Erdmann-Wildon Lie Algebras - Solvable Lie Algebras and a Rough Classification

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1 Notes

1.1 Solvable Lie Algebras

Lemma 4.1 Suppose that I is and ideal of L. Then L/I is abelian iff I contians the derived algebra L'.

We note that the smallest such ideal is L' where L/I is abelian. We denote $L^{(i)}$ as

$$L^{(1)} = L' \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}]$$

and notice $L \supseteq L^{(1)} \supseteq L^{(2)} \dots$.

□

Def 4.2 L is solvable if for $m \geq 1$, $L^{(m)} = 0$.

Lemma 4.3 If L is a Lie algebra with ideals

$$L = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_{m-1} \supseteq I_m = 0$$

s.t I_{k-1}/I_k is abelian for $1 \leq k \leq m$, then L is solvable.

□

The derived series is the fastest descending sequence.

Lemma 4.4 Let L be a Lie Algebra.

(a) If L is solvable, then every subalgebra and and homomorphic image of L are solvable.
(b) With ideal I s.t. I and L/I are solvable, L must be solvable.
(c) I, J solvable ideals of L implies $I + J$ is a solvable ideal of L.

□

Corollary 4.5 Let L be a finite-dimensional Lie Algebra. There is a unique solvable ideal of L containing every solvable ideal of L. This is the *radical* of L is denoted rad$L$.

Def 4.6 A non-zero Lie Algebra L is said to be *semisimple* if it has no non-zero solvable ideals or equivalently if $\text{rad}L = 0$.

Lemma 4.7 If L is a Lie algebra, then the factor algebra $L/\text{rad}L$ is semisimple.

1.2 Nilpotent Lie Algebras

The *lower central series* of a Lie algebra L is

$$L^1 = L' \quad L^k = [L, L^{k-1}]$$

Then $L \supseteq L^1 \supseteq \dots$ and L^k is an ideal of L, and L^k/L^{k+1} is contained in the centre of L/L^{k+1}.

Def 4.8 L is said to be nilpotent if some $m \geq 1$ has $L^m = 0$.

Lemma 4.9 L is a Lie algebra

(a) If L is nilpotent, then any subalgebra is nilpotent.
(b) $L/Z(L)$ is nilpotent implies L is nilpotent.

Remark 4.10 The analogue of 4.4(b) doesn't hold. If L/I and I are nilpotent, then L isn't necessarily.

1.3 A Look Ahead

We note that $\text{rad}L$ is solvable, $L/\text{rad}L$ is semisimple. To understand L it is necessary to understand

(i) an arbitary solvable Lie algebra
(ii) an arbitrary semisimple Lie algebra

In $\mathbb{C}$, (i) results in Lie's Theorem (every solvable Lie algebra appears as a subalgebra of a Lie algebras of upper triangular matrices). (ii) is the direct sum of *simple* Lie algebras.

Def 4.11 L is *simple* if it has no ideals other than 0 and L and is not abelian.

Thm 4.12(Simple Lie Algebras) With 5 exceptions, every finite-dimensional simple Lie algebra over $\mathbb{C}$ is isomorphic to one of the *classical Lie Algebras*

$$\mathbf{sl}(n, \mathbb{C}) \quad \mathbf{so}(n, \mathbb{C}) \quad \mathbf{sp}(2n, \mathbb{C})$$

and the special ones are e_6, e_7, e_8, f_4, g_2. We recall that by deining

$$\mathbf{gl}_S(n, \mathbb{C}) := \{x \in \mathbf{gl}(n, \mathbb{C}) : x^t S = -Sx\}$$

then $\mathbf{so}(2\ell, \mathbb{C}) = \mathbf{gl}_S(2\ell, \mathbb{C})$ where

$$S = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}$$

and $\mathbf{so}(2\ell + 1, \mathbb{C}) = \mathbf{gl}_S(2\ell + 1, \mathbb{C})$ where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$$

and these are called the *orthogonal Lie algebras.* $\mathbf{sp}(2\ell, \mathbb{C}) = \mathbf{gl}_S(2\ell, \mathbb{C})$ where

$$S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$$

and these are called the *symplectic Lie algebras* and are only defined for even dimensions.

2 Exercise

Exercise 4.1 We note that $\varphi(L_1) = L_2$, and proceed by induction where we assume $\varphi(L_1^{(k)}) = L_2^{(k)}$. We see that $\varphi([L_1^{(k)}, L_1^{(k)}]) = [\varphi(L_1)^{(k)}, \varphi(L_1)^{(k)}] = [L_2^{(k)}, L_2^{(k)}] = L_2^{(k+1)}$ and this proves our result.

Exercise 4.2 We note that, if we take some element which is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d have dimension $n/2$, then note that, through calculation

$$\begin{pmatrix} -c^t & a^t \\ -d^t & b^t \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

so it follows that $a = -d^t$ and b, c are symmetric. Therefore, the form

$$x = \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix}$$

where p, q are symmetric, as desired.

Exercise 4.3 We note that ad$x$ is homomorphic to x. By Lemma 4.4 (a), it follows that L is solvable implies that ad$L$ is also solvable. Furthermore, taking a subalgebra implies that we can have an inverse of ad with image L, so ad$L$ is solvable implies L is as well.

This result also holds for being nilpotent since homomorphic images of L are also nilpotent, and we use the same process.

Exercise 4.4 This is obvious since each application of the Lie Bracket with L and L^{k-1} results in one fewer diagonal. I tfollows that for a sufficiently large $k = n$, then $j > n$, so this is impossible.

Exercise 4.5

(i) For some basis e_i and upper triangular mappings μ, λ, we notice that $[\lambda, \mu]e_i$ has no coefficient for e_i or higher. However, a simple surjectivity proof shows that $L' = \mathbf{n}(n, \mathbb{F})$.
(ii) This is apparent as we note that, if we are n off from the diagonal (including that value), we double the distance after applying the Lie Bracket. This can be shown through induction or
(iii) For $k : 2^{k-1} > n$, then we notice that $L^{(k)} = 0$.
(iv) Note that the adjoint of $\mathbf{b}(n, \mathbb{F})$ maps to itself.

Exercise 4.6 We prove the contrapositive (if it has some non-zero solvable ideal iff it has some non-zero abelian ideal).

If we have a solvable ideal I, then we let $I^{(m)} = 0, I^{(m-1)} \neq 0$. Then notice that $I^{(m-1)}$ is an abelian ideal that is not 0.

If we have an abelian ideal I, then we notice that $I^{(1)} = 0$ and this is our non-zero solvable ideal.

Exercise 4.7 We note that, for $i \neq j$, we have $\mathrm{ad} e_{ij}$ maps to all values e_{ik} and e_{kj} for some value k and $e_{ii} - e_{jj}$. Comparing these values notes that they map to all values in $\mathbf{sl}(n, \mathbb{C})$.

Examining values for $\mathrm{ad}(e_{ii} - ejj)$ notes that we have a range of values e_{ik} and e_{kj} and $e_{ij} - e_{ji}$. Lastly, for $\mathrm{ad} e_{nn}$, maps to values of e_{nk} and e_{kn}. This similarly shows that we can't have a non-zero ideal, and furthermore $\mathbf{sl}(n, \mathbb{C})$ is non abelian, so it is simple.

Exercise 4.8

(i) We note that, since $[[a, b + c], b + c] = 0$ implies

$$[[a, b], c] = -[[a, c], b]$$

so it follows that, for $[[x, y], z]$ we note that

$$\begin{aligned} [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \\ \implies 3[[x, y], z] = 0 \end{aligned}$$

and, since $\mathbb{F}$ doesn't have a characteristic of 3, it follows that

$$[[x,y],z]=0$$

and therefore $L^3=0$.

(ii) We note that

$$[[a,b],c]=-[[a,c],b]$$

Furthermore, we note that

$$[[a,b],c]=-[[b,a],c]$$

relatively simply. We note that

$$[[x,y],[z,y]]=[[[x,y],z],t]-[[[x,z],t],z]=2[[[x,y],z],t]=-[[[x,y],z],t]$$

But, we also note that $[[[z,t],x],y]=[[[x,y],z],t]$ and this implies $2[[[x,y],z],t]=0$ and therefore, if F has characteristic 3, then $L^4=0$.

Exercise 4.9

(i) We note that $\det(I+\epsilon A)=\exp\operatorname{tr}\log(I+\epsilon A)$. Therefore, we notice that (after expanding $I+\epsilon A$ with a log Taylor series), that

$$\det(I+\epsilon]'A)=1+\operatorname{tr}(A)\epsilon+\ldots$$

Therefore, note that, if we ignore the later terms, then

$$I+\epsilon X\in\mathrm{SL}(n,\mathbb{C})\iff X\in\mathrm{sl}(n,\mathbb{C})$$

(ii) (a) This is in fact a group as I is int his group under multiplication

(b) We note that, if we have $I+\epsilon X\in V$, then

$$((I+\epsilon X)v,(I+\epsilon X)v)=(v,v)+((Xv,v)+(v,Xv))\epsilon+(Xv,Xv)\epsilon^2$$

and if we ignore the ϵ^2 term, then we note that

$$\begin{aligned}0=(Xv,v)+(v,Xv)&=v^tX^tSv+v^tSXv\\ \iff X^tS=-SX&\iff X\in\mathbf{gl}_S(n,\mathbb{C})\end{aligned}$$

(iii) (a) Similarly, we note that $I\in G_I(n,\mathbb{C})$, and furthermore that this is a group under matrix multiplication as $A,B\in G_I(n,\mathbb{C})$ implies that

$$(AB)^{-1}=B^{-1}A^{-1}=B^tA^t=(AB)^t$$

We note that, if $I + \epsilon A \in G_I(n, \mathbb{C})$, then $(I + \epsilon A)^{-1} = I - \epsilon A + \dots$. Similarly,

$$\begin{aligned} (I + \epsilon A)^t &= (I + \epsilon A)^{-1} \\ \iff I + \epsilon A^t &= I - \epsilon A \end{aligned}$$

and so it follows that $A^t = A$, so A is antisymmetric. The associated Lie algebra, $g_I(n, \mathbb{C})$ is the space of antisymmetric matrices.

(b) Note that $g_I(n, \mathbb{C}) = \mathrm{gl}_I(n, \mathbb{C})$ since that would imply that $x^t = -x$. The mapping $e_i \to e_{n+1-i}$ is the mapping. for even n and $e_i \to e_{n-i}$ and $e_1 \to e_1$ for odd n. By 2.11, $g_I(n, \mathbb{C}) \cong \mathbf{so}(n, \mathbb{C})$.

(iv) Under our $S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$ our value of $v = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives (v, v) as

$$v^t S v = \begin{pmatrix} a^t c - c^t a & a^t d - c^t b \\ b^t c - d^t a & b^t d - d^t b \end{pmatrix} = 0$$

This forms a group, since $I \in V$ and A, B in our group implies that

$$(AB)^t S(AB) = B^t A^t SAB = 0 \implies AB \text{ is in the group}$$

Notice $((I + \epsilon X), (I + \epsilon X)) = (I, I) + ((I, X) + (X, I))\epsilon + \dots$. We finalize by noting that we must have $X^t S + SX = 0$, so the associated Lie algebra is clearly $\mathbf{gl}_S(2\ell, \mathbb{C}) = \mathbf{sp}(2\ell, \mathbb{C})$.

Exercise 4.10 T is a change of basis for each $x \to y$ where $x \in \mathbf{gl}_S(n, \mathbb{C})$ and $y \in \mathbf{gl}_T(n, \mathbb{C})$.