Correlation and Risk Reduction

Portfolio risk and return

- Expected return on a portfolio is a weighted average of expected returns on the securities in the portfolio:

$$\text{Portfolio rate of return } (r_p) = \sum_{i=1}^{M} w_i (r_i)$$

  - - w_i is the fractional share the i^{th} asset represents of the portfolio's total value
  - - r_i is the i^{th} asset's return.

- The risk of the portfolio is NOT a straightforward weighted average of the variances of the securities in the portfolio.

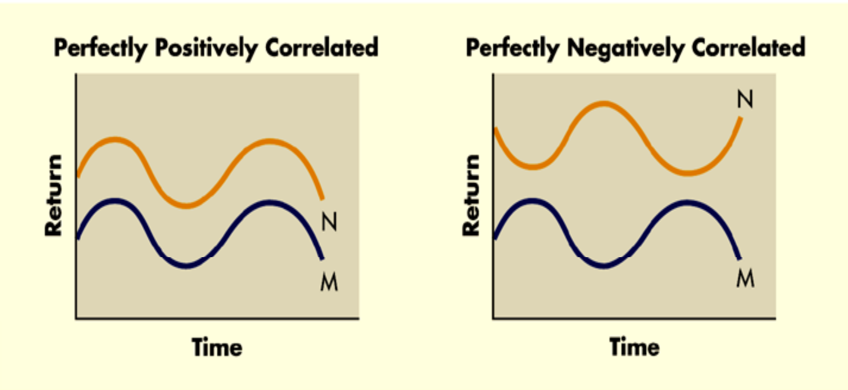
- We have to take into account correlation.

What is correlation?

- The strength and direction of a linear relation between two variables.
  - - Correlation ranges between -1 and 1
  - - Correlation > 0: Returns on two stocks tend to move in the same direction.
  - - Correlation = 0: Returns on two stocks are independent.
  - - Correlation < 0: Returns on two stocks tend to move in opposite directions.

Examples of Correlation

- Patterns of correlation between the returns of two stocks (M and N).



Source: Addison-Wesley, 2001

An Example: 2 risky assets

$E(r_A)$= 9% σ_A = 12% and $E(r_B)$= 18% σ_B = 25% and $\rho_{A,B}$= -0.5

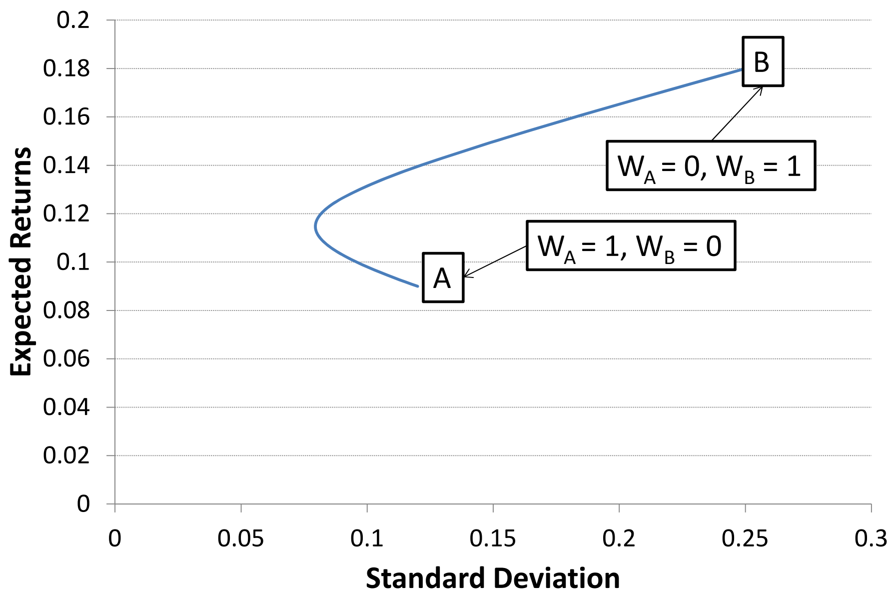
To calculate portfolio risk and return, use the formulae:

$$E(r_P) = w_A E(r_A) + w_B E(r_B) \qquad \text{where } w_A + w_B = 1.$$

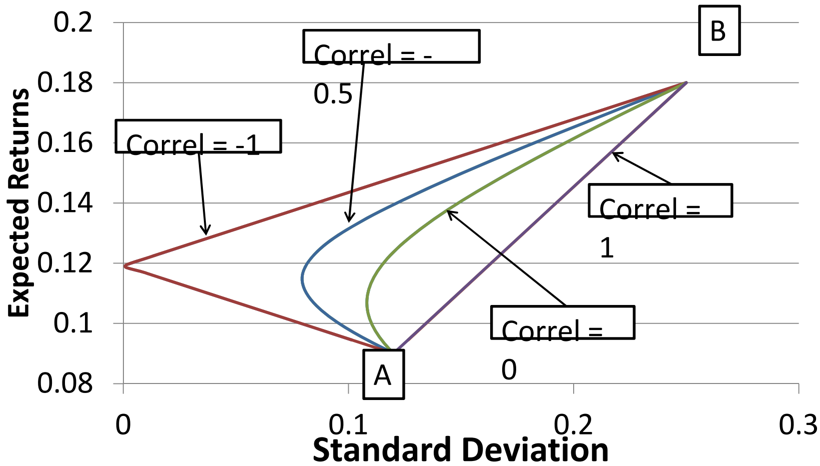
$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + \left(2 w_A \sigma_A w_B \sigma_B \rho_{A,B}\right)$$

Calculate portfolio risk and return if w_A=0.25 and w_B=0.75:

Combining Assets A and B



The Importance of Correlation



Understanding Correlation

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + \left(2w_A\sigma_A w_B\sigma_B\rho_{A,B}\right)$$

$$\text{If } \rho_{A,B} = 1 \rightarrow \sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + \left(2w_A\sigma_A w_B\sigma_B\right)$$

Recall that: $a^2 + b^2 + 2ab = (a + b)^2$

$$\text{If } \rho_{A,B} = 1 \rightarrow \sigma_p^2 = \left(w_A\sigma_A + w_B\sigma_B\right)^2$$

$$\text{If } \rho_{A,B} = 1 \rightarrow \sigma_p = \sqrt{\left(w_A\sigma_A + w_B\sigma_B\right)^2} = w_A\sigma_A + w_B\sigma_B$$

$$\text{If } \rho_{A,B} < 1 \rightarrow \sigma_p < \sqrt{\left(w_A\sigma_A + w_B\sigma_B\right)^2} = w_A\sigma_A + w_B\sigma_B$$

Portfolio Risk and Correlations

- Because returns on assets held in a portfolio are (inevitably) **not perfectly correlated**, portfolio standard deviation is NOT just a weighted average of the standard deviations for each firm.
- Holding a portfolio of stocks helps to smooth out firm-specific risk events in a portfolio.
- The positive performance of some investments will neutralize the negative performance of others.
- The benefits of diversification only hold if the stocks in the portfolio are not perfectly correlated.

Three assets or more

- Consider first a three-asset case

$$\sigma_P^2 = X_1^2\sigma_1^2 + X_2^2\sigma_2^2 + X_3^2\sigma_3^2 + 2X_1X_2\sigma_{12} + 2X_1X_3\sigma_{13} + 2X_2X_3\sigma_{23}$$

- General expression for the variance of a portfolio yields

$$\sigma_P^2 = \sum_{j=1}^{N}\left(X_j^2\sigma_j^2\right) + \sum_{j=1}^{N}\sum_{\substack{k=1 \\ k\neq j}}^{N}\left(X_jX_k\sigma_{jk}\right)$$

- Consider equal investment in *N assets*

$$\sigma_P^2 = \sum_{j=1}^{N}(1/N)^2\sigma_j^2 + \sum_{j=1}^{N}\sum_{\substack{k=1 \\ k\neq j}}^{N}(1/N)(1/N)\sigma_{jk}$$

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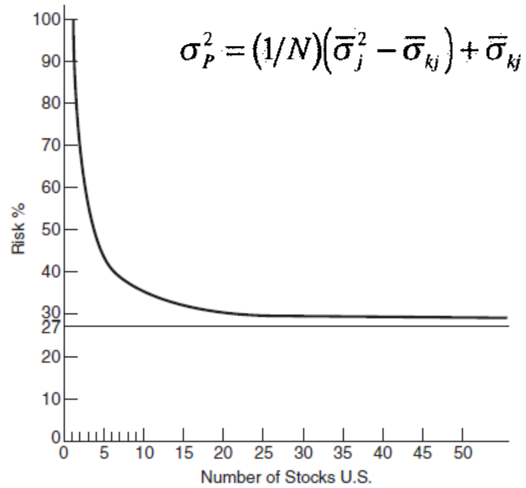
The effect of number of securities on risk of the portfolio in the United States (1975).

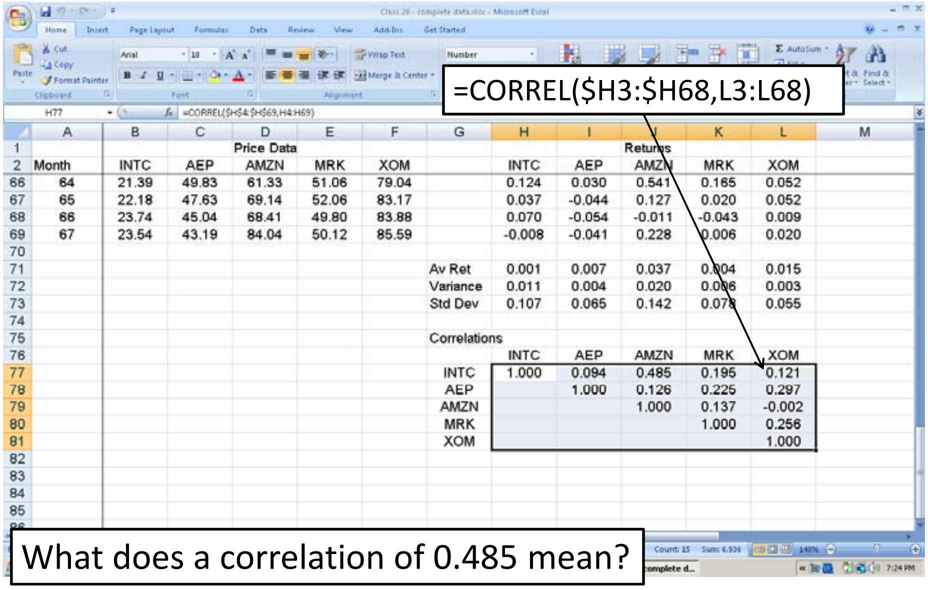


How to derive the equation $\sigma_P^2 = (1/N)\left(\bar{\sigma}_j^2 - \bar{\sigma}_{kj}\right) + \bar{\sigma}_{kj}$

from the general N assets equation?

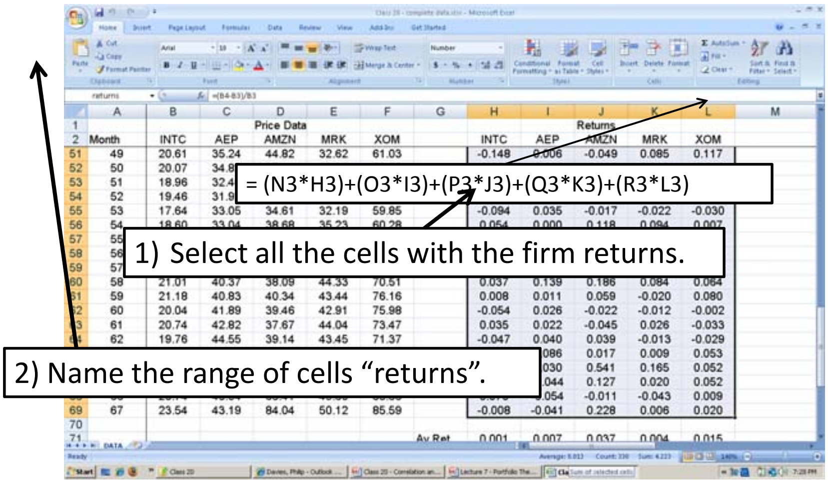
$$\sigma_P^2 = \sum_{j=1}^{N} (1/N)^2 \sigma_j^2 + \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq j}}^{N} (1/N)(1/N)\sigma_{jk}$$

Correlations

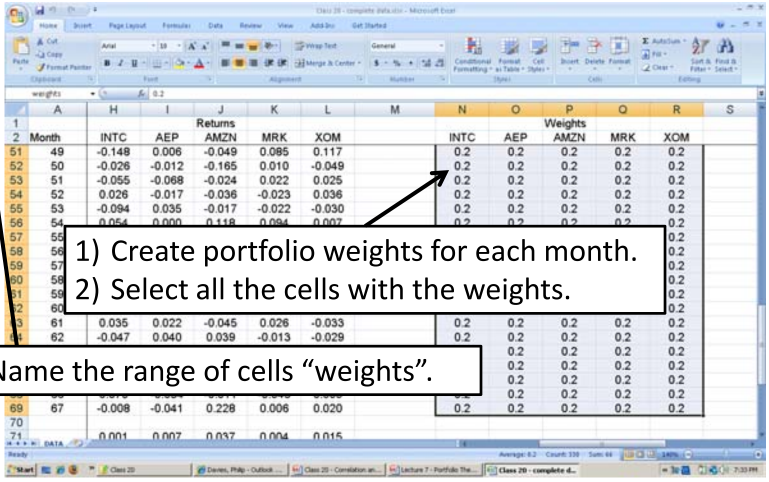


	A	B	C	D	E	F	G	H	I	J	K	L
1				Price Data						Returns		
2	Month	INTC	AEP	AMZN	MRK	XOM		INTC	AEP	AMZN	MRK	XOM
66	64	21.39	49.83	61.33	51.06	79.04		0.124	0.030	0.541	0.165	0.052
67	65	22.18	47.63	69.14	52.06	83.17		0.037	-0.044	0.127	0.020	0.052
68	66	23.74	45.04	68.41	49.80	83.88		0.070	-0.054	-0.011	-0.043	0.009
69	67	23.54	43.19	84.04	50.12	85.59		-0.008	-0.041	0.228	0.006	0.020
71							Av Ret	0.001	0.007	0.037	0.004	0.015
72							Variance	0.011	0.004	0.020	0.006	0.003
73							Std Dev	0.107	0.065	0.142	0.078	0.055
75							Correlations					
76								INTC	AEP	AMZN	MRK	XOM
77							INTC	1.000	0.094	0.485	0.195	0.121
78							AEP		1.000	0.126	0.225	0.297
79							AMZN			1.000	0.137	-0.002
80							MRK				1.000	0.256
81							XOM					1.000

Portfolio Returns: Step 1



Portfolio Returns: Step 2

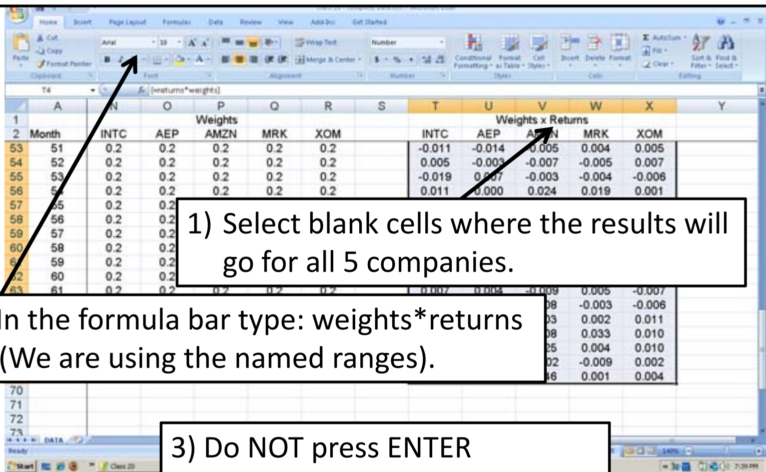


1) Create portfolio weights for each month.
2) Select all the cells with the weights.

3) Name the range of cells "weights".

Portfolio Returns: Step 3

Aim: Calculate the returns for INTC x Weights for INTC. Do the same for other 4 companies.

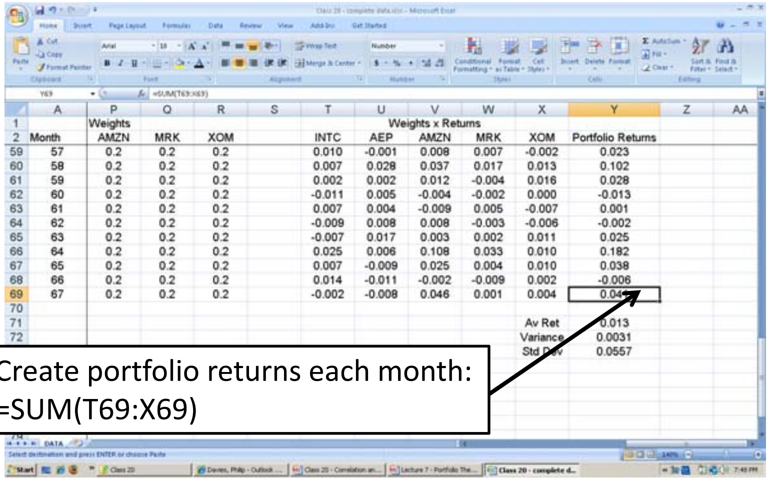


1) Select blank cells where the results will go for all 5 companies.

2) In the formula bar type: weights*returns (We are using the named ranges).

3) Do NOT press ENTER
4) Press CTRL + SHIFT + ENTER

Portfolio Returns: Step 4



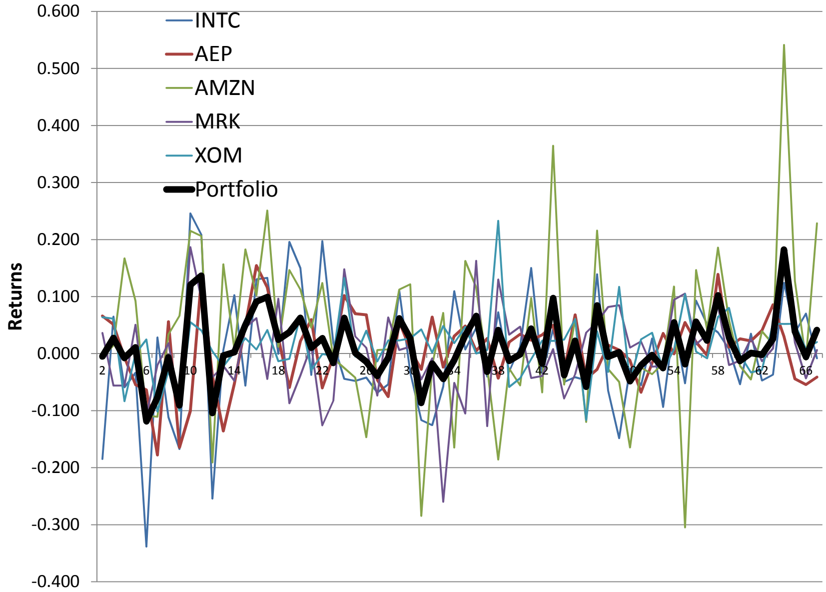
	A	P	Q	R	S	T	U	V	W	X	Y
1		Weights					Weights x Returns				
2	Month	AMZN	MRK	XOM		INTC	AEP	AMZN	MRK	XOM	Portfolio Returns
59	57	0.2	0.2	0.2		0.010	-0.001	0.008	0.007	-0.002	0.023
60	58	0.2	0.2	0.2		0.007	0.028	0.037	0.017	0.013	0.102
61	59	0.2	0.2	0.2		0.002	0.002	0.012	-0.004	0.016	0.028
62	60	0.2	0.2	0.2		-0.011	0.005	-0.004	-0.002	0.000	-0.013
63	61	0.2	0.2	0.2		0.007	0.004	-0.009	0.005	-0.007	0.001
64	62	0.2	0.2	0.2		-0.009	0.008	0.008	-0.003	-0.006	-0.002
65	63	0.2	0.2	0.2		-0.007	0.017	0.003	0.002	0.011	0.025
66	64	0.2	0.2	0.2		0.025	0.006	0.108	0.033	0.010	0.182
67	65	0.2	0.2	0.2		0.007	-0.009	0.025	0.004	0.010	0.038
68	66	0.2	0.2	0.2		0.014	-0.011	-0.002	-0.009	0.002	-0.006
69	67	0.2	0.2	0.2		-0.002	-0.008	0.046	0.001	0.004	0.04[illegible]
70											
71										Av Ret	0.013
72										Variance	0.0031
										Std Dev	0.0557

Create portfolio returns each month:
=SUM(T69:X69)

Portfolio Return and Risk

- The average portfolio return was 1.3% per month.
  - \- A weighted average of the individual stock returns.
- The portfolio standard deviation is only 5.6% per month, while the average standard deviation across all 5 stocks is 8.9.
- Why is the portfolio risk lower than 8.9%?
  - \- The returns of the 5 stocks are not perfectly positively correlated.

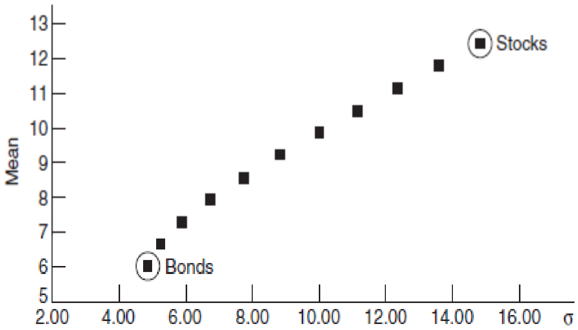
Firm and Portfolio Returns



Two Concluding Examples

- Bond Stock Allocation

$$\bar{R}_{S\&P} = 12.5\% \quad \sigma_{S\&P} = 14.9\% \quad \rho_{S\&P\,B} = 0.45$$
$$\bar{R}_B = 6\% \quad \sigma_B = 4.8\%$$



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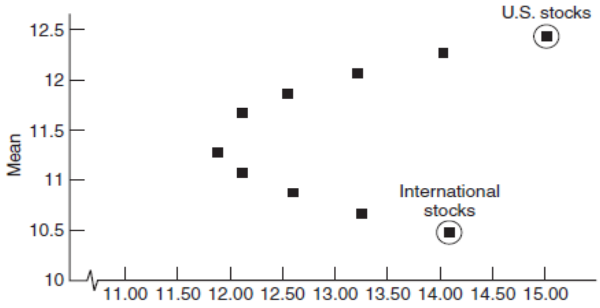
Domestic Foreign Allocation

$\bar{R}_{S\&P} = 12.5\%$ $\sigma_{S\&P} = 14.9\%$ $\rho_{S\&P\ int} = 0.33$

$\bar{R}_{int} = 10.5\%$ $\sigma_{int} = 14.0\%$



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The minimum variance portfolio

- Two asset case
- How to derive it?
- Is the negative correlation coefficient necessary?

Portfolios and Matrices

Portfolios and Matrices

- Previously we calculated portfolio returns by multiplying stock returns by their weights in the portfolio, and summing up these products.
  - \- Calculation takes a long time for 10 stocks, let alone 10,000.
  - \- To overcome this problem we will use matrix algebra.
- An r x c matrix is a matrix which has r rows and c columns.

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

  - \- If a matrix has only 1 row, it is called a row vector. Similarly if a matrix only has 1 column, it is called a column vector.

Matrix Notation: A Review

- Consider matrix A: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

- The transpose of A, A^T:

$$A^T =$$

- If we take the transpose of a matrix, the rows of the original matrix become the columns of the transposed matrix.

Matrix Addition and Subtraction

- You can only add or subtract matrices that have the same dimensions.
- Consider matrix A: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Each component of a matrix is referred to as an element of the matrix.

$$A + A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} =$$

- Now suppose we have matrix B: $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- We cannot calculate A + B or A – B.

Matrix Multiplication

- Consider matrices P and Q:

$$P = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$$

General Matrix Multiplication

- Let D be a m x s matrix, and E and s x k matrix.
- We want to calculate F = DE.
- The dimensions of the multiplication are:
    - m x s by s x k

You can only multiply 2 matrices if the number of columns in the first matrix is equal to the number of rows in the second matrix.

General Matrix Multiplication

- Let D be a m x s matrix, and E and s x k matrix.
- We want to calculate F = DE.
- The dimensions of the multiplication are:
  - m x s by s x k

Matrix F will have the following dimensions: The number of rows will be equal to the number of rows in the first matrix (m), while the number of columns will be equal to the number of columns in the second matrix (k).

An Example

$$D = \begin{bmatrix} 1 & 2 \end{bmatrix} \text{and } E = \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

- Can we do the matrix multiplication DE?
-
-

Matrix Inversion

- Two important properties of numbers.
- 1) a x 1 = a and 1 x a = a, for any number a.
- 2) a^{-1} x a = 1 and a x a^{-1} = 1.

Recall: $a^{-1} = 1/a$

- These properties can be extended to square matrices.
- 1) AI = A and IA = A, where $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

The Identity Matrix

- 2) $A^{-1}A = I$ and $AA^{-1} = I$

A^{-1} denotes the inverse of A

Calculating Portfolio Returns and Variance

- The Two Asset Case:

$$E(r_P) = w_A E(r_A) + w_B E(r_B) \qquad \text{where } w_A + w_B = 1$$

$$\sigma_p^2 = w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2\left(w_A \sigma_A\right)\left(w_B \sigma_B\right) \rho_{A,B}$$

Correlation

A General Formula

- New Notation:

$$Var(r_A) = \sigma_A^2 = \sigma_{AA}$$

$$Cov(r_A, r_B) = \sigma_{AB}$$

$$r_p = \sum_{i=1}^{M} w_i r_i$$

$$\sigma_p^2 = \sum_{i=1}^{M} w_i^2 \sigma_{ii} + 2\sum_{i=1}^{M} \sum_{j=i+1}^{M} w_i w_j Cov(r_i, r_j)$$

$$\rightarrow \sigma_p^2 = \sum_{i=1}^{M} \sum_{j=1}^{M} w_i w_j \sigma_{ij}$$

Does the general formula work?

- The Two Asset Case:

Portfolios and Matrices

- Expected portfolio returns and risk:

$$E(r_p) = \sum_{i=1}^{N} w_i E(r_i) = w^T \mu$$

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = w^T S w$$

Does the general formula work?

- The Two Asset Case:

The Variance-Covariance Matrix, S

- The variance for a stock's returns is calculated as,

$$\sigma_{ii} = \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{i,t} - \bar{r}_i\right)\left(r_{i,t} - \bar{r}_i\right)$$

- where $\bar{r}_i$ denotes average returns for stock i.
- The covariance of the return for asset i and asset j is calculated as,

$$\sigma_{ij} = \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{i,t} - \bar{r}_i\right)\left(r_{j,t} - \bar{r}_j\right)$$

- How can we calculate these variances and covariances efficiently?

The Variance-Covariance Matrix, S

Let A = Matrix of demeaned returns.

M denotes time
N denotes the total number of stocks

$$\begin{array}{cc} & \text{Stocks} \\ \text{Time} & A = \begin{bmatrix} r_{1,1} - \bar{r}_1 & \cdots & r_{N,1} - \bar{r}_N \\ r_{1,2} - \bar{r}_1 & \cdots & r_{N,2} - \bar{r}_N \\ \vdots & & \vdots \\ r_{1,M} - \bar{r}_1 & \cdots & r_{N,M} - \bar{r}_N \end{bmatrix} \end{array} \rightarrow A^T = \begin{bmatrix} r_{1,1} - \bar{r}_1 & r_{1,2} - \bar{r}_1 & \cdots & r_{1,M} - \bar{r} \\ \vdots & \vdots & & \vdots \\ r_{N,1} - \bar{r}_N & r_{N,2} - \bar{r}_N & \cdots & r_{N,M} - \bar{r}_N \end{bmatrix}$$

$$S = \frac{A^T A}{M-1}$$

$$S = \begin{bmatrix} \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{1,t} - \bar{r}_1\right)\left(r_{1,t} - \bar{r}_1\right) & \cdots & \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{1,t} - \bar{r}_1\right)\left(r_{N,t} - \bar{r}_N\right) \\ \vdots & & \vdots \\ \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{N,t} - \bar{r}_N\right)\left(r_{1,t} - \bar{r}_1\right) & \cdots & \frac{1}{M-1}\sum_{t=1}^{M}\left(r_{N,t} - \bar{r}_N\right)\left(r_{N,t} - \bar{r}_N\right) \end{bmatrix}$$

The Two Stock Example

Portfolios and Matrices

- We have monthly returns data for 5 stocks: INTC, AEP, AMZN, MRK, XOM.
- Assume that the average returns, and the variance covariance matrix measured over the past 5 years are good estimates of expected future returns, and the future variance-covariance matrix.
- Aim: Construct an equally weighted portfolio of all 5 stocks and calculate expected returns and risk.

Portfolios and Matrices

- What do we need to calculate?
- A 5 x 1 column vector of expected returns for the 5 stocks, called μ.
- A 5 x 1 column vector of portfolio weights for the 5 stocks, called w.
- A 5 x 5 variance-covariance matrix for the 5 stocks, called S.

Portfolios and Matrices

- Expected portfolio returns and risk:

$$E(r_p) = \sum_{i=1}^{N} w_i E(r_i) = w^T \mu$$

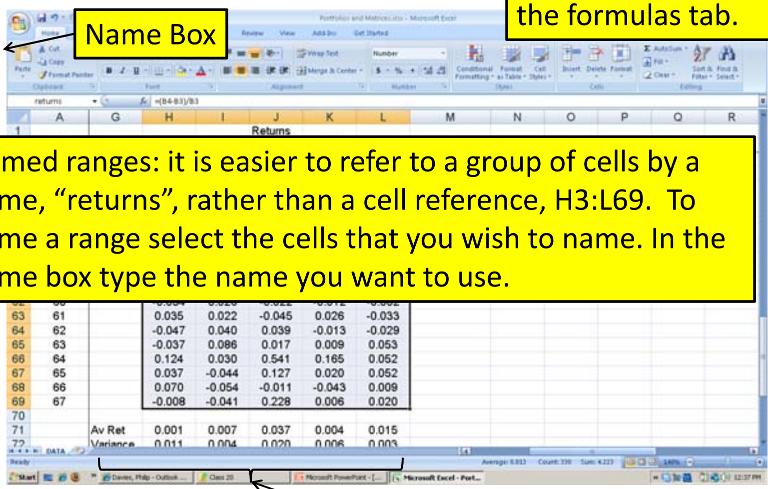
$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_{ij} = w^T S w$$

- How do we do this in Excel?

Excel Matrices Trick: N

To edit, change, or delete names, click on the formulas tab.

Name Box



Named ranges: it is easier to refer to a group of cells by a name, “returns”, rather than a cell reference, H3:L69. To name a range select the cells that you wish to name. In the name box type the name you want to use.

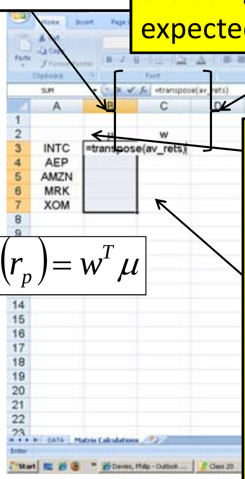
Name the average return row vector “av_rets”.

Expected Portfolio Returns

I have named a range “mu”

We want a column vector of expected returns, μ. We are using average historical returns as our estimate of expected returns.

I have named a range “weights”



$$E(r_p) = w^T \mu$$

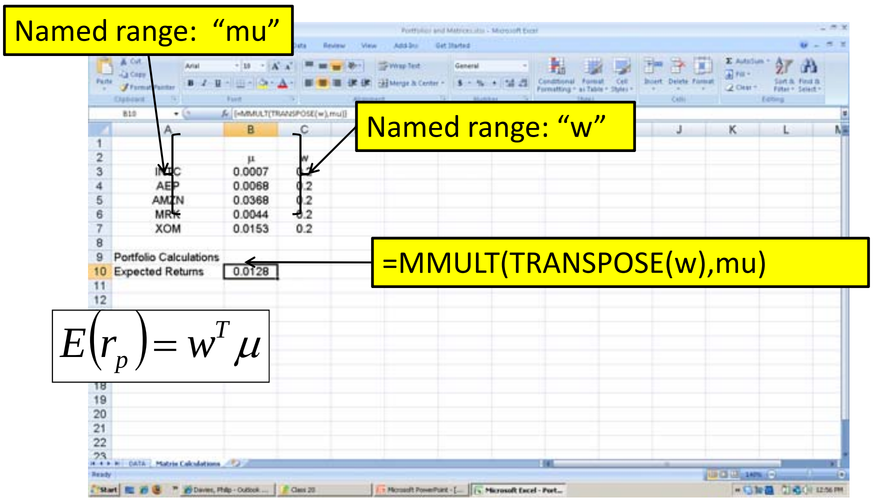
Select 5 cells where expected returns will be:

=TRANSPOSE(av_rets)

Do NOT press enter when you have typed this. To tell Excel that the formula involves matrices you have to press: Ctrl + Shift + Enter

By pressing Ctrl + Shift + Enter Excel will automatically insert curly brackets around the entire expression in the formula bar

Expected Portfolio Returns



$$E(r_p) = w^T \mu$$

The Variance-Covariance Matrix, S

- To calculate portfolios which maximize returns while minimizing risks, we must calculate the variance-covariance matrix from stock return data.
- The variance for a stock's returns is calculated as,

$$\sigma_{ii} = \frac{1}{M-1} \sum_{t=1}^{M} \left(r_{i,t} - \bar{r}_i\right)\left(r_{i,t} - \bar{r}_i\right)$$

- where $\bar{r}_i$ denotes average returns for stock i.
- The covariance of the return for asset i and asset j is calculated as,
- $$\sigma_{ij} = \frac{1}{M-1} \sum_{t=1}^{M} \left(r_{i,t} - \bar{r}_i\right)\left(r_{j,t} - \bar{r}_j\right)$$
- How can we calculate these variances and covariances efficiently?

The Variance-Covariance Matrix, S

Let A = Matrix of demeaned returns.

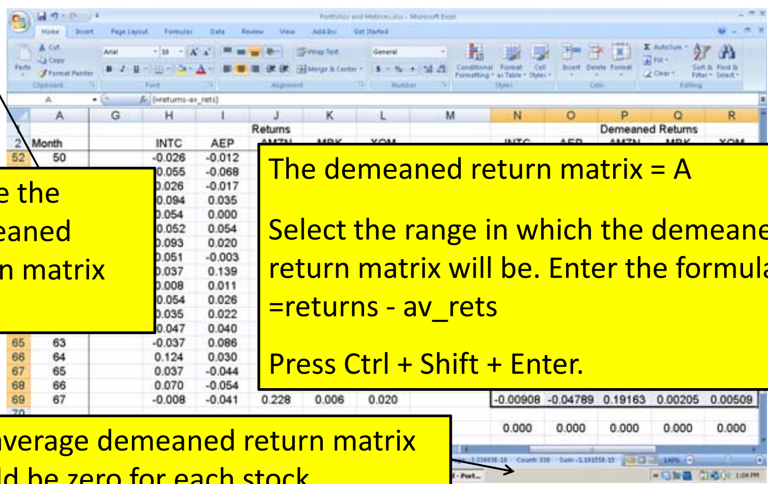
M denotes time
N denotes the total number of stocks

$$
\begin{array}{cc}
 & \text{Stocks} \\
\text{Time} & A=\begin{bmatrix} r_{1,1}-\bar{r}_1 & \cdots & r_{N,1}-\bar{r}_N \\ r_{1,2}-\bar{r}_1 & \cdots & r_{N,2}-\bar{r}_N \\ \vdots & & \vdots \\ r_{1,M}-\bar{r}_1 & \cdots & r_{N,M}-\bar{r}_N \end{bmatrix}
\end{array}
\rightarrow A^T = \begin{bmatrix} r_{1,1}-\bar{r}_1 & r_{1,2}-\bar{r}_1 & \cdots & r_{1,M}-\bar{r} \\ \vdots & \vdots & & \vdots \\ r_{N,1}-\bar{r}_N & r_{N,2}-\bar{r}_N & \cdots & r_{N,M}-\bar{r}_N \end{bmatrix}
$$

$$
S=\frac{A^T A}{M-1}
$$

$$
S=\begin{bmatrix} \frac{1}{M-1}\sum_{t=1}^{M}(r_{1,t}-\bar{r}_1)(r_{1,t}-\bar{r}_1) & \cdots & \frac{1}{M-1}\sum_{t=1}^{M}(r_{1,t}-\bar{r}_1)(r_{N,t}-\bar{r}_N) \\ \vdots & & \vdots \\ \frac{1}{M-1}\sum_{t=1}^{M}(r_{N,t}-\bar{r}_N)(r_{1,t}-\bar{r}_1) & \cdots & \frac{1}{M-1}\sum_{t=1}^{M}(r_{N,t}-\bar{r}_N)(r_{N,t}-\bar{r}_N) \end{bmatrix}
$$

Step 1: Create a demeaned return matrix



Name the demeaned return matrix "A".

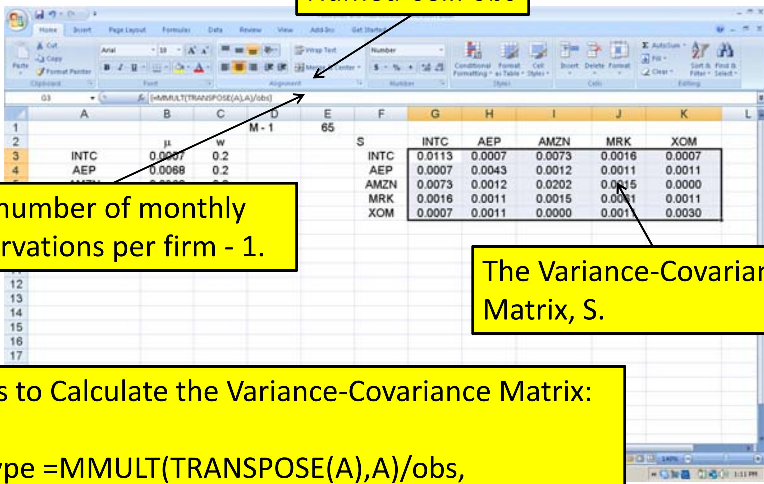
The demeaned return matrix = A

Select the range in which the demeaned return matrix will be. Enter the formula: =returns - av_rets

Press Ctrl + Shift + Enter.

The average demeaned return matrix should be zero for each stock.

Step 2: The Variance-Covariance Matrix



Named Cell: obs

The number of monthly observations per firm - 1.

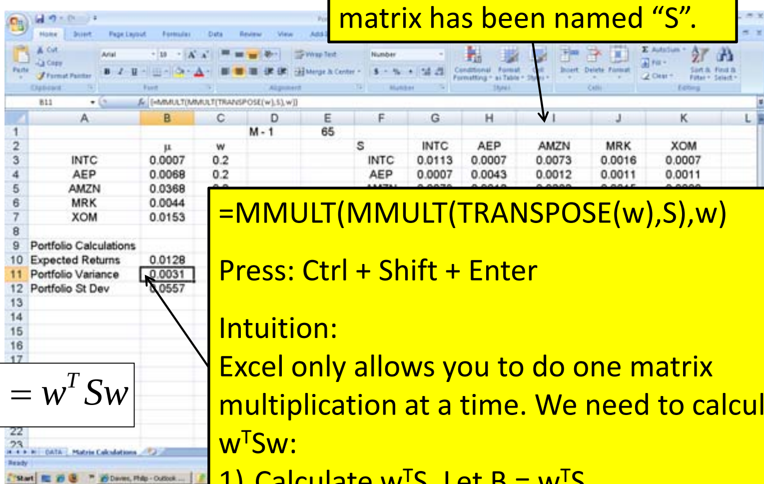
The Variance-Covariance Matrix, S.

Steps to Calculate the Variance-Covariance Matrix:

1) Type =MMULT(TRANSPOSE(A),A)/obs,
2) Press Ctrl + Shift + Enter

Portfolio Variance

The variance-covariance matrix has been named "S".



=MMULT(MMULT(TRANSPOSE(w),S),w)

Press: Ctrl + Shift + Enter

Intuition:
Excel only allows you to do one matrix multiplication at a time. We need to calculate $w^T Sw$:

1) Calculate $w^T S$. Let $B = w^T S$
2) Calculate Bw

$$\sigma_p^2 = w^T Sw$$