Rudin Real and Complex Analysis - Harmonic Functions

Aaron Lou

December 2018

1 Notes

1.1 The Cauchy-Riemann Equations

11.1: The Operators ∂ and $\overline{\partial}$ Suppose f is a complex function defined in a plane open set Ω. Regard f as a transformation which maps Ω into $\mathbb{R}^2$, and assume that f has a differential at some point $z_0 \in \Omega$, in the sense of Def 7.22. For simplicity, suppose $z_0 = f(z_0) = 0$. Our differentiability assumption is then equivalent to the existence of two complex numbers α, β (the partial derivatives of f with respect to x and Y at $z_0 = 0$) s.t.

$$f(z) = \alpha x + \beta y + \eta(z)z$$

where $\eta(z) \to 0$ as $z \to 0$. Since $2x = z + \overline{z}$ and $2iy = z - \overline{z}$, this can be rewritten as

$$f(z) = \frac{\alpha - i\beta}{2}z + \frac{\alpha + i\beta}{2}\overline{z} + \eta(z)z$$

This suggests the introduction of the differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and the $f(z)$ equation becomes

$$\frac{f(z)}{z} = (\partial f)(0) + (\overline{\partial} f)(0) \cdot \frac{\overline{z}}{z} + \eta(z)$$

For real z, $\overline{z}/z = 1$ for pure imaginary z, $\overline{z}z = -1$. Hence $f(z)/z$ has a limit at 0 iff $(\overline{\partial} f)(0) = 0$.

11.2 Thm Suppose f is a complex function in Ω that has a differential at every point of Ω. Then $f \in H(\Omega)$ iff the Cauchy-Riemann equation

$$(\overline{\partial} f)(z) = 0$$

holds for every $z \in \Omega$. In that case we have

$$f'(z) = (\partial f)(z) \quad z \in \Omega$$

If $f = u + iv$, then this splits into the pair of equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

11.3 The Laplacian Let f be a complex function in a plane open set Ω, s.t. f_{xx} and f_{yy} exist at every point of Ω. The *Laplacian* of f is then defined to be

$$\Delta f = f_{xx} + f_{yy}$$

If f is continuous in Ω and if

$$\Delta f = 0$$

at every point of Ω, then f is said to be *harmonic* in Ω. Since the Laplacian of a real function is real, a complex function is harmonic in Ω iff the real and imaginary parts are harmonic. Note that

$$f_{xy} = f_{yx} \implies \Delta f = 4\partial\overline{\partial} f$$

If f is holomorphic, then $\overline{\partial} f = 0$, f has continuous derivatives of all orders.

11.4 Thm Holomorphic functions are harmonic.

1.2 The Poisson Integral

11.5 The Poisson Kernel This is the function

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int} \quad (0 \leq r < 1, t \text{ real})$$

If $z = re^{i\theta}$ for $0 \leq r < 1$, a simple calculation shows that

$$P_r(\theta - t) = \mathrm{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}$$

and from the Poisson Kernel formula, we can see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt = 1$$

It also follows from the special case that

$$P_r(t) < P_r(\delta) \quad (0 < \delta < |t| \leq \pi)$$

$$\lim_{r\to 1} P_r(\delta) = 0 \quad (0 < \delta \leq \pi)$$

The open unit disc $D(0,1)$ is now U. The unit circle - the boundary of U - is T. Note that

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} \quad z \in U, e^{it} \in T$$

11.6 The Poisson Integral If $f \in L^1(T)$ and

$$F(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) dt$$

then the function F so defined in U is called the *Poisson integral* of f (also $P[f]$). If f is real, we have that $P[f]$ is the real part of

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} f(t) dt$$

11.7 Thm If $f \in L^1(T)$ then the Poisson integral $P[f]$ is a harmonic function in U.

11.8 Thm If $f \in C(T)$ and if Hf is defined on the closed unit disc $\overline{U}$ by

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & r = 1 \\ P[f](re^{i\theta}) & 0 \le r < 1 \end{cases}$$

then $Hf \in C(\overline{U})$.

11.9 Thm Suppose u is a continuous real function on the closed unit disc $\overline{U}$, and suppose u is harmonic in U. Then (in U) u is the Poisson integral of its restriction to T, and u is the real part of the holomorphic function

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it}) dt \quad z \in U$$

11.10 We can use apply the previous work to arbitrary circular discs using cov. The results are as follows:

If u is a continuous real function on the boundary of the disc $D(a, R)$ and if u is defined in $D(a, R)$ by the Poisson integral

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u(a + Re^{it}) dt$$

then u is continuous on $\overline{D}(a, R)$ and harmonic on $D(a, R)$.

If u is harmonic and real in an open set Ω and if $\overline{D}(a, R) \subset \Omega$, then u satisfies the above equation in $D(a, R)$ and there is a holomorphic function f defined in $D(a, R)$ whose real part is u. This f is uniquely defined, up to a pure imaginary additive constant.

Every real harmonic function is locally the real part of a holomorphic function.

11.11 Harnack's Thm Let $\{u_n\}$ be a sequence of harmonic functions in a region Ω.

(a) If $u_n \to u$ uniformly on compact subsets of Ω, then u is harmonic in Ω.

(b) If $u_1 \leq u_2 \leq \dots$, then either $\{u_n\}$ converges uniformly on compact subsets of Ω, or $u_n(z) \to \infty$ for every $z \in \Omega$.

1.3 The Mean Value Property

11.12 Def We say that a continuous function u in an open set Ω has the *mean value property* if to every $z \in \Omega$ there corresponds a sequence $\{r_n\}$ s.t. $r_n > 0, r_n \to 0$ as $n \to \infty$, and

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + r_n e^{it}) dt \quad (n = 1, 2, \dots)$$

or $u(z)$ is equal to the mean value of u on the circles of radius r_n with center at z.

11.13 Thm If a continuous function u has the mean value property in an open set Ω, then u is harmonic in Ω.

We denote Π^+ as the upper half plane $\{x + iy : y > 0\}$, the lower half plane is Π^-.

11.14 Thm (The Schwarz reflection principle) Suppose L is a segment of the real axis Ω^+ is a region in Π^+, and every $t \in L$ is the center of an open disc D_t s.t. $\Pi^+ \cap D_T$ lies in Ω^+. Let Ω^- be the reflection of Ω^+:

$$\Omega^- = \left\{z : \overline{z} \in \Omega^+\right\}$$

Suppose $f = u + iv$ is holomorphic in Ω^+, and

$$\lim_{n \to \infty} v(z_n) = 0$$

for every sequence $\{z_n\}$ in Ω^+, which converges to a point of L. Then, there is a function F, holomorphic in $\Omega^+ \cup L \cup \Omega^-$ s.t. $F(z) = f(z)$ in Ω^+ and satisfies

$$F(\overline{z}) = \overline{F(z)} \quad z \in \Omega^+ \cup L \cup \Omega^-$$

1.4 Boundary Behavior of Poisson Integrals

11.15 Our next objective is to find analogues of Thm 11.8 for Poisson integrals of L^p functions and measures on T. For any u in function in U, let u_r be a family of function on T s.t.

$$u_r(e^{i\theta}) = u(re^{i\theta}) \quad (0 \leq r < 1)$$

If $f \in C(T)$ and $F = P[f]$, then $F_r \to f$ uniformly on T as $r \to 1$, or

$$\lim_{r\to 1} \|F_r - f\|_\infty = 0$$

$$\lim_{r\to 1} F_r(e^{i\theta}) = f(e^{i\theta})$$

11.16 Thm If $1 \leq p \leq \infty$, $f \in L^p(T)$ and $u = P[f]$, then

$$\|u_r\|_p \leq \|f\|_p \quad (0 \leq r < 1)$$

If $1 \leq p < \infty$, then

$$\lim_{r\to 1} \|u_r - f\|_p = 0$$

11.17 Poisson Integrals of Measures If μ is a complex measure on T, and if we want to replace integrals over T by integrals over intervals of length 2π in $\mathbb{R}^1$, these intervals have to be taken half open, because of the possible presence of point masses in μ. To avoid this problem, we shall keep integration on the circle in what follows, and will write the Poisson integral $u = P[d\mu]$ of μ in the form

$$u(z) = \int_T P(z, e^{it}) d\mu(e^{it}) \quad z \in U$$

where $P(z, e^{it}) = (1 - |z|^2)/|e^{it} - z|^2$. Thus, u, defined above, is harmonic in U. Setting $\|\mu\| = |\mu|(T)$, the analogue of 11.16 is

$$\|u_r\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^i\theta)| d\theta \leq \|\mu\|$$

11.18 Approach Regions For $0 < \alpha < 1$, we define Ω_α to be the union of the disc $D(0, \alpha)$ and the line segments from $z = 1$ to points of $D(0, \alpha)$. In other words, Ω_α is the smallest convex open set that contains $D(0, \alpha)$ and has the point 1 in its boundary. Near $z = 1$, Ω_α is an angle, bisected by the radius of U that terminates at 1, of opening 2θ, where $\alpha = \sin\theta$. Curves that approach 1 within Ω_α cannot be tangent to T. Therefore Ω_α is called a *nontangential approach* region, with vertex 1. The regions Ω_α expand when α increases. Their union is U, their intersection is the radius $[0, 1)$. Rotated copies of Ω_α, with vertex at e^{it}, will be denoted by $e^{it}\Omega_\alpha$.

11.19 Maximal Functions If $0 < \alpha < 1$ and u is any complex function with domain U, its *nontagential maximal function* $N_\alpha u$ is defined on T by

$$(N_\alpha u)(e^{it}) = \sup\left\{|u(z)| : z \in e^{it}\Omega_\alpha\right\}$$

Similarly, the *radial maximal function* of u is

$$(M_{rad}u)(e^{it}) = \sup\left\{|u(re^{it})| : 0 \le r < 1\right\}$$

If u is continuous and λ is a positive number, then the set where either of these maximal functions is $\le \lambda$ is a closed subset of T. Consequently, $N_\alpha u$ and $M_{rad}u$ are lower semicontinuous on T and are measurable.

Note that $M_{rad}u \le N_\alpha u$ and the latter increases with α. If $u = P[d\mu]$, Thm 11.20 will show that the size of $N_\alpha u$ is, in turn, controlled by the maximal function $M\mu$. We let $\sigma = m/2\pi$, where m is the Lebesgue measure, and is a rotation-invariant positive Borel measure on T, s.t. $\sigma(T) = 1$. We have

$$(M\mu)(e^{i\theta}) = \sup \frac{|\mu|(I)}{\sigma(I)}$$

The supremum is taken over all open arcs $I \subset T$, whose centers are at $e^{i\theta}$, including T itself (even though T is not an arc). The derivative $D\mu$ of a measure μ on T is now

$$(D\mu)(e^{i\theta}) = \lim \frac{\mu(I)}{\sigma(I)}$$

as $I \subset T$ shrink to their center $e^{i\theta}$ and $e^{i\theta}$ is a *Lebesgue point* of $f \in L^(T)$ if

$$\lim \frac{1}{\sigma(I)} \int_I |f - f(e^{i\theta})| d\sigma = 0$$

with these I are as above. If $d\mu = f d\sigma + d\mu_s$ is the Lebesgue decomposition of a complex Borel measure μ on T, where $f \in L^1(T)$ and $\mu_s \perp \sigma$, we see that

$$\sigma\left\{M\mu > \lambda\right\} \le \frac{3}{\lambda}\|\mu\|$$

that almost every point of T is a Lebesgue point of f, and that $D\mu = f, D\mu_s = 0$ a.e. $[\sigma]$.

11.20 Thm Assume $0 < \alpha < 1$. Then there is a constant $c_\alpha > 0$ with the following property: If μ is a positive finite Borel measure on T and $u = P[d\mu]$ is its Poisson integral, then the inequalities

$$c_\alpha(N_\alpha u)(e^{i\theta}) \le (M_{radu})(e^{i\theta}) \le (M\mu)(e^{i\theta})$$

holds at every point $e^{i\theta} \in T$.

11.21 Nontangential Limits A function F, defined in U, is said to have *non-tangential limit* λ at $e^{i\theta} \in T$ if, for each $\alpha < 1$,

$$\lim_{j\to\infty} F(z_j) = \lambda$$

for every sequence $\{z_j\}$ that converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$.

11.22 Thm If μ is a positive Borel measure on T and $(D\mu)(e^{i\theta}) = 0$ for some θ, then its Poisson integral $u = P[d\mu]$ has nontangential limit 0 at $e^{i\theta}$.

11.23 Thm If $f \in L^(T)$, then $P[f]$ has nontangential limit $f(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f.

11.24 Thm If $d\mu = fd\sigma + d\mu_s$ is the Lebesgue decomposition of a complex Borel measure μ on T, where $f \in L^1(T), \mu_s \perp \sigma$, then $P[d\mu]$ has a nontangential limit $f(e^{i\theta})$ at almost all points of T.

11.25 Thm For $0 < \alpha < 1$ and $1 \le p \le \infty$, there are constants $A(\alpha, p) < \infty$ with

(a) If μ is a complex Borel measure on T, and $u = P[d\mu]$, then

$$\sigma\{N_\alpha u > \lambda\} \le \frac{A(\alpha,1)}{\lambda}\|\mu\| \quad 0 < \lambda < \infty$$

(b) If $1 < p \le \infty$, $f \in L^p(T)$ and $u = P[f]$, then

$$\|N_\alpha u\|_p \le A(\alpha, p)\|f\|_p$$

1.5 Representation Theorems

11.26 One can tell whether a harmonic function u in U is a Poisson integral if the L^p-boundedness of the family $\{u_r : 0 \le r < 1\}$ is sufficient (or the previous theorems 11.16, 11.25). Thus, in particular, the boundedness of $\|u_r\|_1$ as $r \to 1$ implies the existence of nontagential limits a.e. on T as u can be represented as the Poisson integral of a measure. This measure will be obtained by taking the "weak limit" of the functions u_r.

11.27 Def Let $\mathcal{F}$ be a collection of complex functions on a metric space X with metric ρ. We say that $\mathcal{F}$ is *equicontinuous* if to every $\epsilon > 0$ corresponds a $\delta > 0$ s.t. $|f(x) - f(y)| < \epsilon$ for every $f \in \mathcal{F}$ and for all pairs of points x, y with $\rho(x, y) < \delta$). We say that $\mathcal{F}$ is *pointwise bounded* if to every $x \in X$ corresponds an $M(X) < \infty$ s.t. $|f(x)| \le M(x)$ for all $f \in \mathcal{F}$.

11.28 Thm (Arzela-Ascoli) Suppose that $\mathcal{F}$ is a pointwise bounded equicontinuous collection of complex functions on a metric space X, and that X contains a countable dense subset E. Every sequence $\{f_n\}$ in $\mathcal{F}$ has then a subsequence that converges uniformly on every compact subset of X.

11.29 Thm Suppose that

(a) X is a separable Banach space

(b) $\{\Lambda_n\}$ is a sequence of linear functionals on X

(c) $\sup_n \|\Lambda_n\| = M < \infty$.

Then there is a subsequence $\{\Lambda_{n_i}\}$ s.t. the limit

$$\Lambda x = \lim_{i\to\infty} \Lambda_{n_i} x$$

exists for every $x \in X$. Moreover, Λ is linear and $\|\Lambda\| \le M$.

11.30 Thm Suppose u is harmonic in U, $1 \le p \le \infty$, and

$$\sup_{0<r<1} \|u_r\|_p = M < \infty$$

(a) If $p = 1$, it follows that there is a unique complex Borel measure μ on T so that $u = P[d\mu]$.

(b) If $p > 1$, it follows that there is a unique $f \in L^p(T)$ so that $u = P[f]$.

(c) Every positive harmonic function in U is the Poisson integral of a unique positive Borel measure on T.

11.31 Since holomorphic functions are harmonic, all of the preceding results (11.16, 11.24, 11.25, 11.30 are the most significant) apply to holomorphic functions in U. This leads to the study of H^p-spaces. For example, for H^∞, which is all bounded holomorphic functions in U, the norm

$$\|f\|_\infty = \sup_{z\in U} |f(z)|$$

turns H^∞ into a Banach space. Let $L^\infty(T)$ be the space of all equivalence classes of essentially bounded functions on T, normed by the essential supremum norm, relative to Lebesgue measure. For $g \in L^\infty(T)$ $\|g\|_\infty$ is the essential supremum of $|g|$.

11.32 Thm To every $f \in H^\infty$ corresponds a function $f^* \in L^\infty(T)$ defined almost everywhere by

$$f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

We also have $\|f\|_\infty = \|f^*\|_\infty$. If $f^*(e^{i\theta}) = 0$ for almost all $e^{i\theta}$ on some arc $I \subset T$, then $f(z) = 0$ for all $z \in U$.

2 Problems

Problem 1 Note that

$$(uv)_{xx} = (u_x v + uv_x)_x = u_{xx}v + u_x v_x + u_x v_x + uv_{xx}$$

so we see that the necessary condition is that

$$u_x v_x + u_y v_y = 0$$

For u^2, we see that the the only possibility is for $u_x^2 + u_y^2 = 0$, so both value must be 0 (since u is a real function and so these values are positive). Therefore, it follows that u is constant on both variables. We see that $|f|^2$ is harmonic iff $|f|$ is constant.

Problem 3 By the Cauchy-Riemann Equations, we can see that the gradients of the real and complex portions are also 0.

Problem 5 Notice that defining $f = u + iv$ means that

$$|f| = \sqrt{u^2 + v^2} \implies \frac{\partial \log |f|}{\partial x} = \frac{uu_x + vv_x}{u^2 + v^2}$$

And there a is a similar derivative for y. Taking the second derivatives, we see that they are similarly

$$\frac{(u^2 + v^2)(uu_{xx} + u_x u_x + vv_{xx} + v_x v_x) - (uu_x + vv_x)(uu_x + vv_x)}{(u^2 + v^2)^2}$$

$$= \frac{u^3 u_{xx} + v^2 uu_{xx} + u^2 vv_{xx} + v^3 v_{xx} + (uv_x - vu_x)^2}{(u^2 + v^2)^2}$$

and putting this together with the second dervivative for y shows that the Laplacian is 0 by the Cauchy-Riemann Equations and the fact that u and v are harmonic. An easier way to see this is that $\frac{f'(z)}{f(z)}$ is holomorphic, and its integral is

$$\log(|f(z)|) + i \arg(f(z))$$

and so $\log |f(z)|$ is the real part of a holomorphic function, and is therefore harmonic.

Problem 7

(a) We differentiate and use the Cauchy-Riemann equations to prove the general formula. In particular, the second derivative with respect to some variable x or y is

$$\psi''(u^2+v^2)\cdot(uu'+vv')^2+\frac{1}{2}\psi'(u^2+v^2)\cdot(u'^2+uu''+v'^2+vv'')$$

and we simplify using the Cauchy-Riemann equations.

(b) This is the same process of differentiating w.r.t some variable and then apply the Cauchy-Riemann equations. This specializes (a) as we let Φ be constant on the absolute value.

Problem 9 Note that

$$u(a)=\frac{2}{R^2}\int_0^R u(a)rdr$$

and we also have

$$u(a)=\frac{1}{\pi R^2}\int_0^R\int_{-\pi}^{\pi}u(a+re^{i\theta})rd\theta dr$$

Therefore we see that

$$\begin{aligned}\int_0^R r\left(u(a)-\frac{1}{2\pi}\int_{-\pi}^{\pi}u(a+re^{i\theta})d\theta\right)dr=0\\ \implies R\left(u(a)-\frac{1}{2\pi}\int_{-\pi}^{\pi}u(a+Re^{i\theta})d\theta\right)=0\\ \implies u(a)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(a+Re^{i\theta})d\theta\end{aligned}$$

and this is the Poisson integral which implies that u is harmonic.

Problem 11 We prove this in a similar manner to Caucy's Thm for triangles. This is because our set $\Omega-I$ is open, so we just need to prove it for when our triangle intersects our line. If only one point of our triangle is in our line $[a,b]$, then again we are done through the same method in the proof of Cauchy's Thm. However, if two points are in our line, say a,b, we can take x and y on $[a,c]$ and $[b,c]$ respectively close to a and b and integrate over the triangles $\{a,b,y\},\{a,y,x\},\{x,y,c\}$. These can be taken arbitrarily small so the integral of the triangle in this case is 0. Similarly, for three points the triangle is degenerate and so the integral is 0. Applying Morera's Thm (as all integrals of triangles are 0), shows that f is holomorphic on all of Ω.

Problem 13 Note that by Thm 11.30 we have

$$f(re^{i\theta})=\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}d\mu(t)$$

for $r=\frac{1}{2}$ and $\theta=0$, we note that we have a value

$$\int_{-\pi}^{\pi} \frac{3}{5-4\cos(t)} d\mu(t)$$

The max and min values depend on μ, so we set μ to be the point mass on the largest and smallest values. For the largest value if 3 and the smallest is $\frac{3}{5}$.

Problem 15 Again we relate this to a Poisson Integral of a Borel measure

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) d\mu(t) = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} d\mu(t)$$

As $r \to 1$ we see that

$$\begin{aligned}
\lim_{r\to 1} u(re^{i\theta}) &= \lim_{r\to 1} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} d\mu(t) \\
&= \int_{-\pi}^{\pi} \lim_{r\to 1} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} d\mu(t)
\end{aligned}$$

We see that the limit is 0 for everything except when $\theta = t$. In particular $\mu\{0\} \neq 0$ and everything else has measure 0. In fact, we see that the value of the integral is (when we set $t = 0$)

$$\mu\{0\} P_r(\theta)$$

Problem 17 Again we relate this to the Poisson Integral of a Borel measure

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(\theta - t) + r^2} d\mu(t)$$

We note that we have, by our condition $\mu([-\pi, \pi]) = 1$. Note that for all Borel Measures with total variation 1, the extreme points of the convex set are the point masses. To see this, note that we can't have two other measure μ_1 and μ_2 which μ is $\frac{\mu_1+\mu_2}{2}$ since $\mu_1(x) < 1$ and $\mu_2(x) < 1$ for $\mu(x) = 1$.

This means that the extremal values of our convex set of positive harmonic functions are $P_r(\theta - t)$ for some $t \in [-\pi, \pi]$.

Problem 19

(a) We note that our value is

$$P_r(\delta) = \frac{1-r^2}{1-2r\cos\delta + r^2}$$

and for ou value of δ where $0 < \cos\delta \leq 1$ we have that

$$\delta P_r(\delta) \geq \delta \frac{1+r}{1-r} = 1 + r > 1$$

for all values of r.

(b) We note that the value is

$$\delta u(r) = \delta \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t)+r^2} d\mu(t) > \int_{-\pi}^{\pi} 1 d\mu(t) > \mu(I_\delta)$$

As $\delta < \pi$. It follows that

$$\frac{\mu(I)}{2\delta} \leq \frac{\delta u(r)}{2\delta} \implies (M\mu)(1) \leq \delta(M_{rad}u)(r) \leq \pi(M_{rad}u)(r)$$

(c) This is a consequence of the fact that $D\mu \leq M\mu$ and we apply 7.15 to show that u goes to infinity because we have a bound below with value ∞.

Problem 21 The only question is for $z = 1$. We claim the value approaches 0 at this point, and can see this as

$$\lim_{z\to 1} \frac{1+z}{1-z} \to \infty \implies \lim e^{-f(z)} \to 0$$

To show that g^* is continuous on T, we note that as $\theta \to 0$ the value similarly approaches 0 for $1-z$ and $e^{-f(z)}$. However, the function g is not bounded as we let $\cos \frac{2y}{(1-x)^2+y^2} = 0$ and to approach the boundary this will cause the value to approach ∞.