Rudin Ch6 - Complex Measures

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June 2018

1 Notes

1.1 Total Variation

6.1 Intro Let $\mathfrak{M}$ be a σ-algebra. A countable collection $\{E_i\}$ of $\mathfrak{M}$ is a partition of E if $E_i \cap E_j = \varnothing$ for unique i, j and $\bigcup E_i = E$. A complex measure μ on $\mathfrak{M}$ is a complex function which satisfies

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$$

for every partition $\{E_i\}$. If we consider a positive measure λ which "dominates" μ, we notice that $\lambda(E) \geq |\mu(E)|$ for all $E \in \mathfrak{M}$. It follows that λ is the supremum of $\sum_{i=1}^{\infty} |\mu(E_i)|$ over all partitions $\{E_i\}$. The set function $|\mu|$ on $\mathfrak{M}$ is defined by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$$

over all partitions $\{E_i\}$. It is called the *total variation* or the *total variation measure*. It is a measure and also $|\mu|(X) < \infty$, and we call it of *bounded variation*.

6.2 Thm The total variation $|\mu|$ of a complex measure μ on $\mathfrak{M}$ is a positive measure.

6.3 Lemma If $z_1, \ldots, z_N$ are complex numbers then there is a subset S of $\{1, \ldots, N\}$ for which

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

6.4 Thm If μ is a complex measure on X, then

$$|\mu(X)| < \infty$$

6.5 If μ and λ are complex measures on the same σ-algebra $\mathfrak{M}$, then $\mu + \lambda$ and $c\mu$ are defined by

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E) \quad (c\mu)(E) = c\mu(E)$$

for scalar c. This is a complex vector space, and we use $||\mu|| = |\mu|(X)$ as a normed linear space.

6.6 Positive and Negative Variations A real measure μ on $\mathfrak{M}$ (called *signed* measures). For $|\mu|$ we define

$$\mu^+ = \frac{1}{2}(|\mu| + \mu) \quad \mu^- = \frac{1}{2}(|\mu| - \mu)$$

These are called the *positive and negative variations* of μ. The *Jordan Decomposition* of μ is $\mu = \mu^+ - \mu^-$.

1.2 Absolute Continuity

6.7 Def Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ and λ be an arbitrary measure on $\mathfrak{M}$. λ is *absolutely continuous* with μ ($\lambda \ll \mu$) if $\mu(E) = 0 \implies \lambda(E) = 0$.

If there is a set A s.t. $\lambda(E) = \lambda(A \cap E)\ \forall E$, then λ is *concentrated* on A.

If λ_1 and λ_2 are measures on $\mathfrak{M}$ s.t. λ_1 is concentrated on A and λ_2 on B for disjoint A and B, then they are *mutually singular* $\lambda_1 \perp \lambda_2$.

6.8 Prop Suppose $\mu, \lambda, \lambda_1, \lambda_2$ are measure on $\mathfrak{M}$ and μ is positive. Then

(a) If λ is concentrated on A, then so is $|\lambda|$.

(b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(c) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

(d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.

(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

6.9 Lemma If μ is a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, then there is a function $w \in L^1(\mu)$ s.t. $0 < w(x) < 1$ for every $x \in X$.

6.10 The Theorem of Lebesgue-Radon-Nikodym Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$.

(a) There is then a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ s.t.

$$\lambda = \lambda_a + \lambda_s \quad \lambda_a \ll \mu \quad \lambda_s \perp \mu$$

If λ is positive and finite, then so are λ_a, λ_s.

(b) There is a unique $h \in L^1(\mu)$ s.t.

$$\lambda_a(E) = \int_E h d\mu$$

for every set $E \in \mathfrak{M}$.

The pair (λ_a, λ_s) is called the *Lebesgue decomposition* of λ relative to μ.

6.11 Thm Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive, and λ is complete. Then the following two conditions are equivalent:

(a) $\lambda \ll \mu$.

(b) To every $\epsilon > 0$ corresponds a $\delta > 0$ s.t. $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.

1.3 Consequences of the Radon-Nikodym Theorem

6.12 Thm Let μ be a complex measurable on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h s.t. $|h(x)| = 1$ for all $x \in X$ and s.t.

$$d\mu = h d|\mu|$$

and this is called the *polar decomposition* of μ.

6.13 Thm Suppose μ is a positive measure on $\mathfrak{M}, g \in L^1(\mu)$, and

$$\lambda(E) = \int_E g d\mu \quad (E \in \mathfrak{M})$$

then

$$|\lambda|(E) = \int_E |g| d\mu (E \in \mathfrak{M})$$

6.14 The Hahn Decomposition Thm Let μ be a real measure on a σ-algebra $\mathfrak{M}$ in a set X. Then there exists sets $A, B \in \mathfrak{M}$ s.t. $A \cup B = X, A \cap B = \varnothing$ s.t. the positive and negative variations of μ^+ and μ^- satisfy

$$\mu^+(E) = \mu(A \cap E) \quad \mu^-(E) = -\mu(B \cap E) \quad (E \in \mathfrak{M})$$

Corollary If $\mu = \lambda_1 - \lambda_2$, where λ_1, λ_2 are positive measures, then $\lambda_1 \geq \mu^+$, $\lambda_2 \geq \mu^-$.

1.4 Bounded Linear Functionals on L^p

6.15 Let μ be a positive measure, suppose $1 \leq p \leq \infty$, and q the exponent conjugate of p, then

$$\Phi_g(f) = \int_X fg d\mu$$

Then this is a bounded linear functional. We can get a bounded linear functional for $1 < p < \infty$, and but is harder for $p = 1$. It works easily for σ-finite measure spaces.

6.16 Thm Suppose $1 \leq p < \infty$, μ is a σ-finite measure on X, and Φ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$ where q is the exponent conjugate to p s.t.

$$\Phi(f) = \int_X fg d\mu \quad (f \in L^p(\mu))$$

Moreover, if Φ and g are related as above, then we have

$$||\Phi|| = ||g||_q$$

so they are isometric.

6.17 Remark This is just a more general case of a previously encounter case ($p = q = 2$). This was based off of the Radon-Nikodym. The special case relies on the fact that it is a Hilbert Space and the bounded linear functions are just inner products.

1.5 The Riesz Representation Theorem

6.18 If X is a locally compact Hausdorff space. We are going to characterize bounded linear functionals (like for positive functionals 2.14). We wish to solve over $C_0(X)$. For every bounded measurable f, we have

$$\int_X fd(\mu + \lambda) = \int_X fd\mu + \int_X fd\lambda$$

6.19 Thm (Riesz Rep) If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique complex Borel measure μ:

$$\Phi f = \int_X fd\mu$$

for every $f \in C_0(X)$. Moreover, the norm of Φ is the total variation

$$||\Phi|| = |\mu|(X)$$

2 Problems

Problem 1 It's clear that

$$\lambda \leq |\mu|$$

and we wish to prove that $|\mu| \leq \lambda$. For any $\epsilon > 0$, we show that $\exists$ a finite partition such that $|\mu| < \sum_{i=1}^{n} |\mu(E_n)| + \epsilon$.

We know that there exists an infinite partition $E'_1, E'_2 \ldots$ such that

$$\sum_{i=1}^{\infty} |\mu(E'_i)| > |\mu| - \frac{\epsilon}{2}$$

and so, WLOG, we assume this set E'_i is ordered decreasingly. It follows that this is a cauchy sequence and we can select a cutoff $n+1$ s.t.

$$\sum_{i=n+1}^{\infty} |\mu(E'_i)| < \frac{\epsilon}{2}$$

and we take the finite sum of $E'_1, \ldots E'_n$ to be our finite partition. Since this holds for all $\epsilon > 0$, it follows that $|\mu| \leq \lambda$, which proves $|\mu| = \lambda$.

Problem 3 We note that for $V \supset E$, $|\mu_1 + \mu_2|(V) \leq |\mu_1 + \mu_2|(E) + |\mu_1|(V \setminus E) + |\mu_2|(V \setminus E)$, which implies that $|\mu_1 + \mu_2|$ is outer regular if $|\mu_1|$ and $|\mu_2|$ are.

Similarly, for $X \subset E$, $|\mu_1 + \mu_2|(X) \geq |\mu_1 + \mu_2|(E) - |\mu_1|(E \setminus X) - |\mu_2|(E \setminus X)$, so this implies inner regularity as well.

a trivial argument shows that multiplication also holds under $M(X)$, so $M(X)$ is a vector space.

We check that $||\cdot||$ is indeed a norm. Clearly $|\mu_1 + \mu_2|(X) \leq |\mu_1|(X) + |\mu_2|(X)$ and $||\alpha\mu|| = |\alpha|||\mu||$. Furthermore, we notice that $|\mu(X)| = 0 \implies \mu(X) = 0$ since $|\mu(E)| \geq 0$. To finish up the proof, we need to show that every Cauchy sequence converges to an element in $M(X)$.

We note that, by definition, if $||\mu - \mu_i|| < \epsilon$ for $\mu_i \in M(X)$, we note that $|\mu - \mu_i|(E) < \epsilon$ for all $E \subset X$ since $|\mu - \mu_i|$ is a measure. It follows that $|\mu_i|(E) - \epsilon < |\mu|(E) < |\mu_i|(E) + \epsilon$. Since this holds for all $\epsilon > 0$, it follows that $\mu \in M(X)$ and that $M(X)$ is a Banach Space.

Problem 5 We note that $L^1(\mu)$ is a one-dimensional vector space since these functions can take a to anything but b must remain 0. We recall that the dual space of a vector space has the same dimension as that vector space. However,

$L^\infty(\mu)$ is two-dimensional as $\sqrt[\infty]{\infty} = 1$ (to see this apply a logarithm and L'Hospital's rule). It follows that $L^\infty(\mu) \neq (L^1(\mu))^*$.

Problem 7 We know that the set of all continuous functions is dense in the trigonometric polynomials, so this holds for these functions. Furthermore, by Lusin's theorem, it holds for all bounded Borel functions, setting up the chain given in the hint.

Since we can find a function $h : |h| = 1, hd\mu = |\mu|$, it follows that the same result should hold for $\int e^{-int} hd|\mu|$. To see this result holds for our $d\mu$, note that e^{-int} is symmetric and that simply switching the value of n doesn't change the absolute value. It follows that $|\hat{\mu}(n)| \to 0$ as $n \to -\infty$ similar to $|\hat{\mu}(n)| \to 0$ as $n \to \infty$.

Problem 9 This is difficult so no.

Problem 11 We note that, by Fatou's Lemma, $\displaystyle\int_X f d\mu < C$. Furthermore, we note that by Jensen's Inequality,

$$\int_X |f_n| d\mu < \sqrt[p]{C} \quad \int_X |f| d\mu < \sqrt[p]{C}$$

By Egoroff's thm, there exists a set F s.t. $\mu(X \setminus F) < \epsilon$ and F has f_n converge uniformly to f. Since f is finite a.e., we can set $\mu(X \setminus F) < \frac{\epsilon}{\sup |f|}$, and it follows that there exists some N s.t. for $n > N$, we have

$$\int_X |f_n - f| d\mu < \int_F |f_n - f| d\mu + \int_{X \setminus F} |f_n| d\mu + \int_{X \setminus F} |f| d\mu < 3\epsilon$$

so this converges to 0.

Problem 13 This is hard. I tried using the Hahn-Banach Theorem to prove that such a functional exists, but am stymied by the norm being equals.

However given this fact, we note that by Thm 5.19 such an $f \in L^\infty(m) \setminus C(I)$ is not in the completion of $C(I)$. It can be proved that $C(I)$ is dense in $L^1(m)$, but that requires Tietze's Extension Thm. Furthermore, since $L^1(\mu)$ is complete, it follows that the completion of $C(I)$ is $L^1(m)$.

Therefore, as $f \notin L^1(m)$, it follows that f can't be expressed with a $g \in L^1(m)$ as it is not in the Hilbert Space $L^1(m)$.