



# Real Analysis

**Author:** Hui Sun

**Date:** January 2022

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We consider the following conjecture:

$$|\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^n e^{ia^3 x} \right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned} \left| \sum_{a=1}^n e^{ia^3 x} \right|^6 &= \left( \sum_{a=1}^n e^{ia^3 x} \right) \left( \sum_{a=1}^n e^{-ib^3 x} \right) \\ &= \sum_{a_1, a_2, a_3, b_1, b_2, b_3} e^{ix(a_1^3 + a_2^3 + a_3^3 - b_1^3 - b_2^3 - b_3^3)} \end{aligned}$$

Hence the integral is 0 if the diophantine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophantine equation.

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## Introduction to decoupling

Now we move to the overview of decoupling.

If we denote a region  $\Omega$  of  $\mathbb{R}^n$  as the Fourier space, and we decompose it into small regions  $\Omega = \bigsqcup \theta$ .

If we assume the function  $f$  whose Fourier transform has support in the region  $\Omega$ , then we can decompose  $\Omega$ , we will now make the definition as follows.

### Definition 0.1 (Decoupling)

Let  $f$  be a sufficiently regular function whose  $\text{supp}(\widehat{f}) \subset \Omega$ , if we define

$$f_\theta = \int_\theta \widehat{f}(\omega) e^{ix\omega} d\omega$$

Then by Fourier inverse formula, we get

$$f = \sum_\theta f_\theta$$



**Proof**  $\sum_\theta f_\theta = \int_\Omega \widehat{f}(\omega) e^{ix\omega} d\omega = f(x)$

One would like to control the norm  $\|f\|_{L^p}$ , using what you know about  $\|f_\theta\|_{L^p}$ . To give a general idea what we are heading towards, we can find a constant  $D_p$ , dependent on  $\Omega, \theta$ , such that the following inequality is achieved.

$$\|f\|_{L^p(\mathbb{R}^n)} \leq D_p(\Omega = \bigsqcup \theta) \left( \sum_\theta \|f_\theta\|_{L^p}^2 \right)^{1/2}$$

Hence if we fix a specific decoupling choice, i.e.  $\Omega = \bigsqcup_\theta \theta$ , we can find an absolute constant such that

$$\|f\|_{L^p} \leq C \left( \sum_\theta \|f_\theta\|_{L^p}^2 \right)^{1/2}$$

And we define the decoupling constant  $D_p(\Omega = \bigsqcup \theta)$  to be the smallest of all  $C$ , for a fixed decoupling choice.

**Remark** Are there different ways of decomposing omega? Is the most intuitive way of decomposing the Fourier space based on frequency?

### Proposition 0.1 (Estimate of sum)

Let  $R > 0$ , and pick Fourier space decomposition  $\Omega = \bigsqcup_j \theta_j$ , and let  $g = \sum_j a_j e^{i\omega_j \cdot x}$ . If  $B_{1/R}(\omega_j) \subset \theta_j$ , for all  $j$ , then for any ball of radius  $R$ ,  $B_R$ , we have

$$\|g\|_{L^p(B_R)} \lesssim D_p \left( \sum_j |a_j|^2 \right)^{1/2} R^{1/p}$$



**Proof** Let  $f = \eta g$ , such that  $\text{supp}(\widehat{\eta}) \subset B_{1/R}$ , such that  $|\eta| \sim 1$  on  $B_R$  and decays rapidly outside of  $B_R$ .

## Lecture 2

Now we begin with some building blocks.

Suppose  $\Omega = [0, N]$ ,  $\theta_j = [j-1, j]$ ,  $\Omega = \bigsqcup_{j=1}^N \theta_j$ . And we ask the question, if we have  $\text{supp}(\widehat{f}) \subset [0, 1]$ , could  $|f|$  look like several narrow peaks and almost 0 elsewhere?

We recall how we decouple the function  $f$ : for  $\text{supp}(\widehat{f}) \subset \Omega$ , define  $f_{\theta_j} = \int_{[j-1, j]} \widehat{f}(\omega) e^{i\omega x} d\omega$ , then  $f = \sum_j f_{\theta_j}$ .

Now we remind ourselves of the height of  $f$ .

### Proposition 0.2

Let  $f \in \mathcal{S}$  be such that  $\text{supp}(\widehat{f}) \subset [0, 1]$ , and we have

$$\|f\|_{L^\infty} \lesssim \|f\|_{L^1}$$

**Proof** We define a cutoff function  $\eta \in \mathcal{S}$  such that  $\eta = 1$  on  $[0, 1]$ , then  $\widehat{f} = \eta \widehat{f}$ , then  $f = f * \check{\eta}$ , also a Schwartz function.

$$\begin{aligned} \|f\|_{L^\infty} &= \|f * \check{\eta}\|_{L^\infty} \\ &\leq \|f\|_{L^1} \|\check{\eta}\|_{L^\infty} \\ &\lesssim \|f\|_{L^1} \end{aligned}$$

Hence the answer is no, because if we have narrow peaks with controlled heights,  $\|f\|_{L^1}$  would be small, which would violate  $\|f\|_{L^\infty} \lesssim \|f\|_{L^1}$ .

Now we ask the following question, can we have flat parts of  $|f|$  where  $\|f\|_{L^1}$  is dominated by the flat parts, but still has narrow peaks? To address that, we introduce an important lemma which allows us to control the height of  $f$  in one interval using its  $L^1$  norm in an even larger interval.

### Proposition 0.3 (Locally Constant Lemma)

If  $\text{supp} \widehat{f}_1 \subset [0, 1]$ , and  $I$  is the unit interval  $[0, 1]$ , then we have

$$\|f\|_{L^\infty(I)} \lesssim \|f\|_{L^1(\omega_I)}$$

Where the weighted  $L^1$  norm is defined to be  $\|f\|_{L^1(\omega_I)} = \int_{\mathbb{R}} |f_1| \omega_I$  where the function  $\omega_I$  satisfies the following:  $\omega_I \geq 0$ ,  $\omega_I \sim 1$  on  $I$ , and  $\omega_I$  decays rapidly off of  $I$ , lastly,  $\omega_I$  is uniform in the sense that  $\omega_{I+a} = \omega_I(\cdot - a)$

**Proof** This follows from the fact that  $\eta \in \mathcal{S}$ , hence  $\check{\eta} \in \mathcal{S}$  as well, i.e. we have

$$|\check{\eta}(y)| \lesssim \left( \frac{1}{1 + |y|} \right)^M$$

for all large  $M$ . Hence we follow the same computation:

$$\begin{aligned} |f(x)| &= \left| \int f(y) \check{\eta}(x - y) dy \right| \\ &\leq \int |f(y)| |\check{\eta}(x - y)| dy \\ &\leq \int |f(y)| \sup_{x \in I} |\check{\eta}(x - y)| dy \end{aligned}$$

And if we define  $\omega_I(y) = \sup_{x \in I} |\check{\eta}(x - y)|$ , surely it satisfies being nonnegative, and by property of  $\check{\eta}$  being Schwartz,  $\check{\eta} \sim 1$  on  $I$ , and decays rapidly if  $|x - y|$  is greater than 0.

In other words, we almost know that  $\|f\|_{L^\infty(I)} \lesssim \|f\|_{L^1(2I)}$ , where  $2I$  is if we stretch the intervals keeping the same center.

**Remark** For  $p = 2, p = \infty$ , the decoupling constant is easier to estimate. For  $p = 2$ , we can apply Plancherel, namely,

$$\|f\|_{L^2} = \left\| \sum_j f_{\theta_j} \right\|_{L^2} = \left\| \sum_j \widehat{f_{\theta_j}} \right\|_{L^2} = \sum_j \|f_{\theta_j}\|_{L^2} = \sum_j \|f_{\theta_j}\|_{L^2}$$

For  $p = \infty$ , we can apply Cauchy Schwartz, namely,

$$\|f\|_{L^\infty} = \left\| \sum_j f_{\theta_j} \right\|_{L^\infty} \leq \sum_j \|f_{\theta_j}\|_{L^\infty} \leq \left( \sum_j \|f_{\theta_j}\|_{L^\infty}^2 \right)^{1/2} N^{1/2}$$

And now we conclude with an example. Consider a function  $f_1$  with height 1, (i.e.  $\|f_1\|_{L^\infty} = 1$ ) and  $f_1(0) = 1$  and it is concentrated on the interval  $[-1, 1]$ . If we define  $f_j(x) = e^{2\pi i(j-1)x} f_1(x)$  and define  $f = \sum_j f_j$ , then we have  $f_j(0) = 1$  and thus  $f(0) = N$ .

We note that  $f_j$  oscillates with frequency  $\frac{1}{j}$  and when  $|x| \leq \frac{1}{10N} \leq \frac{1}{10j}$ , we have  $f_j(x) \sim N$ . Hence, if we consider  $\|f\|_{L^p}$ , we have

$$\|f\|_{L^p}^p = \int |f|^p \geq \int_{|x| \leq \frac{1}{10N}} |f|^p \gtrsim \frac{1}{N} \cdot N^p = N^{p-1}$$

Hence taking the  $1/p$  of both sides, we have  $\|f\|_{L^p} \gtrsim N^{1-1/p}$ .

Now if we would wish to consider the decoupling constant, we now consider  $\|f_j\|_{L^p}$ . Note  $\|f_j\|_{L^p} \sim 1$ , hence  $(\sum_j \|f_j\|_{L^2}^2) \sim N^{1/2}$ . Thus we have  $D_p \gtrsim N^{1/2-1/p}$ .

## Main Obstacle

Consider a function  $f_j$  such that  $|f_j| = 1$  on  $[0, 1]$ , and is  $\frac{1}{N}$  on  $[1, N^3]$ , and 0 elsewhere. Then the  $\|f_j\|_{L^2} \sim N^{1/2}$ , whereas  $\|f_j\|_{L^4} \sim 1$ . (Exactly how one would expect the  $L^p$  norm to behave).

Like the above remark, we note that  $\|f\|_{L^2} \sim \sum_j \|f_j\|_{L^2}^2)^{1/2} \sim N$ , and  $\|f\|_{L^\infty} \leq N^{1/2}(N)^{1/2} = N$ . Now we ask the question, could  $|f(x)| \sim N$  on the unit interval  $[0, 1]$ ? The answer is no.

**Proof** Assume  $|f(x)| \sim N$  on  $[0, 1]$ , then  $\|f\|_{L^4} \gtrsim N$ , however, we know

$$\|f\|_{L^4} \lesssim D_p \left( \sum_j \|f_j\|_{L^4}^2 \lesssim N^{1/4} \cdot N^{1/2} = N^{3/4} \right)$$

Note  $D_p$  arises from our above lower bound given that  $p = 4$ .

Recall the Local Constant Lemma tells us how the height is controlled by the  $L^1$  norm, now we introduce another lemma that connects the  $L^2$  norms, which improves our estimate.

### Lemma 0.1 (Local Orthogonality Lemma)

If  $I$  is a unit interval, and  $f = \sum_{j=1}^N f_j$ , and  $\text{supp } \widehat{f_j} \subset [j-1, j]$ , then we have

$$\|f\|_{L^2(I)}^2 \lesssim \sum_j \|f_j\|_{L^2(\omega_I)}^2$$



**Proof** We choose  $\eta$  such that it preserves  $f$  on the unit interval, and whose fourier transform has support land in  $[-1, 1]$ ,

i.e.  $|\eta| \sim 1$  on  $I$ , and  $\text{supp}(\eta) \subset [-1, 1]$ .

$$\begin{aligned}
 \|f\|_{L^2(I)}^2 &= \int_I |f|^2 \\
 &\leq \int_{\mathbb{R}} |\eta f|^2 \\
 &= \int_{\mathbb{R}} |\hat{\eta} * \hat{f}|^2 \\
 &= \int_{\mathbb{R}} \left| \sum_j \hat{\eta} * \hat{f}_j \right|^2 \\
 &\lesssim \sum_j \int_{\mathbb{R}} |\hat{\eta} * \hat{f}_j|^2 \\
 &= \sum_j \int_{\mathbb{R}} |\eta|^2 |f_j|^2 \\
 &= \sum_j \|f_j\|_{L^2(\omega_I)}^2
 \end{aligned}$$

if we define  $\omega_I = |\eta|^2$ .

We thus obtain this local orthogonality result, in the sense that we can decompose the  $L^2$  norm locally and control the  $L^2$  norm of  $f$  by the sum of the  $L^2$  norm of  $f_j$ .

Now we generalize this to a wide range of  $p$  to obtain our local decoupling result.

**Proposition 0.4 (Local decoupling)**

If  $I$  is a unit interval, for  $2 \leq p \leq \infty$ , for each  $1 \leq j \leq N$ ,  $\text{supp}(f_j) \subset [j-1, j]$ , then we have

$$\|f\|_{L^p(I)} \lesssim N^{1/2-1/p} \left( \sum_{j=1}^N \|f_j\|_{L^p(\omega_I)}^2 \right)^{1/2}$$



**Proof** This follows from the Locally Constant Lemma and the Locally Orthogonality Lemma above.

$$\int |f|^p = \int |f|^2 |f|^{p-2} \leq \|f\|_{L^\infty(I)}^{p-2} \int |f|^2 \leq \sum_j \|f_j\|_{L^2(\omega_I)}^2 \left( \sum_j \|f_j\|_{L^\infty(I)} \right)^{p-2}$$

The last inequality follows from the local orthogonality lemma above which states  $\|f\|_{L^2(I)} \leq \sum_j \|f_j\|_{L^2(\omega_I)}$ .

Then for the second term, local constant lemma states that the height is controlled by the  $L^1$  norm  $\|f_j\|_{L^\infty} \lesssim \|f_j\|_{L^1(\omega_I)} \lesssim \|f_j\|_{L^2}$ , where the last inequality is to match the  $L^2$  norm of the first term. Combining, we have  $\int |f|^p \leq (\sum_j \|f_j\|_{L^2(\omega_I)}^2)(\sum_j \|f_j\|_{L^2(\omega_I)})^{p-2}$ . By Cauchy Schwarz on the second term, we obtain,

$$\int |f|^p \leq \left( \sum_j \|f_j\|_{L^2(\omega_I)}^2 \right) \left( \sum_j \|f_j\|_{L^2(\omega_I)}^2 \right)^{p/2-1} N^{p/2-1} = \left( \sum_j \|f_j\|_{L^2(\omega_I)}^2 \right)^{p/2} N^{p/2-1}$$

If we replace the  $\|f\|_{L^2}$  with  $\|f\|_{L^p}$ , we get the desired result.

**Lemma 0.2**

In finite measure spaces, for  $p \geq q$ , we have

$$\|f\|_{L^p} \lesssim \|f\|_{L^q}$$



**Proof** This follows from Holder's inequality.

$$\int_I |f|^p = \int_I |f|^p \|1\|_{L^{q/p}(I)}^p \lesssim \int_I |f|^p = \|f\|_{L^q}^p$$

Now we prove the parallel decoupling lemma, which basically states that if we decompose two measures as  $\mu = \sum_i \mu_i$ ,  $\omega = \sum_i \omega_i$ , and for each  $i$ , we have the same decoupling constant, then we would be able to keep that decoupling constant when we sum them up. Recall the Minkowski's inequality refers to triangle inequality with respect to the  $L^p$



norm.

**Proposition 0.5 (Parallel Decoupling Lemma)**

For some  $p \geq 2$ , and for any function  $g = \sum_j g_j$ , and any measures  $\mu = \sum_i \mu_i, \omega = \sum_i \omega_i$ , then if for each  $i$ , we have

$$\|g\|_{L^p(\mu_i)} \leq D \left( \sum_j \|g_j\|_{L^p(\omega_i)}^2 \right)^{1/2}$$

then summing up, we would have the combined inequality with the same decoupling constant,

$$\|g\|_{L^p(\mu)} \leq D \left( \sum_j \|g_j\|_{L^p(\omega)}^2 \right)^{1/2}$$



**Proof** The proof uses the Minkowski's inequality.

$$\begin{aligned} \int |g|^p \mu &= \sum_i \int |g|^p \mu_i \\ &\leq D^p \sum_i \left( \sum_j \|g_j\|_{L^p(\omega_i)}^2 \right)^{p/2} \\ &\leq D^p \left\| \sum_j \|g_j\|_{L^p(\omega_i)}^2 \right\|_{l_i^{p/2}}^{p/2} \\ &\leq D^p \left( \sum_j \left\| \|g_j\|_{L^p(\omega_i)}^2 \right\|_{l_i^{p/2}} \right)^{p/2} \\ &= D^p \left( \sum_j \left( \sum_i \|g_j\|_{L^p(\omega_i)}^p \right)^{2/p} \right)^{p/2} \\ &= D^p \left( \sum_j \|g_j\|_{L^p(\omega)}^2 \right)^{p/2} \end{aligned}$$



## Lecture 3

We state Bourgain and Demeter's decoupling theorem for the paraboloid. We define a paraboloid as

$$P = \{\omega \in \mathbb{R}^n : \omega_n = \omega_1^2 + \dots \omega_{n-1}^2, |\omega| \leq 1\}$$

We now define a slightly larger neighborhood of  $P$ , denoted by  $N_{1/R}P$  as the neighborhood of  $P$  of radius  $1/R$  where  $R$  is some large constant. This is the area that we will cut up using rectangles  $\theta$ ,

$$\theta \approx R^{-1/2} \times R^{-1/2} \times \dots \times R^{-1}$$

Now we've fixed a decomposition, we denote its decoupling constant  $D_p(R) = D_p(\omega = \cup \theta)$ , and we are ready to state our decoupling theorem.

### Theorem 0.1 (Bourgain and Demeter)

For  $2 \leq p \leq \frac{2(n+1)}{n-1}$ , for the "cutting up" scheme above, we have

$$D_p(R) \lesssim R^\epsilon$$



Now we define the dual of  $\theta$ , denoted by  $\theta^* = \{x \in \mathbb{R}^n : \omega_\theta + \frac{1}{x} \in \theta\}$ .