Numerical Linear Algebra for Financial Engineering

The Pre-MFE Program at Baruch College

Homework 2

Assigned: February 10; Due: February 17

(1) (i) Let

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ -1 & 3 & 1 & 0 \end{pmatrix}.$$

Compute B^2, B^3, B^4.

(ii) Let $C = I + B$, i.e.,

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & 1 & 1 \end{pmatrix}.$$

Compute C^m, where $m \geq 2$ is a positive integer.

(2) Let u be an $n \times n$ upper triangular matrix with entries equal to 0 on the main diagonal, i.e., with $U(i,i) = 0$ for $i = 1 : n$.

(i) Show that $U^n = 0$;

(ii) Compute $(I+U)^m$ in terms of U, U^2, $\ldots$, U^{n-1}, where $m \geq n$ is a positive integer.

(3) Let

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 2 & -2 & 3 & 0 \\ 2 & 2 & -3 & 4 \end{pmatrix}; \quad U_1 = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -0.5 & 0.5 & 0 \\ 0 & 0 & 1/3 & -1/3 \\ 0 & 0 & 0 & -0.25 \end{pmatrix}.$$

(i) Show that $LU = L_1 U_1$.

(ii) Explain why this does not contradict the uniqueness of the LU decomposition of a matrix.

(4) Let L_1 and L_2 be nonsingular lower triangular matrices and let U_1 and U_2 be nonsingular upper triangular matrices. If $L_1 U_1 = L_2 U_2$, show that there exists a nonsingular diagonal matrix D such that

$$L_2 = L_1 D^{-1} \quad \text{and} \quad U_2 = D U_1.$$

(5) (i) Write the pseudocode for the forward substitution corresponding to a lower triangular banded matrix of band 2, i.e., for solving $Ly = b$ where b is an $n \times 1$ vector and L is an $n \times n$ lower triangular matrix such that

$$L(j,k) = 0, \ \forall\, 1 \leq j,k \leq n \quad \text{with} \quad j - k > 2.$$

The input for the pseudocode are the matrix L and the vector b; the output is the vector y.

What is the operation count for this?

(ii) Write the pseudocode for the backward substitution corresponding to an upper triangular banded matrix of band 2, i.e., for solving $Ux = b$ where b is an $n \times 1$ vector and U is an $n \times n$ upper triangular matrix such that

$$U(j,k) \;=\; 0, \;\forall\, 1 \leq j,k \leq n \quad \text{with} \quad k - j > 2.$$

The input for the pseudocode are the matrix U and the vector b; the output is the vector y.

What is the operation count for this?

(6) The LU decomposition of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -2 & 0 & 1 & -1 \\ 4 & -1 & 0 & 1 \\ 4 & -3 & 0 & 2 \end{pmatrix}$$

is given by $A = LU$, where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 2 & 1 & -1 & 1 \end{pmatrix}; \quad U = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Solve $Ax = b$, where

$$b = \begin{pmatrix} 3 \\ 2 \\ -2 \\ 0 \end{pmatrix}.$$

(7) (i) Let A be an $n \times n$ matrix and let L be a nonsingular lower triangular of size n. Show that if LA is a lower triangular matrix, then A is a lower triangular matrix. Show that if AL is a lower triangular matrix, then A is a lower triangular matrix.

(ii) Let A be an $n \times n$ matrix and let U be a nonsingular upper triangular of size n. Show that, if UA is an upper triangular matrix, then A is an upper triangular matrix. Show that, if AU is an upper triangular matrix, then A is an upper triangular matrix.

(8) The values of the following coupon bonds with face value \$100 are given:

Bond Type	Coupon Rate	Bond Price
5 months semiannual	0	\$98.75
11 months semiannual	4%	\$102.00
17 months semiannual	6%	\$103.50
23 months semiannual	4%	\$105.50

Find the 5 months, 11 months, 17 months, and 23 months discount factors.

(9) The values of the following coupon bonds with face value \$100 are given:

Bond Type	Coupon Rate	Bond Price
10 months semiannual	2%	\$100.60
16 months semiannual	4%	\$103.30
16 months annual	5%	\$107.30
22 months semiannual	5%	\$110.30

(i) List the cash flows and cash flow dates for each bond.

(ii) Find the 4 months, 10 months, 16 months, and 22 months discount factors.