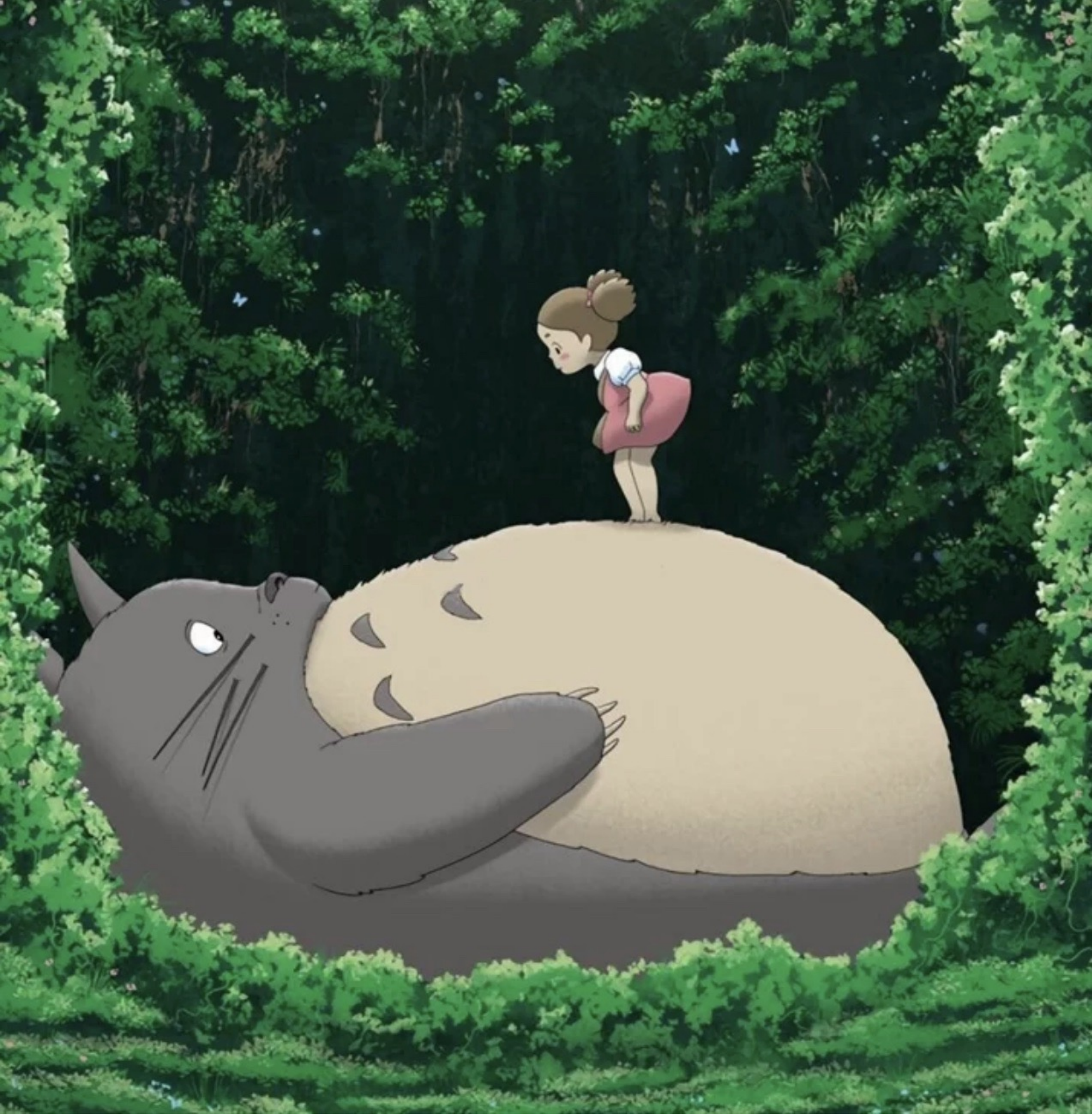

Representation Theory

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What is representation theory

0.1 Introduction

We would like to represent abstract algebraic concepts using linear objects. We will start with an example of a specific type of representation.

Definition 0.1 (Representation on a vector space)

A representation π of a group G on a $\mathbb{K}$-vector space V is a mapping:

$$\pi : G \to Aut(V) = GL(V)$$

It assigns every element in G with an invertible linear map on V. ♣

We will categorize some interesting features of a representation.

Definition 0.2 (faithful representation)

A representation is called faithful if it is injective,i.e., for $g_1, g_2 \in G$, we have

$$\pi(g_1) = \pi(g_2) \text{ implies } g_1 = g_2$$

♣

Remark This means that there is "no loss."

Definition 0.3 (Trivial representation)

The trivial representation of G on V is such that

$$\pi(g) = I$$

Or $\pi(g)v = v$ for all $v \in V$. ♣

Example 0.1 For the $\mathbb{R}$-vector space $\mathbb{C}^4$, we have a faithful representation of a group order 8, with each g sent to each one of the following linear maps:

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \begin{bmatrix}i & 0\\0 & i\end{bmatrix} \begin{bmatrix}i & 0\\0 & -i\end{bmatrix} \begin{bmatrix}0 & i\\i & 0\end{bmatrix} \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

Now we introduce "equivalent" representations.

Definition 0.4 (Intertwiner)

For two representations $(\pi_1, V_1), (\pi_2, V_2)$ of G, a linear isomorphism $\phi : V_1 \to V_2$ is called an Intertwiner if it "intertwines" the action of G:

$$\phi(\pi_1(g)v) = \pi_2(g)(\phi(v))$$

♣

Remark We can think of ϕ as the isomorphism between V_1 and V_2 such that the following diagram commutes:

$$\begin{array}{ccc} V_1 & \xrightarrow{\pi_1(g)} & V_1 \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ V_2 & \xrightarrow{\phi_2(g)} & V_2 \end{array}$$

Now we find the building blocks of representations: invariant subspaces, and irreducible representations.

Definition 0.5 (Invariant subspaces)

Let (π, V) be a representation of G, and let $K \subset V$, then K is called a G-invariant subspace if for all $w \in W$,

$$\pi(g)w \in W, \text{ for all } g \in G$$

♣

Definition 0.6 (Irreducible representations)

A representation (π, V) of G is called irreducible if it contains no proper G-invariant subspaces. (And is call totally reducible if it can be written as a direct sum of irreducible subspaces).

♣