NLA Homework2

Group O

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Question 1

(i) Let

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix}$$

Compute B^2, B^3, B^4.

(ii) Let $C = I + B$, i.e.,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & 1 & 1 \end{pmatrix}$$

Compute C^m, where $m \geq 2$ is a positive integer.

Solutions:

(i)

$$B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 13 & -2 & 0 & 0 \end{pmatrix}$$

$$B^3 = B^2 \cdot B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \\ 13 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 \end{pmatrix}$$

$$B^4 = B^3 \cdot B = 0$$

(ii)

$$\begin{aligned} C^m &= (I+B)^m \\ &= \sum_{k=0}^{m} \binom{m}{k} I^{m-k} \cdot B^k \\ &= \sum_{k=0}^{m} \binom{m}{k} B^k \\ &= B^0 + \binom{m}{1} \cdot B + \binom{m}{2} \cdot B^2 + \binom{m}{3} \cdot B^3 \\ &= I + m \cdot B + \frac{m(m-1)}{2} \cdot B^2 + \frac{m(m-1)(m-2)}{6} \cdot B^3 \end{aligned}$$

$$m \cdot B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4m & 0 & 0 & 0 \\ m & -2m & 0 & 0 \\ -m & 3m & m & 0 \end{pmatrix}$$

$$\frac{m(m-1)}{2} \cdot B^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4m(m-1) & 0 & 0 & 0 \\ \frac{13}{2}m(m-1) & -m(m-1) & 0 & 0 \end{pmatrix}$$

$$\frac{m(m-1)(m-2)}{6} \cdot B^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{4}{3}m(m-1)(m-2) & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore C^m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4m & 1 & 0 & 0 \\ a & -2m & 1 & 0 \\ c & b & m & 1 \end{pmatrix}$$

where $a = -4m^2 + 5m$, $b = -m^2 + 4m$, $c = -\frac{4}{3}m + \frac{21}{2}m^2 - \frac{61}{6}m$

Question 2

Let u be an $n \times n$ upper triangular matrix with entries equal to 0 on the main diagonal, i.e., with $U(i,i) = 0$ for $i = 1 : n$.

(i) Show that $U^n = 0$;

(ii) Compute $(I+U)^m$ in terms of $U, U^2, \ldots, U^{n-1}$, where $m \geq n$ is a positive integer. $(I+U)^m = I + \sum_{k=1}^{n-1} \binom{m}{k} U^k$

Solutions:

(i) $U = \mathrm{col}(u_k)_{k=1:n}$

$$u_k(j) = 0 \quad k \leq j \leq n$$

$$U^2 = \mathrm{col}(U \cdot u_k)_{k=1:n}$$

$$U \cdot u_k = \sum_{j=1}^{n} u_k(j) \cdot u_j = \sum_{j=1}^{k-1} u_k(j) \cdot u_j$$

Let $k-1 \leq i \leq n$

$$U \cdot u_k(i) = \sum_{j=1}^{k-1} u_k(j) \cdot u_j(i) = 0 \quad \text{since } u_j(i) = 0 \quad j \leq i \leq n$$

Denote the k-th column of U^α as $u_k^{(\alpha)}$, therefore,

$$u_k^{(2)} = U \cdot u_k$$

$$u_k^{(2)}(i) = U \cdot u_k(i) = 0 \quad k-1 \leq i \leq n$$

Repeat this process, we have

$$u_k^{(k)}(i) = 0 \quad 1 \leq i \leq n$$

For $n \geq k$, $u_k^{(n)} = 0$

$$U^n = \mathrm{col}(u_k^{(n)})_{k=1:n} = 0$$

(ii)

$$\begin{aligned}(I+U)^m &= \sum_{k=1}^{m} \binom{m}{k} I^{m-k} U^k \\ &= \sum_{k=1}^{m} \binom{m}{k} U^k \\ &= \sum_{k=1}^{n-1} \binom{m}{k} U^k \quad m \geq n\end{aligned}$$

Question 3

Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -2 & 3 & 0 \\ 2 & 2 & -3 & 4 \end{pmatrix}; \quad U_1 = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & -0.25 \end{pmatrix}$$

(i) Show that $LU = L_1U_1$.

(ii) Explain why this does not contradict the uniqueness of the LU decomposition of a matrix.

Solutions:

(i)

By matrix multiplication,

$$LU = L_1U_1 = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

(ii) The uniqueness means when L and L_1 are both lower triangular matrices with diagonal entries equal to 1, then $L = L_1$, $U = U_1$. While the entries of L_1 on the main diagonal are not all equal to 1, this example doesn't contradict the uniqueness.

Question 4

Let L_1 and L_2 be nonsingular lower triangular matrices and let U_1 and U_2 be nonsingular upper triangular matrices. If $L_1U_1 = L_2U_2$, show that there exists a nonsingular diagonal matrix D such that

$$L_2 = L_1D^{-1} \quad \text{and} \quad U_2 = DU_1.$$

Solution:

To show $L_2 = L_1D^{-1}, U_2 = DU_1$ since all matrices are nonsingular, it is equivalent to show $D = L_2^{-1}L_1 = U_2U_1^{-1}$. This is to show that matrices $L_2^{-1}L_1$ and $U_2U_1^{-1}$ are the same, and they're diagonal matrix.

$$L_1U_1 = L_2U_2$$

Multiply L_2^{-1} on the left and U_1^{-1} on the right: $L_2^{-1}(L_1U_1)U_1^{-1} = L_2^{-1}(L_2U_2)U_1^{-1}$

Since $U_1U_1^{-1} = L_2^{-1}L_2 = I$

$$L_2^{-1}L_1 = U_2U_1^{-1}$$

Since the inverse of a lower/upper triangular matrix is still lower/upper triangular, and the product of two lower/upper triangular matrices is also lower/upper triangular

$\therefore$ L_2L_1 is lower triangular, $U_2U_1^{-1}$ is upper triangular. They must be diagonal matrices. Q.E.D.

Question 5

(i) Write the pseudocode for the forward substitution corresponding to a lower triangular banded matrix of band 2, i.e., for solving $Ly = b$ where b is an $n \times 1$ vector and L is an $n \times n$ lower triangular matrix such that

$$L(j,k) = 0, \forall 1 \leq j,k \leq n \quad \text{with} \quad j - k > 2$$

The input for the pseudocode are the matrix L and the vector b; the output is the vector y.

What is the operation count for this?

(ii) Write the pseudocode for the backward substitution corresponding to an upper triangular banded matrix of band 2, i.e., for solving $Ux = b$ where b is an $n \times 1$ vector and U is an $n \times n$ upper triangular matrix such that

$$U(j,k) = 0, \forall 1 \leq j,k \leq n \quad \text{with} \quad k - j > 2$$

The input for the pseudocode are the matrix U and the vector b; the output is the vector y.

What is the operation count for this?

Solutions:

(i)

$$
\begin{aligned}
y(1) &= \frac{b(1)}{L(1,1)} \\
y(2) &= \frac{b(2) - L(2,1)y(1)}{L(2,2)} \\
&\vdots \\
y(j) &= \frac{b(j) - L(j,j-2)y(j-2) - L(j,j-1)y(j-1)}{L(j,j)} \qquad j = 3:n
\end{aligned}
$$

Pseudocode:

```
y(1) = b(1) / L(1,1)
y(2) = (b(2) - L(2,1)*y(1)) / L(2,2)
for j = 3:n
    y(j) = (b(j) - L(j,j-2)*y(j-2) - L(j,j-1)*y(j-1)) / L(j,j)
end
```

Operation counts:

$$1 + 3 + 5(n-2) = 5n - 6$$

(ii)

$$
\begin{aligned}
x(n) &= \frac{b(n)}{U(n,n)} \\
x(n-1) &= \frac{b(n-1) - U(n-1,n)x(n)}{U(n-1,n-1)} \\
&\vdots \\
x(j) &= \frac{b(j) - U(j,j+1)x(j+1) - U(j,j+2)x(j+2)}{U(j,j)} \qquad j = (n-2):1
\end{aligned}
$$

Pseudocode:

```
x(n) = b(n) / U(n,n)
x(n-1) = (b(n-1) - U(n-1,n)*x(n)) / U(n-1,n-1)
for j = (n-2):1
    x(j) = (b(j) - U(j,j+1)*x(j+1) - U(j,j+2)*x(j+2)) / U(j,j)
end
```

Operation counts:

$$1 + 3 + 5(n-2) = 5n - 6$$

Question 6

The LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

is given by $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Solve $Ax = b$, where

$$b = \begin{pmatrix} 3 \\ 2 \\ -2 \\ 0 \end{pmatrix}$$

Solution:

Solve $Ax = b$

$$LUx = b$$

Let $y = Ux$, we first solve $Ly = b$

$$\begin{aligned}
y(1) &= \frac{b(1)}{L(1,1)} = 3 \\
y(2) &= \frac{b(2) - L(2,1) \cdot y(1)}{L(2,2)} = 5 \\
y(3) &= \frac{b(3) - L(3,1) \cdot y(1) - L(3,2) \cdot y(2)}{L(3,3)} = -3 \\
y(4) &= \frac{b(4) - \sum_{k=1}^{3} L(4,k)y(k)}{L(4,4)} = -14
\end{aligned}$$

Then solve $Ux = y$

$$\begin{aligned}
x(4) &= \frac{y(4)}{U(4,4)} = 14 \\
x(3) &= \frac{y(3) - (-1) \cdot (+14)}{1} = 11 \\
x(2) &= \frac{y(2) - 1 \cdot 11}{-1} = 6 \\
x(1) &= \frac{y(1) - (-1) \cdot 6 - 1 \cdot 14}{2} = -\frac{5}{2}
\end{aligned}$$

$$\therefore \quad x = \begin{bmatrix} -5/2 \\ 6 \\ 11 \\ 14 \end{bmatrix}$$

Question 7

(i) Let A be an $n \times n$ matrix and let L be a nonsingular lower triangular of size n. Show that if LA is a lower triangular matrix, then A is a lower triangular matrix. Show that if AL is a lower triangular matrix, then A is a lower triangular matrix.

(ii) Let A be an $n \times n$ matrix and let U be a nonsingular upper triangular of size n. Show that if UA is an upper triangular matrix, then A is an upper triangular matrix. Show that if AU is an upper triangular matrix, then A is an upper triangular matrix.

Solutions:

(i) Let $L_1 = LA$, since L is nonsingular

$$A = L^{-1}L_1$$

By Lemma 1.15 and Lemma 1.17, the inverse of a lower triangular matrix is lower triangular, and the product of two lower triangular matrices is lower triangular

$\therefore A$ is lower triangular

Let $L_2 = AL \Rightarrow A = L_2L^{-1}$

With similar reasoning above, A is lower triangular

(ii) Let $U_1 = UA$, since U is nonsingular

$$A = U^{-1}U_1$$

Then inverse of an upper triangular matrix is upper triangular, and the product of two upper triangular matrices is upper triangular

$\therefore A$ is upper triangular.

Let $U_2 = AU \Rightarrow A = U_2U^{-1}$

With similar reasoning above, A is upper triangular

Question 8

The values of the following coupon bonds with face value $100 are given:

Bond Type	Coupon Rate	Bond Price
5 months semiannual	0	$98.75
11 months semiannual	4%	$102.00
17 months semiannual	6%	$103.50
23 months semiannual	4%	$105.50

Find the 5 months, 11 months, 17 months, and 23 months discount factors.

Solution:

Denote the 5 months, 11 mos, 17 mos. and 23 mos discount factors as d_1, d_2, d_3, d_4, respectively.

$$\begin{aligned} 98.75 &= 100d_1 \\ 102 &= 2d_1 + 102d_2 \\ 103.5 &= 3d_1 + 3d_2 + 103d_3 \\ 105.5 &= 2d_1 + 2d_2 + 2d_3 + 102d_4 \end{aligned}$$

$$L = \begin{pmatrix} 100 & 0 & 0 & 0 \\ 2 & 102 & 0 & 0 \\ 3 & 3 & 103 & 0 \\ 2 & 2 & 2 & 102 \end{pmatrix}, \quad b = \begin{pmatrix} 98.75 \\ 102 \\ 103.5 \\ 105.5 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

By solving $Ld = b$, we get $d = \text{forward\_subset}(L, b) = \begin{pmatrix} 0.9875 \\ 0.980637 \\ 0.94753 \\ 0.977144 \end{pmatrix}$

That is $d_1 = 0.9875, d_2 = 0.980637, d_3 = 0.94753, d_4 = 0.977144$

In [97]:
```python
def forward_subset(L, b):
    """
    Forward Substitution
    Input:
    - L: nonsingular lower triangular matrix of size n
    - b: column vector of size n
    Output:
    - x: solution to Lx=b
    """
    n = len(b)
    x = [0 for _ in range(n)]
    x[0] = b[0] / L[0][0]
    for j in range(1, n):
        ssum = 0
        for k in range(0, j):
            ssum += L[j][k] * x[k]
        x[j] = (b[j] - ssum) / L[j][j]

    for i in range(n):
        x[i] = round(x[i], 6)

    return x
```

In [98]:
```python
L = [[100, 0, 0, 0], [2, 102, 0, 0], [3, 3, 103, 0], [2, 2, 2, 102]]
b = [98.75, 102, 103.5, 105.5]
forward_subset(L, b)
```

Out[98]:
```
[0.9875, 0.980637, 0.94753, 0.977144]
```

Question 9

The values of the following coupon bonds with face value $100 are given:

Bond Type	Coupon Rate	Bond Price
10 months semiannual	2%	$100.60
16 months semiannual	4%	$103.30
16 months annual	5%	$107.30
22 months semiannual	5%	$110.30

(i) List the cash flows and cash flow dates for each bond.

(ii) Find the 4 months, 10 months, 16 months, and 22 months discount factors.

Solutions:

(i)

Type	Cash Flow & Date
10 mos semi-annual	\$1 in 4 months \$101 in 10 mos
16 mos semi-annual	\$2 in 4 mos \$2 in 10 mos \$102 in 16 mos
16 mos annual	\$5 in 4 mos \$105 in 16 mos
22 mos semi-annual	\$2.5 in 4 mos \$2.5 in 10 mos \$2.5 in 16 mos \$102.5 in 22 mos

(ii) Denote the 4 mos, 10 mos, 16 mos and 22 mos discount factors as d_1, d_2, d_3, d_4 respectively.

$$\begin{aligned}100.6 &= d_1 + 101d_2\\ 103.3 &= 2d_1 + 2d_2 + 102d_3\\ 107.3 &= 5d_1 + 105d_3\\ 110.3 &= 2.5d_1 + 2.5d_2 + 2.5d_3 + 102.5d_4\end{aligned}$$

Solve Ad=b, where

$$A = \begin{pmatrix} 1 & 101 & 0 & 0 \\ 2 & 2 & 102 & 0 \\ 5 & 0 & 105 & 0 \\ 2.5 & 2.5 & 2.5 & 102.5 \end{pmatrix}, \quad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}, \quad b = \begin{pmatrix} 100.6 \\ 103.3 \\ 107.3 \\ 110.3 \end{pmatrix}$$

We get

$$d = \begin{pmatrix} 1.017178 \\ 0.985969 \\ 0.973468 \\ 1.003497 \end{pmatrix}$$

That is $d_1 = 1.017178, d_2 = 0.985969, d_3 = 0.973468, d_4 = 1.003497$

In [99]:

```
def lu_no_pivoting(A):
    """
    LU decomposition without pivoting
    Input:
    - A: nonsingular matrix of size n
    Output:
    - L, U
    """
    n = len(A)
    # create empty L, U
    L = [[0 for _ in range(n)] for _ in range(n)]
    U = [[0 for _ in range(n)] for _ in range(n)]
    for i in range(0, n-1):
        for l in range(i, n):
            U[i][l] = A[i][l]
            L[l][i] = A[l][i] / U[i][i]
        for j in range(i+1, n):
            for k in range(i+1, n):
                A[j][k] -= L[j][i] * U[i][k]
    L[n-1][n-1] = 1
    U[n-1][n-1] = A[n-1][n-1]

    for j in range(n):
        for k in range(n):
            L[j][k] = round(L[j][k], 6)
            U[j][k] = round(U[j][k], 6)

    return L, U

def backward_subset(U, b):
    """
    Backward substitution
    Input:
    - U: nonsingular upper triangular matrix of size n
    - b: column vector of size n
    Output:
    - x: solution to Ux=b
    """
    n = len(b)
```

```python
    x = [0 for _ in range(n)]
    x[n-1] = b[n-1] / U[n-1][n-1]
    for j in range(n-2, -1, -1):
        ssum = 0
        for k in range(j+1, n):
            ssum += U[j][k] * x[k]
        x[j] = (b[j] - ssum) / U[j][j]

    for i in range(n):
        x[i] = round(x[i], 6)

    return x
```

In [100…

```python
def linear_solve_LU_no_pivoting(A, b):
    """
    Input:
    - A: nonsingular square matrix of size n with LU decom
    - b: column vector of size n
    Output:
    - x: solution to Ax=b
    """
    L, U = lu_no_pivoting(A)
    y = forward_subset(L, b)  # solve Ly=b
    x = backward_subset(U, y)  # solve Ux=y
    return x
```

In [101…

```python
A = [
    [1, 101, 0, 0],
    [2, 2, 102, 0],
    [5, 0, 105, 0],
    [2.5, 2.5, 2.5, 102.5]
]
b = [100.6, 103.3, 107.3, 110.3]
x = linear_solve_LU_no_pivoting(A, b)
x
```

Out[101…

```
[1.017178, 0.985969, 0.973468, 1.003497]
```