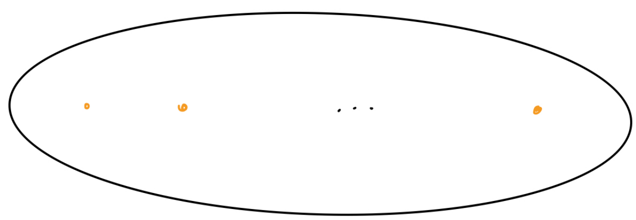
MATH 8200 Homework 4

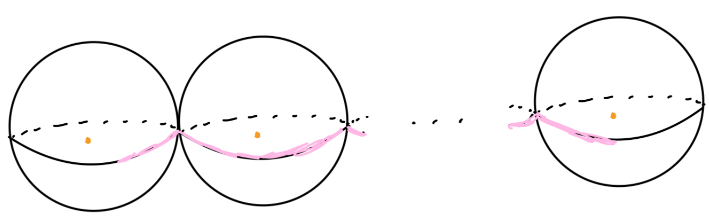
February 10, 2025

Problem 1. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

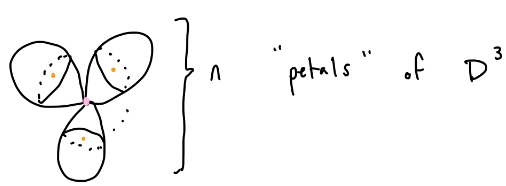
Proof. (Note that the following pictures are only for the $n = 3$ case, but a similar idea is followed for $n > 3$.) We begin by imagining these n missing points in an nth dimensional elipsoid-like shape:



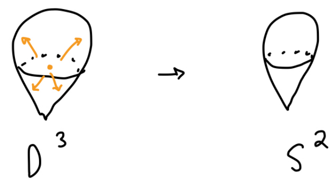
From here, we can "pinch" in between the n missing points to create n D^n's, each missing a point in the center.

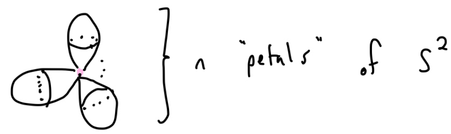


Then, note that the pink line continues on the boundary of all n D^n's, and can be contracted to a point.



Then, in each D^n, the hole in the center can be expanded so that together the D^n and the hole become an S^{n-1}.





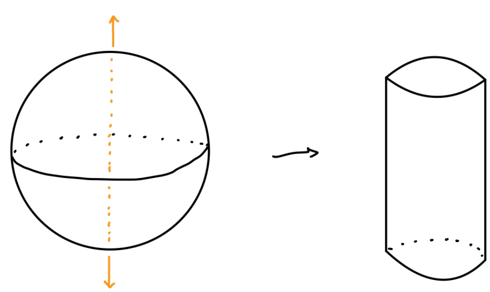
This is clearly path connected, as this is a wedge sum of n S^{n-1}'s. Then, because the fundamental group of S^n is trivial for $n \geq 2$, we know

$$\begin{aligned}
\pi_1(\mathbb{R}^n/\{x_0, x_1, \dots, x_n\}) &\cong \pi_1(\vee^n(S^{n-1})) \\
&\cong \pi_1(S^{n-1}) * \cdots * \pi_1(S^{n-1}) \\
&\cong 0 * \cdots * 0 \\
&\cong 0.
\end{aligned}$$

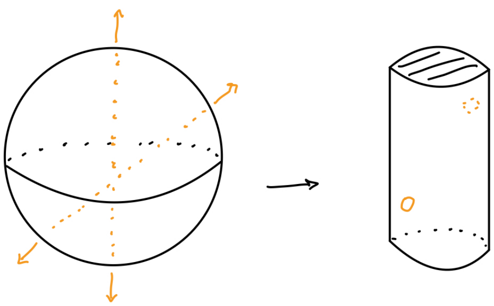
Thus the space is path connected and has a trivial fundamental group, so it is simply-connected. □

Problem 2. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

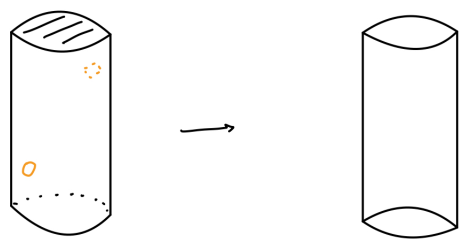
Proof. Once again, I will provide pictures for the $n = 2$ case, but describe any n case. First note that when looking at $\mathbb{R}^3$ as an origin-centered sphere, when a line goes through it, we can deformation retract it to a cylinder:



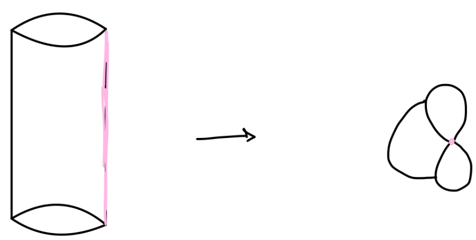
With this same logic, any other lines through the sphere will have 2 intersection points, which create holes in the cylinder. Thus we end up with a cylinder with $2(n-1)$ holes in it.



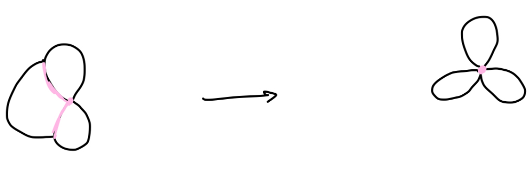
These holes can be expanded, so that we end up with a cylinder with no surface, just $2(n-1)$ lines connecting the top and bottom circles.



One of the lines can be contracted to a point, giving us $2(n-1)$ loops with a line connecting them.



This last line can be contracted to a point, and so we end up with $2(n-1)+1 = 2n-1$ S^1's connected at a point.



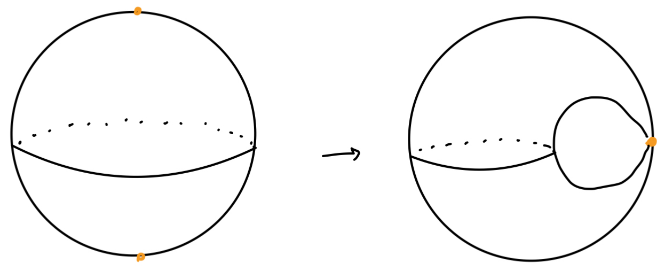
Thus we have

$$\begin{aligned}
\pi_1(\mathbb{R}^3 - X) &\cong \pi_1(\vee^{2n-1}(S^1)) \\
&\cong \pi_1(S^1) * \cdots * \pi_1(S^1) \\
&\cong \mathbb{Z} * \cdots * \mathbb{Z}
\end{aligned}$$

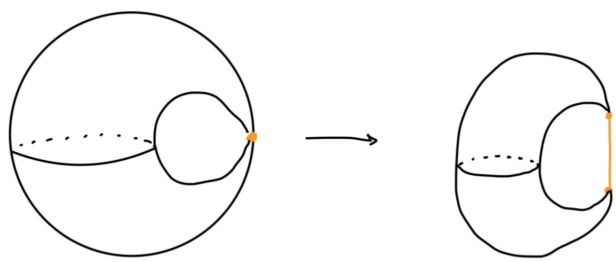
And so $\mathbb{R}^3 - X$ has a fundamental group that is isomorphic to $2n-1$ copies of $\mathbb{Z}$. $\square$

Problem 3. Let X be the quotient space of S^2 obstained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

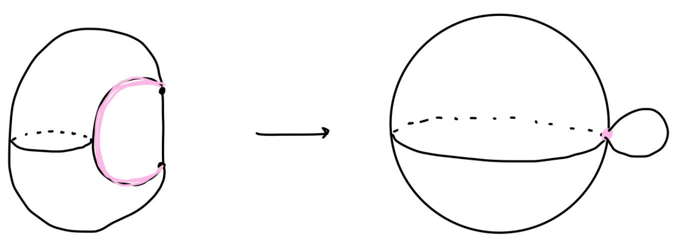
Proof. When we identify these two poles, we create the following shape:



We can take this point and expand it into a line



Then we can take the line on the boundary of the sphere and contract it to a point, creating a loop on the outside.



Thus we have $S^2 \vee S^1$, and in terms of cell complex, an e^0_2 attached to an e^0_1 at an e^0_0. Thus the space's fundamental group can be calculated to be $0 * \mathbb{Z} \cong \mathbb{Z}$. □

Problem 4. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in the torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

Proof. □

Problem 5. The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(x) \to \pi_1(X)$. Do the same when $X = S^1 \times S^1$.

Proof. $\square$