Erdmann-Wildon - Subalgebras of gl(V)

Aaron Lou

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1 Notes

1.1 Nilpotent Maps

$\mathbf{gl}(V)$ has a special property since we can also exploit xy of linear maps. We attempt to figure out the nilpotent maps of $\mathbf{gl}(V)$.

Lemma 5.1 Let $x \in L$. If the linear map $x : V \to V$ is nilpotent, then $\mathrm{ad}x : L \to L$ is also nilpotent.

1.2 5.2 Weights

There are similar definitions to eigenvectors and values in Lie Algebras. We say $v \in V$ is an eigenvector on algebra A if v is an eigenvector every element a ie $a(v) \in \mathbf{Span}\{v\}$ for every $a \in A$.

Example 5.2 Let $A = \mathbf{d}(n, F)$ be the Lie subalgebra of $\mathbf{gl}(n, F)$ consisting of diagonal matrices. Let $\{e_1, \ldots, e_n\}$ be the standard basis of F^n. Then each e_i is an eigenvector of A.

To generalize this, we specify a function $\lambda : A \to F$ where $\lambda(a)$ is an "eigenvector" of a. Note that the corresponding eigenspace is

$$V_\lambda := \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}$$

If V_λ is a non-zero eigenspace for the function $\lambda : A \to F$. let $v \in V_\lambda$ be non-zero, $a, b \in A$, and $\alpha, \beta \in F$. Then

$$(\alpha a + \beta b)v = \alpha(av) + \beta(bv) = \alpha\lambda(a)v + \beta\lambda(b)v = (\alpha + \lambda(a) + \beta\lambda(b))v$$

We can see that λ is linear and so $\lambda \in A^*$, the dual space of linear maps from A to F.

Def 5.3 A *weight* for a Lie subalgebra A of $\mathbf{gl}(V)$ is a linear map $\lambda : A \to F$ s.t.

$$V_\lambda := \{v \in V : a(v) = \lambda(a)v \text{ for all } a \in A\}$$

is a non-zero subspace of V. The vector space V_λ is the *weight space*. Note that V_λ is non-zero if V contains a common eigenvector for the elements of A.

1.3 The Invariance Lemma

In linear algebra, we know that if $a, b : V \to V$ are commuting linear transformations and W is the kernel of a, then W is b-invariant. There is a generalization in $\mathbf{gl}(V)$.

Lemma 5.4 Suppose that A is an ideal of a Lie subalgebra L of $\mathbf{gl}(V)$. Let

$$W = \{v \in V : a(v) = 0 \text{ for all } a \in A\}$$

then W is an L-inveriant subspace of V.

Lemma 5.5 (Invariance Lemma) Assume that F has characteristic zero. Let L be a Lie subalgebra of $\mathbf{gl}(V)$ and let A be an ideal of L. Let $\lambda : A \to F$ be a weight of A. The associated weight space

$$V_\lambda = \{v \in V : av = \lambda(a)v \text{ for all } a \in A\}$$

is an L-invariant subspace of V.

Remark 5.6 This proof would be much easier if it were true that the linear maps y^r belongs to the Lie Algebra L.

1.4 An Application of the Invariance Lemma

Prop 5.7 Let $x, y : V \to V$ be linear maps from a complex vector space V to itself. Suppose that x and y both commute with $[x, y]$. Then $[x, y]$ is a nilpotent map.

2 Exercises

Exercise 5.1
(i) This is a relatively straightforward exercise as

$$a(v_1 + v_2) = a(v_1) + a(v_2) = \lambda(a)v_1 + \lambda(a)v_2 = \lambda(a)(v_1 + v_2)$$

(ii) Letting $V_{\epsilon_i} := \{v \in V : a(v) = \epsilon_i(a)v\}$. We quickly note that this only holds for all e_i, as α_i could be any value. Clearly, this means that V decomposes as a direect sum of V_{ϵ_i} as V decomposes of the spans of e_i.

Exercise 5.2 e_1 is indeed an eigenvector of A, as the corresponding mapping is $\lambda(a) = a_{1,1}$ where $a_{1,1}$ is the top left hand element of a.

The weight space is the span of e_1.

Exercise 5.3 The application of Lemma 5.4 is when A is an abelian Lie Algebra containing a, b (or a, y).

Exercise 5.4
(i) The isomorphism is a simple change os basis from the standard basis to our specialized basis. This implies that L is isomorphic to a subalgebra $\mathbf{n}(n, F)$ as both are strictly upper triangular matrices. It follows from exercise 4.4 that L is nilpotent (it is isomorphic to $\mathbf{n}(n, F)^n = 0$.

(ii) The subalgebra is isomorphic to a subalgebra of $\mathbf{b}(n, F)$ and is solvable by exercise 4.5.

Exercise 5.5
Skipped because I can't find a reasonable way to do this after seeing the solution to Prop 5.7 (or at least it's not clear how they wish for this solution to diverge from the one given).
Exercise 5.6
(i) To see that this is a subalgebra, note that for $x_1, x_2 \in N_L(A)$ that

$$[[x_1, x_2], a] = -[x_1, [x_2, a]] + [x_2, [x_1, a]]$$

and the two parts of the sum are clearly in A, so $[x_1, x_2] \in N_L(A)$. Furthermore, we can clearly see that $A \subset N_L(A)$ as all elements of A satisfy the condition. To see that this is the largest subalgebra in which A is an ideal, we can prove this by contradiction. If we have some superset X with element $x \in X, x \notin N_L(A)$ s.t. A is an ideal of X, note that $[a, x] \in A \implies [x, a] \in A \implies x \in N_L(A)$, contradiction.

(ii) Note that any weight space for A is $N_L(A)$-invariant. Taking our λ to be the maps from $a \to a_{ii}$ for $1 \leq i \leq n$, we note that all spans of e_i are $N_L(A)$-invariant which implies that all elements of $N_L(A)$ are diagonal.

Exercise 5.7 We note that $a_{k+1} = a_k y - y a_k \implies a_k y = y a_k + a_{k+1}$. We now prove the problem through induction. It is trivial to check the base cases for $m = 0$ or $m = 1$. The inductive step is given by

$$ay^{m+1} = (ay^m)y = (y^m a + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k)y = y^m ay + \sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k$$

$$= y^m(ya + [a, y]) + \sum_{k=1}^{m} \binom{m}{k} y^{m-k+1} a_k + y^{m-k} a_{k+1} = y^{m+1} a + \sum_{k=1}^{m+1} \binom{m+1}{k} y^{m-k+1} a_k$$

as desired. Note that $a_k = (-1)^k(\mathrm{ad}y)^k a$ (this can be proven again through basic induction) and so we note that

$$
\begin{aligned}
y^m a - a y^m &= -\sum_{k=1}^{m} \binom{m}{k} y^{m-k} a_k \\
\implies \mathrm{ad}y^m a &= \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\mathrm{ad}y)^k a \\
\implies \mathrm{ad}y^m &= \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} y^{m-k} (\mathrm{ad}y)^k
\end{aligned}
$$

as desired.