Rudin Ch2 - Positive Borel Measure

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1 Notes/Problems

Problem 1
a. This is true. To see this, notice that the set $\{x : (f_1 + f_2)(x) < \alpha\}$ is the union of the uncountably many sets

$$\bigcup_{0 \le \beta_1 \le \alpha} \{x : f_1(x) < \beta_1\} \cap \{x : f_2(x) < \alpha - \beta_1\}$$

b. This is also true. To see this, notice that the set $\{x : (f_1 + f_2)(x) > \alpha\}$ is the union of uncountably many sets

$$\bigcup_{0 \le \beta_1} \{x : f_1(x) > \beta_1\} \cap \{x : f_2(x) > \alpha - \beta_1\}$$

c. This is false. If we define $f_n = \chi_{[\frac{1}{n+1}, \frac{1}{n}]}$, then clearly this is an upper semi-continuous function. However, we notice that $\sum_{i=1}^{\infty} f_n = \chi_{(0,1]} + \chi_X$, where $X = \{x : \frac{1}{x} \in \mathbb{Z}^+\}$. Notice that this value, for an α of 1, is clearly not an open set as it's complement is not closed.

d. This is true. Let $g_k = \sum_{i=1}^{k} f_i$. Clearly, we can check that g_k is lower continuous

$$\bigcup_{\beta_1, \beta_2, \dots \beta_{k-1}} \{x : f_1(x) > \beta_1\} \dots \cap \{x : f_{k-1}(x) > \beta_3\} \cap \{x : f_k(x) > \alpha - \sum_{i=1}^{k-1} \beta_i\}$$

We know that $\sup_k g_k$ is also lower continuous, and, since $f_i \ge 0$, it follows that $\lim_{k \to \infty} g_k = \sup_k g_k$ is lower continuous.

If f_i are not nonnegative, when our proofs for (a), (b), and (c) are still correct. However, our last proof fails because $\sup_k g_k$ is not necessarily $\sum_{i=1}^{\infty} f_i$.

Problem 3
We wish to prove that $\forall \epsilon, x_1, x_2$, there exists $\delta : \rho(x_1, x_2) < \delta \implies |\rho_E(x_1) - \rho_E(x_2)| < \epsilon$.

We let x_1, x_2 be two different points. Let $y_1 \in E$ such that $\rho(x, y_1) = \rho_E(x_1)$. Notice, by triangle inequality and the definition of infimum, we have

$$\rho(x_1, x_2) + \rho(x_1, y) \geq \rho(x_2, y) \geq \rho_E(x_2)$$
$$\implies \rho(x_1, x_2) \geq \rho_E(x_2) - \rho_E(x_1)$$

Similarly, setting $y_2 \in E$ such that $\rho(x_{y2} = \rho_E(x_2)$ gives us the similar result

$$\rho(x_1, x_2) \geq \rho_E(x_1) - \rho_E(x_2)$$

It follows that

$$\rho(x_1, x_2) \geq |\rho_E(x_1) - \rho_E(x_2)|$$

so we can set $\delta = \epsilon$ and obtain the desired result above, as $\rho(x_1, x_2) < \epsilon \implies |\rho_E(x_1) - \rho_E(x_2)| \leq \rho(x_1, x_2) < \epsilon$.

We notice that our X is a locally compact Hausdorff space. Setting $K = B$ and $V = X - A$ gives us the relation described in Urysohn's lemma:

$$K \prec f \prec V$$

Problem 5
To see that the cantor set has measure 0, we can count the measure removed. For C_1, it is $\frac{1}{3}$, for C_2 it is $\frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$, and in general, for C_i it is $\frac{2^{i-1}}{3^i}$. Summing, we have

$$\sum_{i=1}^{\infty} \frac{2^{i-1}}{3^i} = \frac{1}{3} \times \sum_{i=1}^{\infty} \frac{2^i}{3^i} = 1$$

so $m(E) = 0$. Furthermore, we can define a surjective mapping from $f : E \to [0, 1]$. In ternary, all elements of the cantor set can be expressed as a decimal with only 2 and 0. Our function substitutes 1 for 2 and considering the resulting string in binary. Clearly, this is implies that f is uncountable and has the same carnality as $\mathbb{R}$.

Problem 7
Note that $m(\{q\}) = 0$, where q is a rational number. To see this, note that $m(\{q\}) < m(W) = \frac{1}{q^n}$ for all n. Since the rationals are countable, it follows that $m(\{q \in \mathbb{Q} : q \in [0, 1]\}) = 0$.

Our construction of the set is $(0, \epsilon) \cup \{q \in \mathbb{Q} : q \in [0, 1]\}$. We see that this measure is ϵ and that the closure is in fact $[0, 1]$.

Problem 9
We define $f_n(x) = \chi_{A_n}$ where $A_n = \{x : \exists q \in \mathbb{R} \mid qn = x\}$. We note that $\int_{[0,1]} f_n(x)dm = 0$ as the measure of A_n is obviously 0 (since there are countably values). Therefore, it follows that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$$

as this holds for all n. Furthermore, we note that this value doesn't converge for any value, using $\epsilon - \delta$ notation.

Problem 11
Let $K = \bigcap K_\alpha$. Note that $K_\alpha \in \mathfrak{M}$, so we can apply Thm 2.7 to get that

$$K \subset K_i \subset V$$

for all open $V : K \subset V$. To prove that K_i is indeed one of the K_α. Note that $\mu(K) = 1$ since μ is regular and

$$\mu(K) = \inf\{\mu(V) : E \subset V \subset X\} = 1$$

By Thm 2.17

$$\mu(V - K) = 0 \implies \mu(K_i) = \mu(K) = 1$$

It follows that $\mu(K^c) = 1 - \mu(K) = 0$. Furthermore, since K is the intersection of all K_α with measure 1, it follows that it is the largest value.

Problem 13
It clear that the set $\{0\}$ is compact. However, this can't be the support of any continuous function f.

Examining compact sets in general, we find that f must have images both 0 and $y \neq 0$. For this case, we find that these must be open sets in order for the support to be compact. In general, if we have a compact set such that its interior at each discontinuous piece is nonzero, then we can define a continuous function f that approaches zero at the border.

Problem 15
We guess that

$$\lim_{n\to\infty} \int_0^n (1 - \frac{x}{n})^n e^{x/2} dx = 2$$

This is because we assume this value tends to

$$\int_0^\infty e^{-x} \times e^{x/2} dx = \int_0^\infty e^{-x/2} = 2$$

We note this is true because of the dominated convergence theorem, and that $|f_n| = |(1 - \frac{x}{n})^n e^{x/2}| \leq e^{-x/2}$ as $(1 - \frac{x}{n}^n)$ is increasing

Similarly, for

$$\lim_{n\to\infty} \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx = 1$$

This is because we apply the dominated convergence theorem and find that

$$|f_n| = |(1 + \frac{x}{n})^n e^x| \leq e^{-x}$$

since $(1 + \frac{x}{n})^n$ is increasing.

Problem 17
We check the conditions of a metric space.

1) Obviously, non-negativity holds.

2) Furthermore, if $\rho((x_1, y_1), (x_2, y_2)) = 0 \implies x_1 = x_2$ and $|y_1 - y_2| = 0 \implies x_1 = x_2, y_1 = y_2$, so identity of 0 holds.

3) Symmetry obviously holds.

4) Lastly, to prove the triangle inequality. We calculate the individual cases. If $x_1 = x_2 = x_3$, then it holds. If $x_1 = x_2 \neq x_3$, then it holds. Lastly, if $x_1 \neq x_2 \neq x_3$, then it also clearly holds (it adds 1 to the side with two elements).

This is locally compact since, for a neighborhood with radius < 1, then we just have a compact perpendicular line through x.

For our function f and our μ as defined, we notice that

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y) dy = \int_X f d\mu$$

Clearly, we can see that the second value is the integral of f over the support. Therefore, it follows that $\mu(E) = \infty$ (we can notice this by taking the function $f = \lim_{n\to\infty} \chi_{E_n}$ and using the construction $\mu(V) = \sup\{\Lambda f : f \prec V)$. This happens because our value of Λf approaches ∞ as $n \to \infty$. However, for $K \subset E$, $\mu(K) = 0$ because this value is a finite sum of $\displaystyle\int_{-\infty}^{\infty} \chi_{\{0\}} dy$.

Problem 19
First, we notice that V is not only an open subset of X, but is also compact. We can apply this to Steps 1 and 5.

Problem 21
We notice that, $\exists\alpha$ such that $\{x : f(x) < \alpha\} \neq X, \{x : f(x) < \alpha+\epsilon\} = X\ \forall\epsilon > 0$. To see this, it suffices to use the fact that f is upper-semicontinuous and we can pick a value larger than or equal to $\sup_X f(x)$. If we pick $\sup_X f(x)$ as our α, then both equalities are satisfied (or else α wouldn't be sup by definition), so it follows that there exists some x such that $f(x) = \sup_X f(x)$.

Problem 23
I don't have a good solution to this question, so I'm leaving it blank.

Problem 25
(i) We rearrange the formula to find that

$$1 + e^t < e^c e^t \implies \log(1 + e^{-t}) < c$$

The LHS is clearly maximized as $t \to 0$, so we find that $c > \log 2$.

(ii) We apply the dominated convergence theorem to find that

$$\lim_{n\to\infty} \frac{1}{n} \int_0^1 (1 + e^{nf(x)})dx < \lim_{n\to\infty} \frac{1}{n} \int_0^1 \log 2 + nf(x)dx = \int_0^1 f(x)$$

Furthermore, we notice that our equations

$$f_n(x) = \frac{1 + e^{nf(x)}}{n} \to f(x), n \to \infty$$

since the 1 becomes irrelevant. It follows that our integral converges to $\int_0^1 f(x)$.