Algebraic Topology Problem Bank

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1 Homework 1

Exercise 0.2. Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Proof. First note that S^{n-1} is defined to be all the points $(x_1, \dots, x_n)$ such that $\sqrt{x_1^2 + \cdots + x_n^2} = 1$. Let $f_t(x_1, \dots, x_n) = (1 + t(\frac{1}{\sqrt{x_1^2+\cdots+x_n^2}} - 1))(x_1, \dots, x_n)$. Note that this is a continuous function because $(x_1, \dots, x_n) \neq 0$ and it is made up of continuous functions. Then we have

$$\begin{aligned}
f_0(x_1, \dots, x_n) &= (1 + 0)(x_1, \dots, x_n) = (x_1, \dots, x_n) \\
f_1(x_1, \dots, x_n) &= (1 + \frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}} - 1)(x_1, \dots, x_n) \\
&= (\frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}})(x_1, \dots, x_n) \\
&= (\frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}}, \dots, \frac{x_n}{\sqrt{x_1^2 + \cdots + x_n^2}}).
\end{aligned}$$

Notice that because

$$\begin{aligned}
&\sqrt{(\frac{x_1}{\sqrt{x_1^2 + \cdots + x_n^2}})^2 + \cdots + (\frac{x_n}{\sqrt{x_1^2 + \cdots + x_n^2}})^2} \\
&= \sqrt{\frac{x_1^2}{x_1^2 + \cdots + x_n^2} + \cdots + \frac{x_n^2}{x_1^2 + \cdots + x_n^2}} \\
&= \sqrt{\frac{x_1^2 + \cdots + x_n^2}{x_1^2 + \cdots + x_n^2}} = 1,
\end{aligned}$$

we can conclude that $f_0(\mathbb{R}^n - \{0\}) = \mathbb{R}^n - \{0\}$ and $f_1(\mathbb{R}^n - \{0\}) = S^{n-1}$. Finally, let $(x_1, \dots, x_n) \in S^{n-1}$, so that $\sqrt{x_1^2 + \cdots + x_n^2} = 1$. Then

$$f_t(x_1, \dots, x_n) = (1 + \frac{1}{\sqrt{x_1^2 + \cdots + x_n^2}} - 1)(x_1, \dots, x_n) = 1 \cdot (x_1, \dots, x_n) = (x_1, \dots, x_n).$$

So $f_t(x_1, \dots, x_n)\big|_{S^{n-1}} = (x_1, \dots, x_n)$. Thus $f_t(x)$ is a deformation retraction. $\square$

Exercise 0.3. Before starting Exercise 0.3, let me state the following implications.

Proof. First, note that for any continuous map h, $f \cong g \implies h(f) \cong h(g)$:

Assuming $f \cong g$, there exists some continuous θ_t such that $\theta_0 = f$ and $\theta_1 = g$. Let $\theta'_t = h(\theta_t)$, and note that this is continuous because it is composed of continuous functions. Then $\theta'_0 = h(\theta_0) = h(f)$, and similarly, $\theta'_1 = h(\theta_1) = h(g)$. Thus $h(f) \cong h(g)$.

Next, I aim to show that $f \cong g \implies f(h) \cong g(h)$:

Assuming $f \cong g$, there exists some continuous θ_t such that $\theta_0 = f$ and $\theta_1 = g$. Let $\theta'_t = \theta_t(h)$, and note that this is continuous because it is composed of continuous functions. Then $\theta'_0 = \theta_0(h) = f(h)$, and similarly, $\theta'_1 = \theta_1(h) = g(h)$. Thus $f(h) \cong g(h)$. □

Exercise 0.3a. Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

Proof. First, to show the composition holds, assume that there is a homotopy equivalence from $X \to Y$, so there exists a continuous map $f_1 : X \to Y$ such that there exists a continuous map $g_1 : Y \to X$, and $f_1 \circ g_1 \cong \mathrm{id}_y$ and $g_1 \circ f_1 = \mathrm{id}_x$. Similarly, if there is a homotopy equivalence from $Y \to Z$, there exists a continuous map $f_2 : Y \to Z$ such that there exists a continuous map $g_2 : Z \to Y$, and $f_2 \circ g_2 \cong \mathrm{id}_z$ and $g_2 \circ f_2 = \mathrm{id}_y$.

Define the following:

$$f = f_2 \circ f_1(x) = f_2(f_1(x))$$
$$g = g_1 \circ g_2(z) = g_1(g_2(z))$$

Then f and g are continuous maps because they are composed of continuous functions, and

$$\begin{aligned}
(f \circ g)(z) &= f(g(z)) \\
&= f_2(f_1(g_1(g_2(z)))) \\
&= f_2(g_2(z)) = z \\
(g \circ f)(x) &= g(f(x)) \\
&= g_1(g_2(f_2(f_1(x)))) \\
&= g_1(f_1(x)) = x
\end{aligned}$$

Thus f is a homotopy equivalence from $X \to Z$.

Next we show that homotopy equivalence is an equivalence relation.

Reflexivity: Let $f, g : X \to X$ be the identity map. Then f, g are continuous, $f \circ g \cong \mathrm{id}_x$, $g \circ f \cong \mathrm{id}_x$. Thus f is a homotopy equivalence, and $X \cong X$.

Symmetry: Assume $X \cong Y$. Then $f : X \to Y$ is a homotopy equivalence, so there exists $g : Y \to X$ such that $g \circ f \cong \mathrm{id}_x$ and $f \circ g \cong \mathrm{id}_y$. Let $f_0 = g$ and $g_0 = f$, so that $f_0 : Y \to X$ is a continuous map, as is $g_0 : X \to Y$; also, $g_0 \circ f_0 \cong \mathrm{id}_y$ and $f_0 \circ g_0 \cong \mathrm{id}_x$. Thus $Y \cong X$.

Transitivity: Assume $X \cong Y$ and $Y \cong Z$. Because $X \cong Y$, we know there exists $f_1 : X \to Y$, $g_1 : Y \to X$, $f_1 \circ g_1 \cong \mathrm{id}_y$, and $g_1 \circ f_1 \cong \mathrm{id}_x$. Similarly, because $Y \cong Z$, we know there exists $f_2 : Y \to Z$, $g_2 : Z \to Y$, $f_2 \circ g_2 \cong \mathrm{id}_z$, and $g_2 \circ f_2 \cong \mathrm{id}_y$. Then define

$$\begin{aligned} f &= f_2 \circ f_1 \\ g &= g_1 \circ g_2, \end{aligned}$$

and note that both are continuous because they are composed of continuous functions.

Then

$$\begin{aligned} (f \circ g)(z) = f(g(z)) = f_2(f_1(g_1(g_2(z)))) = f_2(g_2(z)) = z \\ (g \circ f)(x) = g(f(x)) = g_1(g_2(f_2(f_1(x)))) = g_1(f_1(x)) = x. \end{aligned}$$

So $X \cong Z$ with f, so that transitivity is true, and a homotopy equivalence is an equivalence relation. □

Exercise 0.3b. Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

Proof. **Reflexivity**: Consider any $f : X \to Y$, and then let $f_t = f$, so that $f_0 = f$ and $f_1 = f$. Thus $f \cong f$.

Symmetry: Assume $f \cong g$, so that there exists a homotopy $f_t(x)$ such that $f_0 = f$ and $f_1 = g$. Define $f'_t(x) = f_{1-t}(x)$, and note that it is continuous. Then $f'_0(x) = f_1(x) = g$ and $f'_1(x) = f_0(x) = f$. Thus, by $f'_t(x)$, $g \cong f$.

Transitivty: Assume $f \cong g$ and $g \cong h$, so that there exists $\psi_t(x)$ such that $\psi_0 = f$, $\psi_1 = g$, and $\theta_t(x)$ such that $\theta_0 = g$ and $\theta_1 = h$. Define $\phi_t(x)$ as $\psi_{2t}(x)$ for $0 \leq t \leq \frac{1}{2}$ and $\theta_{2t-1}(x)$ for $\frac{1}{2} < t \leq 1$. Thus ϕ_t is continuous, because ψ and θ are, and $\lim_{t \to \frac{1}{2}^-} \phi_t(x) = \psi_1(x) = g(x) = \theta_0(x) = \lim_{t \to \frac{1}{2}^+} \phi_t(x)$. Futhermore, $\phi_0 = \psi_0 = f$ and $\phi_1 = \theta_1 = h$. Thus, $f \cong h$.

Thus the relation of homotopy among maps is an equivalence relation. □

Exercise 0.3c. Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Proof. Let $f : X \to Y$ be a homotopy equivalence, and assume it is homotopic to $g : X \to Y$. Because f is a homotopy equivalence, there exists some continuous $h : Y \to X$ such that $f \circ h \cong \mathrm{id}_y$ and $h \circ f \cong \mathrm{id}_x$. Because h is continuous, we can use a previous result to say that because $f \cong g$, then $h(f) \cong h(g)$. Because $f(h) \cong \mathrm{id}_y$, by transitivity proven above, $h(g) \cong \mathrm{id}_y$. Similarly, we can also say that $f(h) \cong g(h)$, and so because $f(h) \cong \mathrm{id}_y$, then $g(h) \cong \mathrm{id}_y$. Thus g is homotopy equivalent. □

Exercise 0.10. Show that a space X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible if and only if every map $f : Y \to X$ is nullhomotopic.

Proof. Assume X is contractible. Then we know any identity map $h : X \to X$ is nullhomotopic. Then we know, because $h \cong g$ for constant function g, that $f(h) \cong f(g)$, and $f(h) = h$ and $f(g) = g_0$, a constant function. Thus f is nullhomotopic for any Y.

Assume every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Then the identity map $f : X \to X$ is nullhomotopic. Thus, by definition, X is contractible.

For the more general statement, first assume X is contractible, so that $h : X \to X$, $h \cong g$ (where g is a constant function). Let $f : Y \to X$ be a map with any space Y. Then there exists $f_t(x)$ such that $f_0 = h$, and $f_1 = g$. Let $f'_t = f_t(f)$ (and note that $f'_t(x)$ is still continuous), so $f'_0 = h(f)$, and $f'_1 \cong g_1(f)$. Then $f \cong g_1(f)$, and $g_1(f)$ is a constant function.

Assume every map $f : Y \to X$ is nullhomotopic, then $f : X \to X$, where f is the identity function, is nullhomotopic, and therefore X is contractible. □

Exercise 0.11. Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \cong \text{id}$ and $hf \cong \text{id}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Proof. Assume there exists $g, h : Y \to X$ such that $fg \cong \text{id}_y$, $hf \cong \text{id}_x$. Then, because $fg \cong \text{id}_y$, we know $fgf \cong f \iff fgf \cong f \circ \text{id}_x$. Then $gf \cong \text{id}_x$, and so for $f : X \to Y$, there exists $g : Y \to X$ such that $fg \cong \text{id}_y$ and $hf \cong \text{id}_x$, so $f : X \to Y$ is a homotopy equivalence.

Assume that there exists $h, g : Y \to X$, and fg and hf are homotopy equivalents. Then we know

$$\begin{aligned} fg \text{ is homotopy equivalent} &\iff \exists g' : Y \to Y, fg \circ g' \cong \text{id}_y, g' \circ fg \cong \text{id}_y \\ hf \text{ is homotopy equivalent} &\iff \exists h' : X \to X, hf \circ h' \cong \text{id}_x, h' \circ hf \cong \text{id}_x \end{aligned}$$

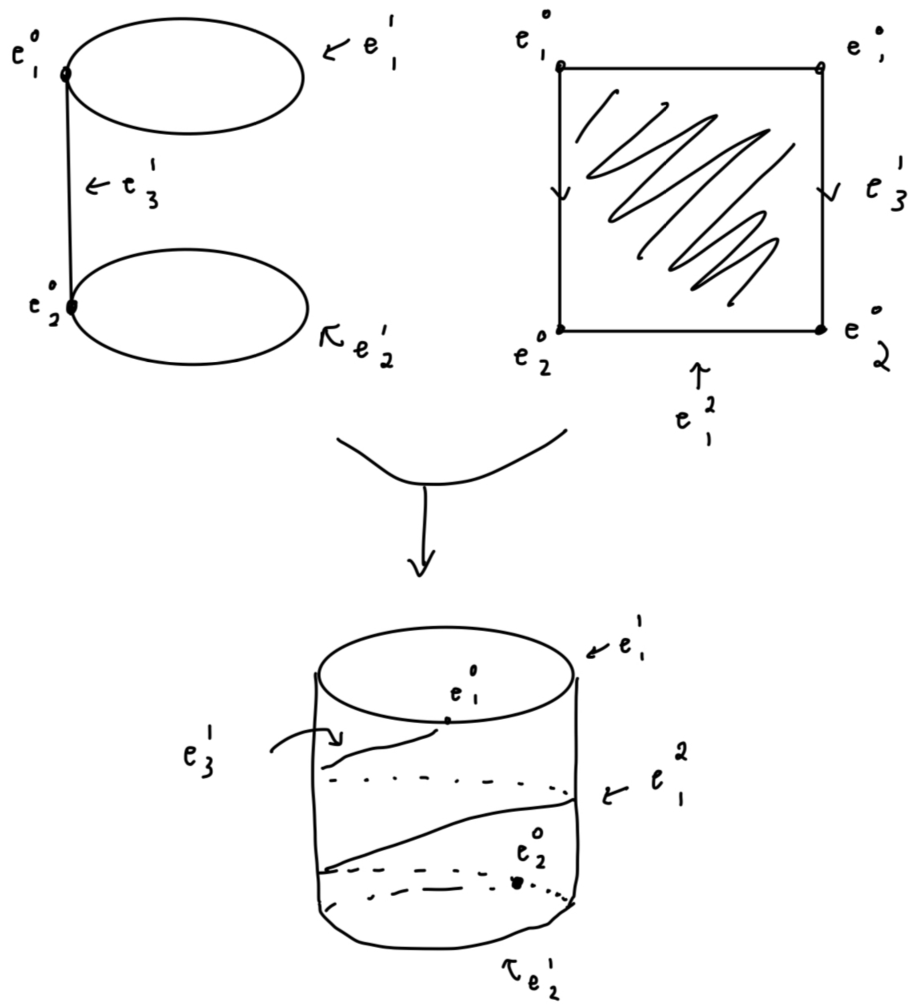
Then we have

$$\begin{aligned} h'hf \cong \text{id}_x &\implies h(fh'h) \cong \text{id}_x \circ h \cong h \cong h \circ \text{id}_x \\ &\implies fh'h \cong \text{id}_x \cong h'hf \end{aligned}$$

So because of $h'h$, f is a homotopy equivalence. □

Exercise 0.17a. Show that the mapping cylinder of every map $f : S^1 \to S^1$ is a CW complex.

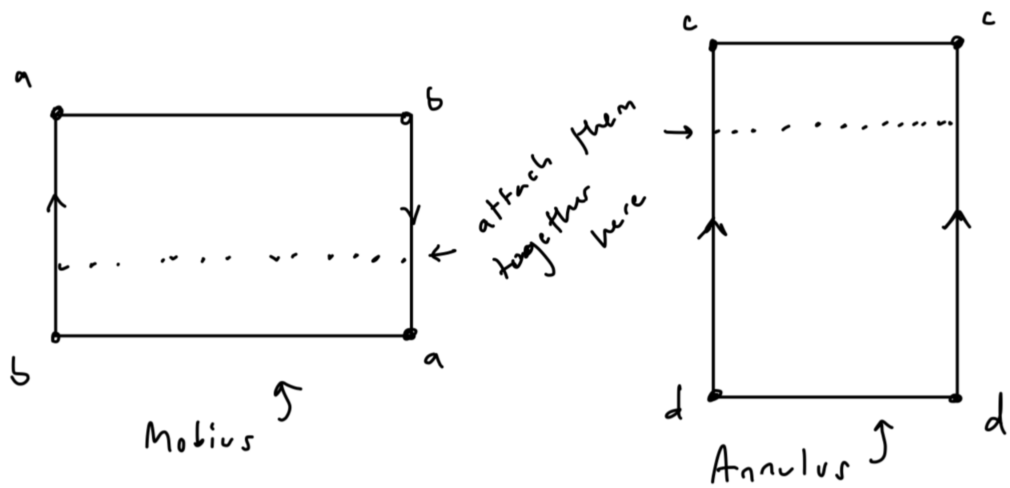
Proof. Let $f : S^1 \to S^1$ be any arbitrary mapping. To construct the CW complex, start with two 0-cells, e_1^0, acting as an x coordinate, and e_2^0, acting as it's y coordinate. Then add the 1-cell e_1^1 as a circle, where e_1^0 is in this circle, and another 1-cell e_2^1 as another (disjoint) circle, this time with e_2^1 containing e_2^0. Then add one more 1-cell, e_3^1, as the graph of f, effectively going around the cylinder. Finally, attach a 2-cell e_1^2 so that it "closes" the cylinder. An attempted diagram is attached below:



□

Exercise 0.17b. Construct a 2-dimensional CW complex that contains both an annulus $S^1 \times I$ and a Mobius band as deformation retracts.

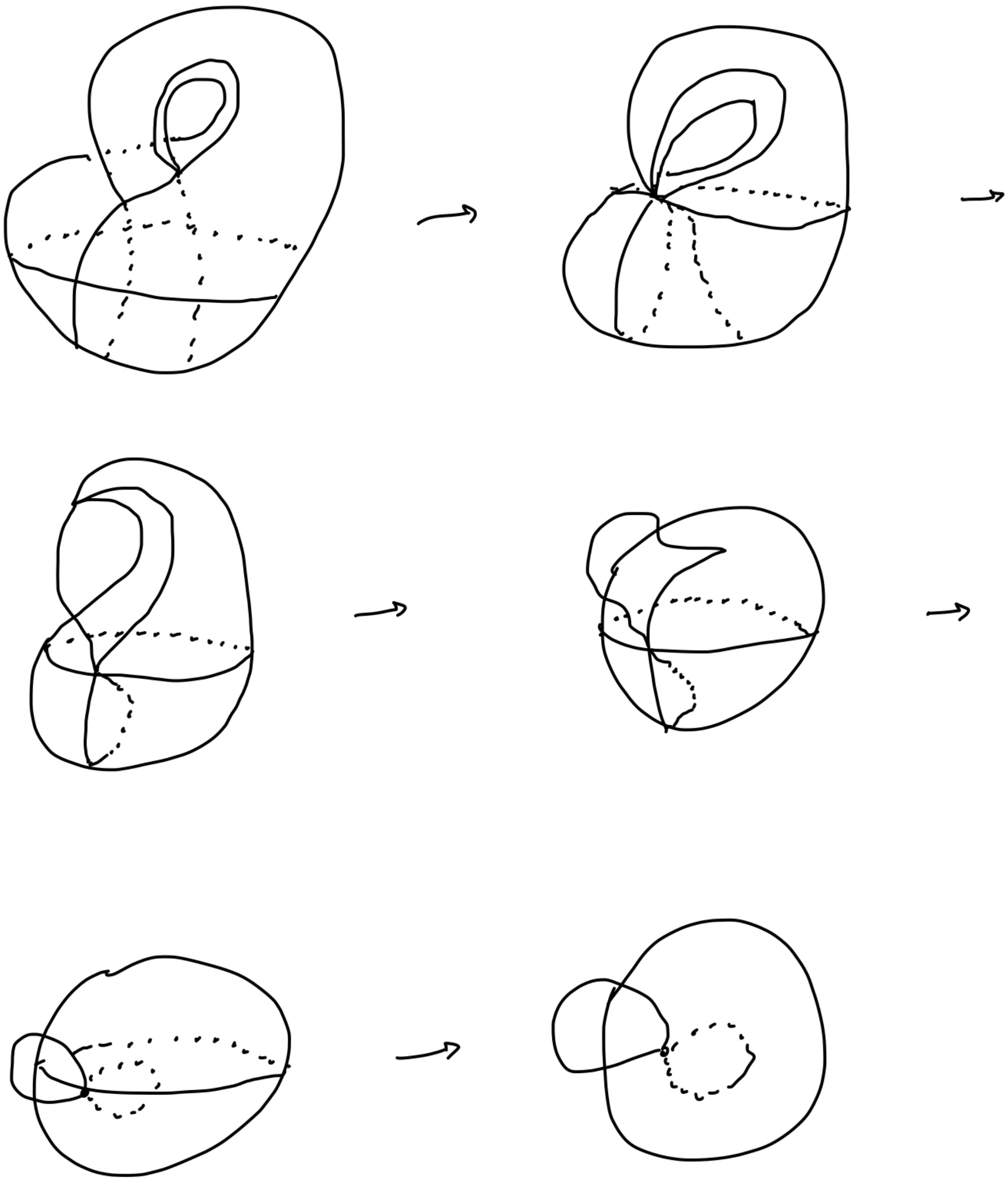
Proof. Both the Mobius band and the annulus can deformation retract onto their middle circles, so if we glue the middle circles together, we can get a 2-dimensional CW complex that can deformation retract to both. Image below:



□

Exercise 0.20. Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $S^1 \vee S^1 \vee S^1$.

Proof. As shown in the picture below, you can condense there the neck meets the rest of the bottle into one point, extend the neck out so there is only a 1-cell coming from that intersection point, until the neck is pushed all the way into a sphere. Then all that is left is a sphere with a circle on the outside and another cirlce on the inside, where the circles intersect at exactly one point. Thus $X \subset \mathbb{R}^3 \cong S^1 \vee S^1 \vee S^1$.



$\square$

2 Homework 2

Exercise 1. Describe a CW complex structure on $\mathbb{C}P^2 \times \mathbb{R}P^2$ and ΣT^2.

Proof. For $\mathbb{C}P^2 \times \mathbb{R}P^2$, we know that $\mathbb{R}P^2$ consists of 1 0-cell, 1 1-cell, and 1 2-cell. Similarly, for $\mathbb{C}P^2$, it consists of 1 1-cell, 1 2-cell, and 1 4-cell. Thus the product of these spaces consists of 1 0-cell, 1 1-cell, 2 2-cell, 1 3-cell, 2 4-cell, 1 5-cell, and 1 6-cell. Visually, we're finding the product of a sphere and a 4th dimensional shape.

For ΣT^2, first note that $T^2 = S^1 \times S^1$, the torus. In Hatcher, it is stated that $\Sigma X = X \wedge S^1$, so in our case, $\Sigma(S^1 \times S^1) = (S^1 \times S^1) \wedge S^1$. Also in Hatcher, $X \wedge Y = X \times Y / X \vee Y$.

So finally, we have $(S^1 \times S^1) \times S^1/(S^1 \times S^1) \vee S^1$. Visually, we can imagine this as a torus crossed with S^1 quotient by a torus touching a circle. Regarding cell complexes, we have $(e^0 \cup e^1 \cup e^1 \cup e^1 \cup e^2 \cup e^2 \cup e^2 \cup e^3)/(e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^1 \cup e^2)$. Thus the reduced suspension of a torus has the CW structure of 1 1-cell, 2 2-cells, and 1 3-cell. □

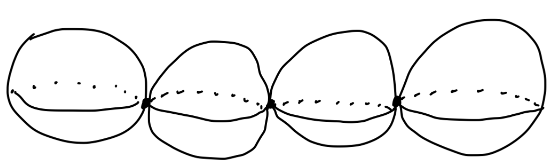
Exercise 2. Let X_n be the topological space obtained by identifying $n > 1$ points on S^2 to a single point. Describe a CW decomposition of X_n.

Proof. When you identify a point with a singular point, you're pinching the S^2 sphere at that singular point, and it creates a "loop" from where the point originally was to the singular point. This occurs for every point identified, so that there are $n-1$ loops (or S^1's) added to the space. Note that we need loops, because if we were to do lines, some lines could intersect. Thus we can say $X_n = S^2 \vee \bigvee_{i=1}^{n-1} S^1$, which is just stating that X_n is a sphere with loops added that intersect at a single point (the singular point). When considering CW decomposition, the S^2 has 1 0-cell and 1 2-cell, while each S^1 has a 0-cell and a 1-cell. However, the singular point can be made the 0-cell for the S^2 and the S^1's, so in total there is 1 0-cell, 1 2-cell, and $n-1$ 1-cells. □

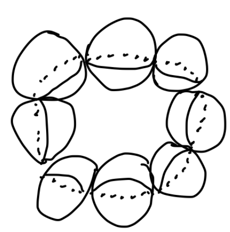
Exercise 3. Hatcher Exercise 0.21: If X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

Proof. First, because the space is connected, we know that there aren't any disjoint 2-spheres. Then, we can imagine this space made up of S^2's which intersect at most 1 point, and because there can't be any disjoint 2-spheres, they must all be in a row like this, or in a loop:

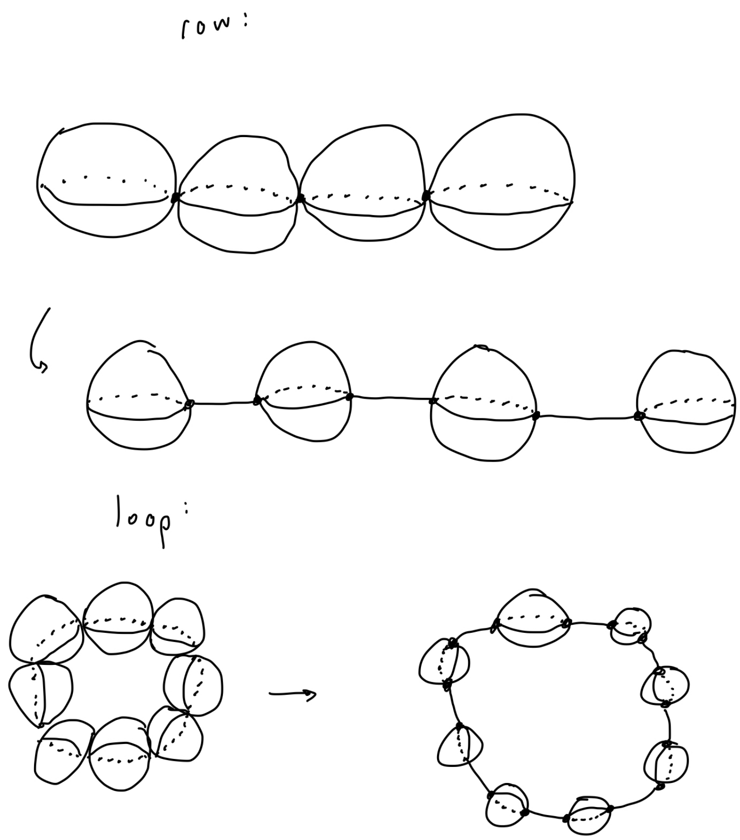
row:



loop:



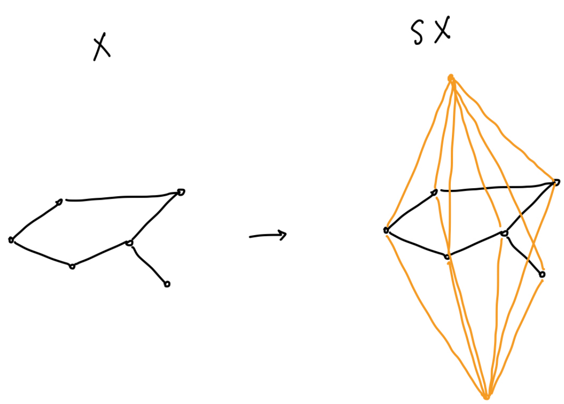
Either way, the intersection points between the spheres can be stretched into lines between them, like this:



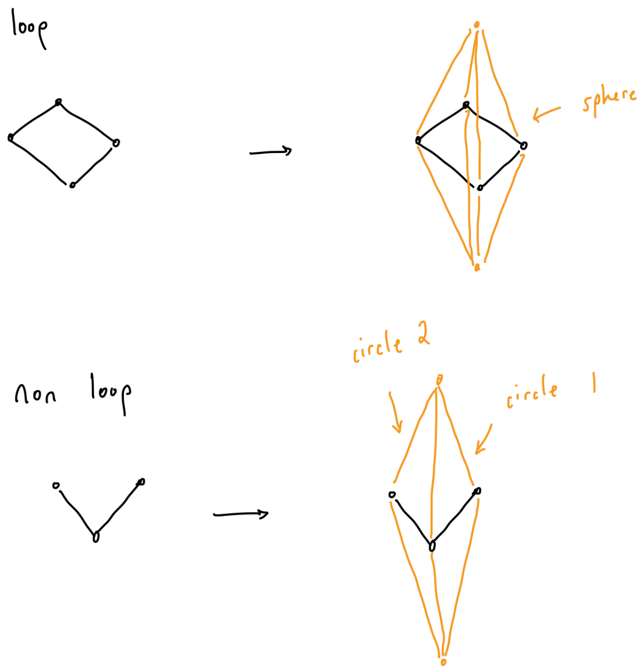
As you can see, the space X can be stretched (and is therefore homotopically equivalent to) to be $\bigvee_{i=1}^{n-1} S^1 \vee \bigvee_{i=1}^{n} S^2$ where n is the number of 2-sphere's in X. $\square$

Exercise 4. Hatcher Exercise 0.25: If X is a CW complex with components X_α, show that the suspension SX is homotopy equivalent to $Y \vee_\alpha SX_\alpha$ for some graph Y. In the case that X is a finite graph, show that SX is homotopy equivalent to a wedge sum of circles and 2-spheres.

Proof. If X is a finite graph, the suspension SX is just taking all the vertices in the graph and connecting them to a point "above" and "below" the graph. This results in something like this:



Note that when you have a closed loop in a graph, when you do the suspension it results in a sphere, and when there is a non-loop, it just creates circles:



Because everything in a graph is either a loop or a nonloop, this accounts for what makes up any graph X. Thus if X is a graph, SX is just a wedge sum of spheres and circles. $\square$