Homework

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1 Stein Problems

Exercise 8.5.1

A holomorphic mapping $f : U \to V$ is a **local bijection** on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z, so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection on U if and only if $f'(z) \neq 0$ for all $z \in U$.

Proof.

Let $U, V \subset \mathbb{C}$ be open and $f : U \to V$ holomorphic. We shall prove that

$$f \text{ is a local bijection on } U \iff f'(z) \neq 0 \text{ for every } z \in U.$$

(⇒) Local bijection $\implies$ $f' \not\equiv 0$.

- Fix $z_0 \in U$. Assume for contradiction that $f'(z_0) = 0$. Write the Taylor expansion at z_0:

$$f(z) = f(z_0) + a_k\,(z - z_0)^k + \cdots, \qquad a_k \neq 0,\ k \geq 2.$$

- Choose $\rho > 0$ so small that the closed disc $\overline{D_\rho}(z_0) \subset U$ and $|f(z) - f(z_0)| < \frac{1}{2}|a_k|\rho^k$ for $|z - z_0| \leq \rho$ (uniform continuity of higher–order terms).

- For $\zeta = e^{2\pi i/k}$ set $z_1 = z_0 + \rho$ and $z_2 = z_0 + \rho\zeta$. Then $|z_1 - z_0| = |z_2 - z_0| = \rho$ but $z_1 \neq z_2$ and

$$f(z_j) = f(z_0) + a_k\rho^k + R_j, \qquad |R_j| < \frac{1}{2}|a_k|\rho^k,$$

  so $|f(z_j) - f(z_0) - a_k\rho^k| < \frac{1}{2}|a_k|\rho^k$ for $j = 1, 2$. By the triangle inequality $f(z_1) = f(z_2)$.

- Hence f is *not* injective on any disc about z_0, contradicting the hypothesis that f is a local bijection. Therefore $f'(z_0) \neq 0$. Since z_0 was arbitrary, $f'(z) \neq 0$ for all $z \in U$.

(⇐) Non-vanishing derivative $\implies$ local bijection.

- Let $z_0 \in U$ with $f'(z_0) \neq 0$. By the *holomorphic inverse function theorem* (or the open mapping theorem plus the implicit function theorem) there exists $r > 0$ such that the restricted map

$$f :\ D_r(z_0) \longrightarrow f(D_r(z_0))$$

  is biholomorphic; in particular it is bijective onto an open disc $f(D_r(z_0)) \subset V$.

- Because z_0 was arbitrary, the same holds around every point of U, so f is a local bijection on U.

Conclusion. A holomorphic map $f : U \to V$ is a local bijection on U *iff* its derivative never vanishes on U. ■

Exercise 8.5.2

Suppose $F(z)$ is holomorphic near $z = z_0$ and $F(z_0) = F'(z_0) = 0$, while $F''(z_0) \neq 0$. Show that there are two curves Γ_1 and Γ_2 that pass through z_0, are orthogonal at z_0, and so that F restricted to Γ_1 is real and has a minimum at z_0, while F restricted to Γ_2 is also real but has a maximum at z_0.

Proof.

■

2 Tie Problems