8200 Homework 6

March 14, 2025

Exercise 1

Suppose X is path connected and $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a path connected covering space of X. Prove that the number of sheets of this covering space is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

Proof.

Let f be any loop with basepoint x_0, so that $\tilde{f}$ is it's lift, where X cooresponds to $\tilde{X}$ and x_0, $\tilde{x_0}$. Let $g \in G = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, so that $g \circ f$ has the lift $\tilde{g} \circ \tilde{f}$. Note that because $\tilde{g}$ is a loop, $\tilde{g} \circ \tilde{f}$ ends at the same point as $\tilde{f}$. Then define a function $\phi : G[f] \to p^{-1}(x)$, where $G[f] \mapsto \tilde{f}(1)$. Because $\tilde{X}$ is path connected, ϕ is surjective. Then note that because $\phi(G[f_1]) = \phi(G[f_2])$ implies that $f_1 \circ \bar{f}_2$ lifts to a loop based at $\tilde{x_0}$, so that $[f_1][f_2]^{-1} \in G$, and $g[f_1] = G[f_2]$, so that ϕ is injective. Thus the number of cosets (index) is equal to the number of sheets. ∎

Exercise 2

Construct nonnormal covering spaces of the Klein Bottle by a Klein bottle and by a torus.

Proof.

For the Klein bottle, we need to find a nonnormal subgroup of the Klein bottle. The Klein bottle's fundamental group is $\langle a, b | aba^{-1}b \rangle$. A nonnormal subgroup of this could be $\langle a, b^2 \rangle$, thus there is a nonnormal covering space that cooresponds to it. We know this covering space is a Klein bottle because the subgroup $\langle a, b^2 \rangle$ is isomorphic to the Klein bottle's fundamental group.

For the torus, it's a little bit trickier because the nonnormal subgroup must be isomorphic to the torus's fundamental group, $\langle a, b | ab = ba \rangle$. If we choose the subgroup $\langle a^2, b^2 \rangle$, it's isomorphic to the torus fundamental group, and nonnormal in the Klein bottle fundamental group, so we get a corresponding nonnormal cover. ∎

Exercise 3

Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Mobius band via a homeomorphism from the boundary circle of the Mobius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover of X, and describe the action of $\pi_1(X)$ on the universal cover. Do the same for the space Y obtained by attaching a Mobius band to $\mathbb{R}P^2$ formed by the 1-skeleton of the usual CW structure on $\mathbb{R}P^2$.

Proof.

Pressed for time and saved this one till the end, so I will just say you have to use Van Kampen to find the fundamental group of this shape, which is just the mobius band wrapping around the torus twice. If I had to guess, the fundamental group would be $\mathbb{Z} * \mathbb{Z}$ or something along those lines. ■

Exercise 4

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R} - \{0\}$. Show this action is a covering space action and compute $\pi_1(X \setminus \mathbb{Z})$. Show the orbit space $X \setminus \mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis.

Proof.

The action of $\mathbb{Z}$ on X is given by:

$$n \cdot (x, y) = \phi^n(x, y) = (2^n x, 2^{-n} y).$$

For each $(x, y) \in X$, choose a neighborhood U small enough so that $\phi^n(U) \cap U = \emptyset$ for all $n \neq 0$. This is possible because ϕ^n scales x by 2^n and y by 2^{-n}, ensuring disjointness for small U. Thus, the action is a covering space action.
Since the action is free and properly discontinuous, the quotient map $X \to X/\mathbb{Z}$ is a covering map with deck transformation group $\mathbb{Z}$. Therefore:

$$\pi_1(X/\mathbb{Z}) \cong \mathbb{Z}.$$

Consider the orbits of $(1, 0)$ and $(0, 1)$. The orbit of $(1, 0)$ is $\{(2^n, 0) \mid n \in \mathbb{Z}\}$, and the orbit of $(0, 1)$ is $\{(0, 2^{-n}) \mid n \in \mathbb{Z}\}$. These orbits accumulate at $(0, 0)$, which is not in X, so their images in $X/\mathbb{Z}$ cannot be separated by disjoint open sets. Thus, $X/\mathbb{Z}$ is non-Hausdorff.
The space X decomposes into four quadrants based on the x-axis and y-axis. Each quadrant is homeomorphic to $S^1 \times \mathbb{R}$, as the angular component corresponds to S^1 and the radial component to $\mathbb{R}$. The action preserves these quadrants, so $X/\mathbb{Z}$ is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$. ■

Exercise 5

For a covering space $p : \tilde{X} \to X$ connected, locally path-connected, and semilocally simply-connected, show:

(a) The components of $\tilde{X}$ are in one-to-one corrspondence with the orbits of the action of $\pi_1(X, x_o)$ on the fiber $p^{-1}(x_0)$.

(b) Under the Galois corrspondence between connected covering spaces of X and subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\tilde{X}$ containing a given lift $\tilde{x_0}$ of x_0 is the *stabilizer* of $\tilde{x_0}$, the subgroup consisting of elements whose action on the fiber leaves $\tilde{x_0}$ fixed.

Proof.

(a) Let $p : \tilde{X} \to X$ be a connected, locally path-connected, and semilocally simply-connected covering space. Fix a basepoint $x_0 \in X$ and consider the fiber $p^{-1}(x_0)$. The fundamental group $\pi_1(X, x_0)$ has the following action on $p^{-1}(x_0)$: for $[\gamma] \in \pi_1(X, x_0)$ and $\tilde{x}_0 \in p^{-1}(x_0)$, the action is defined by lifting γ to a path in $\tilde{X}$ starting at $\tilde{x}_0$ and taking its endpoint. Each component of $\tilde{X}$ is path-connected, and the restriction of p to a component is a covering map. For a fixed $\tilde{x}_0 \in p^{-1}(x_0)$, the orbit of $\tilde{x}_0$ under the action of $\pi_1(X, x_0)$ consists of all points in $p^{-1}(x_0)$ that lie in the same component of $\tilde{X}$ as $\tilde{x}_0$. This is because we have two cases:

- If $\tilde{x}_1$ is in the same component as $\tilde{x}_0$, there is a path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Projecting $\tilde{\gamma}$ to X gives a loop γ in X based at x_0, and the action of $[\gamma]$ on $\tilde{x}_0$ sends it to $\tilde{x}_1$.
- Conversely, if $\tilde{x}_1$ is in the orbit of $\tilde{x}_0$, there exists a loop γ in X such that the lift of γ starting at $\tilde{x}_0$ ends at $\tilde{x}_1$. This implies $\tilde{x}_0$ and $\tilde{x}_1$ are in the same component of $\tilde{X}$.

Thus, the components of $\tilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.

(b) Under the Galois correspondence, connected covering spaces of X correspond to subgroups of $\pi_1(X, x_0)$. Let $\tilde{X}_0$ be the component of $\tilde{X}$ containing a given lift $\tilde{x}_0$ of x_0. The subgroup of $\pi_1(X, x_0)$ corresponding to $\tilde{X}_0$ is the stabilizer of $\tilde{x}_0$. The stabilizer of $\tilde{x}_0$ is the subgroup $H \leq \pi_1(X, x_0)$ consisting of elements $[\gamma]$ such that the lift of γ starting at $\tilde{x}_0$ ends at $\tilde{x}_0$. Thus it is sufficient to show that this subgroup H cooresponds to the covering space $\tilde{X}_0$.

The covering map $p|_{\tilde{X}_0} : \tilde{X}_0 \to X$ has H as its fundamental group. This follows from the lifting criterion: loops in X lift to loops in $\tilde{X}_0$ if and only if they are in H. By Galois correspondence, H is the subgroup associated with $\tilde{X}_0$. Thus, the subgroup corresponding to $\tilde{X}_0$ is the stabilizer of $\tilde{x}_0$.

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Exercise 6

Consider covering spaces $p : \tilde{X} \to X$ with $\tilde{X}$ and X connected CW complexes, the cells of $\tilde{X}$ projecting homeomorphically onto cells of X. Restricting p to the 1-skeleton then gives a covering space $\tilde{X}^1 \to X$ over the 1-skeleton of X. Show:

(a) Two such covering spaces $\tilde{X}_1 \to X$ and $\tilde{X}_2 \to X$ are isomorphic if and only if the restrictions $\tilde{X}_1^1 \to X^1$ and $\tilde{X}_2^1 \to X^1$ are isomorphic.

(b) $\tilde{X} \to X$ is a normal covering if and only if $\tilde{X}^1 \to X^1$ is normal.

(c) The groups of deck transformations of the coverings $\tilde{X} \to X$ and $\tilde{X}^1 \to X^1$ are isomorphic, via the restriction map.

Proof.

(a) Let $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$ be covering spaces with $\tilde{X}_1$ and $\tilde{X}_2$ connected CW complexes, and assume the cells of $\tilde{X}_1$ and $\tilde{X}_2$ project homeomorphically onto cells of X. Let $\tilde{X}_1^1 \to X^1$ and $\tilde{X}_2^1 \to X^1$ be the restrictions to the 1-skeletons.

($\Rightarrow$) If $\tilde{X}_1 \to X$ and $\tilde{X}_2 \to X$ are isomorphic, there exists a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ such that $p_2 \circ f = p_1$. Restricting f to the 1-skeletons gives a homeomorphism $f|_{\tilde{X}_1^1} : \tilde{X}_1^1 \to \tilde{X}_2^1$ that commutes with the covering maps, so $\tilde{X}_1^1 \to X^1$ and $\tilde{X}_2^1 \to X^1$ are isomorphic.

($\Leftarrow$) If $\tilde{X}_1^1 \to X^1$ and $\tilde{X}_2^1 \to X^1$ are isomorphic, let $g : \tilde{X}_1^1 \to \tilde{X}_2^1$ be a homeomorphism such that $p_2 \circ g = p_1|_{\tilde{X}_1^1}$. Since $\tilde{X}_1$ and $\tilde{X}_2$ are CW complexes and the cells project homeomorphically, g extends uniquely to a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ satisfying $p_2 \circ f = p_1$. Thus, $\tilde{X}_1 \to X$ and $\tilde{X}_2 \to X$ are isomorphic.

(b) ($\Rightarrow$) If $\tilde{X} \to X$ is normal, then for any two lifts $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ of a point $x \in X$, there exists a deck transformation $f : \tilde{X} \to \tilde{X}$ such that $f(\tilde{x}_1) = \tilde{x}_2$. Restricting f to $\tilde{X}^1$ gives a deck transformation of $\tilde{X}^1 \to X^1$, so $\tilde{X}^1 \to X^1$ is normal.

($\Leftarrow$) If $\tilde{X}^1 \to X^1$ is normal, then for any two lifts $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$ of a point $x \in X^1$, there exists a deck transformation $g : \tilde{X}^1 \to \tilde{X}^1$ such that $g(\tilde{x}_1) = \tilde{x}_2$. Since $\tilde{X}$ is a CW complex and the cells project homeomorphically, g extends uniquely to a deck transformation $f : \tilde{X} \to \tilde{X}$. Thus, $\tilde{X} \to X$ is normal.

(c) Let $\mathrm{Deck}(\tilde{X} \to X)$ and $\mathrm{Deck}(\tilde{X}^1 \to X^1)$ denote the groups of deck transformations of $\tilde{X} \to X$ and $\tilde{X}^1 \to X^1$, respectively. Define the restriction map:

$$\Phi : \mathrm{Deck}(\tilde{X} \to X) \to \mathrm{Deck}(\tilde{X}^1 \to X^1), \quad f \mapsto f|_{\tilde{X}^1}.$$

Then note that:

- Φ is injective: If $f|_{\tilde{X}^1} = g|_{\tilde{X}^1}$, then $f = g$ because $\tilde{X}$ is a CW complex and f and g agree on the 1-skeleton.
- Φ is surjective: For any deck transformation $g : \tilde{X}^1 \to \tilde{X}^1$, g extends uniquely to a deck transformation $f : \tilde{X} \to \tilde{X}$ because the cells of $\tilde{X}$ project homeomorphically onto cells of X.

Thus, Φ is an isomorphism, and the groups of deck transformations are isomorphic.

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