Derivation of Update Rules for Kalman-Based PLL

In Figure 1 we can see a factor graph representation of a Kalman-based phase-locked loop. The matrices A, C and the observed measurements y_k are given as follows

$$
\begin{aligned}
&\mathbf{A} = R(\omega_0) = \begin{bmatrix} \cos(\omega_0) & -\sin(\omega_0 \\ \sin(\omega_0 & \cos(\omega_0 \end{bmatrix}, \\
&\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \\
&Y_k = \cos(\omega_0 k + \phi) + Z_k, \qquad Z_k \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma^2\right).
\end{aligned}
$$

Since Z_k is a Gaussian random variable it follows that all random variables in the factor graph are Gaussian. The states X_k can therefore be described by its mean vector and covariance matrix.

In every step k, a new sample y_k is observed. Together with this sample and the previous state X_{k-1}, using Gaussian message passing, we can calculate the next state estimate X_k.

We begin the message passing algorithm by computing the message given by the observed samples y_k [1]

$$\overrightarrow{m}_{Z_k} = 0 \qquad \overrightarrow{V}_{Z_k} = \sigma^2 \tag{1}$$

$$\overleftarrow{m}_{Y_k} = y_k \qquad \overleftarrow{V}_{Y_k} = 0. \tag{2}$$

In a next step, we add the noise in Eq. (1) to the observed sample in Eq. (2) to get the message at Y_k

$$\overleftarrow{m}_{Y_k} = y_k \qquad \overleftarrow{V}_{Y_k} = \sigma^2. \tag{3}$$

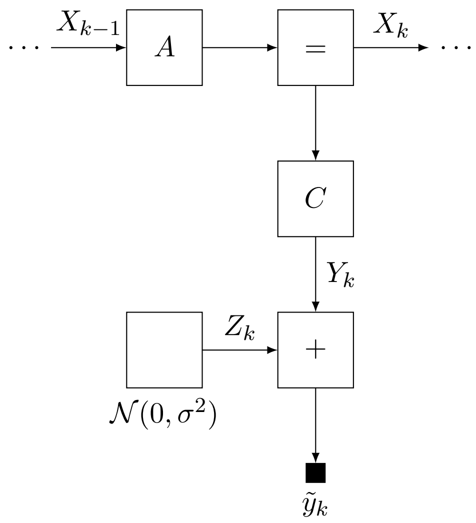

Figure 1: Factor graph representation of a Kalman filter's kth cell.

With these results, we can now compute the messages at the equality constraint

$$
\begin{aligned}
\overrightarrow{m}_{X''_k} &= \mathbf{A}\overrightarrow{m}_{X_{k-1}} \qquad & \overrightarrow{V}_{X''_k} &= \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}_T & (4)\\
\overleftarrow{W}_{X'_k} &= \mathbf{C}^T\overleftarrow{W}_{Y_k}\mathbf{C} \qquad & \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} &= \mathbf{C}^T\overleftarrow{W}_{Y_k}\overleftarrow{m}_{Y_k}, & (5)
\end{aligned}
$$

where $\overleftarrow{W}_{X'_k}$ denotes the precision matrix with the following equality

$$
\overrightarrow{W}_{X_k}^{-1} = \overrightarrow{V}_{X_k},
$$

and $\overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k}$ denotes the weighted mean. The set of equations in (5) can be simplified by using the set of equations in (3)

$$
\overleftarrow{W}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} \qquad \overleftarrow{W}_{X'_k}\overleftarrow{m}_{X'_k} = \mathbf{C}^T\frac{1}{\sigma^2}y_k.
$$

Hence we can characterize X_k by its precision matrix and its weighted mean (update rules). Note that $\mathbf{A}$ is an invertible matrix since

$$
\begin{aligned}
\overrightarrow{W}_{X_k} &= \overrightarrow{W}_{X''_k} + \overleftarrow{W}_{X'_k} && (6)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \mathbf{C}^T\frac{1}{\sigma^2}\mathbf{C} && (7)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1} + \frac{1}{\sigma^2}\mathbf{C}^T\mathbf{C} && (8)\\
&= \mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2} && (9)\\
&= \mathbf{A}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}, && (10)
\end{aligned}
$$

$$
\begin{aligned}
\overrightarrow{W}_{X_k}\overrightarrow{m}_{X_k} &= \overrightarrow{W}_{X''_k}\overrightarrow{m}_{X''_k} + \overrightarrow{W}_{X'_k}\overrightarrow{m}_{X'_k} = && (11)\\
&= \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}y_k && (12)\\
&= \left(\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right)^{-1}\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} + \mathbf{C}^T\frac{1}{\sigma^2}y_k && (13)\\
&= \mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k && (14)\\
&= \mathbf{A}\overrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k, && (15)
\end{aligned}
$$

where the last step in the particular update equations follows from the fact that $\mathbf{A}$ is the rotation matrix and therefore an orthogonal matrix, i.e., its transpose is equal to its inverse

$$\mathbf{A}^T = \mathbf{A}^{-1}.$$

In a next step we try to get to an expression for the covariance matrix and the mean vector by using the Matrix Inversion Lemma (16)

$$(\mathbf{B} + \mathbf{DEF})^{-1} = \mathbf{B}^{-1} - \mathbf{B}^{-1}\mathbf{D}\left(\mathbf{E}^{-1} + \mathbf{FB}^{-1}\mathbf{D}\right)^{-1}\mathbf{FB}^{-1}. \qquad (16)$$

Thus we get the following assignments from equation (6)

$$
\begin{aligned}
\mathbf{B} &= \mathbf{A}^{-T}\overrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}, && (17)\\
\mathbf{D} &= \mathbf{C}^T, && (18)\\
\mathbf{E} &= \frac{1}{\sigma^2}, && (19)\\
\mathbf{F} &= \mathbf{C}. && (20)
\end{aligned}
$$

Using these, the inverse of equation (6), i.e., the covariance matrix can be written as

$$\overleftrightarrow{V}_{X_k} = \left[\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1} + \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2}\right]^{-1} \tag{21}$$

$$= \left(\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1} - \left(\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}\mathbf{C}^T\left(\sigma^2 + \mathbf{C}\left(\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1}\mathbf{C}^T\right)^{-1}\mathbf{C}\left(\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\mathbf{A}^{-1}\right)^{-1} \tag{22}$$

$$= \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\underbrace{\left(\sigma^2 + \mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1}}_{=:G}\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T$$

$$= \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T \tag{23}$$

where

$$G := \left(\sigma^2 + \mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1}.$$

The mean vector $\overrightarrow{m}_{X_k}$ can be retrieved by multiplying the matrix $\overleftrightarrow{V}_{X_k}$ with (11)

$$\overrightarrow{m}_{X_k} = \overleftrightarrow{V}_{X_k}\left(\overleftrightarrow{W}_{X_k}\overrightarrow{m}_{X_k}\right) \tag{24}$$

$$= \left(\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right) \cdot \left(\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}\overrightarrow{m}_{X_{k-1}} + \frac{\mathbf{C}^T}{\sigma^2}y_k\right) \tag{25}$$

$$= \mathbf{A}\underbrace{\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\underbrace{\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{A}^{-T}\overleftrightarrow{W}_{X_{k-1}}}_{\mathbf{I}}\overrightarrow{m}_{X_{k-1}} \tag{26}$$

$$+ \left(\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\right) \cdot \left(\frac{\mathbf{C}^T}{\sigma^2}y_k\right)$$

$$= \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \left(\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T - \mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right) \cdot \left(\frac{1}{\sigma^2}\right) \cdot y_k \tag{27}$$

$$= \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \left(\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\right)\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \left(\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)\underbrace{\left(\mathbf{I} - G\mathbf{C}\mathbf{A}\overleftrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right) \cdot \left(\frac{1}{\sigma^2}\right)}_{=:\lambda} \cdot y_k. \tag{28}$$

The factor λ can further be simplified to

$$\lambda := \left(\mathbf{I} - G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right) \cdot \left(\frac{1}{\sigma^2}\right) \tag{29}$$

$$= \left(GG^{-1} - G\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right) \cdot \left(\frac{1}{\sigma^2}\right) \tag{30}$$

$$= G\left(G^{-1} - \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right) \cdot \left(\frac{1}{\sigma^2}\right) \tag{31}$$

$$= G\left(\sigma^2 + \underbrace{\mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T - \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T}_{=0}\right) \cdot \left(\frac{1}{\sigma^2}\right) \tag{32}$$

$$= G\left(\sigma^2\right) \cdot \left(\frac{1}{\sigma^2}\right) \tag{33}$$

$$= G, \tag{34}$$

where we used the fact, that $G^{-1} = \sigma^2 + \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T$. So we finally find the following condensed expression for $\overrightarrow{m}_{X_k}$

$$\overrightarrow{m}_{X_k} = \mathbf{A}\overrightarrow{m}_{X_{k-1}} - \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\right)\mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}} + \left(\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\right) y_k \tag{35}$$

$$= \mathbf{A}\overrightarrow{m}_{X_{k-1}} + \mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G\left(y_k - \mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}}\right). \tag{36}$$

Amplitude and Phase Estimate and Cost Function

The predicted signal estimate given the measurements $y_1, ..., y_{k-1}$ can be written as

$$\tilde{y}_k = \mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}},$$

and the corrected estimate given the measurements $y_1, ..., y_k$ as

$$\hat{y}_k = \mathbf{C}\overrightarrow{m}_{X_{k-1}}.$$

The difference between the corrected and the predicted estimate can be deduced as

$$\begin{aligned}
\hat{y}_k - \tilde{y}_k &= \mathbf{C}\left(\overrightarrow{m}_{X_k} - \mathbf{A}\overrightarrow{m}_{X_{k-1}}\right) && (37)\\
&= \mathbf{C}\left(\overrightarrow{m}_{X_k} - \overrightarrow{m}_{X_k} + \underbrace{\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G}_{=:\mathbf{B}_k}\left(y_k - \mathbf{C}\mathbf{A}\overrightarrow{m}_{X_{k-1}}\right)\right)\\
&= \mathbf{C}\mathbf{B}_k\left(y_k - \tilde{y}_k\right) && (38)
\end{aligned}$$

Since both the input signal and estimate is known to be sinusoidal the mean vector at time index k can be written as [2]

$$\overrightarrow{m}_{X_k} = \hat{a}_k \begin{bmatrix} \cos\left(\omega_0 k + \hat{\phi}_k\right) \\ \sin\left(\omega_0 k + \hat{\phi}_k\right) \end{bmatrix}$$

and subsequently

$$y_k = a\cos\left(\omega_0 k + \phi\right) + Z_k, \qquad Z_k \overset{i.i.d}{\sim} \mathcal{N}\left(0, \sigma^2\right), \tag{39}$$

$$\tilde{y}_k = \hat{a}_{k-1}\cos\left(\omega_0 k + \hat{\phi}_{k-1}\right), \tag{40}$$

$$\hat{y}_k = \hat{a}_k\cos\left(\omega_0 k + \hat{\phi}_k\right) \tag{41}$$

Now we consider the scenario where only the phase is tracked and the amplitude is assumed to be $\hat{a}_k = \hat{a}_{k-1} = a_k$. If the phase estimation error is small such that $\left|\hat{\phi}_k - \phi\right| \ll 1$ and $\left|\hat{\phi}_{k-1} - \phi\right| \ll 1$, equation (38) can be reduced to

$$a\left(\cos\left(\omega_0 k + \hat{\phi}_k\right) - \cos\left(\omega_0 k + \hat{\phi}_{k-1}\right)\right) = a\alpha_k\left(\cos\left(\omega_0 k + \hat{\phi}_k\right) - \cos\left(\omega_0 k + \hat{\phi}_{k-1}\right)\right) \tag{42}$$

$$\sin\left(\frac{\hat{\phi}_k - \hat{\phi}_{k-1}}{2}\right) \approx \alpha_k \sin\left(\frac{\phi - \hat{\phi}_{k-1}}{2}\right) \tag{43}$$

$$\hat{\phi}_k - \hat{\phi}_{k-1} \approx \alpha_k\phi - \hat{\phi}_{k-1}. \tag{44}$$

Thus follows from equation (**??**) that the phase update can be approximated as

$$\hat{\phi}_k \approx \alpha_k\phi + \left(1 - \alpha_k\right)\hat{\phi}_{k-1}. \tag{45}$$

In a similar way, the amplitudes can be extracted under the condition that $\hat{\phi}_k = \hat{\phi}_{k-1} = \phi$

$$\hat{a}_k\cos\left(\omega_0 k + \phi\right) - \hat{a}_{k-1}\cos\left(\omega_o k + \phi\right) = \alpha_k\left(a\cos\left(\omega_0 k + \phi\right) - \hat{a}_{k-1}\cos\left(\omega_o k + \phi\right)\right) \tag{46}$$

$$\hat{a}_k - \hat{a}_{k-1} = \alpha_k\left(a - \hat{a}_{k-1}\right) \tag{47}$$

$$\hat{a}_k = \alpha_k a + \left(1 - \alpha_k\right)\hat{a}_{k-1}. \tag{48}$$

The cost function for the PLL then can be written as follows

$$J = \mathbb{E}\left[\right] \tag{49}$$

Independence of State Update from Noise

By induction and for a time index $k > 0$, the precision matrix update rule in eq. (6) can be written as follows

$$\overrightarrow{W}_{X_k} = \sum_{j=0}^{k-1} \left(\mathbf{A}^{-T}\right)^j \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2} \left(\mathbf{A}^{-1}\right)^j. \tag{50}$$

Since $\mathbf{A}$ is a rotation matrix it can be diagonalized in an orthonormal basis in $\mathcal{C}$ such that

$$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H, \tag{51}$$

where $\mathbf{Q}$ is a unitary matrix and $\mathbf{Q}^H$ denotes the its Hermitian transpose. The decomposition of the matrix $\mathbf{A}$ can be done as follows

$$\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix},$$
$$\boldsymbol{\Lambda} = \begin{bmatrix} \exp(i\omega_0) & 0 \\ 0 & \exp(-i\omega_0) \end{bmatrix}.$$

In a next step, the new expression for $\mathbf{A}$ can be inserted into equation (49) [2]

$$
\begin{aligned}
\overrightarrow{W}_{X_k} &= \sum_{j=0}^{k-1} \left(\left(\left(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H\right)^H \right)^{-j} \frac{\mathbf{C}^T\mathbf{C}}{\sigma^2} \left(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H\right)^{-j} \right) \\
&= \frac{1}{\sigma^2} \sum_{j=0}^{k-1} \left(\mathbf{Q} \left(\bar{\Lambda}\right)^{-j} \mathbf{Q}^H \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{Q}\mathbf{\Lambda}^{-j}\mathbf{Q}^H \right) && (52) \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\mathbf{Q} \left(\bar{\Lambda}\right)^{-j} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{\Lambda}^{-j} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} \exp(i\omega_0 j) & 0 \\ 0 & \exp(-i\omega_0 j) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \exp(-i\omega_0 j) & 0 \\ 0 & \exp(i\omega_0 j) \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} \exp(i\omega_0 j) & \exp(i\omega_0 j) \\ \exp(-i\omega_0 j) & \exp(-i\omega_0 j) \end{bmatrix} \begin{bmatrix} \exp(-i\omega_0 j) & 0 \\ 0 & \exp(i\omega_0 j) \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} \exp(i\omega_0 j) & 0 \\ 0 & \exp(-i\omega_0 j) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \exp(-i\omega_0 j) & 0 \\ 0 & \exp(i\omega_0 j) \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} 1 & \exp(2i\omega_0 j) \\ \exp(-2i\omega_0 j) & 1 \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \exp(2i\omega_0 j) \\ \exp(-2i\omega_0 j) & 0 \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{1}{\sigma^2} \left(\frac{k}{2}\mathbf{I}_2 + \mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} 0 & \exp(2i\omega_0 j) \\ \exp(-2i\omega_0 j) & 0 \end{bmatrix} \right) \mathbf{Q}^H \right) && (53)
\end{aligned}
$$

where equation (51) follows from the unitary property of $\mathbf{Q}$. Then we rewrite the equation (52) as follows

$$
\begin{aligned}
\sum_{j=0}^{k-1} \exp(2i\omega_0 j) &= \frac{1-\exp(2i\omega_0 k)}{1-\exp(2i\omega_0)} && (54) \\
&= \frac{\exp(i\omega_0 k)\left(\exp(i\omega_0 k) - \exp(-i\omega_0 k)\right)}{\exp(i\omega_0)\left(\exp(i\omega_0) - \exp(-i\omega_0)\right)} \\
&= \exp\left(i\omega_0 (k-1)\right) \frac{\sin(\omega_0 k)}{\sin(\omega_0)} && (55)
\end{aligned}
$$

where we used the geometric series in equation (53)

$$
\sum_{n=0}^{n-1} ar^k = a\frac{1-r^n}{1-r},
$$

and the following identity for the sine (derived from Euler's formula) (see equation (54))

$$\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-}ix\right).$$

In a similar way we get the following equation

$$\sum_{j=0}^{k-1} \exp(-2i\omega_0 j) = \exp(-i\omega_0 (k-1)) \frac{\sin(\omega_0 k)}{\sin(\omega_0)}.$$

Thus the second term in equation (52) can be written as

$$\begin{aligned}
&\mathbf{Q} \sum_{j=0}^{k-1} \left(\begin{bmatrix} 0 & \exp(2i\omega_0 j) \\ \exp(-2i\omega_0 j) & 0 \end{bmatrix} \right) \mathbf{Q}^H \\
&= \frac{\sin(\omega_0 k)}{2\sin(\omega_0)} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} 0 & \exp(i\omega_0 (k-1)) \\ \exp(-i\omega_0 (k-1)) & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{2\sin(\omega_0)} \begin{bmatrix} \exp(-i\omega_0 (k-1)) & \exp(i\omega_0 (k-1)) \\ i\exp(-i\omega_0 (k-1)) & -i\exp(i\omega_0 (k-1)) \end{bmatrix} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{2\sin(\omega_0)} \begin{bmatrix} 2\cos(\omega_0 (k-1)) & 2\sin(\omega_0 (k-1)) \\ 2\sin(\omega_0 (k-1)) & -2\cos(\omega_0 (k-1)) \end{bmatrix} \\
&= \frac{\sin(\omega_0 k)}{\sin(\omega_0)} \mathbf{A}^{k-1}\mathbf{S} \qquad (56)
\end{aligned}$$

where $\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Finally the precision matrix is now reduced to

$$\overrightarrow{W}_{X_k} = \frac{1}{2\sigma^2}\left(k\mathbf{I}_2 + \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\mathbf{A}^{k-1}\mathbf{S}\right)$$

$$\begin{aligned}
\overrightarrow{V}_{X_k} &= \left(\overrightarrow{W}_{X_k}\right)^{-1} \qquad (57) \\
&= \frac{2\sigma^2}{k^2 - \left(\frac{\sin(\omega_0 k)}{\sin(\omega_0)}\right)^2}\left(k\mathbf{I}_2 - \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\mathbf{A}^{k-1}\mathbf{S}\right) \qquad (58)
\end{aligned}$$

$$
\begin{aligned}
\alpha_k &= \mathbf{C}\mathbf{B}_k && (59)\\
&= \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T G && (60)\\
&= \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T \left(\sigma^2 + \mathbf{C}\mathbf{A}\overrightarrow{V}_{X_{k-1}}\mathbf{A}^T\mathbf{C}^T\right)^{-1} && (61)\\
&= \frac{2\left(k - \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\cos(\omega_0(k+1))\right)}{k^2 - \left(\frac{\sin(\omega_0 k)}{\sin(\omega_0)}\right)^2 + 2\left(k - \frac{\sin(\omega_0 k)}{\sin(\omega_0)}\cos(\omega_0(k+1))\right)} && (62)
\end{aligned}
$$

Bibliography

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