Rudin Ch4 - Elementary Hilbert Space Theory

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1 Notes

1.1 Inner Products and Linear Functionals

4.1 Def A complex vector space H is an *inner product space* if to each ordered pair $x, y \in H$, there exists an inner/scalar - product (x, y) which is a complex number s.t.

a. $(x, y) = \overline{(y, x)}$
b. $(x + y, z) = (x, z) + (y, z)$
c. $(\alpha x, y) = \alpha(x, y)$
d. $(x, x) \geq 0$, and (x, x) is real
e. $(x, x) = 0$ iff $x = 0$
The norm is called $||x|| = \sqrt{(x, x)}$.

4.2 Schwarz Inequality

$$|(x, y)| \leq ||x||||y||$$

4.3 Triangle Inequality

$$||x + y|| \leq ||x|| + ||y||$$

4.4 Def It follows that

$$||x - z|| \leq ||x - y|| + ||y - z||$$

and this could be a metric space. If this metric space is complete, then H is a Hilbert space.

4.5 Here are some examples.

a. The set of C^n can have $(x, y) = \sum_{j=1}^{n} \zeta_j \overline{\eta_j}$ where $x = (\zeta_i), y = (\eta_i)$.
b. If μ is a positive measure, then $L^2(\mu)$ is a Hilbert space with

$$(f,g) = \int_X f\bar{g}d\mu$$

c. The vector space of all continuous functions on $[0,1]$ is an inner product space, but not a Hilbert space, if

$$(f,g) = \int_0^1 f(t)\overline{g(t)}dt$$

4.6 Thm For any fixed $y \in H$, the mappings

$$x \to (x,y), y \to (x,y), x \to ||x||$$

are continuous functions on H.

4.7 Subspaces A subset M of a vector space V is called a subspace of V if M is a vector space. A closed subspace of H is a subspace that is a closed set relative to the topology induced by the metric of H. Furthermore, the closure $\overline{M}$ is a subspace as well.

4.8 Convex Sets A set E is convex if, $x, y \in E$ and $0 < t < 1$, then $z_t = (1-t)x + ty \in E$. Furthermore, each translate of E is also convex.

4.9 Orthogonality $(x,y) = (y,x) = 0$ implies that x and y are orthogonal. $x^\perp$ is the set of all orthogonal x. $M^\perp$ is the set of all orthogonal to every $x \in M$. Both are closed subspaces.

4.10 Thm Every nonempty, closed, convex set E in a Hilbert space H contains a unique element of smallest norm.

4.11 Thm Let M be a closed subspace of a Hilbert space H.

(a) Every $x \in H$ has then a unique decomposition

$$x = Px + Qx, Px \in M, Qx \in Q^\perp$$

(b) Px and Qx are the nearest points to x in M and $M^\perp$ respectively.
(c) The mappings $P : H \to M$ and $Q : H \to M^\perp$ are linear.
(d) $||x||^2 = ||Px||^2 + ||Qx||^2$

Corollary: If $M \neq H$ then there exists $y \in H, y \neq 0, y \perp M$.

4.12 Thm If F is a continuous linear functional on H, then there is a unique $y \in H$ s.t.

$$Lx = (x,y)$$

1.2 Orthonormal Sets

4.13 Def If $x_1, \dots x_k \in V$ and if $c_1, \dots c_k$ are scalars, then $\sum_{i=1}^{k} c_i x_i$ is a linear combination. It is independent if $\sum_{i=1}^{k} c_i x_i = 0 \implies c_i = 0$. A set $S \subset V$ is independent if every finite subset of S is independent. $[S]$ is all linear combinations of all finite subsets of S and is called the span.

A set of vectors u_α is called orthonormal if every two are orthogonal and $||u_\alpha|| = 1$. We define $\hat{x}(\alpha) = (x, u_\alpha)$ and these are called the fourier coefficients of x relative to $\{u_\alpha\}$.

4.14 Thm Suppose that $\{u_\alpha : \alpha \in A\}$ is an orthonormal set in H and F is a finite subset of A. Let M_F be the span of $\{u_\alpha : \alpha \in F\}$.

(a) If φ is a complex function on A that is 0 outside F, then there is a vector $y \in M_F$, namely

$$y = \sum_{\alpha \in F} \varphi(\alpha) u_\alpha$$

that has $\hat{y}(\alpha) = \varphi(\alpha)$ for every $\alpha \in A$. Also,

$$||y||^2 = \sum_{\alpha \in F} |\varphi(\alpha)|^2$$

(b) If $x \in H$ and

$$s_F(x) = \sum_{\alpha \in F} \hat{x}(\alpha) u_\alpha$$

then

$$||x - s_F(x)|| < ||x - s||$$

for every $s \in M_F$, except for $s = s_F(x)$, and

$$\sum_{\alpha \in F} |\hat{x}(\alpha)|^2 \leq ||x||^2$$

4.15 Def We define

$$\sum_{\alpha \in A} \varphi(\alpha)$$

to be the supremum of the set of all finite sums $\varphi(\alpha_1) + \cdots + \varphi(\alpha_n)$, where these α_i are distinct elements of A. This is the same as

$$\int_A \varphi d\mu$$

for the counting measure μ. A complex function is in $\ell^2(A)$ (which is the same as $L^2(\mu)$) iff

$$\sum_{\alpha \in A} |\varphi(\alpha)|^2 < \infty$$

The number of elements of A is at most

$$\sum_{\alpha \in A} |n\varphi(\alpha)^2| \leq n^2 \sum_{\alpha \in A} |\varphi(\alpha)|^2$$

4.16 Lemma Suppose that

(a) X and Y are metric spaces, X is complete.
(b) $f : X \to Y$ is continuous.
(c) X has a dense subset X_0 on which f is an isometry, and
(d) $f(X_0)$ is dense in Y.

Then f is an isometry of X onto Y.

4.17 Thm Let $\{u_\alpha : \alpha \in A\}$ be an orthonormal set in H, and let P be the space of all finite linear combinations of the vectors u_α. The inequality

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leq ||x||^2$$

holds then for every $x \in H$, and $x \to \hat{x}$ is a continuous linear mapping of H onto $\ell^2(A)$ whose restriction to the closure $\overline{P}$ of P is an isometry of $\overline{P}$ onto $\ell^2(A)$.

The sum is the *Bessel Inequality* and the mapping is the *Riesz-Fischer Theorem*

4.18 Thm Let $\{u_\alpha : \alpha \in A\}$ be an orthonormal set in H. Each of the following four conditions on $\{u_\alpha\}$ implies the other three:

(i) $\{u_\alpha\}$ is a maximal orthonormal set on H.
(ii) The set P of all finite linear combinations of members of $\{u_\alpha\}$ is dense in H.
(iii) The equality below holds for $x \in H$

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = ||x||^2$$

(iv) The equality below holds for $x, y \in H$

$$\sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = (x, y)$$

These are maximal orthonormal sets, or orthonormal bases.

4.19 Isomorphisms An isomorphism is a bijection that preserves all relevant properties. For Hilbert spaces, our isomorphism Λ has $(\Lambda x, \Lambda y) = (x, y)$.

4.20 Partially Ordered Sets A set $\mathscr{P}$ is said to be partially ordered by a binary relation $\leq$ if

(a) $a \leq b, b \leq c \implies a \leq c$
(b) $a \leq a$ for every $a \in \mathscr{P}$
(c) $a \leq b, b \leq a \implies a = b$

A subset $\mathscr{L}$ of a partially ordered set $\mathscr{P}$ is *totally-ordered or linearly ordered* if $a, b \in \mathscr{L}$ satisfies $a \leq b$ or $b \leq a$.

4.21 The Hausdorff Maximality Thm Every nonempty poset contains a maximal totally ordered set.

4.22 Thm Every orthonormal set in B in a Hilbert space H is contained in a maximal orthonormal set in H.

1.3 Trigonometric Series

4.23 Def Let T be the unit circle in the complex plane. If F is any function on T, and if f is defined on R^1 by

$$f(t) = F(e^{it})$$

Then F is periodic with period 2π. We define $L^P(T)$ for $1 \leq p < \infty$ to be the class of all complex, Lebesgue measurable, 2π-periodic functions on R^1 for which

$$||f||_p = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{1/p} < \infty$$

A trigonometric polynomial is a finite sum of the form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$$

where a_n and b_n are complex numbers. We can rewrite this as

$$f(t) = \sum_{n=-N}^{N} c_n e^{int}$$

We define $u_n(t) = e^{int}$ and we define the inner product in $L^2(T)$ by

$$(f,g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

And we find that

$$(u_n, u_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t} dt = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

Thus $\{u_n : n \in \mathbb{Z}\}$ is an orthonormal set in $L^2(T)$ usually called the trigonometric system.

4.24 The Completeness of the Trigonometric System To show that Trigonometric System is complete, it suffices to prove this for $||f - P||_\infty < \epsilon$.

4.25 Thm If $f \in C(T)$ and $\epsilon > 0$, there is a trig poly P s.t.

$$|f(t) - P(t)| < \epsilon$$

for every real t.

4.26 Fourier Series For any $f \in L^1(T)$, we define the *Fourier Coefficients* of f by the formula

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{int}dt$$

The *Fourier Series* of f is

$$\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$$

and the partial sums are

$$s_N(t) = \sum_{-N}^{N} \hat{f}(n)e^{int}$$

The *Riesz-Fischer theorem* asserts that if $\{c_n\}$ is a sequence of complex numbers, then

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$$

then there exists an $f \in L^2(T)$ s.t.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-int}dt$$

The *Parseval Thm* asserts that

$$\sum n = -\infty^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

whenever $f \in L^2(T)$ and $g \in L^2(T)$; the series on the left converges absolutely. Furthermore, for s_N as in the Fourier Series, we have

$$\lim_{N\to\infty} ||f - s_N||_2 = 0$$

2 Problems

Problem 1
We apply theorem 4.11 to find that every X can be written as

$$x = Px + Qx = Qx + Rx$$

Where $Px \in M, Qx \in M^{\perp}, Rx \in (M^{\perp})^{\perp})$. Noting that these functions take x to the closest points on the respective subspaces, we apply $||x||^2 = ||Px||^2 + ||Qx||^2 = ||Qx||^2 + ||Rx||^2$. Taking $x \in M, \notin (M^{\perp})^{\perp}$ and $x \in (M^{\perp})^{\perp}, \notin M$, we notice that $(M^{\perp})^{\perp} = M$, as desired.

Clearly, if M is not closed, then $M \neq (M^{\perp})^{\perp}$, since one set is closed, and the other is open.

Problem 3 We note that the set of all trigonometric polynomials, which we denote T, is separable. This is because $QC = \mathbb{Q} + i\mathbb{Q}$ is dense and countable in $\mathbb{C}$.

We furthermore note that T is dense in $L^p(T)$ by Theorem 4.25. We note that $T \subset \overline{QC} \implies \overline{T} \subset \overline{QC}$. It follows that $L^p(T) \subset \overline{T} \subset \overline{QC}$, so QC is the countable subset which is dense in $L^p(T)$.

$L^{\infty}(T)$ is not separable. We consider this set isomorphic to $L^{\infty}([0, 2\pi])$ and note that the set

$$S = \{\chi_{\alpha} : \alpha \subset [0, 2\pi]\}$$

is uncountable. If we assume there exists a countable set C which is dense in $L^{\infty}([0, 2\pi])$, we notice that $\overline{C}$ doesn't contain S because, for $\epsilon < \frac{1}{2}$, each element of C is only close to 1 value in S. Since S is uncountable, C is not dense in S which means that, by contradiction, $L^{\infty}(T)$ is not separable.

Problem 5 We note that by Thm 4.12,

$$\exists y : Lx = (x, y)$$

So $M^{\perp}$ is necessarily $(\{y\}^{\perp})^{\perp} = \{y\}$, where $\{y\}$ denotes the vector space generated by y and scalars. It follows that $M^{\perp}$ is a 1-dim vector space.

Problem 7 We let the sets E_k denote disjoint subsets of $\{a_n\}$ with $1 < \sum_{n \in E_k} a_n^2 < \infty$. Necessarily, there are infinite E_k, and we let $E_{k+1} = 2 \times E_k$. We let $c_k = \frac{1}{\sum_{n \in E_k} a_n^2}$. It follows that

$$\sum a_n b_n = \infty, \sum b_n^2 < 1 + \frac{1}{2} + \cdots = 2 < \infty$$

Problem 9 Note that, by definition,

$$\int_A \cos nx dx < \int_0^{2\pi} \cos nx dx$$

and a similar result holds for sin. We calculate

$$\lim_{n \to \infty} \int_0^{2\pi} \cos nx dx = \lim_{n \to \infty} \frac{\sin nx}{n} \Big|_0^{2\pi} = 0$$

$$\lim_{n \to \infty} \int_0^{2\pi} \sin nx dx = \lim_{n \to \infty} -\frac{\cos nx}{n} \Big|_0^{2\pi} = 0$$

and so it follows

$$\lim_{n \to \infty} \int_A \cos nx dx = \lim_{n \to \infty} \int_A \sin nx dx = 0$$

Problem 11 We examine the set

$$E = \{(1 + \frac{1}{n}) u_n : n \in \mathbb{N}\}$$

Notice that, for distinct $\alpha, \beta \in E$, which we denote $(1 + \frac{1}{a}) u_a$ and $(1 + \frac{1}{b}) u_b$ respectively

$$||\alpha - \beta|| = ||u_a - u_b + \frac{u_a}{a} - \frac{u_b}{b}|| = \sqrt{||u_a + \frac{u_a}{a}|| + ||u_b + \frac{u_b}{b}||} \geq \sqrt{2}$$

So there can't be any limit points in the set, which means it's closed. Furthermore, we note that $\inf_{e \in E} = \lim_{n \to \infty} 1 + \frac{1}{n} = 1$, but this is never achieved in E.

Problem 13 We first check $e^{2\pi i k}$ and will prove that this is dense in the general set. If $k = 0$ we note that

$$f(t) = 1 \implies \int_0^1 f(t) dt = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\alpha n) = 1$$

If $k \neq 0$, we note that

$$f(t) = \cos 2\pi kt + i \sin 2\pi kt \implies \int_0^1 f(t)dt = \frac{1}{2\pi k} \sin 2\pi kt - i\frac{1}{2\pi k} \cos 2\pi kt \Big|_0^1 = 0$$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\alpha n) = \lim_{N\to\infty} \frac{1}{N} \frac{e^{2\pi ik(N+1)\alpha} - e^{2\pi ik\alpha}}{e^{2\pi ik\alpha} - 1} \leq \lim_{N\to\infty} \frac{1}{N} \frac{1+i}{e^{2\pi ik\alpha} - 1} = 0$$

And now we prove that this set is dense in the set of all periodic functions of period 1. This is trivial because the set of trigonometric polynomials is dense in $L^2(T)$, and it's isomorphic.

Problem 15 We expand to get

$$\min_{a,b,c} \int_0^\infty (x^6 - 2cx^5 + (c^2 - 2b)x^4 + (2bc - 2a)x^3 + (b^2 + 2ac)x^2 + 2abx + a^2)e^{-x}dx$$

We note that, for all n, we have

$$\int x^n e^{-x} = -e^{-x} \sum_{i=0}^{n} \frac{n!}{i!} x^i$$

and therefore, since L'ohpital's rule removes the case for ∞, we have an integral of $n!$. Expanding our integral, we have

$$\begin{aligned} &720 - 240c + 24c^2 - 48b + 12bc - 12a + 2b^2 + 4ac + 2ab + a^2 \\ &= (a + b + 2c - 6)^2 + (b + 4c - 18)^2 + (2c - 18)^2 + 36 \end{aligned}$$

which clearly takes a minimum value of 36 when $a = 6, b = -18, c = 9$.

Problem 17 We define the mapping as

$$\gamma(x) = \{u_x : 1 \leq x \leq \lceil \frac{1}{n} \rceil\}$$

We notice that, for $0 \leq a \leq b \leq c \leq d \leq 1$, we clearly have $\gamma(b) - \gamma(a) \cap \gamma(d) - \gamma(c) = \varnothing$, so these sets are clearly orthogonal since they are composed of u_x.

Problem 19 Based on the definition of ω as the N root of unity, it follows that ω^k is a root of unity unless $k \equiv 0 \bmod N$.

We now prove that

$$(x, y) = \frac{1}{N} \sum_{n=1}^{N} ||x + \omega^n y|| \omega^n$$

Note that we can simplify this

$$\frac{1}{N}\sum_{n=1}^{N}((x,x)+\omega^n(y,x)+\omega^{-n}(x,y)+(y,y))\omega^n$$
$$=\frac{1}{N}\sum_{n=1}^{N}(x,y)+((x,x)+\omega^n(y,x)+(y,y))\omega^n=(x,y)$$

as shown above. To show this for the integral, note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}||x+e^{i\theta}y||^2e^{i\theta}d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}((x,x)+(y,y))e^{i\theta}+(y,x)e^{2i\theta}+(x,y)d\theta$$
$$=(x,y)$$

as the other terms cancel out.