8150 Homework III

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Stein Problems

Exercise 1

Tie Problems

Exercise 2

Prove that if

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \text{ and } \sum_{n=-\infty}^{\infty} c'_n(z-a)^n$$

are Laurent series expansions of $f(z)$, then $c_n = c'_n$ for all n.

Proof.

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=-\infty}^{\infty} c'_n(z-a)^n$. Then we know, for any integer k,

$$f(z)(z-a)^{-k-1} = \sum_{n=-\infty}^{\infty} c_n(z-a)^{n-k-1} = \sum_{n=-\infty}^{\infty} c'_n(z-a)^{n-k-1}$$

Then let γ be any closed contour in the annulus going around a once, and because it is a compact set of points, the Luarent serieses can be integrated termwise:

$$\sum_{n=-\infty}^{\infty} c_n \oint_\gamma (z-a)^{n-k-1}dz = \sum_{n=-\infty}^{\infty} c'_n \oint_\gamma (z-a)^{n-k-1}dz$$

We know that

$$\oint (z-a)^{n-k-1}dz = 2i\pi \text{ if } n=k \text{ and } 0 \text{ if } n \neq k$$

So then we are left with $2i\pi c_m = 21\pi c'_n$ for any k, which proves the statement. ∎

Exercise 3

Expand $\frac{1}{1-z^2} + \frac{1}{3-z}$ in a series of the form $\sum^{\infty} -\infty a_n z^n$. How many such expansions are there? In which domain is each of them valid?

Proof.

We find that:

$$\begin{aligned}\frac{1}{z-3} &= -\frac{1}{3}\frac{1}{1-3z^{-1}} = -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k \text{ for } |z|<3\\ &= \frac{1}{z}\frac{1}{1-3z^{-1}} = z^{-1}\sum_{k\geq 0} 3^k z^{-k} \text{ for } |z|>3\end{aligned}$$

and:

$$\begin{aligned}\frac{1}{1-z^2} &= \sum_{k\geq 0} z^{2k} \text{ for } |z|<1\\ &= \frac{1}{z^2}\frac{-1}{1-z^{-2}} = -z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z|>1\end{aligned}$$

So we can just list all the possible combinations to find:

$$f(z) = -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1)$$

$$f(z) = -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (1, 3)$$

$$f(z) = z^{-1}\sum_{k\geq 0} 3^k z^{-k} + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1)\cup(3,\infty)$$

$$f(z) = z^{-1}\sum_{k\geq 0} 3^k z^{-k} - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (3, \infty)$$

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Exercise 4

Let $P(z)$ and $Q(z)$ be polynomials with no common zeros. Assume $Q(a)=0$. Find the principal part of $P(z)/Q(z)$ at $z=a$ if the zero a is (i) simple; (ii) double. Express your answers explicitly using P and Q.

Proof.

i.

ii.

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