Rudin Ch1 - Abstract Integration

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1 Notes

1.1 Set-Theoretic Notions and Terminology

1.1 Def This is omitted, but it covers the basic definitions of sets and functions.

1.2 Def We define topology and open sets
(a) A collection τ of subsets of a set X is said to be a topology in X if τ has the following three properties

(i) $\varnothing \in \tau$ and $X \in \tau$.
(ii) If $V_i \in \tau$ for $i = 1, \dots, n$, then $V_1 \cap \cdots \cap V_n \in tau$.
(iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, uncountable), then $\bigcup\limits_{\alpha} V_\alpha \in \tau$.

(b) If τ is a topology in X, then X is called a topological space, and the members of τ are called the open sets in X.

(c) If X and Y are topological spaces and if f is mapping of X into Y, then f is said to be continuous provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

1.3 Def We define σ-algebra and measurability
(a) A collection $\mathfrak{M}$ of subsets of a set X is said to be a σ-algebra in X if $\mathfrak{M}$ has the following properties:

(i) $X \in \mathfrak{M}$.
(ii) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where A^c is the complement of A relative to X.
(iii) If $A = \bigcup\limits_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, \dots$, then $A \in \mathfrak{M}$.

(b) If $\mathfrak{M}$ is a σ-algebra in X, then X is a measurable space and the members of $\mathfrak{M}$ are measurable sets in X.

(c) If X is a measurable space, Y is a topological space, and f is a mapping $X \to Y$, then f is measurable if $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

1.4 Remarks Metric spaces are most familiar topological spaces. A metric space is a set X with a metric ρ s.t.

(a) $0 \leq \rho(x, y) < \infty$ for all x and $y \in X$.
(b) $\rho(x, y) = 0$ iff $x = y$.
(c) $\rho(x, y) = \rho(y, x)$.
(d) $\rho(x, y) \leq \rho(x, y) + \rho(z, y)$ for all x, y, and $z \in X$.

$f : X \to Y$ is continuous at a point $x_0 \in X$ if every neighborhood of $f(x_0)$ there corresponds a neighborhood W of x_0 s.t. $f(W) \subset V$. A neighborhood of x_0 is an open set that contains x_0.

1.5 Prop If X, Y are topological spaces, a mapping $f : X \to Y$ is continuous iff f is continuous at all points.

1.6 Remarks Let $\mathfrak{M}$ be a σ-algebra in a set X. Then
(a) $\varnothing \in \mathfrak{M}$ since $X \in \mathfrak{M}$.
(b) $A_{n+i} = \varnothing, \forall i > 0 \implies A_1 \cup \dot{\cup} A_n \in \mathfrak{M}$ if $A_j \in \mathfrak{M}\ \forall j \in [n]$
(c) $\displaystyle\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$ so all countable intersections are also contained in $\mathfrak{M}$.
(d) $A - B \in \mathfrak{M}$ if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

1.7 Thm Let Y and Z be topological spaces, and let $g : Y \to Z$ be continuous.

(a) If X is a topological space, $f : X \to Y$ is continuous, and $h = g \circ f$, then $h : X \to Z$ is continuous.

(b) If X is a measurable space, $f : X \to Y$ is measurable, and $h = g \circ f$, then $h : X \to X$ is measurable.

1.8 Thm Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h : X \to Y$ is measurable.

1.9 Corr Let X be a measurable space.

(a) $f = u + iv$ where u and v are real measurable functions means that f is a complex measurable function on X.

(b) $f = u + iv$ is a complex measurable function means that $u, v, |f|$ are real measurable functions.

(c) f, g real measurable $\implies$ fg real measurable.

(d) χ_E, the characteristic function on a measurable set E, is measurable.

(e) If f is complex measurable, there exists α complex measurable such that $|\alpha| = 1$ and $f = \alpha|f|$.

1.10 Thm If $\mathscr{F}$ is any collection of subsets of X, there exists a σ-algebra $\mathfrak{M}^*$ in X s.t. $\mathscr{F} \subset \mathfrak{M}^*$.

1.11 Def There exists a smallest σ-algebra $\mathscr{B}$ s.t. every open set in X belongs in $\mathscr{B}$. The members of $\mathscr{B}$ are borel sets. They can take the form closed sets and F_σ, G_δ, or countable unions of closed sets or intersection of open sets.

Borel sets are the measurable sets, and any continuous mapping of X is Borel Measurable. These are called Borel mappings or Borel functions.

1.12 Thm Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.

(a) If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra.

(b) If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.

(c) If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.

(d) If f is measurable, Z a topological space, and $g : Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h : X \to Z$ is measurable.

1.13 Def Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$, and put

$$b_k = \sup\{a_k, a_{k+1}, \dots\} \ (k = 1, 2, 3, \dots)$$
$$\beta = \inf\{b_1, b_2, \dots\}$$

We call β the upper limit on $\{a_n\}$ and write

$$\beta = \limsup_{n\to\infty} a_n$$

The lower limits defined as

$$b_k = \inf\{a_k, a_{k+1}, \dots\} \ (k = 1, 2, 3, \dots)$$

$$\beta = \sup\{b_1, b_2, \dots\}$$

1.14 Thm If $f_n : X \to [-\infty, \infty]$ is measurable, for $n = 1, 2, \dots$ and

$$g = n \underset{\sup}{\geq} 1 f_n \; h = \limsup_{n\to\infty} f_n$$

then g and h are measurable.

The corollaries are

(a) The limit of every pointwise convergent sequence of complex measurable functions if measurable.
(b) If f, g are measurable, (with range in $[-\infty, \infty]$), then so are $\max\{f, g\}$ and $\min\{f, g\}$. In particular, we define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$.

1.15 Prop If $f = g - h, g \geq 0$, and $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.

1.2 Simple Functions

1.16 Def A complex measurable function s on a measurable space X with a finite many points in its range is a simple function. If $\alpha_1, \dots \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = \{x : s(x) = \alpha_i\}$, then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

1.17 Thm Let $f : X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that

(a) $0 \leq s_1 \leq s_2 \cdots \leq f$.
(b) $s_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$.

1.18 Def

(a) A positive measure is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is countably additive.

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

and we assume $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.
(b) A measure space is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.
(c) A complex measure is a complex-valued countably additive function defined on the σ-algebra.

1.19 Thm Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then

(a) $\mu(\varnothing) = 0$.
(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ if A_i are pairwise disjoint members of $\mathfrak{M}$.
(c) $A \subset B \implies \mu(A) \leq \mu(B)$ if $A, B \in \mathfrak{M}$.
(d) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{i=1}^{n} A_n$, $A_n \in \mathfrak{M}$, and

$$A_1 \subset A_2 \subset A_3 \ldots$$

(e) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{i=1}^{n} A_n$, $A_n \in \mathfrak{M}$, and

$$A_1 \supset A_2 \supset A_3 \ldots$$

1.20 Examples Some examples of measures include the counting measure (number of elements) and unit mass concentrated at x_0.

1.21 Comment We can just say X instead of the full information $(X, \mathfrak{M}, \mu)$.

1.3 Arithmetic in $[0, \infty]$

1.22 Def We define arithmetic on ∞. Most notably $0 \cdot \infty = 0$, and everything else if intuitive.

1.4 Integration of Positive Functions

1.23 Def $s : X \to [0, \infty)$ is a simple measurable function

$$s = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}$$

we can define the integral on $E \in \mathfrak{M}$ and μ a measure to be

$$\int_E s d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E)$$

The Lebesgue integral of a function f is

$$\int_E f d\mu = \sup \int_E s d\mu$$

over all s such that $0 \leq s \leq f$.

1.24 Prop The following propositions assume measurability

(a) $0 \leq f \leq g \implies \int_E f d\mu \leq \int_E g d\mu$.
(b) If $A \subset B$ and $f \geq 0$, then $\int_A f d\mu \leq \int_B f d\mu$.
(c) If $f \geq 0$ and c is a constant, $0 \leq c < \infty$, then

$$\int_E cf d\mu = c\int_E f d\mu$$

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f d\mu = 0$, even if $\mu(E) = \infty$.
(e) If $\mu(E) = 0$, then $\int_E f d\mu = 0$ even if $f(x) = \infty$ for every $x \in E$.
(f) If $f \geq 0$, then $\int_E f d\mu = \int_X \chi_E f d\mu$.

1.25 Prop Let s and t be nonnegative measurable simple functions on X. For $E \in \mathfrak{M}$, define

$$\varphi(E) = \int_E s d\mu$$

Then φ is a measure on $\mathfrak{M}$. Also

$$\int_X (s+t) d\mu = \int_X s d\mu + \int_X t d\mu$$

1.26 Lebesgue's Monotone Convergence Thm Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that

(a) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ for every $x \in X$.
(b) $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

Then f is measurable, and

$$\int_X f_n d\mu \to \int_X f d\mu : n \to \infty$$

1.27 Thm If $f_n : X \to [0, \infty]$ is measurable, for $n = 1, 2, \ldots$

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \ (x \in X)$$

$$\implies \int_X f d\mu = \sum_{i=1}^{\infty} \int_X f_n d\mu$$

Corollary: If $a_{i,j} \geq 0$ for i and $j = 1, 2, \ldots$ then

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{i,j} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{i,j}$$

1.28 Fatou's Lemma If $f_n : X \to [0, \infty]$ is measurable, for each positive integer n, then

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu$$

1.29 Thm Suppose $f : X \to [0, \infty]$ is measurable, and

$$\varphi(E) = \int_E f d\mu \ (E \in \mathfrak{M})$$

Then φ is a measure on $\mathfrak{M}$, and

$$\int_X g d\varphi = \int_X g f d\mu$$

for every measurable g on X with range $[0, \infty]$.

1.5 Integration of Complex Functions

1.30 Def We define $L^1(\mu)$ be the collection of all complex measurable functions f on X for which

$$\int_X |f| d\mu < \infty$$

$L^1(\mu)$ are the Lebesgue Integrable functions (on μ) or summable functions.

1.31 Def If $f = u + iv$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f d\mu = \int_E u^+ d\mu - \int_E u^- d\mu + i \int_E v^+ d\mu - i \int_E v^- d\mu$$

1.32 Thm Suppose $f, g \in L^1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$, and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$$

1.33 Thm If $f \in L^1(\mu)$, then

$$\left| \int_X f d\mu \right| \leq \int_X |f| d\mu$$

1.34 Lebesgue's Dominated Convergence Thm Suppose $\{f_n\}$ is a sequence of complex measurable functions on X s.t.

$$f(x) = \lim_{x \to \infty} f_n(x)$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ s.t.

$$|f_n(x)| \leq g(x) \ (n = 1, 2, \ldots; x \in X)$$

then $f \in L^1(\mu)$

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0$$

and

$$\lim_{n\to\infty}\int_X f_n d\mu = \int_X f d\mu$$

1.6 The Role Played by Sets of Measure Zero

1.35 Def Sets of measure 0 are negligible in integration. We denote this "almost everywhere" if this works everywhere except for a set with measure 0.

1.36 Thm Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$ and define $\mu(E) = \mu(A)$. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.

This extended measure μ is called complete. The σ-algebra $\mathfrak{M}^*$ is the μ-completion of $\mathfrak{M}$.

1.37 Def We expand our definition of measurable. A function f is measurable on X if $\mu(E^c) = 0$ and if $f^{-1}(V) \cap E$ is measurable for every open set V.

1.38 Thm Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X s.t.

$$\sum_{n=1}^{\infty}\int_X |f_n| d\mu < \infty$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges for almost all $x, f \in L^1(\mu)$, and

$$\int_X f d\mu = \sum n = 1^{\infty} \int_X f_n d\mu$$

1.39 Thm
(a) Suppose $f : X \in [0, \infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f d\mu = 0$. Then $f = 0$ a.e. on E.
(b) Suppose $f \in L^1(\mu)$ and $\int_E f d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.
(c) Suppose $f \in L^1(\mu)$ and

$$\left|\int_X f d\mu\right| = \int_X |f| d\mu$$

Then there is a constant α s.t. $\alpha f = |f|$ a.e. on X.

1.40 Thm Suppose $\mu(X) < \infty, f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages

$$A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu E > 0$. Then $f(x) \in S$ for almost all $x \in X$.

1.41 Thm Let $\{E_k\}$ be a sequence of measurable sets in X, s.t.

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

2 Problems

Theorem 1.29 Full Solution
To prove φ is a measure on $\mathfrak{M}$, let $E_1, E_2, \ldots$ be disjoint members of $\mathfrak{M}$ such that their union is E. Note that, by definition of σ-algebra, this works. Observe that

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f$$

and that

$$\varphi(E) = \int_X \chi_E f d\mu \quad \varphi(E_j) = \int_X \chi_{E_j} f d\mu$$

By theorem 1.27 (sum version of monotone convergence theorem) we note that

$$\varphi(E) = \sum_{j=1}^{\infty} \varphi(E_j)$$

and we note that, since $\varphi(\varnothing) = 0$ so that means that $\varphi(E)$ is a measure since all individual functions are measures.

To prove the second condition, if g is a χ_E for some value of E, we note that

$$\int_X g d\varphi = \int_X \chi_E d\varphi = \int_E d\varphi = \varphi(E) = \int_E f d\mu = \int_X \chi_E f d\mu$$

Where we note that the third equality comes from the definition of the "1" function. We can therefore build every possible simple function using linear combinations of individual χ_E. From these simple functions, this shows that this holds for every function f that is measurable (by taking the supremum), and every other case can be shown using the monotone convergence theorem.

Problem 1
Suppose that this set is countable. Then the σ-algebra can be divided into the sets

$$A_1, A_2, \ldots \quad A_1^c, A_2^c, \ldots$$

where we exclude $\varnothing$ and X for trivial purposes and let $\bigcap A_i \neq \varnothing$. We notice that

$$A_1 \cap A_2 = A_1^c \cup A_2^c \implies \bigcap_{i=1}^{\infty} A_i \in \mathfrak{M}$$

We denote $B_1 = \bigcap A_i$ and $B_2 = \bigcup A_i^c$. Note that, $B_1 \cup B_2 \in \mathfrak{M}$ but it is clear that these values do not appear in our countable A_i or A_i^c since $A_i \cap B_2 = \varnothing$ and $A_i^c \cap B_1 = \varnothing$. It follows, by contradiction, that the σ-algebra is uncountable.

Problem 3
Notice that this is a more specific case of the previous proof (of $\alpha \in \mathbb{R}$ instead of just rationals). To prove this, we just need to notice that we can define a sequences of real $\alpha_n \to \alpha$ as $n \to \infty$ for all $\alpha \in \mathbb{R}$.

More formally, we can take α_n to be the number β such that $\beta > \alpha$, β is a decimal to the nearest n-th place. This means that,

$$\lim_{n\to\infty} \alpha_n = \alpha \implies \lim_{n\to\infty} \{x : f(x) \geq \alpha_n\} = (\alpha, \infty]$$

and we just apply the previous problem to show that this function is measurable.

Problem 5
a. For the case $f(x) < g(x)$, we note that $\Phi(u, v) = v - u$ is a continuous function which implies that $g - f$ is a measurable function. Furthermore, it therefore follows that $(g - f)^+$ is a measurable function. It follows that the set $\{x : f(x) \leq g(x)\}$ is a measurable set since if is the preimage of $[-\infty, \infty]$ under the measurable function $(g - f)^+$.

For the other case $f(x) = g(x)$, we define the function as $(g - f)^{+-}$, which is clearly $\{x : f(x) = g(x)\}$. $\{x : f(x) < g(x)\}$ follows from $\{x : f(x) \leq g(x)\} - \{x : f(x) = g(x)\}$ and both are measurable sets.

b. We let the functions be $f_1, f_2, \ldots$ converge to f. We also note that the set of all things that converges is $\lim_{\alpha\to\infty}(-\alpha, \alpha)$. We can clearly see that this is $f^{-1}(\bigcup_{\alpha\to\infty}(-\alpha, \alpha)) = \lim_{\alpha\to\infty} \bigcup f^{-1}((-\alpha, \alpha))$. Clearly, this is the union of measurable sets, so we see that the points where f converges to a finite value is a measurable set.

Problem 7
This is a natural deduction using the dominated convergence theorem. Notably, we note that $|f_i| \leq f_1$, and f_1 must be $L^1(\mu)$.

Problem 9

We note that the taylor expansion of $\log(1+x)$ is given by

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}x}{i}$$

As we take $n \to \infty$, we note that the terms of degree $o(x^2)$ are negligible for $\alpha \geq 1$ and this value converges for x. We calculate the value to be

$$\lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) d\mu \leq \lim_{n\to\infty} \int_X \frac{f^\alpha}{n^{\alpha-1}} d\mu$$

and these values converge. If $\alpha = 1$, then this converges to $\int_X f d\mu = c$. If $\alpha > 1$, then we notice that these values converge to 0 since they are the equations $f_n = \frac{f^\alpha}{n^{\alpha-1}}$ are dominated by $f^\alpha \in L^1(\mu)$.

Otherwise, if $0 < \alpha < 1$, we apply Fatou's lemma to find that

$$\int_X \lim_{n\to\infty} \inf f_n d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu$$

where each f_n is

$$f_n = n \log(1 + (f/n)^\alpha)$$

Clearly, we note that, as $n \to \infty$, $\inf f_n \to \infty$ since $n \to \infty$ and $\log(1 + (f/n)^\alpha) \to \infty$. Therefore, it follows that

$$\infty \leq \lim_{n\to\infty} \inf \int_X f_n d\mu \implies \lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) = \infty$$

Since, if the inf (lowest value) is infinity, then the entire function is obviously infinity.

Problem 11

$A \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ because, by definition, every point that is in an infinite number of E_i will be a member of $\bigcup_k^{\infty} E_k$ for all k.

Similarly, $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq A$ since every element of the RHS can't have a finite number of elements (or else we can take k to be the supremeum +1).

To finish the proof, note that

$$\mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = \lim_{k\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) = \mu(\bigcup_{k=n}^{\infty} E_k) - \mu(\bigcup_{k=n}^{\infty} E_k) = 0$$

Problem 13 This is pretty obvious based on the way we define the extended real line and the various operations on them.