Homework 3

February 25, 2025

Problem 1. Show that composition of paths satisfies the following cancellation property: If $f_0 \circ g_0 \cong f_1 \circ g_1$ and $g_0 \cong g_1$, then $f_0 \cong f_1$.

Proof. Let $g_0 \cong g_1$, so that there exists a homotopy G_t such that it is continuous, and $G_0 = g_0$ and $G_1 = g_1$. Similarly, there exists a continuous H_t such that $H_0 = f_0 \circ g_0$ and $H_1 = f_1 \circ g_1$. Then define $G_t^*(x) = G_t(1-x)$ so that $G_0^*(x) = G_0(1-x) = g_0(1-x) = \bar{g_0}(x)$ and $G_1^*(x) = G_1(1-x) = g_1(1-x) = \bar{g_1}(x)$. Clearly, G_t^* is continuous, so $\bar{g_0} \cong \bar{g_1}$.

Note that $g_0 \circ \bar{g_0}$ and $g_1 \circ \bar{g_1}$ are the identity map, because they follow the path g and then go back to the first endpoint when following $\bar{g}$. Then consider continuous $H_t \circ G_t^*$, where $H_0 \circ G_0^* = f_0 \circ g_0 \circ \bar{g_0} = f_0$ and $H_1 \circ G_1^* = f_1 \circ g_1 \circ \bar{g_1} = f_1$. Thus $f_0 \cong f_1$. □

Problem 2. Show that the change-of-basepoint homomorphism β_h depends only on the homotopy class of h.

Proof. Consider $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ and $\beta_{h'} : \pi_1(X, x_1) \to \pi_1(X, x_0)$, where the two paths, h and h', exist in the same homotopy class. Both h and h' must have the same endpoints, x_0 and x_1, and there must exist a homotopy H_t, where $H_0 = h$ and $H_1 = h'$. Thus for all $t \in [0, 1]$, $H_t(x_1) = (x_0)$. Because β_h maps $[f]$ to $[h \circ f \circ \bar{h}]$ and $\beta_{h'}$ maps $[f]$ to $[h' \circ f \circ \bar{h}']$, it is sufficient to show that $[h \circ f \circ \bar{h}] \cong [h' \circ f \circ \bar{h}']$.

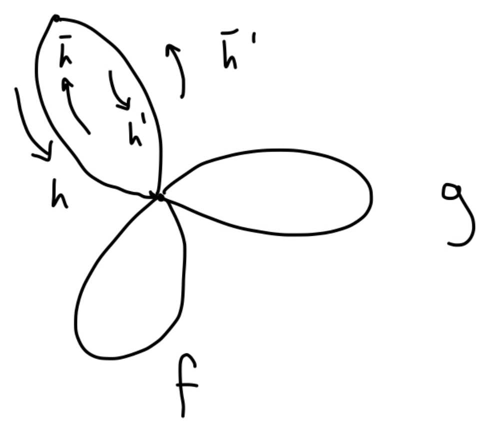
First note that because h and h' are homotopic by H_t, there exists some $\bar{H}_t$ so that $\bar{h}$ and $\bar{h}'$ are homotopic as well. This is because we can define $\bar{H}_t(s) = H_t(1-s)$, so that it is continuous and $\bar{H}_0(s) = H_0(1-s) = h(1-s) = \bar{h}$, and $\bar{H}_1(s) = H_1(1-s) = h'(1-s) = \bar{h}'$. Then define $F_t = H_t(f)$, so that $F_0 = H_0(f) = h \circ f$ and $F_1 = H_1(f) = h' \circ f$, and $h \circ f \cong h' \circ f$. Similarly, we can define $G_t = F_t \circ \bar{H}_t$, so that $G_0 = F_0 \circ \bar{H}_0 = h \circ f \circ \bar{h}$, and $G_1 = F_1 \circ \bar{H}_1 = h' \circ f \circ \bar{h}'$. Thus $h \circ f \circ \bar{h} \cong h' \circ f \circ \bar{h}'$, so that their homotopy classes are equal as well. □

Problem 3. For a path-connected space X, show that $\pi_1(X)$ is abelian iff all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

Proof. ($\Longrightarrow$) Assume that $\pi_1(X)$ is abelian, so that for all $f, g \in \pi_1(X)$, $[f][g] = [f \circ g] = [g \circ f] = [g][f]$. This is equivalent to saying that $f \circ g \cong g \circ f$. Let h and h' be paths with the same endpoints, and let $\bar{h}$ and $\bar{h'}$ be thier respective "reverses", like in problem 1. Then we know

$$\begin{aligned} f \circ g &\cong g \circ \bar{h'} \circ h' \circ f \\ &\cong g \circ \bar{h'} \circ h \circ \bar{h} \circ h' \circ f \\ &\cong (g \circ \bar{h'} \circ h) \circ (\bar{h} \circ h' \circ f) \\ &\cong (\bar{h} \circ h' \circ f) \circ (g \circ \bar{h'} \circ h) \end{aligned}$$

Thus $h \circ f \circ g \circ \bar{h} \cong h' \circ f \circ g \circ \bar{h'}$, and so the basepoint-change homomorphisms only depend on the endpoints. Here is a visual of what is happening:



($\Longleftarrow$) Let h be a constant loop and $\bar{h}$ it's reverse, and let $f, g \in \pi_1(X)$. Then note that $f \cong h \circ f \circ \bar{h}$, but h and g have the same endpoints, because both are loops. Thus $f \cong h \circ f \circ \bar{h} \cong g \circ f \circ \bar{g}$. Thus $\pi_1(X)$ is abelian. □

Problem 4. Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.

(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Proof. First note that we already know D^2 is contractible.

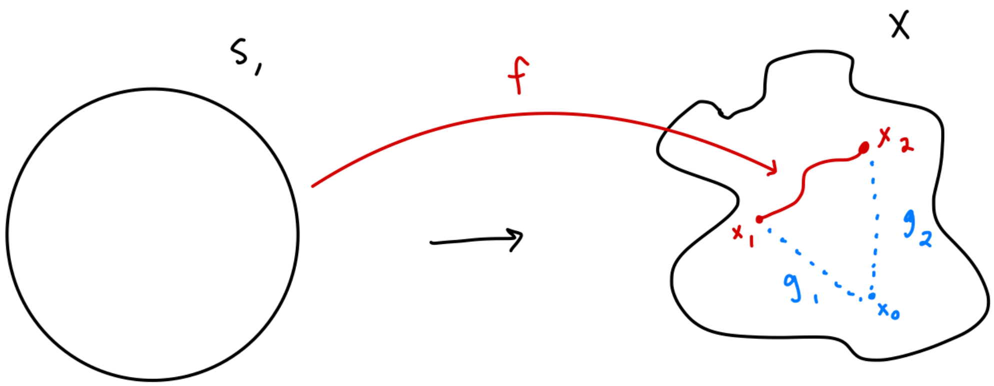
$(a) \implies (b)$ Every map $S^1 \to X$ is homotopic to a constant map, call it $c : S^1 \to x_0$, where $x_0 \in X$. We know D^2 is contractible, so that there exists a homotopy H_t such that $H_0(x) = x$ and $H_1(x) = x_0$. Then we can extend F_t to a map $G_t : D^2 \to X$, where $G_0(x) = f(x)$ and $G_1(x) = c$.

$(b) \implies (c)$ By assumption, any loop on S^1 can be extended to a loop in D^2. Because D^2 is simply connected, we know that any loop in D^2 can be contracted to any point $x_0 \in X$. Thus any loop is homotopic to a constant loop at x_0, and so $\pi_1(X, x_0) = 0$.

$(a) \implies (c)$ First note that the fundamental group being trivial tells us that all loops are homotopic to the constant loop, so all loops are contractible to a point. This is equivalent to saying all maps that are loops are homotopic to a constant function. Thus it is sufficient to show that all maps $S^1 \to X$ are homotopic to loops.

Let f be any map in $S^1 \to X$, with endpoints $f(0) = x_1$ and $f(1) = x_2$. If $x_1 = x_2$, f is a loop, we are done. If $x_1 \neq x_2$, we can simply define $g_1 : [0,1] \to X$ such that $g_1(0) = x_0$ and $g_1(1) = x_1$, and $g_2 : [0,1] \to X$ such that $g_2(0) = x_2$ and $g_2(1) = x_0$. Thus $g_1 \circ f \circ g_2$ is a loop, and by assumption, homotopic to a constant function. Because the homotopy between them must be continuous, f must be homotopic to a constant function as well.

An image to describe what is happening here:



□

Problem 5. Define $f : S^1 \times I \to S^1 \times I$ by $f(\theta, s) = (\theta + 2\pi s, s)$, so f restricts to the identity on the two boundary circles of $S^1 \times I$. Show that f is homotopic to the identity by a homotopy F_t that is stationary on one of the boundary circles, but not by any homotopy F_t that is stationary on both boundary circles.

Proof. □

Problem 6. Show that there are no retractions $\nabla : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.

(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.

(c) $X = S^1 \times D^2$ with A the circle shown in the figure.

(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.

(e) X a disk with two points on its boundary identified with A and its boundary $S^1 \vee S^1$.

(f) X the Mobius band and A its boundary circle.

Proof. $\square$

Problem 7. Let G be a topological group and $e \in G$ be the identity element. Show that $\pi_1(G, e)$ is abelian.

Proof. $\square$

Problem 8. Let $H^1(X) = [X, S^1]$ denote the set of homotopy classes of continuous maps from X to S^1. (There are no basepoints in this discussion)

(a) Recall that S^1 is a topological group. Use the group structure on S^1 to make $H^1(X)$ into a group. Note that this group is abelian for any X.

(b) Compute $H^1(\mathrm{pt})$.

(c) Compute $H^1(S^1)$.

(d) Show that H^1 is functional in the following sense: if $f : X \to Y$ is continuous then there is an induced homomorphism $f^* H^1(Y) \to H^1(X)$. Moreover, if $g : Y \to Z$, then $(g \circ f)^* = f^* \circ g^* : H^1(Z) \to H^1(X)$.

(e) Show that if f g then $f^* = g^*$. Conclude that if $X \cong Y$ then $H^1(X) \cong H^1(Y)$.

(f) Use H^1 to prove that there is no retraction $D^2 \to S^1$, the key step in proving the Brouwer fixed point theorem.

Proof. $\square$