Algebraic Topology Qualifying Exam Cheat Sheet

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Point Set Topology Definitions

Definition

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition

A subset A of a topological space X is said to be **closed** if the set $x - A$ is open.

Definition

The **closure** of A is defined as the intersection of all closed sets containing A.

Casually, I like to think about it as the outer edge of the space, unioned with the space itself. For a closed space, like $[0, 1]$, the closure is $\{0\} \cup \{1\}$, which is contained in the closed set. Note that all closed sets are equal to the closure of the closed sets; this is because the smallest closed set containing a closed set A is A itself.

Notation

The closure of a set A is denoted as $\bar{A}$.

So for closed sets A, $\bar{A} = A$. Similarly, we have:

Definition

The **interior** of A is defined as the union of all open sets contained in A.

Notation

The interior of a set A is denoted as $\mathrm{Int}A$.

For similar logic as before, for open sets A, $\mathrm{Int}A = A$.

Definition

If A is a subset of the topological space X and if x is a point of X. we say that x is a **limit point/ cluster point/ point of accumulation** of A if every neighborhood of x intersects A in some point other than x itself.
You can also say x is a limit point of A if x is in the closure of $A - \{x\}$.

A limit point is just a point on the boundary on a subspace A, although it doesn't necessarily have to be in A itself. The limit points are kind of the boundary part of the closure. Hence the following theorem:

Theorem 0.1

Let A be a subset of the topological space X; let A' be the set of all the limit points of A. Then

$$\bar{A} = A \cup A'.$$

One can then consider the relationship between limit points and a set being closed. If the "boundary" of a set is made up of limit points, one can see that:

Theorem 0.2

A subset of a topological space is closed if and only if it contains all its limit points.

Definition

A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition

A Space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Definition

A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Definition

Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. A space X is said to be **connected** if there does not exist a separation of X.

This is really as simple as it sounds. Examples:
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Definition

A space is called **path connected** if every pair of points in X can be joined by a path in X

Question

So how is there a difference between being path connected and connected? Shouldn't being connected, so that the space has no disjoint parts, be enough to say that a path can be drawn from point to point?

Answer

The biggest example is the **Topologists sine curve**:

$$y(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

This is a curve that is connected, but **not** path connected. Proof of connectedness: I'm not exactly sure why this works, but my thoughts are just because you technically can't find any separation between the two parts, because the limit of $\sin \frac{1}{x}$ as $x \to 0$ from the right is 0; however, because there is not actually any connected, you cannot draw a path. At least that's what I'm getting from this right now.

Algebraic Topology Definitions

Definition

A map is **nullhomotopic** if it is homotopic to a constant map.

Definition

A space is **contractible** if it is homotopically equivalent to a point.

So it's contractible if it can homotopically be squeezed into a point.

Theorem 0.3 (A)

space is contractible if and only if it's identity map is nullhomotopic.

Proof.

($\Longrightarrow$) Assume a space X is contractible, so that X is homotopically equivalent to a point x_0. Then there exists maps $f : X \to x_0$, $g : x_0 \to X$ such that $g \circ f \cong \text{id}_X$. But $g \circ f : X \to X, x \mapsto x_0$, making it a constant map.

($\Longleftarrow$) If the identity map is nullhomotopic, it is homotopic to a constant map. This is equivalent to saying there exists some $F(x, t)$ such that $F(x, 0) = \text{id}_X$ and $F(x, 1) = x_0 \forall x \in X$. This homotopy describes the space X contracting to the point x_0 continuously.

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The backwards direction of this proof seems hand-wavey, so I may need to go back and look at this.

Definition

A space is called **simply connected** if it is path connected and has trivial fundamental group

So a space is simply connected if there's nothing on the inside of it that causes loops to get "caught" on things. The first example that comes to my mind is how S^2 is simply connected, because it is path connected clearly, and any loop on S^2 can be contracted to a point (trivial fundamental group). On the other hand, the torus is path connected, but a loop could be caught around the donut hole in the middle, unable to contract, resulting in a nontrivial fundamental group ($\mathbb{Z} \times \mathbb{Z}$, specifically).