Essentials of Stochastic Processes

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Note: Due to the way the solutions manual was produced (including the entire text but only printing the solutions) the page numbering is somewhat strange.

1.12 Exercises

Understanding the definitions

1.1. A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \ldots$ that are 0 or 1 with probability 1/2 each. For $n \geq 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?

Ans. No. We first argue this intuitively: when $X_n = 1$ the last two results may be $0,1$ or $1,0$. In the first case we may jump only to 2 or 1, while in the second case we may only jump to 0 or 1. Thus it is not enough to know just current state. To get a formal contradiction we note that $X_1 = 2$, $X_2 = 1$ implies that $Y_0 = Y_1 = 1$, $Y_2 = 0$ so

$$P(X_3 = 2 | X_2 = 1, X_1 = 2) = 0 < P(X_3 = 2 | X_2 = 1)$$

5ballE **1.2.** Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

Ans.

	0	**1**	**2**	**3**	**4**	**5**
0	0	1	0	0	0	0
1	1/25	8/25	16/25	0	0	0
2	0	4/25	12/25	9/25	0	0
3	0	0	9/25	12/25	4/25	0
4	0	0	0	16/25	8/25	1/25
5	0	0	0	0	1	0

dicemod6 **1.3.** We repeated roll two four sided dice with numbers 1, 2, 3, and 4 on them. Let Y_k be the sum on the kth roll, $S_n = Y_1 + \cdots + Y_n$ be the total of the first n rolls, and $X_n = S_n \pmod 6$. Find the transition probability for X_n.

Ans.

	0	**1**	**2**	**3**	**4**	**5**
0	3/16	2/16	2/16	2/16	3/16	4/16
1	4/16	3/16	2/16	2/16	2/16	3/16
2	3/16	4/16	3/16	2/16	2/16	2/16
3	2/16	3/16	4/16	3/16	2/16	2/16
4	2/16	2/16	3/16	4/16	3/16	2/16
5	2/16	2/16	2/16	3/16	4/16	3/16

1.4. The 1990 census showed that 36% of the households in the District of Columbia were homeowners while the remainder were renters. During the next decade 6% of the homeowners became renters and 12% of the renters became homeowners. What percentage were homeowners in 2000? in 2010?

Ans. 0.4152, 0.4604

1.5. Consider a gambler's ruin chain with $N = 4$. That is, if $1 \leq i \leq 3$, $p(i, i+1) = 0.4$, and $p(i, i-1) = 0.6$, but the endpoints are absorbing states: $p(0,0) = 1$ and $p(4,4) = 1$ Compute $p^3(1,4)$ and $p^3(1,0)$.

Ans. (a) to go from 1 to 4 in three steps we must go 1,2,3,4 so $p^3(1,4) = (.4)^3 = .064$. (b) to go from 1 to 0 in three steps we may go 1,2,1,0 or 1,0,0,0 so $p^3(1,0) = (.4)(.6)^2 + .6 = .744$

1.6. A taxicab driver moves between the airport A and two hotels B and C according to the following rules. If he is at the airport, he will be at one of the two hotels next with equal probability. If at a hotel then he returns to the airport with probability 3/4 and goes to the other hotel with probability 1/4. (a) Find the transition matrix for the chain. (b) Suppose the driver begins at the airport at time 0. Find the probability for each of his three possible locations at time 2 and the probability he is at hotel B at time 3.

Ans.

(a)	**A**	**B**	**C**
A	0	1/2	1/2
B	3/4	0	1/4
C	3/4	1/4	0

(b) At time 2, A has probability 3/4, while B and C have probability 1/8 each. The probability of B at time 3 is then $(3/4)(1/2) + (1/8)(0) + (1/8)(1/4) = 13/32$.

2stagerain

1.7. Suppose that the probability it rains today is 0.3 if neither of the last two days was rainy, but 0.6 if at least one of the last two days was rainy. Let the weather on day n, W_n, be R for rain, or S for sun. W_n is not a Markov chain, but the weather for the last two days $X_n = (W_{n-1}, W_n)$ is a Markov chain with four states $\{RR, RS, SR, SS\}$. (a) Compute its transition probability. (b) Compute the two-step transition probability. (c) What is the probability it will rain on Wednesday given that it did not rain on Sunday or Monday.

Ans.

(a)	**RR**	**RS**	**SR**	**SS**
RR	.6	.4	0	0
RS	0	0	.6	.4
SR	.6	.4	0	0
SS	0	0	.3	.7

(b)	**RR**	**RS**	**SR**	**SS**
RR	.36	.24	.24	.16
RS	.36	.24	.12	.28
SR	.36	.24	.24	.16
SS	.18	.12	.21	.49

(c) $p^2(SS, RR) + p^2(SS, SR) = .18 + .21 = .39$.

1.8. Consider the following transition matrices. Identify the transient and recurrent states, and the irreducible closed sets in the Markov chains. Give reasons for your answers.

(a)	**1**	**2**	**3**	**4**	**5**
1	.4	.3	.3	0	0
2	0	.5	0	.5	0
3	.5	0	.5	0	0
4	0	.5	0	.5	0
5	0	.3	0	.3	.4

(b)	**1**	**2**	**3**	**4**	**5**	**6**
1	.1	0	0	.4	.5	0
2	.1	.2	.2	0	.5	0
3	0	.1	.3	0	0	.6
4	.1	0	0	.9	0	0
5	0	0	0	.4	0	.6
6	0	0	0	0	.5	.5

$$\begin{array}{cccccc}
(c) & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\
\mathbf{1} & 0 & 0 & 0 & 0 & 1 \\
\mathbf{2} & 0 & .2 & 0 & .8 & 0 \\
\mathbf{3} & .1 & .2 & .3 & .4 & 0 \\
\mathbf{4} & 0 & .6 & 0 & .4 & 0 \\
\mathbf{5} & .3 & 0 & 0 & 0 & .7
\end{array}
\qquad
\begin{array}{ccccccc}
(d) & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} \\
\mathbf{1} & .8 & 0 & 0 & .2 & 0 & 0 \\
\mathbf{2} & 0 & .5 & 0 & 0 & .5 & 0 \\
\mathbf{3} & 0 & 0 & .3 & .4 & .3 & 0 \\
\mathbf{4} & .1 & 0 & 0 & .9 & 0 & 0 \\
\mathbf{5} & 0 & .2 & 0 & 0 & .8 & 0 \\
\mathbf{6} & .7 & 0 & 0 & .3 & 0 & 0
\end{array}$$

Ans. (a) $1 \to 2$ but $2 \not\to 1$ so 1 is transient. $3 \to 2$ but $2 \not\to 3$ so 3 is transient. $5 \to 4$ but $4 \not\to 5$ so 5 is transient. $\{2,4\}$ is an irreducible closed set so all these states are recurrent.
(b) $3 \to 6$ but $6 \not\to 3$ so 3 is transient. $2 \to 1$ but $1 \not\to 2$ so 1 is transient. $\{1,4,5,6\}$ is an irreducible closed set so all these states are recurrent.
(c) $\{1,5\}$ and $\{2,4\}$ are irreducible closed sets so all of these states are recurrent. $3 \to 1$ but $1 \not\to 3$ so 3 is transient.
(d) $\{1,4\}$ and $\{2,5\}$ are irreducible closed sets so all of these states are recurrent. $3 \to 2$ but $2 \not\to 3$ so 3 is transient. $6 \to 1$ but $1 \not\to 6$ so 6 is transient.

1.9. Find the stationary distributions for the Markov chains with transition matrices:

$$\begin{array}{cccc}
(a) & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
\mathbf{1} & .5 & .4 & .1 \\
\mathbf{2} & .2 & .5 & .3 \\
\mathbf{3} & .1 & .3 & .6
\end{array}
\qquad
\begin{array}{cccc}
(b) & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
\mathbf{1} & .5 & .4 & .1 \\
\mathbf{2} & .3 & .4 & .3 \\
\mathbf{3} & .2 & .2 & .6
\end{array}
\qquad
\begin{array}{cccc}
(c) & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
\mathbf{1} & .6 & .4 & 0 \\
\mathbf{2} & .2 & .4 & .2 \\
\mathbf{3} & 0 & .2 & .8
\end{array}$$

Ans. (a) The third row of

$$\begin{pmatrix} .5 & .4 & .1 \\ .2 & .5 & .3 \\ .1 & .3 & .6 \end{pmatrix}^{-1}$$

is 11/47, 19/47, 17/47.
(b) The matrix is doubly stochastic so $\pi(i) = 1/3$, $i = 1, 2, 3$.
(c) This is a birth and death chain so $.4\pi(1) = .2\pi(2)$ and $.4\pi(2) = .2\pi(3)$. Taking $\pi(1) = c$, $\pi(2) = 2c$, $\pi(3) = 4c$ and $c = 1/7$

1.10. Find the stationary distributions for the Markov chains on $\{1, 2, 3, 4\}$ with transition matrices:

$$(a)\begin{pmatrix} .7 & 0 & .3 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .5 & 0 & .5 \\ 0 & .4 & 0 & .6 \end{pmatrix}
\qquad
(b)\begin{pmatrix} .7 & .3 & 0 & 0 \\ .2 & .5 & .3 & 0 \\ .0 & .3 & .6 & .1 \\ 0 & 0 & .2 & .8 \end{pmatrix}
\qquad
(c)\begin{pmatrix} .7 & 0 & .3 & 0 \\ .2 & .5 & .3 & 0 \\ .1 & .2 & .4 & .3 \\ 0 & .4 & 0 & .6 \end{pmatrix}$$

Ans. (a) The fourth row of

$$\begin{pmatrix} -.3 & 0 & .3 & 1 \\ .6 & -1 & .4 & 1 \\ 0 & .5 & -1 & 1 \\ 0 & .4 & 0 & 1 \end{pmatrix}^{-1}$$

is 8/21, 4/21, 4/21, 5/21.
(b) This is a birth and death chain so $.3\pi(1) = .2\pi(2)$, $.3\pi(2) = .3\pi(3)$, and $.1\pi(3) = .2\pi(4)$. Taking $\pi(1) = c$, $\pi(2) = 3c/2$, $\pi(3) = 3c/2$, $\pi(4) = 3c/4$ and $c = 4/19$, making the stationary distribution 4/19, 6/19, 6/19, 3/19.

(c) The matrix is doubly stochastic so $\pi(i) = 1/4$, $i = 1, 2, 3, 4$.

1.11. Find the stationary distributions for the chains in exercises (a) 1.2, (b) 1.3, and (c) 1.7.

Ans. (a) The sixth row of

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 1 \\ 1/25 & -17/25 & 16/25 & 0 & 0 & 1 \\ 0 & 4/25 & -13/25 & 9/25 & 0 & 1 \\ 0 & 0 & 9/25 & -13/25 & 4/25 & 1 \\ 0 & 0 & 0 & 16/25 & -17/25 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}^{-1}$$

is 1/252, 25/252, 100/252, 100/252, 25/252, 1/252. In Exercise 1.46 we will see that

$$\pi(i) = \binom{5}{i}\binom{5}{5-i} \Big/ \binom{10}{5}$$

(b) This chain is doubly stochastic so $\pi(i) = 1/6$ for $i = 0, 1, 2, 3, 4, 5$
(c) The fourth row of

$$\begin{pmatrix} -.4 & .4 & 0 & 1 \\ 0 & -1 & .6 & 1 \\ .6 & .4 & -1 & 1 \\ 0 & 0 & .3 & 1 \end{pmatrix}^{-1}$$

is 9/29, 6/29, 6/29, 8/29.

norev **1.12.** (a) Find the stationary distribution for the transition probability

	1	**2**	**3**	**4**
1	0	2/3	0	1/3
2	1/3	0	2/3	0
3	0	1/6	0	5/6
4	2/5	0	3/5	0

and show that it does not satisfy the detailed balance condition (1.11).
(b) Consider

	1	**2**	**3**	**4**
1	0	a	0	$1-a$
2	$1-b$	0	b	0
3	0	$1-c$	0	c
4	d	0	$1-d$	0

and show that there is a stationary distribution satisfying (1.11) if

$$0 < abcd = (1-a)(1-b)(1-c)(1-d).$$

Ans. (a) The fourth row of

$$\begin{pmatrix} -1 & 2/3 & 0 & 1 \\ 1/3 & -1 & 2/3 & 1 \\ 0 & 1/6 & -1 & 1 \\ 2/5 & 0 & 3/5 & 1 \end{pmatrix}^{-1}$$

is 35/186, 33/186, 58/186, 60/186. To satisfy (1.11) we must have $(2/3)\pi(1) = 1/3)\pi(2)$.
(b) If (1.11) holds

$$\begin{aligned} \pi(2) &= \pi(1)a/(1-b) & \pi(3) &= \pi(1)ab/(1-b)(1-c) \\ \pi(4) &= \pi(1)abc/(1-b)(1-c)(1-d) & \pi(1) &= abcd/(1-b)(1-c)(1-d)(1-a) \end{aligned}$$

1.13. Consider the Markov chain with transition matrix:

	1	**2**	**3**	**4**
1	0	0	0.1	0.9
2	0	0	0.6	0.4
3	0.8	0.2	0	0
4	0.4	0.6	0	0

(a) Compute p^2. (b) Find the stationary distributions of p and all of the stationary distributions of p^2. (c) Find the limit of $p^{2n}(x,x)$ as $n \to \infty$.

Ans.

(a)	**1**	**2**	**3**	**4**
1	.44	.56	0	0
2	.64	.36	0	0
3	0	0	.2	.8
4	0	0	.4	.6

(b) The fourth row of

$$\begin{pmatrix} -1 & 0 & 0.1 & 1 \\ 0 & -1 & 0.6 & 1 \\ 0.8 & 0.2 & -1 & 1 \\ 0.4 & 0.6 & 0 & 1 \end{pmatrix}^{-1}$$

is 8/30, 7/30, 5/30, 10/30. p^2 has two irreducible closed sets. By the formula for the stationary distribution of the two state chain $\pi_1 = (8/15, 7/15, 0, 0)$ and $\pi_2 = (0, 0, 1/3, 2/3)$ are stationary distributions. Since $\pi p = \pi$ is linear, if $\theta \in [0,1]$ then $\theta\pi_1 + (1-\theta)\pi_2$ is a stationary distribution.
(c) 8/15,7/15,1/3/2/3.

1.14. Do the following Markov chains converge to equilibrium?

(a)	**1**	**2**	**3**	**4**
1	0	0	1	0
2	0	0	.5	.5
3	.3	.7	0	0
4	1	0	0	0

(b)	**1**	**2**	**3**	**4**
1	0	1	.0	0
2	0	0	0	1
3	1	0	0	0
4	1/3	0	2/3	0

(c)	**1**	**2**	**3**	**4**	**5**	**6**
1	0	.5	.5	0	0	0
2	0	0	0	1	0	0
3	0	0	0	.4	0	.6
4	1	0	0	0	0	0
5	0	1	0	0	0	0
6	.2	0	0	0	.8	0

Ans. (a) No. The chain moves from $\{1,2\}$ to $\{3,4\}$ and then back, so all states have period 2.
(b) Yes. The chain is irreducible. Starting from 4 you can return to it in 3 or 4 steps so it and all the other states have period 1.
(c) No. The chain moves from $\{1,5\}$ to $\{2,3\}$ to $\{4,6\}$ and then back to $\{1,5\}$ so all states have period 3.

1.15. Find $\lim_{n\to\infty} p^n(i,j)$ for

$$p = \begin{array}{cccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{1} & 1 & 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 2/3 & 0 & 1/3 & 0 \\ \mathbf{3} & 1/8 & 1/4 & 5/8 & 0 & 0 \\ \mathbf{4} & 0 & 1/6 & 0 & 5/6 & 0 \\ \mathbf{5} & 1/3 & 0 & 1/3 & 0 & 1/3 \end{array}$$

You are supposed to do this and the next problem by solving equations. However you can check your answers by using your calculator to find FRAC(p^{100}).

Ans. 1 is an absorbing state. $\{2,4\}$ is a closed irreducible set with stationary distribution 1/3, 2/3. When we leave 3 we go to 1 with probability 1/3 and enter $\{2,4\}$ with probability 2/3. When we leave 5 we go to 1 or 3 with probability 1/2 each, so we get absorbed at 1 with probability $1/2 + (1/2)(1/3) = 2/3$ and in $\{2,4\}$ with probability 1/3. Thus the limit is

$$p = \begin{array}{cccccc} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} \\ \mathbf{1} & 1 & 0 & 0 & 0 & 0 \\ \mathbf{2} & 0 & 1/3 & 0 & 2/3 & 0 \\ \mathbf{3} & 1/3 & 2/9 & 0 & 4/9 & 0 \\ \mathbf{4} & 0 & 1/3 & 0 & 2/3 & 0 \\ \mathbf{5} & 2/3 & 1/9 & 0 & 2/9 & 0 \end{array}$$

1.16. If we rearrange the matrix for the seven state chain in Example 1.14 we get

	2	**3**	**1**	**5**	**4**	**6**	**7**
2	.2	.3	.1	0	.4	0	0
3	0	.5	0	.2	.3	0	0
1	0	0	.7	.3	0	0	0
5	0	0	.6	.4	0	0	0
4	0	0	0	0	.5	.5	0
6	0	0	0	0	0	.2	.8
7	0	0	0	0	1	0	0

Find $\lim_{n\to\infty} p^n(i,j)$.

Ans. The stationary distribution on the closed irreducible set $\{3,4\}$ is $2/3, 1/3$; on $\{5,6,7\}$ is $8/17, 5/17, 4/17$. Starting from 2 we enter $\{3,4\}$ with probability $2/5$ and $\{5,6,7\}$ with probability $3/5$. When we leave 1 we go to 2 with probability $3/8$, enter $\{3,4\}$ with probability $1/8$ and $\{5,6,7\}$ with probability $4/8$, so we get absorbed in $\{3,4\}$ with probability $1/8 + (3/8)(2/5) = 11/40$ and in $\{5,6,7\}$ with probability $4/8 + (3/8)(3/5) = 29/40$. Combining these observations we see that the limit is

	2	**3**	**1**	**5**	**4**	**6**	**7**
2	0	0	11/60	11/120	29/85	29/136	29/170
3	0	0	4/15	2/15	24/85	15/85	12/85
1	0	0	2/3	1/3	0	0	0
5	0	0	2/3	1/3	0	0	0
4	0	0	0	0	8/17	5/17	4/17
6	0	0	0	0	8/17	5/17	4/17
7	0	0	0	0	8/17	5/17	4/17

Two state Markov chains

1.17. Market research suggests that in a five year period 8% of people with cable television will get rid of it, and 26% of those without it will sign up for it. Compare the predictions of the Markov chain model with the following data on the fraction of people with cable TV: 56.4% in 1990, 63.4% in 1995, and 68.0% in 2000. What is the long run fraction of people with cable TV?

Ans. The transition probability is

	Cable	**No**
Cable	0.92	0.08
No	0.26	0.74

Letting $\mu = (0.564, 0.436)$ and computing μp and $(\mu p)p$ we see that the Markov chain predicts 63.2% in 1995 and 67.7% in 2000. Using our formula for the stationary distribution of the two state chain, we see that in the long run $26/34 = 76.47\%$ will have it.

1.18. A sociology professor postulates that in each decade 8% of women in the work force leave it and 20% of the women not in it begin to work. Compare the predictions of his model with the following data on the percentage of women working: 43.3% in 1970, 51.5% in 1980, 57.5% in 1990, and 59.8% in 2000. In the long run what fraction of women will be working?

Ans. The Markov chain predicts 51.2% in 1980, 56.8% in 1990, and 60.9% in 2000. In the long run $20/28 = 71.43\%$ will be working.

1.19. A rapid transit system has just started operating. In the first month of operation, it was found that 25% of commuters are using the system while 75% are travelling by automobile. Suppose that each month 10% of transit users go back to using their cars, while 30% of automobile users switch to the transit system. (a) Compute the three step transition probaiblity p^3. (b) What will be the fractions using rapid transit in the fourth month? (c) In the long run?

Ans.

$$\text{a.}\quad p^3 = \begin{array}{c|cc} & \mathbf{RT} & \mathbf{C} \\ \mathbf{RT} & .804 & .196 \\ \mathbf{C} & .588 & .412 \end{array}$$

(b) 0.642, (c) 0.75

1.20. A regional health study indicates that from one year to the next, 75% percent of smokers will continue to smoke while 25% will quit. 8% of those who stopped smoking will resume smoking while 92% will not. If 70% of the population were smokers in 1995, what fraction will be smokers in 1998? in 2005? in the long run?

Ans. 38%, 25%, $8/33 = 24\%$

1.21. Three of every four trucks on the road are followed by a car, while only one of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Ans. Let $X_n \in \{C, T\}$ denote the type of the nth vehicle. We have a Markov chain with transition probability

	T	**C**
T	1/4	3/4
C	1/5	4/5

A stationary distribution π has $\pi(T) = 1/4\pi(T) + 1/5\pi(C)$, i.e., $\pi(T) = (4/15)\pi(C)$. Since we want the two probabilities to sum to one we must have $\pi(T) = (4/15)/(1 + 4/15) = 4/19$.

1.22. In a test paper the questions are arranged so that 3/4's of the time a True answer is followed by a True, while 2/3's of the time a False answer is followed by a False. You are confronted with a 100 question test paper. Approximately what fraction of the answers will be True.

Ans. The answers are a Markov chain with transition probability

	T	**F**
T	3/4	1/4
F	1/3	2/3

The detailed balance condition tells us that the stationary distribution has $\pi(T)/4 = \pi(H)/3$ so $\pi(T) = 4/7$ and $\pi(H) = 3/7$. Thus we expect about 4/7's of the answers to be true.

1.23. In unprofitable times corporations sometimes suspend dividend payments. Suppose that after a dividend has been paid the next one will be paid with probability 0.9, while after a dividend is suspended the next one will be suspended with probability 0.6. In the long run what is the fraction of dividends that will be paid?

Ans. Writing P for paid and S for suspended we have the following matrix

	P	**S**
P	0.9	0.1
S	0.4	0.6

A stationary distribution π has $\pi(P) = 0.9\pi(P) + 0.4\pi(S)$, i.e., $\pi(P) = 4\pi(S)$. Since we want the two probabilities to sum to one we must have $\pi(P) = 4/5$, $\pi(S) = 1/5$.

1.24. Census results reveal that in the United States 80% of the daughters of working women work and that 30% of the daughters of nonworking women work. (a) Write the transition probability for this model. (b) In the long run what fraction of women will be working?

Ans. Writing W for work and N for not work:

(a)

	W	**N**
W	.8	.2
N	.3	.7

(b) 3/5

1.25. When a basketball player makes a shot then he tries a harder shot the next time and hits (H) with probability 0.4, misses (M) with probability 0.6. When he misses he is more conservative the next time and hits (H) with probability 0.7, misses (M) with probability 0.3. (a) Write the transition probability for the two state Markov chain with state space $\{H, M\}$. (b) Find the long-run fraction of time he hits a shot.

Ans. The transition probability is

	H	**M**
H	.4	.6
M	.7	.3

By our formula the long run probability of H is $.7/(.7 + .6) = 7/13$.

1.26. Folk wisdom holds that in Ithaca in the summer it rains 1/3 of the time, but a rainy day is followed by a second one with probability 1/2. Suppose that Ithaca weather is a Markov chain. What is its transition probability?

Ans. From the information given we must have $p(R, R) = 1/2$ and hence $p(R, S) = 1/2$. Detailed balance for the proposed stationary distribution implies $(1/3)p(R, S) = (2/3)p(S, R)$ so $P(S, R) = P(R, S)/2 = 1/4$ and it follows that $p(S, S) = 3/4$.

Chains with three or more states

1.27. (a) Suppose brands A and B have consumer loyalties of .7 and .8, meaning that a customer who buys A one week will with probability .7 buy it again the next week, or try the other brand with .3. What is the limiting market share for each of these products? (b) Suppose now there is a third brand with loyalty .9, and that a consumer who changes brands picks one of the other two at random. What is the new limiting market share for these three products?

Ans. (a) The transition matrix is

	A	**B**
A	.7	.3
B	.2	.8

Using (1.8), $\pi(A) = 2/5$ and $\pi(B) = 3/5$.
(b) The transition matrix is

	A	**B**	**C**
A	.7	.15	.15
B	.1	.8	.1
C	.05	.05	.9

The third row of

$$\begin{pmatrix} -.3 & .15 & 1 \\ .1 & -.2 & 1 \\ .05 & .05 & 1 \end{pmatrix}^{-1}$$

is 2/11, 3/11, 6/11.

1.28. A midwestern university has three types of health plans: a health maintenance organization (HMO), a preferred provider organization (PPO), and a traditional fee for service plan (FFS). Experience dictates that people change plans according to the following transition matrix

	HMO	**PPO**	**FFS**
HMO	.85	.1	.05
PPO	.2	.7	.1
FFS	.1	.3	.6

In 2000, the percentages for the three plans were HMO:30%, PPO:25%, and FFS:45%. (a) What will be the percentages for the three plans in 2001? (b) What is the long run fraction choosing each of the three plans?

Ans. (a) 0.35, 0.34, 0.31.
(b) The third row of

$$\begin{pmatrix} -.15 & .1 & 1 \\ .2 & -.3 & 1 \\ .1 & .3 & 1 \end{pmatrix}^{-1}$$

is $(18/34, 11/34, 5/34) = (0.5294, 0.3235, 0.1470)$.

1.29. Bob eats lunch at the campus food court every week day. He either eats Chinese food, Quesadila, or Salad. His transition matrix is

	C	**Q**	**S**
C	.15	.6	.25
Q	.4	.1	.5
S	.1	.3	.6

He had Chinese food on Monday. (a) What are the probabilities for his three meal choices on Friday (four days later). (b) What are the long run frequencies for his three choices?

Ans. (a) Using our calculator (you only have to write the first row).

$$p^4 = \begin{array}{c|ccc} & \mathbf{C} & \mathbf{Q} & \mathbf{S} \\ \mathbf{C} & .211 & .286 & .502 \\ \mathbf{Q} & .191 & .315 & .494 \\ \mathbf{S} & .201 & .296 & .503 \end{array}$$

(b) The third row of

$$\begin{pmatrix} -.85 & .6 & 1 \\ .4 & -.9 & 1 \\ .1 & .3 & 1 \end{pmatrix}^{-1}$$

is .2, .3, .5.

1.30. The liberal town of Ithaca has a "free bikes for the people program." You can pick up bikes at the library (L), the coffee shop (C) or the cooperative grocery store (G). The director of the program has determined that bikes move around accroding to the following Markov chain

	L	**C**	**G**
L	.5	.2	.3
C	.4	.5	.1
G	.25	.25	.5

On Sunday there are an equal number of bikes at each place. (a) What fraction of the bikes are at the three locations on Tuesday? (b) on the next Sunday? (c) In the long run what fraction are at the three locations?

Ans. (a) (.39333, .31, .29666), (b) (.394743, .307018, .298238),
(c) The third row of

$$\begin{pmatrix} -.5 & .2 & 1 \\ .4 & -.5 & 1 \\ .25 & .25 & 1 \end{pmatrix}^{-1}$$

is $(45/114, 35/114, 34/114) = (.394736, .307017, .298245)$.

1.31. A plant species has red, pink, or white flowers according to the genotypes RR, RW, and WW, respectively. If each of these genotypes is crossed with a pink (RW) plant then the offspring fractions are

	RR	**RW**	**WW**
RR	.5	.5	0
RW	.25	.5	.25
WW	0	.5	.5

What is the long run fraction of plants of the three types?

Ans. The third row of

$$\begin{pmatrix} -.5 & .5 & 1 \\ .25 & -.5 & 1 \\ 0 & .5 & 1 \end{pmatrix}^{-1}$$

is .25, .5, .25.

1.32. The weather in a certain town is classified as rainy, cloudy, or sunny and changes according to the following transition probability is

	R	**C**	**S**
R	1/2	1/4	1/4
C	1/4	1/2	1/4
S	1/2	1/2	0

In the long run what proportion of days in this town are rainy? cloudy? sunny?

Ans. The third row of

$$\begin{pmatrix} -.5 & .25 & 1 \\ .25 & -.5 & 1 \\ .5 & .5 & 1 \end{pmatrix}^{-1} \qquad \text{is } .4,\ .4,\ .2.$$

1.33. A sociologist studying living patterns in a certain region determines that the pattern of movement between urban (U), suburban (S), and rural areas (R) is given by the following transition matrix.

	U	**S**	**R**
U	.86	.08	.06
S	.05	.88	.07
R	.03	.05	.92

In the long run what fraction of the population will live in the three areas?

Ans. The third row of

$$\begin{pmatrix} -.14 & .08 & 1 \\ .05 & -.12 & 1 \\ .03 & .05 & 1 \end{pmatrix}^{-1} \qquad \text{is } (61/283,\ 94/283,\ 128/283) = (.21554,.33215,.45229)\ .$$

1.34. In a large metropolitan area, commuters either drive alone (A), carpool (C), or take public transportation (T). A study showed that transportation changes according to the following matrix:

	A	**C**	**T**
A	.8	.15	.05
C	.05	.9	.05
S	.05	.1	.85

In the long run what fraction of commuters will use the three types of transportation?

Ans. The third row of

$$\begin{pmatrix} -.2 & .15 & 1 \\ .05 & -.1 & 1 \\ .05 & .1 & 1 \end{pmatrix}^{-1} \qquad \text{is } (0.2,0.55,0.25).$$

1.35. (a) Three telephone companies A, B, and C compete for customers. Each year customers switch between companies according the followinng transition probability

	A	**B**	**C**
A	.75	.05	.20
B	.15	.65	.20
C	.05	.1	.85

What is the limiting market share for each of these companies?

Ans. The third row of

$$\begin{pmatrix} -.2 & .15 & 1 \\ .05 & -.1 & 1 \\ .05 & .1 & 1 \end{pmatrix}^{-1} \quad \text{is } (13/56,11/56,32/56) = (0.2321,\ 0.1964,0.5714).$$

1.36. A professor has two light bulbs in his garage. When both are burned out, they are replaced, and the next day starts with two working light bulbs. Suppose that when both are working, one of the two will go out with probability .02 (each has probability .01 and we ignore the possibility of losing two on the same day). However, when only one is there, it will burn out with probability .05. (i) What is the long-run fraction of time that there is exactly one bulb working? (ii) What is the expected time between light bulb replacements?

Ans. (i) Letting the state be the number of light bulbs that are burnt out, the state space is $\{0, 1, 2\}$, and the transition probability is

	0	**1**	**2**
0	.98	.02	0
1	0	.95	.05
2	1	0	0

The third row of

$$\begin{pmatrix} -.02 & .02 & 1 \\ 0 & -.05 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \quad \text{is } (50/71,20/71,1/71)$$

Thus, in the long run there is one bulb with probability $20/71 = 0.2817$. (ii) To compute the expected time between blub changes, note that the number of replacements is the same as the number of visits to state 2, so by Theorem 1.22:

$$\frac{1}{E_2T_2} = \pi(2) = \frac{1}{71} \quad \text{and hence} \quad E_2T_2 = 71$$

1.37. An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining, she will take an umbrella, if one is there. Otherwise, she gets wet. Assume that independent of the past, it rains on each trip with probability 0.2. To formulate a Markov chain, let X_n be the number of umbrellas at her current location. (a) Find the transition probability for this Markov chain. (b) Calculate the limiting fraction of time she gets wet.

Ans. Taking into account the shift in view point, i.e., if we have 0 umbrellas at the current location, we will always have 3 at the next current location, one can compute $p(i,j)$ as

	0	**1**	**2**	**3**
0	0	0	0	1
1	0	0	0.8	0.2
2	0	0.8	0.2	0
3	0.8	0.2	0	0

The fourth row of

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & .8 & 1 \\ 0 & .8 & -.8 & 1 \\ .8 & .2 & 0 & 1 \end{pmatrix}^{-1} \quad \text{is } (4/19,5/19,5/19,5/19)$$

This means that the fraction of times with no umbrellas is $\pi(0) = 4/19$ and the fraction of time getting wet is $(.2)(4/19) = 0.0421$.

shavechain

1.38. Let X_n be the number of days since David last shaved, calculated at 7:30AM when he is trying to decide if he wants to shave today. Suppose that X_n is a Markov chain with transition matrix

	1	**2**	**3**	**4**
1	1/2	1/2	0	0
2	2/3	0	1/3	0
3	3/4	0	0	1/4
4	1	0	0	0

In words, if he last shaved k days ago, he will not shave with probability 1/(k+1). However, when he has not shaved for 4 days his mother orders him to shave, and he does so with probability 1. (a) What is the long-run fraction of time David shaves? (b) Does the stationary distribution for this chain satisfy the detailed balance condition?

Ans. The fourth row of

$$\begin{pmatrix} -1/2 & 1/2 & 0 & 1 \\ 2/3 & -1 & 1/3 & 1 \\ 3/4 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1} \quad \text{is } (24/41,12/41,4/41,1/41).$$

so the answer is $24/41 = 0.5853$.
(b) $\pi(1)p(1,2) > 0$ while $\pi(2)p(2,1) = 0$ so π does not satisfy the detailed balance condition.

1.39. In a particular county voters declare themselves as members of the Republican, Democrat, or Green party. No voters change directly from the Republican to Green party or vice versa. Other transitions occur according to the following matrix:

	R	**D**	**G**
R	.85	.15	0
D	.05	.85	.10
G	0	.05	.95

In the long run what fraction of voters will belong to the three parties?

Ans. This is a birth and death chain, so its stationary distribution satisfies detailed balance:

$$.15\pi(R) = .05\pi(D) \quad .10\pi(D) = .05\pi(G)$$

Suppose $\pi(R) = c$. These equations imply $\pi(D) = 3c$ and $\pi(G) = 6c$, so we take $c = 1/10$. That is, the long run frequencies are R: 0.1, D: 0.3, G:0.6.

1.40. An auto insurance company classifies its customers in three categories: poor, satisfactory and excellent. No one moves from poor to excellent or from excellent to poor in one year.

	P	**S**	**E**
P	.6	.4	0
S	.1	.6	.3
E	0	.2	.8

What is the limiting fraction of drivers in each of these categories?

Ans. This is a birth and death chain, so its stationary distribution satisfies detailed balance:

$$.4\pi(P) = .1\pi(S) \quad .3\pi(S) = .2\pi(E)$$

Suppose $\pi(P) = c$. These equations imply $\pi(S) = 4c$ and $\pi(E) = 6c$, so we take $c = 1/11$. That is, the long run frequencies are P: 1/11, S: 4/11, E: 6/11.

1.41. *Reflecting random walk on the line.* Consider the points $1, 2, 3, 4$ to be marked on a straight line. Let X_n be a Markov chain that moves to the right with probability 2/3 and to the left with probability 1/3, but subject this time to the rule that if X_n tries to go to the left from 1 or to the right from 4 it stays put. Find (a) the transition probability for the chain, and (b) the limiting amount of time the chain spends at each site.

Ans. (a) The transition probability is

	1	**2**	**3**	**4**
1	1/3	2/3	0	0
2	1/3	0	2/3	0
3	0	1/3	0	2/3
4	0	0	1/3	2/3

This is a birth and death chain, so its stationary distribution satisfies detailed balance:

$$(2/3)\pi(1) = 1/3\pi(2) \quad (2/3)\pi(2) = 1/3\pi(3) \quad (2/3)\pi(3) = 1/3\pi(4)$$

Suppose $\pi(1) = c$. These equations imply $\pi(2) = 2c$, $\pi(3) = 4c$, $\pi(4) = 8c$. The sum of the $\pi(k)$ should be one, so we take $c = 1/15$.

1.42. At the end of a month, a large retail store classifies each of its customer's accounts according to current (0), 30–60 days overdue (1), 60–90 days overdue (2), more than 90 days (3). Their experience indicates that the accounts move from state to state according to a Markov chain with transition probability matrix:

	0	**1**	**2**	**3**
0	.9	.1	0	0
1	.8	0	.2	0
2	.5	0	0	.5
3	.1	0	0	.9

In the long run what fraction of the accounts are in each category?

Ans. The fourth row of

$$\begin{pmatrix} -.1 & .1 & 0 & 1 \\ .8 & -1 & .2 & 1 \\ .5 & 0 & -1 & 1 \\ .1 & 0 & 0 & 1 \end{pmatrix}^{-1} \quad \text{is } (50/61, 5/61, 1/61, 5/61).$$

1.43. At the beginning of each day, a piece of equipment is inspected to determine its working condition, which is classified as state 1 = new, 2, 3, or 4 = broken. We assume the state is a Markov chain with the following transition matrix:

	1	**2**	**3**	**4**
1	.95	.05	0	0
2	0	.9	.1	0
3	0	0	.875	.125

(a) Suppose that a broken machine requires three days to fix. To incorporate this into the Markov chain we add states 5 and 6 and suppose that $p(4,5) = 1$, $p(5,6) = 1$, and $p(6,1) = 1$. Find the fraction of time that the machine is working. (b) Suppose now that we have the option of performing preventative maintenance when the machine is in state 3, and that this maintenance takes one day and returns the machine to state 1. This changes the transition probability to

	1	**2**	**3**
1	.95	.05	0
2	0	.9	.1
3	1	0	0

Find the fraction of time the machine is working under this new policy.

Ans. The sixth row of

$$\begin{pmatrix} -.05 & .05 & 0 & 0 & 0 & 1 \\ 0 & -.1 & .1 & 0 & 0 & 1 \\ 0 & 0 & -.125 & .125 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

is (20/41,10/41,8/41,1/41, 1/41, 1/41) so the machine is working 38/41's or 92.68% of the time.
(b) The third row of

$$\begin{pmatrix} -.05 & .05 & 1 \\ 0 & -.1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \quad \text{is } (20/31, 10/31, 1/31)$$

so the machine is working 30/31's or 96.77%'s of the time.

1.44. *Landscape dynamics.* To make a crude model of a forest we might introduce states 0 = grass, 1 = bushes, 2 = small trees, 3 = large trees, and write down a transition matrix like the following:

	0	**1**	**2**	**3**
0	1/2	1/2	0	0
1	1/24	7/8	1/12	0
2	1/36	0	8/9	1/12
3	1/8	0	0	7/8

The idea behind this matrix is that if left undisturbed a grassy area will see bushes grow, then small trees, which of course grow into large trees. However, disturbances such as tree falls or fires can reset the system to state 0. Find the limiting fraction of land in each of the states.

Ans. The fourth row of

$$\begin{pmatrix} -1/2 & 1/2 & 0 & 1 \\ 1/24 & -1/8 & 1/12 & 1 \\ 1/36 & 0 & -1/9 & 1 \\ 1/8 & 0 & 0 & 7 \end{pmatrix}^{-1} \quad \text{is } (0.1, 0.4, 0.3, 0.2).$$

More Theoretical Exercises

1.45. Consider a general chain with state space $S = \{1, 2\}$ and write the transition probability as

	1	**2**
1	$1-a$	a
2	b	$1-b$

Use the Markov property to show that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)\left\{P(X_n = 1) - \frac{b}{a+b}\right\}$$

and then conclude

$$P(X_n = 1) = \frac{b}{a+b} + (1-a-b)^n \left\{P(X_0 = 1) - \frac{b}{a+b}\right\}$$

This shows that if $0 < a+b < 2$, then $P(X_n = 1)$ converges exponentially fast to its limiting value $b/(a+b)$.

Ans. The Markov property and $P(X_n = 2) = 1 - P(X_n = 1)$ imply that

$$P(X_{n+1} = 1) = (1-a)P(X_n = 1) + bP(X_n = 0) = b + (1-a-b)P(X_n = 1)$$

Subtracting $b/(a+b)$ from each side gives

$$P(X_{n+1} = 1) - \frac{b}{a+b} = b + (1-a-b)\left\{P(X_n = 1) - \frac{b}{a+b}\right\} - b$$

which is the first equality. Iterating we conclude that

$$P(X_{n+1} = 1) - \frac{b}{a+b} = (1-a-b)^n\left\{P(X_0 = 1) - \frac{b}{a+b}\right\}$$

which gives the second equality.

BernLap **1.46.** *Bernoulli–Laplace model of diffusion.* Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is in state i if the first urn contains i black balls and $m - i$ white balls while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain. (a) Compute its transition probability. (b) Verify that the stationary distribution is given by

$$\pi(i) = \binom{b}{i}\binom{2m-b}{m-i}\Big/\binom{2m}{m}$$

(c) Can you give a simple intuitive explanation why the formula in (b) gives the right answer?

Ans. (a) To increase the number of black balls by 1 we have to pick a white from the first and a black from the second. Using similar reasoning on the other cases we have

$$p(i,i+1) = \frac{(m-i)\cdot(b-i)}{m\cdot m} \qquad p(i,i-1) = \frac{i\cdot(m-b+i)}{m\cdot m}$$
$$p(i,i) = \frac{i(b-i) + (m-i)(m-b+i)}{m\cdot m}$$

(b) Detailed balance is equivalent to $\pi(i+1)/\pi(i) = p(i,i+1)/p(i+1,i)$. To check this condition we compute

$$\begin{aligned}\frac{\pi(i+1)}{\pi(i)} &= \frac{b!}{(i+1)!(b-i-1)!}\frac{(2m-b)!}{(m-i-1)!(m-b+i+1)!} \\ &\quad\cdot\frac{i!(b-i)!}{b!}\frac{(m-i)!(m-b+i)!}{(2m-b)!} \\ &= \frac{i!(b-i)!(m-i)!(m-b+i)!}{(i+1)!(b-i-1)!(m-i-1)!(m-b+i+1)!} \\ &= \frac{(b-i)(m-i)}{(i+1)(m-b+i+1)} = \frac{p(i,i+1)}{p(i+1,i)}\end{aligned}$$

(c) If we pick m balls at random from the $2m$ to put in one urn then $\pi(i)$ gives the probability we will get exactly i black balls. If we put the balls in randomly and then switch two we still have a random arrangement so we have constructed a stationary distribution.

1.47. *Library chain.* On each request the ith of n possible books is the one chosen with probability p_i. To make it quicker to find the book the next time, the librarian moves the book to the left end of the shelf. Define the state at any time to be the sequence of books we see as we examine the shelf from left to right. Since all the books are distinct this list is a permutation of the set $\{1, 2, \dots n\}$, i.e., each number is listed exactly once. Show that

$$\pi(i_1, \dots, i_n) = p_{i_1} \cdot \frac{p_{i_2}}{1 - p_{i_1}} \cdot \frac{p_{i_3}}{1 - p_{i_1} - p_{i_2}} \cdots \frac{p_{i_n}}{1 - p_{i_1} - \cdots p_{i_{n-1}}}$$

is a stationary distribution.

Ans. Suppose that the process of requests has been going on forever and let $Y_{-1}, Y_{-2}, Y_{-3}, \dots$ be the sequence of requests. At time 0, Y_{-1} will be in the first position, the first Y_{-j} different from Y_{-1} will be in the second, etc. It is easy to check that the distribution at time 0 is given by π and that the distribution at time 1 is also given by π so π is a stationary distribution.

1.48. *Random walk on a clock.* Consider the numbers $1, 2, \dots 12$ written around a ring as they usually are on a clock. Consider a Markov chain that at any point jumps with equal probability to the two adjacent numbers. (a) What is the expected number of steps that X_n will take to return to its starting position? (b) What is the probability X_n will visit all the other states before returning to its starting position?

Ans. (a) This a random walk on a graph in which each vertex has degree 2, so it has a uniform stationary distribution, i.e., $\pi(i) = 1/12$ for all i. Using (6.6) now we have $E_i T_i = 1/\pi(i) = 12$. (b) Suppose for simplicity that we start at 12 and the first step is to 1. We will visit all the states if and only if the chain get to 11 before it hits 12. This is the same as a random walk on the line starting at 1 and hitting 11 before 0 and hence has probability 1/11 by (6.2).

The next three examples continue Example 1.34. Again we represent our chessboard as $\{(i, j) : 1 \le i, j \le 8\}$. How do you think that the pieces bishop, knight, king, queen, and rook rank in their answers to (b)?

1.49. *King's random walk.* A king can move one squares horizontally, vertically, or diagonally. Let X_n be the sequence of squares that results if we pick one of king's legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner (1,1) when we start there.

Ans. (a) The degrees for the four squares in the upper left corner are

$$\begin{matrix} 3 & 5 & 5 & 5 \\ 5 & 8 & 8 & 8 \\ 5 & 8 & 8 & 8 \\ 5 & 8 & 8 & 8 \end{matrix}$$

Using symmetry, the sum of the degrees for the graph is $36\cdot 8+24\cdot 5+4\cdot 3 = 420$. Dividing by the sum gives the stationary distribution. (b) Theorem 1.22 gives $E_{(1,1)}T_{(1,1)} = 1/\pi(1,1) = 420/3 = 140$.

1.50. *Bishop's random walk.* A bishop can move any number of squares diagonally. Let X_n be the sequence of squares that results if we pick one of bishop's legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner (1,1) when we start there.

Ans. A bishop must stay on squares of the same color so only 32 are possible. (a) The degrees for the graph are

$$\begin{array}{cccccccc}
7 & 0 & 7 & 0 & 7 & 0 & 7 & 0 \\
0 & 9 & 0 & 9 & 0 & 9 & 0 & 7 \\
7 & 0 & 11 & 0 & 11 & 0 & 9 & 0 \\
0 & 9 & 0 & 13 & 0 & 11 & 0 & 7 \\
7 & 0 & 11 & 0 & 13 & 0 & 9 & 0 \\
0 & 9 & 0 & 11 & 0 & 11 & 0 & 7 \\
7 & 0 & 9 & 0 & 9 & 0 & 9 & 0 \\
0 & 7 & 0 & 7 & 0 & 7 & 0 & 7
\end{array}$$

The sum of the degrees for the graph is $13\cdot 2 + 11\cdot 6 + 9\cdot 10 + 7\cdot 14 = 270$. Dividing by the sum gives the stationary distribution. (b) Theorem 1.22 gives $E_{(1,1)}T_{(1,1)} = 1/\pi(1,1) = 270/7 = 38.57$.

1.51. *Queen's random walk.* A queen can move any number of squares horizontally, vertically, or diagonally. Let X_n be the sequence of squares that results if we pick one of queen's legal moves at random. Find (a) the stationary distribution and (b) the expected number of moves to return to corner (1,1) when we start there.

Ans. (a) This is a random walk on a graph so the stationary distribution is proportional to the degrees of the vertices. There are always 14 horizontal or vertical moves from any square, so considering the possibilities for diagonal moves we see that the degrees for the four squares in the upper left corner are

$$\begin{array}{cccc}
21 & 21 & 21 & 21 \\
21 & 23 & 23 & 23 \\
21 & 23 & 25 & 25 \\
21 & 23 & 25 & 27
\end{array}$$

The sum of the degrees for the graph is $28\cdot 21 + 20\cdot 23 + 12\cdot 25 + 4\cdot 27 = 1456$. Dividing by the sum gives the stationary distribution. (b) Using Theorem 1.22 gives $E_{(1,1)}T_{(1,1)} = 1/\pi(1,1) = 1456/21 = 69.333$.

1.52. *Wright–Fisher model.* Consider the chain described in Example 1.7.

$$p(x,y) = \binom{N}{y}(\rho_x)^y(1-\rho_x)^{N-y}$$

where $\rho_x = (1-u)x/N + v(N-x)/N$. (a) Show that if $u, v > 0$, then $\lim_{n\to\infty} p^n(x,y) = \pi(y)$, where π is the unique stationary distribution. There is no known formula for $\pi(y)$, but you can (b) compute the mean $\nu = \sum_y y\pi(y) = \lim_{n\to\infty} E_x X_n$.

Ans. (a) When $u, v > 0$ the chain is irreducible so the conclusion follows from (4.7) and (4.5). (b) By the formula for the mean of the binomial $E_x X_1 = N\rho_x = vN + (1-u-v)x$. Iterating we have

$$\begin{aligned} E_x X_2 &= vN + (1-u-v)vN + (1-u-v)^2 x \\ E_x X_3 &= vN + (1-u-v)vN + (1-u-v)^2 vN + (1-u-v)^3 x \end{aligned}$$

so $\lim_{n\to\infty} E_x X_n = vN \sum_{m=0}^{\infty} (1-u-v)^m = Nv/(u+v)$. A simple way to see this is to note that in the absence of mutation an individual in generation n s the same as a randomly chosen one from generation $n-1$, so its type is dictated by the first mutation encountered as we work backwards and that will be to a 1 with probability $v/(u+v)$.

1.53. *Ehrenfest chain.* Consider the Ehrenfest chain, Example 1.2, with transition probability $p(i, i+1) = (N-i)/N$, and $p(i, i-1) = i/N$ for $0 \le i \le N$. Let $\mu_n = E_x X_n$. (a) Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$. (b) Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2)$$

From this we see that the mean μ_n converges exponentially rapidly to the equilibrium value of $N/2$ with the error at time n being $(1-2/N)^n(x - N/2)$.

Ans. (a) $E_x X_1 = x + (N-x)/N - x/N = 1 + (1-2x)/N$. Taking x to be random with distribution equal to that of X_n the result follows. (b) The formula holds when $n = 0$. If it is true for n then

$$\begin{aligned} \mu_{n+1} &= 1 + (1-2/N)\left(\frac{N}{2} + (1-2/N)^n (x - N/2)\right) \\ &= \frac{N}{2} + \left(1 - \frac{2}{N}\right)^{n+1} (x - N/2) \end{aligned}$$

1.54. Prove that if $p_{ij} > 0$ for all i and j then a necessary and sufficient condition for the existence of a reversible stationary distribution is

$$p_{ij} p_{jk} p_{ki} = p_{ik} p_{kj} p_{ji} \quad \text{for all } i, j, k$$

Hint: fix i and take $\pi_j = c p_{ij}/p_{ji}$.

Ans. To show necessity we note that

$$\begin{aligned} \pi_i p_{ij} p_{jk} p_{ki} &= p_{ji} \pi_j p_{jk} p_{ki} \\ &= p_{ji} p_{kj} \pi_k p_{ki} = p_{ji} p_{kj} p_{ik} \pi_i \end{aligned}$$

For sufficiency, we use the condition to conclude that

$$\pi_j p_{jk} = c \frac{p_{ij} p_{jk}}{p_{ji}} = c \frac{p_{ik} p_{kj}}{p_{ki}} = \pi_k p_{kj}$$

The assumption that $p_{ij} > 0$ in the last result is needed. The chain in Exercise 1.12 has $p^3(i,i) = 0$, but has no reversible stationary distribution.

Exit distributions and times

1.55. The Markov chain associated with a manufacturing process may be described as follows: A part to be manufactured will begin the process by entering step 1. After step 1, 20% of the parts must be reworked, i.e., returned to step 1, 10% of the parts are thrown away, and 70% proceed to step 2. After step 2, 5% of the parts must be returned to the step 1, 10% to step 2, 5% are scrapped, and 80% emerge to be sold for a profit. (a) Formulate a four-state Markov chain with states 1, 2, 3, and 4 where 3 = a part that was scrapped and 4 = a part that was sold for a profit. (b) Compute the probability a part is scrapped in the production process.

Ans. (a) The transition probability is

	1	**2**	**3**	**4**
1	0.2	0.7	0.1	0
2	0.05	0.1	0.05	0.8
3	0	0	1	0
4	0	0	0	1

(b) Let $h(x) = P_x(V_3 < V_4)$. Clearly $h(3) = 1$ and $h(4) = 0$. To compute the values for the other two states we note that

$$h(1) = 0.2h(1) + 0.7h(2) + 0.1 \qquad h(2) = 0.05h(1) + 0.1h(2) + 0.05$$

Solving gives $h(1) = .125/.685$ and $h(2) = .045/.685$.

1.56. A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according to the following transition probability:

	F	**G**	**A**	**B**
F	1	0	0	0
G	.1	.8	.1	0
A	.1	.4	.4	.1
B	0	0	0	1

What fraction of loans in good standing are eventually paid in full? What is the answr for those in arrears?

Ans. Let $h(x)$ be the probability of ending in F starting from state x.

$$h(G) = .1 + .8h(G) + .1h(A) \qquad h(A) = .1 + .4h(G) + .4h(A)$$

The first equation implies $h(G) = 1/2 + h(A)/2$ so the second implies $h(A) = .1 + .2 + .2h(A) + .4h(A)$ so $h(A) = 3/4$ and $h(G) = 7/8$.

1.57. A warehouse has a capacity to hold four items. If the warehouse is neither full nor empty, the number of items in the warehouse changes whenever a new item is produced or an item is sold. Suppose that (no matter when we look) the probability that the next event is "a new item is produced" is 2/3 and that the new event is a "sale" is 1/3. If there is currently one item in the warehouse, what is the probability that the warehouse will become full before it becomes empty.

Ans. Let $h(x)$ be the probability that the warehouse becomes full before it becomes empty when we start with x items. By considering what happens on one step we see that

$$\begin{aligned} h(3) &= (2/3) + (1/3)h(2) \\ h(2) &= (2/3)h(3) + (1/3)h(1) \\ h(1) &= (2/3)h(2) \end{aligned}$$

If we let $h(1) = c$ then the last equation implies $h(2) = 3c/2$. Using the first and then the second we have

$$(2/3) + c/2 = h(3) = (3/2)(3c/2) - (1/2)c = 7c/4$$

i.e., $5c/4 = 2/3$ or $c = 8/15$. Plugging this in we have $h(1) = 8/15$, $h(2) = 12/15$, and $h(3) = 7c/4 = 14/15$.

1.58. Six children (Dick, Helen, Joni, Mark, Sam, and Tony) play catch. If Dick has the ball he is equally likely to throw it to Helen, Mark, Sam, and Tony. If Helen has the ball she is equally likely to throw it to Dick, Joni, Sam, and Tony. If Sam has the ball he is equally likely to throw it to Dick, Helen, Mark, and Tony. If either Joni or Tony gets the ball, they keep throwing it to each other. If Mark gets the ball he runs away with it. (a) Find the transition probability and classify the states of the chain. (b) Suppose Dick has the ball at the beginning of the game. What is the probability Mark will end up with it?

Ans. Indicating each child by their initial and rearranging to make the structure more clear

(a)	**M**	**H**	**D**	**S**	**J**	**T**
M	1	0	0	0	0	0
H	0	0	1/4	1/4	1/4	1/4
D	1/4	1/4	0	1/4	0	1/4
S	1/4	1/4	1/4	0	0	1/4
J	0	0	0	0	0	1
T	0	0	0	0	1	0

States D, H, S communicate with T which does not communicate with them so they are all transient. $\{j, T\}$ and $\{H\}$ are closed irreducible sets so those states are recurrent. (b) Letting p_D, p_H, p_S be the probabilities that Mark ends up with the ball when Dick, Helen, or Sam have it first we have

$$\begin{aligned} p_H &= \frac{1}{4}p_D + \frac{1}{4}p_S \\ p_D &= \frac{1}{4} + \frac{1}{4}p_H + \frac{1}{4}p_S \\ p_S &= \frac{1}{4} + \frac{1}{4}p_H + \frac{1}{4}p_D \end{aligned}$$

Symmetry implies $p_D = p_S = c$. Using this in the first equation gives $p_H = c/2$. Plugging these relations into the second (or third) equation gives $c = 1/4 + c/4 + c/8$ so $5c/8 = 2/8$ and $c = 2/5$.

3coin **1.59.** Use the second solution in Example 1.48 to compute the expected waiting times for the patterns HHH, HHT, HTT, and HTH. Which pattern has the longest waiting time? Which ones achieve the minimum value of 8?

Ans. For any pattern x, $E_x T_x = 1/\pi(x) = 8$. If we use E to denote the expected value starting from nothing

$$\begin{aligned} E_{HHH}T_{HHH} &= (1/2)\cdot 1 + (1/2)(1 + ET_{HHH}) \\ E_{HHT}T_{HHT} &= ET_{HHT} \\ E_{HTT}T_{HTT} &= ET_{HTT} \\ ET_{HTH} &= 2(ET_{HT} + 1) \end{aligned}$$

So $ET_{HHH} = 14$, $ET_{HHT} = ET_{HTT} = 8$, and $ET_{HTH} = 10$. To explain the equaitons: In the first case if the first toss is H, $T_{HHH} = 1$, while if the first toss is T one turn has been wasted. In the second and third cases, the first three tosses provide nothing useful. For the fourth, we note that we first have to wait for HT. If the next toss is H we are done. If not then the game starts again.

1.60. *Sucker bet.* Consider the following gambling game. Player 1 picks a three coin pattern (for example HTH) and player 2 picks another (say THH). A coin is flipped repeatedly and outcomes are recorded until one of the two patterns appears. Somewhat surprisingly player 2 has a considerable advantage in this game. No matter what player 1 picks, player 2 can win with probability $\geq 2/3$. Suppose without loss of generality that player 1 picks a pattern that begins with H:

case	Player 1	Player 2	Prob. 2 wins
1	HHH	THH	7/8
2	HHT	THH	3/4
3	HTH	HHT	2/3
4	HTT	HHT	2/3

Verify the results in the table. You can do this by solving six equations in six unknowns but this is not the easiest way.

Ans. 1. Player 1 wins if the first three tosses are HHH but loses otherwise. 2. Player 1 wins if the first two tosses are HH but loses otherwise. 3. Nothing can happen until the first H. If the next is H then 2 will win. If the next two are TH player 1 wins. If the next two are TT things start over. 4. Nothing can happen until the first H. If the next is H then 2 will win. If the next two are TT player 1 wins. If the next two are TH things start over.

1.61. At the New York State Fair in Syracuse, Larry encounters a carnival game where for one dollar he may buy a single coupon allowing him to play a guessing game. On each play, Larry has an even chance of winning or losing a coupon. When he runs out of coupons he loses the game. However, if he can collect three coupons, he wins a surprise. (a) What is the probability Larry will win the surprise? (b) What is the expected number of plays he needs to win or lose the game.

Ans. (a) Let p_i be the probability he will win when he has i coupons. By considering what happens on one step: $p_1 = p_2/2$, $p_2 = 1/2+p_1/2 = 1/2+p_2/4$ so $p_2 = 2/3$ and $p_1 = 1/3$. (b) Let e_i be the expected duration of the game. By considering what happens on the first step: $e_1 = 1+e_2/2$, $e_2 = 1+e_1/2$. From this it is clear that $e_1 = e_2 = c$. Solving gives $c = 2$.

1.62. The Megasoft company gives each of its employees the title of programmer (P) or project manager (M). In any given year 70% of programmers remain in that position 20% are promoted to project manager and 10% are fired (state X). 95% of project managers remain in that position while 5% are fired. How long on the average does a programmer work before they are fired?

Ans. The transition probability is

	P	**M**	**X**
P	.7	.2	.1
M	0	.95	.05
X	0	0	1

If we let p be the 2×2 matrix of trasnsitions between nonabsorbing states then

$$(I-p)^{-1} = \begin{matrix} 3.333 & 13.333 \\ 0 & 20 \end{matrix}$$

so the expected time is $3.333 + 13.333$.

1.63. At a nationwide travel agency, newly hired employees are classified as beginners (B). Every six months the performance of each agent is reviewed. Past records indicate that transitions through the ranks to intermediate (I) and qualified (Q) are according to the following Markov chain, where F indicates workers that were fired:

	B	**I**	**Q**	**F**
B	.45	.4	0	.15
I	0	.6	.3	.1
Q	0	0	1	0
F	0	0	0	1

(a) What fraction are eventually promoted? (b) What is the expected time until a beginner is fired or becomes qualified?

Ans. (a) Let $h(I)$ be the fraction of intermediates that reach Q. $h(I) = 0.6h(I) + 0.3$, so $h(I) = 3/4$. The fraction of beginners that make it, satisfies $h(B) = 0.45h(B)+0.4h(I)$ so $h(B) = .3/.55 = 6/11$. (b) If we let r be the 2×2 matrix of transitions between nonabsorbing states then

$$(I-p)^{-1} = \begin{pmatrix} 1.8181 & 1.8181 \\ 0 & 2.5 \end{pmatrix}$$

so the expected time is $2(1.8181) = 3.636$.

1.64. At a manufacturing plant, employees are classified as trainee (R), technician (T) or supervisor (S). Writing Q for an employee who quits we model their progress through the ranks as a Markov chain with transition probability

	R	**T**	**S**	**Q**
R	.2	.6	0	.2
T	0	.55	.15	.3
S	0	0	1	0
Q	0	0	0	1

(a) What fraction of recruits eventually make supervisor? (b) What is the expected time until a trainee auits or becomes supervisor?

Ans. (a) Let $h(T)$ be the fraction of technicians that reach S. $h(T) = 0.55h(T) + 0.15$, so $h(T) = 1/3$. The fraction of trainees that make it, satisfies $h(R) = 0.2h(R) + 0.6h(T)$ so $h(R) = (.6/.8)(1/3) = 1/4$. (b) If we let p be the 2×2 matrix of transitions between nonabsorbing states then

$$(I-p)^{-1} = \begin{pmatrix} 1.25 & 1.666 \\ 0 & 2.222 \end{pmatrix}$$

so the expected time is $1.25 + 1.666 = 3.916$.

1.65. Customers shift between variable rate loans (V), thirty year fixed-rate loans (30), fifteen year fixed-rate loans (15), or enter the states paid in full (P), or foreclosed according to the following transition matrix:

	V	**30**	**15**	**P**	**f**
V	.55	.35	0	.05	.05
30	.15	.54	.25	.05	.01
15	.20	0	.75	.04	.01
P	0	0	0	1	0
F	0	0	0	0	1

(a) For each of the three loan types find (a) the expected time until paid or foreclosed. (b) the probability the loan is paid.

Ans. (a) If we let r be the 3×3 matrix of transitions between nonabsorbing states then

$$(I-p)^{-1} = \begin{pmatrix} 5.4437 & 4.1420 & 4.1420 \\ 4.1420 & 5.3254 & 5.3254 \\ 4.3550 & 3.3136 & 7.313 \end{pmatrix}$$

so the expected time is $(I-p)^{-1}\mathbf{1} = 13.7278, 14.7928, 14.9822$. The answer to (b) is

$$(I-p)^{-1} \begin{pmatrix} .05 \\ .05 \\ .04 \end{pmatrix} = \begin{matrix} .6449 \\ .6863 \\ .6759 \end{matrix}$$

brosis **1.66.** *Brother–sister mating.* In this genetics scheme two individuals (one male and one female) are retained from each generation and are mated to give the next. If the individuals involved are diploid and we are interested in a trait with two alleles, A and a, then each individual has three possible states AA, Aa, aa or more succinctly 2, 1, 0. If we keep track of the sexes of the two individuals the chain has nine states, but if we ignore the sex there are just six: 22, 21, 20, 11, 10, and 00. (a) Assuming that reproduction corresponds to picking one letter at random from each parent, compute the transition probability. (b) 22 and 00 are absorbing states for the chain. Show that the probability of absorption in 22 is equal to the fraction of A's in the state. (c) Let $T = \min\{n \geq 0 : X_n = 22 \text{ or } 00\}$ be the absorption time. Find $E_x T$ for all states x.

Ans.

	22	21	20	11	10	00
22	1	0	0	0	0	0
21	1/4	1/2	0	1/4	0	0
20	0	0	0	1	0	0
11	1/16	1/4	1/8	1/4	1/4	1/16
10	0	0	0	1/4	1/2	1/4
00	0	0	0	0	0	1

(b) Let $h(x)$ be the fraction of A's in state x. It is easy to see that $E_x h(X_1) = h(x)$. Since $h(22) = 1$ and $h(00) = 0$, it follows from (6.3) that $P_x(V_{22} < V_{00}) = h(x)$. (c) (a) If we let r be the 4×4 matrix of transitions between nonabsorbing states then

$$(I-p)^{-1} = \begin{pmatrix} 8/3 & 1/6 & 4/3 & 2/3 \\ 4/3 & 4/3 & 8/3 & 4/3 \\ 4/3 & 1/3 & 8/3 & 4/3 \\ 2/3 & 1/6 & 4/3 & 8/3 \end{pmatrix}$$

so the expected time is $(I-p)^{-1}\mathbf{1} = 29/6, 20/3, 17/3, 29/6$.

1.67. Roll a fair die repeatedly and let $Y_1, Y_2, \dots$ be the resulting numbers. Let $X_n = |\{Y_1, Y_2, \dots, Y_n\}|$ be the number of values we have seen in the first n rolls for $n \geq 1$ and set $X_0 = 0$. X_n is a Markov chain. (a) Find its transition probability. (b) Let $T = \min\{n : X_n = 6\}$ be the number of trials we need to see all 6 numbers at least once. Find ET.

Ans. (a) The transition matrix is

	0	**1**	**2**	**3**	**4**	**5**	**6**
0	0	1	0	0	0	0	0
1	0	1/6	5/6	0	0	0	0
2	0	0	2/6	4/6	0	0	0
3	0	0	0	3/6	3/6	0	0
4	0	0	0	0	4/6	2/6	0
5	0	0	0	0	0	5/6	1/6
6	0	0	0	0	0	0	1

(b) Let $h(x) = E_x T_6$. We have, $h(0) = 1 + h(1)$, $h(1) = 1 + (1/6)h(1) + (5/6)h(2)$ so $h(1) = 6/5 + h(2)$. Similarly $h(2) = 6/4 + h(3)$, $h(3) = 6/3 + h(4)$, $h(4) = 6/2 + h(5)$, $h(5) = 6/1 + h(6)$. Since $h(6) = 0$ it follows that

$$h(0) = \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = 13.7$$

1.68. *Coupon collector's problem.* We are interested now in the time it takes to collect a set of N baseball cards. Let T_k be the number of cards we have to buy before we have k that are distinct. Clearly, $T_1 = 1$. A little more thought reveals that if each time we get a card chosen at random from all N possibilities, then for $k \geq 1$, $T_{k+1} - T_k$ has a geometric distribution with success probability $(N-k)/N$. Use this to show that the mean time to collect a set of N baseball cards is $\approx N \log N$, while the variance is $\approx N^2 \sum_{k=1}^{\infty} 1/k^2$.

Ans. The mean and variance of a geometric with success probability p are $1/p$ and $(1-p)/p^2$ respectively. Using this we have

$$\begin{aligned} E_0 T_N &= N \cdot \left\{ \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{2} + \frac{1}{1} \right\} \\ &\sim N \int_1^N \frac{dx}{x} = N \log N \end{aligned}$$

$\text{var}\,(T_N) = \sum_{k=1}^{n} (N/k)^2 - (N/k) \approx N^2 \sum_{k=1}^{\infty} 1/k^2$.

1.69. *Algorthmic efficiency.* The simplex method minimizes linear functions by moving between extreme points of a polyhedral region so that each transition decreases the objective function. Suppose there are n extreme points and they are numbered in increasing order of their values. Consider the Markov chain in which $p(1,1) = 1$ and $p(i,j) = 1/i - 1$ for $j < i$. In words, when we leave j we are equally likely to go to any of the extreme points with better value. (a) Use (1.25) to show that for $i > 1$

$$E_i T_1 = 1 + 1/2 + \cdots + 1/(i-1)$$

(b) Let $I_j = 1$ if the chain visits j on the way from n to 1. Show that for $j < n$

$$P(I_j = 1 | I_{j+1}, \ldots I_n) = 1/j$$

to get another proof of the result and conclude that $I_1, \ldots I_{n-1}$ are independent.

Infinite State Space

1.70. *General birth and death chains.* The state space is $\{0, 1, 2, \ldots\}$ and the transition probability has

$$\begin{aligned} &p(x, x+1) = p_x \\ &p(x, x-1) = q_x \qquad \text{for } x > 0 \\ &p(x, x) = r_x \qquad\quad\ \text{for } x \geq 0 \end{aligned}$$

while the other $p(x,y) = 0$. Let $V_y = \min\{n \geq 0 : X_n = y\}$ be the time of the first visit to y and let $h_N(x) = P_x(V_N < V_0)$. By considering what happens on the first step, we can write

$$h_N(x) = p_x h_N(x+1) + r_x h_N(x) + q_x h_N(x-1)$$

Set $h_N(1) = c_N$ and solve this equation to conclude that 0 is recurrent if and only if $\sum_{y=1}^{\infty} \prod_{x=1}^{y-1} q_x/p_x = \infty$ where by convention $\prod_{x=1}^{0} = 1$.

Ans. Rearranging the equation gives

$$h_N(x+1) - h_N(x) = (q_x/p_x)(h_N(x+1) - h_N(x))$$

Since $h_N(0) = 0$ and we have assumed $h_N(1) = c_N$ we have $h_N(y) - h_N(y-1) = c_N \prod_{x=1}^{y-1} q_x/p_x$. Let $\phi(z) = \sum_{y=1}^{z} \prod_{x=1}^{y-1} q_x/p_x$. Summing we have $h_N(z) = c_N \phi(z)$ so $c_N = 1/\phi(N)$. In order for 0 to be recurrent we must have $P_1(V_N < V_0) \to 0$ as $N \to \infty$ which is equivalent to $\phi(N) \to \infty$.

1.71. To see what the conditions in the last problem say we will now consider some concrete examples. Let $p_x = 1/2$, $q_x = e^{-cx^{-\alpha}}/2$, $r_x = 1/2 - q_x$ for $x \geq 1$ and $p_0 = 1$. For large x, $q_x \approx (1 - cx^{-\alpha})/2$, but the exponential formulation keeps the probabilities nonnegative and makes the problem easier to solve. Show that the chain is recurrent if $\alpha > 1$ or if $\alpha = 1$ and $c \leq 1$ but is transient otherwise.

Ans. Let $\Sigma = \sum_{y=1}^{\infty} \prod_{x=1}^{y-1} q_x/p_x$. $\prod_{x=1}^{y-1} q_x/p_x = \exp(-c \sum_{x=1}^{y-1} x^{-\alpha})$. If $\alpha > 1$ then $\sum_{x=1}^{\infty} x^{-\alpha}$ converges so the product $\prod_{x=1}^{y-1} q_x/p_x$ is bounded away from 0, and $\Sigma = \infty$. If $\alpha = 1$ then comparing the sum with an integral gives

$$c \log y \leq c \sum_{x=1}^{y-1} x^{-1} \leq c(1 + \log y)$$

so $\Sigma = \infty$ if and only if $c \leq 1$. When $\alpha < 1$ a similar comparison shows

$$\sum_{x=1}^{y-1} x^{-\alpha} \geq (1-\alpha)^{-1} y^{1-\alpha}$$

and we conclude that $\Sigma < \infty$.

1.72. Consider the Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probability

$$p(m, m+1) = \frac{1}{2}\left(1 - \frac{1}{m+2}\right) \qquad \text{for } m \geq 0$$
$$p(m, m-1) = \frac{1}{2}\left(1 + \frac{1}{m+2}\right) \qquad \text{for } m \geq 1$$

and $p(0,0) = 1 - p(0,1) = 3/4$. Find the stationary distribution π.

Ans. Detailed balance tells us that the stationary distribution has

$$\pi(m+1) = \frac{p(m, m+1)}{p(m+1, m)} \pi(m) = \frac{(m+1)(m+3)}{(m+2)(m+4)} \pi(m)$$

From this we see $\pi(m) = c/(m+1)(m+3)$. To pick the constant to make the sum 1, we note

$$\sum_{m=0}^{\infty} \frac{1}{(m+1)(m+3)} = \frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{(m+1)} - \frac{1}{(m+3)}$$

and $\frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \cdots = 1 + \frac{1}{2}$ so $c = 4/3$.

1.73. Consider the Markov chain with state space $\{1, 2, \dots\}$ and transition probability

$$\begin{aligned} p(m, m+1) &= m/(2m+2) && \text{for } m \geq 1 \\ p(m, m-1) &= 1/2 && \text{for } m \geq 2 \\ p(m, m) &= 1/(2m+2) && \text{for } m \geq 2 \end{aligned}$$

and $p(1,1) = 1 - p(1,2) = 3/4$. Show that there is no stationary distribution.

Ans. Since the probability of being in $\{1, \dots m\}$ does not change in time, any stationary distribution has $\pi(m+1)p(m+1, m) = \pi(m)p(m, m+1)$. From this it follows that

$$\pi(m+1) = 2\frac{m}{2m+2}\pi(m)$$

and hence $\pi(m) = c/(m+1)$. However, $\sum_m 1/(m+1) = \infty$, so we cannot pick c to make the sum 1.

1.74. Consider the aging chain on $\{0, 1, 2, \dots\}$ in which for any $n \geq 0$ the individual gets one day older from n to $n+1$ with probability p_n but dies and returns to age 0 with probability $1 - p_n$. Find conditions that guarantee that (a) 0 is recurrent, (b) positive recurrent. (c) Find the stationary distribution.

Ans. (a) The probability of avoiding 0 for n steps when we start at 0 is $p_0 p_1 \cdots p_{n-1}$ so a necessary and sufficient condition is that this product tends to 0 as $n \to \infty$. (b) Let T_0 be the time of the first return to 0.

$$P_0(T_0 \geq n) = p_0 p_1 \cdots p_{n-1}$$

so for the expected value to be finite we need $\sum_n P_0(T > n) < \infty$. (c) It is immediate from the formula in (8.2) that the stationary distribution has $\pi(n) = cp_0 p_1 \cdot p_{n-1}$ where $c = 1/E_0 T_0$ is chosen to make the sum 1. Alternatively one can simply check that for $n \geq 1$, $\pi(n-1)p(n-1, n) = \pi(n)$ and hence $\pi p = \pi$.

1.75. The opposite of the aging chain is the renewal chain with state space $\{0, 1, 2, \dots\}$ in which $p(i, i-1) = 1$ when $i > 0$. The only nontrivial part of the transition probability is $p(0, i) = p_i$. Show that this chain is always recurrent but is positive recurrent if and only if $\sum_n np_n < \infty$.

Ans. Clearly $P_0(T_0 < \infty) = 1$ since no matter how far we jump out we will eventually come back home. When we jump n units out it takes $n+1$ units of time to return to 0. The mean return time $E_0 T_0 = \sum_n (n+1)p_n$ but $\sum_n p_n = 1$ so this is finite if and only if the indicated condition holds.

1.76. Consider a branching process as defined in Example 7.2, in which each family has exactly three children, but invert Galton and Watson's original motivation and ignore male children. In this model a mother will have an average of 1.5 daughters. Compute the probability that a given woman's descendents will die out.

Ans. The generating function is $\phi(x) = (1 + 3x + 3x^2 + x^3)/8$ The fixed point equation $(1 - 5x + 3x^2 + x^3)/8 = 0$ is a cubic but using the fact that $x = 1$ is always a root we can reduce it to a quadratic by dividing by $x - 1$:

$$\frac{1}{8}x^2 + \frac{1}{2}x - \frac{1}{8} = 0$$

Using the quadratic formula we have

$$\rho = \frac{-1/2 + \sqrt{1/4 - 4(1/8)(-1/8)}}{2/8} = -2 + \sqrt{5} = .236$$

1.77. Consider a branching process as defined in Example 7.2, in which each family has a number of children that follows a shifted geometric distribution: $p_k = p(1-p)^k$ for $k \geq 0$, which counts the number of failures before the first success when success has probability p. Compute the probability that starting from one individual the chain will be absorbed at 0.

Ans. The mean of the shifted geometric is $1/p - 1$ so results in Section 1.7 imply that extinction is certain when $p \geq 1/2$. To find the extinction probability when $p < 1/2$ we first compute the generating function

$$\phi(x) = \sum_{k=0}^{\infty} p(1-p)^k x^k = \frac{p}{1-(1-p)x}$$

Setting $\phi(x) = x$ leads to the quadratic equation $(1-p)x^2 - x + p = 0$. Recalling that $x = 1$ is always a root we can factor this to

$$(x-1)\{(1-p)x - p\} = 0$$

and conclude that extinction probability is $p/(1-p)$.

2.6 Exercises

Exponential distribution

2.1. Suppose that the time to repair a machine is exponentially distributed random variable with mean 2. (a) What is the probability the repair takes more than 2 hours. (b) What is the probability that the repair takes more than 5 hours given that it takes more than 3 hours.

Ans. (a) e^{-1}, (b) again e^{-1} by the lack of memory property of the exponential.

2.2. The lifetime of a radio is exponentially distributed with mean 5 years. If Ted buys a 7 year-old radio, what is the probability it will be working 3 years later?

Ans. By the lack of memory property this is the same as the probability a new radio lasts 3 years or $e^{-3/5}$.

2.3. A doctor has appointments at 9 and 9:30. The amount of time each appointment lasts is exponential with mean 30. What is the expected amount of time after 9:30 until the second patient has completed his appointment?

Ans. With probability $1 - e^{-1}$ the first patient is done before 9:30 and the expected time is 30 minutes. If not then then the expected time is 60 minutes, so the answer is $(1 - e^{-1})30 + e^{-1}60$.

2.4. Copy machine 1 is in use now. Machine 2 will be turned on at time t. Suppose that the machines fail at rate λ_i. What is the probability that machine 2 is the first to fail?

Ans. Machine 1 must work until time t, then it is a race. $e^{-\lambda_i t}\lambda_2/(\lambda_1 + \lambda_2)$.

2.5. Three people are fishing and each catches fish at rate 2 per hour. How long do we have to wait until everyone has caught at least one fish?

Ans. The time until the first fish is caught is the minimum of three rate 2 exponentials so it is exponential with rate 6 and hence mean 1/6. Once the first person has caught a fish, there are two waiting to catch one so the rate drops to 4 and the mean grows to 1/4. Finally there is only one rate 2 fisherman who needs mean 1/2 time to get his fish. Adding up gives $1/2 + 1/4 + 1/6 = 11/12$.

2.6. Alice and Betty enter a beauty parlor simultaneously, Alice to get a manicure and Betty to get a haircut. Suppose the time for a manicure (haircut) is exponentially distributed with mean 20 (30) minutes. (a) What is the probability Alice gets done first? (b) What is the expected amount of time until Alice and Betty are both done?

Ans. (a) The rates are 1/20 and 1/30 so Alice finishes first with probability

$$\frac{1/20}{1/20 + 1/30} = \frac{30}{30 + 20} = \frac{3}{5}$$

(b) The total service rate is $1/30 + 1/20 = 5/60$, so again the time until the first customer completes service is exponential with mean 12 minutes. With probability 3/5, Alice is done first. When this happens the lack of memory property of the exponential implies that it will take an average of 30 minutes for Betty to complete her haircut. With probability 2/5's Betty is done first and Alice will take an average of 20 more minutes. Combining we see that the total waiting time is

$$12 + (3/5) \cdot 30 + (2/5) \cdot 20 = 12 + 18 + 8 = 38$$

2.7. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S,T\}$ and $V = \max\{S,T\}$. Find (a) EU. (b) $E(V-U)$, (c) EV. (d) Use the identity $V = S+T-U$ to get a different looking formula for EV and verify the two are equal.

Ans. (a) U is exponential with rate $\lambda+\mu$ so $EU = 1/(\lambda+\mu)$. (b) $U = S$ with probability $\lambda/(\lambda+\mu)$. In this case $E(V-U) = 1/\mu$. $U = T$ with probability $\mu/(\lambda+\mu)$. In this case $E(V-U) = 1/\lambda$. Combining gives

$$E(V-U) = \frac{\lambda}{\lambda+\mu} \cdot \frac{1}{\mu} + \frac{\mu}{\lambda+\mu} \cdot \frac{1}{\lambda}$$

(c)
$$EV = EU + E(V-U) = \frac{1}{\lambda+\mu}\left(1 + \frac{\lambda}{\mu} + \frac{\mu}{\lambda}\right)$$

(d)
$$EV = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}$$

Since $\lambda/\mu + \mu/\lambda = (\lambda+\mu)/\mu + (\lambda+\mu)/\lambda - 2$ the two formulas are equal.

2.8. Let S and T be exponentially distributed with rates λ and μ. Let $U = \min\{S,T\}$, $V = \max\{S,T\}$, and $W = V - U$. Find the variances of U, V, and W.

Ans. U is exponential with rate $\lambda+\mu$ so by (2.5) $\text{var}\,(U) = 1/(\lambda+\mu)^2$.
Let $W = V - U$. Using (b) of the previous exercise

$$\begin{aligned}
EW^2 &= \frac{\lambda}{\lambda+\mu} \cdot \frac{2}{\mu^2} + \frac{\mu}{\lambda+\mu} \cdot \frac{2}{\lambda^2} \\
(EW)^2 &= \left(\frac{\lambda}{\lambda+\mu}\right)^2 \cdot \frac{1}{\mu^2} + \left(\frac{\mu}{\lambda+\mu}\right)^2 \cdot \frac{1}{\lambda^2} - 2 \cdot \frac{\lambda}{(\lambda+\mu)\mu} \cdot \frac{\mu}{(\lambda+\mu)\lambda} \\
&= \frac{1}{(\lambda+\mu)^2}\left(\frac{\mu^2}{\lambda^2} + \frac{\lambda^2}{\mu^2} - 2\right)
\end{aligned}$$

U and W are independent, so $\text{var}\,(V) = \text{var}\,(U) + \text{var}\,(W)$.

2.9. In a hardware store you must first go to server 1 to get your goods and then go to a server 2 to pay for them. Suppose that the times for the two activities are exponentially distributed with means 6 minutes and 3 minutes. (a) Compute the average amount of time it take Bob to get his goods and pay if when he comes in there is one customer named Al with server 1 and no one at server 2. (b) Find the answer when times for the two activities are exponentially distributed with rates λ and μ.

Ans. It takes an average of 6 minutes for Al to get his goods. Once this happens Bob starts getting served by server 1 at the same time Al is served at server 2. The time until the first of them is finished is exponential with rate $1/3 + 1/6 = 1/2$ so the mean is two minutes. Al finishes first with probability 2/3. In this case, there is no one ahead of Bob and he will be done in 6+3 more minutes on the average. Bob finishes first with probability 1/3. In this case, Al must wait for Bob to finish and then pay himself so the average time is 3+3 more minutes. Adding things up the total time is

$$6 + 2 + (2/3) \cdot 9 + (1/3) \cdot 6 = 16$$

To check this note that Bob will wait behind Al with probability 1/3 so he loses an average of 1 minute for that. Thus his expected wait is Al's first six minute service time plus his own 9 minutes in service plus the 1 extra minute from having to wait behind Al. So the total time has expected value 6+9+1=16.
(b) Reasoning as at the end of part (a), Bob must always wait an average of $1/\lambda$ minutes for Al to get his order, and will be in service for an average of $1/\lambda + 1/\mu$ minutes. He will wait an additional $1/\mu$ minutes at the cash register if he gets done first, an event with probability $\lambda/(\lambda + \mu)$. Adding up we have

$$\frac{2}{\lambda} + \frac{1}{\mu} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\mu}$$

2.10. Consider a bank with two tellers. Three people, Alice, Betty, and Carol enter the bank at almost the same time and in that order. Alice and Betty go directly into service while Carol waits for the first available teller. Suppose that the service times for each customer are exponentially distributed with mean 4 minutes. (a) What is the expected total amount of time for Carol to complete her businesses? (b) What is the expected total time until the last of the three customers leaves? (c) What is the probability Carol is the last one to leave?

Ans. (a) Each server has rate 1/4, so the time until the first customer completes service is exponential with rate 1/2, and has mean two minutes. At that time Carol enters service and requires an average of 4 minutes of service, for a total of six minutes. (b) The mean time until the first customer departs is 2 minutes by (a). At this point there are two customers in service so on the average another 2 minutes is required until one of them leaves. Once there is only one customer, the time to the final departure has mean 4, so the total is $2 + 2 + 4 = 8$. (c) 1/2. When she enters service she has an equal chance of beating the customer that remains.

2.11. Consider the set-up of the previous problem but now suppose that the two tellers have exponential service times with rates $\lambda \le \mu$. Again, answer questions (a), (b), and (c).

Ans. (a) The total service rate is $\lambda + \mu$, so the time until the first customer completes service is exponential has mean $1/(\lambda+\mu)$. With probability $\lambda/(\lambda+\mu)$, server 1 is done first and Carol's mean service time will be $1/\lambda$. With probability $\mu/(\lambda + \mu)$, server 2 is done first and Carol's mean service time will be $1/\mu$. Combining we see that the total waiting time is

$$\frac{1}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\lambda} + \frac{\mu}{\lambda + \mu} \cdot \frac{1}{\mu} = \frac{3}{\lambda + \mu}$$

(b) The mean time until the first customer departs is $1/(\lambda+\mu)$ minutes by (a). At this point there are two customers in service so on the average another $1/(\lambda+\mu)$ minutes is required until one of them leaves. When there is only one customer, they will be at the slow server with probability $\mu/(\lambda+\mu)$ and at the fast server with probability $\lambda/(\mu+\lambda)$ so the time to the final departure has mean

$$\frac{2}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\cdot\frac{1}{\lambda} + \frac{\lambda}{\lambda+\mu}\cdot\frac{1}{\mu}$$

(c) Carol will enter service at the fast server with probability $\mu/(\lambda+\mu)$ and at the slow server with probability $\lambda/(\lambda+\mu)$. Since again the fast server will finish first with probability $\mu/(\lambda+\mu)$, Carol's probability of leaving last is

$$\frac{\lambda}{\lambda+\mu}\cdot\frac{\mu}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\cdot\frac{\lambda}{\lambda+\mu}$$

2.12. A flashlight needs two batteries to be operational. You start with four batteries numbered 1 to 4. Whenever a battery fails it is replaced by the lowest-numbered working battery. Suppose that battery life is exponential with mean 100 hours. Let T be the time at which there is one working battery left and N be the number of the one battery that is still good. (a) Find ET. (b) Find the distribution of N. (c) Solve (a) and (b) for a general number of batteries.

Ans. (a) Each battery fails at rate 1/100 so one of the two fails at rate 1/50 and the time between failures is exponential with mean 50. It takes three battery failures to leave us with only one working battery so $ET = 150$. (b) In order for 4 to be last it must beat only the battery that is in there when it is added so $P(N=4) = 1/2$. In order for 3 to be last, it must beat the battery it joined and then beat 4, so $P(N=3) = 1/4$. In order for 1 or 2 to be last they must win three races, so $P(N=1) = P(N=2) = 1/8$. (c) With n batteries we need $n-1$ failures so $ET = 50(n-1)$. From the solution of part (b) we see that $P(N=j) = 1/2^{n+1-j}$ for $2 \le j \le n$ (start with $j=n$ and work backwards) while $P(N=1) = P(N=2) = 1/2^{n-1}$.

2.13. A machine has two critically important parts and is subject to three different types of shocks. Shocks of type i occur at times of a Poisson process with rate λ_i. Shocks of types 1 break part 1, those of type 2 break part 2, while those of type 3 break both parts. Let U and V be the failure times of the two parts. (a) Find $P(U>s, V>t)$. (b) Find the distribution of U and the distribution of V. (c) Are U and V independent?

Ans. (a) $\exp(-\lambda_1 s - \lambda_2 t - \lambda_3(s\vee t))$. (b) Letting $t=0$ in (a) shows that U is exponential rate $\lambda_1+\lambda_3$. Letting $s=0$ shows that V is exponential with rate $\lambda_2+\lambda_3$. (c) No, since

$$\begin{aligned}P(U>s,V>t) &= \exp(-\lambda_1 s - \lambda_2 t - \lambda_3(s\vee t)) \\ &> \exp(-(\lambda_1+\lambda_3)s - (\lambda_2+\lambda_3)t) = P(U>s)P(V>t)\end{aligned}$$

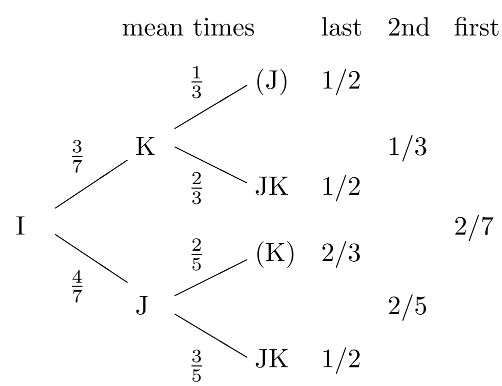
2.14. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. What is the average length of time the boat can remain at sea.

Ans. The rates for the three exponentials are 1, 2/3 and 1/3. Thus the time to the first failure is exponential with rate $2 = 1 + 2/3 + 1/3$, so the mean time to first failure is 1/2. 1/2 of the time part 1 is the first to fail. In this case the time to the next failure has rate $2/3 + 1/3 = 1$ so the mean is 1. Part 2 is the first to fail with probability 2/6. In this case the time to the next failure has rate $1 + 1/3 = 4/3$ or mean 3/4. Part 3 is the first to fail with probability 1/6. In this case the time to the next failure has rate $1 + 2/3 = 5/3$ and mean 3/5. Adding things up the mean time until the second failure is

$$1/2 + (1/2) \cdot 1 + (1/3) \cdot (3/4) + (1/6) \cdot (3/5) = .5 + .5 + .25 + .10 = 1.35 \text{ years}$$

2.15. Excited by the recent warm weather Jill and Kelly are doing spring cleaning at their apartment. Jill takes an exponentially distributed amount of time with mean 30 minutes to clean the kitchen. Kelly takes an exponentially distributed amount of time with mean 40 minutes to clean the bath room. The first one to complete their task will go outside and start raking leaves, a task that takes an exponentially distributed amount of time with a mean of one hour. When the second person is done inside, they will help the other and raking will be done at rate 2. (Of course the other person may already be done raking in which case the chores are done.) What is the expected time until the chores are all done?

Ans. Label the states I = the initial state, J = Jill raking, K = Kelly raking, JK = Jill and Kelly both raking, (J) = raking done and Jill still working, (K) = raking done and Kelly still working. The rates for Jill and Kelly inside are 2 and 3/2, while the raking rate = the number of people.



$$\text{ans} = \frac{2}{7} + \left(\frac{3}{7} \cdot \frac{1}{3} + \frac{4}{7} \cdot \frac{2}{5}\right) + \left(\frac{8}{35} \cdot \frac{2}{3} + \frac{27}{35} \cdot \frac{1}{2}\right)$$
$$= \frac{180 + 90 + 144 + 96 + 243}{630} = \frac{753}{630} = 1.195238$$

RST **2.16.** Ron, Sue, and Ted arrive at the beginning of a professor's office hours. The amount of time they will stay is exponentially distributed with means of 1, 1/2, and 1/3 hour. (a) What is the expected time until only one student remains? (b) For each student find the probability they are the last student left. (c) What is the expected time until all three students are gone?

Ans. (a) When there are three students the rates are $1+2+3$ so the time until the first departure has mean 1/6. At this point there are three cases to consider

who's still there	mean time for next departure	probability
R,S	1/3	3/6
R,T	1/4	2/6
S,T	1/5	1/6

Multiplying and summing up the mean for the second stage is

$$1/6 + 1/12 + 1/30 = (10 + 5 + 2)/60 = 17/60$$

Adding the 10/60 for the first stage the answer is 27 minutes.
(b) There are six orders in which the students can depart. We compute their probabilities in the next table

TSR	$\frac{3}{6}\cdot\frac{2}{3}=\frac{1}{3}=\frac{20}{60}$	STR	$\frac{2}{6}\cdot\frac{3}{4}=\frac{1}{4}=\frac{15}{60}$	TRS	$\frac{3}{6}\cdot\frac{1}{3}=\frac{1}{6}=\frac{10}{60}$
RTS	$\frac{1}{6}\cdot\frac{3}{5}=\frac{1}{10}=\frac{6}{60}$	SRT	$\frac{2}{6}\cdot\frac{1}{4}=\frac{1}{12}=\frac{5}{60}$	RST	$\frac{1}{6}\cdot\frac{2}{5}=\frac{1}{15}=\frac{4}{60}$

Collecting results the probabilities of being last are R:35/60, S:16/60, T:9/60.
(c) Using the result from (b) we see that the time at which there is only one student has mean

$$1\cdot\frac{35}{60}+\frac{1}{2}\cdot\frac{16}{60}+\frac{1}{3}\cdot\frac{9}{60}=\frac{35+8+3}{60}=\frac{46}{60}$$

Adding this to the answer from (a) gives 73 minutes.

2.17. Let T_i, $i = 1, 2, 3$ be independent exponentials with rate λ_i. (a) Show that for any numbers t_1, t_2, t_3

$$\begin{aligned}\max\{t_1,t_2,t_3\} = t_1+t_2+t_3 &- \min\{t_1,t_2\} - \min\{t_1,t_3\}\\ &- \min\{t_2,t_3\} + \min\{t_1,t_2,t_3\}\end{aligned}$$

(b) Use (a) to find $E\max\{T_1, T_2, T_3\}$. (c) Use the formula to give a simple solution of part (c) of Exercise 2.16.

Ans. (a) Since the formula is not changed by permuting the t_i, we can without loss of generality suppose $t_1 > t_2 > t_3$. In this case the formula says

$$t_1 = t_1 + t_2 + t_3 - t_2 - t_3 - t_3 + t_3$$

which is correct. (b) Taking expected value

$$\begin{aligned}E\max\{T_1,T_2,T_3\} &= \frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}\\ &\quad -\frac{1}{\lambda_1+\lambda_2}-\frac{1}{\lambda_1+\lambda_3}-\frac{1}{\lambda_2+\lambda_3}+\frac{1}{\lambda_1+\lambda_2+\lambda_3}\end{aligned}$$

(c) In Exercise 2.16, $\lambda_i = 1/i$ for $i = 1, 2, 3$ so

$$\begin{aligned} E\max\{T_1, T_2, T_3\} &= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \frac{1}{6} \\ &= 1 + \frac{30 - 15 - 12 + 10}{60} = 73 minutes \end{aligned}$$

Poisson approximation to binomial

2.18. Compare the Poisson approximation with the exact binomial probabilities of 1 success when $n = 20$, $p = 0.1$.

Ans. In either case the Poisson approximation is $e^{-2}2^1/1! = 0.2706$. The exact probabilities is $20(.9)^{19}(.1) = .2701$.

2.19. Compare the Poisson approximation with the exact binomial probabilities of no success when (a) $n = 10$, $p = 0.1$, (b) $n = 50$, $p = 0.02$.

Ans. In either case the Poisson approximation is $e^{-1} = 0.3678$. The exact probabilities are (a) $(.9)^{10} = .3486$, (b) $(.98)^{50} = 0.3641$.

2.20. The probability of a three of a kind in poker is approximately 1/50. Use the Poisson approximation to estimate the probability you will get at least one three of a kind if you play 20 hands of poker.

Ans. $1 - e^{-0.4} = 0.3296$

2.21. Suppose 1% of a certain brand of Christmas lights is defective. Use the Poisson approximation to compute the probability that in a box of 25 there will be at most one defective bulb.

Ans. $(1.25)e^{-0.25} = 0.9735$

Poisson processes: Basic properties

2.22. Suppose $N(t)$ is a Poisson process with rate 3. Let T_n denote the time of the nth arrival. Find (a) $E(T_{12})$, (b) $E(T_{12}|N(2) = 5)$, (c) $E(N(5)|N(2) = 5)$.

Ans. (a) $ET_{12} = 12Et_i = 12/3 = 4$. (b) At time 2 we have 5 arrivals, so we need to get 7 more. $ET_7 = 7/3$ so $E(T_{12}|N(2) = 5) = 2 + 7/3 = 41/3$. (c) Independent increments implies that $E(N(5) - N(2)|N(2) = 5) = 9$. Adding $N(2) = 5$ we have $E(N(5)|N(2) = 5) = 14$.

2.23. Customers arrive at a shipping office at times of a Poisson process with rate 3 per hour. (a) The office was supposed to open at 8AM but the clerk Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period? (b) What is the distribution of the amount of time Oscar has to wait until his first customer arrives?

Ans. (a) $e^{-6} = .0025$. (b) By the lack of memory property, this is exponential with rate 3.

2.24. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. (a) What is the probability that fewer (i.e., $<$) than 2 calls came in the first hour? (b) Suppose that 6 calls arrive in the first hour, what is the probability there will be < 2 in the second hour. (c) Suppose that the operator gets to take a break after she has answered 10 calls. How long are her average work periods?

Ans. (a) The probability of 0 or 1 call is $e^{-4} + 4e^{-4} = .0916$.
(b) Increments are independent so the answer to (b) is the same as the answer to (a).
(c) 10/4, since the waiting times between the 10 calls are independent exponential with mean 1/4.

2.25. Traffic on Rosedale Road in Princeton, NJ, follows a Poisson process with rate 6 cars per minute. A deer runs out of the woods and tries to cross the road. If there is a car passing in the next 5 seconds then there will be a collision. (a) Find the probability of a collision. (b) What is the chance of a collision if the deer only needs 2 seconds to cross the road.

Ans. (a) 6 cars per minute is 1/10 of a car per second, so the probability of no collision is $e^{-5/10} = .6066$, and the probability of a collision is .3934. (b) When the time is 2 seconds the probability of no collision is $e^{-2/10} = .8187$, and the probability of a collision is .1813.

DrydenFD **2.26.** Calls to the Dryden fire department arrive according to a Poisson process with rate 0.5 per hour. Suppose that the time required to respond to a call, return to the station, and get ready to respond to the next call is uniformly distributed between 1/2 and 1 hour. If a new call comes before the Dryden fire department is ready to respond, the Ithaca fire department is asked to respond. Suppose that the Dryden fire department is ready to respond now. Find the probability distribution for the number of calls they will handle before they have to ask for help from the Ithaca fire department.

Ans. Let T be exponential with mean 2 and let U be uniform on $(1/2, 1)$. The number of calls N they will answer has $P(N = n) = (1-p)^{n-1}p$ for $n = 1, 2, \ldots$, where

$$p = P(U < T) = \int_{1/2}^{1} e^{-x/2}/2\, dx = -e^{-x/2}\Big|_{1/2}^{1} = e^{-1/4} - e^{-1/2}$$

2.27. A math professor waits at the bus stop at the Mittag-Leffler Institute in the suburbs of Stockholm, Sweden. Since he has forgotten to find out about the bus schedule, his waiting time until the next bus is uniform on (0,1). Cars drive by the bus stop at rate 6 per hour. Each will take him into town with probability 1/3. What is the probability he will end up riding the bus?

Ans. Rides are offered at rate 2 per hour. The probability of no offered ride before the bus comes is

$$\int_0^1 e^{-2u}\, du = -e^{-2u}\Big|_0^1 = 1 - e^{-2} = 0.8647$$

2.28. The number of hours between successive trains is T which is uniformly distributed between 1 and 2. Passengers arrive at the station according to a Poisson process with rate 24 per hour. Let X denote the number of people who get on a train. Find (a) EX, (b) $\text{var}(X)$.

Ans. (a) $E(X|T) = 24T$, so $EX = 36$. (b) It follows from Theorem 2.3 that $E(X^2|T) = 24T + (24T)^2$. Since

$$ET^2 = \int_1^2 t^2\, dt = 7/3$$

$EX^2 = 36 + 1344 = 1380$ and $\text{var}(X) = 1380 - (36)^2 = 84$.

2.29. Consider a Poisson process with rate λ and let L be the time of the last arrival in the interval $[0, t]$, with $L = 0$ if there was no arrival. (a) Compute $E(t - L)$ (b) What happens when we let $t \to \infty$ in the answer to (a)?

Ans. (a) $P(t - L > s) = e^{-\lambda s}$, so differentiating to find the density function and integrating by parts we have

$$\begin{aligned} E(t-L) &= \int_0^t s e^{-\lambda s}\, ds + t e^{-\lambda t} \\ &= -s e^{-\lambda s}\Big|_0^t + \int_0^t e^{-\lambda s}\, ds + t e^{-\lambda t} = \frac{1 - e^{-\lambda t}}{\lambda} \end{aligned}$$

(b) As $t \to \infty$ this $\to 1/\lambda$.

2.30. Customers arrive according to a Poisson process of rate λ per hour. Joe does not want to stay until the store closes at $T =$ 10PM, so he decides to close up when the first customer after time $T - s$ arrives. He wants to leave early but he does not want to lose any business so he is happy if he leaves before T and no one arrives after. (a) What is the probability he achieves his goal? (b) What is the optimal value of s and the corresponding success probability?

Ans. (a) He succeeds if there is exactly one arrival in the s units of time: $e^{-\lambda s}\lambda s$. (b) Differentiating gives $e^{-\lambda s}(-\lambda^2 s + \lambda)$ so $s = 1/\lambda$ and the success probability e^{-1}.

2.31. Customers arrive at a sporting goods store at rate 10 per hour. 60% of the customers are men and 40% are women. Women spend an amount of time shopping that is uniformly distributed on $[0, 30]$ minutes, while men spend an exponentially distributed amount of time with mean 30 minutes. Let M and N be the number of men and women in the store. What is the distribution of (M, N) in equilibrium.

Ans. Men's arrivals are Poisson rate 6 and men stay in the store for a mean 1/2 amount of time. Women's arrivals are Poisson rate 4 and women stay in the store for a mean 1/4 amount of time. By results for the $M/G/\infty$. $M =$ Poisson man 3 and $N =$ Poisson mean 1 are independent.

2.32. Let T be exponentially distributed with rate λ. (a) Use the definition of conditional expectation to compute $E(T|T<c)$. (b) Determine $E(T|T<c)$ from the identity

$$ET = P(T<c)E(T|T<c) + P(T>c)E(T|T>c)$$

Ans. (a) Integrating by parts gives

$$\begin{aligned} E(T;T<c) &= \int_0^c t\cdot\lambda e^{-\lambda t}\,dt = -te^{-\lambda t}\Big|_0^c + \int_0^c e^{-\lambda t}\,dt \\ &= -ce^{-\lambda c} + \frac{1}{\lambda}(1-e^{-\lambda c}) \end{aligned}$$

Dividing by $P(T>c) = 1-e^{-\lambda c}$ now gives

$$E(T|T<c) = \frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}}$$

(b) The lack of memory property implies that $E(T|T>c) = c + (1/\lambda)$. Using the identity it follows that

$$\begin{aligned} E(T;T<c) &= ET - E(T;T>c) = 1/\lambda - (c+1/\lambda)e^{-\lambda c} \\ &= (1-e^{-\lambda c})/\lambda - ce^{-\lambda c} \end{aligned}$$

Dividing by $P(T<c) = 1-e^{-\lambda c}$ now gives the result found in (a).

`chicken` **2.33.** *When did the chicken cross the road?* Suppose that traffic on a road follows a Poisson process with rate λ cars per minute. A chicken needs a gap of length at least c minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time T_{J-1} and complete his journey at time $T_{J-1}+c$. Use the previous exercise to show $E(T_{J-1}+c) = (e^{\lambda c}-1)/\lambda$.

Ans. Conditional on $J=j$, the interarrival times $t_1,\ldots,t_{j-1}$ are independent and have an exponential(λ) distribution conditioned to be $<c$. By (3.1), $ET_{J-1} = E(J-1)E(t_i|t_i<c)$. $t_i>c$ is an event of probability $e^{-\lambda c}$, and J has a geometric distribution with that success probability, so $E(J-1) = e^{\lambda c}-1$. In the previous problem we computed that

$$E(t_1|t_1<c) = \frac{1}{\lambda} - \frac{ce^{-\lambda c}}{1-e^{-\lambda c}}$$

Multiplying top and bottom by $e^{\lambda c}$ in the fraction and then combining with the formula for $E(J-1)$ gives the quoted result.

Random sums

2.34. Edwin catches trout at times of a Poisson process with rate 3 per hour. Suppose that the trout weigh an average of 4 pounds with a standard deviation of 2 pounds. Find the mean and standard deviation of the total weight of fish he catches in two hours.

Ans. Mean is $3 \cdot 2 \cdot 4 = 24$, Variance is $6 \cdot (4^2 + 2^2) = 120$, so the standard deviation is $\sqrt{120} = 10.95$.

2.35. An insurance company pays out claims at times of a Poisson process with rate 4 per week. Writing K as shorthand for "thousands of dollars," suppose that the mean payment is 10K and the standard deviation is 6K. Find the mean and standard deviation of the total payments for 4 weeks.

Ans. Mean is $4 \cdot 4 \cdot 10 = 160K$, Variance is $16 \cdot 36K^2 + 16 \cdot 100K^2 = 16 \cdot 136K^2 = 576K^2$, so the standard deviation is $\sqrt{2176K^2} = 46,647$.

2.36. Customers arrive at an automated teller machine at the times of a Poisson process with rate of 10 per hour. Suppose that the amount of money withdrawn on each transaction has a mean of \$30 and a standard deviation of \$20. Find the mean and standard deviation of the total withdrawals in 8 hours.

Ans. Mean is $8 \cdot 10 \cdot 30 = 2400$, the variance is $80 \cdot (30^2 + 20^2) = 80 \cdot 1300 = 104,000$ so the standard deviation is 322.5

2.37. As a community service members of the Mu Alpha Theta fraternity are going to pick up cans from along a roadway. A Poisson mean 60 members show up for work. 2/3 of the workers are enthusiastic and will pick up a mean of 10 cans with a standard deviation of 5. 1/3 of the workers are lazy and will only pick up an average of 3 cans with a standard deviation of 2. Find the mean and standard deviation of the the number of cans collected.

Ans. The number of enthusiastic workers is Poisson mean 40, while the lazy ones are Poisson mean 20. The mean number of cans picked up is $40 \cdot 10 + 20 \cdot 3$. Using the fact that $\operatorname{var}(S_N) = \lambda E(X_i^2)$ when N is Poisson we have that the variance of the number of cans picked up is

$$40 \cdot (10^2 + 5^2) + 20 \cdot (3^2 + 2^2) = 40 \cdot 125 + 20 \cdot 13 = 5260$$

so the standard deviation is 72.52.

2.38. Let S_t be the price of stock at time t and suppose that at times of a Poisson process with rate λ the price is multiplied by a random variable $X_i > 0$ with mean μ and variance σ^2. That is,

$$S_t = S_0 \prod_{i=1}^{N(t)} X_i$$

where the product is 1 if $N(t) = 0$. Find $ES(t)$ and $\operatorname{var} S(t)$.

Ans. $E(S_t|N(t) = n) = S_0 \mu^n$, so

$$ES_t = S_0 \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \mu^n = S(0) e^{t\lambda(\mu - 1)}$$

To compute the variance we begin with $E(S(t)^2|N(t) = n) = S(0)^2 v^n$ where $v = (\mu^2 + \sigma^2)^n$, so

$$ES_t^2 = S_0^2 \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} v^n = S_0^2 e^{t\lambda(v-1)}$$

and hence $\operatorname{var}(S_t) = S_0^2 (e^{t\lambda(v-1)} - e^{t\lambda(\mu-1)})$.

2.39. Messages arrive to be transmitted across the internet at times of a Poisson process with rate λ. Let Y_i be the size of the ith message, measured in bytes, and let $g(z) = Ez^{Y_i}$ be the generating function of Y_i. Let $N(t)$ be the number of arrivals at time t and $S = Y_1 + \cdot + Y_{N(t)}$ be the total size of the messages up to time t. (a) Find the generating function $f(z) = E(z^S)$. (b) Differentiate and set $z = 1$ to find ES. (c) Differentiate again and set $z = 1$ to find $E\{S(S-1)\}$. (d) Compute $\operatorname{var}(S)$.

Ans. (a) Breaking things down according to the number of arrivals:

$$f(z) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} g(z)^n = \exp(-\lambda t[1 - g(z)])$$

(b) $f'(z) = \lambda g'(z) \exp(-\lambda t[1 - g(z)])$ so $ES = f'(1) = \lambda g'(1) = \lambda EY_i$.

$$\begin{aligned} f''(z) &= \left\{\lambda g''(z) + [\lambda g'(z)]^2\right\} \exp(-\lambda t[1 - g(z)]) \\ ES(S-1) = f''(1) &= \lambda g''(1) + [\lambda g'(1)]^2 = \lambda EY_i(Y_i - 1) + \lambda^2 (EY_i)^2 \end{aligned}$$

(d) As in Exercise 2.1. $\operatorname{var}(S) = ES(S-1) + ES - (ES)^2$. Plugging in our formulas

$$\operatorname{var}(S) = \lambda EY_i(Y_i - 1) + (\lambda EY_i)^2 + \lambda EY_i - (\lambda EY_i)^2 = \lambda EY_i^2$$

2.40. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let $T \geq 0$ be an independent with mean μ and variance σ^2. Find $\operatorname{cov}(T, N_T)$.

Ans. $E(TN_T|T) = TE(N_T|T) = \lambda T^2$, so $E(TN_T) = \lambda(\mu^2 + \sigma^2)$ and

$$\operatorname{cov}(T, N_T) = E(TN_T) - ET \cdot EN_T = \lambda\sigma^2$$

2.41. Let $t_1, t_2, \ldots$ be independent exponential(λ) random variables and let N be an independent random variable with $P(N = n) = (1-p)^{n-1}$. What is the distribution of the random sum $T = t_1 + \cdots + t_N$?

Ans. Suppose we thin the original Poisson process by independently accepting points with probability p. T is the time of the first point accepted so T has an exponential(λp) distribution.

Thinning and conditioning

2.42. Traffic on Snyder Hill Road in Ithaca, NY, follows a Poisson process with rate 2/3's of a vehicle per minute. 10% of the vehicles are trucks, the other 90% are cars. (a) What is the probability at least one truck passes in a hour? (b) Given that 10 trucks have passed by in an hour, what is the expected number of vehicles that have passed by. (c) Given that 50 vehicles have passed by in a hour, what is the probability there were exactly 5 trucks and 45 cars.

Ans. (a) The rate for trucks is $60 \cdot (2/3) \cdot 1/10 = 4$, so the probability is $1 - e^{-4}$. (b) The number of cars is independent of the number of trucks and has mean 36. Adding in the 10 trucks the conditional expectation of the number of vehicles is $36 + 10 = 46$. (c) The number of trucks is binomial$(50, 1/10)$ so the answer is $\binom{50}{5}(.1)^5(.9)^{45}$.

2.43. Rock concert tickets are sold at a ticket counter. Females and males arrive at times of independent Poisson processes with rates 30 and 20 customers per hour. (a) What is the probability the first three customers are female? (b) If exactly 2 customers arrived in the first five minutes, what is the probability both arrived in the first three minutes. (c) Suppose that customers regardless of sex buy 1 ticket with probability 1/2, two tickets with probability 2/5, and three tickets with probability 1/10. Let N_i be the number of customers that buy i tickets in the first hour. Find the joint distribution of (N_1, N_2, N_3).

Ans. (a) $(3/5)^3$, (b) $(3/5)^2$, (c) Independent Poissons. Means 25, 20, and 5.

2.44. Ellen catches fish at times of a Poisson process with rate 2 per hour. 40% of the fish are salmon, while 60% of the fish are trout. What is the probability she will catch exactly 1 salmon and 2 trout if she fishes for 2.5 hours?

Ans. The total number of fish she catches in 2.5 hours is Poisson with mean 5, so the number of salmon and the number of trout are independent Poissons with means 2 and 3. Thus the probability of interest is

$$e^{-2}\frac{2^1}{1!} \cdot e^{-3}\frac{3^2}{2!}$$

2.45. Signals are transmitted according to a Poisson process with rate λ. Each signal is successfully transmitted with probability p and lost with probability $1-p$. The fates of different signals are independent. For $t \geq 0$ let $N_1(t)$ be the number of signals successfully transmitted and let $N_2(t)$ be the number that are lost up to time t. (a) Find the distribution of $(N_1(t), N_2(t))$. (b) What is the distribution of $L =$ the number of signals lost before the first one is successfully transmitted?

Ans. (a) By (4.1) $N_1(t)$ and $N_2(t)$ are independent Poissons with means λpt and $\lambda(1-p)t$ (b) $P(L = k) = (1-p)^k p$ for $k = 0, 1, 2, \ldots$.

2.46. A policewoman on the evening shift writes a Poisson mean 6 number of tickets per hour. 2/3's of these are for speeding and cost \$100. 1/3's of these are for DWI and cost \$400. (a) Find the mean and standard deviation for the total revenue from the tickets she writes in an hour. (b) What is the probability that between 2AM and 3AM she writes 5 tickets for speeding and 1 for DWI. (c) Let A be the event that she writes no tickets between 1AM and 1:30, and N be the number of tickets she writes between 1AM and 2AM. Which is larger $P(A)$ or $P(A|N = 5)$? Don't just answer yes or no, compute both probabilities.

Ans. (a) If $X_i = 100$ with probability 2/3 and 400 with probability 1/3 then $EX_i = 600/3 = 200$ and the variance $E(X_i - 200)^2 = 10,000(2/3) + 40,000(1/3) = 20,000$. $N = \text{Poisson}(6)$ so $S_N = X_1 + \cdots X_N$ has $ES_N = ENEX_i = 6 \cdot 1200$ and

$$\text{var}\,(N) = EN\,\text{var}\,(X_i) + \text{var}\,(N)(EX_i)^2 = 6 \cdot 20,000 + 6 \cdot 40,000 = 360,000$$

so the standard deviation is 600.
(b) The number of speeding tickets S and the number of DWI's D are independent Poissons with means 4 and 2, so

$$P(S = 5, D = 1) = e^{-4}\frac{4^5}{5!} \cdot e^{-2}\frac{2^1}{1!}$$

(c) The number of tickets in 30 minutes is Poisson(3) so $P(A) = e^{-3} = 0.04978$. If we condition on $N = 5$ then the probability none occurs in the first half of the interval is $P(A|N = 5) = (1/2)^5 = 1/32 = 0.03125$.

2.47. Trucks and cars on highway US 421 are Poisson processes with rate 40 and 100 per hour respectively. 1/8 of the trucks and 1/10 of the cars get off on exit 257 to go to the Bojangle's in Yadkinville. (a) Find the probability that exactly 6 trucks arrive at Bojangle's between noon and 1PM. (b) Given that there were 6 truck arrivals at Bojangle's between noon and 1PM, what is the probability that exactly two arrived between 12:20 and 12:40? (c) Suppose that all trucks have 1 passenger while 30% of the cars have 1 passenger, 50% have 2, and 20% have 4. Find the mean and standard deviation of the number of customers are that arrive at Bojangles' in one hour.

Ans. (a) $e^{-5}5^6/6! = 0.1462$. (b) $C_{6,2}(1/3)^2(2/3)^4 = 15(16/729) = 80/243 = 0.3292$. (c) Trucks and cars leaving the highway are Poisson processes with rates 5 and 10 respectively. The mean number of customers is

$$5 \cdot 1 + 10 \cdot [(0.3)1 + (0.5)2 + (0.2)4] = 26$$

The variance is

$$5 \cdot 1 + 10 \cdot [(0.3)1 + (0.5)4 + (0.2)16] = 55$$

so the standard deviation is $\sqrt{60} = 7.746$.

2.48. When a power surge occurs on an electrical line, it can damage a computer without a surge protector. There are three types of surges: "small" surges occur at rate 8 per day and damage a computer with probability 0.001; "medium" surges occur at rate 1 per day and will damage a computer with probability 0.01; "large" surges occur at rate 1 per month and damage a computer with probability 0.1. Assume that months are 30 days. (a) what is the expected number of power surges per month? (b) What is the expected number of computer damaging power surges per month? (c) What is the probability a computer will not be damaged in one month? (d) What is the probability that the first computer damaging surge is a small one?

Ans. (a) $240 + 30 + 1 = 271$, (b) $240(.001) + 30(.01) + 1(.1) = .64$, (c) $e^{-.64} = 0.528$, (d) $0.24/0.64 = 0.375$

2.49. Wayne Gretsky scored a Poisson mean 6 number of points per game. 60% of these were goals and 40% were assists (each is worth one point). Suppose he is paid a bonus of 3K for a goal and 1K for an assist. (a) Find the mean and standard deviation for the total revenue he earns per game. (b) What is the probability that he has 4 goals and 2 assists in one game? (c) Conditional on the fact that he had 6 points in a game, what is the probability he had 4 in the first half?

Ans. (a) He scored a Poisson mean 3.6 goals and had a Poisson mean 2.4 assists. This makes his mean revenue $3.6(3) + 2.4(1) = 13.2$. The variance for the payoff is $3.6(9) + 2.4(1) = 34.8$ (the payoffs are not random and the two types are independent) so the standard deviation is 5.9.

$$\text{(b)} \qquad e^{-3.6}\frac{(3.6)^4}{4!} \cdot e^{-2.4}\frac{(2.4)^2}{2!}$$

(c) By conditioning $C_{6,4}(1/2)^6 = 15/64$.

2.50. A copy editor reads a 200-page manuscript, finding 108 typos. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while from long experience we know that the copyeditor finds 90% of the mistakes that are there. (a) Compute the expected number of typos found as a function of the arrival rate λ. (b) Use the answer to (a) to find an estimate of λ and of the number of undiscovered typos.

Ans. (a) $0.9\lambda \cdot 200$, (b) Setting $180\lambda = 108$ and solving gives $\lambda = 0.6$. We estimate the number of undiscovered typos by $0.1\lambda \cdot 200 = 20\lambda = 12$.

2.51. Two copy editors read a 300-page manuscript. The first found 100 typos, the second found 120, and their lists contain 80 errors in common. Suppose that the author's typos follow a Poisson process with some unknown rate λ per page, while the two copy editors catch errors with unknown probabilities of success p_1 and p_2. Let X_0 be the number of typos that neither found. Let X_1 and X_2 be the number of typos found only by 1 or only by 2, and let X_3 be the number of typos found by both. (a) Find the joint distribution of (X_0, X_1, X_2, X_3). (b) Use the answer to (a) to find an estimates of p_1, p_2 and then of the number of undiscovered typos.

Ans. (a) If $\mu = 300\lambda$ they are independent Poisson with means

$$\mu(1-p_1)(1-p_2), \quad \mu p_1(1-p_2), \quad \mu(1-p_1)p_2, \quad \mu p_1 p_2$$

(b) Here $X_1 = 20$, $X_2 = 40$ and $X_3 = 80$. Since $EX_3/E(X_2 + X_3) = p_1$ we guess that $p_1 = 2/3$, Since $EX_3/E(X_1+X_3) = p_2$ we guess that $p_2 = 0.8$. Since $EX_0/EX_1 = (1-p_1)/p_1 = EX_2/EX_3$ we guess that there are $20 \cdot 40/80 = 10$ typos remaining. Alternatively one can estimate $\mu = 80/p_1p_2 = 150$ and note that $X_1 + X_2 + X_3 = 140$ have been found.

2.52. A light bulb has a lifetime that is exponential with a mean of 200 days. When it burns out a janitor replaces it immediately. In addition there is a handyman who comes at times of a Poisson process at rate .01 and replaces the bulb as "preventive maintenance." (a) How often is the bulb replaced? (b) In the long run what fraction of the replacements are due to failure?

Ans. (a) The superposition of these two processes is Poisson with rate .015 so the time between replacements is 66 2/3's days. (b) Failures occur at rate .005 versus maintenance at rate .01, so in the long run 1/3 of the replacements are due to failures.

2.53. Starting at some fixed time, which we will call 0 for convenience, satellites are launched at times of a Poisson process with rate λ. After an independent amount of time having distribution function F and mean μ, the satellite stops working. Let $X(t)$ be the number of working satellites at time t. (a) Find the distribution of $X(t)$. (b) Let $t \to \infty$ in (a) to show that the limiting distribution is Poisson($\lambda\mu$).

Ans. (a) A satellite launched at time $s < t$ has probability $1-F(t-s)$ to still be in orbit at time t. By our nonhomogeneous thinning result, the number is orbit at time $t$is Poisson with mean $\lambda \int_0^t (1-F(t-s))\,ds$. (b) Changing variables the mean in (a)

$$\lambda \int_0^t (1-F(r))\,dr \to \lambda \int_0^\infty (1-F(r))\,dr = \lambda\mu$$

2.54. Calls originate from Dryden according to a rate 12 Poisson process. 3/4 are local and 1/4 are long distance. Local calls last an average of 10 minutes, while long distance calls last an average of 5 minutes. Let M be the number of local calls and N the number of long distance calls in equilibrium. Find the distribution of (M,N). what is the number of people on the line.

Ans. Thinning implies that local and long distance calls are independent Poisson processes with rates 9 and 3 respectively. By results for the $M/G/\infty$ queue, in equilibrium M and N are independent Poisson with means 90 and 15 respectively.

2.55. Ignoring the fact that the bar exam is only given twice a year, let us suppose that new lawyers arrive in Los Angeles according to a Poisson process with mean 300 per year. Suppose that each lawyer independently practices for an amount of time T with a distribution function $F(t) = P(T \le t)$ that has $F(0) = 0$ and mean 25 years. Show that in the long run the number of lawyers in Los Angeles is Poisson with mean 7500.

Ans. A lawyer entering practice at time $s < t$ has probability $1 - F(t-s)$ to still be working at time t. By our nonhomogeneous thinning result, Theorem 2.12, the number working at time t is Poisson with mean $\lambda \int_0^t (1-F(t-s))\,ds$. Changing variables the mean is

$$\lambda \int_0^t (1-F(r))\,dr \to \lambda \int_0^\infty (1-F(r))\,dr = \lambda\mu$$

as $t \to \infty$.

2.56. Policy holders of an insurance company have accidents at times of a Poisson process with rate λ. The distribution of the time R until a claim is reported is random with $P(R \le r) = G(r)$ and $ER = \nu$. (a) Find the distribution of the number of unreported claims. (b) Suppose each claim has mean μ and variance σ^2. Find the mean and variance of S the total size of the unreported claims.

Ans. (a) By our nonhomogeneous thinning result, Theorem 2.12, the number of claims is Poisson with mean $\lambda \int_0^\infty (1-G(s))\,ds = \lambda\nu$. (b) Using Theorem 2.10 the mean and variance of S are $\lambda\nu\mu$ and $\lambda(\mu^2+\sigma^2)$.

2.57. Suppose $N(t)$ is a Poisson process with rate 2. Compute the conditional probabilities (a) $P(N(3) = 4|N(1) = 1)$, (b) $P(N(1) = 1|N(3) = 4)$.

Ans. (a) $N(3) - N(1)$ is Poisson(4) so

$$P(N(3) = 4|N(1) = 1) = P(N(3) - N(1) = 3|N(1) = 1) \cdot e^{-4}\frac{4^3}{3!}$$

(b) (5.3) implies that conditional on $N(3) = 4$, $N(1)$ is Binomial(4,1/4) so

$$P(N(1) = 1|N(3) = 4) = \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3$$

2.58. For a Poisson process $N(t)$ with arrival rate 2 compute: (a) $P(N(2) = 5)$, (b) $P(N(5) = 8|N(2) = 3$, (c) $P(N(2) = 3|N(5) = 8)$.

Ans. (a) $e^{-4}4^5/5!$, (b) $e^{-6}6^5/5!$, (c) $C_{8,3}(0.4)^3(0.6)^5$.

2.59. Customers arrive at a bank according to a Poisson process with rate 10 per hour. Given that two customers arrived in the first 5 minutes, what is the probability that (a) both arrived in the first 2 minutes. (b) at least one arrived in the first 2 minutes.

Ans. (a) $(2/5)^2$, (b) $1 - (3/5)^2$

2.60. Suppose that the number of calls per hour to an answering service follows a Poisson process with rate 4. Suppose that 3/4's of the calls are made by men, 1/4 by women, and the sex of the caller is independent of the time of the call. (a) What is the probability that in one hour exactly 2 men and 3 women will call the answering service? (b) What is the probability 3 men will make phone calls before 3 women do?

Ans. (a) The Poisson processes of men and women are independent Poisson processes with rates 3 and 1 so the probability of interest is

$$e^{-3}\frac{3^2}{2!} \cdot e^{-1}\frac{1^3}{3!}$$

(b) The event in question is the same as at least three out of the first five calls are from men which has probability

$$\binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + \left(\frac{3}{4}\right)^5$$

2.61. Hockey teams 1 and 2 score goals at times of Poisson processes with rates 1 and 2. Suppose that $N_1(0) = 3$ and $N_2(0) = 1$. (a) What is the probability that $N_1(t)$ will reach 5 before $N_2(t)$ does? (b) Answer part (a) for Poisson processes with rates λ_1 and λ_2.

Ans. (a) We are interested in the probability of 2 arrivals in the first process before 4 arrivals in the second, or what is equivalent the probability of at least 2 arrivals of type 1 in the first 5. Since this is 1 minus the probability of 0 or 1 the answer is

$$1 - \left(\frac{2}{3}\right)^5 - 5\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right) = 1 - \frac{7}{3}\left(\frac{2}{3}\right)^4 = 1 - \frac{112}{243} = \frac{131}{243}$$

(b) Generalizing from the last computation we have

$$1 - \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^5 - 5\left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^4 \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)$$

2.62. Consider two independent Poisson processes $N_1(t)$ and $N_2(t)$ with rates λ_1 and λ_2. What is the probability that the two-dimensional process $(N_1(t), N_2(t))$ ever visits the point (i, j)?

Ans. In order to do this there must be i arrivals of type 1 and j arrivals of type 2 in the first $i + j$ arrivals, an event of probability

$$\binom{i+j}{i} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^i \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^j$$

3.5 Exercises

3.1. The weather in a certain locale consists of alternating wet and dry spells. Suppose that the number of days in each rainy spell is a Poisson distribution with mean 2, and that a dry spell follows a geometric distribution with mean 7. Assume that the successive durations of rainy and dry spells are independent. What is the long-run fraction of time that it rains?

Ans. By Theorem 3.4 it is $2/(2+7) = .222$.

3.2. Monica works on a temporary basis. The mean length of each job she gets is 11 months. If the amount of time she spends between jobs is exponential with mean 3 months, then in the long run what fraction of the time does she spend working?

Ans. By Theorem 3.4 it is $11/(3+11) = .786$.

3.3. Thousands of people are going to a Grateful dead concert in Pauley Pavillion at UCLA. They park their 10 foot cars on several of the long streets near the arena. There are no lines to tell the drivers where to park, so they park at random locations, and end up leaving spacings between the cars that are independent and uniform on $(0, 10)$. In the long run, what fraction of the street is covered with cars?

Ans. Since the mean of a uniform on (0,10) is 5, Theorem 3.4 implies that the answer is $10/(5+10) = 2/3$.

3.4. The times between the arrivals of customers at a taxi stand are independent and have a distribution F with mean μ_F. Assume an unlimited supply of cabs, such as might occur at an airport. Suppose that each customer pays a random fare with distribution G and mean μ_G. Let $W(t)$ be the total fares paid up to time t. Find $\lim_{t\to\infty} E\,W(t)/t$.

Ans. By Theorem 3.3 this is μ_G/μ_F.

3.5. In front of terminal C at the Chicago airport is an area where hotel shuttle vans park. Customers arrive at times of a Poisson process with rate 10 per hour looking for transportation to the Hilton hotel nearby. When 7 people are in the van it leaves for the 36-minute round trip to the hotel. Customers who arrive while the van is gone go to some other hotel instead. (a) What fraction of the customers actually go to the Hilton? (b) What is the average amount of time that a person who actually goes to the Hilton ends up waiting in the van?

Ans. (a) The customer interarrival times have mean $1/10$ hour $= 6$ minutes, so it takes the van an average of $7/10$ hour or 42 minutes to pick up 7 customers. The van drives for exactly 36 minutes so by our results for alternating renewal processes the limit fraction of time picking up is $42/(42+36) = 7/13$ and this is the fraction of customers picked up. (b) The first customer in the van has to wait an average of 36 minutes for the next 6 to come, the second 30 min, the third 24 min, the fourth 18 min, the fifth 12 min, the sixth 6 min, and the seventh 0 min. So the average waiting time of these 7 customers is

$$\frac{36+30+24+18+12+6+0}{7} = \frac{6 \cdot 21}{7} = 18$$

3.6. Three children take turns shooting a ball at a basket. They each shoot until they misses and then it is next child's turn. Suppose that child i makes a basket with probability p_i and that successive trials are independent. (a) Determine the proportion of time in the long run that each child shoots. (b) Find the answer when $p_1 = 2/3$, $p_2 = 3/4$, $p_3 = 4/5$.

Ans. To try to confuse things, let us call missing the target a "success." Child i shoots until she is successful at missing, so the number of shots has a geometric distribution, and the mean number of shots is $1/(1-p_i)$. The three shooters are an alternating renewal process, so child i shoots a fraction of the time

$$\frac{1/(1-p_i)}{1/(1-p_1) + 1/(1-p_2) + 1/(1-p_3)}$$

(b) In this case the mean number of shots for the three children are 3, 4, and 5, so the limiting fractions are 3/12, 4/12, and 5/12.

3.7. A policeman cruises (on average) approximately 10 minutes before stopping a car for speeding. 90% of the cars stopped are given speeding tickets with an \$80 fine. It takes the policeman an average of 5 minutes to write such a ticket. The other 10% of the stops are for more serious offenses, leading to an average fine of \$300. These more serious charges take an average of 30 minutes to process. In the long run, at what rate (in dollars per minute) does he assign fines.

Ans. The time spent writing tickets t_i is 30 minutes with probability 0.1 and 5 minutes with probability 0.9 for mean of 7.5 minutes. Adding the mean 10 minutes spent cruising the average cycle length. The fine r_i is 300 with probability 0.1 and 80 with probability 0.9 so the average fine is \$102. Using Theorem 3.3 we see that the long run rate at which he assigns fines is $102/17.5 = 5.83$ dollars per minute.

3.8. *Counter processes.* As in Example 1.5, we suppose that arrivals at a counter come at times of a Poisson process with rate λ. An arriving particle that finds the counter free gets registered and then locks the counter for an amount of time τ. Particles that arrive while the counter is locked have no effect. (a) Find the limiting probability the counter is locked at time t. (b) Compute the limiting fraction of particles that get registered.

Ans. (a) The counter alternates between being free for an exponential amount of time with rate λ and locked for a fixed amount of time τ, so the long run fraction of time locked is $\tau/(\tau + 1/\lambda)$. (b) The fraction that are registered is equal to the fraction of time the counter is unlocked or $(1/\lambda)/(\tau + 1/\lambda) = 1/(\lambda\tau + 1)$.

3.9. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate λ. The customer and the dealer then disappear from the street for an amount of time with distribution G while the transaction is completed. Customers that arrive during this time go away never to return. (a) At what rate does the dealer make sales? (b) What fraction of customers are lost?

Ans. (a) The dealer alternates between waiting for an exponential amount of time with mean $1/\lambda$ and then servicing a customer for an amount of time with mean μ_G. The average time for a cycle is $\mu_G + 1/\lambda$ so in the long run sales are made at rate $1/(\mu_G + 1/\lambda)$. (b) The long run fraction of lost customers is equal to the long run fraction of time the dealer is busy with a customer, which by Theorem 3.4 is $\mu_G/(\mu_G + 1/\lambda)$.

3.10. One of the difficulties about probability is realizing when two different looking problems are the same, in this case dealing cocaine and fighting fires. In Problem 2.26, calls to a fire station arrive according to a Poisson process with rate 0.5 per hour, and the time required to respond to a call, return to the station, and get ready to respond to the next call is uniformly distributed between 1/2 and 1 hour. If a new call comes before the Dryden fire department is ready to respond, the Ithaca fire department is asked to respond. What fraction of calls must be handled by the Ithaca fire department

Ans. $\mu_G = 3/4$ and $\lambda = 0.5$ so by the solution to the previous problem the answer is $0.75/2.75 = 3/11$.

3.11. A young doctor is working at night in an emergency room. Emergencies come in at times of a Poisson process with rate 0.5 per hour. The doctor can only get to sleep when it has been 36 minutes (.6 hours) since the last emergency. For example, if there is an emergency at 1:00 and a second one at 1:17 then she will not be able to get to sleep until at least 1:53, and it will be even later if there is another emergency before that time.
(a) Compute the long-run fraction of time she spends sleeping, by formulating a renewal reward process in which the reward in the ith interval is the amount of time she gets to sleep in that interval.
(b) The doctor alternates between sleeping for an amount of time s_i and being awake for an amount of time u_i. Use the result from (a) to compute Eu_i.
(c) Solve problem (b) by noting that the doctor trying to sleep is the same as chicken crossing the road in Exercise 2.33.

Ans. (a) $r_i = (t_i - 0.6)^+$, i.e., $t_i - 0.6$ if this is positive, zero otherwise. The lack of memory property of the exponenetial implies that if the doctor gets to sleep she will get to sleep for an average of two hours, so $Er_i = 2P(t_i > .6) = 2e^{-0.3}$ and limiting reward (i.e., sleep) per unit time is $Er_i/Et_i = (2e^{-0.3})/2 = e^{-0.3}$ or 0.741. (b) The lack of memory property implies that her sleep periods are exponential with mean two hours. Theorem 3.4 implies that the limiting fraction of time spent sleeping is $Es_i/(Es_i + Eu_i)$ so $2/(2 + Eu_i) = e^{-0.3}$ and it follows that $Eu_i = 2(e^{0.3} - 1) = .70$. (c) Taking $\lambda = 1/2$ and $c = 0.6$ in the solution to the chicken problem the expected time between naps is $e^{(0.5)0.6} - 1)/0.5 = 2(e^{0.3} - 1)$.

3.12. A worker has a number of machines to repair. Each time a repair is completed a new one is begun. Each repair independently takes an exponential amount of time with rate μ to complete. However, independent of this, mistakes occur according to a Poisson process with rate λ. Whenever a mistake occurs, the item is ruined and work is started on a new item. In the long run how often are jobs completed?

Ans. Consider a renewal reward process in which t_i is the amount of work given to item i and and $r_i = 1$ if the work is completed before a mistake occurs. Results in Section 2.1 imply that t_i is exponential with rate $\lambda+\mu$ while $P(r_i = 1) = \mu/(\mu + \lambda)$. Using Theorem 3.3 now gives

$$\frac{R(t)}{t} \to \frac{\mu/(\mu+\lambda)}{1/(\mu+\lambda)} = \mu$$

This conclusion can be derived directly by imagining a green Poisson process with rate μ and a red Poisson process of mistakes with rate λ and noting that service completions occur at the green points.

3.13. In the Duke versus Miami football game, possessions alternate between Duke who has the ball for an average of 2 minutes and Miami who has the ball for an average of 6 minutes. (a) In the long run what fraction of time does Duke have the ball? (b) Suppose that on each possession Duke scores a touchdown with probability 1/4 while Miami scores with probability one. On the average how many touchdowns will they score per hour?

Ans. (a) $Es_i = 2$ and $Eu_i = 6$ in the alternating renewal process so the answer is $Es_i/(ES_i+Eu_i) = 2/8$. (b) We use the renewal reward theorem twice. When we are counting Duke touch downs $Er_i = 1/4$ while $E(s_i + u_i) = 8/60 hours$ so the long run number of touchdowns per hour is $60/32 = 1.875$. For Miami $Er_i = 1$ so their answer is $60/8 = 7.5$.

3.14. Random Investment. An investor has \$100,000. If the current interest rate is i% (compounded continuously so that the grow per year is $\exp(i/100)$), he invests his money in a i year CD, takes the profits and then reinvests the \$100,000. Suppose that the kth investment leads to an interest rate X_k which is uniform on $\{1, 2, 3, 4, 5\}$. In the long run how much money does he make per year.

Ans. This is a renewal reward system with expected cycle time $Et_i = 3$ and expected reward

$$Er_i = 100K \cdot \frac{1}{5}\sum_{i=1}^{5}[\exp(i^2/100) - 1] = 12,051$$

so the long run average return per year is $12,051/3 = 4,017$ or about 4% per year.

3.15. Consider the set-up of Example 3.4 but now suppose that the car's lifetime $h(t) = \lambda e^{-\lambda t}$. Show that for any A and B the optimal time $T = \infty$. Can you give a simple verbal explanantion?

Ans. Using the lack of memory property

$$\begin{aligned}\int_0^T th(t)\,dt + \int_T^\infty Th(t)\,dt &= \int_0^\infty th(t)\,dt + \int_T^\infty (T-t)h(t)\,dt \\ &= \frac{1}{\lambda} - \frac{1}{\lambda}e^{-\lambda T}\end{aligned}$$

so we have

$$\frac{Er_i}{Et_i} = \frac{A + B(1 - e^{-\lambda T})}{(1/\lambda)(1 - e^{-\lambda T})} = \frac{A}{(1/\lambda)(1 - e^{-\lambda T})} + B\lambda$$

The denominator is increasing in T so the function is decreasing. The result is what we should expect. The lack of memory property implies that the car is always new so we should never sell it.

3.16. A machine tool wears over time and may fail. The failure time measured in months has density $f_T(t) = 2t/900$ for $0 \le t \le 30$ and 0 otherwise. If the tool fails it must be replaced immediately at a cost of \$1200. If it is replaced prior to failure the cost is only \$300. Consider a replacement policy in which the tool is replaces after c months or when it fails. What is the value of c that minimizes cost per unit time.

Ans. If we replace every $c \le 30$ months then the probability of failure is

$$\int_0^c 2t/900 \, dt = c^2/900$$

the expected cycle time is

$$Et_i = \int_0^c t \cdot \frac{2t}{900} \, dt + c\left(1 - \frac{c^2}{900}\right) = \frac{2c^3}{2700} + c - \frac{c^3}{900} = c - \frac{c^3}{2700}$$

Note that this is 0 at $c = 0$, $(2/3)30 = 20$ and increasing since the derivative is $1 - c^2/900$. The expected cost per cycle is

$$Er_i = \frac{c^2}{900} \cdot 1200 + \left(1 - \frac{c^2}{900}\right) \cdot 300 = 300 + c^2$$

The long run cost per year is

$$\frac{Er_i}{Et_i} = \frac{300 + c^2}{c - (c^3/2700)}$$

Differentiating with respect to c we have

$$\frac{2c[c - c^3/2700] - (300 + c^2)[1 - 3c^2/2700)]}{[c - (c^3/2700)]^2}$$

The numerator is

$$-300 - c^2 + 2c^2 + c^2/3 - 2c^4/2700 + 3c^2/2700 = c^2/2700 + 4c^2/3 - 300$$

which is 0 when $c^4 + 4c^2 - 900 = 0$ or $c^2 = (-4 \pm \sqrt{16+4})/(2/900) = (\pm\sqrt{20} - 4)450$ The root we want has $c^2 = .4721(450)$ or $c = 14.576$.

3.17. People arrive at a college admissions office at rate 1 per minute. When k people have arrive a tour starts. Student tour guides are paid \$20 for each tour they conduct. The college estimates that it loses 10 cents in good will for each minute a person waits. What is the optimal tour group size?

Ans. If the tour group has size k then for $j = k-1, \dots 1$, one person waits an average of j minutes for a total of $1 + \cdots + k - 1 = k(k-1)/2$. The long run cost per unit time is $[20 + .05(k^2 - k)]/k$ Differentiating with respect to k we want

$$0 = -\frac{20}{k^2} + .05 \qquad k^2 = 400 \quad \text{or} \quad k = 20$$

3.18. A scientist has a machine for measuring ozone in the atmosphere that is located in the mountains just north of Los Angeles. At times of a Poisson process with rate 1, storms or animals disturb the equipment so that it can no longer collect data. The scientist comes every L units of time to check the equipment. If the equipment has been disturbed then she can usually fix it quickly so we will assume the the repairs take 0 time. (a) What is the limiting fraction of time the machine is working? (b) Suppose that the data that is being collected is worth a dollars per unit time, while each inspection costs $c < a$. Find the best value of the inspection time L.

Ans. (a) Let s_i be independent exponential random variables with rate 1, and let

$$r_i = \begin{cases} s_i & \text{if } s_i < L \\ L & \text{if } s_i \geq L \end{cases}$$

Reasoning as in the concrete example in Example 2.4, or using calculus it is easy to see that $Er_i = 1 - e^{-L}$, thus by (2.4) with the $t_i = L$ the long run fraction of time the machine is working is $(1 - e^{-L})/L$. (b) The long run profit from inspecting every L units of time is

$$\frac{a}{L}(1 - e^{-L}) - \frac{c}{L}$$

This converges to $-\infty$ as $L \to 0$ and is asymptotically $(a - c)/L$ as $L \to \infty$. The condition $a > c$ implies that the function is positive for large L so we will have a global maximum. To find the maximum we differentiate and set

$$\begin{aligned} 0 &= -\frac{1}{L^2}\left\{a(1 - e^{-L}) - c\right\} + \frac{a}{L} \cdot e^{-L} \\ &= \frac{1}{L^2}\left(a(Le^{-L} - 1 + e^{-L}) + c\right) \end{aligned}$$

Solving we want to pick L so that

$$1 - (1 + L)e^{-L} = c/a$$

Differentiating with respect to L shows that the left-hand side increases from 0 at $L = 0$ to 1 as $L \to \infty$. Since $0 < c/a < 1$ we have a unique solution.

Age and Residual Life

3.19. Consider the discrete renewal process with $f_j = P(t_1 = j)$ and $F_i = P(t_1 > i)$. (a) Show that the age chain has transition probability

$$q(j, j+1) = \frac{F_{j+1}}{F_j} \qquad q(j, 0) = 1 - \frac{F_{j+1}}{F_j} = \frac{f_{j+1}}{F_j} \quad \text{for } j \geq 0$$

(b) Show that if $Et_1 < \infty$, the stationary distribution $\pi(i) = P(t_1 > i)/Et_1$.
(c) Let $p(i,j)$ be the transition probability for the renewal chain. Verify that It should be clear by comparing the numerical examples above that there is a close relationship between q is the dual chain of p, i.e., the chain p run backwards. That is,

$$q(i,j) = \frac{\pi(j)p(j,i)}{\pi(i)}$$

Ans. (a) When $A_n = j$ we know that the associated renewal has $t_i > j$. In order to move on to $j+1$, we must have $t_i > j+1$. so $q(j,j+1) = F_{j+1}/F_j$. If the age does not go to $j+1$ it drops to 0 so $q(j,0) = 1 - q(j,j+1)$. (b) To define a stationary measure we will use the cycle trick, Theorem 1.20 with $x = 0$. Starting from 0 the chain will visit a site i at most once before it returns to 0, and this will happen with probability

$$q(0,1)q(1,2)\cdots q(i-1,i) = \frac{F_1}{F_0} \cdot \frac{F_2}{F_1} \cdots \frac{F_i}{F_{i-1}} = F_i$$

so $\pi(i) = P(t_1 > i)/Et_1$. (c) There are two cases to consider. If $i \geq 0$ and $j = i+1$

$$q(i,i+1) = \frac{F_{i+1}}{F_i} = \frac{\pi(i+1)p(i+1,i)}{\pi(i)}$$

since $\pi(j) = F_j/Et_1$. If $j = 0$ then

$$q(i,0) = \frac{f_{i+1}}{F_i} = \frac{\pi(0)p(0,i)}{\pi(i)}$$

3.20. Show that chain in Exercise 1.38 with transition probability is

	1	**2**	**3**	**4**
1	1/2	1/2	0	0
2	2/3	0	1/3	0
3	3/4	0	0	1/4
4	1	0	0	0

is a special case of the age chain. Use this observation and the previous exercise to compute the stationary distribution.

Ans. By the formula in the previous exercise we want

$$F_1/F_0 = 1/2 \quad F_2/F_1 = 1/3 \quad F_3/F_2 = 1/4 \quad F_4/F_3 = 0$$

so $F_1 = 1/2$, $F_2 = 1/6$, $F_3 = 1/24$ and it follows that

$$f_1 = 1/2 \quad f_2 = 1/3 \quad f_3 = 1/8 \quad f_4 = 1/24.$$

The mean $Et_1 = 41/24$ so the limit distribution has

$$\pi_0 = \frac{24}{41} \quad \pi_1 = \frac{12}{41} \quad \pi_2 = \frac{4}{41} \quad \pi_3 = \frac{1}{41}$$

3.21. The city of Ithaca, New York, allows for two-hour parking in all downtown spaces. Methodical parking officials patrol the downtown area, passing the same point every two hours. When an official encounters a car, he marks it with chalk. If the car is still there two hours later, a ticket is written. Suppose that you park your car for a random amount of time that is uniformly distributed on $(0, 4)$ hours. What is the probability you will get a ticket?

Ans. Solution 1. If you park for an amount of time $T > 2$ then you get a ticket if the parking official comes in the first $T - 2$ units of time, an event of probability $(T - 2)/2$. Integrating over the possible values of T that can produce a ticket gives

$$\begin{aligned}\int_2^4 \left(\frac{T}{2} - 1\right) \frac{dt}{4} &= \frac{1}{4}\left(\frac{T^2}{4} - T\right)\Big|_2^4 \\ &= \frac{1}{4}\left(\frac{16}{4} - 4 - \left(\frac{4}{4} - 2\right)\right) = 1/4\end{aligned}$$

Solution 2. The limiting residual life Z^* has a density that is a triangle with height $1/2$ and base 4. From this we see that $P(Z^* > y) = y^2/4^2$. We get a ticket if the residual life is > 2 when the parking official comes by, which has probabilikty $1/4$.

3.22. Each time the frozen yogurt machine at the mall breaks down, it is replaced by a new one of the same type. (a) What is the limiting age distribution for the machine in use if the lifetime of a machine has a gamma(2,λ) distribution, i.e., the sum of two exponentials with mean $1/\lambda$. (b) Find the answer to (a) by thinking about a rate one Poisson process in which arrivals are alternately colored red and blue.

Ans. (a) The distribution function of the gamma(2,λ) is $1 - e^{-\lambda t} - \lambda t e^{-\lambda t}$ since the time of the second arrival in a Poisson process is $\leq t$ unless there have only been 0 or 1 arrivals. Using (3.9) with the fact that $Et_i = 2/\lambda$ the limiting residual life time is

$$\frac{1}{2}\lambda \exp(-\lambda t) + \frac{1}{2}\lambda^2 t \exp(-\lambda t)$$

(b) In the long run the last renewal before t is red or blue with probability $1/2$ each. When it is red then we have to wait until the new renewal, which by the lack of memory property has an exponential distribution. When it is blue we have to wait for two and the distribution is gamma(2,λ).

3.23. While visiting Haifa, Sid Resnick discovered that people who wish to travel from the port area up the mountain frequently take a shared taxi known as a sherut. The capacity of each car is 5 people. Potential customers arrive according to a Poisson process with rate λ. As soon as 5 people are in the car, it departs for The Carmel, and another taxi moves up to accept passengerso on. A local resident (who has no need of a ride) wanders onto the scene. What is the distribution of the time he has to wait to see a cab depart?

Ans. In the long run the number of people in the cab will be 0, 1, 2, 3, or 4 with equal probability. When there are k people in the cab, the time to the next departure will be gamma$(5-k,\lambda)$. Using the formula for the gamma density in (2.12, it follows that the density function of the time to the next departure is

$$\lambda e^{-\lambda t} \cdot \frac{1}{5} \sum_{m=1}^{5} \frac{(\lambda t)^{m-1}}{(m-1)!}$$

3.24. Suppose that the limiting age distribution in (3.9) is the same as the original distribution. Conclude that $F(x) = 1 - e^{-\lambda x}$ for some $\lambda > 0$.

Ans. Let $H(c) = 1 - G(c)$, and $\lambda = 1/Et_i$. (**??**) implies $H'(c) = -\lambda H(c)$. Combining this with the observation $H(0) = 1$ and using the uniqueness of the solution of the differential equation, we conclude that $H(c) = e^{-\lambda c}$.

4.8 Exercises

Atlanta

4.1. A salesman flies around between Atlanta, Boston, and Chicago as follows.

	A	B	C
A	-4	2	2
B	3	-4	1
C	5	0	-5

(a) Find the limiting fraction of time she spends in each city. (b) What is her average number of trips each year from Boston to Atlanta?

Ans. (a) The stationary distribution is the third row of

$$\begin{pmatrix} -4 & 2 & 2 \\ 3 & -4 & 1 \\ 5 & 0 & -5 \end{pmatrix}^{-1}$$

so the limiting fractions of time she spends in the three cities are $A = 1/2$, $B = 1/4$, $C = 1/4$. (b) She spends 1/4 of her year, i.e., 3 months in Boston, and makes trips from Boston to Atlanta at rate $3 = 4 \cdot 3/4$ per month, so the average number of trips per year is 9.

4.2. A small computer store has room to display up to 3 computers for sale. Customers come at times of a Poisson process with rate 2 per week to buy a computer and will buy one if at least 1 is available. When the store has only 1 computer left it places an order for 2 more computers. The order takes an exponentially distributed amount of time with mean 1 week to arrive. Of course, while the store is waiting for delivery, sales may reduce the inventory to 1 and then to 0. (a) Write down the matrix of transition rates Q_{ij} and solve $\pi Q = 0$ to find the stationary distribution. (b) At what rate does the store make sales?

Ans. (a) The Q matrix is

	0	1	2	3
0	-1	0	1	0
1	2	-3	0	1
2	0	2	-2	0
3	0	0	2	-2

Replacing the last column of the Q matrix by 1 and taking the inverse we find

$$\pi(0) = 2/5, \qquad \pi(1) = 1/5, \qquad \pi(2) = 3/10, \qquad \pi(3) = 1/10$$

(b) Sales are made at rate $2(1 - \pi(0)) = 6/5$ per week.

4.3. Consider two machines that are maintained by a single repairman. Machine i functions for an exponentially distributed amount of time with rate λ_i before it fails. The repair times for each unit are exponential with rate μ_i. They are repaired in the order in which they fail. (a) Formulate a Markov chain model for this situation with state space $\{0, 1, 2, 12, 21\}$. (b) Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$. Find the stationary distribution.

Ans. (a) If 0 = both working, 1 = 1 failed, 2 = 2 failed, 12 and 21 indicate both failed in the order indicated then the transition rate matrix is

$$\begin{array}{cccccc} & 0 & 1 & 2 & 12 & 21 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 & 0 \\ 1 & \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 & 0 \\ 2 & \mu_2 & 0 & -(\mu_2+\lambda_1) & 0 & \lambda_1 \\ 12 & 0 & 0 & \mu_1 & -\mu_1 & 0 \\ 21 & 0 & \mu_2 & 0 & 0 & -\mu_2 \end{array}$$

(c) Filling in the given values the previous matrix becomes:

$$\begin{array}{cccccc} & 0 & 1 & 2 & 12 & 21 \\ 0 & -4 & 1 & 3 & 0 & 0 \\ 1 & 2 & -5 & 0 & 3 & 0 \\ 2 & 4 & 0 & -5 & 0 & 1 \\ 12 & 0 & 0 & 2 & -2 & 0 \\ 21 & 0 & 4 & 0 & 0 & -4 \end{array}$$

Replacing the last column of the Q matrix by 1 and taking the inverse we find

$$\pi(0) = \frac{44}{129} \quad \pi(1) = \frac{16}{129} \quad \pi(2) = \frac{36}{129} \quad \pi(12) = \frac{24}{129} \quad \pi(21) = \frac{9}{129}$$

4.4. Consider the set-up of the previous problem but now suppose machine 1 is much more important than 2, so the repairman will always service 1 if it is broken. (a) Formulate a Markov chain model for the this system with state space $\{0, 1, 2, 12\}$ where the numbers indicate the machines that are broken at the time. (b) Suppose that $\lambda_1 = 1$, $\mu_1 = 2$, $\lambda_2 = 3$, $\mu_2 = 4$. Find the stationary distribution.

Ans. (a) The transition rate matrix is given by

$$\begin{array}{ccccc} & 0 & 1 & 2 & 12 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 1 & \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 \\ 2 & \mu_2 & 0 & -(\mu_2+\lambda_1) & \lambda_1 \\ 12 & 0 & 0 & \mu_1 & -\mu_1 \end{array}$$

(b) Filling in the given values the previous matrix becomes

$$\begin{array}{ccccc} & 0 & 1 & 2 & 12 \\ 0 & -4 & 1 & 3 & 0 \\ 1 & 2 & -5 & 0 & 3 \\ 2 & 4 & 0 & -5 & 1 \\ 12 & 0 & 0 & 2 & -2 \end{array}$$

Replacing the last column of the Q matrix by 1 and taking the inverse we find

$$\pi_0 = 20/57 \quad \pi_1 = 4/57 \quad \pi_2 = 18/57 \quad \pi_3 = 15/57$$

4.5. Two people are working in a small office selling shares in a mutual fund. Each is either on the phone or not. Suppose that salesman i is on the phone for an exponential amount of time with rate μ_i and then off the phone for an exponential amount of time with rate λ_i. (a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates who is on the phone. (b) Find the stationary distribution.

Ans. (a) The transition matrix is

$$\begin{array}{ccccc} & 0 & 1 & 2 & 12 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 1 & \mu_1 & -(\mu_1+\lambda_2) & 0 & \lambda_2 \\ 2 & \mu_2 & 0 & -(\mu_2+\lambda_1) & \lambda_1 \\ 12 & 0 & \mu_1 & \mu_2 & -(\mu_1+\mu_2) \end{array}$$

(b) The states of the two salesmen are independent two state Markov chains with stationary probability $\lambda_i/(\lambda_i+\mu_i)$ of being on the phone.

4.6. (a) Consider the special case of the previous problem in which $\lambda_1 = \lambda_2 = 1$, and $\mu_1 = \mu_2 = 3$, and find the stationary probabilities. (b) Suppose they upgrade their telephone system so that a call to one line that is busy is forwarded to the other phone and lost if that phone is busy. Find the new stationary probabilities.

Ans. (a) Plugging into the previous answer gives $\pi_{12} = 1/9$, $\pi_1 = \pi_2 = 2/9$ and $\pi_0 = 4/9$. (b) The new transition matrix is

$$\begin{array}{ccccc} & 0 & 1 & 2 & 12 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 3 & -5 & 0 & 2 \\ 2 & 3 & 0 & -5 & 2 \\ 12 & 0 & 3 & 3 & -6 \end{array}$$

Replacing the last column by 1's and taking the inverse

$$\pi_0 = 9/17, \quad \pi_1 = \pi_2 = 3/17, \quad \pi_{12} = 2/17$$

4.7. Two people who prepare tax forms are working in a store at a local mall. Each has a chair next to his desk where customers can sit and be served. In addition there is one chair where customers can sit and wait. Customers arrive at rate λ but will go away if there is already someone sitting in the chair waiting. Suppose that server i requires an exponential amount of time with rate μ_i and that when both servers are free an arriving customer is equally likely to choose either one. (a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12, 3\}$ where the first four states indicate the servers that are busy while the last indicates that there is a total of three customers in the system: one at each server and one waiting. (b) Consider the special case in which $\lambda = 2$, $\mu_1 = 3$ and $\mu_2 = 3$. Find the stationary distribution.

Ans. (a) The transition matrix is

$$\begin{array}{cccccc} & 0 & 1 & 2 & 12 & 3 \\ 0 & -\lambda & \lambda/2 & \lambda/2 & 0 & 0 \\ 1 & \mu_1 & -(\mu_1+\lambda) & 0 & \lambda & 0 \\ 2 & \mu_2 & 0 & -(\mu_2+\lambda) & \lambda & 0 \\ 12 & 0 & \mu_2 & \mu_1 & -(\mu_1+\mu_2+\lambda) & \lambda \\ 3 & 0 & 0 & 0 & \mu_1+\mu_2 & -(\mu_1+\mu_2) \end{array}$$

(b) In the special case the transition matrix is

$$\begin{array}{cccccc} & 0 & 1 & 2 & 12 & 3 \\ 0 & -2 & 1 & 1 & 0 & 0 \\ 1 & 3 & -5 & 0 & 2 & 0 \\ 2 & 3 & 0 & -5 & 2 & 0 \\ 12 & 0 & 3 & 3 & -8 & 2 \\ 3 & 0 & 0 & 0 & 6 & -6 \end{array}$$

Replacing the last column by 1's and taking the inverse

$$\pi_0 = 27/53, \quad \pi_1 = \pi_2 = 9/53, \quad \pi_{12} = 6/53, \quad \pi_3 = 2/53$$

twoqins **4.8.** *Two queues in series.* Consider a two station queueing network in which arrivals only occur at the first server and do so at rate 2. If a customer finds server 1 free he enters the system; otherwise he goes away. When a customer is done at the first server he moves on to the second server if it is free and leaves the system if it is not. Suppose that server 1 serves at rate 4 while server 2 serves at rate 2. Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates the servers who are busy. In the long run (a) what proportion of customers enter the system? (b) What proportion of the customers visit server 2?

Ans. The transition matrix is

$$\begin{array}{ccccc} & 0 & 1 & 2 & 12 \\ 0 & -2 & 2 & 0 & 0 \\ 1 & 0 & -4 & 4 & 0 \\ 2 & 2 & 0 & -4 & 2 \\ 12 & 0 & 2 & 4 & -6 \end{array}$$

Replacing the last column by 1's and taking the inverse we have

$$\pi(0) = 3/9 \quad \pi(1) = 2/9 \quad \pi(2) = 3/9 \quad \pi(12) = 1/9$$

(a) Customers can only enter the system in states 0 or 2 so the fraction that enter is $\pi(0)+\pi(2) = 6/9$. (b) A customer that enters in state 0 will always visit the second server. A customer that enters in state 2 will only receive service if the person at server 2 gets finished before him. Since this is a race between an exponential with rate 2 and one with rate 4, it is an event of probability 1/3. The answer is thus $\pi(0) + \pi(2)/3 = 4/9$.

Detailed balance

4.9. A hemoglobin molecule can carry one oxygen or one carbon monoxide molecule. Suppose that the two types of gases arrive at rates 1 and 2 and attach for an exponential amount of time with rates 3 and 4, respectively. Formulate a Markov chain model with state space $\{+, 0, -\}$ where $+$ denotes an attached oxygen molecule, $-$ an attached carbon monoxide molecule, and 0 a free hemoglobin molecule and find the long-run fraction of time the hemoglobin molecule is in each of its three states.

Ans. Detailed balance implies $3\pi_+ = \pi_0$ and $4\pi_- = 2\pi_0$. Setting $\pi_0 = c$ we have $\pi_+ = c/3$ and $\pi_- = c/2$. Now pick c to make $1 = c/3 + c + c/2 = 11c/6$, i.e., $c = 6/11$. This means $\pi_+ = 2/11$, $\pi_0 = 6/11$, and $\pi_- = 3/11$.

4.10. A machine is subject to failures of types $i = 1, 2, 3$ at rates λ_i and a failure of type i takes an exponential amount of time with rate μ_i to repair. Formulate a Markov chain model with state space $\{0, 1, 2, 3\}$ and find its stationary distribution.

Ans. Detailed balance implies $\mu_i \pi_i = \lambda_i \pi_0$. Setting $\pi_0 = c$ we have $\pi_i = c\lambda_i/\mu_i$ for $i = 1, 2, 3$. Now pick c to make the π_i sum to 1.

4.11. Solve the previous problem in the concrete case $\lambda_1 = 1/24$, $\lambda_2 = 1/30$, $\lambda_3 = 1/84$, $\mu_1 = 1/3$, $\mu_2 = 1/5$, and $\mu_3 = 1/7$.

Ans. Set $\pi_0 = c$. The equations in the previous solution imply $\pi_1 = c/8$, $\pi_2 = c/6$, and $\pi_3 = c/12$. The sum of the π's is $33c/24$. Setting $c = 24/33$ we have stationary probabilities $\pi_0 = 24/33$, $\pi_1 = 3/33$, $\pi_2 = 4/33$, $\pi_3 = 2/33$.

4.12. Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. Let X_t be the number of frogs in the sun at time t. (a) Find the stationary distribution for X_t. (b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

Ans. X_t is a birth and death chain. The death rates (jumping into the lake) are $\mu_i = i$, while the birth rates (jumping out of the lake) are $\lambda_i = 2(3-i)$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{6}{1} \cdot c = 6c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{4}{2} \cdot 6c = 12c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{2}{3} \cdot 12c = 8c
\end{aligned}$$

Adding up the π's gives $(8 + 12 + 6 + 1) = 27c$ so $c = 1/27$ and we have

$$\pi(3) = \frac{8}{27} \quad \pi(2) = \frac{12}{27} \quad \pi(1) = \frac{6}{27} \quad \pi(0) = \frac{1}{27}$$

(b) Each frog is a two state Markov chain that stays in the sun 2/3's of the time and in the lake 1/3 of the time. Thus the number in the sun should be Binomial(3,2/3). Since the Binomial probabilities are

$$\pi(3) = (2/3)^3 \quad \pi(2) = 3(2/3)^2(1/3) \quad \pi(1) = 3(1/3)^2(2/3) \quad \pi(0) = (1/3)^3$$

this agrees with the previous answer.

4.13. There are 15 lily pads and 6 frogs. Each frog at rate 1 gets the urge to jump and when it does, it moves to one of the 9 vacant pads chosen at random. Find the stationary distribution for the set of occupied lily pads.

Ans. The stationary distribution is uniform over the set of possibilities. To check stationarity we check reversibility which is clear since given any two configurations i and j that differ by one frog jump $q(i,j) = q(j,i) = 1/9$.

4.14. A computer lab has three laser printers, two that are hooked to the network and one that is used as a spare. A working printer will function for an exponential amount of time with mean 20 days. Upon failure it is immediately sent to the repair facility and replaced by another machine if there is one in working order. At the repair facility machines are worked on by a single repairman who needs an exponentially distributed amount of time with mean 2 days to fix one printer. In the long run how often are there two working printers?

Ans. Let X_t be the number of working machines. X_t is a birth and death chain. Since there is one repairman we have constant birth rates $\lambda_i = 1/2$ for $i < 3$. On the other hand the failure rate is proportional to the number of machines in use so $\mu_1 = 1/20$ and $\mu_2 = \mu_3 = 1/10$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{1/2}{1/20} \cdot c = 10c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{1/2}{1/10} \cdot 50c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{1/2}{1/10} \cdot 250c
\end{aligned}$$

Adding up the π's gives $(250 + 50 + 10 + 1)c = 311c$ so $c = 1/311$ and we have

$$\pi(3) = \frac{250}{311} \quad \pi(2) = \frac{50}{311} \quad \pi(1) = \frac{10}{311} \quad \pi(0) = \frac{1}{311}$$

Thus the long run two machines are working $\pi(3) + \pi(2) = 300/311 = 0.9646$ of the time.

4.15. A computer lab has three laser printers that are hooked to the network. A working printer will function for an exponential amount of time with mean 20 days. Upon failure it is immediately sent to the repair facility. There machines are worked on by two repairman who can each repair one printer in an exponential amount of time with mean 2 days. However, it is not possible for two people to work on one printer at once. (a) Formulate a Markov chain model for the number of working printers and find the stationary distribution. (b) How often are both repairmen busy? (c) What is the average number of machines in use?

Ans. Let X_t be the number of working machines. X_t is a birth and death chain. Taking into account the number of repairmen working $\lambda_2 = 1/2$, $\lambda_1 = \lambda_0 = 1$.

The death rate is proportional to the number of machines working so $\mu_1 = 1/20$, $\mu_2 = 2/10$ and $\mu_3 = 3/20$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{1}{1/20} \cdot c = 20c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{1}{2/20} \cdot 20c = 200c \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{1/2}{3/20} \cdot 200c = 2000c/3
\end{aligned}$$

Adding up the π's gives $(2000 + 600 + 60 + 3)c/3 = 2663c/3$ so $c = 3/2663$ and we have

$$\pi(3) = \frac{2000}{2663} \quad \pi(2) = \frac{600}{2663} \quad \pi(1) = \frac{60}{2663} \quad \pi(0) = \frac{3}{2663}$$

(b) $\pi(0) + \pi(1) = 63/2663 = .0237$ of the time. (c) $(6000 + 1200 + 60)/2663 = 7260/2663 = 2.726$.

4.16. A computer lab has 3 laser printers and 5 toner cartridges. Each machine requires one toner cartridges which lasts for an exponentially distributed amount of time with mean 6 days. When a toner cartridge is empty it is sent to a repairman who takes an exponential amount of time with mean 1 day to refill it. (a) Compute the stationary distribution. (b) How often are all three printers working?

Ans. If the state of the system is the number of filled toner cartridges then $\lambda_i = 1$ for $i < 5$, $\mu_i = i/6$ for $i \leq 3$ and $\mu_i = 1/2$ for $i \geq 3$. If $\pi(0) = c$ using detailed balance gives

$$\pi(1) = 6c \quad \pi(2) = 18c \quad \pi(3) = 36c \quad \pi(4) = 72c \quad \pi(5) = 144c$$

The sum is 277 so we have

$$\begin{aligned}
&\pi(5) = 144/277, \quad \pi(4) = 72/277, \quad \pi(3) = 36/277, \\
&\pi(2) = 18/277, \quad \pi(1) = 6/277, \quad \pi(0) = 1/277
\end{aligned}$$

All three printers are working when there are 3 or more cartridges, which occurs with probability $1 - 25/277$.

4.17. Customers arrive at a full-service one-pump gas station at rate of 20 cars per hour. However, customers will go to another station if there are at least two cars in the station, i.e., one being served and one waiting. Suppose that the service time for customers is exponential with mean 6 minutes. (a) Formulate a Markov chain model for the number of cars at the gas station and find its stationary distribution. (b) On the average how many customers are served per hour?

Ans. (a) Detailed balance implies $20\pi_0 = 10\pi_1$ and $20\pi_1 = 10\pi_2$. Setting $\pi_0 = c$ we have $\pi_1 = 2\pi_0 = 2c$ and $\pi_2 = 2\pi_1/ = 4c$. Picking $c = 1/7$ to make the sum 1 we see $\pi_0 = 1/7$, $\pi_1 = 2/7$, $\pi_2 = 4/7$. (b) 4/7's of the time the station is full so on the average $20 \cdot 3/7 = 60/7 = 8.57$ customers are served per hour.

4.18. Solve the previous problem for a two-pump self-serve station under the assumption that customers will go to another station if there are at least four cars in the station, i.e., two being served and two waiting.

Ans. (a) Detailed balance implies $20\pi_0 = 10\pi_1$ and $20\pi_i = 20\pi_{i+1}$ for $1 \le i < 4$. Setting $\pi_0 = c$ we have $2c = \pi_1 = \pi_2 = \pi_3 = \pi_4$. Picking $c = 1/9$ to make the sum 1 we see $\pi_0 = 1/9$, $\pi_i = 2/7$ for $1 \le i \le 4$. (b) 2/9's of the time the station is full so on the average $20 \cdot 7/9 = 140/9 = 15.55$ customers are served per hour.

4.19. Consider a barbershop with two barbers and two waiting chairs. Customers arrive at a rate of 5 per hour. Customers arriving to a fully occupied shop leave without being served. Find the stationary distribution for the number of customers in the shop, assuming that the service rate for each barber is 2 customers per hour.

Ans. Let X_t be the number of customers in the barbershop. X_t is a birth and death chain. $\lambda_i = 5$ when $i < 4$. Taking into account the number of barbers working $\mu_1 = 2$, $\mu_2 = \mu_3 = \mu_4 = 4$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{5}{2} \cdot c = 5c/2 \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{5}{4} \cdot \frac{5c}{2} = 25c/8 \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{5}{4} \cdot \frac{25c}{8} = 125c/32 \\
\pi(4) &= \frac{\lambda_3}{\mu_4} \cdot \pi(3) = \frac{5}{4} \cdot \frac{125c}{32} = 625c/128
\end{aligned}$$

Adding up the π's gives $(625 + 500 + 400 + 320 + 128)c/128 = 1973c/128$ so $c = 128/1973$ and we have

$$\pi(4) = \frac{625}{1973} \quad \pi(3) = \frac{500}{1973} \quad \pi(2) = \frac{400}{1973} \quad \pi(1) = \frac{320}{1973} \quad \pi(0) = \frac{128}{1973}$$

4.20. Consider a barbershop with one barber who can cut hair at rate 4 and three waiting chairs. Customers arrive at a rate of 5 per hour. (a) Argue that this new set-up will result in fewer lost customers than the previous scheme. (b) Compute the increase in the number of customers served per hour.

Ans. The arrival rates and all but one of the service rates are unchanged. The new system has $\mu_1 = 4$ versus $\mu_1 = 2$ in the old one. (b) To compute the improvement in service we have to compute the new stationary distribution.

Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{5}{4} \cdot c = 5c/4 \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{5}{4} \cdot \frac{5c}{4} = 25c/16 \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{5}{4} \cdot \frac{25c}{16} = 125c/64 \\
\pi(4) &= \frac{\lambda_3}{\mu_4} \cdot \pi(3) = \frac{5}{4} \cdot \frac{125c}{64} = 625c/256
\end{aligned}$$

Adding up the π's gives $(625 + 500 + 400 + 320 + 256)c/256 = 2101c/256$ so $c = 256/2101$ and we have

$$\pi(4) = \frac{625}{2101} \quad \pi(3) = \frac{500}{2101} \quad \pi(2) = \frac{400}{2101} \quad \pi(1) = \frac{320}{2101} \quad \pi(0) = \frac{128}{2101}$$

The improvement in $\pi(4)$ is then $625/1973 - 625/2101 = .3168 - .2975 = .0193$ or about .0386 customers per hour.

4.21. There are two tennis courts. Pairs of players arrive at rate 3 per hour and play for an exponentially distributed amount of time with mean 1 hour. If there are already two pairs of players waiting new arrivals will leave. Find the stationary distribution for the number of courts occupied.

Ans. Let X_t be the number of pairs of players that are playing or waiting. X_t is a birth and death chain. $\lambda_i = 3$ when $i < 4$. Taking into account the number of pairs playing $\mu_1 = 1$, $\mu_2 = \mu_3 = \mu_4 = 2$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{3}{1} \cdot c = 3c \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{3}{2} \cdot 3c = 9c/2 \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{3}{2} \cdot \frac{9c}{2} = 27c/4 \\
\pi(4) &= \frac{\lambda_3}{\mu_4} \cdot \pi(3) = \frac{3}{2} \cdot \frac{27c}{4} = 81c/8
\end{aligned}$$

Adding up the π's gives $(81 + 54 + 36 + 24 + 8)c/8 = 203c/8$ so $c = 8/203$ and we have

$$\pi(4) = \frac{81}{203} \quad \pi(3) = \frac{54}{203} \quad \pi(2) = \frac{36}{203} \quad \pi(1) = \frac{24}{203} \quad \pi(0) = \frac{8}{203}$$

4.22. A taxi company has three cabs. Calls come in to the dispatcher at times of a Poisson process with rate 2 per hour. Suppose that each requires an exponential amount of time with mean 20 minutes, and that callers will hang up if they hear there are no cabs available. (a) What is the probability all three cabs are busy when a call comes in? (b) In the long run, on the average how many customers are served per hour?

Ans. Let X_t be the number of cabs in service. X_t is a birth and death chains. The birth or arrival rates are constant: if $i < 3$ then $q(i, i+1) = 2$. On the other hand the rate at which cabs become free is proportional to the number working, so $\mu_1 = 3$, $\mu_2 = 6$ and $\mu_3 = 9$. Setting $\pi(0) = c$ and plugging into the recursion (4.17) gives

$$\begin{aligned}
\pi(1) &= \frac{\lambda_0}{\mu_1} \cdot \pi(0) = \frac{2}{3} \cdot c = 2c/3 \\
\pi(2) &= \frac{\lambda_1}{\mu_2} \cdot \pi(1) = \frac{2}{6} \cdot \frac{2c}{3} = 2c/9 \\
\pi(3) &= \frac{\lambda_2}{\mu_3} \cdot \pi(2) = \frac{2}{9} \cdot \frac{2c}{9} = 4c/81
\end{aligned}$$

Adding up the π's gives $(4 + 18 + 54 + 81)c/81 = 157c/81$ so $c = 81/157$ and we have

$$\pi(3) = \frac{4}{157} \quad \pi(2) = \frac{18}{157} \quad \pi(1) = \frac{54}{157} \quad \pi(0) = \frac{81}{157}$$

From this we see $\pi(3) = 4/157 = .0254$. (b) Customers server per hour $= 2 \cdot (1 - .0254) = 1.9492$.

4.23. *Detailed balance for three state chains.* Consider a chain with state space $\{1, 2, 3\}$ in which $q(i, j) > 0$ if $i \neq j$ and suppose that there is a stationary distribution that satisfies the detailed balance condition. (a) Let $\pi(1) = c$. Use the detailed balance condition between 1 and 2 to find $\pi(2)$ and between 2 and 3 to find $\pi(3)$. (b) What conditions on the rates must be satisfied for there to be detailed balance between 1 and 3?

Ans. (a) $\pi(2) = cq(1,2)/q(2,1)$, $\pi(3) = cq(1,2)q(2,3)/q(3,2)q(2,1)$. (b) in order for $\pi(1)q(1,3) = \pi(3)q(3,1)$ we must have

$$c = c \cdot \frac{q(1,2)q(2,3)q(3,1)}{q(1,3)q(3,2)q(2,1)}$$

That is, $q(1,2)q(2,3)q(3,1) = q(1,3)q(3,2)q(2,1)$.

4.24. *Kolmogorov cycle condition.* Consider an irreducible Markov chain with state space S. We say that the cycle condition is satisfied if given a cycle of states $x_0, x_1, \ldots, x_n = x_0$ with $q(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$, we have

$$\prod_{i=1}^{n} q(x_{i-1}, x_i) = \prod_{i=1}^{n} q(x_i, x_{i-1})$$

(a) Show that if q has a stationary distribution that satisfies the detailed balance condition, then the cycle condition holds. (b) To prove the converse, suppose that the cycle condition holds. Let $a \in S$ and set $\pi(a) = c$. For $b \neq a$ in S let $x_0 = a, x_1 \ldots x_k = b$ be a path from a to b with $q(x_{i-1}, x_i) > 0$ for $1 \leq i \leq k$ let

$$\pi(b) = \prod_{j=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})}$$

Show that $\pi(b)$ is well defined, i.e., is independent of the path chosen. Then conclude that π satisfies the detailed balance condition.

Ans. (a) Detailed balance implies $\pi(x_{i-1})/\pi(x_i) = q(x_i, x_{i-1})/q(x_{i-1}, x_i)$. Taking the product from $i = 1, \dots n$ and using the fact that $x_0 = x_n$ we have

$$1 = \prod_{i=1}^{n} \frac{\pi(x_{i-1})}{\pi(x_i)} = \prod_{i=1}^{n} \frac{q(x_i, x_{i-1})}{q(x_{i-1}, x_i)}$$

(b) Let $x_0 = a, \dots x_k = b$ and $x'_0 = a, \dots x'_\ell = b$ be two paths from a to b. Combine these to get a loop that begins and ends at a.

$$\begin{aligned} x''_0 &= x_0, \dots x''_k = x_k = x'_\ell = b, \\ x''_{k+1} &= x'_{\ell-1}, \dots, x''_{k+\ell} = x'_0 = a \end{aligned}$$

Since $x''_{k+j} = x'_{\ell-j}$ for $1 \le j \le \ell$, letting $h = \ell - j$ we have

$$1 = \prod_{h=1}^{k+\ell} \frac{q(x''_{h-1}, x''_h)}{q(x''_h, x''_{h-1})} = \prod_{i=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} \cdot \prod_{j=1}^{\ell} \frac{q(x'_{\ell-j+1}, x'_{\ell-j})}{q(x'_{\ell-j}, x'_{\ell-j+1})}$$

Changing variables $m = \ell - j + 1$ we have

$$\prod_{i=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} = \prod_{m=1}^{\ell} \frac{q(x'_{m-1}, x'_m)}{q(x'_m, x'_{m-1})}$$

This shows that the definition is independent of the path chosen. To check that π satisfies the detailed balance condition suppose $q(c, b) > 0$. Let $x_0 = a, \dots, x_k = b$ be a path from a to b with $q(x_i, x_{i-1}) > 0$ for $1 \le i \le k$. If we let $x_{k+1} = c$ then since the definition is independent of path we have

$$\pi(b) = \prod_{i=1}^{k} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} \quad \pi(c) = \prod_{i=1}^{k+1} \frac{q(x_{i-1}, x_i)}{q(x_i, x_{i-1})} = \pi(b) \frac{q(b, c)}{q(c, b)}$$

and the detailed balance condition is satisfied.

Hitting times and exit distributions

4.25. Consider the salesman from Problem 4.1. She just left Atlanta. (a) What is the expected time until she returns to Atlanta? (b) Find the answer to (a) by computing the stationary distribution.

Ans. Let $g(i) = E_i T_A$. By (4.21)

$$g = -\begin{pmatrix} -4 & 1 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} .3 \\ .2 \end{pmatrix}$$

She has equal probability of going to B or C when she leaves A, so the expected time is $(0.3 + 0.2)/2 = 0.25$.
(b) The stationary distribution is the third row of

$$\begin{pmatrix} -4 & 2 & 1 \\ 3 & -4 & 1 \\ 5 & 0 & 1 \end{pmatrix}^{-1}$$

which is $(1/2, 1/4, 1/4)$. She spends an average of $1/4$ of a month in Atlanta, so in order the for the long run fraction of time spent there to be $1/2$, the average amount of time she is away must also be $1/4$.

4.26. Consider the two queues in series in Problem 4.8. (a) Use the methods of Section 4.4 to compute the expected duration of a busy period. (b) calculate this from the stationary distribution.

Ans. (a) Let $g(i) = E_i T_0$. The expected duration of the busy period is $g(1)$. By (4.21)

$$g = -\begin{pmatrix} -4 & 4 & 0 \\ 0 & -4 & 2 \\ 2 & 4 & -6 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ .75 \\ 1 \end{pmatrix}$$

(b) $\pi(0) = 1/3$. The chain stays in state 1 for an amount of time with mean $1/2$, so $E_1 T_0 = 1$.

4.27. We now take a different approach to analyzing the Duke Basketball chain, Example 4.11.

	0	**1**	**2**	**3**
0	-3	2	1	0
1	0	-5	5	0
2	1	0	-2.5	1.5
3	6	0	0	-6

(a) Find $g(i) = E_i(V_1)$ for $i = 0, 2, 3$. (b) Use the solution to (a) to show that the number of Duke scores (visits to state 1) by time t has $N_1(t)/t \to 0.6896$ as computed previously. (c) Compute $h(i) = P_i(V_3 < V_1)$ for $i = 0, 2$. (d) Use this to compute the distribution of $X =$ the number of time UNC scores between successive Duke baskets. (e) Use the solution of (d) to conclude that the number of UNC scores (visits to state 3) by time t has $N_3(t)/t \to 0.6206$ as computed previously.

Ans. (a) Removing the row and column for 1:

$$R = \begin{pmatrix} -31 & 0 & \\ 1 & -2.5 & 1.5 \\ 6 & 0 & -6 \end{pmatrix} \qquad -R^{-1}\mathbf{1} = \begin{pmatrix} 3/4 \\ 5/4 \\ 11/12 \end{pmatrix}$$

(b) The expected time to return to state 1 is $1/5 + g(2) = 1.45$ so $N(t)/t \to 1/1.45 = 0.6896$. (c) Using the embedded chain we have

$$h(2) = 0.6 + 0.4h(0) \qquad h(0) = h(2)/3$$

Solving gives $h(2) = 18/26$ and $h(0) = 6/26$. (d) $P(X \geq k) = h(2)h(0)^{k-1}$. (e) The mean $EX = \sum_{k=1}^{\infty} P(X \geq k) = h(2)/(1 - h(0)) = 0.9$. Thinking of the number of baskets as the reward and using the renewal reward theorem $N_3(t)/t \to 0.9(0.6896)$.

4.28. Brad's relationship with his girl friend Angelina changes between Amorous, Bickering, Confusion, and Depression according to the following transition rates when t is the time in months.

	A	**B**	**C**	**D**
A	-4	3	1	0
B	4	-6	2	0
C	2	3	-6	1
D	0	0	2	-2

(a) Find the long run fraction of time he spends in these four states? (b) Does the chain satisfy the detailed balance condition? (c) They are amorous now. What is the expected amount of time until depression sets in?

Ans. (a) The stationary distribution is the last row of

$$\begin{pmatrix} -4 & 3 & 1 & 1 \\ 4 & -6 & 2 & 1 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}^{-1}$$

which is $(.4, .3, .2, .1)$. (b) Yes it satisfies detailed balance. (c) The expected hitting time of state D, $g(i) = E_i V_D$ satisfies

$$\begin{pmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{pmatrix} \begin{pmatrix} g(A) \\ g(B) \\ g(C) \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

Inverting the matrix gives

$$\begin{pmatrix} -4 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & 3 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} -5/2 & -7/4 & -1 \\ -7/3 & -11/6 & -1 \\ -2 & -3/2 & -1 \end{pmatrix}$$

so the answer is $5/2 + 7/4 + 1 = 5.25$.

4.29. A small company maintains a fleet of four cars to be driven by its workers on business trips. Requests to use cars are a Poisson process with rate 1.5 per day. A car is used for an exponentially distributed time with mean 2 days. Forgetting about weekends, we arrive at the following Markov chain for the number of cars in service.

	0	**1**	**2**	**3**	**4**
0	−1.5	1.5	0	0	0
1	0.5	−2.0	1.5	0	0
2	0	1.0	−2.5	1.5	0
3	0	0	1.5	−3	1.5
4	0	0	0	2	−2

(a) Find the stationary distribution. (b) At what rate do unfulfilled requests come in? How would this change if there were only three cars? (c) Let $g(i) = E_i T_4$. Write and solve equations to find the $g(i)$. (d) Use the stationary distribution to compute $E_3 T_4$.

Ans. (a) Using detailed balance condition

$$\pi(1) = \frac{1.5}{0.5}\pi(0) \quad \pi(2) = \frac{1.5}{1}\pi(1) \quad \pi(3) = \frac{1.5}{1.5}\pi(0) \quad \pi(4) = \frac{1.5}{2}\pi(3)$$

Setting $\pi(0) = c$ we have $\pi(1) = 3c$, $\pi(2) = 9c/2$, $\pi(3) = 9c/2$, $\pi(4) = 27c/8$. The sum is $131c/8$ so the stationary distribution is

$$(8/131, 24/131, 36/131, 36/131, 27/131)$$

(b) With four vans the rate of unfulfilled requests is $(1.5)\pi(4) = 27/131 = 0.3091$. If there are only three cars then detailed balance gives:

$$\pi(0) = c \quad \pi(1) = 3c \quad \pi(2) = 9c/2 \quad \pi(3) = 9c/2$$

This time the sum is 13c, so the stationary distribution is

$$(1/13, 3/13, 9/26, 9/26)$$

so the rate of unfulfilled requests is now $(1.5)\pi(3) = 0.5192$.
(c) g satisfies $Rg = -1$ where R is the restriction of G to $\{0, 1, 2, 3\}$. Therefore

$$g = -\begin{pmatrix} -1.5 & 1.5 & 0 & 0 \\ 0.5 & -2.0 & 1.5 & 0 \\ 0 & 1.0 & -2.5 & 1.5 \\ 0 & 0 & 1.5 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 128/27 \\ 110/27 \\ 186/27 \\ 52/27 \end{pmatrix}$$

and we have $E_3T_4 = 52/27 = 1.926$.
(d) The chain is in state 4 for an amount of time s_i and then not in state 4 for for an amount of time u_i, so we have an alternating renewal process. $Es_i = 1/2$, so we have

$$\frac{1/2}{1/2 + E_3T_4} = \pi(4) = \frac{27}{131}$$

and it follows that

$$E_3T_4 = \left(\frac{131}{27} - 1\right)\frac{1}{2} = \frac{52}{27}$$

4.30. A submarine has three navigational devices but can remain at sea if at least two are working. Suppose that the failure times are exponential with means 1 year, 1.5 years, and 3 years. Formulate a Markov chain with states 0 = all parts working, 1,2,3 = one part failed, and 4 = two failures. Compute E_0T_4 to determine the average length of time the boat can remain at sea.

Ans. The transition rate matrix is:

	0	**1**	**2**	**3**	**4**
0	-2	1	2/3	1/3	0
1	0	-1	0	0	1
2	0	0	$-4/3$	0	4/3
3	0	0	0	$-5/3$	5/3

Let $g(i) = E_iT_4$. By (4.21)

$$g = -\begin{pmatrix} -2 & 1 & 2/3 & 1/3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4/3 & 0 \\ 0 & 0 & 0 & -5/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

-1 times =the first row of the inverse is (1/2, 1/2, 1/4, 1/10) which the average amount of time spent in each state before reaching stae 4.

4.31. Excited by the recent warm weather Jill and Kelly are doing spring cleaning at their apartment. Jill takes an exponentially distributed amount of time with mean 30 minutes to clean the kitchen. Kelly takes an exponentially distributed amount of time with mean 40 minutes to clean the bath room. The first one to complete their task will go outside and start raking leaves, a task that takes an exponentially distributed amount of time with a mean of one hour. When the second person is done inside, they will help the other and raking will be done at rate 2. (Of course the other person may already be done raking in which case the chores are done.) What is the expected time until the chores are all done?

Ans. Label the states I = the initial state, J = Jill raking, K = Kelly raking, JK = Jill and Kelly both raking, (J) = raking done and Jill still working, (K) = raking done and Kelly still working, D = done. The rates for Jill and Kelly inside are 2 and 3/2, while the raking rate = the number of people.

	I	**J**	**K**	**JK**	(**J**)	(**K**)	**D**
I	$-5/2$	2	3/2	0	0	0	0
J	0	$-5/2$	0	3/2	0	1	0
K	0	0	-3	2	1	0	0
JK	0	0	0	-2	0	0	2
(**J**)	0	0	0	0	-2	0	2
(**K**)	0	0	0	0	0	$-3/2$	3/2

Dropping the last column and inverting the matrix, we find that -1) times the first row is:

$$2/7, 8/35, 1/7, 11/35, 1/14, 16/105$$

implies that the total time is 1.195238.

Markovian queues

4.32. Consider a taxi station at an airport where taxis and (groups of) customers arrive at times of Poisson processes with rates 2 and 3 per minute. Suppose that a taxi will wait no matter how many other taxis are present. However, if an arriving person does not find a taxi waiting he leaves to find alternative transportation. (a) Find the proportion of arriving customers that get taxis. (b) Find the average number of taxis waiting.

Ans. (a) Let X_t be the number of taxis at time t. This is an M/M/1 queue with arrivals at rate 2 and service at rate 3, so the stationary distribution is shifted geometric with success probability $p = 2/3$, so $\pi(0) = 1/3$ by (4.2), i.e., 2/3's of arriving customers get taxis. (b) The mean of the shifted geometric is $(1/p) - 1$ where $p = 1 - (2/3)$ so the mean number of taxis waiting is 2.

4.33. *Queue with impatient customers.* Customers arrive at a single server at rate λ and require an exponential amount of service with rate μ. Customers waiting in line are impatient and if they are not in service they will leave at rate δ independent of their position in the queue. (a) Show that for any $\delta > 0$ the system has a stationary distribution. (b) Find the stationary distribution in the very special case in which $\delta = \mu$.

Ans. Let X_t be the number of customers in the system. X_t is a birth and death chain with $\lambda_n = \lambda$ for all $n \geq 0$, and $\mu_n = \mu + (n-1)\delta$. It follows from (4.17) that

$$\pi(n+1) = \frac{\lambda_n}{\mu_{n+1}} \cdot \pi(n)$$

If $\delta > 0$, we have $\lambda_n/\mu_{n+1} \to 0$ as $n \to \infty$. Hence if N is large enough and $n \geq N$ then $\lambda_n/\mu_{n+1} \leq 1/2$ and the desired conclusion follows from the argument in Example 3.5. (b) When $\delta = \mu$, $\mu_n = n\mu$ and

$$\frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n} = \frac{\lambda^n}{\mu^n} \cdot \frac{1}{1 \cdot 2 \cdots n} = \frac{(\lambda/\mu)^n}{n!}$$

It follows that the stationary distribution is Poisson with mean λ/μ.

4.34. Customers arrive at the Shortstop convenience store at a rate of 20 per hour. When two or fewer customers are present in the checkout line, a single clerk works and the service time is 3 minutes. However, when there are three or more customers are present, an assistant comes over to bag up the groceries and reduces the service time to 2 minutes. Assuming the service times are exponentially distributed, find the stationary distribution.

Ans. This is a birth and death chain with $\lambda_k = 20$, $\mu_1 = \mu_2 = 20$, $\mu_k = 30$ for $k \geq 3$. The recursion (4.17) implies that $\pi_0 = \pi_1 = \pi_2 = c$ while $\pi_{2+k} = (2/3)^k c$. $\sum_{k=0}^{\infty}(2/3)^k c = 3c$, so $c = 1/5$.

4.35. Customers arrive at a carnival ride at rate λ. The ride takes an exponential amount of time with rate μ, but when it is in use, the ride is subject to breakdowns at rate α. When a breakdown occurs all of the people leave since they know that the time to fix a breakdown is exponentially distributed with rate β. (i) Formulate a Markov chain model with state space $\{-1, 0, 1, 2, \ldots\}$ where -1 is broken and the states $0, 1, 2, \ldots$ indicate the number of people waiting or in service. (ii) Show that the chain has a stationary distribution of the form $\pi(-1) = a$, $\pi(n) = b\theta^n$ for $n \geq 0$.

Ans. (i) $q(n, n+1) = \lambda$ for $n \geq 0$; $q(n, n-1) = \mu$ for $n \geq 1$; $q(n, -1) = \alpha$ for $n \geq 1$; $q(-1, 0) = \beta$. (ii) Considering transitions in and out of the set $[n, \infty)$ for $n \geq 1$ shows that

$$\lambda b\theta^{n-1} = \mu b\theta^n + \alpha b\theta^n/(1-\theta)$$

this will hold if $\lambda = \mu\theta + \alpha\theta/(1-\theta)$. The right hand side is a strictly increasing function of $\theta \in [0, 1)$ which is 0 at 0 and tends to ∞ at $\theta = 1$. Thus there is a unique solution, which can be found explicitly if one wants by solving a quadratic equation. Considering transitions in and out of $[0, \infty)$ now and recalling that $\sum_{n=-1}^{\infty} \pi(n) = 1$ implies $a + b/(1-\theta) = 1$ we have

$$a\beta = \alpha b\theta/(1-\theta) = \alpha\theta(1-a)$$

which can be solved for a. To complete the proof now we observe that if there is no change in the probability of $[k, \infty)$ or of $[k+1, \infty)$ then the probability of k is not changing.

4.36. Customers arrive at a two-server station according to a Poisson process with rate λ. Upon arriving they join a single queue to wait for the next available server. Suppose that the service times of the two servers are exponential with rates μ_a and μ_b and that a customer who arrives to find the system empty will go to each of the servers with probability 1/2. Formulate a Markov chain model for this system with state space $\{0, a, b, 2, 3, \dots\}$ where the states give the number of customers in the system, with a or b indicating there is one customer at a or b respectively. Show that this system is time reversible. Set $\pi(2) = c$ and solve to find the limiting probabilities in terms of c.

Ans. Let $\mu = \mu_a + \mu_b$. The chain on $\{2, 3, \dots\}$ has the same transition probabilities as the M/M/1 queue so we have $\pi_n = \pi_2(\lambda/\mu)^{n-2}$ for $n \geq 2$. Detailed balance between a and 2 and b and 2 imply

$$\lambda\pi_a = \mu_b\pi_2 \qquad \lambda\pi_b = \mu_a\pi_2$$

So if $\pi_2 = c$ we have $\pi_a = c\mu_b/\lambda$ and $\pi_b = c\mu_a/\lambda$. Detailed balance between 0 and a and 0 and b imply

$$(\lambda/2)\pi_0 = \mu_a\pi_a = c\mu_a\mu_b/\lambda \qquad (\lambda/2)\pi_0 = \mu_a\pi_2 = c\mu_b\mu_a/\lambda$$

so even though we had two equations for one unknown they both gave us the same answer.

4.37. At present the Economics department and the Sociology department each have one typist who can type 25 letters a day. Economics requires an average of 20 letters per day, while Sociology requires only average of 15. Assuming Poisson arrival and exponentially distributed typing times find (a) the average queue length and average waiting time in each departments (b) the average overall waiting time if they merge their resources to form a typing pool.

Ans. When the departments are separate letters are an $M/M/1$ queue with $\mu = 25$ and $\lambda = 20$ or 15. The stationary distribution is $(1 - \lambda/\mu)(\lambda/\mu)^n$ for $n = 0, 1, \dots$ so the queue length in equilibrium has

$$L = \frac{1}{1 - \lambda/\mu} - 1 = \frac{\lambda}{\mu - \lambda} \quad \text{which is 4 or 1.5}$$

and the average waiting time is $W = L/\lambda = 1/(\mu - \lambda)$ which is 1/5 or 1/10 of a day. With two typists we have an $M/M/2$ queue in which the total arrival rate is $\lambda = 35$ and the service rate $\mu = 25$ per active server. Using detailed balance and setting $\pi(0) = c$ we conclude that $\pi(1) = c\lambda/\mu$ and for $k \geq 2$

$$\pi(k) = \frac{\lambda}{2\mu}\pi(k-1) = \left(\frac{\lambda}{2\mu}\right)^{k-1}\pi(1)$$

We can combine the formulas to

$$\pi(k) = 2c\left(\frac{\lambda}{2\mu}\right)^k \quad \text{for } k \geq 1$$

where c is a constant that makes the sum equal to 1.

$$\frac{1}{c}\sum_{k=0}^{\infty}\pi(k) = 1 + 2\sum_{k=1}^{\infty}\left(\frac{\lambda}{2\mu}\right)^k = 1 + \frac{\lambda/\mu}{1-\lambda/2\mu}$$

When $\lambda = 35$ and $\mu = 25$ the right-hand side is

$$1 + \frac{35/25}{15/50} = 1 + \frac{14}{3}$$

so $c = 3/17$. The calculate the mean queue length now

$$\begin{aligned}\sum_{k=1}^{\infty} k\pi(k) &= \frac{3}{17}\sum_{k=1}^{\infty} k\cdot 2\left(\frac{\lambda}{2\mu}\right)^k \\ &= \frac{3}{17}\cdot\frac{2}{1-\lambda/2\mu}\cdot\frac{\lambda/2\mu}{1-\lambda/2\mu} = \frac{3\cdot 100\cdot 35}{17\cdot 15\cdot 15} = 2.745\end{aligned}$$

by the formula for the mean of the shifted geometric distribution. Using $W = L/\lambda$ we conclude that the average waiting time is now $0.078843 = 1/12.75$ of a day.

4.38. Consider an $M/M/s$ queue with no waiting room. In words, requests for a phone line occur at a rate λ. If one of the s lines is free, the customer takes it and talks for an exponential amount of time with rate μ. If no lines are free, the customer goes away never to come back. Find the stationary distribution. You do not have to evaluate the normalizing constant.

Ans. Using the stationary distribution for the $M/M/\infty$ queue given in Example 4.5 and using the result on restrictions given in Exercise 4.7, we have that $\pi(n) = c_s e^{-\lambda/\mu}(\lambda/\mu)^n/n!$ where c_s is chosen to make the $\sum_{n=0}^{s}\pi(n) = 1$.

Queueing networks

4.39. Consider a production system consisting of a machine center followed by an inspection station. Arrivals from outside the system occur only at the machine center and follow a Poisson process with rate λ. The machine center and inspection station are each single-server operations with rates μ_1 and μ_2. Suppose that each item independently passes inspection with probability p. When an object fails inspection it is sent to the machine center for reworking. Find the conditions on the parameters that are necessary for the system to have a stationary distribution.

Ans. In this case the equations are

$$r_1 = \lambda + (1-p)r_2 \qquad r_2 = r_1$$

which solves to give that each $r_i = \lambda/p$. For stability we must have that $\lambda/p < \min\{\mu_1, \mu_2\}$.

4.40. Consider a three station queueing network in which arrivals to servers $i = 1, 2, 3$ occur at rates $3, 2, 1$, while service at stations $i = 1, 2, 3$ occurs at rates $4, 5, 6$. Suppose that the probability of going to j when exiting i, $p(i, j)$ is given by $p(1, 2) = 1/3$, $p(1, 3) = 1/3$, $p(2, 3) = 2/3$, and $p(i, j) = 0$ otherwise. Find the stationary distribution.

Ans. In this case the equations

$$r_1 = \lambda_1 = 3 \quad r_2 = \lambda_2 + r_1 p(1,3) = 2 + 3 \cdot (1/3) = 3$$
$$r_3 = \lambda_3 + r_1 p(1,3) + r_2 p(2,3) = 1 + 3 \cdot (1/3) + 3 \cdot (2/3) = 4$$

So the stationary distribution is independent shifted geometrics with failure probabilities 3/4, 3/5, 4/6.

4.41. *Feed-forward queues.* Consider a k station queueing network in which arrivals to server i occur at rate λ_i and service at station i occurs at rate μ_i. We say that the queueing network is feed-forward if the probability of going from i to $j < i$ has $p(i,j) = 0$. Consider a general three station feed-forward queue. What conditions on the rates must be satisfied for a stationary distribution to exist?

Ans. In this case the equations are

$$r_1 = \lambda_1 \quad r_2 = \lambda_2 + r_1 p(1,2) \quad r_3 = \lambda_3 + r_1 p(1,3) + r_2 p(2,3)$$

Solving we have $r_2 = \lambda_2 + \lambda_1 p(1,2)$ and

$$r_3 = \lambda_3 + \lambda_1 p(1,3) + \lambda_2 p(2,3) + \lambda_1 p(1,2) p(2,3)$$

For stability we must have $r_i < \mu_i$ for all i.

4.42. *Queues in series.* Consider a k station queueing network in which arrivals to server i occur at rate λ_i and service at station i occurs at rate μ_i. In this problem we examine the special case of the feed-forward system in which $p(i, i+1) = p_i$ for $1 \le i < k$. In words the customer goes to the next station or leaves the system. What conditions on the rates must be satisfied for a stationary distribution to exist?

Ans. In this case the equations are $r_1 = \lambda_1$ and

$$r_i = \lambda_i + r_{i-1} p(i-1, i)$$

for $i > 1$. Solving we have $r_2 = \lambda_2 + \lambda_1 p(1,2)$ and

$$r_3 = \lambda_3 + \lambda_1 p(1,3) + \lambda_2 p(2,3) + \lambda_1 p(1,2) p(2,3)$$

or in general that

$$r_i = \lambda_i + \lambda_{i-1} p(i-1,i) + \cdots + \lambda_1 p(1,2) \cdots p(i-1,i)$$

For stability we must have $r_i < \mu_i$ for all i.

4.43. At registration at a very small college, students arrive at the English table at rate 10 and at the Math table at rate 5. A student who completes service at the English table goes to the Math table with probability 1/4 and to the cashier with probability 3/4. A student who completes service at the Math table goes to the English table with probability 2/5 and to the cashier with probability 3/5. Students who reach the cashier leave the system after they pay. Suppose that the service times for the English table, Math table, and cashier are 25, 30, and 20, respectively. Find the stationary distribution.

Ans. In this case the equations are

$$r_1 = 10 + 2r_2/5 \quad r_2 = 5 + r_1/4 \quad r_3 = 3r_1/4 + 3r_2/5$$

Inserting the second equation in the first gives $r_1 = 12 + r_1/10$ so $r_1 = 40/3$. Using the second and third equations now we have

$$r_2 = 5 + 10/3 = 25/3 \quad r_3 = 10 + 5$$

The problem told us that $\mu_1 = 25$, $\mu_2 = 30$, and $\mu_3 = 20$. Thus $r_i < \mu_i$ for all i and there is a stationary distribution consisting of independent shifted geometrics with failure probabilities r_i/μ_i.

4.44. At a local grocery store there are queues for service at the fish counter (1), meat counter (2), and café (3). For $i = 1, 2, 3$ customers arrive from outside the system to station i at rate i, and receive service at rate $4 + i$. A customer leaving station i goes to j with probabilities $p(i, j)$ given the following matrix

	1	**2**	**3**
1	0	1/4	1/2
2	1/5	0	1/5
3	1/3	1/3	0

In equilibrium what is the probability no one is in the system, i.e., $\pi(0, 0, 0)$.

Ans. To solve $r_j = \lambda_j + \sum_{i=1}^{3} r_i p(i, j)$ we note that

$$r = \lambda(I - p)^{-1} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1.4 & .625 & .825 \\ .4 & 1.25 & .45 \\ .6 & .625 & 1.425 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

$\pi(0, 0, 0) = (1 - 4/5)(1 - 5/6)(1 - 6/7) = 1/210$.

4.45. Three vendors have vegetable stands in a row. Customers arrive at the stands 1, 2, and 3 at rates 10, 8, and 6. A customer visiting stand 1 buys something and leaves with probability 1/2 or visits stand 2 with probability 1/2. A customer visiting stand 3 buys something and leaves with probability 7/10 or visits stand 2 with probability 3/10. A customer visiting stand 2 buys something and leaves with probability 4/10 or visits stands 1 or 3 with probability 3/10 each. Suppose that the service rates at the three stands are large enough so that a stationary distribution exists. At what rate do the three stands make sales. To check your answer note that since each entering customers buys exactly once the three rates must add up to 10+8+6=24.

Ans. In this case the equations are

$$r_1 = 10 + .3r_2 \quad r_2 = 8 + .5r_1 + .3r_3 \quad r_3 = 6 + .3r_2$$

Plugging the first and third question into the middle one

$$r_2 = 8 + .5(10 + .3r_2) + .3(6 + .3r_2) = 14.8 + .24r_2$$

Thus $r_2 = 14.8/.76 = 19.473$. $r_1 = 10 + .3r_2 = 15.842$, $r_3 = 6 + .3r_2 = 11.842$. The rates at which sales are made by the three vendors are

$$.5r_1 = 7.921 \quad .4r_2 = 7.7892 \quad .7r_3 = 8.2894$$

4.46. Four children are playing two video games. The first game, which takes an average of 4 minutes to play, is not very exciting, so when a child completes a turn on it they always stand in line to play the other one. The second one, which takes an average of 8 minutes, is more interesting so when they are done they will get back in line to play it with probability 1/2 or go to the other machine with probability 1/2. Assuming that the turns take an exponentially distributed amount of time, find the stationary distribution of the number of children playing or in line at each of the two machines.

Ans. This is a special case of Example 7.1 in which the service rates are $\mu_1 = 1/4$, $\mu_2 = 1/8$, and the routing matrix is

$$p(i,j) = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

The stationary distribution has $\pi_1 = 1/3$ and $\pi_2 = 2/3$. These queues are single servers so

$$\psi_1(n) = 1/4^n \qquad \psi_2(n) = 1/8^n$$

Plugging into the formula in (7.3) we have

$$\pi(n, 4-n) = c(4/3)^n\,(16/3)^{4-n} = c'(1/4)^n$$

where c' is chosen to make the sum equal to 1. $\sum_{n=0}^{4}(1/4)^n = 1365/1024$ so $c' = 1024/1365$.

5.6 Exercises

Throughout the exercises we will use our standard notion for hitting times. $T_a = \min\{n \geq 1 : X_n = a\}$ and $V_a = \min\{n \geq 0 : X_n = a\}$.

5.1. *Brother–sister mating.* Consider the six state chain defined in Exercise 1.66. Show that the total number of A's is a martingale and use this to compute the probability of getting absorbed into the 2,2 (i.e., all A's state) starting from each initial state.

Ans. The fact that X_n is a martingale follows from Theorem 5.5 and the computations done in Exercise 1.66. Let $T = \min\{n : X_n = 22 \text{ or } 00\}$. The pedestrian lemma, Lemma 1.3, implies that $P(T < \infty) = 1$. $|X_{T\wedge n}| \leq N$ so it follows from 5.14 that $E_{ij} X_\tau = i + j$ so $P_{ij}(V_{22} < V_{00} = (i + j)/4$.

5.2. Let X_n be the *Wright–Fisher model with no mutation* defined in Example 1.9. (a) Show that X_n is a martingale and use Theorem 5.14 to conclude that $P_x(V_N < V_0) = x/N$. (b) Show that $Y_n = X_n(N - X_n)/(1 - 1/N)^n$ is a martingale. (c) Use this to conclude that

$$(N-1) \leq \frac{x(N-x)(1-1/N)^n}{P_x(0 < X_n < N)} \leq \frac{N^2}{4}$$

Ans. (a) The fact that X_n is a martingale follows from Theorem 5.5 and the computations done in Example 1.42. If we let $T = \min\{n : X_n \notin (0, N)\}$ then T is a stopping time, and the pedestrian lemma (Lemma 1.3) implies that $P(T < \infty) = 1$. $|X_{T\wedge n}| \leq N$ so it follows from 5.14 that $x = E_x X_T = N P_x(V_N < V_0)$. (b) When $X_0 = x$, $X_1 = \text{Binomial}(x/N, N)$, so $E_x X_1 = x$, $E_x X_1^2 = x(1 - x/N) + x^2$, and $E_x(X_1(N - X_1)) = Nx - x + x^2/N - x^2 = x(N - x)(1 - 1/N)$. (b) Since Y_n is a martingale, $E_x X_n(N - X_n) = x(N - x)(1 - 1/N)^n$. The desired result now follows from the observation that when $0 < X_n < N$ we have $N - 1 \leq X_n(N - X_n) \leq N^2/4$.

5.3. *Lognormal stock prices.* Consider the special case of Example 5.5 in which $X_i = e^{\eta_i}$ where $\eta_i = \text{normal}(\mu, \sigma^2)$. For what values of μ and σ is $M_n = M_0 \cdot X_1 \cdots X_n$ a martingale?

Ans. By (5.15) we have $E \exp(\theta\eta_i) = \exp(\sigma^2\theta^2/2 + \theta\mu)$. For M_n to be a martingale we want $EX_n = 1$, i.e., $0 = \sigma^2/2 - \mu$.

5.4. Suppose that in Polya's urn there is one ball of each color at time 0. Let X_n be the fraction of red balls at time n. Use Theorem 5.13 to conclude that $P(X_n \geq 0.9 \text{ for some } n) \leq 5/9$.

Ans. Let $T = \min\{n : X_n \geq 0.9\}$.

$$1/2 = EX_{T\wedge n} \geq 0.9 P(T \leq n)$$

so $P(T \leq n) \leq 5/9$. Now let $n \to \infty$.

5.5. Suppose that in Polya's urn there are r red balls and g green balls at time 0. show that $X = \lim_{n\to\infty} X_n$ has a beta distribution

$$\frac{(g + r - 1)!}{(g - 1)!(r - 1)!} x^{g-1}(1 - x)^{r-1}$$

Ans. The probability that red balls are drawn on the first j draws and then green balls are drawn on the next $n-j$.

$$\frac{g(g+1)\cdots(g+j-1)\cdot r(r+1)\cdots(r+n-j-1)}{(g+r)(g+r+1)\cdots(g+r+n-1)}$$
$$= \frac{(g+r-1)!}{(g-1)!(r-1)!}\frac{(g+j-1)!(n-j+r-1)!}{(g+r+n-1)!}$$

There are $\binom{n}{j} = n!/j!(n-j)!$ ways to choose the j draws on which we get red

$$P(R_n = r+j) = \frac{(g+r-1)!}{(g-1)!(r-1)!}$$
$$\cdot \frac{(j+g-1)\cdots(j+1)(n-j+r-1)\cdots(n-j+1)}{(n+g+r-1)\cdots n+1}$$

Note that there are $g+r-1$ terms in the denominator and $(g-1)+(r-1)$ in the numerator. If j and $n-j$ are both large then the the second line is

$$\approx \frac{j^{g-1}(1-j)^{r-1}}{n^{g+r-2}}\cdot\frac{1}{n+1}$$

If $j/n \to x$ the first term converges to $x^{g-1}(1-x)^{r-1}$, which proves the desired result.

5.6. *An unfair fair game.* Define random variables recursively by $Y_0 = 1$ and for $n \geq 1$, Y_n is chosen uniformly on $(0, Y_{n-1})$. If we let $U_1, U_2, \ldots$ be uniform on $(0,1)$, then we can write this sequence as $Y_n = U_n U_{n-1}\cdots U_0$. (a) Use Example 5.5 to conclude that $M_n = 2^n Y_n$ is a martingale. (b) Use the fact that $\log Y_n = \log U_1 + \cdots + \log U_n$ to show that $(1/n)\log X_n \to -1$. (c) Use (b) to conclude $M_n \to 0$, i.e., in this "fair" game our fortune always converges to 0 as time tends to ∞.

Ans. (a) Since $M_n = M_0 X_1 \cdots X_n$ where the $X_m = 2U_m$ are i.i.d. with mean 1, the desired conclusion follows from Example 5.5. (b) $-\log U_1, -\log U_2, \ldots$ are independent with $P(-\log U_i > x) = P(U_i < e^{-x}) = e^{-x}$, so $-\log U_i$ has an exponential distribution with mean 1. Using the strong law of large numbers gives $(1/n)\log X_n \to -1$. (c) The last conclusion implies $(1/n)\log M_n \to -1 + \log 2 < 0$ so $\log M_n \to -\infty$ and $M_n \to 0$.

5.7. *General birth and death chains.* The state space is $\{0,1,2,\ldots\}$ and the transition probability has

$$\begin{aligned} &p(x,x+1) = p_x \\ &p(x,x-1) = q_x && \text{for } x > 0 \\ &p(x,x) = 1 - p_x - q_x && \text{for } x \geq 0 \end{aligned}$$

while the other $p(x,y) = 0$. Let $V_y = \min\{n \geq 0 : X_n = y\}$ be the time of the first visit to y and let $h_N(x) = P_x(V_N < V_0)$. Let $\phi(z) = \sum_{y=1}^{z}\prod_{x=1}^{y-1} q_x/p_x$. Show that

$$P_x(V_b < V_a) = \frac{\phi(x)-\phi(a)}{\phi(b)-\phi(a)}$$

From this it follows that 0 is recurrent if and only if $\phi(b) \to \infty$ as $b \to \infty$, giving another solution of Exercise 9.46 from Chapter 1.

Ans. The definition of ϕ implies $\phi(x+1) - \phi(x) = (q_x/p_x)(\phi(x) - \phi(x-1)$, so $E_x\phi(X_1) = \phi(x)$ for $a < x < b$. Let $h(x) = (\phi(x) - \phi(a))/(\phi(b) - \phi(a))$. Clearly $E_x h(X_1) = h(x)$ for $a < x < b$, $h(a) = 0$ and $h(b) = 1$, so the desired result follows from (6.3) in Chapter 1. A different solution more in keeping with this chapter is to let $\bar{X}_n$ be X_n modified so that a and b are absorbing states, note that $\phi(\bar{X}_n)$ is a martingale, and imitate the proof of (4.1).

5.8. Let $S_n = X_1 + \cdots + X_n$ where the X_i are independent with $EX_i = 0$ and $\text{var}\,(X_i) = \sigma^2$. (a) Show that $S_n^2 - n\sigma^2$ is a martingale. (b) Let $\tau = \min\{n : |S_n| > a\}$. Use Theorem 5.13 to show that $E\tau \geq a^2/\sigma^2$. For simple random walk $\sigma^2 = 1$ and we have equality.

Ans. (a) Since S_n is a martingale, using the notation from (5.5) and Lemma 5.7 gives

$$E(S_{n+1}^2|A_v) = S_n^2 + E(X_{n+1}|A_v) = S_n^2 + \sigma^2$$

Subtracting $(n+1)\sigma^2$ from both sides gives the desired result.
(b) Using Theorem 5.13, we have

$$\sigma^2 E(\tau \wedge n) = ES_{\tau\wedge n}^2 \geq a^2 P(\tau \leq n)$$

Letting $n \to \infty$ we have $E\tau \geq a^2/\sigma^2$.

5.9. *Wald's second equation.* Let $S_n = X_1 + \cdots + X_n$ where the X_i are independent with $EX_i = 0$ and $\text{var}\,(X_i) = \sigma^2$. Use the martingale from the previous problem to show that if T is a stopping time with $ET < \infty$ then $ES_T^2 = \sigma^2 ET$.

Ans. Using Theorem 5.13 and (5.6) we have

$$E(S_{T\wedge n}^2) = \sum_{k=1}^{n} E(S_{T\wedge k} - S_{T\wedge(k-1)})^2 = \sum_{k=1}^{n} \sigma^2 P(T > k-1)$$

Letting $n \to \infty$ now gives the desired result.

5.10. *Mean time to gambler's ruin.* Let $S_n = S_0 + X_1 + \cdots + X_n$ where $X_1, X_2, \ldots$ are independent with $P(X_i = 1) = p < 1/2$ and $P(X_i = -1) = 1-p$. Let $V_0 = \min\{n \geq 0 : S_n = 0\}$. Use Wald's equation to conclude that if $x > 0$ then $E_x V_0 = x/(1-2p)$.

Ans. The mean movement on one step is $\mu = 2p - 1$ so $S_n - (2p-1)n$ is a martingale. Theorems 5.13 and 5.11 give

$$x = E_x S_{V_0\wedge n} - (2p-1)E_x(V_0 \wedge n)$$

Rearranging we have

$$(1-2p)E_x(V_0 \wedge n) = x - E_x S_{V_0\wedge n} \leq x$$

This shows that $E_x V_0 < \infty$ so we can use Wald's equation to conclude that $-x = (2p-1)E_x V_0$.

5.11. *Variance of the time of gambler's ruin.* Let $\xi_1, \xi_2, \ldots$ be independent with $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = q = 1 - p$ where $p < 1/2$. Let $S_n = S_0 + \xi_1 + \cdots + \xi_n$. In Example 4.3 we showed that if $V_0 = \min\{n \geq 0 : S_n = 0\}$ then $E_x V_0 = x/(1-2p)$. The aim of this problem is to compute the variance of V_0. (a) Show that $(S_n - (p-q)n)^2 - n(1-(p-q)^2)$ is a martingale. (b) Use this to conclude that when $S_0 = x$ the variance of V_0 is

$$x \cdot \frac{1-(p-q)^2}{(p-q)^3}$$

(c) Why must the answer in (b) be of the form cx?

Ans. (a) Since $\xi - (p-q)$ has mean 0 and variance $1-(p-q)^2$ this follows from Example 2.5. (b) Since $(S_n - (p-q)n)^2 = S_n^2 - 2(p-q)nS_n + (p-q)^2n^2$, using the optional stopping theorem at time V_0, noticing $S_{V_0} = 0$, (and not worrying about the details) we have

$$(p-q)^2 E_x V_0^2 - (1-(p-q)^2)E_x V_0 = x^2$$

Using $E_x V_0 = x/(p-q)$ and rearranging gives

$$E_x V_0^2 = x \cdot \frac{1-(p-q)^2}{(p-q)^3} + \frac{x^2}{(p-q)^2}$$

Since the second term is $(E_x V_0)^2$ this gives the desired result. (c) The strong Markov property implies that $V_{x-1}, V_{x-2} - V_{x-1}, \ldots V_0 - V_1$ are independent and have the same distribution.

5.12. *Generating function of the time of gambler's ruin.* Continue with the set-up of the previous problem. (a) Use the exponential martingale and our stopping theorem to conclude that if $\theta \leq 0$, then $e^{\theta x} = E_x(\phi(\theta)^{-V_0})$. (b) Let $0 < s < 1$. Solve the equation $\phi(\theta) = 1/s$, then use (a) to conclude

$$E_x(s^{V_0}) = \left(\frac{1-\sqrt{1-4pqs^2}}{2ps} \right)^x$$

(c) Why must the answer in (b) be of the form $f(s)^x$?

Ans. (a) If $\theta \leq 0$ then $e^{\theta S_n} \leq 1$ when $S_n \geq 0$. If in addition, $p < 1/2$ then

$$\phi(\theta) = pe^{\theta} + (1-p)e^{-\theta} > 1 \quad \text{for } \theta < 0$$

These two facts imply that $\exp(\theta S_n)/\phi(\theta)^n \leq 1$ when $S_n \geq 0$. Applying (3.10) and noting $S(V_0) = 0$ we see that $e^{\theta x} = E(\phi(\theta)^{-V_0}$. (b) If we let $r = e^{\theta}$ then $\phi(\theta) = 1/s$ is equivalent to the quadratic equation $pr^2 - r/s + q = 0$ which has solutions

$$\frac{1/s \pm \sqrt{1/s^2 - 4pq}}{2p} = \frac{1 \pm \sqrt{1-4pqs^2}}{2ps}$$

where in the second step we have multiplied top and bottom by s. (c) The strong Markov property implies that $V_{x-1}, V_{x-2} - V_{x-1}, \ldots V_0 - V_1$ are independent and have the same distribution. The result then follows from the fact that the generating function for the sum of n independent and identically distributed random variables is the nth power of that for one term.

5.13. Consider a favorable game in which the payoffs are -1, 1, or 2 with probability 1/3 each. Use the results of Example 5.12 to compute the probability we ever go broke (i.e, our winnings W_n reach \$0) when we start with \$$i$.

Ans. In order for $1 = \phi(\theta)$ we need

$$1 = (e^{-\theta} + e^{\theta} + e^{2\theta})/3$$

Multiplying by $3e^\theta$ and setting $x = e^\theta$ this becomes $x^3 + x^2 - 3x + 1 = 0$. Since $\theta = 0$ is a solution, $x = 1$ is a root. Dividing the cubic by $x - 1$ we get $x^2 + 2x - 1$, which by the quadratic formula has roots

$$r_1 = \frac{-2 - \sqrt{4 - 4(1)(-1)}}{2} \quad \text{and} \quad r_2 = \frac{-2 + \sqrt{8}}{2} = -1 + \sqrt{2}$$

The first root is negative so it cannot be e^θ. The solution then is to set $\alpha = \ln r_2$. The answer given by (4.7) is then

$$P_0(T_{-i} < \infty) = e^{-\alpha(-i)} = r_2^i = (0.4142)^i$$

5.14. A branching process can be turned into a random walk if we only allow one individual to die and be replaced by its offspring on each step. If the offspring distributions is p_k and the generating function is ϕthen the random walk increments have $P(X_i = k - 1) = p_k$. Let $S_k = 1 + X_1 + \ldots + X_n$ and $T_0 = \min\{n : S_n = 0\}$. Suppose $\mu = \sum_k kp_k > 1$. Use Example 5.12 to show that $P(T_0 < \infty) = \rho$, the solution < 1 of $\phi(\rho) = \rho$.

Ans. If we let $\rho = e^\theta$ then $Ee^{\theta X} = \sum_k p_k \rho^{k-1} = \phi(\rho)/\rho) = 1$.

5.15. Let Z_n be a branching process with offspring distribution p_k with $p_0 > 0$ and $\mu = \sum_k kp_k > 1$. Let $\phi(\theta) = \sum_{k=0}^\infty p_k \theta^k$. (a) Show that $E(\theta^{Z_{n+1}}|Z_n) = \phi(\theta)^{Z_n}$. (b) Let ρ be the solution < 1 of $\phi(\rho) = \rho$ and conclude that $P_k(T_0 < \infty) = \rho^k$

Ans. (a) Z_{n+1} is the sum of Z_n independent random variables ξ_i with distribution p_k so

$$E(\theta^{Z_{n+1}}|Z_n = k) = E(\theta^{\xi_1 + \cdots + \xi_k}) = \phi(\theta)^k$$

(b) Using Theorem 5.13,

$$\rho^k = E\rho^{Z(T_0 \wedge n)} = P_k(T_0 \le n) + E(\rho^{Z(n)}; T_0 > n)$$

Every time the process returns to state k, with probability p_0^k it will hit 0 on the next step. Using the reasoning Lemma 1.3 it follows that Z_n will visit k only finitely many times. Since this is true for any $k > 0$, it follows that $Z_n \to \infty$ on $\{Z_n > 0 \text{ for all } n \ge 0\}$, and letting $n \to \infty$ in the displayed equation proves the desired formula.

5.16. *Hitting probabilities.* Consider a Markov chain with finite state space S. Let a and b be two points in S, let $\tau = V_a \wedge V_b$, and let $C = S - \{a, b\}$. Suppose $h(a) = 1$, $h(b) = 0$, and for $x \in C$ we have

$$h(x) = \sum_y p(x, y)h(y)$$

(a) Show that $h(X_n)$ is a martingale. (b) Conclude that if $P_x(\tau < \infty) > 0$ for all $x \in C$, then $h(x) = P_x(V_a < V_b)$ giving a proof of Theorem 1.27.

Ans. (a) Our assumptions imply that $h(x) = E_x h(X_1)$ so this follows from Theorem 5.5. (b) Lemma 1.3 sows that $P_x(\tau < \infty) > 0$ for all $x \in C$ implies $P_x(\tau < \infty) = 1$ for all $x \in C$. Since $h(X_n)$ is bounded using Theorem **??** now gives $h(x) = E_x h(X_\tau) = P_x(X_\tau = a) = P_x(V_a < V_b)$.

5.17. *Expectations of hitting times.* Consider a Markov chain state space S. Let $A \subset S$ and suppose that $C = S - A$ is a finite set. Let $V_A = \min\{n \geq 0 : X_n \in A\}$ be the time of the first visit to A. Suppose that $g(x) = 0$ for $x \in A$, while for $x \in C$ we have

$$g(x) = 1 + \sum_y p(x,y)g(y)$$

(a) Show that $g(X_{V_A \wedge n}) + (V_A \wedge n)$ is a martingale. (b) Conclude that if $P_x(V_A < \infty) > 0$ for all $x \in C$ then $g(x) = E_x V_A$, giving a proof of Theorem 1.28.

Ans. (a) Our assumptions imply that $g(x) = 1 + E_x g(X_1)$ for $x \in B$ from which the martingale property follows. (b) The pedestrian lemma implies that for some $k < \infty$ and $\alpha > 0$ we have $P_x(V_A > nk) \leq (1-\alpha)^n$ so $P_x(V_A < \infty) = 1$ for all $x \in C$ and $E_x V_A < \infty$. Using (3.6) now at $T = V_A \wedge n$ we have

$$g(x) - E_x g(V_A \wedge n) = E_x(V_A \wedge n)$$

Letting $n \to \infty$ now we have $g(x) = E_x V_A$.

5.18. *Lyapunov functions.* Let X_n be an irreducible Markov chain with state space $\{0, 1, 2, \dots\}$ and let $\phi \geq 0$ be a function with $\lim_{x\to\infty} \phi(x) = \infty$, and $E_x\phi(X_1) \leq \phi(x)$ when $x \geq K$. Then X_n is recurrent. This abstract result is often useful for proving recurrence in many chains that come up in applications and in many cases it is enough to consider $\phi(x) = x$.

Ans. Let $T = \min\{n : X_n \leq K \text{ or } X_n \geq N\}$. Since $\phi(X_{T\wedge n})$ is a supermartingale and $\phi \geq 0$

$$\phi(x) \geq P_x(T \leq n, X_T \geq N) \cdot \min_{y \geq N} \phi(y)$$

Letting $n \to \infty$ we have $P_x(X_T \geq N) \leq \phi(x)/\min_{y\geq N}\phi(y)$. The right hand side tends to 0 as $N \to \infty$ proving that the chain hits $\{0, \dots, K\}$ with probability 1.

5.19. *GI/G/1 queue.* Let $\xi_1, \xi_2, \dots$ be independent with distribution F and Let $\eta_1, \eta_2, \dots$ be independent with distribution G. Define a Markov chain by

$$X_{n+1} = (X_n + \xi_n - \eta_{n+1})^+$$

where $y^+ = \max\{y, 0\}$. Here X_n is the workload in the queue at the time of arrival of the nth customer, not counting the service time of the nth customer, η_n. The amount of work in front of the $(n+1)$th customer is that in front of the nth customer plus his service time, minus the time between the arrival of customers n and $n+1$. If this is negative the server has caught up and the waiting time is 0. Suppose $E\xi_< E\eta_i$ and let $\epsilon = (E\eta_i - E\xi_i)/2$. (a) Show that there is a K so that $E_x(X_1 - x) \leq -\epsilon$ for $x \geq K$. (c) Let $U_k = \min\{n : X_n \leq K\}$. (b) Use the fact that $X_{U_k \wedge n} + \epsilon(U_k \wedge n)$ is a supermartingale to conclude that $E_x U_k \leq x/\epsilon$.

Ans. (a) As $x \to \infty$ $E_x(X_1 - x) = E(x + \eta_0 - \xi_1)^+ - x \to E(\eta_0 - \xi_1) = -2\epsilon$ so eventually this will be $< -\epsilon$. (b) Theorem 5.13 and $X_m \geq 0$ imply that $x \geq \epsilon E(U_k \wedge n)$. Letting $n \to \infty$ gives the desired result.

6.8 Exercises

6.1. A stock is now at \$110. In a year its price will either be \$121 or \$99. (a) Assuming that the interest rate is $r = 0.04$ find the price of a call $(S_1 - 113)+$. (b) How much stock Δ_0 do we need to buy to replicate the option. (c) Verify that having V_0 in cash and Δ_0 in stock replicates the option exactly.

Ans. $u = 1.1$ and $d = 0.9$ so the risk neutral probability $p^* = (1.04 - 0.9)/(1.1 - 0.9) = 0.14/0.2 = 0.7$. So the option price is $5.6/1.04 = 5.385$. (b) $\Delta_0 = V_1(H) - V_1(T)/S_1(H) - S_1(T) = 8/22$. (c) We need to verify

$$\frac{5.6}{1.04} + \frac{8}{22}\left[\frac{121}{1.04} - 110\right] = \frac{8}{1.04}$$
$$\frac{5.6}{1.04} + \frac{8}{22}\left[\frac{99}{1.04} - 110\right] = 0$$

The second equality implies the first. To verify the second we note that

$$\frac{11.2 + 8 \cdot 9}{2(1.04)} = \frac{83.2}{2.08} = 40$$

6.2. A stock is now at \$60. In a year its price will either be \$75 or \$45. (a) Assuming that the interest rate is $r = 0.05$ find the price of a put $(60 - S_1 -)+$. (b) How much stock Δ_0 do we need to sell to replicate the option. (c) Verify that having V_0 in cash and Δ_0 in stock replicates the option exactly.

Ans. $u = 1.25$ and $d = 0.75$ so the risk neutral probability $p^* = (1.05 - 0.75)/(1.25 - 0.75) = 3/5$. So the option price is $6/1.05 = 5.714$. (b) $\Delta_0 = V_1(H) - V_1(T)/S_1(H) - S_1(T) = -1/2$. (c) We need to verify

$$\frac{6}{1.05} - \frac{1}{2}\left[\frac{75}{1.05} - 60\right] = 0$$
$$\frac{6}{1.05} - \frac{1}{2}\left[\frac{45}{1.05} - 60\right] = \frac{15}{1.05}$$

The first equality implies the second. To verify the firxt we note that

$$\frac{12 - 75}{2(1.05)} = -\frac{63}{2.1} = -30$$

6.3. It was crucial for our no arbitrage computations that there were only two possible values of the stock. Suppose that a stock is now at 100, but in one month my be at 130, 110 or 80 in outcomes that we call 1, 2 and 3. (a) Find all the (nonnegative) probabilities p_1, p_2 and $p_3 = 1 - p_1 - p_2$ that make the stock proce a martingale. (b) Find the maximum and minimum values, v_1 and v_0, of the expected value of the the call option $(S_1 - 105)^+$ among the martingale probabilities. (c) Show that we can start with v_1 in cash, buy x_1 shares of stock and we have $v_1 + x_1(S_1 - S_0) \geq (S_1 - 105)^+$ in all three outcomes with equality for 1 and 3. (d) If we start with v_0 in cash, buy x_0 shares of stock and we have $v_0 + x_0(S_1 - S_0) \le (S_1 - 105)^+$ in all three outcomes with equality for 2 and 3. (e) Use (c) and (d) to argue that the only prices for the option consistent with absence of arbitrage are those in $[v_0, v_1]$.

Ans. (a) We must have $30p_1 + 10p_2 - 20p_3 = 0$ and $p_1 + p_2 + p_3 = 1$. $p_1^1 = 2/5$, $p_2^1 = 0$, and $p_3^1 = 3/5$ is one choice, $p_1^2 = 0$, $p_2^2 = 2/3$, and $p_3^2 = 1/3$ is another. The set of solutions is the line segment connecting these points: $\theta p^1 + (1-\theta)p^2$. (b) p^1 yields expected value $(2/5)(25) = 10$, while gives $(2/3)5 = 10/3$. Since the price is linear in the p_i, $v_1 = 5$ and $v_0 = 10/3$. (c) $x_0 = 1/4$. (d) $x_0 = 1/6$. (e) If the price was outside the interval then there would be an arbitrage opportunity.

6.4. The Cornell hockey team is playing a game against Harvard that it will either win, lose, or draw. A gambler offers you the following three payoffs, each for a \$1 bet

	win	lose	draw
Bet 1	0	1	1.5
Bet 2	2	2	0
Bet 3	.5	1.5	0

(a) Assume you are able to buy any amounts (even negative) of these bets. Is there an arbitrage opportunity? (b) What if only the first two bets are available?

Ans. Subtracting the \$1 we bet the results are

	win	lose	draw
Bet 1	-1	0	.5
Bet 2	1	1	-1
Bet 3	-.5	.5	-1

(a) $B_1 + B_2 - B_3$ has payoffs .5,.5,.5. (b) To have a martingale measure we need

$$-p_1 + 0.5p_3 = 0 \qquad p_1 + p_2 - p_3 = 0 \qquad p_1 + p_2 + p_3 = 1$$

The first equation implies $p_3 = 2p_1$, using this in the second $-p_1 + p_2 = 0$, i.e., $p_2 = p_1$. If $p_1 = c$, then $p_2 = c$ and $p_3 = c$ so $c + 1/4$. Since there is a martingale measure there is no arbitrage.

6.5. Suppose Microsoft stock sells for 100 while Netscape sells for 50. Three possible outcomes of a court case will have the following impact on the two stocks.

	Microsoft	Netscape
1 (win)	120	30
2 (draw)	110	55
3 (lose)	84	60

What should we be willing to pay for an option to buy Netscape for 50 after the court case is over? Answer this question two ways: (i) find a probability distribution so that the two stocks are martingales, (ii) show that by using cash and buying Microsoft and Netscape stock one can replicate the option.

Ans. For part (i) we write two equations

$$20p_1 + 10p_2 - 16p_3 = 0 \qquad -20p_1 + 5p_2 + 10p_3 = 0$$

Substituting $p_3 = 1 - p_1 - p_2$ we have

$$36p_1 + 26p_2 = 16 \qquad -30p_1 - 5p_2 = -10$$

Multiplying the second equation by 6/5 and adding it to the first we have $20p_2 = 4$ so $p_2 = .2$. Solving now gives $p_1 = .3$ and $p_3 = .5$. An option to buy Netscape at 50 pays off 0 in case 1, 5 in case 2, and 10 in case 3, so the option is worth $0p_1 + 5p_2 + 10p_3 = 0 + 1 + 5 = 6$.
For (ii) we begin by noting that the change in value of the stocks and the price of the option are given by:

	Microsoft	Netscape	option
1 (win)	20	−20	0
2 (draw)	10	5	5
3 (lose)	−16	10	10

Thus we want to find x, y, and z so that

$$x + 20y - 20z = 0 \qquad x + 10y + 5z = 5 \qquad x - 16y + 10z = 10$$

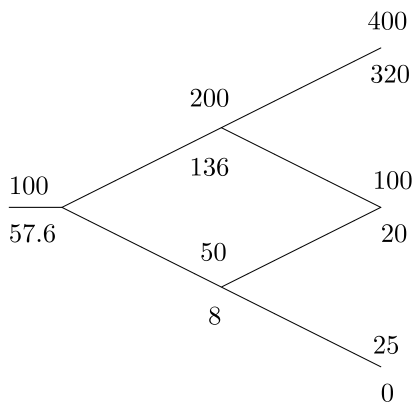
Subtracting the second and the third equations from the first one:

$$10y - 25z = -5 \qquad 36y - 30z = -10$$

The first equation times −3.6 plus the second one gives $60z = 8$. Solving for y now gives $y = -1/6$ and we conclude from the first equation that $x = 6$. This is the amount of cash we need to replicate the option so this is its value.

6.6. Consider the two-period binomial model with $u = 2$, $d = 1/2$ and interest rate $r = 1/4$. and suppose $S_0 = 100$. What is the value of the European call option with strike price 80, i.e., the option with payoff $(S_2 - 80)^+$. Find the stock holdings Δ_0, $\Delta_1(H)$ and $\Delta_1(T)$ need to replicate the option exactly.

Ans. By (6.16) and (6.17) the risk neutral probability has $p^* = 1/2$ and the recursion $V_n(a) = 0.4[V_n(aH) + V_n(aT)]$ we can compute the values which are given below the stock prices in the following diagram.



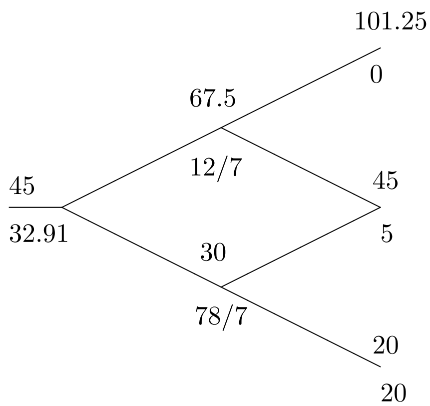
To compute the hedging strategy we note that

$$
\begin{aligned}
\Delta_1(H) &= (320 - 20)/(400 - 100) = 1 \\
\Delta_1(T) &= (20 - 0)/(100 - 25) = 4/15 \\
\Delta_0 &= (136 - 8)/(200 - 50) = 0.8533
\end{aligned}
$$

.

6.7. Consider the two-period binomial model with $u = 3/2$, $d = 2/3$, interest rate $r = 1/6$. and suppose $S_0 = 45$. What is the value of the European call option with strike price 50, i.e., the option with payoff $(50 - S_2)^+$. Find the stock holdings Δ_0, $\Delta_1(H)$ and $\Delta_1(T)$ need to replicate the option exactly.

Ans. As computed in Example 6.4, $p^* = 0.6$ and the recursion $V_n(a) = [3.6V_n(aH) + 2.4V_n(aT)]/7$ so we can compute the values which are given below the stock prices in the following diagram.

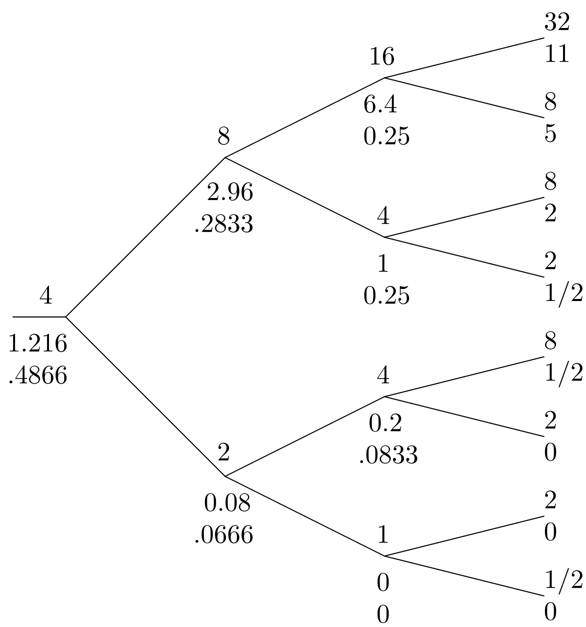


To compute the hedging strategy we note that

$$\begin{aligned}
\Delta_1(H) &= (0 - 5)/(101.25 - 45) = -0.0888 \\
\Delta_1(T) &= (5 - 25)/(45 - 20) = -20/25 = -0.8 \\
\Delta_0 &= (12/7 - 78/7)/(67.5 - 30) = -0.1985.
\end{aligned}$$

6.8. The payoff of the Asian option is based on the average price: $A_n = (S_0 + \cdots + S_n)/(n+1)$. Suppose that the stock follows the binomial model with $S_0 = 4$, $u = 2$, $d = 1/2$, and $r = 1/4$. (a) Compute the value function $V_n(a)$ and the replicating portfolio $\Delta_n(a)$ for the three period call option with strike 4. (b) Check your answer for V_0 by using $V_0 = E^*(V_3/(1+r)^3)$.

Ans. Numbers at the end and above the nodes are the stock price and option value. The values $\Delta_n(a)$ are given below the nodes. Since the risk neutral probabilities are $p^* = q^* = 1/2$, the recursion is $V_n(a) = 0.4[V_{n+1}(aH) + V_{n+1}(aT)]$.



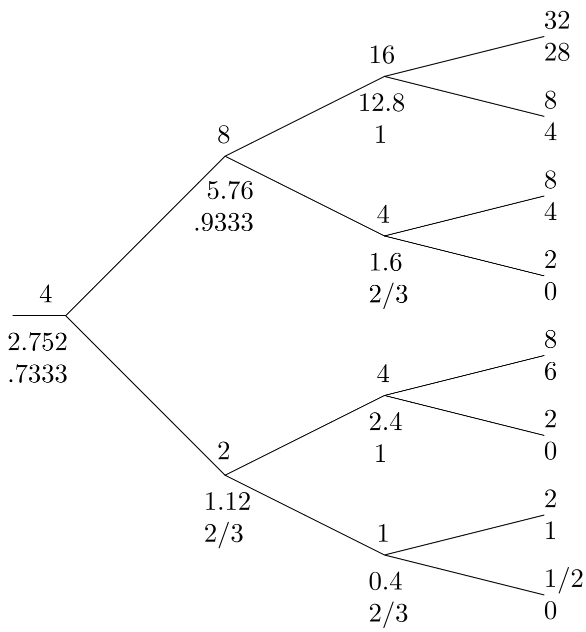
(b) Theorem 6.5 implies that the value is $(4/5)^3(11+5+2+0.5+0.5)/8 = 1.216$.

6.9. In the putback option at time 3 you can buy the stock for the lowest price seen in the past and the sell it at its current price for a profit of

$$V_3 = S_3 - \min_{0 \le m \le 3} S_m$$

Suppose that the stock follows the binomial model with $S_0 = 4$, $u = 2$, $d = 1/2$, and $r = 1/4$. (a) Compute the value function $V_n(a)$ and the replicating portfolio $\Delta_n(a)$ for the three period call option with strike 4. (b) Check your answer for V_0 by using $V_0 = E^*(V_3/(1+r)^3)$.

Ans. Numbers above the nodes are the stock price. Below are the values of $V_n(a)$ and $\Delta_n(a)$. Since the risk neutral probabilities are $p^* = q^* = 1/2$, the recursion is $V_n(a) = 0.4[V_{n+1}(aH) + V_{n+1}(aT)]$.



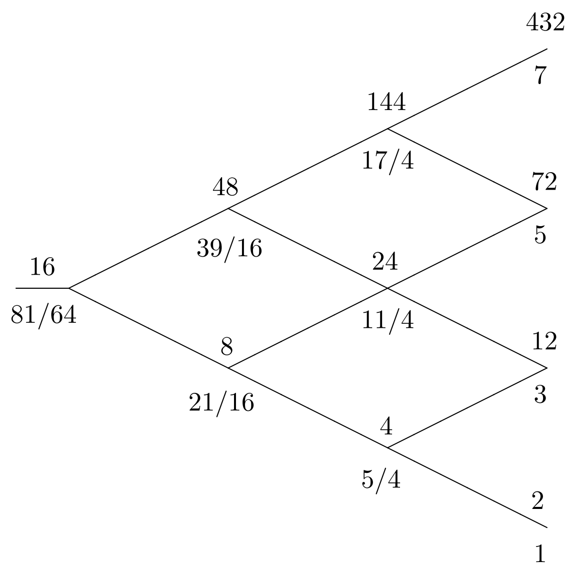
(b) $V_0 = (4/5)^3(28 + 4 + 4 + 6 + 1)/8 = 2.752$

6.10. Consider the three-period binomial model with $u = 3$, $d = 1/2$ and $r = 1/3$ and $S_0 = 16$. The European prime factor option pays off \$1 for each factor in the prime factorization of the stock price at time 3 (when the option expires). For example, if the stock price is $24 = 2^3 3^1$ then the payoff is $4 = 3+1$. Find the no arbitrage price of this option.

Ans. $p^* = (1+r-d)/(u-d) = (4/3-1/2)/(3-1/2) = 1/3$ so the recursion is

$$V_n(a) = \frac{1}{1+r}[p^* V_{n+1}(aH) + (1-p^*)V_{n+1}(aT)] = \frac{1}{4}[V_{n+1}(aH) + 2V_{n+1}(aT)]$$

Stock prices are above the node and option prices below.

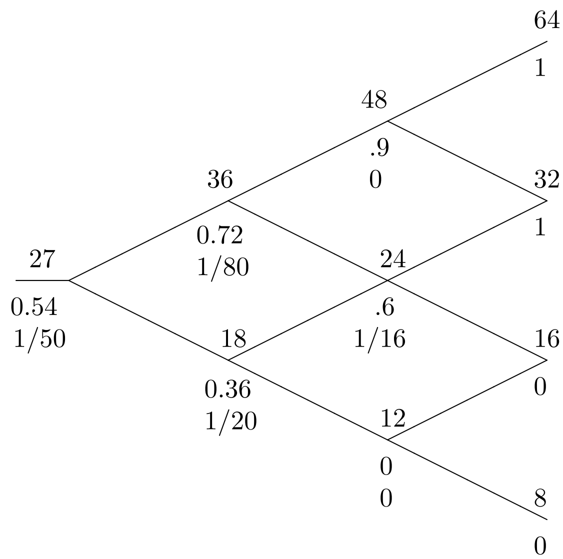


For those who like decimals: $81/64 = 1.265625$, $39/16 = 2.4375$, $21/16 = 1.3125$. One can also compute the value as

$$\begin{aligned} V_0 = E\left(\frac{V_3}{1+r)^3}\right) &= \frac{27}{64}\left[\frac{1}{27}\cdot 7 + \frac{3\cdot 2}{27}\cdot 5 + \frac{3\cdot 4}{27}\cdot 3 + \frac{8}{27}\cdot 1\right] \\ &= \frac{7+30+36+8}{64} = \frac{81}{64} \end{aligned}$$

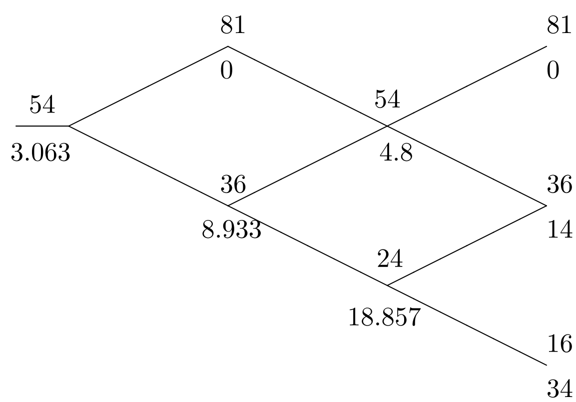
6.11. Suppose $S_0 = 27$, $u = 4/3$, $d = 2/3$ and interest rate $r = 1/9$. The European "cash-or-nothing option" pays \$1 if $S_3 > 27$ and 0 otherwise. Find the value of the option V_n and for the hedge Δ_n.

Ans. The risk neutral probability $p^* = 2/3$. The recursion for the value is: $V_n(a) = 0.6V_{n+1}(AH) + 0.3V_{n+1}(aT)$. On the diagram the stock price is above the node, the value and the hedging strategy are below.



6.12. Assume the binomial model with $S_0 = 54$, $u = 3/2$, $d = 2/3$, and $r = 1/6$. and consider a put $(50 - S_3)^+$ with a knockout barrier at 70. Find the value of the option.

Ans. From Example 6.4, we know that the risk neutral probability is $p^* = 3/5$ the value recursion is $V_n(a) = \frac{6}{7}[.6V_n(aH) + .4V_n(aT)]$, with the extra boundary condition that if the price is ≥ 70 the value is 0.



To check the answer note that the knockout feature eliminates one of the paths to 36 so

$$V_0 = (6/7)^3[(.4)^3 \cdot 34 + 2(.4)^2(.6) \cdot 14] = 3.063$$

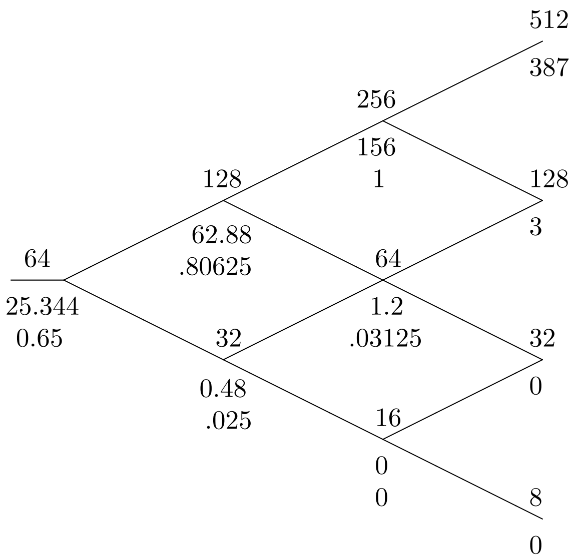
6.13. Consider now a four period binomial model with $S_0 = 32$, $u = 2$, $d = 1/2$, and $r = 1/4$, and suppose we have a put $(50 - S_4)^+$ with a knockout barrier at 100. Show that the knockout option as the same value as an option that pays off $(50 - S_4)^+$ when $S_4 = 2$, 8, or 32, 0 when $S_4 = 128$, and -18 when $S_4 = 512$. (b) Compute the value of the option in (a).

Ans. (a) $p^* = 1/2$, so by symmetry the value starting from 128 at time 2 or 4 is 0 and hence the option price satisfies the boundary condition for the knockout. (b) Taking into account the value of the option at the five endpoints implies that the value is

$$(4/5)^4 \left[\frac{1}{16} \cdot 48 + \frac{4}{16} \cdot 42 + \frac{6}{16} \cdot 18 + \frac{1}{16} \cdot (-18) \right] = 7.8336$$

6.14. Consider the binomial model with $S_0 = 64$, $u = 2$, $d = 1/2$, and $r = 1/4$. (a) Find the value $V_n(a)$ of the call option $(S_3 - 125)^+$ and the hedging strategy $\Delta_n(a)$. (b) Check your answer to (a) by computing $V_0 = E^*(V_3/(1+r)^3)$. (c) Find the value at time 0 of the put option.

Ans. On the diagram the stock price is above the node, the value and the hedging strategy are below. As in Section 6.3, the value recursion is $V_n(a) = 0.4[V_n(aH) + V_n(aT)]$.



(b) $V_0 = (4/5)^3(387 \cdot 1/8 + 3 \cdot 3/8) = 25.344$. (c) $125 = 64 \cdot (5/4)^3$ so by put-call parity, the value of the put is 25.344 at time 0.

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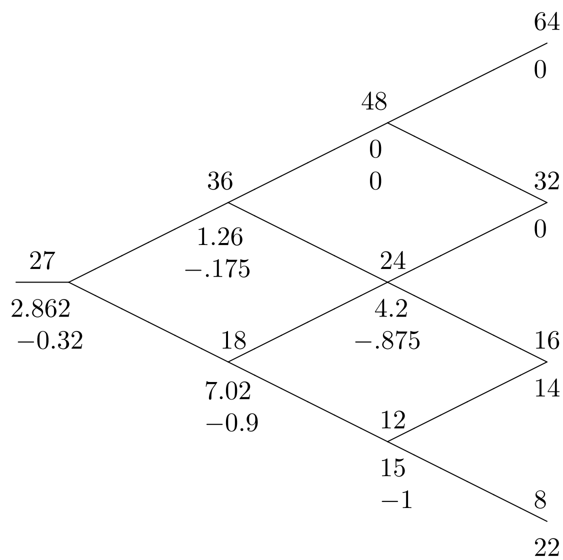
6.15. Consider the binomial model with $S_0 = 27$, $u = 4/3$, $d = 2/3$, and $r = 1/9$. (a) Find the risk neutral probability p^*. (b) Find value $V_n(a)$ of the put option $(30 - S_3)^+$ and the hedging strategy $\Delta_n(a)$. (c) Check your answer to (b) by computing $V_0 = E^*(V_3/(1+r)^3)$.

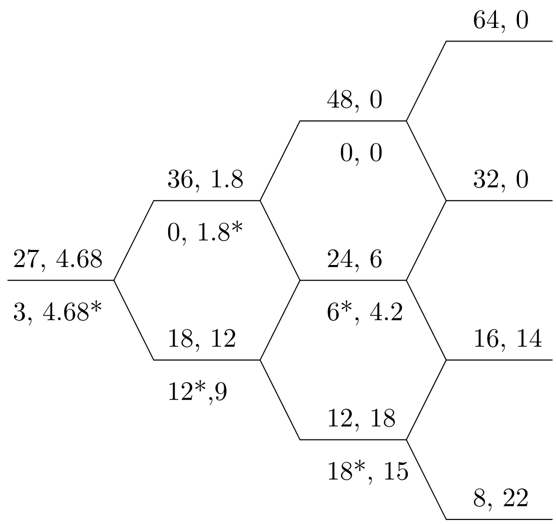
Ans. (a) $p^* = (1+r-d)/(u-d) = (10/9-2/3)/(4/3-2/3) = (4/9)/(2/3) = 2/3$ (b) The value recursion is $V_n(a) = 0.9[(2/3)V_{n+1}(aH) + (1/3)V_{n+1}(aT)] = 0.6V_{n+1}(AH) + 0.3V_{n+1}(aT)$. On the diagram the stock price is above the node, the value and the hedging strategy are below.



(c) $V_0 = (9/10)^3(14 \cdot 6/27 + 22 \cdot 1/27) = 2.862$.

6.16. Consider the binomial model of Problem 6.15 $S_0 = 27$, $u = 4/3$, $d = 2/3$, and $r = 1/9$ but now (a) find value and the optimal exercise strategy for the American put option $(30 - S_3)^+$, and (b) find the value of the American call option $(S_3 - 30)^+$.

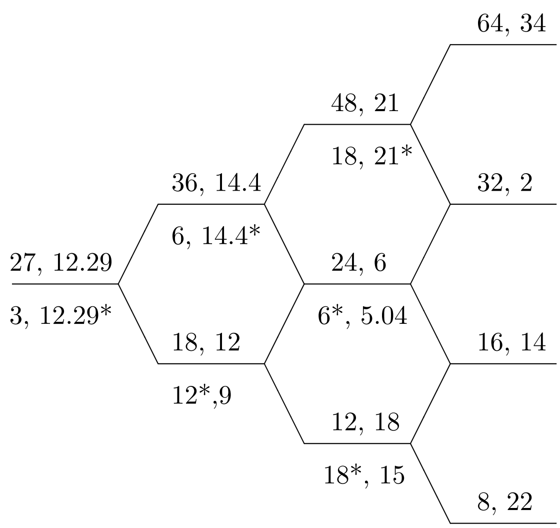
Ans. (a) Numbers above the lines give the stock price and option value. Below the line are the payoffs from stopping or continuing with an asterisk marking the larger. $V_n(a) = 0.6V_{n+1}(AH) + 0.3V_{n+1}(aT)$.



(b) For the American call option, the optimal strategy is to continue so its expected value is $(0.9)^3[34 \cdot 8/27 + 2 \cdot 3 \cdot 4/27] = 7.992$

6.17. Continuing with the model of previous problem $S_0 = 27$, $u = 4/3$, $d = 2/3$, and $r = 1/9$, we are now interested in finding value V_S of the American straddle $|S_3 - 30|$. Comparing with the values V_P and V_C of the call and the put computed in the previous problem we see that $V_S \le V_P + V_C$. Explain why this should be true.

Ans. Numbers above the lines give the stock price and option value. Below the line are the payoffs from stopping or continuing with an asterisk marking the larger. $V_n(a) = 0.6V_{n+1}(AH) + 0.3V_{n+1}(aT)$



$V_P = 4.68$ and $V_C = 7.992$ while $V_S = 12.29$. To see why this should be the case recall that by (6.21) the option value is $\max_\tau E^* g_\tau/(1+r)^\tau$ where the max is over stopping times. Using superscripts to indicate the option we are using

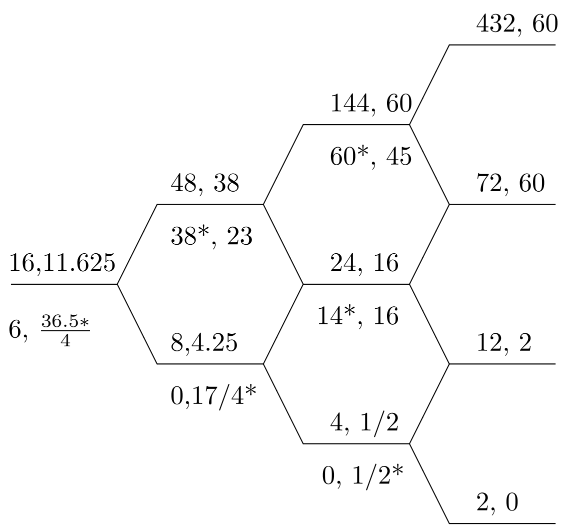
$$\begin{aligned}\max_\tau E^* g_\tau^S(1+r)^\tau &= \max_\tau [E^* g_\tau^C/(1+r)^\sigma + E^* g_\tau^P(1+r)^\tau \\ &\le \max_\sigma E^* g_\sigma^C/(1+r)^\sigma + \max_\rho E^* g_\rho^P(1+r)^\rho\end{aligned}$$

6.18. Consider the three-period binomial model with $S_0 = 16$, $u = 3$, $d = 1/2$ and $r = 1/3$ An American limited liability call option pays $\min\{(S_n - 10)^+, 60\}$ if exercised at time $0 \le n \le 3$. In words it is a call option but your profit is limited to \$60. Find the value and the optimal exercise strategy.

Ans. Numbers above the lines give the stock price and option value. Below the line are the payoffs from stopping or continuing with an asterisk marking the larger.

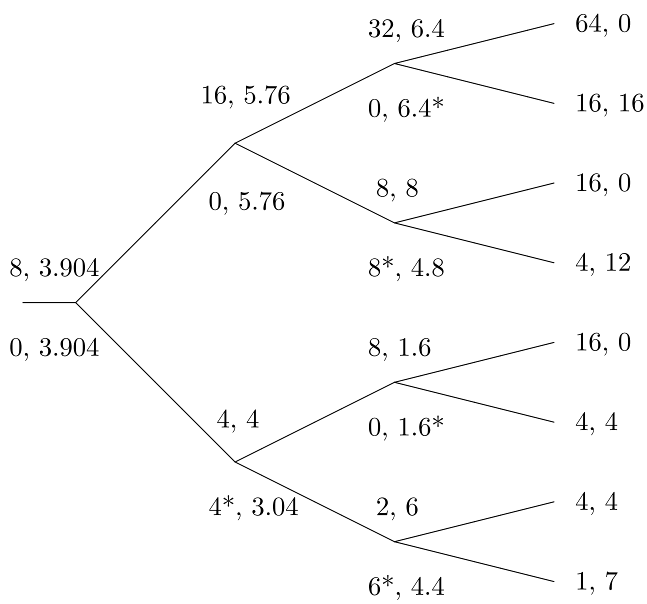
$$p^* = \frac{4/3 - 1/2}{3 - 1/2} = \frac{8-3}{18-3} = 1/3$$

so the payoff from continuing is $(1/4)[V_{n+1}(aH) + 2V_{n+1}(aT)]$.



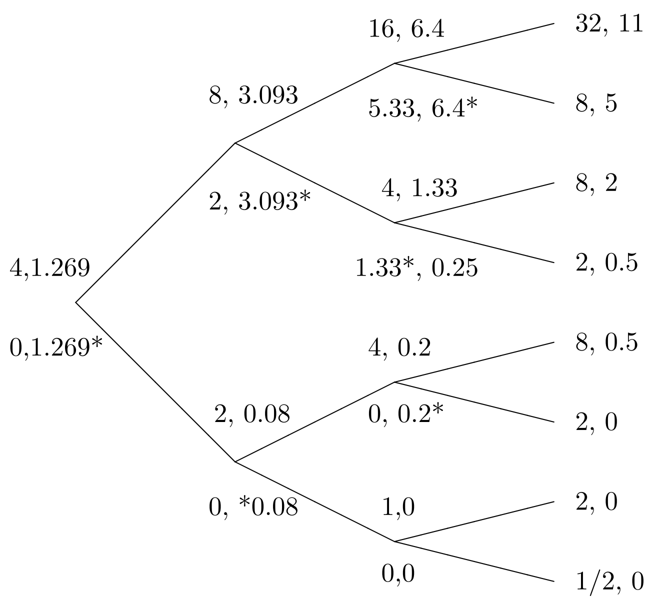
6.19. In the American version of the callback option, you can buy the stock at time n at its current price and then sell it at the highest price seen in the past for a profit of $V_n = \max_{0\le m\le n} S_m - S_n$. Compute the value of the three period version of this option when the stock follows the binomial model with $S_0 = 8$, $u = 2$, $d = 1/2$, and $r = 1/4$.

Ans. Numbers above the lines give the stock price and option value. Below the line are the payoffs from stopping or continuing with an asterisk marking the larger.



6.20. The payoff of the Asian option is based on the average price: $A_n = (S_0 + \cdots + S_n)/(n+1)$. Suppose that the stock follows the binomial model with $S_0 = 4$, $u = 2$, $d = 1/2$, and $r = 1/4$. Find the value of the American version of the three period Asian option, $(S_n - 4)^+$, i.e., when you can exercise the option at any time.

Ans. Numbers at the end and above the nodes are the stock price and option value. The values below are the value if you exercise or if you continue. Since the risk neutral probabilities are $p^* = q^* = 1/2$, the recursion is $V_n(a) = 0.4[V_{n+1}(aH) + V_{n+1}(aT)]$.



6.21. Show that for any a and b, $V(s,t) = as + be^{rt}$ satisfies the Black-Scholes differential equation. What investment does this correspond to?

Ans. $\partial V/\partial t = rbe^{rt}$, $-rV = -raS + rbe^{rt}$, $rs\partial V/\partial x = ras$, $\partial^2 V/\partial x^2 = 0$, so the sum of these four terms is 0. This corresponds to holding a shares of stock and having b in the bank.

6.22. Find a formula for the value (at time 0) of cash-or-nothing option that pays off \$1 if $S_t > K$ and 0 otherwise. What is the value when the strike is the initial value, the option is for 1/4 year, the volatility is $\sigma = 0.3$, and for simplicity we suppose that the interest rate is 0.

Ans. By Theorem 6.10 the value is

$$P(S_0 \exp(-\sigma^2 t/2 + \sigma B_t) > K) = \Phi(\beta) \quad \beta = \frac{\ln(S_0/K) - \sigma^2 t/2}{\sigma}$$

In the special case $\beta = -0.3/8$ so the answer is $\Phi(\beta) = 0.398$.

6.23. On May 22, 1998 Intel was selling at 74.625. Use the Black-Scholes formula to compute the value of a Janaury 2000 call ($t = 1.646$ years) with strike 100, assuming the interest rate was $r = 0.05$ and the volatility $\sigma = 0.375$.

Ans. Plugging into the Black-Scholes formula we have $d_1 = -0.19675$ and $d_2 = -0.67787$ which gives an option price of 8.566

6.24. On December 20, 2011, stock in Kraft Foods was selling at 36.83. (a) Use the Black-Scholes formula to compute the value of a March 12 call ($t = 0.227$ years) with strike 33, assuming an interest rate of $r = 0.01$ and the volatility $\sigma = 0.15$. The volatility here has been chosen to make the price consistent with the bid-ask spread of (3.9,4.0). (b) Is the price of 0.4 for a put with strike 33 consistent with put-call parity.

Ans. (a) Plugging into the Black-Scholes formula we have $d_1 = 1.603$ and $d_2 = 1.531$ which gives an option price of 3.97 (b) $V_C - V_P = 3.55$ while $S_0 - e^{-rt}K = 3.9$.

6.25. On December 20, 2011, stock in Exxon Mobil was selling at 81.63. (a) Use the Black-Scholes formula to compute the value of an April 12 call ($t = 0.3123$ years) with strike 70, assuming an interest rate of $r = 0.01$ and the volatility $\sigma = 0.26$. The volatility here has been chosen to make the price consistent with the bid-ask spread of (12.6,12.7). (b) Is the price of 1.43 for a put with strike 70 consistent with put-call parity.

Ans. (a) Plugging into the Black-Scholes formula we have $d_1 = 1.603$ and $d_2 = 1.531$ which gives an option price of 3.97 (b) $V_C - V_P = 11.22$ while $S_0 - e^{-rt}K = 11.84$.