Rudin Ch8 - Integration of Product Spaces

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1 Notes

1.1 Measurability on Cartesian Products

8.1 Def A *Cartesian Product* is defined trivially. However, if $(X, \mathscr{L})$ and $(Y \mathscr{F})$ are measurable spaces, where $\mathscr{L}, \mathscr{F}$ are σ-algebra on X and Y respectively, then a *measurable rectangle* is $A \times B$ where $A \in \mathscr{L}$ and $B \in \mathscr{F}$. If $Q = R_1 \cup R_2 \cdots \cup R_n$ for measurable rectangles R_i, then $Q \in \mathscr{E}$, an *elementary set.* $\mathscr{L} \times \mathscr{F}$ is the smallest σ-algebra which contains all rectangles of $X \times Y$.

A *monotone class* $\mathfrak{M}$ is a collection of sets with the following properties. $A_i \in \mathfrak{M}, B_i \in \mathfrak{M}, A_i \subset A_{i+1}, B_i \supset B_{i+1}$ for $i = 1, 2 \dots$ and

$$A = \bigcup_{i=1}^{\infty} A_i, B = \bigcap_{i=1}^{\infty} B_i$$

then $A, B \in \mathfrak{M}$. If $E \subset X \times Y$, then

$$E_x = \{y : (x, y) \in E\} \quad E^y = \{x : (x, y) \in E\}$$

and E_x and E^y are called the *x section* and *y section.*

8.2 Thm If $E \in \mathscr{L} \times \mathscr{F}$, then $E_x \in \mathscr{F}$ and $E^y \in \mathscr{L}$.

8.3 Thm $\mathscr{L} \times \mathscr{F}$ is the smallest monotone class which contains all elementary sets.

8.4 Def $f_x(y) = f(x, y)$ and $f^y(x) = f(x, y)$.

8.5 Thm Let f be an $\mathscr{L} \times \mathscr{F}$ function, then

(i) For each $x \in X, f_x$ is $\mathscr{F}$-measurable.
(ii) For each $y \in Y, f^y$ is $\mathscr{L}$-measurable.

1.2 Product Measures

8.6 Thm Let $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be σ-finite measure spaces. Suppose $Q \in \mathscr{L} \times \mathscr{F}$. If

$$\varphi(x) = \lambda(Q_x) \quad \psi(y) = \mu(Q^y)$$

for every x, y. Then φ is $\mathscr{L}$ measurable and ψ is $\mathscr{F}$ measurable and

$$\int_X \varphi d\mu = \int_Y \psi d\lambda$$

8.7 Def If $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{F}, \lambda)$, and if $Q \in \mathscr{L} \times \mathscr{F}$, then

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_Y \mu(Q^y) d\lambda(y)$$

and we call $\mu \times \lambda$ as the *product* of μ and λ, and is σ-finite.

1.3 The Fubini Theorem

8.8 Thm If $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be a σ-finite measure spaces, and let f be an $(\mathscr{L} \times \mathscr{F})$-measurable function on $X \times Y$.

(a) If $0 \leq f \leq \infty$, and if

$$\varphi(x) = \int_Y f_x d\lambda \quad \psi(y) = \int_X f^y d\mu$$

then φ is $\mathscr{L}$-measurable, ψ is $\mathscr{F}$ measurable, and

$$\int_X \varphi d\mu = \int_{X \times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda$$

(b) If f is complex and if

$$\varphi^*(x) = \int_Y |f|_x d\lambda \quad \int_X \varphi^* d\mu < \infty$$

then $f \in L^1(\mu \times \lambda)$.

(c) If $f \in L^1(\mu \times \lambda)$, then $f_x \in L^1(\lambda)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$, then φ and ψ are in $L^1(\mu)$ and $L^1(\lambda)$, and the integrals holds.

Notice that the outside two integrals are called the *iterated integral* and the central one is the *double integral.* Notice that if f is $(\mathscr{L} \times \mathscr{F})$-measurable and if

$$\int_X d\mu(x) \int_Y |f(x,y)| d\lambda(y) < \infty$$

then the two iterated integrals are equal.

8.9 Counterexamples There are counterexamples which show that the conditions for Fubini's theorem are necessary. However, these are omitted for sake of brevity.

1.4 Completion of Product Measures

8.10 If $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ are complete metric spaces, then $(X \times Y, \mathscr{L} \times \mathscr{F}, \mu \times \lambda)$ need not be complete.

8.11 Thm Let m_k denote Lebesgue measure on $\mathbb{R}^k$. If $k = r + s, r \geq 1, s \geq 1$, then m_k is the completion of the product measure $m_r \times m_s$.

8.12 Thm Let $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{F}, \lambda)$ be complete σ-finite measure spaces. Let $(\mathscr{L} \times \mathscr{F})^*$ be the completion of $\mathscr{L} \times \mathscr{F}$, relative to the measure $\mu \times \lambda$. Let f be an $(\mathscr{L} \times \mathscr{F})^*$-measurable function on $X \times Y$. Then all conclusions of Fubini's (Thm 8.8) follows except that f_x is $\mathscr{F}$-measurable only a.e. and φ is defined a.e. on $[\mu]$, and the same result follows for f^y and ψ.

Lemma 1 Suppose v is a positive measure on a σ-algebra $\mathfrak{M}, \mathfrak{M}^*$ is the completion of $\mathfrak{M}$ relative to v, and f is an $\mathfrak{M}^*$-measurable function. Then there exists a $\mathfrak{M}$-measurable function g s.t. $f = g$ a.e. [v].

Lemma 2 Let h be an $(\mathscr{L} \times \mathscr{F})^*$-measurable function on $X \times Y$ s.t. $h = 0$ a.e. $[\mu \times \lambda]$. Then for almost all $x \in X$, $h(x, y) = 0$ for almost all $y \in Y$. h_x is $\mathscr{F}$-measurable for almost all x and h^y is $\mathscr{L}$-measurable for almost all y.

1.5 Convolutions

8.13 To show that a set is not of measure 0, it is often necessary to show that it is "large". One of these cases is to show that its complement has measure 0, in which case Fubini's theorem becomes useful. For example, if $f, g \in L^1(\mathbb{R}^1)$ and $f, g \geq 0$, then

$$h(x) = \int_{-\infty}^{\infty} f(x - t) g(t) dt$$

and $h(x) < \infty$ a.e.

8.14 Thm Suppose $f \in L^1(\mathbb{R}), g \in L^1(\mathbb{R})$. Then

$$\int_{-\infty}^{\infty} |f(x - y) g(y)| dy < \infty$$

for almost all x. For these x, define

$$h(x) = \int_{-\infty}^{\infty} f(x - y) g(y) dy$$

Then $h \in L^1(\mathbb{R}^1)$, and

$$||h||_1 \leq ||f||_1 ||g||_1$$

where

$$||f||_1 = \int_{-\infty}^{\infty} |f(x)| dx$$

h is the *convolution* of f and g and $h = f * g$.

1.6 Distributions

8.15 Def Let μ be a σ-finite measure on some σ-algebra in some set X. Let $f : X \to [0, \infty]$ be measurable. The function that assigns to each $t \in [0, \infty)$ the number

$$\mu\{f > t\} = \mu(\{x \in X : f(x) > t\})$$

is called the distribution function of f.

8.16 Thm Suppose that f and μ are as above, that $\varphi : [0, \infty] \to [0, \infty]$ is monotonic, absolutely continuous on $[0, T]$ for every $T < \infty$ and that $\varphi(0) = 0$ and $\varphi(t) \to \varphi(\infty)$ as $t \to \infty$. Then

$$\int_X (\varphi \circ f) d\mu = \int_0^\infty \mu\{f > t\} \varphi'(t) dt$$

8.17 Recall that Mf, the maximal function, is in weak L^1. Also,

$$||Mf||_\infty \leq ||f||_\infty$$

8.18 Thm If $1 < p < \infty$ and $f \in L^p(\mathbb{R}^k)$, then $Mf \in L^p(\mathbb{R}^k)$.

2 Problems

Problem 1 We let $D_x = \{B_r(x) : r \in \mathbb{R}\}$. Take our set A to be

$$D_1 \cup D_{-1} \cup (\mathbb{R} - D_1) \cup (\mathbb{R} - D_{-1})$$

clearly this set satisfies our conditions, as it is a monotone class which contains complements and $\mathbb{R}$. However, notice that the set $B_{2+\epsilon}(0)$, which is the union of $B_{1+\epsilon}(1)$ and $B_{1+\epsilon}(-1)$, isn't an element of our set.

Problem 3 Notice that the function

$$f(x, y) = \frac{1}{|\frac{1}{2} - x| y^{1 - |\frac{1}{2} - x|}}$$

is quite obviously continuous. Furthermore, the integral is finite (everywhere else the φ is 1). However, at $x = \frac{1}{2}$, the φ is given by

$$\varphi(x) = \int_0^1 \frac{1}{0} dy$$

which diverges.

Problem 5
(a) We note that $\mu * \lambda$ is obviously a measure over E. To see this, for E and E' where $E \cap E' = \varnothing$ we just note that E_2 and E_2' also are disjoint.

Notice that, since $\mathbb{R}_2^1$ is very clearly $\mathbb{R}^2$,

$$||\mu * \lambda||(\mathbb{R}^1) = |\mu * \lambda|(\mathbb{R}^1) = \sup \sum_{i=1}^{\infty} |(\mu * \lambda)(E^i)| = \sup \sum_{i=1}^{\infty} |(\mu \times \lambda)(E_2^i)|$$
$$= ||\mu \times \lambda||$$

where our norm is take over $\mathbb{R}^2$. Notice that our term is less than $||\mu||||\lambda||$ since, for a set A,

$$|\mu \times \lambda|(A) = |\int_y \mu(A^y) d\lambda(y)| \leq \sup |\mu(A^y)| \sup |\lambda(A_x)|$$

which implies that

$$||\mu * \lambda|| \leq ||\mu||||\lambda||$$

(b) Notice that v obviously satisfies the condition. This can be seen as

$$\int f d(\mu * \lambda) = \int f d(\mu \times \lambda) = \int_X f \lambda(\mathbb{R}^1) d\mu(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x+y) dx dy$$

(c) We note that

$$\{(x, y) : x + y \in E\} = \{(y, x) : y + x \in E\} \implies (\mu * \lambda)(E) = (\lambda * \mu)(E)$$

A similar result (decomposition of E into $x + y + z$) proves associativity. Distributivity follows from $\mu \times (h_1 + h_2) = \mu \times h_1 + \mu \times h_2$.

(d) This follows from the definition of the cross

$$(\mu \times \lambda)(E_2) = \int \mu(E_2^y) d\lambda(y)$$

But $E_2^y = E - y$ so our value is $\int \mu(E - y) d\lambda(y)$.

(e) If μ is concentrated at countable A, λ at B, then $\mu * \lambda$ is concentrated at $A \times B$, which means that $\mu * \lambda$ is discrete (since this set is countable).

If μ is continuous and $\lambda \in M$, then notice

$$(\mu \times \lambda)(\{(x,y)\}) = \int_Y \mu(\{x\}) d\lambda(y) = 0$$

so $\mu * \lambda$ is also continuous. Lastly, note that this same argument shows that $\mu \ll m \implies \mu * \lambda \ll m$, where we instead substitute E, where $m(E) = 0$, instead of $\{x, y\}$.

(f) We note that the integral over the 1 function is

$$\int_X \int_Y d\lambda d\mu = \int_X \int_Y f(x)g(y-x)d^2m = \int_{X \times Y} d(\mu \times \lambda) \implies (f*g)dm = d(\mu \times \lambda)$$

(g) This is relatively obvious from the definitions of subalgebra, ideal, and the fact that $f * g = h$ for some $h \in L^1(\mathbb{R}^1)$.

(h) Our identity δ is $\delta(\{0\}) = 0$. This is because

$$(\mu * \delta)(E) = (\mu \times \delta)(E_2) = \int_Y \mu(E_2^y) d\delta(y) \implies \delta(\{0\}) = 1, 0 \text{ otherwise}$$

(i) All properties remain true for $\mathbb{R}^k, T, T^k$, so we can derive similar results.

Problem 7 Notice that

$$(\mu \times \lambda)(E) = \int_X \lambda(E_x) d\mu(x) = \mu(E^y)\lambda(E_x) = \psi(E)$$

Problem 9 Since f^y is continuous, we note that

$$f(x,y) = \lim_{n \to \infty} f(\lfloor nx \rfloor / n, y)$$

and since f^x is measurable, f is the pointwise limit of Borel measurable functions, so it is Borel measurable.

Problem 11 We let

$$\mathcal{B} = \{F \subset \mathbb{R}^n | \mathbb{R}^m \times F \in \mathscr{B}_{m+n}\}$$

We notice that this is a σ-algebra, as $\mathbb{R}^m \times \mathbb{R}^n$ is necessarily open which implies that $\mathbb{R}^m \times F^c \in \mathscr{B}_{m+n}$ and the countable union also follows. A similar result shows that

$$\mathcal{A} = \{E \subset \mathbb{R}^m | E \times \mathbb{R}^n \in \mathscr{B}_{m+n}\}$$

so if we have $a \subset \mathbb{R}^m, b \subset \mathbb{R}^n$ s.t. a, b are Borel, then $a \in \mathcal{A}$ and $b \in \mathcal{B}$. It follows that $(\mathbb{R}^m \times b) \cap (a \times \mathbb{R}^n) = a \times b \in \mathscr{B}_{m+n}$. This shows that

$$\mathscr{B}_m \times \mathscr{B}_n \subset \mathscr{B}_{m+n}$$

However, we notice that all open sets in $\mathbb{R}^{m+n}$ can be represented as the Cartesian product of an open set in $\mathbb{R}^m$ and $\mathbb{R}^n$. It follows that all open sets are in $\mathscr{B}_m \times \mathscr{B}_n$, which implies that it is equivalent to $\mathscr{B}_{m+n}$ since the Borel Set is the smallest such set.

Problem 13 Letting θ be the real measurable function s.t. $d\mu = e^{i\theta} d|\mu|$. Let A_α be the subset of E where $\cos(\theta - \alpha) > 0$. We note that

$$\text{Re}[e^{-i\alpha}\mu(A_\alpha)] = \int_E \cos^+(\theta - \alpha) d|\mu|$$

as we note that

$$e^{-i\alpha}\mu(A_\alpha) = e^{-i\alpha} \int_{A_\alpha} e^{i\theta} d|\mu|$$

We note that, naturally,

$$|\mu(A_\alpha)| \geq \text{Re}[e^{-i\alpha}\mu(A_\alpha)]$$

and we note that the average of $\int_E \cos^+(\theta - \alpha) d|\mu|$ over alpha is at least $\frac{1}{\pi}|\mu|(E)$, so it follows that

$$|\mu(A_\alpha)| \geq \frac{1}{\pi}|\mu|(E)$$

With $i = 1, 2, \ldots$ with $\sin(i\theta)$ for some irrational θ, we set the measure of each discrete $e_i = \sin(i\theta)$. Notice that $|\mu(E)| = 2\pi$. Furthermore, note that the maximal value of $|\mu(A)|$ for some $A \subset E$ is 2 (it forms a degenerate triangle), and the equality follows, which shows that $\frac{1}{\pi}$ is necessary.

Problem 15 We notice that $g(t) = \varphi'(t) = \sin(t)$ for $0 \leq t \leq 2\pi$ and 0 otherwise. Furthermore, note that $h(t) = t - \sin t$ if $0 \leq t \leq 2\pi$, 0 if $t < 0$ and 2π if $t > 2\pi$.

(i) Note

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - t)g(t)dt = \int_0^{2\pi} \sin(t) = 0$$

(ii) We note that

$$\int_{-\infty}^{\infty} g(x-t)h(t)dt = -\varphi(x-t)h(t)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi(x-t)\varphi(t)dt$$

Note that $\varphi(x-t)h(t) = 0$ from $-\infty$ to ∞ as $\varphi(x-t) = 0$. Therefore, $(f*g)(x) = (\varphi * \varphi)(x)$. Note that this value is > 0 on $(0, 4\pi)$ since there exists some t s.t. $0 \leq x-t, t \leq 2\pi$, and, for this value, both φ are positive.

(iii) It follows that $(f * g) * h = 0$ since $f * g = 0$. Furthermore, $f * (g * h) = \int_{-\infty}^{\infty} (g * h)(t)dt > 0$. We note that Fubini's Thm fails because $h(x) \notin L^1(m)$.