Rudin Real and Complex Analysis - Elementary Properties of Holomorphic functions

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1 Notes

1.1 Complex Differentiation

10.1 Def Let

$$D(a,r) = \{z : |z-a| < r\}$$

be the open circular disc with center at a with ratisd r. $\overline{D}(a,r)$ is the closure of $D(a,r)$, and

$$D'(a,r) = \{z : 0 < |z-a| < r\}$$

be the punctured disc. A set E in a topological space X is said to be *not connected* if E is the union of two nonempty sets A and B s.t.

$$\overline{A} \cap B = \emptyset = A \cap \overline{B}$$

If we have V and W the complements of $\overline{A}$ and $\overline{B}$, then

$$A \subset W \quad B \subset V$$

So

$$E \subset V \cup W \quad E \cap V = \emptyset \quad E \cap W = \emptyset \quad E \cap V \cap W = \emptyset$$

If E is closed and not connected, then we see that E is the union of two disjoint nonempty closed sets. If E is open and not connected, then E is the union of two disjoint nonempty open sets. If $x \in E$, the family Φ_x of all connected subsets of E that contain x is therefore not empty. The union of all members of Φ_x is connected and is *the maximal connected subset* of E. The elements of Φ_x are the *components* of E. Any two components of E are disjoint and E is the union of its components.

A *region* is a nonempty connected open subset of the complex plane. Since each open Ω in the plane is a union of discs, and since all discs are connected, each component of Ω is open. Ω will now be a plane open set.

10.2 Def Suppose f is a complex function defined in Ω. if $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and we call this limit $f'(z_0)$ and is the derivative of f at z_0. If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is *holomorphic* (or *analytic*) in Ω. The class of all holomorphic functions in Ω is $H(\Omega)$. In particular,

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \quad z \in D'(z_0, \delta)$$

Thus $f'(z_0)$ is a complex number. Note that f is a mapping of Ω into $\mathbb{R}^2$ and by 7.22 the other mapping of this kind is the linear operator that is multiplication by $f'(z_0)$.

10.3 Remarks If $f \in H(\Omega)$ and $g \in H(\Omega)$ then $f + g \in H(\Omega)$ and $fg \in H(\Omega)$. The usual differentiation applies. Furthermore, $h = g \circ f$ is holomorphic and

$$h'(z_0) = g'(f(z_0))f'(z_0)$$

10.4 Ex For $n = 0, 1, \ldots, z^n$ is holomorphic in the whole plane and the same is true of every polynomial in z. One easily verifies directly that $\frac{1}{z}$ is holomorphic in $\{z : z \neq 0\}$. Hence, taking $g(w) = \frac{1}{w}$ in the chain rule, we see that if $f_1, f_2 \in H(\Omega)$ and Ω_0 is an open subset of Ω in which f_2 has no zero, then $f_1/f_2 \in H(\Omega_0)$. A function that is holomorphic in the whole plane (such functions are called *entire*) is the exponential function defined in the Prologue.

10.5 Power Series From the theory of power series we shall assume only one fact as known, namely, that to each power series

$$\sum_{n=0}^{\infty} c_n (z - a)^n$$

There correpsponds a number $R \in [0, \infty]$ s.t. the series converges absolutely and uniformly in $\overline{D}(a, r)$ for every $r < R$ and diverges if $z \notin \overline{D}(a, R)$ the "radius of convergence" R is given by the root test:

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}$$

A function f in Ω is *representable by power series* in Ω if to every disc $D(a, r) \subset \Omega$ there corresponds a series as above that converges to $f(z)$ for all $z \in D(a, r)$.

10.6 Thm If f is representable by power series in Ω, then $f \in H(\Omega)$ and f' is also representable by power series in Ω. For $z \in D(a,r)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \implies f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

10.7 Thm Suppose μ is a complex (finite) measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z}$$

Then f is representable by power series in Ω.

1.2 Integration over Paths

10.8 Def If X is a topological space, a *curve* in X is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset \mathbb{R}^1$ into X. We call $[\alpha, \beta]$ the *parameter interval* of γ and denote the range of γ by γ^*. Thus γ is a mapping, and γ^* is the set of all points $\gamma(t)$ for $\alpha \le t \le \beta$. If $\gamma(\alpha) = \gamma(\beta)$ then γ is a *closed curved.* A *path* is a piecewise continuously differentiable curve in the plane. More explicitly, a path with parameter interval $[\alpha, \beta]$ is a continuous complex function γ on $[\alpha, \beta]$ s.t. the following holds:

There are finitely many points $s_j, \alpha = s_0 < \cdots < s_n = \beta$ and the restriction of γ to each interval $[s_{j-1}, s_j]$ has a continuous derivative on $[s_{j-1}, s_j]$. However, at the points $s_1, \ldots, s_{n-1}$ the left and right hand derivatives of γ may differ. A *closed path* is a closed curve which is also a path. Now suppose γ is a path, and f is a continuous function on γ^*. The integral of f over γ is defined as an integral over the parameter interval $[\alpha, \beta]$ of γ:

$$\int_\gamma f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt$$

Let φ be a continuously differentiable one-to-one mapping of an interval $[\alpha_1, \beta_1]$ onto $[\alpha, \beta]$ s.t. $\varphi(\alpha_1) = \alpha$ and $\varphi(\beta_1) = \beta$ and but $\gamma_1 = \gamma \circ \varphi$. Then γ_1 is a path with parameter interval $[\alpha_1, \beta_1]$ and the integral of f over γ_1 is

$$\int_{\alpha_1}^{\beta_1} f(\gamma_1(t))\gamma_1'(t)dt = \int_{\alpha_1}^{\beta_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt = \int_\alpha^\beta f(\gamma(s))\gamma'(s)ds$$

So, in particular, the reparametrization does not change the integral.

$$\int_{\gamma_1} f(z)dz = \int_\gamma f(z)dz$$

and we regard γ and γ_1 equivalent. We can also split up the interval to γ_1, γ_2 s.t. they form one path γ and ge that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f$$

for every continuous f on $\gamma^* = \gamma_1^* \cup \gamma_2^*$. If we have a path γ_1 that is the opposite of γ ie $\gamma(t) = \gamma_1(1-t), 0 \leq t \leq 1$ then f continuous on γ^* we have

$$\int_0^1 f(\gamma_1(t))\gamma_1'(t)dt = -\int_0^1 f(\gamma(s))\gamma'(s)ds$$

$$\implies \int_{\gamma_1} f = -\int_\gamma f$$

In particular, we have an inequality relating L^∞ norm and the path integral

$$\left|\int_\gamma f(z)dz\right| \leq \|f\|_{L^\infty} \int_\alpha^\beta |\gamma'(t)|\, dt$$

10.9 Special Cases

(i) If $a \in \mathbb{C}$ and $r > 0$, the path defined by

$$\gamma(t) = a + re^{it}(0 \leq t \leq 2\pi)$$

is called the *partially oriented circle* with center at a and radius r, we have

$$\int_\gamma f(z)dz = ir\int_0^{2\pi} f(a + re^{i\theta})e^{i\theta}d\theta$$

(ii) If a and b are complex numbers, the path γ given by

$$\gamma(t) = a + (b-a)t \quad (0 \leq t \leq 1)$$

is the *oriented interval* $[a, b]$. The length is $|b-a|$ and

$$\int_{[a,b]} f(z)dz = (b-a)\int_0^1 f[a + (b-a)t]dt$$

If

$$\gamma_1(t) = \frac{\alpha(\beta - t) + b(t - \alpha)}{\beta - \alpha}$$

we obtain an equivalent path, which we still denote by $[a, b]$. The path opposite to $[a, b]$ is $[b, a]$.

(iii) Let $\{a, b, c\}$ be an ordered triple of complex numbers, let

$$\Delta = \Delta(a, b, c)$$

be the triangle with vertices of a, b and c (Δ is the smallest convex set which contains a, b and c) and define

$$\int_{\partial\Delta} f = \int_{[a,b]} f + \int_{[b,c]} f + \int_{[c,a]} f$$

for any f continuous on the boundary of Δ. We may regard this as the definition of its left side, or $\partial\Delta$ is a path from $[a, b]$ to $[b, c]$ to $[c, a]$. If $\{a, b, c\}$ is permuted cyclically, we see that the left side is unaffected. If $\{a, b, c\}$ is reaplced by $\{a, c, b\}$ then the left side changes sign.

10.10 Thm Let γ be a closed path, let Ω be the complement of γ^* (relative to the plane), and define

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{d\zeta}{\zeta - z}$$

Then Ind_γ is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω.

We call $\mathrm{Ind}_\gamma(z)$ the index of z with respect to γ. Note that γ^* is compact, hence γ^* lies in a bounded disc DD whose complement D^c is connected. Thus D^c lies in some component of Ω, which means that Ω has precisely one unbounded component.

10.11 Thm If γ is the positively oriented circle with center at a and radius r, then

$$\mathrm{Ind}_\gamma(s) = \begin{cases} 1 & |z - a| < r \\ 0 & |z - a| > r \end{cases}$$

1.3 The Local Cauchy Theorem

10.12 Cauchy's Thm Suppose $F \in H(\Omega)$ and F' is continuous in Ω. Then

$$\int_\gamma F'(z)dz = 0$$

for every closed path γ in Ω.

10.13 Cauchy's Thm for a Triangle Suppose Δ is a closed triangle in a plane open set $\Omega, p \in \Omega, f$ is continuous on Ω and $f \in H(\Omega - \{p\})$. Then

$$\int_{\partial\Delta} f(z)dz = 0$$

10.14 Cauchy's Thm on Convex Set Suppose Ω is a convex open set, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence

$$\int_{\gamma} f(z)dz = 0$$

for every closed path γ in Ω.

10.15 Cauchy's Formula in a Convex Set Suppose γ is a closed path in a convex open set Ω and $f \in H(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then

$$f(z) \cdot \mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi$$

The case of greatest interest is, of course, $\mathrm{Ind}_\gamma(z) = 1$.

10.16 Thm For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by power series in Ω.

Corr $f \in H(\Omega) \implies f' \in H(\Omega)$.

10.17 Morera's Thm Suppose f is a continuous complex function in an open set Ω s.t.

$$\int_{\partial\Delta} f(z)dz = 0$$

for every closed triangle $\Delta \subset \Omega$. Then $f \in H(\Omega)$.

1.4 The Power Series Representation

10.18 Thm Suppose Ω is a region, $f \in H(\Omega)$, and

$$Z(f) = \{a \in \Omega : f(a) = 0\}$$

Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ s.t.

$$f(z) = (z - a)^m g(z)$$

where $g \in H(\Omega)$ and $g(a) \neq 0$. Furthermore, $Z(f)$ is at most countable. The integer m is called the *order* of the zero which f has at the point a. Clearly $Z(f) = \Omega$ iff f is identically 0 in Ω. We call $Z(f)$ the *zero set* of f. Analogous results hold of course for the set of α-points of f, ie, the zero set of $f - \alpha$, where α is any complex number.

Corr: if f and g are holomorphic functions in a region Ω and if $f(z) = g(z)$ for all z in some set which has a limit point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$.

10.19 Def If $a \in \Omega$ and $f \in H(\Omega - \{a\})$ then f is said to have an *isolated singularity* at the point a. If f can be so defined at a that the extended function is holomorphic in Ω, the singularity is said to be *removable*.

10.20 Thm Suppose $f \in H(\Omega - \{a\})$ and f is bounded in $D'(a, r)$ for some $r > 0$. Then f has a removable singularity at a.

10.21 Thm If $a \in \Omega$ and $f \in H(\Omega - \{a\})$ then one of the following three cases must occur:

(a) f has a removable singularity at a.

(b) There are complex numbers $c_1, \dots c_m$, where m is a positive integer and $c_m \neq 0$, s.t.

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

has a removable singularity at a.

(c) If $r > 0$ and $D(a, r) \subset \Omega$, then $f(D'(a, r))$ is dense in the plane.

In case (b), f is said to have a *pole of order* m at a. The function

$$\sum_{k=1}^{m} c_k (z-a)^{-k}$$

a polynomial in $(z-a)^{-1}$ is called the principal part of f at a. It is clear that $|f(z)| \to \infty$ as $z \to a$. In case (c), f is said to have an *essential singularity* at a. A satement equivalent to 9c) is that to each complex number w there corresponds a sequence $\{z_n\}$ s.t. $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.

10.22 Thm If

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad (z \in D(a, R))$$

and if $0 < r < R$, then

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 d\theta$$

10.23 Liouville's Thm Every bounded entire function is constant.

10.24 The Maximum Modulus Thm Suppose Ω is a region, $f \in H(\Omega)$ and $\overline{D}(a,r) \subset \Omega$. Then

$$|f(a)| \leq \max_{\theta} |f(a + re^{i\theta})|$$

Equality occurs in (1) iff f is constant in Ω.

Corr Under the same hypothese,

$$|f(a)| \geq \min_{\theta} |f(a + re^{i\theta})|$$

if f has no zero in $D(a,r)$.

10.25 Thm (FTA) If n is a positive integer, and

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

where $a_0, \ldots, a_{n-1}$ are complex numbers, then P has precisely n zeros in the plane.

10.26 Thm (Cauchy's Thm) If $f \in H(D(a,R))$ and $|f(z)| \leq M$ for all $z \in D(a,R)$, then

$$\left| f^{(n)}(a) \right| \leq \frac{n!M}{R^n} \quad (n = 1, 2, 3, \ldots)$$

10.27 Def A sequence $\{f_j\}$ of functions in Ω is said to converge to f uniformly on compact subsets of Ω if to every compact $K \subset \Omega$ and to every $\epsilon > 0$ there corresponds an $N = N(K, \epsilon)$ s.t. $|f_j(z) - f(z)| < \epsilon$ for all $z \in K$ if $j > N$.

10.28 Thm Suppose $f_j \in H(\Omega)$, for $j = 1, 2, 3, \ldots$ and $f_j \to f$ uniformly on compact subsets of Ω. Then $f \in H(\Omega)$, and $f_j' \to f'$ uniformly on compact subsets of Ω.

1.5 The Open Mapping Theorem

10.29 Lemma If $f \in H(\Omega)$ and g is defined in $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & w \neq z \\ f'(z) & w = z \end{cases}$$

then g is continuous in $\Omega \times \Omega$.

10.30 Thm Suppose $\varphi \in H(\Omega)$, $z_0 \in \Omega$, and $\varphi'(z_0) \neq 0$. Then Ω contains a neighborhood V of z_0 s.t.

(a) φ is one-to-one in V.

(b) $W = \varphi(V)$ is an open set and

(c) If $\psi : W \to V$ is defined by $\psi(\varphi(z)) = z$ then $\psi \in H(W)$.

and ψ is the holomorphic inverse of $\varphi : V \to W$.

10.31 Def For $m = 1, 2, 3, \ldots$ we denote the "m^{th} power function" $z \to z^m$ by π_m.

10.32 Thm Suppose Ω is a region, $f \in H(\Omega)$, f is not constant, $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0. Then there exists a neighborhood V of z_0, $V \subset \Omega$ and there exists $\varphi \in H(V)$, s.t.

(a) $f(z) = w_0 + [\varphi(z)]^m$ for all $z \in V$.

(b) φ' has no zero in V and φ is an invertible mapping of V onto a disc $D(0, r)$.

Thus $f - w_0 = \pi_m \circ \varphi$ in V. It follows that f is an exactly m-to-1 mapping of $V - \{z_0\}$ onto $D'(w_0, r^m)$, and that each $w_0 \in f(\Omega)$ is an interior point of $f(\Omega)$. Hence $f(\Omega)$ is open.

10.33 Thm Suppose Ω is a region, $f \in H(\Omega)$, and f is one-to-one in Ω. Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.

1.6 The Global Cauchy Theorem

10.34 Chains and Cycles Suppose $\gamma_1, \ldots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\tilde{\gamma}_i$ on the vector space $C(K)$, by using the formula

$$\tilde{\gamma}_i(f) = \int_{\gamma_i} f(z)dz$$

Define

$$\tilde{\Gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n$$

We introduce a "formal sum"

$$\Gamma = \gamma_1 \dot{+} \ldots \dot{+} \gamma_n$$

and define

$$\int_\Gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)dz = \tilde{\Gamma}(f)$$

Γ is a *chains*. If each γ_j is a closed path, then Γ is a *cycle*. If each γ_j is a path in some open set Ω, we say that Γ is a *chain* in Ω. We define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*$$

If Γ is a cycle and $\alpha \in \Gamma^*$, we define the *index* of α w.r.t Γ by

$$\mathrm{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z-\alpha}$$

Which implies

$$\mathrm{Ind}_\Gamma(\alpha) = \sum_{i=1}^{n} \mathrm{ind}_{\gamma_i}(\alpha)$$

If each γ_i is replaced by its negative, then we have $-\Gamma$. In particular

$$\int_{-\Gamma} f(z)dz = -\int_\Gamma f(z)dz$$

in particular $\mathrm{Ind}_{-\Gamma}(\alpha) = -\mathrm{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$. Chains can be added and subtracted in the obvious way,

$$\Gamma = \Gamma_1 + \Gamma_2 \text{ means } \int_\Gamma f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$$

Finally, note that a chain may be represented as a sum of paths in many ways. Note that

$$\gamma_1 \dot{+} \ldots \dot{+} \gamma_n = \delta_1 \dot{+} \ldots \dot{+} \delta_k$$

means that

$$\sum_i \int_{\gamma_i} f(z)dz = \sum_j \int_{\delta_j} f(z)dz$$

Cauchy's Theorem Suppose that $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \; \forall \alpha \notin \Omega$$

Then

$$f(z) \cdot \mathrm{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z} dw \quad z \in \Omega \setminus \Gamma^*$$

and

$$\int_\Gamma f(z)dz = 0$$

If Γ_0 and Γ_1 are cycles in Ω s.t.

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha) \quad \forall \alpha \notin \Omega$$

then

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} (z)dz$$

10.36 Remarks

(a) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Theorem 10.14 to $f(z) = (z-\alpha)^{-1}$ shows that $\mathrm{Ind}_\gamma(\alpha) = 0$. Hypothesis (1) of Thm 10.35 is satisfied by every cycle in Ω if Ω is convex.

(b) The last part of Thm 10.35 shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the integral value. If Ω is the plane with three disjoin closed discs D_i removed, $\Gamma, \gamma_1, \gamma_2, \gamma_3$ are positively oriented circles in Ω that Γ surrounds $D_1 \cup D_2 \cup D_3$, and γ_i surrounds D_i, then

$$\int_\Gamma f(z)dz = \sum_{i=1}^{3} \int_{\gamma_i} f(z)dz$$

(c) In order to apply Cauchy's Theorem, we must have a reasonably efficient method of find the index of a point w.r.t a closed path. This is the following theorem, which states that the Ind increases by 1 when the path is crossed "from right to left".

10.37 Thm Suppose γ is a closed path in the plane, with parameter interval $[\alpha, \beta]$. Suppose $\alpha < u < v < \beta$, a, b complex numbers, $|b| = r > 0$, and

(i) $\gamma(u) = a - b, \gamma(v) = a + b$

(ii) $|\gamma(s) - a| < r$ iff $u < s < v$

(iii) $|\gamma(s) - a| = r$ iff $s = u$ or $s = v$

Assume furthermore that $D(a, r) \setminus \gamma^*$ is the union of two regions, D_+ and D_-, labeled so that $a + bi \in \overline{D}_+$ and $a - bi \in \overline{D}_-$. Then

$$\mathrm{Ind}_\gamma(s) = 1 + \mathrm{Ind}_\gamma(w)$$

if $x \in D_+$ and $w \in D_-$. As $\gamma(t)$ traverses $D(a, r)$ from $a - b$ to $a + b$, D_- is "on the right" and D_+ is "on the left" of the path.

10.38 Homotopy Suppose γ_0 and γ_1 are closed curves in a topological space X, both with parameter interval $I = [0, 1]$. We say that γ_0 and γ_1 are X-*homotopic* if there is a continuous mapping H of the unit square $I^2 = I \times I$ into X s.t.

$$H(s, 0) = \gamma_0(s) \quad H(s, 1) = \gamma_1(s) \quad H(0, t) = H(1, t)$$

If γ_0 is X-homotopic to a constant mapping γ_1 (γ_1^* is one point), we say that γ_0 is *null-homotopic* in X. If X is connecte dna dif every closed curve in X is null-homotopic, X is said to be simply connected. An example is every convex region Ω is simply connected as, for γ_0 closed curve

$$H(s,t) = (1-t)\gamma_0(s) + tz_1$$

10.39 Lemma If γ_0 and γ_1 are closed paths with parameter interval $[0,1]$ if α is a complex number, and if

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)|$$

then $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_0}(\alpha)$.

10.40 Thm If Γ_0 and Γ_1 are Ω-homotopic closed paths in region Ω, and if $\alpha \notin \Omega$, then

$$\mathrm{Ind}_{\Gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$$

1.7 The Calculus of Residues

10.41 Def A function f is said to meromorphic in an open set Ω if there is a set $A \subset \Omega$ s.t.

(a) A has no limit point in Ω

(b) $f \in H(\Omega - A)$

(c) f has a pole at each point of A

Note that $f \in H(\Omega)$ means that it is meromorphic in Ω. Furthermore, we have

$$Q(z) = \sum_{k=1}^{m} c_k (z-a)^{-k}$$

is the principal part of f at a, then c_1 is called the *residue* of f at a: $c_1 = \mathrm{Res}(f,a)$. In particular, if Γ is a cycle and $a \notin \Gamma^*$ then

$$\frac{1}{2\pi i}\int_\Gamma Q(z)dz = c_1 \mathrm{Ind}_\Gamma(a) = \mathrm{Res}(Q,a)\mathrm{Ind}_\Gamma(a)$$

10.42 The Residue Theorem Suppose f is a meromorphic function in Ω. Let A be the set of points in Ω for which f has poles. If Γ is a cycle in $\Omega \setminus A$ s.t.

$$\mathrm{Ind}_\Gamma(\alpha) = 0 \quad \forall \alpha \notin \Omega$$

then

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz = \sum_{a \in A} \mathrm{Res}(f,a)\mathrm{Ind}_\Gamma(a)$$

10.43 Thm Suppose γ is a closed path in a region Ω, such that $\mathrm{Ind}_\gamma(\alpha) = 0$ for every α not in Ω. Suppose also that $\mathrm{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega - \gamma^*$, and let Ω_1 be the set of all α with $\mathrm{Ind}_\gamma(\alpha) = 1$. For any $f \in H(\Omega)$ let N_f be the number of zeros of f in Ω_1, counted according to their multiplicites.

(a) If $f \in H(\Omega)$ and f has no zeros on γ^* then

$$N_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \mathrm{Ind}_\Gamma(0)$$

where $\Gamma = f \circ \gamma$.

(b) If also $g \in H(\Omega)$ and

$$|f(z) - g(z)| < |f(z)| \quad \forall z \in \gamma^*$$

then $N_g = N_f$.

Part (b) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in Ω_1, if they are close together on the boundary of Ω_1.

10.44 Problem For real t, find the limit, as $A \to \infty$, of

$$\int_{-A}^{A} \frac{\sin x}{x} e^{ixt} dx$$

Notice that $z^{-1} \cdot \sin z \cdot e^{itz}$ is entire, its integral over $[-A, A]$ equals that over the path Γ_A from $-A$ to -1 along real, -1 to 1 along the lower half of the unit circle, and from 1 to A along the real axis from Cauchy's Thm. Γ_A avoid the origin, and therefore

$$2i \sin z = e^{iz} - e^{-iz}$$

and we see the original equation is equal to $\varphi_A(t+1) - \varphi_A(t-1)$, where

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi i} \int_{\Gamma_A} \frac{e^{isz}}{z} dz$$

Complete Γ_A to a closed path in two ways: first by the semicircl from A to $-Ai$ to $-A$ or the one from up to Ai. The function $\frac{e^{isz}}{z}$ has a single pole, $z = 0$, where the residue is 1. Therefore,

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi} \int_{-\pi}^{0} \exp(isAe^{i\theta}) d\theta$$

and

$$\frac{1}{\pi}\varphi_A(s) = 1 - \frac{1}{2\pi} \int_{0}^{\pi} \exp(isAe^{i\theta}) d\theta$$

Note that

$$\left|\exp(isAe^{i\theta})\right| = \exp(-As\sin\theta)$$

and that this is < 1 and tends to 0 as $A \to \infty$ if s and $\sin\theta$ have the same sign. The dominated convergence theorem shows therefore that the integral in (3) tends to 0 if $s < 0$ and the one in (4) tends to 0 if $s > 0$. Thus

$$\lim_{A\to\infty} \varphi_A(s) = \begin{cases} \pi & s > 0 \\ 0 & s < 0 \end{cases}$$

and if we reapply the above to $s = t + 1$ and $t - 1$ we get

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x} e^{itx} dx = \begin{cases} \pi & -1 < t < 1 \\ 0 & |t| > 1 \end{cases}$$

Since $\varphi_A(0) = \pi/2$, the above limit is $\pi/2$ when $t = \pm 1$.

2 Exercises

Problem 1 Suppose not for sake of contradiction. Then suppose we have $\alpha \in A, \beta_n \in B$ s.t. $|\alpha - \beta_n| \to 0$. Then we note that $\beta_n \to \alpha$ which means that $\alpha \in B$ since B is closed. But, this contradicts the assumption that A and B are disjoint.

Problem 3 No conclusions can be drawn on the derivative. For example, if $g(z)$ is constant 1 and $f(z)$ has some slope. On the other hand, if $f(z) = 0$ and $g(z)$ is some other slope, then we have the opposite on the derivative.

Problem 5 Let $K \subset \Omega$ be compact. We can rewrite

$$f_n(x) - f_m(x) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(z) - f_m(z)}{z - x} dz$$

But note that this value of $f_n - f_m$ is holomorphic so we can expand this to

$$\sum_{n=0}^{\infty} \int_\gamma \frac{f_n(z) - f_m(z)}{(z-a)^{n+1}} dz \cdot (x-a)^n$$

These integrals converge to 0 through dominated convergence. However, for the sum to converge, we note that we can cover K with an open covering of balls of radius $r < 1$. Take a finite subcover (Heine-Borel) and we have an n s.t.

$$\|f_n(x) - f_m(x)\| < \epsilon \;\forall x \in B$$

for all balls B in our compact set K.

Problem 7 If we assume that Γ is closed and $\mathrm{Ind}_\gamma(z) = 1$, $z \in \Omega, z \notin \Gamma^*$. If this is true, then we integrate by parts multiple times (as $\Gamma(\alpha) = \Gamma(\beta)$), and the results show relatively simply.

Problem 8 This is a result of the residue theorem and taking an semicircle large enough such that it contains all zeros of Q. For the actual question, we note that the residues above 0 are $e^{k\pi/4}$ for $k \in \{1, 3\}$. We calculate the residues for these values by noting that these are poles of order 1 and the residues are

$$\lim_{x \to e^{k\pi/4}} \frac{x^2}{1+x^4}(x - e^{k\pi/4})$$

or, in particular,

$$-\frac{i\sqrt{i}}{4} \quad -\frac{\sqrt{i}}{4}$$

This gives us a total integral value of

$$\frac{\pi\sqrt{i}}{2} - \frac{\pi i\sqrt{i}}{2}$$

If we consider the poles below 0, then it is the value of $-2\pi i$ times the sum of these residues.

Problem 9 The poles are $\pm i$, but the only pole above the plane is i. We also calculate the poles

$$\lim_{x \to i} \frac{e^{itx}}{1+x^2}(x - i) = \frac{e^{-t}}{2i}$$

and so the integral is $\frac{\pi}{e^t}$. We can see this through fourier inversion.

Problem 11

Note that we are integrating the holomorphic function

$$(\alpha - z)^{-1}(\alpha - 1/z)^{-1}$$

over the path given by the unit circle. Note that this is 0 by Cauchy's Thm.

Problem 13

We integrate over the curve given by the circle from $0 \to R \to Re^{2\pi i/n} \to 0$. Note that this is a curve, so the integral of this Γ is 0. The integral is given by

$$\int_0^R \frac{dx}{1+x^n} + \int_0^{2\pi/n} \frac{Rie^{it}}{1+(Re^{it})^n} dt + \int_R^0 \frac{e^{2\pi i/n}dx}{1+(e^{2\pi i/n}x)^n} = 0$$

Note that therefore

$$\int_0^R \frac{dx}{1+x^n} = -\int_0^{2\pi/n} \frac{Rie^{it}}{1+(Re^{it})^n}dt + \int_0^R \frac{e^{2\pi i/n}dx}{1+(e^{2\pi i/n}x)^n}$$

and calculating we get that the integral is $\frac{\pi \sin(\pi/n)}{n}$.

Problem 15 The result for zeros of order m comes from the fact that φ has no zeros, and repeatedly differentiating the value gives this as a result. If φ' has a zero of order k at z_0, then we note that

$$g'(x) = f'(\varphi(x))\varphi'(x)$$

and so if g has m zeros, φ' has k zeros, then f' has $m-k$ zeros.

Problem 17

Note that the first function is

$$f(z) = \int_0^1 \frac{dt}{1+tz} = \frac{1}{z}\ln(1+tz)\Big|_0^1 = \frac{\ln(1+z)}{z}$$

and this is holomorphic in Ω as ln and $\frac{1}{z}$ are holomorphic. The second function is defined and holomorphic when t has a real value < 0. The third function is defined when the real value of $z > 0$ and it is also holomorphic here.

Problem 19

Note that this means that the derivatives of $\ln(f)$ and $\ln(g)$ are the same on $\frac{1}{n}$ for $n \in \mathbb{Z}^+$. Therefore, it follows that $\ln(f/g) = 0$ on these same values, or that $f = g$ on these values.

Problem 21

There is no limit on the number of points. Notice that for $f(x) = \frac{1}{2}$ we have an uncountable number of points. However, we will prove that there always exists at least one solution. This is because we must have $|f(z) - z| > 0$ for all z, but we note that $|f(z)| < |z|$ for $|z| = 1$. But, we must have $|f(0)| \geq |0| = 0$ so by intermediate value theorem, they must cross and we will always have a solution $f(z) = z$ as $|f(z) - z| = 0$.

Problem 22

Assume f has no zeros in D, then we examine $1/f$ as a holomorphic function. Note that $\left|\frac{1}{f}\right|$ is the value of 1 at 0 and is $< \frac{1}{2}$ at 1. Therefore, there must be a maximal value somewhere in the disc, since the disc is compact and the value at 0 is greater than the edge values. But, this violated the maximum modulus principle.

Problem 23

For large n the function $P_n(z)$ approaches e^z. This has no zeros but has a low value for extremely small z. Therefore, the zeros of $P_n(z)$ approach $-\infty$ as $n \to \infty$. On the other hand, $Q_n(z)$ approaches $e^z - 1$, which has zeros near 0.

Problem 29 Note that the integral is given by

$$\int \frac{d\theta}{re^{i\theta} + z} = \frac{\theta + i\ln(re^{i\theta} + z)}{z}$$

For $r < |z|$, we note that the integral from $-\pi$ to π is given by

$$\left.\frac{\theta + i\ln(re^{i\theta} + z)}{z}\right|_{-\pi}^{\pi} = \frac{2\pi}{z}$$

as $z - r \neq 0$. For $r > |z|$, we note that the integral from $-\pi$ to π is given by 0 since we incorporate the $\theta = \pi$ term into our value to give us $i\log(z - r)$ since otherwise it will be negative. Therefore, we see that we simply must integrate the part where $r > |z|$ and we have two separate cases for $|z|$. For $|z| < 1$ we have

$$\int_0^{|z|} \frac{2r}{z} dr = \frac{|z|^2}{z} = \overline{z}$$

since we integrate up $|z|$ and for $|z| \geq 1$ we have

$$\int_0^1 \frac{2r}{z} dr = \frac{1}{z}$$

as we only need to integrate to 1 every time.