Rudin Ch5 - Examples of Banach Space Techniques

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1 Notes

1.1 Banach Spaces

5.1 Banach spaces are used when Hilbert Spaces become difficult to work with (ie when orthogonality is hard to achieve)

5.2 Def A complex vector space X is said to be a normed linear space if to each $x \in X$, there exists a norm $||x||$ with the properties

(a) $||x + y|| \leq ||x|| + ||y||$
(b) $||\alpha x|| = |\alpha|||x||$ where α is a complex scalar.
(c) $||x|| = 0 \implies x = 0$.

A Banach space is a normed linear space which is complete in the metric defined by the norm.

5.3 Def The norm of a linear transformation Λ from a normed linear space X to a normed linear space Y by

$$||\Lambda|| = \sup\{||\Lambda x|| : x \in X, ||x|| \leq 1\}$$

If $||\Lambda|| < \infty$, then Λ is a bounded linear transformation. Note that $||\Lambda(\alpha x)|| = |\alpha|||\Lambda x||$.

5.4 Def For a linear transformation Λ from a nls X to Y, each of the following implies the others:

(a) Λ is bounded
(b) Λ is continuous
(c) Λ is continuous at one point of X

1.2 Consequences of Baire's Thm

5.5 The completeness of Banach Spaces implies two important theorems: *Banach-Steinhaus Thm* and *open mapping thm*. The last is the *Hahn-Banach Extension Thm*.

5.6 Baire's Thm If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.

Corollary: In a complete metric space, the intersection of any countable collections of dense G_δ, finite intersection of open sets, is again dense in G_δ.

5.7 Notes Baire's Thm is called Category Theorem. A set $E \subset X$ is called nowhere dense is $\overline{E}$ doesn't contain any nonempty open subset of X. Any countable union of nowhere dense sets is a set of the *First Category*, all others subsets of X are of the *Second Category*. There is no complete metric space of the first category.

5.8 The Banach-Steinhaus Thm If X is a Banach Space, Y is a normed linear space and $\{\Lambda_\alpha\}$ is a collection of bounded linear transformations of X into Y, then there exists M s.t.

$$||\Lambda_\alpha|| < M \ \forall\alpha$$

or we have

$$\sup_{\alpha\in A}||\Lambda_\alpha x|| = \infty$$

for all x belonging to some dense G_δ in X.

5.9 The Open Mapping Thm Let U and V be the open balls of the Banach space X and Y. To every bounded linear transformation Λ of X onto Y, there exists a corresponding $\delta > 0$ s.t.

$$\Lambda(U) \supset \delta V$$

where δV is the the set of $\{y : y \in V, ||y|| < \delta\}$. Another way of stating this is that for every $||y|| < \delta$ there exists an $x : ||x|| < 1$ s.t. $\Lambda x = y$.

5.10 Thm If X, Y are Banach spaces and Λ is a bounded linear transformation $X \to Y$ that is also one-to-one, then there is a $\delta > 0$ s.t.

$$||\Lambda x|| \geq \delta||x||$$

ie δ^{-1} is a bounded linear transformation $Y \to X$.

1.3 Fourier Series of Continuous Functions

5.11 Convergence Problem Recall that the Fourier series of f at a point x is given by

$$s_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt$$

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}$$

In particular, to each real number x, there corresponds a set $E_x \subset C(T)$ which is dense in G_δ in $C(T)$ s.t. $s^*(f,x) = \infty$ for every $f \in E_x$, where $s^*(f,x) = \sup_n |s_n(f,x)|$.

5.12 Thm There is a set $E \subset C(T)$ which is a dense G_δ in $C(T)$ and which has the following properties: For each $f \in E$, the set

$$Q_f = \{x : s^*(f,x) = \infty\}$$

is a dense G_δ in $\mathbb{R}^1$.

5.13 Thm In a complete metric space X which has no isolated points, no countable dense set is a G_δ.

1.4 Fourier Coefficients of L^1-functions

5.14 We associate to every $f \in L^(T)$, a function $\hat{f}$ on Z is defined by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$$

The Riemann-Lebesgue lemma states that $\hat{f}(n) \to 0$ as $n \to \pm\infty$. The converse is not true, and we let c_0 be the space of all complex functions φ s.t. $\varphi(n) \to 0$ as $n \to \pm\infty$

5.15 Thm The mapping $f \to \hat{f}$ is a one-to-one bounded linear transformation of $L^1(T)$ into, but not onto, c_0.

1.5 The Hahn-Banach Theorem

5.16 The Hahn-Banach Thm If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $||F|| = ||f||$.

F is the *extension* of f. The norms are given by

$$||f|| = \sup\{|f(x)| : x \in M, ||x|| \leq 1\},\ ||F|| = \sup\{|F(x)| : x \in X, ||x|| \leq 1\}$$

A real-linear functional φ is linear for all real scales α. A complex linear functional is the same.

5.17 Prop Let V be a complex vector space

(a) If u is the real part of a complex-linear functional on V, then

$$f(x) = u(x) - iu(x), (x \in V)$$

(b) If u is a real-linear functional on V and if f is defined by (1), then f is a complex-linear functional on V.
(c) if V is a normed linear space and f and u are related as in (1), then $||f|| = ||u||$.

5.18 Proof of Thm 5.16 Omitted because long.

5.19 Thm Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then x_0 is in the closure $\overline{M}$ of M iff there is no bounded linear functional f on X s.t. $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$.

5.20 Thm If X is a normed linear space and if $x_0 \in X$, $x_0 \neq 0$, there is a bounded linear functional on X, of norm 1, so that $f(x_0) = ||x_0||$.

5.21 Remarks If X is a normed linear space, then X^*, which is the collection of all linear functionals on X, is a Banach space. X^* separates points on X.

$$||x|| = \sup\{|f(x)| : f \in X^*, ||f|| = 1\}$$

$f \to f(x)$ is a bounded linear functional on X^* of norm $||x||$

1.6 An Abstract Approach to the Poisson Integral

5.22 Def We define $||f||_E = \sup\{|f(x)| : x \in E\}$. If H is a compact subset of K, and $1 \in A = C(K)$ s.t. $||f||_K = ||f||_H$. H is the boundary of K. Specifically, to each $x \in K$, there corresponds a positive measure μ_x on the boundary H which represents x in the sense that

$$f(x) = \int_H f d\mu_x \ (f \in A)$$

holds for every $f \in A$.

5.23 Example If we let U be the open unit disk, $\overline{U}$ is the closed circle, and let T be the boundary. Every function of the form

$$f(z) = \sum_{n=0}^{N} a_n z^n$$

where a_i are complex numbers satisfies

$$||f||_U = ||f||_T$$

5.24 The Poisson Integral If we let everything be as above and let A contain all polynomials s.t.

$$||f||_U = ||f||_T$$

There exists a positive Borel Measure μ_z on T s.t.

$$f(z) = \int_T f d\mu_z$$

If $u_n(w)$ then $u_n \in A$. Look at the real function $P_r(\theta - t) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$ since $f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta - t)dt$. We calculate

$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}$$

5.25 Thm Suppose A is a vector space of continuous complex functions on the closed unit disk $\overline{U}$. If A contains all polynomials, and if

$$\sup_{z\in U}|f(z)| = \sup_{z\in T}|f(z)|$$

for every $f \in A$, then the Poisson integral representation

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} f(e^{it})dt$$

where $z = re^{i\theta}$ is valid for every $f \in A$ and every $z \in U$.

2 Exercises

Problem 1 Drawing are ommited, but we note the special cases of p. If $p \to 0$, the graph becomes more and more of a cross. If $p = 1$, then it is a diamond. If $p = 2$, then it is a circle. If $p \to \infty$ then it goes to a square.

If $\mu\{a\} \neq \mu\{b\}$, then the various graphs are scaled horizontally or vertically (depending on the ratios of $\mu\{a\}$ and $\mu\{b\}$). For example it becomes an oval or a rectangle.

Problem 3 We note that our value for $||h||_p$ is

$$\sqrt[p]{\int_X \frac{(f+g)^p}{2} d\mu} \leq \sqrt[p]{\int_X \frac{f^p + g^p}{2} d\mu}$$

since $f \neq g$, it follows that $||h||_p < 1$ for $p > 1$. For $p = 1$, we note that this is equality (since it is not convex). For $p = \infty$, this is just the max of $f + g$ compared to the max of f or g. Clearly, the equality is reversed in this situation (as well as for $p < 1$).

For $C(X)$, we just need to consider the supremum norm $\sup_{x \in X} |\frac{f(x)+g(x)}{2}|$. If we take $f(x) = 1$ and $g(x)$ is some other function, then we note that $\sup_{x \in X} h(x) = ||h|| = 1$.

Problem 5 It is clear that our set is convex. To see this, take f, g s.t. $\int_0^1 f dt = \int_0^1 g dt = 1$. We find that

$$\int 0^1 (1-t)f + tg dt = (1-t) \int_0^1 f dt + t \int_0^1 g dt = (1-t) + t = 1$$

Furthermore, the norm, which is given by

$$\int_{[0,1]} |f| dt$$

obviously takes a minimum value whenever $|f|$ is all positive on $[0, 1]$. There are an infinite number of these functions, so it follows that there are an infinite number of elements with smallest norm in our set.

Problem 7 Consider the subspace $X \subset \mathbb{C}$ where

$$X = \{x + xi : x \in \mathbb{R}\}$$

and the linear functional

$$fx = \mathrm{Re}(x) + \mathrm{Im}(x)$$

Clearly, $||f|| = \sqrt{2}$. If we take the same linear functional over $\mathbb{C}$, we get $||F|| = \sqrt{2}$. Furthermore, if we take $Fx = \sqrt{2}|x|$, then we also get the same value $||F|| = \sqrt{2}$.

Problem 9
(a) Clearly, this value of Λ is a linear functional on c_0, since the sums are closed under scalars and addition. The sequence which gives the maximal sum can be defined as follows.

Let $x_i = \{1, 1, \ldots 0, 0 \ldots\}$, where the first zero occurs at the i-th position. Clearly, $x_i \in c_0$ and $\lim_{i \to \infty} \Lambda x_i = ||y_i||$, and this is clearly the supremum (based on our sum in the definition of Λ).

To see that every Λ is defined based on such a y, consider the sequences

$$\{1,0\dots\},\{1,1,0\dots\},\dots$$

and note that they define the coefficients of y inductively. Naturally, the spaces $(c_0)^*$ is isomorphic to ℓ^1.

(b) If $y \in \{\eta_i\} \in \ell^\infty$ and $\Lambda x = \sum \zeta_i \eta_i$, notice that Λ is a bounded linear functional on ℓ^1. This is because $\Lambda x \leq \sup(|\eta_i|) \sum \zeta_i < \infty$ by definition of ℓ^1 and ℓ^∞. Similarly, all $\Lambda \in (\ell^1)^*$ can be defined by its output on values

$$\{1,0\dots\},\{0,1,0\dots\},\{0,0,1\dots\},\dots$$

and use these to reconstruct y. It therefore follows that $(\ell^1)^*$ is isomorphic to ℓ^∞.

(c) Obviously each $y \in \ell^1$ induces a bounded linear functional. However, noting the linear functionals $\Lambda_i = x_i$, or the i-th index of our input. Notice that $\lim\limits_{i\to\infty} \Lambda_i x$ is not necessarily true for $x \in \ell^\infty$ but is 0 for $x \in c_0$. This overloads the sequence $y = \{0,0\dots\}$ and therefore breaks equality.

(d) Let $X = \{x_i\}$ where $x_i \in \mathbb{Q}$ and $x_i = 0$ for $i \geq N$ for some value N. It is clear this is the countable cartesian product of $\mathbb{Q}$, so this is countable. Furthermore, we note that this is indeed dense in c_0 and ℓ^1.

To prove that ℓ^∞ is uncountable, consider the subset $S \subset \ell^\infty = \{s_i\}$ where $s_i = 0, 1$. Note that this set is uncountable, and any dense subset of S is uncountable as, for each s, there exists another element s' with $||s - s'||_\infty = 1$.

Problem 11 We examine the first case and our rules for bounded linear spaces. We note that

$$||f+g|| = M_{f+g} + |f(a)+g(a)| = \sup \frac{|f(s)+g(s)-f(t)-g(t)|}{|s-t|^\alpha}$$

$$+|f(a)+g(a)| \leq M_f + M_g + |f(a)| + |g(a)| \leq ||f|| + ||g||$$

$$||\beta f|| = M_{\beta f} + |\beta f(a)| = |\beta| M_f + |\beta||f(a)| = |\beta|||f||$$

$$||f|| = 0 \implies M_f + |f(a)| = 0 \implies M_f = 0, |f(a)| = 0 \implies f = 0$$

We note that the underlying metric space is complete as $||g - f|| < \epsilon \implies ||g|| - ||f|| = M_g - M_f + |g(a)| - |f(a)| < \epsilon \implies M_g < \infty \implies g \in \mathrm{Lip}\alpha$.

For the other case, we note that

$$||f+g|| = M_{f+g} + \sup|f(x)+g(x)| = \sup\frac{|f(s)+g(s)-f(t)-g(t)|}{|s-t|^{\alpha}}$$

$$+\sup|f(x)+g(x)| \leq M_f + M_g + \sup|f(x)| + \sup|g(x)| \leq ||f|| + ||g||$$

$$||\beta f|| = M_{\beta f} + \sup|\beta f(x)| = |\beta| M_f + |\beta|\sup|f(x)| = |\beta|||f||$$

$$||f|| = 0 \implies M_f + \sup|f(x)| = 0 \implies M_f = 0, \sup|f(x)| = 0 \implies f = 0$$

and we note that the same argument implies that metric space is complete.

Problem 13

(a) We let $X_M = \cap_n\{x : |f_n(x)| \leq M\}$ and note that we are looking for $\cup_M X_M \neq \varnothing$. Note that this countable union of closed sets is $= X$ if we take $M = 1, 2, \ldots$. It follows that X is of the second category and there exists $E \subset X$ s.t. E is not nowhere dense, and we take that V as our set.

(b) In a similar fashion, we define $A_N = \{x : |f_m(x) - f_n(x)| \leq \epsilon \ m, n \geq N\}$. $X = \cup_N A_N$ as $\lim f_n = f$ and every converging sequence is a Cauchy sequence. It follows from Baire's thm that X is of the second category and there exists $E \subset X$ s.t. E is not nowhere dense, and we take V to be the corresponding open set.

Problem 15
We first prove the forward direction.

Note that each $a_{ij} \to 0$ as $i \to \infty$. To see this, examine sequence $\{1, \ldots, 1, 0 \ldots\}$ and $\{1, \ldots, 1, 1, 0 \ldots\}$ where the second one has 1 more 1 than the first. They're difference and the fact that both converge to the same limit implies that $\lim a_{ij} = 0$.

Furthermore, we note that $\sup_i \sum_{j=1}^{\infty}|a_{ij}| < \infty$. Otherwise, take the converging sequences

$$x_1 = \{\alpha_1, 0 \ldots\} x_2 = \{\alpha_1, \alpha_2, \ldots\}, \ldots x_i = \{\alpha_1 \ldots \alpha_i, 0, \ldots\}$$

where α is the corresponding complex number that $\alpha a_{ij} = |a_{ij}|$. Let $x = \{x_i\}$. Note that $\sup_i \sum_{j=1}^{\infty}|a_{ij}| = A \sup_{x_i \in x} x_i$, which converges to a value $0 < \infty$.

For the last condition $\lim_{i\to\infty}\sum_{j=0}^{\infty} a_{ij} = 1$, it suffices to plug in the sequence $\{1, 1, \dots\}$ and noting that the resulting limit follows accordingly.

To prove the other way, note that condition (b) implies that $\{\sigma\}$ converges and conditions (c) and (a) imply that

$$\lim_{i\to\infty}\sum_{j=N}^{\infty} a_{ij} = 1$$

If we let N be sufficiently large, we note that $|s - s_n| < \epsilon \ \forall n \geq N$. It follows that

$$|s - \sum_{j=N}^{\infty} a_{ij}s_j| < \epsilon$$

which proves convergence of $\{\sigma\}$ as desired.

For the specific sequences, we note that for the matrix with

$$a_{ij} = \begin{cases} \frac{1}{i+1} & 0 \leq j \leq i \\ 0 & i < j \end{cases}$$

the resulting convergent sequence $\{\sigma_n\}$ must satisfy the property that $\{s_{n+1}\} = \{(n+2)\sigma_{n+1} - (n+1)\sigma_n\}$ is divergent. Note that this is equivalent to $\{(n+1)(\sigma_{n+1} - \sigma_n) + \sigma_{n+1}\}$. If we take $\sigma_n = \frac{1}{\sqrt{n+1}}e^{ni\alpha}$, we note the resulting terms are divergent since the difference $(n+1)(\sigma_{n+1} - \sigma_n)$ is always greater than $\frac{n+1}{\sqrt{n+2}}(2\sin\frac{\alpha}{2})$, which diverges. The other σ_{n+1} becomes irrelevant as $n \to \infty$, so our sequence diverges.

For the matrix given by

$$a_{ij} = (1 - r_i)r_i^j \ r_i \to 1$$

we note that $(1 - r_i)$ converges to 0, so our sequence $\{\sigma\}$ always converges to 0.

Problem 17

We know that $\int_X g^2 d\mu \leq 1$, so it follows that

$$||M_f|| = \sqrt{\int_X (fg)^2 d\mu} \leq \sqrt{(\sup f)^2 \int_X g^2 d\mu} \leq \sup f = ||f||_\infty$$

for all g. In order to always ensure equality, it follows that the measure must be the zero measure $\mu X = 0$, otherwise we can't guarantee the first inequality. For the mapping to be onto, f must be invertible.

Problem 19 Omitted because I didn't do problem 18.

Problem 21 We notice that

$$\bigcup_{e \in E} (e - E)$$

has measure 0. Selecting a t that is not in this set shows that $(t + E) \cap E = \varnothing$, and this always happens as our set is of measure 0. We can set addition by this t to be our homeomorphism, and we are done.