Algebra Qualifying Exam Notes

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1 Group Theory

1.1 Subgroups and Quotient Groups

Definition 1.1

Let G be a group and H a nonempty subset that is closed under the product in G. If H itself is a group under the product in G, then H is said to be a **subgroup** of G. This is denoted by $H \subset G$.

Definition 1.2

A subgroup other than the group itself and the trivial subgroup $\langle e \rangle$ is called a **proper subgroup**

Theorem 1.1

Let H be a nonempty subset of a group G. Then H is a subgroup of G if and only if $ab^{-1} \in H$ for all $a, b \in H$.

Proof.

($\Longrightarrow$) First, assume H is a subgroup of G. Then, by the existence of inverses in subgroups, we know if $b \in H$, $b^{-1} \in H$, and by closure of subgroups, if $a \in H$ as well, then $ab^{-1} \in H$.
($\Longleftarrow$) Next, assume for all $a, b \in H$, $ab^{-1} \in H$. Then, to prove it is a subgroup, we need to show closure, associativity, inverses, and the identity is in H.

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