Numerical Linear Algebra for Financial Engineering

The Pre-MFE Program at Baruch College

Homework 3

Assigned: February 24; Due: March 3

(1) The LU decomposition with row pivoting of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

is given by $PA = LU$, where

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -0.5 & 0.25 & 1 & 0 \\ 0.5 & 0.25 & 0 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 4 & -1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & -0.75 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}.$$

(i) Solve $Ax = b$, where

$$b = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

(ii) Find A^{-1}, the inverse matrix of A.

(2) The following discount factors are obtained by fitting market data:

Date	Discount Factor
2 months	0.9980
5 months	0.9935
11 months	0.9820
15 months	0.9775

The overnight rate is 1%.

(i) What is the linear system that has to be solved for the cubic spline interpolation of the zero rate curve?

(ii) Use cubic spline interpolation to find a zero rate curve for all times less than 15 months matching the discount factors above.

(iii) Find the value of a 13 months quarterly bond with 2.5% coupon rate.

Note: A quarterly coupon bond with face value \$100, coupon rate C, and maturity T pays the holder of the bond a coupon payment equal to $\frac{C}{4} \cdot 100$ every three months, except at maturity. The final payment at maturity T is equal to the face value of the bond plus one coupon payment, i.e., $100 + \frac{C}{4}100$.

(3) The values of the following coupon bonds with face value \$100 are given:

Bond Type	Coupon Rate	Bond Price
10 months semiannual	3%	\$101.30
16 months semiannual	4%	\$102.95
22 months annual	6%	\$107.35
22 months semiannual	5%	\$105.45

(i) List the cash flows and cash flow dates for each bond.

(ii) Identify the matrix and the right hand side vector corresponding to the linear system whose solution are the 4 months, 10 months, 16 months, and 22 months discount factors.

(iii) Find the 4 months, 10 months, 16 months, and 22 months discount factors.

(4) Consider three assets with the following expected rates of return, standard deviations of their rates of return, and correlations of their rates of return:

$$\begin{aligned} \mu_1 &= 0.1; & \sigma_1 &= 0.15; & \rho_{1,2} &= -0.25; \\ \mu_2 &= 0.15; & \sigma_2 &= 0.3; & \rho_{2,3} &= 0.2; \\ \mu_3 &= 0.2; & \sigma_3 &= 0.35; & \rho_{1,3} &= 0.3. \end{aligned}$$

(i) Find the covariance matrix M of the rates of return of the three assets.

(ii) A minimum variance portfolio with 16% expected rate of return can be set up by investing a percentage w_i of the total value of the portfolio in asset i, with $i = 1:3$, where w_i can be found by solving the following linear system:

$$\begin{pmatrix} 2M & \mathbf{1} & \mu \\ \mathbf{1}^t & 0 & 0 \\ \mu^t & 0 & 0 \end{pmatrix} \begin{pmatrix} w \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \mu_P \end{pmatrix}, \tag{1}$$

where $\mu_P = 0.16$,

$$\mu = \begin{pmatrix} 0.1 \\ 0.15 \\ 0.2 \end{pmatrix} \quad \text{and} \quad \mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The matrices from the LU decomposition with row pivoting of the matrix on the left hand side of (1) are

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.0225 & 1 & 0 & 0 & 0 \\ 0.0315 & 0.051852 & 1 & 0 & 0 \\ 0.045 & -0.333333 & 0.038067 & 1 & 0 \\ 0.1 & 0.246914 & 0.400056 & -0.482738 & 1 \end{pmatrix};$$

$$U = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0.2025 & 0.0645 & 1 & 0.15 \\ 0 & 0 & 0.210555 & 0.948148 & 0.192222 \\ 0 & 0 & 0 & 1.297240 & 0.142683 \\ 0 & 0 & 0 & 0 & -0.045059 \end{pmatrix}.$$

Find the weights of each asset in this minimum variance portfolio.

(iii) Compute the standard deviation of the returns of the following portfolios with 16% expected rate of return:

- 30% invested in asset 1, 20% invested in asset 2, 50% invested in asset 3;
- 50% invested in asset 1, 70% invested in asset 3, and short an amount equal to 20% of the value of the portfolio of asset 2.

Eigenvalues and eigenvectors

(1) Let A and B be square matrices of the same size. Show that if v is an eigenvector of both A and B, then v is also an eigenvector of the matrix

$$M = c_1 A + c_2 B,$$

where c_1 and c_2 are constants. What is the eigenvalue of M corresponding to the eigenvector v?

(2) Let A be a square matrix such that $A^2 = A$. Show that any eigenvalue of A is either 0 or 1.

Note: A matrix A with the property that $A^2 = A$ is called an idempotent matrix.

(3) Let A be a square matrix with the property that there exists a positive integer n such that $A^n = 0$. Show that any eigenvalue of A must be equal to 0.

Note: A matrix A with the property that $A^n = 0$ for a positive integer n is called a nilpotent matrix.

(4) Let v be a column vector of size n, and let $A = vv^t$ be an $n \times n$ matrix.

(i) How many non-zero eigenvalues does the matrix A have?

(ii) What are the eigenvalues of A, and what are the corresponding eigenvectors?

(5) Find the eigenvalues and the eigenvectors of the $n \times n$ matrix

$$A = \begin{pmatrix} d & 1 & \dots & 1 \\ 1 & d & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & d \end{pmatrix},$$

where $d \in \mathbb{R}$ is a constant.

(6) Let λ and v be an eigenvalue and the corresponding eigenvector of the square matrix A of size n. Let S be a nonsingular matrix of size n. Show that λ is also an eigenvalue of the matrix $S^{-1}AS$. What is the corresponding eigenvector?

(7) Let A be a square matrix with real entries. If $\lambda = a + ib$ is a complex eigenvalue of A (i.e., with $b \neq 0$), show that $\overline{\lambda} = a - ib$, the complex conjugate of λ, is also an eigenvalue of A.

(8) Let $A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$.

(i) Compute the eigenvalues and the eigenvectors of the matrix A.

(ii) What is the diagonal form of A?

(ii) Compute A^{12}.

(9) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, and let

$$P_A(t) = t^2 - (a+d)t + (ad - bc)$$

be the characteristic polynomial associated to A.

Show that $P_A(A) = 0$, i.e., show that

$$A^2 - (a+d)A + (ad - bc)I = 0.$$

Note: This is the 2×2 case of the Cayley-Hamilton theorem which states that $P_A(A) = 0$ for any square matrix A.

(10) Let A be an $n \times n$ matrix given by

$$\begin{aligned} A(i,i) &= 2, \ \forall\, i = 1 : n; \\ A(i,i-1) &= 1, \ \forall\, i = 2 : n; \\ A(j,k) &= 0, \ \text{otherwise}. \end{aligned}$$

Find the eigenvalues and the eigenvectors of A.

(11) (i) Show that the eigenvalues of the matrix

$$A = \begin{pmatrix} -16 & 6 & -6 & 0 \\ -30 & 11 & -12 & 0 \\ 15 & -6 & 5 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

are -1, with multiplicity 3, and 2, with multiplicity 1.

Show that $\begin{pmatrix} 0 \\ 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are three linearly independent eigenvectors corresponding to the eigenvalue -1, and show that there exists only one linearly independent eigenvector of the matrix A corresponding to the eigenvalue 2, e.g., $\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$.

(ii) Show that the eigenvalues of the matrix

$$B = \begin{pmatrix} 10 & -20 & -32 & -26 \\ 18 & -41 & -68 & -54 \\ -14 & 19 & 26 & 23 \\ 7 & 1 & 9 & 4 \end{pmatrix}$$

are -1, with multiplicity 3, and 2, with multiplicity 1.

Show that there exists only one linearly independent eigenvector of the matrix B corresponding to the eigenvalue -1, e.g., $\begin{pmatrix} 0 \\ -1 \\ -1 \\ 2 \end{pmatrix}$, and show that $\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 2.