

NLA Homework2

Group O

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Question 1

(i) Let

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix}$$

Compute B^2, B^3, B^4 .

(ii) Let $C = I + B$, i.e.,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & 1 & 1 \end{pmatrix}$$

Compute C^m , where $m \geq 2$ is a positive integer.

Solutions:

(i)

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 13 & -2 & 0 & 0 \end{pmatrix}$$

$$B^3 = B^2 \cdot B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 13 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{pmatrix}$$

$$B^4 = B^3 \cdot B = 0$$

(ii)

$$\begin{aligned} C^m &= (I + B)^m \\ &= \sum_{k=0}^m \binom{m}{k} I^{m-k} \cdot B^k \\ &= \sum_{k=0}^m \binom{m}{k} B^k \\ &= B^0 + \binom{m}{1} \cdot B + \binom{m}{2} \cdot B^2 + \binom{m}{3} \cdot B^3 \\ &= I + m \cdot B + \frac{m(m-1)}{2} \cdot B^2 + \frac{m(m-1)(m-2)}{6} \cdot B^3 \end{aligned}$$

$$m \cdot B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4m & 0 & 0 & 0 \\ m & -2m & 0 & 0 \\ -m & 3m & m & 0 \end{pmatrix}$$

$$\frac{m(m-1)}{2} \cdot B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4m(m-1) & 0 & 0 & 0 \\ \frac{13}{2}m(m-1) & -m(m-1) & 0 & 0 \end{pmatrix}$$

$$\frac{m(m-1)(m-2)}{6} \cdot B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{4}{3}m(m-1)(m-2) & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore C^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4m & 1 & 0 & 0 \\ a & -2m & 1 & 0 \\ c & b & m & 1 \end{pmatrix}$$

where $a = -4m^2 + 5m$, $b = -m^2 + 4m$, $c = -\frac{4}{3}m + \frac{21}{2}m^2 - \frac{23}{2}m$

Question 2

Let u be an $n \times n$ upper triangular matrix with entries equal to 0 on the main diagonal, i.e., with $U(i, i) = 0$ for $i = 1 : n$.

(i) Show that $U^n = 0$;

(ii) Compute $(I + U)^m$ in terms of U, U^2, \dots, U^{n-1} , where $m \geq n$ is a positive integer. $(I + U)^m = I + \sum_{k=1}^{n-1} \binom{m}{k} U^k$

Solutions:

(i) $U = \text{col}(u_k)_{k=1:n}$

$$u_k(j) = 0 \quad k \leq j \leq n$$

$$U^2 = \text{col}(U \cdot u_k)_{k=1:n}$$

$$U \cdot u_k = \sum_{j=1}^n u_k(j) \cdot u_j = \sum_{j=1}^{k-1} u_k(j) \cdot u_j$$

Let $k-1 \leq i \leq n$

$$U \cdot u_k(i) = \sum_{j=1}^{k-1} u_k(j) \cdot u_j(i) = 0 \quad \text{since } u_j(i) = 0 \quad j \leq i \leq n$$

Denote the k -th column of U^α as $u_k^{(\alpha)}$, therefore,

$$u_k^{(2)} = U \cdot u_k$$

$$u_k^{(2)}(i) = U \cdot u_k(i) = 0 \quad k-1 \leq i \leq n$$

Repeat this process, we have

$$u_k^{(k)}(i) = 0 \quad 1 \leq i \leq n$$

For $n \geq k$, $u_k^{(n)} = 0$

$$U^n = \text{col}(u_k^{(n)})_{k=1:n} = 0$$

(ii)

$$\begin{aligned} (I + U)^m &= \sum_{k=1}^m \binom{m}{k} I^{m-k} U^k \\ &= \sum_{k=1}^m \binom{m}{k} U^k \\ &= \sum_{k=1}^{n-1} \binom{m}{k} U^k \quad m \geq n \end{aligned}$$

Question 3

Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -2 & 3 & 0 \\ 2 & 2 & -3 & 4 \end{pmatrix}; \quad U_1 = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & -0.25 \end{pmatrix}$$

(i) Show that $LU = L_1U_1$.

(ii) Explain why this does not contradict the uniqueness of the LU decomposition of a matrix.

Solutions:

(i)

By matrix multiplication,

$$LU = L_1U_1 = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

(ii) The uniqueness means when L and L_1 are both lower triangular matrices with diagonal entries equal to 1, then $L = L_1$, $U = U_1$. While the entries of L_1 on the main diagonal are not all equal to 1, this example doesn't contradict the uniqueness.

Question 4

Let L_1 and L_2 be nonsingular lower triangular matrices and let U_1 and U_2 be nonsingular upper triangular matrices. If $L_1U_1 = L_2U_2$, show that there exists a nonsingular diagonal matrix D such that

$$L_2 = L_1D^{-1} \quad \text{and} \quad U_2 = DU_1.$$

Solution:

To show $L_2 = L_1D^{-1}$, $U_2 = DU_1$ since all matrices are nonsingular, it is equivalent to show $D = L_2^{-1}L_1 = U_2U_1^{-1}$. This is to show that matrices $L_2^{-1}L_1$ and $U_2U_1^{-1}$ are the same, and they're diagonal matrix.

$$L_1U_1 = L_2U_2$$

Multiply L_2^{-1} on the left and U_1^{-1} on the right: $L_2^{-1}(L_1U_1)U_1^{-1} = L_2^{-1}(L_2U_2)U_1^{-1}$

Since $U_1U_1^{-1} = L_2^{-1}L_2 = I$

$$L_2^{-1}L_1 = U_2U_1^{-1}$$

Since the inverse of a lower/upper triangular matrix is still lower/upper triangular, and the product of two lower/upper triangular matrices is also lower/upper triangular

$\therefore L_2L_1$ is lower triangular, $U_2U_1^{-1}$ is upper triangular. They must be diagonal matrices. Q.E.D.

Question 5

(i) Write the pseudocode for the forward substitution corresponding to a lower triangular banded matrix of band 2, i.e., for solving $Ly = b$ where b is an $n \times 1$ vector and L is an $n \times n$ lower triangular matrix such that

$$L(j, k) = 0, \forall 1 \leq j, k \leq n \quad \text{with} \quad j - k > 2$$

The input for the pseudocode are the matrix L and the vector b ; the output is the vector y .

What is the operation count for this?

(ii) Write the pseudocode for the backward substitution corresponding to an upper triangular banded matrix of band 2, i.e., for solving $Ux = b$ where b is an $n \times 1$ vector and U is an $n \times n$ upper triangular matrix such that

$$U(j, k) = 0, \forall 1 \leq j, k \leq n \quad \text{with} \quad k - j > 2$$

The input for the pseudocode are the matrix U and the vector b ; the output is the vector y .

What is the operation count for this?

Solutions:

(i)

$$\begin{aligned}
 y(1) &= \frac{b(1)}{L(1,1)} \\
 y(2) &= \frac{b(2) - L(2,1)y(1)}{L(2,2)} \\
 &\vdots \\
 y(j) &= \frac{b(j) - L(j,j-2)y(j-2) - L(j,j-1)y(j-1)}{L(j,j)} \quad j = 3 : n
 \end{aligned}$$

Pseudocode:

```

y(1) = b(1) / L(1,1)
y(2) = (b(2) - L(2,1)*y(1)) / L(2,2)
for j = 3:n
    y(j) = (b(j) - L(j,j-2)*y(j-2) - L(j,j-1)*y(j-1)) / L(j,j)
end

```

Operation counts:

$$1 + 3 + 5(n - 2) = 5n - 6$$

(ii)

$$\begin{aligned}
 x(n) &= \frac{b(n)}{U(n,n)} \\
 x(n-1) &= \frac{b(n-1) - U(n-1,n)x(n)}{U(n-1,n-1)} \\
 &\vdots \\
 x(j) &= \frac{b(j) - U(j,j+1)x(j+1) - U(j,j+2)x(j+2)}{U(j,j)} \quad j = (n-2) : 1
 \end{aligned}$$

Pseudocode:

```

x(n) = b(n) / U(n,n)
x(n-1) = (b(n-1) - U(n-1,n)*x(n)) / U(n-1,n-1)
for j = (n-2):1
    x(j) = (b(j) - U(j,j+1)*x(j+1) - U(j,j+2)*x(j+2)) / U(j,j)
end

```

Operation counts:

$$1 + 3 + 5(n - 2) = 5n - 6$$

Question 6

The LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

is given by $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Solve $Ax = b$, where

$$b = \begin{pmatrix} 3 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

Solution:

Solve $Ax = b$

$$LUx = b$$

Let $y = Ux$, we first solve $Ly = b$

$$\begin{aligned} y(1) &= \frac{b(1)}{L(1,1)} = 3 \\ y(2) &= \frac{b(2) - L(2,1) \cdot y(1)}{L(2,2)} = 5 \\ y(3) &= \frac{b(3) - L(3,1) \cdot y(1) - L(3,2) \cdot y(2)}{L(3,3)} = -3 \\ y(4) &= \frac{b(4) - \sum_{k=1}^3 L(4,k)y(k)}{L(4,4)} = -14 \end{aligned}$$

Then solve $Ux = y$

$$\begin{aligned} x(4) &= \frac{y(4)}{U(4,4)} = 14 \\ x(3) &= \frac{y(3) - (-1) \cdot (+14)}{1} = 11 \\ x(2) &= \frac{y(2) - 1 \cdot 11}{-1} = 6 \\ x(1) &= \frac{y(1) - (-1) \cdot 6 - 1 \cdot 14}{2} = -\frac{5}{2} \end{aligned}$$

$$\therefore x = \begin{bmatrix} -5/2 \\ 6 \\ 11 \\ 14 \end{bmatrix}$$

Question 7

(i) Let A be an $n \times n$ matrix and let L be a nonsingular lower triangular of size n . Show that if LA is a lower triangular matrix, then A is a lower triangular matrix. Show that if AL is a lower triangular matrix, then A is a lower triangular matrix.

(ii) Let A be an $n \times n$ matrix and let U be a nonsingular upper triangular of size n . Show that if UA is an upper triangular matrix, then A is an upper triangular matrix. Show that if AU is an upper triangular matrix, then A is an upper triangular matrix.

Solutions:

(i) Let $L_1 = LA$, since L is nonsingular

$$A = L^{-1}L_1$$

By Lemma 1.15 and Lemma 1.17, the inverse of a lower triangular matrix is lower triangular, and the product of two lower triangular matrices is lower triangular

$\therefore A$ is lower triangular

$$\text{Let } L_2 = AL \Rightarrow A = L_2L^{-1}$$

With similar reasoning above, A is lower triangular

(ii) Let $U_1 = UA$, since U is nonsingular

$$A = U^{-1}U_1$$

Then inverse of an upper triangular matrix is upper triangular, and the product of two upper triangular matrices is upper triangular

$\therefore A$ is upper triangular.

$$\text{Let } U_2 = AU \Rightarrow A = U_2U^{-1}$$

With similar reasoning above, A is upper triangular

Question 8

The values of the following coupon bonds with face value \$100 are given:

Bond Type	Coupon Rate	Bond Price
5 months semiannual	0	\$98.75
11 months semiannual	4%	\$102.00
17 months semiannual	6%	\$103.50
23 months semiannual	4%	\$105.50

Find the 5 months, 11 months, 17 months, and 23 months discount factors.

Solution:

Denote the 5 months, 11 mos, 17 mos. and 23 mos discount factors as d_1, d_2, d_3, d_4 , respectively.

$$\begin{aligned}98.75 &= 100d_1 \\102 &= 2d_1 + 102d_2 \\103.5 &= 3d_1 + 3d_2 + 103d_3 \\105.5 &= 2d_1 + 2d_2 + 2d_3 + 102d_4\end{aligned}$$

$$L = \begin{pmatrix} 100 & 0 & 0 & 0 \\ 2 & 102 & 0 & 0 \\ 3 & 3 & 103 & 0 \\ 2 & 2 & 2 & 102 \end{pmatrix}, \quad b = \begin{pmatrix} 98.75 \\ 102 \\ 103.5 \\ 105.5 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

By solving $Ld = b$, we get $d = \text{forward_subset}(L, b) = \begin{pmatrix} 0.9875 \\ 0.980637 \\ 0.94753 \\ 0.977144 \end{pmatrix}$

That is $d_1 = 0.9875, d_2 = 0.980637, d_3 = 0.94753, d_4 = 0.977144$

```
In [97]: def forward_subset(L, b):
        """
        Forward Substitution
        Input:
        - L: nonsingular lower triangular matrix of size n
        - b: column vector of size n
        Output:
        - x: solution to Lx=b
        """
        n = len(b)
        x = [0 for _ in range(n)]
        x[0] = b[0] / L[0][0]
        for j in range(1, n):
            ssum = 0
            for k in range(0, j):
                ssum += L[j][k] * x[k]
            x[j] = (b[j] - ssum) / L[j][j]

        for i in range(n):
            x[i] = round(x[i], 6)

        return x

In [98]: L = [[100, 0, 0, 0], [2, 102, 0, 0], [3, 3, 103, 0], [2, 2, 2, 102]]
        b = [98.75, 102, 103.5, 105.5]
        forward_subset(L, b)

Out[98]: [0.9875, 0.980637, 0.94753, 0.977144]
```

Question 9

The values of the following coupon bonds with face value \$100 are given:

Bond Type	Coupon Rate	Bond Price
10 months semiannual	2%	\$100.60
16 months semiannual	4%	\$103.30
16 months annual	5%	\$107.30
22 months semiannual	5%	\$110.30

- (i) List the cash flows and cash flow dates for each bond.
- (ii) Find the 4 months, 10 months, 16 months, and 22 months discount factors.

Solutions:

(i)

Type	Cash Flow & Date
10 mos semi-annual	\$1 in 4 months \$101 in 10 mos
16 mos semi-annual	\$2 in 4 mos \$2 in 10 mos \$102 in 16 mos
16 mos annual	\$5 in 4 mos \$105 in 16 mos
22 mos semi-annual	\$2.5 in 4 mos \$2.5 in 10 mos \$2.5 in 16 mos \$102.5 in 22 mos

(ii) Denote the 4 mos, 10 mos, 16 mos and 22 mos discount factors as d_1, d_2, d_3, d_4 respectively.

$$\begin{aligned} 100.6 &= d_1 + 101d_2 \\ 103.3 &= 2d_1 + 2d_2 + 102d_3 \\ 107.3 &= 5d_1 + 105d_3 \\ 110.3 &= 2.5d_1 + 2.5d_2 + 2.5d_3 + 102.5d_4 \end{aligned}$$

Solve $Ad=b$, where

$$A = \begin{pmatrix} 1 & 101 & 0 & 0 \\ 2 & 2 & 102 & 0 \\ 5 & 0 & 105 & 0 \\ 2.5 & 2.5 & 2.5 & 102.5 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad b = \begin{pmatrix} 100.6 \\ 103.3 \\ 107.3 \\ 110.3 \end{pmatrix}$$

We get

$$d = \begin{pmatrix} 1.017178 \\ 0.985969 \\ 0.973468 \\ 1.003497 \end{pmatrix}$$

That is $d_1 = 1.017178, d_2 = 0.985969, d_3 = 0.973468, d_4 = 1.003497$

```
In [99]: def lu_no_pivoting(A):
        """
        LU decomposition without pivoting
        Input:
        - A: nonsingular matrix of size n
        Output:
        - L, U
        """
        n = len(A)
        # create empty L, U
        L = [[0 for _ in range(n)] for _ in range(n)]
        U = [[0 for _ in range(n)] for _ in range(n)]
        for i in range(0, n-1):
            for l in range(i, n):
                U[i][l] = A[i][l]
                L[l][i] = A[l][i] / U[i][i]
            for j in range(i+1, n):
                for k in range(i+1, n):
                    A[j][k] -= L[j][i] * U[i][k]
        L[n-1][n-1] = 1
        U[n-1][n-1] = A[n-1][n-1]

        for j in range(n):
            for k in range(n):
                L[j][k] = round(L[j][k], 6)
                U[j][k] = round(U[j][k], 6)

        return L, U

def backward_subset(U, b):
    """
    Backward substitution
    Input:
    - U: nonsingular upper triangular matrix of size n
    - b: column vector of size n
    Output:
    - x: solution to Ux=b
    """
    n = len(b)
```

```

x = [0 for _ in range(n)]
x[n-1] = b[n-1] / U[n-1][n-1]
for j in range(n-2, -1, -1):
    ssum = 0
    for k in range(j+1, n):
        ssum += U[j][k] * x[k]
    x[j] = (b[j] - ssum) / U[j][j]

for i in range(n):
    x[i] = round(x[i], 6)

return x

```

```

In [100... def linear_solve_LU_no_pivoting(A, b):
    """
    Input:
    - A: nonsingular square matrix of size n with LU decom
    - b: column vector of size n
    Output:
    - x: solution to Ax=b
    """
    L, U = lu_no_pivoting(A)
    y = forward_subset(L, b) # solve Ly=b
    x = backward_subset(U, y) # solve Ux=y
    return x

```

```

In [101... A = [
    [1, 101, 0, 0],
    [2, 2, 102, 0],
    [5, 0, 105, 0],
    [2.5, 2.5, 2.5, 102.5]
]
b = [100.6, 103.3, 107.3, 110.3]
x = linear_solve_LU_no_pivoting(A, b)
x

```

```

Out[101... [1.017178, 0.985969, 0.973468, 1.003497]

```