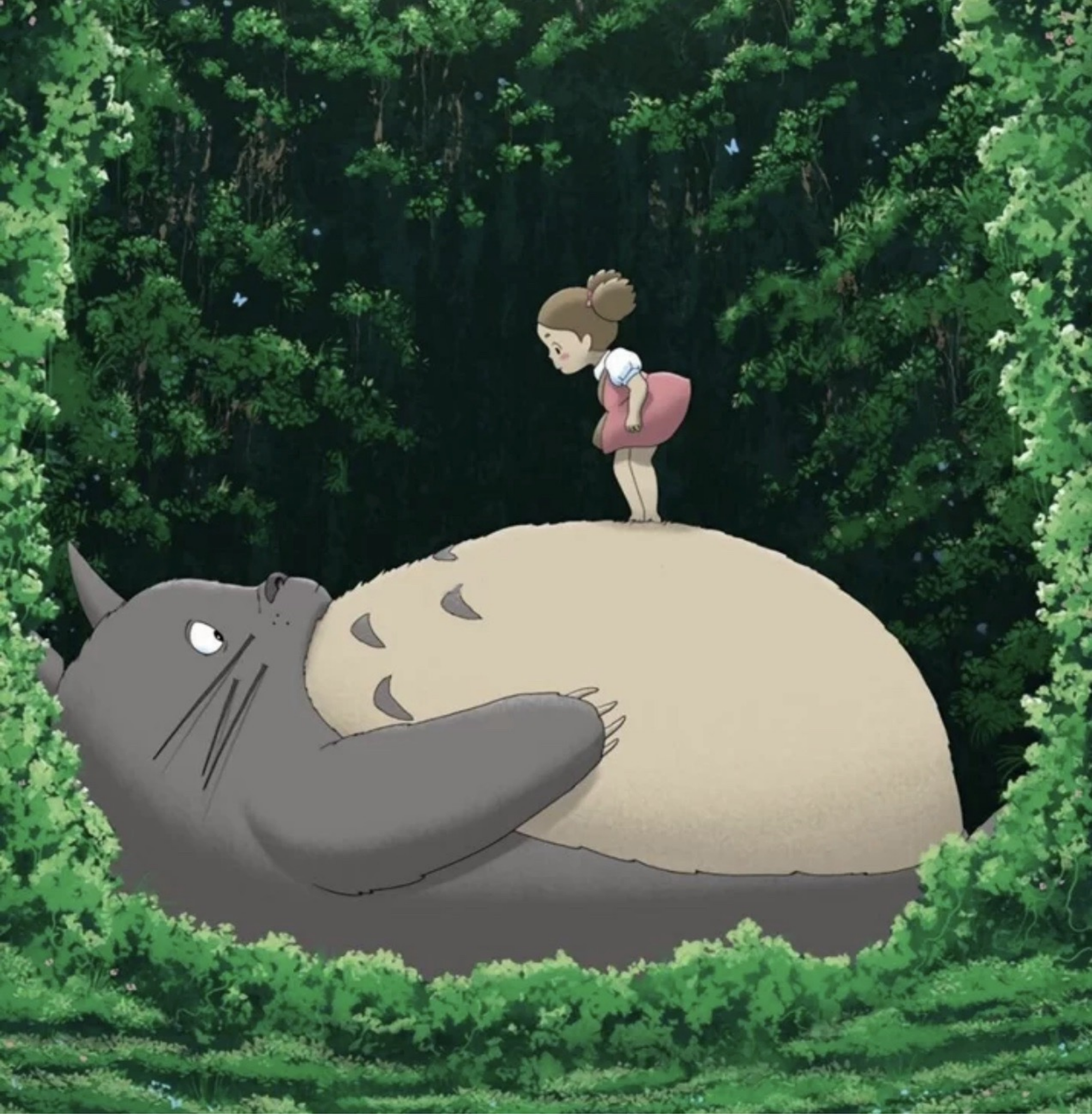

Real Analysis

Author: Hui Sun

Date: January 2022

Contents

We consider the following conjecture:

$$|\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 &= (\sum_{a=1}^{n} e^{ia^3 x})(\sum_{a=1}^{n} e^{-ib^3 x}) \\ &= \sum_{a_1, a_2, a_3, b_1, b_2, b_3} e^{ix(a_1^3 + a_2^3 + a_3^3 - b_1^3 - b_2^3 - b_3^3)} \end{aligned}$$

Hence the integral is 0 if the diophatine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophatine equation.

We consider the following conjecture:

$$|\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 &= (\sum_{a=1}^{n} e^{ia^3 x})(\sum_{a=1}^{n} e^{-ib^3 x}) \\ &= \sum_{a_1, a_2, a_3, b_1, b_2, b_3} e^{ix(a_1^3 + a_2^3 + a_3^3 - b_1^3 - b_2^3 - b_3^3)} \end{aligned}$$

Hence the integral is 0 if the diophatine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophatine equation.

Introduction to decoupling

Now we move to the overview of decoupling.

If we denote a region Ω of $\mathbb{R}^n$ as the Fourier space, and we decompose it into small regions $\Omega = \bigsqcup \theta$.

If we assume the function f whose Fourier transform has support in the region Ω, then we can decompose Ω, we will now make the definition as follows.

Definition 0.1 (Decoupling)

Let f be a sufficiently regular function whose $supp(\widehat{f}) \subset \Omega$, if we define

$$f_\theta = \int_\theta \widehat{f}(\omega) e^{ix\omega} d\omega$$

Then by Fourier inverse formula, we get

$$f = \sum_\theta f_\theta$$

♣

Proof $\sum_\theta f\theta = \int_\Omega \widehat{f}(\omega) e^{ix\omega} d\omega = f(x)$

One would like to control the norm $||f||_{L^p}$, using what you know about $||f_\theta||_{L^p}$. To give a general idea what we are heading towards, we can fine a constant D_p, dependent on Ω, θ, such that the following inequality is achieved.

$$||f||_{L^p}(\mathbb{R}^n) \leq D_p(\Omega = \bigsqcup \theta) \left(\sum_\theta ||f_\theta||_{L^p}^2 \right)^{1/2}$$

Hence if we fix a specific decoupling choice, i.e. $\Omega = \bigsqcup_\theta \theta$, we can find an absolute constant such that

$$||f||_{L^p} \leq C \left(\sum_\theta ||f_\theta||_{L^p}^2 \right)^{1/2}$$

And we define the decoupling constant $D_p(\Omega = \bigsqcup \theta)$ to be the smallest of all C, for a fixed decoupling choice.

Remark Are there different ways of decomposing omega? Is the most intuitive way of decomposing the Fourier space based on frequency?

Proposition 0.1 (Estimate of sum)

Let $R > 0$, and pick Fourier space decomposition $\Omega = \bigsqcup_j \theta_j$, and let $g = \sum_j a_j e^{i\omega_j \cdot x}$. If $B_{1/R}(\omega_j) \subset \theta_j$, for all j, then for any ball of radius R, B_R, we have

$$||g||_{L^p(B_R)} \lesssim D_p \left(\sum_j |a_j|^2 \right)^{1/2} R^{1/p}$$

♠

Proof Let $f = \eta g$, such that $supp(\widehat{\eta}) \subset B_{1/R}$, such that $|\eta| \sim 1$ on B_R and decays rapidly outside of B_R.

Lecture 2

Now we begin with some building blocks.

Suppose $\Omega = [0, N], \theta_j = [j-1.j], \Omega = \bigsqcup_{j=1}^{N} \theta_j$. And we ask the question, if we have $supp(\widehat{f}) \subset [0,1]$, could $|f|$ look like several narrow peaks and almost 0 elsewhere?

We recall how we decouple the function f: for $supp(\widehat{f}) \subset \Omega$, define $f_{\theta_j} = \int_{[j-1,j]} \widehat{f}(\omega) e^{i\omega x} d\omega$, then $f = \sum_j f_{\theta_j}$.

Now we remind ourselves of the height of f.

Proposition 0.2

Let $f \in \mathcal{S}$ be such that $supp(\widehat{f}) \subset [0,1]$, and we have

$$||f||_{L^\infty} \lesssim ||f||_{L^1}$$

Proof We define a cutoff function $\eta \in \mathcal{S}$ such that $\eta = 1$ on $[0,1]$, then $\widehat{f} = \eta \widehat{f}$, then $f = f * \check{\eta}$, also a Schwartz function.

$$\begin{aligned} ||f||_{L^\infty} &= ||f * \check{\eta}||_{L^\infty} \\ &\le ||f||_{L^1} ||\check{\eta}||_{L^\infty} \\ &\lesssim ||f||_{L^1} \end{aligned}$$

Hence the answer is no, because if we have narrow peaks with controlled heights, $||f||_{L^1}$ would be small, which would violate $||f||_{L^\infty} \lesssim ||f||_{L^1}$.

Now we ask the following question, can we have flat parts of $|f|$ where $||f||_{L^1}$ is dominated by the flat parts, but still has narrow peaks? To address that, we introduce an important lemma which allows us to control the height of f in one interval using its L^1 norm in an even larger interval.

Proposition 0.3 (Locally Constant Lemma)

If $supp\widehat{f_1} \subset [0,1]$, and I is the unit interval $[0,1]$, then we have

$$||f||_{L^\infty(I)} \lesssim ||f||_{L^1(\omega_I}$$

Where the weighted L^1 norm is defined to be $||f||_{L^1(\omega_I)} = \int_{\mathbb{R}} |f_1| \omega_I$ where the function ω_I satisfies the following: $\omega_I \ge 0$, $\omega_I \sim 1$ on I, and ω_I decays rapidly off of I, lastly, ω_I is uniform in the sense that $\omega_{I+a} = \omega_I(\cdot - a)$

Proof This follows from the fact that $\eta \in \mathcal{S}$, hence $\check{\eta} \in \mathcal{S}$ as well, i.e. we have

$$|\check{\eta}(y)| \lesssim \left(\frac{1}{1+|y|}\right)^M$$

for all large M. Hence we follow the same computation:

$$\begin{aligned} |f(x)| &= \left| \int f(y) \check{\eta}(x-y) dy \right| \\ &\le \int |f(y)||\check{\eta}(x-y)| dy \\ &\le \int |f(y)| \sup_{x \in I} |\check{\eta}(x-y)| dy \end{aligned}$$

And if we define $\omega_I(y) = \sup_{x\in I} |\check{\eta}(x-y)|$, surely it satisfies being nonnegative, and by property of $\check{\eta}$ being Schwartz, $\check{\eta} \sim 1$ on I, and decays rapidly if $|x-y|$ is greater than 0.

In other words, we almost know that $||f||_{L^\infty(I)} \lesssim ||f||_{L^1(2I)}$, where $2I$ is if we stretch the intervals keeping the same center.

Remark For $p = 2, p = \infty$, the decoupling constant is easier to estimate. For $p = 2$, we can apply Plancherel, namaley,

$$||f||_{L^2} = ||\sum_j f_{\theta_j}||_{L^2} = ||\sum_j \widehat{f}_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2} = \sum_j ||f_{\theta_j}||_{L^2}$$

For $p = \infty$, we can apply Cauchy Schwartz, namely,

$$||f||_{L^\infty} = ||\sum_j f_{\theta_j}||_{L^\infty} \leq \sum_j ||f_{\theta_j}||_{L^\infty} \leq \left(\sum_j ||f_{\theta_j}||^2_{L^\infty}\right)^{1/2} N^{1/2}$$

And now we conclude with an example. Consider a function f_1 with height 1, (i.e. $||f||_{L^\infty} = 1$) and $f_1(0) = 1$ and it is concentrated on the interval $[-1, 1]$. If we define $f_j(x) = e^{2\pi i(j-1)x} f_1(x)$ and define $f = \sum_j f_j$, then we have $f_j(0) = 1$ and thus $f(0) = N$.

We note that f_j oscillates with frequency $\frac{1}{j}$ and when $|x| \leq \frac{1}{10N} \leq \frac{1}{10j}$, we have $f_(x) \sim N$. Hence, if we consider $||f||_{L^p}$, we have

$$||f||^p_{L^p} = \int |f|^p = \geq \int_{|x| \leq \frac{1}{10N}} |f|^p = \gtrsim \frac{1}{N} \cdot N^p = N^{p-1}$$

Hence taking the $1/p$ of both sides, we have $||f||_{L^p} \gtrsim N^{1-1/p}$.

Now if we would wish to consider the decoupling constant, we now consider $||f_j||_{L^p}$. Note $||f_j||_{L^p} \sim 1$, hence $\left(\sum_j ||f_j||^2_{L^2}\right) \sim N^{1/2}$. Thus we have $D_p \gtrsim N^{1/2-1/p}$.

Main Obstacle

Consider a function f_j such that $|f_j| = 1$ on $[0, 1]$, and is $\frac{1}{N}$ on $[1, N^3]$, and 0 elsewhere. Then the $||f_j||_{L^2} \sim N^{1/2}$, whereas $||f_j||_{L^4} \sim 1$. (Exactly how one owuld expect the L^p norm to behave).

Like the above remark, we note that $||f||_{L^2} \sim \sum_j ||f_j||^2_{L^2})^{1/2} \sim N$, and $||f||_{L^\infty} \leq N^{1/2}(N)^{1/2} = N$. Now we ask the question, could $|f(x)| \sim N$ on the unit interval $[0, 1]$? The answer is no.

Proof Assume $|f(x)| \sim N$ on $[0, 1]$, then $||f||_{L^4} \gtrsim N$, however, we know

$$||f||_{L^4} \lesssim D_p \left(\sum_j ||f_j||^2_{L^4} \lesssim N^{1/4} \cdot N^{1/2} = N^{3/4}\right)$$

Note D_p arises from our above lower bound given that $p = 4$.

Recall the Local Constant Lemma tells us how the height is controlled by the L^1 norm, now we introduce another lemma that connects the L^2 norms, which improves our estimate.

Lemma 0.1 (Local Orthogonality Lemma)

If I is a unit interval, and $f = \sum_{j=1}^N f_j$, and supp$\widehat{f_j} \subset [j-1, f]$, then we have

$$||f||^2_{L^2(I)} \lesssim \sum_j ||f_j||^2_{L^2(\omega_I)}$$

Proof We choose η such that it preserves f on the unit interval, and whose fourier transform has support land in $[-1, 1]$, i.e. $|\eta| \sim 1$ on I, and $supp(\eta) \subset [-1,1]$.

$$\begin{aligned}
||f||_{L^2(I)}^2 &= \int_I |f|^2 \\
&\leq \int_{\mathbb{R}} |\eta f|^2 \\
&= \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \int |\sum_j \widehat{\eta} * \widehat{f}|^2 \\
&\lesssim \sum_j \int_{\mathbb{R}} |\widehat{\eta} * \widehat{f}|^2 \\
&= \sum_j \int_{\mathbb{R}} |\eta|^2 |f_j|^2 \\
&= \sum_j ||f||_{L^2(\omega_I)}^2
\end{aligned}$$

if we define $\omega_I = |\eta|^2$.

We thus obtain this local orthogonality result, in the sense that we can decompose the L^2 norm locally and control the L^2 norm of f by the sum of the L^2 norm of f_j.

Now we generalize this to a wide range of p to obtain our local decoupling result.

Proposition 0.4 (Local decoupling)

If I is a unit interval, for $2 \leq p \leq \infty$, for each $1 \leq j \leq N$, $supp(f_j) \subset [j-1, j]$, then we have

$$||f||_{L^p(I)} \lesssim N^{1/2-1/p} \left(\sum_{j=1}^{N} ||f_j||_{L^p(\omega_I)}^2 \right)^{1/2}$$

Proof This follows from the Locally Constant Lemma and the Locally Orthogonality Lemma above.

$$\int |f|^p = \int |f|^2 |f|^{p-2} \leq ||f||_{L^\infty(I)}^{p-2} \int |f|^2 \leq \sum_j ||f_j||_{L^2(\omega_I)}^2 \left(\sum_j ||f_j||_{L^\infty(I)} \right)^{p-2}$$

The last inequality follows from the local orthogonality lemma above which states $||f||_[L^2(I)] \leq \sum_j ||f_j||_{L^2(\omega_I)}$.

Then for the second term, local constant lemma states that the height is controlled by the L^1 norm $||f_j||_{L^\infty} \lesssim ||f_j||_{L^1(\omega_I)} \lesssim ||f_j||_{L^2}$, where the last inequality is to match the L^2 norm of the first term. Combining, we have $\int |f|^p \leq (\sum_j \|f_j\|_{L^2(\omega_I)}^2)(\sum_j \|f\|_{L^2(\omega_I)})^{p-2}$. By Cauchy Schwarz on the second term, we obtain,

$$\int |f|^p \leq \left(\sum_j \|f_j\|_{L^2(\omega_I)}^2 \right) \left(\sum_j \|f_j\|_{L^2(\omega_I)}^2 \right)^{p/2-1} N^{p/2-1} = \left(\sum_j \|f_j\|_{L^2(\omega_I)}^2 \right)^{p/2} N^{p/2-1}$$

If we replace the $\|f\|_{L^2}$ with $\|f\|_{L^p}$, we get the desired result.

Lemma 0.2

In finite measure spaces, for $p \geq q$, we have

$$\|f\|_{L^p} \lesssim \|f\|_{L^q}$$

Proof This follows from Holder's inequality.

$$\int_I |f|^p = \||f|^p\|_{L^{q/p}} \mu(I)^s \lesssim \|f\|_{L^q}^p$$

Now we prove the parallel decoupling lemma, which basically states that if we decompose two measures as $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, and for each i, we have the same decoupling constant, then we would be able to keep that decoupling constant when we sum them up. Recall the Minkowski's inequality refers to triangle inequality with respect to the L^p

norm.

Proposition 0.5 (Parallel Decoupling Lemma)

For some $p \geq 2$, and for any function $g = \sum_j g_j$, and any measures $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$, then if for each i, we have

$$\|g\|_{L^p(\mu_i)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega_i)} \right)^{1/2}$$

then summing up, we would have the combined inequallity with the same decoupling constant,

$$\|g\|_{L^p(\mu)} \leq D \left(\sum_j \|g_j\|^2_{L^p(\omega)} \right)^{1/2}$$

Proof The proof uses the Minkowski's inequality.

$$\begin{aligned}
\int |g|^p \mu &= \sum_i \int |g|^p \mu_i \\
&\leq D^p \sum_i \left(\sum_j ||g_j||^2_{L^p(\omega_i)} \right)^{p/2} \\
&\leq D^p \left\| \sum_j ||g_j||^2_{L^p(\omega_i)} \right\|^{p/2}_{l_i^{p/2}} \\
&\leq D^p \left(\sum_j \left\| ||g_j||^2_{L^p(\omega_i)} \right\|_{l_i^{p/2}} \right)^{p/2} \\
&= D^p \left(\sum_j \left(\sum_i ||g_j||^p_{L^p(\omega_i)} \right)^{2/p} \right)^{p/2} \\
&= D^p \left(\sum_j ||g_j||^2_{L^p(\omega)} \right)^{p/2}
\end{aligned}$$

Lecture 3

We state Bourgain and Demeter's decoupling theorem for the parabaloid. We define a parabaloid as

$$P = \{\omega \in \mathbb{R}^n : \omega_n = \omega_1^2 + ...\omega_{n-1}^2, |\omega| \leq 1\}$$

We now define a slightly larger neighborhood of P, denoted by $N_{1/R}P$ as the neighborhood of P of radius $1/R$ where R is some large constant. This is the area that we will cut up using rectangles θ,

$$\theta \approx R^{-1/2} \times R^{-1/2} \times ... \times R^{-1}$$

Now we've fixed a decomposition, we denote its decoupling constant $D_p(R) = D_p(\omega = \cup\theta)$, and we ar eready to state our decoupling theorem.

Theorem 0.1 (Bourgain and Demeter)

For $2 \leq p \leq \frac{2(n+1)}{n-1}$, for the "cutting up" scheme above, we have

$$D_p(R) \lesssim R^{\epsilon}$$

♡

Now we define the dual of θ, denoted by $\theta^* = \{x \in \mathbb{R}^n : |\omega - \omega_\theta| < \frac{1}{x}, \forall \omega \in \theta\}$, with ω_θ being the center of θ. Now we look at how these are $\widehat{\phi}$ behaves on $\theta*$ and then rapidly decaying.

Lemma 0.3

If ϕ_θ is a smooth bump function supported in θ, then $\check{\phi_\theta} \sim |\theta|$ on $\theta^*|$ and rapidly decaying outside of it. ♡

Proof If ϕ_θ lives on θ, we calculate its inverse Fourier transform to find the corresponding function in the physical space.

$$|\check{\phi_\theta}| = \left|\int e^{2\pi i x\omega}\phi_\theta(\omega)d\omega\right| = \left|e^{2\pi i x\omega_\theta}\int e^{2\pi i x(\omega-\omega_\theta)}\phi_\theta(\omega)d\omega\right| = \left|\int_\theta e^{2\pi i x(\omega-\omega_\theta)}\right| \sim |\theta|$$

If $x \in \theta^*$, then we know $|(\omega - \omega_\theta)x| < 1$, hence the oscillatory integral on the right hand side does not get too much cancellation, hence $|\check{\phi_\theta}| \sim |\theta|$, while outside of θ^*, if one integrate by parts many times, then get rapid decay.

Remark This lemma shows why the dual θ^* is called the dual, i.e., if we consider functions first in the Fourier space, then we can find the corresponding θ^* such that $\check{\phi_\theta}$ is located and the size of it. Moreover,

$$\check{\phi_\theta} \text{ roughly looks like } e^{2\pi i x\omega_\theta}|\theta|\chi_{\theta^*}$$

Based off of the smooth bump function in the Fourier space, we now consider its translation $check\phi_\theta(x - x_0)$ and the sum $\sum a_k\check{\phi_\theta}(x - x_0)$. Recall the locally constant lemma in lecture 2 states that if we consider the a function f whose Fourier support $supp(\widehat{f}) \subset I$, then we have the locally constant lemma which states $\|f\|_{L^\infty(I)} \lesssim \|f\|_{L^1(\omega_I)}$, where ω_I is 1 on I and rapidly decays off of I. Here we present a similar result.

Lemma 0.4 (Locally Constant)

Suppose f is such that $supp(\widehat{f}) \subset \theta$, and T is some translation of θ^*, then the locally constant lemma holds on T as well, namely,

$$\|f\|_{L^\infty(T)} \lesssim \|f\|_{L^1(\omega_T)}$$

where the L^1 norm here is the average L^1 norm.

♡

Proof Just like how we proved the unit interval case, we define a smooth bump function in the Fourier space that captures f. Let η bu such that it is 1 on θ and decays rapidly outside of θ, $|\eta(\omega)| \leq \left(\frac{1}{1+|\omega|}\right)^M$, for all large M. Then we have $\widehat{f} = \eta\widehat{f}$, hence

$$|f(x)| \leq \sup_{x\in T}\int |f(y)\check{\eta}(x-y)|dy \leq \int |f(y)|\sup_{x\in T}|\check{\eta}(x-y)|dy$$

As we noted above, $|\check{\eta}| \sim |\theta| = \frac{1}{|\theta^*|}$ on θ^* and decays rapidly off of it. And we note $\omega_T(y) = \sup_{x\in T}\check{\eta}(x-y)$ also behaves on T like this.

Let's now get in business. Consider single wavepackets f_θ, a wave packet is the inverse Fourier transform of a bump function in the Fourier space, for example, $f_\theta = \check{\eta}$, where $\eta(omega) = 1$ on θ and decays rapidly off of it. Here we consider these wave packets normalized as $f_\theta(0) = 1$, let $f = \sum_\theta f_\theta$.

Then in a small neighborhood of 0, where not much cancellation can happen, we have $|f(0)|$ equals the number of θ, which is $R^{(n-1)/2}$. Hence the L^p norm of f.

$$\|f\|_{L^p} \geq \|f\|_{B_1} \gtrsim R^{(n-1)/2}$$

And now we consider the RHS of the decoupling relation, which is the L^p norm of each wave packet, that is f_θ. Because $\widehat{f_\theta}$ is supported in θ and f_θ is normalized, hence f_θ is 1 on θ^* and decays rapidly off of it. Hence,

$$\|f\|_{L^p} \sim |\theta^*|^{1/p} = R^{(n+1)/(2p)}$$

Hence we have,

$$\left(\sum_\theta \|f\|_{L^p}^2\right)^{1/2} \sim R^{\frac{n-1}{4}+\frac{n+1}{2p}}$$

Combining, we have $R^{(n-1)/2} \lesssim D_p R^{\frac{n-1}{4}+\frac{n+1}{2p}}$, hence $D_p \gtrsim R^{\frac{n-1}{4}-\frac{n+1}{2p}}$. And Bourgain and Demeter claim up to an R^ϵ loss, this is the worst that can happen.

Theorem 0.2 (Bourgain and Demeter)

$D_p(R) \lesssim R^\epsilon \max\{R^{\frac{n-1}{4}-\frac{n+1}{2p}}, 1\}$ ♡