Rudin Ch1

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1 Chapter 1

Theorem 1.29 Full Solution
To prove φ is a measure on $\mathfrak{M}$, let $E_1, E_2, \ldots$ be disjoint members of $\mathfrak{M}$ such that their union is E. Note that, by definition of σ-algebra, this works. Observe that

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f$$

and that

$$\varphi(E) = \int_X \chi_E f d\mu \quad \varphi(E_j) = \int_X \chi_{E_j} f d\mu$$

By theorem 1.27 (sum version of monotone convergence theorem) we note that

$$\varphi(E) = \sum_{j=1}^{\infty} \varphi(E_j)$$

and we note that, since $\varphi(\varnothing) = 0$ so that means that $\varphi(E)$ is a measure since all individual functions are measures.

To prove the second condition, if g is a χ_E for some value of E, we note that

$$\int_X g d\varphi = \int_X \chi_E d\varphi = \int_E d\varphi = \varphi(E) = \int_E f d\mu = \int_X \chi_E f d\mu$$

Where we note that the third equality comes from the definition of the "1" function. We can therefore build every possible simple function using linear combinations of individual χ_E. From these simple functions, this shows that this holds for every function f that is measurable (by taking the supremum), and every other case can be shown using the monotone convergence theorem.

Problem 1
Suppose that this set is countable. Then the σ-algebra can be divided into the sets

$$A_1, A_2, \dots \quad A_1^c, A_2^c, \dots$$

where we exclude $\varnothing$ and X for trivial purposes and let $\bigcap A_i \neq \varnothing$. We notice that

$$A_1 \cap A_2 = A_1^c \cup A_2^c \implies \bigcap_{i=1} A_i \in \mathfrak{M}$$

We denote $B_1 = \bigcap A_i$ and $B_2 = \bigcup A_i^c$. Note that, $B_1 \cup B_2 \in \mathfrak{M}$ but it is clear that these values do not appear in our countable A_i or A_i^c since $A_i \cap B_2 = \varnothing$ and $A_i^c \cap B_1 = \varnothing$. It follows, by contradiction, that the σ-algebra is uncountable.

Problem 3

Notice that this is a more specific case of the previous proof (of $\alpha \in \mathbb{R}$ instead of just rationals). To prove this, we just need to notice that we can define a sequences of real $\alpha_n \to \alpha$ as $n \to \infty$ for all $\alpha \in \mathbb{R}$.

More formally, we can take α_n to be the number β such that $\beta > \alpha$, β is a decimal to the nearest n-th place. This means that,

$$\lim_{n\to\infty} \alpha_n = \alpha \implies \lim_{n\to\infty} \{x : f(x) \geq \alpha_n\} = (\alpha, \infty]$$

and we just apply the previous problem to show that this function is measurable.

Problem 5

a. For the case $f(x) < g(x)$, we note that $\Phi(u, v) = v - u$ is a continuous function which implies that $g - f$ is a measurable function. Furthermore, it therefore follows that $(g - f)^+$ is a measurable function. It follows that the set $\{x : f(x) \leq g(x)\}$ is a measurable set since if is the preimage of $[-\infty, \infty]$ under the measurable function $(g - f)^+$.

For the other case $f(x) = g(x)$, we define the function as $(g - f)^{+-}$, which is clearly $\{x : f(x) = g(x)\}$. $\{x : f(x) < g(x)\}$ follows from $\{x : f(x) \leq g(x)\} - \{x : f(x) = g(x)\}$ and both are measurable sets.

b. We let the functions be $f_1, f_2, \dots$ converge to f. We also note that the set of all things that converges is $\lim_{\alpha\to\infty} (-\alpha, \alpha)$. We can clearly see that this is $f^{-1}(\bigcup_{\alpha\to\infty} (-\alpha, \alpha)) = \lim_{\alpha\to\infty} \bigcup f^{-1}((-\alpha, \alpha))$. Clearly, this is the union of measurable sets, so we see that the points where f converges to a finite value is a measurable set.

Problem 7

This is a natural deduction using the dominated convergence theorem. Notably, we note that $|f_i| \leq f_1$, and f_1 must be $L^1(\mu)$.

Problem 9

We note that the taylor expansion of $\log(1 + x)$ is given by

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}x}{i}$$

As we take $n \to \infty$, we note that the terms of degree $o(x^2)$ are negligible for $\alpha \geq 1$ and this value converges for x. We calculate the value to be

$$\lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) d\mu \leq \lim_{n\to\infty} \int_X \frac{f^\alpha}{n^{\alpha-1}} d\mu$$

and these values converge. If $\alpha = 1$, then this converges to $\int_X f d\mu = c$. If $\alpha > 1$, then we notice that these values converge to 0 since they are the equations $f_n = \frac{f^\alpha}{n^{\alpha-1}}$ are dominated by $f^\alpha \in L^1(\mu)$.

Otherwise, if $0 < \alpha < 1$, we apply Fatou's lemma to find that

$$\int_X \lim_{n\to\infty} \inf f_n d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu$$

where each f_n is

$$f_n = n \log(1 + (f/n)^\alpha)$$

Clearly, we note that, as $n \to \infty$, $\inf f_n \to \infty$ since $n \to \infty$ and $\log(1 + (f/n)^\alpha) \to \infty$. Therefore, it follows that

$$\infty \leq \lim_{n\to\infty} \inf \int_X f_n d\mu \implies \lim_{n\to\infty} \int_X n \log(1 + (f/n)^\alpha) = \infty$$

Since, if the inf (lowest value) is infinity, then the entire function is obviously infinity.

Problem 11

$A \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ because, by definition, every point that is in an infinite number of E_i will be a member of $\bigcup_{k}^{\infty} E_k$ for all k.

Similarly, $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \subseteq A$ since every element of the RHS can't have a finite number of elements (or else we can take k to be the supremeum $+1$).

To finish the proof, note that

$$\mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k) = \lim_{k\to\infty} \mu(\bigcup_{k=n}^{\infty} E_k) = \mu(\bigcup_{k=n}^{\infty} E_k) - \mu(\bigcup_{k=n}^{\infty} E_k) = 0$$

Problem 13 This is pretty obvious based on the way we define the extended real line and the various operations on them.