8200 Homework 9

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Exercise 1

Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ has isomorphic homology groups in all dimensions, but thier universal covering spaces do not.

Proof.

We know the torus $S^1 \times S^1$ is a CW complex with one 0-cell e^0, two 1-cells e^1_a, e^1_b, and one 2-cell e^2. Then we know:

- $H_0(S^1 \times S^1)$ counts path components. Since the torus is connected, $H_0 \cong \mathbb{Z}$.
- $H_1(S^1 \times S^1)$ can be computed by looking at the boundary maps. The only possible nontrivial boundary map involving 1-cells is $\partial_1 : C_1 \to C_0$. However, the single 0-cell means that the boundary of each 1-cell is that same 0-cell. Therefore we end up with two independent loops, so $H_1 \cong \mathbb{Z}^2$.
- For $H_2(S^1 \times S^1)$, the boundary of the 2-cell corresponds to a loop $aba^{-1}b^{-1}$ in the 1-skeleton that is null-homotopic in the 1-skeleton. This ensures that the resulting ∂_2 map is zero in homology, so $H_2 \cong \mathbb{Z}$.
- In higher dimensions ($n > 2$), there are no n-cells, so $H_n(S^1 \times S^1) = 0$ for $n > 2$.

Therefore

$$H_n(S^1 \times S^1) \cong \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $Y = S^1 \vee S^1 \vee S^2$, we can use the following fact about wedge sums:

$$\widetilde{H}_n(X \vee Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Z) \quad \text{for all } n \geq 1.$$

Let $X = S^1 \vee S^1$ and $Z = S^2$. Then

$$\widetilde{H}_n(Y) = \widetilde{H}_n(X \vee Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Z).$$

We know that

$$\widetilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 1, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{H}_n(S^2) = \begin{cases} \mathbb{Z} & n = 2, \\ 0 & \text{otherwise,} \end{cases} \quad \widetilde{H}_n(S^1 \vee S^1) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1).$$

Therefore:

$$\widetilde{H}_n(Y) = \widetilde{H}_n(X \vee Z) = \begin{cases} \mathbb{Z}^2 \oplus 0 = \mathbb{Z}^2 & n = 1, \\ 0 \oplus \mathbb{Z} = \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

So that

$$H_n(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $H_n(S^1 \times S^1) \cong H_n(S^1 \vee S^1 \vee S^2)$.
For the universal covering spaces, we know that the torus's covering space is $\mathbb{R}^2$, and for $S^1 \vee S^1 \vee S^2$, we can use an infinite tree with a 2-sphere at each vertex, which is clearly not homeomorphic to $\mathbb{R}^2$.
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Exercise 2

Show that for every $f : S^n \to S^n$, degree of $Sf : S^{n+1} \to S^{n+1}$ is equal to degree of f. Here, Sf denotes the suspension of f which is the map induced from $f \times \mathrm{id}$: $S^n \times [0,1] \to S^n \times [0,1]$ on $SS^n \cong S^{n+1}$.

Proof.

The degree of a map $g : S^k \to S^k$ can be characterized by its induced map

$$g_* : H_k(S^k) \longrightarrow H_k(S^k).$$

Since $H_k(S^k) \cong \mathbb{Z}$, the map g_* must be multiplication by some integer d. In passing from f to Sf, $H_{n+1}(S^{n+1})$ is essentially determined by how f acts on $H_n(S^n)$. This is because when the suspension is done, collapsing $S^n \times \{0\}$ and $S^n \times \{1\}$ to points does not alter the d integer.
Therefore, $(Sf)_* : H_{n+1}(S^{n+1}) \to H_{n+1}(S^{n+1})$ acts on the generator $[S^{n+1}]$ by the same integer $\deg(f)$. Therefore,

$$\deg(Sf) = \deg(f).$$

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Exercise 3

Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Proof.

- Suppose, for contradiction, that $f(x) \neq x$ and $f(x) \neq -x$ for all $x \in S^{2n}$. We know $\deg(\mathrm{id}) = +1$ on S^k and that the antipodal map a has degree $(-1)^{k+1}$. In particular, for $k = 2n$,

  $$\deg(a) = (-1)^{2n+1} = -1.$$

  If $f(x)$ never equals x or $-x$, then for each x we can travel continuously between x and $f(x)$ (and then on to $-x$) without passing through a point where x and $f(x)$ coincide or become antipodes. Then a map H can be constructed, where H would be a homotopy in S^{2n} between id and a. But id has degree $+1$, while a has degree -1. Since degree is a homotopy invariant, no such homotopy can exist. Thus there must be some point $x \in S^{2n}$ such that $f(x) = x$ or $f(x) = -x$.

- Let $g : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ be an arbitrary continuous map. We can lift g to a map $f : S^{2n} \to S^{2n}$

such that the following diagram commutes with the projection $\pi : S^{2n} \to \mathbb{R}P^{2n}$:

$$\begin{array}{ccc} S^{2n} & \xrightarrow{f} & S^{2n} \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{R}P^{2n} & \xrightarrow{g} & \mathbb{R}P^{2n}. \end{array}$$

By Part 1, there is an $x \in S^{2n}$ with $f(x) = x$ or $f(x) = -x$. Project down via π to find that $\pi(x) = [x]$ in $\mathbb{R}P^{2n}$ and $g(\pi(x)) = g([x]) = \pi(f(x))$.

If $f(x) = x$, then $\pi(f(x)) = [x]$, so $g([x]) = [x]$ is a fixed point. If $f(x) = -x$, then $\pi(-x) = [x]$ again (since $x \sim -x$ in projective space), so $g([x]) = [x]$ still holds. Hence $[x]$ is a fixed point of g in $\mathbb{R}P^{2n}$. Thus every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ must have a fixed point.

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Exercise 4

Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Proof.

- Let $f : S^n \to S^n$ be a map of degree zero. Suppose that $f(x) \neq x$ for every $x \in S^n$. We can try to construct a homotopy H between f and id by continuously "sliding" each point $f(x)$ to x:
$$H(x,t) = \frac{(1-t)\,f(x) + t\,x}{\|(1-t)\,f(x) + t\,x\|}, \quad t \in [0,1].$$
Since $f(x) \neq x$ for all x, this formula never hits 0 in $\mathbb{R}^{n+1}$, so H remains on the sphere. Thus f would be homotopic to id, but $\deg(f) = 0 \neq 1 = \deg(\mathrm{id})$, so no such homotopy can exist. Thus there must exists some $x \in S^n$ such that $f(x) = x$.

  We can use a similar argument for y, because if $f(y) \neq -y$ for all $y \in S^n$, we can attempt the homotopy
$$K(y,t) = \frac{(1-t)\,f(y) \;+\; t\big(-y\big)}{\|(1-t)\,f(y) + y)\|}, \quad t \in [0,1],$$
yielding a homotopy between f and the antipodal map $a(y) = -y$. Since $\deg(a) = (-1)^{n+1} \neq 0$, this too contradicts $\deg(f) = 0$. Hence there must be some y satisfying $f(y) = -y$.

- Suppose F is a continuous vector field on the closed unit ball $D^n \subset \mathbb{R}^n$. Since $F(x) \neq 0$ for every $x \in D^n$, we may restrict to the boundary S^{n-1} and define:
$$f : S^{n-1} \longrightarrow S^{n-1}, \quad f(x) = \frac{F(x)}{\|F(x)\|}.$$
Because $\deg(f) = 0$, by Step 1 there exists an $x \in S^{n-1}$ with $f(x) = x$ and a $y \in S^{n-1}$ with $f(y) = -y$. At x, we have $f(x) = x$, meaning
$$\frac{F(x)}{\|F(x)\|} = x,$$

so $F(x)$ is a positive scalar multiple of x. Thus $F(x)$ is pointing directly outward. At y, we have $f(y) = -y$, meaning

$$\frac{F(y)}{\|F(y)\|} = -y,$$

so $F(y)$ is a negative scalar multiple of y. Thus $F(y)$ is pointing inward.

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Exercise 5

For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map on $H_n\left(\mathbb{R}^n, \mathbb{R}^n - \{0\}\right) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

Proof.

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Exercise 6

A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Proof.

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