# Unit 2.3 Sets and Graphs

Algorithms

EE/NTHU

Mar. 24, 2020

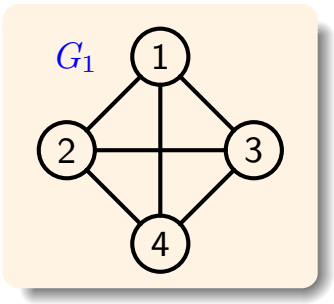


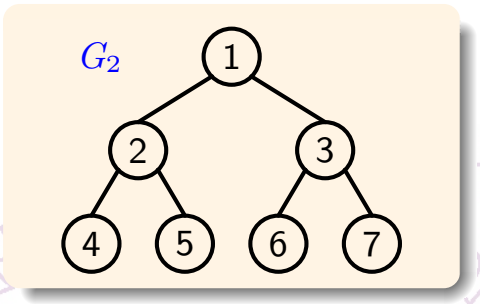
## Graphs

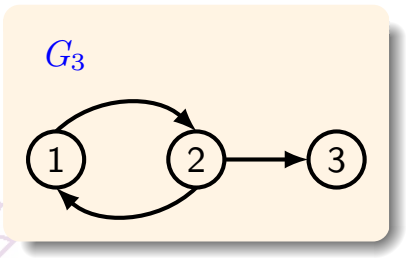
- A graph, $G$, consists of two sets $V$ and $E$.
  - The set $V$ is a finite, nonempty set of vertices.
  - The set $E$ is a set of pairs of vertices; these pairs are called edges.
  - They are also denoted by $V(G)$ and $E(G)$.
  - And the graph is also denoted by $G(V, E)$.
- In an undirected graph the pair of vertices representing any edge is unordered.
  - Thus, the pairs $(u, v)$ and $(v, u)$ represent the same edge.
- In a directed graph each edge is represented by a direct pair $\langle u, v \rangle$; $u$ is the tail and $v$ is the head.
  - And, $\langle u, v \rangle$ and $\langle v, u \rangle$ represent two different edges.
  - In a direct graph the edges are drawn as arrows and they are drawn from tail to head.

# Graph Examples

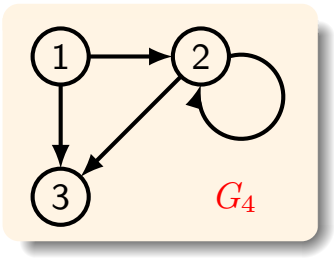
- Examples

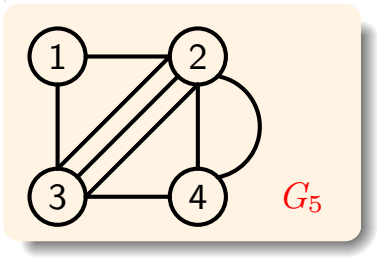



- Note that $G_1$ and $G_2$ are undirected graphs; $G_3$ is a directed graph.
- $G_1$ is a complete graph; $G_2$ is a tree.
- The following graphs are not studied in our classes:
  - Graphs with self-edges, which have edges connecting the same vertex.
  - Multi-graph, which has multiple edges between the same two vertices.






# Adjacency

- The number of distinct unordered pairs $(u, v)$ with $u \neq v$ in a graph with $n$ vertices is $n(n-1)/2$.
  - This is the maximum number of edges in any $n$-vertex, undirected graph.
  - An $n$-vertex, undirected graph with exactly $n(n-1)/2$ edges is said to be complete.
  - In case of a direct graph with $n$ vertices, the maximum number of edges $n(n-1)$.
- If $(u, v)$ is an edge in $E(G)$, the we say vertices $u$ and $v$ are adjacent and edge $(u, v)$ is incident on vertices $u$ and $v$.
- If $\langle u, v \rangle$ is a directed edge, the vertex $u$ is adjacent to $v$ and $v$ is adjacent from $u$.
  - The edge $\langle u, v \rangle$ is incident to $u$ and $v$.
- A subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V'(G') \subseteq V(G)$ and $E'(G') \subseteq E(G)$.

# Paths and Cycles

- A path from vertex $u$ to vertex $v$ in a graph $G = (V, E)$ is a sequence of vertices $u, i_1, i_2, \cdots, i_k, v$, such that $(u, i_1)$, $(i_1, i_2)$, $\cdots$, $(i_k, v)$ are edges in $E(G)$.
  - If $G = (V, E)$ is directed, then the path consists of the edges $\langle u, i_1 \rangle$, $\langle i_1, i_2 \rangle$, $\cdots$, $\langle i_k, v \rangle$ in $E(G)$.
- The length of a path is the number of edges on it.
- A simple path is a path in which all vertices except possibly the first and the last are distinct.
- A cycle is a simple path in which the first and the last vertices are the same.
  - A directed cycle is a cycle in a directed graph.
- In an undirected graph $G = (V, E)$, two vertices $u$ and $v$ are said to be connected if and only if there is a path in $G$ from $u$ to $v$.
- An undirected graph is said to be connected if and only if for every pair of distinct vertices $u$ and $v$ in $V(G)$, there is a path from $u$ to $v$.
- A connected component or simply a component $H$ of an undirected graph is a maximal connected subgraph.
  - By maximal we mean that $G$ contains no other subgraph that is both connected and properly contains $H$.



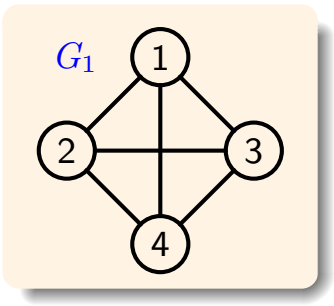
# Graph Degrees

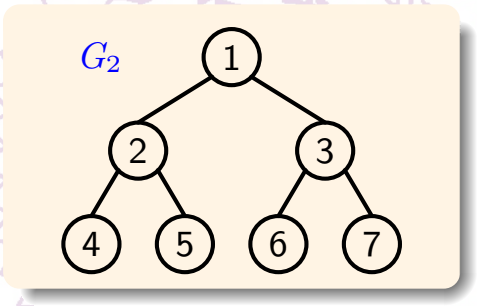
- A tree is a connected acyclic (contain no cycles) graph.
- A directed graph $G = (V, E)$ is said to be strongly connected if and only if for every pair of distinct vertices $u$ and $v$ in $V(G)$, there is a directed path from $u$ to $v$ and also from $v$ to $u$.
- A strongly connected component is a maximal subgraph that is strongly connected.
- The degree of a vertex is the number of edges incident to that vertex.
- If $G = (V, E)$ is a directed graph, we define the in-degree of a vertex $v$ to be the number of edges for which $v$ is the head.
  - The out-degree is defined to be the number of edges for which $v$ is the tail.
- If $d_i$ is the degree of vertex $i$ in a graph $G = (V, E)$ with $n$ vertices, then the number of edge is

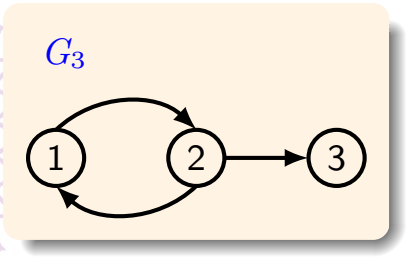
$$e = \left( \sum_{i=1}^{n} d_i \right) /2. \tag{2.3.1}$$

# Graph Representation – Adjacency Matrix

- Let $G = (V, E)$ be a graph with $n$ vertices, $n \geq 1$. The adjacency matrix $A$ is a two-dimensional $n \times n$ matrix with the property that $A[i, j] = 1$ if and only if the edge $(i, j)$ ($\langle i, j \rangle$ for a directed graph) is in $E(G)$. $A[i, j] = 0$ if there is no such edge in $E(G)$.




$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



# Graph Representation – Adjacency Matrix, II

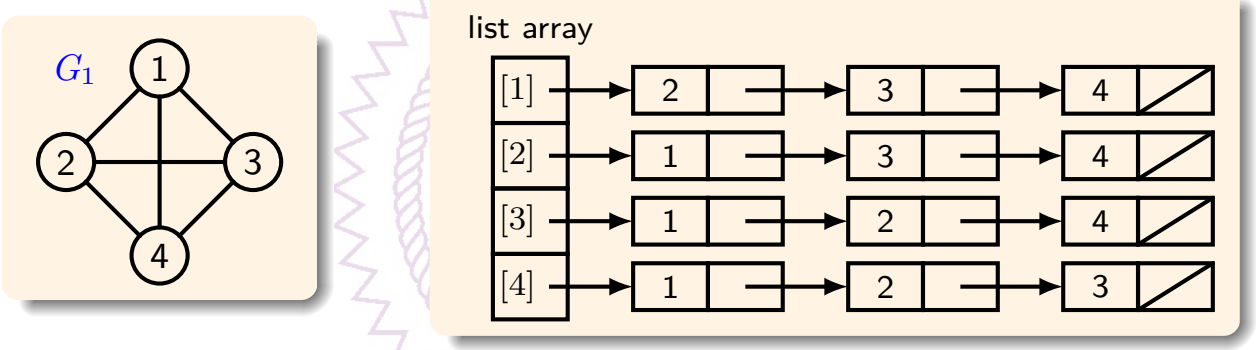
- The adjacency matrix for a undirected graph is symmetric.
  - This is due to that if the edge $(i, j)$ is in $E(G)$ then the edge $(j, i)$ is also in $E(G)$.
- The adjacency matrix for an directed graph may not be symmetric.
- The space needed to represent for a adjacency matrix is $n^2$ bits.
  - The undirected graph needs only half of this space.
- For an undirected graph the degree of any vertex $i$ is its row sum:

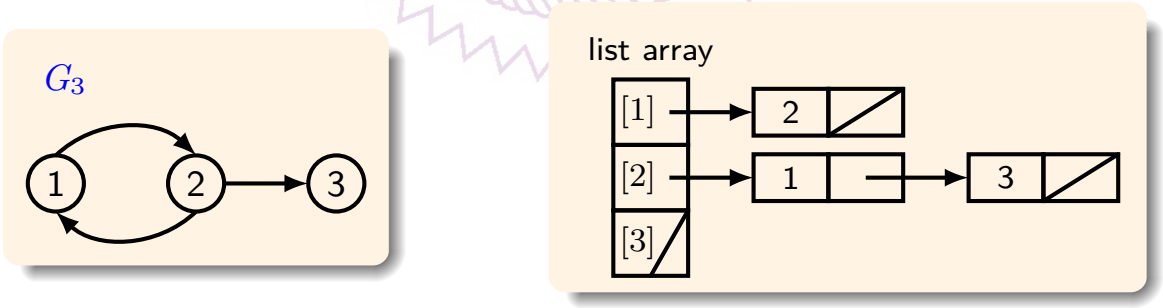
$$\sum_{j=1}^{n} A[i][j]. \tag{2.3.2}$$

- For a directed graph the row sum is the out-degree and the column sum is the in-degree.
- The adjacency matrix approach to represent the graph is not the most efficient way in both space and execution time.
  - It does not take advantage of the sparsity of the graph.
  - For example, the time complexity to find the number of edges of a graph, with $n$ vertices, represented by a adjacency matrix is $\mathcal{O}(n^2)$.

# Graph Representation – Adjacency Lists

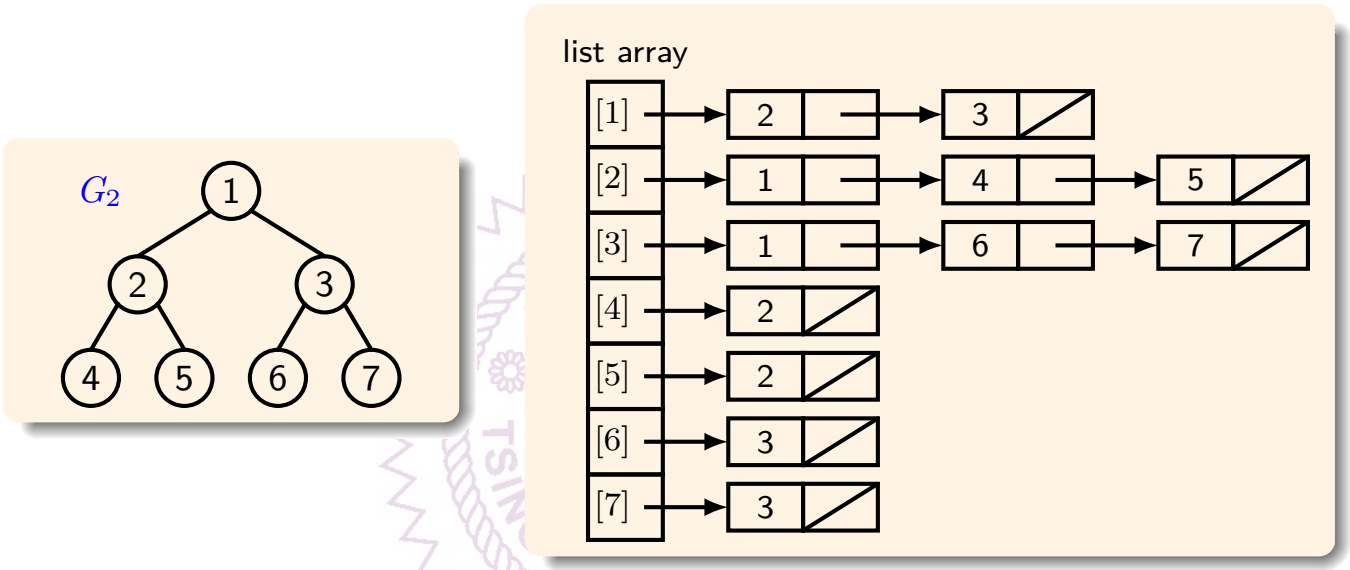
- A graph, $G = (V, E)$, of $n$ vertices, can also be represented by $n$ linked lists.
  - Each vertex has a linked list to represent the adjacent vertices.
- Examples






# Adjacency Lists – Examples




- For an undirected graph with $n$ vertices and $e$ edges, the adjacency list representation requires $n$ head nodes and $2e$ list nodes.
- The degree of any vertex in an undirected graph can be determined by counting the number of nodes in the adjacency list.
  - Hence the total number of edges can be determined in $\mathcal{O}(n + e)$ time.
- For a directed graph, the out-degree of any vertex is again the number of nodes of its adjacency list.
- The in-degree may need to have another inverse adjacency list.

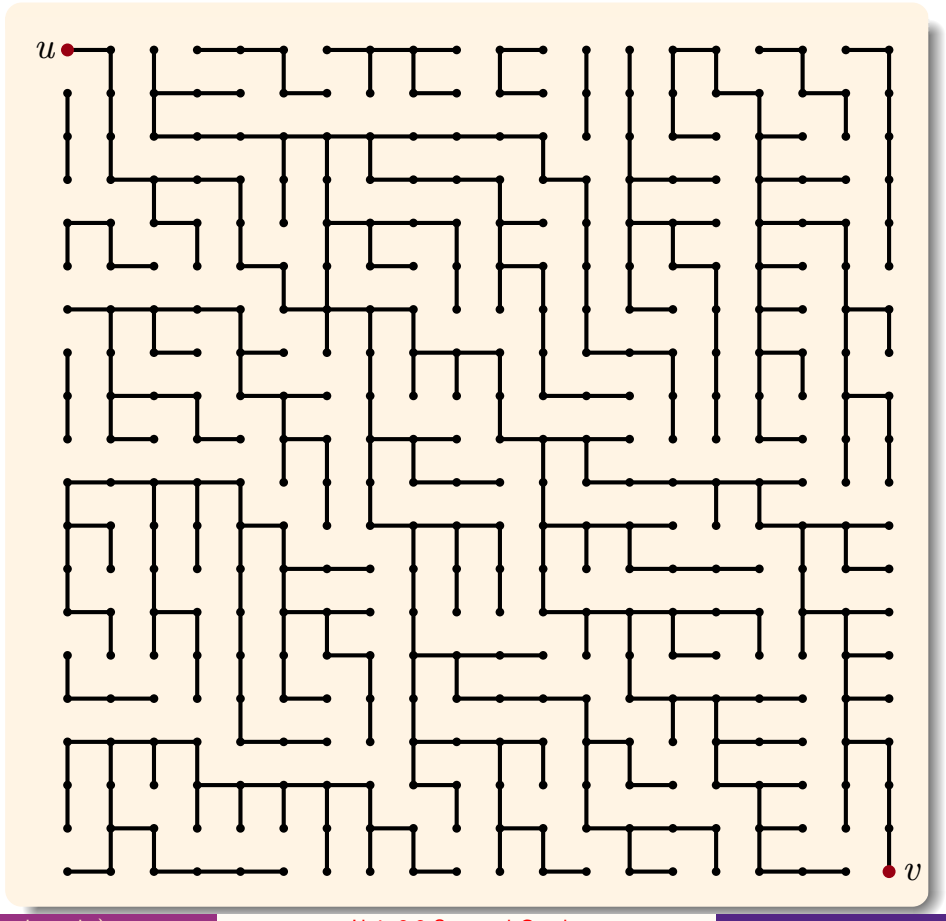
# Weighted Edges

- In many applications, the edges of a graph have weights assigned to them. Thus, the adjacency matrix and adjacency lists need to accommodate these weights information.
- The adjacency matrix can store the weight of edge $\langle i, j \rangle$ to $A[i][j]$ directly.
  - No extra storage is required.
  - Space complexity is $\mathcal{O}(n^2)$.
- For the adjacency lists, each node of the list needs to have an additional field to store the weight.
  - In terms of space complexity, it is still the same as $\Theta(e)$, where $e$ is the number of edges in $G = (V, E)$, Or, in worst-case $\mathcal{O}(n^2)$.

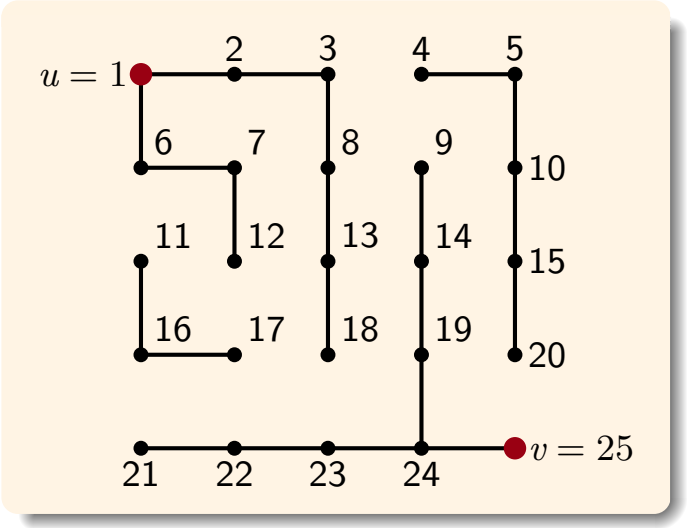


# Network Connectivity Problem

- Given a network below, is node $u$ connected to node $v$?



# Network Connectivity Problem, II




- A smaller instance is shown above.
- One solution approach is to form sets of connected nodes, $S_i$.
- If there is a $S_k$ such that $u, v \in S_k$, then $u$ is connected to $v$.



# Network Connectivity Problem, Algorithm

- A generic algorithm for network connectivity problem is shown below.

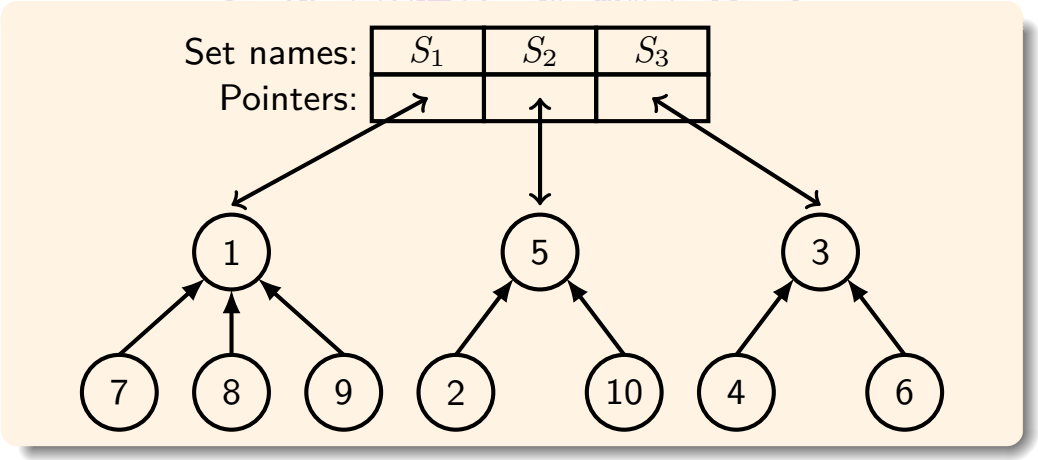
## Algorithm 2.3.1. Connectivity – Generic.

```
   // Given G(V, E) and u, v ∈ V, find if u and v are connected.
   // Input: G, u, v
   // Output: true if connected, false otherwise.
 1 Algorithm Connected(G, u, v)
 2 {
 3       for each v_i∈V do S_i := {v_i} ; // One element for each set.
 4       for each e = (v_i, v_j) do { // Connected vertices
 5           S_i := SetFind(v_i) ;
 6           S_j := SetFind(v_j) ;
 7           S_i := S_i ∪ S_j ; // Set union.
 8       }
 9       if SetFind(u) = SetFind(v) then return true ;
10       return false ;
11 }
```

- The time complexity is dominated by the loop on lines 4-7
  - Iterations: $\mathcal{O}(|E|)$.
  - Two `SetFind` and one `SetUnion` per iteration.

# Disjoint Sets

- Disjoint sets
  - Assume the elements are numbered 1, 2, ..., $n$.
  - Disjoint sets $S_i$, $S_j$ such that $S_i \cap S_j = \emptyset$, $i \neq j$.
  - Forest can be used to represent disjoint sets
- Operations important to set manipulations
  - `Union`: Merge two disjoint sets into one.
  - `Find`$(i)$: Given an element $i$ find the set that contains $i$.
- Example: $S_1 = \{1, 7, 8, 9\}$, $S_2 = \{2, 5, 10\}$, $S_3 = \{3, 4, 6\}$.

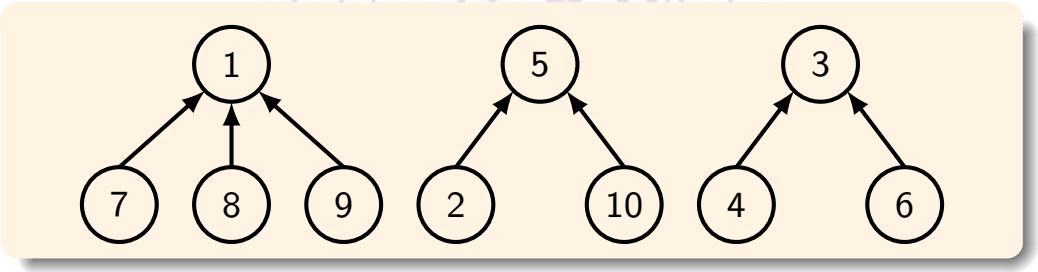



- Note that for the forest the link is pointing from child to parent.
  - In this way, the set name can be found by following the pointers.



# Disjoint Sets – Array Representation

- Simple array can also be used to represent disjoint sets.
- Example: $S_1 = \{1, 7, 8, 9\}$, $S_2 = \{2, 5, 10\}$, $S_3 = \{3, 4, 6\}$.

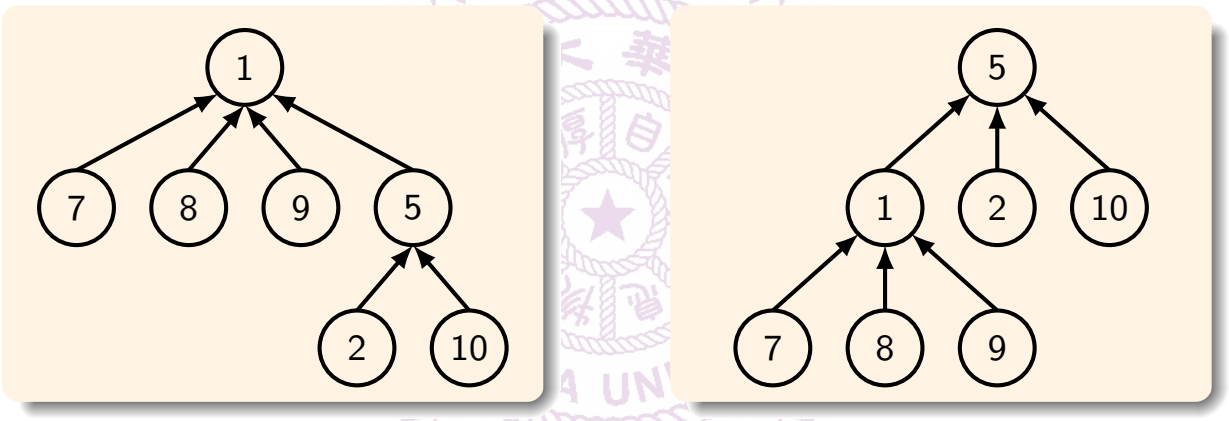



| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | -1 | 5 | -1 | 3 | -1 | 3 | 1 | 1 | 1 | 5 |

- A single array, $p$, can represent the disjoint sets.

# Disjoints Sets – Union

- Two disjoint sets can be united easily.
- Example




$S_1 \cup S_2$ $S_1 \cup S_2$

- Both scenarios are legal and efficient.
- Union of two sets are done by setting one of the roots to be the parent of another root.



# Disjoint Sets – Algorithms

- Using the array $p$ to represent the disjoint sets, then the following algorithms perform the desired operations.

## Algorithm 2.3.2. Set Union.

```
   // Form union of two sets with roots, i and j.
   // Input: roots, i and j
   // Output: none.
1  Algorithm SetUnion(i, j)
2  {
3        p[i] := j;
4  }
```

## Algorithm 2.3.3. Set Find.

```
   // Find the set that element i is in.
   // Input: element i
   // Output: root element of the set.
1  Algorithm SetFind(i)
2  {
3        while (p[i] ≥ 0) do i := p[i];
4        return i;
5  }
```

# Disjoint Sets – Weighting Rule

- Algorithm `SetFind`$(i)$ has the complexity $\mathcal{O}(h)$, where $h$ is the height of the tree the element $i$ is in.
- Algorithm `SetUnion`$(i, j)$ has the time complexity $\mathcal{O}(1)$.
  - However, each union operation increases the height of the tree by 1.
  - Thus, after some union operations the tree might become skewed and the execution time of `SetFind` increases.
  - This issue can be alleviated by using the weighting rule.

## Definition 2.3.4. Weighting rule for set union.

If the number of nodes in the tree with root $i$ is less than the number of nodes in the tree with root $j$, then make $j$ the parent of $i$; otherwise make $i$ the parent of $j$.

- In order to implement the weighting rule, we need to know the number of elements in each set. This can be done using the root location in the $p$ array. Set it to $-count(i)$, $count(i)$ is the number of elements in set $i$.
- Example: the disjoint sets can be represented as

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | -4 | 5 | -3 | 3 | -3 | 3 | 1 | 1 | 1 | 5 |



# Disjoint Sets – Weighted Set Union

## Algorithm 2.3.5. Weighted Set Union.

```
   // Form union of two sets with roots, i and j, using the weighting rule.
   // Input: roots of two sets i, j
   // Output: none.
 1 Algorithm WeightedUnion(i, j)
 2 {
 3       temp := p[i] + p[j]; // Note that temp < 0.
 4       if (p[i] > p[j]) then { // i has fewer elements.
 5           p[i] := j;
 6           p[j] := temp;
 7       }
 8       else { // j has fewer elements.
 9           p[j] := i;
10           p[i] := temp;
11       }
12 }
```

- Using this algorithm, the depth of the union tree can be controlled.

# Weighted Set Union – Complexity

## Lemma 2.3.6.

Assume that we start with a forest of trees, each having one element. Let $T$ be a tree with $m$ nodes created as a result of a sequence of unions each performed using `WeightedUnion` algorithm. The height of $T$ is no greater than $\lfloor \lg m \rfloor + 1$.

**Proof.** The first step is true when two sets of one element are united. Assume the Lemma is true for the first $m-1$ operations, consider the last step of the union operations, `WeightedUnion`$(k, j)$. If set $j$ has $a$ elements, then set $k$ has $m-a$ elements. And, $1 \leq a \leq m/2$. The height of $T$ must be the same as that of $k$ or one more than that of $j$. In the former case, the height of $T$ is $\leq \lfloor \lg(m-a) \rfloor + 1 \leq \lfloor \lg m \rfloor + 1$. In the latter case, the height of $T$ is $\leq \lfloor \lg a \rfloor + 2 \leq \lfloor \lg m/2 \rfloor + 2 \leq \lfloor \lg m \rfloor + 1$. □

- Thus, the union set created using Algorithm `WeightedUnion` has no more than $\lfloor \lg m \rfloor + 1$ levels.
- And the time complexity of `Find` algorithm on the resulting set is $\mathcal{O}(\lg m)$.



# Disjoint Sets – Collapsing Find

- The height of a set may still be improved using the collapsing rule.

## Definition 2.3.7. Collapsing Rule.

If $j$ is an element on the path from $i$ to its root and $p[i] \neq root(i)$, then set $p[j]$ to $root(i)$.

- The `CollapsingFind` algorithm below utilizes this rule.

## Algorithm 2.3.8. Collapsing Find.

```
   // Find the root of i, and collapsing the elements on the path.
   // Input: an element i
   // Output: root of the set containing i.
1  Algorithm CollapsingFind(i)
2  {
3        r := i; // Initialized r to i.
4        while (p[r] > 0) do r := p[r]; // Find the root.
5        while (i ≠ r) do { // Collapse the elements on the path.
6              s := p[i]; p[i] := r; i := s;
7        }
8        return r;
9  }
```

# Ackermann's Function

## Definition 2.3.9. Ackermann's function.

The Ackermann's function is defined as

$$\begin{array}{ll} A(1,j) = 2^j & \text{for } j \geq 1, \\ A(i,1) = A(i-1,2) & \text{for } i \geq 2, \\ A(i,j) = A(i-1, A(i,j-1)) & \text{for } i,j \geq 2. \end{array} \tag{2.3.3}$$

Also define

$$\alpha(p,q) = \min\{z \geq 1 | A(z, \lfloor \frac{p}{q} \rfloor) > \lg q\}, p \geq q \geq 1. \tag{2.3.4}$$

$A(1,1) = 2$ $A(1,2) = 4$ $A(1,3) = 8$ $A(1,4) = 16$
$A(2,1) = 4$ $A(2,2) = 16$ $A(2,3) = 2^{16}$ $A(2,4) = 2^{65536}$
$A(3,1) = 16$ $A(3,2) \gg 2^{65536}$
$A(4,1) \gg 2^{65536}$

- $A$ is very fast growing function and $\alpha$ is a very slow growing function.
- Note that $A(3,1) = 16$, $\alpha(p,q) \leq 3$ for $q < 2^{16} = 65,536$ and $p > q$.
- Since $A(4,1)$ is a very large number, $\alpha(p,q) \leq 4$ for all practical purposes.



# Tarjan and Van Leeuwen Bound

## Lemma 2.3.10. Tarjan and Van Leeuwen bounds.

Assume that we start with a forest of trees, each having one node. Let $T(f,u)$ be the maximum time required to process any intermixed sequence of $f$ finds and $u$ unions. Assume that $u \geq n/2$, then

$$k_1\Big(n + f \cdot \alpha(f+n, n)\Big) \leq T(f,u) \leq k_2\Big(n + f \cdot \alpha(f+n, n)\Big) \tag{2.3.5}$$

for some positive constants $k_1$ and $k_2$.

- Proof please see textbook [Cormen], pp. 575-581.

- Thus, manipulating disjoint sets are rather efficient.
- Though algorithms (2.3.2), (2.3.3), (2.3.5), and (2.3.8) assume the disjoint sets are represented using a simple array, they can be implemented if the disjoint sets are represented using linked lists as well.
- The complexities are the same with either data structure.

# Summary

- Graphs
  - Definitions.
  - Adjacency matrix.
  - Adjacency lists.
- Network connectivity problem
- Disjoint sets.
  - Set union.
  - Set find.
  - Weighted set union.
  - Collapsing set find.