8200 Homework 6

March 7, 2025

Exercise 1

Suppose X is path connected and $p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)$ is a path connected covering space of X. Prove that the number of sheets of this covering space is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x_0}))$ in $\pi_1(X, x_0)$.

Proof.

Let f be any loop with basepoint x_0, so that $\tilde{f}$ is it's lift, where X cooresponds to $\tilde{X}$ and x_0, $\tilde{x_0}$. Let $g \in G = p_*(\pi_1(\tilde{X}, \tilde{x_0}))$, so that $g \circ f$ has the lift $\tilde{g} \circ \tilde{f}$. Note that because $\tilde{g}$ is a loop, $\tilde{g} \circ \tilde{f}$ ends at the same point as $\tilde{f}$. Then define a function $\phi : G[f] \to p^{-1}(x)$, where $G[f] \mapsto \tilde{f}(1)$. Because $\tilde{X}$ is path connected, ϕ is surjective. Then note that because $\phi(G[f_1]) = \phi(G[f_2])$ implies that $f_1 \circ \bar{f}_2$ lifts to a loop based at $\tilde{x_0}$, so that $[f_1][f_2]^{-1} \in G$, and $g[f_1] = G[f_2]$, so that ϕ is injective. Thus the number of cosets (index) is equal to the number of sheets. ∎

Exercise 2

Construct nonnormal covering spaces of the Klein Bottle by a Klein bottle and by a torus.

Proof.

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