Visual Group Theory PS

Reference Book: Visual Group Theory by Nathan Carter

CMPE220 PS, Fall23
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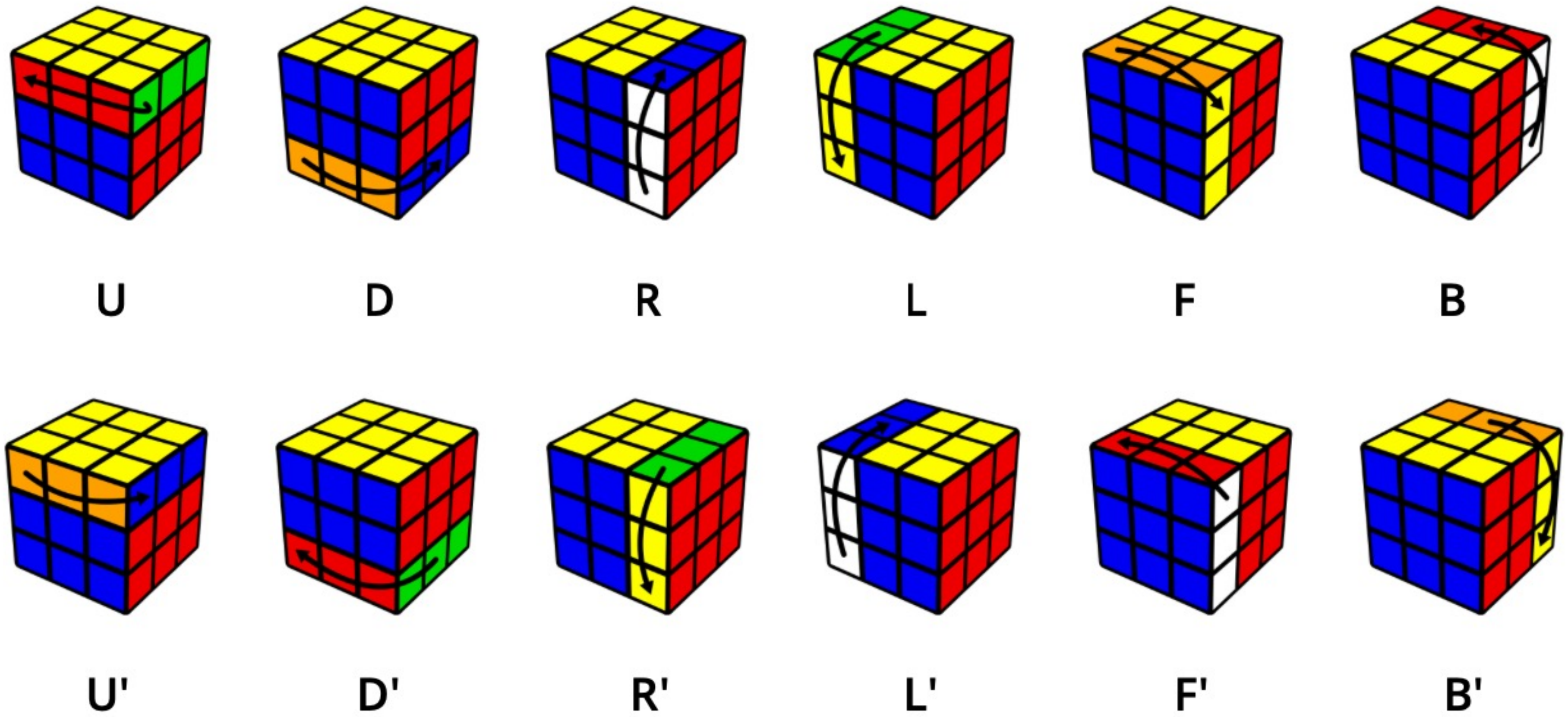
Moves of a Rubiks Cube



Moves of a Rubiks Cube

- Show that movements of a Rubiks cube, as movement concatenation as its operator, form a group.
- Associativity
- Existence of the identity element
- Existence of inverses

Moves of a Rubiks Cube

Generating Sets

- Can you find a finite set of movements such that every movement can be written as a concatenation of some elements of this set?
- A set that satisfies this property is called a generating set of this group.
- Every element of this generating set is called a **generator**.
- If S is a generating set, we denote it by G=<S>.

Moves of a Rubiks Cube

- **Generating Set**
- Let G be a group, and S a subset of G. We say that S generates G (and that S is a set of generators for G) if every element of G can be expressed as a product of elements of S and their inverses.
- Most of the time, point of interest is the minimal generating set.
- If a generating set S is known, a group can be presented in the following form where R is a set relations among the generators

$$\langle S \mid R \rangle.$$

- This form is called group presentation.

Examples

- Exercise 1.3. Imagine that you have five marbles in your left pocket. Consider two actions, moving a marble from your left pocket into your right pocket and moving a marble from your right pocket into your left pocket. Is this a group?
- You have learned in the lectures that if <G, *> is a group, set G is closed under the operation *. Which means if we take an element a from G (Does such an element always exist?), a*a*a*…*a will also be in G. Does that mean every group has infinitely many elements? If not, disprove it with a counter example.
- Give a minimal generating set for multiplicative group of integers modulo 11.

How to Visualize Groups

- Take this two operations on a rectangle, flipping horizontally and vertically.

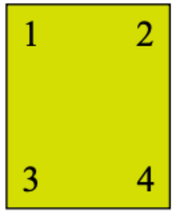


Figure 2.2. A rectangle with its corners numbered

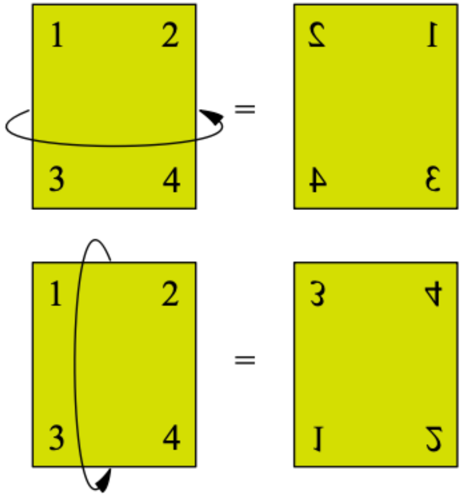


Figure 2.3. The top arrow illustrates a horizontal flip; the upper right rectangle shows the result of such a flip. The bottom arrow illustrates a vertical flip; the lower right rectangle shows the result of such a flip.

How to Visualize Groups

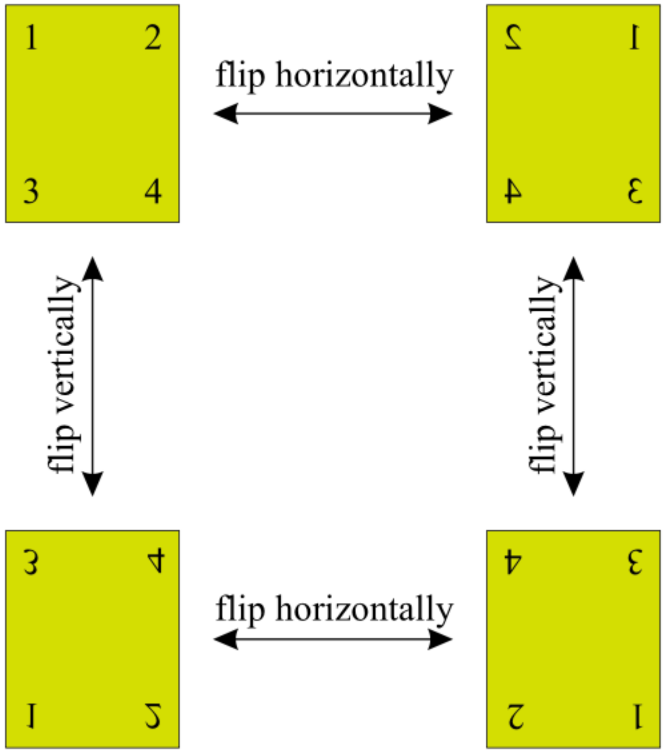


Figure 2.7. Full map of the configurations of the rectangle puzzle

Cayley Graphs

- **Cayley Graphs**
- Every finitely generated group can be represented as a graph, where nodes represent group elements and edges represent generators.
- Cayley graphs clearly show every path between every element.

Cayley Graphs

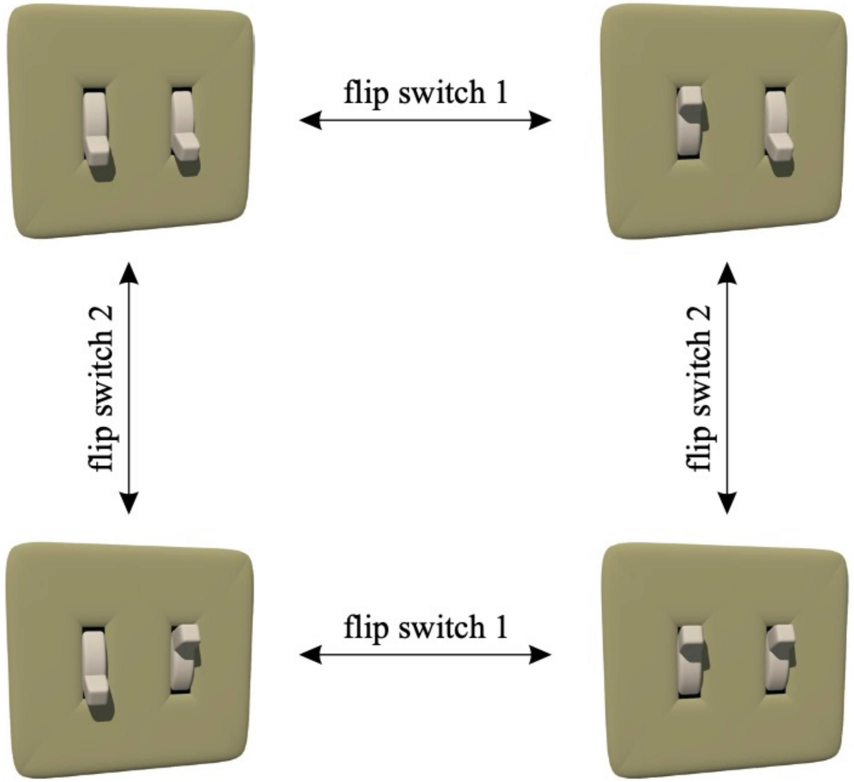


Figure 2.8. Full map of the configurations of the two-light switch group

Cayley Graphs

- Cayley graphs illustrate a group's structure and are helpful for recognizing when two groups share the same structure, essentially being isomorphic.
- Two graphs we presented before are actually isomorphic to Klein-4 group. (a.ka. Klein-vier)

$$\mathrm{V} = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle \,.$$

Cayley Graphs

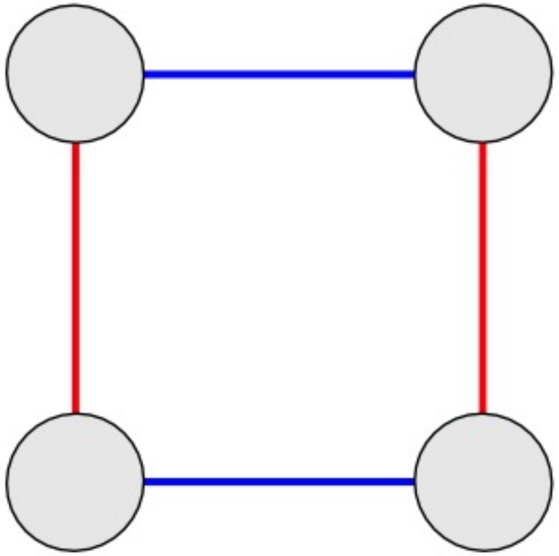


Figure 2.9. Cayley diagram of the Klein 4-group

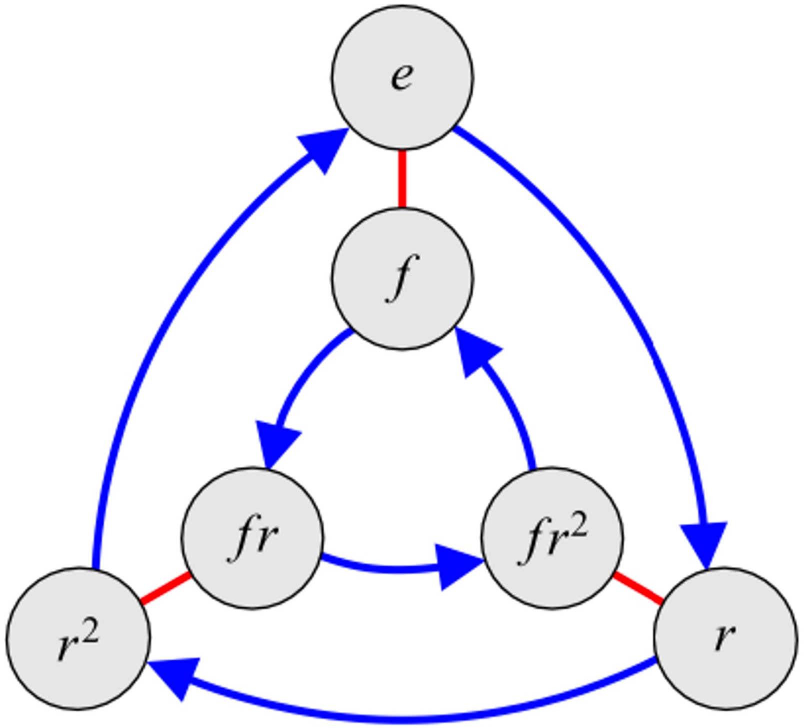
Cayley Graphs

- How to draw Cayley graphs?
- Draw one vertex for every group element, generator or not. (And don't forget the identity!)
- For every generator g, connect vertex a to ag; by a directed edge from a to ag;. Label this edge with the generator.
- Repeat step 2 for every element (i.e. vertex) a in G.

Examples

- Take the symmetry group of a triangle, that is rotations and reflections of a triangle under the operation composition. Show all elements of this group and draw the cayley graph.
- What if instead of a triangle you take a polygon with n sides?

Examples

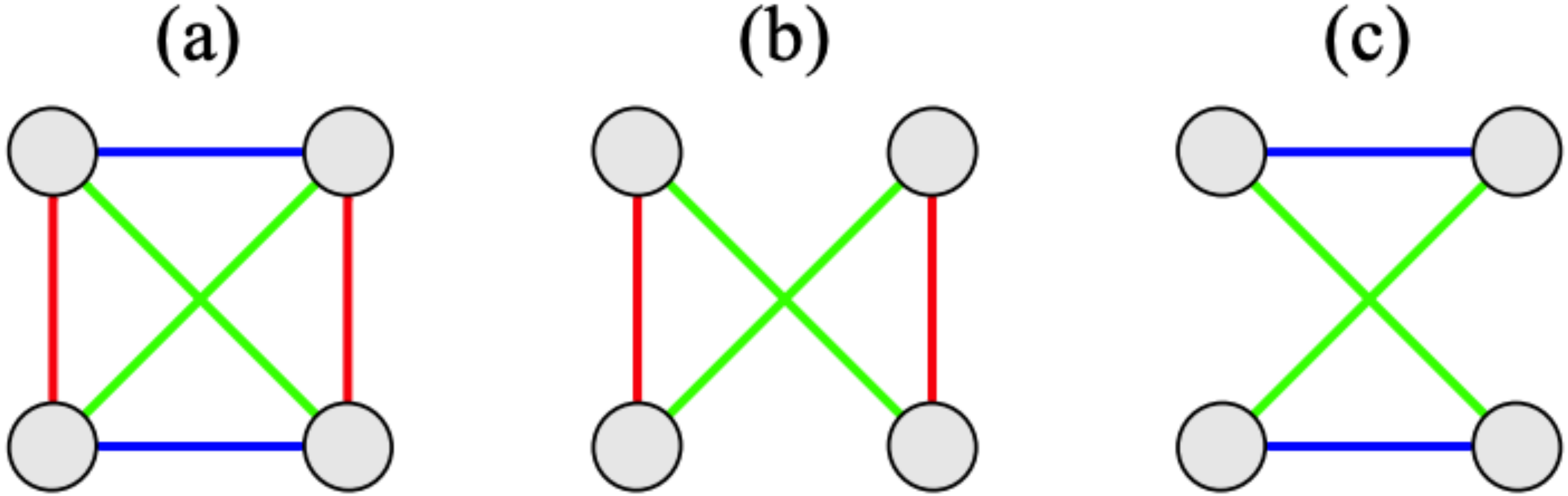


Symmetric group S_3

Examples

- Realize that more than one generating sets can exist for a group. Find two generator sets for Klein-4 and draw their Cayley graphs.

Examples



Cayley graphs of different generating sets for Klein-4

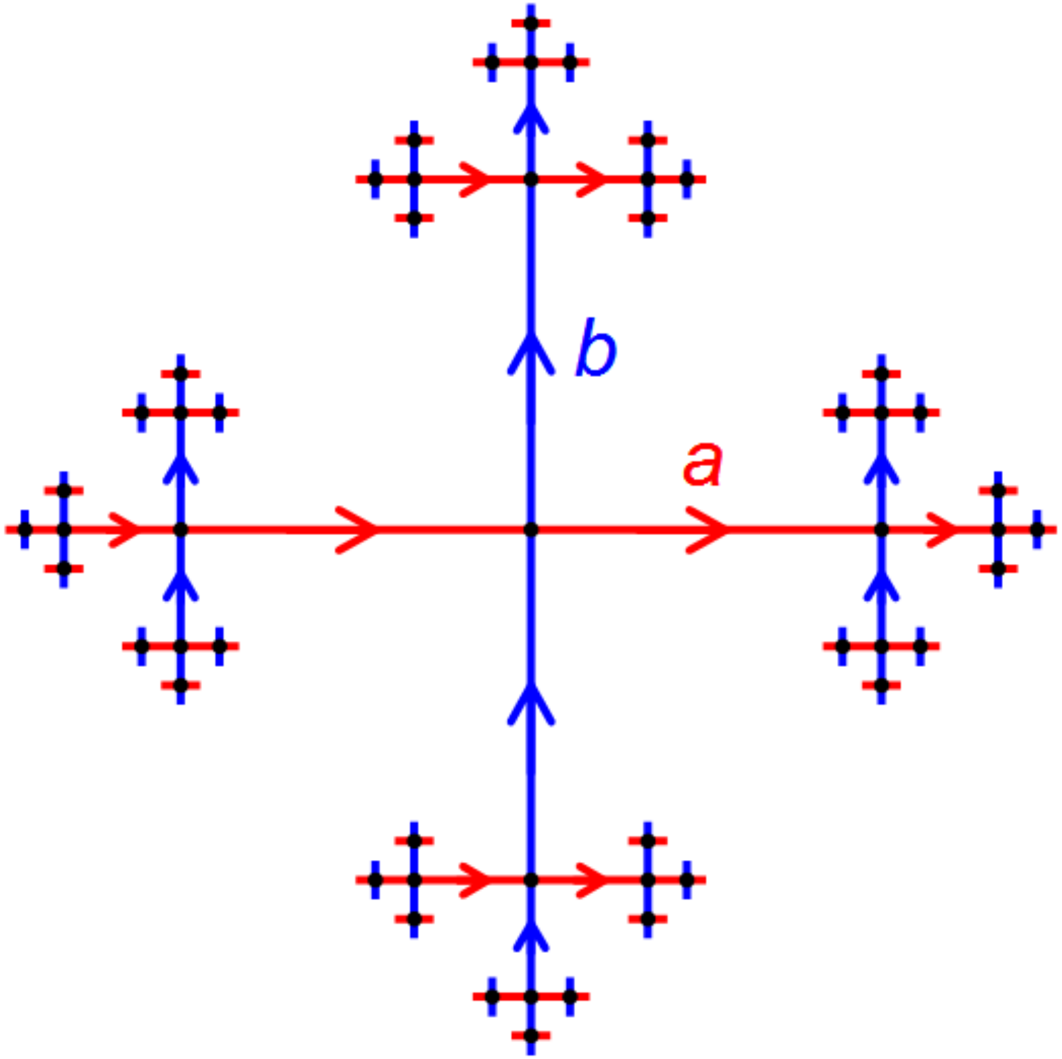
Example

- Exercise 2.3. Can an arrow in a Cayley diagram ever connect a node to itself?

Example

- A group is called a free group if no relation exists between its group generators other than the relationship between an element and its inverse required as one of the defining properties of a group.
- Take the free group F = <a,b>, which is generated by two different generators. Can you find a way to represent its Cayley graph?

Examples



How to Visualize Groups

Group Explorer 3.0

GE 3.6.1

Name	Order	Definition	Cayley diagram	Multiplication table	Object of symmetry	Cycle graph
$\mathbb{Z}_1$	1	$\{e\}$			none	
$\mathbb{Z}_2$	2	$\langle a : a^2 = 1\rangle$				
$\mathbb{Z}_3$	3	$\langle a : a^3 = 1\rangle$				
V_4	4	$\langle a, b : a^2 = b^2 = 1, ab = ba\rangle$				
$\mathbb{Z}_4$	4	$\langle a : a^4 = 1\rangle$				
$\mathbb{Z}_5$	5	$\langle a : a^5 = 1\rangle$				
S_3	6	$\langle a, b : a^3 = b^2 = 1, bab = a^{-1}\rangle$				
$\mathbb{Z}_6$	6	$\langle a : a^6 = 1\rangle$				
$\mathbb{Z}_7$	7	$\langle a : a^7 = 1\rangle$				
D_4	8	$\langle a, b : a^2 = b^4 = 1, bab = a^{-1}\rangle$				

https://nathancarter.github.io/group-explorer/GroupExplorer.html

Multiplication Tables

- Another way to represent groups is the multiplication table of its elements.

	e	a	a^2
e	e	a	a^2
a	a	a^2	e
a^2	a^2	e	a

Cyclic group C_3 (or $\mathbb{Z}_3$)

	e	r	r^2	f	fr^2	fr
e	e	r	r^2	f	fr^2	fr
r	r	r^2	e	fr^2	fr	f
r^2	r^2	e	r	fr	f	fr^2
f	f	fr	fr^2	e	r^2	r
fr^2	fr^2	f	fr	r	e	r^2
fr	fr	fr^2	f	r^2	r	e

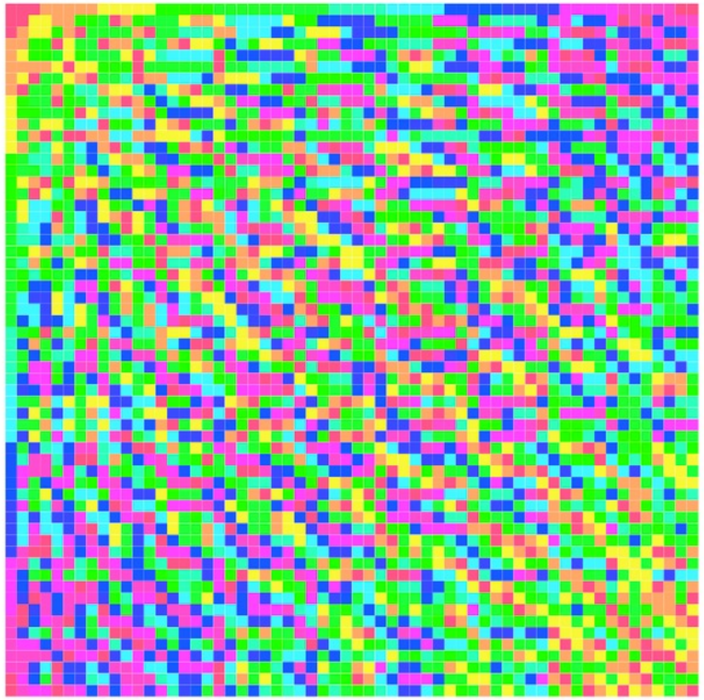
Symmetric group S_3

Multiplication Tables

- Multiplication tables of higher orders can be quite complex.
- Generally, using multiplication tables is not a good way to understand group structure.

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
a	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p
b	b	c	d	e	f	g	h	a	j	k	l	m	n	o	p	i
c	c	d	e	f	g	h	a	b	k	l	m	n	o	p	i	j
d	d	e	f	g	h	a	b	c	l	m	n	o	p	i	j	k
e	e	f	g	h	a	b	c	d	m	n	o	p	i	j	k	l
f	f	g	h	a	b	c	d	e	n	o	p	i	j	k	l	m
g	g	h	a	b	c	d	e	f	o	p	i	j	k	l	m	n
h	h	a	b	c	d	e	f	g	p	i	j	k	l	m	n	o
i	i	l	o	j	m	p	k	n	a	d	g	b	e	h	c	f
j	j	m	p	k	n	i	l	o	b	e	h	c	f	a	d	g
k	k	n	i	l	o	j	m	p	c	f	a	d	g	b	e	h
l	l	o	j	m	p	k	n	i	d	g	b	e	h	c	f	a
m	m	p	k	n	i	l	o	j	e	h	c	f	a	d	g	b
n	n	i	l	o	j	m	p	k	f	a	d	g	b	e	h	c
o	o	j	m	p	k	n	i	l	g	b	e	h	c	f	a	d
p	p	k	n	i	l	o	j	m	h	c	f	a	d	g	b	e

Quasihedral group with 16 elements



Alternating group A_5

Examples

Exercise 4.10. Consider the following multiplication table that displays a binary operation.

	e	A	B
e	e	A	B
A	A	e	e
B	B	e	e

(a) Explain succinctly why the binary operation is not associative. Can you write your answer as one equation?

(b) Does the operation have inverses?

Examples

- Exercise 4.17. Explain why a Cayley diagram must be connected. That is, why must there be a path from every node to every other node?

Examples

- Exercise 4.19. Complete each of the following multiplication tables so that it depicts a group. There is only one way to do so, if we require that 0 be the identity element in each table. Then search Group Explorer's group library to determine the names for the groups the tables represent. (Fun fact with a hint: I don’t like sudoku!)

(a)

	0	1
0		
1		

(b)

	0	1	2
0			
1			
2			

(d)

	0	1	2	3
0				
1		2		
2				
3				

Different Types of Groups

- Groups that can be generated with a single element are called **cyclic groups.**

Figure 5.1. A six-bladed propeller and an eight-foiled pinwheel, whose symmetry groups are the cyclic groups with six and eight elements, respectively.

Different Types of Groups

- **Abelian groups a.k.a commutative groups**
- Show that every cyclic group is also abelian.
- Exercise 5.38. Prove using algebra that if every element in a group has order 2, then the group is abelian.

Different Types of Groups

- A **dihedral group** is the group of symmetries of a regular polygon, which includes rotations and reflections.

$$\begin{aligned} \mathrm{D}_n &= \left\langle r, s \mid \operatorname{ord}(r) = n, \operatorname{ord}(s) = 2, srs^{-1} = r^{-1} \right\rangle \\ &= \left\langle r, s \mid r^n = s^2 = (sr)^2 = 1 \right\rangle . \end{aligned}$$

- List all elements of the dihedral four, symmetries of a square.

Different Types of Groups

- The **symmetric group** defined over any set is the group whose elements are all the bijections from the set to itself, and whose group operation is the composition of functions. These bijections are called permutations of the set.

- Every permutation can be written as compositions of permutation cycles.

Different Types of Groups

- A permutation cycle is a subset of a permutation whose elements trade places with one another. For example, in the permutation {4,2,1,3}, (143) is a 3-cycle and (2) is a 1-cycle. Here, the notation (143) means that starting from the original ordering {1,2,3,4}, the first element is replaced by the fourth, the fourth by the third, and the third by the first, i.e., 1->4->3->1.

- Write every element of symmetric group S3 with cycle notation.

https://mathworld.wolfram.com/PermutationCycle.html

Exercise

- Take D4, symmetries of a square,
  Show that a^2 commutes with all other elements.

$$D_4 = \langle a, b : a^4 = b^2 = e, ab = ba^{-1} \rangle$$

Important Theorems in Group Theory

- Cayley’s theorem states that every group G is isomorphic to a subgroup of its permutation group.
- Lagrange's theorem states that for any finite group G, the order (number of elements) of every subgroup of G divides the order of G.
- Isomorphism theorems