8150 Homework III

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Stein Problems

Exercise 1

Using Euler's Formula
$$\sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2i},$$
show that the complex zeroes of $\sin \pi z$ are exactly the integers, and that they are each of order 1. Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

Proof.

To find the zeroes, set Euler's Formula to zero:
$$\frac{e^{i\pi z} - e^{-i\pi z}}{2i} = 0.$$
From here, we can find that
$$\begin{aligned} e^{i\pi z} - e^{-i\pi z} &= 0 \\ e^{i\pi z} &= e^{-i\pi z} \\ i\pi z &= -i\pi z + 2\pi i k, k \in \mathbb{Z} \\ 2i\pi z &= 2i\pi k, \end{aligned}$$
so that $z = k$, meaning z must be an integer.
To show that they are all of order one, we can find the derivate of $\sin \pi z$ to be $\pi \cos \pi z$, and notice that this is nonzero for any integer.
To find the residue of $1/\sin \pi z$ at z an integer, we can use this formula:
$$\mathrm{Res}(f, a) = \lim_{z \to a} (z - a)(f(z)).$$
So in our case,
$$\mathrm{Res}(\frac{1}{\sin \pi z}, n) = \lim_{z \to n} (z - n) \cdot \frac{1}{\sin \pi z},$$
where $n \in \mathbb{Z}$. We know that near $z = n$, $\sin \pi z$ can be approximated using a Taylor Expansion, so that
$$\frac{1}{\sin \pi z} \approx \frac{1}{\pi(z - n)(-1)^n}.$$
Therefore we find that
$$\mathrm{Res}(\frac{1}{\sin \pi z}, n) = \frac{1}{\pi(-1)^n}.$$

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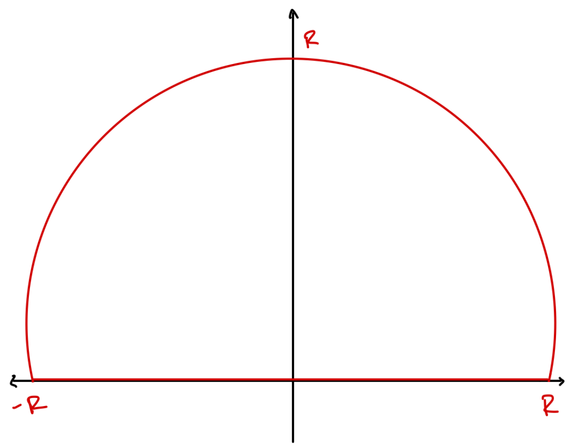
Exercise 2

Evaluate the inegral
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$
Where are the poles of $1/(1+z^4)$?

Proof.

First, to find the poles, we must find where $1+z^4=0$. This is at the points $z_0 = e^{i\pi/4}, z_1 = e^{i3\pi/4}, z_2 = e^{i5\pi/4}, z_3 = e^{i7\pi/4}$. Then, we can contour over a semicircle centered at the origin with radius R:



As $R \to \infty$, the integral vanishes because $\frac{1}{1+z^4}$ decays like $\frac{1}{|z|^4}$ for large enough $|z|$. We're only concerned with the poles in the upper half of the plane, so by the residue theorem,
$$\oint \frac{dz}{1_z^4} = 2\pi i(\operatorname{Res}(z_0) + \operatorname{Res}(z_1)).$$
Calculating the residues, we find:
$$\operatorname{Res}(z_0) = \lim_{z\to z_0} \frac{z-z_0}{1+z^4} = \frac{1}{4z_0^3} = \frac{1}{4}e^{-i3\pi/4}$$
and, similarly,
$$\operatorname{Res}(z_1) = \frac{1}{4z_1^3} = \frac{1}{4}e^{-i9\pi/4} = \frac{1}{4}e^{-i\pi/4}.$$
Therefore we find that
$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{1+x^4} &= 2\pi i\left(\frac{1}{4}\left(e^{-i3\pi/4} + e^{-i\pi/4}\right)\right) \\
&= 2\pi i\left(\frac{1}{4}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right) \\
&= 2\pi i\left(-\frac{i\sqrt{2}}{4}\right) \\
&= \frac{\pi\sqrt{2}}{2}.
\end{aligned}$$
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Exercise 4

Show that
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}, \text{ for all } a > 0.$$

Proof.

First, note that
$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2} dx\right).$$

For this integral, we can use a semicircle contour like we did the last problem. Once again, the integral vanishes as $R \to \infty$. The integrand has poles at $z = \pm ia$, but only $z = ia$ lies in the top half of the graph, so we only need to calculate it's residue:

$$\operatorname{Res}(ia) = \lim_{z \to ia} \frac{(z - ia) z e^{iz}}{z^2 + a^2} = \frac{iae^{-a}}{2ia} = \frac{e^{-a}}{2}.$$

Thus the first integral is equal to $\pi i e^{-a}$ due to the residue theorem. From there,

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \operatorname{Im}(\pi i e^{-a}) = \pi e^{-a}$$

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Exercise 5

Use contour integration to show that

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i e \xi}}{(1 + x^2)^2} dx = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}$$

for all ξ real.

Proof.

The integrand is:
$$f(x) = \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2}.$$

For $\xi > 0$, we can close the contour in the lower half-plane, and for $\xi < 0$, the upper half-plane. The denominator $(1 + x^2)^2$ has double poles at $x = \pm i$. Only one of these poles lies inside the each of the two contours. For $\xi > 0$, the pole is at $x = -i$. Since this is a double pole, the residue is given by:

$$\operatorname{Res}(f, -i) = \lim_{x \to -i} \frac{d}{dx}\left((x + i)^2 \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2}\right).$$

This can be simplified to
$$(x + i)^2 \frac{e^{-2\pi i x \xi}}{(1 + x^2)^2} = \frac{e^{-2\pi i x \xi}}{(x - i)^2}.$$

We can differentiate with respect to x to get

$$\frac{d}{dx}\left(\frac{e^{-2\pi i x \xi}}{(x - i)^2}\right) = \frac{-2\pi i \xi e^{-2\pi i x \xi}(x - i)^2 - 2(x - i)e^{-2\pi i x \xi}}{(x - i)^4},$$

and then evaluate at $x = -i$ to get

$$\operatorname{Res}(f, -i) = \frac{-2\pi i \xi e^{-2\pi i(-i)\xi}(-2i)^2 - 2(-2i)e^{-2\pi i(-i)\xi}}{(-2i)^4}.$$

After some algebra,

$$\operatorname{Res}(f, -i) = \frac{i(1 - 2\pi\xi)e^{-2\pi\xi}}{4}.$$

For $\xi < 0$, the pole is at $x = i$, and the residue calculation is found with a similar process as the previous:

$$\operatorname{Res}(f, i) = \frac{i(1 + 2\pi|\xi|)e^{-2\pi|\xi|}}{4}.$$

Then for $\xi > 0$:

$$\int_{-\infty}^{\infty} f(x)\, dx = -2\pi i \cdot \operatorname{Res}(f, -i) = -2\pi i \cdot \frac{i(1 - 2\pi\xi)e^{-2\pi\xi}}{4} = \frac{\pi(1 - 2\pi\xi)e^{-2\pi\xi}}{2}.$$

For $\xi < 0$:

$$\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \cdot \operatorname{Res}(f, i) = 2\pi i \cdot \frac{i(1 + 2\pi|\xi|)e^{-2\pi|\xi|}}{4} = \frac{\pi(1 + 2\pi|\xi|)e^{-2\pi|\xi|}}{2}.$$

Thus, for all real ξ:

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{(1 + x^2)^2}\, dx = \frac{\pi}{2}(1 + 2\pi|\xi|)e^{-2\pi|\xi|}.$$

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Exercise 6

Show that

$$\int_{\infty}^{\infty} \frac{dx}{(1 + x^2)^{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \pi.$$

Proof.

We once again use a semicircle with radius R to contour integrate this. As $R \to \infty$, the integral vanishes because the integrand decays as $|x|^{-2(n+1)}$ for large $|x|$. The integrand has poles where $(1 + x^2)^{n+1} = 0$, i.e., at $x = \pm i$. Only the pole at $x = i$ lies inside the contour.

The pole at $x = i$ is of order $n + 1$. To compute the residue, we use the formula for the residue of a function $f(x) = \frac{g(x)}{(x-i)^{n+1}}$ at $x = i$:

$$\operatorname{Res}(f, i) = \frac{1}{n!} \lim_{x \to i} \frac{d^n}{dx^n} \left((x - i)^{n+1} f(x)\right) = \frac{1}{n!} \lim_{x \to i} \frac{d^n}{dx^n} \left(\frac{1}{(x + i)^{n+1}}\right).$$

For simplicity, let $h(x) = \frac{1}{(x+i)^{n+1}}$. Then the n-th derivative of $h(x)$ is

$$(-1)^n \frac{(n + 1)(n + 2) \cdots (2n)}{(x + i)^{2n+1}}.$$

At $x = i$, this derivative is

$$(-1)^n \frac{(n + 1)(n + 2) \cdots (2n)}{(2i)^{2n+1}}.$$

Therefore we have :

$$\mathrm{Res}(f,i) = \frac{1}{n!}\cdot(-1)^n\frac{(n+1)(n+2)\cdots(2n)}{(2i)^{2n+1}}$$

After some algebra, and the fact that $(2n)! = 2^n n!\cdot 1\cdot 3\cdot 5\cdots(2n-1)$:

$$\mathrm{Res}(f,i) = \frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot\frac{1}{(2i)^{n+1}}.$$

By the residue theorem, we have:

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = 2\pi i\cdot\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot\frac{1}{2^{n+1}i^{n+1}}.$$

From there, because $i^{n+1} = i^n\cdot i$, and i^n cycles through $1, i, -1, -i$, we have:

$$2\pi\cdot\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdots(2n-1)}{n!}\cdot\frac{1}{2^{n+1}i^n}.$$

For even n, $i^n = (-1)^{n/2}$, and for odd n, $i^n = (-1)^{(n-1)/2}\cdot i$. Therefore, the result simplifies to:

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\pi.$$

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Exercise 7

Show that

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}},\ \text{whenever } a>1.$$

Proof.

First, note that the integral can be rewritten in terms of $z = e^{i\theta}$ so that we have:

$$\cos\theta = \frac{z+z^{-1}}{2},\quad d\theta = \frac{dz}{iz}$$

and

$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = \oint_{|z|=1}\frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\cdot\frac{dz}{iz}.$$

We can do the following algebra to simplify:

$$\begin{aligned}\oint_{|z|=1}\frac{1}{\left(a+\frac{z+z^{-1}}{2}\right)^2}\cdot\frac{dz}{iz} &= \oint_{|z|=1}\frac{1}{\left(\frac{2a+z+z^{-1}}{2}\right)^2}\cdot\frac{dz}{iz}\\ &= \oint_{|z|=1}\frac{4}{(2a+z+z^{-1})^2}\cdot\frac{dz}{iz}\\ &= \oint_{|z|=1}\frac{4z^2}{(2az+z^2+1)^2}\cdot\frac{dz}{iz}\\ &= \frac{4}{i}\oint_{|z|=1}\frac{z}{(z^2+2az+1)^2}\,dz\end{aligned}$$

The denominator is zero at $z = -a \pm \sqrt{a^2-1}$. Since $a > 1$, only the root $z = -a + \sqrt{a^2-1}$ lies inside the unit circle and is of order 2. To compute the residue, we use the formula for the residue of a function $f(z) = \frac{g(z)}{(z-z_0)^2}$ at $z = z_0$:

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz} \left((z-z_0)^2 f(z) \right).$$

Let z_1 be the other root, $-a - \sqrt{a^2-1}$. Then we have

$$f(z) = \frac{z}{(z-z_0)^2 (z-z_1)^2}$$

and

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{d}{dz} \frac{z}{(z-z_1)^2}.$$

We can find $\frac{d}{dz} \frac{z}{(z-z_1)^2}$:

$$\frac{d}{dz} \left(\frac{z}{(z-z_1)^2} \right) = \frac{(z-z_1)^2 \cdot 1 - z \cdot 2(z-z_1)}{(z-z_1)^4} = \frac{(z-z_1) - 2z}{(z-z_1)^3}.$$

At $z = z_0$:

$$\operatorname{Res}(f, z_0) = \frac{(z_0 - z_1) - 2z_0}{(z_0 - z_1)^3} = \frac{-z_0 - z_1}{(z_0 - z_1)^3}.$$

Substitute $z_0 = -a + \sqrt{a^2-1}$ and $z_1 = -a - \sqrt{a^2-1}$:

$$z_0 - z_1 = 2\sqrt{a^2-1}, \quad -z_0 - z_1 = 2a.$$

So we are left with:

$$\operatorname{Res}(f, z_0) = \frac{2a}{(2\sqrt{a^2-1})^3} = \frac{2a}{8(a^2-1)^{3/2}} = \frac{a}{4(a^2-1)^{3/2}}.$$

By the residue theorem, the integral is:

$$\oint_{|z|=1} \frac{z}{(z^2+2az+1)^2} \, dz = 2\pi i \cdot \operatorname{Res}(f, z_0) = 2\pi i \cdot \frac{a}{4(a^2-1)^{3/2}}.$$

Substitute back into the original expression:

$$\frac{4}{i} \oint_{|z|=1} \frac{z}{(z^2+2az+1)^2} \, dz = \frac{4}{i} \cdot 2\pi i \cdot \frac{a}{4(a^2-1)^{3/2}} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

Therefore the final result is:

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2\pi a}{(a^2-1)^{3/2}}.$$

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Exercise 8

Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

if $a > |b|$ and $a, b \in \mathbb{R}$.

Proof.

First, let $z = e^{i\theta}$ and use Euler's formula so that the integral becomes

$$\oint_{|z|=1} \frac{1}{a + b(\frac{z+z^{-1}}{2})} \frac{dz}{iz}.$$

After some algebra, we can simplify this to be

$$\frac{2}{i} \oint_{|z|=1} \frac{1}{bz^2 + 2az + b} dz.$$

Using the quadratic formula, we can find the (simple) poles to be

$$z_1 = \frac{-a + \sqrt{a^2 - b^2}}{b} \text{ and } z_2 = \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Between the two poles, only the first lies inside the unit circle, so we only need to find that residue:

$$\text{Res}(f, z_1) = \frac{1}{b(z_1 - z_2)} = \frac{1}{2\sqrt{a^2 - b^2}}$$

Using the residue theorem, we find that

$$\frac{2}{i} \oint_{|z|=1} \frac{1}{bz^2 + 2az + b} dz = \frac{2}{i} \cdot 2\pi i \cdot \frac{1}{2\sqrt{a^2 - b^2}} = \frac{2\pi}{\sqrt{a^2 - b^2}}.$$

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Exercise 9

Show that

$$\int_0^1 \log(\sin \pi x) dx = -\log 2.$$

Proof.

First, note that because of the symmetry about $x = \frac{1}{2}$,

$$\int_0^1 \log(\sin \pi x) dx = 1 \int_0^{1/2} \log(\sin \pi x) dx.$$

Then let $x = \frac{u}{2}$, so the integral becomes

$$\int_0^1 \log(\sin(\frac{\pi u}{2})) du.$$

We can then use the double angle identity to find that this is equal to

$$\begin{aligned}
&\int_0^1 \left(\log 2 + \log\left(\sin\left(\frac{\pi u}{4}\right)\right) + \log\left(\cos\left(\frac{\pi u}{4}\right)\right)\right) du \\
&= \int_0^1 \log 2 du + \int_0^1 \log\left(\sin\left(\frac{\pi u}{4}\right)\right) du + \int_0^1 \log\left(\cos\left(\frac{\pi u}{4}\right)\right) du
\end{aligned}$$

The first integral is simply $\log 2$, and the second and third are equivalent because of the symmetry of sine and cosine. Using the known fact that $\int_0^\pi \log(\sin x)dx = -\pi \log 2$ and letting $v = \pi u/4$, we find that

$$\begin{aligned}
\int_0^1 \log\left(\sin\left(\frac{\pi u}{4}\right)\right) du &= \frac{4}{\pi}\left(\int_0^{\pi/4} \log(\sin v)dv\right) \\
&= \frac{4}{\pi}\left(-\frac{\pi}{4}\log 2\right) \\
&= -\log 2
\end{aligned}$$

Thus $\int_0^1 \log(\sin \pi x)dx = \log 2 - 2\log 2 = -\log 2$. ■

Exercise 10

Show that if $a > 0$, then

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a}\log a.$$

Proof.

We can integrate this over a keyhole contour Γ consisting of a large semicircle C_R with radius R in the upper half of the plane, a small semicircle C_ϵ of radius ϵ around the origin, and two horizontal line segments that close the keyhole contour. Then the simple poles are $z = \pm ia$, but we only need to find the residue of the positive one, $z = ia$.

We know

$$\oint_\Gamma f(z)dz = 2\pi i \cdot \operatorname{Res}(f, z = ia),$$

so we can find the residue to be

$$\operatorname{Res}(f, ia) = \frac{\log(ia)}{2ia} = \frac{\log(a) + i\frac{\pi}{2}}{2ia}.$$

To evaluate the contour integral, we can break it into four parts: the large semicircle, the small one, and both line segments.

First, for the large semicircle C_R, as $R \to \infty$ the integrand vanishes because it behaves like $\frac{\log z}{z^2}$.

For the small semicircle C_ϵ, as $\epsilon \to 0$, the integrand behaves like $\frac{\log z}{a^2}$, so it must vanish as well.

Along the horizontal line segments,

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx - \int_0^\infty \frac{\log x + 2\pi i}{x^2 + a^2} = -2\pi i \int_0^\infty \frac{1}{x^2 + a^2} dx = \frac{\pi}{2a}.$$

Thus we find that

$$-2\pi i \cdot \frac{\pi}{2a} = -\frac{\pi^2 i}{a}.$$

Therefore, we can combine this with the residue calculation to get

$$\int_0^\infty \frac{\log x}{x^2 + a^2} dx = \frac{\pi}{2a} \log a.$$

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Exercise 14

Prove that all entire functions that are also injective take the form $f(z) = az + b$ with $a, b \in \mathbb{C}$ and $a \neq 0$.

Proof.

Let $f : \mathbb{C} \to \mathbb{C}$ be entire and injective. From here, there are two cases:

1) f is a polynomial. If this is true, if it has degree 2 or greater, then by the fundamental theorem of algebra, it must have at least two roots, and therefore cannot be injective. Thus f must have degree less than two.

2) f is not a polynomial. If this is true, then the function must be an entire transcendental function. These are never injective on $\mathbb{C}$, so we have a contradiction.

Constant functions are clearly not injective, so the function must be linear with $a \neq 0$. ■

Tie Problems

Exercise 1

Prove that if

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n \text{ and } \sum_{n=-\infty}^{\infty} c'_n (z-a)^n$$

are Laurent series expansions of $f(z)$, then $c_n = c'_n$ for all n.

Proof.

Let $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n = \sum_{n=-\infty}^{\infty} c'_n (z-a)^n$. Then we know, for any integer k,

$$f(z)(z-a)^{-k-1} = \sum_{n=-\infty}^{\infty} c_n (z-a)^{n-k-1} = \sum_{n=-\infty}^{\infty} c'_n (z-a)^{n-k-1}$$

Then let γ be any closed contour in the annulus going around a once, and because it is a compact set of points, the Luarent serieses can be integrated termwise:

$$\sum_{n=-\infty}^{\infty} c_n \oint_\gamma (z-a)^{n-k-1} dz = \sum_{n=-\infty}^{\infty} c'_n \oint_\gamma (z-a)^{n-k-1} dz$$

We know that

$$\oint (z-a)^{n-k-1} dz = 2i\pi \text{ if } n = k \text{ and } 0 \text{ if } n \neq k$$

So then we are left with $2i\pi c_m = 21\pi c'_n$ for any k, which proves the statement. ■

Exercise 2

Expand $\frac{1}{1-z^2} + \frac{1}{3-z}$ in a series of the form $\sum^{\infty} -\infty a_n z^n$. How many such expansions are there? In which domain is each of them valid?

Proof.

We find that:

$$\begin{aligned}\frac{1}{z-3} &= -\frac{1}{3}\frac{1}{1-3z^{-1}} = -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k \text{ for } |z| < 3 \\ &= \frac{1}{z}\frac{1}{1-3z^{-1}} = z^{-1}\sum_{k\geq 0} 3^k z^{-k} \text{ for } |z| > 3\end{aligned}$$

and:

$$\begin{aligned}\frac{1}{1-z^2} &= \sum_{k\geq 0} z^{2k} \text{ for } |z| < 1 \\ &= \frac{1}{z^2}\frac{-1}{1-z^{-2}} = -z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| > 1\end{aligned}$$

So we can just list all the possible combinations to find:

$$\begin{aligned}f(z) &= -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1) \\ f(z) &= -\frac{1}{3}\sum_{k\geq 0} 3^{-k}z^k - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (1, 3) \\ f(z) &= z^{-1}\sum_{k\geq 0} 3^k z^{-k} + \sum_{k\geq 0} z^{2k} \text{ for } |z| \in (-\infty, 1) \cup (3, \infty) \\ f(z) &= z^{-1}\sum_{k\geq 0} 3^k z^{-k} - z^{-2}\sum_{k\geq 0} z^{-2k} \text{ for } |z| \in (3, \infty)\end{aligned}$$

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Exercise 3

Let $P(z)$ and $Q(z)$ be polynomials with no common zeros. Assume $Q(a) = 0$. Find the principal part of $P(z)/Q(z)$ at $z = a$ if the zero a is (i) simple; (ii) double. Express your answers explicitly using P and Q.

Proof.

i. If a is a simple zero of $Q(z)$, we can write $Q(z) = (z-a)Q_1(z)$, where $Q_1(a)$ is nonzero. The function $f = \frac{P(z)}{Q(z)}$ has a simple pole at $z = a$, so the principle part is $\frac{\mathrm{Res}(f,a)}{z-a}$. To compute the residue, we can find

$$\mathrm{Res}(f, a) = \lim_{z\to a}(z-a)\frac{P(z)}{Q(z)} = \lim_{z\to a}\frac{P(z)}{Q_1(z)} = \frac{P(a)}{Q_1(a)}.$$

Note that, because $Q'(z) = Q_1(z) + (z-a)Q_1'(z)$, $Q_1(a) = Q'(a)$.

Therefore the principal part of $\frac{P(z)}{Q(z)}$ at $z = a$ is

$$\frac{P(a)}{Q'(a)} \cdot \frac{1}{z-a}.$$

ii. If a is a double zero of $Q(z)$, we can write $Q(z) = (z-a)^2 \cdot Q_2(z)$, where $Q_2(a) \neq 0$; we also know that the principal part of $\frac{P(z)}{Q(z)}$ is of the form $\frac{A}{(z-a)^2} + \frac{B}{z-a}$. First, let's calculate A:

$$A = \lim_{z \to a} (z-a)^2 \cdot \frac{P(z)}{Q(z)} = \lim_{z \to a} \frac{P(z)}{Q_2(z)} = \frac{P(a)}{Q_2(a)}$$

To calculate B, we find:

$$\begin{aligned} B &= \lim_{z \to a} \frac{d}{dz} \left((z-a)^2 \frac{P(z)}{Q(z)} \right) \\ &= \lim_{z \to a} \frac{d}{dz} \left(\frac{P(z)}{Q_2(z)} \right) \\ &= \lim_{z \to a} \frac{P'(z)Q_2(z) - P(z)Q_2'(z)}{Q_2(z)^2} \\ &= \frac{P'(a)Q_2(a) - P(a)Q_2'(a)}{Q_2(a)^2} \end{aligned}$$

We know that $Q'(a) = 0$ and $Q''(a) = 2Q_2(a)$, and from this we can deduce $Q_2'(a) = \frac{Q'''(a)}{6}$. From there,

$$B = \frac{2P'(a)Q''(a) - P(a)Q'''(a)}{3Q''(a)^2}$$

so that the principal part of $\frac{P(z)}{Q(z)}$ is

$$\frac{P(a)}{Q''(a) \cdot \frac{1}{2} \cdot (z-a)^2} + \frac{2P'(a)Q''(a) - P(a)Q'''(a)}{3Q''(a)^2} \cdot \frac{1}{z-a}.$$

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Exercise 4

Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $lim_{n \to \infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$.

Proof.

Assume, for contradiction, that $z = 0$ is a removable singularity. Then f would extend to a holomorphic function over $z = 0$, so that $f(0) = f(\lim z_n) = \lim f(z_n) = 0$. But then f would have to be identically zero, because of the identity principal. This contradicts the fact that f is stated to be non-constant.
Then assume for contradiction that $z = 0$ is a pole. Then $f(z_n) \to \infty$. This is a contradiction because $f(z_n) = 0$ infinitely many times.
Thus $z = 0$ must be an essential singularity. ■

Exercise 5

Let f be entire and suppose that $\lim_{x\to\infty} f(z) = \infty$. Show that f is a polynomial.

Proof.

First, note that because f is unbounded, there must exist some R such that $f(D_R^c) \subset D^c$. Therefore we know that f is nonvanishing on D_R^c. Then we know the zeroes of f, Z_f, is a closed subset of a compact set. Therefore we know it is either finite, or has an accumulation point. If it had an accumulation point, f would have to be identically zero, so Z_f must be finite. We can then define, where n represents the number of zeroes for f,

$$\phi(z) = \Pi_{i\leq n}(z - z_i) \text{ and } F(z) = \frac{\phi(z)}{f(z)}.$$

Then note that F is nonvanishing, entire, and bounded. Thus by Liouville, it has to be constant, so $f(z) = c\phi(z)$. ∎

Exercise 6

(1) Show without using 3.8.9 in the textbook by Stein and Shakarchi that

$$\int_0^{2\pi} \log|1 - e^{i\theta}|d\theta = 0.$$

(2) Show the above identity is equivalent to the one in 3.8.9 of the textbook.

Exercise 7

Evaluate $\int_0^\infty \frac{x^{a-1}}{1+x^3}dx, 0 < a < 4$.

Proof.

Consider a keyhole contour Γ that is made up of a large circle C_R with radius R centered at the origin, a small circle C_ϵ of radius ϵ centered at the origin, and two horizontal line segments just above and below the branch cut on the positive real axis.
The simple poles of the function are $z_0 = e^{i\pi/3}$, $z_1 = e^{i\pi} = -1$, and $z_2 = e^{i5\pi/3}$. Between these three, only z_0 and z_2 contribute to the integral, so we have

$$\oint_\Gamma f(z)dz = 2\pi i(\text{Res}(f, z_0) + \text{Res}(f, z_2)).$$

To calculate the residues, we find that

$$\text{Res}(f, z_0) = \frac{z_0^{a-1}}{3z_0^2} \text{ and } \text{Res}(f, z_2) = \frac{z_2^{a-1}}{3z_2^2}$$

so

$$\oint_\Gamma f(z)dz = 2\pi i\left(\frac{z_0^{a-1}}{3z_0^2} + \frac{z_2^{a-1}}{3z_2^2}\right).$$

Now, to integrate over the contour, we can first integrate over C_R; notice that for large enough R, $f(z)$ acts like z^{a-4}, so the integral vanishes here. For C_ϵ, as $\epsilon \to 0$, $f(z)$ behaves like z^{a-1}, so this part vanishes as well. For the horizontal line segments, we have

$$\int_0^\infty \frac{x^{a-1}}{1+x^4}dx - \int_0^\infty \frac{(e^{2\pi i})^{a-1}}{1+x^3} = \left(1-e^{2\pi i(a-1)}\right)\int_0^\infty \frac{x^{a-1}}{1+x^3}.$$

So we are left with

$$\left(1-e^{2\pi i(a-1)}\right)\int_0^\infty \frac{x^{a-1}}{1+x^3} = 2\pi i\left(\frac{z_0^{a-1}}{3z_0^2}+\frac{z_2^{a-1}}{3z_2^2}\right)$$

We can solve for the integral and do some algebra to find

$$\int_0^\infty \frac{x^{a-1}}{1+x^3}dx = \frac{\pi}{3\sin(\pi a/3)}.$$

■

Exercise 8

(1) Prove the fundamental theorem of algebra using Rouche's theorem.

(2) Prove the fundamental theorem of algebra using the maximum modulus principle.

Proof.

(1) Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. Choose $R > 1$ large enough that for $|z| = R$:

$$|a_{n-1}z^{n-1} + \cdots + a_0| \le |a_{n-1}|R^{n-1} + \cdots + |a_0| < R^n.$$

By Rouché's Theorem, $P(z)$ and z^n have the same number of zeros inside $|z| < R$. Since z^n has n zeros (at 0), $P(z)$ has n roots in $\mathbb{C}$.

(2) Assume, for contradiction, that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Then $f(z) = \frac{1}{P(z)}$ is entire. For $|z| = R$, write $P(z) = z^n + Q(z)$, where $Q(z) = a_{n-1}z^{n-1} + \cdots + a_0$. By the reverse triangle inequality:

$$|P(z)| \ge |z^n| - |Q(z)| = R^n - \sum_{k=0}^{n-1}|a_k|R^k.$$

Choose $R > 1$ sufficiently large such that $\sum_{k=0}^{n-1}|a_k|R^k \le \frac{R^n}{2}$. Then:

$$|P(z)| \ge R^n - \frac{R^n}{2} = \frac{R^n}{2} \quad \text{for } |z| = R.$$

For the closed disk $\overline{D_R} = \{z \in \mathbb{C} : |z| \le R\}$, $|f(z)| = \frac{1}{|P(z)|}$ attains its maximum on the boundary $|z| = R$. We know

$$\max_{|z|\le R}|f(z)| = \max_{|z|=R}|f(z)| \le \frac{2}{R^n}.$$

as $R \to \infty$, $\frac{2}{R^n} \to 0$. Therefore

$$\sup_{z\in\mathbb{C}}|f(z)| = 0 \implies f(z) \equiv 0.$$

Clearly, this contradicts $f(z) = \frac{1}{P(z)} \neq 0$. Thus, $P(z)$ must have at least one root $z_1 \in \mathbb{C}$. Factor $P(z)$ as $P(z) = (z - z_1)Q(z)$, where $Q(z)$ is a polynomial of degree $n-1$. Repeating the argument inductively, $Q(z)$ must also have a root. Continuing this process yields all n roots of $P(z)$.

■

Exercise 9

Assume $f(z)$ is analytic in region D and γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

Exercise 10

Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta}, a > 0$.

Exercise 11

Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

Proof.

- In $|z| < 1$:

  Small: $z^4 + 3$

  Big: $-6z$

- In $|z| = 1$:

  $|m(z)| = |z^4 + 3| \leq |4|^4 + 3 = 4 < 6 = |-6z| = |M(z)|$

- In $|z| < 2$:

  Small: $-6z + 3$

  Big: z^4

- In $|z| = 2$:

  $|m(z)| = |-6z + 3| \leq 6 + 3 = 9 < 2^4 = |M(z)|$

Therefore there is 1 root in $|z| < 1$, and there are 3 zeroes in $1 < |z| < 2$. ■

Exercise 12

Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

Proof.

First note that it is sufficient to prove the existence of exactly one root in Q_1, because conjugate pairs proves the existence in the other open quadrant. We know the polynomial is entire, so we can use the argument principle to count the zeroes. Let γ be made up of

$$\gamma_1 = [0, R]$$
$$\gamma_2 = Re^{it} \text{ for } t \in [0, \pi/2]$$
$$\gamma_3 = i[0, R].$$

Then we can consider

$$Z_f = \frac{1}{2\pi i} \int_\gamma \partial^{\log} f(z) dz = \Delta_\gamma \text{Arg}(f).$$

Then for each part of gamma,

$$\begin{aligned}
\Delta_{\gamma_1}\mathrm{Arg}(f) &= 0 \\
\Delta_{\gamma_2}\mathrm{Arg}(f) &= 4(\frac{\pi}{2}) = 2\pi \\
\Delta_{\gamma_3}\mathrm{Arg}(f) &= 0.
\end{aligned}$$

To prove the last part, consider $f(it) = t^4 - it^3 - 2it + 10 = t^4(1 - it^{-1} - 2it^{-2} + 10t^{-4})$.
Thus $\Delta_\gamma \mathrm{Arg}(f) = 1$, so as $R \to \infty$, there is only 1 zero. ∎

Exercise 13

Prove the equation $z \tan z = a$, $a > 0$, has only real roots in $\mathbb{C}$.

Proof.

Assume for contradiction that there exists a non-real root $z = x + iy$ with $y \neq 0$. We can then also note that

$$\begin{aligned}
\tan z &= \frac{\sin(x+iy)}{\cos(x+iy)} \\
&= \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}.
\end{aligned}$$

So that

$$\begin{aligned}
a &= z \tan z \\
&= (x+iy)\frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}
\end{aligned}$$

We can then multiply by the complex conjugate of the denominator, and then split into real and imaginary parts:

$$\begin{aligned}
\text{Real part:} \quad & \frac{x \sin 2x - y \sinh 2y}{2(\cos^2 x + \sinh^2 y)} = a, \\
\text{Imaginary part:} \quad & \frac{x \sinh 2y + y \sin 2x}{2(\cos^2 x + \sinh^2 y)} = 0.
\end{aligned}$$

We can see, from the imaginary part:

$$x \sinh 2y + y \sin 2x = 0.$$

Thus we are left with two cases:
Case 1: $y > 0$
Then we know $\sinh 2y > 0$, since $\sinh t > 0$ for $t > 0$, and that $x \sinh 2y = -y \sin 2x$. Then the real

part of the equation becomes

$$
\begin{aligned}
a &= \frac{x\sin 2x - y\sinh 2y}{2(\cos^2 x + \sinh^2 y)} \\
&= \frac{\left(-\frac{y\sin 2x}{\sinh 2y}\right)\sin 2x - y\sinh 2y}{2(\cos^2 x + \sinh^2 y)} \\
&= \frac{-y\left(\frac{\sin^2 2x}{\sinh 2y} + \sinh 2y\right)}{2(\cos^2 x + \sinh^2 y)}
\end{aligned}
$$

However, this contradicts $a > 0$.
Case 2: $y < 0$ One again, the real part becomes:

$$\frac{x\sin 2x - y\sinh 2y}{2(\cos^2 x + \sinh^2 y)} = a > 0.$$

Let $y = -|y|$ ($|y| > 0$) and use $\sinh 2y = -\sinh 2|y|$. Then the numerator becomes

$$\left(-\frac{|y|\sin 2x}{\sinh 2|y|}\right)\sin 2x - (-|y|)(-\sinh 2|y|) = -|y|\left(\frac{\sin^2 2x}{\sinh 2|y|} + \sinh 2|y|\right).$$

Once again, the numerator is negative and the denominator is positive, so $a < 0$.
Both cases lead to contradictions, therefore $y = 0$. Hence all solutions must be real. ■

Exercise 14

Let f be analytic on a bounded region Ω and continuous on the closure $\bar{\Omega}$. Assume $f(z) \neq 0$. Show that $f(z) = e^{i\theta}M$ (where θ is a real constant) if $|f(z)| = M$ (a constant) for $z \in \partial\Omega$.

Proof.

Given $|f(z)| = M$ on $\partial\Omega$:

- By the Maximum Modulus Principle, $|f(z)| \leq M$ for all $z \in \overline{\Omega}$
- By the Minimum Modulus Principle (since $f(z) \neq 0$), $|f(z)| \geq M$ for all $z \in \overline{\Omega}$

From here, I do not know where to go. ■