# Math 6120 - Final Paper

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## 1 Introduction

Non-euclidean geometry first arose in the 19th century due to the independent discoveries of famous Geometers such as Gauss, Poincaré and Riemann. Of these hyperbolic geometry is perhaps the most famous and useful. In this paper, we explore various classical results as well as generalizations to higher dimensions.

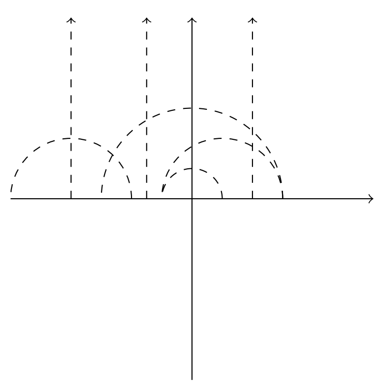
## 2 Two-dimensional Hyperbolic Space

We start by examining various models of and key results of 2-dimensional hyperbolic space. For these models, we specifically work in the complex plane to explore specific properties that don't arise in $\mathbb{R}^2$.

### 2.1 The Poincaré Half Plane Model

The first model of hyperbolic geometry we consider is the upper half plane model. Consider the upper half of the complex plane $\mathbb{U} = \{z : \text{Im}(z) > 0\}$.

We define a *hyperbolic line* to be either a ray perpendicular to the real axis or a semicircle with center on the real axis. Example lines are given below in dashes



As is easily checked, these lines satisfy the first four of Euclid's postulates. However, they don't satisfy the parallel postulate, as can be checked with any hyperbolic line which is not a circle and some point not on the line. However, as this is a seemingly arbitrary definition, it would be fruitful to check that these are in fact the lines, at least in some sense. To do so, we borrow some tools from differential geometry.

First, we consider, the *metric* of $\mathbb{U}$ to be

$$\frac{dx^2 + dy^2}{y^2}$$

Note that the curvature of $\mathbb{U}$ is $-1$, as opposed to the 0 curvature of Euclidean space. We calculate the *geodesics*, or the shortest arc lengths between points. In particular, The above hyperbolic lines are geodesics under integration with the above metric. The distance between two points $a, b \in \mathbb{U}$ is given by

$$d_{\mathbb{U}}(a, b) = 2 \tanh^{-1} \frac{|b - a|}{b - \overline{a}}$$

where $|x|$ is the standard complex norm.

Similar to Euclidean Geometry, Hyperbolic space also exhibits a set of isometries. In the geometric sense, they consist of translations on the real line, reflection across the imaginary axis, dilations, and inversions of circles on the real axis.

To examine these in a more analytical sense, we first consider the set of *Mobius transformations*. These are denoted by Mob and are the set of $m : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$

$$m(z) = \frac{az + b}{cz + d} \text{ or } \frac{a\overline{z} + b}{c\overline{z} + d} \quad a, b, c, d \in \overline{\mathbb{C}}, ad - bc \neq 0$$

where $\overline{\mathbb{C}}$ is the Riemann Sphere. Note that the terms of the form $\frac{az+b}{cz+d}$ are homeomorphic and can be defined at $\infty$. In fact, the Mobius transformations play a much more crucial role in the Riemann sphere.

**Thm 2.1.1** Mob is the set of homeomorphic functions in the Riemann Sphere.

**Pf:** All that needs to be shown is that each homeomorphic function is a Mobius transformation. To do this, we examine a homeomorphic $f$, specifically at the points $f(0), f(1), f(\infty)$ and construct the unique Mobius transformation $m$ which takes $f(0) \to 0, f(1) \to 1, f(\infty) \to \infty$. Let $Z$ be the fixed points of $m \circ f$, and by examining various cases find that $Z$ is dense in $\overline{\mathbb{C}}$. Therefore, $m \circ f = 1 \implies f = m^{-1}$ and $f$ is a Mobius transformation.

$\square$

**Thm 2.1.2** Mob is the set of isometries of $\mathbb{U}$.

**Pf:** Through calculation, we can see that Mob is a subset of isometries of $\mathbb{U}$.

For the reverse inclusion, a basic outline is given. For an arbitrary isometry $f$, we find the unique Mobius transform $m$ s.t. $m(f(x)) = x, m(f(y)) = y$ for set $x, y$. For any $w$ on the hyperbolic line between $x$ and $y$, which we denote $I$, we can see that $m(f(w)) = w$ since it is determined by the distances between $x, y$ and $w$. For any $w$ not on the hyperbolic line, we construct a perpendicular bisector $\ell$ through $w$ that intersect our $I$ at some $z$. As this is a perpendicular bisector and $m(f(I)) = I$, then $m(f(\ell)) = \ell$ and in particular $m(f(w)) = w$. Therefore $m \circ f = 1 \implies f = m^{-1} \in$ Mob.

$\square$

Mobius transformations lend themselves to a natural matrix representation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim m \quad m(z) = \frac{az+b}{cz+d}$$

We can define classes of Mobius Transformations up to conjugacy. In particular $m$ and $n$ are conjugate if $m = p \circ n \circ p^{-1}$ where $p$ is some Mobius transformation. We define the following classes of Mobius Transformations (up to conjugacy)

- *Parabolic:* $m$ is *parabolic* if it is conjugate with some $n$ s.t. $n(z) = 1$. The matrix representation is

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

  There is only one fixed point in the Riemann sphere, namely $\infty$.

- *Elliptic:* $m$ is *elliptic* if it is conjugate with some $n(z) = e^{2i\theta}z$ for $\theta \in (0, \pi)$. The matrix representation is

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

  Note that this fixes at least 1 point.

- *Loxodromic:* $m$ is *loxodromic* if it is conjugate with $n(z) = \rho^2 e^{2i\theta} z$ for $\theta \in (0, \pi)$, $\rho \in \mathbb{R} \setminus 1$. The matrix representation is

$$\begin{bmatrix} \rho e^{i\theta} & 0 \\ 0 & \rho^{-1} e^{-i\theta} \end{bmatrix}$$

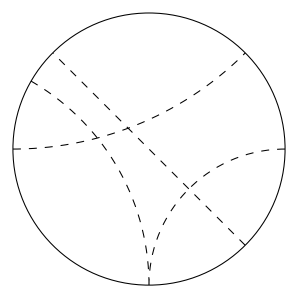
  and Note that this fixes exactly 2 points.

### 2.2 The Poincaré Disc

The poincaré disc, which we denote $\mathbb{D}_P$, is another model of hyperbolic geometry. To construct this model, we first restrict our attention to $\mathbb{D} : \{z : |z| < 1\}$. To construct $\mathbb{D}_P$, we equip this space with another metric. The *metric* we give to $\mathbb{D}_P$ is

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}$$

Note that the curvature is also $-1$. The hyperbolic lines of the Poincaré disc are given by orthogonal arcs inside the circle and diameters. Examples are given below in dashed lines



Note that these lines satisfy the first 4 Euclid postulates, but not the fifth, and are indeed the shortest path between any two points.

Integrating along these curves, we have the *hyperbolic distance* in $\mathbb{D}_P$ is given by

$$d_{\mathbb{D}_p}(a, b) = 2\tanh^{-1} \frac{|b - a|}{|1 - \overline{a}b|}$$

It is interesting to note that this construction of the hyperbolic space in the unit ball is not in fact unique! In fact, a separate model developed earlier by Beltrami in the unit ball exhibits a different metric and geodesics. However, the Poincaré disc is particularly meaningful in the scope of complex analysis, as we will discuss below.

Consider the *Cayley Transform*

$$\mathcal{C}(z) = \frac{z - i}{z + i}$$

With some work, we can see that maps $\mathbb{U} \to \mathbb{D}$, where $\mathbb{D} : \{z : |z| < 1\}$. Furthermore, note that this map is actually a conformal mapping, since it is both holomorphic and bijective with inverse

$$\mathcal{C}^{-1}(z) = i\frac{1+z}{1-z}$$

The Cayley Transform is also more than just a conformal mapping. In the scope of hyperbolic geometry, the Cayley transform also interacts with the distance metrics of the two spaces. In particular,

**Thm 2.2.1** The Cayley Transform is an isometry from $\mathbb{U}$ to $\mathbb{D}_P$. In particular, for $a, b \in \mathbb{U}$

$$d_{\mathbb{U}}(a,b) = d_{\mathbb{D}_P}(\mathcal{C}(a), \mathcal{C}(b))$$

$\square$

This means that not only are $\mathbb{U}$ and $\mathbb{D}$ conformally equivalent, a consequence of the Riemann Mapping Theorem, but they also exhibit the same distance behavior under this mapping.

## 2.3 Other Models

The Poincaré disc and the half-plane are the two most important models, especially as it relates to complex analysis. However, there are several interesting models of hyperbolic geometry that aren't necessarily conformally equivalent or restricted to the complex plane.

The *Beltrami-Klein model* of hyperbolic geometry $\mathbb{D}_K$ is $\mathcal{K}(\mathbb{D}_P)$, where $\mathcal{K} = \frac{2z}{|z|^2+1}$.

Furthermore, this $\mathcal{K}$ is an isometry. Note that $\mathcal{K}$ maps hyperbolic lines of $\mathbb{D}_P$ to straight Euclidean lines, so the hyperbolic lines of $\mathbb{D}_K$ are simply Euclidean lines through the ball. If points $a, b$ lie on line $\overline{CD}$ where $C, D$ correspond to points $c, d \in \mathbb{D}$, then we calculate the hyperbolic distance in $\mathbb{D}_K$ to be

$$d_{\mathbb{D}_K}(a,b) = \frac{1}{2}|\ln(a,b;c,d)|$$

where $(a,b;c,d)$ is the cross-ratio defined by

$$(a,b;c,d) = \frac{(a-c)(b-d)}{(b-c)(a-d)}$$

Note that this model is not conformal as $\mathcal{K}$ is clearly not conformal.

The 2-dimensional *hyperboloid model* of hyperbolic geometry is actually a subset of $\mathbb{R}^3$. Consider

$$\mathbb{H} : \{(x,y,z) : x^2 - y^2 - z^2 = 1, z > 0\}$$

and equip this with a metric

$$dx^2 - dy^2 - dz^2$$

The geodesics are the hyperbolas generated by intersecting the hyperboloid with plane going through the origin. The distance is defined as

$$d_{\mathbb{H}}(a,b) = \cosh^{-1}(B(a,b)) \text{ where } B(a,b) = a_1b_1 - a_2b_2 - a_3b_3$$

# 3 Generalization to higher dimensions

## 3.1 Riemannian Manifolds

Hyperbolic Geometry can be generalized in $\mathbb{R}^n$ and more specifically Riemannian manifolds. To see what this means, we first introduce some concepts.

A *topological manifold* is a second countable Hausdorff space that is locally homeomorphic to an open subset of $\mathbb{R}^n$. We define a *coordinate chart* to be pair $(U_M, \varphi)$ for which $U_M$ is an open subset of the manifold and $\varphi$ is a homeomorphism that maps this to an open subset of $\mathbb{R}^n$, and two coordinate charts $(U, \varphi)$ and $(V, \psi)$ are *smoothly compatible* if $\varphi^{-1} \circ \psi$ applied to $U \cap V$ is smooth. An *atlas* is a collection of charts for which the union of the open sets coves the manifold, and a *smooth atlas* is an atlas for which any two coordinate charts are either disjoint or smoothly compatible. A *smooth manifold (manifold)* is a topological manifold with a smooth atlas.

At each point $p$ of a manifold $M \subseteq \mathbb{R}^n$ we can define the tangent space at that point. We define a linear map $X : C^\infty(M) \to \mathbb{R}$ to be a *derivation* at $p$ if

$$X(fg) = f(p)X(g) + g(p)X(f)$$

and the set of all derivations at $p$ is the *tangent space* to $M$ at point $p$, denoted by $T_pM$.

Now, to introduce a sense of distance in differentiable geometry, we endow a metric on our manifold. A *Riemannian-Metric* is a 2-tensor field $g$ that satisfies

- Symmetry: $g(X,Y) = g(Y,X)$.
- Positive Definite: $g(X,X) > 0$ if $X \neq 0$.

we can consider this as an inner product $\langle X, Y \rangle := g(X,Y)$ for $X, Y \in T_pM$. A *Riemannian manifold* is a manifold with such a metric.

In this framework, we can now consider generalizations to the above models of hyperbolic geometry. In particular, note that Hyperbolic Geometry is a strict subset of Riemmanian manifolds.

- The *Poincaré half-space model* $\mathbb{U}^n$ is the upper half space $\{\mathbf{x} \in \mathbb{R}^n, x_n > 0\}$ with metric

$$\frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

- The *Poincaré ball model* $\mathbb{B}^n$ is the unit ball in $\mathbb{R}^n$. The metric is given by

$$\frac{4(dx_1^2 + \cdots + dx_n^2)}{1 - (x_1^2 + \cdots + x_n^2)^2}$$

- The *Hyperboloid model* $\mathbb{H}^n$ is the sheet of the hyperboloid defined by

$$\{(y, x_1, \ldots, x_n) : y > 0, y^2 - |\mathbf{x}|^2 = 1\}$$

  and is equipped with the *Minkowski metric*

$$-dy^2 + dx_1^2 + \cdots + dx_n^2$$

Note that, in addition to being able to define the higher-dimensional versions of our original models as Riemannian manifolds, we can also scale these by a factor of $R$. The following models are isometric

- The *Poivecncaré half-space model* $\mathbb{U}^n_R$ is the same space with metric

$$R^2 \frac{dx_1^2 + \cdots + dx_n^2}{x_n^2}$$

- The *Poincaré ball model* $\mathbb{B}^n_R$ is the ball of radius $R$ with metric

$$\frac{4R^4(dx_1^2 + \cdots + dx_n^2)}{R^2 - (x_1^2 + \cdots + x_n^2)^2}$$

- The *Hyperboloid model* $\mathbb{H}^n_R$ is the sheet of the hyperboloid defined by

$$\{(y, x_1, \ldots, x_n) : y > 0, y^2 - |\mathbf{x}|^2 = R^2\}$$

  equipped with the same Minkowski metric.

Using tools from Riemannian Geometry, in particular connections and the fact that geodesics have 0 acceleration, we can similarly define the geodesics of these models. These are the curves (with constant speed parametrizations) given below

- $\mathbb{U}^n_R$ has geodesics defined by half lines created by fixing $x_i$ for $1 \leq i < n$ and varying $x_n$ from 0 to infinity and semicircles with centers on $x_n = 0$.

- $\mathbb{B}^n_R$ has geodesics defined by arcs orthogonal to the boundary as well as diameters.
- $\mathbb{H}^n_R$ has geodesics defined great hyperbolas created by intersecting the hyperboloid with a plane through the origin.

Note that these are very similar to the geodesics defined in the lower dimensional case, showing that indeed we have created an accurate generalization to higher dimensional space.

## 3.2 Complex 2-space

Although the above section was restricted to Riemannian Manifolds in $\mathbb{R}^n$, there are some especially interesting generalizations to higher dimensional complex space. We examine the case of $\mathbb{C}^2$.

If we let $A$ be a complex matrix, then we define the *Hermitian transpose* as the transpose and conjugation of all terms of $A$ to be $A^*$. A matrix is *Hermitian* if it is equal to its Hermitian transpose, and we can define the *Hermitian Form* $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is given by $\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* A \mathbf{z}$.

We let $\mathbb{C}^{2,1}$ be $\mathbb{C}^3$ equipped with a Hermitian form of signature $(2, 1)$.. The two canonical Hermitian forms are defined below.

- The *first Hermitian form* is defined to be

$$\langle \mathbf{z}, \mathbf{w} \rangle_1 = z_1 \overline{w_1} + z_2 \overline{w_2} - z_3 \overline{w_3}$$

with Hermitian matrix

$$J_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- The *second Hermitian form* is defined to be

$$\langle \mathbf{z}, \mathbf{w} \rangle_2 = z_1 \overline{w_3} + z_2 \overline{w_2} + z_3 \overline{w_1}$$

with Hermitian matrix

$$J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Note that in $\mathbb{C}^{2,1}$, $\langle z, z\rangle \in \mathbb{R}$ for both cases. We can split up$\mathbb{C}^{2,1}$ into the set

$$
\begin{aligned}
V_- &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle < 0\} \\
V_0 &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle = 0, z \neq 0\} \\
V_+ &= \{z \in \mathbb{C}^{2,1} | \langle z, z\rangle > 0\}
\end{aligned}
$$

And note that these sets are closed under scaling. We define the *projection map* $\mathcal{P}$

$$
\mathcal{P} : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \to (z_1/z_3, z_2/z_3) \in \mathbb{C}^2
$$

for points where $z_3 \neq 0$. Examining the image of the negative lines under the projection map, we define the *projective model* of complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ given by

$$
\mathbf{H}^2_{\mathbb{C}} = \mathcal{P}(V_-) \quad \partial\mathbf{H}^2_{\mathbb{C}} = \mathcal{P}(V_0)
$$

since it closed up to scaling, we just assume that $z_3 = 1$ WLOG when taking our mapping.

Under the first Hermitian form, we obtain the *unit ball model* of $\mathbf{H}^2_{\mathbb{C}}$

$$
\mathbf{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}
$$

and we define the *standard lift* of a point on the ball as $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}$

Under the the second Hermitian form, we obtain the *Siegel domain model* of $\mathbf{H}^2_{\mathbb{C}}$ given by

$$
\mathbf{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : 2\Re(z_1) + |z_2|^2 < 0\}
$$

with *standard lift* $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix}$ and at $\infty$ (which makes the Siegel domain to be compact) we define the standrard lift to be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

We can define the distance as the *Bergman metric* by $\rho(z, w)$ where

$$
\cosh^2\left(\frac{\rho(z, w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w}\rangle \langle \mathbf{w}, \mathbf{z}\rangle}{\langle \mathbf{z}, \mathbf{z}\rangle \langle \mathbf{w}, \mathbf{w}\rangle}
$$

where $\mathbf{z}, \mathbf{w}$ are the standard lifts of $z, w$. This gives the distance between two points in both the ball models and Siegel domain models.

There are many parallels between these models and those of one complex dimension. In particular, we can define a similar class of isometries $PU(2,1)$.

$U(2,1)$ is defined to be group of unitary matrices and define the *projective unitary group* $PU(2,1)$ to be $U(2,1)/U(1)$, where $U(1)$ is the set of rotation matrices.

**Thm 3.2.1** $PU(2,1)$ and elements of $PU(2,1)$ followed by the complex conjugate are the isometries of complex 2-space.

**Pf:** The proof for $PU(2,1)$ is omitted for brevity. The conjugate map is an isometry due as $\frac{\langle \mathbf{z},\mathbf{w}\rangle\langle \mathbf{w},\mathbf{z}\rangle}{\langle \mathbf{z},\mathbf{z}\rangle\langle \mathbf{w},\mathbf{w}\rangle}$ is invariant under complex conjugation.

$\square$

In a similar manner, we can classify the isometries. In particular, they are one of the following categories.

- *Loxodromic* if it fixes exactly two points of $\partial\mathbf{H}^2_{\mathbb{C}}$
- *Parabolic* if it fixes exactly one point of $\partial\mathbf{H}^2_{\mathbb{C}}$
- *Elliptic* if it fixes at least one point fo $\partial\mathbf{H}^2_{\mathbb{C}}$

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