Algebraic Topology Homework 2

January 24, 2025

Exercise 1. Describe a CW complex structure on $\mathbb{C}P^2 \times \mathbb{R}P^2$ and ΣT^2.

Proof. For $\mathbb{C}P^2 \times \mathbb{R}P^2$, we know that $\mathbb{R}P^2$ consists of 1 0-cell, 1 1-cell, and 1 2-cell. Similarly, for $\mathbb{C}P^2$, it consists of 1 1-cell, 1 2-cell, and 1 4-cell. Thus the product of these spaces consists of 1 0-cell, 1 1-cell, 2 2-cell, 1 3-cell, 2 4-cell, 1 5-cell, and 1 6-cell. Visually, we're finding the product of a sphere and a 4th dimensional shape.

For ΣT^2, first note that $T^2 = S^1 \times S^1$, the torus. In Hatcher, it is stated that $\Sigma X = X \wedge S^1$, so in our case, $\Sigma(S^1 \times S^1) = (S^1 \times S^1) \wedge S^1$. Also in Hatcher, $X \wedge Y = X \times Y / X \vee Y$. So finally, we have $(S^1 \times S^1) \times S^1 / (S^1 \times S^1) \vee S^1$. Visually, we can imagine this as a torus crossed with S^1 quotient by a torus touching a circle. Regarding cell complexes, we have $(e^0 \cup e^1 \cup e^1 \cup e^1 \cup e^2 \cup e^2 \cup e^2 \cup e^3)/(e^0 \cup e^0 \cup e^1 \cup e^1 \cup e^1 \cup e^2)$. Thus the reduced suspension of a torus has the CW structure of 1 1-cell, 2 2-cells, and 1 3-cell. □

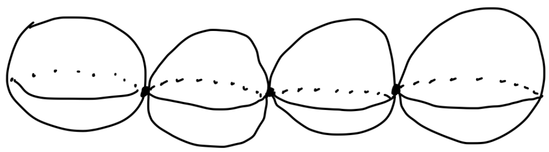
Exercise 2. Let X_n be the topological space obtained by identifying $n > 1$ points on S^2 to a single point. Describe a CW decomposition of X_n.

Proof. When you identify a point with a singular point, you're pinching the S^2 sphere at that singular point, and it creates a "loop" from where the point originally was to the singular point. This occurs for every point identified, so that there are $n-1$ loops (or S^1's) added to the space. Note that we need loops, because if we were to do lines, some lines could intersect. Thus we can say $X_n = S^2 \vee \bigvee_{i=1}^{n-1} S^1$, which is just stating that X_n is a sphere with loops added that intersect at a single point (the singular point). When considering CW decomposition, the S^2 has 1 0-cell and 1 2-cell, while each S^1 has a 0-cell and a 1-cell. However, the singular point can be made the 0-cell for the S^2 and the S^1's, so in total there is 1 0-cell, 1 2-cell, and $n-1$ 1-cells. □

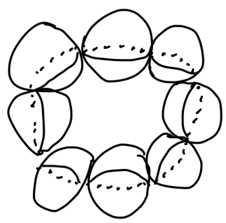
Exercise 3. Hatcher Exercise 0.21: If X is a connected Hausdorff space that is a union of a finite number of 2-spheres, any two of which intersect in at most one point, show that X is homotopy equivalent to a wedge sum of S^1's and S^2's.

Proof. First, because the space is connected, we know that there aren't any disjoint 2-spheres. Then, we can imagine this space made up of S^2's which intersect at most 1 point, and because there can't be any disjoint 2-spheres, they must all be in a row like this, or in a loop:

row:

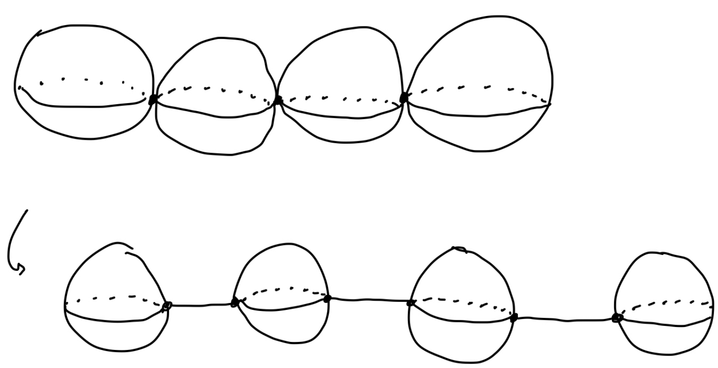


loop:

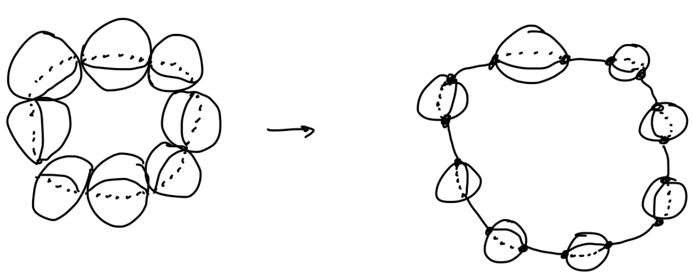


Either way, the intersection points between the spheres can be stretched into lines between them, like this:

row:



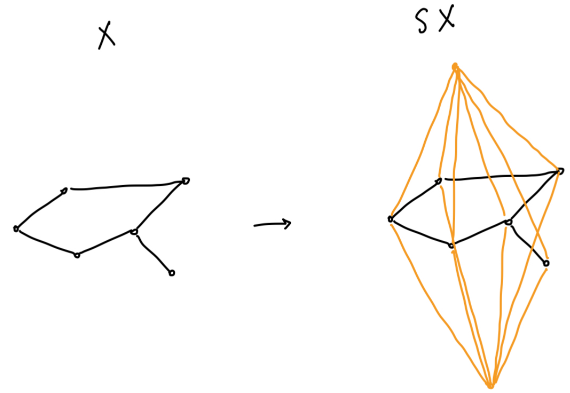
loop:



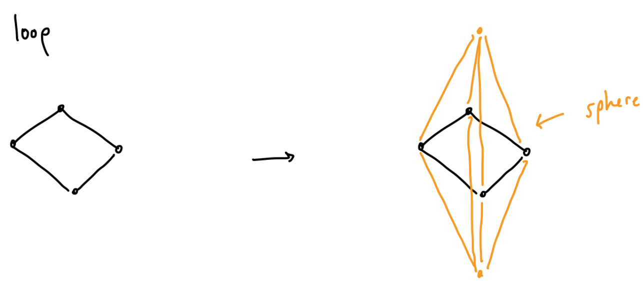
As you can see, the space X can be stretched (and is therefore homotopically equivalent to) to be $\bigvee_{i=1}^{n-1} S^1 \vee \bigvee_{i=1}^{n} S^2$ where n is the number of 2-sphere's in X. $\square$

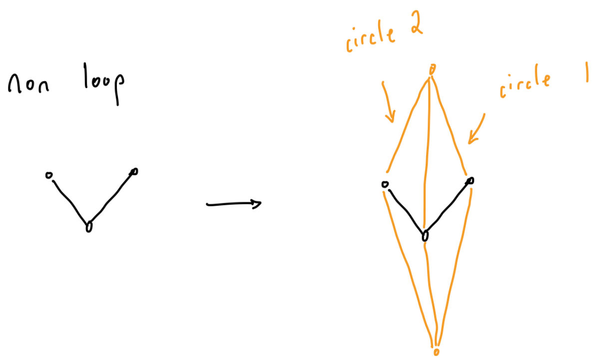
Exercise 4. Hatcher Exercise 0.25: If X is a CW complex with components X_α, show that the suspension SX is homotopy equivalent to $Y \vee_\alpha SX_\alpha$ for some graph Y. In the case that X is a finite graph, show that SX is homotopy equivalent to a wedge sum of circles and 2-spheres.

Proof. If X is a finite graph, the suspension SX is just taking all the vertices in the graph and connecting them to a point "above" and "below" the graph. This results in something like this:



Note that when you have a closed loop in a graph, when you do the suspension it results in a sphere, and when there is a non-loop, it just creates circles:





Because everything in a graph is either a loop or a nonloop, this accounts for what makes up any graph X. Thus if X is a graph, SX is just a wedge sum of spheres and circles. $\square$