Complex Analysis Notes

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Introduction

Let us begin by noting that every complex number z can be written as $z = x + iy$, where $x, y \in \mathbb{R}$.

Definition 0.1

A function is **holomorphic** at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \text{ where } h \in \mathbb{C}$$

exists.

Question

So is this just being differentiable for complex numbers?

Answer

Essentially, however because complex numbers have a value (radius) and an angle, h can approach 0 from infinitely many angles. So holomorphicity is much stronger than differentiablity; In the real case, it is differentiable going left and right. For a function to be holomorphic at a point, it must be differentiable from infinitely many angles.

Fact 0.1

If f is holomorphic in Ω, then for appropriate closed paths in Ω,

$$\int_\gamma f(z)dz = 0.$$

Fact 0.2

If f is holomorphic, then f is indefinitely differentiable.

Question

Why indefinitely differentiable? Why not indefinitely holomorphic?

Fact 0.3

If f and g are holomorphic functions in Ω which are equal in an arbitrarily samll disc in ω, then $f = g$ everywhere in Ω.

Definition 0.2

The **zeta function**,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is holomorphic in the half-plane $\mathrm{Re}(s) > 1$.

Definition 0.3

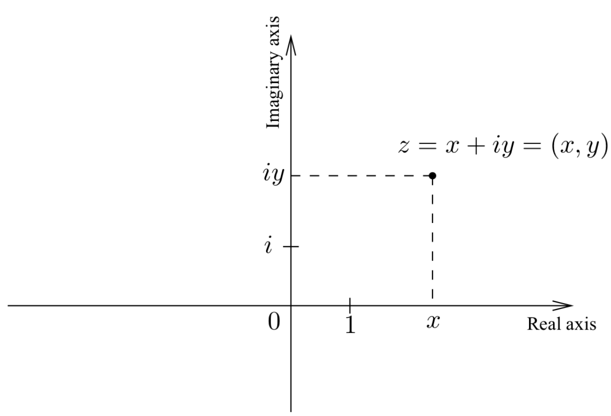
The **theta function** is the following:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

1 Preliminaries to Complex Analysis

1.1 Complex Numbers and the Complex Plane

We can imagine complex numbers as an ordered pair of the two real numbers:



Addition and mulitplication are defined like so:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 * z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

It is easy to prove that commutativity, associativity, and distributivity hold for complex numbers. We can think about addition like adding two vectors in $\mathbb{R}^2$, and mulitplication like a rotation and dilation.

Definition 1.1

The length, or absolute value of a complex number, is defined as the following:

$$|z| = (x^2 + y^2)^{1/2}$$

Note that this is the same as taking the norm, or length, of a vector in $\mathbb{R}^2$, or even finding the length of the hypotenuse that is created by the x and y values.

The triangle equality holds:

Theorem 1.1 (Triangle Inequality)

$$|z + w| \leq |z| + |w|$$

for all $z, x \in \mathbb{C}$.

From the triangle inequality, there comes this helpful fact as well:

Fact 1.1

$$||z| - |w|| \leq |z - w|$$

You can imagine a complex conjugate, $\bar{z} = x - iy$, as a reflection across the real (horizontal) axis.

The following are also useful facts easily deduced:

Fact 1.2

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

Fact 1.3

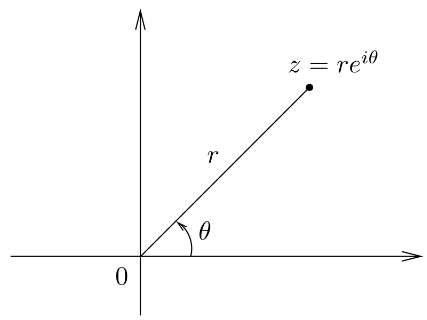
$$|z| = z\bar{z} \text{ and, when } z \neq 0, \ \frac{1}{z} = \bar{z}|z|^2$$

Definition 1.2

A complex number z's **polar form** is written as $z = re^{i\theta}$, where $r > 0$, and θ is referred to as the **argument** of z.

A mathematical fact useful in Complex Analysis is $e^{i\theta} = \cos\theta + i\sin\theta$.

From the two previous statements, we can see that $r = |z|$, the length of z, and θ is the angle.



With this form, we can redefine mulitplication to be:

$$z = re^{i\theta}, w = se^{i\phi} \implies zw = rse^{i(\theta+\phi)}$$

It is easier to see in this definition that mulitplication is simply a rotation $(\theta + \phi)$, and a dilation (rs).

Definition 1.3

A sequence $\{z_1, z_2, \dots\}$ of complex numbers is said to **converge** to $w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0 \text{ or,equivalently, } w = \lim_{n\to\infty} z_n.$$

Question

Is this just the same thing as being convergent in $\mathbb{R}$? Just that the limit is a complex number, not a real one?

Answer

Stein states that z_n converges to w if and only if the sequence of the real and imaginary parts of z_n converge to the real and imaginary parts of w. This makes sense, as in real convergence, it converges to a y value, but in the complex plane, it must converge from the x (real) direction, and the y (imaginary) direction.

Question

Given the previous answer, is it possible for a complex sequence to converge in the real direction much sooner than the imaginary? Does the difference in "convergence speed" matter?

Definition 1.4

A sequence $\{z_n\}$ is said to be a **Cauchy Sequence** if

$$|z_n - z_m| \to 0 \text{ as } n, m \to \infty.$$

Similarly to Real Analysis, a sequence is Cauchy if it's elements get closer and closer together as the sequence reaches infinity.

Question

How can a sequence be convergent, but not Cauchy?

Answer

Being convergent always implies the sequence is Cauchy, because if the elements get farther away from each other, they can never approach a limit. But it is possible for a sequence to be Cauchy and not convergent. This is because being Cauchy doesn't specify a value for the limit; it is possible the Cauchy sequence is approaching a value outside whatever space we're discussing. Thus from our scope (only considering our space missing the limit value), the sequence is not convergent.

Similar to convergence, the sequence $\{z_n\}$ is Cauchy if and only if the real and imaginary parts are.

Definition 1.5

A space is **complete** if every Cauchy sequence converges to a point in the space.

Theorem 1.2

The complex numbers $\mathbb{C}$ is complete.

Proof.

Because $\mathbb{R}$ is complete, we know every real Cauchy sequence has a limit in $\mathbb{R}$. Thus the real and imaginary parts of any Cauchy sequence must converge, so $\{z_n\}$ must converge as well. ■

Definition 1.6

If $z_0 \in \mathbb{C}$ and $r > 0$, we define the **open disc** $D_r(z_0)$ **of radius** r **centered at** z_0 to be the set of all complex numbers that are at absolute value strictly less than r from z_0. Alternatively,

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$$

An open disc is just a circle of radius r, centered at x_0, with no "edge". It's called the "open circle" because it is an open set.

Definition 1.7

Similarly, a **closed disc** is defined as

$$\bar{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$$

This is just a disc including the boundary, so it is a closed set.

Definition 1.8

As you can imagine, the **boundary of a disc** is defined as so:

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$$

Notation

The **unit disc** is defined as the following:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

This is just an open disc centered at the origin of radius one.

Definition 1.9

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an **interior point** of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$.

Definition 1.10

The **interior** of a set consists of all it's interior points.

Question

So is the interior just everything other than the boundary and exterior? Meaning an open disc is equivalent to it's interior?

Answer

Yes. :)

Definition 1.11

A set if **open** if every point in that set is an interior point. Similarly, a set is **closed** if it's complement $\Omega^c = \mathbb{C} - \Omega$ is open.

Note that $\Omega^c = \mathbb{C} - \Omega$ is just the exterior of the set. So if the exterior is open, then Ω must include it's boundary.

Definition 1.12

A point $z \in \mathbb{C}$ is said to be a **limit point** of the set Ω if there exists a sequence of points $z_n \in \Omega$ such that $z_n \neq z$ and $\lim_{n\to\infty} z_n = z$.

So an example of a limit point would be a point in the boundary. It is worth noting that a set is closed if and only if it contains all it's limit points (boundary points).

Definition 1.13

The **closure** of any set Ω is the union of Ω and its limit points, and is often denoted by $\bar{\Omega}$.

So in an open set, the closure would be the set and it's boundary (what could be added to the set to make it closed). In a closed set, the closure is just the set itself.

Definition 1.14

The **boundary** of a set Ω is equal to its closure minus its interior, and is often denotes by $\delta\Omega$.

So the boundary is just the set of limit points.

Definition 1.15

A set Ω is **bounded** if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$.

So it is the same as being bounded in Real Analysis, just instead of a horizontal line at height M, it is a disc with radius M centered at the origin.

Definition 1.16

If Ω is bounded, we define its **diameter** by

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|$$

So the diameter is just the furthest straight line you can draw between any two points.

Definition 1.17

A set Ω is said to be **compact** if it is closed and bounded.

Note that this is the same definition as Real Analysis.

Theorem 1.3

The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

This is just saying that every sequence in the set can be "cut off" at a point and still converge in the set.

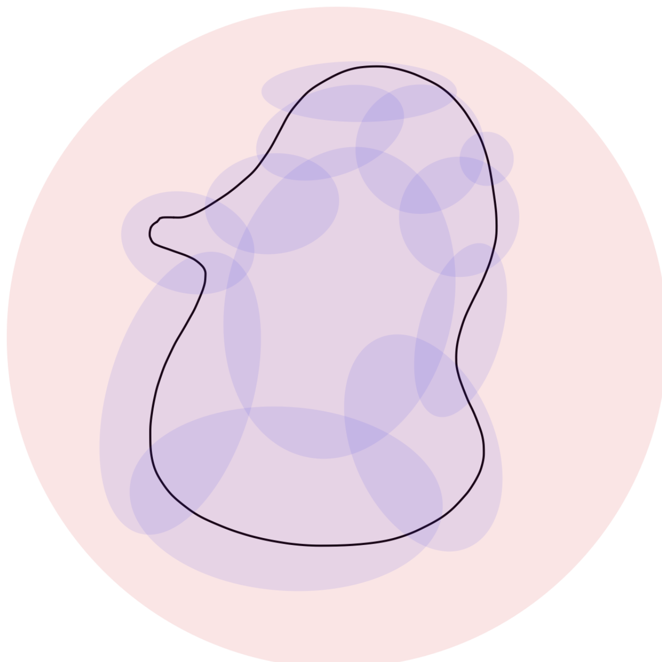
Proof.

This follows from the similar statement in Real Analysis, and the correspondence between convergence in the real and complex plane. ■

Definition 1.18

An **open covering** of Ω is a family of open sets $\{U_\alpha\}$ such that $\Omega \subset \bigcup_\alpha U_\alpha$.

An open covering is just any set of open discs that completely cover the set on the complex plane. Here's how I visualize it:



where both the red and blue parts are different subcovers. Note that a subcover can have uncountably many open sets.

Theorem 1.4

A set Ω is compact if and only if every open covering has a finite subcovering.

This is similar to the statement in Real Analysis, and can be proved using that. But the general idea is that if a set is compact, it's not infinite in size (more accurately, diameter), and it's closed. If this is the case, then clearly you can have a finite subcovering of any open cover.

Question

May want to look back at a proof of this. I'm not understanding why an open cover of infinitely many balls around points would have a finite subcover, if the set itself is infinite.

Theorem 1.5

If $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that $\text{diam}(\Omega_n) \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Visually, this is saying if you have a bunch of circles containing each other that get smaller and smaller, there must be some point in the center, where the smallest circles are, and there must only be one.

Proof.

Let z_n be a point in each Ω_n. Because $\text{diam}(\Omega_n) \to 0$, then $\sup_{z,w \in \Omega_n} |z - w| \to 0$, by the definition of a diameter. Thus $\{z_n\}$ is a Cauchy sequence, because the points are getting closer and closer together. Thus it must converge, and call this value w. Because each set is compact, we must have w (a limit point), be in each set. If there were another point satisfying this, call it w', then the diameter between them would be nonzero, which is a contradiction. ■

Definition 1.19

An open set $\Omega \subset \mathbb{C}$ is said to be **connected** if it is not possible to find two disjoint non-empty open sets Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cap \Omega_2$. Similarly, a closed set F is connected if one cannot write $F = F_1 \cup F_2$ where F_1 and F_2 are disjoint non-empty closed sets.

If a set cannot be broken up into two disjoint open sets, there must be some "overlap". Connectedness is a simple as it sounds.

THIS IS WHERE I LEFT OFF

2 Meromorphic Functions and the Logarithm

2.1 Zeros and Poles

Definition 2.1

A **point singularity** of a function f is a complex number z_0 such that f is defined in a neighborhood of z_0 but not at the point z_0 itself. This is also called an **isolated singularity**.

An exmple of this could be the function f that is defined on the punctured plane by $f(z) = z$. Then the origin is a point of singularity, unless we explicitly define $f(0) = 0$.

Question

So I'm confused– why can't $f(0) = 0$ naturally? What happens at $z = 0$?

Because the "issue" can be solved by explicitly defining the one point, we can call that singularity **removable**.

Another example is $f(z) = \frac{1}{z}$. This singularity (also at $z = 0$) is called a **pole singularity** because the function approaches infinity from the left and the right of the singularity.

A more complicated example is $f(z) = e^{i/z}$, which can not be explained by poles or removable singularities because the limits from the left and right on the real axis are not equal.

Definition 2.2

A complex number z_0 is a **zero** for the holomorphic function f if $f(z_0) = 0$.

Riveting, who could have guessed.

Note that if f is holomorphic in Ω and $f(z_0) = 0$ for some $z_0 \in \Omega$, then there exists an open neighborhood U of z_0 such tht $f(z) \neq 0$ for all $z \in U - \{z_0\}$, unless f is identically zero.

Theorem 2.1

Suppose that f is holomorphic in a connected open set Ω, has a zero at a point $z_0 \in \Omega$, and does not vanish identically in Ω. Then there exists a neighborhood $U \subset \Omega$ of z_0, a nonvanishing holomorphic function g on U, and a unique positive integer n such that

$$f(z) = (z - z_0)^n g(z) \text{ for all } z \in U.$$

So all this is saying is that if there is a zero in a nonzero holomorphic function, the function in a neighborhood of this point can be written in the above form.

Proof.

Bruh go back and fill this out!!!! ■