§ Recurrence Relations

S/ The number of bacteria in a colony doubles every hour. If a colony starts with 5 bacteria, how many will be present in n hours?

model: $a_n = 2a_{n-1}$ and $a_0 = 5$

solution: how to find explicit formula?

Defn 8 A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer.

Ex/ Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every non-negative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$ (True / False)

Soln: Suppose that $a_n = 3n$ So
$$a_n = 2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n \checkmark$$
same

Ex/ Same question $a_n = 5$ is a solution of the rec. rel. $a_n = 2a_{n-1} - a_{n-2}$ (True / False)

Soln: Suppose that $a_n = 5$ So
$$a_n = 2a_{n-1} - a_{n-2} = 2.(5) - (5) = 5 \checkmark$$
same

Ex/ Same question $a_n = 2^n$ is a solution of the rec. rel. $a_n = 2a_{n-1} - a_{n-2}$ (True / False)

Soln: Suppose that $a_n = 2^n$ So
$$a_n = 2a_{n-1} - a_{n-2} = 2.2^{n-1} - 2^{n-2} = 3.2^{n-2}$$
not same.

$a_0 = 1 \quad a_1 = 2 \quad a_2 = 4 \neq 3$

The initial conditions for a seq. specify the terms that precede the first term where the rec. rel. takes effect. The rec. rel. and initial conditions uniquely determine ~~the~~ a seq.

⊛ Modeling with Rec. Rel.

Ex/ Suppose that a person deposits 1000 USD in a saving account at a bank yielding 11 % per year with interest compounded annually. How much will be in the account after 30 years?

Soln: P_n: denote the amount in the account after n years

$$P_n = P_{n-1} + 0.11 \cdot P_{n-1} = 1.11\, P_{n-1}$$

$$P_0 = 1000 \text{ (USD)}$$

$$P_1 = 1.11\, P_0$$
$$P_2 = 1.11\, P_1 = (1.11)^2 P_0$$
$$P_3 = 1.11\, P_2 = (1.11)^3 P_0$$
$$\vdots \qquad \vdots$$
$$P_n = (1.11)^n P_0$$

By Math. Ind. you can show validity of the formula

So $P_n = (1.11)^n \cdot 1000$

$$P_{30} = (1.11)^{30} \cdot 1000 = 22\,892.297$$

Ex/ A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, (assuming that no rabbits ever die)

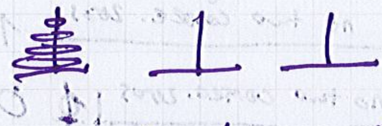
Soln: To find the number of pairs after n months add the number of the prev. month f_{n-1} and the " " " newborn pairs, which is f_{n-2}

So $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$

$f_1 = 1$

$f_2 = 1$

Ex/ Tower of Hanoi :



Initially these disks are placed on the 1st peg in order of size with the largest on the bottom. The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with largest on the bottom.

Let H_n denote the number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a rec. rel. for $\{H_n\}$

Soln: Transfer $n-1$ disks to peg 3 (H_{n-1} moves)

Put largest disk to peg 2

Transfer $n-1$ disks to peg 2 (H_{n-1} moves)

So $H_n = 2H_{n-1} + 1$ initial cond $\rightarrow H_1 = 1$

1 disk directly to peg 2

$\rightarrow$

We can use an iterative approach to solve rec. rel.

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2\underbrace{(2H_{n-2}+1)}+1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3}+1)+2+1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\;\vdots \\
&= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1 \\
&= \underbrace{2^{n-1} + 2^{n-2} + \dots + 2 + 1}_{x} \qquad 2x - x = 2^n - 1 \\
&= 2^n - 1
\end{aligned}$$

Ex/ Find a recur. rel. and give initial cond. for the number of bit strings of length n that do not have two consecutive 0s.

Soln: These bit strings may end with 1 OR 0

Ending with 1 : $\underbrace{\text{no two consec. zeros}}\,1$ $\qquad a_{n-1}$

ending with 0 : $\underbrace{\text{no two consec. zeros}}\,1\,0$ $\qquad a_{n-2}$

Total $\longrightarrow a_n = a_{n-1} + a_{n-2}$ model

$a_1 = 2 \quad \{0, 1\}$

$a_2 = 3 \quad \{01, 10, 11\}$

initial conditions

§ Solving Linear Rec. Rel.s

Defn: A linear homogeneous rec. rel. of degree k with constant coefficients is a rec. rel. of the form.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

where $c_1, c_2, \dots c_k$ are real numbers and $c_k \neq 0$

The rec. rel. in the defn. is ~~linear~~ homogeneous because no terms occur that are not multiples of the a_js

The degree is k because a_n is expressed in terms of the previous k terms of the seqn.

Ex/ $P_n = (1.11) P_{n-1}$ linear homogeneous rec. rel. degree $= 1$

$f_n = f_{n-1} + f_{n-2}$ linear " " " deg $= 2$

$a_n = a_{n-5}$ linear " " " deg $= 5$

$a_n = a_{n-1} + a_{n-2}^2$ not linear

$H_n = 2 H_{n-1} + 1$ not homogeneous

$B_n = n B_{n-1}$ does not have constant coeff.

* Linear Homogeneous rec. rels are studied since they often appear in modeling problems AND they can be solved systematically. *

Basic Idea Approach for solving lin. homogeneous rec. rels

look for a solution of the form $a_n = r^n$ where r is const.

So $$\frac{a_n = r^n}{r^{n-k}} = \frac{c_1 r^{n-1} + c_2 r^{n-2} \dots + c_k r^{n-k}}{r^{n-k}}$$

So $$r^k - c_1 r^{k-1} - c_2 r^{k-2} \dots - c_k = 0$$ char. eqn

We find the characteristic roots of char. eqn

Let's focus on degree $= 2$ //

Thm: Let c_1 c_2 real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then $\{a_n\}$ is a soln. of the rec. rel. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff.
$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 and α_2 are constants

Ex/ what is the solution of the rec. rel. $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Soln: Char. eqn : $r^2 - r - 2 = 0$
Roots $r_1 = 2$ and $r_2 = -1$
Hence $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$ Find α_1 α_2 with using init. cond.s

$$a_0 = 2 = \alpha_1 + \alpha_2$$
$$a_1 = 7 = 2\alpha_1 - \alpha_2$$
$$9 = 3\alpha_1 \qquad \alpha_1 = 3 \qquad \text{so} \qquad \alpha_2 = -1$$

So $\boxed{a_n = 3\, 2^n - (-1)^n}$ //

Ex/ Find an explicit formula for the Fib. Nums $f_n = f_{n-1} + f_{n-2}$
$f_0 = 0$ $f_1 = 1$

Soln: Char eqn : $r^2 - r - 1 = 0$
char. roots : $r_1 = \frac{1+\sqrt{5}}{2}$, $r_2 = \frac{1-\sqrt{5}}{2}$

Hence $a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

$$\left.\begin{aligned} f_0 &= 0 = \alpha_1 + \alpha_2 \\ f_1 &= 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \end{aligned}\right\} \quad \alpha_1 = \frac{1}{\sqrt{5}} \qquad \alpha_2 = \frac{-1}{\sqrt{5}}$$

So $f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$ //

Thm-2: Let c_1 c_2 be real nums with $c_2 \neq 0$. Suppose $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sqce $\{a_n\}$ is a solution of the rec. rel. $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ where α_1, α_2 are constants.

Ex/ what is the soln. of rec rel. $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 1$, $a_1 = 6$

Soln: char. eqn: $r^2 - 6r + 9 = 0$ root $r = 3$

Soln. format: $a_n = \alpha_1 3^n + \alpha_2 n 3^n$

$$\left.\begin{aligned} a_0 &= 1 = \alpha_1 + 0 \\ a_1 &= 6 = \alpha_1 3 + \alpha_2 . 3 \end{aligned}\right\} \alpha_1 = 1 \quad \alpha_2 = 1$$

So $a_n = 3^n + n 3^n$

Thm-3: Let $c_1, c_2, \ldots c_k$ be real nums. Suppose that $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots r_k$. Then a sqce $\{a_n\}$ is a soln. of the rec. rel. $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ iff. $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$ where $\alpha_1, \ldots \alpha_k$ are constants

Ex/ Find soln of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ $a_0 = 2$ $a_1 = 5$ $a_2 = 15$

Soln: char eqn: $r^3 - 6r^2 + 11r - 6 = 0$

roots: $r_1 = 1$ $r_2 = 2$ $r_3 = 3$

Soln format: $a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n$

$$\left.\begin{aligned} a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3 \\ a_1 &= 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3 \\ a_2 &= 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3 \end{aligned}\right\} \begin{aligned} &\alpha_1 = 1 \quad \alpha_2 = -1 \quad \alpha_3 = 2 \\ &\text{So } a_n = 1 - 2^n + 2.3^n \end{aligned}$$

Thm-4: Let $c_1\ c_2 \dots c_k$ be real numbers. Suppose that the char. eqn $r^k - c_1 r^{k-1} - \dots - c_k = 0$ has t distinct roots $r_1, r_2, \dots r_t$ with multiplicities $m_1\ m_2, \dots m_t$ resp. so that $m_1 + m_2 + \dots + m_t = k$, Then a sqce $\{a_n\}$ is a soln. of the rec. rel $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ iff.

$$\begin{aligned} a_n = &(\alpha_{1,0} + \alpha_{1,1} n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + \\ &(\alpha_{2,0} + \alpha_{2,1} n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \dots + \\ &(\alpha_{t,0} + \alpha_{t,1} n + \dots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

where α_{ij} are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$

Ex/ Suppose that the roots of the char. eqn. of a lin. homogeneous rec. rel are 2, 2, 2, 5, 5, and 9. what is the form of the gen. soln.

Soln: $(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)\, 2^n + (\alpha_{2,0} + \alpha_{2,1} n)\, 5^n + \alpha_{3,0}\, 9^n$

Ex/ Find the soln of the rec. rel. $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ with $a_0 = 1 \quad a_1 = -2 \quad a_2 = -1$

Soln: char. eqn: $r^3 + 3r^2 + 3r + 1 = 0 = (r+1)^3$ so $r = -1$

$$a_n = \underline{\alpha_{1,0}} (-1)^n + \underline{\alpha_{1,1}}\, n (-1)^n + \underline{\alpha_{1,2}}\, n^2 (-1)^n$$

$$\left.\begin{aligned} a_0 &= 1 = \alpha_{1,0} + 0 + 0 \\ a_1 &= -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2} \\ a_2 &= -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} \end{aligned}\right\} \quad \begin{aligned} &\alpha_{1,0} = 1 \\ &\alpha_{1,1} = 3 \qquad \alpha_{1,2} = -2 \end{aligned}$$

So $a_n = (-1)^n (1 + 3n - 2n^2)$

⊛ Linear NON-homogeneous Rec. Rel. with Const. Coeffs

Rec. Rel. Form: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$

↳ associated homogeneous rec. rel.

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

Thm-5: If $\{a_n^{(p)}\}$ is a particular soln. of the NON-homogeneous lin. rec. rel. with constant coeffs. $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k} + F(n)$ then every soln. is of the form $\{a_n^{(p)} + a_n^{(h)}\}$ where $\{a_n^{(h)}\}$ is a soln. of the associated homogeneous rec. rel.

Thm-6: Suppose that $\{a_n\}$ satisfies the lin. NON-homogeneous rec. rel $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k} + F(n)$ where $c_1, \dots c_k$ are real numbers and $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_1 n + b_0) s^n$ where $b_0, \dots b_t$ and s are real nums

case 1: (when s is not a root of char. eqn)

$a_n^{(p)}$ is a particular soln. of the form $(p_t n^t + p_{t-1} n^{t-1} + \dots + p_1 n + p_0) s^n$

case 2: (when s is a root of assoc char. eqn with multiplicity m)

$a_n^{(p)}$ is a particular soln of the form $\underline{\underline{n^m}} (p_t n^t + \dots + p_1 n + p_0) s^n$

Ex/ What form does a particular soln. of lin. non-homogeneous rec. rel. $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$

a) $F(n) = 3^n$ b) $F(n) = n 3^n$ c) $F(n) = n^2 \underline{\underline{2}}^n$ d) $F(n) = (n^2+1) 3^n$

Soln: char Eqn: $r^2 - 6r + 9 = 0 = (r-3)^2$ $r = 3$ with multiplicity two

a) $a_n^{(p)}$ format: $p_0 n^2 3^n$ b) $a_n^{(p)}$ format: $n^2 (p_1 n + p_0) 3^n$

c) $a_n^{(p)}$ format: $(p_2 n^2 + p_1 n + p_0) 2^n$ d) $a_n^{(p)}$ format: $n^2 (p_2 n^2 + p_1 n + p_0) 3^n$

! Care must be taken when $s = 1$! Think about it

Ex/ $a_n = \sum_{k=1}^{n} k = a_{n-1} + n$ $\quad a_1 = 1$

$\underbrace{\qquad\qquad\qquad\qquad}$ linear non-homogeneous rec. rel.

Soln: Assoc. lin. homogeneous rec. rel. $a_n = a_{n-1}$

$r - 1 = 0 \Rightarrow r = 1 \quad a_n^{(h)} = c.(1)^n = c$ //

By Thm-6 $F(n) = n = n.(1)^n$

particular soln. form $a_n^{(p)} = n.(p_1 n + p_0) = p_1 n^2 + p_0 n$

Insert into rec. rel.

$p_1 n^2 + p_0 n = p_1(n-1)^2 + p_0(n-1) + n$

So $n(2p_1 - 1) + (p_0 - p_1) = 0$

So $2p_1 - 1 = 0$ and $p_0 - p_1 = 0$

So $p_0 = p_1 = 1/2$

So $a_n^{(p)} = \frac{n^2}{2} + \frac{n}{2} = \frac{n(n+1)}{2}$

So General Soln $= a_n^{(h)} + a_n^{(p)} = \underbrace{c + \frac{n(n+1)}{2} \quad \text{AND } a_1 = 1}_{\text{so } c=0}$

So General Soln $= \frac{n(n+1)}{2}$ //

Divide and Conquer Algorithms and Rec. Rels

- Binary Search - Merge Sort

Divide the problem into smaller size and solve smaller parts and merge the smaller solved instances to get the original size prob soln.

Rec. Rel. Form: $f(n) = a f(n/b) + g(n)$

Ex/ Binary Search search seqce of size n (sorted)
Problem size reduces to half at each iteration

$$f(n) = f(n/2) + \underbrace{2}_{g(n)}$$

Ex/ Merge Sort divide into two and sort them and merge

$$M(n) = 2\,M(n/2) + n$$

Thm-1b: Let f be an increasing func. that satisfies the rec. rel.

$$f(n) = a\,f(n/b) + c \qquad b \in \mathbb{N} \quad b > 1 \quad c \text{ pos. real num}$$

Then $f(n)$ is $\begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$

Furthermore, when $n = b^{k \to \text{pos. integer}}$ $f(n) = C_1\, n^{\log_b a} + C_2$
where $C_1 = f(1) + c/a-1$ $C_2 = -c/a-1$

Ex/ Let $f(n) = 5 f(n/2) + 3$ $f(1) = 7$ Find. $f(2^{k \to \text{pos. int}})$

Soln: $f(n) = \left(f(1) + 3/4\right) \cdot n^{\log_2 5} + \left(-3/4\right)$

$f(2^k) = \;'' \; (2^k)^{\log_2 5} + \left(-3/4\right)$

$$= \frac{31}{4}\, 5^k - 3/4$$

Thm-2b (Master Theorem): Let f be an increasing func. that satisfies the recurrence relation $f(n) = a\,f(n/b) + c\,n^d$, whenever $n = b^k$ where k is a pos. integer, $a \geq 1$, b is an integer $b > 1$ and $c, d \in \mathbb{R}$ c positive and d nonnegative. Then

$$f(n) \text{ is } \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

<u>Defn-1</u>: The <u>generating function</u> for the seqce $a_0, a_1, \ldots a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Ex/ The generating function for the sqce $\{a_k\}$ with

a) $a_k = 3$ b) $a_k = k+1$ c) $a_k = 2^k$

<u>Soln</u>: a) $\sum_{k=0}^{\infty} 3x^k$ b) $\sum_{k=0}^{\infty} (k+1)x^k$ c) $\sum_{k=0}^{\infty} 2^k x^k$

Ex/ The generating function of $1, 1, 1, \ldots$ is $f(x) = 1/(1-x)$

since $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ for $|x| < 1$ //

Ex/ The gen. func. of $1, a, a^2, a^3 \ldots$ is $f(x) = 1/(1-ax)$

since $\frac{1}{1-ax} = 1 + ax + a^2x^2 + a^3x^3 + \cdots$ for $|ax| < 1$

<u>Thm-1c</u>: Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ $g(x) = \sum_{k=0}^{\infty} b_k x^k$ Then

$f(x) + g(x) = \sum_{k=0}^{\infty} (a_k + b_k) x^k$ and $f(x)\,g(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) x^k$

(Thm-1c is valid only for power series that converge in an interval)

<u>Thm-2c</u>: Let x be a real number with $|x| < 1$ and let u be a real num. Then $(1+x)^u = \sum_{k=0}^{\infty} \binom{u}{k} x^k$

Ex/ Solve the rec. rel. $\boxed{a_k = 3a_{k-1}}$ for $k = 1, 2, 3.$ and $a_0 = 2$

<u>Soln</u>: $G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$

$x\,G(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$

$$G(x) - 3x\,G(x) = \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty} \underbrace{(a_k - 3a_{k-1})}_{0} x^k$$

So $G(x) - 3x\,G(x) = (1-3x)\,G(x) = a_0 = 2$

So $G(x) = 2/(1-3x)$

Remember the Identity $\underbrace{1/1-ax = \sum_{k=0}^{\infty} a^k x^k}$

So $G(x) = 2\dfrac{1}{1-3x} = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} \underbrace{2.3^k}_{a_k} x^k$

So $a_k = 2.3^k$ //

Let's check $a_0 = 2$ $\quad a_1 = 3.a_0 = 6$ $\quad a_2 = 3.a_1 = 18$

Ex/ Solve the rec. rel. $a_n = 8a_{n-1} \overbrace{+10^{n-1}}^{F(n)}$ $\quad \cancel{a_1 = 9}$ $\quad a_0 = 1$

Soln: Consider $\underline{a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n}$

Consider the Gen. func. $G(x) = \sum_{n=0}^{\infty} a_n x^n = \underbrace{a_0}_{1} + \underbrace{\sum_{n=1} a_n x^n}$

$$\begin{aligned}
\text{So } G(x) - 1 &= \sum_{n=1}^{\infty} a_n x^n \\
&= \sum_{n=1}^{\infty} 8a_{n-1}x^n + 10^{n-1}x^n \\
&= 8x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x\sum_{n=1}^{\infty} 10^{n-1}x^{n-1} \\
&= 8x\sum_{n=0}^{\infty} a_n x^n + x\sum_{n=0}^{\infty} 10^n x^n \\
&= 8x\,G(x) + x\frac{1}{1-10x}
\end{aligned}$$

So

$G(x) - 1 = 8x\,G(x) + x\dfrac{1}{1-10x} \Rightarrow G(x) = \dfrac{1-9x}{(1-8x)(1-10x)}$

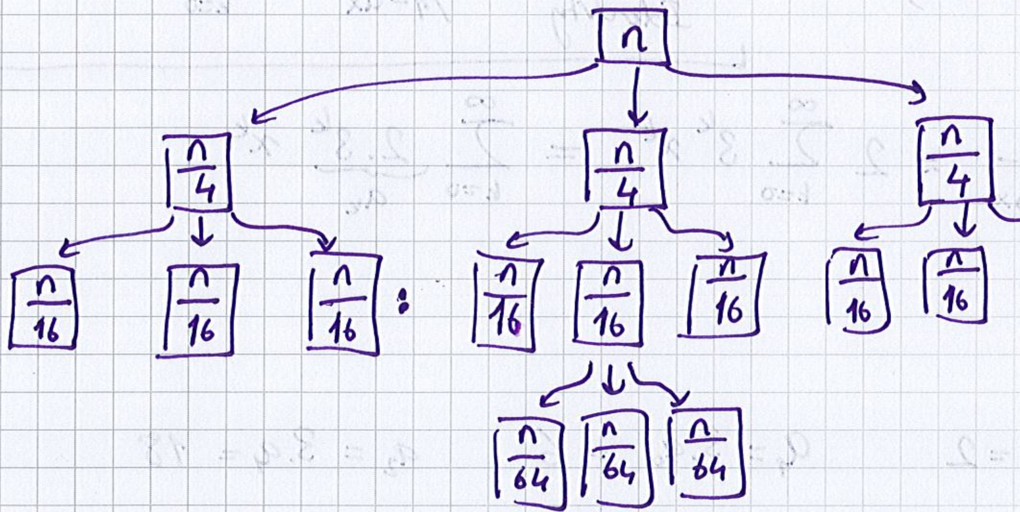
$$\begin{aligned}
\text{So } G(x) &= \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right) \\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) \\
&= \sum_{n=0}^{\infty} \underbrace{\tfrac{1}{2}(8^n + 10^n)}_{a_n} x^n
\end{aligned}$$

Hence $a_n = \frac{1}{2}(8^n + 10^n)$

(*) ~~Substitution~~ Method Recursion Trees

Consider $T(n) = 3\,T(n/4) + n^2$

$T\left(\frac{n}{4}\right) = 3\,T\left(\frac{n}{16}\right) + \frac{n^2}{16}$



$: \rightarrow n^2$ steps

$: \rightarrow \frac{3}{16} n^2$

$: \rightarrow \frac{9}{16^2} n^2$

$$\underbrace{n^2 + \frac{3}{16} n^2 + \frac{9}{16^2} n^2 + \left(\frac{3}{16}\right)^3 n^2 + \cdots}_{\text{Geometric Series}}$$

$$T(n) = n^2\left(1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \left(\frac{3}{16}\right)^3 + \cdots\right)$$

$\frac{3}{16} < 1$ ✓

$$= n^2\left(\frac{1}{1 - \frac{3}{16}}\right) = n^2\left(\frac{16}{13}\right)$$

(*) Substitution Method

Solve $T(n) = 2\,T(n/2) + 4n$ $\quad T(1) = 4$

$$
\begin{aligned}
T(n) &= 2\,T(n/2) + 4n \\
&= 2\left[2\,T(n/2^2) + 4\,n/2\right] + 4n \\
&= 2^2\,T(n/2^2) + 4n + 4n \\
&= 2^2\left[2\,T(n/2^3) + 4\,n/2^2\right] + 4n + 4n \\
&= 2^3\,T(n/2^3) + 4n + 4n + 4n \\
&= 2^3\left[2\,T(n/2^4) + 4\,n/2^3\right] + 4n + 4n + 4n
\end{aligned}
$$

how many tries to get $T(1)$

$\frac{n}{2^i} = 1$

so $i = \log_2 n$

general $= 2^i\,T(n/2^i) + i(4n)$

so

$$T(n) = 2^{\log_2 n}\,T(1) + \log_2 n\,(4n) = 4n + 4n\log_2 n$$