Rudin Ch7 - Differentiation

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1 Notes

1.1 Derivatives of Measures

7.1 Thm Suppose μ is a complex Borel measure on $\mathbb{R}^1$ and

$$f(x) = \mu((-\infty, x)) \quad x \in \mathbb{R}^1$$

If $x \in \mathbb{R}^1$ and A is a complex number, each of the following two statements implies the other:

(a) f is differentiable at x and $f'(x) = A$.

(b) To every $\epsilon > 0$ corresponds a $\delta > 0$ s.t.

$$\left| \frac{\mu(I)}{m(I)} - A \right| < \epsilon$$

for every open I that contains x and whose length is $< \delta$. m is the Lebesgue measure.

7.2 Def Let the open ball with center $x \in \mathbb{R}^k$ and radius $r > 0$ be given by

$$B(x, r) = \{y \in \mathbb{R}^k : |y - x| < r\}$$

The *quotients* associated with μ are

$$(Q_r\mu)(x) = \frac{\mu(B(x,r))}{m(B(x,r))}$$

and the *symmetric derivative* is given by

$$(D\mu)(x) = \lim_{r \to 0} (Q_r\mu)(x)$$

the *maximal function* $M\mu$ for $\mu \geq 0$, is defined by

$$(M\mu)(x) = \sup_{0<r<\infty} (Q_r\mu)(x)$$

and for a complex measure μ M is the same as that of $|\mu|$.

7.3 Lemma If W is the union of a finite collection of balls $B(x_i, r_i)$, $1 \le i \le N$, then there is a set $S \subset \{1, \ldots, N\}$ so that

(a) the balls $B(x_i, r_i), i \in S$ are disjoint.

(b) $W \subset \bigcup_{i \in S} B(x_i, 3r_i)$ and

(c) $m(W) \le 3^k \sum_{i \in S} m(B(x_i, r_i))$.

7.4 Thm If μ is a complex Borel measure on $\mathbb{R}^k$ and λ is a positive number, then

$$m\{M\mu > \lambda\} \le 3^k \lambda^{-1} ||\mu||$$

where $||\mu|| = |\mu|(\mathbb{R}^k)$ and the left side is

$$m(\{x \in \mathbb{R}^k : (M\mu)(x) > \lambda\})$$

7.5 Weak L^1 We notice that, by definition,

$$m\{|f| > \lambda\} \le \lambda^{-1} ||f||_1$$

and we say that $f \in L^1(\mathbb{R}^k)$ is weak L^1 if

$$\lambda \cdot m\{|f| > \lambda\}$$

is a bounded function on $(0, \infty)$. The *maximal function* of f $Mf : \mathbb{R}^k \to [0, \infty]$ is given by

$$(Mf)(x) = \sup_{0 < r < \infty} \frac{1}{m(B_r)} \int_{B(x,r)} |f| dm$$

Similarly, by 7.4, we also have the equality

$$m\{Mf > \lambda\} \le 3^k \lambda^{-1} ||f||_1$$

7.6 Lebesgue Points If $f \in L^1(\mathbb{R}^k)$, any $x \in \mathbb{R}^k$ for which

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| dm(y) = 0$$

are *Lebesgue Points.* These are points in which the average is small on small balls centered at x, ie there is little oscillation.

7.7 Thm If $f \in L^1(\mathbb{R}^k)$, then almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.

7.8 Thm Suppose μ is a complex Borel measure on $\mathbb{R}^k$ and $\mu \ll m$. Let f be the *Radon-Nikodym* derivative of μ with respect to m. Then $D\mu = f$ a.e. $[m]$ and

$$\mu(E) = \int_E (D\mu) dm$$

7.9 Nicely Shrinking Sets Suppose $x \in \mathbb{R}^k$. A sequence $\{E_i\}$ of Borel sets in $\mathbb{R}^k$ *shrinks to x nicely* if there is an $\alpha > 0$ and a sequence of balls $B(x, r_i)$ with $\lim r_i = 0$ s.t. $E_i \subset B(x, r_i)$ and

$$m(E_i) \geq \alpha m(B(x, r_i))$$

7.10 Associate to each $x \in \mathbb{R}^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely, and let $f \in L^1(\mathbb{R}^k)$. Then

$$f(x) = \lim_{i \to \infty} \frac{1}{m(E_i(x))} \int_{E_i(x)} f dm$$

at every Lebesgue point of f. Hence a.e. [m].

7.11 Thm If $f \in L^1(\mathbb{R}^1)$ and

$$F(x) = \int_{-\infty}^{x} f dm \quad (-\infty < x < \infty)$$

then $F'(x) = f(x)$ at every Lebesgue point of f a.e. [m].

7.12 Metric Density Let E be a Lebesgue measurable subset of $\mathbb{R}^k$. The *metric density* of E at point $x \in \mathbb{R}^k$ is defined to be

$$\lim_{r \to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

7.13 Thm Associate to each $x \in \mathbb{R}^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely. If μ is a complex Borel measure and $\mu \perp m$, then

$$\lim_{i \to \infty} \frac{\mu(E_i(x))}{m(E_i(x))} = 0 \quad \text{a.e. [m]}$$

7.14 Thm Suppose that to each $x \in \mathbb{R}^k$ is associated sequences $\{E_i(x)\}$ that shrinks to x nicely and μ is a complex Borel measure on $\mathbb{R}^k$. If we let $\mu = f dm + d\mu_s$ be the Lebesgue decomposition of μ with respect to m, then

$$\lim_{i \to \infty} \frac{\mu(E_i(x))}{m(E_i(x))} = f(x) \quad \text{a.e. [m]}$$

In particular, $\mu \perp m$ iff $(D\mu)(x) = 0$ a.e. [m].

7.15 Thm If μ is a positive Borel measure on $\mathbb{R}^k$ and $\mu \perp m$, then

$$(D\mu)(x) = \infty \quad \text{a.e. [m]}$$

1.2 The Fundamental Theorem of Calculus

7.16 The other way of the fundamental theorem of calculus (derivatives of integral) is given as

$$f(x) - f(a) = \int_a^x f'(t)dt \quad (a \le x \le b)$$

Note that even if f is continuous on $[a, b]$, f is differentiable on $[a, b]$ and $f' \in L^1$ on $[a, b]$. The counterexample uses Cantor's middle third set to disprove it.

7.17 Def A complex function f, defined on an interval $I = [a, b]$ is said to be absolutely continuous on I (shorthand: f is AC on I) if there corresponds to every $\epsilon > 0$ a $\delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \epsilon$$

for any n and any disjoint sets $(\alpha_1, \beta_1) \dots (\alpha_n, \beta_n)$ in I where

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta$$

7.18 Thm Let $I = [a, b]$, let $f : I \to \mathbb{R}^1$ be continuous and nondecreasing. Each of the following three statements about f implies the other two:

(a) f is AC on I
(b) f maps sets of measure 0 to sets of measure 0.
(c) f is differentiable a.e. on I, $f' \in L^1$ and

$$f(x) - f(a) = \int_a^x f'(t)dt \quad (a \le x \le b)$$

7.19 Thm Suppose $f : I \to \mathbb{R}^1$ is AC, $I = [a, b]$. Define

$$F(x) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \quad (a \le x \le b)$$

where the supremum is taken over all N and over all choices of $\{t_i\}$ s.t.

$$a = t_0 < t_1 < \cdots < t_N = x$$

then $F, F + f, F - f$ are nondecreasing and AC on I. F is the *total variation function* and if $F(b) < \infty$ then f has *bounded variation* and $F(b)$ is the *total variation.*

7.20 Thm If f is a complex function that is AC on $I = [a, b]$, then f is differentiable at almost all points of $I, f' \in L^1(m)$, and

$$f(x) - f(a) = \int_a^x f'(t)dt \quad (a \le x \le b)$$

7.21 Thm If $f : [a, b] \to \mathbb{R}^1$ is differentiable at every point of $[a, b]$ and $f' \in L^1$ on $[a, b]$, then

$$f(x) - f(a) = \int_a^x f'(t)dt \quad (a \le x \le b)$$

1.3 Differentiable Transformations

7.22 Def Suppose V is an open set in $\mathbb{R}^k$, T maps V into $\mathbb{R}^k$, and $x \in V$. If there exists a linear operator A on $\mathbb{R}^k$ s.t.

$$\lim_{h \to 0} \frac{|T(x+h) - T(x) - Ah|}{|h|} = 0$$

Then T is *differentiable* and A is the *derivative/ differential.* It is also called the *Jacobian* of T at x and is denote by $J_T(x)$.

7.23 Lemma let $s = \{x : |x| = 1\}$ be the sphere in $\mathbb{R}^k$ that is the boundary of the open unit ball $B = B(0, 1)$. If $F : \overline{B} \to \mathbb{R}^k$ is continuous, $0 < \epsilon < 1$, and

$$|F(x) - x| < \epsilon$$

for all $x \in S$, then $F(B) \supset B(0, 1 - \epsilon)$.

7.24 Thm If

(a) V is open in $\mathbb{R}^k$,
(b) $T : V \to \mathbb{R}^k$ is continuous, and
(c) T is differentiable at some point $x \in V$, then

$$\lim_{r \to 0} \frac{m(T(B(x, r)))}{m(B(x, r))} = \triangle(T'(x))$$

7.25 Lemma Suppose $E \subset \mathbb{R}^k$, $m(E) = 0$, T maps E into $\mathbb{R}^k$, and

$$\limsup \frac{|T(y) - T(x)|}{|y - x|} < \infty$$

for every $x \in E$ as $y \to x$ within E. Then $m(T(E)) = 0$.

7.26 Thm Suppose that

(i) $X \subset V \subset \mathbb{R}^k$, V is open, $T : V \to \mathbb{R}^k$ is continuous.
(ii) X is Lebesgue measurable, T is one-to-one on X, and T is differentiable at every point of X.
(iii) $m(T(V - X)) = 0$.

Then, setting $Y = T(X)$,

$$\int_Y f dm = \int_X (f \circ T)|J_T| dm$$

for every measurable $f : \mathbb{R}^k \to [0, \infty]$.

2 Problems

Problem 1 We note that, for Lebesgue point x, we can apply the triangle inequality and find that

$$\begin{aligned} |f(x)| &\leq \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| dm + \frac{1}{m(B_r)} \int_{B(x,r)} |f(y)| dm \\ &\leq \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| dm + (Mf)(x) \end{aligned}$$

and this shows $|f(x)| \leq (Mf)(x)$ as we take r to 0.

Problem 3 We proceed with the hint and pick $\alpha \in \mathbb{R}^1$ where $F(x) = m(E \cap [\alpha, x])$. Note that

$$F(x + p_i) - F(x - p_i) = F(y + p_i) - F(y - p_i)$$

since we can rearrange this to get

$$m(E \cap (x + p_i, y + p_i]) = m(E \cap (x - p_i, y - p_i])$$

and this is obvious since $E = E + 2p_i$ and m is translation invariant.

Notice that our $F(x)$ is obviously nondecreasing, and, futhermore, setting $\delta = \epsilon$ shows F is AC. It follows by Thm 7.18 that

$$F(x) - F(\alpha) = \int_\alpha^x F'(t) dt \implies F(x) = \int_\alpha^x F'(t) dt$$

It follows that $F' = \chi_E$. Furthermore, we note that

$$F(x + p_i) - F(x - p_i) = \int_{x - p_i}^{x + p_i} \chi_E dm$$

We notice that the metric density at each point of $\mathbb{R}^1$ is 0 or 1 a.e. Furthermore, we know that $F(x+p_i) - F(x-p_i)$ is the same for all x. If $F(x+p_i) - F(x-p_i) = 0$, then it follows that $m(E) = 0$. This can be shown through the telescoping sums

$$F(x+p_i) - F(x-p_i) + F(x+2p_i) - F(x+p_i) + \dots$$

If it is 1 as $i \to \infty$, then it follows that, through the same telescoping sums, that $\int_x^y \chi_E dm = m([x,y])$ through alternating sums, which implies that $\mathbb{R} \setminus E$ has measure 0, as desired.

Problem 5 We note that, by definition of metric density of 1, we for each $\epsilon > 0$ we have some δ s.t.

$$r < \delta \implies |\frac{m(E \cap B(x,r))}{\mu(B(x,r))} - 1| < \epsilon$$

Therefore, for points $a_0 \in A$, $b_0 \in B$ with metric density 1, we have some value of δ_0 for which

$$m(A \cap B(a_0, \delta_0)) = (1-\epsilon) m(B(a_0, \delta_0))$$

It follows that A is present a.e. on $B(a_0, \delta_0)$. Picking our δ_0 s.t. $\delta_0 = |\epsilon| + \delta$ in our hint, we have

$$m(A \cap B_\epsilon) = B_\epsilon$$

so they intersect on B_ϵ a.e. It follows that $B_\epsilon + Bc_0 + \epsilon \subset A + B$, and so it follows that for some ϵ we have $c_0 \pm \epsilon$ is a segment in our $A + B$.

The Cantor set C is all elements of the form

$$0.c_1c_2c_3 \dots \quad c_i \in \{0, 2\}$$

Notice that $\frac{1}{2}C$ is given by

$$0.a_1a_2a_3 \dots \quad a_i \in \{0, 1\}$$

It can be seen that every element of $[0,1]$ is given by $\frac{1}{2}C + \frac{1}{2}C$ as each 0 is $0+0$, 1 is given by $1+0$, and 2 is given by $1+1$ and this separation results in two elements in $\frac{1}{2}C$.

Therefore

$$\frac{1}{2}C + \frac{1}{2}C = [0,1] \implies C + C = [0,2]$$

Problem 7 We enumerate the rationals into $\{q_n\}$ and define functions

$$f_n = \frac{1}{2^n} \chi_{[q_n, \infty)}$$

We set

$$f = \sum_{n=1}^{\infty} f_n$$

Note that this function's derivative is 0 a.e. since the derivatives of f_n are 0 (except on the rationals) and $f = \sum f_n \implies f' = \sum f_n'$. It's monotonic since each function is increasing, and every segment can't have constant value since the rationals are dense in the reals.

Problem 9 WLOG let $x < y$. We notice that the Lipchitz constant

$$\frac{|f(y) - f(x)|}{|y - x|^{\alpha}}$$

takes a maximal value when the segment $[x, y]$ to be is one of the 2^n segments in E_n whose union forms E_n. Under this circumstance, $f_i(y) - f_i(x)$ doesn't decrease as $i \to \infty$, meaning it reaches a maximal value here for a given $|y - x|$.

Simple calculation shows that

$$\frac{|f(y) - f(x)|}{|y - x|^{\alpha}} \leq \frac{2^{-n}}{\frac{1}{t}^{n\alpha}} = \frac{2^{-n}}{2^{-n}} = 1$$

so it follows that our f is indeed Lipα.

Problem 11 We note that our condition that $p \in L^1$ implies that

$$\sqrt[p]{\int_x^y |f'|^p dm} < \infty$$

We notice that this implies that

$$\infty > \sqrt[p]{\int_x^y |f'|^p dm} < \infty = (y - x)^{1/p} \sqrt[p]{\frac{1}{y - x} \int_x^y |f'|^p dm}$$

$$\geq (y - x)^{1/p} (\frac{\int_x^y |f'| dm}{y - x}) \geq \frac{|f(y) - f(x)|}{|y - x|^{\alpha}}$$

where the first inequality is due to x^p being convex and the last equality is due to $|x|$ being convex. It follows that $f \in$ Lipα.

Problem 13
(a) WLOG let f be increasing. Note that $|f(a)|, |f(b)| < \infty$ so $F(b) \leq f(b) - f(a) < \infty$, which implies that $f \in BV$.

(b) We note that if F has bounded variation, then we note then obviously f is bounded. It follows that $F + f$ and $F - f$ are bounded and monotonic, which means they are in BV. Similarly, $\frac{F+f}{2}$ and $\frac{F-f}{2}$ are bounded and monotonic, so $f = \frac{F+f}{2} - \frac{F-f}{2}$ is our desired construction.

(c) Our construction as above works since F is AC and therefore $F + f$, and $F - f$ are left continuous if f is.

(d) If f is AC, then it follows through 7.18 that

$$f(x) - f(a) = \int_a^x f' dm = \mu([a, x))$$

Therefore, if $m(E) = 0$ that obviously means that $\int_E f' dm = 0 \implies \mu(E) = 0$.

The other direction again follows from 7.18.

(e) This follows from the fact that f is AC (required for 7.19) and Theorem 7.18.

Problem 15 The function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is obviously differentiable and continuous (can check using $\epsilon - \delta$ and the fact that it is bounded by x^2. However, we find the derivative to be

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

and therefore the derivative is not continuous at 0 although it exists at every x.

Problem 17 We notice that for a countable collection of pairwise disjoint sets $\{A_i\}$ we have

$$\mu\Big(\bigcup A_i\Big) = \sum_{n=1}^{\infty} \mu_n \Big(\bigcup A_i\Big) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \mu_n(A_i)$$
$$= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} \mu_n(A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

and so it follows that it is a Borel measure. We notice that the Lebesgue decomposition of each $\mu_n = \lambda_a^n + \lambda_s^n$ implies that

$$\mu = \sum_{i=1}^{\infty} \lambda_a^i + \sum_{i=1}^{\infty} \lambda_s^i$$

and notice that these measures satisfy $\sum \lambda_a^i = \lambda_a \ll m$ and $\sum \lambda_s^i = \lambda_s \perp m$. To see this, note that

$$m(E) = 0 \implies \lambda_a^i(E) = 0 \implies \lambda_a(E) = 0$$

and for sets A_i concentrated for λ_s^i we have λ_s concentrated at $\bigcap A_i$.

It follows that

$$(D\mu)(x) = \frac{\mu(B(x,r))}{m(B(x,r))} = \frac{\sum_{n=1}^{\infty} \mu_n(B(x,r))}{m(B(x,r))} = \sum_{n=1}^{\infty} \frac{\mu_n(B(x,r))}{m(B(x,r))} = \sum_{n=1}^{\infty} (D\mu_n)(x)$$

For functions, only the sum holds as M relies on the sup, which is not guaranteed to line up.

Problem 19 We notice that the function $\sup f < \infty$. Considering the individual sections $x \in [\frac{1}{n+1}, \frac{1}{n})$ we notice that these values can take a maximum value $h_c(x) = n^c \sup f$.

Our values for $h_c(x)$ is given by

$$\sum_{n=1}^{\infty} \int_{\frac{1}{n+1}}^{\frac{1}{n}} n^c \sup f = \sup f(\sum_{n=1}^{\infty} \frac{n^c}{n(n+1)})$$

Notice for $0 < c < 1$, our function is dominated by

$$\sup f(\sum_{n=1}^{\infty} \frac{1}{(n+1)^c})$$

and so it converges and is L^1. However, when $c = 1$, our value is given by

$$\sup f(\sum_{n=1}^{\infty} \frac{1}{n+1})$$

which diverges. For λ, the measure of all $x : f(x) > \lambda$ is given as follows. Letting $n = \lceil \frac{\lambda}{\sup f} \rceil$ we notice that our set $\subset (0, \frac{1}{n})$. It follows that $\lambda m\{x : f(x) > \lambda\} = \frac{\lambda}{n} < \sup f$ and so it is weak L^1.

However, when $c > 1$ we notice that our function is defined by $\frac{\lambda}{n}$ for $n = \lceil \sqrt[c]{\frac{\lambda}{\sup f}} \rceil$ so it follows that our function in not bounded above.

Defining f_ϵ which takes a value of 1 on $[\epsilon, 1-\epsilon]$ and is linear on $[0, \epsilon], [1-\epsilon, 1]$ is clearly in our set. However, as $\epsilon \to 0$ the limit of our functions is not in weak L^1.

Problem 21 We denote F to be the total variation of f and G to be the total variation of γ. Notice that

$$
\begin{aligned}
G(1) &= \sup \sum_{i=1}^{N} |t_{i+1} + if(t_{i+1}) - t_i - if(t_i)| \\
&\leq 1 + \sup \sum_{i=1}^{N} |f(t_{i+1}) - f(t_i)| = 1 + F(1)
\end{aligned}
$$

and, the other way is also

$$
G(1) + 1 = \sup \sum_{i=1}^{N} |t_{i+1} + if(t_{i+1}) - t_i - if(t_i)| + \sum_{i=1}^{N} |t_{i+1} - t_i| \geq F(1)
$$

so one being finite implies the other. To prove the arc-length formula, it suffices to note that $G(1)$ is given by

$$
\sup \sum_{i=1}^{N} \sqrt{(t_{i+1}^2 - t_i^2)^2 + (f(t_{i+1}) - f(t_i))^2}
$$

and using the mean value theorem shows that there exists a c_i s.t. $\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i} = f'(c_i)$ and threfore our sum is given by

$$
\sup \sum_{i=1}^{N} (t_{i+1} - t_i) \sqrt{1 + f'(c_i)}
$$

and so it follows that as we take $N \to \infty$ we can consider this is an integral instead of a Riemann sum. Our integral is given by

$$
\int_0^1 \sqrt{1 + (f'(t))^2} dt
$$

Problem 23 Obviously if x is a Lebesgue point of f, then it is also a Lebesuge point of F since $f \in F$. Clearly, it follows that $SF(X) = f(x) \implies f \in F$. Hence, $SF \in F$, and in particular, is the $f \in F$ with maximal number of Lebesgue points, as all Lebesgue points of F must belong to some $f \in F$.