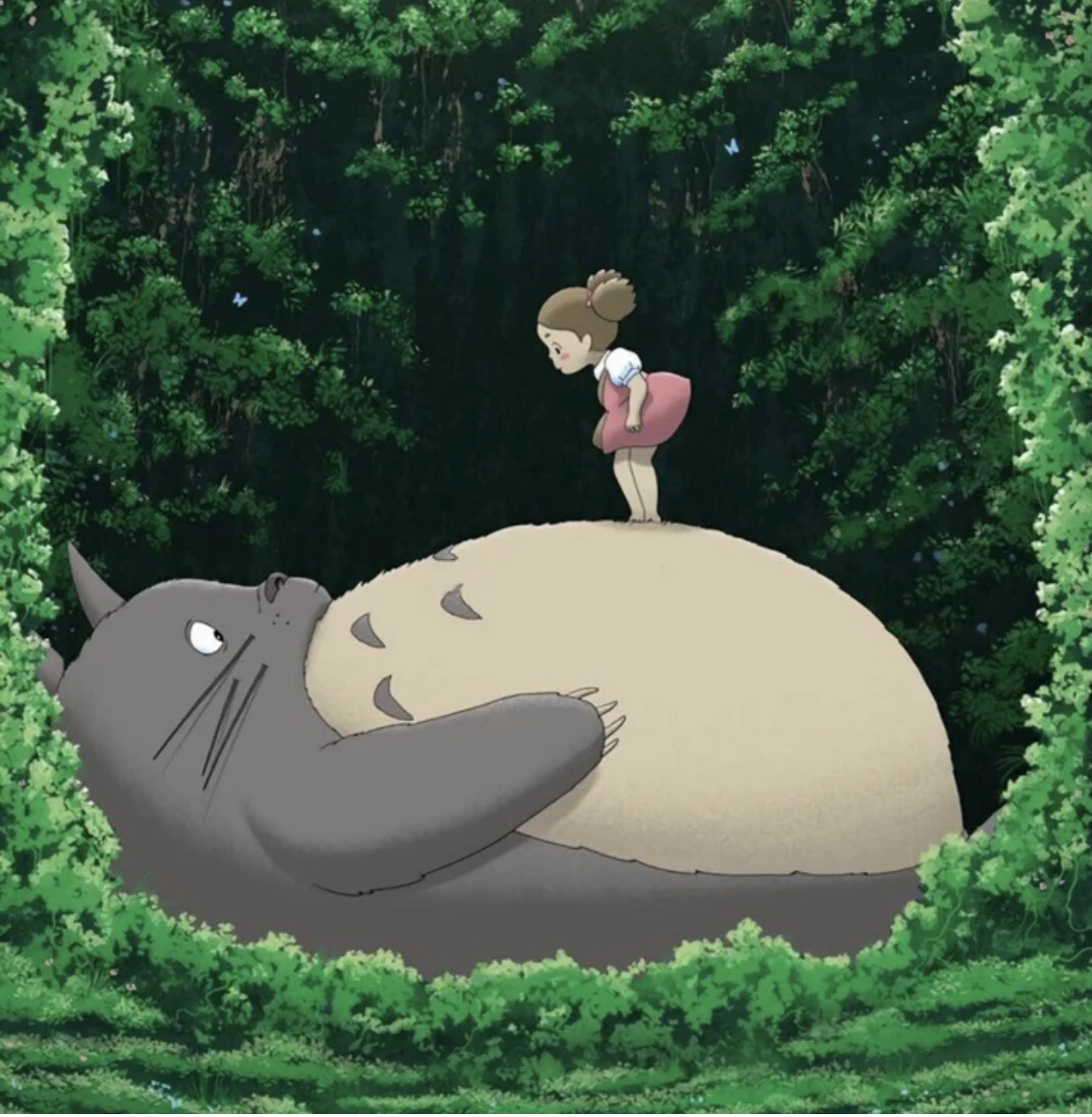

Real Analysis

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Date: January 2022

Contents

We consider the following conjecture:

$$|\{1 \le a_i, b_i \le N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi}\int_0^{2\pi}\left|\sum_{a=1}^{n} e^{ia^3x}\right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned}\left|\sum_{a=1}^{n} e^{ia^3x}\right|^6 &= (\sum_{a=1}^{n} e^{ia^3x})(\sum_{a=1}^{n} e^{-ib^3x}) \\ &= \sum_{a_1,a_2,a_3,b_1,b_2,b_3} e^{ix(a_1^3+a_2^3+a_3^3-b_1^3-b_2^3-b_3^3)}\end{aligned}$$

Hence the integral is 0 if the diophatine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophatine equation.

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Introduction to decoupling

Now we move to the overview of decoupling.

If we denote a region Ω of $\mathbb{R}^n$ as the Fourier space, and we decompose it into small regions $\Omega = \bigsqcup \theta$.

If we assume the function f whose Fourier transform has support in the region Ω, then we can decompose Ω, we will now make the definition as follows.

Definition 0.1 (Decoupling)

Let f be a sufficiently regular function whose supp$(\widehat{f}) \subset \Omega$, if we define

$$f_\theta = \int_\theta \widehat{f}(\omega)e^{ix\omega}d\omega$$

Then by Fourier inverse formula, we get

$$f = \sum_\theta f_\theta$$

♣

Proof $\sum_\theta f\theta = \int_\Omega \widehat{f}(\omega)e^{ix\omega}d\omega = f(x)$

One would like to control the norm $||f||_{L^p}$, using what you know about $||f_\theta||_{L^p}$. To give a general idea what we are heading towards, we can fine a constant D_p, dependent on Ω, θ, such that the following inequality is achieved.

$$||f||_{L^p}(\mathbb{R}^n) \leq D_p(\Omega = \bigsqcup \theta) \left(\sum_{\theta} ||f_\theta||_{L^p}^2 \right)^{1/2}$$

Hence if we fix a specific decoupling choice, i.e. $\Omega = \bigsqcup_\theta \theta$, we can find an absolute constant such that

$$||f||_{L^p} \leq C \left(\sum_{\theta} ||f_\theta||_{L^p}^2 \right)^{1/2}$$

And we define the decoupling constant $D_p(\Omega = \bigsqcup \theta)$ to be the smallest of all C, for a fixed decoupling choice.

Remark Are there different ways of decomposing omega? Is the most intuitive way of decomposing the Fourier space based on frequency?

Proposition 0.1 (Estimate of sum)

Let $R > 0$, and pick Fourier space decomposition $\Omega = \bigsqcup_j \theta_j$, and let $g = \sum_j a_j e^{i\omega_j \cdot x}$. If $B_{1/R}(\omega_j) \subset \theta_j$, for all j, then for any ball of radius R, B_R, we have

$$||g||_{L^p(B_R)} \lesssim D_p \left(\sum_j |a_j|^2 \right)^{1/2} R^{1/p}$$

♠

Proof Let $f = \eta g$, such that $supp(\widehat{\eta}) \subset B_{1/R}$, such that $|\eta| \sim 1$ on B_R and decays rapidly outside of B_R.

Lecture 2

Now we begin with some building blocks.

Suppose $\Omega = [0, N], \theta_j = [j - 1.j], \Omega = \bigsqcup_{j=1}^{N} \theta_j$. And we ask the question, if we have $supp(\widehat{f}) \subset [0, 1]$, could $|f|$ look like several narrow peaks and almost 0 elsewhere?

We recall how we decouple the function f: for $supp(\widehat{f}) \subset \Omega$, define $f_{\theta_j} = \int_{[j-1,j]} \widehat{f}(\omega)e^{i\omega x}d\omega$, then $f = \sum_j f_{\theta_j}$.

Now we remind ourselves of the height of f.

Proposition 0.2

Let $f \in \mathcal{S}$ be such that $supp(\widehat{f}) \subset [0, 1]$, and we have

$$||f||_{L^\infty} \lesssim ||f||_{L^1}$$

Proof We define a cutoff function $\eta \in \mathcal{S}$ such that $\eta = 1$ on $[0, 1]$, then $\widehat{f} = \eta\widehat{f}$, then $f = f * \check{\eta}$, also a Schwartz function.

$$\begin{aligned} ||f||_{L^\infty} &= ||f * \check{\eta}||_{L^\infty} \\ &\leq ||f||_{L^1}||\check{\eta}||_{L^\infty} \\ &\lesssim ||f||_{L^1} \end{aligned}$$

Hence the answer is no, because if we have narrow peaks with controlled heights, $||f||_{L^1}$ would be small, which would violate $||f||_{L^\infty} \lesssim ||f||_{L^1}$.

Now we ask the following question, can we have flat parts of $|f|$ where $||f||_{L^1}$ is dominated by the flat parts, but still has narrow peaks? To address that, we introduce an important lemma which allows us to control the height of f in one interval using its L^1 norm in an even larger interval.

Proposition 0.3 (Locally Constant Lemma)

If $supp\widehat{f_1} \subset [0, 1]$, and I is the unit interval $[0, 1]$, then we have

$$||f||_{L^\infty(I)} \lesssim ||f||_{L^1(\omega_I}$$

Where the weighted L^1 norm is defined to be $||f||_{L^1(\omega_I)} = \int_{\mathbb{R}} |f_1|\omega_I$ where the function ω_I satisfies the following: $\omega_I \geq 0$, $\omega_I \sim 1$ on I, and ω_I decays rapidly off of I, lastly, ω_I is uniform in the sense that $\omega_{I+a} = \omega_I(\cdot - a)$

Proof This follows from the fact that $\eta \in \mathcal{S}$, hence $\check{\eta} \in \mathcal{S}$ as well, i.e. we have

$$|\check{\eta}(y)| \lesssim \left(\frac{1}{1 + |y|}\right)^M$$

for all large M. Hence we follow the same computation:

$$\begin{aligned} |f(x)| &= \left|\int f(y)\check{\eta}(x - y)dy\right| \\ &\leq \int |f(y)||\check{\eta}(x - y)|dy \\ &\leq \int |f(y)| \sup_{x\in I} |\check{\eta}(x - y)|dy \end{aligned}$$

And if we define $\omega_I(y) = \sup_{x\in I} |\check{\eta}(x - y)|$, surely it satisfies being nonnegative, and by property of $\check{\eta}$ being Schwartz, $\check{\eta} \sim 1$ on I, and decays rapidly if $|x - y|$ is greater than 0.

In other words, we almost know that $||f||_{L^\infty(I)} \lesssim ||f||_{L^1(2I)}$, where $2I$ is if we stretch the intervals keeping the same center.

Remark For $p = 2, p = \infty$, the decoupling constant is easier to estimate. For $p = 2$, we can apply Plancherel, namaley,

$$||f||_{L^2} = ||\sum_j f_{\theta_j}||_{L^2} = ||\sum_j \widehat{f}_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2}$$

For $p = \infty$, we can apply Cauchy Schwartz, namely,

$$||f||_{L^\infty} = ||\sum_j f_{\theta_j}||_{L^\infty} \leq \sum_j ||f_{\theta_j}||_{L^\infty} \leq \left(\sum_j ||f_{\theta_j}||_{L^\infty}^2\right)^{1/2} N^{1/2}$$

And now we conclude with an example. Consider a function f_1 with height 1, (i.e. $||f||_{L^\infty} = 1$) and $f_1(0) = 1$ and it is concentrated on the interval $[-1, 1]$. If we define $f_j(x) = e^{2\pi i(j-1)x} f_1(x)$ and define $f = \sum_j f_j$, then we have $f_j(0) = 1$ and thus $f(0) = N$.

We note that f_j oscillates with frequency $\frac{1}{j}$ and when $|x| \leq \frac{1}{10N} \leq \frac{1}{10j}$, we have $f_(x) \sim N$. Hence, if we consider $||f||_{L^p}$, we have

$$||f||_{L^p}^p = \int |f|^p = \geq \int_{|x| \leq \frac{1}{10N}} |f|^p = \gtrsim \frac{1}{N} \cdot N^p = N^{p-1}$$

Hence taking the $1/p$ of both sides, we have $||f||_{L^p} \gtrsim N^{1-1/p}$.

Now if we would wish to consider the decoupling constant, we now consider $||f_j||_{L^p}$. Note $||f_j||_{L^p} \sim 1$, hence $\left(\sum_j ||f_j||_{L^2}^2\right) \sim N^{1/2}$. Thus we have $D_p \gtrsim N^{1/2-1/p}$.

Main Obstacle

Consider a function f_j such that $|f_j| = 1$ on $[0, 1]$, and is $\frac{1}{N}$ on $[1, N^3]$, and 0 elsewhere. Then the $||f_j||_{L^2} \sim N^{1/2}$, whereas $||f_j||_{L^4} \sim 1$. (Exactly how one owuld expect the L^p norm to behave).

Like the above remark, we note that $||f||_{L^2} \sim \sum_j ||f_j||_{L^2}^2)^{1/2} \sim N$, and $||f||_{L^\infty} \leq N^{1/2}(N)^{1/2} = N$. Now we ask the question, could $|f(x)| \sim N$ on the unit interval $[0, 1]$? The answer is no.

Proof Assume $|f(x)| \sim N$ on $[0, 1]$, then $||f||_{L^4} \gtrsim N$, however, we know

$$||f||_{L^4} \lesssim D_p \left(\sum_j ||f_j||_{L^4}^2 \lesssim N^{1/4} \cdot N^{1/2} = N^{3/4}\right)$$

Note D_p arises from our above lower bound given that $p = 4$.

Recall the Local Constant Lemma tells us how the height is controlled by the L^1 norm, now we introduce another lemma that connects the L^2 norms, which improves our estimate.

Lemma 0.1 (Local Orthogonality Lemma)

If I is a unit interval, and $f = \sum_{j=1}^N f_j$, and supp$\widehat{f_j} \subset [j-1, f]$, then we have

$$||f||_{L^2(I)}^2 \lesssim \sum_j ||f_j||_{L^2(\omega_I)}^2$$

♡

Proof We choose η such that it preserves f on the unit interval, and whose fourier transform has support land in $[-1, 1]$, i.e. $|\eta| \sim 1$ on I, and $supp(\eta) \subset [-1, 1]$.

$$\begin{aligned}
||f||_{L^2(I)}^2 &= \int_I |f|^2 \\
&\leq \int_{\mathbb{R}} |\eta f|^2 \\
&= \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \int |\sum_j \widehat{\eta} * \widehat{f}|^2 \\
&\lesssim \sum_j \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \sum_j \int_{\mathbb{R}} |\eta|^2 |f_j|^2 \\
&= \sum_j ||f||_{L^2(\omega_I)}^2
\end{aligned}$$

if we define $\omega_I = |\eta|^2$.

We thus obtain this local orthogonality result, in the sense that we can decompose the L^2 norm locally and control the L^2 norm of f by the sum of the L^2 norm of f_j.

Now we generalize this to a wide range of p to obtain our local decoupling result.

Proposition 0.4 (Local decoupling)

If I is a unit interval, for $2 \leq p \leq \infty$, for each $1 \leq j \leq N$, $supp(f_j) \subset [j-1, j]$, then we have

$$||f||_{L^p(I)} \lesssim N^{1/2-1/p} \left(\sum_{j=1}^{N} ||f_j||_{L^p(\omega_I)}^2 \right)^{1/2}$$

Proof This follows from the Locally Constant Lemma and the Locally Orthogonality Lemma above.

$$\int |f|^p = \int |f|^2 |f|^{p-2} \leq ||f||_{L^\infty(I)}^{p-2} \int |f|^2 \leq \sum_j ||f_j||_{L^2(\omega_I)}^2 \left(\sum_j ||f_j||_{L^\infty(I)} \right)^{p-2}$$

The last inequality follows from the local orthogonality lemma above which states $||f||_[L^2(I)] \leq \sum_j ||f_j||_{L^2(\omega_I)}$.

Then for the second term, local constant lemma states that the height is controlled by the L^1 norm $||f_j||_{L^\infty} \lesssim ||f_j||_{L^1(\omega_I)} \lesssim ||f_j||_{L^2}$, where the last inequality is to match the L^2 norm of the first term. Combining, we have $\int |f|^p \leq (\sum_j \|f_j\|_{L^2(\omega_I)}^2)(\sum_j \|f\|_{L^2(\omega_I)})^{p-2}$. By Cauchy Schwarz on the second term, we obtain,

$$\int |f|^p \leq \left(\sum_j ||f_j||_{L^2(\omega_I)}^2 \right) \left(\sum_j ||f_j||_{L^2(\omega_I)}^2 \right)^{p/2-1} N^{p/2-1} = \left(\sum_j ||f_j||_{L^2(\omega_I)}^2 \right)^{p/2} N^{p/2-1}$$

If we replace the $\|f\|_{L^2}$ with $\|f\|_{L^p}$, we get the desired result.

Lemma 0.2

In finite measure spaces, for $p \geq q$, we have

$$\|f\|_{L^p} \lesssim \|f\|_{L^q}$$

Proof This follows from Holder's inequality.

$$\int_I |f|^p = \||f|^p\|_{L^{q/p}} \mu(I)^s \lesssim \|f\|_{L^q}^p$$

Now we prove the parallel decoupling lemma, which basically states that if we decompose two measures as $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, and for each i, we have the same decoupling constant, then we would be able to keep that decoupling constant when we sum them up. Recall the Minkowski's inequality refers to triangle inequality with respect to the L^p

norm.

Proposition 0.5 (Parallel Decoupling Lemma)

For some $p \geq 2$, and for any function $g = \sum_j g_j$, and any measures $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, then if for each i, we have

$$\|g\|_{L^p(\mu_i)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega_i)} \right)^{1/2}$$

then summing up, we would have the combined inequallity with the same decoupling constant,

$$\|g\|_{L^p(\mu)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega)} \right)^{1/2}$$

Proof The proof uses the Minkowski's inequality.

$$\begin{aligned}
\int |g|^p \mu &= \sum_i \int |g|^p \mu_i \\
&\leq D^p \sum_i \left(\sum_j ||g_j||^2_{L^p(\omega_i)} \right)^{p/2} \\
&\leq D^p \left\| \sum_j ||g_j||^2_{L^p(\omega_i)} \right\|^{p/2}_{l_i^{p/2}} \\
&\leq D^p \left(\sum_j \left\| ||g_j||^2_{L^p(\omega_i)} \right\|_{l_i^{p/2}} \right)^{p/2} \\
&= D^p \left(\sum_j \left(\sum_i ||g_j||^p_{L^p(\omega_i)} \right)^{2/p} \right)^{p/2} \\
&= D^p \left(\sum_j ||g_j||^2_{L^p(\omega)} \right)^{p/2}
\end{aligned}$$

Lecture 3

We state Bourgain and Demeter's decoupling theorem for the paraboloid. We define a parabaloid as

$$P = \{\omega \in \mathbb{R}^n : \omega_n = \omega_1^2 + ...\omega_{n-1}^2, |\omega| \leq 1\}$$

We now define a slightly larger neighborhood of P, denoted by $N_{1/R}P$ as the neighborhood of P of radius $1/R$ where R is some large constant. This is the area that we will cut up using rectangles θ,

$$\theta \approx R^{-1/2} \times R^{-1/2} \times ... \times R^{-1}$$

Now we've fixed a decomposition, we denote its decoupling constant $D_p(R) = D_p(\omega = \cup\theta)$, and we ar eready to state our decoupling theorem.

Theorem 0.1 (Bourgain and Demeter)

For $2 \leq p \leq \frac{2(n+1)}{n-1}$, for the "cutting up" scheme above, we have

$$D_p(R) \lesssim R^\epsilon$$

Now we define the dual of θ, denoted by $\theta^* = \{x \in \mathbb{R}^n : |\omega - \omega_\theta| < \frac{1}{x}, \forall\omega \in \theta\}$, with ω_θ being the center of θ. Now we look at how these are $\widehat{\phi}$ behaves on $\theta*$ and then rapidly decaying.

Lemma 0.3

If ϕ_θ is a smooth bump function supported in θ, then $\check{\phi}_\theta \sim |\theta|$ on $\theta^*|$ and rapidly decaying outside of it.

Proof If ϕ_θ lives on θ, we calculate its inverse Fourier transform to find the corresponding function in the physical space.

$$|\check{\phi}_\theta| = \left|\int e^{2\pi i x\omega}\phi_\theta(\omega)d\omega\right| = \left|e^{2\pi i x\omega_\theta}\int e^{2\pi i x(\omega-\omega_\theta)}\phi_\theta(\omega)d\omega\right| = \left|\int_\theta e^{2\pi i x(\omega-\omega_\theta)}\right| \sim |\theta|$$

If $x \in \theta^*$, then we know $|(\omega - \omega_\theta)x| < 1$, hence the oscillatory integral on the right hand side does not get too much cancellation, hence $|\check{\phi}_\theta| \sim |\theta|$, while outside of θ^*, if one integrate by parts many times, then get rapid decay.

Remark This lemma shows why the dual θ^* is called the dual, i.e., if we consider functions first in the Fourier space, then we can find the corresponding θ^* such that $\check{\phi}_\theta$ is located and the size of it. Moreover,

$$\check{\phi}_\theta \text{ roughly looks like } e^{2\pi i x\omega_\theta}|\theta|\chi_{\theta^*}$$

Based off of the smooth bump function in the Fourier space, we now consider its translation $\check{\phi}_\theta(x - x_0)$ and the sum $\sum a_k\check{\phi}_\theta(x - x_0)$. Recall the locally constant lemma in lecture 2 states that if we consider the a function f whose Fourier support $supp(\widehat{f}) \subset I$, then we have the locally constant lemma which states $\|f\|_{L^\infty(I)} \lesssim \|f\|_{L^1(\omega_I)}$, where ω_I is 1 on I and rapidly decays off of I. Here we present a similar result.

Lemma 0.4 (Locally Constant)

Suppose f is such that $supp(\widehat{f}) \subset \theta$, and T is some translation of θ^*, then the locally constant lemma holds on T as well, namely,

$$\|f\|_{L^\infty(T)} \lesssim \|f\|_{L^1(\omega_T)}$$

where the L^1 norm here is the average L^1 norm.

Proof Just like how we proved the unit interval case, we define a smooth bump function in the Fourier space that captures f. Let η bu such that it is 1 on θ and decays rapidly outside of θ, $|\eta(\omega)| \leq \left(\frac{1}{1+|\omega|}\right)^M$, for all large M. Then we have $\widehat{f} = \eta\widehat{f}$, hence

$$|f(x)| \leq \sup_{x\in T}\int |f(y)\check{\eta}(x-y)|dy \leq \int |f(y)| \sup_{x\in T}|\check{\eta}(x-y)|dy$$

As we noted above, $|\check{\eta}| \sim |\theta| = \frac{1}{|\theta^*|}$ on θ^* and decays rapidly off of it. And we note $\omega_T(y) = \sup_{x\in T}\check{\eta}(x-y)$ also behaves on T like this.

Let's now get in business. Consider single wavepackets f_θ, a wave packet is the inverse Fourier transform of a bump function in the Fourier space, for example, $f_\theta = \check{\eta}$, where $\eta(omega) = 1$ on θ and decays rapidly off of it. Here we consider these wave packets normalized as $f_\theta(0) = 1$, let $f = \sum_\theta f_\theta$.

Then in a small neighborhood of 0, where not much cancellation can happen, we have $|f(0)|$ equals the number of θ, which is $R^{(n-1)/2}$. Hence the L^p norm of f.

$$\|f\|_{L^p} \geq \|f\|_{B_1} \gtrsim R^{(n-1)/2}$$

And now we consider the RHS of the decoupling relation, which is the L^p norm of each wave packet, that is f_θ. Because $\widehat{f_\theta}$ is supported in θ and f_θ is normalized, hence f_θ is 1 on θ^* and decays rapidly off of it. Hence,

$$\|f\|_{L^p} \sim |\theta^*|^{1/p} = R^{(n+1)/(2p)}$$

Hence we have,

$$\left(\sum_\theta \|f\|_{L^p}^2\right)^{1/2} \sim R^{\frac{n-1}{4}+\frac{n+1}{2p}}$$

Combining, we have $R^{(n-1)/2} \lesssim D_p R^{\frac{n-1}{4}+\frac{n+1}{2p}}$, hence $D_p \gtrsim R^{\frac{n-1}{4}-\frac{n+1}{2p}}$. And Bourgain and Demeter claim up to an R^ϵ loss, this is the worst that can happen.

Theorem 0.2 (Bourgain and Demeter)

$D_p(R) \lesssim R^\epsilon \max\{R^{\frac{n-1}{4}-\frac{n+1}{2p}}, 1\}$

Lecture 4

We first start with a warm up problem using projections. Consider a set U in $\mathbb{R}^3$ is bounded,

$$\begin{cases} Area(Proj_{xy-plane}(U)) = A \\ Area(Proj_{xz-plane}(U)) = B \\ Area(Proj_{yz-plane}(U)) = C \end{cases} \quad \text{Then what is } |U| \text{ bounded by?}$$

Proposition 0.6

$$|U| \leq (ABC)^{1/2}$$

Proof We first just consider the projection of U onto the xy-plane, i.e., fix z, we define $U_z = \{(x,y) : (x,y,z) \in U\}$, then for any z, we have $|U_z| \leq A$. Then we cut up U_z to find another bound using B, C. For U_z, fix y, if we define $X_z = \{x \in : (x,y) \in U_z\}$, and $Y_z = \{y \in \mathbb{R} : (x,y) \in U_z\}$. Hence by a simple picture, we have $|U_z| \leq |X_z||Y_z|$. Then we apply Cauchy-Schwarz:

$$|U| \leq \int_z |U_z| dz \leq A^{1/2}(|X_z||Y_z|)^{1/2} \leq A^{1/2}\left(\int_z |X_z|\right)^{1/2}\left(\int_z |Y_z|\right)^{1/2} \leq (ABC)^{1/2}$$

As an immediate corollary, if we consider the boundary of U, because the projection to any plane is $\leq |\partial U|$ (as if you are mapping the whole surface out, which is "maximal" projection)

$$|U| \leq |\partial U|^{3/2}$$

0.0.1 Tubes in different directions

We first introduce the theorem that connects projections and incidence geometry.

Theorem 0.3 (Loomis-Whitney)

For $j = 1, \ldots, n$, define $f_j : \mathbb{R}^{n-1} \to R^+$, and $\pi_j : R^n \to \mathbb{R}^{n-1}$, and the projection by forgetting about the j-th coordinate, then we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n (f_j \circ \pi_j)^{\frac{1}{n-1}} \leq \prod_{j=1}^n \left(\int_{R^{n-1}} f_j\right)^{\frac{1}{n-1}}$$

Now we look at an example, $U \subset \mathbb{R}^n$ is a bounded region and $f_j = \chi_{\pi_j(U)}$ be the characteristic function of $\pi_j(U)$ and Loomis Whitney guarantees that we have,

$$|U| \leq \prod_{j=1}^n (|\pi_j(U)|)^{\frac{1}{n-1}}$$

$l_{j,a} \subset \mathbb{R}^n$ to be lines that are parallel to x_j-axis for $1 \leq j \leq n$, and $1 \leq a \leq N_j$. Let $T_{j,a}$ be the characteristic function of the 1-nbd of $l_{j,a}$. Surprisingly, we can provide some estimates on how these $T_{j,a}$ intersect.

Proposition 0.7 (parallel case)

Let everything be defined as above,i.e. when all lines are parallel to the axis, we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^n N_j^{\frac{1}{n-1}}$$

Proof We need to define the π_j, f_j's. Let π_j be forgetting the j-th coordinate, and let $f_j = \sum_{a=1}^{N_j} \chi_{D_{j,a}}$, where $D_{j,a}$ denotes the disk of radius 1, with center where $l_{j,a}$ intersects R^{n-1} with j-th axis removed ($l_{j,a} \cap \mathbb{R}^{n-1}$). Thus we have

$\sum_{a=1}^{N_j} T_{j,a} = f_j \circ \pi_j$, hence we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} = \int_{\mathbb{R}^n} \prod_{j=1}^{n} (f_j \circ \pi_j)^{\frac{1}{n-1}} \leq \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n}} = \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} \sum_{a=1}^{N_j} \chi_{D_{j,a}} \right)^{\frac{1}{n-1}} \lesssim \prod_{j=1^n} N_j^{\frac{1}{n-1}}$$

Where the last inequality follows from equality holds only when $D_{j,a}$ have no intersections/overlaps.

Example 0.1 Now we look at a cube of side length s, then for each direction j, its "area" is s^{n-1}, hence the number of tubes for each j is s^{n-1}. And and

$$\sum_{a=1}^{N_j} T_{j,a} = 1 \text{ on } E, 0 \text{ elsewhere}$$

Hence we have the LHS of Loomis-Whitney inequality as

$$LHS \sim \int_{Q_s} 1 = s^n, RHS = \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} = s^n$$

This means the inequality in the proposition is quite sharp.

Now we investigate the case where $l_{j,a}$ are not parallel to the x_j-axis, but rather are with an angle no larger than $\frac{1}{100n}$, and will the bound using Loomis-Whitney in the previous proposition still hold? With a simple picture, the answer is yes in $\mathbb{R}^2$.

$$\int_{\mathbb{R}^2} \sum_{a=1}^{N_1} T_{1,a} \sum_{b=1}^{N_2} T_{2,b} = \sum_{a=1}^{N_1} \sum_{b=1}^{N_2} \int_{\mathbb{R}^2} T_{1,a} \cdot T_{2,b} \lesssim N_1 N_2$$

The last inequality follows from the fact that the product of $T_{1,a} \cdot T_{2,b}$ is equal to 1 in an area that may be slightly larger than 1 (going at different angles). The same inequality would hold if we replace lines with curves, and denote the angle at a given point $x \in \gamma_{j,a}$ to be the angle between the tangent line and the x_j-axis and have that less than δ. However, the result is incorrect in $n = 3$ for cuves, but there is a good result about tilting lines by Bennett-Carbery-Tao in 2005.

Theorem 0.4 (Bennett-Carbery-Tao, 2005)

Let $l_{j,a}$ be lines in $\mathbb{R}^n$ with an angle between $l_{j,a}$ and the x_j-axis no larger than $\frac{1}{100n}$. Let $T_{j,a}$ be the characteristic function of 1-neighborhood of $l_{j,a}$ (i.e. a tube of radius 1), then suppose Q_s is a cube of side length s, then for any $\epsilon > 0$, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

♡

In other words, the inequality of Loomis-Whitney holds in $\mathbb{R}^n$ but with an s^ϵ loss.

We introduce the main lemma that is used to prove this theorem.

Lemma 0.5 (Main lemma)

For any $\epsilon > 0$, there exists $\delta > 0$ such that if all angles between $l_{j,a}$ and the x_j-axis are no larger than δ, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

♡

We now prove how the main lemma implies the Theorem above. First we note that the angle between lines are no larger than $1/100n$, which is not our any $\delta > 0$. But we can decompose the area on S^{n-1} that are formed by points of lines going through the origin that are within $1/100n$ from the x_j-axis.

For each j, denote the area on the unit sphere S^{-1} that is formed by points of lines $l_{j,a}$ of angle with x_j-axis no larger than $1/100n$, S_j, then we decompose S_j into smaller portions of $S_j = \bigcup_b S_{j,b}$, where $S_{j,b}$ denotes the area that have diameter no larger than $\delta/10$ hence the angle that S_j subtended is less than δ. In $n = 3$, there would be S_1, S_2, S_3. We further note that e_j lives on S^{n-1} and is the center of each S_j. Note that if for all b $S_{j,b}$ is centered at e_j, then all lines are no more than δ from the x_j-axis, then we would have

If we denote $g_j = \sum_{a=1}^{N_j} T_{j,a}$, and denote $g_{j,b}$ as those if we pick out those in $S_{j,b}$, then $g = \sum_b g_{j,b}$. Then in the theorem, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} = \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{b} g_{j,b} \right)^{\frac{1}{n-1}} \leq \sum_b \int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}} \lesssim \int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}}$$

The last inequality follows from the constant depends on n, δ. Now to use the lemma, we note that if all $g_{j,b}$ are centered at e_j, then all lines are within δ of x_j-axis, but this is not the case. However, if one applies a linear transformation, the determinant is controlled by δ, heence by an epsilon loss. Hence applying the lemma, we get

$$\int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

We now attempt to prove the main lemma now, and we do so by looking at a cube of size $s \leq \delta^{-1}$ and cutting it up into smaller cubes, and getting a good bound on the smaller ones, and finally rescaling $s \leq \delta^{-1}$ to $\leq \delta^{-m}$ for any large m.

Lemma 0.6

Let $s \leq \delta^{-1}$, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}}$$

where $N_j(Q_s) = \{a : l_{j,a} \cap Q_s \neq \emptyset\}$, i.e. it is no larger than N_j. ♡

Proof Since we can get a good bound on parallel case, we note that for each $T_{j,a}$, since $s \leq \delta^{-1}$, we can find $\tilde{T}_{j,a}$ suhc that $T_{j,a} \subset \tilde{T}_{j,a}$, and we use the same $\tilde{T}_{j,a}$ to denote its characteristic function. Hence we have,

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \leq \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} \tilde{T}_{j,a} \right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}}$$

where the last inequality follows from the parallel proposition.

Then we consider smaller cube that are inside this cube. First we introduce the "thickening" of tubes, define $T_{j,a,w}$ to be the w neighborhood of line $l_{j,a}$, namely, when $w = 1, T_{j,a,1} = T_{j,a}$. This allows us to look at "thick" tubes and have them contain smaller ones.

Lemma 0.7

Let $\frac{1}{20n}\delta^{-1} \leq s \leq \frac{1}{10n}\delta^{-1}$, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim_n \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}}$$

♡

Proof By the previous lemma, it suffices to show that

$$\prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}} \lesssim \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}} \right)^{\frac{1}{n-1}}$$

We note that if $s \leq \frac{1}{10n}\delta^{-1}$, then if $T_{j,a} \cap Q_s \neq \emptyset$, then we have $T_{j,a,\delta^{-1}} = 1$ on Q_s, since $T_{j,a,\delta^{-1}}$ is simply a large, fat tube that contains Q_s. Hence we have, on Q_s,

$$N_j(Q_s) \leq \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}}$$

Moreover, because $s \leq \frac{1}{10n}\delta^{-1}$, we have $|Q_s| \lesssim \delta^{-n}$, thus obtaining the inequality. And by the same reasoning, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-(m-1)}} \right)^{\frac{1}{n-1}} \lesssim_n \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-m}} \right)^{\frac{1}{n-1}}$$

And by inducting on m, we get

$$\int_{Q^s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}\le C(n)^m\delta^{nm}\int_{Q^s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}}\right)^{\frac{1}{n-1}}$$

Now we connect the above two lemmas.

Proposition 0.8

Let $s\le\delta^{-1}$, then we have

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}\le C(n)\delta^{n}\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

Proof This essentially follows from cutting Q_s up into smaller cubes of the form in the previous lemma and use the lemma. Let $Q_s=\sum_b Q_{s,b}$,

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}=\sum_b\int_{Q_{s,b}}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}\le C(n)\delta^{n}\sum_b\int_{Q_{s,b}}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

with the RHS $=C(n)\delta^n\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$. And by the same reasoning, going along with what happened right before this proposition, for arbitrary large m, and a cube with $s\le\delta^{-m}$, we have

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}\le C(n)^m\delta^{nm}\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}}\right)^{\frac{1}{n-1}}$$

Finally, we have all the tools to prove the main lemma, i.e. there always exists $\delta>0$ such that

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}}\lesssim s^{\epsilon}\prod_{j=1}^{n}N_j^{\frac{1}{n-1}}$$

Proof Note that $\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}}\le N_j$, and $|Q_s|\le\delta^{-nm}$, hence it suffices to choose $C(n)\le s^{\epsilon}$. We note that

$$C(n)^m\le C(n)^{\frac{\log s}{-\log(\delta)}}=s^{\frac{\log C(n)}{-\log\delta}}$$

Hence it suffices to choose δ small enough such that $\frac{\log C(n)}{-\log\delta}\le\epsilon$ as required.

□

Lecture 5

It's okay. we can make this work. We will see how one can use multilinear restriction kakeya conjecture towards the restriction theory. More specifically, we will try and see how one can say about the size of f given that $supp(\widehat{f} \subset \Omega)$, and knowing the "geometry" of Ω.

We use $f = E\phi$ to denote $f = \int e^{2\pi i x\omega}\phi(\omega)d\omega$, then if $f = E_\Omega\phi$, then $supp(\widehat{f}) \subset \Omega$.

Let's go back to a previous example, where $\theta \subset S^{-1}$ is a $R^{-1/2}$-cap on the unit sphere in $\mathbb{R}^n$. Let ϕ_θ be a smooth bump funciton on θ, $\|\phi\|_{L^p} \sim |\theta|^{1/p} \sim R^{\frac{-(n-1)}{2p}}$, then we have

$$|E\phi_\theta(x)| \sim |\theta| \sim R^{\frac{-(n-1)}{2}} \text{ on } \theta^*$$

Hence

$$\|E\phi_\theta\|_{B_R} \sim R^{\frac{-(n-1)}{2}+\frac{n+1}{2p}}$$

We will now consider the entire S^{-1} and dissect it into spherical caps θ of size $R^{-1/2}$ as above, then the number θ's is $R^{\frac{n-1}{2}}$. Then let ϕ be constant 1 on S^{-1}, and let $\phi = \sum_\theta \phi_\theta$ (each ϕ_θ) is the wave packet like above. If we let B denote where every θ^* intsersect and have small frequency change, then $|E\phi| \sim \sum_\theta |E\phi_\theta| \sim R^{-(n-1)/2} \cdot R^{(n-1)/2} = 1$ on B, but outside of B and places where there are many intersections of θ^*'s, say, $B_R \setminus B_{R/2}$, we have $|E\phi(x)| \sim |\theta| = R^{-(n-1)/2}$ on $B_R \setminus B_{R/2}$. To sum up, $|E\phi_\theta| \sim 1$ where all θ^*'s intsersect and die off as you move away from the center. Hence the larger the p, the more focus on the higher values, hence we have, by interpolation,

$$\|E\phi\|_{L^p(B_R)} \sim \begin{cases} 1, \text{ for } p > \frac{2n}{n-1} \\ R^{\frac{-(n-1)}{2}}|B_R|^{1/p} = R^{\frac{-(n-1)}{2}-\frac{n}{p}}, p < \frac{2n}{n-1} \end{cases}$$

What does the above example of wave packets on S^{n-1} tell us? Recall $\|\phi_\theta\|_{L^q} \sim R^{\frac{-(n-1)}{2q}}$, and the conjecture is that we can control the size of $\|E\phi\|_{L^p(B_R)}$ using the size of $\|\phi\|_{L^q}$. The conjecture is as follows:

$$\frac{\|E\phi\|_{L^p(B_R)}}{\|\phi\|_{L^q}} \lesssim R^\epsilon$$

For $q = 2$, we have Tomas-Stein,

Theorem 0.5 (Tomas-Stein)

If $p \geq \frac{2(n+1)}{n-1}$, then $\|E\phi\|_{L^p(B_R)} \lesssim \|\phi\|_{L^2}$

And this gives one end of the range to put on q, and Stein conjectures the other end:

$$\|E\phi\|_{L^p(B_R)} \lesssim \|\phi\|_{L^\infty}, p > \frac{2n}{n-1}$$

Note: Only $n = 2$ has been proven. Here we will us the wave packet approach. If we let $S^{n-1} = \bigcup \theta$, where θ are $R^{-1/2}$ caps, and let $\phi = \sum \phi_\theta$. By the Locally Constant Lemma, we have

$$|E\phi(x)| \sim \text{ constant on } T$$

where T are translates of θ^*. Hence, if we add things up "horizontally" like this, then we have

$$|E\phi(x)| \sim \sum_T a_T \chi_T$$

where T are parallel to θ^* and are disjoint.

By Plancherel identity, we have

$$\|E\phi\|_{L^2} = \|\phi\|_{L^2}$$

which means

$$\sum a_T^2 \sim \|\phi\|_{L^2}^2 \leq \sum_\theta \|\phi_\theta\|_{L^2}^2 \sim R^{\frac{n-1}{2}} \|\phi_\theta\|_{L^2}^2$$

To prove the decoupling, we have to look at how the tubes T (which are translates of θ^*) intersect. For each $B_{R^{1/2}}$, we define

$$\mu(B_R^{1/2} = \sum_{T\cap B_{R^{1/2}}\neq\emptyset} a_T^2$$

The Kakeya problem concerns with how many $B_{R^{1/2}}$ such that $\mu \sim 2^k$ for some k. The next thing we will do is to estimate $\|E\phi\|_{L^p(B_R)}$.

Proposition 0.9

The inverse Fourier transform of characteristic functions $\check{\chi}_\theta$ doesn't decay fast.

Proof

$$\check{\phi}(x) = \int \phi(\omega)e^{2\pi i x\omega}d\omega = \int_E \phi(\omega)e^{2\pi i x\omega}d\omega = \frac{e^{2\pi i x\omega}}{2\pi i x}\Big|_E$$