8200 Homework 11

May 12, 2025

Exercise 2.2.22

For X a finite CW complex and $p : \widetilde{X} \to X$ an n-sheeted covering space, show that $\chi(\widetilde{X}) = n\chi(X)$.

Proof.

The Euler characteristic is an alternating sum of the cells in the CW Complex

$$\chi(X) = \sum_{k=0}^{\infty}(-1)^k c_k$$

where c_k is the number of k-cells in X. In any covering space, each k-cell in the CW complex must be covered the same amount of times, and this is what gives us our number of sheets. So in an n-sheeted covering space, there are $n \cdot c_k$ k-cells. Thus

$$\chi(\tilde{X}) = \sum_{k=0}^{\infty}(-1)^k n \cdot c_k = n\sum_{k=0}^{\infty}(-1)^k c_k = n\chi(X).$$

■

Exercise 2.2.23

Show that if the closed orientable surface M_g of genus g is a covering space of M_h, then $g = n(h-1)+1$ for some n, namely, n is the number of sheets in the covering. [Conversely, if $g = n(h-1)+1$ then there is an n-sheeted covering $M_g \to M_h$, as we saw in Example 1.41.]

Proof.

We know the Euler characteristic of an orientable surface is $\chi(M_g) = 2 - 2g$ (or for M_h, $\chi(M_h) = 2 - 2h$). From the previous problem, we know that if n is the number of sheets in the covering, $\chi(M_g) = n \cdot \chi(M_h)$. So we are left with

$$\begin{aligned}
2 - 2g &= n(2 - 2h)\\
2 - 2g &= 2n - 2nh\\
1 - g &= n - nh\\
-g &= n - nh - 1\\
g &= nh - n + 1\\
g &= n(h-1) + 1.
\end{aligned}$$

■

Exercise 2.2.28

(a) Use the Mayer–Vietoris sequence to compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

(b) Do the same for the space obtained by attaching a Möbius band to $\mathbb{RP}^2$ via a homeomorphism of its boundary circle to the standard $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

Exercise 2.2.30

For the mapping torus T_f of a map $f : X \to X$, we constructed in Example 2.48 a long exact sequence $\cdots \to H_n(X) \xrightarrow{\mathbb{1}-f_*} H_n(X) \to H_n(T_f) \to H_{n-1}(X) \to \cdots$. Use this to compute the homology of the mapping tori of the following maps:

(a) A reflection $S^2 \to S^2$.

(b) A map $S^2 \to S^2$ of degree 2.

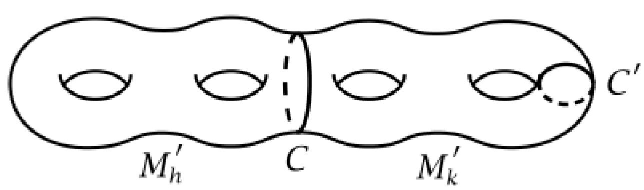
(c) The map $S^1 \times S^1 \to S^1 \times S^1$ that is the identity on one factor and a reflection on the other.

(d) The map $S^1 \times S^1 \to S^1 \times S^1$ that is a reflection on each factor.

(e) The map $S^1 \times S^1 \to S^1 \times S^1$ that interchanges the two factors and then reflects one of the factors.

Exercise 1.2.9

In the surface M_g of genus g, let C be a circle that separates M_g into two compact subsurfaces M_h' and M_k' obtained from the closed surfaces M_h and M_k by deleting an open disk from each. Show that M_h' does not retract onto its boundary circle C, and hence M_g does not retract onto C. [Hint: Abelianize π_1.] But show that M_g *does* retract onto the nonseparating circle C' in the figure.



Proof.

First, we'll show that M_h' does not retract onto its boundary circle C. For contradiction, assume that M_h' does retract onto its boundary circle C with $r : M_h' \to C$. Then there must exist $r_* : \pi_1(M_h') \twoheadrightarrow \pi_1(C)$, but we know $\pi_1(C) \cong \mathbb{Z}$, so $r_* : \pi_1(M_h') \twoheadrightarrow \mathbb{Z}$. We also know $\pi_1(M_h') = \langle a_1, b_1, \ldots, a_h, b_h, c | \prod_{i=1}^{h}[a_i, b_i] = c\rangle$, but after abelianizing, $H_1(M_h') \cong \mathbb{Z}^{2h} \oplus \mathbb{Z}$. But $H_1(C) \cong \mathbb{Z}$, so that M_h' cannot retract onto C.
To show that M_g does retract onto C', we can imagine a cut along C', then collapsing on a segment orthogonal to it, then regluing, so that it can deform onto C'. ■

Exercise 2.C.5

Let M be a closed orientable surface embedded in $\mathbb{R}^3$ in such a way that reflection across a plane P defines a homeomorphism $r : M \to M$ fixing $M \cap P$, a collection of circles. Is it possible to homotope

r to have no fixed points?

Proof.

We can find the Lefshetz number by noting that $H_0(M) \cong \mathbb{Z}$ with trace 1, $H_1(M) \cong \mathbb{Z}^{2g}$ with an unknown trace, call it t, and $H_2(M) \cong \mathbb{Z}$ with trace -1. So the Lefshetz number is $-t$, so it can be zero, but that doesn't guarantee the existence of such a homotopy. Because r is a reflection and $M \cap P$ is fixed, these loops are not hull-homotopic, so that there must exist some fixed points. So no, it is not possible to homotope r to have no fixed points. ■

Exercise 2.C.6

Do an even-genus analog of Example 2C.4 by replacing the central torus by a sphere, letting f be a homeomorphism that restricts to the antipodal map on this sphere.