Rudin Ch9 - Fourier Transforms

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1 Notes

1.1 Formal Properties

9.1 Def In this chapter m will refer to the Lebesgue measure divided by $\frac{1}{\sqrt{2\pi}}$. Similarly, we denote

$$\int_{-\infty}^{\infty} f(x)dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)dx$$

and similarly

$$||f||_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p dm(x) \right\}^{1/p}$$

$$(f * g) = \int_{-\infty}^{\infty} f(x-y)g(y)dm(y)$$

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-ixt}dm(x)$$

$\hat{f}$ is called the *Fourier Transform* and this is also the term for the mapping $f \to \hat{f}$. A *character* φ of $\mathbb{R}^1$ if $|\varphi(t)| = 1$ and

$$\varphi(s+t) = \varphi(s)\varphi(t)$$

9.2 Thm Suppose $f \in L^1$ and α and λ are real numbers

(a) If $g(x) = f(x)e^{i\alpha x}$, then $\hat{g}(t) = \hat{f}(t-\alpha)$.
(b) If $g(x) = f(x-\alpha)$, then $\hat{g}(t) = \hat{f}(t)e^{-i\alpha t}$.
(c) If $g \in L^1$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$.

Thus the Fourier transform converts multiplication by a character into translations, vice-versa, and convolution to pointwise products.

(d) If $g(x) = \overline{f(-x)}$, then $\hat{g}(t) = \overline{\hat{f}(t)}$.
(e) If $g(x) = f(x/\lambda)$ and $\lambda > 0$, then $\hat{g}(t) = \lambda\hat{f}(t)$.

(f) If $g(x) = -ixf(x)$ and $g \in L^1$, then $\hat{f}$ is differentiable and $\hat{f}'(t) = \hat{g}(t)$.

9.3 Remarks

(a) The dominated convergence theorem only applies to countable sequences, but it can be used for uncountable sequences in this case.

(b) Thm 9.2(b) shows that

$$\widehat{[f(x+\alpha) - f(x)]/\alpha} = \hat{f}(t)\frac{e^{i\alpha t} - 1}{\alpha}$$

Similarly, integration by parts shows that $\hat{f}' = it\hat{f}(t)$.

1.2 The Inversion Theorem

9.4 We note that we can find the inversion of some formulas. For example, for the Fourier series,

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \implies f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$$

but this is not necessarily true for pointwise convergence. In particular, we can try (but run into roadblocks) to prove that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{itx}dm(t)$$

but run into

$$\int_{\infty}^{\infty} e^{i(t-s)x}dx$$

9.5 Thm For any function $\mathbb{R}^1$ and every $y \in \mathbb{R}^1$, let f_y be the translate of f defined by

$$f_y(x) = f(x-y)$$

If $1 \leq p < \infty$ and if $f \in L^p$, the mapping

$$y \to f_y$$

is a uniformly continuous mapping of $\mathbb{R}^1$ into $L^p(\mathbb{R}^1)$.

9.6 Thm If $f \in L^1$ then $\hat{f} \in C_0$ and

$$||\hat{f}||_\infty \leq ||f||_1$$

9.7 A Pair of Auxiliary Functions If we put $H(t) = e^{-|t|}$ and define

$$h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t)e^{itx}dm(t)$$

and this calculates to

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}}\frac{\lambda}{\lambda^2 + x^2}$$

so

$$\int_{-\infty}^{\infty} h_\lambda(x)dm(x) = 1$$

and $H(\lambda t) \to 1$ as $\lambda \to 0$.

9.8 Prop If $f \in L^1$, then

$$(f * h_\lambda)(x) = \int_{\infty}^{\infty} H(\lambda t)\hat{f}(t)e^{ixt}dm(t)$$

9.9 Thm If $g \in L^\infty$ and g is continuous at a point x, then

$$\lim_{\lambda\to 0}(g * h_\lambda)(x) = g(x)$$

9.10 Thm If $1 \leq p < \infty$ and $f \in L^p$, then

$$\lim_{\lambda\to 0}||f * h_\lambda - f||_p = 0$$

9.11 The Inversion Thm If $f \in L^1$ and $\hat{f} \in L^1$, and if

$$g(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}dm(t)$$

then $g \in C_0$ and $f(x) = g(x)$ a.e.

9.12 The Uniqueness Thm If $f \in L^1$ and $\hat{f}(t) = 0$ for all $t \in \mathbb{R}^1$, then $f(x) = 0$ a.e.

1.3 The Plancherel Theorem

We have that $L^2 \not\subseteq L^1$, and so we look at $L^1 \cap L^2$. It turns out $||\hat{f_2}|| = ||f||_2$. There exists an extension from $L^1 \cap L^2$ to L^2 (and this defined a Fourier Transform called the *Plancherel Transform*).

9.13 Thm One can associate to each $f \in L^2$ a function $\hat{f} \in L^2$ s.t.

(a) If $f \in L^1 \cap L^2$, then $\hat{f}$ is the previously defined Fourier transform of f.
(b) For every $f \in L^2$, $||\hat{f}||_2 = ||f||_2$.
(c) The mapping $f \to \hat{f}$ is a Hilbert Space Isomorphism of L^2 onto L^2.

(d) The following symmetric relation exists between f and $\hat{f}$: If

$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}dm(x) \quad \psi_A(x) = \int_{-A}^{A} \hat{f}(t)e^{ixt}dm(t)$$

then $||\varphi_A - \hat{f}||_2 \to 0$ and $||\psi_A - f||_2 \to 0$ as $A \to \infty$.

9.14 Thm If $f \in L^2$ and $\hat{f} \in L^1$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}dm(t) \quad \text{a.e.}$$

9.15 Remark If $f \in L^1$, we have

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}dm(t)$$

but, if $f \in L^2$, this is only a.e. which makes it harder for Fourier transforms in L^1 as compared to L^2.

9.16 Translation-Invariant Subspace of L^2 A subspace M of L^2 is said to be *translation invariant* if $f \in M \implies f_\alpha \in M$, where $f_\alpha = f(x - \alpha)$ for every real α. We ask for a description of the closed translation invariant subspaces of L^2.

Using the fourier transform, we find that M is the set of all preimages of $\hat{M}$ under the Fourier Transform which vanish a.e. on a measurable subset E of $\mathbb{R}^1$.

9.17 Thm Associate to each measurable set $E \subset \mathbb{R}^1$ the space M_E of all $f \in L^2$ s.t. $\hat{f} = 0$ a.e. M_E is a closed translation invariant subspace of L^2. $M_A = M_B$ iff

$$m(((A - B) \cup (B - A)) = 0$$

1.4 The Banach Algebra L^1

9.18 Def A Banach Space A is a *Banach Algebra* if there is multiplication defined in A s.t.

$$||xy|| \leq ||x||||y||$$

and the associative and distributive laws are held true for elements, and commutativity also holds for scalars.

9.19 Examples
(a) $A = C(X)$ where X is a compact Hausdorff space with sup norm and $(fg)(x) = f(x)g(x)$. This is a commutative Banach Algebra.

(b) $C_0(\mathbb{R}^1)$ is a commutative Banach Algebra without a unit.

(c) The set of all linear operators in $\mathbb{R}^k$ with a norm as defined by

$$||\Lambda|| = \sup\{||\Lambda x|| : x \in X, ||x|| \leq 1\}$$

and additions and multiplication defined by

$$(A + B)x = Ax + Bx \quad (AB)x = A(Bx)$$

is a Banach Algebra with a unit and is not commutative for $k > 1$.

(d) L^1 is a Banach Algebra if we define multiplication by convolution, as

$$||f * g||_1 \leq ||f||_1||g||_1$$

L^1 is a commutative Banach Algebra that maps L^1 to C_0, so there is no unit.

9.20 Complex Homomorphisms The most important complex functions on a Banach Algebra A are linear functionals that preserve multiplications

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y) \quad \varphi(xy) = \varphi(x)\varphi(y)$$

9.21 Thm If φ is a complex homomorphism on a Banach Algebra A, then the norm φ, as a linear functional, is at most 1.

9.22 The Complex Homomorphisms of L^1 If φ is a complex homomorphism of L^1 of norm 1 where

$$\varphi(f * g) = \varphi(f)\varphi(g)$$

There exists a $\beta \in L^\infty$ s.t.

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)\beta(x)dm(x)$$

and we can see that $\beta = e^{-ixt}$, and this is the Fourier transform.

9.23 Thm To every complex homomorphism φ on L^1 (except $\varphi = 0$) there corresponds $t \in \mathbb{R}^1$ s.t. $\varphi(f) = \hat{f}(t)$.

2 Exercises

Exercise 1 Note that, we have

$$|\hat{f}(y)| = |\int_{-\infty}^{\infty} f(x)e^{-ixy}dm(x)| \leq \int_{-\infty}^{\infty} |f(x)e^{-ixy}|dm(x)$$

$$= \int_{-\infty}^{\infty} f(x)|e^{-ixy}|dm(x) = \hat{f}(0)$$

and note that equality only happens when $e^{-ixy} = 1$ for all x, or when $y = 0$.

Exercise 3 We note that this if just f where $\hat{f}(t) = \frac{\sin(\lambda t)}{t}$. Note that this value is just

$$\int_{-\lambda}^{\lambda} \frac{e^{-ixt}}{2} dm(x)$$

and so we remove other values

$$\int_{-\infty}^{\infty} \frac{1}{2}(\operatorname{sgn}(x+\lambda) - \operatorname{sgn}(x-\lambda))dm(x)$$

and so $f(x) = \frac{1}{2}(\operatorname{sgn}(x+\lambda) - \operatorname{sgn}(x-\lambda))$.

Exercise 5 We note that, by Fubini's Theorem, we have

$$\int_a^b i \int_{-\infty}^{\infty} t\hat{f}(t)e^{ixt}dm(t)dm(x) = \int_{-\infty}^{\infty}\int_a^b it\hat{f}(t)e^{ixt}dm(x)dm(t) = f(b) - f(a)$$

by the inversion theorem. Therefore, it follows that for some g with derivative $i\int_{-\infty}^{\infty} t\hat{f}(t)e^{ixt}dm(t)$ and a proper minimum C, we have $f = g$ a.e.

Exercise 7 We note that, for $|x| \leq 1$, we find that $|f(x)| \leq A_{00}(f)$. Furthermore, for $|x| \geq 1$, we note that

$$|x^2 f(x)| \leq A_{20}(f) \implies |f(x)| \leq \frac{A_{20}(f)}{x^2}$$

let C be the maximum amount $A_{00}(f)$ and $A_{20}(f)$. Note that that implies

$$\int_{-\infty}^{\infty} |f(x)|dx = \int_{-\infty}^{1} |f(x)|dx + \int_{-1}^{1} |f(x)|dx + \int_{1}^{\infty} |f(x)|dx \leq 4C$$

In general, we can define C_{mn} s.t.

$$\int_{-\infty}^{\infty} |x^n D^m f(x)|dx \leq C_{mn}(f)$$

by using the fact that $|x^{n+2}D^m f(x)| \leq A_{(n+2)m}(f)$ and $|x^n D^m f(x)| \leq A_{mn}(f)$ for $|x| \geq 1$ and $|x| \leq 1$ respectively. Now, we note that

$$D^m\hat{f}(x) = D^m \int_{-\infty}^{\infty} f(t)e^{-ixt}dm(t) = \int_{-\infty}^{\infty} (-it)^m f(t)e^{-ixt}dm(t)$$

So we note

$$|x^n D^m \hat{f}(x)| \leq |\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t^{m+n} f(t)e^{-it}dm(t) + \int_{-x}^{x}(-it)^m f(t)e^{-ixt}dm(t)|$$
$$\leq |C_{(m+2)0}(f)| + |C_{m0}(f)|$$

and so we take these values as our $A_{mn}(\hat{f})$, which implies $\hat{f} \in S$. Some possible elements of f are $0, \sin(\frac{1}{x})$.

Exercise 9 Note that, for some ϵ, clearly there must be a N for which $|x| > N \implies |f(x)|^p < \epsilon$ by definition of L^p. Therefore, it follows that,

$$f(x) \leq |f(x)| < \sqrt[p]{\epsilon}$$

so, there exists $\epsilon_0 = \sqrt[p]{\epsilon}$ and $g(x) < \epsilon_0$ for all $|x| > N$, which means it vanishes at ∞. For $p = \infty$, we note that implies $g(x) < \infty$ for all ∞, but not much else.

Exercise 11 We note that the nth Fourier coefficients of F is $\varphi(n)$, as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{ixt}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} f(x+2k\pi)e^{-ixt}dx$$
$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\int_{-\pi}^{\pi} f(x+2k\pi)e^{-ixt}dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ixt}dx$$

only when f is also cyclic. The condition $F = \sum \varphi(n)e^{inx}$ is the summation of the Fourier coefficients. This implies the condition that

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{-\infty}^{\infty}\varphi(n)$$

and the more general condition

$$\sum_{k=-\infty}^{\infty} f(k\beta) = \alpha\sum_{-\infty}^{\infty}\varphi(nx) \quad \alpha\beta = 2\pi$$

follows when we consider different cycles of β. As $\alpha \to 0$, the RHS goes to $\int_{-\infty}^{\infty}\varphi dm$ and this makes sense with the inversion theorem.

Exercise 13
(a) Note that integration by parts taking $u = e^{-cx^2}$ and $dv = e^{-ixt}$ gives us the equation

$$\int_{-\infty}^{\infty} e^{-ct^2}e^{-itx}dt + \int_{-\infty}^{\infty}\frac{2cte^{-ct^2}e^{-itx}}{xi}dt = \frac{e^{-ct^2}e^{-itx}}{-ix}\Bigg|_{-\infty}^{\infty} = 0$$

where the last equality comes from a norm argument. Notice tha the second term (with the x from the denominator removed) is the derivative with respect to x. Rearranging, we can conclude that

$$x\hat{f}_c(x) + 2c\hat{f}_c'(x) = 0$$

as given by the hint. Through first order differential equations, we find that $f(x) = c_1 e^{-\frac{x^2}{2c}}$ for some constant c_1. However, substituting value $x = 0$ gives us a value of $c_1 = \sqrt{\frac{\pi}{c}}$.

(b) Note that our value of c is given by the equation $\sqrt{\frac{\pi}{c}} e^{-\frac{x^2}{2c}} = e^{-cx^2}$, which, under natural log, gives us

$$-\frac{x^2}{2c} \ln \sqrt{\frac{\pi}{c}} = -cx^2$$

and this is just an equation of c which simplifies to

$$4c + \frac{1}{2}\ln c = \frac{1}{2}\ln \pi$$

Note that the LHS has a slope (respective to c) of $4 + \frac{1}{c}$, so it can only be equal to the RHS once. At ∞ the LHS goes to ∞ and nearing 0 we have a value of $-\infty$, and so this must have a solution since the LHS is continuous.

(c) We note that $f_a * f_b$ is given by

$$\int_{-\infty}^{\infty} e^{-a(x-t)^2 - b(t)^2} dt = \int_{-\infty}^{\infty} e^{-(a+b)t^2 + 2axt - ax^2} dt = e^{-ax^2} \int_{-\infty}^{\infty} f_{a+b} e^{2axt} dt$$

and this value can be simplified by completing the square to

$$e^{-abx^2/(a+b)} \int_{-\infty}^{\infty} e^{-(\sqrt{a+b}t - \frac{ax}{\sqrt{a+b}})^2} dt = \sqrt{\frac{\pi}{a+b}} e^{-\frac{abx^2}{a+b}}$$

and so we find that $\gamma = \sqrt{\frac{\pi}{a+b}}$ and $c = \frac{ab}{a+b}$.

(d) This proves that the theta function on β is equal to the theta function of α times α.

Exercise 15 This is outside the purview of this book's material. The book doesn't define the Fourier Transformation for $\mathbb{R}^k$ and integration is difficult for this.

Exercise 17 We note that, if $f(\frac{1}{n}) = \sqrt[n]{f(1)}$, which means that this value get's arbitarily close to 1. Therefore, we note that $f(\alpha \pm \frac{1}{n}) = f(\alpha)f(\frac{1}{n})$ get's arbitraily close to $f(\alpha)$, which proves continuity.

we can take a similar argument to prove for $\mathbb{R}^k$ as we divide by $\frac{1}{n}$ for $n \to \infty$ and at some point the distance get's arbitarily close to 0, as desired.

Exercise 19 We note that $\chi_A * \chi_B$ is obviously continuous, as we can set δ where the total length of all continuous segments of measure $0 < m < \delta$ in $\chi_A * \chi_B$ and $n * \delta$, where n is the number of continuous segments of measure $m < \delta$, is $< \epsilon$. Note that this is possible since this function goes to 0 as $\delta \to 0$.

Furthermore, our function $\chi_A * \chi_B$ is not 0 as, for some measure, we can have an x s.t. the integral isn't 0.

It follows that there is a segment among the domain of $\chi_A * \chi_B$ which has a positive value for all elements. This segment is clearly the sum of values of $a \in A$ and $b \in B$, so it follows that this is the segment of $A + B$ with measure > 0.