Topology Qual Cheat Sheet

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Point Set Topology Definitions

Definition

If X is a topological space with topology $\mathcal{T}$, we say that a subset U of X is an **open set** of X if U belongs to the collection $\mathcal{T}$.

Definition

A subset A of a topological space X is said to be **closed** if the set $x - A$ is open.

Definition

The **closure** of A is defined as the intersection of all closed sets containing A.

Casually, I like to think about it as the outer edge of the space, unioned with the space itself. For a closed space, like $[0, 1]$, the closure is $\{0\} \cup \{1\}$, which is contained in the closed set. Note that all closed sets are equal to the closure of the closed sets; this is because the smallest closed set containing a closed set A is A itself.

Notation

The closure of a set A is denoted as $\bar{A}$.

So for closed sets A, $\bar{A} = A$. Similarly, we have:

Definition

The **interior** of A is defined as the union of all open sets contained in A.

Notation

The interior of a set A is denoted as $\text{Int}A$.

For similar logic as before, for open sets A, $\text{Int}A = A$.

Definition

If A is a subset of the topological space X and if x is a point of X. we say that x is a **limit point/ cluster point/ point of accumulation** of A if every neighborhood of x intersects A in some point other than x itself.
You can also say x is a limit point of A if x is in the closure of $A - \{x\}$.

A limit point is just a point on the boundary on a subspace A, although it doesn't necessarily have to be in A itself. The limit points are kind of the boundary part of the closure. Hence the following theorem:

Theorem 0.1

Let A be a subset of the topological space X; let A' be the set of all the limit points of A. Then

$$\bar{A} = A \cup A'.$$

One can then consider the relationship between limit points and a set being closed. If the "boundary" of a set is made up of limit points, one can see that:

Theorem 0.2

A subset of a topological space is closed if and only if it contains all its limit points.

Definition

A collection $\mathcal{A}$ of subsets of a space X is said to **cover** X, or to be a **covering** of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an **open covering** of X if its elements are open subsets of X.

Definition

A Space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Definition

A topological space X is called a **Hausdorff space** if for each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively that are disjoint.

Definition

Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. A space X is said to be **connected** if there does not exist a separation of X.

This is really as simple as it sounds. Examples:
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Definition

A space is called **path connected** if every pair of points in X can be joined by a path in X

Question

So how is there a difference between being path connected and connected? Shouldn't being connected, so that the space has no disjoint parts, be enough to say that a path can be drawn from point to point?

Answer

The biggest example is the **Topologists sine curve**:

$$y(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } 0 < x < 1, \\ 0 & \text{if } x = 0, \end{cases}$$

This is a curve that is connected, but **not** path connected. Proof of connectedness: I'm not exactly sure why this works, but my thoughts are just because you technically can't find any separation between the two parts, because the limit of $\sin\frac{1}{x}$ as $x \to 0$ from the right is 0; however, because there is not actually any connected, you cannot draw a path. At least that's what I'm getting from this right now.

Algebraic Topology Definitions

Definition

A map is **nullhomotopic** if it is homotopic to a constant map.

Definition

A space is **contractible** if it is homotopically equivalent to a point.

So it's contractible if it can homotopically be squeezed into a point.

Theorem 0.3

A space is contractible if and only if it's identity map is nullhomotopic.

Proof.

($\implies$) Assume a space X is contractible, so that X is homotopically equivalent to a point x_0. Then there exists maps $f : X \to x_0$, $g : x_0 \to X$ such that $g \circ f \cong \mathrm{id}_X$. But $g \circ f : X \to X, x \mapsto x_0$, making it a constant map.

($\impliedby$) If the identity map is nullhomotopic, it is homotopic to a constant map. This is equivalent to saying there exists some $F(x,t)$ such that $F(x,0) = \mathrm{id}_X$ and $F(x,1) = x_0 \forall x \in X$. This homotopy describes the space X contracting to the point x_0 continuously.

■

The backwards direction of this proof seems hand-wavey, so I may need to go back and look at this.

Definition

A space is called **simply connected** if it is path connected and has trivial fundamental group

So a space is simply connected if there's nothing on the inside of it that causes loops to get "caught" on things. The first example that comes to my mind is how S^2 is simply connected, because it is path connected clearly, and any loop on S^2 can be contracted to a point (trivial fundamental group). On the other hand, the torus is path connected, but a loop could be caught around the donut hole in the middle, unable to contract, resulting in a nontrivial fundamental group ($\mathbb{Z} \times \mathbb{Z}$, specifically).

Definition

For a space X to be **locally path-connected**, for each point $x \in X$ and each neighborhood U of x, there must be an open neighborhood $V \subset U$ of x that is path-connected.

A good example of this would be two disjoint discs. Clearly, the space itself isn't path connected, but the two components are path-connected themselves, so the space is locally path-connected.

Question

So why is p from $\tilde{X}$ to X? To me, it seems more natural to assign $x \in X$ to somewhere in the cover.

Answer

A cover could have more than one sheet, so that each $x \in X$ maps to multiple $\tilde{x} \in \tilde{X}$. Thus p would not be a function if it went from X to $\tilde{X}$ for multi-sheeted covering spaces.

Theorem 0.4 (Homotopy Lifting Property)

Given a covering space $p : \tilde{Y} \to Y$, a homotopy $f_t : X \to Y$, and a map $\tilde{f}_0 : X \to \tilde{Y}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t : X \to \tilde{Y}$ of $\tilde{f}_0$ that lifts f_t.

Theorem 0.5 (Lifting Criterion)

Suppose given a covering space $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ and a map $f : (X, x_0) \to (Y, y_0)$ with X path-connected and locally path-connected. Then a lift $\tilde{f} : (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$ of f exists if and only if $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$.

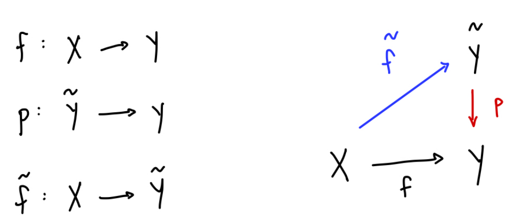
Question

So what is $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$?

Answer

First note that $(\tilde{Y}, \tilde{y}_0)$ is the covering space of Y and it's matching basepoint y_0 in the cover. Obviously, $\pi_1(\tilde{Y}, \tilde{y}_0)$ is the fundamental group of this covering space, and $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ is the map that describes how the covering space covers Y, so p_* is the induced homomorphism of the fundamental groups. So $p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$ are the values in the codomain that are actually mapped to, or $\text{img}(p_*)$. Note that $\text{img}(p_*) \subset \pi_1(Y, y_0)$, just like $f_*(\pi_1(X, x_0)) = \text{img}(f_*) \subset \pi_1(Y, y_0)$, which makes sense.

Given this, we can look at this diagram to see the relationship between X,Y and $\tilde{Y}$:



And there is a similar setup for their respective fundamental groups:

$$f_* : \pi_1(X) \to \pi_1(Y)$$
$$p_* : \pi_1(\tilde{Y}) \to \pi_1(Y)$$
$$\tilde{f}_* : \pi_1(X) \to \pi_1(\tilde{Y})$$

$$\begin{tikzcd} & \pi_1(\tilde{Y}) \arrow[d, "p_*"] \\ \pi_1(X) \arrow[ur, "\tilde{f}_*"] \arrow[r, "f_*"'] & \pi_1(Y) \end{tikzcd}$$

Fact 0.1

By the diagrams above, we can see $f = p \circ \tilde{f}$ and $f_* = p_* \circ \tilde{f}_*$.

Question

So what determines whether or not we can assume that f is this composition? Is there some property to satisfy?

Qualifying Exams

Spring 2025

Problem 1

Prove that any map $\mathbb{R}P^2 \to S^1 \times S^1$ is nullhomotopic. Prove that there exists a map $S^1 \times S^1 \to \mathbb{R}P^2$ which is not nullhomotopic.

Proof.

The universal cover of $S^1 \times S^1$ is $\mathbb{R} \times \mathbb{R}$. So if $f : \mathbb{R}P^2 \to S^1 \times S^1$, the lifted map $\tilde{f} : \mathbb{R}P^2 \to \mathbb{R} \times \mathbb{R}$ is nullhomotopic, since $\mathbb{R} \times \mathbb{R}$ is contractible. If the lifted map $\tilde{f}$ is nullhomotopic, then the map f must be. So all that is left to show is that we can lift the map, so we must meet the lifting criterion.
The fundamental group of $S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z}$, and the fundamental group of $\mathbb{R}P^2$ is $\mathbb{Z}/2\mathbb{Z}$, as can be seen by the following diagrams:
INSERT IPAD DRAWINGS HERE
Then f induces a homomorphism $f_* : \pi_1(\mathbb{R}P^2) \to \pi_1(S^1 \times S^1)$, which is clearly trivial. Then we know $f_*(\pi_1(X, x_0)) \subset p_*(\pi_1(\tilde{Y}, \tilde{y}_0))$, because $f_*(\pi_1(X, x_0)) = f_*(\mathbb{Z}/2\mathbb{Z}) = 0$, and $p_*(\pi_1(\tilde{Y}, \tilde{y}_0)) = p_*(\pi_1(\mathbb{R} \times \mathbb{R})) = p_*(0) = 0$. We must also show that $\mathbb{R}P^2$ is path-connected and locally path-connected, but this is obvious as it is a quotient space of S^2, and S^2 has those properties. Thus the lifting criterion is met, and f must be nullhomotopic. ■

Question

Why do we need the fundamental groups? Why can't we just say that the lifted map is nullhomotopic, so the normal map must be to?

Answer

We have to prove the lift exists, and so we must satisfy the lifting criterion.

Question

Why does any map $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ have to be trivial?

Answer

We know that $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}_2$ has two elements, $\{0, 1\}$. We know because f is a homomorphism, $f(0) = 0$. Then $f(1)$ must satisfy $f(1 + 1) = f(1) + f(1)$, but $1 + 1 = 0$ in $\mathbb{Z}_2$, so we have $f(1) + f(1) = 0$, and because $\mathbb{Z} \times \mathbb{Z}$ is **torsion-free** (there are no non-identity elements that have finite order (so like there are no non-identity elements that can every generate the identity by themselves)), the only values $f(1)$ can take on to follow this restriction is 0. Thus every element in $\mathbb{Z}_2$ maps to 0, and f must be trivial.

Question

So why does the lift being nullhomotopic imply the original is nullhomotopic?

Answer

So we know $tildef \cong c_1$, where c_1 is a constant map. But $f = p \circ \tilde{f} \cong p \circ c_1$, so for f to be nullhomotopic, p must be nullhomotopic too. But because $p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$, and $\mathbb{R} \times \mathbb{R}$ is contractible, p is nullhomotopic as well, so that $p \cong c_2$ for some constant map c_2. Thus $f \cong c_2 \circ c_1$, so that f is nullhomotopic.

Blank Problem Bank

Homework 1

Problem 0.2

Construct an explicit deformation retraction of $\mathbb{R}^n - \{0\}$ onto S^{n-1}.

Problem 0.3

(a) Show that the composition of homotopy equivalences $X \to Y$ and $Y \to Z$ is a homotopy equivalence $X \to Z$. Deduce that homotopy equivalence is an equivalence relation.

(b) Show that the relation of homotopy among maps $X \to Y$ is an equivalence relation.

(c) Show that a map homotopic to a homotopy equivalence is a homotopy equivalence.

Problem 0.6

(a) Let X be the subspace of $\mathbb{R}^2$ consisting of the horizontal segment $[0,1] \times \{0\}$ together with all the vertical segments $\{r\} \times [0, 1-r]$ for r a rational number in $[0,1]$. Show that X deformation retracts to any point in the segment $[0,1] \times \{0\}$, but not to any other point. [See the preceding problem]

(b) Let Y be the subspace of $\mathbb{R}^2$ that is the union of an infinite number of copies of X arranged as in the figure below. Show that Y is contractible but does not deformation retract onto any point.

(c) Let Z be the zigzag subspace of Y homeomorphic to $/R$ indicated by the heavier line. Show there is a deformation retraction in the weak sense (see Exercise 4) of Y onto Z, but no true deformation retraction.

Problem 0.10

Show that a space X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible if and only if every map $f : Y \to X$ is nullhomotopic.

Problem 0.11

Show that $f : X \to Y$ is a homotopy equivalence if there exist maps $g, h : Y \to X$ such that $fg \cong \mathrm{id}$ and $hf \cong \mathrm{id}$. More generally, show that f is a homotopy equivalence if fg and hf are homotopy equivalences.

Problem 0.16

Show that S^∞ is contractible.

Problem 0.17

Construct a 2-dimensional cellcomplex that contains both an annulus $S^1 \times I$ and a Mobius band as deformation retractions.

Problem 0.20

Show that the subspace $X \subset \mathbb{R}^3$ formed by a Klein bottle intersecting itself in a circle is homotopy equivalent to $S^1 \vee S^1 \vee S^1$.