Homework 10

April 23, 2025

Exercise 2.2.10

Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$. Do the same for S^3 with antipodal points of the equatorial $S^2 \subset S^3$ identified.

Proof.

For the first X, because we have 1 0-cell, 1 1-cell, and 2 2-cells, we get the following chain complex:

$$0 \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{\phi} 0$$

So that our homology groups are:

$$\begin{aligned} H_0(X) &\cong \ker(\phi) \ / \ \operatorname{im}(h) \\ H_1(X) &\cong \ker(h) \ / \ \operatorname{im}(g) \\ H_2(X) &\cong \ker(g) \ / \ \operatorname{im}(f) \end{aligned}$$

with all other homology groups for $n > 2$ zero.
Clearly, $\ker(\phi) \cong \mathbb{Z}$, and because h is a map from 1-cells to 0-cells, h is the zero map. This is also evident because the boundary of the 1-cell is zero, because the ends are identified. Thus $\operatorname{im}(h) \cong 0$, and $\ker(h) \cong \mathbb{Z}$.
For the g map, it's easiest to first think about what the map is doing. Because the equator has $x \sim -x$, the 2-cell "wraps around" the equator twice, for both the north and south 2-cells. Thus both generators of $\mathbb{Z} \oplus \mathbb{Z}$, $(1, 0)$ and $(0, 1)$, map to 2. Thus the image of g is the map generated by 2, so $\operatorname{im}(g) \cong 2\mathbb{Z}$. When considering the kernel, the elements are of the form $a(1, 0) + b(0, 1) = 0$, so that $b = -a$. Thus the kernel is wholly generated by $(-1, 1)$, so $\ker(g) \cong \mathbb{Z}$. Finally, we can clearly see that $\operatorname{im}(f) \cong 0$.
Thus we are left with:

$$\begin{aligned} H_0(X) &\cong \ker(\phi) \ / \ \operatorname{im}(h) \cong \mathbb{Z}/0 \cong \mathbb{Z} \\ H_1(X) &\cong \ker(h) \ / \ \operatorname{im}(g) \cong \mathbb{Z}/ \ 2\mathbb{Z} \\ H_2(X) &\cong \ker(g) \ / \ \operatorname{im}(f) \cong \mathbb{Z}/0 \cong \mathbb{Z} \end{aligned}$$

Now letting X be the second space, we have a similar CW Complex structure: 1 0-cell, 1 1-cell, 1 2-cell, 2 3-cells. Thus we have the following chain complex:

$$0 \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\psi} 0$$

So that our homology groups are:

$$
\begin{aligned}
H_0(X) &\cong \ker(\psi) \ / \ \operatorname{im}(\phi) \\
H_1(X) &\cong \ker(\phi) \ / \ \operatorname{im}(h) \\
H_2(X) &\cong \ker(h) \ / \ \operatorname{im}(g) \\
H_3(X) &\cong \ker(g) \ / \ \operatorname{im}(f)
\end{aligned}
$$

with all other homology groups for $n > 2$ zero.
We can use a similar logic as the last time to find everything but the map h. Note that we also have to adjust the map g, because although the 3-cell "wraps around" twice, they are in opposite directions, so that the generators of $\mathbb{Z} \oplus \mathbb{Z}$ actually map to 0, so that g is the 0 map. So we have

$$
\begin{aligned}
H_0(X) &\cong \ker(\psi) \ / \ \operatorname{im}(\phi) \cong \mathbb{Z}/0 \cong \mathbb{Z} \\
H_1(X) &\cong \ker(\phi) \ / \ \operatorname{im}(h) \cong \mathbb{Z}/\operatorname{im}(h) \\
H_2(X) &\cong \ker(h) \ / \ \operatorname{im}(g) \cong \ker(h)/0 \\
H_3(X) &\cong \ker(g) \ / \ \operatorname{im}(f) \cong \mathbb{Z} \oplus \mathbb{Z}/0 \cong \mathbb{Z} \oplus \mathbb{Z}
\end{aligned}
$$

For the map h, as it maps the 2-cell to the 1-cell, it's similar to before where because of the identification, where the 2-cell wraps around the 1-cell twice, so that the degree of h is 2. Because it's from $\mathbb{Z}$ to $\mathbb{Z}$, it's image must be $2\mathbb{Z}$ and it's kernel is 0. Thus we are left with:

$$
\begin{aligned}
H_0(X) &\cong \mathbb{Z} \\
H_1(X) &\cong \mathbb{Z}/2\mathbb{Z} \\
H_2(X) &\cong 0 \\
H_3(X) &\cong \mathbb{Z} \oplus \mathbb{Z}
\end{aligned}
$$

■