Rudin Real and Complex Analysis - The Maximum Modulus Principle

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1 Notes

1.1 Introduction

12.1 The maximum modulus theorem (covered in 10.24) states that the constants are the only holomorphic functions in a region Ω whose absolute values have a local maximum at any point of Ω. Another restatement is that if K is the closure of a bounded region Ω and f is continuous on K and holomorphic in Ω, then

$$|f(z)| \leq \|f\|_{\partial\Omega}$$

$z \in \Omega$. If there is equality at some z, then f is constant. The RHS is the supremum on the boundary of Ω. Note that by Thm 11.32 that $\|f\|_\infty = \|f^*\|_\infty$ and therefore

$$|f(z)| \leq \|f^*\|_\infty \quad \text{on the boundary}$$

1.2 The Schwarz Lemma

12.2 Thm Suppose $f \in H^\infty$ and $\|f\|_\infty \leq 1$ and $f(0) = 0$. Then

$$|f(z)| \leq |z| \quad |f'(0)| \leq 1$$

if equality holds for $z \in U \setminus \{0\}$ or if equality holds in the second then $f(z) = \lambda z$ where λ is a constant and $|\lambda| = 1$. Geometrically, f is moves U closer to the origin or is a rotation if it is holomorphic.

12.3 Def For $\alpha \in U$, define

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$

12.4 Thm Fix $\alpha \in U$. Then φ_α is a one-to-one mapping which carries T onto T, U onto U, and α to 0. The inverse of φ_α is $\varphi_{-\alpha}$. We have

$$\varphi_\alpha'(0) = 1 - |\alpha|^2 \quad \varphi_\alpha'(\alpha) = \frac{1}{1 - |\alpha|^2}$$

12.5 An Extremal Problem Suppose α and β are complex, $|\alpha| < 1$ and $|\beta| < 1$. How large can $|f'(\alpha)|$ be if f is subject to the conditions $f \in H^\infty$, $\|f\| \leq 1$ and $f(\alpha) = \beta$. Note that

$$g = \varphi_\beta \circ f \circ \varphi_{-\alpha}$$

This reduces it to the Schwarz lemma, which has $|g'(0)| \leq 1$. The chain rule gives

$$g'(0) = \varphi_\beta'(\beta) f'(\alpha) \varphi_{-\alpha}'(0)$$

So we get that

$$|f'(\alpha)| \leq \frac{1 - |\beta|^2}{1 - |\alpha|^2}$$

This can only happen iff $|g'(0)| = 1$, when g is a rotation and $f(z) = \varphi_{-\beta}(\lambda \varphi_\alpha(z))$.

12.6 Thm If $f \in H(U)$ and f is one-to-one, $f(U) = U$, $\alpha \in U$ and $f(\alpha) = 0$, then there is a constant $\lambda : |\lambda| = 1$ s.t. $f(z) = \lambda \varphi_\alpha(z)$.

1.3 The Phgramen-Lindelof Method

12.7 For a bounded region Ω, the maximal modulus theorem implies that $\|f\|_\Omega = \|f\|_{\partial\Omega}$. For unbounded regions, this is not true. For example, letting

$$\Omega = \left\{ z = x + yi : -\frac{\pi}{2} < y < \frac{\pi}{2} \right\} \quad f(z) = e^{e^z}$$

Suppose f is an entire function and $|f(z)| < 1 + |z|^{1/2}$. Then f is constant. This follows from the Cauchy estimates since all sequence constants besides the first are 0.

12.8 Thm Suppose

$$\Omega = \{x + iy : a < x < b\} \quad \overline{\Omega} = \{x + iy : a \leq x \leq b\}$$

f is continuous on $\overline{\Omega}$, $f \in H(\Omega)$ and suppose that $|f(z)| < B$ for all $z \in \Omega$ and some fixed $B < \infty$. If

$$M(x) = \sup\{|f(x + iy)| : -\infty < y < \infty\} \quad (a \leq x \leq b)$$

then we have

$$M(x)^{b-a} \leq M(a)^{b-a} M(b)^{x-a}$$

Corr: Under the same hypotheses, $\log M$ is a convex function on (a, b).

12.9 Thm Suppose

$$\Omega = \left\{x + iy : |y| < \frac{\pi}{2}\right\} \quad \overline{\Omega} = \left\{x + iy : |y| \leq \frac{\pi}{2}\right\}$$

Suppose f is continuous on $\overline{\Omega}$, $f \in H(\Omega)$, there are constants $\alpha < 1, A < \infty$ s.t.

$$|f(z)| < \exp\{A \exp(\alpha|x|)\} \quad (z = x + iy \in \Omega)$$

and

$$\left| f\left(x \pm \frac{\pi i}{2}\right)\right| \leq 1 \quad -\infty < x < \infty$$

Then $|f(z)| \leq 1$ for all $z \in \Omega$.

12.10 Lindelof's Thm Suppose Γ is a curve with parameter interval $[0, 1]$ s.t. $|\Gamma(t)| < 1$ if $t < 1$ and $\Gamma(1) = 1$. If $g \in H^\infty$ and

$$\lim_{t \to 1} g(\Gamma(t)) = L$$

then g has radial limit L at 1.

1.4 An Interpolation Theorem

12.11 The convexity theorem in 12.8 can be used to prove that certain linear transformations are bounded w.r.t. certain L^p-norms. Let X be a measure space, with positive measure μ. Suppose $\{\psi_n\}$ is an orthonormal set of functions in $L^2(\mu)$. Furthermore, let us assume it is bounded in $L^\infty(\mu)$.

Then $f \in L^p(\mu)$ where $1 \leq p \leq 2$, the integrals

$$\widehat{f}(n) = \int_X f\overline{\psi}_n d\mu$$

exist and define a function $\widehat{f}$ on the set of all positive integers.We have that

$$\left\|\widehat{f}\right\|_\infty \leq M \left\|f\right\|_1$$

where M is the bound on $|\psi_n(x)|$. The Bessel inequality gives $\left\|\widehat{f}\right\|_2 \leq \|f\|_2$.

The Hausdorff-Young Theorem With the above assumptions, the inequality

$$\left\|\widehat{f}\right\|_q \leq M^{(2-p)/p} \left\|f\right\|_p$$

holds if $1 \leq p \leq 2$ and if $f \in L^p(\mu)$.

1.5 A Converse of the Maximum Modulus Theorem

12.13 Thm Suppose M is a vector space of continuous complex functions on the closed unit disc $\overline{U}$ with the following:

(a) $1 \in M$

(b) If $f \in M$ then so is $jf \in M$

(c) If $f \in M$ then $\|f\|_I = \|f\|_T$

Radós Theorem Assume $f \in C(\overline{U})$, Ω is the set of all $z \in U$ at which $f(z) \neq 0$ and f is holomorphic in Ω. Then f is holomorphic in U.

2 Exercises

Problem 1 Note that the function $(z-a)(z-b)(z-c)$ is holomorphic. By the maximum modulus principle there is no maximum value inside the triangle. The maximum is on the outside, and this occurs when a is in the middle of a segment with value $\frac{\sqrt{3}}{8}$.

Problem 3 If $f \neq 0$ at all points, we can take the multiplicative inverse $1/f$ and the maximum modulus principle states that the maximum of $1/f$ occurs on the boundary, or in this case the minimum of f must occur on the boundary. Therefore, when $f = 0$ at some point $|f|$ can have a local minimum in Ω.

Problem 5 Note that $f_n - f_m$ is holomorphic. Furthermore, note that for any ϵ we can find an N s.t. $f_n - f_m < \epsilon$ for $n, m > N$. Therefore, inside the Ω we have, by the maximum modulus principle, that

$$|(f_n - f_m)(z)| \leq |(f_n - f_m)(z_0)| < \epsilon \quad z \in \Omega, z_0 \in \text{boundary of } \Omega$$

So there is uniform convergence in Ω.

Problem 7 We note that the result must be true. Suppose that $M(x) > 0$ and $M(b) > 0$ (if $M(b) = 0$ then a similar argument to the h_ϵ and rectangle argument proves that $f = 0$). Then if we can appropriately scale the f by some constant λ s.t. $M(x) > M(b)$ and this violates the same h_ϵ and rectangle argument.

Problem 9 We define an $h_\epsilon(z) = \frac{1-\epsilon}{\exp\{|z|^\beta\}}$ for $\alpha < \beta < 1$. Note that we have $|h_\epsilon(z)| < 1$. There exists a r s.t. for $|z| = r$ points outside the semicircle have $|fh_\epsilon(z)| \leq 1$. By maximum modulus principle, it follows that $|fh_\epsilon(z)| \leq 1$ for all $z \in \Pi$. Taking $\epsilon \to 0$ gives us our desired result.

We can let $f = e^x$. Note that this is holomorphic but increases to infinity as $x \to \infty$. If we modify the region to be an arc, the results still hold for $\alpha < 1$ as we can just take the section as our region.

Problem 11 We can define $h_\epsilon(z)$ which similarly separates the region Ω into sections where for $z > |z_\epsilon|$ we have $|fh_\epsilon| \leq M$ and for $|z| \leq |z_\epsilon|$ we also have $|fh_\epsilon| \leq M$. Taking $h_\epsilon \to 1$ as $\epsilon \to 0$ gives us the property that $|f| \leq M$ in Ω.

Problem 13 Note that exp can never reach 0 or infinity and these are asymptotes. This holds on the real line clearly, and on the complex plane this just rotates the value. For sin and cos, the asymptotes are $\pm\infty$.

Problem 15 Suppose not. Then by the identity theorem, we see that there are finitely many zeros in f since otherwise we would either have a zequence z_0 which converges to a circle point or $f = 0$. Then we take some $\frac{p(z)}{f(z)}$ where $p(z)$ is a polynomial with high enough degree s.t. the fraction evaluates to a noninfinity value at zeros of f (using L'Ohpitals), making it holomorphic. But, this contradicts maximum modulus as we can take $p(z)$ to arbitrarily large exponents w.r.t the sum of the coefficients of $p(z)$.

Problem 17 We reapply $\varphi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}$ as in the example to find that $|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}$ and this gives us $M(c) = 1 - c^2, M = 1$ and there is no f with $f'(0) = M$.