Alegbraic Topology Problem Bank

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Homework 1

Hatcher 0.2

Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Hatcher 0.3

(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Hatcher 0.6

(a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem]

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $/R$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z, but no true deformation retraction.

Hatcher 0.10

Show that a space X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible if and only if every map $f : Y \to X$ is nullhomotopic.

Hatcher 0.11

Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \cong \text{id}$ and $hf \cong \text{id}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Hatcher 0.16

Show that S^∞ is contractible.

Hatcher 0.17

Construct a 2-dimensional cellcomplex that contains both an annulus $S^1 \times I$ and a Mobius band as deformation retractions.

Hatcher 0.20

Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $S^1 \vee S^1 \vee S^1$.