8150 Homework III

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Stein Problems

Exercise 1

Using Euler's Formula

$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$

show that the complex zeroes of $\sin \pi z$ are exactly the integers, and that they are each of order 1.
Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

Proof.

To find the zeroes, set Euler's Formula to zero:

$$\frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0$$

. From here, we can find that

$$\begin{aligned} e^{i\pi z} - e^{-i\pi z} &= 0 \\ e^{i\pi z} &= e^{-i\pi z} \\ i\pi z &= -i\pi z + 2\pi i k, k \in \mathbb{Z} \\ 2i\pi z &= 2i\pi k, \end{aligned}$$

so that $z = k$, meaning z must be an integer.
To show that they are all of order one, we can find the derivate of $\sin \pi z$ to be $\pi \cos \pi z$, and notice that this is nonzero for any integer.
To find the residue of $1/\sin \pi z$ at z an integer, we can use this formula:

$$\operatorname{Res}(f, a) = \lim_{z \to a} (z - a)(f(z)).$$

So in our case,

$$\operatorname{Res}(\frac{1}{\sin \pi z}, n) = \lim_{z \to n} (z - n) \cdot \frac{1}{\sin \pi z},$$

where $n \in \mathbb{Z}$. ■

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Exercise 2

Prove that if
$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \text{ and } \sum_{n=-\infty}^{\infty} c'_n(z-a)^n$$
are Laurent series expansions of $f(z)$, then $c_n = c'_n$ for all n.

Proof.

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=-\infty}^{\infty} c'_n(z-a)^n$. Then we know, for any integer k,
$$f(z)(z-a)^{-k-1} = \sum_{n=-\infty}^{\infty} c_n(z-a)^{n-k-1} = \sum_{n=-\infty}^{\infty} c'_n(z-a)^{n-k-1}$$
Then let γ be any closed contour in the annulus going around a once, and because it is a compact set of points, the Luarent serieses can be integrated termwise:
$$\sum_{n=-\infty}^{\infty} c_n \oint_\gamma (z-a)^{n-k-1}dz = \sum_{n=-\infty}^{\infty} c'_n \oint_\gamma (z-a)^{n-k-1}dz$$
We know that
$$\oint (z-a)^{n-k-1}dz = 2i\pi \text{ if } n = k \text{ and } 0 \text{ if } n \neq k$$
So then we are left with $2i\pi c_m = 21\pi c'_n$ for any k, which proves the statement. ■

Exercise 3

Expand $\frac{1}{1-z^2} + \frac{1}{3-z}$ in a series of the form $\sum^\infty -\infty a_n z^n$. How many such expansions are there? In which domain is each of them valid?

Proof.

We find that:

$$\begin{aligned}\frac{1}{z-3} &= -\frac{1}{3}\frac{1}{1-3z^{-1}} = -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k \text{ for } |z|<3\\ &= \frac{1}{z}\frac{1}{1-3z^{-1}} = z^{-1}\sum_{k\geq 0} 3^k z^{-k} \text{ for } |z|>3\end{aligned}$$

and:

$$\begin{aligned}\frac{1}{1-z^2} &= \sum_{k\geq 0} z^{2k} \text{ for } |z|<1\\ &= \frac{1}{z^2}\frac{-1}{1-z^{-2}} = -z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z|>1\end{aligned}$$

So we can just list all the possible combinations to find:

$$\begin{aligned}f(z) &= -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1)\\ f(z) &= -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (1, 3)\\ f(z) &= z^{-1}\sum_{k\geq 0} 3^k z^{-k} + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1)\cup(3,\infty)\\ f(z) &= z^{-1}\sum_{k\geq 0} 3^k z^{-k} - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (3, \infty)\end{aligned}$$

■

Exercise 4

Let $P(z)$ and $Q(z)$ be polynomials with no common zeros. Assume $Q(a) = 0$. Find the principal part of $P(z)/Q(z)$ at $z = a$ if the zero a is (i) simple; (ii) double. Express your answers explicitly using P and Q.

Proof.

i.

ii.

■

Exercise 5

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$.

Proof.

Assume, for contradiction, that $z = 0$ is a removable singularity. Then f would extend to a holomorphic function over $z = 0$, so that $f(0) = f(\lim z_n) = \lim f(z_n) = 0$. But then f would have to be identically zero, because of the identity principal. This contradicts the fact that f is stated to be non-constant.
Then assume for contradiction that $z = 0$ is a pole. Then $f(z_n) \to \infty$. This is a contradiction because $f(z_n) = 0$ infinitely many times.
Thus $z = 0$ must be an essential singularity. ■

Exercise 6

Let f be entire and suppose that $\lim_{x\to\infty} f(z) = \infty$. Show that f is a polynomial.

Proof.

First, note that because f is unbounded, there must exist some R such that $f(D_R^c) \subset D^c$. Therefore we know that f is nonvanishing on D_R^c. Then we know the zeroes of f, Z_f, is a closed subset of a compact set. Therefore we know it is either finite, or has an accumulation point. If it had an accumulation point, f would have to be identically zero, so Z_f must be finite. We can then define, where n represents the number of zeroes for f,

$$\phi(z) = \Pi_{i\leq n}(z - z_i) \text{ and } F(z) = \frac{\phi(z)}{f(z)}.$$

Then note that F is nonvanishing, entire, and bounded. Thus by Liouville, it has to be constant, so $f(z) = c\phi(z)$. ■

Exercise 7

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

Proof.

- In $|z| < 1$:

  Small: $z^4 + 3$

  Big: $-6z$

- In $|z| = 1$:

  $|m(z)| = |z^4 + 3| \leq |4|^4 + 3 = 4 < 6 = |-6z| = |M(z)|$

- In $|z| < 2$:

  Small: $-6z + 3$

  Big: z^4

- In $|z| = 2$:

  $|m(z)| = |-6z + 3| \leq 6 + 3 = 9 < 2^4 = |M(z)|$

Therefore there is 1 root in $|z| < 1$, and there are 3 zeroes in $1 < |z| < 2$. ■

Exercise 8

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

Proof.

First note that it is sufficient to prove the existence of exactly one root in Q_1, because conjugate pairs proves the existence in the other open quadrant. We know the polynomial is entire, so we can use the argument principle to count the zeroes. Let γ be made up of

$$\begin{aligned}\gamma_1 &= [0, R]\\ \gamma_2 &= Re^{it} \text{ for } t \in [0, \pi/2]\\ \gamma_3 &= i[0, R].\end{aligned}$$

Then we can consider

$$Z_f = \frac{1}{2\pi i} \int_\gamma \partial^{\log} f(z) dz = \Delta_\gamma \mathrm{Arg}(f).$$

Then for each part of gamma,

$$\begin{aligned}\Delta_{\gamma_1} \mathrm{Arg}(f) &= 0\\ \Delta_{\gamma_2} \mathrm{Arg}(f) &= 4(\frac{\pi}{2}) = 2\pi\\ \Delta_{\gamma_3} \mathrm{Arg}(f) &= 0.\end{aligned}$$

To prove the last part, consider $f(it) = t^4 - it^3 - 2it + 10 = t^4(1 - it^{-1} - 2it^{-2} + 10t^{-4})$. Thus $\Delta_\gamma \mathrm{Arg}(f) = 1$, so as $R \to \infty$, there is only 1 zero. ■