8200 Homework 6

March 10, 2025

Exercise 1

Suppose X is path connected and $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a path connected covering space of X. Prove that the number of sheets of this covering space is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

Proof.

Let f be any loop with basepoint x_0, so that $\tilde{f}$ is it's lift, where X cooresponds to $\tilde{X}$ and x_0, $\tilde{x_0}$. Let $g \in G = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, so that $g \circ f$ has the lift $\tilde{g} \circ \tilde{f}$. Note that because $\tilde{g}$ is a loop, $\tilde{g} \circ \tilde{f}$ ends at the same point as $\tilde{f}$. Then define a function $\phi : G[f] \to p^{-1}(x)$, where $G[f] \mapsto \tilde{f}(1)$. Because $\tilde{X}$ is path connected, ϕ is surjective. Then note that because $\phi(G[f_1]) = \phi(G[f_2])$ implies that $f_1 \circ \bar{f}_2$ lifts to a loop based at $\tilde{x_0}$, so that $[f_1][f_2]^{-1} \in G$, and $g[f_1] = G[f_2]$, so that ϕ is injective. Thus the number of cosets (index) is equal to the number of sheets. ■

Exercise 2

Construct nonnormal covering spaces of the Klein Bottle by a Klein bottle and by a torus.

Proof.

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Exercise 3

Let X be the space obtained from a torus $S^1 \times S^1$ by attaching a Mobius band via a homeomorphism from the boundary circle of the Mobius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$, describe the universal cover of X, and describe the action of $\pi_1(X)$ on the universal cover. Do the same for the space Y obtained by attaching a Mobius band to $\mathbb{R}P^2$ formed by the 1-skeleton of the usual CW structure on $\mathbb{R}P^2$.

Proof.

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Exercise 4

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\phi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R} - \{0\}$. Show this action is a covering space action and compute $\pi_1(X \setminus \mathbb{Z})$. Show the orbit space $X \setminus \mathbb{Z}$ is non-Hausdorff, and describe how it is a union of four subspaces homeomorphic to $S^1 \times \mathbb{R}$, coming from the complementary components of the x-axis and the y-axis.

Proof.

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Proof.

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Exercise 6

Proof.

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