Rudin Real and Complex Analysis - Approximation by Rational Functions

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1 Notes

1.1 Preparation

13.1 The Riemann Sphere $\mathbb{S}^2$ the *Riemann sphere* is the union of $\mathbb{R}^2$ and ∞. The topology is as follows. For any $r > 0$, let $D'(\infty, r)$ be the set of all complex z s.t. $|z| > r$. $D(\infty, r) = D'(\infty, r) \cup \{\infty\}$. A subset of $\mathbb{S}^2$ is open iff it is the union of discs $D(a, r)$. $\mathbb{S}^2$ is homeomorphic to a sphere. $\varphi(\infty) = (0, 0, 1)$ and

$$\varphi(re^{i\theta}) = \left(\frac{2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{r^2-1}{r^2+1}\right)$$

If f is holomorphic in $D'(\infty, r)$, we say that f has an isolated singularity at ∞. If f is bounded in $D'(\infty, r)$ then $\underset{z\to\infty}{f}(z)$ exists and is complex, where $f(\infty)$ is this value. Let $\tilde{f}(z) = f(1/z)$ and the nature of this singularity is the same for $D'(0, 1/r)$. If $\tilde{f}$ has a pole of order m at 0, then f is said to have a pole of order m at ∞, the *principal part* of f at ∞ is then an ordinary polynomial of deg m. If we subtract this polynomial from f, we obtain a function with a removable singularity at ∞.

If $\tilde{f}$ has an essential singularity at 0, then f has an essential singularity at ∞. Note that "$\mathbb{S}^2 \setminus \Omega$" is connected is not equivalent to the complement of Ω is connected. For example $\Omega : x + iy : 0 < y < 1$ has a complement of two components, but $\mathbb{S}^2 \setminus \Omega$ has one.

13.2 Rational Functions A rational function f is the quotient of two functions P, Q. It follows that every nonconstant polynomial is a product of factors of degree 1. We can also assume that P and Q have no common factors. f has at a pole at each zero of the Q (same order as the zero of Q). If we subtract the corresponding principal parts, there is a rational function whose singularity is at ∞ and is a polynomial. Every $f = P/Q$ has thus a representation of the form

$$f(z) = A_0(z) + \sum_{j=1}^{k} A_j((z-a_j)^{-1})$$

where $A_0, \dots, A_k$ are polynomials, $A_1, \dots A_k$ have no constant term, and $a_1, \dots, a_k$ are the distinct zeros of Q. The above is the *partial fractions decomposition* of f.

13.3 Thm Every open set Ω in the plane is the union of a sequence $\{K_n\}$, $n = 1, 2, \dots$ of compact sets s.t.

(a) K_n lies in the interior of K_{n+1} for some n.

(b) Every compact subset of Ω lies in some K_n.

(c) Every component of $\mathbb{S}^2 \backslash K_n$ contains a component of $\mathbb{S}^2 \backslash \Omega$ for $n = 1, 2, \dots$.

13.4 Sets of Oriented Intervals Let Φ be a finite collection of oriented intervals in the plane. For each point p, let $m_I(p)[m_E(p)]$ be the number of members of Φ that have initial point [end point] p. if $m_I(p) = m_E(p)$ for every p, we say that Φ is *balanced*.

If Φ is balanced and nonempty, then we can construct as follows. Pick $\gamma_1 = [a_0, a_1] \in \Phi$. Assume $k \geq 1$ and $\gamma_1 \dots \gamma_k$ of Φ have been chosen s.t. $\gamma_i = [a_{i-1}, a_i]$ for $1 \leq i \leq k$. If $a_k = a_0$, stop. if not, and if r of the intervals have a_k as end point, then $r-1$ have a_k as initial point. Since Φ is balanced, Φ contains another intervals γ_{k+1} whose initial point is a_k. Since Φ is finite, we must return to a_0 at the nth step. Then $\gamma_1, \dots, \gamma_n$ join to form a closed path.

The remaining members of Φ still form a balanced collection to which the above construction can be applied. It follows that the members of Φ can be so numbered that they form finitely many closed paths. Thus, we see that if $\Phi = \{\gamma_1, \dots, \gamma_N\}$ is a balanced collection of oriented intervals, and if

$$\Gamma = \gamma_1 \dot{+} \dots \dot{+} \gamma_N$$

then γ is a cycle.

13.5 Thm If K is a compact subset of a plane open set $\Omega \neq \emptyset$, then there is a cycle Γ in $\Omega \setminus K$ s.t. the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

holds for every $f \in H(\Omega)$ and for every $z \in K$.

1.2 Runge's Theorem

13.6 Thm Suppose K is a compact set in the plane and $\{\alpha_j\}$ is a set which contains one point in each component of $\mathbb{S}^2 \setminus K$. If Ω is open $\Omega \supset K$, $f \in H(\Omega)$, and $\epsilon > 0$, there exists a rational function R, all of whose poles lie in the prescribed set $\{\alpha_j\}$ s.t.

$$|f(z) - R(z)| < \epsilon$$

for every $z \in K$.

13.7 Thm Suppose K is a compact set in the plane $\mathbb{S}^2 \setminus K$ is connected and $f \in H(\Omega)$, where Ω is some open set containing K. Then there is a sequence P_n of polynomials s.t. $P_n(z) \to f(z)$ uniformly on K.

13.8 Remark The preceding result is false for *every* compact K in the plane s.t. $\mathbb{S}^2 \setminus K$ is not connected. There exists a bounded component V. For $\alpha \in V$, $f(z) = (z - \alpha)^{-1}$ and $m = \max\{|z - \alpha| : z \in K\}$. If P is a polynomial s.t. $|P(z) - f(z)| < 1/m$ for all $z \in K$, then

$$|(z - \alpha)P(z) - 1| < 1$$

However for $z = \alpha$, we have $1 < 1$.

13.9 Thm Let Ω be an open set in the plane, let A be a set which has one point in each component of $\mathbb{S}^2 \setminus \Omega$ and assume $f \in H(\Omega)$. Then there is a sequence R_n of rational functions, with poles only in A, s.t. $R_n \to f$ uniformly on compact subsets of Ω.

In the special case in which $\mathbb{S}^2 \setminus \Omega$ is connected, we may take $A = \{\infty\}$ and thus obtain polynomials P_n s.t. $P_n \to f$ uniformly on compact subsets of Ω. Note that $\mathbb{S}^2 \setminus \Omega$ may have uncountably many components, for instance $\{\infty\} \cup C$ where C is the cantor set.

1.3 The Mittag-Leffler Theorem

13.10 Thm Suppose Ω is an open set in the plane, $A \subset \Omega$, A has no limit point in Ω, and to each $\alpha \in A$ there are associated a positive integer $m(\alpha)$ and a rational function

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} c_{j,\alpha}(z - \alpha)^{-j}$$

Then there exists a meromorphic function f in Ω whose principal part at each $\alpha \in A$ is P_α and which has no other poles in Ω.

1.4 Simply Connected Regions

13.11 Thm For a plane region Ω, each of the following nin conditions implies all the others.

(a) Ω is homemorphic to the open unit disc U.

(b) Ω is simply connected.

(c) $\mathrm{Ind}_\gamma(\alpha) = 0$ for every closed path γ in Ω and for every $\alpha \in \mathbb{S}^2 \setminus \Omega$.

(d) $\mathbb{S}^2 \setminus \Omega$ is connected.

(e) Every $f \in H(\Omega)$ can be approximated by polynomials, uniformly on compact subsets of Ω.

(f) For every $f \in H(\Omega)$ and every closed path γ in Ω

$$\int_\gamma f(z)dz = 0$$

(g) To every $f \in H(\Omega)$ corresponds an $F \in H(\Omega)$ s.t. $F' = f$.

(h) If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $g \in H(\Omega)$ s.t. $f = \exp(g)$.

(i) If $f \in H(\Omega)$ and $1/f \in H(\Omega)$,t here exists a $\varphi \in H(\Omega)$ s.t. $f = \varphi^2$.

13.12 Thm If $f \in H(\Omega)$, where Ω is any open set in the plane, and if f has no zero in Ω, then $\log |f|$ is harmonic in Ω.

2 Exercises

Problem 1 Note that there are finitely many poles of h, say $z_1, \dots z_n$. Then $h(z)(z - z_1)\dots(z - z_n)$ is an entire function. If $w_1 \dots w_m$ are the zeros, then we have

$$h(z)(z - z_1)\dots(z - z_n) = (w - w_1)\dots(w - w_m)f(z)$$

for some f. Note that f is an entire function and the Riemann Sphere is compact, so by Liouville's theorem f is constant, which implies that h is a rational function.

Problem 3 This is false. We take K an arbitrary compact set not including 0 and Ω to be $D'(0, r)$ for some $r > 0$ which K is contained in. Note that $f = 0$ is holomorphic in $\mathbb{S}^2 \setminus \Omega$ and therefore there is no sequence of polynomials converging to f since $\mathbb{S}^2 \setminus \Omega$ is not connected.

Problem 5 Construct the discs Δ_n to be the closed disc $\overline{D}(0, r_n)$ where $r_n = \frac{n-1}{n}$ and let the arcs L_n be the circle of radius $\frac{2n-1}{2n}$. Set P_n to be $+2$ if n is even and -2 if n is odd on L_n, but is $\frac{1}{2^n}$ on Δ_n. Then we see that f is holomorphic but on the edge it oscillates.

Problem 7 This works since each component $\mathbb{S}^2 \setminus K_n$ is a strict superset of the corresponding component of $\mathbb{S}^2 \setminus \Omega$. In particular, it is a superset of the closure (it is compact while Ω is open), which means that if a point in the closure of A is in the component of $\mathbb{S}^2 \setminus \Omega$, a point of A is in th $\mathbb{S}^2 \setminus K_n$.

Problem 9 Note that $1/f$ is holomorphic. Note that we have a h s.t. $f = \exp(h)$. Set $g = \exp(\frac{1}{n}h)$.