Exercise 3: DCM for EEG

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3.1 Dynamic equations

$$v(t) = \int_{-\infty}^{t} h(t-\tau)\sigma(\tau)d\tau$$

With

$$h(t) = \begin{Bmatrix} H\kappa t e^{-\kappa t}, & t \geq 0 \\ 0, & otherwise \end{Bmatrix}$$

derive: $\ddot{v}(t) = H\kappa\sigma(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t)$

According to *Leibniz' rule:*

$$\frac{d}{dt}\int_{a(t)=-\infty}^{b(t)=t} f(t,\tau)d\tau = f\big(t,b(t)\big)\frac{d}{dt}b(t) - f\big(t,a(t)\big)\frac{d}{dt}a(t) + \int_{a(t)=-\infty}^{b(t)=t}\frac{d}{dt}f(t,\tau)d\tau$$

$$\Rightarrow \dot{v}(t) = h(0)\sigma(t)\cdot 1 + \int_{-\infty}^{t}\dot{h}(t-\tau)\sigma(\tau)d\tau$$

$$\Rightarrow \dot{v}(t) = \int_{-\infty}^{t}\dot{h}(t-\tau)\sigma(\tau)d\tau$$

$$\dot{h}(t) = H\kappa(1-\kappa t)e^{-\kappa t} = H\kappa e^{-\kappa t} - \kappa h(t)$$

$$\Rightarrow \dot{v}(t) = \int_{-\infty}^{t} H\kappa e^{-\kappa(t-\tau)}\sigma(\tau)d\tau - \int_{-\infty}^{t}\kappa h(t-\tau)\sigma(\tau)d\tau$$

$$= \int_{-\infty}^{t} H\kappa e^{-\kappa(t-\tau)}\sigma(\tau)d\tau - \kappa v(t)$$

$$\Rightarrow \ddot{v}(t) = H\kappa\frac{d}{dt}\int_{-\infty}^{t} e^{-\kappa(t-\tau)}\sigma(\tau)d\tau - \kappa\dot{v}(t)$$

Again, plug term $\frac{d}{dt}\int_{-\infty}^{t} e^{-\kappa(t-\tau)}\sigma(\tau)d\tau$ into *Leibniz' rule:*

$$\frac{d}{dt}\int_{-\infty}^{t} e^{-\kappa(t-\tau)}\sigma(\tau)d\tau = H\kappa\left(\sigma(t) + \int_{-\infty}^{t}\sigma(\tau)e^{\kappa\tau}\frac{d}{dt}e^{-\kappa t}d\tau\right)$$

$$= H\kappa\left(\sigma(t) - \kappa\int_{-\infty}^{t}\sigma(\tau)e^{-\kappa(t-\tau)}d\tau\right)$$

$$= H\kappa\big(\sigma(t) - \kappa v(t)\big)$$

$$\Rightarrow \ddot{v}(t) = H\kappa\sigma(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t)$$

Q.E.D.

3.2 Coupled harmonic oscillator

a) Convert the 2nd order ODE of harmonic oscillator (HO) to a 1st order system.

We have

$$\ddot{x} = -f\dot{x} - \kappa^2 x + u(t)$$

Let

$$v = \dot{x}$$
$$\dot{v} = \ddot{x}$$

Then we get a equation set

$$\begin{Bmatrix} \dot{x} = 0 * x + 1 * v \\ \dot{v} = -\kappa^2 x - fv + u(t) \end{Bmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa^2 & -f \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ u(t) \end{pmatrix}$$

i.e.,

$$\dot{\vec{x}} = A\vec{x} + \vec{u}(t)$$

Q.E.D.

b) Coupled dynamic system: HO x is driven by a second HO $z(t)$, i.e., $u(t) = a \cdot z(t)$

$$\ddot{z} = -f_z\dot{z} - \kappa_z^2 z + u_z(t)$$
$$\ddot{x} = -f\dot{x} - \kappa^2 x + a \cdot z$$

Similarly, let:

$$v_z = \dot{z}$$

Keeping notation v in (a), we get:

$$\begin{Bmatrix} \dot{x} = 0 \cdot x + 0 \cdot z + 1 \cdot v + 0 \cdot v_z \\ \dot{z} = 0 * x + 0 \cdot z + 0 \cdot v + 1 \cdot v_z \\ \dot{v} = -\kappa^2 x + a \cdot z - fv + 0v_z + a \cdot z \\ \dot{v}_z = 0 * x - \kappa_z^2 z + 0 * v - f_z v_z + u_z(t) \end{Bmatrix}$$

$$\Rightarrow \begin{pmatrix} \dot{x} \\ \dot{z} \\ \dot{v} \\ \dot{v}_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa^2 & 0 & -f & 0 \\ 0 & -\kappa_z^2 & 0 & -f_z \end{pmatrix} \begin{pmatrix} x \\ z \\ v \\ v_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_z(t) \end{pmatrix}$$

i.e.,

$$\dot{\vec{x}} = A\vec{x} + \vec{u}(t)$$

Where:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa^2 & 0 & -f & 0 \\ 0 & -\kappa_z^2 & 0 & -f_z \end{pmatrix}, \qquad \vec{u}(t) = \begin{pmatrix} 0 \\ 0 \\ u(t) \\ u_z(t) \end{pmatrix}$$

c) Reconsider equation

$$\ddot{v}(t) = H\kappa\sigma(t) - 2\kappa\dot{v}(t) - \kappa^2 v(t) \qquad (*)$$

Assume:

$$\sigma(t) = a \cdot s\big(v_z(t)\big) + u(t)$$

$v_z(t)$: dynamics of a different population (also follows Eq. $(*)$)

$$s(v) = \frac{1}{1 + \exp(-rv)} - \frac{1}{2}$$

Transform Eq. $(*)$ to 1st order linear equation system by linearizing $s(v)$ $v \cong 0$.

Around $v = 0$, with Taylor expansion, we have:

$$s(v) = s(0) + \frac{s'(0)}{1} \cdot v + o(v)$$
$$s(v) = \frac{r}{4} \cdot v + o(v)$$

Then:

$$\sigma(t) = a \cdot \frac{r}{4} v_z(t) + u(t)$$

Eq. $(*)$ becomes:

$$\ddot{v}(t) = H\kappa \left(a \cdot \frac{r}{4} v_z(t) + u(t) \right) - 2\kappa \dot{v}(t) - \kappa^2 v(t)$$
$$\ddot{v} = \frac{H\kappa a r}{4} v_z - 2\kappa \dot{v} - \kappa^2 v + H\kappa u(t)$$

Meanwhile:

$$\ddot{v}_z(t) = H_z \kappa_z \sigma(t) - 2\kappa_z \dot{v}_z(t) - \kappa_z^2 v_z(t)$$

Let

$$\begin{Bmatrix} w = \dot{v} \\ w_z = \dot{v}_z \end{Bmatrix}$$
$$\Rightarrow \begin{Bmatrix} \dot{w} = \frac{H\kappa a r}{4} v_z - 2\kappa w - \kappa^2 v + H\kappa u(t) \\ \dot{w}_z = -2\kappa w_z - \kappa^2 v_z + H_z \kappa_z \sigma(t) \\ \dot{v} = w \\ \dot{v}_z = w_z \end{Bmatrix}$$
$$\begin{pmatrix} \dot{v} \\ \dot{v}_z \\ \dot{w} \\ \dot{w}_z \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\kappa^2 & \frac{H\kappa a r}{4} & 2\kappa & 0 \\ 0 & -\kappa^2 & 0 & -2\kappa \end{pmatrix} \begin{pmatrix} v \\ v_z \\ w \\ w_z \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ u(t) \\ H_z \kappa_z \sigma(t) \end{pmatrix}$$

d)

3.3 Inference on NN structure

a) Integration of the system

$$\dot{x} = Ax + Cu$$
$$x(t) = 0, t < 0$$
$$u(t) = normpdf(t, \mu, \sigma)$$

with A and C as given in the exercise sheet.

The solution for a) and b) is implemented in Matlab and uploaded in Moodle.
The declaration of the parameters and the integration with Euler methods (step size: 0.001) is shown in the code snippet.

```
A = [0, 1, 0, 0; -80^2, -50, 3000, 0; 0, 0, 0, 1; 1000, 0, -50^2,
C = [0 0 0 1];

% Euler's Method
h = 0.001;
t = 0:h:0.2;  % range of t

% workaround to get size of t to create empty x array
x_col = size(t);
x = zeros(4,x_col(2));

% start values for x
x(:,1) = [0 0 0 0]; % x(t) = 0; t<0

n = numel(t);  % number of x values

% euler's method
for i=1:n-1
    f = A * x(:,i) + (normpdf(t(i),0.05,0.01) * C)';
    x(:,i+1) = x(:,i) + h * f;
end
```

To verify the integrated states (x) correspond to the data in x_condition_1, we calculated the absolute difference between the single values. The maximal difference $2.7756 * 10^{-17}$.

```
diff = abs(x - x_condition_1)
max_error = max(diff(:))
 max_error = 2.7756e-17
```

```
% verfication for x1 and x3
diff_row1 = abs(x(1,:) - x_condition_1(1,:))
 diff_row1 = 1×201
 10^-18 ×
        0        0        0        0        0        0        0
max(diff_row1)
 ans = 1.0842e-19
diff_row3 = abs(x(3,:) - x_condition_1(3,:))
 diff_row3 = 1×201
      0    0    0    0    0    0    0    0    0    0    0    0
max(diff_row3)
 ans = 0
```

Specifically, we verified x1 and x3:
The maximum errors between all integrated states and the given data were still negligibly small and for x3 even 0.

b) Parameter Grid search

For the grid search of the 4 parameters we used as a first range + / - the half of the standard value in a reasonable step size.

```
K1 = linspace(40,120,80); % step size: 1
K2 = linspace(25,75,50); % step size: 1
Ab = linspace(500,1500,100); % step size: 10
Af = linspace(2000,4000,200); % step size: 10
```

By comparing the maximum
The grid search showed quick that the explained variance is the best for changing the k_1 parameter. Listed below are the maximum sums and mean values of the explained variances for all grid search parameter estimations.

```
max(sum(eXs_Af,1))
 ans = 3.6192
max(sum(eXs_Ab,1))
 ans = 0
max(sum(eXs_K1,1))
 ans = 3.9999
max(sum(eXs_K2,1))
 ans = 0
```

```
max(mean(eXs_Af,1))
 ans = 0.9048
max(mean(eXs_Ab,1))
 ans = 0
max(mean(eXs_K1,1))
 ans = 1.0000
max(mean(eXs_K2,1))
 ans = 0
```

The estimations for x_1 to x_4 are shown in the table by applied grid search for k_1.

eXs_K1 × | eXs_K2 × | eXs_Af × | eXs_Ab ×

4x80 double

	56	57	58	59	60	61	62	63	64	65	66	67	68
1	0.9855	0.9917	0.9962	0.9988	1.0000	0.9996	0.9979	0.9950	0.9910	0.9860	0.9801	0.9734	0.966
2	0.9794	0.9882	0.9945	0.9983	0.9999	0.9994	0.9969	0.9926	0.9866	0.9790	0.9700	0.9597	0.948
3	0.9997	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.99
4	0.9997	0.9998	0.9999	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9997	0.9996	0.9995	0.99

The estimated k_1 value with the best explained variance was 99.74 (index 60). For this value we got explained variances of 1.0, 0.999, 1.0 and 1.0 for x_1 to x_4. This means that for all variables the explained variance reached 99%. The implemented matrix A with the parameter estimate is shown below.

```
best_K1 = K1(60)

best_K1 = 99.7468

best_eXs_K1 = eXs_K1(:,60)

best_eXs_K1 = 4x1
      1.0000
      0.9999
      1.0000
      1.0000

A_best = [0, 1, 0, 0; -best_K1^2, -50, af, 0; 0, 0, 0, 1; ab, 0, -k2^2, -50]

A_best = 4x4
10^3 x
           0    0.0010         0         0
     -9.9494   -0.0500    3.0000         0
           0         0         0    0.0010
      1.0000         0   -2.5000   -0.0500
```

When taking the absolute difference between the x_condition_2 and the predicted x values, the highest difference was 0.0014.

```
diff = abs(x - x_condition_2)

diff = 4x201
           0         0         0         0         0         0    0.0000    0.0000    0.0000    0.0000
           0         0         0         0         0    0.0000    0.0000    0.0000    0.0000    0.0000
           0         0         0         0         0         0         0         0    0.0000    0.0000
           0         0         0         0         0         0         0    0.0000    0.0000    0.0000

max_error = max(diff(:))

max_error = 0.0014
```

So our final model and the ensuing parameters that best explain the data are:

$$\dot{x} = Ax + Cu$$
$$x(t) = 0, t < 0$$
$$u(t) = normpdf(t, \mu, \sigma)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -{k_1}^2 & -f_1 & af & 0 \\ 0 & 0 & 0 & 1 \\ a_b & 0 & -{k_2}^2 & -f_2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ c \end{pmatrix}$$

$\kappa_1 = 99.74$, $\kappa_2 = 50$

$f_1 = 50$, $f_2 = 50$

$a_f = 3000$

$a_b = 1000$

$c = 1$

$\mu = 0.05$

$\sigma = 0.01$