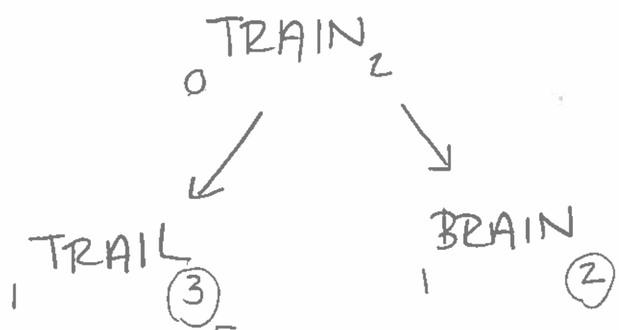
Heuristic Search

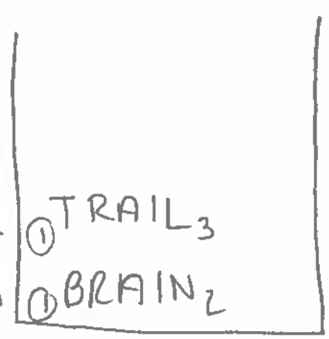
① So why were we making these guesses anyway?



3 letters are different from PRAWN, so let's guess we can reach PRAWN in 3 steps

2 letters are different from PRAWN, so let's guess we can reach PRAWN in 2 steps

② So far, the most "intelligent" container we had was a priority queue that prioritized nodes by g-value:



but these are the same

Why wouldn't we want to factor in our guess about how much it should cost to complete our solution?

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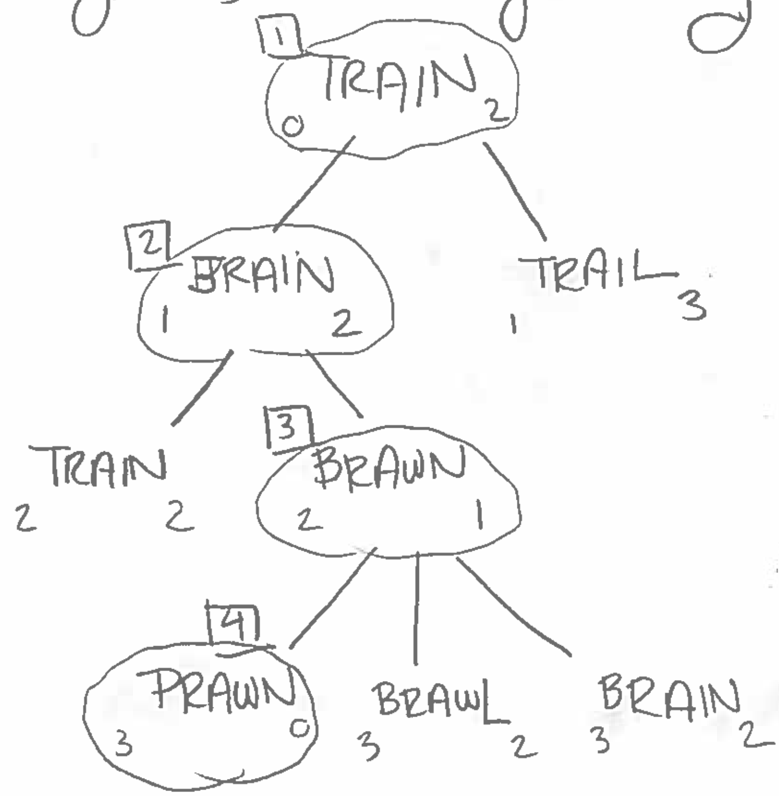
3) It would seem to make sense to have a priority queue that prioritizes nodes by:

the cost so far + the cost to reach the goal from this point on (guessed)

g-value

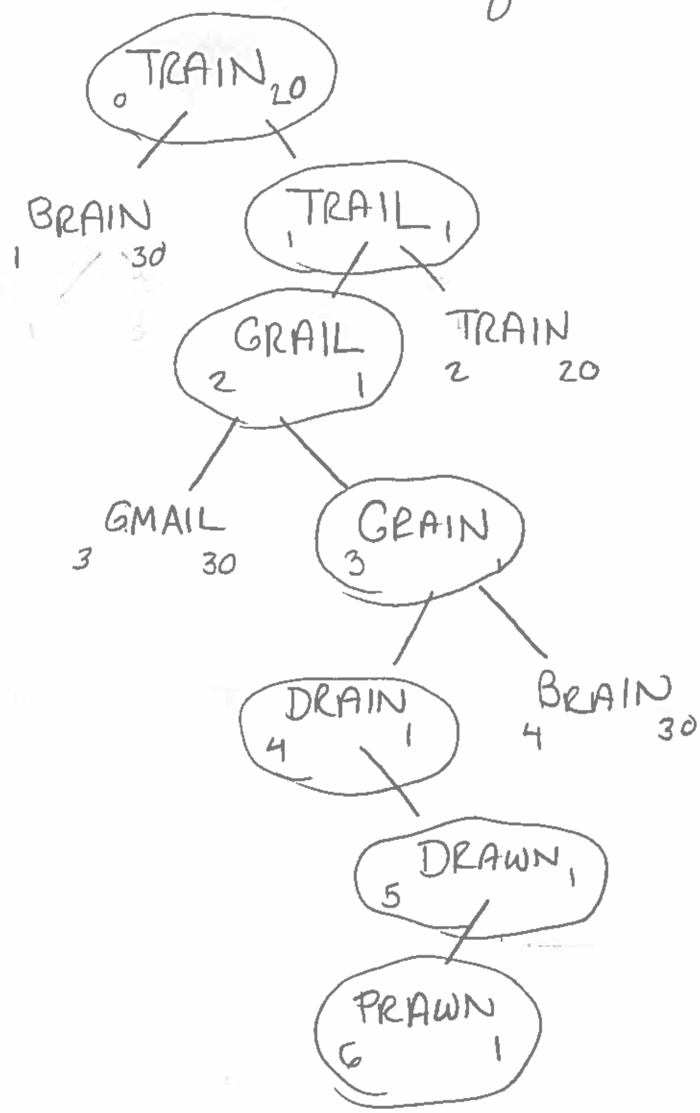
h-value

4) If we use a priority queue that prioritizes search nodes based on $g + h$, something amazing happens



We go straight to the optimal solution!

5) If our guesses are terrible, bad things can happen:



6) How do we choose a heuristic function such that amazing things, as opposed to bad things, happen?

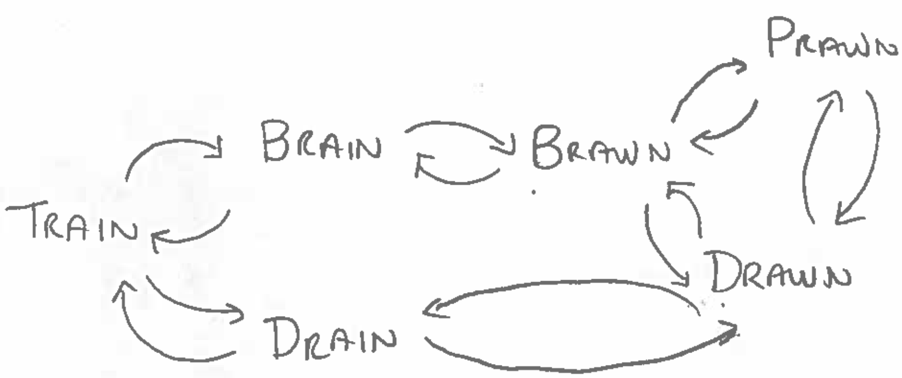
(7) To study this question, it will help to have some extra terminology.

Suppose we have a weighted state machine $M = (Q, \Sigma, \Delta, q_0, F, \omega)$.

Define a <u>completion path</u> from state $q \in Q$ as a sequence $\langle \delta_1, \ldots, \delta_K \rangle$ of transitions where:

- $\forall 1 \le i \le K$, $\delta_i \in \Delta$ and $\delta_i = (q_i, \sigma_i, q_{i+1})$
- $q_1 = q$
- $q_{k+1} \in F$

e.g.



If $q_0 =$ TRAIN and $F = \{$PRAWN$\}$, then the following are examples of completion paths from DRAIN:

< (DRAIN, 4W, DRAWN), (DRAWN, 1P, PRAWN) >

< (DRAIN, 4W, DRAWN), (DRAWN, 1B, BRAWN), (BRAWN, 1P, PRAWN) >

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8) The cost of a completion path is the sum of the weights of its transitions.

The optimal completion path (from state q) is the completion path of minimal cost.

Define $H^*(q)$ as the cost of the optimal completion path.

e.g. $H^*(\text{DRAIN}) = 2$ (i.e. the cost of completion path

< DRAIN, 4W, DRAWN>; < DRAWN, 1P, PRAWN>)

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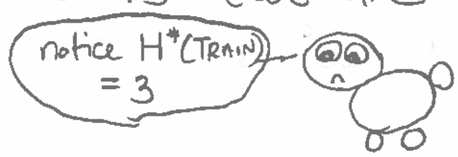
9) Let's consider what happens if we use an "optimistic" heuristic function H, in other words, one that never overestimates the cost of the optimal completion cost.

In other, other words:

$$H(q) \leq H^*(q) \quad \text{for all } q \in Q$$

An example (for word ladder) would estimate the cost of completion to be the number of letters that are different from the solution word!



TRAIN (T: different, I: different), so $H(\text{TRAIN}) = 2$

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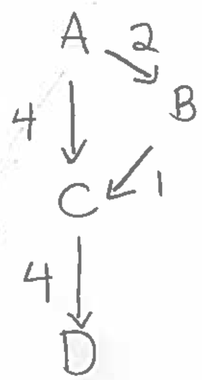
(10) Such a heuristic is called <u>admissible</u>. Let's show a simple example. Consider the state machine:



$Q = \{A, B, C, D\}$

$q_0 = A$

$F = \{D\}$

Suppose our heuristic function is defined:

$H(A) = 7$

$H(B) = 5$

$H(C) = 1$

$H(D) = 0$

We can check easily that it is admissible:

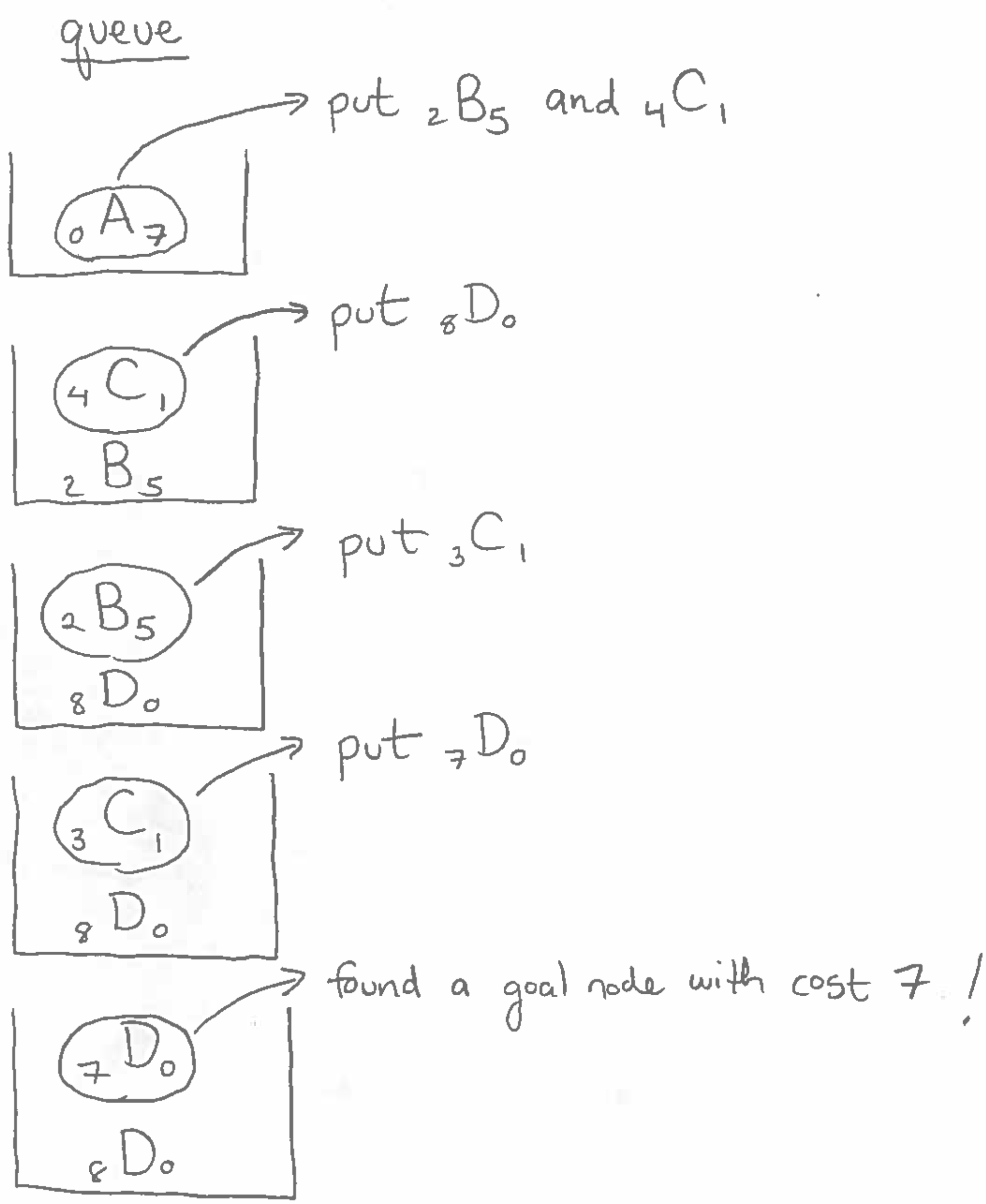
$H^*(A) = 8 \geq 7 = H(A)$

$H^*(B) = 5 \geq 5 = H(B)$

$H^*(C) = 4 \geq 1 = H(C)$

$H^*(D) = 0 \geq 0 = H(D)$

(11) Let's see how the master SEARCH algorithm performs if we use this heuristic, and a priority queue prioritized by $g+h$:



It returns the correct (optimal) solution! In fact, this is true in general. If H is admissible, and we run SEARCH using a priority queue prioritized by $g+h$, then we are guaranteed the optimal solution.

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(12) Guaranteed, that is, as long as we <u>don't</u> memoize.
Suppose we run MemoizedSearch with the same container and heuristic.

container		visited
${}_0A_7$	put ${}_2B_5$ and ${}_4C_1$	A
${}_4C_1$, ${}_2B_5$	put ${}_8D_0$	A, C
${}_2B_5$, ${}_8D_0$	put ${}_3C_1$	A, B, C
${}_3C_1$, ${}_8D_0$	IGNORE; already visited	A, B, C
${}_8D_0$	found a goal node with cost 8	

Clearly admissibility is not enough to ensure optimality for MemoizedSearch.

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(13) Turns out we need a slightly stronger condition for MEMOIZEDSEARCH.

A heuristic function H is <u>consistent</u> if $H(q) = 0 \ \forall q \in F$ and

$$H(q) \leq w(q, \sigma, q') + H(q')$$

for all $q, q' \in Q$, $\langle q, \sigma, q' \rangle \in \Delta$.

In our simple example, H is not consistent.

$$H(A) = 7 > 5 = \underbrace{w(A, \cdot, C)}_{4} + \underbrace{H(C)}_{1}$$

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(14) We can prove that if H is consistent, then both SEARCH and MEMOIZEDSEARCH, using a priority queue prioritized by g+h, will be optimal.

First, observe that is node $n' \in \text{SUCCESSORS}_{M,H}(n)$, then:

$$\begin{aligned} g(n') + h(n') &= \left[g(n) + w(q(n), \sigma, q(n'))\right] + h(n') \\ &= g(n) + \left[w(q(n), \sigma, q(n')) + h(n')\right] \\ &\geq g(n) + \left[w(q(n), \sigma, q(n')) + H(q(n'))\right] \\ &\geq g(n) + H(q(n)) \quad \text{[b/c H is consistent]} \\ &= g(n) + h(n) \end{aligned}$$

(15) Next, let's show that if $n_1, n_2, \ldots, n_k$ are the search nodes visited by SEARCH or MEMOIZEDSEARCH (in order), then $j > i \Rightarrow g(n_j) + h(n_j) \geq g(n_i) + h(n_i)$.

Assume (for contradiction) that there exists $j > i$ for which $g(n_j) + h(n_j) < g(n_i) + h(n_i)$.

When we visit node n_i, some ancestor of node n_j must still be in the container (or else n_j will never be visited). Call this ancestor n.

From (14), we know
$$g(n) + h(n) \leq g(n_j) + h(n_j) < g(n_i) + h(n_i)$$

So the priority queue should have visited n, not n_i. Contradiction!

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(16) Finally, consider the goal nodes. Suppose goal node n_i is visited before goal node n_j. From (15):

$$i < j \Rightarrow g(n_i) + h(n_i) \leq g(n_j) + h(n_j)$$
$$\Rightarrow g(n_i) \leq g(n_j)$$

So we visit goal nodes in order of their cost. Thus the algorithm is optimal!

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(17) The master SEARCH algorithm whose container is a priority queue prioritized by g+h is called A*.

What we've asserted so far:

- if heuristic H is <u>admissible</u>, then the nonmemoized A* returns the optimal solution
- if heuristic H is <u>consistent</u>, then A* (memoized or nonmemoized) returns the optimal solution