TDA HW1

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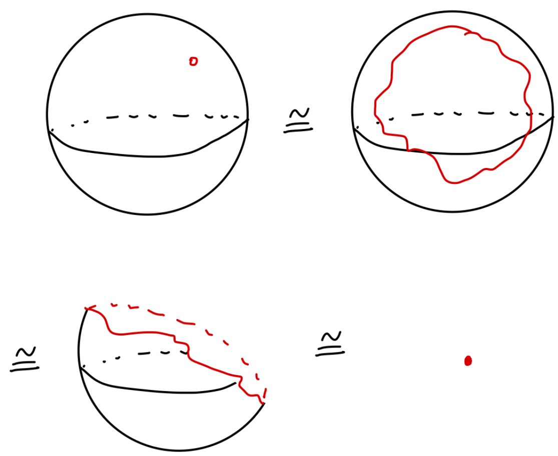
Problem 1. Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map.

Proof. If f is not surjective, then there exists some $x \in \mathbb{S}^2$ such that $x \notin f(\mathbb{S}^1)$. So we can consider $f(\mathbb{S}^1)$ to be $\mathbb{S}^2 \setminus \{x\}$, which is homotopy equivalent to a point:



Thus every map in $f(\mathbb{S}^1)$ is homotopic to a constant map. $\square$

Problem 2. Let X and Y be two homeomorphic topological spaces. Show that if X has dimension n, then Y also has dimension n.

Proof. Consider a homeomorphism $f : Y \to X$, and then let $y \in Y$ with $x = f(y)$. Because $x \in X$, x must have dimension n, so that there eists an open set of X, call it O, containing x with a homeomorphism $h : O \to \mathbb{R}^n$. Then, we can let $O' = f^{-1}(O)$, and note that it is an open set of Y containing y. We also know that $h \circ g : O' \to \mathbb{R}^n$ is a homeomorphism, and so Y has dimension n. $\square$

Problem 3. Let $(G, +)$ be a group. Prove that

$$\forall g \in G, g + g = 0 \Rightarrow G \text{ is commutative.}$$

Proof. Let $g_1, g_2 \in G$, and note that $g + g = 0 \iff g = -g$. Then

$$\begin{aligned}
(g_1 + g_2) + (g_1 + g_2) &= 0 \\
g_1 + g_2 &= -(g_1 + g_2) \\
g_1 + g_2 &= (-g_2) + (-g_1) \\
g_1 + g_2 &= g_2 + g_1
\end{aligned}$$

Thus the group is commutative. □

Problem 4. Characterize the two surfaces depicted in Figure 1 in terms of genus, boundary, and orientability.

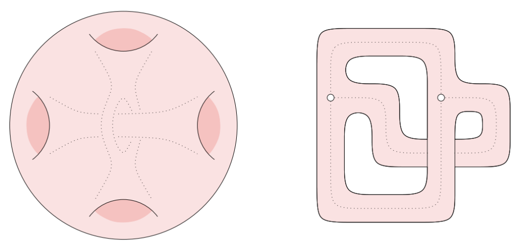
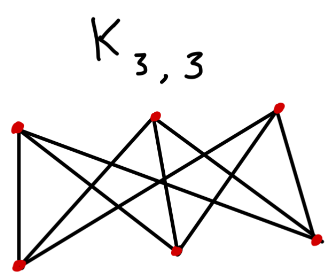


Figure 1: Left: a 2-manifold without boundary obtained by adding tunnels inside the sphere. Wee see four tunnel openings and one tunnel passing though a fork of the other. Right: a 2-manifold with boundary obtained by thickening a graph.

Proof. □

Problem 5. Is every graph that can be embedded on the Mobius strip planar?

Proof. No. A counterexample would be $K_{3,3}$, which is not planar:



However, it can be embedded on the Mobius strip. □