MATH 8150 Homework 2

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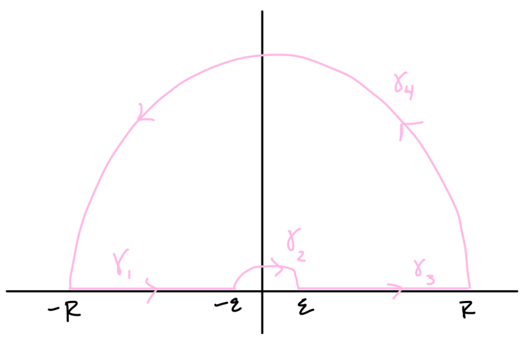
Problem (Stein) 2. Show that

$$\int_0^\infty \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

Proof. First, consider the following function $f(z) = \frac{e^{iz}}{z}$ and contour γ:



Because the function is holomorphic on the closed contour, we can apply Cauchy's theorem to find that

$$\oint_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz = 0.$$

First, we evaluate the integrals over real numbers:

$$\int_{\gamma_1} f(z)dz + \int_{\gamma_3} f(z)dz = \int_{-R}^{-\varepsilon} f(z)dz + \int_\varepsilon^R f(z)dz$$

Using u-subsitution and switch variables, we get

$$\begin{aligned}
\int_\varepsilon^R \frac{e^{ix}}{x} dx - \int_\varepsilon^R \frac{e^{-ix}}{x} dx &= \int_\varepsilon^R \frac{e^{ix} - e^{-ix}}{x} dx \\
&= 2i \int_\varepsilon^R \frac{e^{ix} - e^{-ix}}{(2i)x} dx \\
&= 2i \int_\varepsilon^R \frac{\sin(x)}{x} dx \\
&\to 2i \int_0^\infty \frac{\sin(x)}{x} dx \text{ as } R \to \infty \text{ and } \varepsilon \to 0.
\end{aligned}$$

Next, we evaluate the γ_2 integral with the following parametrization:

$$\begin{aligned} z &= \gamma(t) = \varepsilon e^{i(\pi - t)} \text{ where } t \in [0, \pi] \\ dz &= \gamma'(t)dt \\ &= -i\varepsilon e^{i(\pi-t)}dt \\ &= -i\gamma(t)dt \end{aligned}$$

So then

$$\begin{aligned} \int_{\gamma_2} \frac{e^{iz}}{z} dz &= \int_0^\pi \frac{e^{i\gamma(t)}}{\gamma(t)} dt \\ &= -i \int_0^\pi e^{i\gamma(t)} dt \\ &= -i \int_0^\pi e^{i\varepsilon e^{i(\pi-t)}} dt \end{aligned}$$

As $\varepsilon \to 0$, this approaches

$$-i \int_0^\pi 1 \cdot dt = -i\pi.$$

For the last integral, use the following parametrization:

$$\begin{aligned} z &= \gamma(t) = Re^{it} \\ dz &= \gamma'(t)dt \\ &= Re^{it}dt \\ &= \gamma(t)dt \end{aligned}$$

So we have

$$\begin{aligned} \int_{\gamma_4} f(z)dz &= \int_0^\pi \frac{e^{i\gamma(t)}}{\gamma(t)} \gamma(t)dt \\ &= \int_0^\pi e^{i\gamma(t)} dt \\ &= \int_0^\pi e^{iRe^{it}} dt \\ &= \int_0^\pi e^{iR(\cos(t) + i\sin(t))} dt. \end{aligned}$$

Then note that

$$\begin{aligned} \left| \int_0^\pi e^{iR(\cos(t)+i\sin(t))} dt \right| &\leq \int_0^\pi \left| e^{iR(\cos(t)+i\sin(t))} \right| dt \\ &= \int_0^\pi \left| e^{iR\cos(t)} \right| \left| e^{-R\sin(t)} \right| dt \\ &= \int_0^\pi 1 \cdot \left| e^{-R\sin(t)} \right| dt \\ &= \int_0^\pi e^{-R\sin(t)} dt \\ &\to 0 \text{ as } R \to \infty. \end{aligned}$$

So we find that this integral must be 0. Finally, we have

$$\begin{aligned}\oint_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + \int_{\gamma_3} f(z)dz + \int_{\gamma_4} f(z)dz \\ &= 2i\int_0^\pi \frac{\sin(x)}{x}dx - i\pi + 0 \\ &= 0.\end{aligned}$$

Thus,

$$\int_0^\pi \frac{\sin(x)}{x}dx = \frac{\pi}{2}.$$

□

Problem (Tie) 2. Let f be any power series centered at the origin. Prove that f has a power series expansion around any point in it's disc of convergence.

Proof. By Cauchy's integral formula, we know that

$$f(z) = \int \frac{f(\xi)}{\xi - z}d\xi.$$

We can rearrange this to be

$$\int f(\xi) \cdot \frac{1}{\xi - z + z_0 - z_0}$$

where z_0 is any point in the disc of convergence of f.

Let $w = \frac{z-z_0}{\xi-z_0}$ so that

$$\int \frac{f(\xi)}{\xi - z_0} \cdot \frac{1}{1-w} = \int \frac{f(\xi)}{\xi - z_0} \sum_{n\geq 0} w^n d\xi.$$

From here, we can bring the integral into the sum:

$$\sum_{n\geq 0} \int \frac{f(\xi)}{\xi - z_0}d\xi \cdot w^n$$

Then note that

$$\begin{aligned}\sum_{n\geq 0} \int \frac{f(\xi)}{\xi - z_0}d\xi \cdot w^n &= \sum_{n\geq 0} \int \frac{f(\xi)}{\xi - z_0}d\xi \cdot \frac{(z-z_0)^n}{(\xi - z_0)^n} \\ &= \sum_{n\geq 0} \int \frac{f(\xi)}{(\xi - z_0)^{k+1}}d\xi \cdot (z-z_0)^k\end{aligned}$$

Thus we are left with a power series centered at z_0, an arbitrary point in the disc of convergence. □

Problem (Tie) 3. Prove the following:

(a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

(b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

(c) The power series $\sum^{\infty} n = 1 \frac{z^n}{n}$ converges at every point of the unit circle except at $z = 1$.

Proof. (a) If the point is on the unit circle, we know that $|z| = 1$. Thus

$$|nz^n| = |n| \to \infty.$$

so that the series does not converge.

(b) Because

$$\begin{aligned}\left|\sum \frac{z^n}{n^2}\right| &\leq \sum \left|\frac{z^n}{n^2}\right| \\ &= \sum \left|\frac{1}{n^2}\right| \cdot |z^n| \\ &= \sum \left|\frac{1}{n^2}\right| \\ &< \infty,\end{aligned}$$

we know the series converges absolutely, which implies convergence.

(c)

$\square$

Problem (Tie) 4. Don't use the Cauchy integral formula. Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi 1}{\alpha - \beta}$$

where γ denotes the circle centered at the origin, of radius r, with positive orientation.

Proof. First, we can rewrite the integral like so using partial fractions:

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{1}{a-b}\left(\int_\gamma \frac{dz}{z-a} - \int_\gamma \frac{dz}{z-b}\right)$$

Then because α is inside γ, we can say

$$\int_\gamma \frac{dz}{z-\alpha} = 2\pi i.$$

Similarly, because β is outside γ, we know

$$\int_\gamma \frac{dz}{z-\beta} = 0$$

Thus

$$\int_\gamma \frac{dz}{(z-\alpha)(z-\beta)} = \frac{2\pi 1}{\alpha - \beta}.$$

$\square$

Note: I started this assignment way too late, I have definitely learned my lesson :(