8150 Homework 4

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1 Stein Problems

Exercise 3.9.2

Let u be a harmonic function in the unit disc that is continuous on its closure. Deduce Poisson's integral formula

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2} u(e^{i\theta})d\theta \text{ for } |z_0| < 1$$

from the special case $z_0 = 0$ (the mean value theorem). Show that if $z_0 = re^{i\phi}$, then

$$\frac{1-|z_0|^2}{|e^{i\theta}-z_0|^2} = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2} = P_r(\theta-\phi)$$

and we recover the expression for the Poisson kernel derived in the exercises of the previous chapter. (Hint: Set $u_0(z) = u(T(z))$ where

$$T(z) = \frac{z_0 - z}{1-\bar{z}_0 z}.$$

Prove that u_0 is harmonic. Then apply the mean value theorem to u_0, and make a change of variables in the integral.)

Proof.

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Exercise 3.9.3

If $f(z)$ is holomorphic in the deleted neighborhood $\{0 < |z-z_0|r\}$ and has a pole of order k at z_0, then we can write

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \cdots + \frac{a_{-1}}{(z-z_0)} + g(z)$$

where g is holomorphic in the disc $\{|z-z_0| < r\}$. There is a generalization of this expansion that holds even if z_0 is an essential singularity. This is a special case of the **Laurent series expansion**, which is valid in an even more general settings.
Let f be holomorphic in a region containing the annulus $\{z : r_1 \leq |z-z_0| \leq r_2\}$ where $0 < r_1 < r_2$. Then,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

where the series converges absolutely in the interior of the annulus. To prove this, it suffices to write

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_2}} \frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\int_{C_{r_1}} \frac{f(\xi)}{\xi - z}d\xi$$

when $r_1 < |z-z_0| < r_2$, and argue as in the proof of Theorem 4.4, Chapter 2. Here C_{r_1} and C_{r_2} are the circles bounding the annulus.

Proof.

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Exercise 4.4.1

Proof.

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Exercise 4.4.2

Proof.

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Exercise 4.4.3

Proof.

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Exercise 4.4.6

Proof.

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Exercise 4.4.7

Proof.

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Exercise 4.4.8

Proof.

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2 Tie Problems

Exercise 1

Proof.

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Exercise 2

Proof. ■

Exercise 3

Proof. ■

Exercise 4

Proof. ■