MATH 8150 Homework 2

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Problem 1. Prove that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}.$$

Proof. First note that

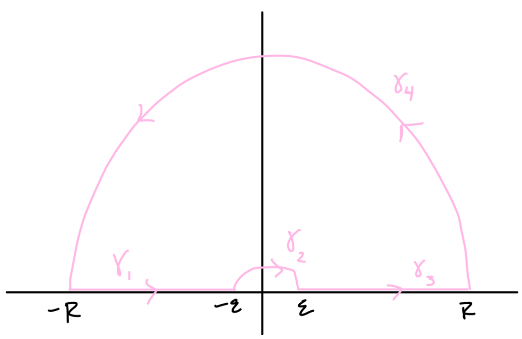
$$\int_0^\infty e^{ix^2} = \int_0^\infty \cos(x^2)dx + i\int_0^\infty \sin(x^2)dx.$$

Then by the hint, we know that □

Problem 2. Show that

$$\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

Proof. First, consider the following function $f(z) = \frac{e^{iz}}{z}$ and contour γ:



Because the function is holomorphic on the closed contour, we can apply Cauchy's theorem to find that

$$\oint_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz = 0.$$

First, we evaluate the integrals over real numbers:

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = \int_{-R}^{-\varepsilon} f(z)dz + \int_{\varepsilon}^{R} f(z)dz$$

Using u-subsitution and switch variables, we get

$$\begin{aligned}
\int_{\varepsilon}^{R} \frac{e^{ix}}{x} dx - \int_{\varepsilon}^{R} \frac{e^{-ix}}{x} dx &= \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{x} dx \\
&= 2i \int_{\varepsilon}^{R} \frac{e^{ix} - e^{-ix}}{(2i)x} dx \\
&= 2i \int_{\varepsilon}^{R} \frac{\sin(x)}{x} dx \\
&\to 2i \int_{0}^{\infty} \frac{\sin(x)}{x} dx \text{ as } R \to \infty \text{ and } \varepsilon \to 0.
\end{aligned}$$

Next, we evaluate the γ_2 integral with the following parametrization:

$$\begin{aligned}
z &= \gamma(t) = \varepsilon e^{i(\pi - t)} \text{ where } t \in [0, \pi] \\
dz &= \gamma'(t) dt \\
&= -i\varepsilon e^{i(\pi - t)} dt \\
&= -i\gamma(t) dt
\end{aligned}$$

So then

$$\begin{aligned}
\int_{\gamma_2} \frac{e^{iz}}{z} dz &= \int_{0}^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)} dt \\
&= -i \int_{0}^{\pi} e^{i\gamma(t)} dt \\
&= -i \int_{0}^{\pi} e^{i\varepsilon e^{i(\pi - t)}} dt
\end{aligned}$$

As $\varepsilon \to 0$, this approaches

$$-i \int_{0}^{\pi} 1 \cdot dt = -i\pi.$$

For the last integral, use the following parametrization:

$$\begin{aligned}
z &= \gamma(t) = Re^{it} \\
dz &= \gamma'(t) dt \\
&= Re^{it} dt \\
&= \gamma(t) dt
\end{aligned}$$

So we have

$$\begin{aligned}
\int_{\gamma_4} f(z) dz &= \int_{0}^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)} \gamma(t) dt \\
&= \int_{0}^{\pi} e^{i\gamma(t)} dt \\
&= \int_{0}^{\pi} e^{iRe^{it}} dt \\
&= \int_{0}^{\pi} e^{iR(\cos(t) + i\sin(t))} dt.
\end{aligned}$$

Then note that

$$\begin{aligned}
\left|\int_0^{\pi} e^{iR(\cos(t)+i\sin(t))}dt\right| &\leq \int_0^{\pi} \left|e^{iR(\cos(t)+i\sin(t))}\right| dt \\
&= \int_0^{\pi} \left|e^{iR\cos(t)}\right| \left|e^{-R\sin(t)}\right| dt \\
&= \int_0^{\pi} 1 \cdot \left|e^{-R\sin(t)}\right| dt \\
&= \int_0^{\pi} e^{-R\sin(t)} dt \\
&\to 0 \text{ as } R \to \infty.
\end{aligned}$$

So we find that this integral must be 0. Finally, we have

$$\begin{aligned}
\oint_{\gamma} f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz \\
&= 2i\int_0^{\pi} \frac{\sin(x)}{x}dx - i\pi + 0 \\
&= 0.
\end{aligned}$$

Thus,

$$\int_0^{\pi} \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

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