TDA Homework 1

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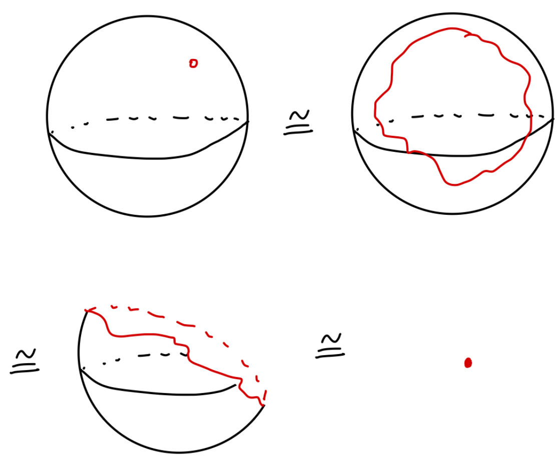
Exercise 1

Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be a continuous map which is not surjective. Prove that it is homotopic to a constant map.

Proof.

If f is not surjective, then there exists some $x \in \mathbb{S}^2$ such that $x \notin f(\mathbb{S}^1)$. So we can consider $f(\mathbb{S}^1)$ to be $\mathbb{S}^2 \setminus \{x\}$, which is homotopy equivalent to a point:



Thus every map in $f(\mathbb{S}^1)$ is homotopic to a constant map. ■

Exercise 2

Let X and Y be two homeomorphic topological spaces. Show that if X has dimension n, then Y also has dimension n.

Proof.

Consider a homeomorphism $f : Y \to X$, and then let $y \in Y$ with $x = f(y)$. Because $x \in X$, x must have dimension n, so that there eists an open set of X, call it O, containing x with a homeomorphism $h : O \to \mathbb{R}^n$. Then, we can let $O' = f^{-1}(O)$, and note that it is an open set of Y containing y. We also know that $h \circ g : O' \to \mathbb{R}^n$ is a homeomorphism, and so Y has dimension n. ■

Exercise 3

Let $(G, +)$ be a group. Prove that

$$\forall g \in G, g + g = 0 \Rightarrow G \text{ is commutative.}$$

Proof.

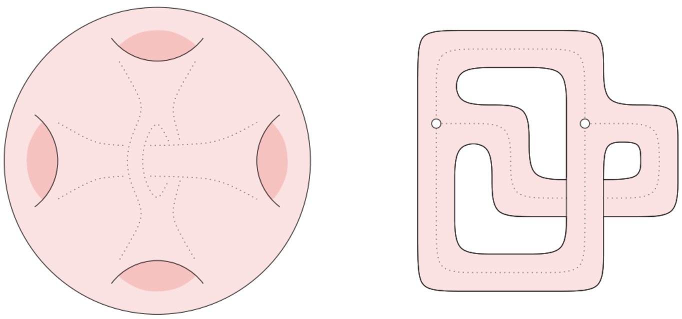
Let $g_1, g_2 \in G$, and note that $g + g = 0 \iff g = -g$. Then

$$\begin{aligned}(g_1 + g_2) + (g_1 + g_2) &= 0\\ g_1 + g_2 &= -(g_1 + g_2)\\ g_1 + g_2 &= (-g_2) + (-g_1)\\ g_1 + g_2 &= g_2 + g_1\end{aligned}$$

Thus the group is commutative. ■

Exercise 4

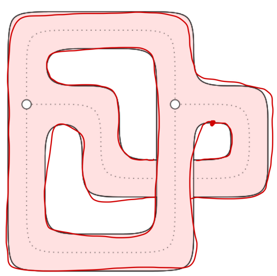
Characterize the two surfaces depicted in Figure 1 in terms of genus, boundary, and orientability.



Proof.

For the first surface, it clearly has no boundary and is non orientable. It's genus is 3, which can be found by using the Euler characteristic, or by seeing the 3 handles, which in the figure presented, are the tunnels in the middle.

For the second surface, it has boundary because, as this diagram shows,



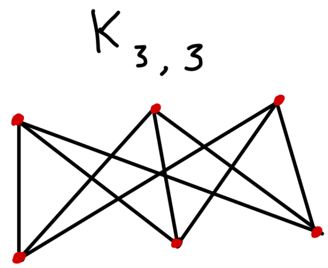
you can see there is only one continuous line on the entire boundary. The surface is orientable as well, as there is a distinct top and bottom. As for the genus, we can use the equation $\chi = 2 - n - b$, where χ is the Euler characteristic, n is the genus, and b is the boundary. We know χ to be $V - E + F = 2 - 3 + 0 = -1$ and b to be 1, so the genus must be 2. ■

Exercise 5

Is every graph that can be embedded on the Mobius strip planar?

Proof.

No. A counterexample would be $K_{3,3}$, which is not planar:



However, it can be embedded on the Mobius strip. ■