Derivation of Update Rules for Kalman-Based PLL

In Figure 1 we can see a factor graph representation of a Kalman-based phase-locked loop. The matrices A, C and the observed measurements y_k are given as follows

$$
\begin{aligned}
&\mathbf{A} = R(\omega_0) = \begin{bmatrix} \cos{(\omega_0)} & -\sin{(\omega_0} \\ \sin{(\omega_0} & \cos{(\omega_0} \end{bmatrix}, \\
&\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
&Y_k = \cos(\omega_0 k + \phi) + Z_k, \qquad Z_k \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma^2\right).
\end{aligned}
$$

Since Z_k is a Gaussian random variable it follows that all random variables in the factor graph are Gaussian. The states X_k can therefore be described by its mean vector and covariance matrix.

In every step k, a new sample y_k is observed. Together with this sample and the previous state X_{k-1}, using Gaussian message passing, we can calculate the next state estimate X_k.

We begin the message passing algorithm by computing the message given by the observed samples y_k [1]

$$\overrightarrow{m}_{Z_k} = 0 \qquad \overrightarrow{V}_{Z_k} = \sigma^2 \tag{1}$$

$$\overleftarrow{m}_{Y_k} = y_k \qquad \overleftarrow{V}_{Y_k} = 0. \tag{2}$$

In a next step, we add the noise in Eq. (1) to the observed sample in Eq. (2) to get the message at Y_k

$$\overleftarrow{m}_{Y_k} = y_k \qquad \overleftarrow{V}_{Y_k} = \sigma^2. \tag{3}$$

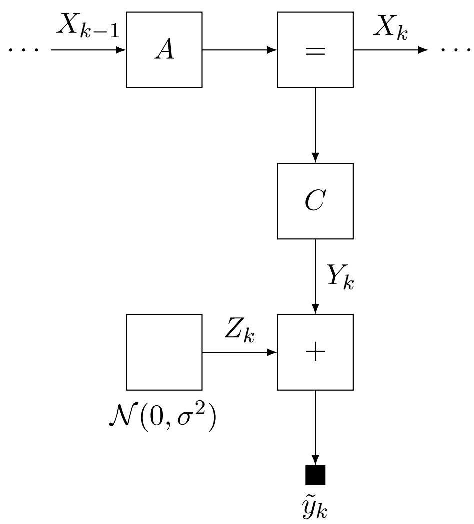

Figure 1: Factor graph representation of a Kalman filter's kth cell.

With these results, we can now compute the messages at the equality constraint

$$\overrightarrow{m}_{X''_k} = \mathbf{A}\overrightarrow{m}_{X_{k-1}} \qquad \overrightarrow{V}_{X''_k} = \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}_T \tag{4}$$

$$\overleftarrow{W}_{X'_k} = \mathbf{C}^T\overleftarrow{W}_{Y_k}\mathbf{C} \qquad \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} = \mathbf{C}^T\overleftarrow{W}_{Y_k}\overleftarrow{m}_{Y_k}, \tag{5}$$

where $\overleftarrow{W}_{X'_k}$ denotes the precision matrix with the following equality

$$\overrightarrow{W}^{-1}_{X_k} = \overrightarrow{V}_{X_k},$$

and $\overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k}$ denotes the weighted mean. The set of equations in (5) can be simplified by using the set of equations in (3)

$$\overleftarrow{W}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} \qquad \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}y_k.$$

Hence we can characterize X_k by its precision matrix and its weighted mean (update rules). Note that $\mathbf{A}$ is an invertible matrix since

$$\begin{aligned}
\overrightarrow{W}_{X_k} &= \overrightarrow{W}_{X''_k} + \overleftarrow{W}_{X'_k} && (6)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} && (7)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \frac{1}{\sigma^2}\mathbf{C}^T\mathbf{C} && (8)\\
&= \mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2} && (9)\\
&= \mathbf{A}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}, && (10)
\end{aligned}$$

$$\begin{aligned}
\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k} &= \overrightarrow{W}_{X''_k}\overrightarrow{m}_{X''_k} + \overrightarrow{W}_{X'_k}\overrightarrow{m}_{X'_k} = && (11)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}y_k && (12)\\
&= \left(\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}y_k && (13)\\
&= \mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k && (14)\\
&= \mathbf{A}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k, && (15)
\end{aligned}$$

where the last step in the particular update equations follows from the fact that $\mathbf{A}$ is the rotation matrix and therefore an orthogonal matrix, i.e., its transpose is equal to its inverse

$$\mathbf{A}^T = \mathbf{A}^{-1}.$$

What is still missing with this implementation, is the ability of our estimator to react to abrupt signal changes, that are not modeled with our state space model. Such changes serve the purpose to physically transmit information with a waveform (e.g., Phase Shift Keying (PSK)). The ability is modeled with a so-called *decay factor* $\gamma \in (0, 1)$. In every iteration step k, $\overrightarrow{W}_{X_{k-1}}$ is scaled with γ which increases the uncertainty of past estimates and therefore puts more emphasis on new measurements. The new update rules are then given by

$$\overrightarrow{W}_{X_k} = \gamma\mathbf{A}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}, \tag{16}$$

and

$$\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k} = \gamma \mathbf{A}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k. \tag{17}$$

In a next step we try to get to an expression for the covariance matrix and the mean vector by using the Matrix Inversion Lemma (18)

$$(\mathbf{B}+\mathbf{DEF})^{-1} = \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}\left(\mathbf{E}^{-1}+\mathbf{FB}^{-1}\mathbf{D}\right)^{-1}\mathbf{FB}^{-1}. \tag{18}$$

Thus we get the following assignments from equation (16)

$$\mathbf{B} = \gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}, \tag{19}$$
$$\mathbf{D} = \mathbf{C}^T, \tag{20}$$
$$\mathbf{E} = \frac{1}{\sigma^2}, \tag{21}$$
$$\mathbf{F} = \mathbf{C}. \tag{22}$$

Using these, the inverse of equation (6), i.e., the covariance matrix can be written as

$$\overrightarrow{V}_{X_k} = \left[\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}\right]^{-1} \tag{23}$$
$$= \left(\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}$$
$$- \left(\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}\mathbf{C}^T\left(\sigma^2 + \mathbf{C}\left(\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}\mathbf{C}^T\right)^{-1}\mathbf{C}\left(\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1} \tag{24}$$
$$= \frac{1}{\gamma}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \frac{1}{\gamma^2}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\left(\sigma^2 + \frac{1}{\gamma}\mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1}\mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T$$
$$= \frac{1}{\gamma}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \frac{1}{\gamma}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\underbrace{\left(\gamma\sigma^2 + \mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1}}_{=:G}\mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T \tag{25}$$
$$= \frac{1}{\gamma}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \frac{1}{\gamma}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T \tag{26}$$

where

$$G := \left(\gamma\sigma^2 + \mathbf{CA}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1}.$$

The mean vector $\overrightarrow{m}_{X_k}$ can be retrieved by multiplying the matrix $\overrightarrow{V}_{X_k}$ with (17)

$$
\begin{aligned}
\overrightarrow{m}_{X_k} &= \overrightarrow{V}_{X_k}\left(\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k}\right) && (27)\\
&= \frac{1}{\gamma}\left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right) && (28)\\
&\quad\cdot\left(\gamma\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k\right)\\
&= \mathbf{A}\underbrace{\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} - \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\underbrace{\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} && (29)\\
&\quad+\frac{1}{\gamma}\left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)\cdot\left(\frac{\mathbf{C}^T}{\sigma^2}y_k\right)\\
&= \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} && (30)\\
&\quad+\frac{1}{\gamma}\left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T - \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\cdot\left(\frac{1}{\sigma^2}\right)\cdot y_k\\
&= \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\right)\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} && (31)\\
&\quad+\frac{1}{\gamma}\left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\underbrace{\left(\mathbf{I} - G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\cdot\left(\frac{1}{\sigma^2}\right)}_{=:\lambda}\cdot y_k.
\end{aligned}
$$

The factor λ can further be simplified to

$$
\begin{aligned}
\lambda &:= \left(\mathbf{I} - G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\cdot\left(\frac{1}{\sigma^2}\right) && (32)\\
&= \left(GG^{-1} - G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\cdot\left(\frac{1}{\sigma^2}\right) && (33)\\
&= G\left(G^{-1} - \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\cdot\left(\frac{1}{\sigma^2}\right) && (34)\\
&= G\left(\gamma\sigma^2 + \underbrace{\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T - \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T}_{=0}\right)\cdot\left(\frac{1}{\sigma^2}\right) && (35)\\
&= G\left(\gamma\sigma^2\right)\cdot\left(\frac{1}{\sigma^2}\right) && (36)\\
&= \gamma G, && (37)
\end{aligned}
$$

where we used the fact, that $G^{-1} = \gamma\sigma^2 + \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T$. So we finally find the following condensed expression for $\overrightarrow{m}_{X_k}$

$$\overrightarrow{m}_{X_k} = \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\right)\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \frac{1}{\gamma}\left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\gamma G\right) y_k \tag{38}$$

$$= \mathbf{A}\overrightarrow{m}_{X_{k-1}} + \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\left(y_k - \mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}}\right). \tag{39}$$

Amplitude and Phase Estimate and Cost Function

The predicted signal estimate given the measurements $y_1, ..., y_{k-1}$ can be written as

$$\tilde{y}_k = \mathbf{CA}\overrightarrow{m}_{X_{k-1}},$$

and the corrected estimate given the measurements $y_1, ..., y_k$ as

$$\hat{y}_k = \mathbf{C}\overrightarrow{m}_{X_{k-1}}.$$

The difference between the corrected and the predicted estimate can be deduced as

$$\begin{aligned}
\hat{y}_k - \tilde{y}_k &= \mathbf{C}\left(\overrightarrow{m}_{X_k} - \mathbf{A}\overrightarrow{m}_{X_{k-1}}\right) && (40)\\
&= \mathbf{C}\left(\overrightarrow{m}_{X_k} - \overrightarrow{m}_{X_k} + \underbrace{\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G}_{=:\mathbf{B}_k}\left(y_k - \mathbf{CA}\overrightarrow{m}_{X_{k-1}}\right)\right)\\
&= \mathbf{CB}_k\left(y_k - \tilde{y}_k\right) && (41)
\end{aligned}$$

Since both the input signal and estimate is known to be sinusoidal the mean vector at time index k can be written as [2]

$$\overrightarrow{m}_{X_k} = \hat{a}_k \begin{bmatrix} \cos\left(\omega_0 k + \hat{\phi}_k\right) \\ \sin\left(\omega_0 k + \hat{\phi}_k\right) \end{bmatrix}$$

and subsequently

$$y_k = a\cos\left(\omega_0 k + \phi\right) + Z_k, \qquad Z_k \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma^2\right), \tag{42}$$
$$\tilde{y}_k = \hat{a}_{k-1}\cos\left(\omega_0 k + \hat{\phi}_{k-1}\right), \tag{43}$$
$$\hat{y}_k = \hat{a}_k\cos\left(\omega_0 k + \hat{\phi}_k\right) \tag{44}$$

Now we consider the scenario where only the phase is tracked and the amplitude is assumed to be $\hat{a}_k = \hat{a}_{k-1} = a_k$. If the phase estimation error is small such that $\left|\hat{\phi}_k - \phi\right| \ll 1$ and $\left|\hat{\phi}_{k-1} - \phi\right| \ll 1$, equation (41) can be reduced to

$$a\left(\cos\left(\omega_0 k + \hat{\phi}_k\right) - \cos\left(\omega_0 k + \hat{\phi}_{k-1}\right)\right) = a\alpha_k\left(\cos\left(\omega_0 k + \hat{\phi}_k\right) - \cos\left(\omega_0 k + \hat{\phi}_{k-1}\right)\right) \tag{45}$$
$$\sin\left(\frac{\hat{\phi}_k - \hat{\phi}_{k-1}}{2}\right) \approx \alpha_k \sin\left(\frac{\phi - \hat{\phi}_{k-1}}{2}\right) \tag{46}$$
$$\hat{\phi}_k - \hat{\phi}_{k-1} \approx \alpha_k\phi - \hat{\phi}_{k-1}. \tag{47}$$

Thus follows from equation (47) that the phase update can be approximated as

$$\hat{\phi}_k \approx \alpha_k\phi + \left(1 - \alpha_k\right)\hat{\phi}_{k-1}. \tag{48}$$

In a similar way, the amplitudes can be extracted under the condition that $\hat{\phi}_k = \hat{\phi}_{k-1} = \phi$

$$\hat{a}_k\cos\left(\omega_0 k + \phi\right) - \hat{a}_{k-1}\cos\left(\omega_o k + \phi\right) = \alpha_k\left(a\cos\left(\omega_0 k + \phi\right) - \hat{a}_{k-1}\cos\left(\omega_o k + \phi\right)\right) \tag{49}$$
$$\hat{a}_k - \hat{a}_{k-1} = \alpha_k\left(a - \hat{a}_{k-1}\right) \tag{50}$$
$$\hat{a}_k = \alpha_k a + \left(1 - \alpha_k\right)\hat{a}_{k-1}. \tag{51}$$

The cost function for the PLL then can be written as follows

$$J = \mathbb{E}\left[\right] \tag{52}$$

Independence of State Update from Noise

The update rules in Equations (16) and (17) are still iterative methods to compute the precision matrix and the weighted mean vector. Equations (53) and (54) provide explicit formulas to compute the $\overrightarrow{W}_{X_k}$ and $\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k}$ respectively

$$\overrightarrow{W}_{X_k} = \frac{1}{\sigma^2}\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T\mathbf{C}\left(\mathbf{A}^{-1}\right)^j, \tag{53}$$

$$\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k} = \frac{1}{\sigma^2}\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T y_{k-j}. \tag{54}$$

The equality of (53) and (16) as well as the equality of (54) and (17) can easily be verified by induction. The independence of $\overrightarrow{m}_{X_k}$ from the output noise variance σ^2 can now be proved by combining the two equations, which yields

$$\begin{aligned}
\overrightarrow{m}_{X_k} &= \left(\overrightarrow{W}_{X_k}\right)^{-1}\left(\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k}\right) && (55)\\
&= \left(\frac{1}{\sigma^2}\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T\mathbf{C}\left(\mathbf{A}^{-1}\right)^j,\right)^{-1}\cdot\left(\frac{1}{\sigma^2}\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T y_{k-j}\right) && (56)\\
&= \left(\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T\mathbf{C}\left(\mathbf{A}^{-1}\right)^j,\right)^{-1}\cdot\left(\sum_{j=0}^{k-1}\gamma^j\left(\mathbf{A}^{-T}\right)^j\mathbf{C}^T y_{k-j}\right) && (57)
\end{aligned}$$

As we can see, the noise σ^2 cancels out in the calculation above, i.e. our estimator does not need any information about the Gaussian noise process.

Steady State Precision Matrix

Decay factor $0 < \gamma < 1$

Since $\mathbf{A}$ is a rotation matrix it can be diagonalized in an orthonormal basis in $\mathcal{C}$ such that

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H, \tag{58}$$

where $\mathbf{Q}$ is a unitary matrix and $\mathbf{Q}^H$ denotes its Hermitian transpose. The decomposition of the matrix $\mathbf{A}$ can be done as follows

$$\mathbf{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix},$$
$$\boldsymbol{\Lambda} = \begin{bmatrix} \exp\left(i\omega_0\right) & 0 \\ 0 & \exp\left(-i\omega_0\right) \end{bmatrix}.$$

In a next step, the new expression for $\mathbf{A}$ can be inserted into equation (53) [2]

$$
\begin{aligned}
\overrightarrow{W}_{X_k} &= \sum_{l=0}^{k-1}\left(\gamma^l\left(\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\right)^H\right)^{-l}\frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\right)^{-l}\right)\\
&= \frac{1}{\sigma^2}\sum_{l=0}^{k-1}\left(\gamma^l\mathbf{Q}\left(\bar{\boldsymbol{\Lambda}}\right)^{-l}\mathbf{Q}^H\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{Q}\boldsymbol{\Lambda}^{-l}\mathbf{Q}^H\right) \qquad (59)\\
&= \frac{1}{2\sigma^2}\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\left(\bar{\boldsymbol{\Lambda}}\right)^{-l}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\boldsymbol{\Lambda}^{-l}\right)\mathbf{Q}^H\\
&= \frac{1}{2\sigma^2}\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\begin{bmatrix}\exp(i\omega_0 l) & 0\\ 0 & \exp(-i\omega_0 l)\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\exp(-i\omega_0 l) & 0\\ 0 & \exp(i\omega_0 l)\end{bmatrix}\right)\mathbf{Q}^H\\
&= \frac{1}{2\sigma^2}\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\begin{bmatrix}\exp(i\omega_0 l) & \exp(i\omega_0 l)\\ \exp(-i\omega_0 l) & \exp(-i\omega_0 l)\end{bmatrix}\begin{bmatrix}\exp(-i\omega_0 l) & 0\\ 0 & \exp(i\omega_0 l)\end{bmatrix}\right)\mathbf{Q}^H\\
&= \frac{1}{2\sigma^2}\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\begin{bmatrix}1 & \exp(2i\omega_0 l)\\ \exp(-2i\omega_0 l) & 1\end{bmatrix}\right)\mathbf{Q}^H\\
&= \frac{1}{2\sigma^2}\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\gamma^l\begin{bmatrix}0 & \exp(2i\omega_0 l)\\ \exp(-2i\omega_0 l) & 0\end{bmatrix}\right)\mathbf{Q}^H\\
&= \frac{1}{2\sigma^2}\left(\frac{1-\gamma^k}{1-\gamma}\mathbf{I}_2+\mathbf{Q}\sum_{l=0}^{k-1}\left(\gamma^l\begin{bmatrix}0 & \exp(2i\omega_0 l)\\ \exp(-2i\omega_0 l) & 0\end{bmatrix}\right)\mathbf{Q}^H\right), \qquad (60)
\end{aligned}
$$

where equation (59) follows from the unitary property of $\mathbf{Q}$. Moreover, we used the geometric series in equation (60)

$$
\sum_{n=0}^{n-1} ar^k = a\frac{1-r^n}{1-r},
$$

where we implicitly assumed that $0 < \gamma < 1$. Now we rewrite equation (60) as follows

$$
\begin{aligned}
\sum_{l=0}^{k-1} \gamma^l \exp\left(2i\omega_0 l\right) &= \sum_{l=0}^{k-1} \left(\gamma \exp\left(2i\omega_0\right)\right)^l && (61)\\
&= \frac{1-\gamma^k \exp\left(2i\omega_0 k\right)}{1-\gamma\exp\left(2i\omega_0\right)}\\
&= \frac{\left(1-\gamma^k\right)+\gamma^k-\gamma^k\exp\left(2i\omega_0 k\right)}{(1-\gamma)+\gamma-\gamma\exp\left(2i\omega_0\right)}\\
&= \frac{\left(1-\gamma^k\right)-\gamma^k\exp\left(i\omega_0 k\right)\left(\exp\left(i\omega_0 k\right)-\exp\left(-i\omega_0 k\right)\right)}{(1-\gamma)-\gamma\exp\left(i\omega_0\right)\left(\exp\left(i\omega_0\right)-\exp\left(-i\omega_0\right)\right)}\\
&= \frac{\left(1-\gamma^k\right)-2\gamma^k\exp\left(i\omega_0 k\right)\sin\left(\omega_0 k\right)}{(1-\gamma)-2\gamma\exp\left(i\omega_0\right)\sin\left(\omega_0\right)} && (62)
\end{aligned}
$$

where we made again use of the geometric series and the following identity for the sine (derived from Euler's formula) (see equation (62))

$$
\sin(x) = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right).
$$

Similarly we get

$$
\begin{aligned}
\sum_{j=0}^{k-1} \gamma^l \exp\left(-2i\omega_0 j\right) &= \frac{1-\gamma^k\exp\left(-2i\omega_0 k\right)}{1-\gamma\exp\left(-2i\omega_0\right)} && (63)\\
&= \frac{\left(1-\gamma^k\right)+2\gamma^k\exp\left(-i\omega_0 k\right)\sin\left(\omega_0 k\right)}{(1-\gamma)+2\gamma\exp\left(-i\omega_0\right)\sin\left(\omega_0\right)}. && (64)
\end{aligned}
$$

We are now interested in the long-term behavior of the precision matrix $\overrightarrow{W}_{X_k}$, i.e., in the special case where $k \to \infty$. We will call the resulting matrix $\overrightarrow{W}_{ss}$ steady state precision matrix. It is formally defined as follows

$$
\overrightarrow{W}_{ss} := \lim_{k\to\infty}\left(\overrightarrow{W}_{X_k}\right).
$$

As k tends to infinity, the second term of equation (60) simplifies to

$$\lim_{k\to\infty} \mathbf{Q} \sum_{l=0}^{k-1} \left(\gamma^l \begin{bmatrix} 0 & \exp(2i\omega_0 l) \\ \exp(-2i\omega_0 l) & 0 \end{bmatrix} \right) \mathbf{Q}^H \tag{65}$$

$$\begin{aligned}
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 0 & (1-\gamma\exp(2i\omega_0))^{-1} \\ (1-\gamma\exp(-2i\omega_0))^{-1} & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} (1-\gamma\exp(-2i\omega_0))^{-1} & (1-\gamma\exp(2i\omega_0))^{-1} \\ i\,(1-\gamma\exp(-2i\omega_0))^{-1} & -i\,(1-\gamma\exp(2i\omega_0))^{-1} \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{1}{2} \frac{1}{(1-\gamma\exp(2i\omega_0))\,(1-\gamma\exp(-2i\omega_0))} \\
&\cdot \begin{bmatrix} 2-\gamma\,(\exp(2i\omega_0)+\exp(-2i\omega_0)) & -i\gamma\,(\exp(2i\omega_0)-\exp(-2i\omega_0)) \\ -i\gamma\,(\exp(2i\omega_0)-\exp(-2i\omega_0)) & -2+\gamma\,(\exp(2i\omega_0)+\exp(-2i\omega_0)) \end{bmatrix} \\
&= \frac{1}{2} \frac{1}{1+\gamma^2-2\gamma\cos(2\omega_0)} \begin{bmatrix} 2\,(1-\gamma\cos(2\omega_0)) & 2\gamma\sin(2\omega_0) \\ 2\gamma\sin(2\omega_0) & -2\,(1-\gamma\cos(2\omega_0)) \end{bmatrix}.
\end{aligned} \tag{66}$$

The steady state precision matrix is therefore given by

$$\overrightarrow{W}_{ss} = \frac{1}{2\sigma^2} \left(\frac{1}{1-\gamma} \mathbf{I}_2 + \frac{1}{1+\gamma^2-2\gamma\cos(2\omega_0)} \begin{bmatrix} 1-\gamma\cos(2\omega_0) & \gamma\sin(2\omega_0) \\ \gamma\sin(2\omega_0) & -1+\gamma\cos(2\omega_0) \end{bmatrix} \right).$$

Decay factor $\gamma = 1$

For the special case where $\gamma = 1$, equations (62) and (63) simplify to

$$\sum_{l=0}^{k-1} \exp(2i\omega_0 l) = \exp(i\omega_0(k-1)) \frac{\sin(\omega_0 k)}{\sin(\omega_0)}$$

and

$$\sum_{l=0}^{k-1} \exp(2i\omega_0 l) = \exp(-i\omega_0(k-1)) \frac{\sin(\omega_0 k)}{\sin(\omega_0)}$$

respectively. The identical calculation as in 65 then yields

$$
\begin{aligned}
&\mathbf{Q}\sum_{l=0}^{k-1}\left(\begin{bmatrix}0 & \exp\left(2i\omega_0 l\right)\\ \exp\left(-2i\omega_0 l\right) & 0\end{bmatrix}\right)\mathbf{Q}^H\\
&=\frac{\sin\left(\omega_0 k\right)}{2\sin\left(\omega_0\right)}\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}\begin{bmatrix}0 & \exp\left(i\omega_0\left(k-1\right)\right)\\ \exp\left(-i\omega_0\left(k-1\right)\right) & 0\end{bmatrix}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}\\
&=\frac{\sin\left(\omega_0 k\right)}{2\sin\left(\omega_0\right)}\begin{bmatrix}\exp\left(-i\omega_0\left(k-1\right)\right) & \exp\left(i\omega_0\left(k-1\right)\right)\\ i\exp\left(-i\omega_0\left(k-1\right)\right) & -i\exp\left(i\omega_0\left(k-1\right)\right)\end{bmatrix}\begin{bmatrix}1 & i\\ 1 & -i\end{bmatrix}\\
&=\frac{\sin\left(\omega_0 k\right)}{2\sin\left(\omega_0\right)}\begin{bmatrix}2\cos\left(\omega_0\left(k-1\right)\right) & 2\sin\left(\omega_0\left(k-1\right)\right)\\ 2\sin\left(\omega_0\left(k-1\right)\right) & -2\cos\left(\omega_0\left(k-1\right)\right)\end{bmatrix}\\
&=\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\mathbf{A}^{k-1}\mathbf{S} \qquad (67)
\end{aligned}
$$

where $\mathbf{S}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$. The precision matrix is now reduced to

$$\overrightarrow{W}_{X_k}=\frac{1}{2\sigma^2}\left(k\mathbf{I}_2+\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\mathbf{A}^{k-1}\mathbf{S}\right).$$

As we can see $\overrightarrow{W}_{X_k}$ will never end up in a steady state, i.e., for $k\to\infty$, the precision matrix will get bigger with linear divergence speed. The corresponding covariance matrix looks as follows

$$
\begin{aligned}
\overrightarrow{V}_{X_k}&=\left(\overrightarrow{W}_{X_k}\right)^{-1} &(68)\\
&=\frac{2\sigma^2}{k^2-\left(\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\right)^2}\left(k\mathbf{I}_2-\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\mathbf{A}^{k-1}\mathbf{S}\right). &(69)
\end{aligned}
$$

The Kalman gain is then defined as

$$
\begin{aligned}
\alpha_k&=\mathbf{C}\mathbf{B}_k &(70)\\
&=\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G &(71)\\
&=\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\left(\sigma^2+\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1} &(72)\\
&=\frac{2\left(k-\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\cos\left(\omega_0\left(k+1\right)\right)\right)}{k^2-\left(\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\right)^2+2\left(k-\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)}\cos\left(\omega_0\left(k+1\right)\right)\right)}. &(73)
\end{aligned}
$$

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