Introduction to Measure Theory

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1 The Exterior Measure

Def: Let E $\subset \mathbb{R}^d$. The exterior measure of E is defined as

$$m_*(E) = \inf\{\sum_{i=1}^{\infty} |Q_i| : E \subset \cup_{i=1}^{\infty} Q_i\} \tag{1}$$

where the infimum is taken over the countable covering

$$\cup_{i=1}^{\infty} Q_i \supset E \tag{2}$$

by cubes $Q_i \in \mathbb{R}^d$.

Remarks:

1. $m_* : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$.
2. finite sums in the definition is not enough.

In this case we get the outer Jordan measure which in general is bigger than our outer measure.
Take. $E = \mathbb{Q} \cap [0,1] \implies J_*(E) = 1$

Note, for our setting, $\mathbb{Q} \cap [0,1] \implies m_*(E) = 0$

1. $m_*(x) = 0$, and same of the emqpty set.
2. if Q $\subset \mathbb{R}^p$ is a closed club, then $m_*(Q) = |Q|$

Since $Q supset Q \implies m_*(Q) \leq |Q|$.

Proof. Let

$$Q \subset \cup_i^{\infty} Q_j$$

Let $\epsilon > 0$ and choose an open cube $S_j \supset Q_j$ such that $|\bar{S_j}| \leq (1+\epsilon)|Q_j|$ Since

$$\bigcup_{j=1}^{\infty}$$

is an open coverign of the compact set Q, there is a finite subcovering such that $Q \subset \cup_{j=1}^{n} S_j$

Applying lemma 2, we deduce that

$$|Q| \leq |\bar{S_j}| + \cdots + |\bar{S_{j_k}}| \leq (1+\epsilon)(|Q_{j_i}| + \cdots + |Q_{j_k}|) \leq (1+\epsilon)\sum_{j=1}^{\infty} |Q_j| \tag{3}$$

for all $\epsilon > 0$

Take $\epsilon \to 0$ to get the result.

If Q is an open cube then $m_*(Q) = |Q|$ Since $Q \subset Q \implies m_*(Q) \leq |\bar{Q}| = |Q|$ Let $\epsilon > 0$ and choose a closed cube $Q_o \subset Q$ such that $|Q| \leq (1+\epsilon)|\bar{S}|$ Then $|Q_o \geq |Q| - \varepsilon \implies |Q| - \varepsilon \geq |\bar{Q_o}| = m_*(Q) \leq m_*(Q)$ for any epsilon If epsilon goes to zero we get $|Q| \leq m_*(Q)$ ☐

If we use a closed rectangle, we see that the intial part of the proof applies the same way. We just need to show that $M_*(R) \leq |R|$ The Key theorem infolved to prove both examples and both inequalities is the Heine-Borel theorem.

2 Properties of the Exterior Measure

Let $E \subset \mathbb{R}^d$. Then $(\forall\varepsilon > 0)$ there exists a countable collection of cubes $\{Q_i\}$ such that
E $\subset \cup_{i=1}^{\infty} Q_i$ and

$$m_*(E) + \varepsilon \geq \sum_{i=1}^{\infty} |Q_i| \leq m_*(E) + \varepsilon, \tag{4}$$

if $m_*(E) < \infty$

If $m_*(E) = \infty, \sum_{i=1}^{\infty} |Q_i| = \infty$
The exterior measure has the following properties:

1. Monotonicity: If $E_1 \subset E_2 \subset \mathbb{R}^d$, then $m_*(E_1) \leq m_*(E_2)$.

$$\text{E is bounded, } \implies m_*(E) < \infty. \tag{5}$$

2. Countable subadditivity:

$$E = \cup_{i=1}^{\infty} E_i, \implies m_*(E) \leq \sum_{i=1}^{\infty} m_*(E_i) \tag{6}$$

$$E = \cup_{i=1}^{n} E_i, \implies m_*(E) \leq \sum_{i=1}^{n} m_*(E_i) \tag{7}$$

Proof. Assume $m_*(E_j) < \infty \, \forall j \in \mathbb{N}$
Let $\varepsilon > 0$ and choose a countable collection of cubes $\{Q_{ij}\}$ such that $E_j \subset \cup_{i=1}^{\infty} Q_{ij}$ such that

$$m_*(E_j) + \frac{\varepsilon}{2^j} \geq \sum_{i=1}^{\infty} |Q_{ij}| \geq m_*(E_j) - \frac{\varepsilon}{2^j} \tag{8}$$

Then $E \subset \cup_{i,j}^{infty} Q_{ij}$ isa covering of closed cubes and $m_*(E) \leq \sum_{i,j} |Q_{ij}| \leq \sum_j m_*(E_j) + \frac{\varepsilon}{2^j}$
Let $\varepsilon \to 0$ to get the result:

$$m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j) \tag{9}$$

□

Proposition:
If $E \subset \mathbb{R}^d$ and $m_*(E) = inf\{m_*(U) : E \subset U, U \text{ is open}\}$ Then, by Monotonicity, $m_*(E) \leq inf\,\{m_*(U) : E \subset U, U \text{ is open}\}$ We need to show that the infimum is less than or equal to $m_*(E)$
It is sufficent to show that $\forall\varepsilon > 0$ there exists an open set U such that $E \subset U$ and $m_*(U) \leq m_*(E) + \varepsilon$ Let $\varepsilon > 0$ and choose a countable collection of cubes $\{Q_i\}$ such that $E \subset \cup_{i=1}^{\infty} Q_i$ and

$$m_*(E) + \varepsilon \leq \sum_{i=1}^{\infty} |Q_i| \leq m_*(E) - \frac{\varepsilon}{2} \tag{10}$$

For each j, let $\tilde{Q}_j$ denote an open cube such that $Q_j \subset \tilde{Q}_j$ and $|\tilde{Q}_j| \leq |Q_j| + \frac{\varepsilon}{2^{j+1}}$ Then U $= \cup_{j=1}^{\infty} \tilde{Q}_j$ is open and, due to subadditivity, we have:

$$m_*(U) \leq \sum_{j=1}^{\infty} m_*(\tilde{Q}) = \sum_{j=1}^{\infty} |\tilde{Q}_j| \leq \sum_{j=1}^{\infty} |Q_j| + \frac{\varepsilon}{2^{j+1}} = \sum_{j=1}^{\infty} |Q_j| + \frac{\varepsilon}{2} \leq m_*(E) + \varepsilon \tag{11}$$

Proposition:
If $E = E_1 \bigcup E_2$ and $d(E_1, E_2) > 0$ then $m_*(E) = m_*(E_1) + m_*(E_2)$

Proof. Due to subadditivity, we have $m_*(E) \leq m_*(E_1) + m_*(E_2)$
We must show that $m_*(E) \geq m_*(E_1) + m_*(E_2)$
Choose a convering of Closed Cubes such that

$$m_*(E) \leq \sum_{i=1}^{\infty} |Q_i| \leq m_*(E) + \varepsilon \tag{12}$$

Then

$$E_1 \subset \cup_{i=1}^{\infty} Q_i \implies m_*(E_1) \le \sum_{i=1}^{\infty} |Q_i| \le m_*(E_1) + \varepsilon \tag{13}$$

Then

$$E_2 \subset \cup_{i=1}^{\infty} Q_i \implies m_*(E_2) \le \sum_{i=1}^{\infty} |Q_i| \le m_*(E_2) + \varepsilon \tag{14}$$

Then

$$E \subset \cup_{i=1}^{\infty} Q_i \implies m_*(E) \le \sum_{i=1}^{\infty} |Q_i| \le m_*(E) + \varepsilon \tag{15}$$

Subdividing the cubes, we can assume that each Q_j has a diameter less than $d(E_1, E_2)$
Let $J_1 := \{j \in \mathbb{N} : Q_j \cap E_1 \neq \emptyset\}$
and
Let $J_2 := \{j \in \mathbb{N} : Q_j \cap E_2 \neq \emptyset\}$
Then J_1 and J_2 are disjoint and $E_1 \subset \cup_{j \in J_1} Q_j$ and $E_2 \subset \cup_{j \in J_2} Q_j$ Then $E \subset \cup_{j \in J_1} Q_j \cup \cup_{j \in J_2} Q_j$ Then

$$m_*(E) \le \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \le m_*(E) + \varepsilon \tag{16}$$

□

Proposition:
If $E \subset \mathbb{R}^d$ isa countable union of almost disjoint cubes, then $m_*(E) = \sum_{i=1}^{\infty} |Q_i|$

Proof. Let $\tilde{Q}_j$ be a cube strictly contained in Q_j such that $|Q_j| < |\tilde{Q}_j| + \frac{\varepsilon}{2^j}$ for all $\tilde{Q}_j$ a positive distance away from eachother
Applying the previous proposition, we obtain:

$$m_*(\cup_{j=1}^{infty} Q_j) = \sum_{j=1}^{N} |\tilde{Q}_j| \ge \sum_{j=1}^{N} |Q_j| - \frac{\varepsilon}{2^j} \ge \sum_{j=1}^{N} |Q_j| - \varepsilon \tag{17}$$

which implies

$$m_*(\cup_{j=1}^{infty} Q_j) \ge \sum_{j=1}^{N} |Q_j| - \varepsilon, \forall N \text{and} \forall \varepsilon > 0 \tag{18}$$

Take $N \to \infty$ and $\varepsilon \to 0$ to get the result:

$$m_*(E) \ge \sum_{j=1}^{\infty} |Q_j| \tag{19}$$

□

Remark:
If U is an open set in $\mathbb{R}^d$ then $m_*(\cup_{i=1}^{\infty} Q_i) = \sum_{i=1}^{\infty} |Q_i|$ is a decomposition of U into almost disjoint cubes.

3. Translation invariance:

$$E \subset \mathbb{R}^d, \implies m_*(E + x) = m_*(E) \forall x \in \mathbb{R}^d \tag{20}$$

3 Lebesgue Measure and Lebesgue Measurable Sets

Definition:

1. A set E $\subset \mathbb{R}^d$ is said to be Lebesgue measurable if $\forall \varepsilon > 0$ there exists an open set U such that $E \subset U$ and $m_*(U \backslash E) < \varepsilon$

2. If E is Lebesgue measurable, then the Lebesgue measure of E is defined as $m(E) = m_*(E)$

Proposition:
Every open set in $\mathbb{R}^d$ is Lebesgue measurable.
Properties of the Lebesgue Measure:

1. If $E \subset \mathbb{R}^d$ is Lebesgue measurable, then $m(E) = 0$
   In particular, if $F \subset E$ and $m(E) = 0$ then F is measurable

   Proof. Recall $m_*(E) = inf\{m_*(U) : E \subset U, U \text{ is open}\}$
   since $m_*(E) = 0$, we have $\forall \varepsilon > 0$ there exists an open set U such that $E \subset U$ and $m_*(U) < \varepsilon$
   Since $U \backslash E \subset U$, we have $m_*(U \backslash E) \leq m_*(U) < \varepsilon$
   If $F \subset E$ and $m(E) = 0$ then $m_*(E) = 0$ and $m_*(F) \leq m_*(E) = 0$
   Then $m_*(F) = 0$ and F is measurable □

   Remark:
   The Cantor Tenary set is measureable and has measure zero.

   Proof. trivial

   □

2. A countabe union of measureable sets is measureable.

   Proof. Let $E = \bigcup_{i=1}^{\infty} E_i$ and E_i is measureable.
   Given $\varepsilon > 0$, let U_j be open such that $O_j \supset E_j$ and $m_*(U_j \backslash E_j) < \frac{\varepsilon}{2^j}$
   Then $U = \bigcup_{j=1}^{\infty} U_j$ is open and $E \subset U$
   Then
   $$U \backslash E = \cup_{j=1}^{\infty} U_j \backslash E_j$$

   Then $O \backslash E \subset \bigcup_{j=1}^{\infty} U_j \backslash E_j$
   Then
   $$m_*(U \backslash E) \leq \sum_{j=1}^{\infty} m_*(U_j \backslash E_j) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon \tag{21}$$
   Then E is measureable □

   Lemma:
   If $F \subset \mathbb{R}^d$ is closed and $K \subset \mathbb{R}^d$ is compact and $F \bigcap K =$, then $d(F, K) > 0$

   Proof. Assume that the $d(F, K) = 0$
   Then $\exists$ a sequence $\{x_n\}$ in F and a sequence $\{y_n\}$ in K such that $d(x_n, y_n) < 1\frac{1}{n}$ since K is compact, due to Heine-Borel Theorem, K is bounded and closed. Then, by Bolzano-Weierstrauss Theorem, $\{y_n\}$ has a convergent subsequence $\{y_{n_k}\}$ such that $y_{n_k} \to y \in K$ Since
   $$d(x_{n_k}, y_{n_k}) < \frac{1}{n_k} \to 0 \tag{22}$$
   Then $x_{n_k} \to y$ Then $x_n \in F$ where F is closed. Therefore $y \in F$ Then $y \in F \bigcap K$ Then $d(F, K) = 0$, which contradicts our lemma that $K \bigcup F =$. □

3. Closed sets are measureable.

   Proof. It is enough to prove that compact sets are measureable: If F is closed, then $F = \bigcup_{n=1}^{\infty} F \bigcap B_n(0)$ then applie the previous property. Assume F is compact, then F is closed and bounded.
   Then $m_*(F) < \infty$
   $\forall \varepsilon > 0$, since $m_*(F) = inf\{m_*(U) : U \supset F\, U = \text{open}\}$ We can choose an U such that $F \subset U$ and $m_*(U) \leq m_*(F) + \varepsilon$ Then $U \backslash F := U \cap F^c$ is open, $U \backslash F = \bigcup_{j=1}^{\infty} Q_j$ a countable union of disjoint cubes. It is enough to prove
   $$\sum_{j=1}^{N} m_*(Q_j) < \varepsilon$$
   Let $K = \bigcup_{j=1}^{N} Q_j$ Then K is compact and $F \bigcap K =$ and $F \cup K \subset U$ Since K and F are compact, we can use the previous lemma to conclude that $d(F, K) > 0$ Now
   $$m_*(U) \geq m_*(F \cup K) = m_*(F) + m_*(K) = m_*(F) + \sum_{j=1}^{N} m_*(Q_j) \tag{23}$$

Then

$$m_*(F) + \sum_{j=1}^{N} m_*(Q_j) \leq m_*(U) - m_*(F) < \varepsilon \tag{24}$$

Then $m_*(F) < \varepsilon$
Then F is measureable □

SHOW THAT THE LEMMA FAILS IF F AND K ARE ONLY CLOSED AND NOT NECESARILY COMPACT WHERE THEY SHARE NO POINTS

4. The complement of a measureable set is measureable.

   Proof. Assume E is measureable. Then $\forall n \in \mathbb{N}, \exists O_n$ open such that $E \subset O_n$ and $m_*(O_n \backslash E) < \frac{1}{n}$ Since O_n^c is a closed set, it is measureable. Consequently, $S := \bigcap_{n=1}^{\infty} O_n^c$ is also measurable. Then $S \subset E^c$ and $E^c \backslash S \subset E \backslash O_n$ $E^c \cap S^c \subset O_n \cap E^c$ because the $\cap_k O_k \subset O_n$ Hence $m_*(E^c \backslash S) \leq m_*(E \backslash O_n) < \frac{1}{n}$
   Take $n \to \infty$ to get the result. Therefore $E^c \backslash S$ is measureable. To conclude, note that $E^c = S \cup (E^c \backslash S)$
   Then E^c is measureable. □

5. The union of two measureable sets is measureable.

   Proof. Assume a squence of sets $\{E_n\}$ is measureable. Then using DeMorgan's Law we have $\bigcup_{j=1}^{\infty} E_j = (\bigcap_{j=1}^{\infty} E_j^c)^c$ Then $\bigcap_{j=1}^{\infty} E_j^c$ is measureable and hence $\bigcup_{j=1}^{\infty} E_j$ is measureable. □

4 Theorem

If $E_1, E_2, \ldots,$ are disjoing measureable sets and $E = \bigcup_{j=1}^{\infty} E_j$ then $m(E) = \sum_{j=1}^{\infty} m(E_j)$

Proof. Case 1:
Assume that each E_j is bounded. we have

$$\sum_{j=1}^{N} m(E_j) \leq m(\bigcup_{j=1}^{N} E_j) \leq m(E) \tag{25}$$

Since E_j^c is measurable $\exists U_j \supset_{open} E_j^c$ such that $m(U_j \backslash E_j^c) < \frac{\varepsilon}{2^j}$
Let $F_j = U_j^c$ and note that F_j is closed and $E_j \backslash F_j = F_j^c \backslash E_j^c = U_j \backslash E_j$
Conesquently, $m(E_j \backslash F_j) = m_*(U_j^c \backslash E_j^c) < \frac{\varepsilon}{2^j}$
Note, F_j is compact by Heine-Borel Theorem
Let $N \in \mathbb{N}$ and since $\{F_n\}_{j=1}^{N}$ are disjoint, we have we have

$$m(\bigcup_{j=1}^{N} F_j) = \sum_{j=1}^{N} m(F_j) \tag{26}$$

since $\cup_{j=1}^{N} F_j \subset E = \cup_{j=1}^{\infty} E_j$ we have

$$\Sigma_{j=1}^{N} m(F_j) \leq m(E) \tag{27}$$

On the otherhand, we have $F_j \cup (E_j \backslash F_j) = E_j$. Hence we deduce that

$$m_*(E_j) \leq m_*(F_j) + m_*(E_j \backslash F_j) = m_*(E_j) + \frac{varepsilon}{2} \tag{28}$$

Giving us

$$\Sigma_{j=1}^{N} m(E_j) \leq \Sigma_{j=1}^{N} m(F_j) + \Sigma_{j=1}^{N} \frac{1}{2^j} \leq m(E) + \varepsilon \tag{29}$$

Take $N \to \infty$ and $\varepsilon \to 0$ to get the result

$$m_*(E) \leq \Sigma_{j=1}^{\infty} m(E_j) \tag{30}$$

Case 2:
E is not necessarily bounded.
Let $\{Q_k\}_{k=1}^{\infty}$ be a sequence of closed cubes, such that Q_k covers $\mathbb{R}^d$ set $S = Q_1$ and $S_k = Q_k \backslash Q_{k-1}$ and $E = \bigcup_{j,k=1}^{\infty} E_{j,k}$

where $E_j = \cup_{k=1}^{\infty} E_{j,k}$
Applying case 1, yields

$$m(E) = \sum_{j,k=1}^{\infty} m(E_{j,k}) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} m(E_{j,k}) = \sum_{j=1}^{\infty} m(E_j) \tag{31}$$

□

5 Corollary

Let E_1, E_2 be measurable sets.

1. If E_1 increases to E, then E is measurable and $m(E) = \lim_{n\to\infty} m(E_n)$
2. If E_1 decreases to E, then E is measurable and $m(E) = \lim_{n\to\infty} m(E_n)$

Proof. Note that E $= \bigcup_{n=1}^{\infty} E_n = E_1 \bigcup (E_2 \setminus E_1) \bigcup (E_3 \setminus E_2) \bigcup \ldots$ where each set equals and element from G_k where $E = \cup_{k=1}^{\infty} G_k$ and G_k is a disjoint set. Hence $m(E) = \sum_{k=1}^{\infty} m(G_k) = \sum_{k=1}^{\infty} m(E_k)$

$$m(E) = \lim_{n\to\infty} m(G_n) = \lim_{n\to\infty} m(E_n) \tag{32}$$

□