pow(d, no_steps);
   for (int i=1; i<=no_steps; ++i) prices[i] = uu*prices[i-1];

   vector<double> values (no_steps+1);
   for (int i=0; i<=no_steps; ++i) values[i] = generic_payoff(prices[i],K);

   for (int CurrStep=no_steps-1 ; CurrStep>=1; --CurrStep) {
      for (int i=0; i<=CurrStep; ++i) {
          prices[i] = d*prices[i+1];
          values[i] = (pDown*values[i]+pUp*values[i+1])*Rinv;
          values[i] = max(values[i], generic_payoff(prices[i],K));
      };
   };
   double delta = (values[1]-values[0])/(S*u-S*d);
   return delta;
};
```

C++ Code 17.5: Generic binomial calculation of delta

Exercise 17.1.

The generic routines discussed in this chapter have been for American options. How would you modify them to account for European options?

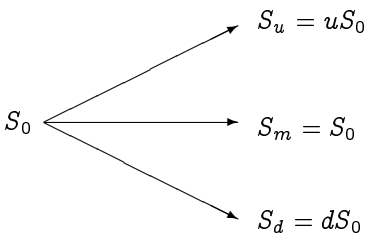
Chapter 18

Trinomial trees

Contents

18.1 Intro

A trinomial tree is similar to a binomial tree, just with one more branch. At each point of time there are three possible future values of the underlying S.



18.2 Implementation

A trinomial tree can be implemented with various parameterizations. We will show the calulation using the following parameterization:

$$u = e^{\sigma\sqrt{3\Delta t}}$$

$$e = \frac{1}{u}$$

$$p_u = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - q - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

We calculate the option price using the usual roll back procedure with the continuation value

$$e^{-r\Delta t}\left(p_u f_u + p_m f_m + p_d f_d\right)$$

```
#include <vector>
#include <cmath>
using namespace std;

double option_price_put_american_trinomial( const double& S,
                                            const double& K,
                                            const double& r,
                                            const double& q,
                                            const double& sigma,
                                            const double& t,
                                            const int& steps) {
    double delta_t = t/steps;
    double Rinv = exp(-r*(delta_t));
    double sigma_sqr=pow(sigma,2);

    double u = exp(sigma*sqrt(3.0*delta_t));
    double d = 1.0/u;
    double p_u = 1.0/6.0 + sqrt(delta_t/(12.0*sigma_sqr)) * (r-q-0.5*sigma_sqr);
    double p_m = 2.0/3.0;
    double p_d = 1.0/6.0 - sqrt(delta_t/(12.0*sigma_sqr)) * (r-q-0.5*sigma_sqr);

    vector< vector<double> > Stree; // price of underlying in a tree
    vector<double> Svec;
    Svec.push_back(S);
    for (int step=1;step<=steps;++step){
        Stree.push_back(Svec);
        Svec.insert(Svec.begin(),Svec[0]*d); // use the fact that only the extreme values change.
        Svec.push_back(Svec[Svec.size()-1]*u);
    };
    int m = int(Svec.size());
    vector<double> values_next = vector<double>(m); // value of option next step
    for (int i=0; i<m; ++i) values_next[i] = max(0.0, K-Svec[i]); // call payoffs at maturity
    vector<double> values;
    for (int step=steps-1; step>=0; --step) {
        m = int(Stree[step].size());
        values = vector<double> (m);  // value of option
        for (int i=0; i<m; ++i) {
            values[i] = (p_u*values_next[i+2]+p_m*values_next[i+1] + p_d*values_next[i])*Rinv;
            values[i] = max(values[i],K-Stree[step][i]);    // check for exercise
        };
        values_next=values;
    };
    return values[0];
};
```

C++ Code 18.1: Price of american put using a trinomial tree

Example

You are given the following information about an option: $S = 100$, $K = 100$, $r = 0.1$, $\sigma = 0.25$ and time to maturity is 1 year. Price an American put using a trinomial approximation with 100 steps.

```
function P = opt_price_trinom_am_put(S, K, r, q, sigma, t, steps)
  delta_t = t/steps;
  sigma_sqr=sigma^2;
  Rinv = exp(-r*delta_t);
  u = exp(sigma*sqrt(3.0*delta_t));
  d = 1.0/u;
  p_u = 1/6 + sqrt(delta_t/(12*sigma_sqr)) * (r-q-0.5*sigma_sqr);
  p_m = 2/3;
  p_d = 1/6 - sqrt(delta_t/(12*sigma_sqr)) * (r-q-0.5*sigma_sqr);

  Svec = [ S ];
  Stree = [ Svec ];
  Su = S;
  Sd = S;
  for step=1:steps+1
    Su = Su*u;
    Sd = Sd*d;
    Svec = [ Sd; Svec; Su ];
    Stree=[ [Stree; zeros(2,step)] Svec ];
  end
  values_next = max(0,K-Svec);
  for step = steps:-1:0
    m = 2*step+1;
    values = Rinv*(p_u*values_next(3:m+2)+p_m*values_next(2:m+1)+p_d*values_next(1:m));
    values = max(values, K-Stree(1:m,step+1));
    values_next = values;
  end
  P = values;
end
```

Matlab **Code 18.1**: Price of american put using a trinomial tree

C++ program:

```
double S = 100.0;  double K = 100.0;
double r = 0.1;    double q = 0;    double sigma = 0.25;
double time=1.0;   int no_steps = 100;
cout << " american put = " << option_price_put_american_trinomial(S,K,r,q,sigma,time,no_steps) << endl;
```

Output from C++ program:

```
 american put = 6.52719
```

Exercise 18.1.

In the code for the trinomial tree the price of the underlying is put into a triangular data structure. This can be collapsed into a single array, since the only changes when you add one step is to add one price at the top and at the bottom. If you keep track of the top and bottom of the current array you can access the prices of the underlying through some clever indexing into the single vector of prices of the underlying.

Exercise 18.2.

Similarly to the binomial case, one can build a generic trinomial tree where the payoff function is passed as a parameter to the routine. Implement such a generic trinomial tree.

18.3 Further reading

Hull (2011)

Chapter 19

Alternatives to the Black Scholes type option formula

Contents

A large number of alternative formulations to the Black Scholes analysis has been proposed. Few of them have seen any widespread use, but we will look at some of these alternatives.

19.1 Merton's Jump diffusion model.

Merton (1976) has proposed a model where in addition to a Brownian Motion term, the price process of the underlying is allowed to have *jumps*. The risk of these jumps is assumed to not be priced.

In the following we look at an implementation of a special case of Merton's model, described in (Hull, 1993, pg 454), where the size of the jump has a normal distribution. λ and κ are parameters of the jump distribution. The price of an European call option is given in **Formula 19.1** and implemented in **C++ Code 19.1**

$$c = \sum_{n=0}^{\infty} \frac{e^{\lambda'\tau}(\lambda'\tau)^n}{n!} C_{BS}(S, X, r_n, \sigma_n^2, T-t)$$

where

$$\tau = T - t$$

$$\lambda' = \lambda(1+\kappa)$$

$C_{BS}(\cdot)$ is the Black Scholes formula, and

$$\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$$

$$r_n = r - \lambda\kappa + \frac{n\ln(1+\kappa)}{\tau}$$

Formula 19.1: The option pricing formula of the Merton (1976) model

In implementing this formula, we need to terminate the infinite sum at some point. But since the

factorial function is growing at a much higher rate than any other, that is no problem, terminating at nabout $n = 50$ should be on the conservative side. To avoid numerical difficulties, use the following method for calculation of

$$\frac{e^{\lambda'\tau}(\lambda'\tau)^n}{n!} = exp\left(\ln\left(\frac{e^{\lambda'\tau}(\lambda'\tau)^n}{n!}\right)\right) = exp\left(-\lambda'\tau + n\ln(\lambda'\tau) - \sum_{i=1}^{n}\ln i\right)$$

```
#include <cmath>
#include "fin_recipes.h"

double option_price_call_merton_jump_diffusion( const double& S,
                                                const double& X,
                                                const double& r,
                                                const double& sigma,
                                                const double& time_to_maturity,
                                                const double& lambda,
                                                const double& kappa,
                                                const double& delta) {
    const int MAXN=50;
    double tau=time_to_maturity;
    double sigma_sqr = sigma*sigma;
    double delta_sqr = delta*delta;
    double lambdaprime = lambda * (1+kappa);
    double gamma = log(1+kappa);
    double c = exp(-lambdaprime*tau)*option_price_call_black_scholes(S,X,r-lambda*kappa,sigma,tau);
    double log_n = 0;
    for (int n=1;n<=MAXN; ++n) {
        log_n += log(double(n));
        double sigma_n = sqrt( sigma_sqr+n*delta_sqr/tau );
        double r_n = r-lambda*kappa+n*gamma/tau;
        c += exp(-lambdaprime*tau+n*log(lambdaprime*tau)-log_n)*
            option_price_call_black_scholes(S,X,r_n,sigma_n,tau);
    };
    return c;
};
```

C++ Code 19.1: Mertons jump diffusion formula

Example

Price an option using Merton's jump diffusion formula, using the parameters

$S = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\lambda = 0.5$, $\kappa = 0.5$, $\delta = 0.5$.

time to maturity=1.

C++ program:

```
double S=100; double K=100; double r=0.05;
double sigma=0.3; double time_to_maturity=1;
double lambda=0.5; double kappa=0.5; double delta=0.5;
cout << " Merton Jump diffusion call = "
     << option_price_call_merton_jump_diffusion(S,K,r,sigma,time_to_maturity,lambda,kappa,delta)
     << endl;
```

Output from C++ program:

```
 Merton Jump diffusion call = 23.2074
```

19.2 Hestons pricing formula for a stochastic volatility model

Heston (1993) relaxes the Black-Scholes assumption of a constant volatility by introducing a stochastic volatility. He finds exact solutions for European options.

Let S be the stock price and v the volatility. These two variables are assumed to follow joint stochastic processes.

$$\left[\begin{array}{l} dS(t) = \mu S dt + \sqrt{v(t)} S dz_1(t) \\ d\sqrt{v(t)} = -\beta \sqrt{v(t)} dt + \delta v(t) S dz_2(t) \end{array} \right]$$

The two processes z_1 and z_2 have correlation ρ. Rewrite the process for the volatility as

$$dv(t) = \kappa \left(\phi - v(t) \right) dt + \sigma \sqrt{v(t)} dz_2$$

Let $\lambda(S, v, t)$ be the price of volatility risk. Under a constant interest rate r

$$P(t, T) = e^{-r(T-t)}$$

Consider a call option with strike price K. Its price is given by **Formula 19.2**.

Price of Call option using Hestons formula The option price is

$$C(s, v, t) = SP_1 - KP(t, T)P_2$$

where

$$P_j(x, v, T, \ln(K)) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{e^{-i\phi \ln(K)} f_j(x, v, T, \phi)}{i\phi} d\phi \right]$$

$$f_j(x, v, T, \phi) = e^{C(\tau, \phi) + D(\tau, \phi) v + i\phi x}$$

$$C(\tau, \phi) = r\phi i\tau + \frac{a}{\sigma^2} \left\{ (b_j - \rho\sigma\phi i + d)\tau - 2 \ln \left[\frac{1 - ge^{d\tau}}{1 - g} \right] \right\}$$

$$D(\tau, \phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2} \left[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}} \right]$$

$$u_1 = \frac{1}{2}, \quad u_2 = -\frac{1}{2}$$

$$a = \kappa\theta$$

$$b_1 = \kappa + \lambda - \rho\sigma \quad b_2 = \kappa + \lambda$$

$$x = \ln(S)$$

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d}$$

$$d = \sqrt{(\rho\sigma\phi i - b_j)^2 - \sigma^2 (2u_j \phi i - \phi^2)}$$

Notation: S: price of undelying security. K: exercise price.

The implementation of this pricing formula has some instructive C++ features. First, it illustrates calculations of of complex variables. Complex numbers is part of the C++ standard, and are accessed by including the

```
#include <complex>
```

statement. Complex numbers are templated, it is necessary to specify what type of floating point type to use, such as `complex<double>` or `complex<double double>`.

To evaluate the price it is also necessary to do a numerical integration. In the calculation this is solved by a call to an external routine. We use a routine provided by the Gnu GSL project.[1]

Example

Given the following set of parameters: $S = 100$, $K = 100$, $r = 0.01$, $v = 0.01$, $\tau = 0.5$, $\rho = 0$, $\kappa = 2$, $\lambda = 0$, $\theta = 0.01$ and $\sigma = 0.01$, price a call option using the Heston formula

C++ program:

```
double S=100; double K=100; double r=0.01; double v=0.01;
double tau=0.5; double rho=0; double kappa=2; double lambda=0.0;
double theta=0.01; double sigma=0.01;
cout << "heston call price "
     << heston_call_option_price( S, K, r, v, tau, rho, kappa, lambda, theta, sigma) << endl;
```

Output from C++ program:

```
heston call price 3.06944
```

[1] An example alternatives would have been Numerical Recipes in C++'s quadrature.

```
#include <iostream>
#include <cmath>
#include <complex>
using namespace std;
#include "gsl/gsl_integration.h"

struct heston_parms {double K; double x; double r; double v; double tau; double kappa; double theta;
                     double rho; double sigma; double lambda; int j;};

extern "C"{
double heston_integrand_j(double phi, void *p){
    struct heston_parms* parms = (struct heston_parms*)p;
    double K = (parms->K); double x = (parms->x);
    double v = (parms->v); double r = (parms->r);
    double kappa = (parms->kappa);
    double theta = (parms->theta);
    double rho = (parms->rho);
    double sigma = (parms->sigma);
    double lambda = (parms->lambda);
    double tau = (parms->tau);
    double j = (parms->j);
    double sigma_sqr = pow(sigma,2);
    double uj;     double bj;
    if (j==1){   uj=0.5; bj=kappa+lambda-rho*sigma; }
    else {       uj=-0.5; bj=kappa+lambda; };
    complex <double> i(0,1);
    double a = kappa*theta;
    complex<double> d = sqrt( pow(rho*sigma*phi*i-bj,2) - sigma_sqr*(2*uj*phi*i-pow(phi,2)) );
    complex<double> g = (bj - rho*sigma*phi*i+d)/(bj-rho*sigma*phi*i-d);
    complex<double> C = r*phi*i*tau+(a/sigma_sqr)*((bj-rho*sigma*phi*i+d)*tau-2.0*log((1.0-g*exp(d*tau))/(1.0-g)));
    complex<double> D = (bj-rho*sigma*phi*i+d)/sigma_sqr * ( (1.0-exp(d*tau))/(1.0-g*exp(d*tau)) );
    complex<double> f1 = exp(C+D*v+i*phi*x);
    complex<double> F = exp(-phi*i*log(K))*f1/(i*phi);
    return real(F);
};};

inline double heston_Pj(double S, double K, double r, double v, double tau, double sigma,
                 double kappa, double lambda, double rho, double theta, int j){
    double x=log(S);
    struct heston_parms parms = { K, x, r, v, tau, kappa, theta, rho, sigma, lambda, j};
    size_t n=10000;
    gsl_integration_workspace* w = gsl_integration_workspace_alloc(n);
    gsl_function F;
    F.function = &heston_integrand_j;
    F.params=&parms;
    double result, error;
    gsl_integration_qagiu(&F,0,1e-7,1e-7,n,w,&result,&error); // integral to infinity starting at zero
    return 0.5 + result/M_PI;
};

double heston_call_option_price(const double& S, const double& K, const double& r, const double& v,
                                const double& tau, const double& rho, const double& kappa,
                                const double& lambda, const double& theta, const double& sigma){
    double P1 = heston_Pj(S,K,r,v,tau,sigma,kappa,lambda,rho,theta,1);
    double P2 = heston_Pj(S,K,r,v,tau,sigma,kappa,lambda,rho,theta,2);
    double C=S*P1-K*exp(-r*tau)*P2;
    return C;
};
```

C++ Code 19.2: Hestons pricing formula for a stochastic volatility model

Chapter 20

Pricing of bond options, basic models

Contents

The area of fixed income securities is one where a lot of work is being done in creating advanced mathematical models for pricing of financial securities, in particular fixed income derivatives. The focus of the modelling in this area is on modelling the term structure of interest rates and its evolution over time, which is then used to price both bonds and fixed income derivatives. However, in some cases one does not need the machinery of term structure modelling which we'll look at in later chapters, and price derivatives by modelling the evolution of the bond price directly.

Specifically, suppose that the price of a Bond follows a Geometric Brownian Motion process, just like the case we have studied before. This is not a particularly realistic assumption for the long term behaviour of bond prices, since any bond price converges to the bond face value at the maturity of the bond. The Geometric Brownian motion may be OK for the case of short term options on long term bonds.

20.1 Black Scholes bond option pricing

Given the assumed Brownian Motion process, prices of European Bond Options can be found using the usual Black Scholes formula, as shown in **C++ Code 20.1** for a zero coupon bond and **C++ Code 20.2** for the case of an option on a coupon bond.

```
#include <cmath>
#include "normdist.h"

double bond_option_price_put_zero_black_scholes(const double& B,
                                                const double& X,
                                                const double& r,
                                                const double& sigma,
                                                const double& time){
    double time_sqrt = sqrt(time);
    double d1 = (log(B/X)+r*time)/(sigma*time_sqrt) + 0.5*sigma*time_sqrt;
    double d2 = d1-(sigma*time_sqrt);
    double p = X * exp(-r*time) * N(-d2) - B * N(-d1);
    return p;
};
```

C++ Code 20.1: Black scholes price for European put option on zero coupon bond

```
#include <cmath>
#include <vector>
using namespace std;
#include "normdist.h"
#include "fin_recipes.h"

double bond_option_price_put_coupon_bond_black_scholes( const double& B,
                                                        const double& X,
                                                        const double& r,
                                                        const double& sigma,
                                                        const double& time,
                                                        const vector<double> coupon_times,
                                                        const vector<double> coupon_amounts){
    double adjusted_B=B;
    for (unsigned int i=0;i<coupon_times.size();i++) {
        if (coupon_times[i]<=time) {
            adjusted_B -= coupon_amounts[i] * exp(-r*coupon_times[i]);
        };
    };
    return bond_option_price_put_zero_black_scholes(adjusted_B,X,r,sigma,time);
};
```

C++ Code 20.2: Black scholes price for European put option on coupon bond

20.2 Binomial bond option pricing

Since we are in the case of geometric Brownian motion, the usual binomial approximation can be used to price American options, where the bond is the underlying security. **C++ Code 20.3** shows the calculation of a put price

```
#include <cmath>          // standard mathematical library
#include <algorithm>          // defining the max() operator
#include <vector>         // STL vector templates
using namespace std;

double bond_option_price_put_american_binomial( const double& B, // Bond price
                                                const double& K,    // exercise price
                                                const double& r,    // interest rate
                                                const double& sigma, // volatility
                                                const double& t,    // time to maturity
                                                const int& steps){ // no steps in binomial tree
   double R = exp(r*(t/steps));         // interest rate for each step
   double Rinv = 1.0/R;                 // inverse of interest rate
   double u = exp(sigma*sqrt(t/steps)); // up movement
   double uu = u*u;
   double d = 1.0/u;
   double p_up = (R-d)/(u-d);
   double p_down = 1.0-p_up;
   vector<double> prices(steps+1);   // price of underlying
   vector<double> put_values(steps+1); // value of corresponding put

   prices[0] = B*pow(d, steps); // fill in the endnodes.
   for (int i=1; i<=steps; ++i) prices[i] = uu*prices[i-1];
   for (int i=0; i<=steps; ++i) put_values[i] = max(0.0, (K-prices[i])); // put payoffs at maturity
   for (int step=steps-1; step>=0; --step) {
      for (int i=0; i<=step; ++i) {
         put_values[i] = (p_up*put_values[i+1]+p_down*put_values[i])*Rinv;
         prices[i] = d*prices[i+1];
         put_values[i] = max(put_values[i],(K-prices[i]));  // check for exercise
      };
   };
   return put_values[0];
};
```

C++ Code 20.3: Binomial approximation to american put bond option price

Example

Parameters: $B = 100$, $K = 100$, $r = 0.05$, $\sigma = 0.1$, time=1.

There is also a coupon bond with the the same bond price, but paying coupon of 0.5 at date 1.

1. Price a an European put option on the zero coupon bond using Black Scholes.
2. Price a an European put option on the coupon coupon bond using Black Scholes.
3. Price a an European put option on the zero coupon bond using binomial approximation with 100 steps.

C++ program:

```
double B=100;
double K=100;
double r=0.05;
double sigma=0.1;
double time=1;
cout << " zero coupon put option price = "
     << bond_option_price_put_zero_black_scholes(B,K,r,sigma,time) << endl;

vector<double> coupon_times; coupon_times.push_back(0.5);
vector<double> coupons; coupons.push_back(1);
cout << " coupon bond put option price = "
     << bond_option_price_put_coupon_bond_black_scholes(B,K,r,sigma,time,coupon_times,coupons);
cout << endl;

int steps=100;
cout << " zero coupon american put option price, binomial = "
     << bond_option_price_put_american_binomial(B,K,r,sigma,time,steps) << endl;
```

Output from C++ program:

```
zero coupon put option price = 1.92791
coupon bond put option price = 2.22852
zero coupon american put option price, binomial = 2.43282
```

Chapter 21

Credit risk

Contents

Option pricing has obvious applications to the pricing of risky bonds.

21.1 The Merton Model

This builds on the Black and Scholes (1973) and Merton (1973) framework to find the value of the debt issued by the firm. The ideas were already in Black and Scholes, who discussed the view of the firm as a call option.

Assume debt structure: There is a single debt issue. Debt is issued as a zero coupon bond. The bond is due on a given date T.

Assuming the firm value V follows the usual Brownian motion proces, debt is found as a closed form solution, similar in structure to the Black Scholes equation for a call option.

Easiest seen from the interpretation of firm debt as the price of risk free debt, minus the value of a put option.

Price debt by the price B of risk free debt, and then subtract the price of the put, using the Black Scholes formula.

The Black Scholes formula for a call option is

$$c = S \cdot N(d_1) - K \cdot e^{-r(T-t)} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$N(\cdot) = \text{ The cumulative normal distribution}$$

$$p = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

In the context here, reinterpret S as V, firm value. The put is priced as

$$p = Ke^{-r(T-t)}N(-d_2) - V_t N(-d_1)$$

where

$$d_1 = \frac{\ln\left(\frac{V_t}{K}\right) + (r + \frac{1}{2}\sigma)(T-t)}{\sigma\sqrt{T-t}}$$

Note on interpretation: The spread between risky and risk free debt determined solely by the price of the put option.

Example

The current value of the firm $V = 100$. The firm has issued one bond with face value 90, which is due to be paid one year from now. The risk free interest rate is 5% and the volatility of the firms value is 25%. Determine the value of the debt.

C++ program:

```
cout << " Credit Risk Calculation " << endl;
double V=100; double F=90; double r=0.05; double T=1; double sigma=0.25;
double p = option_price_put_black_scholes(V,F,r,sigma,T);
cout << " Debt value = " << exp(-r*T)*F - p << endl;
```

Output from C++ program:

```
Credit Risk Calculation
Debt value = 81.8592
```

21.2 Issues in implementation

- Firm value and firm volatility is unobservable.
- The model assumes a simple debt structure, most debt structures tend to be more complex.

Chapter 22

Term Structure Models

Contents

We now expand on the analysis of the term structure in chapter 5. As shown there, the term structure is best viewed as an abstract class providing, as functions of term to maturity, the prices of zero coupon bonds (discount factors), yield on zero coupon bonds (spot rates) or forward rates. In the earlier case we considered two particular implementations of the term structure: A flat term structure or a term structure estimated by linear interpolations of spot rates. We now consider a number of alternative term structure models. The focus of this chapter is empirical, we consider ways in which on one can specify a term structure in a lower dimensional way. Essentially we are looking at ways of doing curve-fitting, of estimating a nonlinear relationship between time and discount factors, or between time and spot rates. Since the relationship is nonlinear, this is a nontrivial problem. One has to choose a functional form to estimate, which allows enough flexibility to "fit" the term structure, but not so flexible that it violates the economic restrictions on the term structure. Here are some considerations.

- Discount factors must be positive. ($d_t > 0$). This is because they are prices, negative prices allow for abritrage.
- Discount factors must be a nonincreasing function of time. ($d_t \geq d_{t+k} \ \forall \ k > 0$). Again, this is to avoid arbitrage.
- Nominal interest rates can not be negative. ($r_t \geq 0 \ \forall \ t$) This is another implication of the absence of arbitrage opportunities.
- Both discount factors and interest rates must be smooth functions of time.
- The value of a payment today is the payment today. $d_0 = 1$.

A number of alternative ways of estimating the term structure has been considered. Some are purely used as interpolation functions, while others are fully specified, dynamic term structure models. Of the models that follow, the approximating function proposed in Nelson and Siegel (1987) and the cubic spline used by e.g. McCulloch (1971) are examples of the first kind, and the term structure models of Cox, Ingersoll, and Ross (1985) and Vasicek (1977) are examples of the second kind.

What is the typical use of the functions we consider here? One starts with a set of fixed income securities, typically a set of treasury bonds. Observing the prices of these bonds, one asks: What set of discount factors is most likely to have generated the observed prices. Or: What term structure approximations provides the "best fit" to this set of observed bond prices.

22.1 The Nelson Siegel term structure approximation

Nelson and Siegel (1987) proposes the parameterization shown in FormulaRefns

$$r(t) = \beta_0 + (\beta_1 + \beta_2)\left[\frac{1 - e^{-\frac{t}{\lambda}}}{\frac{t}{\lambda}}\right] + \beta_2\left[e^{-\frac{t}{\lambda}}\right]$$

Notation: t: Time to maturity. r spot interest rate, $\beta_0, \beta_1, \beta_2$ and λ: constants.

Formula 22.1: Nelson and Siegel (1987) parameterization of term structure

The implementation of this calculation is shown in **C++ Code 22.1**.

```
#include <cmath>
using namespace std;

double term_structure_yield_nelson_siegel(const double& t,
                                          const double& beta0,
                                          const double& beta1,
                                          const double& beta2,
                                          const double& lambda) {
   if (t==0.0) return beta0;
   double tl = t/lambda;
   double r = beta0 + (beta1+beta2) * ((1-exp(-tl))/tl) + beta2 * exp(-tl);
   return r;
};
```

C++ Code 22.1: Calculation of the Nelson and Siegel (1987) term structure model

This is wrapped in a term structure *class* as shown in **Header File 22.1** and **C++ Code 22.2**.

```
class term_structure_class_nelson_siegel : public term_structure_class {
private:
    double beta0_, beta1_, beta2_, lambda_;
public:
    term_structure_class_nelson_siegel(const double& beta0,
                                       const double& beta1,
                                       const double& beta2,
                                       const double& lambda);
    virtual double yield(const double& T) const;
};
```

Header file 22.1: Header file defining a term structure class wrapper for the Nelson Siegel approximation

```
#include "fin_recipes.h"

term_structure_class_nelson_siegel::term_structure_class_nelson_siegel( const double& b0,
                                                                        const double& b1,
                                                                        const double& b2,
                                                                        const double& l) {
   beta0_=b0; beta1_=b1; beta2_=b2; lambda_=l;
};

double term_structure_class_nelson_siegel::r(const double& t) const {
   if (t<=0.0) return beta0_;
   return term_structure_yield_nelson_siegel(t,beta0_,beta1_,beta2_,lambda_);
};
```

C++ Code 22.2: Defining a term structure class wrapper for the Nelson Siegel approximation

Example

Using the parameters $\beta_0 = 0.01$, $\beta_1 = 0.01$, $\beta_2 = 0.01$, $\lambda = 5.0$ and $t = 1$ in the Nelson Siegel approximation, find the 1 year discount factor and spot rate, and the forward rate between years 1 and 2.

C++ program:

```
double beta0=0.01; double beta1=0.01; double beta2=0.01; double lambda=5.0;
double t=1.0;
cout << "Example calculations using the Nelson Siegel term structure approximation"
    << endl;
cout << " direct calculation, yield = "
       << term_structure_yield_nelson_siegel(t,beta0,beta1,beta2,lambda) << endl;
term_structure_class_nelson_siegel ns(beta0,beta1,beta2,lambda);
cout << " using a term structure class" << endl;
cout << " yield (t=1) = " << ns.r(t) << endl;
cout << " discount factor (t=1) = " << ns.d(t) << endl;
cout << " forward rate (t1=1, t2=2) = " << ns.f(1,2) << endl;
```

Output from C++ program:

```
Example calculations using the Nelson Siegel term structure approximation
 direct calculation, yield = 0.0363142
 using a term structure class
 yield (t=1) = 0.0363142
 discount factor (t=1) = 0.964337
 forward rate (t1=1, t2=2) = 0.0300602
```

22.2 Extended Nelson Siegel models

The Nelson and Siegel (1987) model is simple, with parameters with clear obvious economic interpretations. It does have the problem that the term structure shapes that it allows is limited. To allow for more complex shapes, such as humped shapes, it has been extended in various ways. A popular approximation was introduced by Lars Svensson, parameterized as shown in **Formula 22.2**

$$r(t) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau}} \right) + \beta_2 \left(\frac{1 - e^{-\frac{t}{\tau_1}}}{\frac{t}{\tau_1}} - e^{-\frac{t}{\tau_1}} \right) + \beta_3 \left(\frac{1 - e^{-\frac{t}{\tau_2}}}{\frac{t}{\tau_2}} - e^{-\frac{t}{\tau_2}} \right)$$

Notation: t: Time to maturity. r spot interest rate, $\beta_0, \beta_1, \beta_2$ and λ: constants.

Formula 22.2: Svensson's extension of the Nelson and Siegel (1987) parameterization of term structure

This is wrapped in a term structure class as shown in **Header File 22.2** and **C++ Code 22.4**.

```
#include <cmath>
using namespace std;

double term_structure_yield_svensson(const double& t,
                                     const double& beta0,
                                     const double& beta1,
                                     const double& beta2,
                                     const double& beta3,
                                     const double& tau1,
                                     const double& tau2){
   if (t==0.0) return beta0;
   double r = beta0;
   r += beta1* ((1-exp(-t/tau1))/(t/tau1)) ;
   r += beta2 * ( ((1-exp(-t/tau1))/(t/tau1)) - exp(-t/tau1) );
   r += beta3 * ( ((1-exp(-t/tau2))/(t/tau2)) - exp(-t/tau2) );
   return r;
};
```

C++ Code 22.3: Calculation of Svensson's extended Nelson and Siegel (1987) term structure model

```
class term_structure_class_svensson:public term_structure_class {
private:
    double beta0_, beta1_, beta2_, beta3_, tau1_, tau2_;
public:
    term_structure_class_svensson(const double& beta0,
                                  const double& beta1,
                                  const double& beta2,
                                  const double& beta3,
                                  const double& tau1,
                                  const double& tau2);
    virtual double yield(const double& T) const;
};
```

Header file 22.2: Header file defining a term structure class wrapper for the Svensson model

```
#include "fin_recipes.h"

term_structure_class_svensson::term_structure_class_svensson( const double& b0, const double& b1, const double& b2, const double& b3,
                                                              const double& tau1, const double& tau2) {
    beta0_=b0; beta1_=b1; beta2_=b2; beta3_=b3; tau1_=tau1; tau2_=tau2;
};

double term_structure_class_svensson::r(const double& t) const {
    if (t<=0.0) return beta0_;
    return term_structure_yield_svensson(t,beta0_,beta1_,beta2_,beta3_,tau1_,tau2_);
};
```

C++ Code 22.4: Defining a term structure class wrapper for the Svensson model

22.3 Cubic spline.

Cubic splines are well known for their good interpolation behaviour. The cubic spline parameterization was first used by McCulloch (1971) to estimate the nominal term structure. He later added taxes in McCulloch (1975). The cubic spline was also used by Litzenberger and Rolfo (1984). In this case the qubic spline is used to approximate the *discount factor*, not the yields.

$$d(t) = 1 + b_1 t + c_1 t^2 + d_1 t^3 + \sum_{j=1}^{K} F_j (t - t_j)^3 1_{\{t<t_j\}}$$

Here $1_{\{A\}}$ is the indicator function for an event A, and we have K *knots*.
To estimate this we need to find the $3 + K$ parameters:

$$\{b_1, c_1, d_1, F_1, \cdots, F_K\}$$

If the spline *knots* are known, this is a simple linear regression. **C++ Code 22.5** shows the calculation using this approximation.

```
#include <cmath>
#include <vector>
using namespace std;

double term_structure_discount_factor_cubic_spline(const double& t,
                                                   const double& b1,
                                                   const double& c1,
                                                   const double& d1,
                                                   const vector<double>& f,
                                                   const vector<double>& knots){
    double d = 1.0 + b1*t + c1*(pow(t,2)) + d1*(pow(t,3));
    for (int i=0;i<knots.size();i++) {
        if (t >= knots[i]) { d += f[i] * (pow((t-knots[i]),3)); }
        else { break; };
    };
    return d;
};
```

C++ Code 22.5: Approximating a discount function using a cubic spline

Header File 22.3 and **C++ Code 22.6** wraps this calculations into a term structure class.

```
#include "fin_recipes.h"
#include <vector>
using namespace std;

class term_structure_class_cubic_spline : public term_structure_class {
private:
    double b_; double c_; double d_;
    vector<double> f_; vector<double> knots_;
public:
    term_structure_class_cubic_spline(const double& b, const double& c, const double& d,
                                      const vector<double>& f, const vector<double> & knots);
    virtual ~term_structure_class_cubic_spline();
    virtual double d(const double& T) const;
};
```

Header file 22.3: Term structure class wrapping the cubic spline approximation

```
#include "fin_recipes.h"

term_structure_class_cubic_spline::
term_structure_class_cubic_spline ( const double& b, const double& c, const double& d,
                                    const vector<double>& f, const vector<double>& knots) {
    b_ = b;  c_ = c;  d_ = d; f_.clear();  knots_.clear();
    if (f.size()!=knots.size()){ return; };
    for (int i=0;i<f.size();++i) {
        f_.push_back(f[i]);
        knots_.push_back(knots[i]);
    };
};

double term_structure_class_cubic_spline::d(const double& T) const {
   return term_structure_discount_factor_cubic_spline(T,b_,c_,d_,f_,knots_);
};
```

C++ Code 22.6: Term structure class wrapping the cubic spline approximation

Example

Using the parameters $b = 0.1$ $c = 0.1$, $d = -0.1$,

$$\mathbf{f} = \begin{bmatrix} 0.01 \\ 0.01 \\ -0.01 \end{bmatrix}$$

$$\mathbf{knots} = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}$$

Find short rates and discount factors for 1 year, and the forward rate between 1 and 2 years.

C++ program:

```
cout << "Example term structure calculations using a cubic spline " << endl;
double b=0.1; double c=0.1; double d=-0.1;
vector<double> f;   f.push_back(0.01); f.push_back(0.01); f.push_back(-0.01);
vector<double> knots; knots.push_back(2); knots.push_back(7); knots.push_back(12);
cout << " direct calculation, discount factor (t=1) "
     << term_structure_discount_factor_cubic_spline(1,b,c,d,f,knots) << endl;
cout << " Using a term structure class " << endl;
term_structure_class_cubic_spline cs(b,c,d,f,knots);
cout << " yield (t=1) = " << cs.r(1) << endl;
cout << " discount factor (t=1) = " << cs.d(1) << endl;
cout << " forward (t1=1, t2=2) = " << cs.f(1,2) << endl;
```

Output from C++ program:

```
Example term structure calculations using a cubic spline
 direct calculation, discount factor (t=1) 1.1
 Using a term structure class
 yield (t=1) = -0.0953102
 discount factor (t=1) = 1.1
 forward (t1=1, t2=2) = 0.318454
```

22.4 Cox Ingersoll Ross.

The Cox et al. (1985) model is the best known example of a continuous time, general equilibrium model of the term structure. It is commonly used in academic work because it is a general equilibrium model that still is "simple enough" to let us find closed form expressions for derivative securities.

The short interest rate.

$$dr(t) = \kappa(\theta - r(t))dt + \sigma\sqrt{r(t)}dW$$

The discount factor for a payment at time T.

$$d(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$\gamma = \sqrt{(\kappa+\lambda)^2 + 2\sigma^2}$$

$$A(t,T) = \left[\frac{2\gamma e^{\frac{1}{2}(\kappa+\lambda+\gamma)(T-t)}}{(\gamma+\kappa+\lambda)(e^{\lambda(T-t)}-1)+2\gamma}\right]^{\frac{2\kappa\theta}{\sigma^2}}$$

and

$$B(t,T) = \frac{2e^{\gamma(T-t)}-1}{(\gamma+\kappa+\lambda)(e^{\lambda(T-t)}-1)+2\gamma}$$

Five parameters: r, the short term interest rate, κ, the mean reversion parameter, λ, the "market" risk parameter, θ the long–run mean of the process and σ, the variance rate of the process.

```
#include <cmath>
using namespace std;

double term_structure_discount_factor_cir(const double& t,
                                          const double& r,
                                          const double& kappa,
                                          const double& lambda,
                                          const double& theta,
                                          const double& sigma){
    double sigma_sqr=pow(sigma,2);
    double gamma = sqrt(pow((kappa+lambda),2)+2.0*sigma_sqr);
    double denum = (gamma+kappa+lambda)*(exp(gamma*t)-1)+2*gamma;
    double p=2*kappa*theta/sigma_sqr;
    double enum1= 2*gamma*exp(0.5*(kappa+lambda+gamma)*t);
    double A = pow((enum1/denum),p);
    double B = (2*(exp(gamma*t)-1))/denum;
    double dfact=A*exp(-B*r);
    return dfact;
};
```

C++ Code 22.7: Calculation of the discount factor using the Cox et al. (1985) model

```
#include "fin_recipes.h"

class term_structure_class_cir : public term_structure_class {
private:
    double r_;                      // interest rate
    double kappa_;                  // mean reversion parameter
    double lambda_;                 // risk aversion
    double theta_;                  // long run mean
    double sigma_;                  // volatility
public:
    ~term_structure_class_cir();
    term_structure_class_cir(const double& r, const double& k, const double& l,
                             const double& th, const double& sigma);
    virtual double d(const double& T) const;
};
```

Header file 22.4: Class definition, Cox et al. (1985) model, header file

```
#include "fin_recipes.h"

//term_structure_class_cir::~term_structure_class_cir(){;};

term_structure_class_cir::term_structure_class_cir(const double& r, const double& k, const double& l,
                                                   const double& th, const double& sigma) {
    r_=r; kappa_=k; lambda_=l; theta_=th; sigma_=sigma;
};

double term_structure_class_cir::d(const double& T) const{
    return term_structure_discount_factor_cir(T,r_,kappa_,lambda_,theta_,sigma_);
};
```

C++ Code 22.8: Class definition, Cox et al. (1985) model

Example

Parameters: $r = 0.05$, $\kappa = 0.01$, $\sigma = 0.1$, $\theta = 0.08$ and $\lambda = 0.0$. Use the CIR term structure model. Find short rate and discount factor for $t = 1$, and forward rate between years 1 and 2.

C++ program:

```
cout << "Example calculations using the Cox Ingersoll Ross term structure model " << endl;
double r = 0.05; double kappa=0.01; double sigma=0.1; double theta=0.08; double lambda=0.0;
cout << " direct calculation, discount factor (t=1): "
    << term_structure_discount_factor_cir(1, r, kappa, lambda, theta, sigma) << endl;
cout << " using a class " << endl;
term_structure_class_cir cir(r,kappa,lambda,theta,sigma);
cout << " yield (t=1) = " << cir.r(1) << endl;
cout << " discount factor (t=1) = " << cir.d(1) << endl;
cout << " forward (t1=1, t2=2) = " << cir.f(1,2) << endl;
```

Output from C++ program:

```
Example calculations using the Cox Ingersoll Ross term structure model
 direct calculation, discount factor (t=1): 0.951166
 using a class
 yield (t=1) = 0.0500668
 discount factor (t=1) = 0.951166
 forward (t1=1, t2=2) = 0.0498756
```

22.5 Vasicek

```
#include <cmath>
using namespace std;

double term_structure_discount_factor_vasicek(const double& time,
                                              const double& r,
                                              const double& a,
                                              const double& b,
                                              const double& sigma){
    double A,B;
    double sigma_sqr = sigma*sigma;
    double aa = a*a;
    if (a==0.0){
        B = time;
        A = exp(sigma_sqr*pow(time,3))/6.0;
    }
    else {
        B = (1.0 - exp(-a*time))/a;
        A = exp( ((B-time)*(aa*b-0.5*sigma_sqr))/aa -((sigma_sqr*B*B)/(4*a)));
    };
    double dfact = A*exp(-B*r);
    return dfact;
}
```

C++ Code 22.9: Calculating a discount factor using the Vasicek functional form

```
#include "fin_recipes.h"

class term_structure_class_vasicek : public term_structure_class {
private:
    double r_; double a_; double b_; double sigma_;
public:
    term_structure_class_vasicek(const double& r, const double& a, const double& b, const double& sigma);
    virtual double discount_factor(const double& T) const;
};
```

Header file 22.5: Class definition, Vasicek (1977) model

```
#include "fin_recipes.h"

term_structure_class_vasicek::term_structure_class_vasicek(const double& r, const double& a,
                                                           const double& b, const double& sigma) {
    r_=r; a_=a; b_=b; sigma_=sigma;
};

double term_structure_class_vasicek::d(const double& T) const{
    return term_structure_discount_factor_vasicek(T,r_,a_,b_,sigma_);
};
```

C++ Code 22.10: Class definition, Vasicek (1977) model

Example

Parameters $r = 0.05$, $a = -0.1$, $b = 0.1$, $\sigma = 0.1$ Use the Vasicek term structure model. Find short rate and discount factor for $t = 1$, and forward rate between years 1 and 2.

C++ program:

```
cout << "Example term structure calculation using the Vasicek term structure model"
     << endl;
double r=0.05; double a=-0.1; double b=0.1; double sigma=0.1;
cout << " direct calculation, discount factor (t=1): "
     << term_structure_discount_factor_vasicek(1, r, a, b, sigma) << endl;
term_structure_class_vasicek vc(r,a,b,sigma);
cout << " using a term structure class " << endl;
cout << " yield (t=1) = " << vc.r(1) << endl;
cout << " discount factor (t=1) = " << vc.d(1) << endl;
cout << " forward rate (t1=1, t2=2) = " << vc.f(1,2) << endl;
```

Output from C++ program:

```
Example term structure calculation using the Vasicek term structure model
 direct calculation, discount factor (t=1): 0.955408
 using a term structure class
 yield (t=1) = 0.0456168
 discount factor (t=1) = 0.955408
 forward rate (t1=1, t2=2) = 0.0281476
```

22.6 Readings

The methods in this chapter I first studied in my dissertation at Carnegie Mellon University in 1992, which lead to the paper published as Green and Ødegaard (1997). A textbook treatment of estimation and fitting of term structure models can be found in (Martinelli, Priaulet, and Priaulet, 2003, Ch 4)

Chapter 23

Binomial Term Structure models

Contents

Pricing bond options with the Black Scholes model, or its binomial approximation, as done in chapter 20, does not always get it right. For example, it ignores the fact that at the maturity of the bond, the bond volatility is zero. The bond volatility decreases as one gets closer to the bond maturity. This behaviour is not captured by the assumptions underlying the Black Scholes assumption. We therefore look at more complicated term structure models, the unifying theme of which is that they are built by building *trees* of the interest rate.

23.1 The Rendleman and Bartter model

The Rendleman and Bartter approach to valuation of interest rate contingent claims (see Rendleman and Bartter (1979) and Rendleman and Bartter (1980)) is a particular simple one. Essentially, it is to apply the same binomial approach that is used to approximate options in the Black Scholes world, but the random variable is now the interest rate. This has implications for multiperiod discounting: Taking the present value is now a matter of choosing the correct sequence of spot rates, and it may be necessary to keep track of the whole "tree" of interest rates. Such a tree can then be used to price various fixed income securities. In the next chapter we illustrate this more generally, here we show a direct implementation of the original model. **C++ Code 23.1** implements the original algorithm for a call option on a (long maturity) zero coupon bond.

```
#include <cmath>
#include <algorithm>
#include <vector>
using namespace std;

double bond_option_price_call_zero_american_rendleman_bartter(const double& X,
                                                              const double& option_maturity,
                                                              const double& S,
                                                              const double& M,  // term structure paramters
                                                              const double& interest, // current short interest rate
                                                              const double& bond_maturity, // time to maturity for underlying bond
                                                              const double& maturity_payment,
                                                              const int& no_steps) {
    double delta_t = bond_maturity/no_steps;

    double u=exp(S*sqrt(delta_t));
    double d=1/u;
    double p_up = (exp(M*delta_t)-d)/(u-d);
    double p_down = 1.0-p_up;

    vector<double> r(no_steps+1);
    r[0]=interest*pow(d,no_steps);
    double uu=u*u;
    for (int i=1;i<=no_steps;++i){ r[i]=r[i-1]*uu;};
    vector<double> P(no_steps+1);
    for (int i=0;i<=no_steps;++i){ P[i] = maturity_payment; };
    int no_call_steps=int(no_steps*option_maturity/bond_maturity);
    for (int curr_step=no_steps;curr_step>no_call_steps;--curr_step) {
        for (int i=0;i<curr_step;i++) {
            r[i] = r[i]*u;
            P[i] = exp(-r[i]*delta_t)*(p_down*P[i]+p_up*P[i+1]);
        };
    };
    vector<double> C(no_call_steps+1);
    for (int i=0;i<=no_call_steps;++i){ C[i]=max(0.0,P[i]-X); };
    for (int curr_step=no_call_steps;curr_step>=0;--curr_step) {
        for (int i=0;i<curr_step;i++) {
            r[i]   = r[i]*u;
            P[i] = exp(-r[i]*delta_t)*(p_down*P[i]+p_up*P[i+1]);
            C[i]=max(P[i]-X, exp(-r[i]*delta_t)*(p_up*C[i+1]+p_down*C[i]));
        };
    };
    return C[0];
};
```

C++ Code 23.1: RB binomial model for European call on zero coupon bond

Example

Parameters:$K = 950$, $S = 0.15$ and $M = 0.05$ The interest rate is 10%, The option matures in 4 years, the bond matures in year 5, with a bond maturity payment of 1000. Price the option on the zero coupon bond using a Randleman–Bartter approximation with 100 steps.

C++ program:

```
double K=950; double S=0.15; double M=0.05; double interest=0.10;
double option_maturity=4; double bond_maturity=5; double bond_maturity_payment=1000;
int no_steps=100;
cout << " Rendleman Bartter price of option on zero coupon bond: ";
cout << bond_option_price_call_zero_american_rendleman_bartter( K, option_maturity, S, M,
                                                    interest, bond_maturity,
                                                              bond_maturity_payment, no_steps);
```

Output from C++ program:

```
 Rendleman Bartter price of option on zero coupon bond: 0.00713661
```

23.2 Readings

General references include Sundaresan (2001).

Rendleman and Bartter (1979) and Rendleman and Bartter (1980) are the original references for building standard binomial interest rate trees.

Chapter 24

Interest rate trees

Contents

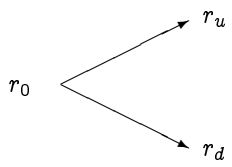
In this chapter we show a way of building interest rate trees and apply it to the pricing of various fixed income securities. The first need in such a procedure is to specify the future evolution of interest rates.

24.1 The movement of interest rates

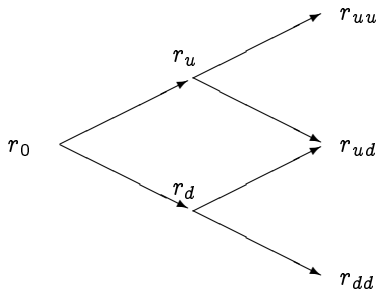
We will assume that interest rates follow a Geometric Brownian Motion process

$$dr = \mu r dt + \sigma r dZ$$

This is the same assumption which was used for stock prices in the Black Scholes case. This leads to the same binomial approximation, for one period:



or several periods:



When using this approach it turns out to be useful to keep the whole tree around, we therefore separate the building of the tree in one routine, as shown in **C++ Code 24.1**.

```
#include <vector>
#include <cmath>
using namespace std;

vector<vector<double> > interest_rate_trees_gbm_build(const double& r0,
                                                      const double& u,
                                                      const double& d,
                                                      const int& n){
    vector< vector<double> > tree;
    vector<double> r(1); r[0]=r0;
    tree.push_back(r);
    for (int i=1;i<=n;++i) {
        double rtop=r[r.size()-1]*u;
        for (int j=0;j<i;++j){
            r[j] = d*r[j];
        };
        r.push_back(rtop);
        tree.push_back(r);
    };
    return tree;
};
```

C++ Code 24.1: Building interest rate tree

```
function tree = interest_rate_trees_gbm_build(r0,u,d,n)
    r=[r0];
    tree=[r];
    rtop=r0;
    for i=1:n
       rtop = u * rtop;
       r = [rtop;d*r];
       tree=[ [zeros(1,i);tree] r ];
    endfor
endfunction
```

`Matlab` **Code 24.1**: Building interest rate tree

Example

Parameters: $r_0 - 10\%$, $u = 1.02$, $d = 0.99$. Build a 2 period interest rate tree.

C++ program:

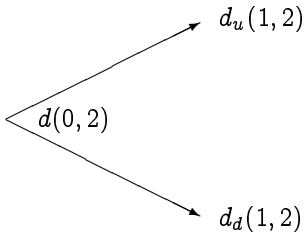
```
vector< vector<double> > tree = interest_rate_trees_gbm_build(0.1,1.02,0.99,3);
cout << " Interest rate tree: " << endl;
cout << " Time 0: " << tree[0][0] << endl;
cout << " Time 1: " << tree[1][0] << " " << tree[1][1] << endl;
cout << " Time 2: " << tree[2][0] << " " << tree[2][1] << " " << tree[2][2] << endl;
```

Output from C++ program:

```
Interest rate tree:
Time 0: 0.1
Time 1: 0.099  0.102
Time 2: 0.09801  0.10098  0.10404
```

24.2 Discount factors

We want to price bonds and other fixed income securities, which contains sets of future cash flows. Instead of interest rates we need prices, not interest rates, i.e. discount factors. The interest rate tree therefore needs to be used to generate discount factors, which needs to be specified at every point of the tree, and for all relevant maturities. Let $d(t,T)$ be the discount factor at time t for a time T payment. If we at time 0 want to price cash flows at time 2, we need the following set of discount factors



In constructing trees of discount we need one additional piece of information, q, which are used as follows:

$$d(0,2) = e^{-r_0}\left(qd_u(1,2) + (1+q)d_d(1,2)\right)$$

The parameter q serves the same purpose as the state price probability, but it is found differently.

Exercise 24.1.

Given the prices of two discount bonds, with maturities 1 and 2, how would you back out q?

Exercise 24.2.

Suppose you have q and the tree of short interest rates. How would you calculate the t-period spot rate?

24.3 Pricing bonds

Pricing straight bonds is then just a matter of the usual recursive valuation

Example

Parameters: $r_0 - 10\%$, $u = 1.02$, $d = 0.99$. Let $q = 0.5$. Determine the price of a 3 period coupon bond with coupon of 10 and face value of 100.

```
#include <vector>
#include <cmath>
using namespace std;

double interest_rate_trees_gbm_value_of_cashflows(const vector<double>& cflow,
                                                  const vector< vector<double> >& r_tree,
                                                  const double& q){
    int n = int(cflow.size());
    vector< vector<double> > values(n);
    vector<double> value(n);
    for (int i=0;i<n;i++){ value[i]=cflow[n-1]; };
    values[n-1]=value;
    for (int t=n-1;t>0;--t){
        vector<double> value(t,0.0);
        for (int i=0;i<t;++i){
            value[i]=cflow[t-1]+exp(-r_tree[t-1][i])*(q*values[t][i]+(1-q)*values[t][i+1]);
        };
        values[t-1]=value;
    };
    return values[0][0];
};
```

C++ Code 24.2: Valuing cash flows

C++ program:

```
double r0=0.1;
double u=1.02; double d=0.99;
int n=3;
double q=0.5;
vector< vector<double> > tree = interest_rate_trees_gbm_build(r0,u,d,n);
vector<double> cashflows;
cashflows.push_back(0); cashflows.push_back(10); cashflows.push_back(10); cashflows.push_back(110);
cout << "Bond price B = " << interest_rate_trees_gbm_value_of_cashflows(cashflows,tree,q);
```

Output from C++ program:

```
Bond price B = 98.5997
```

Exercise 24.3.

The example just presented assumes cash flows dates matches the dates of interest rate changes. How would you modify the code to allow for differences in timing of interest rate changes and cashflows. Or can you do this alternatively, by adjusting the inputs to the routine?

24.4 Callable bond

A slightly more involved example is a callable bond.

```
#include <vector>
#include <cmath>
using namespace std;

double interest_rate_trees_gbm_value_of_callable_bond(const vector<double>& cflows,
                                                      const vector< vector<double> >& r_tree,
                                                      const double& q,
                                                      const int& first_call_time,
                                                      const double& call_price){
    int n = int(cflows.size());
    vector< vector<double> > values(n);
    vector<double> value(n);
    for (int i=0;i<n;i++){ value[i]=cflows[n-1]; };
    values[n-1]=value;
    for (int t=n-1;t>0;--t){
        vector<double> value(t,0.0);
        for (int i=0;i<t;++i){
            value[i]=cflows[t-1]+exp(-r_tree[t-1][i])*(q*values[t][i]+(1-q)*values[t][i+1]);
            if (t>=first_call_time){ value[i]=min(value[i],call_price); };
        };
        values[t-1]=value;
    };
    return values[0][0];
};
```

C++ Code 24.3: Valuing callable bond

Example

Construct a short rate lattice for periods (years) 0 through 9 with an initial rate of $r_0 = 6\%$ and with successive rates determined by a multiplicative factors $u = 1.2$ or $d = 0.9$. Assign $q = 0.5$.

1. Using this lattice, find the value of a 10-year 6% bond.
2. Suppose this bond can be called by the issuing party at any time after 5 years. (When the bond is called, the face value plus the currently due coupon are paid at that time and the bond is canceled.) What is the fair value of this bond?

The interest rate lattice:

step:	0	1	2	3	4	5	6	7	8	9
nodes:										31.0
									25.8	23.2
								21.5	19.3	17.4
							17.9	16.1	14.5	13.1
						14.9	13.4	12.1	10.9	9.8
					12.4	11.2	10.1	9.1	8.2	7.3
				10.4	9.3	8.4	7.6	6.8	6.1	5.5
			8.6	7.8	7.0	6.3	5.7	5.1	4.6	4.1
		7.2	6.5	5.8	5.2	4.7	4.3	3.8	3.4	3.1
	6.0	5.4	4.9	4.4	3.9	3.5	3.2	2.9	2.6	2.3

1. Valuing the noncallable bond

   The cash flow lattice

step:	0	1	2	3	4	5	6	7	8	9	10
nodes:											106
										6	106
									6	6	106
								6	6	6	106
							6	6	6	6	106
						6	6	6	6	6	106
					6	6	6	6	6	6	106
				6	6	6	6	6	6	6	106
			6	6	6	6	6	6	6	6	106
		6	6	6	6	6	6	6	6	6	106
	0	6	6	6	6	6	6	6	6	6	106

The value lattice:

step:	0	1	2	3	4	5	6	7	8	9	10
nodes:											106.00
										83.78	106.00
									73.14	90.04	106.00
								68.67	82.27	95.06	106.00
							67.84	79.28	89.93	99.02	106.00
						69.38	79.32	88.46	96.19	102.11	106.00
					72.59	81.45	89.45	96.14	101.20	104.49	106.00
				77.07	85.09	92.19	98.04	102.39	105.15	106.32	106.00
			82.57	89.88	96.24	101.37	105.10	107.37	108.21	107.71	106.00
		88.89	95.59	101.29	105.79	108.96	110.78	111.29	110.57	108.77	106.00
	89.90	102.03	107.13	111.05	113.70	115.09	115.26	114.32	112.38	109.56	106.00

The value of the bond is $B = 89.90$.

2. The callable will be called when the value of the bond is above par, because the issuing company can issue a bond at a lower interest rate.

   The following set of values is calculated

step:	0	1	2	3	4	5	6	7	8	9	10
nodes:											106.00
										83.78	106.00
									73.14	90.04	106.00
								68.67	82.27	95.06	106.00
							67.84	79.28	89.93	99.02	106.00
						69.38	79.32	88.46	96.19	102.11	106.00
					72.59	81.45	89.45	96.14	101.20	104.49	106.00
				77.07	85.08	92.18	98.01	102.32	105.00	106.00	106.00
			82.53	89.80	96.08	101.03	104.42	106.00	106.00	106.00	106.00
		88.70	95.21	100.56	104.39	106.36	106.00	106.00	106.00	106.00	106.00
	89.35	101.05	105.44	108.22	109.19	108.31	106.00	106.00	106.00	106.00	106.00

The value of the bond is $B = 89.35$.

C++ program:

```
double r0=0.06;
double u=1.2; double d=0.9;
int n=10;
double q=0.5;
vector< vector<double> > tree = interest_rate_trees_gbm_build(r0,u,d,n);
vector<double> cashflows;
cashflows.push_back(0);
for (int t=1;t<=9;++t){ cashflows.push_back(6); };
cashflows.push_back(106);
cout << "Straight bond price = " << interest_rate_trees_gbm_value_of_cashflows(cashflows,tree,q) << endl;
int first_call_time = 6;
double call_price = 106;
cout << "Callable bond price = "
     << interest_rate_trees_gbm_value_of_callable_bond(cashflows,tree,q, first_call_time, call_price) << endl;
```

Output from C++ program:

```
Straight bond price  = 89.9017
Callable bond price = 89.2483
```

Exercise 24.4.

How would you price a call option on a coupon bond in this setting?

24.5 Readings

General references include Sundaresan (2001).

Rendleman and Bartter (1979) and Rendleman and Bartter (1980) are the original references for building standard binomial interest rate trees.

Chapter 25

Building term structure trees using the Ho and Lee (1986) approach

25.1 Intro

In this section we build interest rate trees following the orgiginal paper of Ho and Lee (1986). We will follow the analysis in the paper, and use it to illustrate how yu can build trees of more complex term structures than the simple binomial trees of the interest rate. The selling point of the original paper was that one could fit an initial term structure and then specify an evolution of the term structure consistent with this initial term structure.

25.2 Building trees of term structures

25.3 Ho Lee term structure class

```
#include "fin_recipes.h"

class term_structure_class_ho_lee : public term_structure_class {
private:
    term_structure_class* initial_term_;
    int n_;
    int i_;
    double delta_;
    double pi_;
public:
    term_structure_class_ho_lee(term_structure_class* fitted_term,
                                const int & n,
                                const int & i,
                                const double& lambda,
                                const double& pi);
    double d(const double& T) const;
};

vector< vector<term_structure_class_ho_lee> >
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
                                                const int& no_steps,
                                                const double& delta,
                                                const double& pi);

double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial,
                                                       const double& delta,
                                                       const double& pi,
                                                       const vector<double>& underlying_bond_cflow_times,
                                                       const vector<double>& underlying_bond_cflows,
                                                       const double& K,
                                                       const double& option_time_to_maturity);
```

Header file 25.1: Term structure class for Ho-Lee

```
#include "fin_recipes.h"

term_structure_class_ho_lee::term_structure_class_ho_lee(term_structure_class* fitted_term,
                                                         const int & n,
                                                         const int & i,
                                                         const double& delta,
                                                         const double& pi){
    initial_term_=fitted_term;
    n_=n;
    i_=i;
    delta_=delta;
    pi_=pi;
};
```

C++ Code 25.1: Term structure class for Ho-Lee

```
#include "fin_recipes.h"
//#include "term_structure_class_ho_lee.h"

inline double hT(const double& T, const double& delta, const double& pi){
    return (1.0/(pi+(1-pi)*pow(delta,T)));
};

double term_structure_class_ho_lee::d(const double& T) const{
    double d=(*initial_term_).d(T+n_)/(*initial_term_).d(n_);
    for (int j=1;j<n_;++j){
        d *= hT(T+(n_-j),delta_,pi_) / hT(n_-j,delta_,pi_) ;
    };
    d *= hT(T,delta_,pi_)*pow(delta_,T*(n_-i_));
    return d;
};
```

C++ Code 25.2: Term structure class for Ho-Lee, calculation of discount function

```
#include "fin_recipes.h"

vector< vector<term_structure_class_ho_lee> >
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
                                                const int& no_steps,
                                                const double& delta,
                                                const double& pi){
    vector< vector<term_structure_class_ho_lee> > hl_tree;
    for (int t=0;t<5;++t){
        hl_tree.push_back(vector<term_structure_class_ho_lee>());
        for (int j=0;j<=t;++j){
            term_structure_class_ho_lee hl(initial,t,j,delta,pi);
            hl_tree[t].push_back(hl);
        };
    };
    return hl_tree;
};
```

C++ Code 25.3: Building a term structure tree

25.4 Pricing things

We now have access to what we need to do pricing through the recursive relationship

$$C(n,i) = \left[\pi C(n+1,i+1) + (1-\pi)C(n+i,i)\right] P_i^{(n)}(1)$$

where $C(n,i)$ is the value of a security at time n at node i.

What we are pricing are typically state and time-contingent claims to cash flows. Let us illustrate pricing of an (European) call option on some underlying bond. Suppose this bond is risk free. Its cash flows at each future date does not depend on the state, but the timing of cash flows changes as you move in the tree. It is therefore necessary to some way figure out at date t: What are the future cash flows when you want to price the underlying bond at that date. We build a small class that contains this information, and use it, together with the term structures in the individual nodes, to find the bond price at each node. The value of the bond at the different nodes change because the term structure you use for discounting is changing. This bond price is then used to find the option value at each node.

```
#include "fin_recipes.h"

const double EPSILON=0.000001;

class time_contingent_cash_flows{
public:
    vector<double> times;
    vector<double> cash_flows;
    time_contingent_cash_flows(const vector<double>& in_times, const vector<double>& in_cflows){
        times=in_times; cash_flows=in_cflows;
    };
    int no_cflows(){ return int(times.size()); };
};

vector<time_contingent_cash_flows>
build_time_series_of_bond_time_contingent_cash_flows(const vector<double>& initial_times,
                                                     const vector<double>& initial_cflows){
    vector<time_contingent_cash_flows> vec_cf;
    vector<double> times = initial_times;
    vector<double> cflows = initial_cflows;
    while (times.size()>0){
        vec_cf.push_back(time_contingent_cash_flows(times,cflows));
        vector<double> tmp_times;
        vector<double> tmp_cflows;
        for (int i=0;i<times.size();++i){
            if (times[i]-1.0>=0.0) {
                tmp_times.push_back(times[i]-1);
                tmp_cflows.push_back(cflows[i]);
            };
        };
        times = tmp_times; cflows = tmp_cflows;
    };
    return vec_cf;
};

double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial, const double& delta, const double& pi,
                                                       const vector<double>& underlying_bond_cflow_times,
                                                       const vector<double>& underlying_bond_cflows,
                                                       const double& K, const double& time_to_maturity){
    int T = int(time_to_maturity+EPSILON);
    vector<vector<term_structure_class_ho_lee> > hl_tree
        = term_structure_ho_lee_build_term_structure_tree(initial,T+1,delta,pi);
    vector<time_contingent_cash_flows> vec_cf
        = build_time_series_of_bond_time_contingent_cash_flows(underlying_bond_cflow_times, underlying_bond_cflows);

    vector<double> values(T+1);
    for (int i=0;i<=T;++i){
        values[i]=max(0.0,bonds_price(vec_cf[T+1].times, vec_cf[T+1].cash_flows, hl_tree[T+1][i]) - K);
    };
    for (int t=T;t>=0;--t){
        vector<double> values_this(t+1);
        for (int i=0;i<=t;++i){ values_this[i]=(pi*values[i+1]+(1.0-pi)*values[i])*hl_tree[t][i].d(1); };
        values=values_this;
    };
    return values[0];
};
```

C++ Code 25.4: Pricing of European call option on straight bond using Ho-Lee

Example

You are pricing options on a 5 year zero coupon risk free bond. The options are European calls with a time to maturity of 3 years.

You will price the options using a Ho-Lee approach with parameters $\pi = 0.5$ and $\delta = 0.98$.

Price the option using two different assumptions about the current term structure:

1. The term structure is flat with an interest rate of 10% (continously compounded).
2. The current term structure has been estimated using a Nelson Siegel parameterization with parameters $\beta_0 = 0.09$, $\beta_1 = 0.01$, $\beta_2 = 0.01$ and $\lambda = 5.0$.

C++ program:

```
double delta=0.98;
double pi=0.5;
double r=0.1;
term_structure_class* initial=new term_structure_class_flat(r);
vector<double> times; times.push_back(5.0);
vector<double> cflows; cflows.push_back(100);
double K=80;
double time_to_maturity=3;
cout << " Flat term structure " << endl;
cout << " c= " << price_european_call_option_on_bond_using_ho_lee(initial,delta, pi, times,cflows,K,time_to_maturity);
cout << endl;
delete (initial);
double beta0=0.09; double beta1=0.01; double beta2=0.01; double lambda=5.0;
initial = new term_structure_class_nelson_siegel(beta0,beta1,beta2,lambda);
cout << " Nelson Siegel term structure " << endl;
cout << " c= " << price_european_call_option_on_bond_using_ho_lee(initial,delta, pi, times,cflows,K,time_to_maturity);
cout << endl;
```

Output from C++ program:

```
Flat term structure
c= 6.4857
Nelson Siegel term structure
c= 6.35411
```

Exercise 25.1.

What changes do you need to make to **C++ Code 25.4** to price an American call option instead of an European call?

Exercise 25.2.

If you for example want to price a bond, you need to keep track of intermediate payments of coupon. Implement such a procedure. To see that it is correct us it to price a (straight) bond and then compare the value calculated in the tree with the value using the current term structure. Then modify the code to build in a callable bond feature. What additional informaiton must be kept track of?

Exercise 25.3.

In the Ho and Lee (1986) paper there is a typo in equation (22). Can you see what it is?

25.5 References

The discussion in this chapter follows closely the original paper Ho and Lee (1986)

Chapter 26

Term Structure Derivatives

Contents

26.1 Vasicek bond option pricing

If the term structure model is Vasicek's model there is a solution for the price of an option on a zero coupon bond, due to Jamshidan (1989).

Under Vacisek's model the process for the short rate is assumed to follow.

$$dr = a(b - r)dt + \sigma dZ$$

where a, b and σ are constants. We have seen earlier how to calculate the discount factor in this case. We now want to consider an European Call option in this setting.

Let $P(t, s)$ be the time t price of a zero coupon bond with a payment of \$1 at time s (the discount factor). The price at time t of a European call option maturing at time T on on a discount bond maturing at time s is (See Jamshidan (1989) and Hull (1993))

$$P(t, s)N(h) - XP(t, T)N(h - \sigma_P)$$

where

$$h = \frac{1}{\sigma_P} \ln \frac{P(t, s)}{P(t, T)X} + \frac{1}{2}\sigma_P$$

$$\sigma_P = v(t, T)B(T, s)$$

$$B(t, T) = \frac{1 - e^{-a(T-t)}}{a}$$

$$v(t, T)^2 = \frac{\sigma^2(1 - e^{-a(T-t)})}{2a}$$

In the case of $a = 0$,

$$v(t, T) = \sigma\sqrt{T - t}$$

$$\sigma_P = \sigma(s - T)\sqrt{T - t}$$

Example

Parameters: $a = 0.1$, $b = 0.1$, $\sigma = 0.02$, $r = 0.05$, $X = 0.9$. Price a Vacicek call on a zero.

```
#include "normdist.h"
#include "fin_recipes.h"
#include <cmath>
using namespace std;

double bond_option_price_call_zero_vasicek(const double& X, // exercise price
                                           const double& r, // current interest rate
                                           const double& option_time_to_maturity,
                                           const double& bond_time_to_maturity,
                                           const double& a, // parameters
                                           const double& b,
                                           const double& sigma){
    double T_t = option_time_to_maturity;
    double s_t = bond_time_to_maturity;
    double T_s = s_t-T_t;
    double v_t_T;
    double sigma_P;
    if (a==0.0) {
        v_t_T = sigma * sqrt ( T_t ) ;
        sigma_P = sigma*T_s*sqrt(T_t);
    }
    else {
        v_t_T = sqrt (sigma*sigma*(1-exp(-2*a*T_t))/(2*a));
        double B_T_s = (1-exp(-a*T_s))/a;
        sigma_P = v_t_T*B_T_s;
    };
    double h = (1.0/sigma_P) * log (term_structure_discount_factor_vasicek(s_t,r,a,b,sigma)/
        (term_structure_discount_factor_vasicek(T_t,r,a,b,sigma)*X) )
        + sigma_P/2.0;
    double c = term_structure_discount_factor_vasicek(s_t,r,a,b,sigma)*N(h)
        -X*term_structure_discount_factor_vasicek(T_t,r,a,b,sigma)*N(h-sigma_P);
    return c;
};
```

C++ Code 26.1: Bond option pricing using the Vasicek model

C++ program:

```
    double a = 0.1; double b = 0.1; double sigma = 0.02; double r = 0.05; double X=0.9;
    cout << " Vasicek call option price "
        << bond_option_price_call_zero_vasicek(X,r,1,5,a,b,sigma) << endl;
```

Output from C++ program:

```
 Vasicek call option price 0.000226833
```

Chapter 27

Date (and time) revisited - the BOOST libraries

In the introduction we used the construction of a *date* class as an example of building a nontrivial (and useful) class. However, even if we follow the hints in the exercises, and use the operating system utilities for dealing with dates, at some point such a "homespun" date class needs to be replaced with something more reliable.

The date class that is most likely to spring to the programmers' minds these days is one provided by the Boost C++ library. Let us therefore use this opportunity to introduce the Boost libraries, and show some examples of date usage. The Boost libraries are collections of C++ libraries, many of which may at some point become part of the C++ standard.[1] One of the most useful of the Boost libraries are complete libraries for date (`gregorian`) and date and time (`date_time`.

We are not going to go into detail here, we limit ourselves to showing some examples of usage:

[1] Ten Boost libraries actually became part of the latest C++ standard (C++-11). If you have a recent C++ implementation, the Boost libraries are typically included.

C++ program:

```
#include "boost/date_time/gregorian/gregorian.hpp"
#include "boost/date_time/posix_time/posix_time.hpp"
#include <iostream>
using namespace std;
using namespace boost::gregorian;
using namespace boost::posix_time;

void ex_date_time(){
    boost::gregorian::date d(2014,4,1); // creating a date
    cout << " date: " << d << endl;

    boost::posix_time::ptime dt;
    stringstream ss("2014-mar-01 10:01:01.01");
    ss >> dt;
    cout << " date time just created: " << dt << endl;

    ptime now = second_clock::local_time(); //get the current time from the clock
    date today = now.date(); //Get the date part out of the time
    cout << " today: " << today << endl;
    date tommorrow = today + days(1); // add one date
    ptime tommorrow_start(tommorrow); //midnight
    time_duration remaining = tommorrow_start - now;
    std::cout << "Time left till midnight: " << to_simple_string(remaining) << std::endl;
}
```

Output from C++ program:

```
 date: 2014-Apr-01
 date time just created: 2014-Mar-01 10:01:01.010000
 today: 2024-Apr-09
Time left till midnight: 00:35:26
```

27.1 References

The Boost homepage is at `http://www.boost.org/`

Appendix A

Normal Distribution approximations.

Contents

We will in general not go into detail about more standard numerical problems not connected to finance, there are a number of well known sources for such, but we show the example of calculations involving the normal distribution.

A.1 The normal distribution function

The nurmal distribution function

$$n(x) = e^{-\frac{x^2}{2}}$$

is calculated as

```
#include <cmath>  // c library of math functions
using namespace std; // which is part of the standard namespace

// most C compilers define PI, but just in case it doesn't
#ifndef PI
#define PI 3.14159265358979323846264338
#endif

double n(const double& z) { // normal distribution function
    return (1.0/sqrt(2.0*PI))*exp(-0.5*z*z);
};
```

C++ Code A.1: The normal distribution function

A.2 The cumulative normal distribution

The solution of a large number of option pricing formulas are written in terms of the cumulative normal distribution. For a random variable x the cumulative probability is the probability that the outcome is lower than a given value z. To calculate the probability that a normally distubuted random variable with mean 0 and unit variance is less than z, $N(z)$, one have to evaluate the integral

$$\text{Prob}(x \leq z) = N(z) = \int_{-\infty}^{z} n(x)dx = \int_{-\infty}^{z} e^{-\frac{x^2}{2}}dx$$

There is no explicit closed form solution for calculation of this integral, but a large number of well known approximations exists. Abramowiz and Stegun (1964) is a good source for these approximations. The following is probably the most used such approximation, it being pretty accurate and relatively fast. The arguments to the function are assumed normalized to a (0,1) distribution.

```
#include <cmath>  // math functions.
using namespace std;

double N(const double& z) {
    if (z >  6.0) { return 1.0; }; // this guards against overflow
    if (z < -6.0) { return 0.0; };

    double b1 =  0.31938153;
    double b2 = -0.356563782;
    double b3 =  1.781477937;
    double b4 = -1.821255978;
    double b5 =  1.330274429;
    double p  =  0.2316419;
    double c2 =  0.3989423;

    double a=fabs(z);
    double t = 1.0/(1.0+a*p);
    double b = c2*exp((-z)*(z/2.0));
    double n = ((((b5*t+b4)*t+b3)*t+b2)*t+b1)*t;
    n = 1.0-b*n;
    if ( z < 0.0 ) n = 1.0 - n;
    return n;
};
```

C++ Code A.2: The cumulative normal

A.3 Multivariate normal

The normal distribution is also defined for several random variables. We then characterise the *vector* of random variables

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

A probability statement about this vector is a joint statement about all elements of the vector.

A.4 Calculating cumulative bivariate normal probabilities

The most used multivariate normal calculation is the bivariate case, where we let x and y be bivariate normally distributed, each with mean 0 and variance 1, and assume the two variables have correlation of ρ. By the definition of correlation $\rho \in [-1, 1]$. The cumulative probability distribution

$$\begin{aligned} P(x < a, y < b) &= N(a, b, \rho) \\ &= \int_{-\infty}^{a} \int_{-\infty}^{b} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2}\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right) dxdy \end{aligned}$$

There are several approximations to this integral. We pick one such, discussed in (Hull, 1993, Ch 10), shown in **C++ Code A.3**.

If one has more than two correlated variables, the calculation of cumulative probabilites is a nontrivial problem. One common method involves Monte Carlo estimation of the definite integral. We will consider this, but then it is necessary to first consider simulation of random normal variables.

Example

Calculate N(0) and N(0,0,0)

C++ program:

```
    cout << " N(0) = " << N(0) << endl;
    cout << " N(0,0,0) = " << N(0,0,0) << endl;
```

Output from C++ program:

```
N(0) = 0.5
N(0,0,0) = 0.25
```

```
#include <cmath> // include the standard library mathematics functions
using namespace std; // which are in the standard namespace
#include <iostream>

double N(const double&); // define the univariate cumulative normal distribution as a separate function

#ifndef PI
const double PI=3.14159265358979323846264 3;
#endif

inline double f(const double& x, const double& y,
                const double& aprime, const double& bprime,
                const double& rho) {
    double r = aprime*(2*x−aprime) + bprime*(2*y−bprime) + 2*rho*(x−aprime)*(y−bprime);
    return exp(r);
};

inline double sgn(const double& x) { // sign function
    if (x>=0.0) return 1.0;
    return −1.0;
};

double N(const double& a, const double& b, const double& rho) {
    if ( (a<=0.0) && (b<=0.0) && (rho<=0.0) ) {
        double aprime = a/sqrt(2.0*(1.0−rho*rho));
        double bprime = b/sqrt(2.0*(1.0−rho*rho));
        double A[4]={0.3253030, 0.4211071, 0.1334425, 0.006374323};
        double B[4]={0.1337764, 0.6243247, 1.3425378, 2.2626645 };
        double sum = 0;
        for (int i=0;i<4;i++) {
            for (int j=0; j<4; j++) {
                sum += A[i]*A[j]* f(B[i],B[j],aprime,bprime,rho);
            };
        };
        sum = sum * ( sqrt(1.0−rho*rho)/PI);
        return sum;
    }
    else  if ( a * b * rho <= 0.0 ) {
        if ( ( a<=0.0 ) && ( b>=0.0 ) && (rho>=0.0) ) {
            return N(a) − N(a, −b, −rho);
        }
        else if ( (a>=0.0) && (b<=0.0) && (rho>=0.0) ) {
            return N(b) − N(−a, b, −rho);
        }
        else if ( (a>=0.0) && (b>=0.0) && (rho<=0.0) ) {
            return N(a) + N(b) − 1.0 + N(−a, −b, rho);
        };
    }
    else  if ( a * b * rho >= 0.0 ) {
        double denum = sqrt(a*a − 2*rho*a*b + b*b);
        double rho1 = ((rho * a − b) * sgn(a))/denum;
        double rho2 = ((rho * b − a) * sgn(b))/denum;
        double delta=(1.0−sgn(a)*sgn(b))/4.0;
        return N(a,0.0,rho1) + N(b,0.0,rho2) − delta;
    }
    else {
        cout << " unknown " << endl;
    }
    return −99.9; // should never get here, alternatively throw exception
};
```

C++ Code A.3: Approximation to the cumulative bivariate normal

A.5 Simulating random normal numbers

Generation of random numbers is a large topic and is treated at length in such sources as Knuth (1997). The generated numbers can never be truly random, only "pseudo"-random, they will be generated according to some reproducible algorithm and after a (large) number of random number generations the sequence will start repeating itself. The number of iterations before replication starts is a measure of the quality of a random number generator. For anybody requiring high-quality random number generators the `rand()` function provided by the standard C++ library should be avoided, but for not getting into involved discussion of random number generations we use this function as a basis for the generation of uniformly distributed numbers in the interval $[0, 1)$, as shown in **C++ Code A.4**.

```
#include <cstdlib>
using namespace std;

double random_uniform_0_1(void){
    return double(rand())/double(RAND_MAX); // this uses the C library random number generator.
};
```

C++ Code A.4: Pseudorandom numbers from an uniform $[0, 1)$ distribution

Exercise A.1.

Replace the `random_uniform` function here by an alternative of higher quality, by looking into what numerical libraries is available on your computing platform, or by downloading a high quality random number generator from such places as `mathlib` or `statlib`.

These uniformly distributed distributed random variates are used as a basis for the polar method for normal densities discussed in Knuth (1997) and inplemented as shown in **C++ Code A.5**.

```
#include <cmath>
#include <cstdlib>
using namespace std;

double random_uniform_0_1(void);

double random_normal(void){
    double U1, U2, V1, V2;
    double S=2;
    while (S>=1) {
        U1 = random_uniform_0_1();
        U2 = random_uniform_0_1();
        V1 = 2.0*U1-1.0;
        V2 = 2.0*U2-1.0;
        S = pow(V1,2)+pow(V2,2);
    };
    double X1=V1*sqrt((-2.0*log(S))/S);
    return X1;
};
```

C++ Code A.5: Pseudorandom numbers from a normal $(0, 1)$ distribution

Example

1. Generate 5 random uniform numbers (0,1)
2. Generate 5 random N(0,1) numbers.

C++ program:

```
    cout << " 5 random uniform numbers between 0 and 1: " << endl;
    cout << " ";
    for (int i=0;i<5;++i){ cout << " " << random_uniform_0_1(); }; cout << endl;
    cout << " 5 random normal(0,1) numbers: " << endl;
    cout << " ";
    for (int i=0;i<5;++i){ cout << " " << random_normal() ; }; cout << endl;
```

Output from C++ program:

```
 5 random uniform numbers between 0 and 1:
     0.840188 0.394383 0.783099 0.79844 0.911647
 5 random normal(0,1) numbers:
     -1.07224 0.925946 2.70202 1.36918 0.0187313
```

A.6 Cumulative probabilities for general multivariate distributions

When moving beyond the bivariate case calculation of probability integrals become more of an exercise in general numerical integration. A typical tool is Monte Carlo integration, but that is not the only possibility.

A.7 Implementation in C++11

The latest standard of the C++ language, C++11, includes a library for generating random numbers. See (Stroustrup, 1997a, Section 5.6.3).

A.8 References

Tong (1990) discusses the multivariate normal distribution, and is a good reference. For the ultimate source, see Knuth (1997).

Appendix B

C++ concepts

This chapter contains a listing of various C/C++ concepts and some notes on each of them.

`accumulate()` Function accumulating the elements of a sequence.

`bool` Boolean variable, taking on the two values `true` and `false`. For historical reasons one can also use the values zero and one for `false` and `true`.

`class` (C++ keyword).

`const` (qualifyer to variable in C++ function call).

`double` (basic type). A floating point number with high accuracy.

`exp(x)` (C function). Defined in <cmath>. Returns the natural exponent e to the given power x, e^x.

`fabs`

`float` (basic type). A floating point number with limited accuracy.

`for` Loop

`header file`

`if`

Indexation (in vectors and matrices). To access element number `i` in an array `A`, use `A[i-1]`. Well known trap for people coming to `C` from other languages. Present in `C` for historical efficiency reasons. Arrays in `C` were implemented using pointers. Indexing was done by finding the first element of the array, and then adding pointers to find the indexed element. The first element is of course found by adding nothing to the first elment, hence the first element was indexed by zero.

`include`

`inline` (qualifyer to C++ function name). Hint to the optimizer that this function is most efficiently implemented by *inlining* it, or putting the full code of the function into each instance of its calling. Has the side effect of making the function local to the file in which it is defined.

`int` (basic type). An integer with a limited number of significant digits.

`log(x)` (C function). Defined in <cmath>. Calculates the natural logarithm $\ln(x)$ of its argument.

`long` (basic type). An integer that can contain a large number.

`min_element()` (C++ function)

`max_element` (C++ function)

`namespace` (C++ concept)

`return`

`standard namespace` (C++ concept)

`string` (C++ basic type)

`using` (C++ concept)

`vector` (C++ container class). Defined in <`vector`>

`while`

Appendix C

Interfacing to external libraries

Contents

In various places we have used routines from other public domain packages. In this appendix there are some notes about the libraries which are referenced, and some comments about linking to them. These are all routines I have actively used in solving finance problems.

C.1 Boost

Large collection of peer-reviewed code. Particularly useful: the *date_time* library.
Homepage `http://www.boost.org/`

C.2 Matrix (Linear Algebra) utilities

A particularly useful library is one that allows one to write linear algebra in a compact syntax. It does so by defining vectors and matrices as objects, together with operations on the objects, allowing one to write code that looks much like matlab or similar matrix handlers. I have used three matrix libraries in the previous, Newmat, IT++ and Armadillo. Of the three, Armadillo is the most recent, and the one I would recommend, as it seems currently (2014) to be the one that is most actively maintained. I leave it as exercises to the reader to replace the other two matrix libraries with the usage of Armadillo.

C.2.1 Newmat

Homepage: `http://www.robertnz.net`

C.2.2 IT++

IT++ is a large library, from which I use the matrix part.

Homepage: `http://itpp.sourceforge.net/`

C.2.3 Armadillo

Homepage: `http://arma.sourceforge.net/`

C.3 GSL

The Gnu Scientific Library, a large collection of routines for numerical mathematical operations. Let me show one example usage, using integration to evaluate a trivariate normal.

C.3.1 The evaluation of N_3

In the calculation of the American Put approximation of Geske and Johnson (1984), a trivariate normal needs to be evaluated. Following the discussion in the paper, this is evaluated as

$$N_3(h,k,j;\rho_{12},\rho_{13},\rho_{23}) = \int_{-\infty}^{j} n(z) N_2\left(\frac{k-\rho_{23}z}{\sqrt{1-\rho_{23}^2}}, \frac{h-\rho_{13}z}{\sqrt{1-\rho_{13}^2}}, \frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}^2}\sqrt{1-\rho_{23}^2}}\right)$$

This is an univariate integral, and can be evaluated using quadrature.

Homepage: `http://www.gnu.org/software/gsl/`

C.4 NLOPT

A collection of optimizers, various algorithms for nonlinear optimization of multivariate functions, with constraints.

`http://ab-initio.mit.edu/wiki/index.php/NLopt`

C.5 GLPK

The GNU Linear Programming Kit,

Homepage `http://www.gnu.org/software/glpk/`

```
#include <cmath>
#include <iostream>
using namespace std;

#include "gsl/gsl_integration.h"
#include "normdist.h"

struct n3_parms {double h; double k; double rho12; double rho13; double rho23; };

extern "C"{
double f3(double z, void *p){
    struct n3_parms* parms = (struct n3_parms*)p;
    double h = (parms->h);
    double k = (parms->k);;
    double rho12 = (parms->rho12);
    double rho13 = (parms->rho13);
    double rho23 = (parms->rho23);
    double f = n(z);
    f*=N( (k-rho23*z)/sqrt(1.0-rho23*rho23),
          (h-rho13*z)/(sqrt(1.0-rho13*rho13)),
          (rho12-rho13*rho23)/(sqrt(1.0-rho13*rho13)*sqrt(1.0-rho23*rho23)));
    return f;
};
};

double N3(const double& h, const double& k, const double& j,
          const double& rho12, const double& rho13, const double& rho23){
    struct n3_parms parms = { h, k, rho12, rho13, rho23};
    size_t n=1000;
    gsl_integration_workspace* w = gsl_integration_workspace_alloc(n);
    gsl_function F;
    F.function = &f3;
    F.params=&parms;
    double result, error;
    gsl_integration_qags(&F,-20.0,j,1e-7,1e-7,n,w,&result,&error);
    return result;
};
```

C++ Code C.1: Approximating $N_3()$ using the method of Geske and Johnson (1984)

Appendix D

Summarizing routine names

In many of the algorithms use is made of *other* routines. To simplify the matter all routines are summarised in one header file, `fin_recipes.h`. This appendix shows this file.

```
// file: fin_recipes.h
// author: Bernt Arne Oedegaard
//   defines all routines in the financial numerical recipes "book"

#ifndef _FIN_RECIPES_H_
#define _FIN_RECIPES_H_

#include <vector>
#include <cmath>
using namespace std;

//////// present value ////////////////////////////////////
// discrete coumpounding
///////////////////////////////
// discrete, annual compounding

double cash_flow_pv_discrete ( const vector<double>& cflow_times, const vector<double>& cflow_amounts,
                               const double& r);
double cash_flow_irr_discrete(const vector<double>& cflow_times, const vector<double>& cflow_amounts);
bool cash_flow_unique_irr(const vector<double>& cflow_times, const vector<double>& cflow_amounts);
double bonds_price_discrete(const vector<double>& cashflow_times, const vector<double>& cashflows,
                            const double& r);
double bonds_yield_to_maturity_discrete(const vector<double>& times,
                                        const vector<double>& amounts,
                                        const double& bondprice);
double bonds_duration_discrete(const vector<double>& times,
                               const vector<double>& cashflows,
                               const double& r);
double bonds_duration_macaulay_discrete(const vector<double>& cashflow_times,
                                        const vector<double>& cashflows,
                                        const double& bond_price);
double bonds_duration_modified_discrete (const vector<double>& times,
                                         const vector<double>& amounts,
                                         const double& bond_price);
double bonds_convexity_discrete(const vector<double>& cflow_times,
                                const vector<double>& cflow_amounts,
                                const double& r);
////////////////////////////////
// continous compounding.
double cash_flow_pv(const vector<double>& cflow_times,const vector<double>& cflow_amounts,const double& r);
double cash_flow_irr(const vector<double>& cflow_times, const vector<double>& cflow_amounts);
double bonds_price(const vector<double>& cashflow_times, const vector<double>& cashflows, const double& r);
double bonds_price(const vector<double>& coupon_times, const vector<double>& coupon_amounts,
                   const vector<double>& principal_times, const vector<double>& principal_amounts,
                   const double& r);
```

```
double bonds_duration(const vector<double>& cashflow_times, const vector<double>& cashflows,
                      const double& r);
double bonds_yield_to_maturity(const vector<double>& cashflow_times,const vector<double>& cashflow_amounts,
                               const double& bondprice);
double bonds_duration_macaulay(const vector<double>& cashflow_times, const vector<double>& cashflows,
                               const double& bond_price);
double bonds_convexity(const vector<double>& cashflow_times, const vector<double>& cashflow_amounts,
                       const double& y );

/// term structure basics

double term_structure_yield_from_discount_factor(const double& dfact, const double& t);
double term_structure_discount_factor_from_yield(const double& r, const double& t);
double term_structure_forward_rate_from_discount_factors(const double& d_t1, const double& d_t2,
                                                         const double& time);
double term_structure_forward_rate_from_yields(const double& r_t1, const double& r_t2,
                                               const double& t1, const double& t2);
double term_structure_yield_linearly_interpolated(const double& time,
                                                  const vector<double>& obs_times,
                                                  const vector<double>& obs_yields);
// a term structure class

class term_structure_class {
public:
    virtual ~term_structure_class();
    virtual double r(const double& t) const; // short rate, yield on zero coupon bond
    virtual double d(const double& t) const; // discount_factor
    virtual double f(const double& t1, const double& t2) const; // forward_rate
};

class term_structure_class_flat : public term_structure_class {
private:
    double R_;                          // interest rate
public:
    term_structure_class_flat(const double& r);
    virtual ~term_structure_class_flat();
    virtual double r(const double& t) const;
    void set_int_rate(const double& r);
};

class term_structure_class_interpolated : public term_structure_class {
private:
    vector<double> times_; // use to keep a list of yields
    vector<double> yields_;
    void clear();
public:
    term_structure_class_interpolated();
    term_structure_class_interpolated(const vector<double>& times, const vector<double>& yields);
    virtual ~term_structure_class_interpolated();
    term_structure_class_interpolated(const term_structure_class_interpolated&);
    term_structure_class_interpolated operator= (const term_structure_class_interpolated&);

    int no_observations() const { return int(times_.size()); };
    virtual double r(const double& T) const;
    void set_interpolated_observations(vector<double>& times, vector<double>& yields);
};

// using the term structure classes

double bonds_price(const vector<double>& cashflow_times,
                   const vector<double>& cashflows,
                   const term_structure_class& d);

double bonds_duration(const vector<double>& cashflow_times,
                      const vector<double>& cashflow_amounts,
                      const term_structure_class& d);
```

```
double bonds_convexity(const vector<double>& cashflow_times,
                       const vector<double>& cashflow_amounts,
                       const term_structure_class& d);

//// Futures pricing
double futures_price(const double& S, const double& r, const double& time_to_maturity);

/// Binomial option pricing

    // one periode binomial
double option_price_call_european_binomial_single_period( const double& S, const double& K, const double& r,
                                                          const double& u, const double& d);
// multiple periode binomial

double option_price_call_european_binomial_multi_period_given_ud( const double& S, const double& K, const double& r,
                                                                  const double& u, const double& d, const int& no_periods);

// multiple periode binomial
vector< vector<double> > binomial_tree(const double& S0, const double& u,const double& d,
                                       const int& no_steps);

/// Black Scholes formula ////////////////////////////////////////

double option_price_call_black_scholes(const double& S, const double& K, const double& r,
                                       const double& sigma, const double& time) ;
double option_price_put_black_scholes (const double& S, const double& K, const double& r,
                                       const double& sigma, const double& time) ;

double
option_price_implied_volatility_call_black_scholes_newton( const double& S, const double& K,
                                                           const double& r, const  double& time,
                                                           const double& option_price);
double
option_price_implied_volatility_put_black_scholes_newton( const double& S, const double& K,
                                                          const double& r, const  double& time,
                                                          const double& option_price);

double option_price_implied_volatility_call_black_scholes_bisections( const double& S, const double& K,
                                                                      const double& r, const double& time,
                                                                      const double& option_price);
double option_price_implied_volatility_put_black_scholes_bisections( const double& S, const double& K,
                                                                     const double& r, const double& time,
                                                                     const double& option_price);
double option_price_delta_call_black_scholes(const double& S, const double& K, const double& r,
                                             const double& sigma, const double& time);
double option_price_delta_put_black_scholes (const double& S, const double& K, const double& r,
                                             const double& sigma, const double& time);
void option_price_partials_call_black_scholes(const double& S, const double& K, const double& r,
                                              const double& sigma, const double& time,
                                              double& Delta, double& Gamma, double& Theta,
                                              double& Vega, double& Rho);
void option_price_partials_put_black_scholes(const double& S, const double& K, const double& r,
                                             const double& sigma, const double& time,
                                             double& Delta, double& Gamma, double& Theta,
                                             double& Vega, double& Rho);

/// warrant price
double warrant_price_adjusted_black_scholes(const double& S, const double& K,
                                            const double& r, const double& sigma,
                                            const double& time,
                                            const double& no_warrants_outstanding,
                                            const double& no_shares_outstanding);

double warrant_price_adjusted_black_scholes(const double& S, const double& K,
                                            const double& r, const double& q,
                                            const double& sigma, const double& time,
                                            const double& no_warrants_outstanding,
```

```
                                        const double& no_shares_outstanding);

/// Extensions of the Black Scholes model ///////////////

double option_price_european_call_payout(const double& S, const double& K, const double& r,
                                         const double& b, const double& sigma, const double& time);
double option_price_european_put_payout (const double& S, const double& K, const double& r,
                                         const double& b, const double& sigma, const double& time);
double option_price_european_call_dividends(const double& S, const double& K, const double& r,
                                            const double& sigma, const double& time,
                                            const vector<double>& dividend_times,
                                            const vector<double>& dividend_amounts );
double option_price_european_put_dividends( const double& S, const double& K, const double& r,
                                            const double& sigma,const double& time,
                                            const vector<double>& dividend_times,
                                            const vector<double>& dividend_amounts);
double option_price_american_call_one_dividend(const double& S, const double& K, const double& r,
                                               const double& sigma,
                                               const double& tau, const double& D1, const double& tau1);
double futures_option_price_call_european_black(const double& F, const double& K, const double& r,
                                                const double& sigma, const double& time);
double futures_option_price_put_european_black(const double& F, const double& K, const double& r,
                                               const double& sigma, const double& time);
double currency_option_price_call_european(const double& S, const double& K, const double& r,
                                           const double& r_f, const double& sigma, const double& time);
double currency_option_price_put_european(const double& S, const double& K, const double& r,
                                          const double& r_f, const double& sigma, const double& time);
double option_price_american_perpetual_call(const double& S, const double& K, const double& r,
                                            const double& q, const double& sigma);
double option_price_american_perpetual_put(const double& S, const double& K, const double& r,
                                           const double& q, const double& sigma);

// binomial option approximation ////////////////

double option_price_call_european_binomial(const double& S, const double& K, const double& r,
                                           const double& sigma, const double& t, const int& steps);
double option_price_put_european_binomial (const double& S, const double& K, const double& r,
                                           const double& sigma, const double& t, const int& steps);
double option_price_call_american_binomial(const double& S, const double& K, const double& r,
                                           const double& sigma, const double& t, const int& steps);
double option_price_put_american_binomial (const double& S, const double& K, const double& r,
                                           const double& sigma, const double& t, const int& steps);
double option_price_call_american_binomial(const double& S, const double& K,
                                           const double& r, const double& y,
                                           const double& sigma, const double& t, const int& steps);
double option_price_put_american_binomial (const double& S, const double& K, const double& r,
                                           const double& y, const double& sigma,
                                           const double& t, const int& steps);

double option_price_call_american_discrete_dividends_binomial( const double& S, const double& K,
                                                               const double& r,
                                                               const double& sigma, const double& t,
                                                               const int& steps,
                                                               const vector<double>& dividend_times,
                                                               const vector<double>& dividend_amounts);

double option_price_put_american_discrete_dividends_binomial(const double& S, const double& K,
                                                             const double& r,
                                                             const double& sigma, const double& t,
                                                             const int& steps,
                                                             const vector<double>& dividend_times,
                                                             const vector<double>& dividend_amounts);

double option_price_call_american_proportional_dividends_binomial(const double& S, const double& K,
                                                                  const double& r, const double& sigma,
                                                                  const double& time, const int& no_steps,
                                                                  const vector<double>& dividend_times,
```

```
                                                    const vector<double>& dividend_yields);

double option_price_put_american_proportional_dividends_binomial( const double& S, const double& K, const double& r,
                                                    const double& sigma, const double& time, const int& no_steps,
                                                    const vector<double>& dividend_times,
                                                    const vector<double>& dividend_yields);

double option_price_delta_american_call_binomial(const double& S, const double& K, const double& r,
                                                 const double& sigma, const double& t, const int& no_steps);
double option_price_delta_american_put_binomial(const double& S, const double& K, const double& r,
                                                const double& sigma, const double& t, const int& no_steps);
void option_price_partials_american_call_binomial(const double& S, const double& K, const double& r,
                                                  const double& sigma, const double& time, const int& no_steps,
                                                  double& delta, double& gamma, double& theta,
                                                  double& vega, double& rho);

void option_price_partials_american_put_binomial(const double& S, const double& K, const double& r,
                                                 const double& sigma, const double& time, const int& no_steps,
                                                 double& delta, double& gamma, double& theta,
                                                 double& vega, double& rho);

double futures_option_price_call_american_binomial(const double& F, const double& K, const double& r, const double& sigma,
                                                   const double& time, const int& no_steps);

double futures_option_price_put_american_binomial( const double& F, const double& K, const double& r, const double& sigma,
                                                   const double& time, const int& no_steps);

double currency_option_price_call_american_binomial( const double& S, const double& K, const double& r, const double& r_f,
                                                     const double& sigma, const double& t, const int& n);

double currency_option_price_put_american_binomial( const double& S, const double& K, const double& r, const double& r_f,
                                                    const double& sigma, const double& t, const int& n);

////////////////////// finite differences ///////////////////

double option_price_call_american_finite_diff_explicit( const double& S, const double& K, const double& r,
                                                        const double& sigma, const double& time,
                                                        const int& no_S_steps, const int& no_t_steps);

double option_price_put_american_finite_diff_explicit( const double& S, const double& K, const double& r,
                                                       const double& sigma, const double& time,
                                                       const int& no_S_steps, const int& no_t_steps);

double option_price_call_european_finite_diff_explicit( const double& S, const double& K, const double& r,
                                                        const double& sigma, const double& time,
                                                        const int& no_S_steps, const int& no_t_steps);

double option_price_put_european_finite_diff_explicit( const double& S, const double& K, const double& r,
                                                       const double& sigma, const double& time,
                                                       const int& no_S_steps, const int& no_t_steps);

double option_price_call_american_finite_diff_implicit( const double& S, const double& K, const double& r,
                                                        const double& sigma, const double& time,
                                                        const int& no_S_steps, const int& no_t_steps);

double option_price_put_american_finite_diff_implicit( const double& S, const double& K, const double& r,
                                                       const double& sigma, const double& time,
                                                       const int& no_S_steps, const int& no_t_steps);

double option_price_call_european_finite_diff_implicit( const double& S, const double& K, const double& r,
                                                        const double& sigma, const double& time,
                                                        const int& no_S_steps, const int& no_t_steps);

double option_price_put_european_finite_diff_implicit( const double& S, const double& K, const double& r,
                                                       const double& sigma, const double& time,
                                                       const int& no_S_steps, const int& no_t_steps);
```

```
//////////////////////// simulated option prices ////////////////////////////////////
// Payoff only function of terminal price
double option_price_call_european_simulated(const double& S, const double& K,
                                            const double& r, const double& sigma,
                                            const double& time_to_maturity, const int& no_sims);
double option_price_put_european_simulated(const double& S, const double& K,
                                           const double& r, const double& sigma,
                                           const double& time_to_maturity, const int& no_sims);
double option_price_delta_call_european_simulated(const double& S, const double& K,
                                                  const double& r, const double& sigma,
                                                  const double& time_to_maturity, const int& no_sims);
double option_price_delta_put_european_simulated(const double& S, const double& K,
                                                 const double& r, const double& sigma,
                                                 const double& time_to_maturity, const int& no_sims);
double simulate_lognormal_random_variable(const double& S, const double& r, const double& sigma,
                                          const double& time);

double
derivative_price_simulate_european_option_generic( const double& S, const double& K,
                                                   const double& r, const double& sigma,
                                                   const double& time,
                                                   double payoff(const double& S, const double& K),
                                                   const int& no_sims);
double
derivative_price_simulate_european_option_generic_with_control_variate(const double& S, const double& K,
                                                                       const double& r,const double& sigma,
                                                                       const double& time,
                                                                       double payoff(const double& S,
                                                                                     const double& K),
                                                                       const int& no_sims);

double
derivative_price_simulate_european_option_generic_with_antithetic_variate(const double& S, const double& K,
                                                                          const double& r,
                                                                          const double& sigma,
                                                                          const double& time,
                                                                          double payoff(const double& S,
                                                                                        const double& K),
                                                                          const int& no_sims);
//////////////////////////////
// payoffs of various options, to be used as function arguments in above simulations
double payoff_call(const double& S, const double& K);
double payoff_put (const double& S, const double& K);
double payoff_cash_or_nothing_call(const double& S, const double& K);
double payoff_asset_or_nothing_call(const double& S, const double& K);

/////////// approximated option prices ////////////////////////
double option_price_american_put_approximated_johnson( const double& S, const double& X, const double& r,
                                                       const double& sigma, const double& time );

double option_price_american_call_approximated_baw(const double& S, const double& K,
                                                   const double& r, const double& b,
                                                   const double& sigma, const double& time);
double option_price_american_put_approximated_baw(const double& S, const double& K,
                                                  const double& r, const double& b,
                                                  const double& sigma, const double& time);

double option_price_american_put_approximated_geske_johnson( const double& S, const double& X,
                                                             const double& r, const double& sigma,
                                                             const double& time );

double option_price_american_call_approximated_bjerksund_stensland( const double& S,
                                                                    const double& X,
                                                                    const double& r,
                                                                    const double& q,
                                                                    const double& sigma,
                                                                    const double& time );
```

```
double option_price_american_put_approximated_bjerksund_stensland( const double& S,
                                                                   const double& X,
                                                                   const double& r,
                                                                   const double& q,
                                                                   const double& sigma,
                                                                   const double& T );

////////////// path dependent and other exotic options ////////////////////////////////

double option_price_call_bermudan_binomial(const double& S, const double& K, const double& r,
                                           const double& q, const double& sigma, const double& time,
                                           const vector<double>& potential_exercise_times,
                                           const int& steps);

double option_price_put_bermudan_binomial( const double& S, const double& K, const double& r,
                                          const double& q, const double& sigma, const double& time,
                                          const vector<double>& potential_exercise_times,
                                          const int& steps);

double option_price_european_lookback_call(const double& S, const double& Smin, const double& r,
                                           const double& q, const double& sigma, const double& time);

double option_price_european_lookback_put(const double& S, const double& Smin, const double& r,
                                          const double& q, const double& sigma, const double& time);


double
option_price_asian_geometric_average_price_call(const double& S, const double& K, const double& r,
                                                const double& q, const double& sigma, const double& time);

vector<double> simulate_lognormally_distributed_sequence(const double& S, const double& r,
                                                         const double& sigma, const double& time, const int& no_steps);
double
derivative_price_simulate_european_option_generic( const double& S, const double& K, const double& r,
                                                   const double& sigma, const double& time,
                                                   double payoff(const vector<double>& S,
                                                                 const double& K),
                                                   const int& no_steps, const int& no_sims);


double
derivative_price_simulate_european_option_generic_with_control_variate(const double& S, const double& K,
                                                                       const double& r, const double& sigma,
                                                                       const double& time,
                                                                       double payoff(const vector<double>& S,
                                                                                     const double& K),
                                                                       const int& nosteps, const int& nosims);


//////////////////////////////
// payoffs of various options, to be used as function arguments in above simulations

double payoff_arithmetric_average_call(const vector<double>& prices, const double& K);
double payoff_geometric_average_call(const vector<double>& prices, const double& K);
double payoff_lookback_call(const vector<double>& prices, const double& unused_variable);
double payoff_lookback_put(const vector<double>& prices, const double& unused_variable);

////////////////////////////////////
// generic binomial trees

double option_price_generic_binomial( const double& S, const double& K,
                                      double generic_payoff(const double& S, const double& K),
                                      const double& r, const double& sigma, const double& t, const int& steps);

double payoff_binary_call(const double& S, const double& K);
double payoff_binary_put(const double& S, const double& K);
```

```
//////////////////////////////////////
// trinomial trees

double option_price_call_american_trinomial( const double& S, const double& K, const double& r, const double& q,
                                             const double& sigma, const double& t, const int& steps) ;

double option_price_put_american_trinomial( const double& S, const double& K, const double& r, const double& q,
                                            const double& sigma, const double& t, const int& steps) ;


////////////////// alternative stochastic processes ////////////////

double option_price_call_merton_jump_diffusion( const double& S, const double& K, const double& r,
                                                const double& sigma, const double& time_to_maturity,
                                                const double& lambda, const double& kappa, const double& delta);

double heston_call_option_price(const double& S, const double& K, const double& r, const double& v, const double& tau,
                                const double& rho, const double& kappa, const double& lambda, const double& theta,
                                const double& sigma);

// fixed income derivatives, GBM assumption on bond price

double bond_option_price_call_zero_black_scholes(const double& B, const double& K, const double& r,
                                                 const double& sigma, const double& time);
double bond_option_price_put_zero_black_scholes(const double& B, const double& K, const double& r,
                                                const double& sigma, const double& time);
double bond_option_price_call_coupon_bond_black_scholes(const double& B, const double& K, const double& r,
                                                        const double& sigma, const double& time,
                                                        const vector<double> coupon_times,
                                                        const vector<double> coupon_amounts);
double bond_option_price_put_coupon_bond_black_scholes(const double& B, const double& K, const double& r,
                                                       const double& sigma, const double& time,
                                                       const vector<double> coupon_times,
                                                       const vector<double> coupon_amounts);
double bond_option_price_call_american_binomial( const double& B, const double& K, const double& r,
                                                 const double& sigma, const double& t, const int& steps);
double bond_option_price_put_american_binomial( const double& B, const double& K, const double& r,
                                                const double& sigma, const double& t, const int& steps);

//////////////////////////////////////////////////////////////////////////////
// term structure models
/// formulas for calculation

double term_structure_yield_nelson_siegel(const double& t,
                                          const double& beta0, const double& beta1, const double& beta2,
                                          const double& lambda );

double term_structure_yield_svensson(const double& t,
                                     const double& beta0, const double& beta1, const double& beta2, const double& beta3,
                                     const double& tau1, const double& tau2 );

double term_structure_discount_factor_cubic_spline(const double& t,
                                                   const double& b1,
                                                   const double& c1,
                                                   const double& d1,
                                                   const vector<double>& f,
                                                   const vector<double>& knots);

double term_structure_discount_factor_cir(const double& t, const double& r,
                                          const double& kappa,
                                          const double& lambda,
                                          const double& theta,
                                          const double& sigma);

double term_structure_discount_factor_vasicek(const double& time,
                                              const double& r,
```

```
                                        const double& a,const double& b, const double& sigma);

/// defining classes wrapping the above term structure approximations

class term_structure_class_nelson_siegel : public term_structure_class {
private:
    double beta0_, beta1_, beta2_, lambda_;
public:
    term_structure_class_nelson_siegel(const double& beta0, const double& beta1,
                                       const double& beta2, const double& lambda);
    virtual double r(const double& t) const;
};

class term_structure_class_svensson:public term_structure_class {
private:
    double beta0_, beta1_, beta2_, beta3_, tau1_, tau2_;
public:
    term_structure_class_svensson(const double& beta0, const double& beta1, const double& beta2, const double& beta3,
                                  const double& tau1, const double& tau2);
    virtual double r(const double& T) const;
};

class term_structure_class_cubic_spline : public term_structure_class {
private:
    double b_; double c_; double d_; vector<double> f_; vector<double> knots_;
public:
    term_structure_class_cubic_spline(const double& b, const double& c, const double& d,
                                      const vector<double>& f, const vector<double> & knots);
    virtual double d(const double& t) const; // discount factor
};

class term_structure_class_cir : public term_structure_class {
private:
    double r_;  double kappa_; double lambda_; double theta_; double sigma_;
public:
    term_structure_class_cir(const double& r, const double& k, const double& l,
                             const double& th,const double& sigma);
    virtual double d(const double& t) const; // discount factor
};

class term_structure_class_vasicek : public term_structure_class {
private:
    double r_; double a_; double b_; double sigma_;
public:
    term_structure_class_vasicek(const double& r, const double& a, const double& b, const double& sigma);
    virtual double d(const double& T) const;
};

//////////////////
/// binomial term structure models
/// bond option, rendlemann bartter (binomial)

double
bond_option_price_call_zero_american_rendleman_bartter(const double& K, const double& option_maturity,
                                                       const double& S, const double& M,
                                                       const double& interest,
                                                       const double& bond_maturity,
                                                       const double& maturity_payment,
                                                       const int& no_steps);

vector< vector<double> > interest_rate_trees_gbm_build(const double& r0,
                                                       const double& u,
                                                       const double& d,
                                                       const int& n);

double interest_rate_trees_gbm_value_of_cashflows(const vector<double>& cflow,
                                                  const vector< vector<double> >& r_tree,
```

```
                                                const double& q);

double interest_rate_trees_gbm_value_of_callable_bond(const vector<double>& cflows,
                                                      const vector< vector<double> >& r_tree,
                                                      const double& q,
                                                      const int& first_call_time,
                                                      const double& call_price);

double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial,
                                                       const double& delta,
                                                       const double& pi,
                                                       const vector<double>& underlying_bond_cflow_times,
                                                       const vector<double>& underlying_bond_cflows,
                                                       const double& K,
                                                       const double& option_time_to_maturity);
////////////////////////////////////////////
// ho and lee modelling

class term_structure_class_ho_lee : public term_structure_class {
private:
    term_structure_class* initial_term_;
    int n_;
    int i_;
    double delta_;
    double pi_;
public:
    term_structure_class_ho_lee(term_structure_class* fitted_term,
                                const int & n,
                                const int & i,
                                const double& lambda,
                                const double& pi);
    double d(const double& T) const;
};

vector< vector<term_structure_class_ho_lee> >
term_structure_ho_lee_build_term_structure_tree(term_structure_class* initial,
                                                const int& no_steps,
                                                const double& delta,
                                                const double& pi);

double price_european_call_option_on_bond_using_ho_lee(term_structure_class* initial,
                                                       const double& delta,
                                                       const double& pi,
                                                       const vector<double>& underlying_bond_cflow_times,
                                                       const vector<double>& underlying_bond_cflows,
                                                       const double& K,
                                                       const double& option_time_to_maturity);


////////////////////////////////
// term structure derivatives, analytical solutions

double bond_option_price_call_zero_vasicek(const double& X, const double& r,
                                           const double& option_time_to_maturity,
                                           const double& bond_time_to_maturity,
                                           const double& a, const double& b, const double& sigma);

double bond_option_price_put_zero_vasicek(const double& X, const double& r,
                                          const double& option_time_to_maturity,
                                          const double& bond_time_to_maturity,
                                          const double& a, const double& b, const double& sigma);
#endif
```

Appendix E

Installation

The routines discussed in the book are available for download.

E.1 Source availability

The algorithms are available from my home page as a ZIP file containing the source code. These have been tested with the latest version of the GNU C++ compiler. As the algorithms in places uses code from the Standard Template Library, other compilers may not be able to compile all the files directly. If your compiler complains about missing header files you may want to check if the STL header files have different names on your system. The algorithm files comply with the current ANSI standard for C++ libraries. If the compiler is more than a couple of years old, it will not have STL. Alternatively, the GNU compiler gcc is available for free on the internet, for most current operating systems.

List of C++ Codes

List of Matlab Codes

Appendix F

Acknowledgements.

After this paper was put up on the net, I've had quite a few emails about them. Some of them has pointed out bugs and other inaccuracies.

Among the ones I want to say thanks to for making improving suggestions and pointing out bugs are

Ariel Almedal
Andrei Bejenari
Steve Bellantoni
Jean-Paul Beveraggi
Lars Gregori
Daniel Herlemont
Lorenzo Isella
Jens Larsson
Garrick Lau
Steven Leadbeater
Michael L Locher
Lotti Luca, Milano, Italy
Tuan Nguyen
Michael R Wayne

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