8200 Homework 9

April 13, 2025

Exercise 1

Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ has isomorphic homology groups in all dimensions, but thier universal covering spaces do not.

Proof.

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Exercise 2

Show that for every $f : S^n \to S^n$, degree of $Sf : S^{n+1} \to S^{n+1}$ is equal to degree of f. Here, Sf denotes the suspension of f which is the map induced from $f \times \mathrm{id}$: $S^n \times [0,1] \to S^n \times [0,1]$ on $SS^n \cong S^{n+1}$.

Proof.

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Exercise 3

Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Proof.

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Exercise 4

Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Proof.

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Exercise 5

For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map on $H_n\left(\mathbb{R}^n, \mathbb{R}^n - \{0\}\right) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

Proof.

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Exercise 6

A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Proof.

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