Homework 2
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Question 1 (25 pts) - (Leaving empty = 5 pts)

Let $f : \mathbb{N} \to 2^{\mathbb{N}}$ be a function such that $\forall n \in \mathbb{N}$.
$f(n) = \{d \mid d \text{ divides } n (d \in \mathbb{N})\}$
Let Δ be an operation on $2^{\mathbb{N}}$ such that $\forall X, Y \in 2^{\mathbb{N}}$
$X \,\Delta\, Y = \{x.y \mid x \in X, y \in Y\}$ = set of multiplications of all possible pairs from set X and Y
Prove or disprove $f(n)\Delta f(m) = f(n.m)$

Answer:

We need to show that $f(n)\Delta f(m) \supseteq f(n.m) \wedge f(n)\Delta f(m) \subseteq f(n.m)$

1) $f(n)\Delta f(m) \supseteq f(n.m)$

Let $d \in f(n.m)$ be arbitrary.
So d divides $n.m$
So $\exists\, a, b \in \mathbb{N} \mid a.b = d \quad a|n \,\wedge\, b|m$
So $a \in f(n)$ and $b \in f(m)$
So $a.b = d \in f(n)\Delta f(m)$

2) $f(n)\Delta f(m) \subseteq f(n.m)$

Let $d \in f(n)\Delta f(m)$ be arbitrary.
So $\exists\, a, b \in \mathbb{N} \mid a.b = d \quad a \in f(n)\ b \in f(m)$
So $a|n$ and $b|m$
So $\exists\, k, z \in \mathbb{N} \mid a.k = n \quad b.z = m$
So $m.n = a.k.b.z$
So $m.n = (a.b).(k.z) = d.(k.z)$
So $d \mid m.n$
So $d \in f(m.n)$

$\therefore f(n)\Delta f(m) = f(n.m)$

Question 2 (25 pts) - (Leaving empty = 5 pts)

Definition

The even-concatenation operation, denoted as A^{**}, is applied to a set A, representing the set of all possible even-length combinations formed by concatenating zero or more elements from A. It includes the empty string and all even-length combinations of elements from A. If $A = \{0, 1\}$, then A^{**} includes strings like "", "00", "01", "10", "11", "0000", "0011", "1100", etc. Notice that A^{**} is an infinite set.

Question 2.1

B is a finite set. Prove or disprove that B^{**} is countable.

Question 2.2

C is a countably infinite set. Prove or disprove that C^{**} is countable. (Hint: You may use unique prime factorization)

Answer 2.1

Let $\mathbb{E}$ be the set of even natural numbers except 0.
Let B_i denote the i-lenght combinations of elements from B.
Because every spot has $card(B) = n \in \mathbb{N}$ possible characters, there are n^i possible combinations for length i.
So $card(B_i) = n^i$
So each B_i is finite.
Then by definition, $B^{**} = \bigcup_{i \in \mathbb{E}} B_i$.
So if $\mathbb{E}$ is countable, we can say B^{**} is countable because, union of countably many finite sets is still countable.
So we only need to show $\mathbb{E}$ is countable.
We can do it by showing a bijection $f : \mathbb{N} \to \mathbb{E}$.
We define for $n \in \mathbb{N}, f(n) = 2n$.
We need to show this is both one-to-one and onto to show it is a bijection.

<u>One-to-one: $f(x_1) = f(x_2) \implies x_1 = x_2$</u>
By definition $f(x_1) = 2x_1$ and $f(x_2) = 2x_2$
So if $f(x_1) = f(x_2)$ then $2x_1 = 2x_2$
So $x_1 = x_2$

<u>Onto: $\forall e \in \mathbb{E}\ \exists n \in \mathbb{N} \mid f(n) = e$</u>
Let $e \in \mathbb{E}$ be arbitrary.
If we choose $n = e/2$, we will have $f(n) = 2n = e$

So this function is one-to-one and onto. It is a bijection.
So $\mathbb{E}$ is countable.
Hence our proof is complete.

Question 3 (25 pts) - (Leaving empty = 5 pts)

Definition

We will define a new number system, called **birkanians** and denote it with $\mathbb{B}$. Every birkanian number x is constructed with two subsets of birkanian numbers, X_L and X_R such that $x = <X_L|X_R>$. More precisely,

$$\mathbb{B} = \{<X_L|X_R>|X_L, X_R \subseteq \mathbb{B} \land \forall x_L \in X_L, \forall x_R \in X_R \neg(x_R \leq_B x_L)\}$$

where that order relation $\leq_B$ is defined as follows: $x = <X_L|X_R> \in \mathbb{B}$ and $y = <Y_L|Y_R> \in \mathbb{B}$

$$x \leq_B y \iff \forall x_L \in X_L \neg(y \leq_B x_L) \land \forall y_R \in Y_R \neg(y_R \leq_B x)$$

Notice that this definition is a recursive definition. But when did it all begin? How was the first birkanian born?

In the beginning, there was only emptiness, $\emptyset$. From this void emerged the first birkanian, born from nothingness, $<\emptyset|\emptyset>$. Following this, all other birkanians came to be, formed by those who came before them. This marks the completion of the Genesis of Birkanians.

Question 3.1

Show that $<\emptyset|\emptyset>$ is a birkanian number. We will denote this number with 0_B, $<\emptyset|\emptyset> = 0_B$.

Question 3.2

Show that $<\{0_B\}|\emptyset>$ and $<\emptyset|\{0_B\}>$ are also birkanian numbers. Using the order relation defined above, find which of these statements are TRUE. Note that 0_B is defined as $<\emptyset|\emptyset>$ above.

- $<\{0_B\}|\emptyset> \ \leq \ <\emptyset|\{0_B\}>$ (TRUE/FALSE)
- $<\emptyset|\{0_B\}> \ \leq \ <\{0_B\}|\emptyset>$ (TRUE/FALSE)

Answer 3.1

We will show that $<\emptyset|\emptyset>$ is a birkanian number by showing that it satisfies the definition.

- $\underline{X_L, X_R \subseteq \mathbb{B}:}$
  $X_L = \emptyset$, $X_R = \emptyset$ and $\emptyset$ is the subset of all sets. So they are also the subset of $\mathbb{B}$.
- $\underline{\forall x_L \in X_L, \forall x_R \in X_R \neg(x_R \leq_B x_L)}$
  Since $X_L = X_R = \emptyset$, this order relation holds for all x_R, x_L. In other words, there exists no x_R, x_L such that the condition does not hold.

$\therefore <\emptyset|\emptyset>$ is a birkanian number.

Answer 3.2

We will show that $<\{0_B\}|\emptyset>$ and $<\emptyset|\{0_B\}>$ are also birkanian numbers.
First we will show that $<\{0_B\}|\emptyset>$ is a birkanian number.

- $\underline{X_L, X_R \subseteq \mathbb{B}:}$
  $X_L = \{0_B\}$, $X_R = \emptyset$
  $\{0_B\} \subseteq \mathbb{B}$ because $0_B \in \mathbb{B}$
  $\emptyset \subseteq \mathbb{B}$ because empty set is the subset of all sets.
  So $X_L, X_R \subseteq \mathbb{B}$

- $\forall x_L \in X_L, \forall x_R \in X_R \neg(x_R \leq_B x_L)$
  Since $X_R = \emptyset$, this order relation holds for all x_R, x_L. In other words, there exists no x_R, x_L such that the order relation does not hold.

$\therefore$ $<\{0_B\}|\emptyset>$ is a birkanian number.

By using symmetry arguments, $<\emptyset|\{0_B\}>$ is also a birkanian number.

$<\{0_B\}|\emptyset> \ \leq_B \ <\emptyset|\{0_B\}>$ (TRUE/FALSE)
To show that this statement is false, we will put them into the definition of the order relation and see that the conditions don't hold.

$<\{0_B\}|\emptyset> \ \leq \ <\emptyset|\{0_B\}> \iff \forall a \in \{0_B\} \neg(<\emptyset|\{0_B\}> \leq_B a) \wedge \forall b \in \{0_B\} \neg(b \leq_B <\{0_B\}|\emptyset>)$

- $\forall a \in \{0_B\} \neg(<\emptyset|\{0_B\}> \leq_B a)$

  $\{0_B\}$ has only one element, 0_B.
  So we need to check if $(<\emptyset|\{0_B\}> \leq_B O_B)$
  $(<\emptyset|\{0_B\}> \leq_B O_B) \iff \forall a \in \{\emptyset\} \neg(O_b \leq_B a) \wedge \forall b \in \emptyset \neg(b \leq_B <\emptyset|\{0_B\}>)$
  Since X_L and Y_R are $\emptyset$ in this case, there exists no $a \in X_L$, $b \in Y_R$ that does not satisfy this condition.
  So $(<\emptyset|\{0_B\}> \leq_B O_B)$ is true.
  So its negation is false.

- $\forall b \in \{0_B\} \neg(b \leq_B <\{0_B\}|\emptyset>)$

  $\{0_B\}$ has only one element, 0_B.
  So we need to check if $(O_B \leq_B <\{0_B\}|\emptyset>)$
  $(O_B \leq_B <\{0_B\}|\emptyset>) \iff \forall a \in \emptyset \ \neg(<\{0_B\}|\emptyset> \leq_B a) \wedge \forall b \in \emptyset \neg(b \leq_B O_B)$
  Since X_L and Y_R are $\emptyset$ in this case, there exists no $a \in X_L$, $b \in Y_R$ that does not satisfy this condition.
  So $(O_B \leq_B <\{0_B\}|\emptyset>)$ is true.
  So its negation is false.

$\therefore$ The statement is FALSE.

$<\emptyset|\{0_B\}> \ \leq_B \ <\{0_B\}|\emptyset>$ (TRUE/FALSE)
To show that this statement is true, we will put them into the definition of the order relation and see that the conditions hold.

$<\emptyset|\{0_B\}> \ \leq_B \ <\{0_B\}|\emptyset> \iff \forall a \in \emptyset \neg(<\{0_B\}|\emptyset> \leq_B a) \wedge \forall b \in \emptyset \neg(b \leq_B <\emptyset|\{0_B\}>)$

Since X_L and Y_R are $\emptyset$ in this case, there exists no $a \in X_L$, $b \in Y_R$ that does not satisfy this condition.

$\therefore$ The statement is TRUE.

Question 4 (25 pts) - (Leaving empty = 5 pts)

Everyone in Arstotzka lives with these principles:

- Friend of my friend is my friend.
- Enemy of my enemy is my friend.
- Everyone is my friend or my enemy.
- Who sees me as a friend is my friend & who sees me as an enemy is my enemy.

As an officer of the Ministry of Information, your task is to find answers to these questions:

Question 4.1

In which cases the number of friend pairs can be equal to the number of enemy pairs? Come up with a general approach. (or you will be fired!)

Question 4.2

Can everyone's number of friends be equal to number of his enemies?

Glory to Arstotzka

Let A be the set of people living in Arstotzka.
$R = \{(x, y) \in A \times A \mid$ x is a friend of y $\}$
The properties listed above tell us respectively that:

- R is transitive,
- $\forall x, y, z \in X \ (x \not R\, y) \wedge (y \not R\, z) \implies x \, R \, y$,
- This is already an innate property of relations,
- R is symmetric.

<u>Claim:</u> There are only 2 friend groups in Arstotzka. i.e. There exists two disjunct subsets of A such that their unions give A and being in the same subset means being friends. So there cannot be more than 2 friend groups.
<u>Proof:</u> Let's assume for contradiction there are more than 2 friend groups. Let X, Y, Z be three arbitrary friend groups. Let $x \in X, y \in Y, z \in Z$ be arbitrary people. Since x and y are not in the same friend group, they are not friends. Also since y and z are not in the same friend group, they are not friends. By the second property of our relation, we deduce that x and z are friends. However, x and z are not in the same subset, so they are not friends. Hence, our contradiction tells us that there are only 2 friend groups. ■

Answer 4.1

Let there be $x \in \mathbb{N}$ people in the first friend group denoted by X and $y \in \mathbb{N}$ people in the second friend group denoted by Y.

So the number of friend pairs in X and Y is equal to $\binom{x}{2}$ and $\binom{y}{2}$.
And the number of enemy pairs between X and Y is equal to $x.y$.

So we want:

$$\binom{x}{2} + \binom{y}{2} = x.y$$

So:

$$a.(a-1) + b.(b-1) = 2.a.b$$

This yields true when:

$$a + b = (a - b)^2$$

Answer 4.2

Let there be $x \in \mathbb{N}$ people in the first friend group denoted by X and $y \in \mathbb{N}$ people in the second friend group denoted by Y.

Let the proposition in Q4.2 be true for a contradiction.
Let $x_1 \in X$ and $y_1 \in Y$ be arbitrary.
So x_1 has $x - 1$ friends.
So x_1 has $x - 1$ enemies.
So there are $x - 1$ people in Y.
So y_1 has $x - 2$ friends.
So y_1 has $x - 2$ enemies.
y_1 has x enemies because there are x people in X.
So $x - 2 = x$.
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So that is not possible.