Complex Analysis Notes

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Introduction

Let us begin by noting that every complex number z can be written as $z = x + iy$, where $x, y \in \mathbb{R}$.

Definition 0.1

A function is **holomorphic** at the point $z \in \mathbb{C}$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \text{ where } h \in \mathbb{C}$$

exists.

Question

So is this just being differentiable for complex numbers?

Answer

Essentially, however because complex numbers have a value (radius) and an angle, h can approach 0 from infinitely many angles. So holomorphicity is much stronger than differentiablity; In the real case, it is differentiable going left and right. For a function to be holomorphic at a point, it must be differentiable from infinitely many angles.

Fact 0.1

If f is holomorphic in Ω, then for appropriate closed paths in Ω,

$$\int_\gamma f(z)dz = 0.$$

Fact 0.2

If f is holomorphic, then f is indefinitely differentiable.

Question

Why indefinitely differentiable? Why not indefinitely holomorphic?

Fact 0.3

If f and g are holomorphic functions in Ω which are equal in an arbitrarily samll disc in ω, then $f = g$ everywhere in Ω.

Definition 0.2

The **zeta function**,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is holomorphic in the half-plane $\mathrm{Re}(s) > 1$.

Definition 0.3

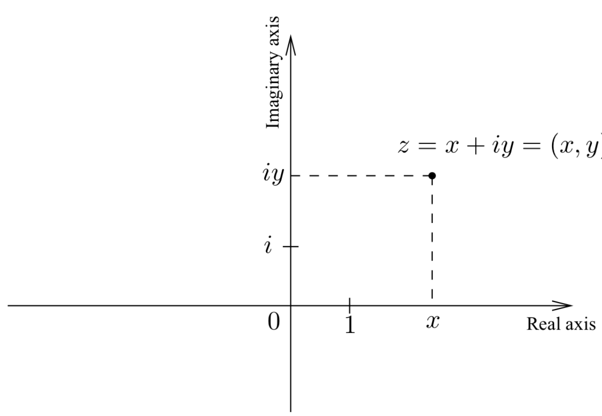
The **theta function** is the following:

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

1 Preliminaries to Complex Analysis

1.1 Complex Numbers and the Complex Plane

We can imagine complex numbers as an ordered pair of the two real numbers:



Addition and mulitplication are defined like so:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$z_1 * z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

It is easy to prove that commutativity, associativity, and distributivity hold for complex numbers. We can think about addition like adding two vectors in $\mathbb{R}^2$, and mulitplication like a rotation and dilation.

Definition 1.1

The length, or absolute value of a complex number, is defined as the following:

$$|z| = (x^2 + y^2)^{1/2}$$

Note that this is the same as taking the norm, or length, of a vector in $\mathbb{R}^2$, or even finding the length of the hypotenuse that is created by the x and y values.

The triangle equality holds:

Theorem 1.1 (Triangle Inequality)

$$|z + w| \leq |z| + |w|$$

for all $z, x \in \mathbb{C}$.

From the triangle inequality, there comes this helpful fact as well:

Fact 1.1

$$||z| - |w|| \leq |z - w|$$

You can imagine a complex conjugate, $\bar{z} = x - iy$, as a reflection across the real (horizontal) axis.

The following are also useful facts easily deduced:

Fact 1.2

$$\text{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

Fact 1.3

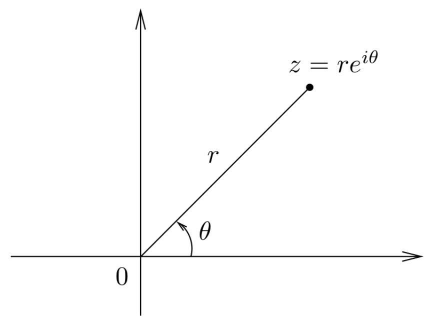
$$|z| = z\bar{z} \text{ and, when } z \neq 0, \ \frac{1}{z} = \bar{z}|z|^2$$

Definition 1.2

A complex number z's **polar form** is written as $z = re^{i\theta}$, where $r > 0$, and θ is referred to as the **argument** of z.

A mathematical fact useful in Complex Analysis is $e^{i\theta} = \cos\theta + i \sin\theta$.

From the two previous statements, we can see that $r = |z|$, the length of z, and θ is the angle.



With this form, we can redefine mulitplication to be:

$$z = re^{i\theta}, w = se^{i\phi} \implies zw = rse^{i(\theta+\phi)}$$

It is easier to see in this definition that mulitplication is simply a rotation $(\theta + \phi)$, and a dilation (rs).

Definition 1.3

A sequence $\{z_1, z_2, \dots\}$ of complex numbers is said to **converge** to $w \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - w| = 0 \text{ or,equivalently, } w = \lim_{n\to\infty} z_n.$$