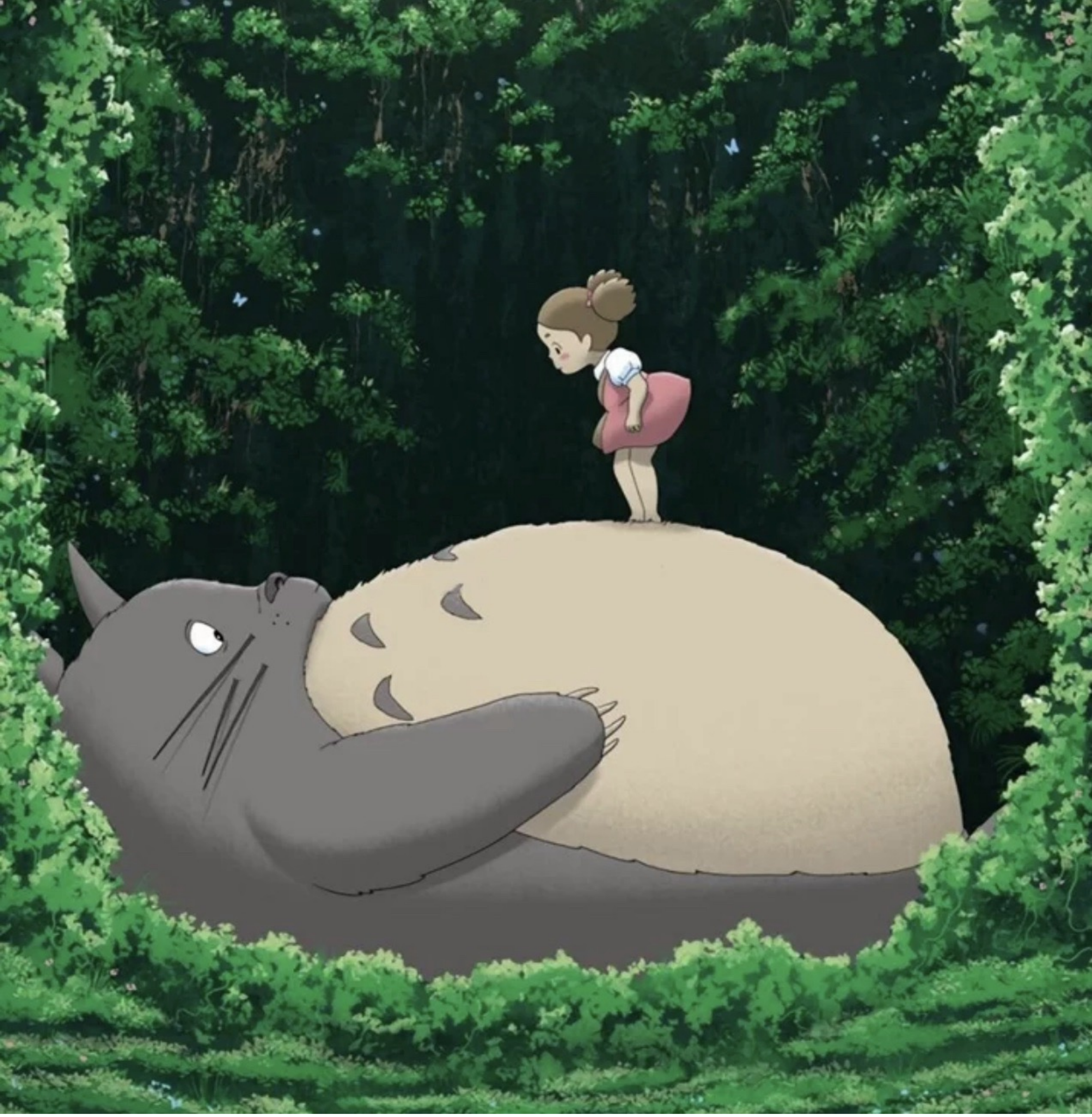

Real Analysis

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We consider the following conjecture:

$$|\{1 \leq a_i, b_i \leq N : a_1^3 + a_2^3 + a_3^3 = b_1^3 + b_2^3 + b_3^3\}| \lesssim N^{3+\epsilon}$$

This follows from the natural Strichartz estimate.

We observe the following integral.

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 dx$$

The RHS is equal to the number of solutions to the diophantine equation above.

$$\begin{aligned} \left| \sum_{a=1}^{n} e^{ia^3 x} \right|^6 &= (\sum_{a=1}^{n} e^{ia^3 x})(\sum_{a=1}^{n} e^{-ib^3 x}) \\ &= \sum_{a_1, a_2, a_3, b_1, b_2, b_3} e^{ix(a_1^3 + a_2^3 + a_3^3 - b_1^3 - b_2^3 - b_3^3)} \end{aligned}$$

Hence the integral is 0 if the diophatine is satisfied, and 0 otherwise. Hence the integral evaluates exactly the number of diophatine equation.

Introduction to decoupling

Now we move to the overview of decoupling.

If we denote a region Ω of $\mathbb{R}^n$ as the Fourier space, and we decompose it into small regions $\Omega = \bigsqcup \theta$.

If we assume the function f whose Fourier transform has support in the region Ω, then we can decompose Ω, we will now make the definition as follows.

Definition 0.1 (Decoupling)

Let f be a sufficiently regular function whose $supp(\widehat{f}) \subset \Omega$, if we define

$$f_\theta = \int_\theta \widehat{f}(\omega) e^{ix\omega} d\omega$$

Then by Fourier inverse formula, we get

$$f = \sum_\theta f_\theta$$

Proof $\sum_\theta f\theta = \int_\Omega \widehat{f}(\omega) e^{ix\omega} d\omega = f(x)$

One would like to control the norm $||f||_{L^p}$, using what you know about $||f_\theta||_{L^p}$. To give a general idea what we are heading towards, we can fine a constant D_p, dependent on Ω, θ, such that the following inequality is achieved.

$$||f||_{L^p}(\mathbb{R}^n) \leq D_p(\Omega = \bigsqcup \theta) \left(\sum_\theta ||f_\theta||_{L^p}^2 \right)^{1/2}$$

Hence if we fix a specific decoupling choice, i.e. $\Omega = \bigsqcup_\theta \theta$, we can find an absolute constant such that

$$||f||_{L^p} \leq C \left(\sum_\theta ||f_\theta||_{L^p}^2 \right)^{1/2}$$

And we define the decoupling constant $D_p(\Omega = \bigsqcup \theta)$ to be the smallest of all C, for a fixed decoupling choice.

Remark Are there different ways of decomposing omega? Is the most intuitive way of decomposing the Fourier space based on frequency?

Proposition 0.1 (Estimate of sum)

Let $R > 0$, and pick Fourier space decomposition $\Omega = \bigsqcup_j \theta_j$, and let $g = \sum_j a_j e^{i\omega_j \cdot x}$. If $B_{1/R}(\omega_j) \subset \theta_j$, for all j, then for any ball of radius R, B_R, we have

$$||g||_{L^p(B_R)} \lesssim D_p \left(\sum_j |a_j|^2 \right)^{1/2} R^{1/p}$$

Proof Let $f = \eta g$, such that $supp(\widehat{\eta}) \subset B_{1/R}$, such that $|\eta| \sim 1$ on B_R and decays rapidly outside of B_R.

Lecture 2

Now we begin with some building blocks.

Suppose $\Omega = [0, N], \theta_j = [j - 1.j], \Omega = \bigsqcup_{j=1}^{N} \theta_j$. And we ask the question, if we have $supp(\widehat{f}) \subset [0, 1]$, could $|f|$ look like several narrow peaks and almost 0 elsewhere?

We recall how we decouple the function f: for $supp(\widehat{f}) \subset \Omega$, define $f_{\theta_j} = \int_{[j-1,j]} \widehat{f}(\omega) e^{i\omega x} d\omega$, then $f = \sum_j f_{\theta_j}$.

Now we remind ourselves of the height of f.

Proposition 0.2

Let $f \in \mathcal{S}$ be such that $supp(\widehat{f}) \subset [0, 1]$, and we have

$$||f||_{L^\infty} \lesssim ||f||_{L^1}$$

Proof We define a cutoff function $\eta \in \mathcal{S}$ such that $\eta = 1$ on $[0, 1]$, then $\widehat{f} = \eta \widehat{f}$, then $f = f * \check{\eta}$, also a Schwartz function.

$$\begin{aligned} ||f||_{L^\infty} &= ||f * \check{\eta}||_{L^\infty} \\ &\leq ||f||_{L^1} ||\check{\eta}||_{L^\infty} \\ &\lesssim ||f||_{L^1} \end{aligned}$$

Hence the answer is no, because if we have narrow peaks with controlled heights, $||f||_{L^1}$ would be small, which would violate $||f||_{L^\infty} \lesssim ||f||_{L^1}$.

Now we ask the following question, can we have flat parts of $|f|$ where $||f||_{L^1}$ is dominated by the flat parts, but still has narrow peaks? To address that, we introduce an important lemma which allows us to control the height of f in one interval using its L^1 norm in an even larger interval.

Proposition 0.3 (Locally Constant Lemma)