EE 3980 Algorithms

Homework 3. Heap Sort

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1. Introduction

In this homework, we implemented heap sort and compared its performance with selection sort, insertion sort, bubble sort and shaker sort by sorting lists containing a different number of randomly-ordered English words. We not only measured their average performance but also rearranged the input data according to our analysis to test their best-case and worst-case execution time. In the end, we plot out the result and observe the growth of CPU time to verify our derivation of their time complexity.

2. Analysis & Implementation

2.1 Selection Sort

```
// Sort A[1 : n] into nondecreasing order.
// Input: array A with n elements
// Output: array A sorted
Algorithm SelectionSort(A, n)
{
    for i := 1 to n do {
        j := i; // initialize j to be i
        // find the smallest in A[i + 1 : n]
        for k := i + 1 to n do
            //if found, record the index
```

```
                if (A[k] < A[j]) then j := k;
            // swap A[i] and A[j]
            t := A[i]; A[i] := A[j]; A[j] := t;
        }
}
```

In selection sort, we can see that the program has to go through the same number of comparisons and swaps no matter how the input data distributes. Therefore, the worst-case and best-case performance are the same, which is

$$\sum_{i=1}^{n}\sum_{k=i+1}^{n} 1 = \sum_{i=1}^{n}(n - i) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} = \mathcal{O}(n^2)$$

Also, selection sort requires $\mathcal{O}(n)$ space for storing the array to be sorted, and same space complexity applies for the rest of sorting algorithms as well.

Best-case time complexity: $\mathcal{O}(n^2)$

Average time complexity: $\mathcal{O}(n^2)$

Worst-case time complexity: $\mathcal{O}(n^2)$

Space complexity: $\mathcal{O}(n)$

2.2 Insertion Sort

```
// Sort A[1 : n] into nondecreasing order.
// Input: array A with n elements
// Output: array A sorted
Algorithm InsertionSort(A, n)
{
    // assume A[1 : j - 1] already sorted
    for j := 2 to n do {
        item := A[j]; // move A[j] to its proper place
        i := j - 1; // initialize i to be j - 1
```

```
10.         // find i such that A[i] <= A[j]
11.         while ((i >= 1) and (item < A[i])) do {
12.             // move A[i] up by one position
13.             A[i + 1] := A[i];
14.             i := i - 1;
15.         }
16.         A[i + 1] := item; // move A[j] to A[i + 1]
17.     }
18. }
```

In insertion sort, we notice that the while loop at line 11 goes on when the index is not out of bound and $item < A[i]$. Hence, the while loop will execute at most $j - 1$ times when $item < A[i], \forall i = 1, 2, \ldots, j - 1$ given $item = A[j]$, i.e. $A[j]$ is lesser than all the elements to its left. If this condition holds for $j = 2, \ldots, n$, the worst-case input will be in nonincreasing order. Therefore, if we count the number of times line 13 being executed, it will be

$$\sum_{j=2}^{n} \sum_{i=1}^{j-1} 1 = \sum_{j=2}^{n} j - 1 = \frac{n(n-1)}{2} = \mathcal{O}(n^2)$$

times; while the best case occurs when

$$A[j] > A[i], \forall i = 1, 2, \ldots, j - 1, \forall j = 2, \ldots, n$$

which requires the input to be in nondecreasing order and the while loop will only execute one time. Therefore, the best-case time complexity is $\mathcal{O}(n)$. On average, we expect the time complexity of the while loop to be bounded by $\mathcal{O}(n)$ rather than $\mathcal{O}(1)$, which makes average time complexity of insertion sort $\mathcal{O}(n^2)$.

Best-case time complexity: $\mathcal{O}(n)$

Average time complexity: $\mathcal{O}(n^2)$

Worst-case time complexity: $\mathcal{O}(n^2)$

Space complexity: $\mathcal{O}(n)$

2.3 Bubble Sort

```
// Sort A[1 : n] into nondecreasing order.
// Input: array A with n elements
// Output: array A sorted
Algorithm BubbleSort(A, n)
{
    // find the smallest item for A[i]
    for i := 1 to n - 1 do {
        for j := n to i + 1 step -1 do {
            if (A[j] < A[j - 1]) {
                // swap A[j] and A[j - 1]
                t := A[j];
                A[j] := A[j - 1];
                A[j - 1] := t;
            }
        }
    }
}
```

In bubble sort, what the inner loop does is moving the smallest element in $A[i, \ldots, n]$ to index i by swapping. Therefore, if the input data is in nonincreasing order, it will require most swaps, thus making the worst-case performance to be $\mathcal{O}(n^2)$. On the contrary, if the input data is already in nondecreasing order, then no swap is needed. However, in our implementation, we still need $\mathcal{O}(n^2)$ of comparisons.

Best-case time complexity: $\mathcal{O}(n^2)$

Average time complexity: $\mathcal{O}(n^2)$

Worst-case time complexity: $\mathcal{O}(n^2)$

Space complexity: $\mathcal{O}(n)$

2.4 Shaker Sort

```
// Sort A[1 : n] into nondecreasing order.
// Input: array A with n elements
// Output: array A sorted
Algorithm ShakerSort(A, n)
{
    l := 1; r := n;
    while l <= r do {
        // element exchange from r to l
        for j := r to l + 1 step -1 do {
              if (A[j] < A[j - 1]) {
                  // swap A[j] and A[j - 1]
                  t := A[j];
                  A[j] := A[j - 1];
                  A[j - 1] := t;
              }
          }
          l := l + 1;
          // element exchange from l to r
          for j := l to r - 1 do {
              if (A[j] > A[j + 1]) {
                  // swap A[j] and A[j + 1]
                  t := A[j];
                  A[j] := A[j + 1];
                  A[j + 1] := t;
              }
          }
          r := r - 1;
      }
```

```
}
```

In shaker sort, the first for loop moves the smallest element in $[l, \dots, r]$ to l and the second for loop moves the greatest element in $[l, \dots, r]$ to r. Therefore, to make the execution time longest, the input data should order as follows:

greatest	2^{nd} greatest	…	2^{nd} smallest	smallest

which is in nonincreasing order; the execution time will be the shortest when in input data is in nondecreasing order, since it requires least swap. Nevertheless, just like bubble sort, the number of comparisons is deterministic, which makes both best-case and worst-case time complexity $O(n^2)$.

Best-case time complexity: $O(n^2)$

Average time complexity: $O(n^2)$

Worst-case time complexity: $O(n^2)$

Space complexity: $O(n)$

2.5 Heap Sort

```
// To enforce max heap property for n-element heap A
// with root i.
// Input: size n max heap array A with root at i
// Output: updated A
Algorithm Heapify(A, i, n)
{
    j := i * 2; // init A[j] to be the lchild of A[i]
    item := A[i]; // copy the original A[i]
    done := false;
      while ((j <= n) and (not done)) do {
```

```
        if ((j < n) and (A[j] < A[j + 1])) then
            j := j + 1; // make A[j] the larger child
        // if A[j] is larger than the original A[i]
        if (item > A[j]) then
            done = true; // end the loop
        else {
            // overwrite A[j]'s parent with A[j]
            A[⌊j / 2⌋] := A[j];
            j := j * 2; // move down to lchild
        }
    }
    // move the original A[i] to proper place
    A[⌊j / 2⌋] = item;
}

// Sort A[1 : n] into nondecreasing order.
// Input: array A with n elements
// Output: array A sorted
Algorithm HeapSort(A, n)
{
    // initialize A[1 : n] to be a max heap
    for i := ⌊n / 2⌋ to 1 step -1 do
        Heapify(A, i, n);

    for i := n to 2 step -1 do {
        // move maximum to the end
        t := A[1]; A[i] := A[1]; A[i] := t;
        // make A[1 : i - 1] a max heap
        Heapify(A, 1, i - 1);
    }
}
```

To make the implementation of heap sort easier, we separate the operation *Heapify* from the main algorithm. What *Heapify* does is moving the root element to the proper place by swapping with its larger child. In *HeapSort*'s first for loop, we execute *Heapify* from the lowest non-leave node all the way to the root. In this way, we can make sure all the subtrees follow the max heap property, thus making array A a max heap. As for the second loop of *HeapSort*, we swap the root element, which is the maximum, with the last element of the array and *Heapify* the array with size decrease by one to make the next maximum be at the root again. Eventually, the element in A will be sorted in nondecreasing order.

In *Heapify*, the while loop ends when *j* is greater than *n* and *j* doubles whenever the root element is smaller than $A[j]$. Therefore, *Heapify* has worst-case time complexity $O(lg\ \frac{n}{i})$ in some scenarios, e.g. when the root element is the minimum of the subtree. *Heapify* has best-case time complexity $O(1)$ if the root element is greater than its children.

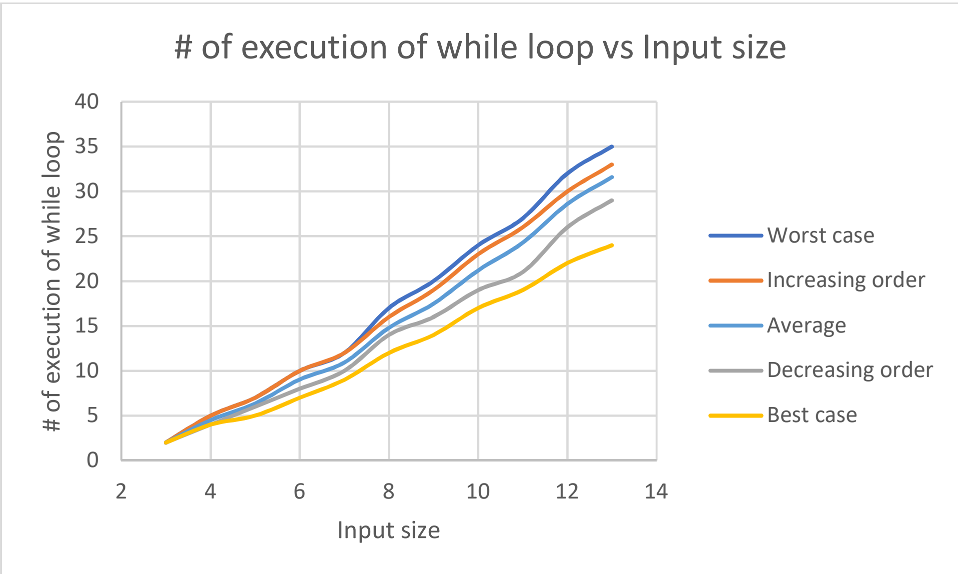
In *HeapSort*, if the input data is in min heap order, worst case will occur in every execution of *Heapify* in first for loop. Thus, it will result in time complexity

$$\sum_{i=1}^{\lfloor n\,/\,2 \rfloor} lg\ \frac{n}{i} \leq \sum_{i=1}^{\lfloor n\,/\,2 \rfloor} lg\ n = (\lfloor n\,/\,2 \rfloor) lg\ n = O(nlg\ n)$$

However, if the input data is already in max heap, then the first loop will only contribute $O(n)$ since every *Heapify* only takes $O(1)$.

For the second for loop in *HeapSort*, the worst-case time complexity is bounded by $O(nlg\ n)$. Therefore, the worst-case complexity of heap sort is $O(nlg\ n)$. However, it is hard to analyze the best-case and average time complexity explicitly. As a result, I can only upper-bounded them by $O(nlg\ n)$.

If we want to test the best-case and worst-case performance, using input with nonincreasing and nondecreasing order respectively may be justified. The graph below is the number of while loop executed versus the number of elements to be sorted. For an array with n distinct elements, there are $n!$ permutations. I enumerate all permutations and execute the heap sort to find the maximal/minimal number of while loop executed. We can treat the input with increasing order to be the lower bound of worst case and input with increasing order to be upper bound of best case.



Best-case time complexity: $O(nlg\ n)$

Average time complexity: $O(nlg\ n)$

Worst-case time complexity: $O(nlg\ n)$

Space complexity: $O(n)$

3. Result and Observation

To measure the best-case, average and worst-case performance, we execute each sorting algorithm with their corresponding rearranged input according to our previous analysis. Also, each data point in the table is the average result of 500 executions.

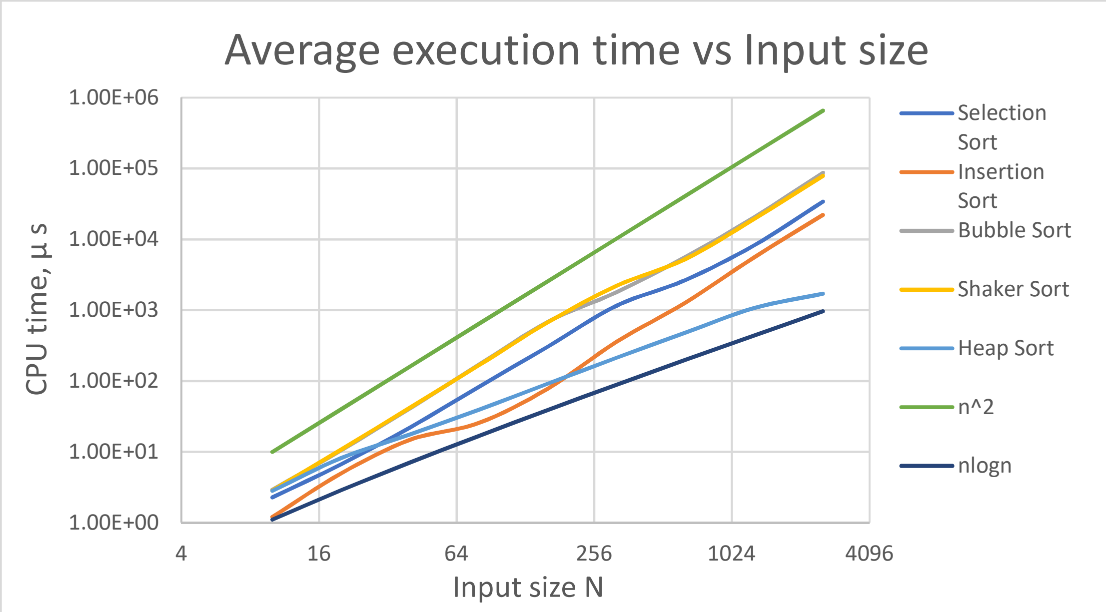
	Selection sort	*Insertion sort*	*Bubble sort*	*Shaker sort*	*Heap sort*
Best-case	Nondecreasing				Nonincreasing
Average	Original data (randomly-ordered)				
Worst-case	Nonincreasing				Nondecreasing

Table 1. Types of input to measure the performance for sorting algorithms

N	**Average**				
	Selection sort	*Insertion sort*	*Bubble sort*	*Shaker sort*	*Heap sort*
10	2.28357	1.21212	2.92778	2.85769	2.83003
20	6.68621	4.98009	10.54	10.8159	8.19206
40	22.172	14.8222	41.3857	42.2621	17.6578
80	82.8619	25.6281	169.11	166.694	39.4602
160	305.348	78.0401	684.662	673.282	91.7859
320	1154.24	357.7	1798.58	2205.04	213.186
640	2660.69	1275.13	5642.05	5257.94	481.65

1280	8227.41	5581.26	20420.3	19429.4	1072.71
2560	34145.1	22112.4	87123.1	78771.4	1709.87

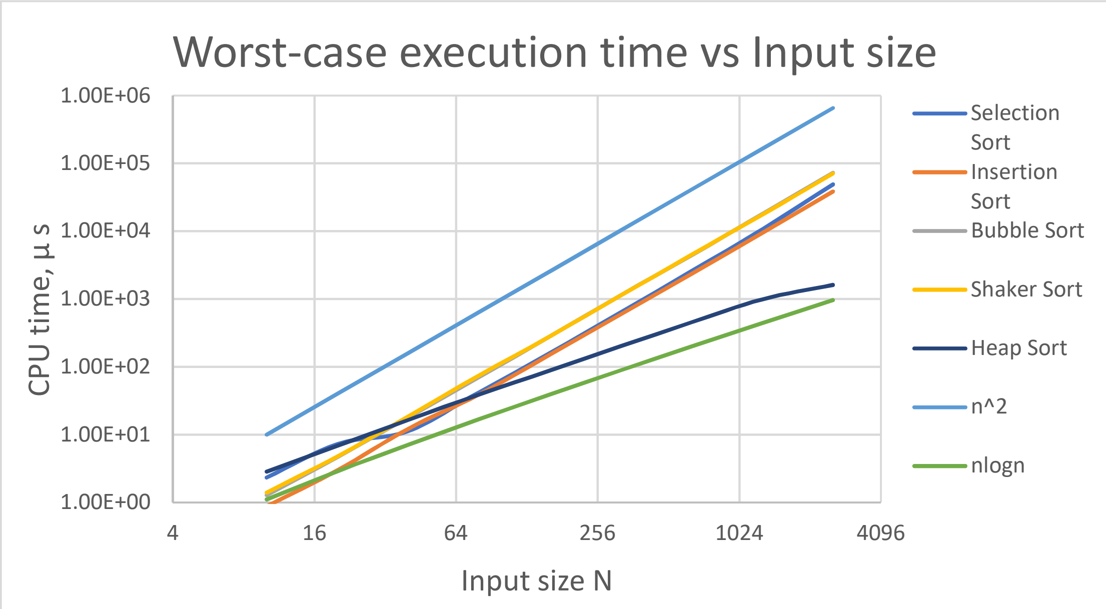
Table 2. Average CPU time [μ s] vs input data size N



N	**Worst Case**				**Lower Bound of Worst Case**
	Selection sort	*Insertion sort*	*Bubble sort*	*Shaker sort*	*Heap sort*
10	2.32601	0.87595	1.28031	1.39809	2.85387
20	7.26175	2.97165	4.68588	4.76599	6.84404
40	11.2519	12.1999	18.024	18.5838	16.4618
80	42.1681	39.0759	70.5919	74.8219	39.1397
160	159.4	150.846	279.596	281.164	88.2602
320	636.564	591.236	1131.56	1123.68	200.886

640	2579.73	2338.45	4399.62	4487.31	452.89
1280	10645.9	9373.83	17959.5	17540	990.148
2560	49081.6	38519.5	72413.9	70900.9	1615.98

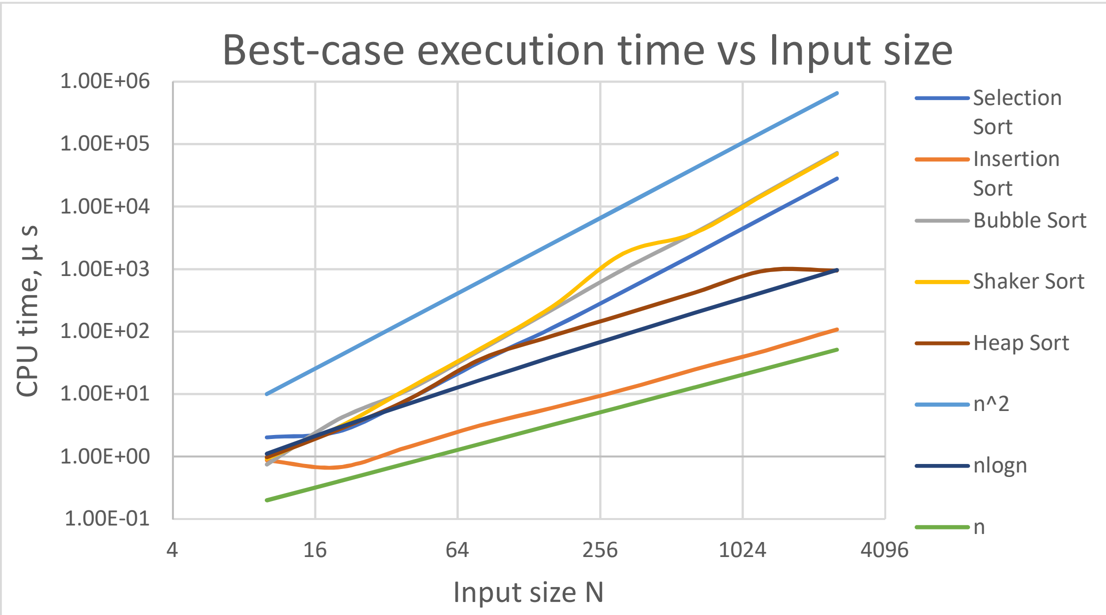
Table 3. Worst-case CPU time [μ s] vs input data size N



N	**Best Case**				**Upper Bound of Best Case**
	Selection sort	*Insertion sort*	*Bubble sort*	*Shaker sort*	*Heap sort*
10	2.03371	0.874043	0.748158	0.92411	0.976086
20	2.52008	0.674248	3.93009	2.93636	2.68412
40	8.38614	1.44815	11.8718	12.6138	8.3518
80	32.9399	3.17574	50.5142	53.412	36.1724
160	113.562	5.99384	221.974	248.068	85.0878

320	435.574	11.7278	988.59	1768.83	187.258
640	1720.92	24.6601	3762.97	3764.88	419.666
1280	7004.51	49.8462	16679.9	16191	950.098
2560	28182.4	108.496	72331.7	69787	947.914

Table 4. Best-case CPU time [μ s] vs input data size N



For the average case performance, heap sort exhibits excellent performance due to its at most $O(nlg\ n)$ time complexity. When it comes to worst-case performance, heap sort's $O(nlg\ n)$ time complexity still dominates other quadratic sorting algorithms because all of them have $O(n^2)$. However, in the best-case scenario, according to our analysis, insertion sort has $O(n)$ time complexity and it is the only case that beats heap sort. In our best-case analysis for heap sort, we only bound the time complexity for $O(nlg\ n)$ which seems reasonable. However, whether the best-case time complexity can be bounded tighter by $O(n)$ is hard to answer by solely observing the plot because $O(nlg\ n)$ and $O(n)$ have a quite similar trend.

For selection sort, although it has the same time complexity $O(n^2)$ for best-case and worst-case. The actual execution time is quite different, which is caused by whether it executes the one operation in the for block.

For bubble sort and shaker sort, they behave very similarly across best-case, average and worst-case, which validates our argument that shaker sort is bubble sort executing in different directions. With proper rearrangement for the input, we expect shaker sort to have same the performance as bubble sort on average or for extreme cases.