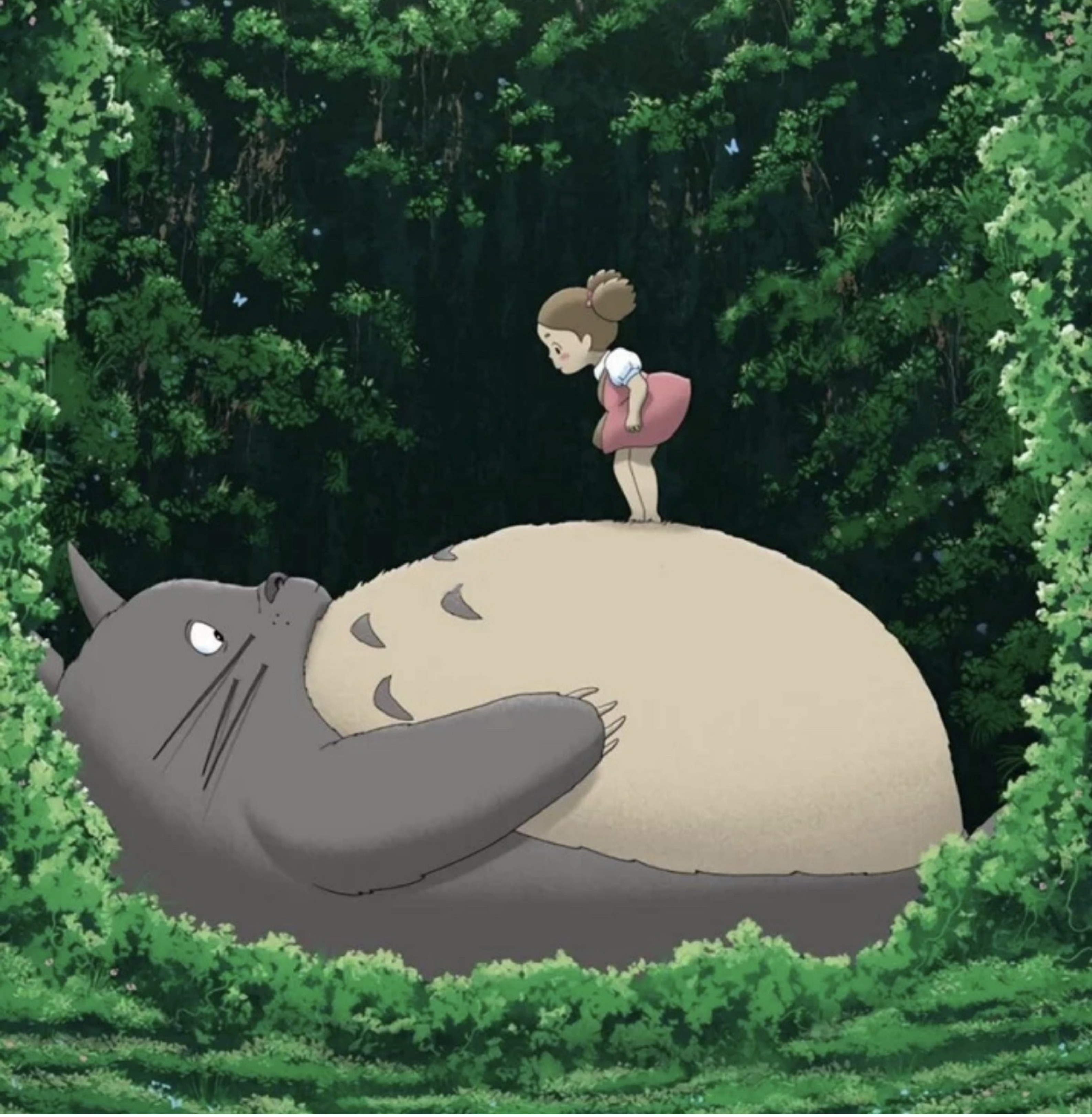

Real Analysis

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Contents

We consider the following conjecture:

$$|\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 &= (\sum_{a=1}^{n} e^{ia^3 x})(\sum_{a=1}^{n} e^{-ib^3 x}) \\ &= \sum_{a_1,a_2,a_3,b_1,b_2,b_3} e^{ix(a_1^3+a_2^3+a_3^3-b_1^3-b_2^3-b_3^3)} \end{aligned}$$

Hence the integral is 0 if the diophatine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophatine equation.

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Introduction to decoupling

Now we move to the overview of decoupling.

If we denote a region Ω of $\mathbb{R}^n$ as the Fourier space, and we decompose it into small regions $\Omega = \bigsqcup \theta$.

If we assume the function f whose Fourier transform has support in the region Ω, then we can decompose Ω, we will now make the definition as follows.

Definition 0.1 (Decoupling)

Let f be a sufficiently regular function whose supp$(\widehat{f}) \subset \Omega$, if we define

$$f_\theta = \int_\theta \widehat{f}(\omega) e^{ix\omega} d\omega$$

Then by Fourier inverse formula, we get

$$f = \sum_\theta f_\theta$$

♣

Proof $\sum_\theta f\theta = \int_\Omega \widehat{f}(\omega) e^{ix\omega} d\omega = f(x)$

One would like to control the norm $||f||_{L^p}$, using what you know about $||f_\theta||_{L^p}$. To give a general idea what we are heading towards, we can fine a constant D_p, dependent on Ω, θ, such that the following inequality is achieved.

$$||f||_{L^p}(\mathbb{R}^n) \leq D_p(\Omega = \bigsqcup \theta) \left(\sum_{\theta} ||f_\theta||_{L^p}^2 \right)^{1/2}$$

Hence if we fix a specific decoupling choice, i.e. $\Omega = \bigsqcup_\theta \theta$, we can find an absolute constant such that

$$||f||_{L^p} \leq C \left(\sum_{\theta} ||f_\theta||_{L^p}^2 \right)^{1/2}$$

And we define the decoupling constant $D_p(\Omega = \bigsqcup \theta)$ to be the smallest of all C, for a fixed decoupling choice.

Remark Are there different ways of decomposing omega? Is the most intuitive way of decomposing the Fourier space based on frequency?

Proposition 0.1 (Estimate of sum)

Let $R > 0$, and pick Fourier space decomposition $\Omega = \bigsqcup_j \theta_j$, and let $g = \sum_j a_j e^{i\omega_j \cdot x}$. If $B_{1/R}(\omega_j) \subset \theta_j$, for all j, then for any ball of radius R, B_R, we have

$$||g||_{L^p(B_R)} \lesssim D_p \left(\sum_j |a_j|^2 \right)^{1/2} R^{1/p}$$

Proof Let $f = \eta g$, such that $supp(\widehat{\eta}) \subset B_{1/R}$, such that $|\eta| \sim 1$ on B_R and decays rapidly outside of B_R.

Lecture 2

Now we begin with some building blocks.

Suppose $\Omega = [0, N], \theta_j = [j-1.j], \Omega = \bigsqcup_{j=1}^{N} \theta_j$. And we ask the question, if we have $supp(\widehat{f}) \subset [0,1]$, could $|f|$ look like several narrow peaks and almost 0 elsewhere?

We recall how we decouple the function f: for $supp(\widehat{f}) \subset \Omega$, define $f_{\theta_j} = \int_{[j-1,j]} \widehat{f}(\omega) e^{i\omega x} d\omega$, then $f = \sum_j f_{\theta_j}$.

Now we remind ourselves of the height of f.

Proposition 0.2

Let $f \in \mathcal{S}$ be such that $supp(\widehat{f}) \subset [0,1]$, and we have

$$||f||_{L^\infty} \lesssim ||f||_{L^1}$$

Proof We define a cutoff function $\eta \in \mathcal{S}$ such that $\eta = 1$ on $[0,1]$, then $\widehat{f} = \eta \widehat{f}$, then $f = f * \check{\eta}$, also a Schwartz function.

$$\begin{aligned} ||f||_{L^\infty} &= ||f * \check{\eta}||_{L^\infty} \\ &\leq ||f||_{L^1} ||\check{\eta}||_{L^\infty} \\ &\lesssim ||f||_{L^1} \end{aligned}$$

Hence the answer is no, because if we have narrow peaks with controlled heights, $||f||_{L^1}$ would be small, which would violate $||f||_{L^\infty} \lesssim ||f||_{L^1}$.

Now we ask the following question, can we have flat parts of $|f|$ where $||f||_{L^1}$ is dominated by the flat parts, but still has narrow peaks? To address that, we introduce an important lemma which allows us to control the height of f in one interval using its L^1 norm in an even larger interval.

Proposition 0.3 (Locally Constant Lemma)

If $supp\widehat{f_1} \subset [0,1]$, and I is the unit interval $[0,1]$, then we have

$$||f||_{L^\infty(I)} \lesssim ||f||_{L^1(\omega_I}$$

Where the weighted L^1 norm is defined to be $||f||_{L^1(\omega_I)} = \int_{\mathbb{R}} |f_1| \omega_I$ where the function ω_I satisfies the following: $\omega_I \geq 0$, $\omega_I \sim 1$ on I, and ω_I decays rapidly off of I, lastly, ω_I is uniform in the sense that $\omega_{I+a} = \omega_I(\cdot - a)$

Proof This follows from the fact that $\eta \in \mathcal{S}$, hence $\check{\eta} \in \mathcal{S}$ as well, i.e. we have

$$|\check{\eta}(y)| \lesssim \left(\frac{1}{1+|y|}\right)^M$$

for all large M. Hence we follow the same computation:

$$\begin{aligned} |f(x)| &= \left| \int f(y) \check{\eta}(x-y) dy \right| \\ &\leq \int |f(y)||\check{\eta}(x-y)| dy \\ &\leq \int |f(y)| \sup_{x \in I} |\check{\eta}(x-y)| dy \end{aligned}$$

And if we define $\omega_I(y) = \sup_{x\in I} |\check{\eta}(x-y)|$, surely it satisfies being nonnegative, and by property of $\check{\eta}$ being Schwartz, $\check{\eta} \sim 1$ on I, and decays rapidly if $|x-y|$ is greater than 0.

In other words, we almost know that $||f||_{L^\infty(I)} \lesssim ||f||_{L^1(2I)}$, where $2I$ is if we stretch the intervals keeping the same center.

Remark For $p=2, p=\infty$, the decoupling constant is easier to estimate. For $p=2$, we can apply Plancherel, namaley,

$$||f||_{L^2} = ||\sum_j f_{\theta_j}||_{L^2} = ||\sum_j \widehat{f}_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2}$$

For $p = \infty$, we can apply Cauchy Schwartz, namely,

$$||f||_{L^\infty} = ||\sum_j f_{\theta_j}||_{L^\infty} \le \sum_j ||f_{\theta_j}||_{L^\infty} \le \left(\sum_j ||f_{\theta_j}||^2_{L^\infty}\right)^{1/2} N^{1/2}$$

And now we conclude with an example. Consider a function f_1 with height 1, (i.e. $||f||_{L^\infty} = 1$) and $f_1(0) = 1$ and it is concentrated on the interval $[-1, 1]$. If we define $f_j(x) = e^{2\pi i(j-1)x} f_1(x)$ and define $f = \sum_j f_j$, then we have $f_j(0) = 1$ and thus $f(0) = N$.

We note that f_j oscillates with frequency $\frac{1}{j}$ and when $|x| \le \frac{1}{10N} \le \frac{1}{10j}$, we have $f_(x) \sim N$. Hence, if we consider $||f||_{L^p}$, we have

$$||f||^p_{L^p} = \int |f|^p = \ge \int_{|x|\le \frac{1}{10N}} |f|^p = \gtrsim \frac{1}{N} \cdot N^p = N^{p-1}$$

Hence taking the $1/p$ of both sides, we have $||f||_{L^p} \gtrsim N^{1-1/p}$.

Now if we would wish to consider the decoupling constant, we now consider $||f_j||_{L^p}$. Note $||f_j||_{L^p} \sim 1$, hence $\left(\sum_j ||f_j||^2_{L^2}\right) \sim N^{1/2}$. Thus we have $D_p \gtrsim N^{1/2-1/p}$.

Main Obstacle

Consider a function f_j such that $|f_j| = 1$ on $[0, 1]$, and is $\frac{1}{N}$ on $[1, N^3]$, and 0 elsewhere. Then the $||f_j||_{L^2} \sim N^{1/2}$, whereas $||f_j||_{L^4} \sim 1$. (Exactly how one owuld expect the L^p norm to behave).

Like the above remark, we note that $||f||_{L^2} \sim \sum_j ||f_j||^2_{L^2})^{1/2} \sim N$, and $||f||_{L^\infty} \le N^{1/2}(N)^{1/2} = N$. Now we ask the question, could $|f(x)| \sim N$ on the unit interval $[0, 1]$? The answer is no.

Proof Assume $|f(x)| \sim N$ on $[0, 1]$, then $||f||_{L^4} \gtrsim N$, however, we know

$$||f||_{L^4} \lesssim D_p \left(\sum_j ||f_j||^2_{L^4} \lesssim N^{1/4} \cdot N^{1/2} = N^{3/4}\right)$$

Note D_p arises from our above lower bound given that $p = 4$.

Recall the Local Constant Lemma tells us how the height is controlled by the L^1 norm, now we introduce another lemma that connects the L^2 norms, which improves our estimate.

Lemma 0.1 (Local Orthogonality Lemma)

If I is a unit interval, and $f = \sum_{j=1}^N f_j$, and $supp\widehat{f_j} \subset [j-1, j]$, then we have

$$||f||^2_{L^2(I)} \lesssim \sum_j ||f_j||^2_{L^2(\omega_I)}$$

♡

Proof We choose η such that it preserves f on the unit interval, and whose fourier transform has support land in $[-1, 1]$, i.e. $|\eta| \sim 1$ on I, and $supp(\eta) \subset [-1, 1]$.

$$\begin{aligned}
||f||^2_{L^2(I)} &= \int_I |f|^2 \\
&\le \int_{\mathbb{R}} |\eta f|^2 \\
&= \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \int | \sum_j \widehat{\eta} * \widehat{f}|^2 \\
&\lesssim \sum_j \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \sum_j \int_{\mathbb{R}} |\eta|^2 |f_j|^2 \\
&= \sum_j ||f||^2_{L^2(\omega_I)}
\end{aligned}$$

if we define $\omega_I = |\eta|^2$.

We thus obtain this local orthogonality result, in the sense that we can decompose the L^2 norm locally and control the L^2 norm of f by the sum of the L^2 norm of f_j.

Now we generalize this to a wide range of p to obtain our local decoupling result.

Proposition 0.4 (Local decoupling)

If I is a unit interval, for $2 \le p \le \infty$, for each $1 \le j \le N$, $supp(f_j) \subset [j-1, j]$, then we have

$$||f||_{L^p(I)} \lesssim N^{1/2-1/p} \left(\sum_{j=1}^{N} ||f_j||^2_{L^p(\omega_I)} \right)^{1/2}$$

Proof This follows from the Locally Constant Lemma and the Locally Orthogonality Lemma above.

$$\int |f|^p = \int |f|^2 |f|^{p-2} \le ||f||^{p-2}_{L^\infty(I)} \int |f|^2 \le \sum_j ||f_j||^2_{L^2(\omega_I)} \left(\sum_j ||f_j||_{L^\infty(I)} \right)^{p-2}$$

The last inequality follows from the local orthogonality lemma above which states $||f||_[L^2(I)] \le \sum_j ||f_j||_{L^2(\omega_I)}$.

Then for the second term, local constant lemma states that the height is controlled by the L^1 norm $||f_j||_{L^\infty} \lesssim ||f_j||_{L^1(\omega_I)} \lesssim ||f_j||_{L^2}$, where the last inequality is to match the L^2 norm of the first term. Combining, we have $\int |f|^p \le (\sum_j \|f_j\|^2_{L^2(\omega_I)})(\sum_j \|f\|_{L^2(\omega_I)})^{p-2}$. By Cauchy Schwarz on the second term, we obtain,

$$\int |f|^p \le \left(\sum_j ||f_j||^2_{L^2(\omega_I)} \right) \left(\sum_j ||f_j||^2_{L^2(\omega_I)} \right)^{p/2-1} N^{p/2-1} = \left(\sum_j ||f_j||^2_{L^2(\omega_I)} \right)^{p/2} N^{p/2-1}$$

If we replace the $\|f\|_{L^2}$ with $\|f\|_{L^p}$, we get the desired result.

Lemma 0.2

In finite measure spaces, for $p \ge q$, we have

$$\|f\|_{L^p} \lesssim \|f\|_{L^q}$$

Proof This follows from Holder's inequality.

$$\int_I |f|^p = \||f|^p\|_{L^{q/p}} \mu(I)^s \lesssim \|f\|^p_{L^q}$$

Now we prove the parallel decoupling lemma, which basically states that if we decompose two measures as $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, and for each i, we have the same decoupling constant, then we would be able to keep that decoupling constant when we sum them up. Recall the Minkowski's inequality refers to triangle inequality with respect to the L^p

norm.

Proposition 0.5 (Parallel Decoupling Lemma)

For some $p \geq 2$, and for any function $g = \sum_j g_j$, and any measures $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, then if for each i, we have

$$\|g\|_{L^p(\mu_i)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega_i)} \right)^{1/2}$$

then summing up, we would have the combined inequallity with the same decoupling constant,

$$\|g\|_{L^p(\mu)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega)} \right)^{1/2}$$

Proof The proof uses the Minkowski's inequality.

$$\begin{aligned}
\int |g|^p \mu &= \sum_i \int |g|^p \mu_i \\
&\leq D^p \sum_i \left(\sum_j ||g_j||^2_{L^p(\omega_i)} \right)^{p/2} \\
&\leq D^p \left\| \sum_j ||g_j||^2_{L^p(\omega_i)} \right\|^{p/2}_{l_i^{p/2}} \\
&\leq D^p \left(\sum_j \left\| ||g_j||^2_{L^p(\omega_i)} \right\|_{l_i^{p/2}} \right)^{p/2} \\
&= D^p \left(\sum_j \left(\sum_i ||g_j||^p_{L^p(\omega_i)} \right)^{2/p} \right)^{p/2} \\
&= D^p \left(\sum_j ||g_j||^2_{L^p(\omega)} \right)^{p/2}
\end{aligned}$$

Lecture 3

We state Bourgain and Demeter's decoupling theorem for the paraboloid. We define a parabaloid as

$$P = \{\omega \in \mathbb{R}^n : \omega_n = \omega_1^2 + ...\omega_{n-1}^2, |\omega| \leq 1\}$$

We now define a slightly larger neighborhood of P, denoted by $N_{1/R}P$ as the neighborhood of P of radius $1/R$ where R is some large constant. This is the area that we will cut up using rectangles θ,

$$\theta \approx R^{-1/2} \times R^{-1/2} \times ... \times R^{-1}$$

Now we've fixed a decomposition, we denote its decoupling constant $D_p(R) = D_p(\omega = \cup\theta)$, and we ar eready to state our decoupling theorem.

Theorem 0.1 (Bourgain and Demeter)

For $2 \leq p \leq \frac{2(n+1)}{n-1}$, for the "cutting up" scheme above, we have

$$D_p(R) \lesssim R^{\epsilon}$$

♡

Now we define the dual of θ, denoted by $\theta^* = \{x \in \mathbb{R}^n : |\omega - \omega_\theta| < \frac{1}{x}, \forall \omega \in \theta\}$, with ω_θ being the center of θ. Now we look at how these are $\widehat{\phi}$ behaves on $\theta*$ and then rapidly decaying.

Lemma 0.3

If ϕ_θ is a smooth bump function supported in θ, then $\check{\phi_\theta} \sim |\theta|$ on $\theta^*|$ and rapidly decaying outside of it.

♡

Proof If ϕ_θ lives on θ, we calculate its inverse Fourier transform to find the corresponding function in the physical space.

$$|\check{\phi_\theta}| = \left|\int e^{2\pi i x\omega}\phi_\theta(\omega)d\omega\right| = \left|e^{2\pi i x\omega_\theta}\int e^{2\pi i x(\omega-\omega_\theta)}\phi_\theta(\omega)d\omega\right| = \left|\int_\theta e^{2\pi i x(\omega-\omega_\theta)}\right| \sim |\theta|$$

If $x \in \theta^*$, then we know $|(\omega - \omega_\theta)x| < 1$, hence the oscillatory integral on the right hand side does not get too much cancellation, hence $|\check{\phi_\theta}| \sim |\theta|$, while outside of θ^*, if one integrate by parts many times, then get rapid decay.

Remark This lemma shows why the dual θ^* is called the dual, i.e., if we consider functions first in the Fourier space, then we can find the corresponding θ^* such that $\check{\phi_\theta}$ is located and the size of it. Moreover,

$$\check{\phi_\theta} \text{ roughly looks like } e^{2\pi i x\omega_\theta}|\theta|\chi_{\theta^*}$$

Based off of the smooth bump function in the Fourier space, we now consider its translation $\check{\phi_\theta}(x - x_0)$ and the sum $\sum a_k\check{\phi_\theta}(x - x_0)$. Recall the locally constant lemma in lecture 2 states that if we consider the a function f whose Fourier support $supp(\widehat{f}) \subset I$, then we have the locally constant lemma which states $\|f\|_{L^\infty(I)} \lesssim \|f\|_{L^1(\omega_I)}$, where ω_I is 1 on I and rapidly decays off of I. Here we present a similar result.

Lemma 0.4 (Locally Constant)

Suppose f is such that $supp(\widehat{f}) \subset \theta$, and T is some translation of θ^*, then the locally constant lemma holds on T as well, namely,

$$\|f\|_{L^\infty(T)} \lesssim \|f\|_{L^1(\omega_T)}$$

where the L^1 norm here is the average L^1 norm.

♡

Proof Just like how we proved the unit interval case, we define a smooth bump function in the Fourier space that captures f. Let η bu such that it is 1 on θ and decays rapidly outside of θ, $|\eta(\omega)| \leq \left(\frac{1}{1+|\omega|}\right)^M$, for all large M. Then we have $\widehat{f} = \eta\widehat{f}$, hence

$$|f(x)| \leq \sup_{x\in T}\int |f(y)\check{\eta}(x-y)|dy \leq \int |f(y)| \sup_{x\in T}|\check{\eta}(x-y)|dy$$

As we noted above, $|\check{\eta}| \sim |\theta| = \frac{1}{|\theta^*|}$ on θ^* and decays rapidly off of it. And we note $\omega_T(y) = \sup_{x\in T}\check{\eta}(x-y)$ also behaves on T like this.

Let's now get in business. Consider single wavepackets f_θ, a wave packet is the inverse Fourier transform of a bump function in the Fourier space, for example, $f_\theta = \check{\eta}$, where $\eta(omega) = 1$ on θ and decays rapidly off of it. Here we consider these wave packets normalized as $f_\theta(0) = 1$, let $f = \sum_\theta f_\theta$.

Then in a small neighborhood of 0, where not much cancellation can happen, we have $|f(0)|$ equals the number of θ, which is $R^{(n-1)/2}$. Hence the L^p norm of f.

$$\|f\|_{L^p} \geq \|f\|_{B_1} \gtrsim R^{(n-1)/2}$$

And now we consider the RHS of the decoupling relation, which is the L^p norm of each wave packet, that is f_θ. Because $\widehat{f_\theta}$ is supported in θ and f_θ is normalized, hence f_θ is 1 on θ^* and decays rapidly off of it. Hence,

$$\|f\|_{L^p} \sim |\theta^*|^{1/p} = R^{(n+1)/(2p)}$$

Hence we have,

$$\left(\sum_\theta \|f\|_{L^p}^2\right)^{1/2} \sim R^{\frac{n-1}{4}+\frac{n+1}{2p}}$$

Combining, we have $R^{(n-1)/2} \lesssim D_p R^{\frac{n-1}{4}+\frac{n+1}{2p}}$, hence $D_p \gtrsim R^{\frac{n-1}{4}-\frac{n+1}{2p}}$. And Bourgain and Demeter claim up to an R^ϵ loss, this is the worst that can happen.

Theorem 0.2 (Bourgain and Demeter)

$D_p(R) \lesssim R^\epsilon \max\{R^{\frac{n-1}{4}-\frac{n+1}{2p}}, 1\}$ ♡

Lecture 4

We first start with a warm up problem using projections. Consider a set U in $\mathbb{R}^3$ is bounded,

$$\begin{cases} Area(Proj_{xy-plane}(U)) = A \\ Area(Proj_{xz-plane}(U)) = B \\ Area(Proj_{yz-plane}(U)) = C \end{cases}$$ Then what is $|U|$ bounded by?

Proposition 0.6

$$|U| \leq (ABC)^{1/2}$$

Proof We first just consider the projection of U onto the xy-plane, i.e., fix z, we define $U_z = \{(x,y) : (x,y,z) \in U\}$, then for any z, we have $|U_z| \leq A$. Then we cut up U_z to find another bound using B, C. For U_z, fix y, if we define $X_z = \{x \in : (x,y) \in U_z\}$, and $Y_z = \{y \in \mathbb{R} : (x,y) \in U_z\}$. Hence by a simple picture, we have $|U_z| \leq |X_z||Y_z|$. Then we apply Cauchy-Schwarz:

$$|U| \leq \int_z |U_z| dz \leq A^{1/2}(|X_z||Y_z|)^{1/2} \leq A^{1/2}\left(\int_z |X_z|\right)^{1/2}\left(\int_z |Y_z|\right)^{1/2} \leq (ABC)^{1/2}$$

As an immediate corollary, if we consider the boundary of U, because the projection to any plane is $\leq |\partial U|$ (as if you are mapping the whole surface out, which is "maximal" projection)

$$|U| \leq |\partial U|^{3/2}$$

0.0.1 Tubes in different directions

We first introduce the theorem that connects projections and incidence geometry.

Theorem 0.3 (Loomis-Whitney)

For $j = 1, ..., n$, define $f_j : \mathbb{R}^{n-1} \to R^+$, and $\pi_j : R^n \to \mathbb{R}^{n-1}$, and the projection by forgetting about the j-th coordinate, then we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n (f_j \circ \pi_j)^{\frac{1}{n-1}} \leq \prod_{j=1}^n \left(\int_{R^{n-1}} f_j\right)^{\frac{1}{n-1}}$$

Now we look at an example, $U \subset \mathbb{R}^n$ is a bounded region and $f_j = \chi_{\pi_j(U)}$ be the characteristic function of $\pi_j(U)$ and Loomis Whitney guarantees that we have,

$$|U| \leq \prod_{j=1}^n (|\pi_j(U)|)^{\frac{1}{n-1}}$$

$l_{j,a} \subset \mathbb{R}^n$ to be lines that are parallel to x_j-axis for $1 \leq j \leq n$, and $1 \leq a \leq N_j$. Let $T_{j,a}$ be the characteristic function of the 1-nbd of $l_{j,a}$. Surprisingly, we can provide some estimates on how these $T_{j,a}$ intersect.

Proposition 0.7 (parallel case)

Let everything be defined as above,i.e. when all lines are parallel to the axis, we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^n \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^n N_j^{\frac{1}{n-1}}$$

Proof We need to define the π_j, f_j's. Let π_j be forgetting the j-th coordinate, and let $f_j = \sum_{a=1}^{N_j} \chi_{D_{j,a}}$, where $D_{j,a}$ denotes the disk of radius 1, with center where $l_{j,a}$ intersects R^{n-1} with j-th axis removed ($l_{j,a} \cap \mathbb{R}^{n-1}$). Thus we have

$\sum_{a=1}^{N_j} T_{j,a} = f_j \circ \pi_j$, hence we have

$$\int_{\mathbb{R}^n} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} = \int_{\mathbb{R}^n} \prod_{j=1}^{n} (f_j \circ \pi_j)^{\frac{1}{n-1}} \leq \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} f_j \right)^{\frac{1}{n}} = \prod_{j=1}^{n} \left(\int_{\mathbb{R}^{n-1}} \sum_{a=1}^{N_j} \chi_{D_{j,a}} \right)^{\frac{1}{n-1}} \lesssim \prod_{j=1^n} N_j^{\frac{1}{n-1}}$$

Where the last inequality follows from equality holds only when $D_{j,a}$ have no intersections/overlaps.

Example 0.1 Now we look at a cube of side length s, then for each direction j, its "area" is s^{n-1}, hence the number of tubes for each j is s^{n-1}. And and

$$\sum_{a=1}^{N_j} T_{j,a} = 1 \text{ on } E, 0 \text{ elsewhere}$$

Hence we have the LHS of Loomis-Whitney inequality as

$$LHS \sim \int_{Q_s} 1 = s^n, RHS = \prod_{j=1}^{n} N_j^{\frac{1}{n-1}} = s^n$$

This means the inequality in the proposition is quite sharp.

Now we investigate the case where $l_{j,a}$ are not parallel to the x_j-axis, but rather are with an angle no larger than $\frac{1}{100n}$, and will the bound using Loomis-Whitney in the previous proposition still hold? With a simple picture, the answer is yes in $\mathbb{R}^2$.

$$\int_{\mathbb{R}^2} \sum_{a=1}^{N_1} T_{1,a} \sum_{b=1}^{N_2} T_{2,b} = \sum_{a=1}^{N_1} \sum_{b=1}^{N_2} \int_{\mathbb{R}^2} T_{1,a} \cdot T_{2,b} \lesssim N_1 N_2$$

The last inequality follows from the fact that the product of $T_{1,a} \cdot T_{2,b}$ is equal to 1 in an area that may be slightly larger than 1 (going at different angles). The same inequality would hold if we replace lines with curves, and denote the angle at a given point $x \in \gamma_{j,a}$ to be the angle between the tangent line and the x_j-axis and have that less than δ. However, the result is incorrect in $n = 3$ for cuves, but there is a good result about tilting lines by Bennett-Carbery-Tao in 2005.

Theorem 0.4 (Bennett-Carbery-Tao, 2005)

Let $l_{j,a}$ be lines in $\mathbb{R}^n$ with an angle between $l_{j,a}$ and the x_j-axis no larger than $\frac{1}{100n}$. Let $T_{j,a}$ be the characteristic function of 1-neighborhood of $l_{j,a}$ (i.e. a tube of radius 1), then suppose Q_s is a cube of side length s, then for any $\epsilon > 0$, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

♡

In other words, the inequality of Loomis-Whitney holds in $\mathbb{R}^n$ but with an s^ϵ loss.

We introduce the main lemma that is used to prove this theorem.

Lemma 0.5 (Main lemma)

For any $\epsilon > 0$, there exists $\delta > 0$ such that if all angles between $l_{j,a}$ and the x_j-axis are no larger than δ, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a} \right)^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

♡

We now prove how the main lemma implies the Theorem above. First we note that the angle between lines are no larger than $1/100n$, which is not our any $\delta > 0$. But we can decompose the area on S^{n-1} that are formed by points of lines going through the origin that are within $1/100n$ from the x_j-axis.

For each j, denote the area on the unit sphere S^{-1} that is formed by points of lines $l_{j,a}$ of angle with x_j-axis no larger than $1/100n$, S_j, then we decompose S_j into smaller portions of $S_j = \bigcup_b S_{j,b}$, where $S_{j,b}$ denotes the area that have diameter no larger than $\delta/10$ hence the angle that S_j subtended is less than δ. In $n = 3$, there would be S_1, S_2, S_3. We further note that e_j lives on S^{n-1} and is the center of each S_j. Note that if for all b $S_{j,b}$ is centered at e_j, then all lines are no more than δ from the x_j-axis, then we would have

If we denote $g_j = \sum_{a=1}^{N_j} T_{j,a}$, and denote $g_{j,b}$ as those if we pick out those in $S_{j,b}$, then $g = \sum_b g_{j,b}$. Then in the theorem, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} = \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{b} g_{j,b}\right)^{\frac{1}{n-1}} \le \sum_b \int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}} \lesssim \int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}}$$

The last inequality follows from the constant depends on n, δ. Now to use the lemma, we note that if all $g_{j,b}$ are centered at e_j, then all lines are within δ of x_j-axis, but this is not the case. However, if one applies a linear transformation, the determinant is controlled by δ, heence by an epsilon loss. Hence applying the lemma, we get

$$\int_{Q^s} \prod_{j=1}^{n} g_{j,b}^{\frac{1}{n-1}} \lesssim s^{\epsilon} \prod_{j=1}^{n} N_j^{\frac{1}{n-1}}$$

We now attempt to prove the main lemma now, and we do so by looking at a cube of size $s \le \delta^{-1}$ and cutting it up into smaller cubes, and getting a good bound on the smaller ones, and finally rescaling $s \le \delta^{-1}$ to $\le \delta^{-m}$ for any large m.

Lemma 0.6

Let $s \le \delta^{-1}$, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}}$$

where $N_j(Q_s) = \{a : l_{j,a} \cap Q_s \neq \emptyset\}$, i.e. it is no larger than N_j. ♡

Proof Since we can get a good bound on parallel case, we note that for each $T_{j,a}$, since $s \le \delta^{-1}$, we can find $\tilde{T}_{j,a}$ suhc that $T_{j,a} \subset \tilde{T}_{j,a}$, and we use the same $\tilde{T}_{j,a}$ to denote its characteristic function. Hence we have,

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \le \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} \tilde{T}_{j,a}\right)^{\frac{1}{n-1}} \lesssim \prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}}$$

where the last inequality follows from the parallel proposition.

Then we consider smaller cube that are inside this cube. First we introduce the "thickening" of tubes, define $T_{j,a,w}$ to be the w neighborhood of line $l_{j,a}$, namely, when $w = 1, T_{j,a,1} = T_{j,a}$. This allows us to look at "thick" tubes and have them contain smaller ones.

Lemma 0.7

Let $\frac{1}{20n}\delta^{-1} \le s \le \frac{1}{10n}\delta^{-1}$, then we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \lesssim_n \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

♡

Proof By the previous lemma, it suffices to show that

$$\prod_{j=1}^{n} N_j(Q_s)^{\frac{1}{n-1}} \lesssim \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

We note that if $s \le \frac{1}{10n}\delta^{-1}$, then if $T_{j,a} \cap Q_s \neq \emptyset$, then we have $T_{j,a,\delta^{-1}} = 1$ on Q_s, since $T_{j,a,\delta^{-1}}$ is simply a large, fat tube that contains Q_s. Hence we have, on Q_s,

$$N_j(Q_s) \le \sum_{a=1}^{N_j} T_{j,a,\delta^{-1}}$$

Moreover, because $s \le \frac{1}{10n}\delta^{-1}$, we have $|Q_s| \lesssim \delta^{-n}$, thus obtaining the inequality. And by the same reasoning, we have

$$\int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-(m-1)}}\right)^{\frac{1}{n-1}} \lesssim_n \delta^n \int_{Q^s} \prod_{j=1}^{n} \left(\sum_{a=1}^{N_j} T_{j,a,\delta^{-m}}\right)^{\frac{1}{n-1}}$$

And by inducting on m, we get

$$\int_{Q^s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} \leq C(n)^m\delta^{nm}\int_{Q^s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}}\right)^{\frac{1}{n-1}}$$

Now we connect the above two lemmas.

Proposition 0.8

Let $s \leq \delta^{-1}$, then we have

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} \leq C(n)\delta^{n}\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

Proof This essentially follows from cutting Q_s up into smaller cubes of the form in the previous lemma and use the lemma. Let $Q_s = \sum_b Q_{s,b}$,

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} = \sum_b\int_{Q_{s,b}}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} \leq C(n)\delta^{n}\sum_b\int_{Q_{s,b}}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$$

with the RHS $= C(n)\delta^n\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-1}}\right)^{\frac{1}{n-1}}$. And by the same reasoning, going along with what happened right before this proposition, for arbitrary large m, and a cube with $s \leq \delta^{-m}$, we have

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} \leq C(n)^m\delta^{nm}\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}}\right)^{\frac{1}{n-1}}$$

Finally, we have all the tools to prove the main lemma, i.e. there always exists $\delta > 0$ such that

$$\int_{Q_s}\prod_{j=1}^{n}\left(\sum_{a=1}^{N_j}T_{j,a}\right)^{\frac{1}{n-1}} \lesssim s^{\epsilon}\prod_{j=1}^{n}N_j^{\frac{1}{n-1}}$$

Proof Note that $\sum_{a=1}^{N_j}T_{j,a,\delta^{-m}} \leq N_j$, and $|Q_s| \leq \delta^{-nm}$, hence it suffices to choose $C(n) \leq s^{\epsilon}$. We note that

$$C(n)^m \leq C(n)^{\frac{\log s}{-\log(\delta)}} = s^{\frac{\log C(n)}{-\log\delta}}$$

Hence it suffices to choose δ small enough such that $\frac{\log C(n)}{-\log\delta} \leq \epsilon$ as required.

□

Lecture 5

It's okay. we can make this work. We will see how one can use multilinear restriction kakeya conjecture towards the restriction theory. More specifically, we will try and see how one can say about the size of f given that $supp(\widehat{f} \subset \Omega)$, and knowing the "geometry" of Ω.

We use $f = E\phi$ to denote $f = \int e^{2\pi i x\omega}\phi(\omega)d\omega$, then if $f = E_\Omega\phi$, then $supp(\widehat{f}) \subset \Omega$.

Let's go back to a previous example, where $\theta \subset S^{-1}$ is a $R^{-1/2}$-cap on the unit sphere in $\mathbb{R}^n$. Let ϕ_θ be a smooth bump funciton on θ, $\|\phi\|_{L^p} \sim |\theta|^{1/p} \sim R^{\frac{-(n-1)}{2p}}$, then we have

$$|E\phi_\theta(x)| \sim |\theta| \sim R^{\frac{-(n-1)}{2}} \text{ on } \theta^*$$

Hence

$$\|E\phi_\theta\|_{B_R} \sim R^{\frac{-(n-1)}{2}+\frac{n+1}{2p}}$$

We will now consider the entire S^{-1} and dissect it into spherical caps θ of size $R^{-1/2}$ as above, then the number θ's is $R^{\frac{n-1}{2}}$. Then let ϕ be constant 1 on S^{-1}, and let $\phi = \sum_\theta \phi_\theta$ (each ϕ_θ) is the wave packet like above. If we let B denote where every θ^* intsersect and have small frequency change, then $|E\phi| \sim \sum_\theta |E\phi_\theta| \sim R^{-(n-1)/2} \cdot R^{(n-1)/2} = 1$ on B, but outside of B and places where there are many intersections of θ^*'s, say, $B_R \setminus B_{R/2}$, we have $|E\phi(x)| \sim |\theta| = R^{-(n-1)/2}$ on $B_R \setminus B_{R/2}$. To sum up, $|E\phi_\theta| \sim 1$ where all θ^*'s intsersect and die off as you move away from the center. Hence the larger the p, the more focus on the higher values, hence we have, by interpolation,

$$\|E\phi\|_{L^p(B_R)} \sim \begin{cases} 1, \text{ for } p > \frac{2n}{n-1} \\ R^{\frac{-(n-1)}{2}}|B_R|^{1/p} = R^{\frac{-(n-1)}{2}-\frac{n}{p}}, p < \frac{2n}{n-1} \end{cases}$$

What does the above example of wave packets on S^{n-1} tell us? Recall $\|\phi_\theta\|_{L^q} \sim R^{\frac{-(n-1)}{2q}}$, and the conjecture is that we can control the size of $\|E\phi\|_{L^p(B_R)}$ using the size of $\|\phi\|_{L^q}$. The conjecture is as follows:

$$\frac{\|E\phi\|_{L^p(B_R)}}{\|\phi\|_{L^q}} \lesssim R^\epsilon$$

For $q = 2$, we have Tomas-Stein,

Theorem 0.5 (Tomas-Stein)

If $p \geq \frac{2(n+1)}{n-1}$, then $\|E\phi\|_{L^p(B_R)} \lesssim \|\phi\|_{L^2}$

♡

And this gives one end of the range to put on q, and Stein conjectures the other end:

$$\|E\phi\|_{L^p(B_R)} \lesssim \|\phi\|_{L^\infty}, p > \frac{2n}{n-1}$$

Note: Only $n = 2$ has been proven. Here we will us the wave packet approach. If we let $S^{n-1} = \bigcup \theta$, where θ are $R^{-1/2}$ caps, and let $\phi = \sum \phi_\theta$. By the Locally Constant Lemma, we have

$$|E\phi(x)| \sim \text{ constant on } T$$

where T are translates of θ^*. Hence, if we add things up "horizontally" like this, then we have

$$|E\phi(x)| \sim \sum_T a_T \chi_T$$

where T are parallel to θ^* and are disjoint.

By Plancherel identity, we have

$$\|E\phi\|_{L^2} = \|\phi\|_{L^2}$$

which means

$$\sum a_T^2 \sim \|\phi\|_{L^2}^2 \leq \sum_\theta \|\phi_\theta\|_{L^2}^2 \sim R^{\frac{n-1}{2}}\|\phi_\theta\|_{L^2}^2$$

To prove the decoupling, we have to look at how the tubes T (which are translates of θ^*) intersect. For each $B_{R^{1/2}}$, we define

$$\mu(B_R^{1/2}) = \sum_{T\cap B_{R^{1/2}}\neq\emptyset} a_T^2$$

The Kakeya problem concerns with how many $B_{R^{1/2}}$ such that $\mu \sim 2^k$ for some k. The next thing we will do is to estimate $\|E\phi\|_{L^p(B_R)}$.

We now state two versions of multilinear restriction that we will give sketches/proofs for. Recall we define $f_j = E\phi_j$ to be the extension operator, i.e., taking the inverse Fourier transform.

Theorem 0.6 (Multilinear Kakeya, first version)

Let Σ_j be spherical caps that are $\frac{1}{100n}$-neighborhood of e_j. Let $\phi_j : \Sigma_j \to \mathbb{C}$ and $f_j = E\phi_j$, then we have

$$\|\prod_j |E\phi_j|^{\frac{1}{n}}\|_{L^p(B_R)} \lesssim R^\epsilon \prod_j \|\phi_j\|_{L^2}^{\frac{1}{n}}$$

♡

The first version of the restriction concerns with how to control the size of $E\phi_j$ using bounds on $\|\phi_j\|$. And now the second version of the multilinear problem concerns with how we control the size of $E\phi_j$ using $E\phi_j$.

Theorem 0.7 (Multilinear Kakeya)

If $T_{j,a}$ is a characteristic function of 1-neighborhood of line $l_{j,a}$ and the angle between $l_{j,a}$ and x_j-axis is no more than $\frac{1}{100n}$, then we have

$$\int_{Q^s} \prod_j \left(\sum_{a=1}^{N_j} T_{j,a}\right)^{\frac{1}{n-1}} \lesssim s^\epsilon \prod_j N_j^{\frac{1}{n-1}}$$

Hence if we have $g_j = \sum_{a=1}^{N_j} T_{j,a}$, then we have

$$\int_{Q^s} \prod_j g_j^{\frac{1}{n-1}} \lesssim s^\epsilon \prod_{j=1}^{n} \left(\int_{Q_s} g_j\right)^{\frac{1}{n-1}}$$

♡

Theorem 0.8 (Multilinear restriction, second version)

Suppose $\Sigma_1, \Sigma_2, ..., \Sigma_n$ are C^2 hyperspaces in $\mathbb{R}^n$, with diameter ≤ 1 and $|curvature| \lesssim 1$, then if the angle between any the normal vector of $\omega_j \in \Sigma_j$ at ω_j and e_j are $\leq \frac{1}{100n}$. Let $f_j = E\phi_j$, we have support supp$\widehat{f_j} \subset N_{1/k}\Sigma_j$. Then we have, for $p = \frac{2n}{n-1}$,

$$\|\prod_j |f_j|^{\frac{1}{n}}\|_{L^p} \lesssim R^\epsilon \prod_{j=1}^{n} \|f_j\|_{\underline{L}^2(\omega_{B_R})}^{\frac{1}{n}}$$

♡

We give the sketch of this using the lemmas that we already know. Consider a cover of B_R using smaller balls of radisu $R^{1/2}$, i.e., $B_R = \bigcup \theta$, then by the local orthogonality lemma, we have

$$\|f_j\|_{L^p(B_{R^{1/2}})}^2 \lesssim \sum_\theta \|f_{j,\theta}\|_{L^2(\omega_{B_{R_{1/2}}})}^2$$

And the "white-lie" version of this lemma would be to replace the inequality with roughly equal to.

$$\|f_j\|_{L^p(B_{R^{1/2}})} \sim \sum_\theta \|f_{j,\theta}\|_{L^2(\omega_{B_{R^{1/2}}})}$$

And we state something similar for the Locally Constant Lemma: let T be a translation of θ^*, for $x_1, x_2 \in T$, we have

$$|f_{j,\theta}(x_1)| \sim |f_{j,\theta}(x_2)|$$

We now would like to prove the multilinear restriction, second version. Let's begin. Here we use B to denote B_R and $B_{1/2}$ to denote $B_{R^{1/2}}$.

Proof We will begin with the LHS, $\int \prod_j |f_j|^{1/n\cdot 2n/(n-1)}$, we would like to have $R^\epsilon \prod_j \|f_j\|_{L^2}^{\frac{2}{n-1}}$ on the RHS. This we can do by using three key ingredientS: Local Constant Lemma, Local Orthogonality Lemma, and the Multilinear Kakeya estiamtes. Here we go,

$$\begin{aligned}
\int \prod_j |f_j|^{\frac{2}{n-1}} &\sim |average| \int_{B_{1/2}} \prod_j |f_j|^{\frac{2}{n-1}} \\
&\lesssim |a| R^{\frac{\epsilon n}{n-1}} \prod_j \|f_j\|_{L^{\frac{2}{n-1}}(B_{1/2})} \\
&= R^{\epsilon} |a| \left(\int_{B_{1/2}} \prod_j |f_j|^2 \right)^{1/(n-1)} \\
&\sim |a| R^{\epsilon} \left(\int_{B_{1/2}} \prod_j \sum_\theta |f_{j,\theta}|^2 \right)^{1/(n-1)} \\
&\sim |a| R^{\epsilon} \int_{B_{1/2}} \prod_j \left(\sum_\theta |f_{j,\theta}|^2 \right)^{1/(n-1)} \\
&= R^{\epsilon} \int_B \prod_j \left(\sum_\theta |f_{j,\theta}|^2 \right)^{(1/(n-1)} \\
&\sim R^{\epsilon} \prod_j \left(\int_B \sum_\theta |f_{j,\theta}|^2 \right)^{\frac{1}{n-1}} \\
&\sim R^{\epsilon} \prod_j \left(\|f_j\|_{L^2}^2 \right)^{\frac{1}{n-1}}
\end{aligned}$$

and we are done!

Lecture 6

We continue our discussion about multilinear restriction, and we gave a sketch of the proof for the second version, now we turn to the first version. (remember, the restriction conjecture concerns with bounding the size of $\|E\phi\|_{L^p}$ using $\|\phi\|$).

We now look at the idea of "tiling" going from the fourier space to the physical space. If we know a function whose fourier transform is supported on a region, can we disect the region, and look at the dual rectangles/tubes, and how they intersect, to bound the size of the original function f. To achieve, we need to familiarize ourselves with the idea of induction on scale.

Let's reiterate our multilinear restriction theorem.

Theorem 0.9 (Multilinear restriction)

$$\left\| \prod_{j=1}^{n} |f_j|^{1/n} \right\|_{L^p_{avg}} (B_R) \leq R^{\epsilon} \prod_{j=1}^{n} \left(\|f_j\|_{L^2_{avg}(\omega B_R)} \right)^{1/n}$$

where $p = 2n/(n-1)$.

Let's first denote $\Sigma_1, ..., \Sigma_n$ as hypersurfaces in $\mathbb{R}^n$, and then decompose $N_{1/R}\Sigma_j$ into disjoint unions os $R^{-1/2}$-caps θ. Thus, we have $f_{j,\theta}$ as the restriction of f whose fourier transform is supported on θ of Σ_j. The dimension of the $R^{-1/2}$ cap θ is $R^{-1/2} \times R^{-1/2} \times ... \times R^{-1/2} \times R$, so its dual tube θ^* has dimension $R^{1/2} \times ... \times R^{1/2} \times R$. We know $|f_{j,\theta}$ is roughly constant on each θ^*, as well as the translation of θ^*, the tubes T's. If we reexamine the multilinear restriction, then we get the LHS is estimating how the tubes overlap and the bound that we can obtain on the size the overlaps.

Now we introduce the induction on scales idea. If we consider a bigger region in the fourier space, that corresponds to a smaller region in the physical space. Hence, if we consider a fatter cap $R^{-1/4}$ cap τ, and denote $f_\tau = \sum_{\theta \subset \tau} f_{j,\theta}$, then if we look at the dual τ^*, it is a smaller tube enclosed in the intersection of T's and has dimension $R^{1/4} \times ... \times R^{1/4} \times R$. How τ^*'s intersect tells us how energy is distributed/concentrated on these tubes.

We apply lcoal orthogonality to $\theta \subset \tau$.

$$\|f_{j,\tau}\|^2_{L^2(B_{R_{1/2}}} \lesssim \sum_{\theta \subset \tau} \|f_{j,\theta}\|^2_{L^2(\omega)}$$

The reverse inequality for the local orthogonality is generality not true, if we write $f_{j,\theta} = \chi_\theta \widehat{f_{j,\tau}}$, then we have

$$f_{j,\theta} = \chi_\theta * f_{j,\tau}$$

if energy of $f_{j,\tau}$ outside $B_{R^{1/2}}$ may contribute a lot after the convolution.

Proposition 0.9

The inverse Fourier transform of characteristic functions $\check{\chi}_\theta$ *doesn't decay fast.*

Proof

$$\check{\phi}(x) = \int \phi(\omega) e^{2\pi i x \omega} d\omega = \int_E \phi(\omega) e^{2\pi i x \omega} d\omega = \frac{e^{2\pi i x \omega}}{2\pi i x}\Big|_E$$

□

The local orthogonality lemma lets us keep track of the size of the following expressions:

$$\sum |f_{j,\theta}(x)|^2, \sum |f_{j,\tau}(x)|^2, \sum_{biggercaps} |f_{j,\tau'}(x)|^2, ..., \text{ until } \sum_{\Sigma_j} |f_{j,\Sigma_j}(x)|^2$$

We can think of $\sum |f_{j,\theta}(x)|^2$ as the energy density of $f_{j,\theta}$ and it concentrates on tubes (T) in the physical space. As we move to the right side, the tubes are getting finer and may move perpendicular to x_j-axis. Now we look at functions whose fourier support is precisely Σ_j.

Now let's define an extension operator over Σ.

Definition 0.2 (Extension operator over Σ)

For any smooth $\phi(x) \in C^\infty(\Sigma)$, we define an extension operator on Σ of ϕ is

$$E_\Sigma \phi(x) = \int_\Sigma e^{2\pi i \omega x} \phi(x) dvol_\Sigma(\omega)$$

♣

We restate the first version of the multilinear restriction, as well as its asusmptions, since they are important in our proof for the restriction conjecture in $n = 2$ below. If τ are spherical caps of $\frac{1}{100n}$-nbd of e_j. Let $\phi_j : \Sigma_j \to \mathbb{C}$, and $f_j = E\phi_j$, then we have the following inequality.

Theorem 0.10 (Multilinear restriction, first version)

$$\left\| \prod_{j=1}^{n} |E_{\Sigma_j} \phi_j|^{1/n} \right\|_{L^p(B_R)} \leq R^\epsilon \prod_{j=1}^{n} \|\phi_j\|_{L^2(\Sigma_j)}^{1/n}$$

where $p = 2n/(n-1)$

♡

Though not included in the restriction conjecture, we first prove the following:

Proposition 0.10

$$\|E\phi\|_{L^2(B_R)} \lesssim R^{1/2} \|\phi\|_{L^2}$$

♠

Proof This L^2 case is a lot simpler due to Plancherel. And we prove this by first proving the following lemma regarding integrating over hypersurfaces:

Lemma 0.8

Let Π be a hyperplane perpendicular to e_j, for some j, we then have

$$\int_\Pi |E_j \phi|^2 \sim \int_{\Sigma_j} |\phi|^2$$

♡

Proof We take e_j to be e_n for convenience. Then for $x \in \Pi$, we have $x = (x_1, ..., x_{n-1}, t)$ for fixed t for all $x \in \Pi$. And Σ_n is within a small neighborhood that is normal to e_n, hence we have for all $\omega \in \Sigma_n$, we have $\omega_n = h(\omega') = h(\omega_1, ...\omega_{n-1})$. And because the integral operator integrates $dvol_{\Sigma_n(\omega)}$, let J be the Jacobian determinant, we have

$$dvol_{\Sigma_n} = J d\omega'$$

And we would want to write $|E_j \phi|$ as a function or the fourier transform of a function to apply Plancherel. We have,

$$E_{\Sigma_n} \phi(x) = \int_{\Sigma_n} e^{2\pi i x \omega} \phi(\omega) dvol_{\Sigma_n}(\omega) = \int_{\mathbb{R}^{n-1}} e^{2\pi i x' \omega'} e^{2\pi i t h(\omega')} \phi(\omega') J d\omega'$$

Then we define $g : \Sigma_n \to \mathbb{C}$ as $g(\omega') = e^{2\pi i t h(\omega')} \phi(\omega') J$, then we have $E_{\Sigma_n} = \int e^{2\pi i x' \omega'} g(\omega') d\omega' = \check{g}$ Hence by Plancherel, we have

$$\int_\Pi |E_{\Sigma_n} \phi|^2 = \int_\Pi |\check{g}|^2 = \int_\Pi |g|^2 = \int_\Pi |\phi|^2 J^2 \sim \int_P i J |\phi|^2 = \int_{\Sigma_n} |\phi|^2 dvol_{\Sigma_n}$$

□

And this lemma directly implies our bound on $\|E\phi\|_{L^2}$ since the only think left for us to do is to integrate over x_n, i.e.

$$\|E\phi\|_{L^2}^2 = \int_{-R}^{R} \int_\Pi |E_j \phi|^2 \sim \int_{-R}^{R} \int_{\Sigma_n} |\phi|^2 \leq R \|\phi\|_{L_2(\Sigma)}^2$$

Taking the square root gives us $\|E\phi\|_{L^2(B_R)} \lesssim R^{1/2} \|\phi\|_{L^2}$.

□

We now restate our conjecture given our $p = 2n/(n-1)$ from the restriction theorem. We expect

$$\|E\phi\|_{L^p(B_R)} \lesssim R^\epsilon \|\phi\|_{L^p(\Sigma)}$$

Remark Note for a general conjecture, one can put $L^q(\Sigma)$ on the RHS, for all $p \leq q \leq \infty$. But any case where $q < p$ would have a counterexample using a single wave packet.

We will now prove the case where $n = 2, p = 4$.

Theorem 0.11

For $n = 2$, and thus $p = 4$, let $\Sigma = S^1$, the unit circle in $\mathbb{R}^2$. We have,

$$\|E\phi\|_{L^4(B_R)} \lesssim R^\epsilon \|\phi\|_{L^4}$$

♡

Proof This is a long one and we will introduce lemmas and definitions along the way, here we go. We first state how may one use the multilinear restriction theorem. Let $\Sigma = \bigcup \tau$, where τ are $1/K$ caps, and there are roughly K τ's.

$$\int |E\phi|^p = \int |\sum_\tau E\phi_\tau|^p = \int \prod_{j=1}^n |\sum_\tau E\phi_\tau|^{p/n} \leq \int \prod_{j=1}^n \sum_\tau |E\phi_\tau|^{p/n} K^{O(1)} = K^{O(1)} \sum_\tau \int \prod_{j=1}^n |E\phi_\tau|^{p/n}$$

The term on the RHS is where we can apply our multilinear restriction theorem, and we have to make sure our τ's are small neighborhoods of e_j. We translate the above equation in terms of $n = 2$.

$$\int |E\phi|^4 \leq K^{O(1)} \sum_{\tau_1,\tau_2} \int |E\phi_{\tau_1}|^2 |E\phi_{\tau_2}|^2$$

However, we can't always apply the multilinear restriction if we don't have our τ_1, τ_2 nicely close to e_1, e_2. Hence we categorize them and then deal with them separately.

Definition 0.3 (transverse caps)

We say $(\tau_1, ..., \tau_n)$ are transverse if there is a linear change of variables L, with $\det(L) \lesssim K^{O(1)}$, such that $(L\tau_1, ..., L\tau_n)$ are $\frac{1}{100n}$ neighborhoods of e_j.

♣

Remark A sequence $(\tau_1, ..., \tau_n)$ are not transverse if they all lie within $O(\frac{1}{k})$ neighborhood of the equator on S^{n-1}. In $n = 2$, this means τ_1, τ_2 are transverse if they don't lie"directly across from each other" on the unit circle.

Recall, we decompose $S^1 = \bigsqcup \tau$ into K^{-1} caps where $K \sim \log(R)$ and hence there are $\sim K$ caps. We pick out the τ's that contribute a lot at any given point $|E\phi(x)|$. And that would be

$$S(x) = \{\tau : |E\phi_\tau(x)| \geq \frac{1}{100K} |E\phi(x)|\}$$

Then we have,

$$\sum_{\tau \notin S(x)} |E\phi_\tau(x)| \leq \frac{1}{100K} |E\phi_\tau(x)| K \leq \frac{1}{10} |E\phi(x)|$$

This means $\left|\sum_{\tau \in S(x)} |E\phi_\tau|\right| \geq \frac{9}{10} |E\phi(x)|$, hence we have

$$\left|\sum_{\tau \in S(x)} |E\phi_\tau(x)|\right| \sim |E\phi(x)|$$

Now we just need to look at τ's for each x that are in $S(x)$.

Definition 0.4 (Broad and narrow points)

We call x is broad, if there exists τ_1, τ_2 such that (τ_1, τ_2) are transverse and they are narrow if they are not transverse.

♣

To evaluate the broad portion, we note that we can make the linear change of variable such that

$$\int_{B_R \cap Broad} |E\phi|^4 \leq K^{O(1)} \sum_{\tau_1,\tau_2,transverse} \int |E\phi_{\tau_1}|^2 |E\phi_{\tau_2}|^2 \leq K^{O(1)} R^\epsilon \sum_{\tau_1,\tau_2} \|\phi_{\tau_1}\|_{L^2}^2 \|\phi_{\tau_2}\|_{L^2}^2$$

We continue with another line:

$$K^{O(1)} R^\epsilon \sum \|\phi_{\tau_1}\|_{L^2}^2 \|\phi_{\tau_2}\|_{L^2}^2 \sim K^{O(1)} R^\epsilon \left(\|\phi_{\tau_1}\|_{L^2}^2 + \|\phi_{\tau_2}\|_{L^2}^2\right)^2 \sim K^{O(1)} R^\epsilon \|\phi\|_{L^2}^4$$

Hence we are done with the broad the x's, in this case, we know all τ_1, τ_2 are right across from each other, i.e. all the

τ are contained in the $O(\frac{1}{K})$ neighborhood of the equator, then we have $|S(x)| \lesssim 1$. Then we would like to estimate $\int_{B_R \cap narrow} = \int |E\phi|^4$.

$$\int_{B_R \cap narrow} |E\phi|^4 = \int |\sum_\tau E\phi_\tau|^4 \leq \sum_\tau \int |E\phi_\tau|^4$$

Again, the last line is by Holder's inequality. We are not able to use multilinear restriction anymore, but we can use our powerful tool of induction on scale. Remember we'd like to have $\|\phi\|_{L^2}^4$ on the RHS, now the question is how would we get that from the RHS above. We establish the following inequality;

$$\|E\phi\|_{L^4(B_R)} \leq C(R)\|\phi\|_{L^4}$$

This can be seen as follows:

$$\int |E\phi|^4 \leq |B_R| \|\phi\|_{L^\infty}^4 \leq |B_R| \|\phi\|_{L^1}^4 \leq |B_R||S^1|^{1/r}\|\phi\|_{L^4}^4$$

Hence we define $C(R)$ as the smallest integer that we can place in the inequality above, such that

$$\|E\phi\|_{L^4(B_R)} \leq C(R)\|\phi\|_{L^4}$$

Now again, we would like to show that $C(R) \lesssim R^\epsilon$. Clearly the above bound is not good enough, and we would like to inprove that based on induction on scale. This is done by the following lemma. We are looking at $E\phi$ on a smaller scale τ.

Lemma 0.9

We connect $\|E\phi_\tau\|_{L^4(B_R)}$ with $\|\phi_\tau\|_{L^4}$

$$\|E\phi_\tau\|_{L^4(B_R)} \lesssim C(R/K)\|\phi_\tau\|_{L^4}$$

♡

Proof We now look at the proof by change of variables. For each τ of dimension $K^{-1} \times K^{-2}$, we do a change of variable such that $\tilde{\tau}$ is of dimension 1×1. Because we've scaled it up by K, the dimension of the dual rectangle is scaled down by K^{-1}, hence we look at $\|E\phi_{\tilde{\tau}}\|_{L^4(B_{R/K})}$. By definition, we have

$$\|E\phi_{\tilde{\tau}}\|_{L^4(B_{R/K})} \leq C\left(\frac{R}{K}\right)\|\phi_{\tilde{\tau}}\|_{L^4}$$

The linear change of variables does not change the constant that we place here.

□

Now we go back to the previous step where we have $\sum_\tau \int |E\phi_\tau|^4$ on the RHS.

$$\int_{B_R \cap narrow} |E\phi|^4 \leq \sum_\tau \int |E\phi_\tau|^4 \leq C\left(\frac{R}{K}\right)^4 \sum_\tau \int |\phi_\tau|^4 = C\left(\frac{R}{K}\right)^4 \sum_\tau \|\phi_\tau\|_{L^4}^4 \sim C\left(\frac{R}{K}\right)^4 \|\phi\|_{L^4}^4$$

Hence combining the narrow and broad part, we have

$$C(R) \lesssim K^{O(1)}R^\epsilon + C\left(\frac{R}{K}\right)$$

By induction, we look at $C(R/K)$, remember the above K comes from $K \sim \log(R)$, hence becomes $\log(R/K)^{O(1)}$ in this case,

$$C\left(\frac{R}{K}\right) \lesssim \left(\frac{R}{K}\right)^\epsilon$$

Hence we have

$$C(R) \leq C_2(K)R^\epsilon + C_2C_1\left(\frac{R}{K}\right)^\epsilon$$

Hence the R^ϵ term dominates, and is what we want for $C(R)$.

□

We restate what we just proved.

$$\|E\phi\|_{L^4} \lesssim R^\epsilon \|\phi\|_{L^4}$$

Lecture 7

We will prove a weaker version of the decoupling theorem for the paraboloid. First we consider the paraboloid $P = \{x \in \mathbb{R}^n : x_n = x_1^2 + ...x_n^2\}$, and let Ω denote the $1/R$-nbd of P, i.e. $\Omega = N_{1/R}P$, and let θ be $R^{-1/2}$-caps of decomposition of Ω. We again denote the decoupling constant as $D_p(R)$.

Theorem 0.12 (Decoupling, weaker)

Let $2 \leq p \leq \frac{2n}{n-1}$, we then have

$$\|f\|_{L^p} \lesssim R^\epsilon \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}$$

♡

We recall the stronger version, and the only difference is that now we push the exponent p to be $2 \leq p \leq \frac{2(n+1)}{n-1}$. The weaker version avoids more technicalities since $2n/(n-1)$ is the exponent in the multilinear restriction theorem. The proof is in three parts: induction on scales and multiscale tools; multilinear restriction implies multilinear decoupling; finally, the decoupling theorem from multilinear decoupling.

Multiscale tools

We now state the two main tools that make our multiscale arguments feasible.

Lemma 0.10 (Linear change of variables preserves D_p)

Fix $L : \mathbb{R}^n \to \mathbb{R}^n$ a linear change of variables, then for any decomposition of Ω, we have

$$D_p(\Omega = \bigsqcup_\theta \theta) = D_p(L\Omega = \bigsqcup L\theta)$$

♡

Proof In other words, this means we can "stretch" these decompositions, if we decompose an area using boxes of size l, then if we double the size of area we are trying to do decoupling on, then we decompose using boxes of size $2l$ to preserve the decoupling constant. Fix f, we want to construct $\tilde{f}$ such that when $supp(\widehat{f}) \subset \Omega$, then we have $supp(\widehat{\tilde{f}}) \subset L\Omega$.

$$\tilde{f}(x) = f((L^*)^{-1}x)$$

We thus have, for the Fourier transform support of f,

$$\int e^{2\pi i\omega\cdot x} f((L^*)^{-1}x) = \int e^{2\pi i\omega\cdot L^* y} f(y)|\det(L^*)|dy = |\det(L^*)| \int e^{2\pi i L\omega\cdot y} f(y)dy = |\det(L^*)|\widehat{f}(L\omega)$$

Hence we've stretched the domain of $\widehat{f}$ from Ω to $L\Omega$. By definition of the decoupling constant, we have

$$\begin{aligned}
\|f\|_{L^p} &= |\det(L^*)|^{-1/p}\|\tilde{f}\|_{L^p} \\
&\leq |\det(L^*)|^{-1/p} D_p(L\Omega = \bigsqcup L\theta) \left(\sum_\theta \|\tilde{f}_\theta\|_{L^p}^2\right)^{1/2} \\
&\leq D_p(L\Omega = \bigsqcup L\theta) \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}
\end{aligned}$$

Hence we have $D_p(\Omega = \bigsqcup \theta) \leq D_p(L\Omega = \bigsqcup L\theta)$. And if we start from the domain $L\Omega$ and let $f(x) = \tilde{f}(((L^{-1})^*)^{-1}x)$, then we get the reserve direction.

□

As an immediate corollary to this, we have the following (rough) equality.

Corollary 0.1 (Two decompositions)

If we write $R = R_1 \cdot R_2$, then define θ as $R^{-1/2}$ caps and τ as $R_1^{-1/2}$ caps, then τ are larger caps that enclose θ

(or θ is their refinement), then we have the following:

$$D_p(\tau = \bigsqcup_{\theta \subset \tau} \theta) = D_p(R_2)$$

♡

Proof τ are $R_1^{-1/2}$ caps, while θ are $R_1^{-1/2}R_2^{-1/2}$ caps, then if we first center τ are the origin, and multiply the coordinate by $R_1^{-1/2}$, then τ is linearly transformed roughly into Ω, while θ have become caps of size $R_2^{-1/2}$. This is what exactly the corollary means. We define an explicit map as follows:

$$L_i(\omega) = \begin{cases} R_1^{1/2}(\omega_i - \alpha_i), 1 \le i \le n \\ R_1(\omega_n - \alpha_n) - 2\sum_{j=1}^{n-1} \alpha_j(\omega_j - \alpha_j), i = n \end{cases}$$

□

The following proposition is quite indepdent of the previous lemmas.

Lemma 0.11

Suppose $R = R_1 \cdot R_2$, we partition Ω first into $R_1^{-1/2}$ caps τ, then refine them to $R_1^{-1/2}R_2^{-1/2}$ caps θ. Then we have

$$D_p(\Omega = \bigsqcup \theta) \lesssim D_p(\Omega = \bigsqcup \tau) \cdot D_p(R_2)$$

Or in another form, $D_p(R) \lesssim D_p(R_1) \cdot D_p(R_2)$.

♡

Proof

$$\begin{aligned} \|f\|_{L^p} &\le D_p(R_1)\left(\sum_\tau \|f_\tau\|_{L^p}^2\right)^{1/2} \\ &\lesssim D_p(R_1)D_p(R_2)\left(\sum \tau \sum \theta \subset \tau \|f_\theta\|_{L^p}^2\right)^{1/2} \\ &= D_p(R_1)D_p(R_2)\left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2} \end{aligned}$$

□

Here we conclude the section which connects different scales and we will see how they are used in the later sections.

Multilinear Decoupling

Just like how we used multilinear restriction to prove the restriction conjecture in $n = 2$, we will use a multilinear version of the decoupling theorem to prove the actual decoupling theorem. And note how the multilinear decoupling theorem uses the multilinear restriction theorem (where we used the critical exponent of $n = 2n/(n-1)$).

We first define the multilinear decoupling theorem.

1. $P_1, ..., P_n \subset P$ are transverse.
2. $\Omega_j = N_{1/R}P_j$
3. $\Omega_j = \bigsqcup \theta_j$
4. Let $MD(R)$ be the smallest constant such that $supp(\widehat{f_j}) \subset P_j$ and $f_j = \sum_\theta f_{j,\theta}$ such that the following inequality holds:

$$\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p_{avg}(B_R)} \le MD(R) \prod_{j=1}^n \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p(\omega B_R)}^{1/n} \right)^{\frac{1}{2} \cdot \frac{1}{n}}$$

We first derive a simple relationship between $MD(R)$ and $D(R)$.

Proposition 0.11 (MD(R) is no larger than $D(R)$)

If we decompose P and Ω like defined above, then we have

$$MD(R) \le D(R)$$

Proof This can be done using Holder's inequality.

$$\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p} \le \prod_{j=1}^n \| f_j^{1/n} \|_{L^{pn}} = \prod_{j=1}^n \|f_j\|_{L^p}^{1/n}$$

And for the RHS, we have

$$\prod_{j=1}^n \|f_j\|_{L^p}^{1/n} \le \prod_{j=1}^n \left(D_p(R) \sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{n}\cdot\frac{1}{2}}$$

With $MD(R)$ being the smallest constant one could put there, we have $MD(R) \le D(R)$.

□

Next, we prove another claim that we will use in the next section, which is the multilinear decoupling constant is the constant that we want for the decoupling constant.

Proposition 0.12 ($MD(R)$ is what we want)

For $2 \le p \le \frac{2n}{n-1}$, we have

$$MD_{p,n}(R) \lesssim R^\epsilon$$

Proof We will use Holder and multilinear restriction. First by definition, we need to have $\| \prod \|_{L^p}$ on the LHS, then by Holder, for the average L^p norm, we can replace p with $2n/(n-1)$, then using multilinear restriction, to get L^2 on the RHS and again using local orthoganlity on small Ω_j, and then getting back to L^p on the RHS using Holder. The R^ϵ term arises from the multilinear restriction and all other inequalities do not come with constants. Now we translate that into some math.

$$\begin{aligned}
\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p} &\le \left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^{2n/(n-1)}} \\
&\lesssim R^\epsilon \prod_{j=1}^n \|f_j\|_{L^2}^{1/n} \\
&\lesssim R^\epsilon \prod_{j=1}^n \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^2}^2 \right)^{\frac{1}{n}\cdot\frac{1}{2}} \\
&\le R^\epsilon \prod_{j=1}^n \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{n}\cdot\frac{1}{2}}
\end{aligned}$$

Here I've skipped some notations, the multilinear restriction gives us $L^2(\omega B_R)$, which aligns with the definition in multilinear decoupling constant.

□

So far, we've moven the multilinear decoupling theorem, which states for $2 \le p \le 2n/(n-1)$, we have

$$\left\| \prod_{j=1}^n |f_j|^{1/n} \right\|_{L^p(B_R)} \lesssim R^\epsilon \prod_{j=1}^n \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{n}\cdot\frac{1}{2}}$$

Now the only thing left for us to do would be to connect the previous two sections to derive the decoupling theorem for the original large paraboloid.

Remark In the full decoupling theorem, we extended the exponent and that is gained from two Holder's inequlaities used above.

Decoupling Theorem

Recall how we proved

$$\|E\phi\|_{L^4(B_R)} \lesssim R^\epsilon \|\phi\|_{L^4}$$

We separated the points that are "broad" and "narrow", and noted the multilinear restriciton applies when we are dealing with the broad, but not narrow points; and for the narrow points, we gain information about where the normal vector lies and use that to derive a bound using induction on scales.

We now introduce the main theorem that we will prove about decoupling (which follows from multilinear decoupling), and also directly implies the decoupling constant $D(R) \lesssim R^\epsilon$ by induction on scale and dimension.

Theorem 0.13 (Bound on $D_{p,n}(R)$)

For any $K \geq 1$, we have

$$D_{p,n}(R) \lesssim K^{O(1)} M D_{p,n}(R) + D_{p,n-1}(K^2) D_{p,n}(R/K^2)$$

♡

Now we should how this implies $D_{p,n}(R) \lesssim R^\epsilon$.

Lemma 0.12

Given the bound on $D_{p,n}$ above, we have $D_{p,n}(R) \lesssim R^\epsilon$.

♡

Proof Taking $K \sim \log(R)$, we have

$$D_{p,n}\left(\frac{R}{K}\right) \lesssim \log(R/K)^{O(1)} \left(\frac{R}{K}\right)^\epsilon + \left(\frac{R}{K}\right)^\epsilon \lesssim \left(\frac{R}{K}\right)^\epsilon$$

Hence in the equation above in the theorem, we have the dominating term being R^ϵ.

Remark I don't know what is going on here.

Going back to the theorem, let us examine what we have: we have the first term coming from the braod region, where we have caps that can be arranged up to a factor of $K^{O(1)}$ to transverse positions. And the second term comes from the narrow region.

We now give two lemmas, one regarding the broad, and one about the narrow region, so that we know where we are heading towards. And we will show these two lemmas imply the main theorem on the bound on $D_{p,n}(R)$. Let τ be the K^{-1} caps, and $\Omega = \bigsqcup \tau$. We specify the τ's that we care about, which is defined as follows. Let $S(B)$ denote the significant set of τ for B, where $B = B_r$ is a ball of radius r in the physical space. Recall $f = \sum_\tau f_\tau$,

$$S(B) = \{\tau : \|f_\tau\|_{L^p(B)} \geq \frac{1}{100|\tau|} \|f\|_{L^p(B)}\}$$

And just like the $n = 2$ case for the restriction conjecture, we can show that $f_B = \sum_{\tau \in S(B)} f_\tau$ has $\|f_B\|_{L^p} \sim \|f\|_{L^p}$.

Basically we are decomposing the large ball B_R which we are integrating over into balls B of radius $B = B_r$. Then we categorize these balls as broad and narrow like previously.

Definition 0.5 (Broad and Even balls)

We define a ball B to be broad if for the signiciant τ set, $S(B)$, we can find $\tau_1, ..., \tau_n$ such that they are transverse. (note here the number is the dimension). If B is not broad, it is narrow.

♣

We now state the main two lemmas for broad and narrow estimates.

Lemma 0.13 (Broad)

The broad region is composed of disjoint B that are broad, denoted as $Broad = \bigsqcup_{B, Broad} B$, and we have

$$\|f\|_{L^p(Broad)} \leq r^{O(1)} M D_{p,n}(R) \left(\sum_\theta \|f_\theta\|_{L^p}^2\right)^{1/2}$$

♡

And for the narrow one. Here we deal with individual balls B_r that are narrow.

Lemma 0.14 (Narrow)

For each $B = B_r$, we take $r = K^2$, we have

$$\|f_B\|_{L^p(B)} \lesssim D_{p,n-1}(K^2)\left(\sum_{\tau\in S(B)} \|f_\tau\|^2_{L^p(B)}\right)^{1/2}$$

Although here the narrow lemma is stated for each individual balls B, we can combine them as in "parallel decoupling" to get the same decoupling constant for the entire narrow region. By parallel decoupling, since $D_{p,n-1}$ is the decoupling constant holds for all $B \subset Narrow$, we thus have $Narrow = \bigsqcup_{B,narrow} B$,

$$\|f\|_{L^p(Narrow)} \lesssim D_{p,n-1}(K^2)\left(\sum_{\tau\in S(B)} \|f_\tau\|^2_{L^p}\right)^{1/2}$$

Now we show how the two equations regarding broad and narrow show the main theorem on $D_{p,n}(R)$.

Proposition 0.13

The above two lemmas imply the main theorem.

Proof By observing the main theorem, we notice that we still need to convert our narrow estimate into $\sum_\theta \|f_\theta\|_{L^p}$ And this can be done by thinking about τ as large caps and their refinements are θ's. Then previously we've shown, if $R = R_1R_2$, with τ being $R^{-1/2}$ caps, then our $R_1 = K^2$, hence $D(R) \lesssim D(K^2)D(\frac{R}{K^2})$,

$$\|f\|_{L^p} \lesssim D(K^2)D(R/K^2)\left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{1/2}$$

And we know $D(K^2) \geq \|f\|_{L^p}\left(\sum_\tau \|f_\tau\|^2_{L^p}\right)^{-1/2}$, hence plugging in, we have

$$\|f\|_{L^p(Narrow)} \lesssim D_{p,n-1}(K^2)\left(\sum_\tau \|f_\tau\|^2_{L^p}\right)^{1/2} \lesssim D_{p,n-1}(K^2)D(R/K^2)\left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{1/2}$$

Combining, we have

$$\begin{aligned}
\int |f|^p &= \int_{Broad} |f|^p + \int_{Narrow} |f|^p \\
&\leq K^{O(1)p} M D(R)^p \left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{p/2} + D^p_{p,n-1}(K^2)D^p(R/K^2)\left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{p/2} \\
&= \left(K^{O(1)p} M D(R)^p + D^p_{p,n-1}(K^2)D^p(R/K^2)\right)\left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{p/2} \\
&\lesssim \left(K^{O(1)} M D(R) + D_{p,n-1}(K^2)D(R/K^2)\right)^p \left(\sum_\theta \|f_\theta\|^2_{L^p}\right)^{p/2}
\end{aligned}$$

□

We start with the narrow estimate. Again, we are trying to show,

$$\|f\|_{L^p} \lesssim D_{p,n-1}\left(\sum_\tau \|f_\tau\|^2_{L^p}\right)^{1/2}$$

But the parallel decoupling lemma, it suffices to show

$$\|f\|_{L^p(B)} \lesssim D_{p,n-1}\left(\sum_\tau \|f_\tau\|^2_{L^p(B)}\right)^{1/2}$$

Now we claim, it suffices to show the following lemma.

Lemma 0.15 (Lemma for the Narrow estimate)

For the narrow balls B, let Π^* be the hyperplane such that $nor(\tau)$ is close to Π^* for all $\tau \in S(B)$. Again, we set $f_B = \sum_{\tau \in S(B)} f_\tau$, and for any Π parallel to Π^*, we have

$$\|f\|_{L^p(B\cap\Pi)} \lesssim D_{p,n-1}(K^2)\left(\sum_\tau \|f_\tau\|^2_{L^p(\omega B\cap\Pi)}\right)^{1/2}$$

♡

Assuming this lemma, we prove the narrow estimate for each B.

Proof Here we go.

$$\begin{aligned}
\|f\|^p_{L^p(B)} &\sim \|f_B\|^p_{L^p(B)} \\
&= \int_t \|f\|^p_{L^p(B\cap\Pi)} dt \\
&\lesssim D^p \int \left(\sum_\tau \|f_\tau\|^2_{L^p(\omega B\cap\Pi)}\right)^{p/2} dt \\
&= D^p \int \left(\sum_\tau \|f_\tau\|^2_{L^p(\omega B\cap\Pi)}\right) \cdot \left(\sum_\tau \|f_\tau\|^2_{L^p(\omega B\cap\Pi)}\right)^{\frac{p-2}{2}} dt \\
&\le D^p \sum_{\tau\in S(B)} \int \|f_\tau\|^2_{L^p(B\cap\Pi)} \left(\sum_\tau \|f_\tau\|^2_{L^p(\omega B\cap\Pi)}\right)^{\frac{p-2}{2}} \\
&\le D^p \sum_{\tau\in S(B)} \|(\|f_\tau\|^2_{L^p(B\cap\Pi)})\|_{L^{p/2}} \|(\sum_\tau \|f_\tau\|^2_{L^p})^{(p-2/)/2}\|_{L^(p/(p-2))} \\
&= D^p \sum_{\tau\in S(B)} \left(\int_t \|f_\tau\|^p_{L^p(B\cap\Pi)}\right)^{2/p} dt \cdot \left(\int_t \left(\sum \|f_\tau\|^2_{L^p(B\cap\Pi)}\right)^{p/2}\right)^{(p-2)/p} \\
&\le D^p \sum_{\tau\in S(B)} \left(\int_t \|f_\tau\|^p_{L^p(B\cap\Pi)}\right)^{2/p} dt \cdot \left(\int_t \|f_\tau\|^p_{L^p(B\cap\Pi)}\right)^{-2/p} \int_t \left(\sum \|f_\tau\|^2_{L^p(B\cap\Pi)}\right)^{p/2} \\
&= D^p \left(\sum_\tau \|f_\tau\|^2_{L^p(B)}\right)^{p/2}
\end{aligned}$$

Taking the pth root, we get

$$\|f\|_{L^p(B)} \lesssim D_{p,n-1}\left(\sum_\tau \|f_\tau\|^2_{L^p(B)}\right)^{1/2}$$

□

Hence to show the narrow estimate, it suffices to show the lemma that was introduced. **Remark** I feel like I don't know enough geometry to fully understand this lemma, will come back to this.

We shall proceed to the Broad estimate.

0.1 Lecture 9

We restate our goal and continue with the Broad estimate.

Theorem 0.14 (Decoupling, Bourgain)

Suppose f is such that $\widehat{f} \subset \Sigma$, where Σ is the $N_{1/R}$-nbhd of the truncated paraboloid,

Let's recall the multilinear decoupling theorem. Let $\Sigma = \bigsqcup_j \Sigma_j$, such that for each j, we have the normal of Σ_j to be close to the x_j axis. And we then separate each Σ_j into θ which are our $R^{-1/2}$ caps. And multilinear decoupling is as follows:

Theorem 0.15 (Multilinear Decoupling)

Let $f_{j,\theta}$ be such that $\widehat{f}_{j,\theta} \subset \theta \subset \Sigma_j$, and we have

$$\|\prod_{j=1}^{n} |f_j|^{1/n}\|_{L^p} \lesssim R^{\epsilon} \prod_{j=1}^{n} \left(\sum_{\theta} \|f_{j,\theta}\|_{L^p}^2\right)^{\frac{1}{2}\frac{1}{n}}$$

We state the main lemma which contains the broad and narrow estimate as two terms.

Lemma 0.16 (Main)

For $2 \leq p \leq \frac{2n}{n-1}$, we have

$$D_{p,n}(R) \lesssim K^{O(1)} M D_{p,n}(R) + D_{p,n-1}(K^2) D_{p,n}(R/K^2)$$

Hence by indunction on both scale R and dimension n, note the allowable range for p increases as n decreases, hence by induction, we are able to conclude the main theorem.

For the broad estimate, we would like to show

$$\|f\|_{L^p(Broad)} \lesssim K^{O(1)} R^{\epsilon} \left(\sum \|f_\theta\|_{L^p}^2\right)^{1/2}$$

And for the narrow estimate, we've shown that for τ, which are K^{-1} caps that contain θ caps, we have

$$\|f\|_{L^p(Narrow)} \lesssim D_{p,n-1}(K^2) \left(\sum \|f_\tau\|_{L^p}^2\right)^{1/2}$$

And by decoupling at the scale of R/K^2, hence if we rescale τ to the paraboloid, we would get $K/R^{1/2}$, hence we have

$$\|f_\tau\|_{L^p} \lesssim D_{p,n}(R/K^2) \left(\sum_{\theta \subset \tau} \|f_\theta\|_{L^p}^2\right)^{1/2}$$

We then would have recovered the main lemma.

Now we show the broad estimate.

Lecture 12

We will recall the setup and will now introduce a fourth tool in induction on scale to prove the full decoupling theorem by Bourgain and Demeter.

Let P denote the paraboloid in $\mathbb{R}^n$, and let θ be the $R^{-1/2}$ caps of decomposition of Ω. And we have the following estimate:

Theorem 0.16 (Bourgain and Demeter)

f has the Fourier support $\widehat{f}$ in Ω, then we have, for $2 \le p \le \frac{2(n+1)}{n-1}$,

$$\|f\|_{L^p} \lesssim R^{\epsilon} \left(\sum \|f_\theta\|_{L^p}^2\right)^{1/2}$$

In other words, the decoupling constant $D_{p,n} \lesssim R^{\epsilon}$.

We now define a new notation that encompasses all the information that we have gathered to pass from one scale to another.

$$M_{p,q}(r,\sigma) = Avg_{B_r \subset B_R} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|^2_{L^q_{avg}(B_r)} \right)^{\frac{1}{2}\frac{1}{n}p}$$

In the Fourier space, we disect Ω_j into σ^{-1} caps of θ; then we come back to the physical space and take $\sigma = r^{1/2}$ to disect the physical space, and divide B_R into finitely overlapping unions of balls B_r, and note $f_{j,\theta}$ is roughly constant on $r^{1/2} \times r$-tubes pointing in the normal direction of θ.

We consider two important cases. The first one is if we take $\sigma = r = 1$, then we would have

$$M_{p,q}(1,1) = Avg_{B_1 \subset B_R} \prod_{j=1}^{n} \|f_j\|^{\frac{1}{n}p}_{L^q_a(B_1)} = \fint_{B_R} \prod_{j=1}^{n} |f_j|^{\frac{p}{n}}$$

This is because B_1 is small enough to invoke the locally constant property, we have $|f_j|$ being constant on B_1. Hence we have

$$\prod_{j=1}^{n} \|f_j\|^{\frac{1}{n}p}_{L^q_a(B_R)} \sim \prod |f_j|^{\frac{1}{n}p} \sim \fint_{B_1} \prod_{j=1}^{n} |f_j|^{\frac{1}{n}p}$$

Let's recall the multilinear decoupling inequality:

$$\left\| \prod_{j=1}^{n} |f_j|^{1/n} \right\|_{L^p_{avg}(B_R)} \lesssim R^{\epsilon} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|^{1/n}_{L^p(\omega B_R)} \right)^{\frac{1}{2}\frac{1}{n}}$$

This means $M_{p,q}(1,1)$ is the LHS of the multilinear decoupling inequality (raised to the p-th power), then now let's see the RHS. Our second example would be to take $r = R$, and $\sigma = R^{1/2}$, then we would have

$$M_{p,q}(R, R^{1/2}) = \prod_{j=1}^{n} \left(\sum_{\theta : R^{-1/2} caps} \|f_{j,\theta}\|^2_{L^q_a(B_R)} \right)^{\frac{1}{n}\frac{1}{2}p}$$

The above equation is the RHS of the multilinear decoupling inequality.

Now it is clear what we have to do, that is to increase r, σ to go from $M_{p,q}(1,1)$ to $M_{p,q}(R, R^{1/2})$.

We now introduce and prove the main tools that prove the full decoupling theorem. The first one concerns with local orthoganality and it says the finer caps are actually worse in approximating.

Lemma 0.17 (Orthoganlity)

If $\sigma \le r$, then we have

$$M_{p,2}(r,\sigma) \lesssim M_{p,2}(r,r)$$

Proof We are fixing $q = 2$, hence we have

$$M_{p,2}(r,\sigma) = Avg_{B_r \subset B_R} \prod_{j=1}^{n} \left(\sum_{\tau \subset \Omega_j} \|f_{j,\tau}\|_{L^2}^2 \right)^{\frac{1}{n}\frac{1}{2}p}$$

Then we decompose each τ into even smaller r^{-1} caps, given $\sigma \leq r$, then by orthoganlity inequality, we have

$$\|f_{j,\tau}\|_{L^2} \lesssim \left(\sum_{\theta \subset \tau} \|f_{j,\theta}\|_{L^2}^2 \right)^{1/2}$$

Hence combining, we have

$$M_{p,2}(r,\sigma) \lesssim Avg_{B_r \subset B_R} \prod_{j=1}^{n} \left(\sum_{r \subset \Omega_j} \|f_{j,\theta}\|_{L^2}^2 \right)^{\frac{1}{2}\frac{1}{n}p} = M_{p,2}(r,r)$$

□

Now we prove a result using multilinear Kakeya.

Lemma 0.18 (MK)

For $p = \frac{2n}{n-1}$, if $r \leq R^{1/2}$, then we have the following:

$$M_{p,2}(r,r) \lesssim r^{\epsilon} M_{p,2}(r^2, r)$$

♡

Remark Let's interpret this lemma. This states the if the dissection of the Fourier space is small enough, we then would have it bounded up by not so big value of a a really large ball?

To prove this, we first realize for $p = \frac{2n}{n-1}$, the RHS exponent becomes $\frac{1}{n-1}$ which is the multilinear Kakeya exponent. Recall we have, if we have a weighted characterstic function g_j such that

$$g_j = \sum_a T_{j,a} W_{j,a}$$

where $W_{j,a} \geq 0$, $T_{j,a}$ is a tube close to x_j for all a. We can invoke the multilinear Kakeya in this case.

$$\int_{Q_s} \prod_{j=1}^{n} |g_j|^{\frac{1}{n-1}} \lesssim S^{\epsilon} \prod_{j=1}^{n} \left(\int_{Q_s} |g_j| \right)^{\frac{1}{n-1}}$$

In this case, if we have $g_j = \sum_\theta \|f_{j,\theta}\|_{L^2(B_r)}^2$, for $x \in B_r$, then g_j is a characteristic function with tubes of size $r \times r^2$ pointing near the normal direction of Σ_j with positive weights. We thus have

$$\begin{aligned}
M_{p,2}(r,r) &= Avg_{B_{r^2} \subset B_R} Avg_{B_r \subset B_{r^2}} \prod_{j=1}^{n} \left(\sum_{\theta} \|f_{j,\theta}\|_{L^2(B_r)}^2 \right)^{\frac{p}{2n}} \\
&\sim Avg_{B_{r^2} \subset B_R} \fint \prod_{j=1}^{n} |g_j|^{\frac{1}{n-1}} \\
&\lesssim Avg_{B_{r^2} \subset B_R} r^{\epsilon} \prod_{j=1}^{n} \left(\fint |g_j| \right)^{\frac{1}{n-1}} \\
&\lesssim r^{\epsilon} Avg_{B_{r^2} \subset B_R} \prod_{j=1}^{n} \left(\sum_{\theta} \|f_{j,\theta}\|_{L^2(B_{r^2})}^2 \right)^{\frac{1}{n-1}} \\
&= r^{\epsilon} M_{p,2}(r^2, r)
\end{aligned}$$

□

This completes our proof.

We make a note on the difference between full decoupling theorem and the critical exponent of $p = \frac{2n}{n-1}$, if $p > \frac{2n}{n-1}$, then we can prove an estimate of the form

$$M_{p,2}(r,r) \lesssim r^{\alpha} M_{p,2}(r^2, r)$$

This makes it hard to increase the scale and to go from r to R, as we introduce a nontrivial power of r^{α} each time. This means we will need a stronger multilinear decoupling theorem stated as follows:

Theorem 0.17 (MK2)

If $r \leq R^{1/2}$, then we have

$$M_{p,\frac{(n-1)p}{n}\geq 2}(r,r) \lesssim r^{\epsilon} M_{p,\frac{(n-1)p}{n}\geq 2}(r^2,r)$$

♡

We will prove this shortly, but observe once we have this, we can mimic the proof for $p \leq \frac{2n}{n-1}$.

Lemma 0.19

If $p = \frac{2(n+1)}{n-1}$, then we have

$$M_{p,2}(1,1) \lesssim R^{O(\delta)} M_{p,2}(r^2,r)^{1/2} M_{p,p}(r^2,r)^{1/2}$$

♡

Notice we have two terms on the RHS, hence we now introduce the two Holder inequalities that we need (we will just take them for granted, they are not hard to prove). If we have $q_1 \leq q_2$, then

$$M_{p,q_1}(r,\sigma) \leq M_{p,q_2}(r,\sigma)$$

Next we have, if $\|f\|_{L^q} \leq \|f\|_{L^{q_1}}^{\alpha_1} \|f\|_{L^{q_2}}^{\alpha_2}$, then we have

$$M_{p,q}(r,\sigma) \leq M_{p,q_1}^{\alpha_1}(r,\sigma) M_{p,q_2}^{\alpha_2}(r,\sigma)$$

0.2 Lecture 19

We dissect slabs using arcs.

Lemma 0.20

We have $r_1 \leq r_2$, then we have

$$M_{p,p}(r_1, \sigma) \leq M(_{p,p} r_2, \sigma)$$

Proof By definition, we have

$$M_{p,p}(r_1, a) = Avg_{B_{r_1} \subset B_R} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}\frac{1}{k}p} = Avg_{B_2 \subset B_R} Avg_{B_1 \subset B_2} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}\frac{1}{k}p}$$

Then we examine

$$Avg_{B_1 \subset B_2} \prod_{j=1}^{n} \left(\sum_{\theta \subset \Omega_j} \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}\frac{1}{k}p} \lesssim \prod_{j=1}^{n} \left(Avg_{B_1 \subset B_2} \left(\sum \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}p} \right)^{\frac{1}{k}} \lesssim \prod_{j=1}^{n} \left(\sum \|f_{j,\theta}\|_{L^p}^2 \right)^{\frac{1}{2}\frac{1}{k}p}$$