# Rudin Ch2 - Positive Borel Measure

Aaron Lou

May 2018

## 1 Notes

### 1.1 Vector Spaces

**2.1 Def** A complex vector space has vectors with scalars and sums. Linear transformations are as expected, and if mapped to the scalars, we call $\Lambda$ a linear functional. $R^k$ is the only real vector space.

**2.2 Integration as a Linear Functional** To every positive linear functional $\Lambda$ on $C$ corresponds a finite positive Borel measure $\mu$ on $I$ s.t.

$$\Lambda f = \int_I f d\mu$$

### 1.2 Topological Preliminaries

**2.3 Def** Let $X$ be a topological space

(a) A set $E \subset X$ is closed if its complement $E^c$ is open.
(b) The closure $\overline{E}$ of a set $E \subset X$ is the smallest closed set in $X$ which contains $E$.
(c) A set $K \subset X$ is compact if every open of $K$ contains a finite subcover. Specifically, if $\{V_\alpha\}$ is a collection of open sets whose union contains $K$, then the union of finite sub-collection of $\{V_\alpha\}$ also covers $K$.
(d) A neighborhood of a point $p \in X$ is any open subset of $X$ which contains $p$.
(e) $X$ is a Haussdorff space if $\forall p, q \in X, p \neq q$ and $p$ has a nbhd $U$ and $q$ has a nbhd $V$ s.t. $U \cap V = \varnothing$.
(f) $X$ is locally compact if every point of $X$ has a nbhd whose closure is compact.

By Heine-Borel, compact subsets are closed and bounded in $\mathbb{R}^k$.

**2.4 Thm** Suppose $K$ is compact and $F$ is closed, in a topological space $X$. If $F \subset K$, then $F$ is compact.

Corollary: If $A \subset B$ and $B$ has compact closure, then so does $A$.

**2.5 Thm** Suppose $X$ is a Hausdorff space, $K \subset X$, $K$ is compact, and $p \in K^c$. Then there are open sets $U$ and $W$ s.t. $p \in U$, $K \subset W$, and $U \cap W = \varnothing$.

Corollaries:

(a) Compact subsets of Hausdorff spaces are closed.
(b) If $F$ is closed and $K$ is compact in a Hausdorff space, then $F \cap K$ is compact.

**2.6 Thm** If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if $\bigcap\limits_\alpha K_\alpha = \varnothing$, then some finite subcollection of $\{K_\alpha\}$ also has empty intersection.

**2.7 Thm** Suppose $U$ is open in a locally compact Hausdorff space in $X$, $K \subset U$ is compact. Then there is an open set $V$ with compact closure s.t.

$$K \subset V \subset \overline{V} \subset U$$

**2.8 Def** Let $f$ be a real (or extended-real) function on a topological space. If

$$\{x : f(x) > \alpha\}$$

is open for every real $\alpha$, $f$ is lower semicontinuous. If

$$\{x : f(x) < \alpha\}$$

is open for every real $\alpha$, $f$ is upper semicontinuous.

The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infinum of any collection of upper semicontinuous functions is upper semicontinuous.

**2.9 Def** The support of a complex function $f$ on a topological space $X$ is the closure of the set

$$\{x : f(x) \neq 0\}$$

The collection of functions whose support is compact is $C_c(x)$.

Corr: The range of any $f \in C_c(X)$ is a compact subset.

**2.10 Def** Let $X$ and $Y$ be topological spaces, and $f : X \to Y$ be continuous. If $K$ is a compact subset of $X$, then $f(K)$ is compact.

**2.11 Not** We denote $K \prec f$ implies that $K$ is a compact subset of $X$, $f \in C_c(X), 0 \leq f \leq 1$, and $f(x) = 1 \forall x \in K$.

We denote $f \prec V$ means that $V$ is open, $f \in C_c(X), 0 \leq f \leq 1$, and the support of $f$ lies in $V$.

**2.12 Urysohn's Lemma** Suppose that $X$ is a locally compact Hausdorff space, $V$ is open in $X$, $K \subset V$, and $K$ is compact. Then there exists an $f \in C_c(X)$ s.t.

$$K \prec f \prec V$$

**2.13 Thm** Suppose $V_1, \dots V_n$ are open locally compact subsets of a locally compact Hausdorff space $X, K$ is compact, and

$$K \subset V_1 \cup \cdots \cup V_n$$

Then there exists a function $h_i \prec V_i (i = 1, \dots, n)$ s.t.

$$h_1(x) + \cdots + h_n(x) = 1$$

$\{h_1, \dots, h_n\}$ is a called a partition of unity on $K$ subordinate to the cover $\{V_1, \dots, V_n\}$.

## 1.3 The Riesz Representation Thm

**2.14 Thm** Let $X$ be a locally compact Hausdorff space, and let $\Lambda$ be a positive linear functional on $C_c(X)$. Then there exists a $\sigma$-algebra on $\mathfrak{M}$ in $X$ which contains all Borel sets in $X$, and there exists a unique positive measure $\mu$ on $\mathfrak{M}$ which represents $\Lambda$ in the sense that

(a) $\Lambda f = \int_X f d\mu$ for every $f \in C_c(X)$.
(b) $\mu(K) < \infty$ for every compact set $K \subset X$.
(c) For every $E \in \mathfrak{M}$, we have

$$\mu(E) = \inf\{\mu(V) : E \subset V, V \text{ open}\}$$

(d) The relation

$$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$$

(e) If $E \in \mathfrak{M}, A \subset E$, and $\mu(E) = 0$, then $A \in \mathfrak{M}$.

## 1.4 Regularity Properties of Borel Measures

**2.15 Def** A measure $\mu$ on the $\sigma$-algebra of all Borel Sets ($\mathfrak{B}$) in a locally compact Hausdorff space $X$ is called a Borel Measure of $X$. $\mu$ is outer regular if it is the infimum of all larger open sets, and $\mu$ is inner regular if it is the supremum of all smaller compact sets.

**2.16 Def** A set $E$ in topological space is called $\sigma$-compact if $E$ is a countable union of compact sets. It is said to have $\sigma$-finite measure if $E$ is a countable union of $E_i$ with $\mu(E_i) < \infty$.

**2.17 Thm** Suppose $X$ is a locally compact, $\sigma$-compact Hausdorff space. If $\mathfrak{M}$ and $\mu$ are as described in the statement of Thm 2.14, then $\mathfrak{M}$ and $\mu$ have the following properties:

(a) If $E \in \mathfrak{M}$ and $\epsilon > 0$, there is a closed set $F$ and an open set $V$ s.t. $F \subset E \subset V$ and $\mu(V - F) < \epsilon$.

(b) $\mu$ is a regular Borel measure of $X$.

If $E \in \mathfrak{M}$ there are sets $A$ and $B$ s.t. $A \in F_\sigma$, $B \in G_\delta$, $A \subset E \subset B$, and $\mu(B - A) = 0$.

**2.18 Thm** Let $X$ be a locally compact Hausdorff space in which every open set is $\sigma$-compact. Let $\lambda$ be any positive Borel measure on $X$ s.t. $\lambda(K) < \infty$ for every compact set $K$. Then $\lambda$ is regular.

## 1.5 Lebesgue Measure

**2.19 Euclidean Space** Euclidean space is defined as $R^k$, with operations as normal. The translate of a set $E + x$ is also as expected. We define a $k - cell$ to be any set of the form

$$W = \{x : \alpha_i < \zeta_i < \beta_i, 1 \leq i \leq k\}$$

and we can use $\leq$ instead of $<$. A $\delta$-box with corner at $a$ is defined as

$$Q(a; \delta) = \{x : \alpha_i \leq \zeta_i < \alpha_i + \delta, 1 \leq i \leq k\}$$

Let $\Omega_n$ be the set of all $2^{-n}$ boxes with corners at integral multiples of $2^{-n}$. The following are obvious.

(a) Points can only lie in one
(b) Larger values of $n$ are entirely within other boxes.
(c) There are $2^{(n-r)k}$ points smaller cubes $r$ in a larger cube of $n$.
(d) Each open set is a countable union of disjoint boxes.

**2.20 Thm** There exists a positive complete measure $m$ on a $\sigma$-algebra $\mathfrak{M}$ in $\mathbb{R}^k$ with the following properties (volume):

(a) $m(W) = vol(W)$ for every $k$-cell $W$.
(b) $\mathfrak{M}$ contains all Borel Sets, $m$ is regular. More precisely, $E \in \mathfrak{M}$ iff $\exists, A$ is $F_\sigma, B$ is $G_\delta$, $A \subset E \subset B, m(B - A) = 0$.
(c) $m$ is translation invariant.
(d) If $\mu$ is any positive, translation-invariant measure where $\mu(K) < \infty$ for every compact $K$, then $\mu(E) = cm(E)$ for all Borel Sets $E$.

(e) Every linear transformation has a unique value (the determinant) which maps the volume.

The member of $\mathfrak{M}$ are Lebesgue measurable. $m$ is the Lebesgue measure.

**2.21 Remarks** We often write $L^1(R^k)$ instead of $L^1(m)$. We can consider a new measure space on the measurable subsets of a set $E$.

**2.22 Thm** If $A \subset R^1$ and every subset of $A$ is Lebesgue Measurable, then $m(A) = 0$.

Corr: Every set of positive measure has nonmeasurable subsets.

**2.23 Det** This is omitted but determinants.

## 1.6 Continuity Properties of Measurable Functions

**2.24 Lusin's Thm** Suppose $f$ is a complex measurable function on $X$, $\mu(A) < \infty$, $f(x) = 0$. If $x \notin A$, and $\epsilon > 0$. Then there exists a $g \in C_c(X)$ s.t.

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon$$

We can also arrnage it so that

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|$$

**2.25 The Vitali-Carathéodary Thm** Suppose $f \in L^1(\mu)$, $f$ is real-valued, and $\epsilon > 0$. Then there exist functions $u$ and $v$ on $X$ s.t. $u \leq f \leq v$. $u$ is upper semicontinuous and bounded above, $v$ is lower semicontinuous and bounded below, and

$$\int_X (v - u) d\mu < \epsilon$$

# 2 Problems

**Problem 1**
a. This is true. To see this, notice that the set $\{x : (f_1 + f_2)(x) < \alpha\}$ is the union of the uncountably many sets

$$\bigcup_{0\leq\beta_1\leq\alpha} \{x : f_1(x) < \beta_1\} \cap \{x : f_2(x) < \alpha - \beta_1\}$$

b. This is also true. To see this, notice that the set $\{x : (f_1 + f_2)(x) > \alpha\}$ is the union of uncountably many sets

$$\bigcup_{0\leq\beta_1}\{x : f_1(x) > \beta_1\} \cap \{x : f_2(x) > \alpha - \beta_1\}$$

c. This is false. If we define $f_n = \chi_{[\frac{1}{n+1},\frac{1}{n}]}$, then clearly this is an upper semi-continuous function. However, we notice that $\sum_{i=1}^{\infty} f_n = \chi_{(0,1]} + \chi_X$, where $X = \{x : \frac{1}{x} \in \mathbb{Z}^+\}$. Notice that this value, for an $\alpha$ of 1, is clearly not an open set as it's complement is not closed.

d. This is true. Let $g_k = \sum_{i=1}^{k} f_i$. Clearly, we can check that $g_k$ is lower continuous

$$\bigcup_{\beta_1,\beta_2,\ldots\beta_{k-1}}\{x : f_1(x) > \beta_1\} \cdots \cap \{x : f_{k-1}(x) > \beta_3\} \cap \{x : f_k(x) > \alpha - \sum_{i=1}^{k-1}\beta_i\}$$

We know that $\sup_k g_k$ is also lower continuous, and, since $f_i \geq 0$, it follows that $\lim_{k\to\infty} g_k = \sup_k g_k$ is lower continuous.

If $f_i$ are not nonnegative, when our proofs for (a), (b), and (c) are still correct. However, our last proof fails because $\sup_k g_k$ is not necessarily $\sum_{i=1}^{\infty} f_i$.

**Problem 3**
We wish to prove that $\forall\epsilon, x_1, x_2$, there exists $\delta : \rho(x_1, x_2) < \delta \implies |\rho_E(x_1) - \rho_E(x_2)| < \epsilon$.

We let $x_1$, $x_2$ be two different points. Let $y_1 \in E$ such that $\rho(x, y_1) = \rho_E(x_1)$. Notice, by triangle inequality and the definition of infimum, we have

$$\rho(x_1, x_2) + \rho(x_1, y) \geq \rho(x_2, y) \geq \rho_E(x_2)$$
$$\implies \rho(x_1, x_2) \geq \rho_E(x_2) - \rho_E(x_1)$$

Similarly, setting $y_2 \in E$ such that $\rho(x_{y2} = \rho_E(x_2)$ gives us the similar result

$$\rho(x_1, x_2) \geq \rho_E(x_1) - \rho_E(x_2)$$

It follows that

$$\rho(x_1, x_2) \geq |\rho_E(x_1) - \rho_E(x_2)|$$

so we can set $\delta = \epsilon$ and obtain the desired result above, as $\rho(x_1, x_2) < \epsilon \implies |\rho_E(x_1) - \rho_E(x_2)| \leq \rho(x_1, x_2) < \epsilon$.

We notice that our $X$ is a locally compact Hausdorff space. Setting $K = B$ and $V = X - A$ gives us the relation described in Urysohn's lemma:

$$K \prec f \prec V$$

**Problem 5**
To see that the cantor set has measure 0, we can count the measure removed. For $C_1$, it is $\frac{1}{3}$, for $C_2$ it is $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$, and in general, for $C_i$ it is $\frac{2^{i-1}}{3^i}$. Summing, we have

$$\sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = \frac{1}{3} \times \sum_{i=1}^{\infty} \frac{2^i}{3^i} = 1$$

so $m(E) = 0$. Furthermore, we can define a surjective mapping from $f : E \to [0,1]$. In ternary, all elements of the cantor set can be expressed as a decimal with only 2 and 0. Our function substitutes 1 for 2 and considering the resulting string in binary. Clearly, this is implies that $f$ is uncountable and has the same carnality as $\mathbb{R}$.

**Problem 7**
Note that $m(\{q\}) = 0$, where $q$ is a rational number. To see this, note that $m(\{q\}) < m(W) = \frac{1}{q^n}$ for all $n$. Since the rationals are countable, it follows that $m(\{q \in \mathbb{Q} : q \in [0,1]\}) = 0$.

Our construction of the set is $(0, \epsilon) \cup \{q \in \mathbb{Q} : q \in [0,1]\}$. We see that this measure is $\epsilon$ and that the closure is in fact $[0,1]$.

**Problem 9**
We define $f_n(x) = \chi_{A_n}$ where $A_n = \{x : \exists q \in \mathbb{R} \mid qn = x\}$. We note that $\int_{[0,1]} f_n(x)dm = 0$ as the measure of $A_n$ is obviously 0 (since there are countably values). Therefore, it follows that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

as this holds for all $n$. Furthermore, we note that this value doesn't converge for any value, using $\epsilon - \delta$ notation.

**Problem 11**
Let $K = \bigcap K_\alpha$. Note that $K_\alpha \in \mathfrak{M}$, so we can apply Thm 2.7 to get that

$$K \subset K_i \subset V$$

for all open $V : K \subset V$. To prove that $K_i$ is indeed one of the $K_\alpha$. Note that $\mu(K) = 1$ since $\mu$ is regular and

$$\mu(K) = \inf\{\mu(V) : E \subset V \subset X\} = 1$$

By Thm 2.17

$$\mu(V-K)=0 \implies \mu(K_i)=\mu(K)=1$$

It follows that $\mu(K^c)=1-\mu(K)=0$. Furthermore, since $K$ is the intersection of all $K_\alpha$ with measure 1, it follows that it is the largest value.

**Problem 13**
It clear that the set $\{0\}$ is compact. However, this can't be the support of any continuous function $f$.

Examining compact sets in general, we find that $f$ must have images both 0 and $y \neq 0$. For this case, we find that these must be open sets in order for the support to be compact. In general, if we have a compact set such that its interior at each discontinuous piece is nonzero, then we can define a continuous function $f$ that approaches zero at the border.

**Problem 15**
We guess that

$$\lim_{n\to\infty}\int_0^n (1-\frac{x}{n})^n e^{x/2}dx = 2$$

This is because we assume this value tends to

$$\int_0^\infty e^{-x}\times e^{x/2}dx = \int_0^\infty e^{-x/2} = 2$$

We note this is true because of the dominated convergence theorem, and that $|f_n| = |(1-\frac{x}{n})^n e^{x/2}| \leq e^{-x/2}$ as $(1-\frac{x}{n}^n)$ is increasing

Similarly, for

$$\lim_{n\to\infty}\int_0^n (1+\frac{x}{n})^n e^{-2x}dx = 1$$

This is because we apply the dominated convergence theorem and find that

$$|f_n| = |(1+\frac{x}{n})^n e^{x}| \leq e^{-x}$$

since $(1+\frac{x}{n})^n$ is increasing.

**Problem 17**
We check the conditions of a metric space.

1) Obviously, non-negativity holds.

2) Furthermore, if $\rho((x_1,y_1),(x_2,y_2))=0 \implies x_1=x_2$ and $|y_1-y_2|=0 \implies x_1=x_2, y_1=y_2$, so identity of 0 holds.

3) Symmetry obviously holds.

4) Lastly, to prove the triangle inequality. We calculate the individual cases. If $x_1 = x_2 = x_3$, then it holds. If $x_1 = x_2 \neq x_3$, then it holds. Lastly, if $x_1 \neq x_2 \neq x_3$, then it also clearly holds (it adds 1 to the side with two elements).

This is locally compact since, for a neighborhood with radius $< 1$, then we just have a compact perpendicular line through $x$.

For our function $f$ and our $\mu$ as defined, we notice that

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y) dy = \int_X f d\mu$$

Clearly, we can see that the second value is the integral of $f$ over the support. Therefore, it follows that $\mu(E) = \infty$ (we can notice this by taking the function $f = \lim\limits_{n\to\infty} \chi_{E_n}$ and using the construction $\mu(V) = \sup\{\Lambda f : f \prec V)$. This happens because our value of $\Lambda f$ approaches $\infty$ as $n \to \infty$. However, for $K \subset E$, $\mu(K) = 0$ because this value is a finite sum of $\displaystyle\int_{-\infty}^{\infty} \chi_{\{0\}} dy$.

**Problem 19**
First, we notice that $V$ is not only an open subset of $X$, but is also compact. We can apply this to Steps 1 and 5.

**Problem 21**
We notice that, $\exists \alpha$ such that $\{x : f(x) < \alpha\} \neq X, \{x : f(x) < \alpha+\epsilon\} = X \; \forall \epsilon > 0$. To see this, it suffices to use the fact that $f$ is upper-semicontinuous and we can pick a value larger than or equal to $\sup_X f(x)$. If we pick $\sup_X f(x)$ as our $\alpha$, then both equalities are satisfied (or else $\alpha$ wouldn't be sup by definition), so it follows that there exists some $x$ such that $f(x) = \sup_X f(x)$.

**Problem 23**
I don't have a good solution to this question, so I'm leaving it blank.

**Problem 25**
(i) We rearrange the formula to find that

$$1 + e^t < e^c e^t \implies \log(1 + e^{-t}) < c$$

The LHS is clearly maximized as $t \to 0$, so we find that $c > \log 2$.

(ii) We apply the dominated convergence theorem to find that

$$\lim_{n\to\infty}\frac{1}{n}\int_0^1 (1+e^{nf(x)})dx < \lim_{n\to\infty}\frac{1}{n}\int_0^1 \log 2 + nf(x)dx = \int_0^1 f(x)$$

Furthermore, we notice that our equations

$$f_n(x) = \frac{1+e^{nf(x)}}{n} \to f(x), n \to \infty$$

since the 1 becomes irrelevant. It follows that our integral converges to $\int_0^1 f(x)$.