Erdmann-Wildon Lie Algebras - Low-Dimensional Lie Algebras

Aaron Lou

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1 Notes

1.1 Dimensions 1 and 2

Any one dimensional Lie Algebra is necessarily abelian.

If L is a non-abelian algebra of dimension 2, then notice that L' can't have dimension more than 1 as $[x, y]$ spans L' where x, y are the bases. In particular, the vector space has

$$[x, y] = x$$

Thm 3.1 Let F be any field. Then there is a unique two-dimensional non-abelian algebra over F up to (isomorphism) $[x, y] = x$.

1.2 Dimension 3

1.2.1 The Heisenberg Algebra

If L' is one-dimensional and $L' \subset Z(L)$, then the *Heisenberg Algebra* is the unique Lie Algebra with basis f, g, z s.t. $[f, g] = z$ and $z \in Z(L)$.

1.3 Another Lie Algebra where dim $L' = 1$

The direct sum $L_1 \oplus L_2$ where L_1 is the two dimensional non-abelian Lie Algebra and L_2 is a one dimensional Lie Algebra. We note that

$$L' = L_1' \oplus L_2' \quad Z(L) = Z(L_1) \oplus Z(L_2)$$

so it is one dimensional and L' is not contained in the center.

Thm 3.2 Let F be any field. There is a unique 3-dimensional Lie Algebra over F s.t. L' is a 1-dim and L' is not contained in $Z(L)$, then this is the direct sum of the 2-dim non-abelian Lie Algebra with the 1-dim Lie Algebra.

1.4 Lie Algebras with a 2-Dimensional Derived Algebra

Examining it for $\mathbb{C}$, for dim$L = 3$ and dim$L' = 2$, there are infinitely many non-isomorphic Lie Algebras. We note that

Lemma 3.3
(a) The derived algebra L' is abelian.
(b) The linear map ad$x : L' \to L'$ is an isomorphism.

There are many ways to classify the complex Lie Algebras.

Case 1: If we have $x \notin L'$ s.t. ad$x : L' \to L'$ is diagonalisable, we let y and z be eigenvectors. Then (after much rescaling) we find that the map ad$x$ has matrix (with respect to y, z).

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}$$

for $\mu \neq 0$. We call this L_μ.

Case 2: If our map ad$x$ is not diagonalisable, we must have an eigenvector y, and, after rescaling $[x, y] = y$. We have some other vector z s.t. $\{y, z\}$ is a basis of L'. We have, after more rescaling, $[x, y] = y + \mu z$ and the matrix of ad$x$ relatively to L' is given by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & \mu \end{pmatrix}$$

and we find that $\mu = 1$ since A can't be diagonalizable.

1.4.1 Lie Algebras where $L' = L$

We already know that we have the Lie Algebra $L = \mathbf{sl}(2, \mathbb{C})$. Up to isomorphism, it is the only one.

Step 1: For $x \neq 0$, we claim ad$x$ has rank 2. Notice that this is obvious as ad$x$ applied to y, z are linearly independent.

Step 2: If we have $h \in L$ s.t. ad$h : L \to L$ has an eigenvector with a non-zero eigenvalue, then the Jordan form of ad$h$ when $h = x$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Similarly, $h = y$ has x with an eigenvalue of -1.

Step 3: If we have $[h, x] = \alpha x \neq 0$ then the eigenvalues are $\alpha, 0, -\alpha$, where $[h, y] = -\alpha y$. $\{h, x, y\}$ is a basis of L.

Step 4: We attempt to find $[x, y]$. Note that $[h, [x, y]] = 0$ and therefore, since ad$h$ has image of dim 2, $[x, y] = \lambda h$ and we can scale this down (since $\lambda \neq 0$). We can scale h with respect to α^2 and, with $\alpha = 2$, we find that our values are

$$[h, x] = 2x \quad [h, y] = -2y \quad [x, y] = h$$

And there is only one such 3-dim complex Lie Algebra with $L' = L$.

2 Exercises

Ex 3.1 We can check that our mapping is indeed bilinear, alternative, and satisfies the Jacobi identity rather trivially.

Note that L' is constructed based on $\varphi(y), y \in V$. It necessarily follows that dimL' is the rank of φ as this would be the dimension of the image of φ.

Ex 3.2 For μ and ν, we have variables

$$[x_1, y_1] = y_1, [x_1, z_1] = \mu z_1$$
$$[x_2, y_2] = y_2, [x_2, z_2] = \nu z_2$$

If $\mu = \nu$, then we simply set $x_1 = x_2, y_1 = y_2, z_1 = z_2$. If $\mu = \nu^{-1}$, then we set $x_1 = \mu x_2, y_1 = z_2, z_1 = y_2$. To prove the other way, we note that

$$[\varphi(x_1), \varphi(y_1)] = \varphi([x_1, y_1]) = \varphi(y_1)$$
$$[\varphi(x_1), \varphi(z_1)] = \varphi([x_1, z_1]) = \mu\varphi(z_1)$$

We notice that $\varphi(x_1), \varphi(y_1), \varphi(z_1)$ has $\{\varphi(y_1), \varphi(z_1)\}$ as a basis of L'_ν. Based on the eigenvalues, we have either $\mu = \nu$ or $\mu\nu = 1$, as desired.

Ex 3.3

(i) We note that, for a matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then we calculate

$$a = e = i = 0, b + d = 0, c - g = 0, f - h = 0$$

and so the matrix is given by

$$\begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ \beta & \gamma & 0 \end{pmatrix}$$

and notice that for the basis

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

And we have $[x, y] = -z$, $[x, z] = y$, and $[y, z] = x$ and we can see that this is the case when $L' = L$ where $\alpha = 1$.

(ii) We notice that

$$[u, v] = (\lambda - \nu)v, [u, w] = (\mu - \nu)w, [v, w] = 0$$

If both are 0, then L is the abelian 3-dim Lie Algebra. If one is not 0, then it is the Lie Algebra with L' dimension 1. Otherwise this is the case where L' has dimension 2 and is given by $L_{\frac{\lambda-\nu}{\mu-\nu}}$.

(iii) We define the basis

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and find that $[x, z] = y, [x, y] = 0, [y, z] = 0$. This is the Heisenberg Algebra.

(iv) The basis is

$$x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} y = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and we find that this is the Abelian 3-dim Lie Algebra.

Ex 3.4 We note that, if $[x, y] = 0$ then this is just the Abelian algebra. If $[x, y] \neq 0$, then WLOG let $[x, y] = 0$. Note that $[x, [x, y]] + [x, [y, x]] + [y, [x, x]] = 0$, as desired. Therefore, L is a Lie Algebra.

Ex 3.5 We notice that $\mathrm{sl}(2,\mathbb{R})$ can be given by the basis e, f, h in problem 1.12 where

$$[e,f]=h,[e,h]=-2e,[f,h]=2f$$

Notice for $\mathrm{ad}h$, the matrix is diagonalizable. Examining $\mathbb{R}^3_\wedge$ reveals that a basis

$$x=\begin{pmatrix}1\\0\\0\end{pmatrix}\ y=\begin{pmatrix}0\\1\\0\end{pmatrix}\ z=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

has the Lie Map based on

$$[x,y]=z,[x,z]=-y,[y,z]=x$$

and we notice that no map $\mathrm{ad}x$ is diagonalizable (since bases are mapped to different bases). We can clearly conclude that no such mapping is diagonalizable, so it can't be isomorphic $\mathrm{sl}(2,\mathbb{R})$.

Ex 3.6 Letting $\mathrm{ad}x$ have eigenvalues $0, \alpha, \overline{\alpha}$ where α is a complex number. In particular, this can be a multiple of i. The matrix of $\mathrm{ad}x$ is given by

$$\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}$$

and this is isomorphic to $R^3_\wedge$. The other one is $\mathrm{sl}(2,\mathbb{R})$.

Ex 3.7 This follows relatively quickly, at there must be x, y s.t. $[x,y]\neq 0$. These must be linearly independent (or else it would be 0) and we simply extend this to a basis of L. It follows that $\dim Z(L)\leq \dim L-2$.

Ex 3.8 We notice that if we take

$$D(f)=af+bg+cz \quad D(g)=df+eg+fz$$

then the only condition that implies is $D(z)=(a+e)z$ and everything else works. Therefore, the Derivations corresponding with the the bases a, b, c, d, e, f form the 6-dim basis of Der L.

We note that inner derivations of L are

$$\mathrm{ad}fg=z \quad \mathrm{ad}gf=-z \quad \mathrm{ad}z=0$$

and $\mathrm{Der}L/\mathrm{IDer}L$ is the basis is based on $D(f)=\{f,g\}, D(g)=\{f,g\}$. After much calculation, this works out to be isomorphic to $\mathrm{gl}(2,\mathbb{R})$, with bases

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

and so they are isomorphic.

Ex 3.9
(i) We notice that $\theta([s,t])x = [[s,t],x]$ can be simplifies as

$$\begin{aligned}
[[s,t],x] &= [s,[t,x]] + [t,[x,s]] \\
&= \theta(s) \circ \theta(t)x - \theta(t) \circ \theta(s)x \\
&= [\theta(s), \theta(t)]x
\end{aligned}$$

which proves that θ is a homomorphism.

(ii) It is clear that bilinearity and alternativity are satisfied. To prove the Jacobi identity, notice that

$$[(s_1,x_1),[(s_2,x_2),(s_3,x_3)]] = [(s_1,x_1),([s_2,s_3],[x_2,x_3] + \theta(s_2)x_3 - \theta(s_3)x_2)]$$

$$= ([s_1,[s_2,s_3]],[x_1,[x_2,x_3] + \theta(s_2)x_3 - \theta(s_3)x_2]$$

$$+\theta(s_1)([x_2,x_3] + \theta(s_2)x_3 - \theta(s_3)x_2) - \theta([s_2,s_3])x_1)$$

$$= (\theta(s_1) + \mathrm{ad}x_1)([x_2,x_3] + \theta(s_2)x_3 - \theta(s_3)x_2) - \theta(s_2) \circ \theta(s_3)x_1 + \theta(s_3) \circ \theta(s_2)x_1$$

and we notice that as we cycle through, the first sum of s goes to 0 (under the Jacobi Identity of S), the solely x terms go to 0 (under the Jacobi identity of X), and the remaining terms sum up to 0 (since they are various permutation of $\theta(x_i) \circ \theta(x_j)x_k$.

To show that this is a semiproduct, note that it is a direct sum and also $L = IS$, as desired.

(iii) This is pretty obvious as we take $\{x\}$ to be our s and $\varphi = \theta(x)$ and we construct $y = (0,y), y \in V$, $x = (x,0)$ and note that $[y,z] = 0, y,z \in V$ is our Abelian Lie Algebra base.

(iv) This occurs when I and S are isomorphic between the various Semidirect products.

Ex 3.10 The basic approach is to examine when $L' \not\subset Z(L)$, $L' = Z(L)$ and $L' \subset Z(L)$ and this is strict.

In the first case, we use the same construction as in Theorem 3.2 to show that, for

$$[x,y] = x \quad \{a_i, \ldots a_n\}$$

we can construct $\{a_i \ldots a_n\}$ s.t. they are Abelian. This can be shown by taking b_i s.t.

$$[x, b_i] = p_i x \quad [y, b_i] = q_i x$$

and noticing for $a_i = \lambda_1 x + \lambda_2 y + \mu_1 b_1 + \dots \mu_{i-1} b_{i-1}$, we can construct a_i to be as desired. This shows that we have a direct sum of an Abelian Lie Algebra and a two-dimensional non-abelian one.

If $L' = Z(L)$ then we have

$$\{x, f_1, g_1, \dots f_n, g_n\}$$

as our basis. This can be shown through induction, where $[f_i, g_i] = x$, $[f_i, g_j] = 0$, $[f_i, f_j] = 0$ and $[g_i, g_j] = 0$.

The last case is when $L' \subset Z(L)$. In this case it is obvious that there is a subspace K s.t. $K' = \{x\}$, $\{x\} = Z(K)$ and that $L = K \oplus A$, where A is an Abelian Lie Algebra.