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Option Pricing and Replication with Transactions Costs

HAYNE E. LELAND*

ABSTRACT

Transactions costs invalidate the Black-Scholes arbitrage argument for option pricing, since continuous revision implies infinite trading. Discrete revision using Black-Scholes deltas generates errors which are correlated with the market, and do not approach zero with more frequent revision when transactions costs are included. This paper develops a modified option replicating strategy which depends on the size of transactions costs and the frequency of revision. Hedging errors are uncorrelated with the market and approach zero with more frequent revision. The technique permits calculation of the transactions costs of option replication and provides bounds on option prices.

OPTION PRICING THEORY AS developed by Black and Scholes [1] rests on an arbitrage argument: by continuously adjusting a portfolio consisting of a stock and a risk-free bond, an investor can exactly replicate the returns to any option on that stock. The value of the option must, therefore, equal the value of the replicating portfolio.

The assumption of continuous portfolio adjustment is awkward in the presence of nonzero transactions costs. Because diffusion processes have infinite variation, continuous trading would be ruinously expensive, no matter how small transactions costs might be as a percentage of turnover.

Formally, the arbitrage argument used by Black-Scholes to price options no longer can be used: because replicating the option by a dynamic strategy would be infinitely costly, no effective option price bounds are implied.[1]

The natural defense of the Black-Scholes approach is to assume that trading takes place only at discrete intervals. This will bound the transactions costs of the replicating strategy. And, if trading takes place reasonably frequently, hedging errors may be relatively small. Black and Scholes and Boyle and Emanuel [2] argue that these errors will be uncorrelated with the market return, and therefore can be ignored if revision is reasonably frequent.

There are some problems with this defense in the presence of transactions costs. First, hedging errors exclusive of transactions costs will not be small unless portfolio revision is frequent. But transactions costs will rise (without limit) as the revision interval becomes shorter: it may be very costly to assure a given

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[1] The advantage of the arbitrage approach is that it does *not* depend upon investor preferences. An alternative approach is to use preference-based arguments to price options (e.g., Rubinstein [8]). In this latter approach, transactions costs may not affect option price bounds.

degree of accuracy in the replicating strategy before transactions costs. Paradoxically, we could find that the total cost of the replicating strategy exceeds that of the stock itself, even though the stock returns dominate the option return.

A second, and perhaps more important problem, is that transactions costs themselves are random, and will add significantly to the error of the Black-Scholes replicating strategy. Let us briefly focus on this concern.

The cost of a replicating strategy must clearly include transactions costs. If we wish to continue to use an arbitrage argument to bound option prices, we are forced to consider the *maximum* transactions cost rather than simply the average. But transactions costs associated with replicating strategies are *path-dependent*: they depend not only on the initial and final stock prices, but also on the entire sequence of stock prices in between. Computation of the maximum transactions costs is a nontrivial problem. And, because the maximum transactions costs will substantially exceed the average, the bounds on option prices will not be very tight.[2]

Perhaps we could be more modest and look at only at *expected* transactions costs from following the Black-Scholes replicating portfolio in discrete time.[3] This could be justified if transaction costs, like hedging errors, were uncorrelated with the market. But in general, this is not the case.[4]

A final problem exists because of path dependency and unboundedness of transactions costs: the uncertainty of transactions costs will *not* become small as the period of revision is shorter. One cannot hope for an arbitrarily good replication (no matter how expensive) by shortening the revision period. While replication errors exclusive of transactions costs will fall, they will not fall when transactions costs are included.

These considerations lead us to pose the following question:

> In the presence of transactions costs, is there an *alternative* to the Black-Scholes replicating strategy which will overcome these problems?

In this paper, we show that there is an alternative replicating strategy. This strategy depends upon the level of transactions costs and upon the revision

[2] We can readily compute an upper bound on the maximum transactions costs. The maximum move between the underlying stock and the riskless asset is 100% at each revision period. If readjustment occurs weekly, e.g., the maximum round trip turnover would be $0.5 \times 52 \times 100\% = 2600\%$. While highly unlikely, enormous weekly swings in the underlying stock price could lead to a maximum turnover approaching this level. Note that such a maximum turnover greatly exceeds expected turnover, which from Table III does not exceed 115%.

[3] In a recent paper, Gilster and Lee [5] estimate expected transactions costs for Black-Scholes strategies. Their analysis is flawed by the fact that their Equation (1) is *not* satisfied by the Black-Scholes strategy, as they themselves admit. This precludes an appropriate use of the Cox-Ross "risk-neutral" approach. However, as a heuristic approach, their results may be of interest.

[4] This can be seen by considering an in-the-money call option written on a stock which is (say) positively correlated with the market. It is easily shown that the replicating portfolio starts with almost one share of stock (since the call option is in-the-money) and will move to one share if the stock price expires at or above its initial price. Thus, there will be low transactions costs associated with upward movements of the stock price (and on average, with upward movements of the market). Major downward movements of the stock price will lead to a replicating portfolio expiring with zero shares of stock, and consequently larger turnover. In short, we conclude that in-the-money options will have replicating transactions costs which are negatively correlated with the market, while out-of-the-money options will have positively correlated transactions costs.

interval, as well as upon the option to be replicated and the environment. The alternative strategy has the following properties:

1. Transactions costs remain bounded as the revision period becomes short.
2. The strategy replicates the option return *inclusive* of transactions costs, with an error which is uncorrelated with the market and approaches zero as the revision period becomes short.
3. Expected turnover and associated transactions costs of the modified replicating strategy can easily be calculated, given the revision interval. Since the error inclusive of transactions costs is uncorrelated with the market, these transactions costs put bounds on option prices.

The modified strategy, therefore, can be used to replicate option returns inclusive of transactions costs, with accuracy that increases as the revision period becomes short. As one would expect, the modified strategy converges to the Black-Scholes strategy as transactions costs become arbitrarily small.

I. Discrete-Time Replicating Strategies with Zero Transactions Costs: A Review

We assume a Black-Scholes world where the stock (or portfolio) value follows a stationary logarithmic diffusion process

$$\frac{dS}{S} = \mu dt + \sigma z \sqrt{dt} \tag{1}$$

where z is a normally distributed random variable with $E(z) = 0$ and $E(z^2) = 1$. The riskless asset pays a continuous rate of return r.

Over a small (but noninfinitesimal) interval, Δt, it can be shown that

$$\frac{\Delta S}{S} = \mu \Delta t + \sigma z \sqrt{\Delta t} + O(\Delta t^{3/2}), \tag{2}$$

where a function $h(x)$ is said to be $0(g(x))$ if $\lim_{x\to 0} | h(x)/g(x) | < \infty$.

Let $C(S; K, T, r, \sigma^2)$ be the value of a call option when the current stock price is S, the striking price is K, the time to maturity is T, the interest rate is r, and the stock's rate of return has variance σ^2. In the absence of transactions costs but with possible continuous trading, Black and Scholes show that

$$C = SN(d_1) - Ke^{-rT}N(d_1 - \sigma\sqrt{T}) \tag{3}$$

where $d_1 = \ln(S/Ke^{-rT})/\sigma\sqrt{T} + \tfrac{1}{2}\, \sigma\sqrt{T}$. It follows that C satisfies the partial differential equation,

$$\tfrac{1}{2}C_{SS}S^2\sigma^2 + C_t - r[C - C_S S] = 0 \tag{4}$$

where subscripts indicate partial derivatives, and the boundary condition,

$$C[S; K, 0, r, \sigma^2] = \max[S - K, 0]. \tag{5}$$

Consider now holding a fixed portfolio of D shares of stock and Q dollars of the risk-free security over the interval, Δt. The length of the interval, Δt, will be termed the *revision interval.*

Over the interval, Δt, the return to this portfolio will be

$$\Delta P = DS\left(\frac{\Delta S}{S}\right) + rQ\Delta t + O(\Delta t^2), \tag{6}$$

when the $O(\Delta t^2)$ term comes from the continuous compounding of interest.

Over the same interval Δt, the change in value of a call option $C[S; K, T, r, \sigma^2]$ will be

$$\Delta C = C_S S\left(\frac{\Delta S}{S}\right) + C_t\Delta t + \frac{1}{2}C_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 + O(\Delta t^{3/2}), \tag{7}$$

using a Taylor series expansion for C and noting that both $\left(\frac{\Delta S}{S}\right)^3$ and $\left(\frac{\Delta S}{S}\right)\Delta t$ are $O(\Delta t^{3/2})$, from (2).

The difference, ΔH, between the change in value of the portfolio and the call option is given by

$$\begin{aligned}\Delta H &= \Delta P - \Delta C\\ &= (DS - C_S S)\left(\frac{\Delta S}{S}\right) + (rQ - C_t)\Delta t\\ &\quad - \frac{1}{2}C_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 + O(\Delta t^{3/2}).\end{aligned} \tag{8}$$

We define a *replicating portfolio* as one for which, at the beginning of each interval $0, \Delta t, 2\Delta t, \cdots, T$,

$$D = C_S, \quad \text{and} \tag{9}$$

$$Q = C - C_S S. \tag{10}$$

Since $P = DS + Q = C$ at each time period, this portfolio yields the option return $\max[S - K, 0]$ at T. However, the portfolio will not be self-financing, since $\Delta P \neq \Delta C$. ΔH_t is a measure of the additional contribution needed over the period $(t, t + \Delta t)$. Substituting (9) and (10) into (8) and using (4) yields

$$\Delta H = \frac{1}{2}C_{SS}S^2\left[\sigma^2\Delta t - \left(\frac{\Delta S}{S}\right)^2\right] + O(\Delta t^{3/2}) \tag{11}$$

where we have suppressed the time subscript, t. Taking expectations, using (2), and ignoring terms of $O(\Delta t^{3/2})$ gives

$$E[\Delta H] = \frac{1}{2}C_{SS}S^2 E\left[\sigma^2\Delta t - \left(\frac{\Delta S}{S}\right)^2\right] = 0. \tag{12}$$

Note that $C_{SS}S^2$ is $O(1)$, and $\sigma^2\Delta t - \left(\frac{\Delta S}{S}\right)^2$ is $O(\Delta t)$, implying ΔH is $O(\Delta t)$. Thus, ΔH is a random variable with mean zero and variance of $O(\Delta t)^2$ whose distribution is considered by Boyle and Emanuel [2].[5] Since the $\left(\frac{\Delta S}{S}\right)$ are

[5] Boyle and Emanuel [2] also show ΔH will be uncorrelated with the market return.

independent and, therefore, the ΔH are uncorrelated across each interval, Δt, the variance of the sum of the $T/\Delta t$ random variables, ΔH_t, $t = 0, \Delta t, 2\Delta t, \cdots$, $T - \Delta t$ will be $O\left[\frac{T}{\Delta t} \times (\Delta t)^2\right] = O(\Delta t)$. As $\Delta t \to 0$, the Law of Large Numbers for Martingales implies that the hedging error will almost surely be zero.[6,7] That is, the payoff $\max[S - K, 0]$ will be delivered almost surely as $\Delta t \to 0$.

While limiting results are suggestive, practical considerations (including possible transactions costs) prevent the interval Δt from becoming "arbitrarily" small. In Table I, we examine the accuracy of replicating strategies for periods of revision of one, four, and eight weeks. We consider a (currently) typical environment where the risk-free rate is 10%, the expected return on the "stock" is 16%, with standard deviation 20%. The latter figure reflects the risk of a market portfolio.

[6] The Law of Large Numbers for Martingales states that if

$$S_n = \textstyle\sum_{k=1}^{n} X_k, \quad n = 1, \cdots \quad \text{is a martingale.}$$

and

$$\textstyle\sum_{k=1}^{\infty} \frac{1}{k^2} E_0(X_k^2) < \infty,$$

then with probability one, $n^{-1}S_n \to 0$ (see Feller [3]).
Let

$$X_k \equiv \Delta H_{k\Delta t}/\Delta t, \quad n \equiv T/\Delta t.$$

Then

$$\textstyle\sum_{k=1}^{\infty} \frac{1}{k^2} E(X_k^2) = \sum_{k=1}^{\infty} \frac{1}{k^2 \Delta t^2} E_0(\Delta H_{k\Delta t}^2).$$

Now (dropping time subscripts on ΔH)

$$\begin{aligned} E_0(\Delta H)^2 &= E_0[C_{SS}S^2]E_0\left(\left(\frac{\Delta S}{S}\right)^2 - \sigma^2\Delta t\right)^2 \\ &\leq 2M\sigma^4\Delta t^2, \end{aligned}$$

where $M < \infty$ is an upper bound on $E_0[C_{SS}S^2]$, over all $t \in [0, T]$.

It follows that

$$\textstyle\sum_{k=1}^{\infty} \frac{1}{k^2} E(X_k^2) \leq 2M\sigma^4 \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

implying that

$$n^{-1} \textstyle\sum_{k=1}^{\infty} X_k = \frac{\Delta t}{T} \times \frac{1}{\Delta t} \sum \Delta H \to 0. \quad \text{a.s.}$$

which in turn implies

$$\textstyle\sum \Delta H \to 0 \quad \text{a.s.}$$

[7] Boyle and Emanuel [2] examine the distribution of errors of ΔH for short intervals [1–5 days] between trades, ignoring terms of $O(\Delta t^{3/2})$. Note that if the terms of $O(\Delta t^{3/2})$ do not have expected value zero, then over a finite interval, T, they will be of the same order as the errors in $\sum \Delta H$ that are included. Of course, this does not affect our limiting result that the replicating portfolio precisely replicates the option as $\Delta t \to 0$.

Table I

Hedging Errors with No Transactions Costs

Revision Period		Time Until Expiration 12 mos.	6 mos.	3 mos.	Approximate Annualized
1 week	$E[\Delta H]$	0.000[a]	0.000	0.000	0.000
	$\sigma[\Delta H]$	0.091[a]	0.099	0.115	0.830
4 weeks	$E[\Delta H]$	−0.003	−0.002	−0.002	−0.0300
	$\sigma[\Delta H]$	0.368	0.402	0.471	1.700
8 weeks	$E[\Delta H]$	−0.009	−0.008	−0.008	−0.540
	$\sigma[\Delta H]$	0.744	0.821	0.981	2.500

[a] Expected errors and standard deviation are expressed in dollar terms, where $S_0 = 100$, $K = 100$. The cost of the call option being replicated is 12.99.

The accuracy of the replicating strategy is path-dependent and tends to be the least good when the price of the stock is close to the striking price just before expiration, but then jumps up or down over the last interval. Of course, paths having these characteristics occur relatively infrequently.

Table I considers a one-year, "at-the-money" call option $[K = S(0)]$. We consider three different revision periods: one week, four weeks, and eight weeks. For each revision period, we report expected errors and standard deviation of errors (as seen from the initial period) of the replicating portfolio over the next revision interval, when three months, six months, and one year remain in the option's life.[8]

In Table I, we note that expected errors are small. Annualized standard deviations are smaller when the revision period is shorter, as we would expect from the theory. Indeed, halving the revision period reduces the standard deviation by a factor of almost exactly $1/\sqrt{2}$.

II. Discrete-Time Replicating Strategies with Positive Transactions Costs

Thus far, we have ignored the impact of transactions costs on the performance of the replicating portfolio. One possible approach to replicating options with transactions costs would be to follow the Black-Scholes strategy [Equations (9)

[8] To be more explicit, consider the entries $E[\Delta H]$ and $\sigma[\Delta H]$ when $T = 6$ months and the revision interval is four weeks. From Table I, we see $E[\Delta H] = 0.002\%$. This is computed as follows: for a given $S(6)$, where $S(6)$ is the stock price when six months remain (and six months have passed), we compute

$$\Delta H(S(6), \Delta S) = \{C[S(6) + \Delta S, 5] - C[S(6), 6]\}$$

$$- \{C_S[S(6), 6]\Delta S + \hat{r}[C[S(6), 6] - C_S[S(6), 6]S(6)]\}$$

where the first term in braces is the actual (exact) change in the call price given $S(6)$ and ΔS over the next four weeks, and the second term in braces is the return to the replicating portfolio, with $\hat{r}$ the interest rate over the 4-week (1-month) revision period. Using a lognormal distribution for both $S(6)$ and $\Delta S/S(6)$—which are independent—we then compute $E[\Delta H]$ and $\sigma[\Delta H]$.

and (10)], but add on an amount to the initial cost of the option which reflects the expected transactions costs.

This approach can be criticized on three grounds. First, the expected transactions cost is a difficult computation with no known closed form solution. Second, the transactions costs will, in general, be correlated with the change in stock price, and therefore with the market. Finally, transactions costs and their uncertainty will become arbitrarily large as $\Delta t \to 0$: in contrast to the case with no transactions costs, accuracy of replicating the option (inclusive of transactions costs) will *not* increase as $\Delta t \to 0$.

We develop an alternative approach to the problem, where the hedging strategy itself depends on the percent transactions cost and the revision interval.

Let k represent the *round trip* transaction cost, measured as a fraction of the volume of transactions.

Define

$$\begin{aligned}\hat{\sigma}^2(\sigma^2, k, \Delta t) &= \sigma^2\left[1 + kE\left|\frac{\Delta S}{S}\right| \Big/ \sigma^2 \Delta t\right] \\ &= \sigma^2[1 + \sqrt{(2/\pi)}\, k/\sigma\sqrt{\Delta t}] \qquad (13)\end{aligned}$$

since

$$E\left|\frac{\Delta S}{S}\right| = \sqrt{(2/\pi)}\,\sigma\sqrt{\Delta t}.^9 \qquad (14)$$

Let

$$\hat{C}(S; K, \sigma^2, r, T, k, \Delta t) = SN(\hat{d}_1) - Ke^{-rT}N(\hat{d}_1 - \hat{\sigma}\sqrt{T}), \qquad (15)$$

where

$$\hat{d}_1 = \ln[S/Ke^{-rT}]/\hat{\sigma}\sqrt{T} + \tfrac{1}{2}\,\hat{\sigma}\sqrt{T}.$$

That is, $\hat{C}$ is the Black-Scholes option price based on the modified variance (13). We now show that following the modified replicating strategy

$$D = \hat{C}_S, \quad \text{and} \qquad (16)$$

$$Q = \hat{C} - \hat{C}_S S \qquad (17)$$

will for small Δt yield an expected payoff of

$$\max[S - K, 0] \qquad (18)$$

inclusive of transactions costs. Furthermore, the hedging errors, ΔH, including transactions costs will almost surely approach zero as $\Delta t \to 0$. This latter result implies that, despite the path dependence of transactions costs, following the modified replicating strategy yields, as $\Delta t \to 0$, a path-independent net result, which with probability one is equal to the desired option return.

[9] This is derived using the assumption that $\Delta S/S$ is normally distributed with mean zero. Allowing for a drift term and for lognormality will create an additional term of $O(\Delta t^{3/2})$, which here and in the subsequent analysis is ignored.

THEOREM. *Following the replicating strategy* $\{D = \hat{C}_S,\ Q = \hat{C} - \hat{C}_S S\}$ *where* $\hat{C}$ *is the modified Black-Scholes price* (15) *will yield* $\max[S - K, 0]$ *almost surely inclusive of transactions costs, as* $\Delta t \rightarrow 0$.

Proof: Consider the after-transactions-cost error, ΔH, of the replicating portfolio over the interval, Δt, where

$$\Delta H = \Delta P - \Delta \hat{C} - TC \tag{19}$$

with

$$\begin{aligned} \Delta P &= D\Delta S + Qr\Delta t + O(\Delta t^2) \\ &= \hat{C}_S S\left(\frac{\Delta S}{S}\right) + (\hat{C} - \hat{C}_S S)r\Delta t + O(\Delta t^2) \end{aligned} \tag{20}$$

using (16) and (17); and

$$\begin{aligned} \Delta \hat{C} &= \hat{C}(S + \Delta S, t + \Delta t) - \hat{C}(S, t) \\ &= \hat{C}_S S\left(\frac{\Delta S}{S}\right) + \frac{1}{2}\hat{C}_{SS}S^2\left(\frac{\Delta S}{S}\right)^2 + \hat{C}_t\Delta t + O(\Delta t^{3/2}) \end{aligned} \tag{21}$$

$$\begin{aligned} TC &= \frac{1}{2}k\,|\Delta D(S + \Delta S)| \\ &= \frac{1}{2}k\,|[\hat{C}_S(S + \Delta S, t + \Delta t) - \hat{C}_S(S, t)](S + \Delta S)| \\ &= \frac{1}{2}k\,|\hat{C}_{SS}(S, t)\Delta S(S + \Delta S)| + O(\Delta t^{3/2}) \\ &= \frac{1}{2}k\hat{C}_{SS}S^2\left|\frac{\Delta S}{S}\right| + O(\Delta t^{3/2}), \end{aligned} \tag{22}$$

where line 3 of (22) utilizes a first-order Taylor series approximation for $\Delta\hat{C}_S$ and line 4 of (22) relies on the fact that $\hat{C}_{SS}S^2 > 0$.

It is important to note that $\hat{C}_{SS}S^2$ as well as $\hat{C}_{SSS}$ and $\hat{C}_{St}$ are $O(\Delta t^{1/2})$.[10,11] This explains the $O(\Delta t^{3/2})$ terms in (22), and implies $\frac{1}{2}k\hat{C}_{SS}S^2\left|\frac{\Delta S}{S}\right|$ is $O(\Delta t)$.

[10] Recall $\hat{C}_{SS}S^2 = SN'(\hat{d}_1)/\hat{\sigma}(T-t)^{1/2}, \quad t < T$
where

$$\hat{d}_1 = [\ln(S/K) + (r + \tfrac{1}{2}\hat{\sigma}^2)(T - t)]/\hat{\sigma}(T - t)^{1/2}$$

and

$$N'(\hat{d}_1) = \exp(-\tfrac{1}{2}\hat{d}_1^2)/(2\pi)^{1/2}.$$

Recall also from (14) that

$$\hat{\sigma} = \sigma(1 + \sqrt{2/\pi}\,k/\sigma\sqrt{\Delta t})^{1/2} \sim O(\Delta t^{-1/4}).$$

From the definition of $\hat{d}_1$, it can be seen that

$$\hat{d}_1 \rightarrow \tfrac{1}{2}\hat{\sigma}(T - t)^{1/2} \text{ as } \hat{\sigma} \rightarrow \infty,$$

$$\text{i.e., as } \Delta t \rightarrow 0.$$

Substituting Equations (20) to (22) in (19) gives

$$\Delta H = (\hat{C} - \hat{C}_S S) r \Delta t - \frac{1}{2} \hat{C}_{SS} S^2 \left(\frac{\Delta S}{S}\right)^2 - \hat{C}_t \Delta t - \frac{1}{2} k \hat{C}_{SS} S^2 \left| \frac{\Delta S}{S} \right| + O(\Delta t^{3/2}). \quad (23)$$

Since $\hat{C}$ satisfies

$$\tfrac{1}{2} \hat{C}_{SS} S^2 \hat{\sigma}^2 + \hat{C}_t - r[\hat{C} - \hat{C}_S S] = 0, \quad (4')$$

we may substitute for the first right-hand term in (23) to give

$$\begin{aligned}
\Delta H &= \frac{1}{2} \hat{C}_{SS} S^2 \left[\hat{\sigma}^2 \Delta t - \left(\frac{\Delta S}{S}\right)^2 - k \left| \frac{\Delta S}{S} \right| \right] + O(\Delta t^{3/2}) \\
&= \frac{1}{2} \hat{C}_{SS} S^2 \left[\sigma^2 \Delta t + kE \left| \frac{\Delta S}{S} \right| - \left(\frac{\Delta S}{S}\right)^2 - k \left| \frac{\Delta S}{S} \right| \right] + O(\Delta t^{3/2}) \\
&= \frac{1}{2} \hat{C}_{SS} S^2 \left[\sigma^2 \Delta t - \left(\frac{\Delta S}{S}\right)^2 + k \left[E \left| \frac{\Delta S}{S} \right| - \left| \frac{\Delta S}{S} \right| \right] \right] + O(\Delta t^{3/2}). \quad (24)
\end{aligned}$$

Taking expectations yields

$$E(\Delta H) = O(\Delta t^{3/2}) \rightarrow 0. \quad (25)$$

Thus, $E \sum_{t=0}^{T-\Delta t} \Delta H_t$, the expected hedging error over the entire program is $O(\Delta t^{1/2})$ and approaches zero as Δt becomes small. Along the lines of Boyle and Emanuel [2], it can be shown that ΔH is uncorrelated with the market return.

In contrast with $\sigma^2 \Delta t - \left(\frac{\Delta S}{S}\right)^2$ in (11), note that $E \left| \frac{\Delta S}{S} \right| - \left| \frac{\Delta S}{S} \right|$ in (24) is $O(\Delta t^{1/2})$ rather than $O(\Delta t)$, and transactions costs error will dominate as $\Delta t \rightarrow 0$. But we now use the fact that $\hat{C}_{SS} S^2$ is $O(\Delta t^{1/2})$ [whereas $C_{SS} S^2$ in (11) is 0(1)]. It is easily shown from (24) that as before

$$\text{var}(\Delta H) = O(\Delta t^2). \quad (26)$$

The total hedging error over the period [0, T] is $\sum_{t=0}^{T-\Delta t} \Delta H_t$. Since the ΔH's are

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Thus

$$\hat{d}_1 \sim O(\Delta t^{-1/4}), \quad \text{and}$$

$$N'(d_1) \sim O[\exp(-\tfrac{1}{2}\Delta t^{-1/2})].$$

It follows that

$$\hat{C}_{SS} S^2 \sim O[\exp(-\tfrac{1}{2}\Delta t^{-1/2}) \cdot \Delta t^{1/4}].$$

Since

$$\lim_{\Delta t \rightarrow 0} [\exp(-\tfrac{1}{2}\Delta t^{-1/2}) \Delta t^{1/4} / \Delta t^{1/2}] = 0,$$

it follows by definition that $\hat{C}_{SS} S^2$ is $O(\Delta t^{1/2})$, and therefore $O(\Delta t^{1/2})$.

[11] Alternatively, we could have assumed that transactions costs, k, were "small"—of $O(\Delta t^{1/2})$—not utilized the fact that $\hat{C}_{SS} S^2$ is $O(\Delta t^{1/2})$. This approach does not permit convergence analysis with a fixed transaction cost, k, however.

uncorrelated,

$$\mathrm{var}\left(\sum_{t=0}^{T-\Delta t} \Delta H_t\right) = \sum_{t=0}^{T-\Delta t} \mathrm{var}(\Delta H_t) = O(\Delta t). \tag{27}$$

Thus, the replicating strategy yields $\max[S - K, 0]$ almost surely as $\Delta t \to 0$.[12] Q.E.D.

Our result is encouraging: in the limit, as the readjustment interval becomes small, the modified hedging strategy [(16) and (17)] yields the option result almost surely, *inclusive* of transactions costs. But it remains to be seen, for realistic values of k and Δt, just how accurate the hedging strategy is. Table II gives results equivalent to Table I, when we follow the modified hedge strategy but include transactions costs. Note that the errors increase only slightly when transactions costs are included. Take, for example, a one-week revision period when transactions costs are 1%. The expected error remains zero, the same as the no-transactions-cost case. The annualized standard deviation rises from 0.830 to 0.887.

What if we had followed the Black-Scholes strategy, but paid the transactions costs which resulted from that strategy? Of course, one would expect the average error to be negative, reflecting the fact that transactions costs are not "covered" by the initial option price. This average error reflects the average turnover, which will in most cases be higher than the turnover associated with the modified hedging strategy, since the latter is based on a higher volatility which tends to "smooth" the required trading. (Tables III to VI, discussed in Section III, detail the difference in turnover and, therefore, in expected transactions costs.)

Perhaps a more interesting question is the *accuracy* of option replication, inclusive of transactions costs, of the modified strategy versus the Black-Scholes strategy. Table VII details the hedging errors of alternative strategies. The first strategy is the Black-Scholes strategy when there are no transactions costs (the entries here simply reproduce Table I). The second strategy is the Black-Scholes strategy when there are transactions costs of 1%. The third strategy is the modified strategy when there are similar transactions costs (the entries here are from Table II).

Consider the increment in standard deviation of error from the "base case" of zero transactions costs. In every circumstance, the accuracy of the modified strategy exceeds the accuracy of the Black-Scholes strategy when transactions costs are present. The incremental standard deviation of the modified strategy averages about half that of the Black-Scholes strategy. The modified strategy performs relatively better as the revision interval becomes shorter and as the time to expiration is greater.

III. Estimating Turnover and Transactions Costs of Replicating Strategies

Subject to the (small) hedging errors studied in the previous sections, we have shown that:

[12] As in the previous section, the variance of the error approaching zero implies that $\sum \Delta H_t$ converges in probability to zero. The stronger "almost surely" result follows from the Law of Large Numbers for Martingales. See footnote 6.

Table II

Hedging Errors with Transactions Costs

Transactions Cost (k)	Revision Period		Time Until Expiration 12 mos.	6 mos.	3 mos.	Approximate Annualized
	1 week	$E[\Delta H]$	0.000	0.000	0.000	0.000
		$\sigma[\Delta H]$	0.092	0.101	0.117	0.844
0.25%	4 weeks	$E[\Delta H]$	−0.003	−0.002	−0.003	−0.035
		$\sigma[\Delta H]$	0.370	0.406	0.475	1.713
	8 weeks	$E[\Delta H]$	−0.010	−0.009	−0.010	−0.063
		$\sigma[\Delta H]$	0.748	0.826	0.988	2.519
	1 week	$E[\Delta H]$	0.000	0.000	0.000	0.000
		$\sigma[\Delta H]$	0.095	0.106	0.123	0.887
1.00%	4 weeks	$E[\Delta H]$	−0.004	−0.003	−0.004	−0.048
		$\sigma[\Delta H]$	0.377	0.416	0.489	1.763
	8 weeks	$E[\Delta H]$	−0.012	−0.012	−0.016	−0.088
		$\sigma[\Delta H]$	0.759	0.843	1.010	2.575
	1 week	$E[\Delta H]$	−0.001	−0.001	−0.002	−0.068
		$\sigma[\Delta H]$	0.107	0.126	0.149	1.074
4.00%	4 weeks	$E[\Delta H]$	−0.006	−0.008	−0.011	−0.130
		$\sigma[\Delta H]$	0.403	0.461	0.547	1.972
	8 weeks	$E[\Delta H]$	−0.019	−0.026	−0.043	−0.195
		$\sigma[\Delta H]$	0.803	0.912	1.102	2.810

Table III

Transactions Costs and Implied Annual Turnover of Option Replicating Strategies: 1 Year Options, Weekly Revision

Striking Price	Call Price	Total TC	Turnover (%)
	Transactions Costs = 0.00%		
80	27.67	0.000	28.01
90	19.68	0.000	63.18
100	12.99	0.000	97.16
110	7.97	0.000	113.25
120	4.55	0.000	107.59
	Transactions Costs = 0.25%		
80	27.67	0.070	28.15
90	19.68	0.156	62.45
100	12.99	0.240	95.81
110	7.97	0.280	112.20
120	4.55	0.267	106.89
	Transactions Costs = 1.00%		
80	27.67	0.300	29.96
90	19.68	0.621	62.12
100	12.99	0.922	92.18
110	7.97	1.069	106.91
120	4.55	1.027	102.68
	Transactions Costs = 4.00%		
80	27.67	1.352	33.80
90	19.68	2.377	59.43
100	12.99	3.259	81.47
110	7.97	3.694	92.34
120	4.55	3.616	90.40

Note: Revision interval = 1.0 weeks; standard deviation = 20%; horizon = 1.0 years; interest = 10%; TC = transactions cost.

(i) The strategy $\{D = C_S,\ Q = C - C_S S\}$ with initial cost $C[S_0; K, r, \sigma, T]$ provides $\max[S - K, 0]$ at the terminal date, T, when there are no transactions costs.

(ii) The strategy $\{D = \hat{C}_S,\ Q = \hat{C} - \hat{C}_S S\}$ with initial cost $\hat{C} = C[S_0; K, r, \hat{\sigma}, T]$ [where $\hat{\sigma}$ is defined in (13)] provides $\max[S - K, 0]$ at the terminal date, T, inclusive of transaction costs.

It follows directly that the difference between the two initial option prices,

$$Z = \hat{C}_0 - C_0 \tag{28}$$

is a valid measure of the total transactions costs associated with the replicating strategy.

Table IV

Transactions Costs and Implied Annual Turnover of Option Replicating Strategies: 1 Year Options, Monthly Revision

Striking Price	Call Price	Total TC	Turnover (%)
	Transactions Costs = 0.00%		
80	27.67	0.000	14.31
90	19.68	0.000	31.59
100	12.99	0.000	47.68
110	7.97	0.000	56.03
120	4.55	0.000	53.94
	Transactions Costs = 0.25%		
80	27.67	0.035	13.90
90	19.68	0.078	31.24
100	12.99	0.121	48.23
110	7.97	0.141	56.59
120	4.55	0.135	53.83
	Transactions Costs = 1.00%		
80	27.67	0.144	14.40
90	19.68	0.312	31.19
100	12.99	0.473	47.27
110	7.97	0.552	55.16
120	4.55	0.527	52.71
	Transactions Costs = 4.00%		
80	27.67	0.634	15.86
90	19.68	1.227	30.68
100	12.99	1.761	44.03
110	7.97	2.023	50.57
120	4.55	1.958	48.96

Note: Revision interval = 4.0 weeks; standard deviation = 20%; horizon = 1.0 years; interest = 10%; TC = transactions cost.

Z is easily computed—it is the difference between two Black-Scholes option values with only the volatility adjusted. Since the volatility adjustment depends on the transactions cost rate, k, and the revision interval, Δt, these parameters [as well as the environmental parameters (r, σ^2) and the option parameters (K, T)] will importantly affect the total transactions cost, Z.

As $\Delta t \to 0$, $\hat{\sigma} \to \infty$, and $\hat{C}_0 \to S_0$. Thus, Z is bounded above by $S_0 - C_0$, implying transactions costs are bounded as $\Delta t \to 0$. This, of course, is in contrast with Black-Scholes hedging.[13] Note that the *actual* transactions costs over any partic-

[13] The reader might be tempted into the following paradox: as $\Delta t \to 0$ and $\hat{\sigma} \to \infty$, $\hat{C}_0 \to S_0$, $\hat{C}_s \to 1$, and $\hat{C}_{ss} \to 0$, implying that in the limit, the option-replicating strategy is simply to hold a share of stock and do no trading. (In addition, one might conclude that this strategy, which costs $S_0 = \hat{C}_0$, gives a final return S which strictly dominates the option return, $\max[S - K, 0]$.)

Table V

Transactions Costs and Implied Annual Turnover of Option Replicating Strategies: 1 Year Options, Bi-monthly Revision

Striking Price	Call Price	Total TC	Turnover (%)
	Transactions Costs = 0.00%		
80	27.67	0.000	10.13
90	19.68	0.000	22.65
100	12.99	0.000	34.57
110	7.97	0.000	39.34
120	4.55	0.000	38.15
	Transactions Costs = 0.25%		
80	27.67	0.024	9.79
90	19.68	0.055	22.09
100	12.99	0.085	34.17
110	7.97	0.100	40.11
120	4.55	0.095	38.14
	Transactions Costs = 1.00%		
80	27.67	0.101	10.05
90	19.68	0.221	22.07
100	12.99	0.337	33.69
110	7.97	0.394	39.39
120	4.55	0.376	37.57
	Transactions Costs = 4.00%		
80	27.67	0.435	10.88
90	19.68	0.874	21.86
100	12.99	1.278	31.96
110	7.97	1.476	36.90
120	4.55	1.423	35.57

Note: Revision interval = 8.0 weeks; standard deviation = 20%; horizon = 1.0 years; interest = 10%; TC = transactions costs.

ular stock price path will *not* in general equal Z—even when Δt is small. But whatever the transactions costs actually are, the modified hedge strategy will deliver $\max[S - K, 0]$ to $O(\Delta t^{1/2})$. Thus, Z can be thought of as the cost of an insurance policy guaranteeing coverage of transactions costs, whatever those may actually be.

This paradox follows from an incorrect interchange of limits. While it is true that the average trade *size* approaches zero as $\Delta t \rightarrow 0$, the *number* of trades approaches infinity. Total transactions equal the product of these two, which we have shown to be increasing as $\Delta t \rightarrow 0$ but bounded above. Also, for $\Delta t > 0$, the cost, $\hat{C}_0$, of delivering $\max[S - K, 0]$ is less than S_0, the cost of delivering S. While most investors might not want to pay the high cost of achieving the option, one can show that a sufficiently risk-averse investor will pay the cost. This is in contrast to following the Black-Scholes strategy, where the cost inclusive of transactions costs increases without bound as $\Delta t \rightarrow 0$, implying the cost of delivering $\max[S - K, 0]$ could exceed the cost, S_0, of a share of stock.

Table VI

Transactions Costs and Implied Annual Turnover of Option Replicating Strategies: 5 Year Options, Monthly Revision

Striking Price	Call Price	Total TC	Turnover (%)
	Transactions Costs = 0.00%		
80	51.11	0.000	5.13
90	45.61	0.000	8.05
100	40.45	0.001	11.27
110	35.69	0.001	14.42
120	31.33	0.001	17.17
	Transactions Costs = 0.25%		
80	51.11	0.066	5.25
90	45.61	0.101	8.08
100	40.45	0.140	11.19
110	35.69	0.179	14.34
120	31.33	0.216	17.30
	Transactions Costs = 1.00%		
80	51.11	0.271	5.43
90	45.61	0.410	8.19
100	40.45	0.560	11.20
110	35.69	0.710	14.21
120	31.33	0.851	17.02
	Transactions Costs = 4.00%		
80	51.11	1.182	5.91
90	45.61	1.686	8.43
100	40.45	2.214	11.07
110	35.69	2.729	13.64
120	31.33	3.202	16.01

Note: Revision interval = 4.0 weeks; standard deviation = 20%; horizon = 5.0 years; interest = 10%; TC = transactions costs.

Once Z has been computed, round trip turnover estimates follow immediately:

$$\text{Turnover} = Z/kS_0. \tag{29}$$

Note that turnover will depend upon k and the revision interval, Δt, as well as upon the option being replicated.

Tables III to VI present the transactions costs, Z, and turnover for a variety of different options, environments, transactions costs, and revision period assumptions. While full comparative static results depend upon the behavior of $\hat{C} - C$, we can get a "feel" for the effect of parametric changes on transactions costs when k is small by the Taylor series approximation

$$Z \simeq C_\sigma(\hat{\sigma} - \sigma), \tag{30}$$

Table VII

Hedging Errors: Black-Scholes Vs. Modified Replicating Strategies

Revision Period	Strategy[a]		Time Until Expiration 12 mos.	6 mos.	3 mos.
1 week	A:	$E[\Delta H]$	0.001	0.000	0.000
		$\sigma[\Delta H]$	0.091	0.099	0.115
	B:	$E[\Delta H]$	−0.019	−0.018	−0.018
		$\sigma[\Delta H]$	0.104	0.114	0.132
	C:	$E[\Delta H]$	0.000	0.000	0.000
		$\sigma[\Delta H]$	0.095	0.106	0.123
4 weeks	A:	$E[\Delta H]$	−0.003	−0.002	−0.002
		$\sigma[\Delta H]$	0.368	0.402	0.471
	B:	$E[\Delta H]$	−0.041	−0.040	−0.040
		$\sigma[\Delta H]$	0.394	0.431	0.502
	C:	$E[\Delta H]$	−0.004	−0.003	−0.004
		$\sigma[\Delta H]$	0.377	0.416	0.489

[a] Strategy A = Black-Scholes exclusive of transactions costs; Strategy B = Black-Scholes inclusive of transactions costs (1%); and Strategy C = modified strategy inclusive of transactions costs (1%).

where

$$C_\sigma = \frac{\partial C}{\partial \sigma} = S_0 N'(d_1)\sqrt{T}, \tag{31}$$

with $N'(d_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} d_1^2\right)$ and d_1 as defined in (3). Now for small $\hat{\sigma} - \sigma$,

$$\hat{\sigma} - \sigma \simeq k/\sqrt{2\pi\Delta t}, \tag{32}$$

implying

$$Z = kS_0 N'(d_1)\sqrt{T}/\sqrt{2\pi\Delta t}. \tag{33}$$

PROPOSITION I. *Transactions costs are roughly proportional to rate, k and are inversely proportional to the square root of the revision period, Δt.*

This proposition follows directly from (33), noting that d_1 does not depend on Δt or k.

Note that Z is approximately proportional to k only when $\hat{\sigma} - \sigma$ is small—it can be seen from Table III that Z grows less than proportionately with k when Z becomes sizable, reflecting the fact that the replicating strategy itself has changed.

We may also show the following:

$$\frac{\partial C\sigma}{\partial K} = \frac{S_0}{\sqrt{2\pi T}\sigma} \exp\left(-\frac{1}{2} d_1^2\right) d_1. \tag{34}$$

which has the sign of d_1. This generates:

PROPOSITION II. *Transactions costs will increase with the striking price, K, when $K < K^*$ and decrease with K when $K > K^*$, where $K^* = S_0 e^{(r+1/2\sigma^2)T}$.*

Proof: Follows from (34) by noting $d_1 > 0$ iff

$$K < S_0 e^{(r+1/2\sigma^2)T}.$$

It follows immediately from Proposition II that turnover is greatest when the striking price is K^*, i.e., an option whose present value of striking price is slightly in the money. Finally, we examine the impact of changes in the horizon, T, on the annualized transactions costs:

PROPOSITION III. *There exist striking prices, K_1 and K_2, such that annualized transactions costs decrease with horizon, T, for $K \in [K_1, K_2]$ and increase for $K \notin [K_1, K_2]$. As $T \rightarrow \infty$, $[K_1, K_2] \rightarrow [0, \infty]$, implying annualized transactions costs decrease for all contracts when the horizon is distant.*

Proof: $C_\sigma/T = SN'(d_1)T^{-1/2}$, and

$$\frac{\partial}{\partial T}[C_\sigma/T] = -\frac{1}{2}SN'(d_1)T^{-3/2} + SN'(d_1)\frac{dd_1}{dT}T^{-1/2}$$

$$= -\frac{1}{2}SN'(d_1)T^{-3/2} + \frac{1}{2}SN'(d_1)RT^{-3/2},$$

where $R = \left[\left(\ln\left(\frac{S_0}{K}\right)\right)^2 - \left(\left(r + \frac{1}{2}\sigma^2\right)T\right)^2\right] \Big/ \sigma^2 T.$

Therefore,

$$\frac{\partial[C_\sigma/T]}{\partial T} \le 0 \quad \text{iff} \quad R \le 1.$$

Now, $R \le 1$ implies $\left(\ln\left(\frac{S_0}{K}\right)\right)^2 \le \left(\sigma^2 T + \left(\left(r + \frac{1}{2}\sigma^2\right)T\right)^2\right)$

which, in turn, will be satisfied for $K \,\varepsilon\, [K_1, K_2]$, where

$$K_1 = S_0\exp[-(\sigma^2 T + ((r + ½\,\sigma^2)T)^2)^{1/2}]$$

$$K_2 = S_0\exp[(\sigma^2 T + ((r + ½\,\sigma^2)T)^2)^{1/2}].$$

Note, as $T \rightarrow \infty$, $K_1 \rightarrow 0$ and $K_2 \rightarrow \infty$.

PROPOSITION IV. *Expected turnover for the Black-Scholes replicating strategy is given by*

$$N'(d_1)\sqrt{T}/\sqrt{2\pi\Delta t}.$$

This proposition follows from (33) and (29), noting that the Black-Scholes strategy is optimal in the limit as $k \rightarrow 0$. Compare the simplicity of this formula with the formulation of Gilster and Lee [5], which requires numerical approximation techniques.

Since Propositions I to III were derived with the assumption that $k \approx 0$, the results of these propositions hold for Black-Scholes replicating strategies as well.

IV. Option Pricing Bounds

Clearly, $\hat{C}$ puts an upper bound on the price of an option, since if the price exceeded that amount the option could be constructed by the replicating strategy. On the other hand, the option can be shown never to have a price less than $\underline{C}$, where $\underline{C}$ is the Black-Scholes price using volatility $\underline{\sigma}^2 = \sigma^2 - kE \left| \frac{\Delta S}{S} \right| \Big/ \Delta t.$

Otherwise, an investor could buy the option, "undo" it by following the offsetting replicating strategy, and make a return after transactions costs which exceeded the risk-free rate. (We note that as before, the strategy will produce a hedging error after transactions costs; however, this hedging error will be uncorrelated with the market and will almost surely approach zero as Δt becomes small.)

When transactions costs are small enough for the approximation (30) to hold, the size of the bound will be given by

$$\begin{aligned} \hat{C} - \underline{C} &\approx 2C_\sigma k/\sqrt{2\pi\Delta t} \\ &\approx 2kS_0N'(d_1)\sqrt{T}/\sqrt{2\pi\Delta t}, \qquad \text{using (33).} \end{aligned}$$

Propositions I to III indicate how the bound, $\hat{C} - \underline{C}$, will respond to changes in the option under consideration. Proposition I, for example, suggests that the widest price bounds will occur for options whose striking price has a present value about equal to the current stock price. The width of the price bound as a *percent* of the call option price increases as the call option becomes more out-of-the-money, which may help explain the empirical observation that the Black-Scholes formula prices these options less exactly.

V. Conclusions

This paper has developed a technique for replicating option returns in the presence of transactions costs. The strategy depends upon the level of transactions costs and the time period between portfolio revision, in addition to the standard variables of option pricing. However, these additional parameters enter in a very simple way, through adjustment of the volatility in the Black-Scholes formula. The "pure" Black-Scholes strategy holds only in the limiting case of zero transactions costs.

There is an intuitive explanation for our results. Inclusive of transactions costs, the net price of purchasing stock is slightly higher than the price without transactions costs. Similarly, the net price of selling stock is slightly lower. This accentuation of up or down movements of the stock price can be modelled as if the volatility of the actual stock price was higher.[14]

[14] To see intuitively where the adjustment (13) to variance comes from, we note that vibration alone incurs an expected cost over the interval, Δt, of

$$\tfrac{1}{2}\hat{C}_{SS}S^2\sigma^2\Delta t.$$

Our methodology enabled us to develop simple formulae for the expected turnover and transactions costs associated with replicating arbitrary options.[15] These formulae, in turn, enabled us to put bounds on option prices.

We might note that our analysis can easily be extended to include a fixed cost of transactions, as well as the variable cost studied. Total fixed cost would simply be the sum of the $T/\Delta t$ fixed charges and would be added to the "Z" cost derived in the paper. But this remark points out an important assumption of our analysis, that the portfolio is revised every Δt periods. A more complete model would allow for the revision interval to be a choice variable. It may well be stochastic, depending upon the change in the underlying stock price or upon the size of the revision required. Work by Magill and Constantinides [7], Constantinides [3], and Kandel and Ross [6] begins to explore this important area.

Transactions costs are an additional cost from price variation (vibration). Expected transactions costs over Δt are given by (22):

$$\frac{1}{2}\, k\hat{C}_{SS}S^2E\left|\frac{\Delta S}{S}\right| = \frac{1}{2}\, k\hat{C}_{SS}S^2(\sqrt{2/\pi})\sigma\sqrt{\Delta t}.$$

Combining costs of vibration and transactions costs gives

$$\tfrac{1}{2}\hat{C}_{SS}S^2[\sigma^2 + k\sqrt{2/\pi}\;\sigma/\sqrt{\Delta t}]\Delta t = \tfrac{1}{2}\hat{C}_{SS}S^2\hat{\sigma}^2\Delta t$$

which is the same as the cost of vibration if the true variance were $\hat{\sigma}^2$ and there were no transactions costs.

[15] Since any contingent payoff pattern can be represented as a portfolio of options, this technique can be used to estimate the transactions costs of any pattern of returns which is either (globally) convex or concave in the underlying stock return. For a convex pattern, we estimate the transactions costs by the additional cost of the portfolio of options, when the higher volatility, σ^2, is used instead of σ^2. (For concavity, we adjust volatility downward.) This straightforward technique is not applicable to functions which are convex and concave over different regions, since the transactions related to the replicating options will cancel rather than be additive in this case.

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