Assignment 2

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1 $\frac{\partial \hat{y}}{\partial \hat{x}}$, given that $\hat{x}$ is a function of $\hat{z}$

Proof. Let x = (kx1) column vector such that $\frac{\partial}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{pmatrix}$.

$$\frac{\partial \hat{y}}{\partial \hat{x}} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \cdots \\ \frac{\partial}{\partial x_k} \end{pmatrix}_{1\,x\,m} \quad \begin{pmatrix} y_{x_n} & y_{x_n} & \vdots & y_{x_n} \end{pmatrix}_{n\,x\,1}.$$

Then by properties of the Kronocker product, denoted:

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{pmatrix} \otimes \begin{pmatrix} y_{x_n} & y_{x_n} & \vdots & y_{x_n} \end{pmatrix}.$$

which results in a (nxm) jacobian matrix:

$$\begin{pmatrix} x_1 y_1^{'} & x_2 y_1^{'} & \dots & x_n y_1^{'} \\ x_1 y_2^{'} & x_2 y_2^{'} & \dots & x_n y_2^{'} \\ \vdots & & \ddots & \\ x_1 y_n^{'} & x_2 y_n^{'} & \dots & x_n y_n^{'} \end{pmatrix}$$

Let this (mxn) matrix be A.

Then, because $\hat{x}$ was a function of $\hat{z}$, by properties of the chain rule wrt to matrix operations, we multiply A by the derivative of $\hat{x}$ wrt $\hat{z}$

$$\therefore \frac{\partial \hat{y}}{\partial \hat{x}} = A \frac{\partial \hat{x}}{\partial \hat{z}}$$

$\square$

2 $\frac{\partial \alpha}{\partial \hat{x}}$ given the scalar matrix $\alpha = \hat{y}^T A \hat{x}$.

It is sufficient to show the case of $\frac{\partial \alpha}{\partial \hat{x}}$ as the case for $\frac{\partial \alpha}{\partial \hat{y}}$ follows closely:

Proof. Replace $\alpha = \hat{y}^T A \hat{x} \implies \frac{\partial (\hat{y}^T A \hat{x})}{\partial \hat{x}}$, and let A be a constant (mxn) constant matrix:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

let $\hat{y}^T$ be a (1xm) row vector, $\begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}$ and let $\hat{x}$ be (kx1) column vector, such that $\frac{\partial}{\partial \hat{x}} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \dots & \frac{\partial}{\partial x_k} \end{pmatrix}$

Then by the product rule wrt matrix operations,

$$\frac{\partial (\hat{y}^T A \hat{x})}{\partial \hat{x}}$$
$$= (y^T \frac{\partial \hat{x}}{\partial \hat{x}} + x^T \frac{\partial \hat{y}^T}{\partial \hat{x}})A$$

$\frac{\partial \hat{x}}{\partial \hat{x}}$ results in the Identity Matrix, I and $\frac{\partial \hat{y}^T}{\partial \hat{x}} = 0$ given by,

$$\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix} = 0$$

Therefore our result gives,

$$\frac{\partial \alpha}{\partial \hat{x}} = y^T A$$
$$= \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$\square$

3 $\frac{\partial \alpha}{\partial \hat{x}}$ given $\alpha = \hat{x}^T A \hat{x}$

Proof. Given $\alpha = \hat{x}^T A \hat{x}$, A is a constant (mxn) matrix and x is a (kx1) column vector such that $\frac{\partial}{\partial x} = \begin{pmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \dots & \frac{\partial}{\partial x_k} \end{pmatrix}$ and x^T is a (1xk) row vector.

Then by properties of the product rule wrt to matrix operations, we have: $\frac{\partial x^T}{\partial x} A + x^T A \frac{\partial \hat{x}}{\partial \hat{x}}$

$$\left(\begin{pmatrix} x_1 & x_2 & \dots & x_k \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \dots \\ \frac{\partial}{\partial x_k} \end{pmatrix} \right) \begin{pmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,n} \\ \vdots & & \ddots & \\ a_{m\,1} & a_{m\,2} & \dots & a_{m\,n} \end{pmatrix}$$

$$+ \begin{pmatrix} x_1 & x_2 & \dots & x_k \end{pmatrix} \begin{pmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,m} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,m} \\ \vdots & & \ddots & \\ a_{n\,1} & a_{n\,2} & \dots & a_{n\,m} \end{pmatrix} \left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{pmatrix} \right)$$

gives,

$$\begin{pmatrix} x_1^{'} & 0 & 0 & \dots \\ 0 & x_2^{'} & 0 & \dots \\ \vdots & & \ddots & \\ 0 & 0 & \dots & x_k^{'} \end{pmatrix} \otimes \begin{pmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,n} \\ \vdots & & \ddots & \\ a_{m\,1} & a_{m\,2} & \dots & a_{m\,n} \end{pmatrix}$$

$$\oplus \begin{pmatrix} x_1 & x_2 & \dots & x_k \end{pmatrix} \otimes \begin{pmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,m} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,m} \\ \vdots & & \ddots & \\ a_{n\,1} & a_{n\,2} & \dots & a_{n\,m} \end{pmatrix} \begin{pmatrix} I \end{pmatrix}$$

without loss of generality, factoring x^T gives $x^T(A + A^T)$

Therefore $\frac{\partial \alpha}{\partial \hat{x}} = x^T$ gives $x^T(A + A^T)$

In the case that A=A^T we see that $x^T(A + A^T)$ gives $2\,x^T A$. $\square$