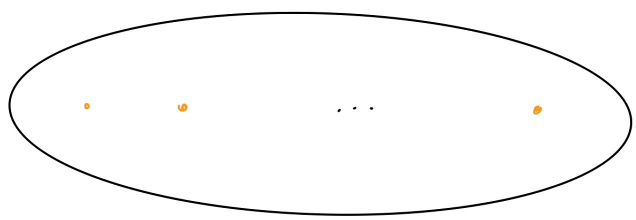
MATH 8200 Homework 4

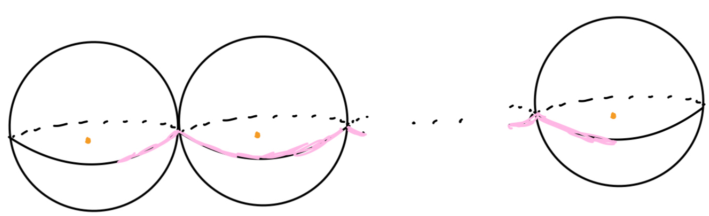
February 11, 2025

Problem 1. Show that the complement of a finite set of points in $\mathbb{R}^n$ is simply-connected if $n \geq 3$.

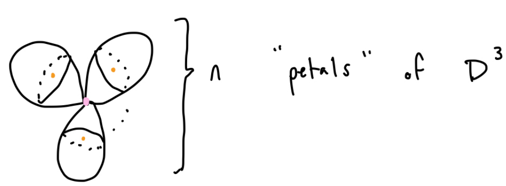
Proof. (Note that the following pictures are only for the $n = 3$ case, but a similar idea is followed for $n > 3$.) We begin by imagining these n missing points in an *nth* dimensional elipsoid-like shape:



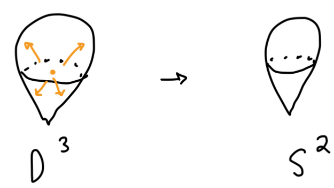
From here, we can "pinch" in between the n missing points to create n D^n's, each missing a point in the center.

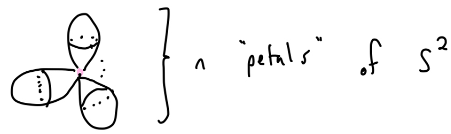


Then, note that the pink line continues on the boundary of all n D^n's, and can be contracted to a point.



Then, in each D^n, the hole in the center can be expanded so that together the D^n and the hole become an S^{n-1}.





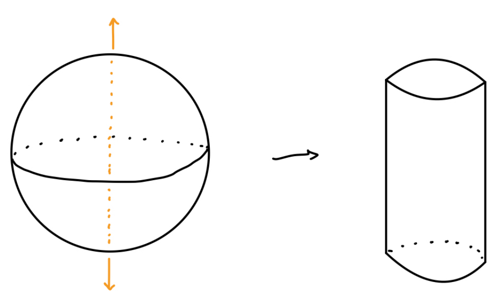
This is clearly path connected, as this is a wedge sum of n S^{n-1}'s. Then, because the fundamental group of S^n is trivial for $n \geq 2$, we know

$$\begin{aligned}\pi_1(\mathbb{R}^n/\{x_0, x_1, \ldots, x_n\}) &\cong \pi_1(\vee^n(S^{n-1})) \\ &\cong \pi_1(S^{n-1}) * \cdots * \pi_1(S^{n-1}) \\ &\cong 0 * \cdots * 0 \\ &\cong 0.\end{aligned}$$

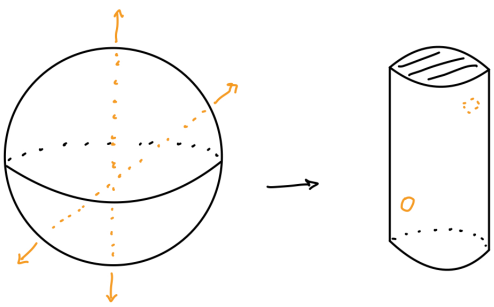
Thus the space is path connected and has a trivial fundamental group, so it is simply-connected. □

Problem 2. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 - X)$.

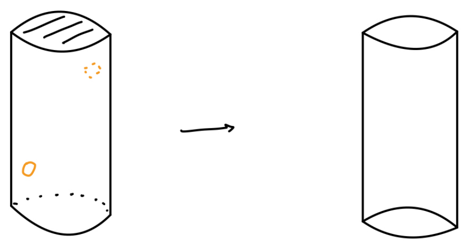
Proof. Once again, I will provide pictures for the $n = 2$ case, but describe any n case. First note that when looking at $\mathbb{R}^3$ as an origin-centered sphere, when a line goes through it, we can deformation retract it to a cylinder:



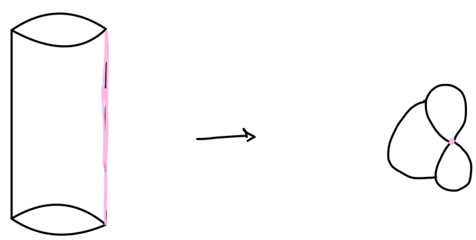
With this same logic, any other lines through the sphere will have 2 intersection points, which create holes in the cylinder. Thus we end up with a cylinder with $2(n-1)$ holes in it.



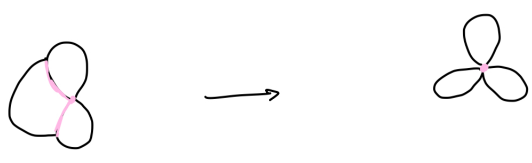
These holes can be expanded, so that we end up with a cylinder with no surface, just $2(n-1)$ lines connecting the top and bottom circles.



One of the lines can be contracted to a point, giving us $2(n-1)$ loops with a line connecting them.



This last line can be contracted to a point, and so we end up with $2(n-1)+1 = 2n-1$ S^1's connected at a point.



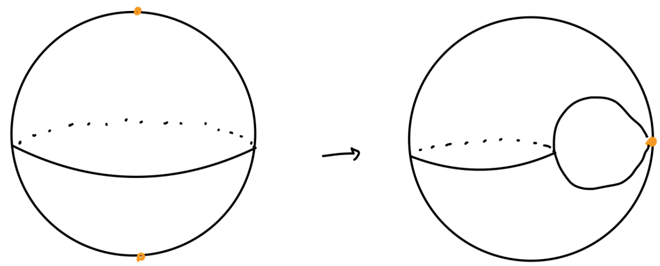
Thus we have

$$\begin{aligned} \pi_1(\mathbb{R}^3 - X) &\cong \pi_1(\vee^{2n-1}(S^1)) \\ &\cong \pi_1(S^1) * \cdots * \pi_1(S^1) \\ &\cong \mathbb{Z} * \cdots * \mathbb{Z} \end{aligned}$$

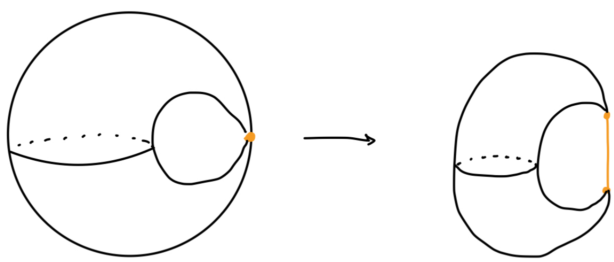
And so $\mathbb{R}^3 - X$ has a fundamental group that is isomorphic to $2n-1$ copies of $\mathbb{Z}$. $\square$

Problem 3. Let X be the quotient space of S^2 obstained by identifying the north and south poles to a single point. Put a cell complex structure on X and use this to compute $\pi_1(X)$.

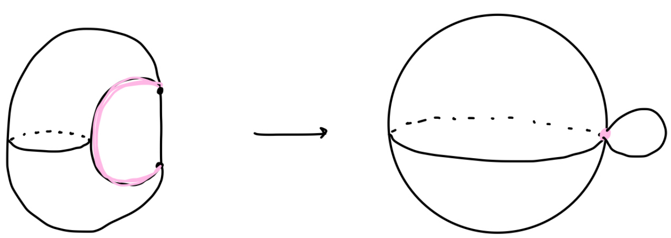
Proof. When we identify these two poles, we create the following shape:



We can take this point and expand it into a line



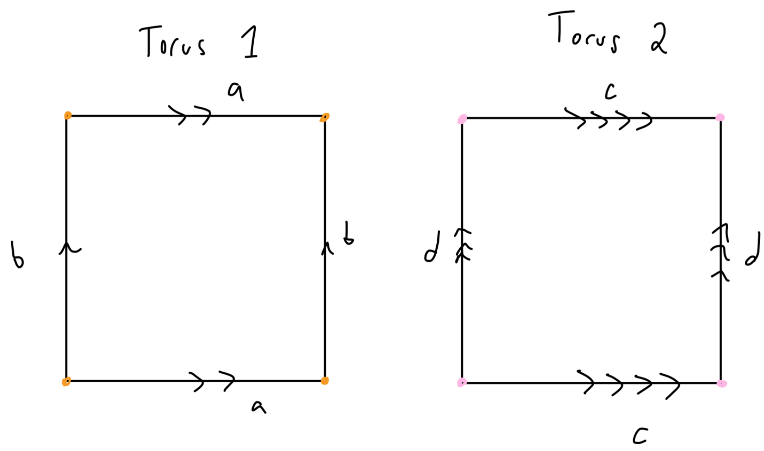
Then we can take the line on the boundary of the sphere and contract it to a point, creating a loop on the outside.



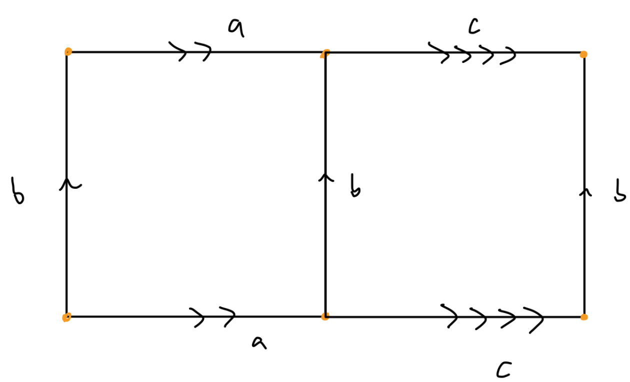
Thus we have $S^2 \vee S^1$, and in terms of cell complex, an e_2^0 attached to an e_1^0 at an e_0^0. Thus the space's fundamental group can be calculated to be $0 * \mathbb{Z} \cong \mathbb{Z}$. □

Problem 4. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in the torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

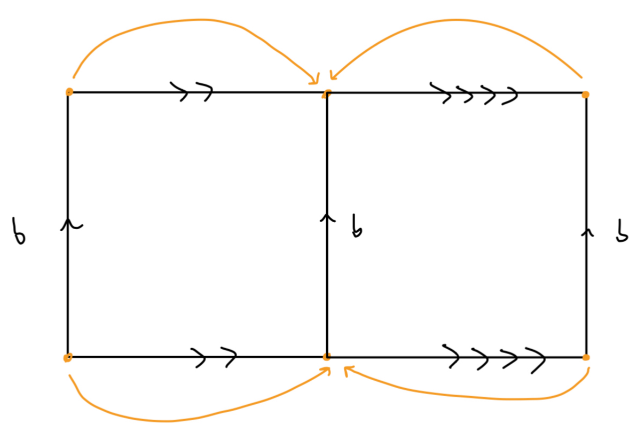
Proof. First, to get an idea of what the shape looks like, we can start with two tori in the following sense:

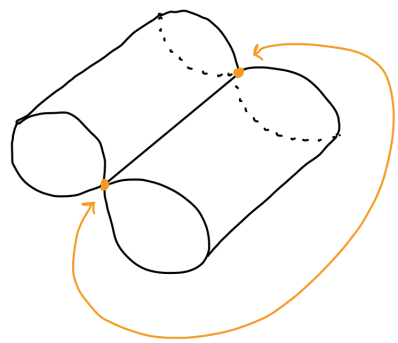


When identifying one circle with another, it is equivalent to say that $b = d$. Thus the diagram transforms into this:

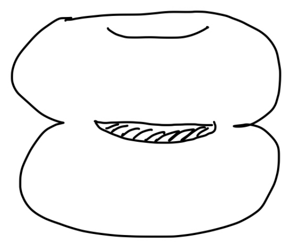


Now, all 6 points are identified to each other, so we can start by gluing the 4 on the corners with the 2 in the center. Once this is done, we can glue the two remaining points together.





Thus we have two tori stacked on top of each other, where they only intersect on one circle on the bottom of the top one, and on the top of the bottom one.



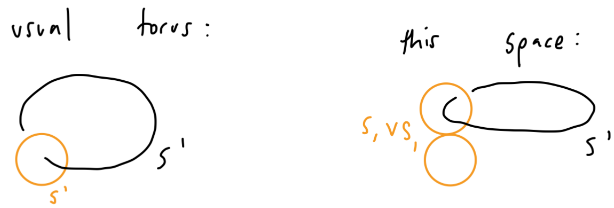
Although it may be worth considering trying this problem with Van Kampen, where the two spaces are each a torus with the intersection circle and an open neighborhood around it, it may prove simpler to look at the group presentation. If we can define each separate torus like so

$$\pi_1(T_1) = \{a, b \mid aba^{-1}b^{-1} = 1\}$$
$$\pi_1(T_2) = \{c, d \mid cdc^{-1}d^{-1} = 1\},$$

then, as we saw in the second diagram for this problem, we are simply asked to set $b = d$. Thus the group reprsentation for this space is

$$\pi_1(X) = \{a, b, c, d \mid aba^{-1}b^{-1} = 1, cdc^{-1}d^{-1} = 1, b = d\}$$

An even simpler way to think about it would be looking at the picture; it is simply another product like the torus, but instead of S^1 being crossed with another S^1, its a wedge sum of two S^1's. A visual reprsentation of this is below:



Thus we can conlude:

$$\begin{aligned}\pi_1(X) &\cong \pi_1(S^1 \times (S^1 \vee S^1)) \\ &\cong \pi(S^1) \times \pi_1(S^1 \vee S^1) \\ &\cong \mathbb{Z} \times (\mathbb{Z} * \mathbb{Z})\end{aligned}$$

□

Problem 5. The mapping torus T_f of a map $f : X \to X$ is the quotient of $X \times I$ obtained by identifying each point $(x, 0)$ with $(f(x), 1)$. In the case $X = S^1 \vee S^1$ with f basepoint-preserving, compute a presentation for $\pi_1(T_f)$ in terms of the induced map $f_* : \pi_1(x) \to \pi_1(x)$. Do the same when $X = S^1 \times S^1$.

Proof. □