8200 Homework 9

April 14, 2025

Exercise 1

Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ has isomorphic homology groups in all dimensions, but thier universal covering spaces do not.

Proof.

We know the torus $S^1 \times S^1$ is a CW complex with one 0-cell e^0, two 1-cells e^1_a, e^1_b, and one 2-cell e^2. Then we know:

- $H_0(S^1 \times S^1)$ counts path components. Since the torus is connected, $H_0 \cong \mathbb{Z}$.
- $H_1(S^1 \times S^1)$ can be computed by looking at the boundary maps. The only possible nontrivial boundary map involving 1-cells is $\partial_1 : C_1 \to C_0$. However, the single 0-cell means that the boundary of each 1-cell is that same 0-cell. Therefore we end up with two independent loops, so $H_1 \cong \mathbb{Z}^2$.
- For $H_2(S^1 \times S^1)$, the boundary of the 2-cell corresponds to a loop $aba^{-1}b^{-1}$ in the 1-skeleton that is null-homotopic in the 1-skeleton. This ensures that the resulting ∂_2 map is zero in homology, so $H_2 \cong \mathbb{Z}$.
- In higher dimensions ($n > 2$), there are no n-cells, so $H_n(S^1 \times S^1) = 0$ for $n > 2$.

Therefore

$$H_n(S^1 \times S^1) \;\cong\; \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $Y = S^1 \vee S^1 \vee S^2$, we can use the following fact about wedge sums:

$$\widetilde{H}_n(X \vee Z) \;\cong\; \widetilde{H}_n(X) \;\oplus\; \widetilde{H}_n(Z) \quad \text{for all } n \geq 1.$$

Let $X = S^1 \vee S^1$ and $Z = S^2$. Then

$$\widetilde{H}_n(Y) \;=\; \widetilde{H}_n(X \vee Z) \;\cong\; \widetilde{H}_n(X) \;\oplus\; \widetilde{H}_n(Z).$$

We know that

$$\widetilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \widetilde{H}_n(S^2) = \begin{cases} \mathbb{Z} & n = 2, \\ 0 & \text{otherwise,} \end{cases} \qquad \widetilde{H}_n(S^1 \vee S^1) \cong \widetilde{H}_n(S^1) \oplus \widetilde{H}_n(S^1).$$

Therefore:

$$\widetilde{H}_n(Y) \;=\; \widetilde{H}_n(X \vee Z) \;=\; \begin{cases} \mathbb{Z}^2 \oplus 0 = \mathbb{Z}^2 & n = 1, \\ 0 \oplus \mathbb{Z} = \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

So that

$$H_n(S^1 \vee S^1 \vee S^2) \;\cong\; \begin{cases} \mathbb{Z} & n = 0, \\ \mathbb{Z}^2 & n = 1, \\ \mathbb{Z} & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $H_n(S^1 \times S^1) \;\cong\; H_n(S^1 \vee S^1 \vee S^2)$.
For the universal covering spaces, we know that the torus's covering space is $\mathbb{R}^2$, and for $S^1 \vee S^1 \vee S^2$, we can use an infinite tree with a 2-sphere at each vertex, which is clearly not homeomorphic to $\mathbb{R}^2$.
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Exercise 2

Show that for every $f : S^n \to S^n$, degree of $Sf : S^{n+1} \to S^{n+1}$ is equal to degree of f. Here, Sf denotes the suspension of f which is the map induced from $f \times \text{id}$: $S^n \times [0, 1] \to S^n \times [0, 1]$ on $SS^n \cong S^{n+1}$.

Proof.

The degree of a map $g : S^k \to S^k$ can be characterized by its induced map

$$g_* : H_k(S^k) \;\longrightarrow\; H_k(S^k).$$

Since $H_k(S^k) \cong \mathbb{Z}$ (generated by the fundamental class $[S^k]$), the map g_* must be multiplication by some integer d, which is precisely $\deg(g)$.
Key Observation for Suspensions: In passing from f (an n-dimensional map) to Sf (an $(n+1)$-dimensional map), the "top homology class" in $H_{n+1}(S^{n+1})$ is essentially determined by how f acts on $H_n(S^n)$.
One way to see this rigorously is via the *cylindrical* construction: inside S^{n+1}, view an "equatorial region" $S^n \times (0, 1)$ mapped according to $f \times \text{id}$, and notice that attaching the "caps" at the two ends (collapsing $S^n \times \{0\}$ and $S^n \times \{1\}$ to points) does not alter the integer by which the fundamental class is multiplied.
Consequently, $(Sf)_* : H_{n+1}(S^{n+1}) \to H_{n+1}(S^{n+1})$ acts on the generator $[S^{n+1}]$ by the *same* integer $\deg(f)$. Therefore,

$$\deg(Sf) \;=\; \deg(f).$$

Hence for every $f : S^n \to S^n$, the degree of its suspension $Sf : S^{n+1} \to S^{n+1}$ is equal to the degree of f. ■

Exercise 3

Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvectors.

Proof.

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Exercise 4

Let $f : S^n \to S^n$ be a map of degree zero. Show that there exist points $x, y \in S^n$ with $f(x) = x$ and $f(y) = -y$. Use this to show that if F is a continuous vector field defined on the unit ball D^n in $\mathbb{R}^n$ such that $F(x) \neq 0$ for all x, then there exists a point on ∂D^n where F points radially outward and another point on ∂D^n where F points radially inward.

Proof.

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Exercise 5

For an invertible linear transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ show that the induced map on $H_n\,(\mathbb{R}^n, \mathbb{R}^n - \{0\}) \cong \tilde{H}_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$ is $\mathbb{1}$ or $-\mathbb{1}$ according to whether the determinant of f is positive or negative.

Proof.

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Exercise 6

A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbb{C} \to \mathbb{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f} : S^2 \to S^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree of $\hat{f}$ at a root of f is the multiplicity of the root.

Proof.

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