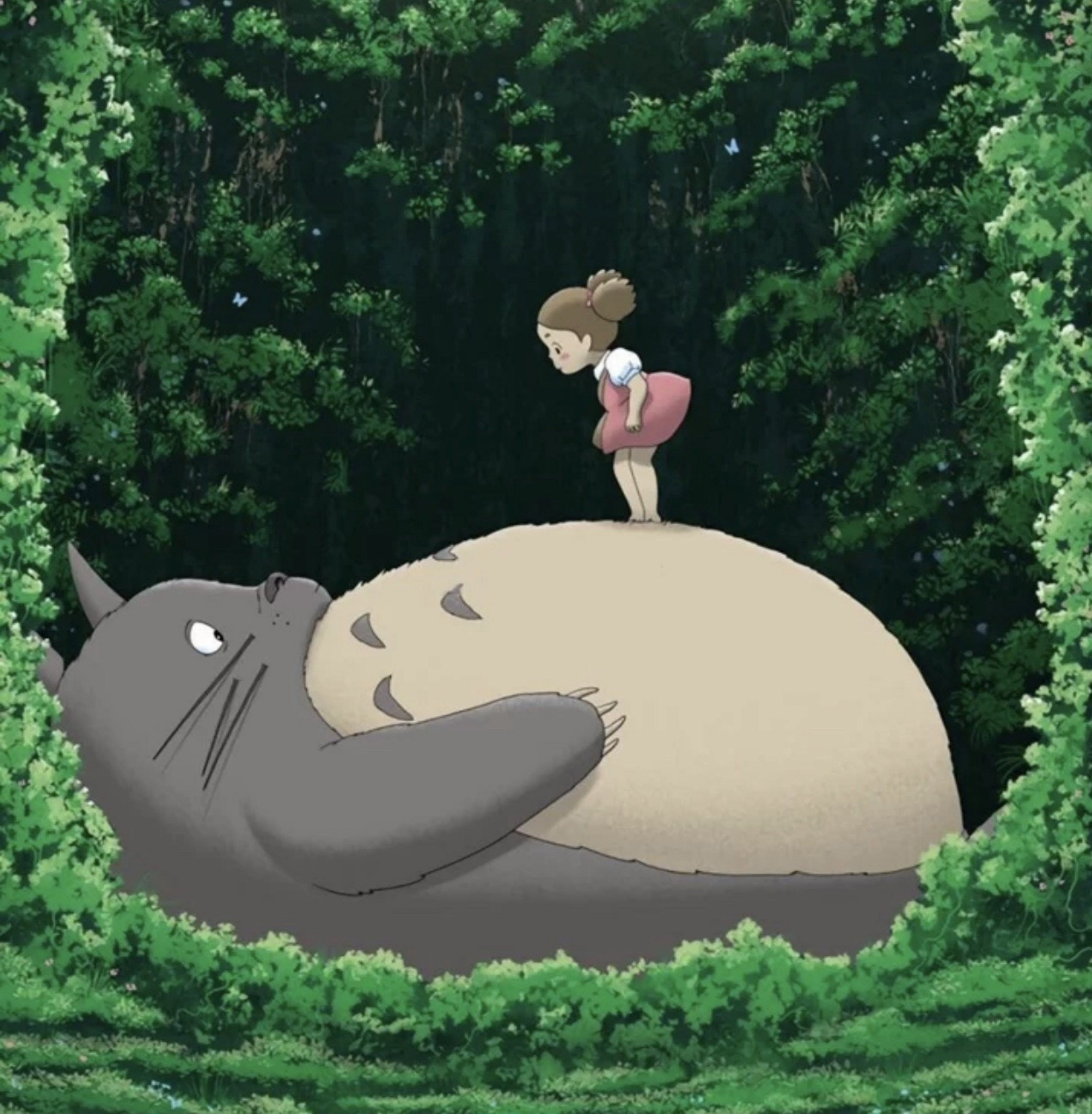

Fourier Analysis

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Date: August 26, 2023

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Chapter 1 Fourier Series and Integrals

We will go through the book's notes in this document. Chapter 1 is organized as follows:

1. Fourier coefficients and Series
2. Criteria for pointwise convergence
3. Convergence in norm
4. summability methods
5. The fourier transform of L^1 functions
6. Schwartz class and tempered distributions
7. Fourier transform on L^p, for $1 < p \leq 2$
8. Convergence and summability of Fourier Integrals
9. Further results

Some Notations The Lebesgue measure in $\mathbb{R}^n$ will be denoted using dx, and on the unit sphere S^{n-1} will be $d\sigma$. Let $a = (a_1, ..., a_n) \in \mathbb{N}^n$ be a multiindex, and $f : \mathbb{R}^n \to \mathbb{C}$, then

$$D^a f = \frac{\partial^{|a|} f}{\partial_{x_1}^{a_1} ... \partial_{x_n}^{a_n}}$$

where $|a| = a_1 + ... + a_n$.

Theorem 1.1 (Minkowski's integral inequality.)

Given $(X, \mu), (Y, \nu)$ as σ-finite measure spaces, we have the following inequality

$$\left(\int_X \left| \int_Y f(x,y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y)$$

♡

Taking ν to be the counting measure over a two point set $S = 1, 2$ gives the usual Minkowski inequality

$$\|f_1 + f_2\|_{L^p} \leq \|f_!\|_{L^p} + \|f_2\|_{L^p}$$

We will use $\mathcal{D}$ to denote the space of test functions, i.e. C_c^∞, and $\mathcal{S}$ to denote the space of Schwartz functions. Recall the dual of $\mathcal{D}$, denoted as $\mathcal{D}'$ is the space of distrubitons, and $\mathcal{S}'$ is the space of temperate distributions.

Definition 1.1 (Convolution of distribution)

Let $T \in \mathcal{D}'$, and $f \in \mathcal{D}$, then we define

$$T * f(x) = \langle T, \tau_x \tilde{f} \rangle$$

where $\tilde{f}(y) = f(-y)$, and $\tau_x f(y) = f(x + y)$. Hence it can be read as $T * f(x) = \langle T, f(x - y) \rangle$

♣

1.0.1 1.1 Review of definitions

We now do some math. If f is a trigonometric series, of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

Then we find c_n for any fixed n by multiplying $f(x)$ by $e^{-2\pi i n x}$, and integrate. Namely, we have

$$\int_0^1 \sum_n c_n e^{2\pi i n x} \cdot e^{-2\pi i n x} = \int_0^1 c_n = c_n$$

We denote the additive group of $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$, which gives $[0, 1)$, and naturally identifies with S^1. Hence, saying a function f defined on $\mathbb{T}$ is the same as saying f is defined on $\mathbb{R}$ with period 1.

Definition 1.2 (Fourier coefficients)

Fix $f \in L^1(\mathbb{T})$, we associate the sequence $\{\hat{f}(n)\}$ of f defined by

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi inx}dx$$

And its Fourier series defined as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

♣

1.0.2 1.2