Portfolio Allocation:
The Mechanics

1

The investor's problem

- Investors are trying to maximize their expected utility.
- Investors face a difficult problem:
  - - Which stocks should they invest in?
  - - How much should they invest in each stock?
- Harry Markowitz was the first person to solve the investor's problem.
  - - H Markowitz, 1952, "Portfolio Selection", Journal of Finance.

2

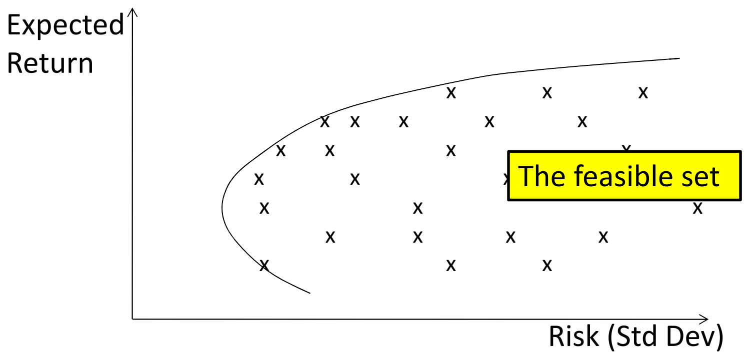
Solving the investor's problem

- Markowitz made several assumptions:
  - \- Investors are risk averse.
    - Investors like expected returns.
    - Investors dislike risk as measured by variance.
  - \- All investors have the same estimates of expected returns, variances, and correlations.
    - Homogeneous Expectations.
  - \- Perfect Markets
    - Investors can borrow and lend as much as they want at the riskless interest rate.
    - No transaction costs.
    - Investors can buy or sell as much as they want, but individual trades do not affect prices.

3

Risk and Expected Return

- There are 1000s of risky investment opportunities.
- Every possible combination of assets can be plotted in risk-return space.
- -The collection of these combinations is called the feasible set.



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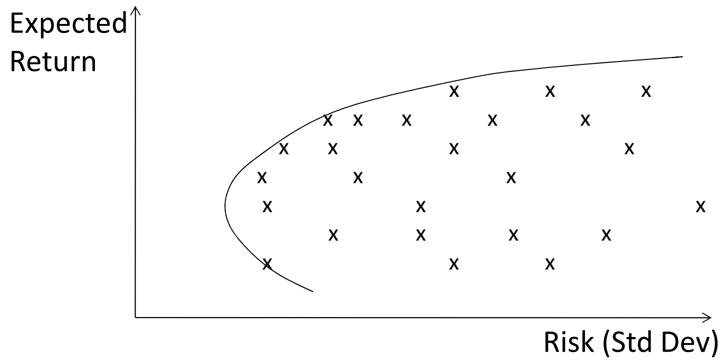
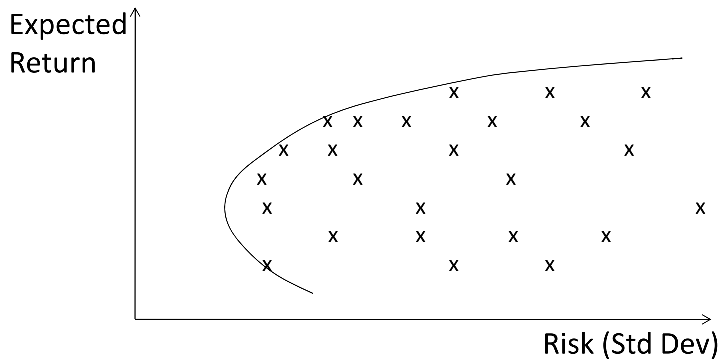
Where is the lowest risk portfolio?



Efficient portfolios

- An efficient portfolio has the maximum expected return for any given risk level.
- Alternatively:
- An efficient portfolio has the lowest risk for any given level of expected return.
- The combination of all efficient portfolios is called the efficient frontier.

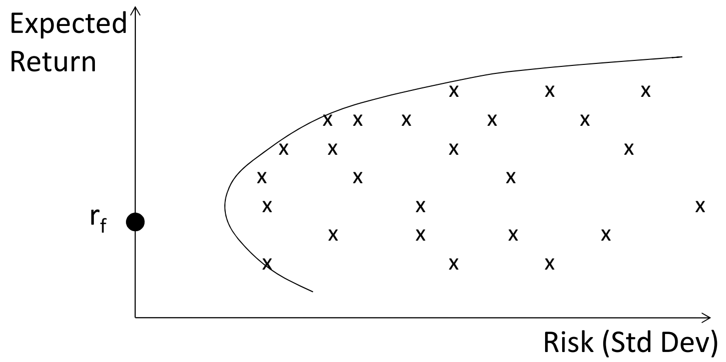
Where is the efficient frontier?



What type of portfolio would all risk averse investors choose to invest in?

7

Adding a risk free asset with return r_f.

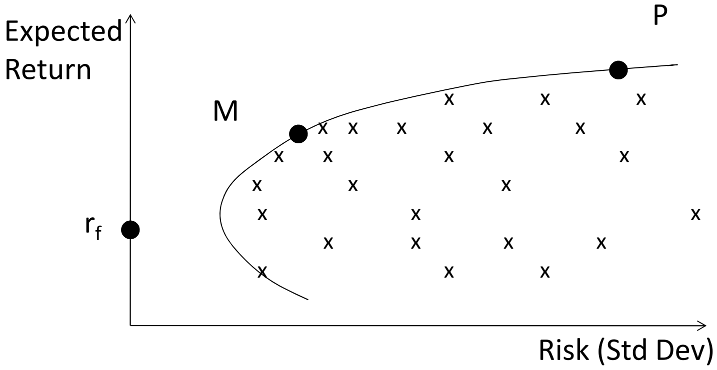


You can invest in any combination of risky assets and the riskless asset.

Which portfolio of risky assets would you invest in?

8

Adding a risk free asset with return r_f.

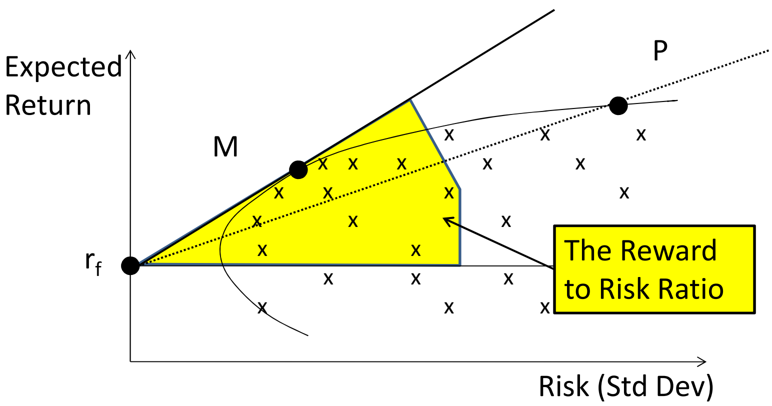


You can invest in any combination of risky assets and the riskless asset.

Which portfolio of risky assets would you invest in?

9

Adding a risk free asset with return r_f.



The OPTIMAL portfolio of risky assets, M, in combination with the riskless asset maximizes the reward to risk ratio.

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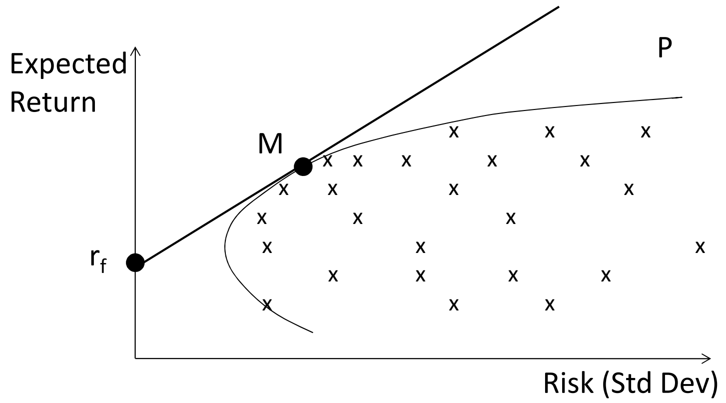
How would you invest?



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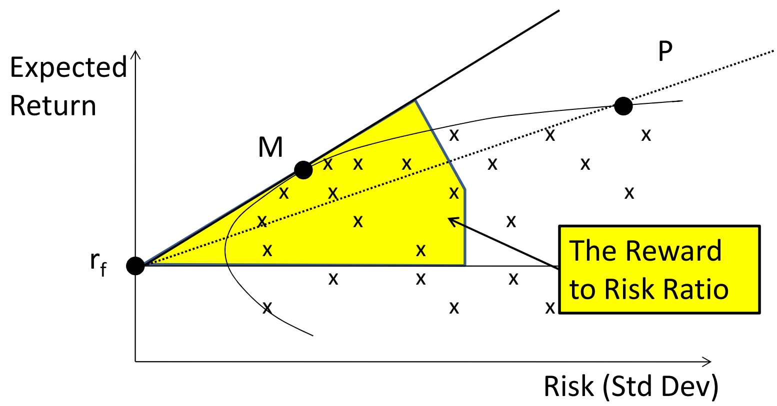
How would you invest?



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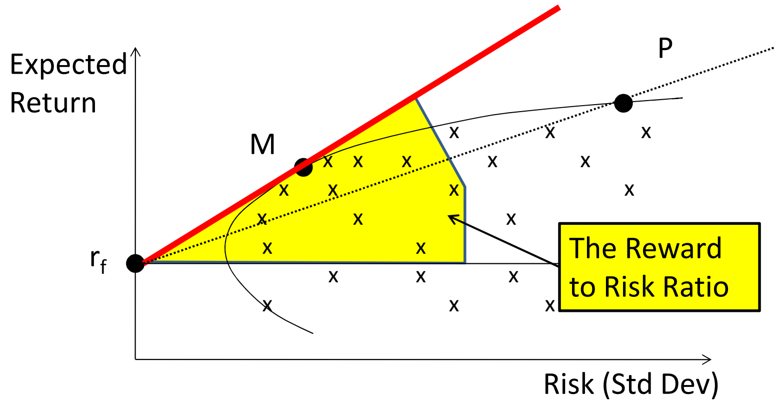
How would you invest?

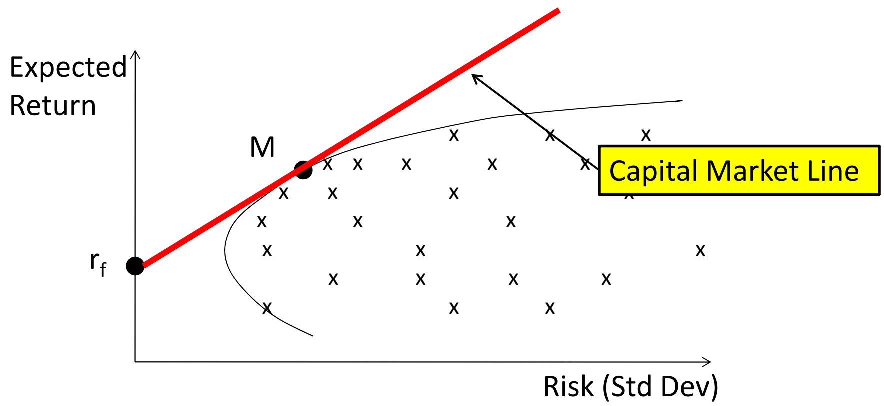


ALL investors will choose some combination of the OPTIMAL portfolio and the riskless asset.

Two Fund Separation

Choice of M is independent of investor preferences.

- ALL risk averse investors will find that M is the OPTIMAL portfolio.



Equilibrium: Risk and Return

All investors hold some combination of the riskless asset and the optimal portfolio, M.

There is a linear relation between expected portfolio returns and risk (std dev) when there is a risk free asset in the economy.

The linear relation is given by the Capital Market Line.

NB: There is not a linear relation between expected returns and risk if there is no risk free asset. The efficient frontier is not linear.

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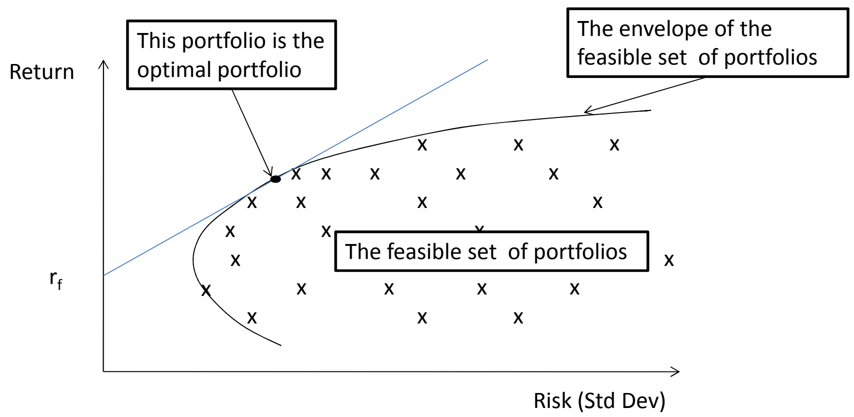
Finding the Optimal Portfolio

- The optimal portfolio lies on the tangency of the line connecting the point r_f on the vertical axis to the efficient frontier.



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Utility functions and the Optimal Portfolio

- An investor's expected utility is a function of the expected returns on an investment and the risk of the investment.

$$E\left[U\left(x\right)\right] = E\left(r_x\right) - \lambda\sigma_x^2$$

- If an investor wants to find the optimal portfolio they must maximize their expected utility: $Max: E\left[U\left(P\right)\right] = E\left(r_p - r_f\right) - \lambda\sigma_{r_p}^2$

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What is the Choice Variable?

- The investor is trying to find how much to invest in each stock to maximize their utility.

$$Max: U\left(P\right) = E\left(r_p - r_f\right) - \lambda\sigma_{r_p}^2$$

$$\rightarrow$$

- There is 1 constraint on this problem. The weights must add up to 1: $w^T 1 = 1$.
- - There is a trick to get around this constraint.

We will ignore the constraint for the time being.

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Reviewing the Notation

- Assume there are M risky assets each of which has an expected return, denoted μ_m.

$$\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_M \end{bmatrix}$$

- Let S be the M x M variance-covariance matrix.

$$S = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1M} \\ \vdots & & \vdots \\ \sigma_{M1} & \cdots & \sigma_{MM} \end{bmatrix}$$

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Reviewing the Notation

- To construct a portfolio we need to have a column vector of portfolio weights, w.

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_M \end{bmatrix} \text{ where } \sum_{i=1}^{M} w_i = 1 \rightarrow w^T 1 = 1$$

- The portfolio's expected return and variance are given by:

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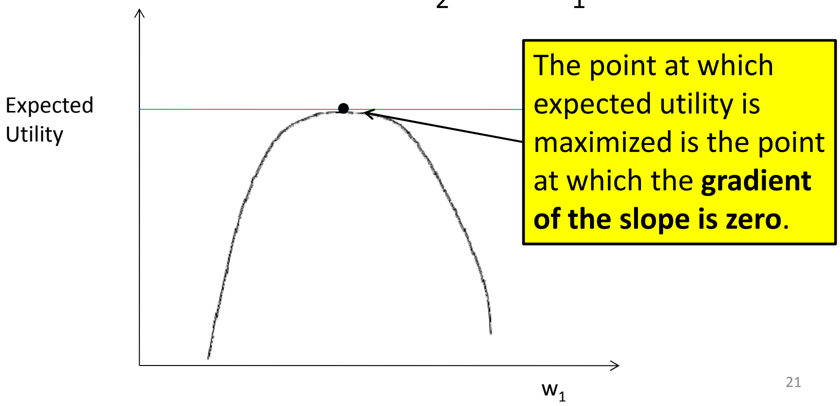
The Two Asset Case

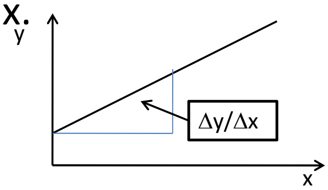
- Our aim is to find the values for w_1 and w_2 which maximize expected utility:

$$Max_w : w^T(\mu - r_f) - \lambda w^T S w$$

- NB: In the two asset case $w_2 = 1 - w_1$.



Optimization and Differentiation

- The slope, or gradient, of a line is calculated as the change in y divided by the corresponding change in x.
- 

- We can calculate the slope of a curve using differentiation.
  - \- The basic rule of differentiation:

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}$$

Differentiation and the Optimal Portfolio

- Aim:
  - - Maximize expected utility by changing the amount invested in different assets.
- How can we use differentiation to find the optimal portfolio?
- The investor's expected utility will be maximized when the gradient of the slope on the utility function is equal to zero.
  - - Use differentiation to calculate the gradient of the slope.

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The Maths Part

- To find the optimal portfolio we must maximize expected utility with respect to portfolio weights, w:

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Finding the Efficient Frontier

- We have found the optimal portfolio. This is a portfolio which lies on the efficient frontier.
- We can also calculate any portfolio on the efficient frontier, or more generally, on the envelope of the feasible set using a similar approach.
- To find the envelope of the feasible set all we need to do is to replace r_f with different values, c, in the maximization problem.
- The efficient frontier will consist of all points on the envelope above the minimum variance portfolio.

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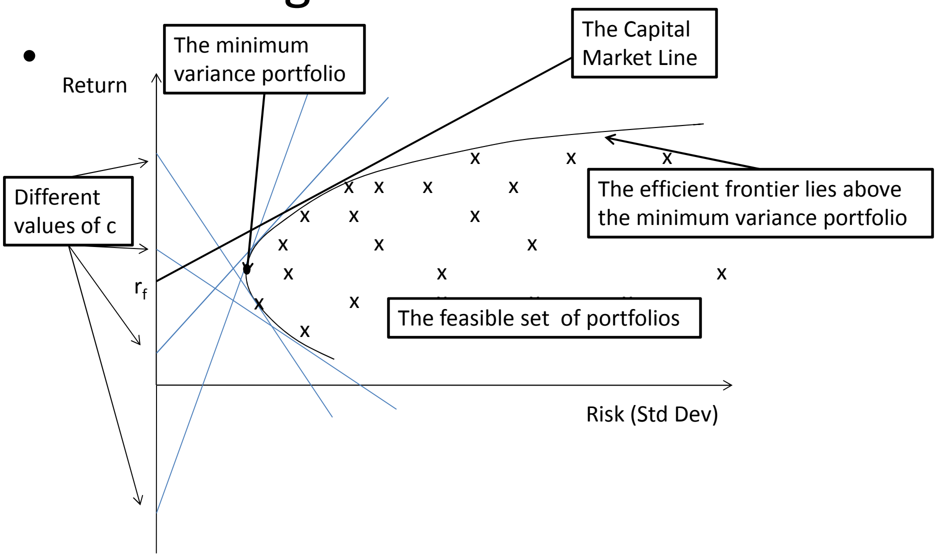
Finding the Efficient Frontier



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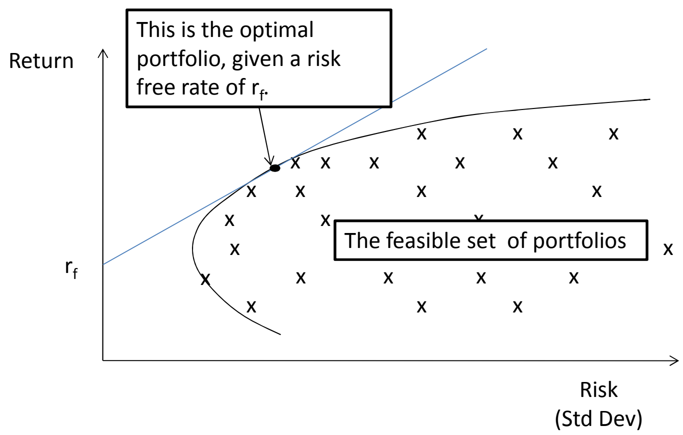
Portfolio Allocation and Excel

The Optimal Portfolio



Finding the Optimal Portfolio: Theory

- To find the optimal portfolio we must maximize expected utility:

$$\text{Max}_{\text{w}} : E\left(r_p - r_f\right) - \lambda \sigma_{r_p}^2$$

$$\rightarrow \text{Max}_{\text{w}} : w^T \left(\mu - r_f\right) - \lambda w^T S w$$

Differentiate with respect to w and set equal to zero:

$$\rightarrow \left(\mu - r_f\right) = Sz \text{ where } z = 2\lambda w$$

$$\rightarrow z = S^{-1}\left(\mu - r_f\right)$$

$$\rightarrow w = \frac{z}{1^T z}$$

3

Portfolio Allocation Inputs

- Portfolio theory focuses on expected returns and the expected variance-covariance matrix.
  - We do not observe expected returns.
  - We do not observe the expected variance-covariance matrix.

- Practitioners must make estimates for all of these inputs.

4

Portfolio Allocation Inputs

- Estimating Expected Returns
  - \- Use historical data:

$$r_{i,t} = \mu_i + \varepsilon_{i,t} \text{ where } \varepsilon_{i,t} \sim N\left(0, \sigma_i^2\right)$$

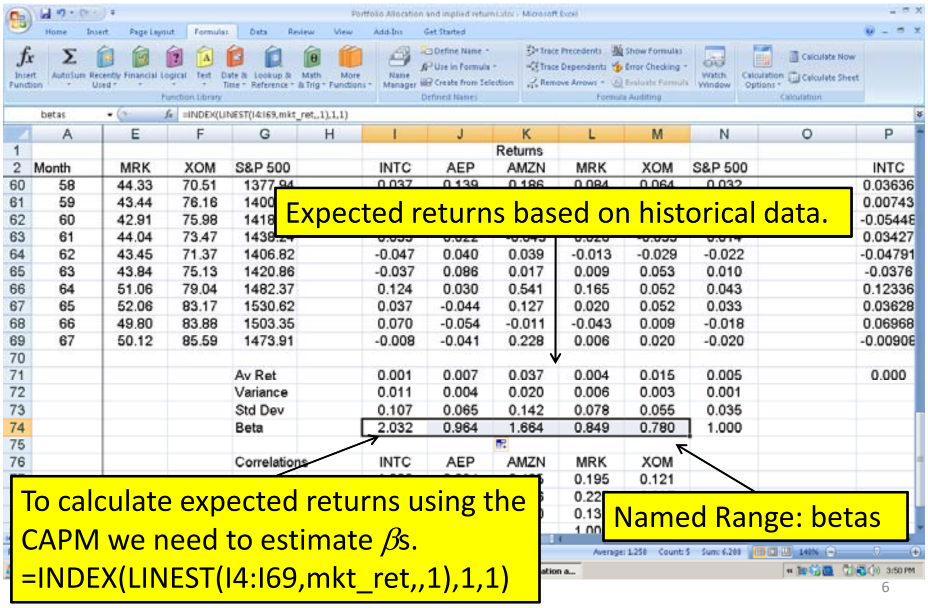
$$\rightarrow \frac{1}{N}\sum_{t=1}^{N} r_{i,t} = \frac{1}{N}\sum_{t=1}^{N} \mu_i + \frac{1}{N}\sum_{t=1}^{N} \varepsilon_{i,t} \approx \mu_i$$

  - \- Use CAPM:

$$E(r_i) = r_f + \beta_i\left(E(r_m) - r_f\right)$$

- Estimating the Variance-Covariance Matrix
  - \- Use historical data.

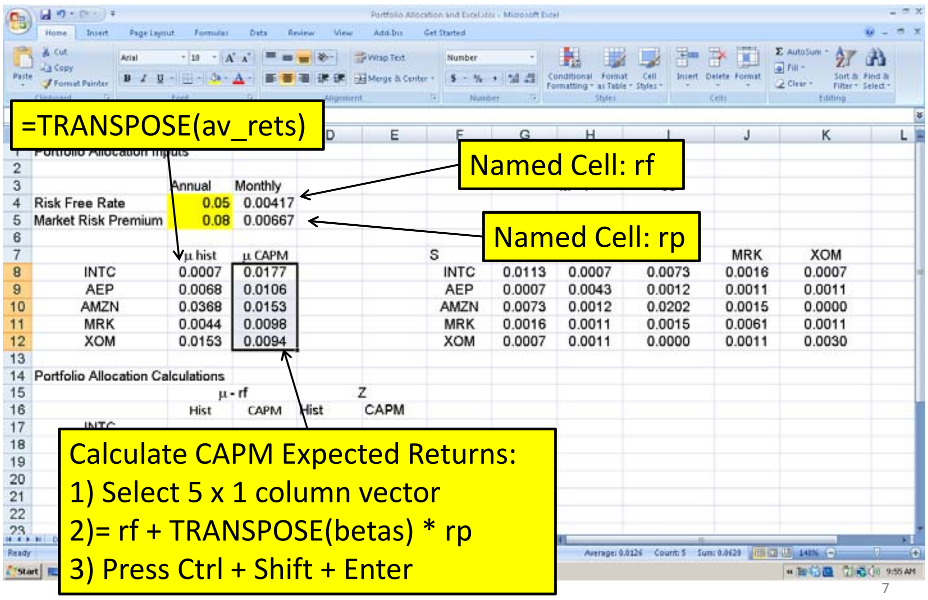
Setting up the inputs in Excel



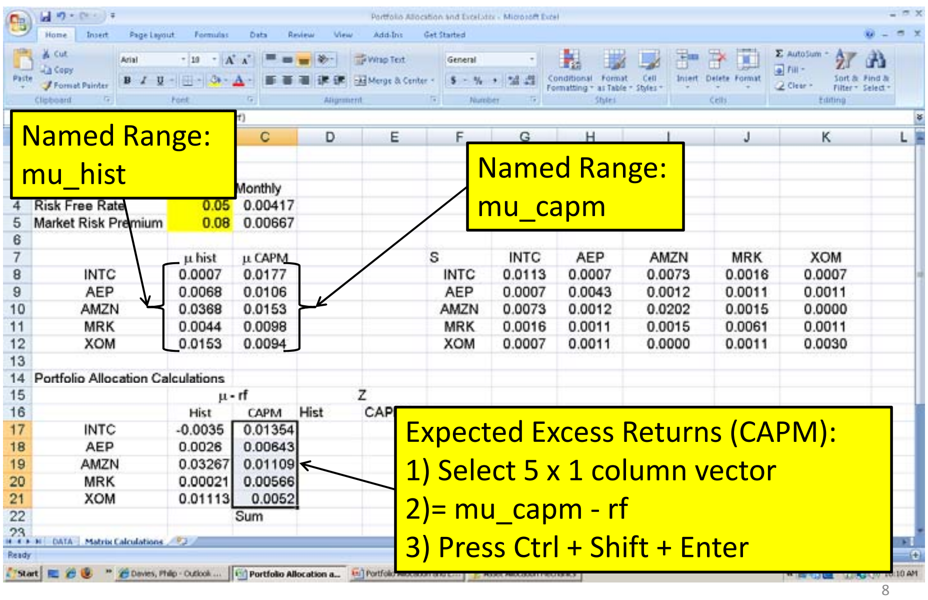
betas | =INDEX(LINEST(I4:I69,mkt_ret,,1),1,1)

	A	E	F	G	H	I	J	K	L	M	N	O	P
1								Returns					
2	Month	MRK	XOM	S&P 500		INTC	AEP	AMZN	MRK	XOM	S&P 500		INTC
60	58	44.33	70.51	1377.94		0.037	0.139	0.186	0.084	0.064	0.032		0.03636
61	59	43.44	76.16	1400 [illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		0.00743
62	60	42.91	75.98	1418 [illegible]		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		-0.05446
63	61	44.04	73.47	1438.24		[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]		0.03427
64	62	43.45	71.37	1406.82		-0.047	0.040	0.039	-0.013	-0.029	-0.022		-0.04791
65	63	43.84	75.13	1420.86		-0.037	0.086	0.017	0.009	0.053	0.010		-0.0376
66	64	51.06	79.04	1482.37		0.124	0.030	0.541	0.165	0.052	0.043		0.12336
67	65	52.06	83.17	1530.62		0.037	-0.044	0.127	0.020	0.052	0.033		0.03628
68	66	49.80	83.88	1503.35		0.070	-0.054	-0.011	-0.043	0.009	-0.018		0.06968
69	67	50.12	85.59	1473.91		-0.008	-0.041	0.228	0.006	0.020	-0.020		-0.00906
70													
71				Av Ret		0.001	0.007	0.037	0.004	0.015	0.005		0.000
72				Variance		0.011	0.004	0.020	0.006	0.003	0.001		
73				Std Dev		0.107	0.065	0.142	0.078	0.055	0.035		
74				Beta		2.032	0.964	1.664	0.849	0.780	1.000		
75													
76				Correlations		INTC	AEP	AMZN	MRK	XOM			
77						[illegible]	[illegible]	[illegible]	0.195	0.121			
78						[illegible]	[illegible]	[illegible]	0.22 [illegible]	[illegible]			
79						[illegible]	[illegible]	[illegible]	0.13 [illegible]	[illegible]			
80						[illegible]	[illegible]	[illegible]	1.00 [illegible]	[illegible]			

Setting up the inputs in Excel



Setting up the inputs in Excel

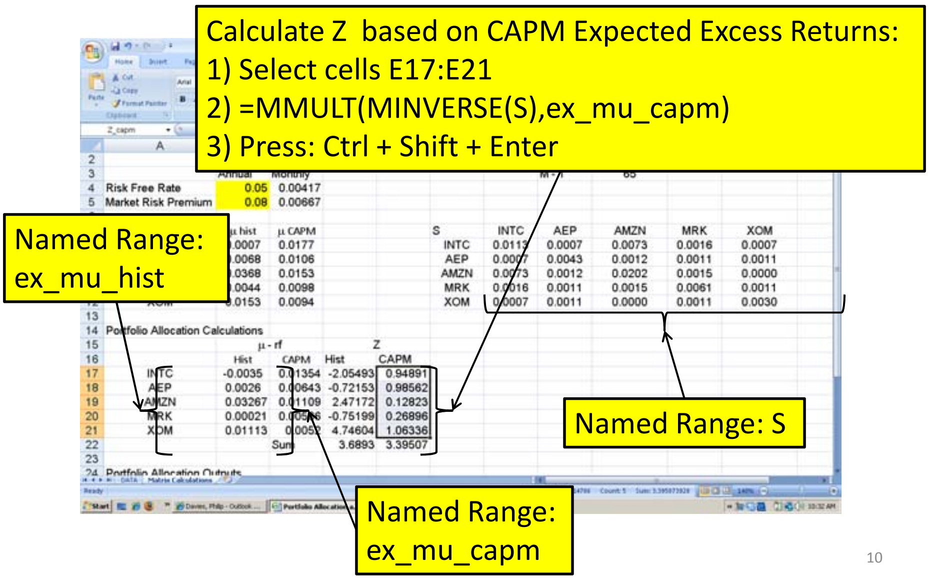


Finding the Optimal Portfolio: Step 1

- We have to solve the matrix algebra problem to find z: $z = S^{-1}\left(\mu - r_f\right)$
- To calculate the inverse of a square matrix in Excel we have to use the matrix function:
- =MINVERSE()

9

Finding the Optimal Portfolio



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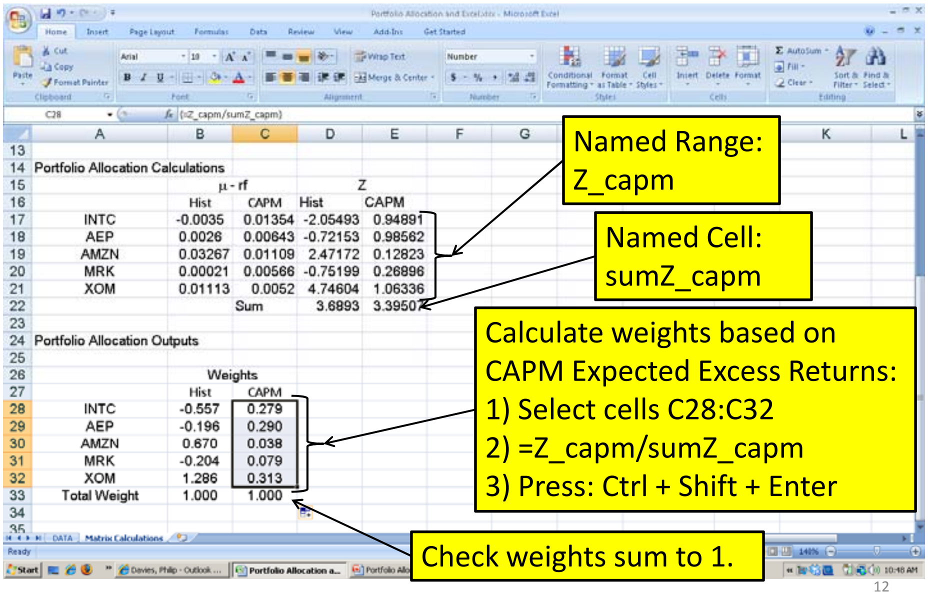
Finding the Optimal Portfolio: Step 2

- We have to add up all the elements of Z, and then divide each element of Z by the sum to calculate the portfolio weights:

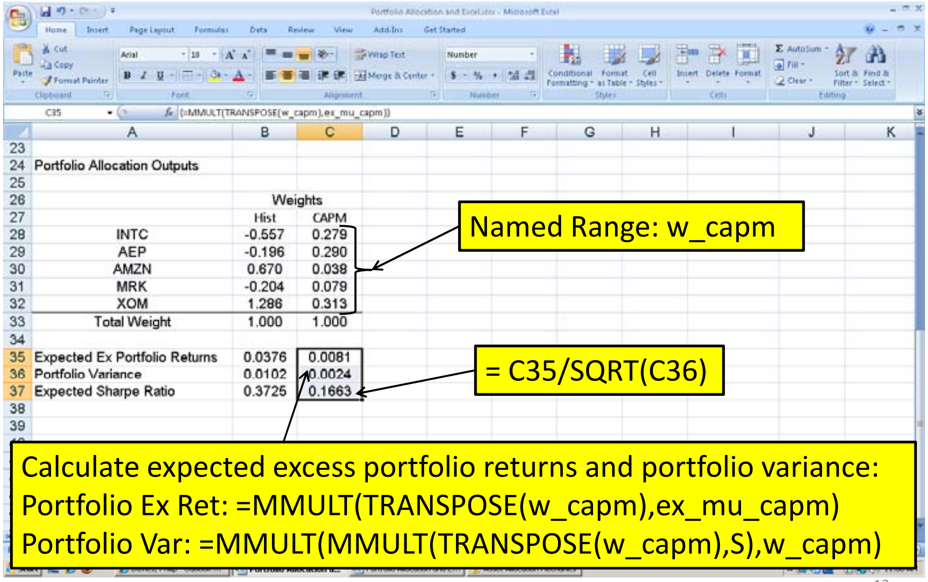
$$w = \frac{z}{1^T z}$$

Finding the Optimal Portfolio: Step 2


C28 {=Z_capm/sumZ_capm}

	A	B	C	D	E
13					
14	Portfolio Allocation Calculations				
15		μ - rf		Z	
16		Hist	CAPM	Hist	CAPM
17	INTC	-0.0035	0.01354	-2.05493	0.94891
18	AEP	0.0026	0.00643	-0.72153	0.98562
19	AMZN	0.03267	0.01109	2.47172	0.12823
20	MRK	0.00021	0.00566	-0.75199	0.26896
21	XOM	0.01113	0.0052	4.74604	1.06336
22			Sum	3.6893	3.3950
23					
24	Portfolio Allocation Outputs				
25					
26		Weights			
27		Hist	CAPM		
28	INTC	-0.557	0.279		
29	AEP	-0.196	0.290		
30	AMZN	0.670	0.038		
31	MRK	-0.204	0.079		
32	XOM	1.286	0.313		
33	Total Weight	1.000	1.000		

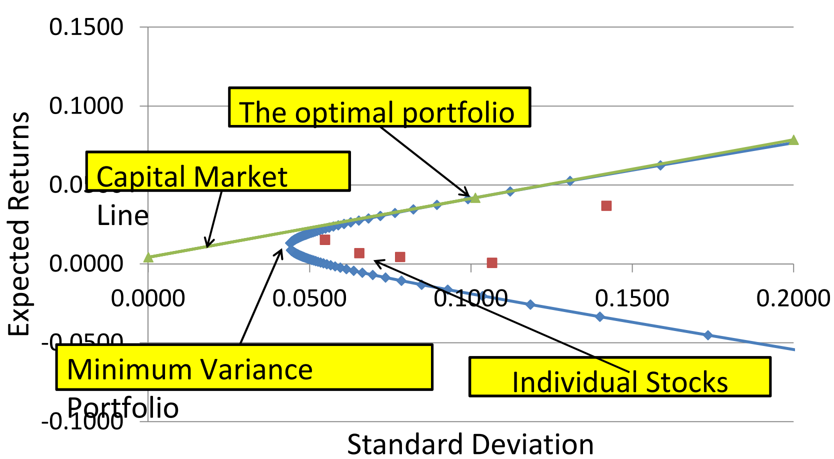
Finding the Optimal Portfolio: Step 2



	A	B	C
24	Portfolio Allocation Outputs		
26		Weights	
27		Hist	CAPM
28	INTC	-0.557	0.279
29	AEP	-0.196	0.290
30	AMZN	0.670	0.038
31	MRK	-0.204	0.079
32	XOM	1.286	0.313
33	Total Weight	1.000	1.000
35	Expected Ex Portfolio Returns	0.0376	0.0081
36	Portfolio Variance	0.0102	0.0024
37	Expected Sharpe Ratio	0.3725	0.1663

13

The Efficient Frontier (using historical returns)



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An Optimal Portfolio without Short Sales

- In the analysis so far we have assumed that short sales are allowed. This means that the weight of a stock can be negative in the optimal portfolio.
- While some institutions may be able to short sell stocks without incurring large costs, many investors cannot, or are restricted from short selling.
- How do we find the optimal portfolio when the holdings of all stocks must be greater than or equal to zero?

Incorporating Short Sales Constraints

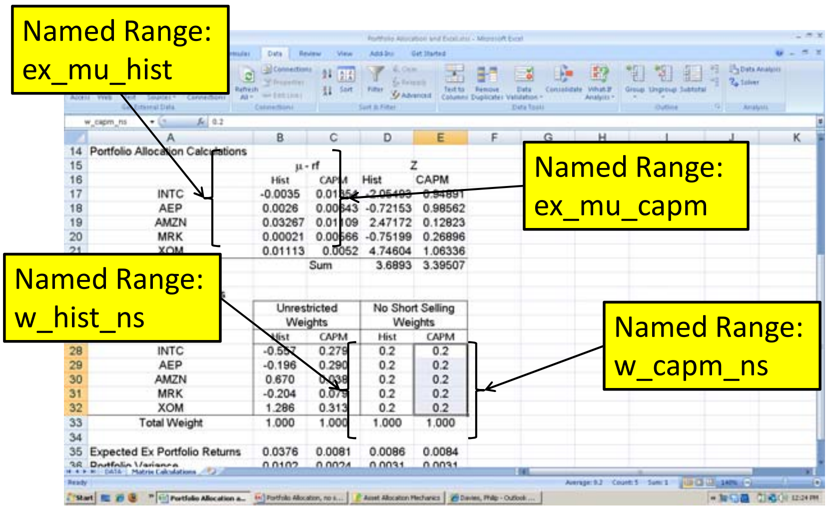
- We want to maximize expected utility. The only difference is that we have an additional constraint.
- No stocks can have negative weights in the optimal portfolio.

$$\text{Max}_w : \frac{E\left(r_p - r_f\right)}{\sigma_p}$$

Maximize the reward to risk (Sharpe) ratio.

$$\text{Subject to: } \sum_{i=1}^{N} w_i = 1 \text{ and } w_i \geq 0 \text{ for all i} = 1,\ldots,\text{N}$$

Setting up the Problem in Excel



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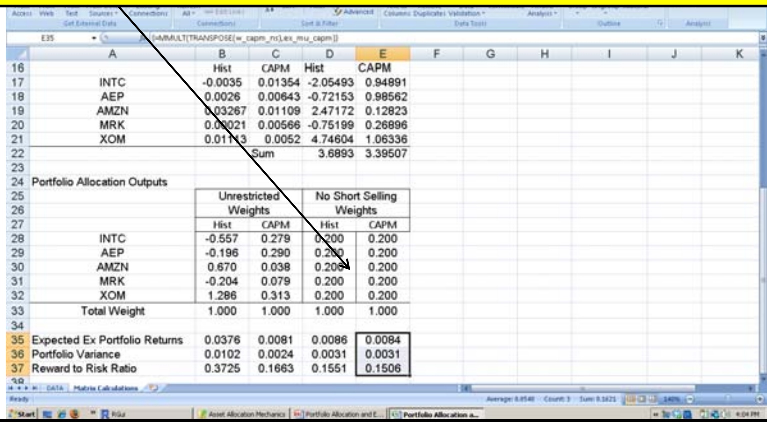
Setting up the Problem in Excel

Calculate the expected excess portfolio return, the portfolio variance, and the Sharpe ratio:
Exp Ex Port Ret: =MMULT(TRANSPOSE(w_capm_ns),ex_mu_capm)
Port Var: =MMULT(MMULT(TRANSPOSE(w_capm_ns),S), w_capm_ns)
Reward to Risk (Sharpe) Ratio: =E35/SQRT(E36)



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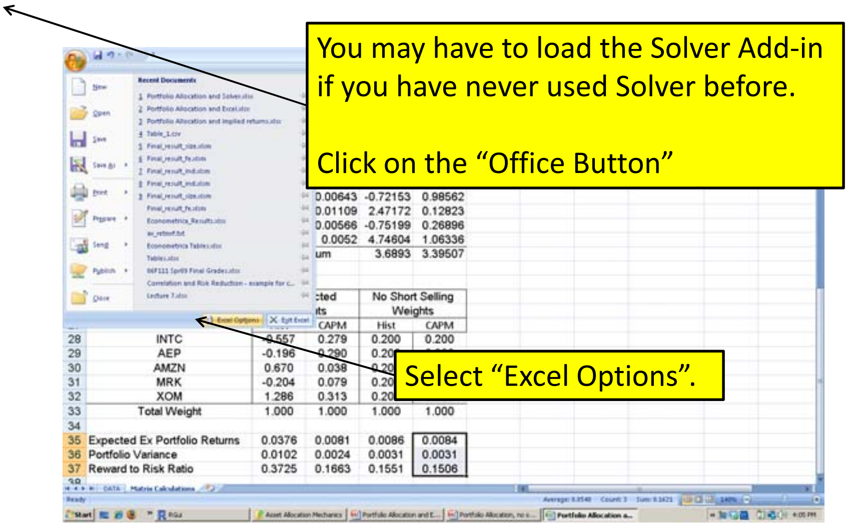
Loading Solver in Excel



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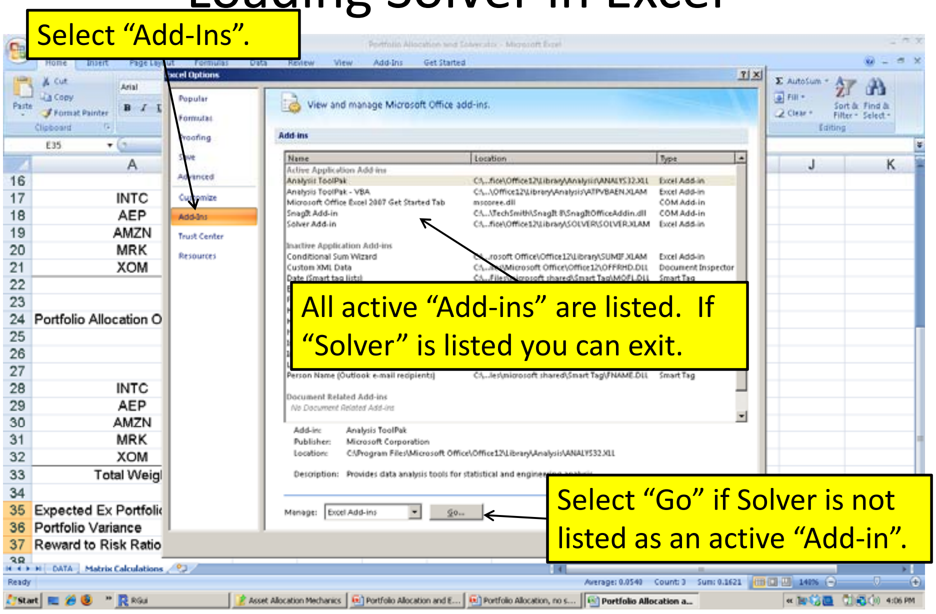
Loading Solver in Excel



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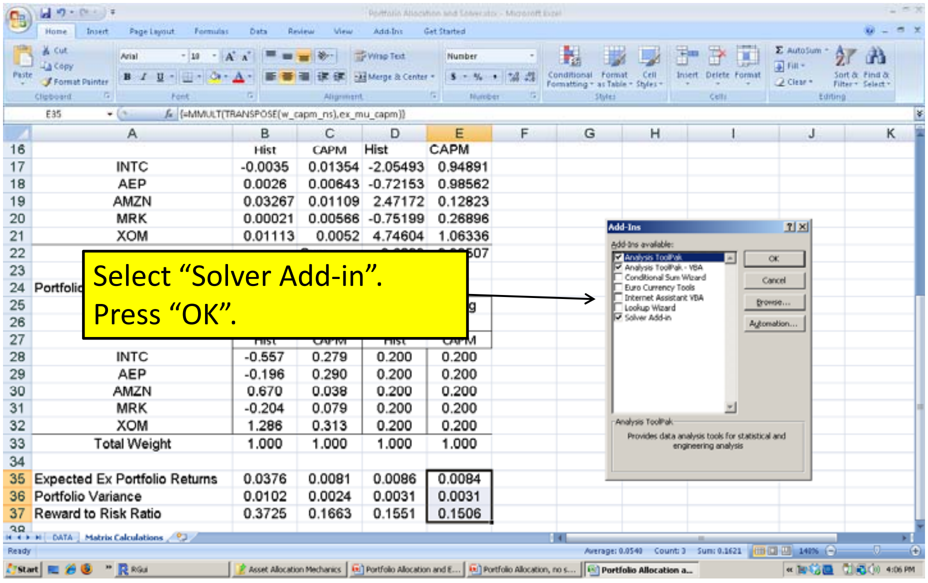
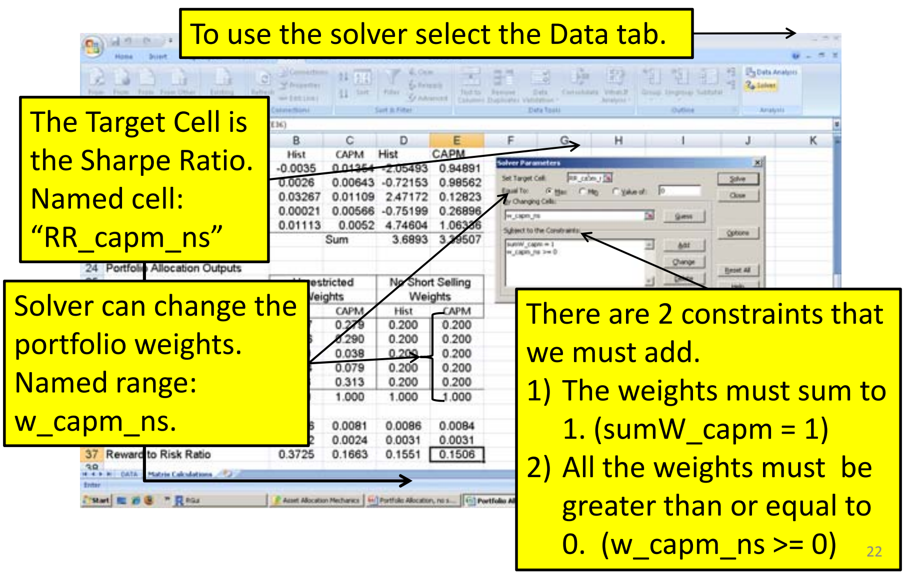
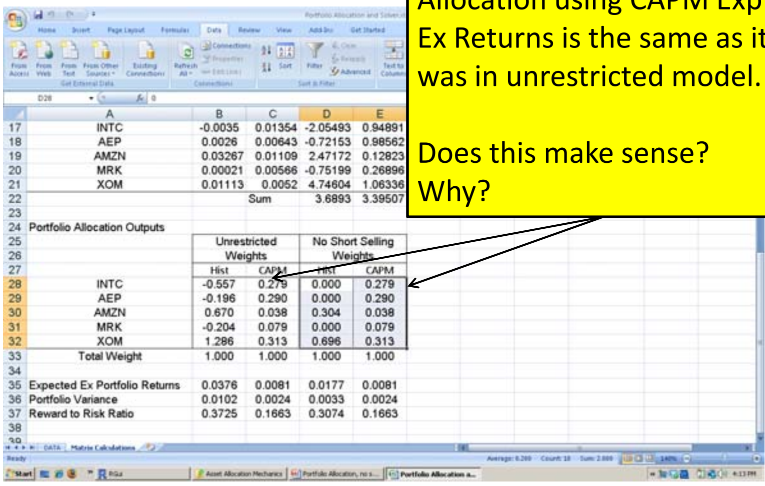
Loading Solver in Excel



Using Solver to Find the Optimal Portfolio



The Optimal Portfolio with No Short Sales



	A	B	C	D	E
17	INTC	-0.0035	0.01354	-2.05493	0.94891
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19	AMZN	0.03267	0.01109	2.47172	0.12823
20	MRK	0.00021	0.00566	-0.75199	0.26896
21	XOM	0.01113	0.0052	4.74604	1.06336
22			Sum	3.6893	3.39507
23					
24	Portfolio Allocation Outputs				
25		Unrestricted Weights		No Short Selling Weights	
27		Hist	CAPM	Hist	CAPM
28	INTC	-0.557	0.279	0.000	0.279
29	AEP	-0.196	0.290	0.000	0.290
30	AMZN	0.670	0.038	0.304	0.038
31	MRK	-0.204	0.079	0.000	0.079
32	XOM	1.286	0.313	0.696	0.313
33	Total Weight	1.000	1.000	1.000	1.000
34					
35	Expected Ex Portfolio Returns	0.0376	0.0081	0.0177	0.0081
36	Portfolio Variance	0.0102	0.0024	0.0033	0.0024
37	Reward to Risk Ratio	0.3725	0.1663	0.3074	0.1663

Allocation using CAPM Exp Ex Returns is the same as it was in unrestricted model.

Does this make sense? Why?

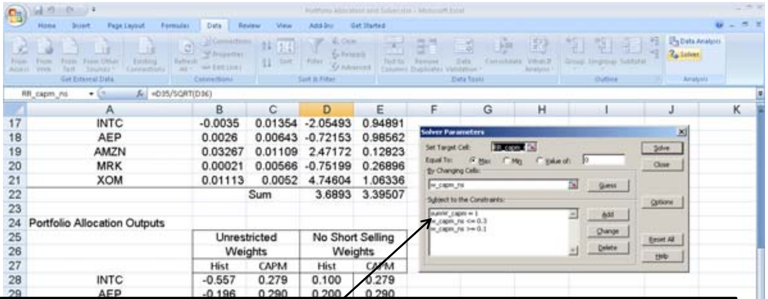
23

Incorporating different constraints

- Many funds have investment "charters" that provide guidelines on what percentage of funds can be allocated to a specific stock or industry.
- Suppose we have the following constraints:
- - Hold at least 10% in each stock.
- - Hold no more than 30% in any stock.
- How do we find the optimal portfolio with these extra constraints?

24

Incorporating different constraints



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23					
24	Portfolio Allocation Outputs				
25		Unrestricted		No Short Selling	
26		Weights		Weights	
27		Hist	CAPM	Hist	CAPM
28	INTC	-0.557	0.279	0.100	0.279
29	AEP	-0.196	0.290	0.200	0.290

There are 3 constraints that we must add.
1) The weights must sum to 1. (sumW_capm = 1)
2) All the weights must be greater than or equal to 10%. (w_capm_ns >= 0.10)
3) All the weights must be less than or equal to 30%. (w_capm_ns <= 0.30)

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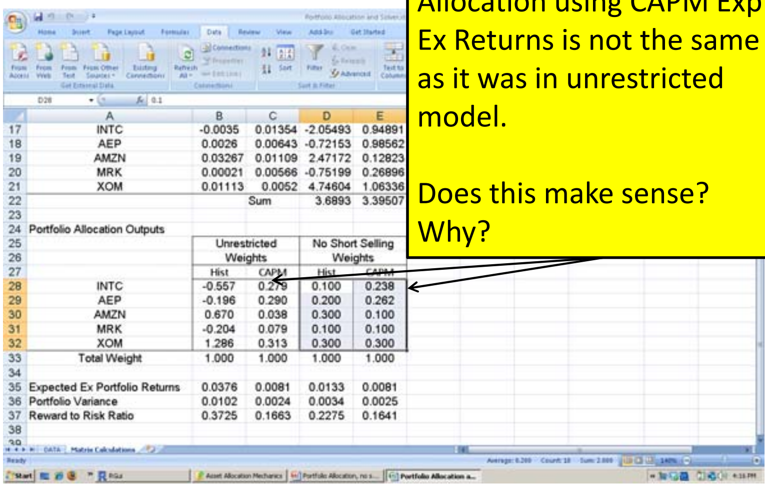
Incorporating different constraints



	A	B	C	D	E
17	INTC	-0.0035	0.01354	-2.05493	0.94891
18	AEP	0.0026	0.00643	-0.72153	0.98562
19	AMZN	0.03267	0.01109	2.47172	0.12823
20	MRK	0.00021	0.00566	-0.75199	0.26896
21	XOM	0.01113	0.0052	4.74604	1.06336
22			Sum	3.6893	3.39507
23					
24	Portfolio Allocation Outputs				
25		Unrestricted		No Short Selling	
26		Weights		Weights	
27		Hist	CAPM	Hist	CAPM
28	INTC	-0.557	0.279	0.100	0.238
29	AEP	-0.196	0.290	0.200	0.262
30	AMZN	0.670	0.038	0.300	0.100
31	MRK	-0.204	0.079	0.100	0.100
32	XOM	1.286	0.313	0.300	0.300
33	Total Weight	1.000	1.000	1.000	1.000
34					
35	Expected Ex Portfolio Returns	0.0376	0.0081	0.0133	0.0081
36	Portfolio Variance	0.0102	0.0024	0.0034	0.0025
37	Reward to Risk Ratio	0.3725	0.1663	0.2275	0.1641
38					

Allocation using CAPM Exp Ex Returns is not the same as it was in unrestricted model.

Does this make sense? Why?

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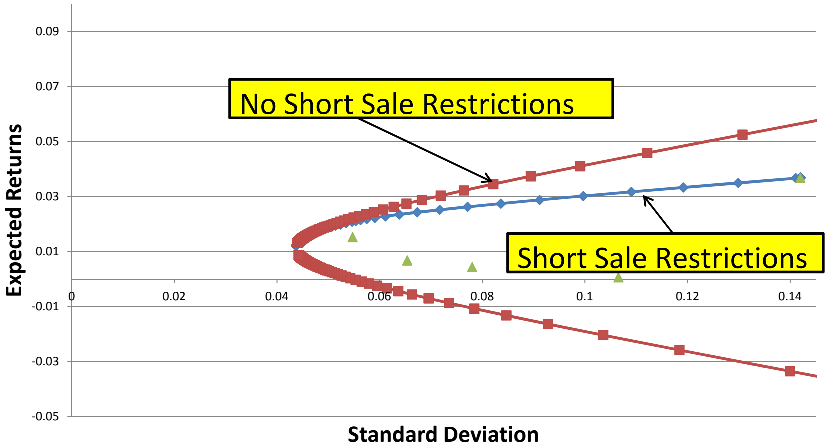
The Different Efficient Frontiers

Why does the efficient frontier in which short sales are allowed plot above the efficient frontier in which short sales are not allowed?



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