Erdmann-Wildon - Engel's Thm and Lie's Thm

Aaron Lou

August 2018

1 Notes

1.1 Engel's Theorem

Thm 6.1 (Engels' Theorem) Let V be a vector space. Suppose that L is a Lie subalgebra of $\mathbf{gl}(V)$ such that every element of L is a nilpotent linear transformation of V. Then there is a basis of V in which every element of L is represented by a strictly upper-triangular matrix.

Prop 6.2 Suppose that L is a Lie subalgebra of $\mathbf{gl}(V)$, where V is a non-zero, such that every element of L is a nilpotent linear transformation. Then there is some non-zero $v \in V$ s.t. $xv = 0$ for all $x \in L$.

1.2 Proof of Engel's Theorem

Given Prop 6.1, we continue the proof as follows. Induct on $\dim V \geq 1$. There is some non-zero vector $v \in V$ s.t. $x(v) = 0$ for all $x \in L$. Let $U = \text{Span}\{v\}$ and let $\overline{V}$ be the quotient vector space V/U. Any $x \in L$ induces a linear transformation $\overline{x}$ of $\overline{V}$. The map $L \to \mathbf{gl}(\overline{V})$ given by $x \to \overline{x}$ is easily checked to b e aLie algebra homomorphism.

The image of L under this homomorphism is a subalgebra of $\mathbf{gl}(\overline{V})$ which satisfies the hypothesis of Engel's Theorem. Moreover, $\dim(\overline{V}) = n - 1$, so by the inductive hypothesis there is a basis of $\overline{V}$ such that with respect to this basis all $\overline{x}$ are strictly upper-triangular. If this basis is $\{v_i + U : 1 \leq i \leq n-1\}$, then $\{v, v_1, \ldots, v_{n-1}\}$ is a basis for V. As $x(v) = 0$ for each $x \in L$, the matrices of elements of L with respect to this basis are strictly upper triangular.

1.3 Another Point of View

Thm 6.3 (Engel's Theorem, second version) A Lie algebra L is nilpotent iff for all $x \in L$ the linear map $\text{ad}x : L \to L$ is nilpotent.

Remark 6.4 (A trap in Engel's Thm) Note that the claim that a Lie subalgebra L of $\mathbf{gl}(V)$ is nilpotent iff there is a basis of V s.t. the elements of L are represented by strictly upper triangular matrices.
This is because the forwards direction fails. For example, the identity I is nilpotent, but this is clearly not upper-triangular.

1.4 Lie's Theorem

Thm 6.5 (Lie's Theorem) Let V be an n-dimensional complex vector space and let L be a solvable Lie subalgebra of $\mathbf{gl}(V)$. Then there is a basis of V in which every element of L is represented by an upper triangular matrix.

Prop 6.6 Let V be a non-zero complex vector space. Suppose that L is a solvable Lie subalgebra of $\mathbf{gl}(V)$. Then there is some non-zero $v \in V$ which is a simultaneous eigenvector for all $x \in L$.

Remark 6.7 (Generalisations of Lie's Theorem) Lie's Theorem holds for more general fields. Certainly we need the field to be algebraically closed, but it also needs to have characteristic zero.

2 Exercises

Exercise 6.1
(i) Suppose not. Then it is not nilpotent as $x^k v = x^k v_0 \neq 0$ for $v_0 \in V$.

(ii) The fact that x induces another nilpotent map $\overline{x}$ is obvious since we just have $x(a + \lambda v) = \overline{x}(a) = x(a)/U$ and taking $\overline{x}^m(a) = x^m(a)/U = 0$. This induction is obvious as for each iteration, we simply have the new "zero vector" v_n has image only consisting of elements $\{v_1 \dots v_{n-1}\}$ and we form the matrix from this order.

Exercise 6.2
(i) Examine the elements

$$(v, x^2 v, \dots x^n v)$$

for some nonzero v. Clearly, we must have, for α_i where α_i are not all zero

$$\sum_{i=0}^{n} \alpha_i x^i v = 0$$

This is a polynomial $\sum_{i=0}^{n} \alpha_i x^i = 0$ and so we can factorize this. Furthermore, this must be a nonzero polynomial since one α_i is non zero. It follows that

$$c(x - \lambda_1) \dots (x - \lambda_m) v = 0$$

and so it follows that x has an eigenvalue and v is an eigenvector.

(ii) This is very similar to 6.1 (ii). Note that the fact that $\overline{x}$ is induced is obvious and is just $x(a + bv) = \overline{x}(a)$. Take another v_1 such that it is an eigenvalue of $\overline{x}$. By induction, we have $\{v, v_1, \ldots v_{n-1}\}$ a basis with an upper triangular matrix for x since each v_i has $x(v_i)$ is composed of $v_j : j < i$.

Exercise 6.3 By Engel's Thm, L is nilpotent iff $x \in L \implies$ ad$x$ is also nilpotent. Every 2-dimensional subalgebra of L has to be abelian. Otherwise, take a subalgebra with basis $\{x, y\}$ s.t. $[x, y] = \alpha x + \beta y \neq 0$. Note that either $\alpha \neq 0$ or $\beta \neq 0$. WLOG, is $\alpha \neq 0$ then ad$y(x) = \alpha x$ and is not nilpotent.

Exercise 6.4 Note that xy is the same structrue as x except the values (by row) are $1, 2, \ldots p - 1, 0$ while yx is the same structure as x except with values (by row) of $0, 1, \ldots p - 2, p - 1$. It follows that $[x, y] = x$. Furthermore, these values span a solvable Lie Algebra as $L^{(2)} = 0$.

x and y don't share a common eigenvector as the eigenvectors of y are vectors spanned by $\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$. The eigenvectors of x are vectors spanned by $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$.

Prop 6.6 fails since there is no common eigenvector, Lie's Theorem fails since no basis has both matrices upper triangular (since one would break the other).

Exercise 6.5

(i) Note that (by Ex 4.5) all elements of L' are representable by a strictly upper triangular matrix (in the same basis as L being upper triangular). This implies that they are nilpotent (Ex 4.4).

(ii) The forward direction is proven using (i) and the second version's of Engel's Theorem. To prove the backwards notation, note that L' being nilpotent means that it is also solvable, so L is therefore solvable.

Exercise 6.6 Use the previous exercise. Note that the subalgebra of $\mathbf{gl}(V)$ spanned by x, y is solvable as x, y commute with $[x, y]$. Therefore, L' is nilpotent, which implies that $[x, y]$ is also nilpotent.