Homework 5

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1 Stein Problems

Exercise 8.5.1

A holomorphic mapping $f : U \to V$ is a **local bijection** on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

Proof.

($\Rightarrow$) Fix $z_0 \in U$. Assume for contradiction that $f'(z_0) = 0$. Write the Taylor expansion at z_0:

$$f(z) = f(z_0) + a_k (z - z_0)^k + \cdots, \qquad a_k \neq 0,\ k \geq 2$$

Choose $\rho > 0$ so small that the closed disc $\overline{D_\rho}(z_0) \subset U$ and $|f(z) - f(z_0)| < \frac{1}{2}|a_k|\rho^k$ for $|z - z_0| \leq \rho$. For $\zeta = e^{2\pi i/k}$ set $z_1 = z_0 + \rho$ and $z_2 = z_0 + \rho\zeta$. Then $|z_1 - z_0| = |z_2 - z_0| = \rho$ but $z_1 \neq z_2$ and

$$f(z_j) = f(z_0) + a_k\rho^k + R_j, \qquad |R_j| < \frac{1}{2}|a_k|\rho^k,$$

so $|f(z_j) - f(z_0) - a_k\rho^k| < \frac{1}{2}|a_k|\rho^k$ for $j = 1, 2$. By the triangle inequality $f(z_1) = f(z_2)$. Hence f is not injective on any disc about z_0, contradicting the hypothesis that f is a local bijection. Therefore $f'(z) \neq 0$ for all $z \in U$.

($\Leftarrow$) Let $z_0 \in U$ with $f'(z_0) \neq 0$. Then there exists $r > 0$ such that the restricted map

$$f :\ D_r(z_0) \longrightarrow f(D_r(z_0))$$

is biholomorphic, and it's bijective onto an open disc $f(D_r(z_0)) \subset V$. Thus f is a local bijection on U. ∎

Exercise 8.5.2

Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0, and so that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

Proof.

Let F be holomorphic near z_0 with $F(z_0) = F'(z_0) = 0, F''(z_0) \neq 0$. Write the second–order Taylor expansion

$$F(z_0 + w) = \frac{F''(z_0)}{2} w^2 + o(|w|^2), \qquad w := z - z_0.$$

Denote $a := \frac{1}{2}F''(z_0) \neq 0$ and let $\varphi \in [0, \pi)$ satisfy $e^{-2i\varphi}a \in \mathbb{R}^+$. Then the rays

$$\gamma_1 :\ w = t\, e^{i\varphi}, \qquad \gamma_2 :\ w = t\, e^{i(\varphi+\pi/2)}, \qquad t \in \mathbb{R},$$

form an orthogonal cross at $w = 0$. Along these directions

$$F(z_0 + t\,e^{i\varphi}) = a\,t^2 + o(t^2) \in \mathbb{R}, \quad F(z_0 + t\,e^{i(\varphi+\pi/2)}) = -\,a\,t^2 + o(t^2) \in \mathbb{R}.$$

Define the real–analytic functions

$$G_1(z) := \operatorname{Im}\big(e^{-2i\varphi}F(z)\big), \qquad G_2(z) := \operatorname{Im}\big(e^{-2i(\varphi+\pi/2)}F(z)\big).$$

Because $G_j(z_0) = 0$ and $\nabla G_j(z_0) \neq 0$, each level set $G_j^{-1}(0)$ is, by the implicit function theorem, a real 1-dimensional smooth curve through z_0 whose tangent direction is that of γ_j. Call these curves Γ_1 and Γ_2; they meet orthogonally at z_0.
Along Γ_1 we have $e^{-2i\varphi}F \in \mathbb{R}$, hence F is real-valued; Taylor's formula gives

$$F(z_0 + t\,e^{i\varphi}) = a\,t^2 + o(t^2) \quad (t \to 0),$$

which is ≥ 0 for t small, with equality only at $t = 0$. Thus $F|_{\Gamma_1}$ attains a minimum at z_0.
Similarly, along Γ_2 we obtain

$$F(z_0 + t\,e^{i(\varphi+\pi/2)}) = -\,a\,t^2 + o(t^2),$$

which is ≤ 0 near $t = 0$; hence $F|_{\Gamma_2}$ is real-valued and achieves a maximum at z_0. ∎

Exercise 8.5.8

Find a harmonic function u in the open first quadrant that extends continuously up to the boundary except at the points 0 and 1, and that takes on the following boundary values: $u(x, y) = 1$ on the half-lines $\{y = 0,\ x > 1\}$ and $\{x = 0,\ y > 0\}$, and $u(x, y) = 0$ on the segment $\{0 < x < 1,\ y = 0\}$.
[**Hint:**] Find conformal maps $F_1, F_2, \ldots, F_5$ indicated in Figure 11. Note that

$$\frac{1}{\pi}\arg(z)$$

is harmonic on the upper half-plane, equals 0 on the positive real axis, and 1 on the negative real axis.

Proof.

Let $z = x + iy$, $\qquad Q = \{x > 0,\ y > 0\} \setminus \{0, 1\}$. Set $w = F_1(z) = z^2$. Because $\arg z \in (0, \pi/2)$ in Q, we have $\arg w \in (0, \pi)$, so F_1 sends Q conformally onto $H = \{\operatorname{Im} w > 0\}$. Then

$$\begin{aligned} z \in (0,1) \quad &\longmapsto \quad w \in (0,1), \\ z \in (1,\infty) \quad &\longmapsto \quad w \in (1,\infty), \\ z = iy,\ y > 0 \quad &\longmapsto \quad w = -y^2 \in (-\infty, 0). \end{aligned}$$

Thus the three boundary pieces become, respectively, $(0, 1)$, $(1, \infty)$, and $(-\infty, 0)$ on the real axis of H. Then use the map $\zeta = F_2(w) = \frac{w}{1-w}$, $w \in H$, so that if $w \in (0, 1)$ then $\zeta > 0$; if $w > 1$ or $w < 0$ then $\zeta < 0$. Because F_2 is real-analytic and preserves H, it is conformal on H.
A harmonic function on H with the required boundary values is

$$U(\zeta) = \frac{1}{\pi}\operatorname{Arg}\zeta, \qquad 0 < \operatorname{Arg}\zeta < \pi.$$

Clearly, $U = 0$ on the positive real axis and $U = 1$ on the negative real axis. Define

$$u(z) = U\big(F_2(F_1(z))\big) = \frac{1}{\pi}\operatorname{Arg}\left(\frac{z^2}{1-z^2}\right), \qquad z \in Q.$$

Because F_1 and F_2 are conformal, u is harmonic in Q and extends continuously (except at $z = 0, 1$) to the boundary. Then we have

$$\begin{aligned} z \in (0,1) &: \frac{z^2}{1-z^2} > 0, \text{ so } \operatorname{Arg} = 0 \text{ and } u = 0. \\ z \in (1,\infty) &: \frac{z^2}{1-z^2} < 0, \text{ so } \operatorname{Arg} = \pi \text{ and } u = 1. \\ z = iy, y > 0 &: \frac{z^2}{1-z^2} < 0, \text{ so } \operatorname{Arg} = \pi \text{ and } u = 1. \end{aligned}$$

Hence u satisfies all the required boundary values:

$$u = 1 \text{ on } \{y = 0,\ x > 1\} \text{ and } \{x = 0,\ y > 0\}, \qquad u = 0 \text{ on } \{0 < x < 1,\ y = 0\}.$$

Thus the function

$$u(x,y) = \frac{1}{\pi}\operatorname{Arg}\left(\frac{(x+iy)^2}{1-(x+iy)^2}\right)$$

works. ■

Exercise 8.5.9

Prove that the function u defined by

$$u(x,y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right) \quad \text{and} \quad u(0,1) = 0$$

is harmonic in the unit disc and vanishes on its boundary. Note that u is not bounded in $\mathbb{D}$.

Proof.

Let $z = x + iy$ and define

$$u(x,y) = \operatorname{Re}\left(\frac{i+z}{i-z}\right), \quad u(0,1) = 0.$$

First, we show that u is harmonic in the unit disc. Let

$$f(z) = \frac{i+z}{i-z}.$$

This is holomorphic on all of $\mathbb{D}$, since $i \notin \overline{\mathbb{D}}$. The real part of a holomorphic function is harmonic wherever the function is holomorphic, so $u = \operatorname{Re}(f(z))$ is harmonic on $\mathbb{D}$.
To show that u vanishes on $\partial\mathbb{D}$, let $z = e^{i\theta}$ with $\theta \in [0, 2\pi)$. Then

$$f(z) = \frac{i+e^{i\theta}}{i-e^{i\theta}}.$$

We claim that $f(z)$ is purely imaginary for $|z| = 1$. Observe that f maps $\mathbb{D}$ conformally onto the right half-plane, and thus maps $\partial\mathbb{D}$ to the boundary of that half-plane, which is the imaginary axis. So $\operatorname{Re}(f(z)) = 0$ for all $|z| = 1$, and hence $u(x,y) = 0$ on $\partial\mathbb{D}$. ■

Exercise 8.5.10

Let $F : \mathbb{H} \to \mathbb{C}$ be a holomorphic function that satisfies

$$|F(z)| \leq 1 \quad \text{and} \quad F(i) = 0.$$

Prove that

$$|F(z)| \leq \left| \frac{z-i}{z+i} \right| \quad \text{for all } z \in \mathbb{H}.$$

Proof.

Define

$$\phi(z) = \frac{z-i}{z+i}.$$

This maps the upper half-plane $\mathbb{H}$ conformally onto the unit disc $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. Clearly,

- ϕ is holomorphic on $\mathbb{H}$
- For all $z \in \mathbb{H}$, $\operatorname{Im}(z) > 0$, so $z + i \neq 0$ and ϕ is well-defined
- The image lies in $\mathbb{D}$ because

$$|\phi(z)| = \left| \frac{z-i}{z+i} \right| < 1 \quad \text{for all } z \in \mathbb{H}.$$

Define the composition

$$G(w) = F\big(\phi^{-1}(w)\big).$$

Then $G : \mathbb{D} \to \mathbb{C}$ is holomorphic, because ϕ^{-1} is holomorphic and maps $\mathbb{D}$ to $\mathbb{H}$. Since $|F(z)| \leq 1$ on $\mathbb{H}$, we have

$$|G(w)| = |F(\phi^{-1}(w))| \leq 1 \quad \text{for all } w \in \mathbb{D}.$$

Also, note that $\phi(i) = 0$, so

$$G(0) = F(\phi^{-1}(0)) = F(i) = 0.$$

Therefore, G is a holomorphic function on $\mathbb{D}$, bounded by 1 in modulus, and vanishing at 0. By the Schwarz Lemma,

$$|G(w)| \leq |w| \quad \text{for all } w \in \mathbb{D}.$$

Apply this inequality to $w = \phi(z)$:

$$|F(z)| = |G(\phi(z))| \leq |\phi(z)| = \left| \frac{z-i}{z+i} \right|.$$

Hence,

$$|F(z)| \leq \left| \frac{z-i}{z+i} \right| \quad \text{for all } z \in \mathbb{H}.$$

■

Exercise 8.5.11

Show that if $f : D(0, R) \to \mathbb{C}$ is holomorphic, with $|f(z)| \leq M$ for some $M > 0$, then

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

[Hint:] Use the Schwarz lemma.

Proof.

Define a new function

$$F(z) = \frac{f(z)}{M},$$

so that $|F(z)| \leq 1$ on $D(0, R)$ and F is holomorphic. Let $w = F(z)$ and $w_0 = F(0) = a/M$. Since $|F(z)| \leq 1$, we can define a transformation that sends w_0 to 0:

$$\phi(w) = \frac{w - w_0}{1 - \overline{w_0} w}.$$

This map sends the unit disc to itself, is holomorphic, and satisfies $\phi(w_0) = 0$. Now define the composition

$$g(z) = \phi(F(z)) = \frac{F(z) - F(0)}{1 - \overline{F(0)} F(z)}.$$

Then g is holomorphic on $D(0, R)$, maps into the unit disc, and satisfies $g(0) = 0$. Define

$$h(z) = g(Rz),$$

which is holomorphic on $D(0, 1)$ with $h(0) = 0$ and $|h(z)| < 1$ for all $|z| < 1$. By the Schwarz Lemma,

$$|h(z)| \leq |z| \quad \Rightarrow \quad |g(w)| \leq \frac{|w|}{R}.$$

So for all $z \in D(0, R)$,

$$\left| \frac{F(z) - F(0)}{1 - \overline{F(0)} F(z)} \right| \leq \frac{|z|}{R}.$$

We know $F(z) = f(z)/M$, so that

$$\frac{F(z) - F(0)}{1 - \overline{F(0)} F(z)} = \frac{f(z)/M - a/M}{1 - \overline{a}/M \cdot f(z)/M} = \frac{f(z) - a}{M^2 - \overline{a} f(z)}.$$

Therefore,

$$\left| \frac{f(z) - f(0)}{M^2 - \overline{f(0)} f(z)} \right| \leq \frac{|z|}{MR}.$$

■

Exercise 8.5.12

A complex number $w \in \mathbb{D}$ is a **fixed point** for the map $f : \mathbb{D} \to \mathbb{D}$ if $f(w) = w$.

(a) Prove that if $f : \mathbb{D} \to \mathbb{D}$ is analytic and has two distinct fixed points, then f is the identity, that is, $f(z) = z$ for all $z \in \mathbb{D}$.

(b) Must every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ have a fixed point? **[Hint:]** Consider the upper half-plane.

Proof.

Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. (a) Suppose f has two distinct fixed points $z_1, z_2 \in \mathbb{D}$ with $z_1 \neq z_2$, so that

$$f(z_1) = z_1, \quad f(z_2) = z_2.$$

Fix $z_0 \in \mathbb{D}$. Define an automorphism of $\mathbb{D}$:

$$\phi(z) = \frac{z - z_1}{1 - \overline{z_1} z}, \quad \text{so that } \phi(z_1) = 0.$$

Then define the conjugated function

$$F(z) = \phi \circ f \circ \phi^{-1}(z).$$

Since ϕ is an automorphism of $\mathbb{D}$ and $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, the function $F : \mathbb{D} \to \mathbb{D}$ is holomorphic. Now, since $f(z_1) = z_1$, we have $F(0) = 0$. Let $w = \phi(z_2) \neq 0$, since $z_2 \neq z_1$. Then

$$F(w) = \phi(f(\phi^{-1}(w))) = \phi(\phi^{-1}(w)) = w,$$

so F has two distinct fixed points: 0 and $w \neq 0$. By the Schwarz lemma, if a holomorphic map $g : \mathbb{D} \to \mathbb{D}$ fixes 0 and is not the identity, then

$$|g(z)| < |z| \quad \text{for all } z \neq 0.$$

But F fixes 0 and $w \neq 0$, so this strict inequality fails. Hence $F(z) = z$ for all $z \in \mathbb{D}$. Therefore,

$$f = \phi^{-1} \circ F \circ \phi = \phi^{-1} \circ \phi = \mathrm{id},$$

so $f(z) = z$ for all $z \in \mathbb{D}$.
(b) Not every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ has a fixed point. Consider the map

$$f(z) = \frac{z + a}{1 + \overline{a} z}, \quad a \in \mathbb{D} \setminus \{0\}.$$

This is an automorphism of $\mathbb{D}$ that maps 0 to a. Then

$$f(0) = a \neq 0, \quad f(a) = \frac{2a}{1 + |a|^2} \neq a,$$

and unless $a = 0$, this function has no fixed point in $\mathbb{D}$. ■

Exercise 8.5.13

Exercise 8.5.14

Prove that all conformal mappings from the upper half-plane $\mathbb{H}$ to the unit disc $\mathbb{D}$ take the form

$$e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}, \quad \theta \in \mathbb{R} \text{ and } \beta \in \mathbb{H}.$$

Proof.

We know that all conformal maps between simply connected domains (other than $\mathbb{C}$) are given by the Riemann Mapping Theorem, and all automorphisms of the unit disc $\mathbb{D}$ are of the form

$$\phi_a(w) = e^{i\theta} \cdot \frac{w-a}{1-\overline{a}w}, \quad |a| < 1,\ \theta \in \mathbb{R}.$$

Also, the Möbius map

$$T(z) = \frac{z-i}{z+i}$$

is a conformal bijection from $\mathbb{H}$ to $\mathbb{D}$. Any conformal map $f : \mathbb{H} \to \mathbb{D}$ can be written as a composition

$$f(z) = \phi(T(z)),$$

where $T(z) = \frac{z-i}{z+i}$ and $\phi : \mathbb{D} \to \mathbb{D}$ is a disc automorphism. So,

$$f(z) = e^{i\theta} \cdot \frac{T(z) - a}{1 - \overline{a}T(z)}, \quad \text{for some } a \in \mathbb{D},\ \theta \in \mathbb{R}.$$

Now substitute the explicit expression for $T(z)$:

$$f(z) = e^{i\theta} \cdot \frac{\dfrac{z-i}{z+i} - a}{1 - \overline{a} \cdot \dfrac{z-i}{z+i}}.$$

Multiply numerator and denominator by $z + i$:

$$f(z) = e^{i\theta} \cdot \frac{(z-i) - a(z+i)}{(z+i) - \overline{a}(z-i)}.$$

Simplify numerator:

$$(z-i) - a(z+i) = z(1-a) - i(1+a),$$

Simplify denominator:

$$(z+i) - \overline{a}(z-i) = z(1-\overline{a}) + i(1+\overline{a}).$$

So the full expression is a Möbius transformation of the form

$$f(z) = e^{i\theta} \cdot \frac{Az+B}{Cz+D}.$$

Now, since f maps $\mathbb{H}$ to $\mathbb{D}$ conformally, it must be a Möbius transformation that maps the upper half-plane to the disc. A standard form of such a map is:

$$f(z) = e^{i\theta} \cdot \frac{z-\beta}{z-\overline{\beta}}, \quad \beta \in \mathbb{H}.$$

For $\beta \in \mathbb{H}$, note that $f(\beta) = 0$, $f(\overline{\beta}) = \infty$, $|f(z)| < 1$ for all $z \in \mathbb{H}$, and f is a Möbius transformation that maps $\mathbb{H}$ onto $\mathbb{D}$. Therefore, any conformal map $f : \mathbb{H} \to \mathbb{D}$ has the form

$$f(z) = e^{i\theta} \cdot \frac{z-\beta}{z-\overline{\beta}}, \quad \beta \in \mathbb{H},\ \theta \in \mathbb{R}.$$

■

Exercise 8.5.15

Here are two properties enjoyed by automorphisms of the upper half-plane.

(a) Suppose Φ is an automorphism of $\mathbb{H}$ that fixes three distinct points on the real axis. Then Φ is the identity.

(b) Suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two pairs of three distinct points on the real axis with

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3.$$

Prove that there exists (a unique) automorphism Φ of $\mathbb{H}$ so that $\Phi(x_j) = y_j$, $j = 1, 2, 3$. The same conclusion holds if $y_3 < y_1 < y_2$ or $y_2 < y_3 < y_1$.

Proof.

(a) We know that automorphisms of $\mathbb{H}$ are of the form

$$\Phi(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$. Suppose Φ fixes $x_1, x_2, x_3 \in \mathbb{R}$ with x_1, x_2, x_3 distinct. Since $\Phi(x_j) = x_j$ for $j = 1, 2, 3$, and Möbius transformations are determined by their values on three distinct points, it follows that $\Phi(z) = z$ for all $z \in \mathbb{H}$. Thus, Φ is the identity.
(b) Now, suppose (x_1, x_2, x_3) and (y_1, y_2, y_3) are two ordered triples of distinct real numbers satisfying

$$x_1 < x_2 < x_3 \quad \text{and} \quad y_1 < y_2 < y_3,$$

To prove one exists, note that automorphisms of $\mathbb{H}$ are precisely those Möbius transformations with real coefficients and positive determinant, and they preserve $\mathbb{H}$. Thus, there exists a Möbius transformation Φ with real coefficients, mapping $x_j \mapsto y_j$ for $j = 1, 2, 3$. To prove uniqueness, suppose there were two automorphisms Φ_1 and Φ_2 satisfying $\Phi_1(x_j) = \Phi_2(x_j) = y_j$ for $j = 1, 2, 3$. Then $\Phi_2^{-1} \circ \Phi_1$ would be an automorphism of $\mathbb{H}$ fixing three distinct points, hence the identity by part (a). Thus $\Phi_1 = \Phi_2$, and the automorphism is unique. ■

Exercise 8.5.16

Exercise 8.5.17

2 Additional Problems

Exercise 1

Let $\Omega = \{z : |z - 1| < \sqrt{2},\ |z + 1| < \sqrt{2}\}$. Find a bijective conformal map from Ω to the upper half-plane $\mathbb{H}$.

Proof.

Note that Ω is the intersection of two open disks of radius $\sqrt{2}$ centered at 1 and -1, respectively.

Let us define

$$f(z) = \frac{z-i}{z+i}.$$

This transformation maps:

- $z = i$ to 0,
- $z = -i$ to ∞,
- the unit circle to the real line,
- the upper half of the unit circle (where $\text{Im}(z) > 0$) to the upper half-plane $\mathbb{H}$.

We claim that f maps Ω onto $\mathbb{H}$. To justify this, note that the boundaries of Ω are arcs of the circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$. These two circles intersect orthogonally at $z = \pm i$, and the map $f(z) = \frac{z-i}{z+i}$ maps any pair of circles intersecting orthogonally at $z = \pm i$ to rays meeting at the origin in the complex plane. In particular, it maps the circular arcs bounding Ω to intervals on the real line, and the domain between them to the upper half-plane. Therefore, f maps Ω conformally and bijectively onto $\mathbb{H}$. ■

Exercise 2

Find the fractional linear transformation that maps the circle $|z| = 2$ into $|z+1| = 1$, the point -2 into the origin, and the origin into i.

Proof.

A fractional linear transformation is uniquely determined by the images of three distinct points. Define $z_1 = -2$, $z_2 = 0$, $z_3 = 2$ (three points on $|z| = 2$), and let their images be $w_1 = 0$, $w_2 = i$, and $w_3 = \infty$ (since $z_3 = 2$ lies on $|z| = 2$, it is reasonable to map it to a point at infinity to map the circle to a line or another circle). Then the desired transformation is the unique function satisfying:

$$f(-2) = 0, \quad f(0) = i, \quad f(2) = \infty.$$

Assume

$$f(z) = \lambda \cdot \frac{z+2}{z-2}$$

so that $f(-2) = \lambda \cdot \frac{-2+2}{-2-2} = 0$ and $f(2) = \lambda \cdot \frac{4}{0} = \infty$. Now choose λ to satisfy $f(0) = i$:

$$f(0) = \lambda \cdot \frac{0+2}{0-2} = \lambda \cdot \left(\frac{2}{-2}\right) = -\lambda.$$

So for $f(0) = i$, we need $-\lambda = i$, or $\lambda = -i$. Therefore, the desired transformation is

$$f(z) = -i \cdot \frac{z+2}{z-2}.$$

To check that $|z| = 2$ maps to $|z+1| = 1$, let z be on the circle $|z| = 2$. Then write $z = 2e^{i\theta}$, and we can find

$$f(z) = -i \cdot \frac{2e^{i\theta}+2}{2e^{i\theta}-2} = -i \cdot \frac{2(e^{i\theta}+1)}{2(e^{i\theta}-1)} = -i \cdot \frac{e^{i\theta}+1}{e^{i\theta}-1}.$$

So it sends the unit circle (in this case, $|z| = 2$ scaled) onto a circle centered at -1 of radius 1, i.e., the circle $|w+1| = 1$. Therefore, the final answer is:

$$f(z) = -i \cdot \frac{z+2}{z-2}.$$

■

Exercise 3

Let $\Omega = \mathbb{D} \setminus (-1, -1/2]$. Find a bijective conformal map from Ω to the unit disk $\mathbb{D}$. How do you find the most general form of all such maps (you don't have to explicitly describe the general form, just explain the strategy for obtaining it)?

Proof.

Let

$$\phi(z) = i \cdot \frac{1+z}{1-z},$$

which maps $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H} = \{\mathrm{Im}(w) > 0\}$. Then $\phi(\Omega)$ is the upper half-plane with the slit $(\phi(-1), \phi(-1/2)]$ removed. We find that

$$\phi(-1) = i \cdot \frac{1-1}{1+1} = 0, \quad \phi(-1/2) = i \cdot \frac{1-1/2}{1+1/2} = i \cdot \frac{1/2}{3/2} = \frac{i}{3}.$$

So $\phi(\Omega) = \mathbb{H} \setminus \left[0, \frac{i}{3}\right]$, a vertical slit segment from 0 to $i/3$ removed from $\mathbb{H}$.
We now look for a conformal map $\psi : \mathbb{H} \setminus [0, i/3] \to \mathbb{H}$. We know that Ω can be mapped conformally onto the upper half-plane using the square root function:

$$\psi(w) = \sqrt{w-c}, \quad \text{where } c \text{ is the endpoint of the slit.}$$

In this case, the slit ends at $i/3$, so we instead first rotate the slit to lie on the real axis. Define

$$T(w) = -iw.$$

Then T rotates the vertical segment $[0, i/3]$ to the real segment $[0, 1/3]$. Now define

$$\psi(w) = \sqrt{w}.$$

So the composition $\sqrt{-iw}$ maps $\mathbb{H} \setminus [0, i/3]$ onto a quadrant (or half-plane). Further composing with another Möbius map sends the result back to $\mathbb{D}$.
Let

$$f(z) = \chi \circ \psi \circ T \circ \phi(z),$$

where $\phi(z) = i \cdot \frac{1+z}{1-z}$ maps Ω to $\mathbb{H} \setminus [0, i/3]$, $T(w) = -iw$ rotates the slit to the positive real axis, $\psi(w) = \sqrt{w}$ removes the branch and straightens the domain, and χ is a final Möbius transformation mapping the resulting domain back to $\mathbb{D}$. Thus, $f(z)$ is a bijective conformal map from Ω to $\mathbb{D}$.
Generally, once you have a specific bijective conformal map $f : \Omega \to \mathbb{D}$, every other such map is of the form

$$g(z) = M(f(z)),$$

where M is an automorphism of the unit disk. That is, to find all conformal maps from Ω to $\mathbb{D}$, find one such map and precompose with all automorphisms of the image domain $\mathbb{D}$. ■

Exercise 4

Let $\Omega \neq \mathbb{C}$ be an unbounded region. Is there an analytic isomorphism from Ω to $\mathbb{C}$? If yes, exhibit one such isomorphism. If no, explain why.

Proof.

No, and suppose for contradiction that there exists a bijective holomorphic map $f : \Omega \longrightarrow \mathbb{C}$. Then f would be a nonconstant entire function (because Ω is open and connected, and if f is holomorphic and surjective onto $\mathbb{C}$, it must be entire). However, f must be injective, and we have previously shown that any entire injective function must be linear. But an affine linear function is surjective on all of $\mathbb{C}$, so f must have domain $\mathbb{C}$ — not just $\Omega \neq \mathbb{C}$. Since $\Omega \subsetneq \mathbb{C}$, f cannot extend to an entire function on all of $\mathbb{C}$. Therefore, there cannot exist a bijective conformal map from Ω to $\mathbb{C}$ if $\Omega \neq \mathbb{C}$. ■

Exercise 5

Let $\Omega = \{z = x + iy : 0 < x < 1,\ y \in \mathbb{R}\}$. Is there an analytic isomorphism from Ω to $\mathbb{C}$? If yes, exhibit one such isomorphism. If no, explain why.

Proof.

Yes. Consider

$$g(z) = \frac{1}{\exp(2\pi i z)},$$

which maps Ω onto $\mathbb{C} \setminus \{0\}$. Now, since $\mathbb{C} \setminus \{0\}$ is simply connected minus a point, and Ω is simply connected, the Riemann mapping theorem implies that such an analytic isomorphism exists. ■

Exercise 6

Let $\Omega = \mathbb{C} \setminus [0, \infty)$. Is there an analytic isomorphism from Ω to $\mathbb{C}$? If yes, exhibit one such isomorphism. If no, explain why.

Proof.

Yes. Define

$$f(z) = \sqrt{z},$$

where $\sqrt{z}$ is the principal branch of the square root: the branch cut is taken along $[0, \infty)$ so that $\sqrt{z}$ is holomorphic on Ω. Then f is holomorphic on Ω, f is injective on Ω, and f maps Ω onto $\mathbb{C} \setminus (-\infty, 0]$, a slit plane. Then, applying log or another conformal map, we can move from the slit plane to $\mathbb{C}$. ■