# Erdmann-Wildon Lie Algebras - Introduction

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## 1 Notes

### 1.1 Definition of Lie Algebras

Let $F$ be a field. A Lie-algebra over $F$ is an $F$ vector space $L$ with a bilinear map *(Lie Bracket)* : $L \times L \to L, (x, y) \to [x, y]$ s.t.

$$[x, x] = 0 \ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

The Lie Bracket is commonly called the *commutator* and the second condition is known as the *Jacobi Identity*

### 1.2 Examples

(1) Setting $x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_3y_2 - x_2y_3)$ denotes the Lie Algebra $\mathbb{R}^3_\wedge$ over $\mathbb{R}^3$.

(2) The *Abelian Lie Algebra* is given by $[x, y] = 0$ for all $x, y \in L$

(3) We write $gl(V)$ for the set of all linear maps from $V \to V$. This is a vector space over $F$, and is called the *general lie algebra.* The Lie Bracket is defined by

$$[x, y] := x \circ y - y \circ x \text{ for } x, y \in gl(V)$$

(3') If we write $gl(n, F)$ for the vector space of all $n \times n$ matrices over $F$ with the Lie Bracket defined by

$$[x, y] := xy - y$$

We also note that, trivially,

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$$

where $\delta$ is the Kronecker delta, $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$

(4) Let $sl(n, F)$ is all matrices with traces of 0. This is a special linear algebra, and everything follows from (3').

(5) Let $b(n, F)$ be he upper triangular matrices. This is the same as case (4).

## 1.3 1.3 Subalgebras and Ideals

A *Lie Subalgebra* is defined intuitively (a vector subspace). An *ideal* is a subspace $I$ of $L$ s.t.

$$[x, y] \in I \ \forall x \in L, y \in I$$

The Lie Algebra and $\{0\}$ are the trivial ideals. The centre of $L$ is another ideal and is

$$Z(L) := \{x \in L : [x, y] = 0 \ \forall y \in L\}$$

## 1.4 Homomorphisms

A map $\varphi : L_1 \to L_2$ is a *homomorphism* if it is linear and

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \ \forall x, y \in L_1$$

It is an *isomorphism* if it is also bijective. The *adjoint homomorphism* is

$$ad : L \to gl(V) \ (adx)(y) := [x, y]$$

$adx$ is linear, and $x \to adx$ is also linear.

## 1.5 Algebras

An *algebra* over field $F$ is a vector space $A$ with map $A \times A \to A$ $(x, y) \to xy$ (called the *product*). It is *associative* if $(xy)z = x(yz)$ and *unital* if there exists an identity.

## 1.6 Derivations

A *derivation* of algebra $A$ is an $F$-linear map $D : A \to A$ s.t.

$$D(a, b) = aD(b) + D(a)b \ \forall a, b \in A$$

Der$A$ is the set of derivations of $A$, and is a Lie subalgebra of $gl(A)$.

Eamples are given below

(1) Let $A = C^\infty R$ be the vector space of infinitely differentiable functions $R \to R$. $(fg)(x) = f(x)g(x)$. $A$ is an associaive algebra with $Df = f'$

(2) The map $adx : L \to L$ is a derivation of $L$ (Lie Algebra).

### 1.7 Structure Constants

if $L$ is a Lie algebra over a field $F$ with basis $(x_1, \dots, x_n)$ then $[-,-]$ is completely determined by the products $[x_i, x_j]$. The scalars $a_{ij}^k \in F$ s.t.

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k$$

these are called the *structure constants.*

## 2 Exercises

**Ex 1.1**

(i) This is clear since $0 \times v = 0$ and the fact that it's bilinear.
(ii) Suppose we have $ax + by = 0$ for sake of contradiction. Then $x = -\frac{b}{a}y$ and it follows that

$$[x, y] = -\frac{b}{a}[y, y] = 0$$

Therefore, by contradiction, it follows that $[x, y] \neq 0$ means tha they are linearly independent.

**Ex 1.2** We check scalars and addition:

$$(\alpha x)\wedge = (\alpha x_1, \alpha x_2, \alpha x_3) \wedge (y_1, y_2, y_3)$$
$$= (\alpha x_2 y_3 - \alpha x_3 y_2, \alpha x_3 y_1 - \alpha x_1 y_3, \alpha x_2 y_1 - \alpha x_2 y_1) = \alpha(x \wedge y)$$

$$(x + y) \wedge z$$
$$= ((x_2 + y_2)z_3 - (x_3 + y_3)z_2, (x_3 + y_3)z_1 - (x_1 + y_1)z_3, (x_1 + y_1)z_2 - (x_2 + y_2)z_1)$$
$$(x \wedge z) + (y \wedge z)$$

and the other $x \wedge (y + z)$ follows naturally. To prove the Jacobi identity, we note that

$$x \wedge (y \wedge z) = x \wedge (y_2 z_3 - y_3 z_2, y_3 z_1 - y_1 z_3, y_1 z_2 - y_2 z_1)$$
$$= (x_2(y_1 z_2 - y_2 z_1) - x_3(y_3 z_1 - y_1 z_3), x_3(y_2 z_3 - y_3 z_2) - x_1(y_1 z_2 - y_2 z_1),$$
$$x_1(y_3 z_1 - y_1 z_3) - x_2(y_2 z_3 - y_3 z_2)) = (x \cdot z)y - (x \cdot y)z$$

Notice that

$$x \wedge (y \wedge z) + y \wedge (z \wedge x) + z \wedge (x \wedge y) =$$
$$(x \cdot z)y - (x \cdot y)z + (z \cdot x)y - (y \cdot z)x + (z \cdot y)x - (z \cdot x)y = 0$$

**Ex 1.3** We bash it out

$$\begin{gathered}[x, y\circ z - z\circ y] + [y, z\circ x - x\circ z] + [z, x\circ y - y\circ x] \\ = x\circ y\circ z - x\circ z\circ y - y\circ z\circ x + z\circ y\circ x + y\circ z\circ x - y\circ x\circ z \\ -z\circ x\circ y + x\circ z\circ y + z\circ x\circ y - z\circ y\circ x - x\circ y\circ z + y\circ x\circ z \\ = 0\end{gathered}$$

**Ex 1.4** This is trivial since the multiplication of two upper-$\triangle$ matrices is upper triangles, and so it addition and subtraction of these. It follows that $xy - yx$ in $b$ and $n$.

**Ex 1.5** We calculate

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}\begin{bmatrix} a & b \\ c & -a \end{bmatrix} - \begin{bmatrix} a & b \\ c & -a \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = 0$$

$$\implies \begin{bmatrix} xc - yb & (w-z)b - 2xa \\ 2ya + (z-w)c & yb - xc \end{bmatrix} = 0$$

so it follows that $b = 0$, $c = 0$ and $2a = 0$. If the characteristic of $F$ is finite and divisible by 2, then $a = \text{char}/2$, otherwise $a = 0$.

**Ex 1.6** Let $K$ be the kernel of $\varphi$, then notice $\varphi([x,k]) = [\varphi(x), \varphi(k)] = 0$ for $k \in K$, which means that $[x,k] \in K$, so $K$ is the ideal. Similarly, let $I$ be the image of $\varphi$. If we have $i_1, i_2 \in I$ with respective $x_1, x_2 : \varphi(x_1) = i_1$ and $\varphi(x_2) = i_2$, then $\varphi([x_1, x_2]) = [\varphi(x_1), \varphi(x_2)] = [i_1, i_2] \in I$, so it is a Lie subalgebra of $L_2$.

**Ex 1.7** By the Jacobi identity, we have

$$[x, [y, z]] + [y, [z, x]] = [[x, y], z]$$

$[x, [y, z]] = [[x, y], z] \implies [y, [z, x]] = 0$, which means that $[a, b] \in Z(L)$. If $[a, b] \in Z(L)$ then the other way is apparent.

**Ex 1.8**

(i) We calculate

$$\begin{gathered}[D, E](ab) = D\circ E(ab) - E\circ D(ab) = D(aE(b) - E(a)b) - E(aD(b) - D(a)b) \\ = a(D\circ E)(b) - D(a)E(b) - E(a)D(b) + (D\circ E)(a)b - a(E\circ D)(b) + \\ E(a)D(b) + D(a)E(b) - (E\circ D)(a)b \\ = a(D\circ E - E\circ D)(b) + (D\circ E - E\circ D)(a)b = a[D, E](b) + [D, E](a)b\end{gathered}$$

so it is a derivation, as desired.

(ii) This happens when $D(a)E(b) + E(a)D(b) = 0$, which doesn't normally happen.

**Ex 1.9**
Let the respective bases be denoted as $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$ and the structure constants are $a_{ij}^k$.

The isomorphic mapping $\varphi : L_1 \to L_2$ means that

$$\varphi[x_i, x_j] = \varphi(\sum_{k=1}^{n} a_{ij}^k x_k) = \sum_{k=1}^{n} a_{ij}^k \varphi(x_k) = [\varphi(x_i), \varphi(x_j)]$$

and we take $y_i = \varphi(x_i)$ and see that the structure constants remain the same. If we're given the bases, then we notice that we just define the mapping $\varphi(x_i) = y_i$ and the structure constants follow.

**Ex 1.10**
We calculate

$$[x_i[x_j, x_k]] = [x_i, \sum_{l=1}^{n} a_{jk}^l x_l] = \sum_{l=1}^{n} a_{jk}^l [x_i, x_l] = \sum_{l=1}^{n} \sum_{m=1}^{n} a_{jk}^l a_{il}^m x_m$$

It follows that, for all $i, j, k \in F$. It follows that

$$a_{jk}^l a_{il}^m + a_{ki}^l a_{jl}^m + a_{ij}^l a_{kl}^m = 0$$

**Ex 1.11**
Obviously, if $L_1$ and $L_2$ are isomorphic, it means that they have the same dimension. To prove the other way, consider an arbitrary basis of $L_1$ $x_1 \ldots x_n$ and $L_2$ $y_2 \ldots y_n$. Since $[x_i, x_j] = 0$ and $[y_i, y_j] = 0$, then we just match $x_i \to y_i$ and this is our isomorphism.

**Ex 1.12**
We calculate

$$[e, f] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[e, h] = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}$$

$$[f, h] = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

and so our coefficients are $a_{ef}^h = 1, a_{eh}^e = -2, a_{fh}^f = 2$ and the reverse coefficients are $a_{fe}^h = -1, a_{he}^e = 2, a_{hf}^f = -2$.

**Ex 1.13**
Examine the basis for $sl(2,\mathbb{C})$. It is given by, as shown above

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Notice that $[e, f] = h, [e, h] = -2e, [f, h] = 2f$. Furthermore, note that, in an ideal which contains some one or two of $e, f, h$ must contain all three, meaning there is no non-trivial ideal.

**Ex 1.14**
(i) The basis we wish to examine for the antisymmetric matrices of $gl(3,\mathbb{C})$ are

$$x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and we calculate that the structure constants are equivalent. This implies the isomorphism between the two Lie Algebras.

(ii) We first set $\varphi(h) = cx$ for some constant $x$. Taking $f$ to be $\alpha x + \beta y + \gamma z$, we find that

$$\varphi([f, h]) = [\varphi(f), \varphi(h)] = [\alpha x + \beta y + \gamma z, cx] = \gamma \bar{c} y - \beta \bar{c} z = 2\alpha x + 2\beta y + 2\gamma z$$

solving we find that $\gamma = \pm\beta i$, and that we can set $\alpha = 0, \gamma = 1$, $\beta = i$, and $c = 2i$. Similarly, we can calculate the coefficients for $e$ and find that

$$\varphi(h) = 2iz, \varphi(f) = ix + y, \varphi(e) = -ix + y$$

as our isomorphic mapping.

**Ex 1.15**
(i) For $x, y \in gl_S(n, F)$, we calculate that

$$[x, y]S = (xy - yx)^t S = (y^t x^t - x^t y^t)S = y^t x^t S - x^t y^t S = Syx - Sxy = -S[x, y]$$

so it follows that $[x, y] \in gl_S(n, F)$ and that it is a Lie sub-algebra.

(ii) We find that

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies c = 0, b = 0, a + d = 0$$

so our subspace is of dimension 1 and has a basis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(iii) No. To see this, we denote $S = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$. If we let $x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, we find that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = - \begin{bmatrix} w & x \\ y & z \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

$$\implies \begin{bmatrix} 2aw & (a+b)x \\ (a+b)y & 2bz \end{bmatrix} = 0 \implies w, x, y, z = 0$$

But this means holds for all matrices, not just the diagonal matrices.

(iv) We wish to prove $gl_S(3, \mathbb{R})$ is isomorphic to the Lie subalgebra of $gl(3, \mathbb{C})$ consisting of antisymmetric matrices. We'll just prove that $gl_S(3, \mathbb{R})$ is the same as $gl(3, \mathbb{C})$.

We examine the bases

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

and find that, if $S = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, that $a = e = i$ and $b + d = 0, c + g = 0, h + f = 0$. Further examining for bases

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we find that no value is 0. We see that any matrix, such as

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

satisfies our requirement.

**Ex 1.16** This is obvious as it would not necessarily imply that $[x, x] = 0$ since $[x, x] = 0/2$.

**Ex 1.17**
We apply a change of variables to diagonalize the matrix. We have $x = P^{-1}AP$ for some diagonal eigenvalue matrix $A$. We also write $B = PYP^{-1}$ and find that $(adx)y = xy - yx = P^{-1}(AB - BA)P$.

$y$ is an eigenvector precisely when $AB - BA$ is $\lambda B$ for some eigenvalue $\lambda$. If we let $B$ to be a matrix that is zero except for some non zero value at $i, j$ then our eigenvector is precisely $\lambda_i - \lambda_j$.

These $y$ form a basis of $L$ and it follows that we can diagonalize $adx$ with eigenvectors $\lambda_i - \lambda_j$.

**Ex 1.18**
We suppose we are given derivation $D$ and $adx$. We find that

$$[D, adx]y = D[x, y] - [x, Dy] = D(xy - yx) - (xDy - Dyx)$$
$$= xDy + Dxy - yDx - Dyx - xDy + Dyx = ad(Dx) \in \mathrm{IDer}L$$

so it follows that $\mathrm{IDer}L$ is an ideal of $\mathrm{Der}L$.

**Ex 1.19**
We prove this by induction.

For $n = 1$, it is clear that $\delta(xy) = x\delta(y) + \delta(x)y$.

For the inductive step, assume the equality holds for value $n$. we see that

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y)$$
$$\implies \delta^{n+1}(xy) = \delta(\sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y))$$
$$= \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r+1}(y) + \sum_{r=0}^{n} \binom{n}{r} \delta^{r+1}(x)\delta^{n-r}(y)$$
$$= \sum_{r=0}^{n+1} \binom{n+1}{r} \delta^r(x)\delta^{n+1-r}(y)$$

as desired.