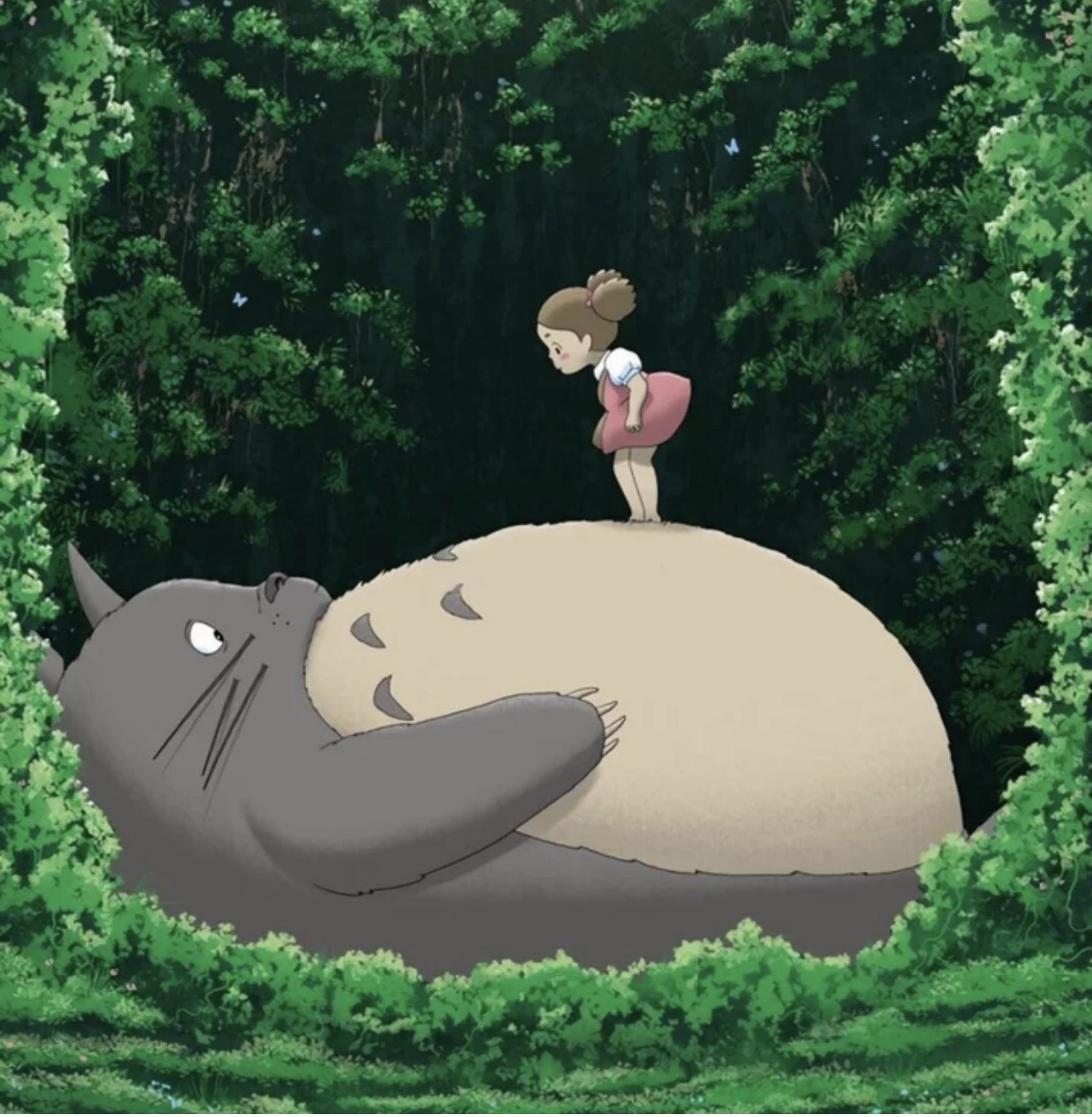

Harmonic Analysis

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Contents

Chapter 1 Preface

1.1 Lecture 1

Here we go.

1.1.1 Logistics

OH: Wednesday 1-2pm virtual, 2-4pm Evans 813.

Textbook: Fourier Analysis by J. Duoandikoetxea (plan to cover chapter 1-6, and sections 1-4 of chapter 8); other texts: *Introduction to Fourier Analysis on Euclidean Spaces* by Stein and Weiss, *Singular Integrals and Differential Properties of Functions* by Stein, *Harmonic Analysis* by Stein.

Topics: Fourier series, Fourier transform, maximal functions, Hilbert transform, singular integrals, Littlewood-Paley theorem, multipliers, oscillatory integrals

Grading: The grading will be entirely dependent on 3 problem sets given throughout the semester (with an ample amount of optional problems).

1.1.2 Course Overview

We will begin by defining what "Harmonic" means in the context of Math 258: to us, this word harmonic refers to "Euclidean Fourier analysis." And more specifically, we will study Fourier analysis on the n-dimensional torus, in $\mathbb{R}^n$. One justification for studying on/in these spaces is that many are equipped with translation invariance, which among other things, gives us nice behaving eigenfunctions.

Consider the function $e(x) := e^{2\pi inx}$, and consider the translation operator $f_t(x) = f(x+t)$, we have

$$f_t(e(x)) = e^{2\pi in(x+t)} = e^{2\pi int} \cdot e^{2\pi inx}$$

Here, $e^{2\pi inx}$ can be seen as an eigenfunction of translations. Another obvious example is differentiation. Consider the differentiation operator on $e(x)$, we have

$$\partial_x(e(x)) = 2\pi in(e^{2\pi inx})$$

Again, $e^{2\pi inx}$ is an eigenfunction. In the forseeable future, we will see $\{e^{2\pi inx}\}_{n\in\mathbb{N}}$ forms a basis of funcitons on the 1-dim torus, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ i.e. functions on the circle. Likewise, we have $\{e^{2\pi i\sum n_i x_i}\}_{n_i\in\mathbb{Z}}$ as the basis of functions in the n-dim torus, defined as $\mathbb{T}^n = (\mathbb{R}/\mathbb{Z})^n$. They have the nice properties of diagonalizing translation, differentiation operators as they are eigenfunctions. Similarly, we have $\{e^{2\pi i\sum n_i x_i}\}_{n_i\in\mathbb{Z}}$ for $\mathbb{R}^n$, and we say they are "almost in L^2," or L^2-wannabes as they are not far from L^2, but not quite in $L^2(\mathbb{R}^n)$.

Remark This property gives them the importance of in studying differential operators with constant coefficients

We will go through various technical things along the way, one of them being "cancellation." In the most general sense, using triangle inequality for everything is quite of a waste, for example, for highly oscillatory functions. We would like to exploit whenever we can, such as the oscillations of functions, kernels of operators, etc. More importantly, we will use different methods for different parts, to treat different issues. In other words, one should go to the dentist when they broke their ankle.

We will first study the question when do partial sums of a Fourier series (of functions on the cirlce) or Fourier transform of functions in the Euclidean space converge, and converge in what sense. Convergence usually has two "senses:" pointwise convergence and L^p norm convergence. We will study both.

Chapter 2 Fourier Series and Fourier transform

2.1 Lecture 2

More logistics: OH's have been updated as follows: Wednesday 10:30am-11:30am, 2-3pm.

We now begin Chapter 1 of our text.

Recall we define the 1-d torus as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and the functions on the torus are naturally identified with the functions on the unit interval. Fourier analysis began when Fourier asked the following question: given a function f on the circle, can we find a_k, b_k such that the following is true:

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(2\pi kx) + b_k \sin(2\pi kx)$$

The modern Fourier analysis asks the following, can we find c_k such the following is true:

$$f(x) = \sum_{k=0}^{\infty} c_k e^{2\pi ikx}$$

The above two questions are identical if we take $a_k = c_k, b_k = ic_k$.

One intuitioin for having $2\pi k$ in the $\cos, \sin$ is that we would like to have periodic functions with period 1 to approximate f. Now we introduce the Fourier coefficients, and we first motivate this using trigonometric polynomials of the form

$$f(x) = \sum_{k=0}^{N} c_k e^{2\pi ikx}$$

We only know f is a finite sum of $e^{2\pi ikx}$, yet we would like to know the c_k's. And we do this by exploiting the orthogonality of $\{e^{2\pi ikx}\}$. Notice we have the following:

$$\int_0^1 e^{2\pi ik_1x}\overline{e^{2\pi ik_2x}}dx = \begin{cases} 1, k_1 = k_2 \\ 0, k_1 \neq k_2 \end{cases}$$

We therefore have, for any fixed k,

$$\int_0^1 f(x)e^{-2\pi ikx}dx = \int_0^1 \left(\sum_{k=0}^{n} c_k e^{2\pi ikx}\right) e^{-2\pi ikx}dx = c_k$$

Definition 2.1 (Fourier coefficients, Fourier series)

Given $f \in L^1(\mathbb{T})$, or any periodic function on $\mathbb{R}$ with period 1 that is locally integrable, we define its k-th Fourier coefficient as follows:

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi ixk}dx$$

Given the Fourier coefficients, we write f's Fourier series as follows:

$$f(x) \sim \sum_{k\in\mathbb{Z}} \hat{f}(k)e^{2\pi ikx}$$

♣

For trigonometric polynomials, as we saw, its Fourier series agrees exactly with itself, and its Fourier series will only have finitely many terms. However, more often than not, arbitrary f's Fourier series will have infinitely many terms, and now we go back to the question: when will f's Fourier series converge and converge in what sense.

We define partial sums of Fourier series as $S_N(f)(x) = \sum_{|k|\leq N} \hat{f}e^{2\pi ikx}$, and pointwise convergence asks the question: for fixed x, when do we have

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

We now introduce two theorems on pointwise convergence (that will be proved in the next lecture).

Theorem 2.1 (Dini's criterion)

Fix $x \in \mathbb{T}$, if we have

$$\int_{|t|<\delta} \left| \frac{f(x+t)f(x)}{t} \right| < \infty$$

Then we have

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

♡

For example, Lipshitz functions would have their Fourier series pointwisely converge. More generally, if a function blows up around a point only slightly, then we would have its Fourier series converge to it at that point.

Theorem 2.2 (Jordan's criterion)

Fix $x \in \mathbb{T}$, and some $\delta > 0$, if f is of bounded variation in $(x-\delta, x+\delta)$, then we have

$$\lim_{N\to\infty} S_N(f)(x) = f(x)$$

♡

Again, we delay the proof till next time.

To study pointwise convergence, we first note that $S_N(f)$ is a convolution operator.

$$\begin{aligned} S_N(f)(x) &= \sum_{|k|\leq N} \hat{f}(k)e^{2\pi ikx} \\ &= \sum_k \left(\int_0^1 f(t)e^{-2\pi ikt}dt \right) e^{2\pi ikx} \\ &= \int_0^1 f(t) \sum_k e^{2\pi ik(x-t)}dt \\ &= f(x) * D_N(x) \end{aligned}$$

where $D_N(x) = \sum_{|k|\leq N} e^{2\pi ikx}$, and this kernel is called the Dirichlet kernel.

Remark To study the pointwise convergence of functions on the circle, it suffices to study the point $x = 0$, and by translation invariance, we have the same conclusion hold for $x = x_0$.

The convolution can be thought of as "redistribution of mass," and we show the total mass of the Dirichlet kernel is 1.

$$\int_0^1 D_N(t)dt = \int_0^1 e^{2\pi ikt}dt = 1$$

We now introduce a simple expression for the Dirichlet's kernel.

Proposition 2.1 (Dirichelt kernel)

We have

$$D_N(t) = \frac{\sin(2N+1)\pi t}{\sin(\pi t)}$$

♠

Proof $D_N(t) = \sum_{k=-N}^{N} e^{2\pi ikt}$ is a geometric series, with the ratio$= e^{-2\pi it}$, hence by the formula of partial sums, we have

$$LHS = \frac{e^{2\pi i2Nt}(1-e^{-2\pi it(2N+1)})}{1-e^{-2\pi it}}$$

Now we examine the RHS. $\sin(t) = (e^{it} - e^{-it})/2i$, hence we have

$$RHS = \frac{e^{(2N+1)\pi it} - e^{-(2N+1)\pi it}}{e^{i\pi t} - e^{-i\pi t}}$$

Dividing top and bottom by $e^{i\pi t}$ gives us the LHS.

□

Can we comment on the bound of D_N? If one draws out a picture, then it is clear that D_N have a blow-up at $t = 0$, and $|D_N(t)| \to 2N+1$ as $t \to 0$, one could also see this using the expression above. Also we have the following:

$$|D_N(t)| \leq \frac{1}{\sin(\pi t)}, 0 < t \leq \frac{1}{2}$$

To prove the above two theorems, we introduce some tools.

Lemma 2.1 (Riemann-Lebesgue Lemma)

If $f \in L^1(\mathbb{T})$, then as $|k| \to \infty$, the Fourier coefficent tends to 0, i.e. we have

$$\lim_{|k|\to\infty} |\hat{f}(k)| = \lim_{|k|\to\infty} \left| \int_0^1 f(x) e^{-2\pi i k x} \right| = 0$$

♡

Proof For $f \in L^1(\mathbb{T})$, there exists g, h such that $f = g + h$, where g is a simple function of the form $g = \sum_{i=1}^n c_i \chi_{E_i}$, and $\|h\|_{L^1} < \epsilon$.

For g simple function, the Fourier coefficent decays with rate $O(1/k)$ by integration by parts.

$$|\hat{g}(k)| = \left| \int_0^1 \sum_i c_i \chi_{E_i} e^{-2\pi i k x} \right| \leq \sum |c_i| \chi_{E_i} \left| \int_0^1 e^{-2\pi i k x} \right| = \sum |c_i| \chi_{E_i} \frac{1}{2\pi i k} \left| \int_0^1 x e^{-2\pi i k x} \right| \lesssim O(1/|k|)$$

For $\hat{h}(k)$, we have,

$$\left| \hat{h}(k) \right| = \left| \int_0^1 h(x) e^{-2\pi i k x} \right| \leq \int_0^1 |h(x)| dx < \epsilon$$

Hence we have $|\hat{f}(k)| \leq O(1/k)$, and as $|k| \to \infty$, we have $\hat{f}(k) \to 0$ if $f \in L^1(\mathbb{T})$.

□

Next we a result that guarantees pointwise convergence.

Theorem 2.3 (Riemann Localization principle)

If $f = 0$ in a neighborhood of x, say, $(x - \delta, x + \delta)$, then we have

$$\lim_{N\to\infty} S_N(f)(x) = 0 = f(x)$$

♡

Proof If we write $S_N(f)$ as $D_N * f$, where $D_N(t) = \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}$, then we see the badness of the denominator is avoided around x by $f = 0$ around x, and if we are far away from x, then nice bound ensues. We now do explicit computation.

$$\begin{aligned}
S_N(f) &= \int_0^1 D_N(t) f(x-t) dt \\
&\int_{\delta \leq t \leq 1} \frac{\sin((2N+1)\pi t)}{\sin(\pi t)} f(x-t) dt \\
&= \int_{\delta \leq t \leq 1} \frac{f(x-t)}{\sin(\pi t)} \frac{e^{i(2N+1)\pi t} - e^{i(2N+1)\pi t}}{2i} dt \\
&= \frac{1}{2i} \widehat{\frac{f(x-t)}{\sin(\pi t)} e^{i\pi t}}(-N) + \widehat{\frac{f(x-t)}{\sin(\pi t)} e^{-i\pi t}}(N)
\end{aligned}$$

And we note that both $\frac{f(x-t)}{\sin(\pi t)} e^{i\pi t}$ and $\frac{f(x-t)}{\sin(\pi t)} e^{-i\pi t}$ are in $L^1(\mathbb{T})$. Hence by the Riemann-Lebesgue lemma, their Fourier coefficients tend to 0 as $N \to \infty$.

□

2.2 Lecture 3

Recall Riemann localization principle.

Remark Given the localization principle, if we'd like to examine whether Fourier series converges at a particular point, we can assume $f = 0$ outside a small neighborhood of that point.

We now will prove Dini's criterion.

Theorem 2.4 (Dini's Criterion)

If $x \in \mathbb{T}$, $\delta > 0$ such that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| < \infty$$

Then the partial sum converges at x.

Proof Assume $\delta < \frac{1}{2}$, and we would like to show $S_N(f)(x) - f(x) \to 0$.

$$\begin{aligned} S_N(f(x) - f(x)) &= \int_{|t|\le 1/2} (f(x-t) - f(x)) D_N(t) dt \\ &= \int_{|t|\le\delta} (f(x-t) - f(x)) D_N(t) dt + \int_{\delta\le|t|\le 1/2} (f(x-t) - f(x)) D_N(t) dt \end{aligned}$$

We notice that

$$\left| \int_{|t|\le\delta} (f(x-t) - f(x)) D_N(t) dt \right| \le \int_{|t|\le\delta} \frac{|f(x-t) - f(x)|}{|\sin(t)|} < \infty$$

By the condition that the integral is ∞, we can choose a even smaller $\delta_1 < \delta$ such that $\int_{|t|<\delta} < \epsilon$. Finishing the proof.

□

Remark For Holder continuous functions, they all satisfy the the assumptions of Dini. Note, a function being continuosu does not guarantee that the partial sum of Fourier series would converge.

Now we begin Jordan's Criterion.

Theorem 2.5 (Jordan's Criterion)

If $x \in \mathbb{T}$, and f is of bounded variation in $(x - \delta, x + \delta)$, then we have

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}(f(x+) + f(x-))$$

Remark We don't expect f to be continuous or even defined at $f(x)$.

Proof WLOG, we assume $\delta < 1/2$, and by the Riemann localization theorem, we assume $f = 0$ outside of $(x-\delta, x+\delta)$. (What happens outside of a neighborhood of δ does not matter).

Using the symmetry of the kernel. We have

$$S_N f(x)(x) = \int_{|t|\le 1/2} f(x-t) D_N(t) dt = \int_0^{1/2} (f(x+t) + f(x-t)) D_N(t) dt$$

Let $g(t) = f(x+t) + f(x-t)$, and we would want to show that $\int_0^{1/2} g(t) D_N(t) dt \to \frac{1}{2} g(0)$.

We can also assume g is monotonic. And we want to show that $\int_{|t|\le 1/2} g(t) D_N(t) dt$ tends to $\frac{1}{2} g(0)$. Again we separate the integral into two parts and the second part is 0, and we use the second mean value theorem of definite integrals.

Lemma 2.2

For f continuous and g monotonic, we have

$$\int_a^b fg = g(a+) \int_a^c f(t) dt + g(b-) \int_c^b f(t) dt$$

We would want to show that $g(\delta-) \int_\nu^\delta f(t) dt$ tends to 0.

$$\begin{aligned} \left| \int_\nu^\delta D_N(t) dt \right| &= \left| \int_\nu^\delta \frac{\sin((2N+1)\pi t)}{\sin(\pi t)} \right| \\ &\le \left| \int \sin((2N+1)\pi t)(1/\sin(\pi t) - 1/\sin(t)) \right| + \left| \int \sin((2N+1)\pi t) \frac{1}{t} \right| \end{aligned}$$

The first term is bounded by a constant, and the second term, by integration by parts, is also bounded.

Remark If we take δ_1 arbitrarily small, then we get that the integral is arbitrarily small.

Hence we can take δ to be arbitrarily small, to finish the proof.

□

2.3 Lecture 4

We will now prove continuity is too rough of a condition to ensure Fourier series converges to f. Namely, we will show the existence of continuous functions whose Fourier series diverges at a particular point.

Theorem 2.6

There exists a continuous function whose ourier series diverges at a point.

Proof WLOG, we take this point to be $x = 0$. We shall use UBP. Let $X = C(\mathbb{T}), Y = \mathbb{C}$, and equip X with the $\|\cdot\|_\infty$ norm. Hence X is a Banach space.

Then we define T_N on X as follows:

$$T_N(f) = S_N f(0) = \int_{|t|\leq 1/2} f(t)D(t)dt$$

Hence it suffices to show $\|T_N\| = \infty$ as $N \to \infty$. We define a new quantity L_N as follows:

$$L_N = \int_{|t|\leq 1/2} |D_N(t)|dt$$

We would like to show that $\|T_N\| = L_N$, and then prove L_N tends to ∞ as $N \to \infty$. We first note that we have

$$|T_N(f)| \leq \int |f(t)||D_N(t)|dt \leq \|f\|_\infty L_N$$

Then for the reverse direction, we note that $|D_N(t)| = D_N(t) \cdot sgn(D_N)(t)$, and by noting that $D_N(t)$ has only finitely many zeros, we can modify $sgn(D_N)$ into a function f with $\|f\|_\infty$, and $f \in C(\mathbb{T})$ and also that $|T_N(f)| \geq L_N - \epsilon$. Then we get

$$\|T_N\| \geq L_N - \epsilon$$

Hence $\|T_N\| = L_N$. We next show, via computation, that $L_N \to \infty$ as $N \to \infty$.

Lemma 2.3

Define $L_N = \int_{-1/2}^{1/2} |D_N(t)|dt$, then we have

$$L_N = \frac{4}{\pi^2}\log(N) + O(1)$$

Proof

$$\begin{aligned}
L_N &= 2\int_0^{1/2} \left|\frac{\sin((2N+1)\pi t)}{\sin \pi t}\right| dt \\
&= 2\int_0^{1/2} \left|\frac{\sin((2N+1)\pi t)}{\pi t}\right| dt + O(1) \\
&= 2\int_0^{N+1/2} \left|\frac{\sin(\pi t)}{\pi t}\right| dt + O(1) \\
&= \frac{2}{\pi}\sum_{k=0}^{N-1}\int_k^{k+1} \left|\frac{\sin(\pi t)}{\pi t}\right| dt + O(1) \\
&= \frac{2}{\pi}\sum_{k=0}^{N-1}\int_k^{k+1} \left|\frac{\sin(\pi t)}{k+1}\right| dt + O(1) \\
&= \frac{2}{\pi}\log(N) \cdot \frac{2}{\pi} + O(1)
\end{aligned}$$

□

Now we've shown $\|T_N\| \to \infty$ as $N \to \infty$, then by UBP, there exists $f \in C(\mathbb{T})$ such that $|T_N(f)| \to \infty$, the Fourier series diverges.

$\square$

Now we would like to address the next two questions on convergence in L^p norm. Namely, do we necessarily have if $f \in L^p, 1 \le p < \infty$, then

1. Do we have

$$\lim_{N\to\infty} \|S_N(f) - f\|_{L^p} = 0$$

2. Do we have

$$\lim_{N\to\infty} S_N(f)(x) = f(x), \text{ for almost every } x$$

We will address Q1.

Lemma 2.4

Let $1 \le p < \infty$, for all $f \in L^p$, we have $\lim_{N\to\infty} \|S_N(f) - f\|_{L^p} = 0$ if and only if there exists a constant C_p such that

$$\|S_N(f)\|_{L^p} \le C_p \|f\|_{L^p}$$

Proof $(\Rightarrow)$ This follows from UBP. If $\|S_N\|$ is unbounded, then by UBP, there exists an f such that $\|S_N(f)\|_{L^p} \to \infty$. Hence would not converge to f in the L^p sense.

$(\Leftarrow)$ We note that trigonometric polynomials are dense in L^p, hence we can find g a trig polynomial such that $\|f - g\|_{L^p} < \epsilon$. Hence we have,

$$\|S_N(f) - f\|_{L^p} \le \|S_N(f) - S_N(g)\|_{L^p} + \|S_N(g) - g\|_{L^p} + \|f - g\|_{L^p}$$

We note the first term is

$$\|S_N(f-g)\|_{L^p} \lesssim \|f - g\|_{L^p} < \epsilon$$

Hence we get convergence in the L^p norm.

$\square$

We note that when $1 < p < \infty$, we indeed have

$$\|S_N(f)\|_{L^p} \le C_p \|f\|_{L^p}$$

but for $p = 1$, we have $\|S_N(f)\|_{L^1} = L_N$ as above, hence there is no convergence for $S_N(f)$ to f in the L^p sense. We will reformulate this theorem as follows.

Theorem 2.7 (Convergence of $S_N(f)$ for $1 < p < \infty$)

If $1 < p < infty$, for all $f \in L^p$, we have

$$\lim_{N\to\infty} \|S_N(f) - f\|_{L^p} = 0$$

2.4 Lecture 5

We started late today.

Recall last night, we mentioned in order to ensure $S_N(f)$ converges to f in the L^p sense, we need to have uniform boundedness. specifically for $p = 2$, the functions $e^{2\pi i n x}$ form an orthonormal basis (by trig polynomials dense in L^p), hence we get

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Last time we asked for $1 \le p < \infty$, and $f \in L^p$, whether we have

$$S_N(f)(x) \to f(x) \text{ almost everywhere}$$

We won't discuss in detail the answer to this question. For $p = 1$, we know that this is not true, by Kolmogov constructing an integrable function whose Fourier series diverges almost everywhere. And we know that the answer is yes for all $p > 1$.

We now begin section 1.5 of the book.

Definition 2.2 (Cesaro sum)

Define the N-th Cesaro sum of Fourier series as follows:

$$\sigma_N(f) = \frac{1}{N+1}\sum_{k=0}^{N} S_k(f)$$

♣

One can show, via the following,

$$\begin{aligned}\sigma_N(f)(x) &= \frac{1}{N+1}\sum_[k=0]^N S_k(f)(x) \\ &= \int_0^1 f(t)\left(\frac{1}{N+1}\sum_{k=0}^{N} D_k(x-t)\right)dt \\ &= \int_0^1 f(t)F_N(x-t)dt\end{aligned}$$

where $F_n(t) = \frac{1}{N+1}\sum_{k=0}^{N} D_k(t)$, we call this the Fejer kernel, and one can show that

$$F_N(t) = \frac{1}{N+1}\left(\frac{\sin((N+1)\pi t)}{\sin(\pi t)}\right)^2$$

We now note a few important and nice properties of $F_N(t)$. We have

1. $F_N \geq 0$
2. $\int_0^1 F_N(t)dt = \|F_N\|_{L^1} = 1$
3. $\lim_{N\to\infty}\int_{\delta\leq|t|\leq 1/2} F_N(t)dt = 0$, for all $\delta > 0$.

Note that the third property follows from the fact that

$$F_N(t) \leq \frac{1}{(N+1)(\sin(\pi t)^2)}$$

Hence it's nicely behaved when staying away from 0.

Theorem 2.8

For the following two cases,

1. *$\forall f \in L^p$, where $1 \leq p < \infty$*
2. *$f \in C(\mathbb{T})$, for $p = \infty$.*

we get convergence in L^p norm for the Cesaro sum:

$$\lim_{N\to\infty}\|\sigma_N(f) - f\|_{L^p} = 0$$

♡

Proof [sketch] For $f \in L^p$, where $1 \leq p < \infty$, we have

$$\int_{|t|<1/2}\|f(x-t) - f(x)\|_{L^p}F_N(t)dt$$

Hence separating into $\int_{|t|<\delta}$, and $\int_{\delta\leq|t|<1/2}$, the second one tends to 0, where the first one also tends to zero as we choose δ to be arbitrarily small and by $f \in L^p$. Hence the entire integral tends to 0. Similar proof for $f \in C(\mathbb{T})$, and $p = \infty$.

□

2.5 Lecture 6

From the above theorem, we are above to conclude the following statements regarding denseness of trig polynomials and the Fourier coefficients uniquely determine f, if f is integrable.

Corollary 2.1

For $1 \leq p < \infty$, trigonometric polynomials are dense in L^p.

Proof Note that $\sigma_N(f)$ is a finite average of S_k's, which are trigonometric polynomials.

Corollary 2.2

For $f \in L^1$, if $\widehat{f}(k) = 0$, for all k, then f is identically 0.

We now introduce a different summability method, by treating Fourier series as the formal limit on the unit circle. Finish Poisson kernel!

2.6 Lecture 7

We now define FT on the most intuitive class of functions.

Definition 2.3 (FT on L^1)

For $f \in L^1(\mathbb{R}^n)$, we define its Fourier transform to be the following transformation:

$$\widehat{f}(\xi) = \int f(x)e^{-2\pi i x\cdot\xi}dx$$

where $x \cdot \xi = x_1\xi_1 + ...x_n\xi_n$.

FT has nice following properties that are easily verified.

Proposition 2.2

1. *FT is linear.*
$$\mathcal{F}(\alpha f + \beta g) = \alpha\hat{f} + \beta\hat{g}$$
2. *Clearly $\|f\|_\infty \leq \|f\|_1$, and by DCT, $\hat{f}$ is continuous.*
3. *Riemann-Lebesgue states that*
$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0$$
4. *Convolution in the physical space is pointwise multiplication in the Fourier space. $\widehat{f * g} = \hat{f}\hat{g}$*
5. *Translation in the physical space is phase shift in the Fourier space; phase shift in the physical space is translation in the Fourier space.*
$$\widehat{f(x+h)}(\xi) = e^{2\pi i h\dot{\xi}}\hat{f}(\xi)$$
$$\widehat{e^{2\pi i h\cdot x}f}(\xi) = \hat{f}(\xi - h)$$
6. *If ρ is an orthogonal transformation, then*
$$\widehat{f(\rho)}(\xi) = \hat{f}(\rho\xi)$$
7. *FT under scaling. If $g(x) = \lambda^{-n}f(\lambda^{-1}x)$, then $\hat{g}(\xi) = \hat{f}(\lambda\xi)$*
8. *FT with derivative. $\widehat{\partial_{x_j f}} = 2\pi i\xi_j\hat{f}$.*
9. *FT with multiplication by x_j. $\widehat{x_j f} = 2\pi i\partial_{\xi_j}\hat{f}$.*

Note that for finite dimensional measure spaces $\mathbb{T}$, by Holder's inequality $L^p(\mathbb{T})$ embeds in $L^1(\mathbb{T})$ for $p > 1$. However, the same embedding does not hold for $L^p(\mathbb{R}^n)$, and $L^1(\mathbb{R}^n)$. We've only defined the Fourier transform on $L^1(\mathbb{R}^n)$, but not all other $p > 1$ yet. To define FT on these spaces, we start by defining it on a nicer class, the Schwartz functions.

Definition 2.4

f is in the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$ if $f \in C^\infty$, and if f and all its derivatives decrease rapidly at infinity. Rigorously, it means for all $\alpha, \beta \in \mathbb{N}^n$, we have

$$\sup_x |x^\alpha D^\beta f(x)| < \infty$$

♣