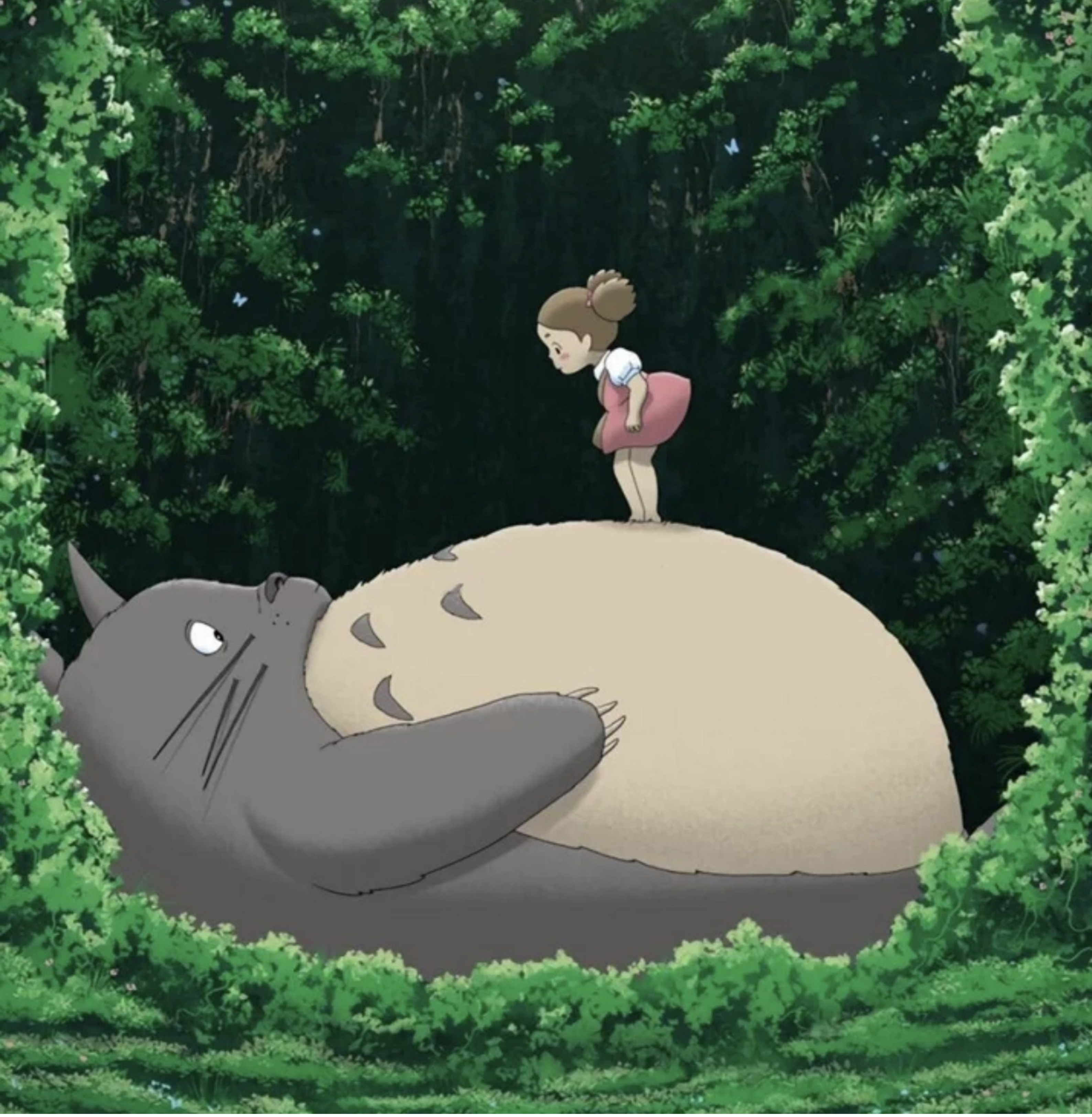

Functional Analysis

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Contents

Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

Remark Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \tag{1.1}$$

♠

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_{\mathcal{H}}) \cong C(X) \tag{1.2}$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on B or an action of the group G on B.

♣

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B, generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B, hence no need to include it.

Elements of the form $\Sigma_{z_x} \alpha_x, z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x\in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many x, $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f\alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f\alpha_g = \sum_x \sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f * g)(y)\alpha_y$$

where we define $f * g(y) = \sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f\alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \| \sum f(x)\alpha_x\| \leq \sum |f(x)|\|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x\xi(y) = \xi(x^{-1}y), \alpha_x$ being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α.

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X. This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X, by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let $\mathcal{A}$ be an algebra on $\mathbb{R}$ or $\mathbb{C}$, is a normed algebra if it has a norm $\|\|$, as a vector space, such that for for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity. ♣

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm. ♣

Given $x \in X$, define $\varphi_x : C(X) \to \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \to \mathbb{R}$ or $C(X) \to \mathbb{C}$. This simply implies

$$\varphi_x(f+g) = (f+g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$. ♠

Proof Main Claim: given a multiplicative linear functional φ, there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f(x) \neq 0$, but $\varphi(f) = 0$.

Set $g_x = \overline{f}_x f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f}_x) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is $x_1, ..., x_n$ such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + ... g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f, consider $f - \varphi(f) \cdot 1$, apply φ, we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field. ♣

Remark We have $|\varphi(f)| \leq \|\varphi\|\|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

Thss is not always true for normed algebra, Let

$$\mathcal{A} := Poly \subset C([0,1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If $\mathcal{A}$ is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word "unital" is key here.

Proposition 1.4

Let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} =? \sum_{n=0}^{\infty} a^n, a^0 = 1_{\mathcal{A}}$ You could look at the partial sums. $S_m = \sum_{n=0}^{m} a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m\to\infty} S_m = \frac{1}{1-a}$.

To prove this is a cauchy sequence:

$$\|S_n - S_m\| = \|\sum_{j=m+1}^{n} a^j\| \leq \sum_{m+1}^{n} \|a^j\| \leq \sum_{m+1}^{n} \|a\|^j$$

And the fact that $\|a\| < 1$, we have the sum bounded by ϵ, hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n\to\infty} S_n(1-a) = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = \lim_{n\to\infty}(1 - a^{n+1}) = 1$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$.

1.2 Lecture 3

We now begin.

Let $\mathcal{A}$ be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Newmann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1-a\| < 1$, then a is invertible.

Proof $a = 1 - (1-a)$.

Proposition 1.5

The set of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$.

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of $\mathcal{A}$ onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d, homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in Range(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.

Proof Let $f \in C(X)$, and if $\alpha \in$ range of f, so $\alpha = f(x_0)$ for some x_0. then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(-\alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in$ range of f.

□

Definition 1.5 (spectrum of an element)

For any unital algebra $\mathcal{A}$ over some field $\mathbb{F}$, for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible } \}$$

is called the spectrum of a, denoted as $\sigma(a)$.

Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$

Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda(1 - \frac{a}{\lambda})$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\alpha \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on $\mathcal{A}$, i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of $\mathbb{R}, \mathbb{C}$.

Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of $\mathcal{A}$ is open, hence

$$\sigma(a) = \phi^{-1}(\text{ noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{ invertible })$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of $\mathcal{A}'$, which denotes the dual vector space of continuous linear transformations.

On $\mathcal{A}'$, we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak- topology.*

Proof let $\{\varphi_\lambda\}$ be a net of elemnts of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \to \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V, the closed unit ball of V' is compact in the weak- topology.*

As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak- toplogy.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto $\mathcal{A}$.

For $\mathcal{A}$ unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ into $C(\mathcal{A})$.

Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{a}\widehat{b})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

□

1.3 Lecture 4

Today we talk about the structure of $\widehat{l^1(S)}, \widehat{l^1(G)}$, where S, G are semigroups and groups, and how they naturally identify with the unit disk $\mathbb{D}$, and the unit circle $\mathbb{T}$.

Let S be a commutative discrete semigroups, for example $\mathbb{N} \cup \{0\}$, and $f \in C_c(S)$, then we can write $f = \sum_{x \in S} f(x)\delta_x$, where we define $\delta_x\delta_y = \delta_{xy}$. Note that $C_c(S)$ is dense in $l^1(S)$.

Definition 1.6 (Convolution)

Take any $f, g \in C_c(S)$, we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{x \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left(\sum_{xy=z} f(x)g(y) \right) \delta_z$$

where we define the convolution between two functions

$$f * g(z) = \sum_{x,y,xy=z} f(x)g(y)$$

And under this convolution operation, we have $l^1(S), *$ as a Banach algebra.

Example 1.1 If we consider polynomials of the form $f(x) = \sum_{n=0}^{\infty} f(n)x^n$, and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right)\left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f * g)(p)$$

And let $f \in C_c(S)$, where $S = \mathbb{N}$. we define $\|f\|_{l^1} = \sum_{x \in S} |f(x)|$.

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ denote the set of unital homomorphisms from $\mathcal{A}$ to $\mathbb{R}, \mathbb{C}$. Note that $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$.

Note that we know $(l^1(S))' = l^\infty(S)$, hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. Note that we have $\|\varphi\| = 1$, hence if we $\varphi \in l^\infty(S)$, we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for $z \in S$, $\|z\| \leq 1$, we have $|\varphi(z)| \leq 1$.

Proposition 1.12

We naturally identify $\widehat{l^1(S)}$ with $Hom(S, \mathbb{D})$, i.e. $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$.

Proof Given $f \in \widehat{l^1(S)}$, we know it's multiplicative, unital, hence all these transfer when viewing $\varphi \in l^\infty(S)$. This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here xy denotes the operation on S between x, y, for example, could be $x + y$. Hence naturally, if $\varphi \in \widehat{l^1(S)}$, φ can also be viewed as $\varphi : S \to \mathbb{D}$, and thus is in l^∞, with $|\varphi(s)| \leq 1$.

□

Furthermore, we can identify elements in $\widehat{l^1(S)}$ with the unit disk. Take $S = \mathbb{N}$.

Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Proof We motivate this by noticing $\mathbb{N}$ is generated by 1, and thus viewing $\varphi \in \widehat{l^1(\mathbb{N})}$ as $\varphi \in l^\infty(\mathbb{N})$, we have φ is determined by $\varphi(1)$. And denote $\varphi(1) = z_0$, then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for $z \in \mathbb{D}$,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space.

□

Proposition 1.14

The standard topology on $\mathbb{D}$ coincides with the weak- topology on $\widehat{l^1(\mathbb{N})}$.*

$$D_{std} \cong D_{weak-*}$$

Proof We just need to associate an element in $\mathbb{D}$ with a function $\varphi \in \widehat{l^1(\mathbb{N})}$. And we do this by

$$z \mapsto \sum_{n\in\mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

1.3.1 On groups

We let G denote a discrete commutative group, and we see everything above follows, with one extra property.

Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where $\mathbb{T}$ *denotes the unit circle* $\{z \in \mathbb{C} : |z| = 1\}$.

Proof For $\varphi \in \widehat{l^1(G)}$, we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because $|\varphi(x)| \leq 1, \forall x$, Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have $\widehat{l^1(G)}$ naturally identifies with $\mathbb{T}$. Like what we described above, we have what is desired.

$\square$

Remark Take $G = \mathbb{Z}$, if we denote $z \in \mathbb{T}$ as $z = e^{2\pi it}$, then we naturally identify with

$$\sum_{n\in\mathbb{Z}} f(m)e^{2\pi int}$$

we denote this mapping as $\widehat{f}$, i.e.

$$\widehat{f}(z) = \sum_{m\in\mathbb{Z}} f(m)e^{2\pi int}$$

This is the Fourier transform.

1.4 Lecture 5

Last time, we talked about if we denote $\mathcal{A} = l^1(G)$, equipped with $\|\cdot\|_{l^1}$, under convolution, we have

$$\widehat{\mathcal{A}} \text{ “=” } Hom(G, \mathbb{T})$$

If we take $G = (\mathbb{Q}, +)$, one can ask the question if $\widehat{\mathcal{A}}$ is big enough. And we will se later in the course, the answer is yes.

For pointwise multiplication, $\widehat{G}$ forms a group, and in fact $\widehat{G}$ is a compact topological group.

For any compact commutative group G, for exapmle $\mathbb{R}^n$ under $+$. Define

$$\widehat{G} = \text{ continuous homomorphisms into } \mathbb{T}$$

Remark We now require continuous with this general G (previously was not required for discrete group G).

Proposition 1.16

Let G be a locally compact and commutative group, we have $\widehat{G}$ as a locally compact, commutative group.

We define the pairing between G and $\widehat{G}$ as follows: $x \in G, \varphi \in \widehat{G}$,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the folliwng map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let G, H denote locally compact groups, and $\phi : G \to H$ bet a continuous homomorphism. Note we have the following diagram:

$$G \xrightarrow{\phi} H$$

$$\widehat{G} \xleftarrow{\phi} \widehat{H}$$

If we take an element $\psi \in \widehat{H}$, we consider $\psi \circ \phi$. We get $\psi \circ \phi \in \widehat{G}$.

Definition 1.7 (category, functor)

A category is specified by

1. *a set of objects*
2. *morphisms between objects*
   (a). *X, Y, Z are objects, and if*

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

   (b). *For each object X, there is an identity morphism 1_X.*

And a functor is defined to be such a morphism between categories. ♣

Example 1.2 For category of finite vector spaces V, passing from vector space to its dual V' is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$V \xrightarrow{T} W$$

$$V' \xLeftarrow{T^t} W'$$

$$V'' \xrightarrow{T^{tt}} W''$$

The map going in the same directions $V \to W$, and $V'' \to W''$ is called covariant, whereas $V' \leftarrow W'$ is called contravariant.

Example 1.3 For category of locally compact groups G, H, assigning the dual group is a functor:

$$G \to H$$

$$\widehat{G} \leftarrow \widehat{H}$$

$$\widehat{\widehat{G}} \to \widehat{\widehat{H}}$$

Example 1.4 Now let X be a compact space. Given Φ continuous map between $X \to Y$.

$$X \xrightarrow{\Phi} Y$$

$$C(X) \leftarrow C(\Phi)C(Y)$$

For $f \in C(Y)$, we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$X \xrightarrow{\varphi}\xrightarrow{\phi} Z$$

$$C(X) \xleftarrow{C(\varphi)} C(Y) \xleftarrow{C(\phi)} C(Z)$$

where for $f \in C(Y), C(\varphi)(f) = f \circ \varphi$, and $g \in C(Z), C(\phi) = g \circ \phi$. This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between $C(X)$ and $C(Y)$, there eixsts a map from X to Y.

Proposition 1.17

Suppose X, Y are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism $\check{F} : X \to Y$.

Proof Define $\varphi_x : C(X) \to \mathbb{C}$ as the evalutation map: take $f \in C(X)$,

$$\varphi_x(f) = f(x)$$

Then $\varphi_x \circ F \in \widehat{C(Y)}$. And we know that any element in $\widehat{C(Y)}$ is a point evaluation, i.e. there exists $y \in Y$ such that

$$\varphi_y = \varphi_x \circ F$$

We thus define $\check{F}(x) = y$ as such that it satisfies the above equation. We need to show $\check{F}$ is continuous. Note that X, Y are compact Hausdorff spaces, and the topology on Y is the coarest topology making all functions $g \in C(Y)$ continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by F, g being continuous, we have $\check{F}$ is also continuous.

□

There is a natural bijection between the continuous functiosn from X to Y, and the unital algebra homomorphism from $C(X)$ to $C(Y)$.

A quick reminder:

Remark For X compact, the weak-* topology coincides with the standard topology.

1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let $\mathcal{A}$ be a unital Banach algebra.

We write $GL_n(\mathcal{A})$ to denote the general linear group, the group formed by $n \times n$ matrices with entries from $\mathcal{A}$.

The less standard notation is $GL_I(\mathcal{A})$ is the group of invertible elements in $\mathcal{A}$. As we have shown previously, this is a closed subset of $\mathcal{A}$. This is the notation that we will use.

Remark It is easy to see that the product is jointly continuous.

Proposition 1.18

The following map is continuous.

$$a \mapsto a^{-1}$$

Proof Given $\|a - b\| < \delta$, we would like to show $\|a^{-1} - b^{-1}\| < \epsilon$. We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\|\|b - a\|\|b^{-1}\|$$

Take $\delta = \epsilon / \|a^{-1}\|\|b^{-1}\|$ would suffice.

□

Proposition 1.19

Fix $a \in GL(\mathcal{A})$, there exists a neighborhood O of a and a constant K such that for all $y \in O$, we have

$$\|c^{-1}\| < K$$

Proof Let $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$, then d is invertible and

$$d^{-1} = \sum_{n=0}^{\infty}(1-d)^n$$

We thus have

$$\|d^{-1}\| \leq \frac{1}{1 - \|1-d\|} \leq \frac{1}{1-1/2} = 2$$

We then identify what our O should be. Let $O = aV$, then we want to show that every ad has an inverse with bounded norm. Because a, d are both invertible, ad is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \leq \|d^{-1}\|\|a^{-1}\| \leq 2\|a^{-1}\|$$

□

Remark For each invertible element, we can find a neighborhood of invertible elements around it, and using that $(1-d)$ is bounded, then d is invertible, we can bound $\|d^{-1}\|$.

Definition 1.8

Fix $a \in \mathcal{A}$, the resolvent set of $\mathcal{A}$ is the complement of spectrum of $\mathcal{A}$, i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible }\}$$

Hence the resolvent set is an open, unbounded suset of $\mathbb{C}$ or $\mathbb{R}$.

Definition 1.9 (Resolvent function)

On the resolvent set, $\{\lambda \in \mathbb{F} : a - \lambda 1 \text{ is invertible }\}$ is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that a is fixed, and λ is the variable here.

Now we note that this $R_a(\lambda)$ function is nicely behaved.

Proposition 1.20

The resolvent function $R_a(z)$ is analytic on the resolvent set, and vanishes as $z \to \infty$.

Proof We first define the notation of analyticity on an open subset of $\mathbb{R}, \mathbb{C}$: this means for every point in the open set O, we can find a power series expansion of the function such that its radius of convergence > 0.

Fix z_0 in the resolvent set. We know $z_0 1_{\mathcal{A}} - a$ is invertible. We consider $(z1_{\mathcal{A}} - a)$, for z in the resolvent set. We will omit the $1_{\mathcal{A}}$ for simplicity.

$$z1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a)\left(1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a}\right)$$

We know the the latter term is invertible if $\|\frac{z_0 - z}{z_0 - a}\| < 1$ has norm, hence we have

$$(z-a)^{-1} = \sum_{n=0}^{\infty}\left(\frac{z_0 - z}{z_0 - a}^n\right)(z_0 - a)^{-1}$$

What happens when we let $z \to \infty$, we consider $R_a(1/z)$, and let $z \to 0$. Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1 - az}{z}\right)^{-1} = z(1-az)^{-1}$$

Let $z \to 0$ makes $R_a(1/z)$ go to zero.

□

Now given that $R_a(z)$ is analytic and bounded at ∞, we can state the following important theorem.

Theorem 1.2 (Nonemptyness of spectrum)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, then for any $a \in \mathcal{A}$, we have $\sigma(a) \neq \emptyset$.

Proof Assume there exists $a \in \mathcal{A}$, such that $\sigma(a) = \emptyset$. If $\mathcal{A} = \mathcal{C}$, then we would have $R_a(\lambda)$ be a bounded entire, complex-valued function defined on all of $\mathbb{C}$. By Liouville's theorem, we must have $R_a(z)$ a constant function, but we know $z \to \infty$, $R_a \to 0$, hence $R_a(z)$ is constantly 0, but this cannot be true.

If our $\mathcal{A}$ is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where ϕ is some nonzero element in $\mathcal{A}'$, guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at ∞. Again, the function is constantly 0.

With the nonemptyness of spectrum theorem, we now state the Gelfand-Mazur theorem.

Theorem 1.3 (Gelfand-Mazur)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, if any nonzero element of $\mathcal{A}$ is invertible, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

Proof For any $a \in \mathcal{A}$, we know $\sigma(a) \neq \emptyset$, hence there exists λ such that $\lambda 1_{\mathcal{A}} - a$ is invertible, i.e. $a = \lambda 1_{\mathcal{A}}$, hence establishing an isomorphism between $\mathcal{A}$ and $\mathbb{C}$. In other words, $\mathcal{A} = \mathbb{C}1_{\mathcal{A}}$. □

1.5.1 Functional Calculus

Proposition 1.21

Let $a \in \mathcal{A}$, then if $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges for $|z| < r$, where $r > \|a\|$, then $\sum_{n=0}^{\infty} \alpha_n a^n$ converges as well.

We first start with proving the following statement.

Lemma 1.1

Let f be a polynomial, $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, $f = \sum_{n=0}^{k} a_n x^n$, then for $a \in \mathcal{A}$, we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of a under f is exactly the spectrum of f evaluated at a.

Proof ($\Leftarrow$). We take $\lambda \in \sigma(a)$, and we would like to show $f(\lambda)$ is in the spectrum of $f(a)$. We note that if $\lambda \in \sigma(a)$, then $a = \lambda 1_{\mathcal{A}}$, and $f(\lambda 1_{\mathcal{A}}) = f(a)$, hence by definition, $f(a) - f(\lambda)1_{\mathcal{A}}$ is not invertible implying $f(\lambda)$ is in the spectrum of $f(a)$. Note that this also implies $f(a) - f(\lambda) = (a - \lambda)Q(z)$ for some polynomial $Q(z)$.

($\Rightarrow$). We take $\lambda \in \sigma(f(a))$, i.e. $f(a) = \lambda 1_{\mathcal{A}}$. we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. If f is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1)...(z - c_n)$$

Plugging in a we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}})...(a - c_n 1_{\mathcal{A}})$$

If $f(a) - \lambda$ is not invertible, then there exists j such that $(a - c_j 1_{\mathcal{A}})$ is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. In fact, we have $\lambda = f(c_j)$ by knowing $f(c_j) - \lambda = 0$. □

Now let $f(z) = z^n$, and if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$

This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

Definition 1.10 (spectral radius)

Fix $a \in \mathcal{A}$, we define the spectral radius of a, denoted by $r(a)$,

$$r(a) = \sup_\lambda \{|\lambda| : \lambda \in \sigma(a)\}$$

♣

Next we introduce an equivalent definition of the spectral radius which connects to the Gelfand transform.

Proposition 1.22

For $\mathcal{A}$ a Banach algebra, we have the following relationship:

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\Gamma(a)\|_\infty$$

♠

Example 1.5 Note it we have a self-adjoint operator T, then the spectral radius of T would be the absolute value of the largest eigenvalue, $|\lambda|$.

Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$

♡

Proof From the previous remark that $|\lambda| \leq \|a^n\|^{1/n}$, hence this follows.

1.6 Lecture 7

I have not typed up for this?

1.7 Lecture 8

Let $\mathcal{A}$ be a unital Banach algebra. Then for $a \in \mathcal{A}$, and we look at the resolvent of a, $R_a(\lambda)$, we've noted that as $\lambda \to \infty$, we have

$$\lim_{\lambda \to \infty} R_a(\lambda) = \lim_{\lambda \to \infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda \to \infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for $|\lambda| \geq \|a\|$.

Recall that we define the spectral raidus, $r(a)$, as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

Proposition 1.23 (Gelfand-Beurling)

$$r(a) = \lim \|a^n\|^{1/n}$$

♠

Proof

If we let $\lambda = 1/z$, then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for $|z| \leq \|a\|^{-1}$, but maybe?? also for $|z| < r(a)^{-1}$?

For $r > r(a)$, i.e. $|z| \leq r^{-1}$, we know $\sum_n a^n r^n$ converges for $r > r(a)$.
know $z \sum a^n z^n$ converges absolutely. In particular,

$$a^n z^n \to 0$$

Hence there exists M such that for $n \geq M$, we have

$$\|a^n r^{-n}\| \leq 1$$

This implies that

$$\|a^n\| \leq r^n \Rightarrow \|a^n\|^{1/n} \leq r$$

for all $n \geq M$.

This implies that

$$\limsup \|a^n\|^{1/n} \leq r$$

And note that r is arbitrary close to the spectral radius $r(a)$. Hence we have

$$\limsup \|a^n\|^{1/n} \leq r(a) \leq \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

□

For each $\varphi \in \mathcal{A}'$, consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n)\lambda^{-n}$$

This series converges for $r > r(a)$. We can apply the same process, to argue that there exists M_φ such that

$$\|\varphi(a^n)r^{-n}\| \leq M_\varphi$$

for all $n \geq 0$. Note that M_φ could be different for all φ.

Note that

$$\mathcal{A} \to \mathcal{A}' \to \mathcal{A}''$$

there is a natural injection of $a \mapsto \widehat{a} \in \mathcal{A}''$.

For each n, definite $F_n \in \mathcal{A}''$, by $F_n(\varphi) = |\varphi(a^n r^{-n})| \leq M_\varphi$. Applying the UBP, we have

$$|F_n(\varphi)| \leq M \Rightarrow |\varphi(a^n)r^{-n}| \leq M$$

This implies that

$$|\varphi(a^n)| \leq r^n M$$

Note that by Hahn-Banach, for any $b \in \mathcal{A}$, we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking n-th root of both sides, we gets

$$\|a^n\| \leq r^n M \Rightarrow \|a^n\|^{1/n} \leq rM^{1/n} \to r$$

Hence we again obtain the same result.

□

Recall UBP.

Theorem 1.4 (Uniform Boudnedness Principle)

Let X be Banach, and Y be normed, let $T_n : X \to Y$ be a family of linear operators, and if for all $x \in X$, we have

$$\|T_n(x)\| < \infty$$

Then for all n, we have

$$\|T_n\| < \infty$$

♡

Note that if $\mathcal{A}$ is unital, and if $\mathcal{A} \subset \mathcal{B}$ with some unit. For $a \in \mathcal{A}$, if a is not invertible in A, then it might be invertible in $\mathcal{B}$. Hence if we use $\sigma_{\mathcal{A}}(a)$ to denote the spectrum of a in $\mathcal{A}$.

Proposition 1.24

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

Example 1.6 Let $\mathcal{B} = l^1(\mathbb{Z})$, and let $\mathcal{A} = l^1(\mathbb{N})$, equipped with convolution.

Clearly $\mathcal{A} \subset \mathcal{B}$. And note that the delta function at 1, δ_1 is not invertible in $\mathcal{A}$ but it has an inverse δ_{-1} in $\mathcal{B}$. Hence we see $0 \in \sigma_{\mathcal{A}}(a)$, but $0 \notin \sigma_{\mathcal{B}}(a)$.

Proposition 1.25 (Spectral radius is preserved)

For $\mathcal{A} \subset \mathcal{B}$, we have

$$r_{\mathcal{A}}(a) = \lim \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

This proposition tells us that the spectral radius of an element $a \in \mathcal{A}$ is independent of the Banach algebra it is considered in, but rather only depends on itself.

Proposition 1.26

Let X be compact, and let $\mathcal{A} = C(X)$. Then for $f \in C(X)$, we have

$$\|f^2\|_\infty = \|f\|_\infty^2$$

Proof Look at where f takes $\|f\|_\infty$, and sqaure it, since when X is compact, you can actually obtain the point where $|f(x)| = \|f\|_\infty$.

Remark The same property holds for f in any unital subalgebra of $C(X)$, for example, if $X \subset \mathbb{C}$, and let $\mathcal{A}$=functions that are holomorphic on an open subset of $\mathbb{C}$ that are in X.

Proposition 1.27

Let $\mathcal{A}$ be a unital Banach algebra such that for $a \in \mathcal{A}$, we have

$$\|a^2\| = \|a\|^2$$

Then we have

$$r(a) = \|a\|$$

Proof If we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any n, we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking $1/2^n$-root of both sides, we get that the spectral radius of $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

□

Let $\mathcal{H}$ be a Hilbert space, over $\mathbb{C}$, and let $\mathcal{A} = B(H)$, i.e. the bounded linear operators on $\mathcal{H}$, and equip with the operation of taking adjoint. $T \mapsto T^*$.

Proposition 1.28

For any $T \in B(\mathcal{H})$, we have

$$\|T^*T\| = \|T\|^2$$

Proof We know that $\|T^*\| = \|T\|$. And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let $\xi \in \mathcal{H}$, then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi\rangle = \langle \xi, T^*T\xi\rangle \leq \|T^*T\|\|\xi\|^2$$

where the last inequality follows form Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2}\|\xi\|$$

which by definition ($\|T\|$ is the smallest constant for the inequality), gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result.

$\square$

Note *We used the inner product to justify $\|T\|^2 \leq \|T^*T\|$, which we cannot necessarily do in a non-Hilbert space.*

Corollary 1.4

If $T^* = T$, then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements.

Note that for general T, we have T^*T is always self-adjoint,

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the *-algebra structure.

1.8 Lecture 9

Let $\mathcal{H}$ be a hilbert space over $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ with $\|\cdot\|_\infty$, and closed under taking involutions. If $T \in \mathcal{B}(\mathcal{H})$, then

$$\|T^*T\| = \|T\|^2$$

So if $T^* = T$, then we have

$$\sigma(T) = \|T\|$$

Definition 1.11 (Concrete C^*-algebra)

A concrete C^*-algebra is a norm-closed sub-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, for some $\mathcal{H}$ such that is ***self-adjoint****, i.e., if $T \in \mathcal{A}$, then $T^* \in \mathcal{A}$. We call $\mathcal{A}$ is unital if $1_{\mathcal{H}} \in \mathcal{A}$.*

Corollary 1.5

If $\mathcal{A}$ is a C^* algebra, then for all $a \in \mathcal{A}$,

$$\|a^*a\| = \|a\|^2$$

If $\mathcal{A}$ is unital C^*-algebra, and if $a^* = a$, then $r(a) = \|a\|$.

This follows from our discussio above. Next we say a bit about the Gelfand transform.

Definition 1.12

Let $\mathcal{A}$ be a unital Banach algebra, and commutative, then we have the Gelfand transform $\Gamma : \mathcal{A} \to C(\widehat{A})$:

$$\Gamma(a)(\varphi) = \varphi(a)$$

then he said something of homomor to the complex numbers or something

Note that if $a \in \mathcal{A}$, and $\varphi \in \widehat{A}$, then $\varphi(a) \in \sigma(a)$, then we have

$$|\varphi(a)| \leq \|a\| = r(a)$$

then $\|\widehat{a}\|_\infty \leq r(a)$. Now we would like to show

$$\|\widehat{a}\|_\infty = r(a)$$

Theorem 1.5

$r(a)$ is the spectral radius of a, which is defined as $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$, and we have

$$r(a) = \|\widehat{a}\|_\infty$$

Note *This gives us a correspondance between φ and non-invertible elements of $\mathcal{A}$. This says if a is not invertible, then we can find some φ such that $\varphi(a) = 0$, i.e. a φ kills the non-invertible element.*

We will now dedicate the next 50 minutes of our life to proving this theorem.

Theorem 1.6 (φ and maximal ideal)

If $\lambda \in \sigma(a)$, then there exists $\varphi \in \widehat{A}$ such that $\varphi(a) = \lambda$. This is equiavalent to saying: if $(a - \lambda 1)$ is not invertible, then there is a $\varphi \in \widehat{\mathcal{A}}$, such that

$$\varphi(a - \lambda 1) = 0$$

This means if a is not invertible, then there exists $\varphi \in \widehat{\mathcal{A}}$ such that $\varphi(a) = 0$.

Proof Suppose $a \in \mathcal{A}$, and consider

$$a\mathcal{A} = \{ab : b \in \mathcal{A}\}$$

The set $a\mathcal{A}$ does not contain the identity element i.e. $1_\mathcal{A} \notin a\mathcal{A}$ (otherwise it would imply it has an inverse). And $a\mathcal{A}$ is a two-sided proper ideal (by $\mathcal{A}$ being commutative).

We now introduce a fact that we will use.

Definition 1.13

An ideal I is maximal if I is proper in R, and not contained in any bigger proper ideals.

Lemma 1.2

Let R be a unital ring commutatitve, every proper ideal is contained in a maximal ideal (by Zorn's lemma).

Lemma 1.3

For $\mathcal{A}$ unital commutative Banach algebra, if I is a proper ideal, then its closure is a proper ideal.

Proof We have seen that $GL(\mathcal{A})$, the set of invertible elements, is open. Hence its complement is closed. Any proper ideal does not contain any elements in $GL(\mathcal{A})$, hence its closure is closed inside a closed set.

Remark Let X be locally compact, but compact, such as $\mathbb{R}$, then we have $C_c(X) \subset C_\infty(X)$. Note that $C_c(X)$ is a proper ideal of $C_\infty(X)$, but it's dense in $C_\infty(X)$, hence its closure is the entire space, hence no longer proper. This tells us the closure of a proper ideal is not always proper, if $\mathcal{A}$ is not unital.

Theorem 1.7

Every maximal ideal of $\mathcal{A}$ is closed.

Proof The closure of any proper ideal is closed in unital algebras, hence its closure is itself.

First let V be a normed vector space. Let W be a closed subspace, and form the quotient space V/W. There is a natrual way to equip V/W with a norm

$$\|\dot{v}\| = \inf\{\|v - w\| : w \in W\}$$

i.e. the distance between v to W. This is a norm. Further if V is complete, so is V/W.

Let $\mathcal{A}$ be a normed algebra, commutative, unital. Let I be a closed ideal.

Proposition 1.29

For $a, b \in \mathcal{A}$, we have

$$\|\dot{a}\dot{b}\| = \|\dot{ab}\| \leq \|\dot{a}\|\|\dot{b}\|$$

so that $\mathcal{A}/I$ is a normed algebra.

Proof Let $c, d \in I$, and

$$(a - c)(b - d) = ab - (ad + cb - cd)$$

Note that $(ad + cb - cd) \in I$, hence

$$\|\dot{ab}\| \leq \|ab - (ad + cb - cd)\| = \|(a - c)(b - d)\| \leq \|(a - c)\|\|(b - d)\|$$

Taking infimum over all c, d, we get

$$\|(\dot{ab})\| \leq \|\dot{a}\|\|\dot{b}\|$$

□

Proposition 1.30

If $\mathcal{A}$ is a Banach algebra and if I is a closed ideal, then $\mathcal{A}/I$ is a Banach algebra for the norm $\|\dot{a}\|$ defined above.

Let $\mathcal{A}$ be a unital commutative Banach algebra over $\mathbb{C}$, let I be a maximal ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is a Banach algebra, if $\mathcal{A}/I$ has a proper ideal, then you can put this ideal back in $\mathcal{A}$ such that it contains I. By I already being ideal, this implies that $\mathcal{A}/I$ does not contain any proper ideals.

Now coming back. Let nonzero element in $\mathcal{A}/I$ is invertible. The Gelfand Mazur theorem tells us

$$\mathcal{A}/I \cong \mathbb{C}$$

Moreover, $\mathcal{A}/I = 1_{\mathcal{A}/I}\mathbb{C}$. Then the quotient map

$$\mathcal{A} \to \mathcal{A}/I \cong \mathbb{C}$$

is an element of $\mathcal{A}$, i.e. an algebra homomorphism φ, with the property $\varphi(I) = 0$.

If $a\mathcal{A} \subset I$, then for $y \in a\mathcal{A}$, we have

$$\varphi(y) = 0$$

And we are therefore finally done. We thus have

$$\|\widehat{a}\| = r(a)$$

□

Corollary 1.6

We have

$$\sigma(a) = Range(\widehat{a})$$

1.9 Lecture 10

Consider $C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$, and $C_\infty(\mathbb{R})$ is an ideal of $C_b(\mathbb{R})$.

Definition 1.14 (Abstract C^*-algebra)

An abstract C^*-algebra is a Banach algebra with an involution, such that

$$\|a^*a\| = \|a\|^2$$

Remark Zorn's lemma states that $C_\infty(\mathbb{R})$ is contained in a maximal ideal, in a commutative Banach algebras, maximal ideals give rise to bounded multiplcative linear functionals.

Remark There is ideals of $C_b(\mathbb{R})$ that are bigger than $C_\infty(\mathbb{R})$.

There exists linear functionals that are 0 on $C_\infty(\mathbb{R})$, but nonzero on $C_b(\mathbb{R})$, but such functional is not "constructable."

We also have $c_0 \subset l^\infty(\mathbb{N})$, where c_0 are sequences that converge to 0 at infinity. Again, nonzero linear functionals exist on $l^\infty(\mathbb{N})$, and is identically zero on c_0, but such is also not constructable.

Definition 1.15

$\prod_{j=1}^\infty \mathbb{Z}_5$= *the sequences of elements of* $\mathbb{Z}_5$. $\bigoplus \mathbb{Z}_5$ *all sequences that are 0 except for finitely many number of entries.*

Note that $\bigoplus \mathbb{Z}_5$ is an ideal of $\prod^\infty \mathbb{Z}_5$.

Proposition 1.31

$l^\infty(\mathbb{N})$ is not separable, $\prod^\infty \mathbb{Z}_5$ is not separable, nor is it finitely generated.

Proposition 1.32

If $\mathcal{A}$ is a unital commutative Banach algebra over $\mathbb{C}$, which is separable, and if I is a closed ideal, then one can construct a maximal ideal containing I by countable what

Proof Let $\{a_n\}$ be a countable subset of $\mathcal{A}$, whose linear span is dense.

Lemma 1.4

Note that $\mathcal{A}/I$ contains noninvertible elements if and only if I is not maximal.

If I is not maximal, then you can find the first a_n such that $a_n \in \mathbb{C}1_\mathcal{A}$ such that $a_n \notin \mathcal{A}/I$, then

$$\overline{a_n\mathcal{A}/I} \text{ is a proper ideal of } \mathcal{A}$$

hence it generates a proper ideal in $\mathcal{A}$, we denote it as I_1. If I_1 is not maximal, then repeat the process. By $\{a_n\}$ being countable, and that they are dense, we have a countable addition, which gives a maximal ideal containing I by countable inclusions.

□

Remark $C_\infty(\mathbb{R})$ is separable, and $C_b(\mathbb{R})$ is not separable.

Let's look at $L^\infty([0,1], m)$, where m denotes the Lebesgue measure. This is a C^*-algebra, commutaitve, unital.

Could you exhibit any linear functionals on L^∞

Note that $L^2([0,1], m)$, a Hilbert space, as an algebra on L^2, L^∞ is clsoed for the strong operator topology on $\mathcal{B}(\mathcal{H})$ by the seminorm,

$$T \in \mathcal{B}(\mathcal{H}), T \to \|t\xi\|, \text{ for } \xi \in \mathcal{H}$$

For example, figure 1. We say that $L^\infty([0,1])$ is a von Neumann algebra. Every commutative von Neumann algebra looks like some $L^2([0,1], \mu)$. But note that noncommutative ones are quite interesting.

For a commutative Banach algebra over $\mathbb{C}$, the Gelfand transform

$$a \mapsto \widehat{a}$$

We have

$$\|\widehat{a}\|_\infty = r(a)$$

Proposition 1.33 (Gelfand isometric condition)

If $\|a^2\| = \|a\|^2$, then the Gelfand transform is isometric. Thus we have

$$\|\widehat{a}\|_\infty = \|a\|$$

Definition 1.16 (Involution $*$)

For an involution on an algebra over $\mathbb{C}$, is a map $*$ from $\mathcal{A} \to \mathcal{A}$, with the properties

1. $(a^*)^* = a$
2. $(a+b)^* = a^* + b^*$
3. $(\alpha a)^* = \overline{\alpha} a^*$ *for* $\alpha \in \mathbb{C}$.
4. $(ab)^* = b^* a^*$

Definition 1.17 (Banach $*$ algebra)

If $\mathcal{A}$ has a norm, then we say it is a $*$ normed if

$$\|a^*\| = \|a\|$$

If $\mathcal{A}$ is complete, then it is called a Banach $*$ algebra.

Let G be a discrete group, let $\mathcal{H}$ be a Hilbert space, then $Aut(\mathcal{H}) = U(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ to itself. By a unitary representation of G on $\mathcal{H}$, we mean

Definition 1.18 (Unitary representation)

A unitary representation of G on $\mathcal{H}$ is a homomorphism $\pi : G \to U(\mathcal{H})$.

We note that $C_c(G) \subset l^1(G)$, and

$$\pi_f = \sum_{x \in G} f(x)\pi_x, \pi_f \pi_g = \pi_{f*g}$$

Now we ask what is $(\pi_f)^*$?

$$(\pi_f)^* = \sum \overline{f(x)}\pi_x^* = \sum_{x\in G} \overline{f(x)}\pi_{x^{-1}} = \sum \overline{f(x^{-1})}\pi_x$$

So we gets

$$(f^*)(x) = \overline{f(x^{-1})}$$

This defines an involution on $l^1(G)$. And it is easy to check that

$$\|f^*\|_1 = \|f\|_1$$

Remark The same process would not work for semingroups without the presence of x^{-1} necessarily.

We would like to think of this involution as some sort of complex conjugation.

Definition 1.19

Let $\mathcal{A}$ be a Banach $*$-algebra is symmetric if whenever $a \in \mathcal{A}$, and $a^* = a$, then

$$\sigma(a) \subset \mathbb{R}$$

If one looks at l^1 over noncommutative groups, some are symmetric, some are not.

Example 1.7 Let $\mathcal{A} = \mathbb{C} \bigoplus \mathbb{C}$, with $\|\cdot\|_\infty$. We define an involution:

$$(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$$

This is a well-defined involution. However, this is not symmetric under this involution.

Proposition 1.34

If G is commutative, then $l^1(G)$ is symmetric.

$$\widehat{\mathcal{A}} = \{ \text{ set of homomorphisms } G \to \mathbb{T} \}$$

This is symmetric.

Proposition 1.35

Let $\mathcal{A}$ be an abstract, unital C^*-algebra, then $\mathcal{A}$ is symmetric.

This is quite strong! (Every C^*-algebra is symmetric).

Lecture 11

Let $\mathcal{A}$ be a *-Banach algebra.

Definition 1.20 (symmetric $*$-algebra)

$\mathcal{A}$ is symmetric if for any $a \in \mathcal{A}$, we have

$$a^* = a$$

we have $\sigma(a) \in \mathbb{R}$.

Note that a C^* algebra is necessarily a *-algebra. Hence we have

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$$

And we apply it to a^*, hence we get

$$\|a^*\| \leq \|a\|$$

Hence the involution property is satisfied.

Proposition 1.36 (C^*-algebras are symmetric)

Let $\mathcal{A}$ be a unital C^*-algebra, i.e. we have

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

Then $\mathcal{A}$ is symmetric, i.e. $a^* = a$.

In the Gelfand Noimark paper (1943),

Proof [Arens Truck, 1946] Given $a \in \mathcal{A}$ with $a^* = a$, for any $t \in \mathbb{R}$, let $b = a + it$, we look at $b^*b = (a - it)(a + it) = a^2 + t^2$. So we have

$$\|b^*b\| \leq \|a^2\| + t^2$$

Now let $\lambda \in \sigma(a)$, with $\lambda = r + is$, we would like to show $s = 0$. Note we have $\lambda + it \in \sigma(b)$, this gives

$$\lambda + it = r + i(s + t) \in \sigma(b)$$

Then we have

$$|r + i(s + t)| \leq \|b\|$$

Hence

$$r^2 + (s + t)^2|r + i(s + t)|^2 \leq \|b^2\| = \|b\|^2 = \|b^*b\| \leq \|a^2\| + t^2$$

We thus have

$$r^2 + s^2 + 2st \leq \|a^2\|, \text{ for all } t$$

This gives $s = 0$.

□

Let's step back. Let $\mathcal{A}$ be a commutative symmetric Banach *-algebra. Then if $a \in \mathcal{A}$, and if $a^* = a$, so $\sigma(a) \subset \mathbb{R}$. But $\sigma(a) = Range(\widehat{a})$, so $\mathcal{A}$ is an $\mathbb{R}$-valued function on $\mathcal{A}$.

For any $a \in \mathcal{A}$, we have

$$a = \frac{a + a^*}{2} + i\frac{a - a^*}{2i} = a_r + ia_i$$

then $\widehat{a} = \widehat{a_r} + i\widehat{a_i}$, note $a^* = a_r - ia_i$, then we have

$$\widehat{a^*} = \widehat{a_r} - i\widehat{a_i} = \overline{\widehat{a}}$$

Thus, we have

$$\widehat{a^*} = \overline{\widehat{a}}$$

So $a \mapsto \widehat{a}$ is a *-algebra homomorphism of $\mathcal{A}$ into $C(\widehat{\mathcal{A}})$.

Definition 1.21 (separation of points by functions)

A collection of functions $\{f\}_j$ defined on X is said to separate points if for all $x, y \in X$, such that $x \neq y$, there exists f such that we have

$$f(x) \neq f(y)$$

Proposition 1.37

For any unital commutative Banach algebra $\mathcal{A}$, then Gelfand transform $a \mapsto \widehat{a}$, separates the points of $\widehat{\mathcal{A}}$.

Proof We prove the contrapositive, if we assume for all $\widehat{a}$, we have $\widehat{a}(\varphi) = \widehat{a}(\psi)$, then we would like to show $\varphi = \psi$. If $\varphi, \psi \in \widehat{\mathcal{A}}$, and $\widehat{a}(\varphi) = \widehat{a}(\psi)$, then

$$\varphi(a) = \psi(a), \text{ for all } a$$

Hence $\varphi = \psi$.

□

Proposition 1.38

*If $\mathcal{A}$ is a unital symmetric Banach *-algebra, then the image of Γ is dense in $C(\widehat{\mathcal{A}})$.*

Proof [Key ingredient: Stone-Weierstrass] $\{\Gamma(\varphi) : \varphi \in \widehat{\mathcal{A}}\}$ is a unital subalgebra of $C(\widehat{\mathcal{A}})$ that separates the points of $\mathcal{A}$, and is closed under taking complex conjugates, so Stone-Weierstrass theorem applies (a compact space and a unital subalgebra of continuous functions that separates the points of the space, and closed under complex conjugation, then this algebra is dense for the $\|\cdot\|_\infty$ norm).

□

Theorem 1.8 (Little Galfand-Naimark theorem)

Let $\mathcal{A}$ be a unital commutative C^*-algebra (abstract, which doesn't include hilbert space), then the Gelfand transform

$$\widehat{a}(\varphi) = \varphi(a)$$

*is an isometric *-isomorphism of $\mathcal{A}$ into $C(\widehat{\mathcal{A}})$, i.e. $\|\widehat{a}\| = \|a\|$.*

Proof Since $\mathcal{A}$ is symmetric, the range of the Gelfand transform is dense. We also saw that $\|a^{2^n}\| = \|a\|^{2^n}$, so the spectral radius of a, $r(a) = \|a\| = \|\Gamma(a)\|$.

Therefore $a \mapsto \Gamma(a)$ is isometric, and the range of $\Gamma(a)$ is norm-closed.

□

Remark In a commutative Banach algebra, we have $r(a) = \|\widehat{a}\|_\infty$, and have $r(ab) \leq r(a)r(b)$, hence $\|a^*a\| = \|a\|^2, r(a^*a) \leq r(a)^2$, we have

$$\|a\|^2 \leq r(a)^2 \leq \|a\|^2$$

For $T \in \mathcal{B}(\mathcal{H})$, and $T = T^*$, then

$$C^*(T, I) =\cong C(\sigma(T))$$

Proposition 1.39

Let G be a commutative group, then $l^1(G)$ with its $*$ is symmetric.

Proof Let $\mathcal{A} = l^1(G)$, then $\widehat{\mathcal{A}}$ is isomorphic to the homomorphisms of G into T.

If $\varphi \in \widehat{\mathcal{A}}$, then

$$\varphi(f) := \sum_{x \in G} f(x)\varphi(x), f \in l^1(G)$$

Then

$$\varphi(f^*) = \sum f^*(x)\varphi(x) = \sum \overline{f(x^{-1})}\varphi(x) = \sum \overline{f(x)}\varphi(x^{-1}) = \sum \overline{f(x)\varphi(x)} = \overline{\varphi(f)}$$

Note how we might have homomorphisms not mapping into $\mathbb{T}$ if G is not commutative.

□

Proposition 1.40

For G commutative, the range of the Gelfand transform Γ, which is the Fourier transform (for $f \in l^1(\mathbb{Z})$, we have $\widehat{f}(e^{i\theta}) = \sum f(n)e^{i\theta^n}$, note $\widehat{\mathbb{Z}} = \mathbb{T}$) in this setting, this is dense in $C(\widehat{G})$. However, this is not isometric, and the range is not norm-closed unless G is finite.

1.10 Lecture 12

Recall last time, we proved the Little-Galfand-Naimark theorem.

Theorem 1.9

Let $\mathcal{A}$ be a unital commutative C^*-algebra then we have

$$\mathcal{A} \cong C(\widehat{\mathcal{A}})$$

And $\widehat{\mathcal{A}}$ is comacpt.

Definition 1.22

Let C be a category, we think of objects as categories X, Y, with morphisms in between X, Y, the abstract dual of C, which is another category, has the same objects, but you reverse all the morphisms (arrows).

Next we introduce the important summary of the things we've been doing.

Theorem 1.10

*The category of unitial commutative C^*-algebras, with unital *-homomorphisms is a concrete realization of dual of the category of compact Hausdorff spaces.*

$$X \to y$$

$$C(X) \leftarrow C(Y)$$

X is normal? if $C(X)$ contains no proper projects, i.e. elemnets like P such that

$$P^2 = P = P^*$$

We now look at the following.

Proposition 1.41

Let $\mathcal{A}$ be a unital Banach algebra, and let $a_0 \in \mathcal{A}$, suppose $\mathcal{A}$ is generated by a_0, i.e.e the norm closure of all the polynomials in a_0, with the identity $1_{\mathcal{A}}$. Then the $\widehat{A}$ is homeomorphic to $\sigma(a_0)$, via $\varphi \in \widehat{\mathcal{A}}$,

$$\varphi \mapsto \varphi(a_0) \in \sigma(a_0)$$

Proof φ is entirely determined by $\varphi(a_0)$, hence the map above is one-to-one. note that for every element in $\lambda \in \sigma(a_0)$, there exist φ such that $\varphi(a_0) = \lambda$. Hence the map is also surjective.

Let $\mathcal{A}$ be a unital C^*-algebra, and let $a \in \mathcal{A}$ and $a^* = a$, then $C^*(a, 1_{\mathcal{A}})$ is commutative, and unital, and generated by a so that

$$\widehat{B} = \sigma_B(a)$$

such that $\mathcal{B} = C(\widehat{\mathcal{B}}) = C(\sigma(a))$, hence we have

$$C^*(a, 1) \cong C(\sigma(a)), a \mapsto \widehat{a}$$

The continuous functional calculus (dealt with operators on a Hilbert space). Still we assume $a^* = a$.

Given $f \in C(\sigma(a))$, then there exists a $b \in C^*(a, 1)$, such that

$$\widehat{b} = f$$

we denote b as $f(a)$, then

$$f \mapsto f(a)$$

is a *-homomorphism of $C(\sigma(a))$ onto $C^*(a, 1)$, and so into $\mathcal{A}$.

Corollary 1.7

If $T \in \mathcal{B}(\mathcal{H})$, and if $T^* = T$, then for any function $f \in C(\sigma(T))$, we can form $f(T)$. This is part of the spectral theorem. ♡

Definition 1.23

In any unital C^*-algebra, and $a \in \mathcal{A}$, we say that a is normal if $a^* \in C^*(a, 1)$, and a^* and a commute. ♣

We introduce the spectral permanents)

Theorem 1.11

Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{B}$ be a unital C^*-subalgebra of $\mathcal{A}$. Then for any $b \in \mathcal{B}$, we have

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

Note that we are not requiring commutativity for $\mathcal{A}$ or $\mathcal{B}$. ♡

Proof Note that we already have $\sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b)$ (an element not invertible in $\mathcal{B}$ may be invertible in $\mathcal{A}$). Then to show the other inclusion. We will first assume that $b^* = b$. If b has an inverse in $\mathcal{A}$, then it has a inverse in $\mathcal{B}$. So suppose c is an inverse in $\mathcal{A}$ for b, then

$$cb = 1_{\mathcal{A}} = bc$$

Now it suffices to consider $(b - \lambda 1)^{-1}$. Assume b is not invertible in $\mathcal{B}$ Consider the C^* algebra generated by $b, 1$, then $C^*(b, 1) := \mathcal{B}_1 \subset \mathcal{B}$. If b is not invertible in $\mathcal{B}$, then b is not invertible in $\mathcal{B}_1$, hence $\widehat{b}$ has no inverse in $C(\sigma_{\mathcal{B}}(\mathcal{H}))$: which has a criterion that it takes nonzero elements to 0.

Thus $\widehat{b}$ takes value 0 at some $\lambda \in \sigma_{\mathcal{B}}(b)$, and note that $\widehat{b}$ is continuous, there is an open neighborhood of λ_0 such that

$$|\widehat{b}(\lambda)| \leq \frac{1}{2\|c\|}$$

So by Urysohn's lemma, there is a $g \in C(\sigma(b))$ such that $supp(g) \subset O$, and $g = 0$ outside of O, with $\|g\|_\infty = 1$.

Then for $d := g(b)$, so for the Gelfand transform,

$$\widehat{g(b)} = g$$

we have $\|\widehat{b}g\|_\infty \leq \frac{1}{2\|c\|}$, then

$$\|db\| \leq \frac{1}{\|c\|}$$

then

$$1 = \|g\|_\infty = \|d\| = \|(cb)d\| \leq \|c\|\|bd\| \leq \|c\|\frac{1}{2\|c\|} = \frac{1}{2}$$

Hence we've reached a contradiction. Hence the self-adjoint case is done.

□

1.11 Lecture 13

Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{B}$ be a unital C^*-subalgebra (implying the same unit), and in previous lecture, we saw if $b \in \mathcal{B}$, and $b^* = b$, and if b has an inverse in $\mathcal{A}$, then it has an inverse in $\mathcal{B}$. This implies that

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

Proposition 1.42

Let $\mathcal{A}$ be a C^*-algebra, and $\mathcal{B}$ subalgebra, if b is invertible in $\mathcal{A}$, then b is invertible in $\mathcal{B}$.

Proof (We already know that this holds for $b^* = b$. For general $b \in \mathcal{B}$, (we no longer require $b^* = b$), then if b has an inverse in $\mathcal{A}$, then so does b^*, then b^*b has an inverse in $\mathcal{A}$, hence b^*b has an inverse in $\mathcal{B}$. Hence b has a left inverse $a(b^*b) = (b^*b)a = 1$, and bb^* is also invertible in $\mathcal{A}$, hence invertible in $\mathcal{B}$, hence b has a right inverse.

□

if we look at the shift operators on $l^2(\mathbb{N})$,

$$Se_n = e_{n+1} \tag{1.3}$$

Then the adjoint of this would be

$$S^*(e_n) = \begin{cases} e_{n-1}, n \geq 2 \\ 0, n = 1 \end{cases}$$

Then we have

$$S^*S = I_{\mathcal{H}}, SS^* = I - P_{e_1}$$

where P_{e_1} is the projection onto e_1.

Let $\mathcal{A} = l^1(\mathbb{N}_{\geq 0})$, and let $\mathcal{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Theorem 1.12

If $f \in l^1(\mathbb{N})$, and if $\hat{f} \in C(D)$, then nowhere takes value 0, then hat$f$ is invertible as a function in $C(D)$, and $\hat{f}$ has absolutely convergence power series, then $\frac{1}{\hat{f}}$ also has an absolutely convergence power series.

Note *If we take $\mathcal{A} = l^1(\mathbb{Z})$, so $\widehat{\mathcal{A}} = \mathbb{T}$, and if $\hat{f}$, which is the Fourier series, nowhere takes value 0 on $\mathbb{T}$, then the function $\frac{1}{\hat{f}}$ has absolutely convergent Fourier series. (This is much harder to prove).*

Let G be a group and let $(\mathcal{H}, U)$ be a unitary representation of G on $\mathcal{H}$, and let $U : G \to U(\mathcal{H})$, and $l^1(G)$ with convolution, for $f \in l^1(G)$, and for

$$U_f = \sum f(x)U_x \in \mathcal{B}(\mathcal{H})$$

with $\|U_f\| \leq \|f\|_{L^1}$. We then define

$$f^*(x) := \overline{f(x^{-1})}, U_f^* = (U^*)_f$$

We take $G = SL(n, \mathbb{Z})$, the $n \times n$ matrices with $\det(T) = 1, T \in SL(n, \mathbb{Z})$.

Problem 1.1 What are the unitary representations of G on $l^1(G)$

G acts on G by left translations α (actions), and acts on G/H (sets of cosets) for H any subgroups. Note that G/H are often called homogenous spaces. If G acts on a set M, consider $l^p(M)$, then G acts on isometries on $l^p(M)$, by

$$(\alpha)x\xi(y) = \xi(\alpha_x^{-1(y)})$$

$$X \to Y, C(X) \leftarrow C(Y)$$

In particular, this action on $l^2(M)$ is unitary.

Definition 1.24

The representaiton of G on $l^2(G)$ is called the left regular representation of G if we define

$$(U_x\xi)(y) = \xi(x^{-1}y)$$

U_f is said to have "integrated form" if U_f for $f \in l^1(G)$, $U_f = \sum_{x\in G} f(x)U_x$ (and we naturally replace with $U_f = \int f(x)U_x$ if our group is not discrete).

Definition 1.25 (Reduced C^*-algebra)

The operator norm closure of $\{U_f : f \in l^1(G)\} \in \mathcal{B}(l^1(G))$ is called the reduced C^*-algebra of G, denoted as $C_r^*(G)$.

Note *Again, the defining property of a C^*-algebra is $\|a^*a\| = \|a\|^2$*

In 1975, we see that $C_r^*(F_2)$ is simple, and has no proper ideals, note F_2 can be thought of the group generated by a, b, a^{-1}, b^{-1} with unit. Note that the trivial representation is not continuous for $\|\|_{C_r^*}$.

Definition 1.26 (amenable groups)

G is amenable if the integrated form of trivial representation if continuous for $\|\|_{C_r^*}$.

Remark This implies that the integrated form of all unitary representations of G are continuous for $\|\|_{C_r^*(G)}$. There are many equivalent properies of amenable using the geometric properties of G.

Note *All commutative groups are amenable.*

Definition 1.27 (faithful representations)

The left representation is faithful if whenever $U_f = 0$, then we have

$$f = 0, f \in l^1(G)$$

Proof $l^1(G)$, with convolution, with identity δ_e. note that $\delta_e \in l^2(G)$. If we consider $\delta_e \in l^2(G)$, and we look at

$$U_f\delta_e = f \in l^2(G)$$

Because we have the embedding $l^1 \subset l^2$.

Now assume that G is commutative, then we consider $l^1(G)$ acts on $l^2(G)$, and

$$C_r^*(G) = C^*(G) = C(\widehat{G})$$

note that we still have $\|f\|_{C_r^*(G)} \leq \|f\|_{L^1}$.

1.12 Lecture 13

Let G be commutative, and $(l^2(G), U)$ be the left regular representation. We have the interal form $U_f, hf \in l^1(G)$, and

$$U : l^1(G) \to \mathcal{B}(l^2(G))$$

and we let

$$C_r^*(G) = \{U_f : f \in l^1(G)\}^-$$

where we take the closure with respect to the operator norm.

For G commutative, we have $C_r^*(G)$ is a commutative C^*-algebra hence

$$\mathcal{A} := C_r^*(G) \cong C(\widehat{\mathcal{A}})$$

We've shown last time, that U is injective from $l^1(G)$ to $C_r^*(G)$. Each $\varphi \in \widehat{\mathcal{A}}$ a multiplicative linear functional, we have

$$f \mapsto \varphi(\pi_f) \in (l^1(G))\widehat{}$$

For any $f \in l^1(G)$, and $f \neq 0$, we have $U_f = 0$, and the map is injective, hence $\pi_f = 0$, we have, there exists φ such that $\varphi(\pi_f) \neq 0$.

Corollary 1.8 (largeness of the dual group)

$\widehat{G}$ is big enough given that $f, g \in l^1(G)$, if $f \neq g$, then there exists $\varphi \in \widehat{G}$ succh that

$$\widehat{f}(\varphi) \neq \widehat{g}(\varphi)$$

In other words $\varphi(f) \neq \varphi(g)$.

We now state the big Galfand-Naimark theorem.

Theorem 1.13 (Big Gelfand-Naimark)

*Let $\mathcal{A}$ be an abstract C^*-algebra, e.g. a Banach *-algebra such that*

$$\|a^* a\| = \|a\|^2, \forall a \in \mathcal{A}$$

*Then there exists a *-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ which is isometric, i.e.*

$$\|\pi(a)\| = \|a\|$$

And

$$\mathcal{A} \cong \{\pi(a) : a \in \mathcal{A}\}$$

Note *In little Galfand-Naimark, $C(\widehat{\mathcal{A}})$ is explicitly determined, but for big, it is not determined.*

For X a compact space, $C(X)$ is a C^*-algebra.

Consider X_d, which is discrete, and take $l^1(X_d)$ with the counting measure, is called the atomic representation.

Take any Borel measure μ on X, we have

$$L^2(X, \mu)$$

If μ has full support, then map of $C(X)$ is isometric.

Let $\mathcal{A}$ be a *-algebra, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a *-representation. Let's take any $\xi \in \mathcal{H}$, and $\xi \neq 0$, define

$$\varphi_\xi(\mathcal{A}) = \langle \pi(a)\xi, \xi \rangle$$

then we look at

$$\varphi(a^* a) = \langle \pi(a^* a)\xi, \xi \rangle = \langle \pi(a)\xi, \pi(a)\xi \rangle \geq 0$$

Note *For $f \in C(X)$, we always have $f^* f \geq 0$.*

Definition 1.28 (Positive linear functionals)

If φ is a linear functional on $C(X)$, such that for all f,

$$\varphi(f^* f) \geq 0$$

Remark They are always continuous. And they give rise to a measure μ_φ on X.

You think of $a^* a$ as being "positive."

Proposition 1.43

If $\mathcal{A}$ is a C^*-algebra, if $a, b \in \mathcal{A}$, there exists c such that

$$a^* a + b^* b = c^* c$$

Definition 1.29 (positive linear functionals)

*It $\mathcal{A}$ is a *-algebra, and if φ is a linear functional on $\mathcal{A}$, if for all $a \in \mathcal{A}$, we have*

$$\varphi(a^* a) \geq 0$$

then we call φ ***positive****.*

Proposition 1.44

If φ, ψ are positive, then $r\varphi + s\psi$, with $r, s \in \mathbb{R}^+$, then the positive linear functionals form a ***cone****.*

Let $\mathcal{A}$ be a *-algebra, and let φ be a positive linear functional on $\mathcal{A}$, then define a pre-inner product on $\mathcal{A}$ by

$$\langle a, b\rangle_\varphi = \varphi(b^* a)$$

Then we have

$$\langle a, a\rangle_\varphi = \varphi(a^* a) \geq 0$$

Exercise 1.1 finish this We have $\overline{\langle a, b\rangle_\varphi} = \langle b, a\rangle_\varphi$

Let $\eta_\varphi = \{a \in \mathcal{A} : \langle a, a\rangle_\varphi = 0\}$, then if $b \in \mathcal{A}, a \in \eta_\varphi$, then we have

$$|\langle ba, ba\rangle| = |\langle b^* ba, a\rangle| \leq \langle a, a\rangle^{1/2} = 0$$

So if η_φ is an ideal of $\mathcal{A}$, so form $\mathcal{A}/\eta_\varphi$, then

$$\langle , \rangle_\varphi$$

drops to an inner product on $\mathcal{A}/\eta_\varphi$, denote its complement by $L^2(a, \varphi)$, if we are given $c \in \mathcal{A}$, let π be the left regular representation of $\mathcal{A}$ on $\mathcal{A}$ via

$$\pi_c a = ca$$

then we have

$$\langle \pi_c a, b\rangle_\varphi = \langle ca, b\rangle = \varphi(b^* ca) = \varphi((c^* b)^* a) = \langle a, c^* b\rangle = \langle a, \pi(c^*)\rangle b$$

Then the left regular representation "is" a *-representation, this drops to a *-representation on $\mathcal{A}/\eta_\varphi$.

Next time: we show π_c is not continuous, and we will use polynomials. Now if you assume $\mathcal{A}$ is Banach *-algebra, then this π_c are always bounded. **GNS representation**, where S is Siegel.