



# Functional Analysis

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# Chapter 1 Lecture 1

Here we go.

## 1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

**Textbook:** an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable number of seminorms).

**Remark** Let  $\mathcal{H}$  be a Hilbert space, then the dual space  $\mathcal{H}^*$  is itself.  $\mathcal{H} = \mathcal{H}^*$ . Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themselves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

## 1.0.2 Motivation

Let  $X$  be a compact Hausdorff space. Let  $C(X) = \{f : X \rightarrow \mathbb{R}, f \text{ continuous}\}$  be the algebra of continuous functions on  $X$  mapping in to  $\mathbb{R}$  or  $\mathbb{C}$ . Define the norm as the sup norm  $\|\cdot\|_{L^\infty}$ .

We will develop the spectral theorem of operators on the Hilbert space, i.e. self-adjoint operators can be diagonalized.

If  $T$  is a self-adjoint operator on a Hilbert space, then we take the product of  $T$  (polynomials of  $T$ ), let  $C^*(T, I_{\mathcal{H}})$  be the sub-algebra of operators generated by  $T$  and  $I$  the identity operator, then take the closure, i.e. making it closed in the operator norm.

**Remark** The  $*$  is to remind us,  $T$  is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

### Proposition 1.1

*We have the next two algebra isomorphic to each other.*

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \quad (1.1)$$

This is what we are aiming for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If  $T_1, \dots, T_n$  is a collection of self-adjoint operators on  $\mathcal{H}$ , and such all commute with each other, then we also have

$$C^*(T_1, \dots, T_n, I_{\mathcal{H}}) \cong C(X) \quad (1.2)$$

## 1.0.3 Groups

Let  $G$  be a group,  $B$  be a Banach space, for example, groups of automorphisms. Let

$$\text{Aut}(B) = \{T : T \text{ is isometric, onto, invertible on } B\}$$

### Definition 1.1

*Suppose that  $\alpha$  is a group homomorphism, and  $\alpha : G \rightarrow \text{Aut}(B)$ , is called a representation on  $B$  or an action of the group  $G$  on  $B$ .*

Then we can consider the subalgebra  $\mathcal{L}(B)$ , consisting of the bounded linear operators on  $B$ , generated by

$$\{\alpha_x : x \in G\}$$

**Remark** The identity on  $G$  should be mapped into the identity operator on  $B$ , hence no need to include it.

Elements of the form  $\sum_{z \in \Sigma} \alpha_x z_x \in \mathbb{C}$ , (where  $\Sigma$  is a finite sum.)

Let's introduce,  $f \in C_c(G)$  are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x \in G} f(x) \alpha_x = \alpha_f$$

note for except finitely many  $x$ ,  $f(x) = 0$ .

Let  $f, g \in C_c(G)$ , then for

$$\alpha_f \alpha_g = \left( \sum f(x) \alpha_x \right) \left( \sum g(y) \alpha_y \right) = \sum_{x,y} f(x) g(y) \alpha_x \alpha_y = \sum_{x,y} f(x) g(y) \alpha_{xy}$$

The last inequality follows from  $\alpha$  being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f \alpha_g = \sum_x \sum_y f(x) g(x^{-1}y) \alpha_y = \sum (f * g)(y) \alpha_y$$

where we define  $f * g(y) = \sum f(x) g(x^{-1}y)$  as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f * g}$$

This is how we define convolution on  $C_c(G)$  Notice we have, by  $\|\alpha_x\| = 1$ ,

$$\|\alpha_f\| = \left\| \sum f(x) \alpha_x \right\| \leq \sum |f(x)| \|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We get  $l^1(G)$  is an algebra with ??

For  $G$  commutative, it is easily connected with the Fourier transform.

Consider  $l^2(G)$  with the counting measure on the group. For  $x \in G$ , let  $\xi \in l^2(G)$  define  $\alpha_x \xi(y) = \xi(x^{-1}y)$ ,  $\alpha_x$  being unitary.  $l^1(G)$  acts on operators in  $l^2(G)$  via  $\alpha$ .

If  $G$  is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where  $X$  is some compact space. Note that  $C_c(G)$  operators on  $l^2(G)$ , and  $\|\alpha_f\| \leq \|f\|_{l^1}$ .

## 1.1 Lecture 2

Let's do some math.

Let  $X$  be a Hausdorff compact space, and let  $C(X)$  denote the space of continuous functions defined on  $X$ . This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm  $\|\cdot\|_{L^\infty}$ . Note  $X$ , by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from  $C(X)$ .

### Definition 1.2 (Normed Algebra)

Let  $\mathcal{A}$  be an algebra on  $\mathbb{R}$  or  $\mathbb{C}$ , is a normed algebra if it has a norm  $\|\cdot\|$ , as a vector space, such that for  $a, b \in \mathcal{A}$ , we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity.

### Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm.

Given  $x \in X$ , define  $\varphi_x : C(X) \rightarrow \mathbb{C}$  the evaluation map such that

$$\varphi_x(f) = f(x)$$

$\varphi_x$  is an algebra homomorphisms between  $C(X) \rightarrow \mathbb{R}$  or  $C(X) \rightarrow \mathbb{C}$ . This simply implies

$$\varphi_x(f + g) = (f + g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that,  $C(X)$  has an identity element, which is the constant function 1, under multiplication. Hence  $C(X)$  is a unital algebra. Note that  $\varphi_x$  defined above is a unital homomorphism, meaning that it sends identity to identity.

Note  $\varphi_x$  is also a multiplicative linear functional, also unital.

### Proposition 1.2

Every multiplicative linear functional on  $C(X)$  is of the form  $\varphi_x$  for some  $x \in X$ .

**Proof** Main Claim: given a multiplicative linear functional  $\varphi$ , there exists a point  $x_0$  and if we have some  $f \in C(X)$ , we have  $\varphi(f) = 0$ , then we have  $f(x_0) = 0$ . To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each  $x \in X$ , there is an  $f_x \in C(X)$  such that  $f_x(x) \neq 0$ , but  $\varphi(f_x) = 0$ .

Set  $g_x = \overline{f_x} f_x$ , then we have  $g_x(x) > 0$ , but  $\varphi(g_x) = \varphi(f_x) \varphi(\overline{f_x}) = 0$ , then there is an open set  $O_x$  such that  $x \in O_x$ , and  $g_x(y) > 0$  for all  $y \in O_x$ . Now by compactness, there is  $x_1, \dots, x_n$  such that  $X = \bigcup_{j=1}^n O_{x_j}$ , let  $g = g_{x_1} + \dots + g_{x_n}$ , then we have  $g(y) > 0$  for all  $y \in X$ , and  $\varphi(g) = 0$ . Note that  $g$  is a continuous function, and  $g$  is invertible, and also  $re(\frac{1}{g}) \in C(X)$ , but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists  $x_0 \in X$  such that if  $\varphi(f) = 0$ , this means  $f(x_0) = 0$ . For any  $f$ , consider  $f - \varphi(f) \cdot 1$ , apply  $\varphi$ , we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies  $f(x_0) = \varphi(f)$  which implies  $\varphi(f) = \varphi_{x_0}(f)$ .

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.

For  $\mathcal{A} = C(X)$ , and  $\varphi \in \widehat{\mathcal{A}}$ .

### Definition 1.4

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.

**Remark** We have  $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$ , since  $\varphi$  is unital, we have  $\|\varphi\| = 1$ .

This is not always true for normed algebra, Let

$$\mathcal{A} := \text{Poly} \subset C([0, 1])$$

We define  $\varphi(p) = p(2)$ ,  $p$  is a polynomial. This is not continuous, nor is the  $\|\varphi\| = 1$ .

### Proposition 1.3

If  $\mathcal{A}$  is a unital commutative Banach algebra, and if  $\phi \in \widehat{\mathcal{A}}$ , then we have  $\|\varphi\| = 1$ .

The word “unital” is key here.

### Proposition 1.4

Let  $\mathcal{A}$  be a unital Banach algebra (not necessarily commutative), then if  $a \in \mathcal{A}$ , and  $\|a\| < 1$ , then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

**Proof** For this, we use completeness.  $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ ,  $a^0 = 1_{\mathcal{A}}$ . You could look at the partial sums.  $S_m = \sum_{n=0}^m a^n$ , you want to show that  $\{S_m\}$  is a Cauchy sequence, and use completeness of Banach algebras.  $\lim_{m \rightarrow \infty} S_m = \frac{1}{1-a}$ .

To prove this is a Cauchy sequence:

$$\|S_n - S_m\| = \left\| \sum_{j=m+1}^n a^j \right\| \leq \sum_{j=m+1}^n \|a^j\| \leq \sum_{j=m+1}^n \|a\|^j$$

And the fact that  $\|a\| < 1$ , we have the sum bounded by  $\epsilon$ , hence  $\{S_n\}$  is a Cauchy sequence. Let  $b = \sum_{n=0}^{\infty} a^n$ , we want to show that  $b(1-a) = 1$ .

$$b(1-a) = \lim_{n \rightarrow \infty} S_n(1-a) = \lim_{n \rightarrow \infty} \left( \sum_{n=0}^{\infty} a^n \right) (1-a) = \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 1$$

The last inequality follows from  $\|a^{n+1}\| \leq \|a\|^{n+1} \rightarrow 0$ .

## 1.2 Lecture 3

We now begin.

Let  $\mathcal{A}$  be a unital Banach algebra, and if  $a \in \mathcal{A}$  and  $\|a\| < 1$ , then we have  $(1-a)$  has an inverse and if  $\mathcal{A} = \mathcal{B}(B)$ , where  $B$  is some Banach space, then  $T \in \mathcal{A}$ , and  $\|T\| < 1$ , then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Neumann series.

Now we have the following corollary.

### Corollary 1.1

If  $a \in \mathcal{A}$  and  $\|1-a\| < 1$ , then  $a$  is invertible.

**Proof**  $a = 1 - (1-a)$ .

### Proposition 1.5

The set of invertible elements of  $\mathcal{A}$  is an open subset of  $\mathcal{A}$ .

**Proof** The open ball about 1 consists of invertible elements. If  $d$  is any invertible element, then we define  $a \mapsto da$ . This map is continuous, i.e. it is the left representation  $L_b(a) = ab$  for all  $a \in \mathcal{A}$ . If  $d$  is invertible, then the inverse is also continuous, hence it is a homeomorphism of  $\mathcal{A}$  onto itself.

Denote the unit ball about 1 as  $B_1(1)$ , and let  $d$  be some invertible element, under  $L_d$ , homeomorphism,  $O \mapsto d \cdot O$ , this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.



□

**Proposition 1.6**

Let  $C(X)$  be the unital Banach algebra, and for  $f \in C(X)$ , we have  $\alpha \in \text{Range}(f)$  if and only if  $(f - \alpha \cdot 1)$  is not invertible.



**Proof** Let  $f \in C(X)$ , and if  $\alpha \in \text{range of } f$ , so  $\alpha = f(x_0)$  for some  $x_0$ . then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $(f - \alpha \cdot 1)$  is not invertible. Conversely, if we have  $f - \alpha \cdot 1$  is not invertible, then there exists  $x_0 \in X$  such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $f(x_0) = \alpha$ , i.e.,  $\alpha \in \text{range of } f$ .

□

**Definition 1.5 (spectrum of an element)**

For any unital algebra  $\mathcal{A}$  over some field  $\mathbb{F}$ , for any  $a \in \mathcal{A}$ , the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible} \}$$

is called the spectrum of  $a$ , denoted as  $\sigma(a)$ .



Interpret this in our familiar linear map:  $\lambda$  is called an eigenvalue, i.e. is in the spectrum of  $T$  if we have  $T - \lambda I$  is not invertible.

**Proposition 1.7**

Let  $\mathcal{A}$  be a unital Banach algebra, and let  $a \in \mathcal{A}$ , then if  $\lambda \in \sigma(a)$ , then

$$|\lambda| \leq \|a\|$$



**Proof** Suppose  $|\lambda| > \|a\|$ , then  $\lambda \neq 0$ , then

$$a - \lambda \cdot 1 = -\lambda \left(1 - \frac{a}{\lambda}\right)$$

And by assumption,  $\|a/\lambda\| \leq 1$ , hence  $(1 - a/\lambda)$  is invertible. Hence  $a - \lambda \cdot 1$  is invertible (product of two invertible elements), meaning  $\lambda \notin \sigma(a)$ .

□

**Proposition 1.8**

Let  $\varphi$  be a multiplicative linear functional on  $\mathcal{A}$ , i.e.  $\varphi \in \widehat{\mathcal{A}}$ , and then  $\varphi(a) \in \sigma(a)$ , and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$



**Proof**  $\varphi(a - \varphi(a) \cdot 1) = 0$ . Hence  $a - \varphi(a)1$  is not invertible.

□

**Proposition 1.9**

$\sigma(a)$  is a closed subset of  $\mathbb{R}, \mathbb{C}$ .



**Proof** Define the map  $\phi : \lambda \mapsto a - \lambda 1$ , the map  $\phi$  is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of  $\mathcal{A}$  is open, hence

$$\sigma(a) = \phi^{-1}(\text{noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{invertible})$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{invertible}))^c$$

Hence the spectrum of an element is closed.

□

Let  $\varphi \in \widehat{\mathcal{A}}$  then  $\|\varphi\| = 1$ . So  $\widehat{\mathcal{A}}$  is a subset of the unit ball of  $\mathcal{A}'$ , which denotes the dual vector space of continuous linear transformations.

On  $\mathcal{A}'$ , we can equip the weak-\* topology, i.e. the weakest topology, making the map  $\psi \mapsto \psi(a)$  continuous.

#### Proposition 1.10

$\widehat{\mathcal{A}}$  is closed for the weak-\* topology.



**Proof** let  $\{\varphi_\lambda\}$  be a net of elements of  $\widehat{\mathcal{A}}$ , that converges to some  $\psi \in \mathcal{A}'$  in the weak-\* topology, i.e., for every  $a \in \mathcal{A}$ ,  $\varphi_\lambda(a) \rightarrow \psi(a)$  for all  $a \in \mathcal{A}$ .

Then  $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$ .

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$ .

#### Theorem 1.1 (Alaoglu's theorem)

For any normed vector space  $V$ , the closed unit ball of  $V'$  is compact in the weak-\* topology.



As an immediate corollary, we have the following.

#### Corollary 1.2

$\widehat{\mathcal{A}}$  is compact with respect to the weak-\* topology.



**Proof**  $\widehat{\mathcal{A}}$  is a closed subset of a compact set, hence is also compact. □

Let  $\mathcal{A} = C(X)$ , and  $\widehat{\mathcal{A}}$ , we define  $x \mapsto \varphi_x$  is a bijection. The weak-\* topology in  $\widehat{\mathcal{A}}$  makes  $\varphi_x \mapsto \varphi_x(f) = f(x)$  continuous. Such  $x \mapsto \varphi_x$  is a homomorphism of  $X$  onto  $\mathcal{A}$ .

For  $\mathcal{A}$  unital Banach algebra, commutative, for any  $a \in \mathcal{A}$ , define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

#### Proposition 1.11

The map  $a \mapsto \widehat{a}$  is a unital algebra homomorphism from  $\mathcal{A}$  into  $C(\mathcal{A})$ .



**Proof** we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{ab})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$



## 1.3 Lecture 4

Today we talk about the structure of  $\widehat{l^1(S)}, \widehat{l^1(G)}$ , where  $S, G$  are semigroups and groups, and how they naturally identify with the unit disk  $\mathbb{D}$ , and the unit circle  $\mathbb{T}$ .

Let  $S$  be a commutative discrete semigroups, for example  $\mathbb{N} \cup \{0\}$ , and  $f \in C_c(S)$ , then we can write  $f = \sum_{x \in S} f(x)\delta_x$ , where we define  $\delta_x\delta_y = \delta_{xy}$ . Note that  $C_c(S)$  is dense in  $l^1(S)$ .

#### Definition 1.6 (Convolution)

Take any  $f, g \in C_c(S)$ , we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{y \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left( \sum_{xy=z} f(x)g(y) \right) \delta_z$$



where we define the convolution between two functions

$$f * g(z) = \sum_{x,y, xy=z} f(x)g(y)$$

And under this convolution operation, we have  $l^1(S), *$  as a Banach algebra.

**Example 1.1** If we consider polynomials of the form  $f(x) = \sum_{n=0}^{\infty} f(n)x^n$ , and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right) \left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f * g)(p)$$

And let  $f \in C_c(S)$ , where  $S = \mathbb{N}$ . we define  $\|f\|_{l^1} = \sum_{x \in S} |f(x)|$ .

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We let  $\mathcal{A} = l^1(S)$ , and  $\widehat{\mathcal{A}}$  denote the set of unital homomorphisms from  $\mathcal{A}$  to  $\mathbb{R}, \mathbb{C}$ . Note that  $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$ .

Note that we know  $(l^1(S))' = l^\infty(S)$ , hence  $\widehat{\mathcal{A}} \subset \mathcal{A}'$ . Note that we have  $\|\varphi\| = 1$ , hence if we  $\varphi \in l^\infty(S)$ , we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for  $z \in S, \|z\| \leq 1$ , we have  $|\varphi(z)| \leq 1$ .

#### Proposition 1.12

We naturally identify  $\widehat{l^1(S)}$  with  $\text{Hom}(S, \mathbb{D})$ , i.e.  $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$ .

**Proof** Given  $f \in \widehat{l^1(S)}$ , we know it's multiplicative, unital, hence all these transfer when viewing  $\varphi \in l^\infty(S)$ . This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here  $xy$  denotes the operation on  $S$  between  $x, y$ , for example, could be  $x + y$ . Hence naturally, if  $\varphi \in \widehat{l^1(S)}$ ,  $\varphi$  can also be viewed as  $\varphi : S \rightarrow \mathbb{D}$ , and thus is in  $l^\infty$ , with  $|\varphi(s)| \leq 1$ . □

Furthermore, we can identify elements in  $\widehat{l^1(S)}$  with the unit disk. Take  $S = \mathbb{N}$ .

#### Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where  $\mathbb{D}$  denotes the unit disk in  $\mathbb{C}$ .

**Proof** We motivate this by noticing  $\mathbb{N}$  is generated by 1, and thus viewing  $\varphi \in \widehat{l^1(\mathbb{N})}$  as  $\varphi \in l^\infty(\mathbb{N})$ , we have  $\varphi$  is determined by  $\varphi(1)$ . And denote  $\varphi(1) = z_0$ , then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for  $z \in \mathbb{D}$ ,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space. □

#### Proposition 1.14

The standard topology on  $\mathbb{D}$  coincides with the weak-\* topology on  $\widehat{l^1(\mathbb{N})}$ .

$$D_{std} \cong D_{weak-*}$$

**Proof** We just need to associate an element in  $\mathbb{D}$  with a function  $\varphi \in \widehat{l^1(\mathbb{N})}$ . And we do this by

$$z \mapsto \sum_{n \in \mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

### 1.3.1 On groups

We let  $G$  denote a discrete commutative group, and we see everything above follows, with one extra property.

#### Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where  $\mathbb{T}$  denotes the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ .

**Proof** For  $\varphi \in \widehat{l^1(G)}$ , we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because  $|\varphi(x)| \leq 1, \forall x$ , Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have  $\widehat{l^1(G)}$  naturally identifies with  $\mathbb{T}$ . Like what we described above, we have what is desired. □

**Remark** Take  $G = \mathbb{Z}$ , if we denote  $z \in \mathbb{T}$  as  $z = e^{2\pi it}$ , then we naturally identify with

$$\sum_{n \in \mathbb{Z}} f(n)e^{2\pi int}$$

we denote this mapping as  $\widehat{f}$ , i.e.

$$\widehat{f}(z) = \sum_{m \in \mathbb{Z}} f(m)e^{2\pi imt}$$

This is the Fourier transform.

## 1.4 Lecture 5

Last time, we talked about if we denote  $\mathcal{A} = l^1(G)$ , equipped with  $\|\cdot\|_{l^1}$ , under convolution, we have

$$\widehat{\mathcal{A}} \cong \text{Hom}(G, \mathbb{T})$$

If we take  $G = (\mathbb{Q}, +)$ , one can ask the question if  $\widehat{\mathcal{A}}$  is big enough. And we will see later in the course, the answer is yes.

For pointwise multiplication,  $\widehat{G}$  forms a group, and in fact  $\widehat{G}$  is a compact topological group.

For any compact commutative group  $G$ , for example  $\mathbb{R}^n$  under  $+$ . Define

$$\widehat{G} = \text{continuous homomorphisms into } \mathbb{T}$$

**Remark** We now require continuous with this general  $G$  (previously was not required for discrete group  $G$ ).

#### Proposition 1.16

Let  $G$  be a locally compact and commutative group, we have  $\widehat{G}$  as a locally compact, commutative group.

We define the pairing between  $G$  and  $\widehat{G}$  as follows:  $x \in G, \varphi \in \widehat{G}$ ,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the following map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let  $G, H$  denote locally compact groups, and  $\phi : G \rightarrow H$  be a continuous homomorphism. Note we have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \widehat{G} & \xleftarrow{\phi} & \widehat{H} \end{array}$$

If we take an element  $\psi \in \widehat{H}$ , we consider  $\psi \circ \phi$ . We get  $\psi \circ \phi \in \widehat{G}$ .

#### Definition 1.7 (category, functor)

A category is specified by

1. a set of objects
  2. morphisms between objects
- (a).  $X, Y, Z$  are objects, and if

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

- (b). For each object  $X$ , there is an identity morphism  $1_X$ .

And a functor is defined to be such a morphism between categories.



**Example 1.2** For category of finite vector spaces  $V$ , passing from vector space to its dual  $V'$  is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ V' & \xleftarrow{T^t} & W' \\ V'' & \xrightarrow{T^{tt}} & W'' \end{array}$$

The map going in the same directions  $V \rightarrow W$ , and  $V'' \rightarrow W''$  is called covariant, whereas  $V' \leftarrow W'$  is called contravariant.

**Example 1.3** For category of locally compact groups  $G, H$ , assigning the dual group is a functor:

$$\begin{array}{ccc} G & \rightarrow & H \\ \widehat{G} & \leftarrow & \widehat{H} \\ \widehat{\widehat{G}} & \rightarrow & \widehat{\widehat{H}} \end{array}$$

**Example 1.4** Now let  $X$  be a compact space. Given  $\Phi$  continuous map between  $X \rightarrow Y$ .

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ C(X) & \leftarrow C(\Phi) & C(Y) \end{array}$$

For  $f \in C(Y)$ , we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Z \\ C(X) & \xleftarrow{C(\varphi)} C(Y) & \xleftarrow{C(\phi)} C(Z) \end{array}$$

where for  $f \in C(Y)$ ,  $C(\varphi)(f) = f \circ \varphi$ , and  $g \in C(Z)$ ,  $C(\phi) = g \circ \phi$ . This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between  $C(X)$  and  $C(Y)$ , there exists a map from  $X$  to  $Y$ .

**Proposition 1.17**

Suppose  $X, Y$  are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism  $\check{F} : X \rightarrow Y$ .



**Proof** Define  $\varphi_x : C(X) \rightarrow \mathbb{C}$  as the evaluation map: take  $f \in C(X)$ ,

$$\varphi_x(f) = f(x)$$

Then  $\varphi_x \circ F \in \widehat{C(Y)}$ . And we know that any element in  $\widehat{C(Y)}$  is a point evaluation, i.e. there exists  $y \in Y$  such that

$$\varphi_y = \varphi_x \circ F$$

We thus define  $\check{F}(x) = y$  as such that it satisfies the above equation. We need to show  $\check{F}$  is continuous. Note that  $X, Y$  are compact Hausdorff spaces, and the topology on  $Y$  is the coarsest topology making all functions  $g \in C(Y)$  continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by  $F, g$  being continuous, we have  $\check{F}$  is also continuous.

□

There is a natural bijection between the continuous functions from  $X$  to  $Y$ , and the unital algebra homomorphism from  $C(X)$  to  $C(Y)$ .

A quick reminder:

**Remark** For  $X$  compact, the weak-\* topology coincides with the standard topology.

## 1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let  $\mathcal{A}$  be a unital Banach algebra.

We write  $GL_n(\mathcal{A})$  to denote the general linear group, the group formed by  $n \times n$  matrices with entries from  $\mathcal{A}$ .

The less standard notation is  $GL_I(\mathcal{A})$  is the group of invertible elements in  $\mathcal{A}$ . As we have shown previously, this is a closed subset of  $\mathcal{A}$ . This is the notation that we will use.

**Remark** It is easy to see that the product is jointly continuous.

**Proposition 1.18**

The following map is continuous.

$$a \mapsto a^{-1}$$



**Proof** Given  $\|a - b\| < \delta$ , we would like to show  $\|a^{-1} - b^{-1}\| < \epsilon$ . We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\| \|b - a\| \|b^{-1}\|$$

Take  $\delta = \epsilon / \|a^{-1}\| \|b^{-1}\|$  would suffice.

□

**Proposition 1.19**

Fix  $a \in GL(\mathcal{A})$ , there exists a neighborhood  $O$  of  $a$  and a constant  $K$  such that for all  $y \in O$ , we have

$$\|c^{-1}\| < K$$



**Proof** Let  $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$ , then  $d$  is invertible and

$$d^{-1} = \sum_{n=0}^{\infty} (1 - d)^n$$

We thus have

$$\|d^{-1}\| \leq \frac{1}{1 - \|1 - d\|} \leq \frac{1}{1 - 1/2} = 2$$

We then identify what our  $O$  should be. Let  $O = aV$ , then we want to show that every  $ad$  has an inverse with bounded norm. Because  $a, d$  are both invertible,  $ad$  is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \leq \|d^{-1}\| \|a^{-1}\| \leq 2\|a^{-1}\|$$

□

**Remark** For each invertible element, we can find a neighborhood of invertible elements around it, and using that  $(1 - d)$  is bounded, then  $d$  is invertible, we can bound  $\|d^{-1}\|$ .

**Definition 1.8**

Fix  $a \in \mathcal{A}$ , the resolvent set of  $\mathcal{A}$  is the complement of spectrum of  $\mathcal{A}$ , i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$$



Hence the resolvent set is an open, unbounded subset of  $\mathbb{C}$  or  $\mathbb{R}$ .

**Definition 1.9 (Resolvent function)**

On the resolvent set,  $\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$  is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that  $a$  is fixed, and  $\lambda$  is the variable here.



Now we note that this  $R_a(\lambda)$  function is nicely behaved.

**Proposition 1.20**

The resolvent function  $R_a(z)$  is analytic on the resolvent set, and vanishes as  $z \rightarrow \infty$ .



**Proof** We first define the notation of analyticity on an open subset of  $\mathbb{R}, \mathbb{C}$ : this means for every point in the open set  $O$ , we can find a power series expansion of the function such that its radius of convergence  $> 0$ .

Fix  $z_0$  in the resolvent set. We know  $z_0 1_{\mathcal{A}} - a$  is invertible. We consider  $(z 1_{\mathcal{A}} - a)$ , for  $z$  in the resolvent set. We will omit the  $1_{\mathcal{A}}$  for simplicity.

$$z 1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a) \left( 1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a} \right)$$

We know the latter term is invertible if  $\|\frac{z_0 - z}{z_0 - a}\| < 1$  has norm, hence we have

$$(z - a)^{-1} = \sum_{n=0}^{\infty} \left( \frac{z_0 - z}{z_0 - a} \right)^n (z_0 - a)^{-1}$$

What happens when we let  $z \rightarrow \infty$ , we consider  $R_a(1/z)$ , and let  $z \rightarrow 0$ . Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1 - az}{z}\right)^{-1} = z(1 - az)^{-1}$$

Let  $z \rightarrow 0$  makes  $R_a(1/z)$  go to zero.

□

Now given that  $R_a(z)$  is analytic and bounded at  $\infty$ , we can state the following important theorem.

**Theorem 1.2 (Nonemptiness of spectrum)**

Let  $\mathcal{A}$  be a unital Banach algebra over  $\mathbb{C}$ , then for any  $a \in \mathcal{A}$ , we have  $\sigma(a) \neq \emptyset$ .



**Proof** Assume there exists  $a \in \mathcal{A}$ , such that  $\sigma(a) = \emptyset$ . If  $\mathcal{A} = \mathbb{C}$ , then we would have  $R_a(\lambda)$  be a bounded entire, complex-valued function defined on all of  $\mathbb{C}$ . By Liouville's theorem, we must have  $R_a(z)$  a constant function, but we know  $z \rightarrow \infty$ ,  $R_a \rightarrow 0$ , hence  $R_a(z)$  is constantly 0, but this cannot be true.

If our  $\mathcal{A}$  is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where  $\phi$  is some nonzero element in  $\mathcal{A}'$ , guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at  $\infty$ . Again, the function is constantly 0.

With the nonemptiness of spectrum theorem, we now state the Gelfand-Mazur theorem.

**Theorem 1.3 (Gelfand-Mazur)**

Let  $\mathcal{A}$  be a unital Banach algebra over  $\mathbb{C}$ , if any nonzero element of  $\mathcal{A}$  is invertible, then  $\mathcal{A}$  is isomorphic to  $\mathbb{C}$ .



**Proof** For any  $a \in \mathcal{A}$ , we know  $\sigma(a) \neq \emptyset$ , hence there exists  $\lambda$  such that  $\lambda 1_{\mathcal{A}} - a$  is invertible, i.e.  $a = \lambda 1_{\mathcal{A}}$ , hence establishing an isomorphism between  $\mathcal{A}$  and  $\mathbb{C}$ . In other words,  $\mathcal{A} = \mathbb{C} 1_{\mathcal{A}}$ .

□

**1.5.1 Functional Calculus****Proposition 1.21**

Let  $a \in \mathcal{A}$ , then if  $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$  converges for  $|z| < r$ , where  $r > \|a\|$ , then  $\sum_{n=0}^{\infty} \alpha_n a^n$  converges as well.



We first start with proving the following statement.

**Lemma 1.1**

Let  $f$  be a polynomial,  $\mathcal{A}$  is a unital Banach algebra over  $\mathbb{C}$ ,  $f = \sum_{n=0}^k a_n x^n$ , then for  $a \in \mathcal{A}$ , we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of  $a$  under  $f$  is exactly the spectrum of  $f$  evaluated at  $a$ .



**Proof** ( $\Leftarrow$ ). We take  $\lambda \in \sigma(a)$ , and we would like to show  $f(\lambda)$  is in the spectrum of  $f(a)$ . We note that if  $\lambda \in \sigma(a)$ , then  $a = \lambda 1_{\mathcal{A}}$ , and  $f(\lambda 1_{\mathcal{A}}) = f(a)$ , hence by definition,  $f(a) - f(\lambda) 1_{\mathcal{A}}$  is not invertible implying  $f(\lambda)$  is in the spectrum of  $f(a)$ . Note that this also implies  $f(a) - f(\lambda) = (a - \lambda)Q(z)$  for some polynomial  $Q(z)$ .

( $\Rightarrow$ ). We take  $\lambda \in \sigma(f(a))$ , i.e.  $f(a) = \lambda 1_{\mathcal{A}}$ . we would like to show  $\lambda = f(y)$ , where  $y \in \sigma(a)$ . If  $f$  is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1) \dots (z - c_n)$$

Plugging in  $a$  we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}}) \dots (a - c_n 1_{\mathcal{A}})$$

If  $f(a) - \lambda$  is not invertible, then there exists  $j$  such that  $(a - c_j 1_{\mathcal{A}})$  is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show  $\lambda = f(y)$ , where  $y \in \sigma(a)$ . In fact, we have  $\lambda = f(c_j)$  by knowing  $f(c_j) - \lambda = 0$ .

□

Now let  $f(z) = z^n$ , and if  $\lambda \in \sigma(a)$ , then  $\lambda^n \in \sigma(a^n)$  by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$



This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

#### Definition 1.10 (spectral radius)

Fix  $a \in \mathcal{A}$ , we define the spectral radius of  $a$ , denoted by  $r(a)$ ,

$$r(a) = \sup_{\lambda} \{|\lambda| : \lambda \in \sigma(a)\}$$



Next we introduce an equivalent definition of the spectral radius which connects to the Gelfand transform.

#### Proposition 1.22

For  $\mathcal{A}$  a Banach algebra, we have the following relationship:

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\Gamma(a)\|_{\infty}$$



**Example 1.5** Note it we have a self-adjoint operator  $T$ , then the spectral radius of  $T$  would be the absolute value of the largest eigenvalue,  $|\lambda|$ .

#### Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$



**Proof** From the previous remark that  $|\lambda| \leq \|a^n\|^{1/n}$ , hence this follows.

## 1.6 Lecture 7

I have not typed up for this?

## 1.7 Lecture 8

Let  $\mathcal{A}$  be a unital Banach algebra. Then for  $a \in \mathcal{A}$ , and we look at the resolvent of  $a$ ,  $R_a(\lambda)$ , we've noted that as  $\lambda \rightarrow \infty$ , we have

$$\lim_{\lambda \rightarrow \infty} R_a(\lambda) = \lim_{\lambda \rightarrow \infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for  $|\lambda| \geq \|a\|$ .

Recall that we define the spectral radius,  $r(a)$ , as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

#### Proposition 1.23 (Gelfand-Beurling)

$$r(a) = \lim \|a^n\|^{1/n}$$



**Proof**

If we let  $\lambda = 1/z$ , then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for  $|z| \leq \|a\|^{-1}$ , but maybe?? also for  $|z| < r(a)^{-1}$ ?

For  $r > r(a)$ , i.e.  $|z| \leq r^{-1}$ , we know  $\sum_n a^n r^n$  converges for  $r > r(a)$ .  
 know  $\sum a^n z^n$  converges absolutely. In particular,

$$a^n z^n \rightarrow 0$$

Hence there exists  $M$  such that for  $n \geq M$ , we have

$$\|a^n r^{-n}\| \leq 1$$

This implies that

$$\|a^n\| \leq r^n \Rightarrow \|a^n\|^{1/n} \leq r$$

for all  $n \geq M$ .

This implies that

$$\limsup \|a^n\|^{1/n} \leq r$$

And note that  $r$  is arbitrary close to the spectral radius  $r(a)$ . Hence we have

$$\limsup \|a^n\|^{1/n} \leq r(a) \leq \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

□

For each  $\varphi \in \mathcal{A}'$ , consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n) \lambda^{-n}$$

This series converges for  $r > r(a)$ . We can apply the same process, to argue that there exists  $M_\varphi$  such that

$$\|\varphi(a^n) r^{-n}\| \leq M_\varphi$$

for all  $n \geq 0$ . Note that  $M_\varphi$  could be different for all  $\varphi$ .

Note that

$$\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}''$$

there is a natural injection of  $a \mapsto \hat{a} \in \mathcal{A}''$ .

For each  $n$ , define  $F_n \in \mathcal{A}''$ , by  $F_n(\varphi) = |\varphi(a^n r^{-n})| \leq M_\varphi$ . Applying the UBP, we have

$$|F_n(\varphi)| \leq M \Rightarrow |\varphi(a^n) r^{-n}| \leq M$$

This implies that

$$|\varphi(a^n)| \leq r^n M$$

Note that by Hahn-Banach, for any  $b \in \mathcal{A}$ , we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking  $n$ -th root of both sides, we gets

$$\|a^n\| \leq r^n M \Rightarrow \|a^n\|^{1/n} \leq r M^{1/n} \rightarrow r$$

Hence we again obtain the same result.

□

Recall UBP.

#### Theorem 1.4 (Uniform Boudnedness Principle)

Let  $X$  be Banach, and  $Y$  be normed, let  $T_n : X \rightarrow Y$  be a family of linear operators, and if for all  $x \in X$ , we have

$$\|T_n(x)\| < \infty$$

Then for all  $n$ , we have

$$\|T_n\| < \infty$$

♥

Note that if  $\mathcal{A}$  is unital, and if  $\mathcal{A} \subset \mathcal{B}$  with some unit. For  $a \in \mathcal{A}$ , if  $a$  is not invertible in  $\mathcal{A}$ , then it might be invertible

in  $\mathcal{B}$ . Hence if we use  $\sigma_{\mathcal{A}}(a)$  to denote the spectrum of  $a$  in  $\mathcal{A}$ .

**Proposition 1.24**

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

**Example 1.6** Let  $\mathcal{B} = l^1(\mathbb{Z})$ , and let  $\mathcal{A} = l^1(\mathbb{N})$ , equipped with convolution.

Clearly  $\mathcal{A} \subset \mathcal{B}$ . And note that the delta function at 1,  $\delta_1$  is not invertible in  $\mathcal{A}$  but it has an inverse  $\delta_{-1}$  in  $\mathcal{B}$ . Hence we see  $0 \in \sigma_{\mathcal{A}}(a)$ , but  $0 \notin \sigma_{\mathcal{B}}(a)$ .

**Proposition 1.25 (Spectral radius is preserved)**

For  $\mathcal{A} \subset \mathcal{B}$ , we have

$$r_{\mathcal{A}}(a) = \lim \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

This proposition tells us that the spectral radius of an element  $a \in \mathcal{A}$  is independent of the Banach algebra it is considered in, but rather only depends on itself.

**Proposition 1.26**

Let  $X$  be compact, and let  $\mathcal{A} = C(X)$ . Then for  $f \in C(X)$ , we have

$$\|f^2\|_{\infty} = \|f\|_{\infty}^2$$

**Proof** Look at where  $f$  takes  $\|f\|_{\infty}$ , and square it, since when  $X$  is compact, you can actually obtain the point where  $|f(x)| = \|f\|_{\infty}$ .

**Remark** The same property holds for  $f$  in any unital subalgebra of  $C(X)$ , for example, if  $X \subset \mathbb{C}$ , and let  $\mathcal{A}$ =functions that are holomorphic on an open subset of  $\mathbb{C}$  that are in  $X$ .

**Proposition 1.27**

Let  $\mathcal{A}$  be a unital Banach algebra such that for  $a \in \mathcal{A}$ , we have

$$\|a^2\| = \|a\|^2$$

Then we have

$$r(a) = \|a\|$$

**Proof** If we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any  $n$ , we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking  $1/2^n$ -root of both sides, we get that the spectral radius of  $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

□

Let  $\mathcal{H}$  be a Hilbert space, over  $\mathbb{C}$ , and let  $\mathcal{A} = B(\mathcal{H})$ , i.e. the bounded linear operators on  $\mathcal{H}$ , and equip with the operation of taking adjoint.  $T \mapsto T^*$ .

**Proposition 1.28**

For any  $T \in B(\mathcal{H})$ , we have

$$\|T^*T\| = \|T\|^2$$

**Proof** We know that  $\|T^*\| = \|T\|$ . And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let  $\xi \in \mathcal{H}$ , then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi \rangle = \langle \xi, T^*T\xi \rangle \leq \|T^*T\| \|\xi\|^2$$

where the last inequality follows from Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2} \|\xi\|$$

which by definition ( $\|T\|$  is the smallest constant for the inequality), gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result. □

 **Note** We used the inner product to justify  $\|T\|^2 \leq \|T^*T\|$ , which we cannot necessarily do in a non-Hilbert space.

#### Corollary 1.4

If  $T^* = T$ , then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements. ♥

Note that for general  $T$ , we have  $T^*T$  is always self-adjoint,

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the  $*$ -algebra structure.

## 1.8 Lecture 9

Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{C}$ , and  $\mathcal{B}(\mathcal{H})$  with  $\|\cdot\|_\infty$ , and closed under taking involutions. If  $T \in \mathcal{B}(\mathcal{H})$ , then

$$\|T^*T\| = \|T\|^2$$

So if  $T^* = T$ , then we have

$$\sigma(T) = \|T\|$$

#### Definition 1.11 (Concrete $C^*$ -algebra)

A concrete  $C^*$ -algebra is a norm-closed sub-algebra  $\mathcal{A}$  of  $\mathcal{B}(\mathcal{H})$ , for some  $\mathcal{H}$  such that is **self-adjoint**, i.e., if  $T \in \mathcal{A}$ , then  $T^* \in \mathcal{A}$ . We call  $\mathcal{A}$  is unital if  $1_{\mathcal{H}} \in \mathcal{A}$ . ♣

#### Corollary 1.5

If  $\mathcal{A}$  is a  $C^*$  algebra, then for all  $a \in \mathcal{A}$ ,

$$\|a^*a\| = \|a\|^2$$

If  $\mathcal{A}$  is unital  $C^*$ -algebra, and if  $a^* = a$ , then  $r(a) = \|a\|$ . ♥

This follows from our discussion above. Next we say a bit about the Gelfand transform.

**Definition 1.12**

Let  $\mathcal{A}$  be a unital Banach algebra, and commutative, then we have the Gelfand transform  $\Gamma : \mathcal{A} \rightarrow C(\widehat{\mathcal{A}})$ :

$$\Gamma(a)(\varphi) = \varphi(a)$$

then he said something of homomorphisms to the complex numbers or something

Note that if  $a \in \mathcal{A}$ , and  $\varphi \in \widehat{\mathcal{A}}$ , then  $\varphi(a) \in \sigma(a)$ , then we have

$$|\varphi(a)| \leq \|a\| = r(a)$$

then  $\|\widehat{a}\|_\infty \leq r(a)$ . Now we would like to show

$$\|\widehat{a}\|_\infty = r(a)$$

**Theorem 1.5**

$r(a)$  is the spectral radius of  $a$ , which is defined as  $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$ , and we have

$$r(a) = \|\widehat{a}\|_\infty$$



**Note** This gives us a correspondence between  $\varphi$  and non-invertible elements of  $\mathcal{A}$ . This says if  $a$  is not invertible, then we can find some  $\varphi$  such that  $\varphi(a) = 0$ , i.e.  $a$  kills the non-invertible element.

We will now dedicate the next 50 minutes of our life to proving this theorem.

**Theorem 1.6 ( $\varphi$  and maximal ideal)**

If  $\lambda \in \sigma(a)$ , then there exists  $\varphi \in \widehat{\mathcal{A}}$  such that  $\varphi(a) = \lambda$ . This is equivalent to saying: if  $(a - \lambda 1)$  is not invertible, then there is a  $\varphi \in \widehat{\mathcal{A}}$ , such that

$$\varphi(a - \lambda 1) = 0$$

This means if  $a$  is not invertible, then there exists  $\varphi \in \widehat{\mathcal{A}}$  such that  $\varphi(a) = 0$ .

**Proof** Suppose  $a \in \mathcal{A}$ , and consider

$$a\mathcal{A} = \{ab : b \in \mathcal{A}\}$$

The set  $a\mathcal{A}$  does not contain the identity element i.e.  $1_{\mathcal{A}} \notin a\mathcal{A}$  (otherwise it would imply it has an inverse). And  $a\mathcal{A}$  is a two-sided proper ideal (by  $\mathcal{A}$  being commutative).

We now introduce a fact that we will use.

**Definition 1.13**

An ideal  $I$  is maximal if  $I$  is proper in  $R$ , and not contained in any bigger proper ideals.

**Lemma 1.2**

Let  $R$  be a unital ring commutative, every proper ideal is contained in a maximal ideal (by Zorn's lemma).

**Lemma 1.3**

For  $\mathcal{A}$  unital commutative Banach algebra, if  $I$  is a proper ideal, then its closure is a proper ideal.

**Proof** We have seen that  $GL(\mathcal{A})$ , the set of invertible elements, is open. Hence its complement is closed. Any proper ideal does not contain any elements in  $GL(\mathcal{A})$ , hence its closure is closed inside a closed set.

**Remark** Let  $X$  be locally compact, but compact, such as  $\mathbb{R}$ , then we have  $C_c(X) \subset C_\infty(X)$ . Note that  $C_c(X)$  is a proper ideal of  $C_\infty(X)$ , but it's dense in  $C_\infty(X)$ , hence its closure is the entire space, hence no longer proper. This tells us the closure of a proper ideal is not always proper, if  $\mathcal{A}$  is not unital.

**Theorem 1.7**

Every maximal ideal of  $\mathcal{A}$  is closed.

**Proof** The closure of any proper ideal is closed in unital algebras, hence its closure is itself.

First let  $V$  be a normed vector space. Let  $W$  be a closed subspace, and form the quotient space  $V/W$ . There is a natural way to equip  $V/W$  with a norm

$$\|v\| = \inf\{\|v - w\| : w \in W\}$$


i.e. the distance between  $v$  to  $W$ . This is a norm. Further if  $V$  is complete, so is  $V/W$ .

Let  $\mathcal{A}$  be a normed algebra, commutative, unital. Let  $I$  be a closed ideal.

**Proposition 1.29**

For  $a, b \in \mathcal{A}$ , we have

$$\|\dot{a}\dot{b}\| = \|\dot{a}\dot{b}\| \leq \|\dot{a}\|\|\dot{b}\|$$

so that  $\mathcal{A}/I$  is a normed algebra. 

**Proof** Let  $c, d \in I$ , and

$$(a - c)(b - d) = ab - (ad + cb - cd)$$

Note that  $(ad + cb - cd) \in I$ , hence

$$\|\dot{a}\dot{b}\| \leq \|ab - (ad + cb - cd)\| = \|(a - c)(b - d)\| \leq \|(a - c)\|\|(b - d)\|$$

Taking infimum over all  $c, d$ , we get

$$\|(\dot{a}\dot{b})\| \leq \|\dot{a}\|\|\dot{b}\|$$

□

**Proposition 1.30**

If  $\mathcal{A}$  is a Banach algebra and if  $I$  is a closed ideal, then  $\mathcal{A}/I$  is a Banach algebra for the norm  $\|\dot{a}\|$  defined above. 

Let  $\mathcal{A}$  be a unital commutative Banach algebra over  $\mathbb{C}$ , let  $I$  be a maximal ideal of  $\mathcal{A}$ , then  $\mathcal{A}/I$  is a Banach algebra, if  $\mathcal{A}/I$  has a proper ideal, then you can put this ideal back in  $\mathcal{A}$  such that it contains  $I$ . By  $I$  already being ideal, this implies that  $\mathcal{A}/I$  does not contain any proper ideals.

Now coming back. Let nonzero element in  $\mathcal{A}/I$  is invertible. The Gelfand Mazur theorem tells us

$$\mathcal{A}/I \cong \mathbb{C}$$

Moreover,  $\mathcal{A}/I = 1_{\mathcal{A}/I}\mathbb{C}$ . Then the quotient map

$$\mathcal{A} \rightarrow \mathcal{A}/I \cong \mathbb{C}$$

is an element of  $\mathcal{A}$ , i.e. an algebra homomorphism  $\varphi$ , with the property  $\varphi(I) = 0$ .

If  $a\mathcal{A} \subset I$ , then for  $y \in a\mathcal{A}$ , we have

$$\varphi(y) = 0$$

And we are therefore finally done. We thus have

$$\|\hat{a}\| = r(a)$$

□

**Corollary 1.6**

We have

$$\sigma(a) = \text{Range}(\hat{a})$$

## 1.9 Lecture 10

Consider  $C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$ , and  $C_\infty(\mathbb{R})$  is an ideal of  $C_b(\mathbb{R})$ .



**Definition 1.14 (Abstract  $C^*$ -algebra)**

An abstract  $C^*$ -algebra is a Banach algebra with an involution, such that

$$\|a^*a\| = \|a\|^2$$



**Remark** Zorn's lemma states that  $C_\infty(\mathbb{R})$  is contained in a maximal ideal, in a commutative Banach algebras, maximal ideals give rise to bounded multiplicative linear functionals.

**Remark** There is ideals of  $C_b(\mathbb{R})$  that are bigger than  $C_\infty(\mathbb{R})$ .

There exists linear functionals that are 0 on  $C_\infty(\mathbb{R})$ , but nonzero on  $C_b(\mathbb{R})$ , but such functional is not “constructable.”

We also have  $c_0 \subset l^\infty(\mathbb{N})$ , where  $c_0$  are sequences that converge to 0 at infinity. Again, nonzero linear functionals exist on  $l^\infty(\mathbb{N})$ , and is identically zero on  $c_0$ , but such is also not constructable.

**Definition 1.15**

$\prod_{j=1}^\infty \mathbb{Z}_5$  = the sequences of elements of  $\mathbb{Z}_5$ .  $\bigoplus \mathbb{Z}_5$  all sequences that are 0 except for finitely many number of entries.



Note that  $\bigoplus \mathbb{Z}_5$  is an ideal of  $\prod_{j=1}^\infty \mathbb{Z}_5$ .

**Proposition 1.31**

$l^\infty(\mathbb{N})$  is not separable,  $\prod_{j=1}^\infty \mathbb{Z}_5$  is not separable, nor is it finitely generated.

**Proposition 1.32**

If  $\mathcal{A}$  is a unital commutative Banach algebra over  $\mathbb{C}$ , which is separable, and if  $I$  is a closed ideal, then one can construct a maximal ideal containing  $I$  by countable *what*



**Proof** Let  $\{a_n\}$  be a countable subset of  $\mathcal{A}$ , whose linear span is dense.

**Lemma 1.4**

Note that  $\mathcal{A}/I$  contains noninvertible elements if and only if  $I$  is not maximal.



If  $I$  is not maximal, then you can find the first  $a_n$  such that  $a_n \in \mathbb{C}1_{\mathcal{A}}$  such that  $a_n \notin \mathcal{A}/I$ , then

$$\overline{a_n \mathcal{A}/I} \text{ is a proper ideal of } \mathcal{A}$$

hence it generates a proper ideal in  $\mathcal{A}$ , we denote it as  $I_1$ . If  $I_1$  is not maximal, then repeat the process. By  $\{a_n\}$  being countable, and that they are dense, we have a countable addition, which gives a maximal ideal containing  $I$  by countable inclusions.

□

**Remark**  $C_\infty(\mathbb{R})$  is separable, and  $C_b(\mathbb{R})$  is not separable.

Let's look at  $L^\infty([0, 1], m)$ , where  $m$  denotes the Lebesgue measure. This is a  $C^*$ -algebra, commutative, unital.

Could you exhibit any linear functionals on  $L^\infty$

Note that  $L^2([0, 1], m)$ , a Hilbert space, as an algebra on  $L^2$ ,  $L^\infty$  is closed for the strong operator topology on  $\mathcal{B}(\mathcal{H})$  by the seminorm,

$$T \in \mathcal{B}(\mathcal{H}), T \rightarrow \|t\xi\|, \text{ for } \xi \in \mathcal{H}$$

For example, figure 1. We say that  $L^\infty([0, 1])$  is a von Neumann algebra. Every commutative von Neumann algebra looks like some  $L^\infty([0, 1], \mu)$ . But note that noncommutative ones are quite interesting.

For a commutative Banach algebra over  $\mathbb{C}$ , the Gelfand transform

$$a \mapsto \hat{a}$$

We have

$$\|\hat{a}\|_\infty = r(a)$$

**Proposition 1.33 (Gelfand isometric condition)**

If  $\|a^2\| = \|a\|^2$ , then the Gelfand transform is isometric. Thus we have

$$\|\widehat{a}\|_\infty = \|a\|$$

**Definition 1.16 (Involution  $*$ )**

For an involution on an algebra over  $\mathbb{C}$ , is a map  $*$  from  $\mathcal{A} \rightarrow \mathcal{A}$ , with the properties

1.  $(a^*)^* = a$
2.  $(a + b)^* = a^* + b^*$
3.  $(\alpha a)^* = \overline{\alpha} a^*$  for  $\alpha \in \mathbb{C}$ .
4.  $(ab)^* = b^* a^*$

**Definition 1.17 (Banach  $*$  algebra)**

If  $\mathcal{A}$  has a norm, then we say it is a  $*$  normed if

$$\|a^*\| = \|a\|$$

If  $\mathcal{A}$  is complete, then it is called a Banach  $*$  algebra.

Let  $G$  be a discrete group, let  $\mathcal{H}$  be a Hilbert space, then  $\text{Aut}(\mathcal{H}) = U(\mathcal{H})$  is the group of unitary operators on  $\mathcal{H}$  to itself. By a unitary representation of  $G$  on  $\mathcal{H}$ , we mean

**Definition 1.18 (Unitary representation)**

A unitary representation of  $G$  on  $\mathcal{H}$  is a homomorphism  $\pi : G \rightarrow U(\mathcal{H})$ .

We note that  $C_c(G) \subset l^1(G)$ , and

$$\pi_f = \sum_{x \in G} f(x) \pi_x, \pi_f \pi_g = \pi_{f * g}$$

Now we ask what is  $(\pi_f)^*$ ?

$$(\pi_f)^* = \sum \overline{f(x)} \pi_x^* = \sum_{x \in G} \overline{f(x)} \pi_{x^{-1}} = \sum \overline{f(x^{-1})} \pi_x$$

So we get

$$(f^*)(x) = \overline{f(x^{-1})}$$

This defines an involution on  $l^1(G)$ . And it is easy to check that

$$\|f^*\|_1 = \|f\|_1$$

**Remark** The same process would not work for semigroups without the presence of  $x^{-1}$  necessarily.

We would like to think of this involution as some sort of complex conjugation.

**Definition 1.19**

Let  $\mathcal{A}$  be a Banach  $*$ -algebra is symmetric if whenever  $a \in \mathcal{A}$ , and  $a^* = a$ , then

$$\sigma(a) \subset \mathbb{R}$$

If one looks at  $l^1$  over noncommutative groups, some are symmetric, some are not.

**Example 1.7** Let  $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ , with  $\|\cdot\|_\infty$ . We define an involution:

$$(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$$

This is a well-defined involution. However, this is not symmetric under this involution.

**Proposition 1.34**

If  $G$  is commutative, then  $l^1(G)$  is symmetric.

$$\widehat{\mathcal{A}} = \{ \text{set of homomorphisms } G \rightarrow \mathbb{T} \}$$

This is symmetric.

**Proposition 1.35**

Let  $\mathcal{A}$  be an abstract, unital  $C^*$ -algebra, then  $\mathcal{A}$  is symmetric.

This is quite strong! (Every  $C^*$ -algebra is symmetric).

## Lecture 11

Let  $\mathcal{A}$  be a  $*$ -Banach algebra.

**Definition 1.20 (symmetric  $*$ -algebra)**

$\mathcal{A}$  is symmetric if for any  $a \in \mathcal{A}$ , we have

$$a^* = a$$

we have  $\sigma(a) \in \mathbb{R}$ .

Note that a  $C^*$  algebra is necessarily a  $*$ -algebra. Hence we have

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|$$

And we apply it to  $a^*$ , hence we get

$$\|a^*\| \leq \|a\|$$

Hence the involution property is satisfied.

**Proposition 1.36 ( $C^*$ -algebras are symmetric)**

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, i.e. we have

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

Then  $\mathcal{A}$  is symmetric, i.e.  $a^* = a$ .

In the Gelfand Noimark paper (1943),

**Proof** [Arens Truck, 1946] Given  $a \in \mathcal{A}$  with  $a^* = a$ , for any  $t \in \mathbb{R}$ , let  $b = a + it$ , we look at  $b^*b = (a - it)(a + it) = a^2 + t^2$ . So we have

$$\|b^*b\| \leq \|a^2\| + t^2$$

Now let  $\lambda \in \sigma(a)$ , with  $\lambda = r + is$ , we would like to show  $s = 0$ . Note we have  $\lambda + it \in \sigma(b)$ , this gives

$$\lambda + it = r + i(s + t) \in \sigma(b)$$

Then we have

$$|r + i(s + t)| \leq \|b\|$$

Hence

$$r^2 + (s + t)^2 |r + i(s + t)|^2 \leq \|b^2\| = \|b\|^2 = \|b^*b\| \leq \|a^2\| + t^2$$

We thus have

$$r^2 + s^2 + 2st \leq \|a^2\|, \text{ for all } t$$

This gives  $s = 0$ .

□

Let's step back. Let  $\mathcal{A}$  be a commutative symmetric Banach  $*$ -algebra. Then if  $a \in \mathcal{A}$ , and if  $a^* = a$ , so  $\sigma(a) \subset \mathbb{R}$ . But  $\sigma(a) = \text{Range}(\widehat{a})$ , so  $\mathcal{A}$  is an  $\mathbb{R}$ -valued function on  $\mathcal{A}$ .

For any  $a \in \mathcal{A}$ , we have

$$a = \frac{a + a^*}{2} + i \frac{a - a^*}{2i} = a_r + ia_i$$

then  $\widehat{a} = \widehat{a}_r + i\widehat{a}_i$ , note  $a^* = a_r - ia_i$ , then we have

$$\widehat{a^*} = \widehat{a}_r - i\widehat{a}_i = \overline{\widehat{a}}$$

Thus, we have

$$\widehat{a^*} = \overline{\widehat{a}}$$

So  $a \mapsto \widehat{a}$  is a  $*$ -algebra homomorphism of  $\mathcal{A}$  into  $C(\widehat{\mathcal{A}})$ .

**Definition 1.21 (separation of points by functions)**

A collection of functions  $\{f\}_j$  defined on  $X$  is said to separate points if for all  $x, y \in X$ , such that  $x \neq y$ , there exists  $f$  such that we have

$$f(x) \neq f(y)$$

**Proposition 1.37**

For any unital commutative Banach algebra  $\mathcal{A}$ , then Gelfand transform  $a \mapsto \widehat{a}$ , separates the points of  $\widehat{\mathcal{A}}$ .

**Proof** We prove the contrapositive, if we assume for all  $\widehat{a}$ , we have  $\widehat{a}(\varphi) = \widehat{a}(\psi)$ , then we would like to show  $\varphi = \psi$ . If  $\varphi, \psi \in \widehat{\mathcal{A}}$ , and  $\widehat{a}(\varphi) = \widehat{a}(\psi)$ , then

$$\varphi(a) = \psi(a), \text{ for all } a$$

Hence  $\varphi = \psi$ . □

**Proposition 1.38**

If  $\mathcal{A}$  is a unital symmetric Banach  $*$ -algebra, then the image of  $\Gamma$  is dense in  $C(\widehat{\mathcal{A}})$ . ♥

**Proof** [Key ingredient: Stone-Weierstrass]  $\{\Gamma(\varphi) : \varphi \in \widehat{\mathcal{A}}\}$  is a unital subalgebra of  $C(\widehat{\mathcal{A}})$  that separates the points of  $\mathcal{A}$ , and is closed under taking complex conjugates, so Stone-Weierstrass theorem applies (a compact space and a unital subalgebra of continuous functions that separates the points of the space, and closed under complex conjugation, then this algebra is dense for the  $\|\cdot\|_\infty$  norm). □

**Theorem 1.8 (Little Gelfand-Naimark theorem)**

Let  $\mathcal{A}$  be a unital commutative  $C^*$ -algebra (abstract, which doesn't have to include hilbert space), then the Gelfand transform

$$\widehat{a}(\varphi) = \varphi(a)$$

is an isometric  $*$ -isomorphism of  $\mathcal{A}$  into  $C(\widehat{\mathcal{A}})$ , i.e.  $\|\widehat{a}\| = \|a\|$ . ♥

**Proof** Since  $\mathcal{A}$  is symmetric, the range of the Gelfand transform is dense. We also saw that  $\|a^{2^n}\| = \|a\|^{2^n}$ , so the spectral radius of  $a$ ,  $r(a) = \|a\| = \|\Gamma(a)\|$ .

Therefore  $a \mapsto \Gamma(a)$  is isometric, and the range of  $\Gamma(a)$  is norm-closed. □

**Remark** In a commutative Banach algebra, we have  $r(a) = \|\widehat{a}\|_\infty$ , and have  $r(ab) \leq r(a)r(b)$ , hence  $\|a^*a\| = \|a\|^2$ ,  $r(a^*a) \leq r(a)^2$ , we have

$$\|a\|^2 \leq r(a)^2 \leq \|a\|^2$$

For  $T \in \mathcal{B}(\mathcal{H})$ , and  $T = T^*$ , then

$$C^*(T, I) = \overline{\text{span}}\{T^n : n \in \mathbb{Z}\} \cong C(\sigma(T))$$

**Proposition 1.39**

Let  $G$  be a commutative group, then  $l^1(G)$  with its  $*$  is symmetric.



**Proof** Let  $\mathcal{A} = l^1(G)$ , then  $\widehat{\mathcal{A}}$  is isomorphic to the homomorphisms of  $G$  into  $\mathbb{T}$ .

If  $\varphi \in \widehat{\mathcal{A}}$ , then

$$\varphi(f) := \sum_{x \in G} f(x)\varphi(x), f \in l^1(G)$$

Then

$$\varphi(f^*) = \sum f^*(x)\varphi(x) = \sum \overline{f(x^{-1})}\varphi(x) = \sum \overline{f(x)}\varphi(x^{-1}) = \sum \overline{f(x)\varphi(x)} = \overline{\varphi(f)}$$

Note how we might have homomorphisms not mapping into  $\mathbb{T}$  if  $G$  is not commutative.

**Proposition 1.40**

For  $G$  commutative, the range of the Gelfand transform  $\Gamma$ , which is the Fourier transform (for  $f \in l^1(\mathbb{Z})$ , we have  $\widehat{f}(e^{i\theta}) = \sum f(n)e^{i\theta n}$ , note  $\widehat{\mathbb{Z}} = \mathbb{T}$ ) in this setting, this is dense in  $C(\widehat{G})$ . However, this is not isometric, and the range is not norm-closed unless  $G$  is finite.



## 1.10 Lecture 12

Recall last time, we proved the Little-Gelfand-Naimark theorem.

**Theorem 1.9**

Let  $\mathcal{A}$  be a unital commutative  $C^*$ -algebra then we have

$$\mathcal{A} \cong C(\widehat{\mathcal{A}})$$

And  $\widehat{\mathcal{A}}$  is compact.

**Definition 1.22**

Let  $\mathcal{C}$  be a category, we think of objects as categories  $X, Y$ , with morphisms in between  $X, Y$ , the abstract dual of  $\mathcal{C}$ , which is another category, has the same objects, but you reverse all the morphisms (arrows).



Next we introduce the important summary of the things we've been doing.

**Theorem 1.10**

The category of unital commutative  $C^*$ -algebras, with unital  $*$ -homomorphisms is a concrete realization of dual of the category of compact Hausdorff spaces.

$$X \rightarrow Y$$

$$C(X) \leftarrow C(Y)$$



$X$  is **normal**? if  $C(X)$  contains no proper projects, i.e. elements like  $P$  such that

$$P^2 = P = P^*$$

We now look at the following.

**Proposition 1.41**

Let  $\mathcal{A}$  be a unital Banach algebra, and let  $a_0 \in \mathcal{A}$ , suppose  $\mathcal{A}$  is generated by  $a_0$ , i.e. the norm closure of all the polynomials in  $a_0$ , with the identity  $1_{\mathcal{A}}$ . Then the  $\widehat{\mathcal{A}}$  is homeomorphic to  $\sigma(a_0)$ , via  $\varphi \in \widehat{\mathcal{A}}$ ,

$$\varphi \mapsto \varphi(a_0) \in \sigma(a_0)$$



**Proof**  $\varphi$  is entirely determined by  $\varphi(a_0)$ , hence the map above is one-to-one. note that for every element in  $\lambda \in \sigma(a_0)$ , there exist  $\varphi$  such that  $\varphi(a_0) = \lambda$ . Hence the map is also surjective.

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $a \in \mathcal{A}$  and  $a^* = a$ , then  $C^*(a, 1_{\mathcal{A}})$  is commutative, and unital, and generated by  $a$  so that

$$\widehat{B} = \sigma_B(a)$$

such that  $\mathcal{B} = C(\widehat{B}) = C(\sigma(a))$ , hence we have

$$C^*(a, 1) \cong C(\sigma(a)), a \mapsto \widehat{a}$$

The continuous functional calculus (dealt with operators on a Hilbert space). Still we assume  $a^* = a$ .

Given  $f \in C(\sigma(a))$ , then there exists a  $b \in C^*(a, 1)$ , such that


$$\widehat{b} = f$$

we denote  $b$  as  $f(a)$ , then


$$f \mapsto f(a)$$

is a  $*$ -homomorphism of  $C(\sigma(a))$  onto  $C^*(a, 1)$ , and so into  $\mathcal{A}$ .

### Corollary 1.7

If  $T \in \mathcal{B}(\mathcal{H})$ , and if  $T^* = T$ , then for any function  $f \in C(\sigma(T))$ , we can form  $f(T)$ . This is part of the spectral theorem. 

### Definition 1.23

In any unital  $C^*$ -algebra, and  $a \in \mathcal{A}$ , we say that  $a$  is normal if  $a^* \in C^*(a, 1)$ , and  $a^*$  and  $a$  commute. 

We introduce the spectral permanents)

### Theorem 1.11

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and let  $\mathcal{B}$  be a unital  $C^*$ -subalgebra of  $\mathcal{A}$ . Then for any  $b \in \mathcal{B}$ , we have

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

Note that we are not requiring commutativity for  $\mathcal{A}$  or  $\mathcal{B}$ . 

**Proof** Note that we already have  $\sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b)$  (an element not invertible in  $\mathcal{B}$  may be invertible in  $\mathcal{A}$ ). Then to show the other inclusion. We will first assume that  $b^* = b$ . If  $b$  has an inverse in  $\mathcal{A}$ , then it has a inverse in  $\mathcal{B}$ . So suppose  $c$  is an inverse in  $\mathcal{A}$  for  $b$ , then

$$cb = 1_{\mathcal{A}} = bc$$

Now it suffices to consider  $(b - \lambda 1)^{-1}$ . Assume  $b$  is not invertible in  $\mathcal{B}$ . Consider the  $C^*$  algebra generated by  $b, 1$ , then  $C^*(b, 1) := \mathcal{B}_1 \subset \mathcal{B}$ . If  $b$  is not invertible in  $\mathcal{B}$ , then  $b$  is not invertible in  $\mathcal{B}_1$ , hence  $\widehat{b}$  has no inverse in  $C(\sigma_{\mathcal{B}}(\mathcal{H}))$ : which has a criterion that it takes nonzero elements to 0.

Thus  $\widehat{b}$  takes value 0 at some  $\lambda \in \sigma_{\mathcal{B}}(b)$ , and note that  $\widehat{b}$  is continuous, there is an open neighborhood of  $\lambda_0$  such that

$$|\widehat{b}(\lambda)| \leq \frac{1}{2\|c\|}$$

So by Urysohn's lemma, there is a  $g \in C(\sigma(b))$  such that  $\text{supp}(g) \subset O$ , and  $g = 0$  outside of  $O$ , with  $\|g\|_{\infty} = 1$ .

Then for  $d := g(b)$ , so for the Gelfand transform,

$$\widehat{g(b)} = g$$

we have  $\|\widehat{bg}\|_{\infty} \leq \frac{1}{2\|c\|}$ , then

$$\|db\| \leq \frac{1}{\|c\|}$$

then

$$1 = \|g\|_{\infty} = \|d\| = \|(cb)d\| \leq \|c\| \|bd\| \leq \|c\| \frac{1}{2\|c\|} = \frac{1}{2}$$




Hence we've reached a contradiction. Hence the self-adjoint case is done.  $\square$

## 1.11 Lecture 13

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra, and  $\mathcal{B}$  be a unital  $C^*$ -subalgebra (implying the same unit), and in previous lecture, we saw if  $b \in \mathcal{B}$ , and  $b^* = b$ , and if  $b$  has an inverse in  $\mathcal{A}$ , then it has an inverse in  $\mathcal{B}$ . This implies that

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

### Proposition 1.42

Let  $\mathcal{A}$  be a  $C^*$ -algebra, and  $\mathcal{B}$  subalgebra, if  $b$  is invertible in  $\mathcal{A}$ , then  $b$  is invertible in  $\mathcal{B}$ . 

**Proof** We already know that this holds for  $b^* = b$ . For general  $b \in \mathcal{B}$ , (we no longer require  $b^* = b$ ), then if  $b$  has an inverse in  $\mathcal{A}$ , then so does  $b^*$ , then  $b^*b$  has an inverse in  $\mathcal{A}$ , hence  $b^*b$  has an inverse in  $\mathcal{B}$ . Hence  $b$  has a left inverse  $a(b^*b) = (b^*b)a = 1$ , and  $bb^*$  is also invertible in  $\mathcal{A}$ , hence invertible in  $\mathcal{B}$ , hence  $b$  has a right inverse.  $\square$

if we look at the shift operators on  $l^2(\mathbb{N})$ ,

$$Se_n = e_{n+1} \tag{1.3}$$

Then the adjoint of this would be

$$S^*(e_n) = \begin{cases} e_{n-1}, n \geq 2 \\ 0, n = 1 \end{cases}$$


Then we have

$$S^*S = I_{\mathcal{H}}, SS^* = I - P_{e_1}$$

where  $P_{e_1}$  is the projection onto  $e_1$ .

Let  $\mathcal{A} = l^1(\mathbb{N}_{\geq 0})$ , and let  $\mathcal{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ .

### Theorem 1.12

If  $f \in l^1(\mathbb{N})$ , and if  $\hat{f} \in C(D)$ , then nowhere takes value 0, then  $\hat{f}$  is invertible as a function in  $C(D)$ , and  $\hat{f}$  has absolutely convergence power series, then  $\frac{1}{\hat{f}}$  also has an absolutely convergence power series. 



**Note** If we take  $\mathcal{A} = l^1(\mathbb{Z})$ , so  $\widehat{\mathcal{A}} = \mathbb{T}$ , and if  $\hat{f}$ , which is the Fourier series, nowhere takes value 0 on  $\mathbb{T}$ , then the function  $\frac{1}{\hat{f}}$  has absolutely convergent Fourier series. (This is much harder to prove).

Let  $G$  be a group and let  $(\mathcal{H}, U)$  be a unitary representation of  $G$  on  $\mathcal{H}$ , and let  $U : G \rightarrow U(\mathcal{H})$ , and  $l^1(G)$  with convolution, for  $f \in l^1(G)$ , and for

$$U_f = \sum f(x)U_x \in \mathcal{B}(\mathcal{H})$$

with  $\|U_f\| \leq \|f\|_{L^1}$ . We then define

$$f^*(x) := \overline{f(x^{-1})}, U_f^* = (U^*)_f$$

We take  $G = SL(n, \mathbb{Z})$ , the  $n \times n$  matrices with  $\det(T) = 1, T \in SL(n, \mathbb{Z})$ .

**Problem 1.1** What are the unitary representations of  $G$  on  $l^1(G)$

$G$  acts on  $G$  by left translations  $\alpha$  (actions), and acts on  $G/H$  (sets of cosets) for  $H$  any subgroups. Note that  $G/H$  are often called homogenous spaces. If  $G$  acts on a set  $M$ , consider  $l^p(M)$ , then  $G$  acts on isometries on  $l^p(M)$ , by

$$(\alpha)x\xi(y) = \xi(\alpha_x^{-1}(y))$$

$$X \rightarrow Y, C(X) \leftarrow C(Y)$$

In particular, this action on  $l^2(M)$  is unitary.

**Definition 1.24 (Left regular representation)**

The representation  $U$  of  $G$  on  $l^1(G)$  is called the left regular representation of  $G$  if we define, for  $x, y \in G$ ,  $\xi \in l^2(G)$ , we have

$$(U_x \xi)(y) = \xi(x^{-1}y)$$

We now, naturally define the integrated form of the representation of  $G$  on  $l^1(G)$ :

**Definition 1.25 (integrated form of a representation)**

We define  $U_f \in \text{End}(l^1(G))$ , we define  $U_f$  naturally as follows: let  $\xi \in l^1(G)$ , and

$$U_f(\xi) = \sum_{g \in G} f(g) U_g(\xi)$$

i.e.  $U_f = \sum_{x \in G} f(x) U_x$ , where  $U_x \in \text{End}(V)$  by

$$x \xrightarrow{U} U_x$$

where  $U$  is the map  $: G \rightarrow \text{End}(l^1(G))$ .

**Definition 1.26 (Reduced  $C^*$ -algebra)**

The operator norm closure of  $\{U_f : f \in l^1(G)\} \in \mathcal{B}(l^1(G))$  is called the reduced  $C^*$ -algebra of  $G$ , denoted as  $C_r^*(G)$ .



**Note** Again, the defining property of a  $C^*$ -algebra is  $\|a^*a\| = \|a\|^2$

In 1975, we see that  $C_r^*(F_2)$  is simple, and has no proper ideals, note  $F_2$  can be thought of the group generated by  $a, b, a^{-1}, b^{-1}$  with unit. Note that the trivial representation is not continuous for  $\|\cdot\|_{C_r^*}$ .

**Definition 1.27 (amenable groups)**

$G$  is amenable if the integrated form of trivial representation is continuous for  $\|\cdot\|_{C_r^*}$ .

**Remark** This implies that the integrated form of all unitary representations of  $G$  are continuous for  $\|\cdot\|_{C_r^*(G)}$ . There are many equivalent properties of amenable using the geometric properties of  $G$ .



**Note** All commutative groups are amenable.

**Definition 1.28 (Non-degenerate representation)**

Let  $\pi$  be a representation of  $\mathcal{A}$  on  $V$ ,  $\pi : \mathcal{A} \rightarrow V$ , then  $\pi$  is non-degenerate if

the linear span of  $\{\pi(a)v : a \in \mathcal{A}, v \in V\}$  is dense in  $V$

**Definition 1.29 (faithful representations)**

(In short, it's a representation that is injective.) The left representation is faithful if whenever  $U_f = 0$ , then we have

$$f = 0, f \in l^1(G)$$

**Proof**  $l^1(G)$ , with convolution, with identity  $\delta_e$ . note that  $\delta_e \in l^2(G)$ . If we consider  $\delta_e \in l^2(G)$ , and we look at

$$U_f \delta_e = f \in l^2(G)$$

Because we have the embedding  $l^1 \subset l^2$ .

Now assume that  $G$  is commutative, then we consider  $l^1(G)$  acts on  $l^2(G)$ , and

$$C_r^*(G) = C^*(G) = C(\widehat{G})$$

note that we still have  $\|f\|_{C_r^*(G)} \leq \|f\|_{L^1}$ .

## 1.12 Lecture 13

Let  $G$  be commutative, and  $(l^1(G), U)$  be the left regular representation. We have the interal form  $U_f, hf \in l^1(G)$ , and

$$U : l^1(G) \rightarrow \mathcal{B}(l^1(G))$$

and we let

$$C_r^*(G) = \{U_f : f \in l^1(G)\}^-$$

where we take the closure with respect to the operator norm.

For  $G$  commutative, we have  $C_r^*(G)$  is a commutative  $C^*$ -algebra hence

$$\mathcal{A} := C_r^*(G) \cong C(\widehat{\mathcal{A}})$$

We've shown last time, that  $U$  is injective from  $l^1(G)$  to  $C_r^*(G)$ . Each  $\varphi \in \widehat{\mathcal{A}}$  a multiplicative linear functional, we have

$$f \mapsto \varphi(\pi_f) \in (l^1(G))^\wedge$$

For any  $f \in l^1(G)$ , and  $f \neq 0$ , we have  $U_f \neq 0$ , and the map is injective, hence  $\pi_f \neq 0$ , we have, there exists  $\varphi$  such that  $\varphi(\pi_f) \neq 0$ .

### Corollary 1.8 (largeness of the dual group)

$\widehat{G}$  is big enough given that  $f, g \in l^1(G)$ , if  $f \neq g$ , then there exists  $\varphi \in \widehat{G}$  such that

$$\widehat{f}(\varphi) \neq \widehat{g}(\varphi)$$

In other words  $\varphi(f) \neq \varphi(g)$ .



We now state the big Gelfand-Naimark theorem.

### Theorem 1.13 (Big Gelfand-Naimark)

Let  $\mathcal{A}$  be an abstract  $C^*$ -algebra, e.g. a Banach  $*$ -algebra such that

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

Then there exists a  $*$ -representation  $\pi$  of  $\mathcal{A}$  on a Hilbert space  $\mathcal{H}$  which is isometric, i.e.

$$\|\pi(a)\| = \|a\|$$

And

$$\mathcal{A} \cong \{\pi(a) : a \in \mathcal{A}\}$$



**Note** In Little Gelfand-Naimark,  $C(\widehat{\mathcal{A}})$  is explicitly determined, but for big, it is not determined.

For  $X$  a compact space,  $C(X)$  is a  $C^*$ -algebra.

Consider  $X_d$ , which is discrete, and take  $l^1(X_d)$  with the counting measure, is called the atomic representation.

Take any Borel measure  $\mu$  on  $X$ , we have

$$L^2(X, \mu)$$

If  $\mu$  has full support, then map of  $C(X)$  is isometric.

**Now to introduce positive linear functionals on  $\mathcal{A}$ .** Let  $\mathcal{A}$  be a  $*$ -algebra, and  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$  a  $*$ -representation.

Let's take any  $\xi \in \mathcal{H}$ , and  $\xi \neq 0$ , define

$$\varphi_\xi(\mathcal{A}) = \langle \pi(a)\xi, \xi \rangle$$

then we look at

$$\varphi(a^*a) = \langle \pi(a^*a)\xi, \xi \rangle = \langle \pi(a)\xi, \pi(a)\xi \rangle \geq 0$$



**Note** For  $f \in C(X)$ , we always have  $f^*f \geq 0$ .

**Definition 1.30 (Positive linear functionals)**

If  $\varphi$  is a linear functional on  $C(X)$ , such that for all  $f$ ,

$$\varphi(f^*f) \geq 0$$



**Remark** They are always continuous. And they give rise to a measure  $\mu_\varphi$  on  $X$ .

You think of  $a^*a$  as being “positive.”

**Proposition 1.43**

If  $\mathcal{A}$  is a  $C^*$ -algebra, if  $a, b \in \mathcal{A}$ , there exists  $c$  such that

$$a^*a + b^*b = c^*c$$

**Definition 1.31 (positive linear functionals)**

If  $\mathcal{A}$  is a  $*$ -algebra, and if  $\varphi$  is a linear functional on  $\mathcal{A}$ , if for all  $a \in \mathcal{A}$ , we have

$$\varphi(a^*a) \geq 0$$

then we call  $\varphi$  **positive**.

**Proposition 1.44**

If  $\varphi, \psi$  are positive, then  $r\varphi + s\psi$ , with  $r, s \in \mathbb{R}^+$ , then the positive linear functionals form a **cone**.



Let  $\mathcal{A}$  be a  $*$ -algebra, and let  $\varphi$  be a positive linear functional on  $\mathcal{A}$ , then define a pre-inner product on  $\mathcal{A}$  by

$$\langle a, b \rangle_\varphi = \varphi(b^*a)$$

Then we have

$$\langle a, a \rangle_\varphi = \varphi(a^*a) \geq 0$$

**Proposition 1.45**

For  $\langle a, b \rangle_\varphi = \varphi(b^*a)$  defines a pre-inner product on  $\mathcal{A}$ . (A pre-inner product does not require  $\langle \xi, \xi \rangle = 0$  implies  $\xi = 0$ .)



**Proof** We have  $\overline{\langle a, b \rangle_\varphi} = \langle b, a \rangle_\varphi$ .

**1.12.1 GNS construction**

Now for a positive linear functional, we attempt to associate a cyclic representation with it: we will define a pair  $(\pi, \mathcal{H})$  such that it satisfies:

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle$$

Let  $\eta_\varphi = \{a \in \mathcal{A} : \langle a, a \rangle_\varphi = 0\}$ , then if  $b \in \mathcal{A}$ ,  $a \in \eta_\varphi$ , then we have

$$|\langle ba, ba \rangle| = |\langle b^*ba, a \rangle| \leq \langle a, a \rangle^{1/2} = 0$$

So we get that  $\eta_\varphi$  is an ideal of  $\mathcal{A}$ , so form  $\mathcal{A}/\eta_\varphi$ , then

$$\langle, \rangle_\varphi$$

drops to an inner product on  $\mathcal{A}/\eta_\varphi$ , denote its complement by  $L^2(a, \varphi)$ , if we are given  $c \in \mathcal{A}$ , let  $\pi$  be the left regular representation of  $\mathcal{A}$  on  $\mathcal{A}$  via

$$\pi_c a = ca$$

then we have

$$\langle \pi_c a, b \rangle_\varphi = \langle ca, b \rangle_\varphi = \varphi(b^*ca) = \varphi((c^*b)^*a) = \langle a, c^*b \rangle_\varphi = \langle a, \pi(c^*)b \rangle_\varphi$$

Then the left regular representation “is” a  $*$ -representation on  $\mathcal{A}$ , this drops to a  $*$ -representation on  $\mathcal{A}/\eta_\varphi$ .

Next time: we show  $\pi_c$  is not continuous, and we will use polynomials. Now if you assume  $\mathcal{A}$  is Banach  $*$ -algebra, then this  $\pi_c$  are always bounded. **GNS representation**, where  $S$  is Siegel.

## 1.13 Lecture 15

Recall the GNS construction. Given a  $*$ -normed unital algebra  $\mathcal{A}$ , and a positive linear functional  $\mu$  on  $\mathcal{A}$ , we mean

$$\mu(a^*a) \geq 0, \forall a \in \mathcal{A}$$

Now we define a pre-inner product on  $\mathcal{A}$  by

$$\langle a, b \rangle_\mu = \mu(b^*a)$$

And one could check this is indeed a pre-inner product.

1.  $\overline{\langle a, b \rangle_\mu} = \langle b, a \rangle_\mu$
2.  $\langle a + b, a + b \rangle_\mu \geq 0$

If we let  $\eta_\mu = \{a \in \mathcal{A} : \langle a, a \rangle = 0\}$ , then by Cauchy-Schwarz inequality, we would show that  $\eta_\mu$  is a left ideal. Form  $\mathcal{A}/\eta_\mu$ , then  $\langle, \rangle$  becomes an inner product on  $\mathcal{A}/\eta_\mu$ .

One could complete  $\mathcal{A}/\eta_\mu$  to get a Hilbert space  $L^2(\mathcal{A}, \mu)$ . Then we have

$$\langle f, g \rangle = \int f \bar{g} d\mu$$

### Definition 1.32 (left regular representation)

For  $c \in \mathcal{A}$ , define the left regular representation  $\pi$  on  $\mathcal{A}$  and on  $\mathcal{A}/\eta_\mu$  via

$$\pi_c a = ca, a \in \mathcal{A}$$

Then we have

$$\langle \pi_c a, b \rangle_\mu = \langle ca, b \rangle_\mu = \langle a, \pi_c^* b \rangle$$

Hence  $\pi$  is a  $*$ -representation.

Now let  $P$  be the algebra of polynomials with complex coefficients, viewed as functions on  $\mathbb{R}$ . Define  $\mu$  on  $P$  as follow:

$$\mu(p) = \int_{-\infty}^{\infty} p(t) e^{-t^2} dt$$

Then our  $\mu$  is a positive linear functional. Then we identify this as  $L^2(\mathbb{R}, e^{-t^2} dt)$ .

### Proposition 1.46

Let  $\mathcal{A}, \mu$ , assume  $\mathcal{A}$  is complete, let  $1$  denote the unit of the algebra. And  $\mu$  is continuous with  $\|\mu\| = \mu(1)$ .

**Remark** By completeness, you immediately get continuity in some cases.

**Proof** Now we introduce the main lemma.

unfinished

## 1.14 Lecture 16

We now discuss classical physics and quantum physics.

All possible “states” for the system. Then you have observables.  $\mathbb{R}$ -valued for the phase space  $P$ .

If the configuration space is  $\mathbb{R}$ , the velocities are given as  $\mathbb{R}$  numbers, then the phase space is the cartesian product  $\mathbb{R} \times \mathbb{R}$ , and the Poisson bracket is given by

$$\{f, g\} = \left( \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} \right) (p)$$

In quantum physics, there is no configuration space that we work with. The phase space becomes a  $C^*$ -algebra, usually non-commutative. And the observables are the self-adjoint elements of  $\mathcal{A}$ , where  $\mathcal{A} = \mathcal{B}(\mathcal{H})$ , and such choice of Hilbert space is not unique. Possible values for an observable  $a = a^*$  form  $\sigma(a)$ .

When you observe a system in a certain state, and you measure an observable  $a$  several times, you can get different values (all in  $\sigma(a)$ ).

For the system in a given state, there is a probability measure on  $\sigma(a)$  that gives you the probability of getting a particular value of  $a$ . As the system evolves, this probably changes.

The **state** (this is why we call them states, i.e. the set of linear transformations that have norm 1) space is the set of all positive linear transformations on  $\mathcal{A}$  of norm 1,  $\mu$ , then for any observable  $a$ , form

$$C^*(a, 1) \cong C(\sigma(a))$$

Then the restriction of  $\mu$  to  $C^*(a, 1)$  give you a probability measure on  $\sigma(a)$ .

Experiments show that for certain pairs of observables  $a, b$ , it is impossible to measure both simultaneously at high accuracy.

The way this is modeled is that it happens exactly if  $ab \neq ba$ , i.e. non-commutativity makes us impossible to measure them simultaneously.

### Definition 1.33 (Poisson bracket)

For  $a, b$  observables such that they are noncommutative, we define the Poisson bracket as follows:

$$i[a, b] = ab - ba$$

For the evolution of a quantum system given by  $\mathbb{R} \mapsto \text{Aut}(\mathcal{A})$ , and  $t \mapsto \alpha_t$ , and states evolve by  $\mu \mapsto \mu \circ \alpha_t$ .

If we let this Planck's constant  $\hbar$  go to 0, then we have

$$\hbar[a, b] \rightarrow \text{Poisson bracket}$$

i.e. getting back from quantum back to the classical system.

Let  $\mathcal{A}$  be a  $*$ -normed unital algebra, and  $\mu$  is a continuous positive linear functional, then  $L^2(\mathcal{A}, \mu)$  and  $\pi^\mu : \mathcal{A} \rightarrow \mathcal{B}(L^2(\mathcal{A}, \mu))$ .

Let  $\xi_\mu = 1_{\mathcal{A}}$ , viewed as an element of  $L^2(\mathcal{A}, \mu)$ , and

$$\{\pi_\mu(a)\xi_\mu : a \in \mathcal{A}\} = \mathcal{A}/\pi_\mu$$

viewed in  $L^2(\mathcal{A}, \mu)$  is dense in  $L^2(\mathcal{A}, \mu)$ .

### Definition 1.34

If we have  $\mathcal{A}$  some representation  $(\mathcal{H}, \pi)$ , and a vector  $\xi \in \mathcal{H}$  is said to be a cyclic vector of  $\{\pi_\mu \xi : a \in \mathcal{A}\}$  is dense in  $\mathcal{H}$ .

## 1.15 Lecture 17

If you look at  $\mathcal{A}$  a  $*$ -normed unital algebra, and  $\mu$  a continuous linear functional, and the GHS representation:  $(\mathcal{H}_\mu, \pi_\mu, \xi_\mu)$ , where  $\overline{\mathcal{A}/\eta_\mu}$ , where  $\xi_\mu$  is the unital element  $1_{\mathcal{A}}$  in the Hilbert space.

Given any  $*$ -representation of  $\mathcal{A}$  on  $K$ , and any vector  $\xi \in K$ , let

$$\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$$

Now we ask what positive linear functional comes from the cyclic vector  $\xi_\mu$ , and we look at

$$\langle \pi_\mu u(a)\xi_\mu, \xi_\mu \rangle_\mu$$

where it is essentially

$$\langle a1_{\mathcal{A}}, 1 \rangle = \mu(a)$$

Every state is a vector state, for some representation, namely, the GNS representation.



Consider the set  $\{\pi(a) : a \in \mathcal{A}\}$ , or for a unitary representation  $U$  of a group  $G$ , consider the set

$$\{U_x : x \in G\}$$

More importantly, given a Hilbert space  $\mathcal{H}$ , let  $S$  be a subset of the bounded operators on  $\mathcal{H}$ ,  $\mathcal{B}(\mathcal{H})$  such that if  $a \in S$ , then  $a^* \in S$ , and  $I_{\mathcal{H}} \in S$ .

#### Definition 1.35

A subspace  $K \subset \mathcal{H}$  is said to be  $S$ -invariant (where  $S \subset \mathcal{B}(\mathcal{H})$ ), if whenever  $\xi \in K$ , and  $a \in S$ , and  $a\xi \in K$ . You could write  $aK \subset K$ .

#### Proposition 1.47

If  $K$  is  $S$ -invariant, then so is  $K^\perp$ .

**Proof** Let  $\xi \in K^\perp$ ,  $a \in S$ , want to show that  $a\xi \in K^\perp$ .

Let  $\eta \in K$ , then  $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle = 0$ . This is true for all  $\eta$ , hence  $a\xi \in K^\perp$ . □

Let  $\mathcal{A}$  be  $*$ -normed unital algebra, and let  $(\mathcal{H}, \pi)$  be a  $*$ -representation of  $\mathcal{A}$  on  $\mathcal{H}$ . Choose  $\xi \in \mathcal{H}$ , and let

$$K_\xi = \overline{\{\pi(a)\xi : a \in \mathcal{A}\}} \subset \mathcal{H}$$

This is a  $\{\pi(a) : a \in \mathcal{A}\}$ -invariant subspace. If we look at  $K_\xi^\perp$ , this is also  $\{\pi(a) : a \in \mathcal{A}\}$ -invariant.

#### Definition 1.36

If  $(\mathcal{H}, \pi)$  a representation of  $\mathcal{A}$ , and  $\mathcal{H}$  contains no proper nonempty  $\pi$ -invariant closed subspaces, we say then  $(\mathcal{H}, \pi)$  is irreducible. (This gives the name “simple module.”)

If  $K_\xi^\perp$  is not the zero subspace, then you can choose  $\xi_1 \in K_\xi^\perp$ ,  $\xi_1 \neq 0$ .

Set  $K_{\xi_1}$  to be the cyclic subspace generated by  $\xi_1$ , and  $\pi$ -invariant, if  $K_{\xi_1}^\perp$  is  $\pi$ -invariant. And if it's not the zero subspace, then you could repeat this process.

If  $\mathcal{H}$  is finite-dimensional, then can what we said above to get

$$\mathcal{H} = K_1 \oplus K_2 \oplus \dots \oplus K_n$$

And for each  $K_j$ , we have  $\pi$  on these is irreducible. You could decompose any  $\mathcal{H}$  into a direct sum of irreducible representations.

What happens when the space is infinite-dimensional? Group  $G$  has a representation on  $l^2(G)$ . Given any  $(\mathcal{H}, \pi)$  with  $\mathcal{H}$  infinite-dimensional, you can apply the above process, and may not get any irreducible representations, but you could keep getting **cyclic representations**.

For any Hilbert space, pick unit vector, look at orthogonal subspace, pick a unit vector in that, etc... What happens when you keep on going. Note that **when you keep going, you use Zorn's lemma**.

Via Zorn's lemma, you get the following.

#### Proposition 1.48

Every representation is a possibly infinite direct sum of cyclic representations.

We need to define infinite sum of representations.

Let  $\{\mathcal{H}_\lambda\}_{\lambda \in \Lambda}$  be a collection of Hilbert spaces. Set the direct sum of these Hilbert spaces  $\bigoplus \mathcal{H}_\lambda$ , and functions on these to be  $\xi : \Lambda \rightarrow \bigcup \mathcal{H}_\lambda : \xi(\lambda) \in \mathcal{H}_\lambda$  for all  $\lambda$  and  $\sum_{\lambda \in \Lambda} \|\xi_\lambda\|^2 < \infty$ , hence we have  $\xi_\lambda = 0$  except for a finite number of  $\lambda$ 's.

And define  $\langle \xi, \eta \rangle = \sum_{\lambda \in \Lambda} \langle \xi_\lambda, \eta_\lambda \rangle$ , such that the sum  $\sum \|\xi_n\|^2 < \infty$ , and  $\langle \xi, \eta \rangle = \sum \xi_\lambda \bar{\eta}_\lambda$ .

You know how to show  $\mathcal{H}$  is complete under  $\langle \xi, \eta \rangle = \sum \xi_\lambda \bar{\eta}_\lambda$ , then you know how to show  $\bigoplus \mathcal{H}_\lambda$  is complete under  $\langle \xi, \eta \rangle = \sum_{\lambda \in \Lambda} \langle \xi_\lambda, \eta_\lambda \rangle$ .

If  $\{T_\lambda : T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)\}$ , and  $\bigoplus_{\lambda \in \Lambda} T_\lambda$ , and if we define

$$\left(\bigoplus T_\lambda\right)(\xi) := \{T_\lambda \xi_\lambda\}_{\lambda \in \Lambda}$$

and we would like to put these in  $\bigoplus \mathcal{H}_\lambda$ , there is a constant  $c$  such that  $\|T_\lambda\| \leq c$  for all  $\lambda$ , i.e.  $T_\lambda$  is bounded in norm.

Hence

$$\|T_\lambda \xi_\lambda\| \leq \|T_\lambda\| \cdot \|\xi_\lambda\| \leq c \|\xi_\lambda\|$$

If  $\mathcal{A}$  is a  $*$ -normed unital algebra, and  $(\mathcal{H}_\lambda, \pi_\lambda)$  is a collection of  $*$ -representation of  $\mathcal{A}$ , continuous, then for  $\bigoplus_{\lambda \in \Lambda} \mathcal{H}_\lambda$ , and for  $a \in \mathcal{A}$ , set

$$\bigoplus (\pi_\lambda)(a)\xi = \{\pi_\lambda(a)\xi_\lambda : \|\pi_\lambda(a)\| \leq \|a\|\}$$

and  $\{\xi_\lambda\} \in \bigoplus_{\lambda} \mathcal{H}_\lambda$ .

By Zorn's lemma, every Hilbert space is a direct sum of cyclic representations.

Next: spectral theorem for self-adjoint operators.

## 1.16 Lecture 18

### Definition 1.37 (Equivalent representations)

Let  $\mathcal{A}$  be a (group)  $*$ -algebra, and let  $(\mathcal{H}, \pi)$ , and  $(K, \rho)$  are two  $*$ -representations of  $\mathcal{A}$ , then these representations are equivalent, if there is a unitary operator  $U : H \rightarrow K$ , ( $U^{-1} : K \rightarrow H$ ), such that

$$U\pi(a) = \rho(a)U$$

$U$  is the representation that equivalates  $\pi$  and  $\rho$ . “ $U$  intertwines  $\pi$  and  $\rho$ ”



### Proposition 1.49

Let  $\mathcal{A}$  be a unital  $*$ -normed algebra, and let  $(\mathcal{H}, \pi, \xi)$  and  $(K, \rho, \eta)$ , where  $\xi, \eta$  are cyclic vectors, and they each determine a positive linear functional on  $\mathcal{A}$ .

Let  $\mu_\xi$ , and  $\mu_\eta$  be the correspondonig positive linear functionals, and if  $\mu_\xi = \mu_\eta$ , then  $(\mathcal{H}, \pi)$  and  $(K, \rho)$  are unitarily equivalent via a unitary operator such that  $U\xi = \eta$ .



**Proof** Since we want that  $U\xi = \eta$ , then we want

$$U(\pi(a)\xi) = U\pi(a)(\xi) = \rho(a)U\xi = \rho(a)\eta$$

Start by try and define  $U$  by

$$U(\pi(a)\xi) = \rho(a)\xi$$

However, this raises the question, is this well-defined? If  $\pi(a)\xi = \pi(b)\xi$ , for some  $a, b$ , then do we have  $\rho(a)\eta = \rho(b)\eta$ ? (This would imply  $U$  is well-defined).

**Remark** In linear situations like this, if  $\pi(a - b)\xi = 0$ , is it true that  $\rho(a - b)\eta = 0$ ?

In other words, **it suffices to show that** if  $\pi(\xi) = 0$ , we have  $\rho(c) = 0$ ?

$$\|\rho(c)\eta\|^2 = \langle \rho(c)\eta, \rho(c)\eta \rangle = \langle \rho(c^*c)\eta, \eta \rangle = \mu_\eta(c^*c) = \mu_\xi(c^*c) = \langle \pi(c)\xi, \pi(c)\xi \rangle = 0$$

We note that

$$\|\rho(c)\eta\|^2 = \|\pi(c)\xi\|^2$$

The operator preserves norm on a dense subspace, we further have  $U$  is unitary.

We now see that  $U$  intertwines  $\pi, \rho$ , we have


$$U\pi(a)(\pi(c)\xi) = U\pi(ac)\xi = \rho(ac)\eta = \rho(a)\rho(c)\eta = \rho(a)U(\pi(c)\xi)$$

From this we see

$$U\pi(a) = \rho(a)U$$



### Corollary 1.9

There is a bijection between the positive linear functionals on  $\mathcal{A}$ , and the pointed cyclic representations on  $\mathcal{A}$ . 

Let  $\mathcal{A}$  be a commutative unital  $*$ -normed algebra and let  $(\mathcal{H}, \mu)$  be a cyclic  $*$ -representation with cyclic vector  $\xi$ . Then let  $\mathcal{B} = \overline{\pi(\mathcal{A})}$  be the norm closure is a  $C^*$ -algebra. So  $\mathcal{B} = C(X)$ .

Then  $\mu_\xi$  is a positive linear functional on  $\mathcal{B} = C(X)$ . Hence  $\mu_\xi$  gives a finite regular Borel measure on  $X$ ,  $\tilde{\mu}_\xi$ , so you can form the  $L^2(X, \tilde{\mu}_\xi)$  with  $\mathcal{B}$  acting on  $L^2(X, \tilde{\mu}_\xi)$  by pointwise multiplication. **This is basically the GNS representation.** Then you have the cyclic vector as the constant function 1. And find that  $\mu_1 = \mu_\xi$ , so

$$(\mathcal{H}, \pi, \xi) \cong (L^2(X, \tilde{\mu}_\xi), 1)$$

via the unitary  $U$  such that  $U\xi = 1$ .

Given  $C(X)$  and any  $*$ -representation of it,  $(H, \pi)$  and you decompose  $(\mathcal{H}, \pi)$  into direct sum of cyclic representations,  $\bigoplus (H_\lambda, \pi_\lambda, \xi_\mu)$  where each  $\xi_\mu$  is cyclic.

Each one  $(H_\lambda, \pi_\lambda, \xi_\mu)$  is isomorphic to some  $L^2(X, \mu_\lambda, \xi_\lambda)$ , with  $C(X)$  acts pointwise on each  $L^2(X, \mu_\lambda, \xi_\lambda)$ .

**Example 1.8** Let  $X = [0, 1]$ , and  $L^2(X, m)$ , where  $m$  is the Lebesgue measure. And let

$$\nu = \delta_{\frac{1}{4}} + \delta_{\frac{1}{2}} + \delta_{\frac{1}{5}}$$

All the measures above and mutually exclusive to  $m$ , and  $L^2(X, m + \nu)$ .

This gives the Borel functional calculus. For any bounded Borel function on  $X$  (Borel functions are those that are measurable with respect to all possible Borel measures), it acts on each  $L^2(X, \mu_\lambda)$  by pointwise multiplication, hence acts on  $\mathcal{H}$ . If you have a given Hilbert space  $\mathcal{H}$ ,  $T \in \mathcal{B}(\mathcal{H})$ , and  $T^* = T$ , and

$$C^*(T, I_{\mathcal{H}}) = C(\sigma(T))$$

so  $C(\sigma(T))$  acts on  $\mathcal{H}$ , hence every Borel function acts on  $\mathcal{H}$ .

Again we have  $C(X)$  acting on  $\mathcal{H}$  and the bounded Borel functions acting on  $\mathcal{H}$ , let  $E$  be any Borel subset of  $X$ , so  $\chi_E$  acts on  $\mathcal{H}$  and is a projection. So we have  $E \mapsto \chi_E \in \mathcal{B}(\mathcal{H})$  via  $\nu$ , and this is a projection-valued measure on  $X$ .

If  $E, F$  are disjoint, then

$$\chi_E \chi_F = \chi_{E \cap F}, \chi_E \oplus \chi_F = \chi_{E \cup F}$$

**Remark** This is somewhat countably additive in a weak sense.

## 1.17 Lecture 19

Let  $\mathcal{A}$  be a unital commutative  $C^*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ , and let  $(\mathcal{H}, \pi, \xi)$  be a cyclic representation (note this is not unique at all, like choosing an orthonormal basis of  $\mathcal{H}$ ). And  $\mathcal{A} \cong C(X)$ , and we've said

$$\mathcal{H} \cong \bigoplus_{\lambda \in \Lambda} L^2(X, \mu_\lambda)$$

If  $f$  is a  $\mathbb{C}$ -valued bounded Borel function on  $X$ , where  $X$  is a compact space, then  $f$  determines an operator  $T$  on each  $L^2(X, \mu)$  by pointwise multiplication. with  $\|T\| \leq \|f\|_\infty$ . Then this gives a bounded operator on  $\mathcal{H}$ . Let  $\mathcal{B}$  be the  $C^*$ -algebra of bounded Borel functions, then you get a representations of  $\mathcal{B}$  on  $\mathcal{H}$ .

For each Borel subset  $E$  of  $X$ ,  $\chi_E$  goes to a projection operator on  $\mathcal{H}$ .

### Definition 1.38

Let  $\Sigma$  be a  $\sigma$ -algebra of subsets of a set  $X$  (don't quite need a topology), and let  $\mathcal{H}$  be a Hilbert space by a projection-valued measure on  $\Sigma$ , we mean an assignment

$$E \mapsto \mu(E)$$

where  $\mu(E)$  is a projection operator on  $\Sigma$  that satisfies

1.  $\mu(\emptyset) = 0, \mu(X) = I_{\mathcal{H}}$  the identity operator

2. If  $E, F \in \Sigma$ , then  $\mu(E \cap F) = \mu(E)\mu(F)$ .
3. If  $\{E_j\}$  is a countable collection of disjoint elements in  $\Sigma$ , then  $\mu(\bigoplus_j E_j) = \sum_{j=1}^{\infty} \mu(E_j)$ , for either the strong or the weak operator topology.



**Note** Every here looks like a mesure, and we have to be careful about how the sum is well-defined.

Let  $X$  be a compact set, and  $B(X)$  denote the set of Borel sets, let  $E_j \subset B(X)$  be disjoint. These viewed in  $L^2(X, \mu)$ , we have

$$\sum_{j=1}^{\infty} \chi_{E_j} = \lim_{n \rightarrow \infty} \sum_{j=1}^n \chi_{E_j} = \lim_{j=1}^n \chi_{\bigoplus_{j=1}^n E_j} = \chi_{\bigoplus_{j=1}^{\infty} E_j}$$

And note as  $j \rightarrow \infty$ , this is an increasing function, this is not converging to the uniform form. But these do converge, for the  $L^1$  or  $L^2$  norm by Dominated Converge theorem.

If  $\xi \in L^2(X, \mu)$ , and we look at

$$\chi_{\bigcup_{j=1}^n E_j} \xi \rightarrow \chi_{\bigcup_{j=1}^{\infty} E_j} \xi \text{ for the } L^2 - \text{norm}$$

If any  $\eta \in L^2(X, \mu)$ ,

$$\langle \chi_{\bigcup_{j=1}^n E_j} \xi, \eta \rangle = \int \chi_{\bigcup_{j=1}^n E_j} \xi \bar{\eta} \rightarrow \langle \chi_{\bigcup_{j=1}^{\infty} E_j} \xi, \eta \rangle$$

where the  $L^2$  case follows from that  $\xi \bar{\eta} \in L^2(X, \mu)$ .

#### Definition 1.39

The strong operator topology on  $\mathcal{B}(\mathcal{H})$  is the topology defined by the seminorms  $T \rightarrow \|T\xi\|$  for all the  $\xi \in \mathcal{H}$ . If  $\{T_\lambda\}$  is a bounded net of operators on  $\mathcal{B}(\mathcal{H})$ , then it converges to  $T$  in the strong operator norm if for all  $\xi$ ,

$$\|T_\lambda \xi - T\xi\| \xrightarrow{\lambda} 0$$



Then the weak operator topology on  $\mathcal{B}(\mathcal{H})$  is defined by the seminorms, for all  $\xi, \eta$

$$T \mapsto |\langle T\xi, \eta \rangle|$$

If  $\{T_\lambda\}$  is a bounded net, then we say it converges to  $T$  if

$$|\langle T_\lambda \xi, \eta \rangle - \langle T\xi, \eta \rangle| \rightarrow 0$$

**Remark** Strong operator topology implies the weak operator topology, not the reverse.

**Remark** If you look at the group of unitary operators, then the strong and weak operator topologies are equivalent.

Let  $\mathcal{A}$  be a  $C^*$ -subalgebra of  $\mathcal{B}(\mathcal{H})$ , and  $\mathcal{H} = \bigoplus (X, \mu_\lambda)$ , and for  $E \in Borel(X)$ ,  $\chi_E$  actives on each one of  $(X, \mu_\lambda)$ , and

$$E \mapsto \chi_E$$

acting on  $\mathcal{H}$  is a projection-valued measure, and is called the spectral measure for action of  $\mathcal{A}$  on  $\mathcal{H}$ .

For  $T \in \mathcal{B}(\mathcal{H})$ ,  $T^* = T$ , we denote  $C^*(T, I) = \mathcal{A}$ , and  $\widehat{\mathcal{A}} = \sigma(\mathcal{A})$ , then  $C(\widehat{\mathcal{A}}) = \mathcal{A}$ . So far each  $E \in Borel(\sigma(T))$ , you get an operator.

Let  $P_r$  be the projection  $\chi_{(-\infty, r)}$ , and for  $r > \|T\|$ , we have  $P_r = I_{\mathcal{H}}$ . These are called the “resolution of the identity” for  $T$ .

You could make sense of the operator  $T$  as follows:

$$T = \int_{-\infty}^{\infty} t dP_t$$

Let  $G$  be a locally compact group. And the product is jointly continuous  $G \times G \rightarrow G, (x, y) \rightarrow xy$ , and taking inverses is also continuous.

**Example 1.9**  $\mathbb{R}, \mathbb{R}^n, \mathbb{T}, \mathbb{T}^n, GL(n, \mathbb{R})$

We know need is a projection measure on  $G$  that is invariant for left translations.

## 1.18 Lecture 20

We will discuss the Haar measure today.

Let  $G$  be a group, and  $M$  a topological space, and action of  $G$  on  $M$  is:

$$\alpha : G \rightarrow \text{Hom}(G)$$

And if  $f$  is a  $\mathbb{C}$ -valued function on  $M$ , then the set

$$\alpha_x(f)m := f(L_x^{-1}(m))$$

So  $\alpha : G \rightarrow \text{Aut}(\text{Functionson}M)$ . If  $M$  is locally compact,  $f \in C_c(M)$ . For example,  $G = \mathbb{R}$ .

Now let  $G$  be a compact group, then  $G$  acts on itself by left translations:  $\lambda$ :

$$\lambda_x(y) = xy$$

We can get an action on functions:

$$\lambda_x(f)(y) = f(x^{-1}y)$$

We then talk about a measure on a group, which is also a topological space in a sense, and would like to assign finite measure on compact sets. The goal is to construct such a measure that is translation invariant with respect to the left translations.

### Definition 1.40 (Translation-invariant measures)


For  $M$  locally compact, a measure  $\mu$  on  $M$  is  $\alpha$ -invariant, if

$$\mu(\alpha_x(E)) = \mu(E), \text{ for all } x \in G$$

where  $E$  is a Borel subset of  $M$ .

**Remark** On discrete groups, the counting measures are invariant under translations.

In 1933, Haar proved that every locally compact, second countable group (i.e. a countable base for the topology) has a left-(translation)invariant (nonzero) Borel measure, which we call the Haar measure. This is unique up to positive scalar multiplication.

 **Note** The Lebesgue measure is a Haar measure on the real numbers  $\mathbb{R}$ .

Weil: worked instead with positive linear functionals  $\varphi$  on  $C_c(G)$ , and he proved that we always have a  $\varphi$  that is left invariant, i.e.

$$\varphi(\lambda_x f) = \varphi(f), \forall x \in G$$

Sketch: choose  $f_0 \in C_c^+(G)$ , so that  $\varphi(f_0) = 1$ , and . For a small nbhd  $O$  of the identity element  $e$ , choose a function  $g_O = g$  supported in  $O$ , a smooth bump function.

Let  $(f : g_O) = \inf\{\sum_{j=1}^n c_j, c_j > 0 : \text{there exists } x_1, \dots, x_j \in G \text{ with } f \leq \sum_j c_j(\lambda_{x_j} g)\}$ , This expression here  $f : g_O$  is left translation-invariant, now we consider

$$\frac{(f : g_O)}{(f_0 : g_O)}$$

You show that this converges to a linear functional on  $C_c(G)$ , which necessarily gives rise to a linear functional that is left translation-invariant.

If for  $f \in C_c(G)$ , define  $\tilde{f}(x) = f(x^{-1})$ , then the function

$$f \mapsto \varphi(\tilde{f})$$

is a right invariant positive linear functional on  $C_c(G)$ .

Sometimes  $\varphi(f) = \int_G f(x) dx$ , given  $x_0$ , if we look at

$$\int f(xx_0) dx = \int \rho_{x_0}(f) dx$$

This is also left-invariant, hence there exists  $\delta(x_0)$  such that

$$\int f(xx_0) dx = \Delta(x_0) \int f(x) dx$$

**Proposition 1.50**

$\Delta : G \rightarrow \mathbb{R}^+$ , a group under multiplication, is and a group homomorphism, and later we will show this is continuous. And this  $\Delta$  is called the “modular function” of  $G$ .

**Definition 1.41**

$G$  is said to be unimodular if  $\delta \equiv 1$ , i.e. the left Haar measure is right-invariant.

**Example 1.10** The discrete group, commutative groups, compact groups, and nilpotent Lie groups, semi-simple lie groups.

But many are lie groups are not unimodular.

$$\int f(x^{-1})\Delta(x^{-1})dx = \int f(x)dx$$

Let  $H$  be a closed subgroup of  $G$ , let  $G/H$  be the corresponding homogenous space, and  $G$  acts on  $G/H$ , we now ask if there is an invariant measure that satisfies  $\Delta_H = \Delta_G|_H$ .

**Definition 1.42**

For a topological group  $G$ , and an action which we call a representation  $\pi : G \rightarrow \text{Aut}(V)$  of  $G$  on a Banach space  $V$ .

**Example 1.11**  $\mathbb{R}$  acts on  $L^2(\mathbb{R})$  by translations  $U_t$ . One could ask if this is continuous, i.e. as  $t \rightarrow 0$ , do we get

$$\|U_t \rightarrow U_0\| \rightarrow 0$$

You could have a little bump around 0 as your  $L^2$  function, and when you translate it, the supports are disjoint and hence the  $L^2$  norm gets preserved.

But there exists a continuous one for the strong operator topology. (The rep via translation itself does not converge, but when you apply it to something, it is indeed convergent).

**Example 1.12** If  $f \in C_c(X)$ , and  $\alpha$ , if you have  $\alpha_x(f) \rightarrow f$ , and  $x \rightarrow e$ .

## 1.19 Lecture 21

Let  $G$  be a locally compact group, and let  $M$  be a locally compact space, and let  $\alpha$  be an action of  $G$  on  $M$ , and

$$\alpha : G \rightarrow \text{Hom}(M)$$

This is with  $\alpha$  jointly continuous, i.e.  $G \times M$  equipped with the product topology, and the mapping  $G \times M \rightarrow M$

$$(x, m) \mapsto \alpha_x(m)$$

is continuous.

One classic example is  $G$  acting by left translation on  $G$ .

**Theorem 1.14**

The action of  $G$  on  $C_\infty(M)$  via

$$\alpha_x(f)(m) = f(\alpha_x^{-1}m)$$

is strongly continuous, for the strong operator topology.

**Proposition 1.51**

If  $\alpha$  is an action of a topology  $G$ , on a Banach space, by isometries, (i.e. each  $\alpha_x$  is an isometry on the Banach space), and if there is a dense subspace  $V_0 \subset V$ , (or the vectors whose span is a dense subspace) such that on  $V_0$ ,

this action is strongly continuous, i.e. for all  $v \in V_0$ , then function

$$x \mapsto \alpha_x(v) \text{ is continuous}$$

Then  $\alpha$  is strongly continuous on  $V$ .



**Proof** You do the standard approximation argument by  $\frac{\epsilon}{3}$ -argument.

Now back to the theorem, it suffices to prove this for  $C_c(M)$ , since it is a dense subset of  $C_\infty(M)$  with  $\|\cdot\|_\infty$  norm. And it suffices to prove continuity at the identity element at  $e_G$  (by uniformity).

$$\|\alpha_x(v) - \alpha_y(v)\| = \|\alpha_{y^{-1}x}(\alpha_x(v) - \alpha_y(v))\| = \|\alpha_{y^{-1}x}(v) - v\|$$

**Proof** Let  $f \in C_c(G)$ , and want to show that the map

$$x \mapsto \alpha_x(f) \text{ is strongly continuous}$$

suffices to show at  $x = e_G$ . Let  $\epsilon > 0$  be given, let  $K = \text{supp}(f)$ , and choose a compact neighborhood  $O$  of  $e_G$ . By joint continuity of  $G \times M \xrightarrow{\alpha} M$ , then we have

$$O \times K \mapsto \alpha_O(K) := OK \in G \text{ is compact}$$

Now because  $f$  is continuous, for each  $m \in OK$ , there is an open neighborhood  $U_m$  of  $m$  such that if  $n \in U_m$ , then

$$\|f(n) - f(m)\| < \epsilon/2$$

So there is an open neighborhood  $W_m$  of the identity element, such that  $O_m W_m \subset U_m$ , you can even required  $O_m \subset O$  (what we started with, ( $O_m$ 's are in  $G$ , and  $W_m$ 's are in  $M$ )). The  $W_m$ 's form an open cover of  $OK$  so there is a finite subcover  $W_{m_1}, \dots, W_{m_n}$  and let  $O^* = \cap_{j=1}^n O_j$ ,

Now we claim that for  $x \in O^*$ , we have that

$$\|\alpha_x(f) - f\| < \epsilon$$

For every  $m$ , if  $\alpha_x(f)(m) - f(m) \neq 0$ , then

$$f(\alpha_x^{-1}m) - f(n) \neq 0$$

So each  $m \in K$ ,  $m \in O^*K$ , and have  $m \in O^*K \subset OK$ , so there exists  $j$  such that  $m \in W_{m_j}$ ,  $x \in O^* \subset O_j$ , and so

$$\alpha_x(m) \in U_{m_j}$$

This gives

$$\|f(\alpha_x(m_j)) - f(m_j)\| < \epsilon/2, \|f(m) - f(m_j)\| < \epsilon/2$$

Let  $\alpha$  be an action of  $G$  on  $M$ , and suppose there is an  $\alpha$ -invariant Borel (or even Radon measure) measure on  $M$ . This gives an action of  $G$  on  $L^p(M, \nu)$ , and

$$\alpha_x(f)(m) = f(\alpha_x^{-1}m)$$

where  $\|\alpha_x(f)\|_{L^p} = \|f\|_{L^p}$  by  $\alpha$ -invariant.

### Proposition 1.52

This action is strongly continuous.



**Proof** Check this is true on  $C_c(M) \subset L^p(M, \nu)$ , where  $C_c(M)$  is a dense subspace.

## 1.20 Lecture 21

$G$  locally compact group, and  $X$  locally compact space, and  $\alpha$  a jointly continuous action of  $G$  on  $X$ , then the corresponding action of  $G$  on  $C_\infty(X)$  is strongly continuous.

For any  $f \in C_\alpha(X)$ , and

$$x \mapsto \alpha_x f \text{ is continuous for } \|\cdot\|_\infty$$

Assume that  $\mu$  is an  $\alpha$ -invariant measure on  $X$ , i.e.

$$\mu(\alpha_x f) = \mu(f), \text{ for all } x, f$$

We then can form  $L^p(X, \mu)$ , with the  $\|\cdot\|_{L^p}$  norm:

$$\|f\|_{L^p} = (\mu(|f|^p))^{\frac{1}{p}}$$

And  $L^p(X)$  contains  $C_c(X)$ , and what we want is the action  $\alpha$  on the  $L^p$  is again strongly continuous. (One could take  $f$  such that  $\|f\|_\infty$  is small, but the  $L^p$  norm is large.)

The action  $\alpha$  of  $G$  on  $C_c(X)$  is continuous for the “inductive limit topology.” We shall define it now.

**Definition 1.43 (Inductive limit topology)**

If  $O \subset X$ , and  $\overline{O}$  is compact, then you can view  $C_c(O)$ , equipped with  $\|\cdot\|_\infty$ , as a subset of  $C_c(X)$  (simply by extending by 0), then the inductive limit topology is the strongest topology on  $C_c(X)$  making the inclusion continuous.

**Definition 1.44 (Convergence in ILT)**

If a net  $\{f_\lambda\}$  converges to  $f$  uniformly, and eventually, and there is a compact set  $K$  such that eventually all  $f_\lambda$  are supported in  $K$ , then this  $\{f_\lambda\}$  converges to  $f$  in the inductive limit topology.

If  $\mu$  is a Radon measure on  $C_c(X)$ , then the net  $\{f_\lambda\}$  converges to  $f$  for  $\|\cdot\|_{L^p}$  of  $L^p(X, \mu)$ . We can find  $f_\lambda$  converges uniformly to  $f$  in the  $\|\cdot\|_\infty$  sense.

$$(\|f - f_\lambda\|_{L^p})^p = \int_X |f - f_\lambda|^p d\mu \leq \|f - f_\lambda\|_\infty^p \mu(K)$$

And  $\mu(K) < \infty$ , and  $\|f - f_\lambda\|_\infty < \epsilon$ . Everything is of finite measure, and uniform convergence gives you convergence in the  $L^p$  norm.

To show that  $x \mapsto \alpha_x(f)$  is continuous for  $\|\cdot\|_{L^p}$ , it suffices to show that it is continuous at  $e$ , let  $O$  be a compact neighborhood of  $E$ , and let  $K = \text{supp}(f)$ , then for  $x \in O$ , then  $\alpha_x(f)$  is compactly supported in  $\alpha_O(K)$ , and  $\|f - \alpha_x(f)\|_\infty \rightarrow 0$ , hence is strongly continuous for the  $\|\cdot\|_{L^p}$  norm.

Let  $G$  be a locally compact and let  $\alpha$  be an action by isometries on a Banach space  $V$ , we define the integrated form of  $\alpha$ ,  $f \in C_c(G)$ , and  $v \in V$ :

$$\alpha_f(v) = \int_G f(x) \alpha_x(v) dx, \alpha_x(v) \in V$$

where  $dx$  is a choice of left-invariant Haar measure.

Given  $\varphi \in V^*$ ,

$$\int f(x) \langle \alpha_x(v), \varphi \rangle dx$$

the above form is called the “weak integral.” and  $\text{range}(f)$  is a normed-closed compact subset of  $V$ .

Consider,  $C_c(X, \mathbb{R})$ , and  $V$  is over  $\mathbb{R}$ . and

$$(X, \mathbb{R}) \otimes V = \left\{ \sum_{i=1}^n f_i \otimes v_i, f_i \in C_c(X, \mathbb{R}), v_i \in V \right\}$$

and viewed as an element of  $C_c(X, V)$ , and  $(x) = \sum_i f_i(x) v_i \in V$ . Given a Radon measure  $\mu$ , we have

$$\int \sum_{j=1}^n f_j \otimes v_j d\mu = \sum \mu(f_j) v_j$$

The above is well-defined, and apply to  $V^*$ , if we denote  $F = \sum_{j=1}^n f_j \otimes v_j$ , then

$$\varphi\left(\int F(x) d\mu(x)\right) = \int \varphi(F(x)) d\mu(x)$$

One should think of these  $F$  as simple functions  $f = \sum_j \chi_{E_j} v_j$ .



Note that

$$\left\{ \sum_{j=1}^n f_j \otimes v_j \right\} \text{ is dense for the inductive limit topology}$$

Given  $F \in C_c(X, V)$ , and some  $\epsilon > 0$ , let  $K = \text{supp}(F)$ , and  $C$  be a compact neighborhood of  $K$ . Then, for each  $x \in C$ , there is an open neighborhood  $O_x$  with

$$\|f(y) - f(x)\| < \epsilon \text{ if } y \in O_x$$

And by  $C$  being compact, there exists a finite subcover  $O_{x_1}, \dots, O_{x_n}$ . Choose a partition of unity, subordinate to the  $\{O_{x_j}\}_{j=1}^n$ . This means  $F_j \in C_c(X, \mathbb{R})$ , and  $0 \leq f_j \leq 1$ , and  $\text{supp}(f_j) \subset O_{x_j}$ , such that  $\sum \varphi_j = 1$  on  $K$ . Then the set

$$F_\epsilon = \sum f_j \otimes F(x_j)$$

and  $\|F - F_\epsilon\|_\infty < \epsilon$ .

Let  $\alpha$  be an action on  $V$ , if we write  $\alpha_f(v) = \int_G f \alpha_x(v) dx$ , and  $\alpha_f \alpha_g = \alpha_{f * g}$ , then

$$(f * g)(x) = \int f(y) g(y^{-1}x) dy$$

Fubini's theorem for Radon measures, use  $C_c(X) \otimes C_c(Y)$ , dense in  $C_c(X \times Y)$

$$\sum_{j=1}^n f_j \otimes g_j$$

for the inductive limit topology. This is the Stone-Weierstrass theorem. Now we ask the question, what is  $(\pi_f)^*$ .

## 1.21 Lecture 22

Let  $G$  be locally compact, and if  $(\mathcal{H}, \pi)$  is a unitary representation of  $G$ , with the integrated form:

$$f \mapsto \pi_f, f \in C_c(G) \subset L^1(G)$$

we have

$$(\pi_f)^* = \left( \int f(x) \pi_x dx \right)^* = \int \overline{f(x)} \pi_x^* dx = \int \overline{f(x)} \pi_{x^{-1}} dx = \int \overline{f(x^{-1})} \Delta(x) \pi_{x^{-1}} dx$$

### Proposition 1.53

If  $G$  is locally compact, but not discrete, then  $L^1(G)$  does not have an identity.

**Example 1.13**  $C_\infty(X)$ .



**Note** Many Banach algebras have an “approximate” identity elements.

**Example 1.14** Let's look at  $C_\infty(\mathbb{R})$ . You could take a function that is a smooth characteristic function, whose support is contained in  $[-N, N]$ , and pushing  $N \rightarrow \infty$ . If you take  $N$  large enough, you eventually approximate the function itself.

### Definition 1.45 (Approximate identities)

For a normed algebra  $\mathcal{A}$ , let  $a$  be a left approximate identity, we mean a net  $e_\lambda$  such that for any  $a \in \mathcal{A}$ , we have that  $e_\lambda a \rightarrow a$  in norm. A right approximate identity is one such that  $a e_\lambda \rightarrow a$  in norm. A two-sided approximate identity is one such that it satisfies both. And a bounded approximate identity is one such that  $\|e_\lambda\| \leq c$  for all  $\lambda$  and such that  $e_\lambda a \rightarrow a$  in norm. (There is therefore, left bounded approximate identity, and right, and two-sided). A norm 1 approximate identity is such that  $\|e_\lambda\| = 1$ .

If  $\mathcal{A}$  has a  $*$  operation, then you can define a self-adjoint approximate identity, if

$$e_\lambda^* = e_\lambda \text{ for all } \lambda$$

If  $\mathcal{A}$  is a  $C^*$ -algebra, a positive approximate identity is a net with  $e_\lambda > 0$  for all  $\lambda$ .

**Proposition 1.54**

For any locally compact group,  $L^1(G)$  has a approximate identity of norm 1.



**Note** The picture should look like a tall bump function around the identity  $e_G$ .

**Proof** Let  $\Gamma$  be a collection of compact neighborhoods of  $e_G$ , and for each  $O$ , choose  $e_O \in C_c(G)$  with  $\text{supp}(e_O) \subset O$ , and  $\int e_O(x)dx = 1$ .

$f * g(x) = \int f(y)g(y^{-1}x)dy$  with integrated form of the action of  $G$  on  $L^1(G)$  by translation  $g(x) \rightarrow g(y^{-1}x)$ . In fact, for any representation of a Banach space  $V$  of  $G$  by  $\pi_{e_\lambda} v \rightarrow v$  in norm. We would like the following to go to zero.

$$\left\| \int e_O(x) \pi_x v - v \right\| = \left\| \int e_O(x) \pi_x v - \int e_O(x) dx v \right\| = \left\| \int e_O(x) (\pi_x v - v) \right\|$$

For any  $\epsilon > 0$ , choose  $O_\epsilon$  and that for  $x \in O_\epsilon$ , we have  $\|\pi_x v - v\| < \epsilon$  (by SOT). For any neighborhood smaller than  $O_\epsilon$ , we still have this property,  $O \subset O_\epsilon$ . So for the above equation, we get that

$$\left\| \int e_O(x) \pi_x(v) - \int e_O(x) dx v \right\| \leq \int O e_O(x) \|\pi_x v - v\| \leq \epsilon$$

□

**Proposition 1.55**

For any  $C^*$ -algebra  $\mathcal{A}$  has a positive approximate identity of norm 1.



**Proof** Let  $\Gamma$  be the collection of finite subsets of  $\mathcal{A}$ , and for  $\lambda \in \Gamma$ , let the finite sum be as follows:

$$b_\lambda = \sum \{b^* b : b \in \lambda\}$$

let

$$e_\lambda = \frac{b_\lambda}{\frac{1}{n} + b_\lambda} < 1, n = |\lambda| = \text{the number of elements in } \lambda$$

□

For any Banach normed algebra  $\mathcal{A}$ , we can get an identity element by

$$\tilde{\mathcal{A}} = \{(a, z) : a + z1_{\mathcal{A}}\}$$

If  $\mathcal{A}$  has an approximate identity of norm 1, and if  $\mu$  is continuous positive  $\mu(a^*a) \geq 0$  linear functional, on  $\mathcal{A}$ , but no identity element, but does have a norm  $\|\mu\|$ . Now if you have an approximate identity, then  $\mu$  extends to a positive linear functional on  $\tilde{\mathcal{A}}$ , where  $\|\mu\| = 1$ . Then you could apply GNS to this  $\mu$ .



**Note** Any  $C^*$ -algebra  $\mathcal{A}$  has a positive approximate identity of norm 1. For example  $G$ ,  $L^2(G)$ , and  $\pi$

$$\overline{\{\pi_f : f \in \mathcal{A}\}} = D_r^*(f)$$

## 1.22 Lecture 23

**Proposition 1.56**

Let  $G$  be a locally compact group, the left-regular representation of  $L^1(G)$  on  $L^2(G)$  is faithful.



**Proof** let  $f \in L^1(G)$ , and  $f \neq 0$ , with  $L^1(G)$  has a norm-1 approximate identity of functions in  $C_c(G)$ . Then there exists  $g \in C_c(G)$  such that  $f * g \neq 0$ , and

$$f * g = \int_G f(x) \lambda_x g dx \in C_\infty(G) \cap L^1(G) \cap L^2(G)$$

so now  $g$  is an element of  $L^2(G)$ ,

$$\lambda_f(g) = f * g \in L^2(G)$$


we have that

$$\|f * g\|_{L^2}^2 = \int |f * g|^2 d\mu > 0$$

So  $\lambda_f \neq 0$ . For  $G$  commutative, the norm closure

$$\overline{\{\lambda_f : f \in L^1(G)\}} = C_r^*(G)$$

**Proposition 1.57**

Let  $\mathcal{A}$  be a  $C^*$ -algebra, without an identity element. Let  $\tilde{\mathcal{A}}$  be  $\mathcal{A}$  with 1 adjoint. (Side note:  $\|a + \alpha 1\| = \|a\| + |\alpha|$ , and note such norm is not good for  $\mathcal{A} = C_\infty(X)$ .) For the left regular representation on  $\mathcal{A}$ , i.e. if  $b \in \tilde{\mathcal{A}}$ , set  $\|b\| := \sup\{\|ba\| : \|a\| \leq 1\}$ . With this norm  $\|b\|$ , we have that  $\tilde{\mathcal{A}}$  is a  $C^*$ -algebra. 

This is to check that  $\|b^*b\| = \|b\|^2$ . We already have  $\|b^*b\| \leq \|b\|^2$ . And  $\|ba\|^2$ , we have that

$$\|ba\|^2 = \|(ba)^*(ba)\| \leq \|a^*\| \|b^*ba\| \leq \|b^*b\| \|a\| = \|a\|^2 \|b^*b\|$$

And it is easy to see that if  $\mathcal{A}$  is complete, then  $\tilde{\mathcal{A}}$  is also complete.

If  $\mathcal{A}$  is a commutative  $C^*$ -algebra without identity, and  $\tilde{\mathcal{A}} = C(X)$ , and this is the nonzero homomorphisms of  $\mathcal{A}$  onto  $\mathbb{C}$ , with  $\{\infty\}$ . So  $\tilde{\mathcal{A}} = X \setminus \{\infty\}$ , and  $\mathcal{A} \cong C_\infty(X \setminus \{\infty\})$ , where the space  $X \setminus \{\infty\}$  is locally compact.

Let  $G$  be a commutative, discrete group, where

$$C_r^*(G) \cong C_\infty(X)$$

where each point of  $X$  gives a nonzero homeomorphism of  $L^1(G)$  into  $\mathbb{C}$ . Then get for any  $f \in L^1(G)$ , there is a continuous  $*$ -homomorphism of  $L^1(G)$ , then there exists a  $\varphi \in \widehat{L^1(G)}$  such that  $\varphi(f) \neq 0$ . We want to describe all  $\varphi \in \widehat{L^1(G)}$ , let  $f, g \in L^1(G)$ ,  $g \neq 0$ . We have that

$$\varphi(f)\varphi(g) = \varphi(f * g) = \varphi\left(\int f(x)\lambda_x g dx\right)$$

So

$$\varphi(f) = \int f(x)\varphi(\lambda_x g)/\varphi(g) dx$$

Define  $\omega_\varphi(x) = \varphi(\lambda_x g)/\varphi(g)$ , is continuous. hence we have that

$$\varphi_f \varphi_g = f(x)\varphi(\lambda_x g) dx$$

We then have

$$\varphi(g)\omega_\varphi(x) = \varphi(\lambda_x g)$$

If we look at

$$\omega_\varphi(xy)\varphi(g) = \varphi(\lambda_{xy}g) = \varphi(\lambda_x(\lambda_y g)) = \varphi(\lambda_y g)\omega_\varphi(y)\omega_\varphi(x)$$

Because we have  $\varphi(g) \neq 0$ , we divide both sides, then we have

$$\omega_\varphi(xy) = \omega_\varphi(x)\omega_\varphi(y)$$

So  $\omega_\varphi \in \widehat{G}$  into  $T$ .

For  $f \in L^1(G)$ , define

$$\widehat{f}(\sigma) = \int f(x)\sigma(x)dx \in \widehat{G} \in C_\infty(\widehat{G})$$

where we equip  $\widehat{G}$  with the weak- $*$  topology, and  $\sigma \in \widehat{G}$ . And the above defines the Fourier transform.

**Corollary 1.10**

The Fourier transform defined above is injective. 

For any  $f \in L^1(G)$ , if  $f \neq 0$ , then  $\widehat{f} \neq 0$ . And we have

$$\widehat{f^*} = \overline{\widehat{f}}$$

Next we state a fact:

**Proposition 1.58**

*The weak-\* topology on  $\widehat{G}$  agrees with the topology of uniform convergence on compact subsets of  $G$ . Moreover, under this uniform topology on compact subsets of  $G$ , we can see that  $\widehat{G}$  is a topological group.*



Next time: unbounded operators on Hilbert spaces! (And the spectral theorem for those).