

# Rabbits and Recurrence Relations

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## Problem

A **sequence** is an ordered collection of objects (usually numbers), which are allowed to repeat. Sequences can be finite or infinite. Two examples are the finite sequence  $(\pi, -\sqrt{2}, 0, \pi)$  and the infinite sequence of odd numbers  $(1, 3, 5, 7, 9, \dots)$ . We use the notation  $a_n$  to represent the  $n$ -th term of a sequence.

A **recurrence relation** is a way of defining the terms of a sequence with respect to the values of previous terms. In the case of Fibonacci's rabbits from the [introduction](https://rosalind.info/problems/fib/), any given month will contain the rabbits that were alive the previous month, plus any new offspring. A key observation is that the number of offspring in any month is equal to the number of rabbits that were live two months prior. As a result, if  $F_n$  represents the number of rabbit pairs alive after the  $n$ -th month, then we obtain the Fibonacci sequence having terms  $F_n$  that are defined by the recurrence relation  $F_n = F_{n-1} + F_{n-2}$  (with  $F_1 = F_2 = 1$  to initiate the sequence). Although the sequence bears Fibonacci's name, it was known to Indian mathematics over two millennia ago.

When finding the  $n$ -th term of a sequence defined by recurrence relation, we can simply use the recurrence relation to generate terms for progressively larger values of  $n$ . This problem introduces us to the computation technique of **dynamic programming**, which successively builds up solutions by using the answers to smaller cases.

- **Given:** Positive integers  $n \leq 40$  and  $k \leq 5$ .
- **Return:** The total number of rabbit pairs that will be present after  $n$  months, if we begin with 1 pair and in each generation, every pair of reproduction-age rabbits produces a litter of  $k$  rabbit pairs (instead of only 1 pair).

## Sample Dataset

5 3

## Sample Output

19

## Intuition

Express the recurrence relationship using an algebraic equation. In this case, a rabbit pair at reproduction age produces  $k$  pairs of offspring. The length of a generation is a single month. The offspring would need to mature an entire generation before producing its own progeny. We can illustrate how a rabbit population (starting with 1 pair) propagates after 5 generations: Each box represents a rabbit pair where blue corresponds to sexually-immature rabbit pairs and orange to sexually-mature rabbit pairs. Arrows connect parents to their offsprings.

The first two months will always have a single rabbit pair therefore we only update the values after two months. In the third, fourth, and fifth months, the total amount of rabbits can be computed as:

$$p_3 = p_1 + k * p_2$$

$$p_4 = p_2 + k * p_3$$

$$p_5 = p_3 + k * p_4$$

where  $p_i$  is the total rabbit pairs in the  $i^{\text{th}}$  generation. The formula can be generalized as:

$$p_n = p_{n-2} + k * p_{n-1}$$

From the example input, we expect 19 rabbits after 5 months, assuming that each mature pair will produce 3 pairs of offspring per month.

## Solution

```
def fibunnyci(n: int, k: int) -> int:
    """Return the number of rabbit pairs after `n` generations."""
    a, b = 1, 1
    for i in range(2, n):
        a, b = b, k*a + b
    return b
```

```
n, k = 5, 3
result = fibunnyci(n, k)
print(result)
```

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