

# Topology Qual Cheat Sheet

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Spring 2025

## Point Set Topology Definitions

### Definition

If  $X$  is a topological space with topology  $\mathcal{T}$ , we say that a subset  $U$  of  $X$  is an **open set** of  $X$  if  $U$  belongs to the collection  $\mathcal{T}$ .

### Definition

A subset  $A$  of a topological space  $X$  is said to be **closed** if the set  $x - A$  is open.

### Definition

The **closure** of  $A$  is defined as the intersection of all closed sets containing  $A$ .

Casually, I like to think about it as the outer edge of the space, unioned with the space itself. For a closed space, like  $[0, 1]$ , the closure is  $\{0\} \cup \{1\}$ , which is contained in the closed set. Note that all closed sets are equal to the closure of the closed sets; this is because the smallest closed set containing a closed set  $A$  is  $A$  itself.

### Notation

The closure of a set  $A$  is denoted as  $\bar{A}$ .

So for closed sets  $A$ ,  $\bar{A} = A$ . Similarly, we have:

### Definition

The **interior** of  $A$  is defined as the union of all open sets contained in  $A$ .

### Notation

The interior of a set  $A$  is denoted as  $\text{Int}A$ .

For similar logic as before, for open sets  $A$ ,  $\text{Int}A = A$ .

### Definition

If  $A$  is a subset of the topological space  $X$  and if  $x$  is a point of  $X$ . we say that  $x$  is a **limit point/ cluster point/ point of accumulation** of  $A$  if every neighborhood of  $x$  intersects  $A$  in some point other than  $x$  itself.

You can also say  $x$  is a limit point of  $A$  if  $x$  is in the closure of  $A - \{x\}$ .

A limit point is just a point on the boundary on a subspace  $A$ , although it doesn't necessarily have to be in  $A$  itself. The limit points are kind of the boundary part of the

closure. Hence the following theorem:

**Theorem 0.1**

Let  $A$  be a subset of the topological space  $X$ ; let  $A'$  be the set of all the limit points of  $A$ . Then

$$\bar{A} = A \cup A'.$$

One can then consider the relationship between limit points and a set being closed. If the "boundary" of a set is made up of limit points, one can see that:

**Theorem 0.2**

A subset of a topological space is closed if and only if it contains all its limit points.

**Definition**

A collection  $\mathcal{A}$  of subsets of a space  $X$  is said to **cover**  $X$ , or to be a **covering** of  $X$ , if the union of the elements of  $\mathcal{A}$  is equal to  $X$ . It is called an **open covering** of  $X$  if its elements are open subsets of  $X$ .

**Definition**

A Space  $X$  is said to be **compact** if every open covering  $\mathcal{A}$  of  $X$  contains a finite subcollection that also covers  $X$ .

**Definition**

A topological space  $X$  is called a **Hausdorff space** if for each pair  $x_1, x_2$  of distinct points of  $X$ , there exist neighborhoods  $U_1$  and  $U_2$  of  $x_1$  and  $x_2$  respectively that are disjoint.

**Definition**

Let  $X$  be a topological space. A **separation** of  $X$  is a pair  $U, V$  of disjoint nonempty open subsets of  $X$  whose union is  $X$ . A space  $X$  is said to be **connected** if there does not exist a separation of  $X$ .

This is really as simple as it sounds. Examples:

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**Definition**

A space is called **path connected** if every pair of points in  $X$  can be joined by a path in  $X$

**Question**

So how is there a difference between being path connected and connected? Shouldn't being connected, so that the space has no disjoint parts, be enough to say that a path can be drawn from point to point?

**Answer**

The biggest example is the **Topologists sine curve**:

$$y(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

This is a curve that is connected, but **not** path connected. Proof of connectedness:

I'm not exactly sure why this works, but my thoughts are just because you technically can't find any separation between the two parts, because the limit of  $\sin \frac{1}{x}$  as  $x \rightarrow 0$  from the right is 0; however, because there is not actually any connected, you cannot draw a path. At least that's what I'm getting from this right now.

# Algebraic Topology Definitions

## Definition

A map is **nullhomotopic** if it is homotopic to a constant map.

## Definition

A space is **contractible** if it is homotopically equivalent to a point.

So it's contractible if it can homotopically be squeezed into a point.

## Theorem 0.3

A space is contractible if and only if its identity map is nullhomotopic.

## Proof.

- ( $\Rightarrow$ ) Assume a space  $X$  is contractible, so that  $X$  is homotopically equivalent to a point  $x_0$ . Then there exists maps  $f : X \rightarrow x_0$ ,  $g : x_0 \rightarrow X$  such that  $g \circ f \cong \text{id}_X$ . But  $g \circ f : X \rightarrow X, x \mapsto x_0$ , making it a constant map.
- ( $\Leftarrow$ ) If the identity map is nullhomotopic, it is homotopic to a constant map. This is equivalent to saying there exists some  $F(x, t)$  such that  $F(x, 0) = \text{id}_X$  and  $F(x, 1) = x_0 \forall x \in X$ . This homotopy describes the space  $X$  contracting to the point  $x_0$  continuously.

■

The backwards direction of this proof seems hand-wavey, so I may need to go back and look at this.

## Definition

A space is called **simply connected** if it is path connected and has trivial fundamental group

So a space is simply connected if there's nothing on the inside of it that causes loops to get "caught" on things. The first example that comes to my mind is how  $S^2$  is simply connected, because it is path connected clearly, and any loop on  $S^2$  can be contracted to a point (trivial fundamental group). On the other hand, the torus is path connected, but a loop could be caught around the donut hole in the middle, unable to contract, resulting in a nontrivial fundamental group ( $\mathbb{Z} \times \mathbb{Z}$ , specifically).

## Definition

For a space  $X$  to be **locally path-connected**, for each point  $x \in X$  and each neighborhood  $U$  of  $x$ , there must be an open neighborhood  $V \subset U$  of  $x$  that is path-connected.

A good example of this would be two disjoint discs. Clearly, the space itself isn't path

connected, but the two components are path-connected themselves, so the space is locally path-connected.

### Question

So why is  $p$  from  $\tilde{X}$  to  $X$ ? To me, it seems more natural to assign  $x \in X$  to somewhere in the cover.

### Answer

A cover could have more than one sheet, so that each  $x \in X$  maps to multiple  $\tilde{x} \in \tilde{X}$ . Thus  $p$  would not be a function if it went from  $X$  to  $\tilde{X}$  for multi-sheeted covering spaces.

### Theorem 0.4 (Homotopy Lifting Property)

Given a covering space  $p : \tilde{Y} \rightarrow Y$ , a homotopy  $f_t : X \rightarrow Y$ , and a map  $\tilde{f}_0 : X \rightarrow \tilde{Y}$  lifting  $f_0$ , then there exists a unique homotopy  $\tilde{f}_t : X \rightarrow \tilde{Y}$  of  $\tilde{f}_0$  that lifts  $f_t$ .

### Theorem 0.5 (Lifting Criterion)

Suppose given a covering space  $p : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$  and a map  $f : (X, x_0) \rightarrow (Y, y_0)$  with  $X$  path-connected and locally path-connected. Then a lift  $\tilde{f} : (X, x_0) \rightarrow (\tilde{Y}, \tilde{y}_0)$  of  $f$  exists if and only if  $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ .

### Question

So what is  $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ ?

### Answer

First note that  $(\tilde{Y}, \tilde{y}_0)$  is the covering space of  $Y$  and it's matching basepoint  $y_0$  in the cover. Obviously,  $\pi_1(\tilde{Y}, \tilde{y}_0)$  is the fundamental group of this covering space, and  $p : (\tilde{Y}, \tilde{y}_0) \rightarrow (Y, y_0)$  is the map that describes how the covering space covers  $Y$ , so  $p_*$  is the induced homomorphism of the fundamental groups. So  $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$  are the values in the codomain that are actually mapped to, or  $\text{img}(p_*)$ . Note that  $\text{img}(p_*) \subset \pi_1(Y, y_0)$ , just like  $f_*(\pi_1(X, x_0)) = \text{img}(f_*) \subset \pi_1(Y, y_0)$ , which makes sense.

Given this, we can look at this diagram to see the relationship between  $X, Y$  and  $\tilde{Y}$ :

$$\begin{array}{lcl}
 f : X & \rightarrow & Y \\
 p : \tilde{Y} & \rightarrow & Y \\
 \tilde{f} : X & \rightarrow & \tilde{Y}
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \tilde{Y} & \\
 \tilde{f} \nearrow & & \downarrow p \\
 X & \xrightarrow{f} & Y
 \end{array}$$

And there is a similar setup for their respective fundamental groups:

$$\begin{array}{lcl}
 f_* : \pi_1(X) & \rightarrow & \pi_1(Y) \\
 p_* : \pi_1(\tilde{Y}) & \rightarrow & \pi_1(Y) \\
 \tilde{f}_* : \pi_1(X) & \rightarrow & \pi_1(\tilde{Y})
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{\tilde{f}_*} & \pi_1(\tilde{Y}) \\
 \pi_1(X) & \xrightarrow{f_*} & \pi_1(Y) \\
 & \searrow p_* &
 \end{array}$$

**Fact 0.1**

By the diagrams above, we can see  $f = p \circ \tilde{f}$  and  $f_* = p_* \circ \tilde{f}_*$ .

**Question**

So what determines whether or not we can assume that  $f$  is this composition? Is there some property to satisfy?

# Qualifying Exams

Spring 2025

## Problem 1

Prove that any map  $\mathbb{R}P^2 \rightarrow S^1 \times S^1$  is nullhomotopic. Prove that there exists a map  $S^1 \times S^1 \rightarrow \mathbb{R}P^2$  which is not nullhomotopic.

## Proof.

The universal cover of  $S^1 \times S^1$  is  $\mathbb{R} \times \mathbb{R}$ . So if  $f : \mathbb{R}P^2 \rightarrow S^1 \times S^1$ , the lifted map  $\tilde{f} : \mathbb{R}P^2 \rightarrow \mathbb{R} \times \mathbb{R}$  is nullhomotopic, since  $\mathbb{R} \times \mathbb{R}$  is contractible. If the lifted map  $\tilde{f}$  is nullhomotopic, then the map  $f$  must be. So all that is left to show is that we can lift the map, so we must meet the lifting criterion.

The fundamental group of  $S^1 \times S^1$  is  $\mathbb{Z} \times \mathbb{Z}$ , and the fundamental group of  $\mathbb{R}P^2$  is  $\mathbb{Z}/2\mathbb{Z}$ , as can be seen by the following diagrams:

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Then  $f$  induces a homomorphism  $f_* : \pi_1(\mathbb{R}P^2) \rightarrow \pi_1(S^1 \times S^1)$ , which is clearly trivial. Then we know  $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ , because  $f_*(\pi_1(X, x_0)) = f_*(\mathbb{Z}/2\mathbb{Z}) = 0$ , and  $p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) = p_*(\pi_1(\mathbb{R} \times \mathbb{R})) = p_*(0) = 0$ . We must also show that  $\mathbb{R}P^2$  is path-connected and locally path-connected, but this is obvious as it is a quotient space of  $S^2$ , and  $S^2$  has those properties. Thus the lifting criterion is met, and  $f$  must be nullhomotopic. ■

## Question

Why do we need the fundamental groups? Why can't we just say that the lifted map is nullhomotopic, so the normal map must be to?

## Answer

We have to prove the lift exists, and so we must satisfy the lifting criterion.

## Question

Why does any map  $f : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  have to be trivial?

## Answer

We know that  $\mathbb{Z}/2\mathbb{Z}$ , or  $\mathbb{Z}_2$  has two elements,  $\{0, 1\}$ . We know because  $f$  is a homomorphism,  $f(0) = 0$ . Then  $f(1)$  must satisfy  $f(1 + 1) = f(1) + f(1)$ , but  $1 + 1 = 0$  in  $\mathbb{Z}_2$ , so we have  $f(1) + f(1) = 0$ , and because  $\mathbb{Z} \times \mathbb{Z}$  is **torsion-free** (there are no non-identity elements that have finite order (so like there are no non-identity elements that can every generate the identity by themselves)), the only values  $f(1)$  can take on to follow this restriction is 0. Thus every element in  $\mathbb{Z}_2$  maps to 0, and  $f$  must be trivial.

**Question**

So why does the lift being nullhomotopic imply the original is nullhomotopic?

**Answer**

So we know  $\tilde{f} \cong c_1$ , where  $c_1$  is a constant map. But  $f = p \circ \tilde{f} \cong p \circ c_1$ , so for  $f$  to be nullhomotopic,  $p$  must be nullhomotopic too. But because  $p : \mathbb{R} \times \mathbb{R} \rightarrow S^1 \times S^1$ , and  $\mathbb{R} \times \mathbb{R}$  is contractible,  $p$  is nullhomotopic as well, so that  $p \cong c_2$  for some constant map  $c_2$ . Thus  $f \cong c_2 \circ c_1$ , so that  $f$  is nullhomotopic.



# Blank Problem Bank

## Homework 1

### Problem 0.2

Construct an explicit deformation retraction of  $\mathbb{R}^n - \{0\}$  onto  $S^{n-1}$ .

### Problem 0.3

- (a) Show that the composition of homotopy equivalences  $X \rightarrow Y$  and  $Y \rightarrow Z$  is a homotopy equivalence  $X \rightarrow Z$ . Deduce that homotopy equivalence is an equivalence relation.
- (b) Show that the relation of homotopy among maps  $X \rightarrow Y$  is an equivalence relation.
- (c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

### Problem 0.6

- (a) Let  $X$  be the subspace of  $\mathbb{R}^2$  consisting of the horizontal segment  $[0, 1] \times \{0\}$  together with all the vertical segments  $\{r\} \times [0, 1 - r]$  for  $r$  a rational number in  $[0, 1]$ . Show that  $X$  deformation retracts to any point in the segment  $[0, 1] \times \{0\}$ , but not to any other point. [See the preceding problem]
- (b) Let  $Y$  be the subspace of  $\mathbb{R}^2$  that is the union of an infinite number of copies of  $X$  arranged as in the figure below. Show that  $Y$  is contractible but does not deformation retract onto any point.
- (c) Let  $Z$  be the zigzag subspace of  $Y$  homeomorphic to  $/R$  indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of  $Y$  onto  $Z$ , but no true deformation retraction.

### Problem 0.10

Show that a space  $X$  is contractible if and only if every map  $f : X \rightarrow Y$ , for arbitrary  $Y$ , is nullhomotopic. Similarly, show  $X$  is contractible if and only if every map  $f : Y \rightarrow X$  is nullhomotopic.

### Problem 0.11

Show that  $f : X \rightarrow Y$  is a homotopy equivalence if there exist maps  $g, h : Y \rightarrow X$  such that  $fg \cong \text{id}$  and  $hf \cong \text{id}$ . More generally, show that  $f$  is a homotopy equivalence if  $fg$  and  $hf$  are homotopy equivalences.

**Problem 0.16**

Show that  $S^\infty$  is contractible.

**Problem 0.17**

Construct a 2-dimensional cellcomplex that contains both an annulus  $S^1 \times I$  and a Mobius band as deformation retractions.

**Problem 0.20**

Show that the subspace  $X \subset \mathbb{R}^3$  formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to  $S^1 \vee S^1 \vee S^1$ .

