Homework 10

April 22, 2025

Exercise 2.2.10

Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

Proof.

For the first X, because we have 1 0-cell, 1 1-cell, and 2 2-cells, we get the following chain complex:

$$0 \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{\phi} 0$$

So that our homology groups are:

$$\begin{aligned} H_0(X) &\cong \ker(\phi) \ / \ \mathrm{im}(h) \\ H_1(X) &\cong \ker(h) \ / \ \mathrm{im}(g) \\ H_2(X) &\cong \ker(g) \ / \ \mathrm{im}(f) \end{aligned}$$

with all other homology groups for $n > 2$ zero.
Clearly, $\ker(\phi) \cong \mathbb{Z}$, and because h is a map from 1-cells to 0-cells, h is the zero map. This is also evident because the boundary of the 1-cell is zero, because the ends are identified. Thus $\mathrm{im}(h) \cong 0$, and $\ker(h) \cong \mathbb{Z}$.
For the g map, it's easiest to first think about what the map is doing. Because the equator has $x \sim -x$, the 2-cell "wraps around" the equator twice, for both the north and south 2-cells. Thus both generators of $\mathbb{Z} \oplus \mathbb{Z}$, $(1,0)$ and $(0,1)$, map to 2. Thus the image of g is the map generated by 2, so $\mathrm{im}(g) \cong 2\mathbb{Z}$. When considering the kernel, the elements are of the form $a(1,0) + b(0,1) = 0$, so that $b = -a$. Thus the kernel is wholly generated by $(-1,1)$, so $\ker(g) \cong \mathbb{Z}$. Finally, we can clearly see that $\mathrm{im}(f) \cong 0$.
Thus we are left with:

$$\begin{aligned} H_0(X) &\cong \ker(\phi) \ / \ \mathrm{im}(h) \cong \mathbb{Z}/0 \cong \mathbb{Z} \\ H_1(X) &\cong \ker(h) \ / \ \mathrm{im}(g) \cong \mathbb{Z}/ \ 2\mathbb{Z} \\ H_2(X) &\cong \ker(g) \ / \ \mathrm{im}(f) \cong \mathbb{Z}/0 \cong \mathbb{Z} \end{aligned}$$

Now letting X be the second space, we have a similar CW Complex structure: 1 0-cell, 1 1-cell, 1 2-cell, 2 3-cells. Thus we have the following chain complex:

$$0 \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\psi} 0$$

So that our homology groups are:

$$\begin{aligned} H_0(X) &\cong \ker(\psi) \ / \ \mathrm{im}(\phi) \\ H_1(X) &\cong \ker(\phi) \ / \ \mathrm{im}(h) \\ H_2(X) &\cong \ker(h) \ / \ \mathrm{im}(g) \\ H_3(X) &\cong \ker(g) \ / \ \mathrm{im}(f) \end{aligned}$$

with all other homology groups for $n > 2$ zero.
We can use a similar logic as the last time to find everything but the map h (note that although we have increased dimension, the 3-cell still "wraps around" the S^2 equator twice, similar to before). So we have

$$\begin{aligned} H_0(X) &\cong \ker(\psi) \ / \ \mathrm{im}(\phi) \cong \mathbb{Z}/0 \cong \mathbb{Z} \\ H_1(X) &\cong \ker(\phi) \ / \ \mathrm{im}(h) \cong \mathbb{Z}/\mathrm{im}(h) \\ H_2(X) &\cong \ker(h) \ / \ \mathrm{im}(g) \cong \ker(h)/2\mathbb{Z} \\ H_3(X) &\cong \ker(g) \ / \ \mathrm{im}(f) \cong \mathbb{Z}/0 \cong \mathbb{Z} \end{aligned}$$

For the map h, as it maps the 2-cell to the 1-cell, ■