Derivation of Update Rules for Kalman-Based PLL

In Figure 1 we can see a factor graph representation of a Kalman-based phase-locked loop. The matrices A, C and the observed measurements $\tilde{y}_k$ are given as follows

$$\begin{aligned}
&\mathbf{A} = R(\omega_0) = \begin{bmatrix} \cos(\omega_0) & -\sin(\omega_0 \\ \sin(\omega_0 & \cos(\omega_0 \end{bmatrix}, \\
&\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
&\tilde{y}_k = \cos(\omega_0 k + \phi) + Z_k, \qquad Z_k \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma^2\right).
\end{aligned}$$

Since Z_k is a Gaussian random variable it follows that all random variables in the factor graph are Gaussian. The states X_k can therefore be described by its mean vector and covariance matrix.

In every step k, a new sample y_k is observed. Together with this sample and the previous state X_{k-1}, using Gaussian message passing, we can calculate the next state estimate X_k.

We begin the message passing algorithm by computing the message given by the observed samples $\tilde{y}_k$ [1]

$$\vec{m}_{Z_k} = 0 \qquad \vec{V}_{Z_k} = \sigma^2 \tag{1}$$

$$\overleftarrow{m}_{\tilde{Y}_k} = \tilde{y}_k \qquad \overleftarrow{V}_{\tilde{Y}_k} = 0. \tag{2}$$

In a next step, we add the noise in Eq. (1) to the observed sample in Eq. (2) to get the message at Y_k

$$\overleftarrow{m}_{Y_k} = \tilde{y}_k \qquad \overleftarrow{V}_{Y_k} = \sigma^2. \tag{3}$$

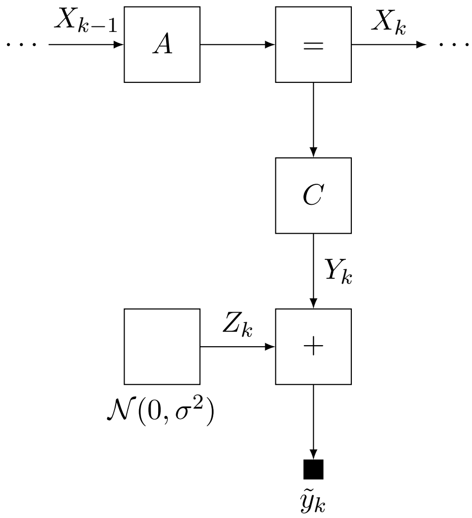

Figure 1: Factor graph representation of a Kalman filter's kth cell.

With these results, we can now compute the messages at the equality constraint

$$\vec{m}_{X''_k} = \mathbf{A}\vec{m}_{X_{k-1}} \qquad \vec{V}_{X''_k} = \mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}_T \tag{4}$$

$$\overleftarrow{W}_{X'_k} = \mathbf{C}^T\overleftarrow{W}_{Y_k}\mathbf{C} \qquad \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} = \mathbf{C}^T\overleftarrow{W}_{Y_k}\overleftarrow{m}_{Y_k}, \tag{5}$$

where $\overleftarrow{W}_{X'_k}$ denotes the precision matrix with the following equality

$$\vec{W}_{X_k}^{-1} = \vec{V}_{X_k},$$

and $\overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k}$ denotes the weighted mean. The set of equations in (5) can be simplified by using the set of equations in (3)

$$\overleftarrow{W}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} \qquad \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}\tilde{y}_k.$$

Hence we can characterize X_k by its precision matrix and its weighted mean (update rules). Note that $\mathbf{A}$ is an invertible matrix since

$$
\begin{aligned}
\vec{W}_{X_k} &= \vec{W}_{X''_k} + \overleftarrow{W}_{X'_k} && (6)\\
&= \left(\mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} && (7)\\
&= \left(\mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \frac{1}{\sigma^2}\mathbf{C}^T\mathbf{C} && (8)\\
&= \mathbf{A}^{-T}\vec{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}, && (9)
\end{aligned}
$$

$$
\begin{aligned}
\vec{W}_{X_k}\vec{m}_{X_k} &= \vec{W}_{X''_k}\vec{m}_{X''_k} + \vec{W}_{X'_k}\vec{m}_{X'_k} = && (10)\\
&= \left(\mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\mathbf{A}\vec{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}\tilde{y}_k && (11)\\
&= \left(\vec{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{I}}\vec{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}\tilde{y}_k && (12)\\
&= \mathbf{A}^{-T}\vec{W}_{X_{k-1}}\vec{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}\tilde{y}_k. && (13)
\end{aligned}
$$

In a next step we try to get to an expression for the covariance matrix and the mean vector by using the Matrix Inversion Lemma (14)

$$
(\mathbf{B}+\mathbf{DEF})^{-1} = \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}\left(\mathbf{E}^{-1}+\mathbf{FB}^{-1}\mathbf{D}\right)^{-1}\mathbf{FB}^{-1}. \qquad (14)
$$

Thus we get the following assignments for the equation (6)

$$
\begin{aligned}
\mathbf{B} &= \left(\mathbf{A}^{-1}\right)^T\vec{W}_{X_{k-1}}\mathbf{A}^{-1}, && (15)\\
\mathbf{D} &= \mathbf{C}^T, && (16)\\
\mathbf{E} &= \frac{1}{\sigma^2}, && (17)\\
\mathbf{F} &= \mathbf{C}. && (18)
\end{aligned}
$$

Finally it follows the update rules for the covariance matrix and the mean vector

$$
\begin{aligned}
\vec{V}_{X_k} &= \left[\mathbf{A}^{-T}\vec{W}_{X_{k-1}}\mathbf{A}^{-1}\right] && (19)\\
&= \left(\vec{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}\mathbf{A}^T\\
&= \mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\mathbf{GCA}\vec{V}_{X_{k-1}}\mathbf{A}^T\\
& && (20)
\end{aligned}
$$

with matrix $\mathbf{G}$ given as follows

$$\mathbf{G} = \left(\sigma^2 + \mathbf{C}\mathbf{A}\vec{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right).$$

and where the last step (see equation (20)) follows from the fact that $\mathbf{A}$ is the rotation matrix and therefore an orthogonal matrix, i.e., its transpose is equal to its inverse

$$\mathbf{A}^T = \mathbf{A}^{-1}.$$

Amplitude and Phase estimate

Independence of State Update from Noise

By induction and for a time index $k > 0$, the precision matrix update rule in eq. (6) can be written as follows

$$\vec{W}_{X_k} = \sum_{j=0}^{k-1} \left(\mathbf{A}^{-T}\right)^j \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2} \left(\mathbf{A}^{-1}\right)^j . \tag{21}$$

Since $\mathbf{A}$ is a rotation matrix it can be diagonalized in an orthonormal basis in $\mathcal{C}$ such that

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H, \tag{22}$$

where $\mathbf{Q}$ is a unitary matrix and $\mathbf{Q}^H$ denotes the its Hermitian transpose. The decomposition of the matrix $\mathbf{A}$ can be done as follows

$$\begin{aligned}
\mathbf{Q} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \\
\mathbf{\Lambda} &= \begin{bmatrix} \exp\left(i\omega_0\right) & 0 \\ 0 & \exp\left(-i\omega_0\right) \end{bmatrix}.
\end{aligned}$$

In a next step, the new expression for $\mathbf{A}$ can be inserted into equation (21) [2]

$$
\begin{aligned}
\vec{W}_{X_k} &= \sum_{j=0}^{k-1}\left(\left(\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\right)^H\right)^{-j}\frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}\left(\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H\right)^{-j}\right) \\
&= \frac{1}{\sigma^2}\sum_{j=0}^{k-1}\left(\mathbf{Q}\left(\bar{\Lambda}\right)^{-j}\mathbf{Q}^H\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\mathbf{Q}\boldsymbol{\Lambda}^{-j}\mathbf{Q}^H\right) \qquad (23)\\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\mathbf{Q}\left(\bar{\Lambda}\right)^{-j}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\boldsymbol{\Lambda}^{-j}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}\exp\left(i\omega_0 j\right) & 0\\ 0 & \exp\left(-i\omega_0 j\right)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\exp\left(-i\omega_0 j\right) & 0\\ 0 & \exp\left(i\omega_0 j\right)\end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}\exp\left(i\omega_0 j\right) & \exp\left(i\omega_0 j\right)\\ \exp\left(-i\omega_0 j\right) & \exp\left(-i\omega_0 j\right)\end{bmatrix}\begin{bmatrix}\exp\left(-i\omega_0 j\right) & 0\\ 0 & \exp\left(i\omega_0 j\right)\end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}\exp\left(i\omega_0 j\right) & 0\\ 0 & \exp\left(-i\omega_0 j\right)\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\exp\left(-i\omega_0 j\right) & 0\\ 0 & \exp\left(i\omega_0 j\right)\end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}1 & \exp\left(2i\omega_0 j\right)\\ \exp\left(-2i\omega_0 j\right) & 1\end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & \exp\left(2i\omega_0 j\right)\\ \exp\left(-2i\omega_0 j\right) & 0\end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{1}{\sigma^2}\left(\frac{k}{2}\mathbf{I}_2+\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix}0 & \exp\left(2i\omega_0 j\right)\\ \exp\left(-2i\omega_0 j\right) & 0\end{bmatrix}\right)\mathbf{Q}^H\right) \qquad (24)
\end{aligned}
$$

where equation (23) follows from the unitary property of $\mathbf{Q}$. Then we rewrite the equation (24) as follows

$$
\begin{aligned}
\sum_{j=0}^{k-1}\exp\left(2i\omega_0 j\right) &= \frac{1-\exp\left(2i\omega_0 k\right)}{1-\exp\left(2i\omega_0\right)} \qquad (25)\\
&= \frac{\exp\left(i\omega_0 k\right)\left(\exp\left(i\omega_0 k\right)-\exp\left(-i\omega_0 k\right)\right)}{\exp\left(i\omega_0\right)\left(\exp\left(i\omega_0\right)-\exp\left(-i\omega_0\right)\right)} \\
&= \exp\left(i\omega_0\left(k-1\right)\right)\frac{\sin\left(\omega_0 k\right)}{\sin\left(\omega_0\right)} \qquad (26)
\end{aligned}
$$

where we used the geometric series in equation (25)

$$
\sum_{n=0}^{n-1} ar^k = a\frac{1-r^n}{1-r},
$$

and the following identity for the sine (derived from Euler's formula) (see equation (26))

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-}ix\right).$$

In a similar way we get the following equation

$$\sum_{j=0}^{k-1} \exp(-2i\omega_0 j) = \exp(-i\omega_0(k-1))\frac{\sin(\omega_0 k)}{\sin(\omega_0)}.$$

Thus the second term in equation (24) can be written as

$$\begin{aligned}
&\mathbf{Q}\sum_{j=0}^{k-1}\left(\begin{bmatrix} 0 & \exp(2i\omega_0 j) \\ \exp(-2i\omega_0 j) & 0 \end{bmatrix}\right)\mathbf{Q}^H \\
&= \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} 0 & \exp(i\omega_0(k-1)) \\ \exp(-i\omega_0(k-1)) & 0 \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\begin{bmatrix} \exp(-i\omega_0(k-1)) & \exp(i\omega_0(k-1)) \\ i\exp(-i\omega_0(k-1)) & -i\exp(i\omega_0(k-1)) \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\begin{bmatrix} 2\cos(\omega_0(k-1)) & 2\sin(\omega_0(k-1)) \\ 2\sin(\omega_0(k-1)) & -2\cos(\omega_0(k-1)) \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\mathbf{A}^{k-1}\mathbf{S} \qquad (27)
\end{aligned}$$

where $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Finally the precision matrix is now reduced to

$$\sum_{j=0}^{k-1} \exp(-2i\omega_0 j) = \exp(-i\omega_0(k-1))\frac{\sin(\omega_0 k)}{\sin(\omega_0)}.$$

Cost function

[1]

Bibliography

[1] H.-A. Loeliger. *Signal and Information Processing: Modeling, Filtering, Learning.* Signal and Information Processing Laboratory, ETH Zurich, 2013.

[2] Erik Hampus Malmberg. Kalman filter-based phase-locked loops for harmonic signals. semester thesis. ISI, ETH Zurich, 2013.