Homework 8

April 10, 2025

Exercise 1

Determine whether there exists a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \bigoplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$. More generally, determine which abelian groups A fit into a short exact sequence $0 \to \mathbb{Z}_{p^m} \to A \to \mathbb{Z}_{p^n} \to 0$ with p prime. What about the case of short exact sequences $0 \to \mathbb{Z} \to A \to \mathbb{Z}_n \to 0$?

Proof.

- For an exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \bigoplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$, we know $f : \mathbb{Z}_4 \to \mathbb{Z}_8 \bigoplus \mathbb{Z}_2$ is injective, and $g : \bigoplus \mathbb{Z}_2 \to \mathbb{Z}_4$ is surjective. Because f is injective, we know $\text{im} f \cong \mathbb{Z}_4$, so $\ker g \cong \mathbb{Z}_4$. By the first isomorphism theorem, because g is surjective, we know $\mathbb{Z}_4 \cong \mathbb{Z}_8 \bigoplus \mathbb{Z}_2 / \ker f = \mathbb{Z}_8 \bigoplus \mathbb{Z}_2 / \mathbb{Z}_4$. But $\mathbb{Z}_4$ isn't a subgroup of $\mathbb{Z}_2$, so we're really just modding out from $\mathbb{Z}_8$, so we are left with
$$\mathbb{Z}_4 \cong \mathbb{Z}_8 \bigoplus \mathbb{Z}_2 / \mathbb{Z}_4 = \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$$
which is clearly false. Thus, there next exists such an exact sequence.

- From a similar setup to last time, the condition that must be met is
$$\mathbb{Z}_{p^n} \cong A / \mathbb{Z}_{p^m}$$
So A must be isomorphic to $\mathbb{Z}_{p^{n+m}}$, and also must have a subgroup isomorphic to $\mathbb{Z}_{p^m}$.

- Using the above part, A must contain Z as a subgroup with index n:
  - $\mathbb{Z}$ (when $n = 1$)
  - $\mathbb{Z} \oplus \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_k}$ where $d_1 \cdots d_k = n$

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Exercise 2

- Show that $H_0(X, A) = 0$ iff A meets each path-component of X.
- Show that $H_1(X, A) = 0$ iff $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.

Proof.

- The long exact sequence for (X, A) gives:
$$H_0(A) \xrightarrow{i_*} H_0(X) \to H_0(X, A) \to 0.$$
($\Longrightarrow$) If $H_0(X, A) = 0$, then i_* is surjective, so A meets every path-component of X.
($\Longleftarrow$) Conversely, if A meets every path-component, i_* is surjective, implying $H_0(X, A) = 0$.

- From the long exact sequence:

$$H_1(A) \xrightarrow{i_*} H_1(X) \to H_1(X,A) \to H_0(A) \xrightarrow{i_*} H_0(X).$$

($\implies$) If $H_1(X,A) = 0$, then:

  - $i_* : H_1(A) \to H_1(X)$ is surjective, and
  - $i_* : H_0(A) \to H_0(X)$ is injective.

($\impliedby$) If the conditions hold, surjectivity of $H_1(A) \to H_1(X)$ forces $H_1(X,A) = 0$, and injectivity of $H_0(A) \to H_0(X)$ follows from (ii).

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Exercise 3

Show that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$ for all n, where SX is the suspension of X. More generally, thinking of SX as the union of two cones CX with their bases identified, compute the reduced homology groups of the union of any finite number of cones CX with their bases identified.

Proof.

- Consider the suspension SX with $U = SX \setminus \{\text{north pole}\}$ and $V = SX \setminus \{\text{south pole}\}$, so that $U \cap V \simeq X$ and U, V contractible.

  The long exact sequence for reduced homology of the pair (SX, X) gives:

$$\cdots \to \tilde{H}_{n+1}(SX) \to \tilde{H}_{n+1}(SX/X) \to \tilde{H}_n(X) \to \tilde{H}_n(SX) \to \cdots$$

  Since SX/X is a wedge of two spheres, and $\tilde{H}_n(SX) \cong \tilde{H}_n(CX_+) \oplus \tilde{H}_n(CX_-) = 0$ (cones are contractible), we get:

$$0 \to \tilde{H}_{n+1}(SX) \to \tilde{H}_{n+1}(S^1 \vee S^1) \to \tilde{H}_n(X) \to 0$$

  From this short exact sequence, we find that $\tilde{H}_n(X) \cong \tilde{H}_{n+1}(SX)$.

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Exercise 4

Making the preceding problem more concrete, construct explicit chain maps $s : C_n(X) \to C_{n+1}(SX)$ inducing isomorphisms $\tilde{H}_n(X) \to \tilde{H}_{n+1}(SX)$.

Proof.

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Exercise 5

Prove by induction on dimension the following facts about the homology of a finite-dimensional CW complex X, using the observation that X^n/X^{n-1} is a wedge sum of n-spheres:

- If (X) has dimension n then $H_i(X) = 0$ for $i > n$ and $H_n(X)$ is free
- $H_n(X)$ is free with basis in bijective correspondence with the n-cells if there are no cells of dimension $n-1$ or $n+1$.
- If X has k n-cells, then $H_n(X)$ is generated by at most k elements.

Proof.

- *Base case:* For X^0 (0-dimensional complex), $H_i(X^0) = 0$ for $i > 0$ and $H_0(X^0)$ is free abelian on the path components.
  Inductive step: Assume true for $(n-1)$-dimensional complexes. Consider the long exact sequence for (X^n, X^{n-1}):
  $$\cdots \to H_{i+1}(X^n/X^{n-1}) \to H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n/X^{n-1}) \to \cdots$$
  Since X^n/X^{n-1} is a wedge of n-spheres, $H_i(X^n/X^{n-1}) = 0$ for $i \neq n$. For $i > n$, we get:
  $$0 \to H_i(X^{n-1}) \to H_i(X^n) \to 0$$
  By induction, $H_i(X^{n-1}) = 0$, so $H_i(X^n) = 0$. For $i = n$, the sequence gives:
  $$0 \to H_n(X^n) \to H_n(X^n/X^{n-1}) \to H_{n-1}(X^{n-1})$$
  Since $H_n(X^n/X^{n-1})$ is free and $H_n(X^n)$ is its subgroup, $H_n(X^n)$ is free.
- When there are no $(n-1)$-cells or $(n+1)$-cells, the exact sequence becomes:
  $$0 \to H_n(X^n) \to \mathbb{Z}^k \to 0$$
  where k is the number of n-cells, because $X^{n-1} = X^{n-2}$. This gives an isomorphism $H_n(X) \cong \mathbb{Z}^k$, with basis corresponding to the n-cells. The absence of $(n-1)$-cells means no relations come from below, and no $(n+1)$-cells means no relations are added from above.
- Let X have k n-cells:
  $$0 \to \mathrm{im}(H_n(X^n)) \to \mathbb{Z}^k \to H_{n-1}(X^{n-1})$$
  Thus $H_n(X^n)$ is isomorphic to a subgroup of $\mathbb{Z}^k$, which is free abelian of rank $\leq k$. Even when $(n+1)$-cells are present, they can only make this subgroup smaller, so $H_n(X)$ must be generated by $\leq k$ elements.

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