

# Functional Analysis

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# Chapter 1 Lecture 1

Here we go.

## 1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

**Textbook:** an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable number of seminorms).

**Remark** Let  $\mathcal{H}$  be a Hilbert space, then the dual space  $\mathcal{H}^*$  is itself.  $\mathcal{H} = \mathcal{H}^*$ . Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themselves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

## 1.0.2 Motivation

Let  $X$  be a compact Hausdorff space. Let  $C(X) = \{f : X \rightarrow \mathbb{R}, f \text{ continuous}\}$  be the algebra of continuous functions on  $X$  mapping in to  $\mathbb{R}$  or  $\mathbb{C}$ . Define the norm as the sup norm  $\|\cdot\|_{L^\infty}$ .

We will develop the spectral theorem of operators on the Hilbert space, i.e. self-adjoint operators can be diagonalized.

If  $T$  is a self-adjoint operator on a Hilbert space, then we take the product of  $T$  (polynomials of  $T$ ), let  $C^*(T, I_{\mathcal{H}})$  be the sub-algebra of operators generated by  $T$  and  $I$  the identity operator, then take the closure, i.e. making it closed in the operator norm.

**Remark** The  $*$  is to remind us,  $T$  is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

### Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \quad (1.1)$$

This is what we are aiming for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If  $T_1, \dots, T_n$  is a collection of self-adjoint operators on  $\mathcal{H}$ , and such all commute with each other, then we also have

$$C^*(T_1, \dots, T_n, I_{\mathcal{H}}) \cong C(X) \quad (1.2)$$

## 1.0.3 Groups

Let  $G$  be a group,  $B$  be a Banach space, for example, groups of automorphisms. Let

$$\text{Aut}(B) = \{T : T \text{ is isometric, onto, invertible on } B\}$$

### Definition 1.1

Suppose that  $\alpha$  is a group homomorphism, and  $\alpha : G \rightarrow \text{Aut}(B)$ , is called a representation on  $B$  or an action of the group  $G$  on  $B$ .

Then we can consider the subalgebra  $\mathcal{L}(B)$ , consisting of the bounded linear operators on  $B$ , generated by

$$\{\alpha_x : x \in G\}$$

**Remark** The identity on  $G$  should be mapped into the identity operator on  $B$ , hence no need to include it.

Elements of the form  $\sum_{z \in \Sigma} \alpha_x z_x \in \mathbb{C}$ , (where  $\Sigma$  is a finite sum.)

Let's introduce,  $f \in C_c(G)$  are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x \in G} f(x) \alpha_x = \alpha_f$$

note for except finitely many  $x$ ,  $f(x) = 0$ .

Let  $f, g \in C_c(G)$ , then for

$$\alpha_f \alpha_g = \left( \sum f(x) \alpha_x \right) \left( \sum g(y) \alpha_y \right) = \sum_{x,y} f(x) g(y) \alpha_x \alpha_y = \sum_{x,y} f(x) g(y) \alpha_{xy}$$

The last inequality follows from  $\alpha$  being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f \alpha_g = \sum_x \sum_y f(x) g(x^{-1}y) \alpha_y = \sum (f * g)(y) \alpha_y$$

where we define  $f * g(y) = \sum f(x) g(x^{-1}y)$  as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f * g}$$

This is how we define convolution on  $C_c(G)$  Notice we have, by  $\|\alpha_x\| = 1$ ,

$$\|\alpha_f\| = \left\| \sum f(x) \alpha_x \right\| \leq \sum |f(x)| \|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We get  $l^1(G)$  is an algebra with ??

For  $G$  commutative, it is easily connected with the Fourier transform.

Consider  $l^2(G)$  with the counting measure on the group. For  $x \in G$ , let  $\xi \in l^2(G)$  define  $\alpha_x \xi(y) = \xi(x^{-1}y)$ ,  $\alpha_x$  being unitary.  $l^1(G)$  acts on operators in  $l^2(G)$  via  $\alpha$ .

If  $G$  is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where  $X$  is some compact space. Note that  $C_c(G)$  operators on  $l^2(G)$ , and  $\|\alpha_f\| \leq \|f\|_{l^1}$ .

## 1.1 Lecture 2

Let's do some math.

Let  $X$  be a Hausdorff compact space, and let  $C(X)$  denote the space of continuous functions defined on  $X$ . This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm  $\|\cdot\|_{L^\infty}$ . Note  $X$ , by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from  $C(X)$ .

### Definition 1.2 (Normed Algebra)

Let  $\mathcal{A}$  be an algebra on  $\mathbb{R}$  or  $\mathbb{C}$ , is a normed algebra if it has a norm  $\|\cdot\|$ , as a vector space, such that for  $a, b \in \mathcal{A}$ , we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity.

### Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm.

Given  $x \in X$ , define  $\varphi_x : C(X) \rightarrow \mathbb{C}$  the evaluation map such that

$$\varphi_x(f) = f(x)$$

$\varphi_x$  is an algebra homomorphisms between  $C(X) \rightarrow \mathbb{R}$  or  $C(X) \rightarrow \mathbb{C}$ . This simply implies

$$\varphi_x(f + g) = (f + g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that,  $C(X)$  has an identity element, which is the constant function 1, under multiplication. Hence  $C(X)$  is a unital algebra. Note that  $\varphi_x$  defined above is a unital homomorphism, meaning that it sends identity to identity.

Note  $\varphi_x$  is also a multiplicative linear functional, also unital.

### Proposition 1.2

Every multiplicative linear functional on  $C(X)$  is of the form  $\varphi_x$  for some  $x \in X$ .

**Proof** Main Claim: given a multiplicative linear functional  $\varphi$ , there exists a point  $x_0$  and if we have some  $f \in C(X)$ , we have  $\varphi(f) = 0$ , then we have  $f(x_0) = 0$ . To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each  $x \in X$ , there is an  $f_x \in C(X)$  such that  $f_x(x) \neq 0$ , but  $\varphi(f_x) = 0$ .

Set  $g_x = \overline{f_x} f_x$ , then we have  $g_x(x) = 0$ ,  $g_x \geq 0$ , but  $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f_x}) = 0$ , then there is an open set  $O_x$  such that  $x \in O_x$ , and  $g_x(y) > 0$  for all  $y \in O_x$ . Now by compactness, there is  $x_1, \dots, x_n$  such that  $X = \bigcup_{j=1}^n O_{x_j}$ , let  $g = g_{x_1} + \dots + g_{x_n}$ , then we have  $g(y) > 0$  for all  $y \in X$ , and  $\varphi(g) = 0$ . Note that  $g$  is a continuous function, and  $g$  is invertible, and also  $re(\frac{1}{g}) \in C(X)$ , but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists  $x_0 \in X$  such that if  $\varphi(f) = 0$ , this means  $f(x_0) = 0$ . For any  $f$ , consider  $f - \varphi(f) \cdot 1$ , apply  $\varphi$ , we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies  $f(x_0) = \varphi(f)$  which implies  $\varphi(f) = \varphi_{x_0}(f)$ .

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.

For  $\mathcal{A} = C(X)$ , and  $\varphi \in \widehat{\mathcal{A}}$ .

**Definition 1.4 (spectra of  $\mathcal{A}$ )**

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.



**Remark** We have  $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$ , since  $\varphi$  is unital, we have  $\|\varphi\| = 1$ .

This is not always true for normed algebra, Let

$$\mathcal{A} := \text{Poly} \subset C([0, 1])$$

We define  $\varphi(p) = p(2)$ ,  $p$  is a polynomial. This is not continuous, nor is the  $\|\varphi\| = 1$ .

**Proposition 1.3**

If  $\mathcal{A}$  is a unital commutative Banach algebra, and if  $\phi \in \widehat{\mathcal{A}}$ , then we have  $\|\phi\| = 1$ .

**Proposition 1.4**

Let  $\mathcal{A}$  be a unital Banach algebra (not necessarily commutative), then if  $a \in \mathcal{A}$ , and  $\|a\| \leq 1$ , then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$



**Proof** For this, we use completeness.  $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ ,  $a^0 = 1_{\mathcal{A}}$  You could look at the partial sums.  $S_m = \sum_{n=0}^m a^n$ , you want to show that  $\{S_m\}$  is a Cauchy sequence, and use completeness of Banach algebras.  $\lim_{m \rightarrow \infty} S_m = \frac{1}{1-a}$ .

To prove this is a Cauchy sequence:

$$\|S_n - S_m\| = \left\| \sum_{j=m+1}^n a^j \right\| \leq \sum_{j=m+1}^n \|a^j\| \leq \sum_{j=m+1}^n \|a\|^j$$

And the fact that  $\|a\| \leq 1$ , we have the sum bounded by  $\epsilon$ , hence  $\{S_n\}$  is a Cauchy sequence. Let  $b = \sum_{n=0}^{\infty} a^n$ , we want to show that  $b(1-a) = 1$ .

$$b(1-a) = \lim_{n \rightarrow \infty} S_n(1-a) = \lim_{n \rightarrow \infty} \left( \sum_{n=0}^{\infty} a^n \right) (1-a) = \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 1$$

The last inequality follows from  $\|a^{n+1}\| \leq \|a\|^{n+1} \rightarrow 0$ .

## 1.2 Lecture 3

We now begin.

Let  $\mathcal{A}$  be a unital Banach algebra, and if  $a \in \mathcal{A}$  and  $\|a\| < 1$ , then we have  $(1-a)$  has an inverse and if  $\mathcal{A} = \mathcal{B}(B)$ , where  $B$  is some Banach space, then  $T \in \mathcal{A}$ , and  $\|T\| < 1$ , then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Neumann series.

Now we have the following corollary.

**Corollary 1.1**

If  $a \in \mathcal{A}$  and  $\|1-a\| < 1$ , then  $a$  is invertible.



**Proof**  $a = 1 - (1-a)$ .

the open ball about 1 consists of invertible elements. If  $d$  is any invertible element, then we define  $a \mapsto ad$  is continuous. Then this operation is a continuous homeomorphism of  $\mathcal{A}$  onto itself.

Denote the unit ball about 1 as  $O$ , and let  $d$  be some invertible element, under the above homeomorphism,  $O \mapsto Od$ , is open and consists of invertible elements.

**Proposition 1.5**

The set of invertible elements of  $\mathcal{A}$  is an open subset of  $\mathcal{A}$ .





Let  $f \in C(X)$ , and if  $\alpha \in \text{range of } f$ , so  $\alpha = f(x_0)$  for some  $x_0$ . then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $f - \alpha \cdot 1$  is not invertible.

Conversely, if we have  $f - \alpha \cdot 1$  is not invertible, then there exists  $x_0 \in X$  such that


$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $f(x_0) = \alpha$ , i.e.,  $\alpha \in \text{range of } f$ .

#### Definition 1.5 (spectrum of an element)

For any unital algebra  $\mathcal{A}$  over some field  $\mathbb{F}$ , for any  $a \in \mathcal{A}$ , the set

$$\{\alpha : a - \alpha 1_{\mathcal{A}} \text{ is not invertible}\}$$

is called the spectrum of  $a$ , denoted as  $\sigma(a)$ . 

#### Proposition 1.6

Let  $\mathcal{A}$  be a unital Banach algebra, and let  $a \in \mathcal{A}$ , then if  $\alpha \in \sigma(a)$ , then

$$|\alpha| \leq \|a\|$$


**Proof** Suppose  $|\alpha| > \|a\|$ , then  $\alpha \neq 0$ , then

$$\alpha - \alpha \cdot 1 = -\alpha(1 - \frac{a}{\alpha})$$

And by assumption,  $\|a/\alpha\| \leq 1$ , hence  $(1 - a/\alpha)$  is invertible. Hence  $\alpha \notin \sigma(a)$ .

let  $\varphi$  be a multiplicative linear functional, i.e.  $\varphi \in \widehat{\mathcal{A}}$ , and then  $\varphi(a) \in \sigma(a)$ , and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

**Proof**  $\varphi(a - \varphi(a) \cdot 1) = 0$

Hence  $a - \varphi(a)1$  is not invertible.

#### Proposition 1.7

$\sigma(a)$  is a closed subset of  $\mathbb{R}, \mathbb{C}$ . 

**Proof**  $\lambda \mapsto a - \lambda 1$  is not invertible, and the map is continuous. And  $\sigma(a)$  is the preimage of the complement of the set of invertible elements, and the set of invertible elements is open. And the preimage of a closed set is closed.

Let  $\varphi \in \widehat{\mathcal{A}}$  then  $\|\varphi\| = 1$ . So  $\widehat{\mathcal{A}}$  is a subset of the unit ball of  $\mathcal{A}'$ , which denotes the dual vector space of continuous linear transformations.

On  $\mathcal{A}'$ , we can equip the weak-\* topology, i.e. the weakest topology, making the map  $\psi \mapsto \psi(a)$  continuous.

#### Proposition 1.8

$\widehat{\mathcal{A}}$  is closed for the weak-\* topology. 

**Proof** let  $\{\varphi_\lambda\}$  be a net of elements of  $\widehat{\mathcal{A}}$ , that converges to some  $\psi \in \mathcal{A}'$  in the weak-\* topology, i.e., for every  $a \in \mathcal{A}$ ,  $\varphi_\lambda(a) \rightarrow \psi(a)$  for all  $a \in \mathcal{A}$ .

Then  $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$ .

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$ .

#### Theorem 1.1 (Alaoglu's theorem)

For any normed vector space  $V$ , the closed unit ball of  $V'$  is compact in the weak-\* topology. 

As an immediate corollary, we have the following.

**Corollary 1.2**

$\widehat{\mathcal{A}}$  is compact with respect to the weak-\* topology.



Let  $\mathcal{A} = C(X)$ , and  $\widehat{\mathcal{A}}$ , we define  $x \mapsto \varphi_x$  is a bijection. The weak-\* topology in  $\widehat{\mathcal{A}}$  makes  $\varphi_x \mapsto \varphi_x(f) = f(x)$  continuous. Such  $x \mapsto \varphi_x$  is a homeomorphism of  $X$  onto  $\widehat{\mathcal{A}}$ .

For  $\mathcal{A}$  unital Banach algebra, commutative, for any  $a \in \mathcal{A}$ , define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{ab})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

we have  $a \mapsto \widehat{a}$  is a unital algebra homeomorphism from  $\mathcal{A}$  into  $C(\widehat{\mathcal{A}})$ .