Computational Fluid Dynamics

SG2212 Project Report

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Question 1

The kronecker operator (**kron**) is an operator for tensor product, which falls convinient when doing finite differences in higher dimensions than the 1D case. As we study the 2D case in this lid-driven cavity project the 5 point stencil is considered:

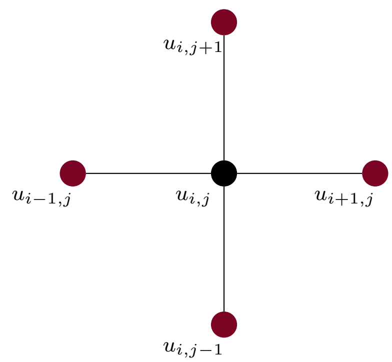

Figure 1: The five point stencil in 2D.

If we construct a finite difference scheme for the laplace operator we get:

$$\nabla^2 u = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} \tag{1}$$

$$\left\{ \text{and if } h_x = h_y = h \implies \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2} \right\} \tag{2}$$

The above expression we can transform into a vector form in size $(N_x, N_y) \mapsto N_x \cdot N_y$, using the indexing that one step in y-direction is equivalent to one row in the matrix *i.e.* $+N_x$ steps.

We can now define the second partial derivative in the separate directions:

$$S_x = \frac{1}{h_x^2}\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix} \qquad S_y = \frac{1}{h_y^2}\begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & \ddots & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}$$

Forming a matrix by the size $(N_x \cdot N_y) \times (N_x \cdot N_y)$ is now done by the kronecker product.

The operation done by the kronecker product essentially does the following in this case. Consider the identity matrix in the shape $N_y \times N_y$ (I_y), and the S_x from above. We get:

$$\text{kron}(I_y, S_x) = I_y \otimes S_x = \begin{bmatrix} [S_x] & 0 & \dots & \\ 0 & [S_x] & 0 & \dots \\ \vdots & & \ddots & \\ & & & [S_x] \end{bmatrix} \quad \text{shape is } (N_y \cdot N_x) \times (N_y \cdot N_x)$$

and for the other term in the sum of kronecker products we have:

$$S_y \otimes I_x = \begin{bmatrix} S_y^{1,1}[I_x] & 0 & \dots & \\ 0 & S_y^{2,2}[I_x] & 0 & \dots \\ \vdots & & \ddots & \\ & & & S_y^{N_y,N_y}[I_x] \end{bmatrix} \quad \text{shape is } (N_y \cdot N_x) \times (N_y \cdot N_x)$$

We can then see that in the simplified example where $N_x = N_y = N$ and $h_x = h_y = h = 1$ we get:

$$I_y \otimes S_x + S_y \otimes I_x = \begin{bmatrix} -4 & 1 & \dots & 1 & \dots & & & \\ 1 & -4 & 1 & \dots & 1 & \dots & & \\ 0 & \ddots & \ddots & \ddots & & & & \\ & & & & & & & \\ & & & 1 & \dots & 1 & -4 \end{bmatrix} \tag{3}$$

which describes the case stated in Equation 2. Note that the 1's in matrix in Equation 3 are N_x values apart, with zero-values in between.

Question 2

Experimental stability condition for Δt (with parameters $N_x = N_y = 30$, $L_y = L_x = 1$ and $Re = 25$) was empirically found to be

$$\Delta t_{\max} \approx 0.006971$$

when integrated up to $T = 50$.

Question 3

Question 4

Question 5

Question 6

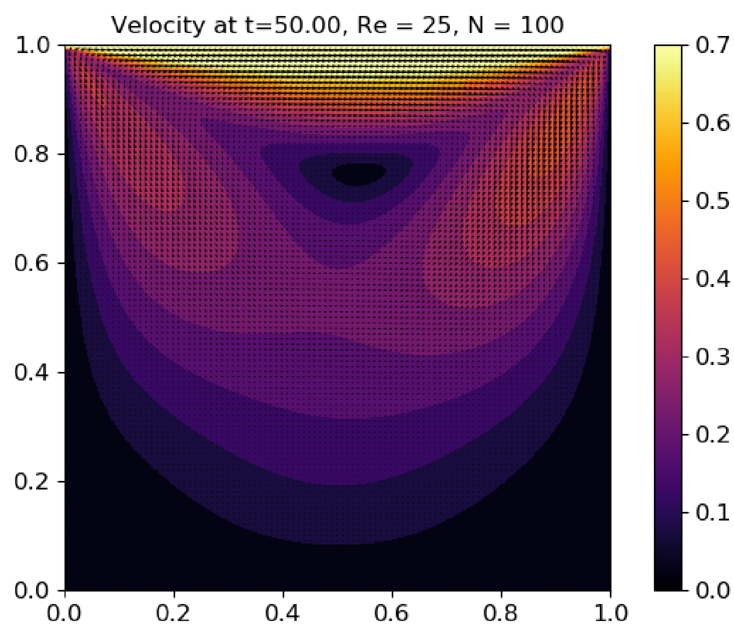

Figure 2: Case A

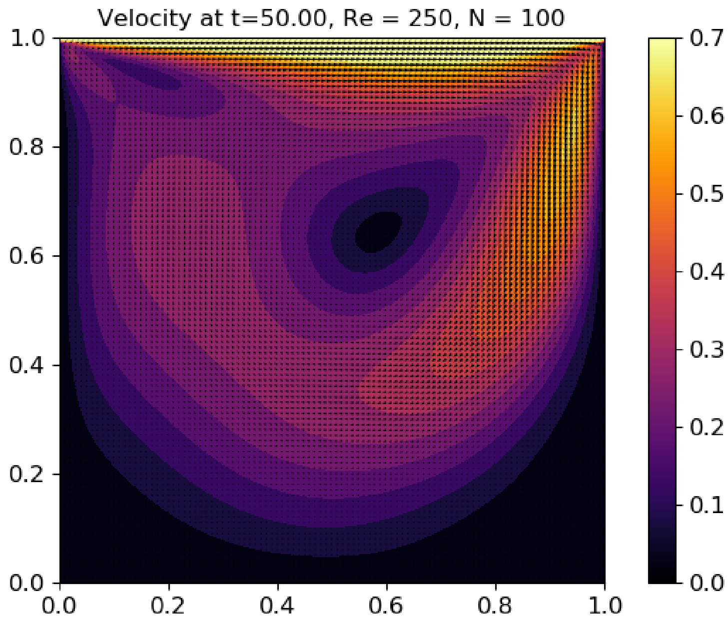

Figure 3: Case B

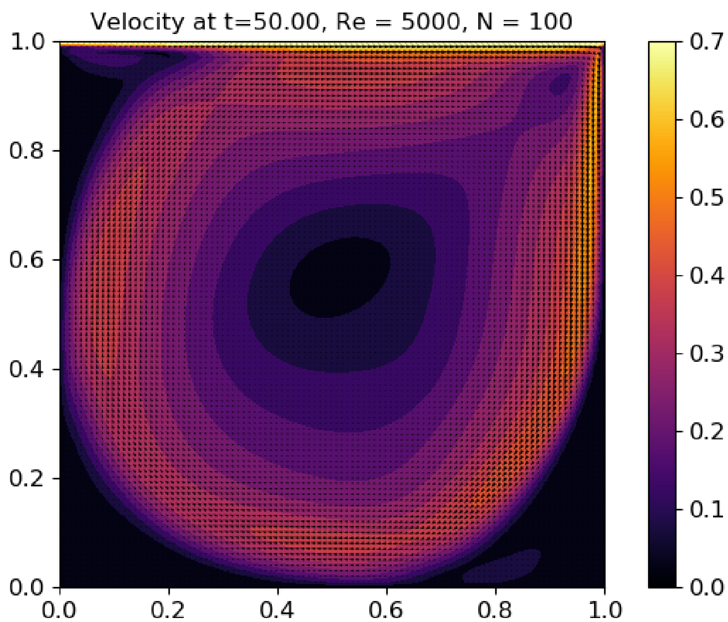

Figure 4: Case C

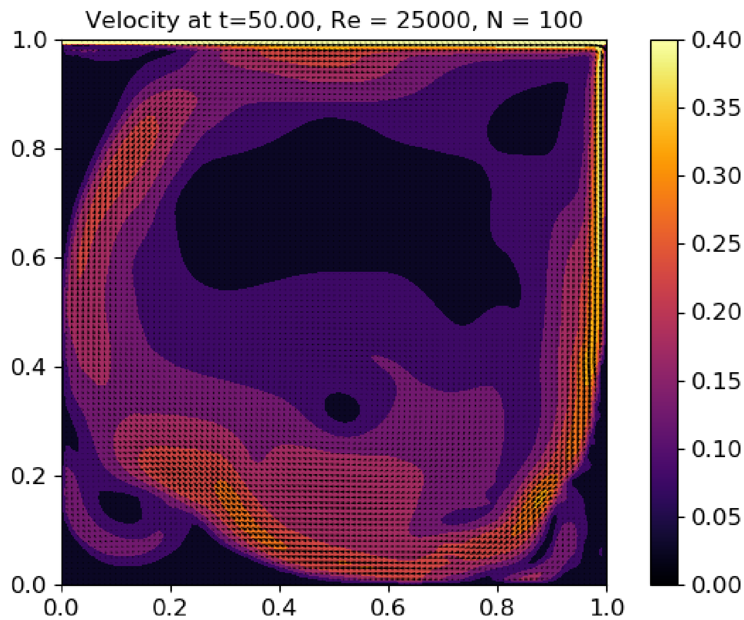

Figure 5: Bonus case with even higher Re