Complex Analysis Notes

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March 5, 2025
Spring 2025

Introduction

Let us begin by noting that every complex number z can be written as $z = x + iy$, where $x, y \in \mathbb{R}$.

Definition 0.1

A function is **holomorphic** at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \text{ where } h \in \mathbb{C}$$

exists.

Question

So is this just being differentiable for complex numbers?

Answer

Essentially, however because complex numbers have a value (radius) and an angle, h can approach 0 from infinitely many angles. So holomorphicity is much stronger than differentiablity; In the real case, it is differentiable going left and right. For a function to be holomorphic at a point, it must be differentiable from infinitely many angles.

Fact 0.1

If f is holomorphic in Ω, then for appropriate closed paths in Ω,

$$\int_\gamma f(z) dz = 0.$$

Fact 0.2

If f is holomorphic, then f is indefinitely differentiable.

Question

Why indefinitely differentiable? Why not indefinitely holomorphic?

Fact 0.3

If f and g are holomorphic functions in Ω which are equal in an arbitrarily samll disc in ω, then $f = g$ everywhere in Ω.

Definition 0.2

The **zeta function**,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is holomorphic in the half-plane $\mathrm{Re}(s) > 1$.

Definition 0.3

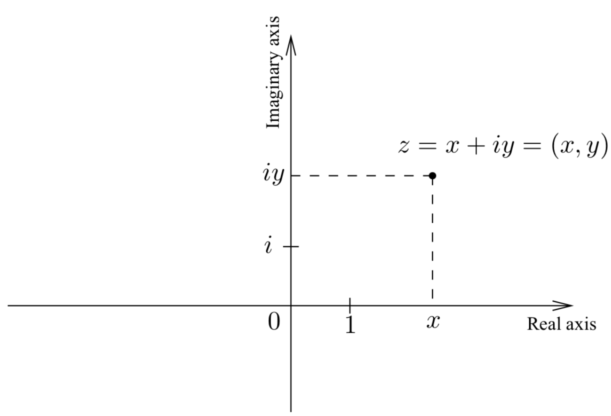
The **theta function** is the following:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

1 Preliminaries to Complex Analysis

1.1 Complex Numbers and the Complex Plane

We can imagine complex numbers as an ordered pair of the two real numbers:



Addition and mulitplication are defined like so:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 * z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

It is easy to prove that commutativity, associativity, and distributivity hold for complex numbers. We can think about addition like adding two vectors in $\mathbb{R}^2$, and mulitplication like a rotation and dilation.

Definition 1.1

The length, or absolute value of a complex number, is defined as the following:

$$|z| = (x^2 + y^2)^{1/2}$$

Note that this is the same as taking the norm, or length, of a vector in $\mathbb{R}^2$, or even finding the length of the hypotenuse that is created by the x and y values.

The triangle equality holds:

Theorem 1.1 (Triangle Inequality)

$$|z + w| \leq |z| + |w|$$

for all $z, x \in \mathbb{C}$.

From the triangle inequality, there comes this helpful fact as well:

Fact 1.1

$$||z| - |w|| \leq |z - w|$$

You can imagine a complex conjugate, $\bar{z} = x - iy$, as a reflection across the real (horizontal) axis.

The following are also useful facts easily deduced:

Fact 1.2

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

Fact 1.3

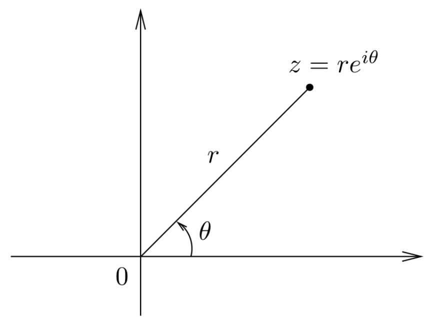
$$|z| = z\bar{z} \text{ and, when } z \neq 0, \ \frac{1}{z} = \bar{z}|z|^2$$

Definition 1.2

A complex number z's **polar form** is written as $z = re^{i\theta}$, where $r > 0$, and θ is referred to as the **argument** of z.

A mathematical fact useful in Complex Analysis is $e^{i\theta} = \cos\theta + i\sin\theta$.

From the two previous statements, we can see that $r = |z|$, the length of z, and θ is the angle.



With this form, we can redefine mulitplication to be:

$$z = re^{i\theta}, w = se^{i\phi} \implies zw = rse^{i(\theta+\phi)}$$

It is easier to see in this definition that mulitplication is simply a rotation $(\theta + \phi)$, and a dilation (rs).

Definition 1.3

A sequence $\{z_1, z_2, \dots\}$ of complex numbers is said to **converge** to $w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0 \text{ or,equivalently, } w = \lim_{n\to\infty} z_n.$$

Question

Is this just the same thing as being convergent in $\mathbb{R}$? Just that the limit is a complex number, not a real one?

Answer

Stein states that z_n converges to w if and only if the sequence of the real and imaginary parts of z_n converge to the real and imaginary parts of w. This makes sense, as in real convergence, it converges to a y value, but in the complex plane, it must converge from the x (real) direction, and the y (imaginary) direction.

Question

Given the previous answer, is it possible for a complex sequence to converge in the real direction much sooner than the imaginary? Does the difference in "convergence speed" matter?

Definition 1.4

A sequence $\{z_n\}$ is said to be a **Cauchy Sequence** if

$$|z_n - z_m| \to 0 \text{ as } n, m \to \infty.$$

Similarly to Real Analysis, a sequence is Cauchy if it's elements get closer and closer together as the sequence reaches infinity.

Question

How can a sequence be convergent, but not Cauchy?

Answer

Being convergent always implies the sequence is Cauchy, because if the elements get farther away from each other, they can never approach a limit. But it is possible for a sequence to be Cauchy and not convergent. This is because being Cauchy doesn't specify a value for the limit; it is possible the Cauchy sequence is approaching a value outside whatever space we're discussing. Thus from our scope (only considering our space missing the limit value), the sequence is not convergent.

Similar to convergence, the sequence $\{z_n\}$ is Cauchy if and only if the real and imaginary parts are.

Definition 1.5

A space is **complete** if every Cauchy sequence converges to a point in the space.

Theorem 1.2

The complex numbers $\mathbb{C}$ is complete.

Proof.

Because $\mathbb{R}$ is complete, we know every real Cauchy sequence has a limit in $\mathbb{R}$. Thus the real and imaginary parts of any Cauchy sequence must converge, so $\{z_n\}$ must converge as well. ■

Definition 1.6

If $z_0 \in \mathbb{C}$ and $r > 0$, we define the **open disc** $D_r(z_0)$ **of radius** r **centered at** z_0 to be the set of all complex numbers that are at absolute value strictly less than r from z_0. Alternatively,

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

An open disc is just a circle of radius r, centered at x_0, with no "edge". It's called the "open circle" because it is an open set.

Definition 1.7

Similarly, a **closed disc** is defined as

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

This is just a disc including the boundary, so it is a closed set.

Definition 1.8

As you can imagine, the **boundary of a disc** is defined as so:

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

Notation

The **unit disc** is defined as the following:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

This is just an open disc centered at the origin of radius one.

Definition 1.9

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an **interior point** of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$.

Definition 1.10

The **interior** of a set consists of all it's interior points.

Question

So is the interior just everything other than the boundary and exterior? Meaning an open disc is equivalent to it's interior?

Answer

Yes. :)

Definition 1.11

A set if **open** if every point in that set is an interior point. Similarly, a set is **closed** if it's complement $\Omega^c = \mathbb{C} - \Omega$ is open.

Note that $\Omega^c = \mathbb{C} - \Omega$ is just the exterior of the set. So if the exterior is open, then Ω must include it's boundary.

Definition 1.12

A point $z \in \mathbb{C}$ is said to be a **limit point** of the set Ω if there exists a sequence of points $z_n \in \Omega$ such that $z_n \neq z$ and $\lim_{n\to\infty} z_n = z$.