Erdmann-Wildon Lie Algebras - Ideals and Homomorphisms

Aaron Lou

June 2018

1 Notes

1.1 Construction With Ideals

We can see (ex: 2.1) that for I and J ideals, then

$$I + J := \{x + y : x \in I, y \in J\}$$

is also an ideal. So is

$$[I, J] := \mathrm{Span}\{[x, y] : x \in I, y \in J\}$$

Remark 2.1 It is necessary for $[I, J]$ to be the span rather than just the set of such commutators. We write $L' = [L, L]$, which is defined as the *derived algebra* or *commutator algebra*.

1.2 Quotient Algebras

An ideal I is a subspace of L, so we may consider the cosets $z+I = \{z+x : x \in I\}$ for $z \in L$ and the quotient vector space

$$L/I = \{z + I : z \in L\}$$

The Lie bracket on L/I is defined by

$$[w + I, z + I] := [w, z] + I \quad w, z \in L$$

L/I is called the *quotient/factor algebra* of L by I.

Thm 2.2 (Isomorphism Thms)

(a) Let $\varphi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then $\ker \varphi$ is an ideal of L_1 and $\mathrm{im}\varphi$ is a subalgebra of L_2, and

$$L_1 / \ker \varphi \cong \mathrm{im}\varphi$$

(b) If I and J are ideals of Lie algebra, then $(I+J)/J \cong I/(I \cap J)$.

(c) Suppose that I and J are ideals of a Lie algebra L s.t. $I \subseteq J$ then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.

and notice that these are the same as the isomorphism theorems as for vector spaces in linear algebra.

Example 2.3 The operator for trace tr : $\mathbf{gl}(n, F) \to F$ is a Lie algebra homomorphism. tr is surjective, it's kernel is $\mathbf{sl}(n, F)$ and applying the first isomorphism theorem gives

$$\mathbf{gl}(n, F)/\mathbf{sl}(n, F) \cong F$$

specifically $x + \mathbf{sl}_n(F)$ is all $n \times n$ matrices whose trace is $\mathrm{tr}x$.

Example 2.4 If L is a Lie Algebra and I is an ideal s.t. L/I is abelian. The subspaces of L/I are the ideals of L/I by the ideal correspondence.

1.3 Correspondence between Ideals

There is a bijection between the ideals of L/I and the ideals of L that contain I. If J is an ideal containing I, then J/I is an ideal of L/I. If K is an ideal of L/I, then set $J := \{z \in L : z + I \in K\}$.

2 Problems

Ex 2.1 We note that, for $x \in I, y \in J$ we have $[z, x+y] = [z, x] + [z, y]$. Note that $[z, x] \in I, [z, y] \in J \implies [z, x+y] \in I + J$.

Ex 2.2 For matrices $x, y \in \mathbf{sl}(2, \mathbb{C})$, we note that $[x, y] \in \mathbf{sl}(2, \mathbb{C})$. Examining the basis from problem 1.13 reveals that $e, f, h \in \mathbf{sl}(2, \mathbb{C})'$, so it follows that $\mathbf{sl}(2, \mathbb{C})' = \mathbf{sl}(2, \mathbb{C})$.

Ex 2.3
(i) Bilinearity can be proved relatively straightforwardly by the definition of the quotient vector space. Simply notice that

$$[a+I+b+I, c+I] = [(a+b)+I, c+I] = [a+b, c]+I = ([a, c]+[b, c])+I$$

$$= [a, c]+I+[b, c]+I = [a+I, c+I]+[b+I, c+I]$$

where this comes from the fact that I is a vector space and we can represent each term of $(a+b)+I$ as $a+0+b+i$ and vice-versa. Similarly

$$[\alpha(a+I), b+I] = [\alpha a + I, b+I] = [\alpha a, b] + I = \alpha[a,b] + I = \alpha[a+I, b+I]$$

where this again comes from the fact that I is a vector space and we can represent each term of $\alpha a + I$ as $\alpha(a + \frac{i}{\alpha})$ for nonzero α and the case where $\alpha = 0$ is trivial.

(ii) We note that

$$\pi([x,y]) = [x,y] + I = [x+I, y+I] = [\pi(x), \pi(y)]$$

as desired. Therefore, it is a homomorphism.

Ex 2.4 This follows if we take the map $x \to \mathrm{ad}x$. The kernel of this map is clearly the center $Z(L)$. Similarly, the image is $\mathbf{gl}(L)$. Under the first isomorphism theorem, we have

$$L/Z(L) \cong \mathbf{gl}(L)$$

Ex 2.5 We notice that, for $z \in [x,y]$, we have

$$\mathrm{tr}(\mathrm{ad}[x,y]) = \mathrm{tr}([\mathrm{ad}x, \mathrm{ad}y]) = 0$$

Ex 2.6
(i) We note that $[x,y] \in \mathbf{sl}(2,\mathbb{C})$ for $x, y \in \mathbf{gl}(2,\mathbb{C})$. The direct isomorphism is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{bmatrix} + \frac{a+d}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{bmatrix}, \frac{a+d}{2} \right)$$

and we note the isomorphism follows as the commutator is bilinear.

(ii) The center follows trivially. Similarly, the commutator algebra also follows trivially. In general, for

$$L = \bigoplus_{i=1}^{n} L_i \implies Z(L) = \bigoplus_{i=1}^{n} Z(L_i) \quad L' = \bigoplus_{i=1}^{n} L_i'$$

(iii) The summands aren't unique as the bases aren't uniquely determined in the direct sum of the underlying vector spaces.

Ex 2.7 The question is formulated incorrectly. $x_2 \in L_2$. To see that they are isomorphic, note that the homomorphism $l_1 \in L_1 \to (l_1, 0)$ is clearly bijective, and a similar result holds for $(l_2, 0)$. The projections are homomorphisms as well, as

$$p_1[(x_1,x_2),(y_1,y_2)] = p_1([x_1,y_1],[x_2,y_2]) = [x_1,y_1] = (p_1(x_1,x_2),p_1[y_1,y_2])$$

and a similar result holds for p_2.To see that these are ideals, note that

$$[(x_1,0),(y_1,y_2)] = ([x_1,y_2],0)$$

and a similar result holds for x_2.

(ii) We claim each (l_1,l_2) is uniquely determined by one of it's elements. This follows since if we have, for example

$$(l_1,l_2),(l_1,l_3) \in J \implies (0,l_3-l_2) \in J \implies J \cap L_2 \neq 0$$

and a similar result for L_1 shows that the terms l_1 and l_2 do indeed uniquely determine the pair. It follows that there is an isomorphism, as each (l_1,l_2) is bijective with each l_1.

(iii) The two ideals are

$$\{(l_1,0),l_1 \in L_1\} \quad \{(0,l_2),l_2 \in L_2\}$$

To prove that these are the only non trivial ideals, we start by enumerating the possibilities of (l_1,l_2). If both are not 0, then since L_1 and L_2 have no ideals, then l_1,l_2 can be anything. If this is not the non-proper ideal L, then we note that

$$(l_1,l_2) \in I,(l_1,l_3) \in I \implies (l_1,l_4) \in I \quad \forall l_4$$

since there are no nontrivial proper ideals. It therefore follows there must be an isomorphic mapping $\varphi(l_1) \to l_2$, as otherwise we have two elements mapped to 1. Therefore, it follows that, for not isomorphic Lie Algebras L_1 and L_2, our two Lie Algebras are the only ones.

(iv) We define the ideals

$$I_i = \{(l_1,iy) : l_1 \in L_1, l_2 \in L_2\}$$

These are indeed ideals. Note that

$$[(x_1,iy_1),(x_2,y_2)] = ([x_1,x_2],i[y_1,y_2]) \in I_i$$

since there are an infinite number of possible values for i, it follows that there an infinite number of ideals.

Ex 2.8
(a) This is trivial. We note that

$$\varphi(L_1') = \varphi([L_1,L_1]) = [\varphi(L_1),\varphi(L_1)] = [L_2,L_2] = L_2'$$

(b) Notice that this is similar. We have

$$\varphi(\{x : [x, y] = 0\}) = \{\varphi(x) : [x, y] = 0\}$$

and we have $[\varphi(x), y] = 0$ since $\exists y_0 : \varphi(y_0) = y$ and $\varphi([x, y_0]) = \varphi(0)$ as φ is surjective.

(c) We limit our domain so that our mapping is injective, which makes φ is a bijection. We find that

$$\mathrm{ad}\varphi(h)i = [\varphi(h), i] = \varphi([h, \varphi^{-1}(i)]) = \varphi \circ \mathrm{ad}h \circ \varphi^{-1}$$

Since ad$h$ can be given by $P^{-1}DP$ for Diagonal D, our map ad$\varphi(h)$ is also diagonalizable since $(P\varphi^{-1})^{-1}D(P\varphi^{-1})$.

If φ is bijective, then we can remove the need to trim the domain in (b) and (c).

Ex 2.9 We already know that $\mathbb{R}^3_\wedge$ and $L = \{x \in \mathbf{gl}(3, \mathbb{R}) : x^t = -x\}$ are isomorphic.

We note that $\mathbf{b}(2, \mathbb{R})$ under the commutator is mapped to the set of strictly upper-triangular matrices of dim 1. Similarly, the set of strictly upper-triangular matrices of dim 3 get's mapped to the set of all matrices with basis

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and it follows pretty easily that this is isomorphic to $\mathbf{b}(2, \mathbb{R})$.

Ex 2.10 We note that $\mathbf{sl}(n, F)$ has a basis given by

$$e_{ij}, e_{ii} - e_{i+1,i+1}$$

We note that the formula is given by

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$$
$$[e_{i1}, e_{1j}] = \delta_{11}e_{ij} - e_{11}$$
$$[e_{i,i+1}, e_{i+1,i}] = e_{i+1,i+1} - e_{ii}$$

so it follows that our basis of $\mathbf{gl}(n, F)'$ is given by e_{ij} where $i \neq j$ and $e_{i+1,i+1} - e_{ii}$. Therefore $\mathbf{gl}(n, F)' \supseteq \mathbf{sl}(n, F)$.

Furthermore, we note that all elements of $\mathbf{gl}(n, F)'$ have trace 0. This means that $\mathbf{gl}(n, F)' \subseteq \mathbf{sl}(n, F)$. This implies equality, as desired.

Ex 2.11 We note that

$$x^tP^tSP = -P^tSPx \implies Px^tP^tSP = -SPx \implies Px^tP^tS = -SPxP^{-1}$$

so each x in $\mathbf{gl}_S(n,F)$ is isomorphic to each x in $\mathbf{gl}_T(n,F)$ and the mapping is $x \to PxP^{-1}$.

Ex 2.12 We note that, by definition

$$x^tS = -Sx \implies \mathrm{tr}(x^tS) + \mathrm{tr}(Sx) = 0 \implies \mathrm{tr}(Sx^t) + \mathrm{tr}(Sx) = 0$$
$$\implies \mathrm{tr}(S(x + x^t)) = 0$$

Since S is symmetric and invertible, we can diagonalize S and $x^t + x$ into D and d and find that

$$\mathrm{tr}(Dd) = 0 \implies \mathrm{tr}(d) = 0$$

since they are diagonal matrices and D can't have a 0 entry. Therefore, it follows that

$$\mathrm{tr}(x^t + x) = 2\mathrm{tr}(x) = 0 \implies \mathrm{tr}(x) = 0$$

Ex 2.13 We notice that, by (i), $I \cap B = 0$ trivially. We prove that for all $l \in L, l = i + b, i \in I, b \in B$.

We notice that ad$l$ is a derivation on I (since I is an ideal), but, by (ii), we note that ad$l$ = adi_1 for some $i_1 \in I$. Simple calculation shows that $l - i_1 \in C_L(I)$, so we can achieve a decomposition as desired. It follows that $L = I \bigoplus B$, as desired.

Ex 2.14
(i) We can trivially see that it is bilinear, alternative, and the Jacobi Identity holds (namely because $[x,[y,z]] = 0$).

(ii) We can simply calculate

$$\left[\begin{pmatrix} 0 & f_1(x) & h_1(x,y) \\ 0 & 0 & g_1(y) \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & f_2(x) & h_2(x,y) \\ 0 & 0 & g_2(y) \\ 0 & 0 & 0 \end{pmatrix}\right]$$
$$= \begin{pmatrix} 0 & 0 & f_1(x)g_2(y) - f_2(x)g_1(y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

hence L' is the set of all matrices of form $A(0,0,h(x,y))$.

Suppose it is a commutator, then we have functions $f_1(x), f_2(x), g_1(y), g_2(y)$ s.t.

$$x^2 + xy + y^2 = f_1(x)g_2(y) - f_2(x)g_1(y)$$

Inspection reveals that terms outside of degree two are superfluous (since they wouldn't affect our lower terms), so we restrict our attention to quadratic f and g. Notice that, if $f_1(x) = a_1x^2 + b_1x + c_1, f_2(x) + a_2x^2 + b_2x + c_2$, then we examine the various derivatives to find that

$$a_1g_1(y) - a_2g_2(y) = 1$$
$$b_1g_1(y) - b_2g_2(y) = y$$

But this is impossible if $g_1(y)$ and $g_2(y)$ are of degree two. But, this would remove the possibility of y^2, so it follows there are no polynomials that satisfy our equations.