Computational Fluid Dynamics

SG2212 Project Report

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Question 1

The kronecker operator (**kron**) is an operator for tensor product, which falls convinient when doing finite differences in higher dimensions than the 1D case. As we study the 2D case in this lid-driven cavity project the 5 point stencil is considered:

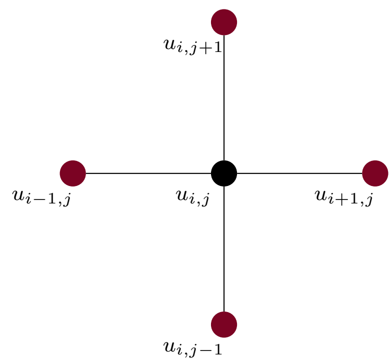


Figure 1: The five point stencil in 2D.

If we construct a finite difference scheme for the laplace operator we get:

$$\nabla^2 u = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h_x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h_y^2} \tag{1}$$

$$\left\{ \text{and if } h_x = h_y = h \implies \frac{u_{i+1,j} + u_{i-1,j} - 4u_{i,j} + u_{i,j+1} + u_{i,j-1}}{h^2} \right\} \tag{2}$$

The above expression we can transform into a vector form in size $(N_x, N_y) \mapsto N_x \cdot N_y$, using the indexing that one step in y-direction is equivalent to one row in the matrix *i.e.* $+N_x$ steps.

We can now define the second partial derivative in the separate directions:

$$S_x = \frac{1}{h_x^2} \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \end{bmatrix} \qquad S_y = \frac{1}{h_y^2} \begin{bmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & \ddots & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \end{bmatrix}$$

Forming a matrix by the size $(N_x \cdot N_y) \times (N_x \cdot N_y)$ is now done by the kronecker product.

The operation done by the kronecker product essentially does the following in this case. Consider the identity matrix in the shape $N_y \times N_y$ (I_y), and the S_x from above. We get:

$$\text{kron}(I_y, S_x) = I_y \otimes S_x = \begin{bmatrix} [S_x] & 0 & \dots & \\ 0 & [S_x] & 0 & \dots \\ \vdots & & \ddots & \\ & & & [S_x] \end{bmatrix} \quad \text{shape is } (N_y \cdot N_x) \times (N_y \cdot N_x)$$

and for the other term in the sum of kronecker products we have:

$$S_y \otimes I_x = \begin{bmatrix} S_y^{1,1}[I_x] & 0 & \dots & \\ 0 & S_y^{2,2}[I_x] & 0 & \dots \\ \vdots & & \ddots & \\ & & & S_y^{N_y,N_y}[I_x] \end{bmatrix} \quad \text{shape is } (N_y \cdot N_x) \times (N_y \cdot N_x)$$

We can then see that in the simplified example where $N_x = N_y = N$ and $h_x = h_y = h = 1$, and no specific boundaries are set, we get:

$$I_y \otimes S_x + S_y \otimes I_x = \begin{bmatrix} -4 & 1 & \dots & 1 & \dots & & \\ 1 & -4 & 1 & \dots & 1 & \dots & \\ 0 & \ddots & \ddots & \ddots & & & \\ & & & & & & \\ & & & 1 & \dots & 1 & -4 \end{bmatrix} \tag{3}$$

which describes the case stated in Equation 2. Note that the 1's in matrix in Equation 3 are N_x values apart, with zero-values in between.

Question 2

Experimental stability condition for Δt (with parameters $N_x = N_y = 30$, $L_y = L_x = 1$ and $Re = 25$) was empirically found to be

$$\Delta t_{\max} \approx 0.006971$$

when integrated up to $T = 50$.

Question 3

Question 4

Placing a probe in the domian centre gives the following velocity U along the time line to $T = 100$:

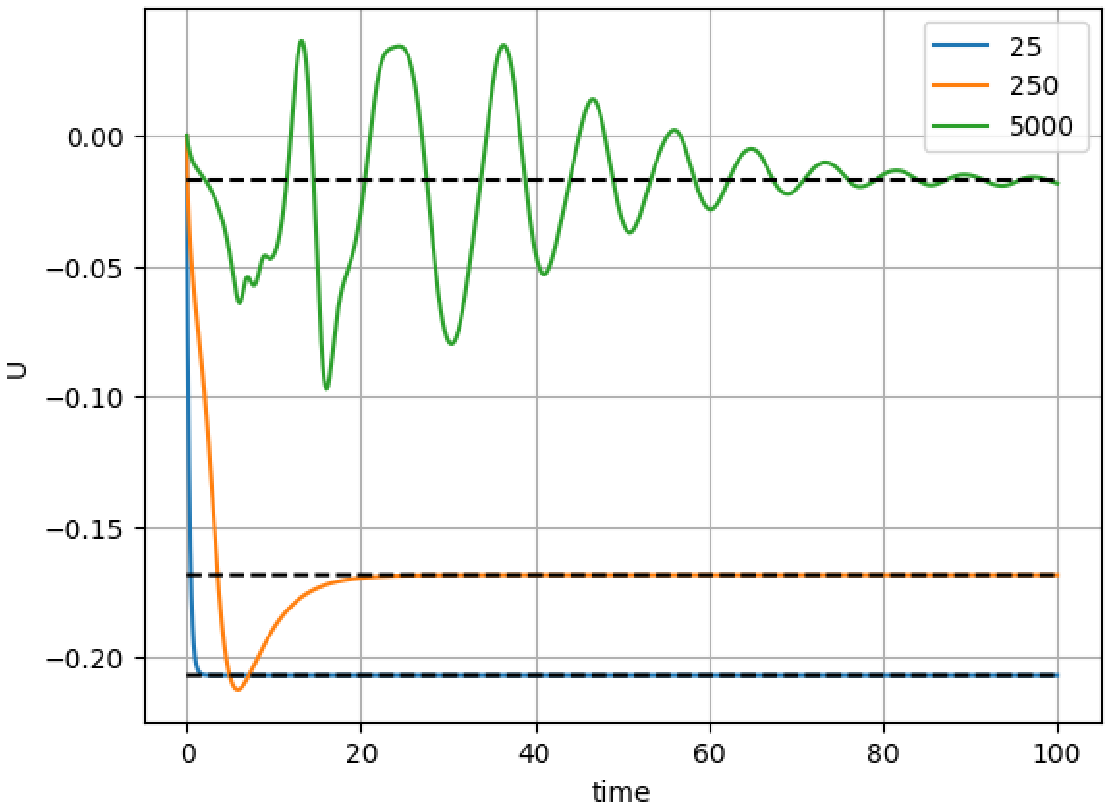


Figure 2: U in middle of domain for different Re-values

Clearly the cases with lower Reynold's number converges to a steady flow quicker than the cases with a higher number. This is due to the amount of turbulance is greater in the case with higher Re; the intertial forces are greater in comparison to the viscous forces when the Re increases.

To in a more concrete way decide when a steady flow is reached in the location of the probe, the derivative of U w.r.t t is taken into account. When setting up two conditions, which essentially measures when the relative derivative and deviation from the median doesn't overshoot 5%. With the following lines of code:

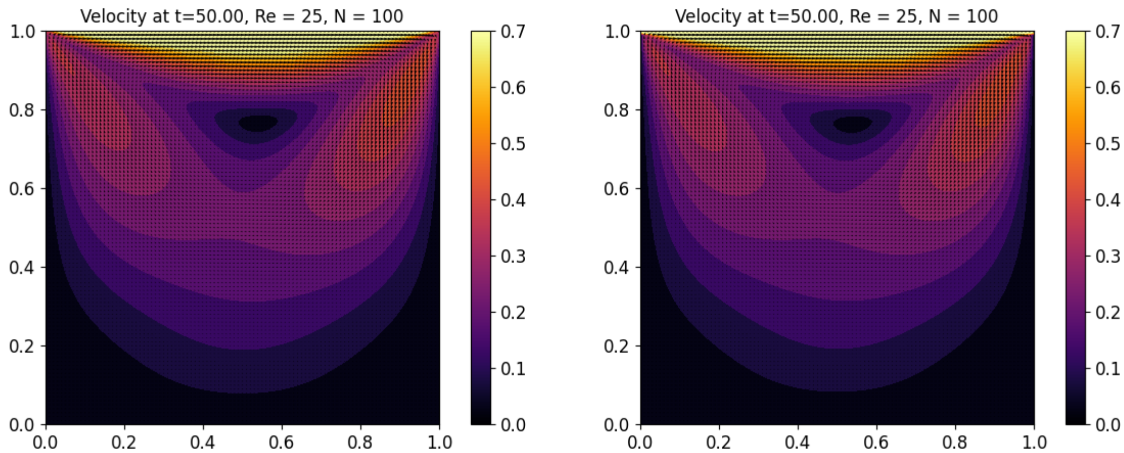
```
du_vec = np.diff(u_vec, axis = 0)/np.diff(t)[0]
rel_derivative = du_vec/np.abs(np.median(u_vec, axis = 0))

for col in range(u_vec.shape[1]):
    cond1 = np.abs(u_vec[1:, col]-np.median(u_vec, axis = 0)[col]) < \
        0.05 * np.abs(np.median(u_vec, axis = 0)[col])
    cond2 = np.abs(rel_derivative[:, col]) < 0.05

    cond = cond1 * cond2
```

The following times for steady flow was estimated, when looking at the first time point when both conditions were fulfilled:

```
Time to steady flow when Re = 25: 1.1998011998011997 s
Time to steady flow when Re = 250: 13.217413217413217 s
Time to steady flow when Re = 5000: 96.35719635719636 s
```

Question 5

Using OpenFoam in each of the cases A-C the following data where obtained: Figure 3 shows both a solution from the Python implementation as well as OpenFOAM solution with the data visualized using matplotlib.



(a) OpenFoam solution

(b) Python Solution

Figure 3: Case A

For a more detailed comparison we can observe the velocity magnitude along the diagonal of the domain, so called a *plot over line* visualization. For case A we have the following:

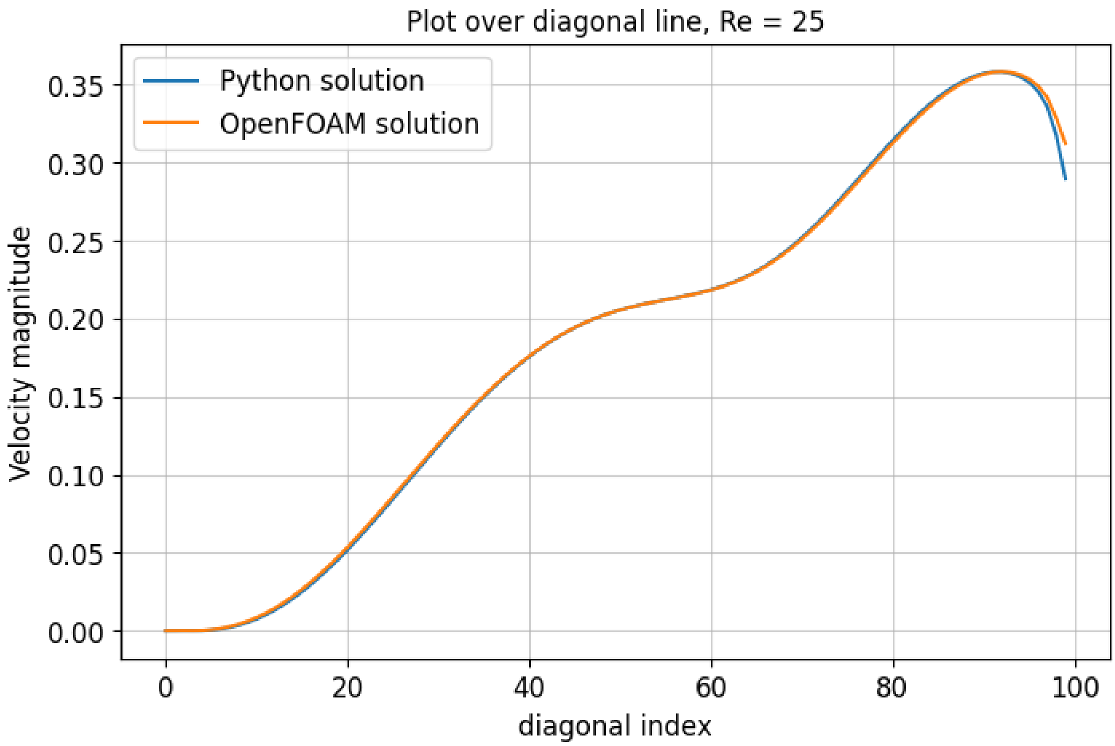
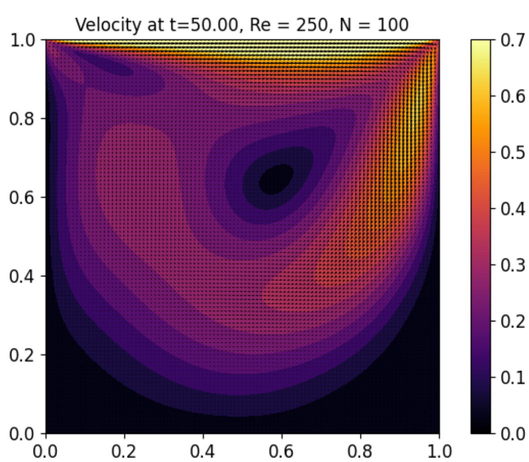
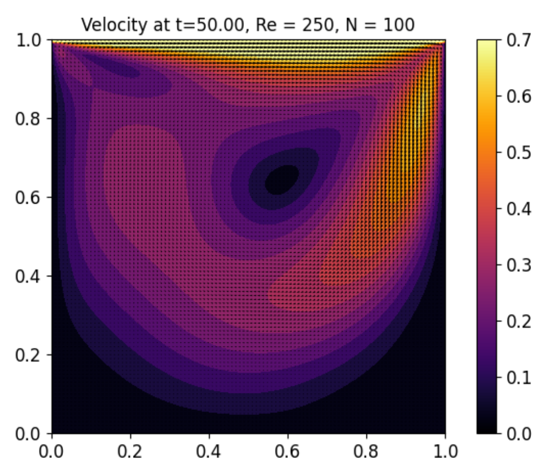


Figure 4: Plot over line (Case A): Python against OpenFOAM

Likewise we can compare case B and C in a similar manner:

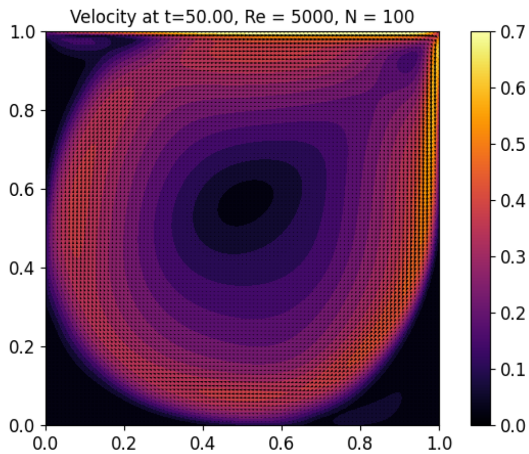


(a) OpenFoam solution

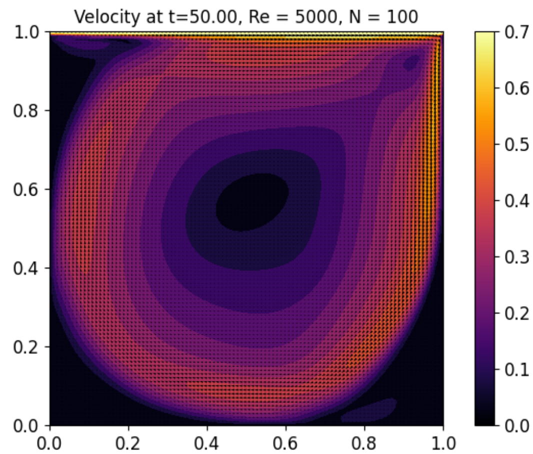


(b) Python Solution

Figure 5: Case B



(a) OpenFoam solution



(b) Python Solution

Figure 6: Case C

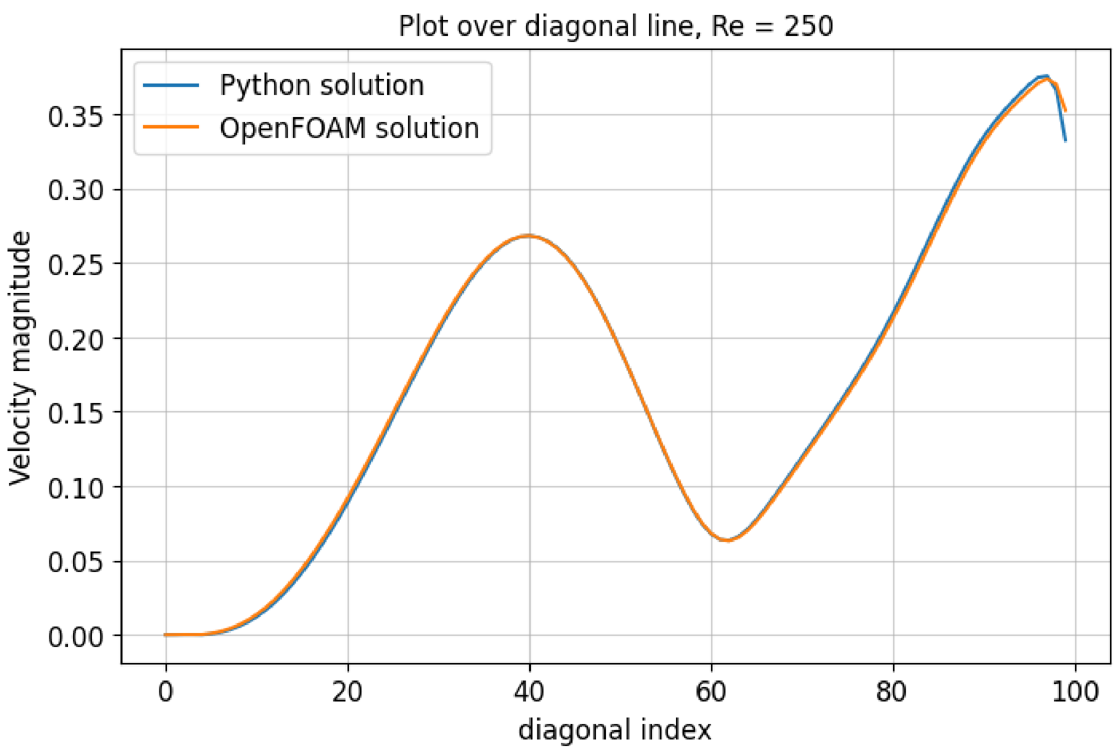


Figure 7: Plot over line (Case B): Python against OpenFOAM

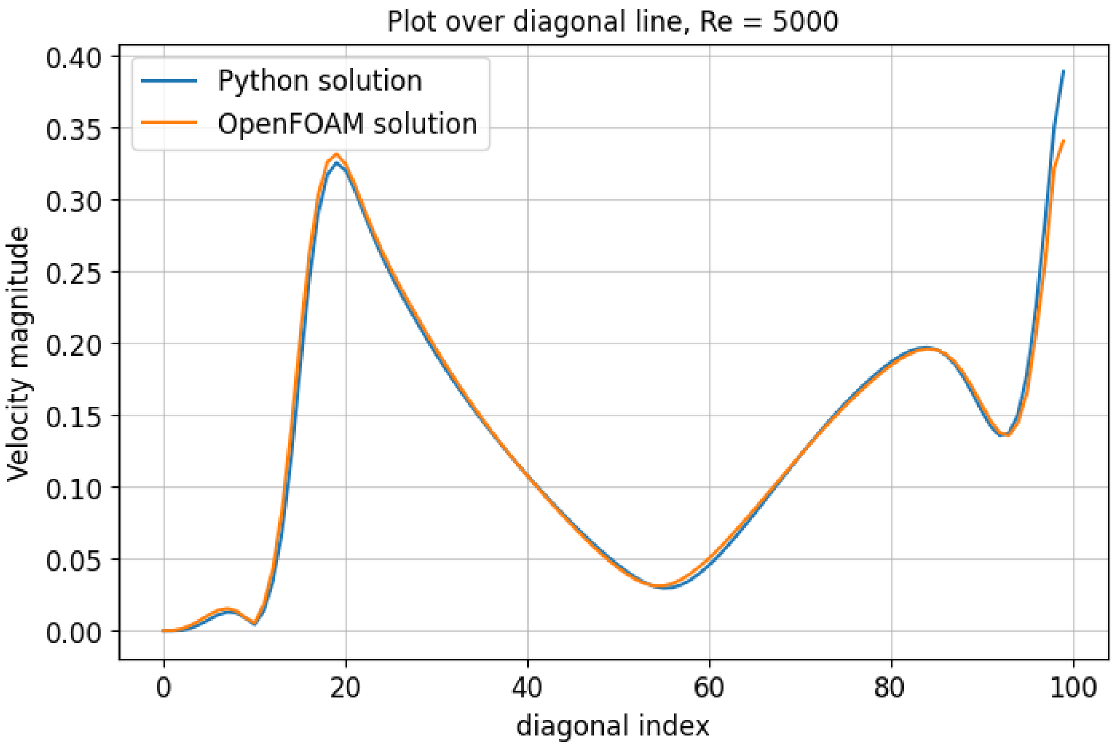


Figure 8: Plot over line (Case C): Python against OpenFOAM

We can conclude that the Python and OpenFOAM solutions converges, and notably the largest differences was found close to the boundaries.

Comparing the L_2-norm between the different steady solutions we got the following data:

```
Running case for Re = 25
Relative error: 0.04912371466615295

Running case for Re = 250
Relative error: 0.05397593894314182

Running case for Re = 5000
Relative error: 0.09147688319684655
```

Question 6

When also including a lower wall with the same velocity as the upper lid, the following solution was obtained at $T = 50$:

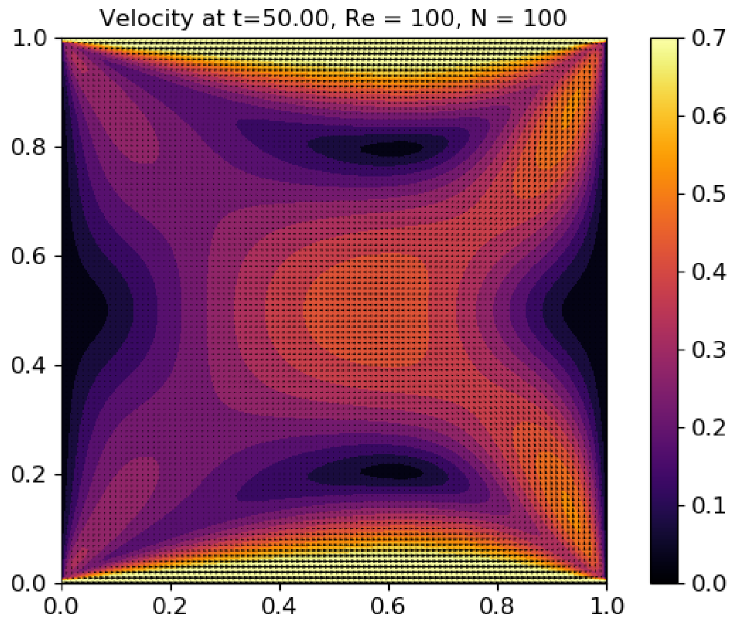


Figure 9: Case D