Algebraic Topology Problem Bank

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1 Revised

2 Not Revised

2.1 Homework 1

Problem 1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z-1|=1$

(b) $|z-1|=2|z-2|$

(c) $1/z=\bar{z}$

(d) $\mathrm{Re}(z)=3$

(e) $\mathrm{Im}(z)=a$ with $a\in\mathbb{R}$

(f) $\mathrm{Re}(z)>a$ with $a\in\mathbb{R}$

(g) $|z-1|<2|z-2|$

Proof. (a) This describes all the points on the complex plane that are 1 distance away from $(1,0)$. Thus this creates a circle with radius 1, centered at $(1,0)$.

(b) This is describing all the points where the distance from $(1,0)$ is twice the distance from $(2,0)$. We can do the following algebra to find the equation of this circle:

$$\begin{aligned}
|z-1| &= 2|z-2| \\
|(x+iy)-(1+i\cdot 0)| &= 2|(x+iy)-(2+i\cdot 0)| \\
|(x-1)+iy| &= 2|(x-2)=iy| \\
((x-1)^2+y^2)^{\frac{1}{2}} &= 2((x-2)^2+y^2)^{\frac{1}{2}} \\
(x-1)^2+y^2 &= 4((x-2)^2+y^2) \\
x^2-2x+1+y^2 &= 4(x^2-4x+4+y^2) \\
x^2-2x+1+y^2 &= 4x^2-16x+16+4y^2 \\
-3x^2+14x-3y^2 &= 15 \\
-3\left(x^2-\frac{14}{3}x+y^2\right) &= -3(-5) \\
x^2-\frac{14}{3}x+y^2 &= -5 \\
x^2-\frac{14}{3}x+\frac{49}{9}+y^2 &= -5+\frac{49}{9} \\
\left(x-\frac{7}{3}\right)^2+y^2 &= \frac{4}{9}.
\end{aligned}$$

Thus this must represent a circle centered at $(\frac{7}{3},0)$ with radius $\frac{2}{3}$.

(c) According to the following algebra:

$$\begin{aligned}
\frac{1}{z} &= \bar{z} \\
\frac{1}{x+iy} &= x-iy \\
1 &= (x-iy)(x+iy) \\
1 &= x^2-xiy+xiy-i^2y2 \\
1 &= x^2+y^2,
\end{aligned}$$

we know that this represents a circle of radius 1 centered at $(0,0)$.

(d) Re(3) is all the complex numbers with 3 as the real component, so it is a straight vertical line at $x=3$.

(e) $\text{Im}(z)=a$, where $a\in\mathbb{R}$, is every complex number with a as it's y value. Thus it is a straight horizontal line at $y=a$.

(f) Similar to part (d), instead of this being a vertical line at a, this would be everything to right of a, not including the vertical line at a itself.

(g) This will be the circle from part (b), $(x-\frac{7}{3})^2+y^2=\frac{4}{9}$, but instead of the boundary of this circle, it will be everything outside of it, not including the inside of it, or the boundary itself.

$\square$

Problem 2. Prove that $|z_1 + z_2| \geq ||z_1| - |z_2||$ and explain when equality holds.

Proof. First let us prove the following: Given any two complex numbers z_1 and z_2,

$$|z_1| \leq |z_1 - z_2| + |z_2|$$
$$|z_2| \leq |z_2 - z_1| + |z_1|.$$

Note that, because these represent distances, $|z_1 - z_2| = |z_2 - z_1|$. Thus we find that

$$|z_1 - z_2| \geq |z_1| - |z_2|$$
$$|z_2 - z_1| = |z_1 - z_2| \geq |z_2| - |z_1| \implies -|z_1 - z_2| \leq |z_1| - |z_2|.$$

Putting both equations together, we get

$$-|z_1 - z_2| \leq |z_1| - |z_2| \leq |z_1 - z_2| \implies |z_1 - z_2| \geq ||z_1| - |z_2||.$$

We will use this fact in the problem.

We proceed by cases:

Case 1: Let $z_1, z_2 \geq 0$. Then $|z_1 + z_2| \geq |z_1 + z_2| \geq ||z_1| - |z_2||$.

Case 2: Let $z_1, z_2 \leq 0$. Then $|z_1 + z_2| = ||z_1| + |z_2|| \geq ||z_1| - |z_2||$.

Case 3: Let $z_1 > 0$, $z_2 < 0$. Then $|z_1 + z_2| = |z_1 - |z_2|| = ||z_1| - |z_2||$.

Case 4: Let $z_1 < 0$, $z_2 > 0$. Then $|z_1 + z_2| = |-|z_1| + |z_2|| = ||z_1| - |z_2||$.

Note that equality holds in cases 3 and 4, or any case where one of the z_i's is 0. □

Problem 3. Prove that the equation $z^3 + 2z + 4 = 0$ has roots outside the unit circle.

Proof. Assume $|z| \leq 1$, and that z is a root so that $z^3 + 2z + 4 = 0$. From $|z| \leq 1$, we know that $|z^3| \leq 1$ and $|2z| \leq 2$. Then we have

$$z^3 + 2z + 4 = 0 \implies z^3 + 2z = -4$$

so that $|z^3 + 2z| = |-4|$. By the triangle inequality, we know that $|z^3 + 2z| \leq |z^3| + |2z|$, so then

$$4 = |-4| = |z^3 + 2z| \leq |z^3| + |2z| \leq 1 + 2 = 3.$$

Thus we have found a contradiction, so for all the roots of the equation, $|z| > 1$ so that it lies outside the unit circle. □

Problem 4. (a) Prove that the if $|w_1| = c|w_2|$ where $c > 0$, then $|w_1 - c^2 w_2| = c|w_1 - w_2|$.

(b) Prove that if $c > 0$, $c \neq 1$, and $z_1 \neq z_2$, then $|\frac{z - z_1}{z - z_2}| = c$ represents a circle. Find it's center and radius.

Proof. (a) Assume that $|w_1| = c|w_2|$, where $w_1 = a + bi$ and $w_2 = e + fi$. Then:

$$|w_1| = c|w_2| \implies$$
$$\sqrt{a^2+b^2} = c\sqrt{e^2+f^2}\text{so that}$$
$$\sqrt{a^2+b^2} = \sqrt{c^2e^2+c^2f^2} \implies$$
$$a^2+b^2 = c^2e^2+c^2f_2^w$$

Then we know that

$$\begin{aligned}
|w_1 - c^2w_2| &= |(a+bi) - c^2(e+fi)| = |(a-c^2e) + (b-c^2f)i| \\
&= \sqrt{(a-c^2e)^2 + (b-c^2f)^2} \\
&= \sqrt{(a^2 - 2ac^2e + c^4e) + b^2 - 2c^2bf + c^4f^2} \\
&= \sqrt{(a^2+b^2) + c^2(c^2e^2+c^2f^2) - 2ac^2e - 2c^2bf} \\
&= \sqrt{c^2e^2 + c^2f^2 + c^2a^2 + c^2b^2 - 2ac^2e - 2c^2bf} \\
&= \sqrt{(ca-ce)^2 + (cb-cf)^2} \\
&= \sqrt{c^2(a-e)^2 + c^2(b-f)^2} \\
&= c\sqrt{(a-e)^2 + (b-f)^2} \\
&= c|w_1 - w_2|.
\end{aligned}$$

(b) First, note that

$$|\frac{z-z_1}{z-z_2}| = \frac{|z-z_1|}{|z-z_2|} = c \implies |(z-z_1) - c^2(z-z_2)| = c|(z-z_1) - (z-z_2)| = c|z_2 - z_1|.$$

Then we can find that

$$\begin{aligned}
\frac{|z-z_1||z-z_1|}{|z-z_2||z-z_2|} = c \implies |(z-z_1)^2| &= c|z_2 - z_1| \\
&= |(z-z_1) - c^2(z-z_2)|.
\end{aligned}$$

Thus

$$\begin{aligned}
|z-z_1| &= |1 - c^2\frac{z-z_2}{z-z_1}| \\
&= |1 - c^2 \cdot c^{-1}| \\
&= 1 - c.
\end{aligned}$$

Therefore we have a circle of center z_1, and radius $1-c$.

□