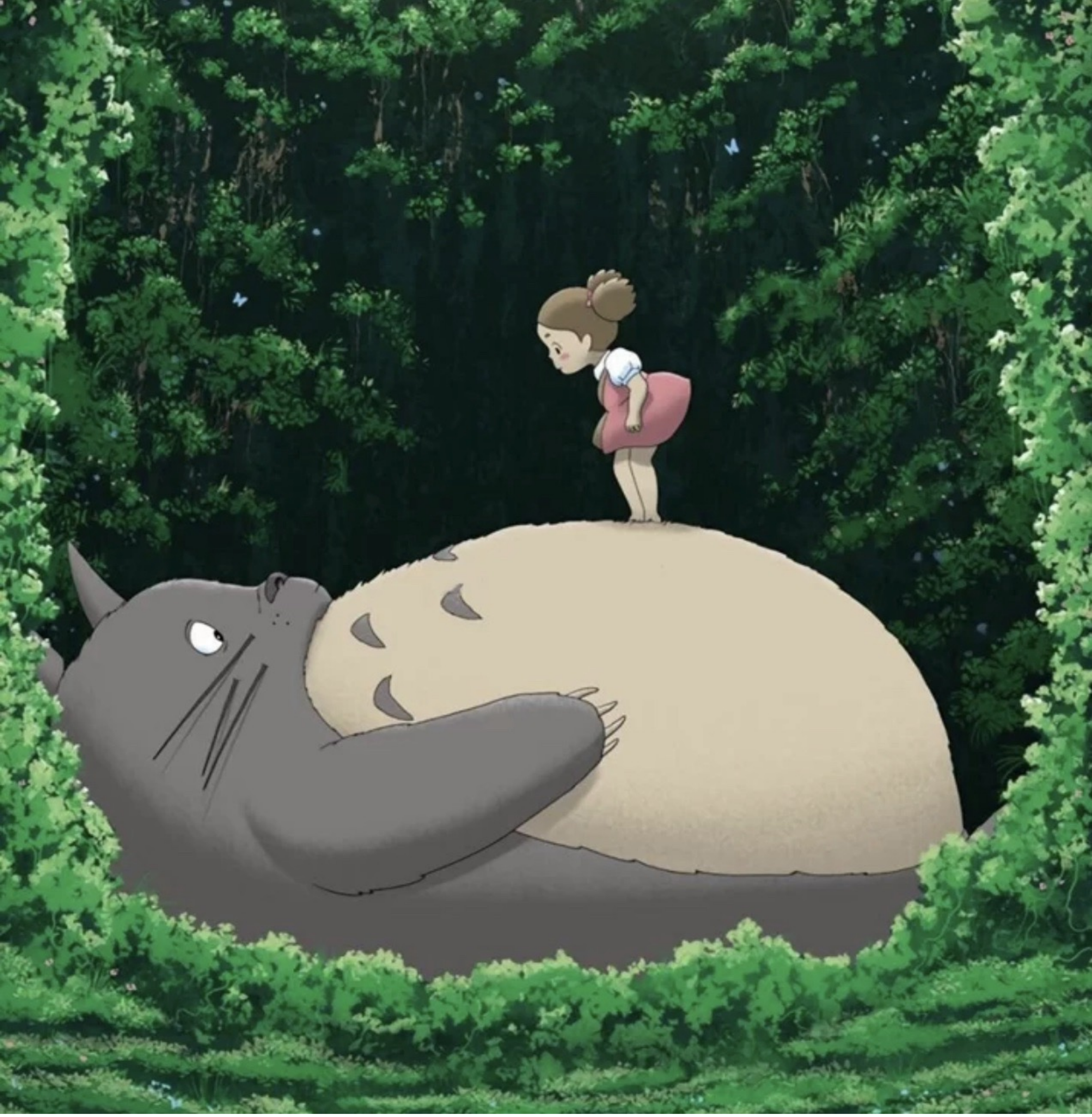

Functional Analysis

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Contents

Chapter 1 Prep work

We will start from the beginning and take baby steps. It's going to be okay.

An algebra is a vector space (with addition and scalar multiplication, usually over $\mathbb{R}, \mathbb{C}$), with an extra multiplication operation such that it is associative, and distributive. Then a normed algebra is an algebra with a sub-multiplicative norm, such that for all $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

A Banach algebra is a normed algebra that is complete under the metric induced by the norm. And we can form a Banach algebra by starting with a normed algebra and form its completion and by uniform continuity of addition and multiplicatoin extend to the completion of the algebra to form a Banach algebra.

We will begin with some important examples of Banach algebras. Let X be a compact topological space, and let $C(X)$ be the space of continuous functions, equip it with $\|\cdot\|_{L^\infty}$ norm, then $(C(X), \|\cdot\|_{L^\infty})$ is a Banach algebra. Similarly, if X is only locally compact, then $C_b(X)$, the space of bounded continuous functions under the $\|\cdot\|_{L^\infty}$ norm is also a Banach algebra.

1.0.1 Some Banach algebra examples

Another important example is that let X be a Banach space, and the space of all bounded/continuous operators on X, denoted by $\mathcal{B}(X)$ is a Banach algebra with the operator norm. Any closed subalgebra of $B(X)$ is also Banach.

If X is a Hilbert space, then we also have the operation of taking adjoints, namely $\|T\| = \|T^*\|$.

Definition 1.1

A C^* algebra is a closed subalgebra of the space of bounded (equivalently) functions defined on a Hilbert space, $\mathcal{B}(\mathcal{H})$.

♣

Remark The space of continuous/bdd operators on a Hilbert space, under the operator norm, then closed under the norm topology and taking adjoints of the operators. On wikipedia, C* algebra is defined to be a Banach algebra equipped with an involution that acts like a adjoint.

One of the goals of this course is to develop the following theorem.

Theorem 1.1

*Let $\mathcal{A}$ be a commutative C^*-algebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is isometrically and *-algebraically isomorphic to some $C(X)$, where X is some locally compact space.*

♡

Proposition 1.1

Multiplication is continuous in Banach algebras.

♠

Proof Multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, hence if we have x_n, y_n such that $x_n \to x, y_n \to y$, then we have

$$\|x_n y_n - xy\| \leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| < \epsilon$$

Hence multiplication is continuous.

Definition 1.2 (Unital Banach algebra and invertibility)

A Banach algebra (let's repeat, a complete vector space with addition, scalar multiplicatin, and multiplication such that the norm is sub-multiplicative) is called unital if there exists a multiplicative inverse.

An element $a \in \mathcal{A}$ is called invertible if there exists an element $a^{-1} \in \mathcal{A}$ such that

$$aa^{-1} = a^{-1}a = e$$

♣

Regarding invertibility, we can determine whether an element is invertible by knowing a related element's norm.

Proposition 1.2

Let $\mathcal{A}$ be a unital Banach algebra, and if $\|a\| < 1$, then $(1-a)$ is invertible.

Proof We would like to use the fact that every Cauchy sequence converges. Define

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

where $a^0 = 1$ by definition. We first show that this geometric series converges to an element in $\mathcal{A}$, and we will show that the quantity defined above is indeed the inverse of $(1-a)$.

Note that we define the partial sum $S_N = \sum_{n=0}^{N} a^n$, then

$$\|S_N - S_M\| \leq \sum_{M+1}^{N} \|a\|^n < \epsilon$$

Hence $\{S_N\}$ is a cauchy sequence, hence converges to some element which we denoted as $(1-a)^{-1} \in \mathcal{A}$. Now

$$(1-a) \cdot (1-a)^{-1} = (1-a) \cdot \lim_{N \to \infty} S_N = (1-a) \cdot \frac{1}{1-a} = 1$$

Likewise for the other side. Notice our $(1-a)^{-1}$ is a defined quantity, while $\frac{1}{1-a}$ is the sum of geometric series.

□

Corollary 1.1

Let $\mathcal{A}$ be a unital Banach algebra, then if $\|(1-a)\| < 1$, then we have, a is invertible.

The implication of this corollary is interesting.

Corollary 1.2

The open ball of radius 1 around the identity element $1_{\mathcal{A}}$ consists of invertible elements.

$$\|1-a\| < 1 \Rightarrow a \in B_1(1_{\mathcal{A}})$$

And we know a is invertible.

Proposition 1.3

The set of invertible elements of a unital Banach algebra is an open subset.

Proof We use the fact that $B_1(1_{\mathcal{A}})$ is an open set. Note that for any invertible element d, we define the map, for all $a \in \mathcal{A}$,

$$L_d(a) = da$$

We observe this map is continuous, and by d be invertible, the inverse is also continuous, hence a homeomorphism. Bijectivity follows from $da = db \Rightarrow a = b$, and for every $c \in \mathcal{A}$, we can find $a = d^{-1}c$ such that $L_d(a) = c$.

Hence for every d invertible, we have $d \cdot O$ an open ball of invertible elements, and taking all union of these open balls give us the set of invertible elements, which is an open set.

□

Proposition 1.4

For $f \in C(X)$, we have α is in the range of f if and only if $(f - 1 \cdot \alpha)$ is invertible.

Proof Refer to the lecture notes. In function spaces, the word **invertible** means having trivial kernel, i.e. $f(x) = 0$ implies $x = 0$.

□

1.0.2 Algebra homomorphisms on $C(X)$

Definition 1.3 (Algebra homomorphism)

An algebra homomorphism is a homomorphism between two algebras. For example, consider X a compact space, and $C(X)$ the space of continuous functions, hence if we define the evalutation map as follows:

$$\varphi_x(f) = f(x)$$

This is an algebra homomorphism between $C(X)$ and $(\mathbb{C})$. Namely, the homomorphism property is justified as: (under both addition and multiplication)

$$\varphi_x(f+g) = f + g(x) = f(x) + g(x) = \varphi_x(f) + \varphi_x(g)$$
$$\varphi_x(fg) = (fg)(x) = f(x)g(x) = \varphi_x(f)\varphi_x(g)$$

And of course, same thing follows for scalar multiplication. ♣

Remark We need to check all three conditions to make sure such φ preserves the structures between the algebras.

An algebra homomorphism is called unital if if maps the (multiplicative identity) unity to unity. In the above example, a unital homomorphism would be $\varphi(1) = 1$, where the left 1 is the constant 1 function, and the right 1 is the number.

Now we will introduce the proposition that every multiplicative linear functional on $C(X)$. Note we can use algebra homomorphism and multiplicative linear functional synonomously on $C(X)$, hence they entail the same information.

Proposition 1.5

Let φ be a multiplicative linear functional on $C(X)$, i.e. a nontrivial algebra homomorphism, then $\varphi(f) = f(x_0)$ for some $x_0 \in X$. In other words, a multiplicative linear functional always takes this form. ♠

Proof It suffices to show the following lemma:

Lemma 1.1

There exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. ♡

We will first show how the lemma implies $\varphi(f) = f(x_0)$. Consider the function $f - \varphi(f) \cdot 1$, then we know

$$\varphi(f - \varphi(f) \cdot 1) = 0$$

Then there exists x_0 such that $f(x_0) - \varphi(f) = 0$, this gives $\varphi(f) = f(x_0)$.

Now we prove the lemma.

Proof Our claim is that there exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. Assume the contrary, which states for all x, there exists an f_x such that $\varphi(f_x) = 0$, but $f(x) \neq 0$. We define a nonnegative function $g_x = f_x\overline{f_x}$. And by multiplicativity, we have $\varphi(g_x) = 0$. We now note that because g is continuous, in a small nbd of x, denoted by O_x, we have $g(y) > 0$ for all $y \in O_x$.

Now using compactness, we can write X as a finite union of small neighborhoods $X = \bigcup_{j=1}^n O_{x_j}$, and define

$$g = g_{x_1} + ... + g_{x_n}$$

Then for each $y \in X$, $y \in O_{x_j}$ for some j, hence $g(y) > 0$ for all $y \in X$. This implies that g is invertible hence we have

$$\varphi(g \cdot 1/g) = 1$$

This contradicts with the fact that $\varphi(g) = 0$. And we are done.

□

Hence we have the following corollary.

Corollary 1.3

Let X be compact, and $C(X)$ the space of continuous functions, then φ is a multiplicative linear functional (i.e. a algebra homomorphism with $\mathbb{C}$) if and only if it is a point evaluation. ♡

Definition 1.4 ($\widehat{\mathcal{A}}$)

Given a unital commutative (or Banach) algebra, for example, $C(X)$ with $\|\cdot\|_{L^\infty}$, we define the set of unital homomorphisms, i.e., nonzero unital multiplicative linear functionals on $\mathcal{A}$ as $\widehat{\mathcal{A}}$.

Proposition 1.6

If $\mathcal{A}$ is a unital algebra, then for $\varphi \in \widehat{\mathcal{A}}$, we have $\|\varphi\| = 1$

Proof We have

$$\|\varphi\| = \sup\{|\varphi(f)| : \|f\|_{L^\infty} = 1\}$$

Because $|\varphi(f)| = |f(x_0)|$ for some x_0, we always have $\|\varphi\| \leq 1$, but with the unity, we have $|\varphi(e)| = 1$, and taking the sup we have $\|\varphi\| = 1$.

□

1.0.3 Spectrum

We now define the spectrum of an element in a Banach algebra.

Definition 1.5 (spectrum)

Let $\mathcal{A}$ be a Banach algebra, fix $a \in \mathcal{A}$, we define the following set to be the spectrum of a, denoted by $\sigma(a)$.

$$\sigma(a) = \{\lambda \in \mathbb{F} : a - \lambda \cdot 1_{\mathcal{A}} \text{ is not invertible } \}$$

We have a bound on the size of λ given $\|a\|$.

Proposition 1.7

For $\lambda \in \sigma(a)$, we have

$$|\lambda| \leq \|a\|$$

Proof Assume the contrary, we have $|\lambda| > \|a\|$, then a/λ has norm $\|a/\lambda\| < 1$. Thus, $(1 - a/\lambda)$ is invertible.

$$a - \lambda \cdot 1 = -\lambda(1 - a/\lambda)$$

Because the product of two invertible elements is again, invertible, we get that $\lambda \notin \sigma(a)$. Hence a contradiction.

□

Proposition 1.8

Let $\mathcal{A}$ be a unital Banach algebra, and let $\varphi \in \widehat{\mathcal{A}}$, then we have

$$\varphi(a) \in \sigma(a)$$

Proof It suffices to show that $a - \varphi(a) \cdot 1$ is not invertible. Assuming that it is, denote its inverse by $(a - \varphi(a))^{-1}$, then

$$\varphi\left((a - \varphi(a)1)\frac{1}{a - \varphi(a)}\right) = 1$$

However, $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence a contradiction.

□

Remark To prove an element $a \in \mathcal{A}$ is not invertible, it suffices to prove $\varphi(a) = 0$.

Corollary 1.4

For the above, $|\varphi(a)| \leq \|a\|$, and again, $\|\varphi\| = 1$.

Remark This is to say, every unital homomorphism $\varphi \in \mathcal{A}$ is continuous.

We now show that the spectrum of an element is always closed.

Proposition 1.9

Let $a \in \mathcal{A}$, then $\sigma(a)$ is closed.

Proof We define a map $\phi : \mathbb{F} \to \mathcal{A}$ as

$$\phi(\lambda) = a - \lambda \cdot 1$$

The map is continuous, and we notice that the $\sigma(a)$ is the complement of the preimage of invertible elements under ϕ, i.e.

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Using the fact that the set of invertible elements is open, we get $\sigma(a)$ is closed.

□

1.0.4 Weak-* topology

We now do some topology. Fix $\mathcal{A}$, Recall the weak-* topology is defined on $\mathcal{A}'$ and it is the weakest topology such that the map $\psi \in \mathcal{A}'$,

$$\psi \mapsto \psi(a) \text{ continuous}$$

We first note that if $\varphi \in \widehat{\mathcal{A}}$, then $\|\varphi\| = 1$. Hence $\widehat{\mathcal{A}}$ is a subset of the closed unit ball in $\mathcal{A}'$. Now with respect to the weak-* topology, we have some nice properties.

Theorem 1.2

$\widehat{\mathcal{A}}$ is closed with respect to the weak- topology.*

Proof Let $\{\varphi_\lambda\}$ be a net that converges to some φ in the weak-* topology, which is a linear functional, i.e. $\varphi \in \mathcal{A}'$. Weak-* convergence implies for all $a \in \mathcal{A}$, we have

$$\varphi_\lambda(a) \to \varphi(a)$$

We show that φ is multiplicative.

$$\varphi(ab) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a) \lim \varphi_\lambda(b) = \varphi(a)\varphi(b)$$

Now it remains to show that $\|\varphi\| = 1$ to show that it is closed. It suffices to show φ is unital.

$$\varphi(1) = \lim \varphi_\lambda(1) = 1$$

Hence $|\varphi(1)| \leq \|\varphi\|$, hence $\|\varphi\| = 1$.

□

Now we recall Alaoglu's theorem.

Theorem 1.3 (Alaoglu's)

The closed unit ball is compact in the weak- topology.*

Hence as an immediate corollary,

Corollary 1.5

$\widehat{\mathcal{A}}$ is compact with respect to the weak- topology.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□

Let S be a semiroup with unity e, and $l^1(S)$ with convolution is a Banach algebra, hence we denote $\mathcal{A} = l^1(S)$.

Example 1.1 Let the positive integers including 0 be the semigroup S, then we have $f = \sum_{n \in S} f(n)\delta_n$.

Now we try to find out what $\widehat{\mathcal{A}}$ looks like. Recall $\widehat{\mathcal{A}}$ is the space of nonzero unital homomorphisms, $\varphi : S \to \mathbb{C}$. For any $a \in \mathcal{A} = l^1(S)$, we know that

$$\varphi(a) \in \sigma(a)$$

and we have $|\varphi(a)| \leq \|a\|$, hence we have $\|\varphi\| \leq 1$. If we view φ as an element in l^∞, then we have

$$\|\varphi\|_{l^\infty} \leq 1$$

Hence this is a unit disk in the space of homomorphisms from $l^1(S)$ to $\mathcal{C}$.

We now extend to the double dual of $\mathcal{A}$, which is $\mathcal{A}''$. For any $a \in \mathcal{A}$. we define

$$\widehat{a}(\varphi) = \varphi(a)$$

Now we attempt to define a Banach algebra of functions on a semigroup. A semigroup is with associative product, but not necessarily an inverse.

Example 1.2 For example, the set of natural numbers with 0, under addition is a semigroup. We will define $\mathbb{N}_{\geq 0} = S$.

We let $l^(\mathbb{N}_{\geq 0})$ denote the set of functions defined on $\mathbb{N}_{\geq 0}$ such that if $f \in C_c(S)$,

$$f(x) = \sum_{n \in S} f(n)\delta_n$$

We define $\delta_x \delta_y = \delta_{xy}$, and we thus have

$$\left(\sum_n f(n)\delta_n\right)\left(\sum_y g(y)\delta_y\right) = \sum_z \left(\sum_{xy=z} f(x)g(y)\right)\delta_z$$

Example 1.3 If we consider polynomials of the form $\sum f(n)x^n$, then we note that

$$\left(\sum f(m)x^m\right)\left(\sum g(n)x^n\right) = \sum_p \left(\sum_{mn=p} f(m)g(n)x^p\right)$$

Hence naturally we have $\delta_m \delta_n = \delta_{mn}$, which agrees with $x^m x^n = x^{m+n}$.

It is also easy to check $\|f * g\|_{L^1} \leq \|f\|_1 \|g\|_1$.

And we define $f \in l^1(S)$ if we have $\sum_{n \in S} |f(n)| < \infty$.

Then we note that $l^1(S)$ is a Banach algebra under the convolution defined as follows: let

$$f = \sum_{n \in S} f(n)\delta_n, g = \sum_{n \in S} g(n)\delta_n$$

Definition 1.6 (Convolution)

We will define a convlution between two functiosn of the above form as

$$f * g(x) = \sum_{x=yz} f(y)g(z)$$

♣

Then we have δ_e as our identity function, in this case δ_1.

$$f * \delta_e(x) = \sum_{x=yz} f(y)\delta_1(z) = f(x)$$

Let's now discuss a specific example. Let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ is the set of unital homomorphisms from $\mathcal{A} \to \mathbb{C}$. Hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. And from previous knowledge, we know

$$\mathcal{A}' = l^\infty(S)$$

Hence let $\varphi \in \widehat{\mathcal{A}}$, and we view it as an element in $l^\infty(S)$, then we define a pairing between φ and the $f \in l^1(S)$ that it acts on. We have

$$\langle f, \varphi \rangle = \varphi(f) = \sum_{x \in S} f(x)\varphi(x)$$

1.0.5 On semigroups

Let S be a discrete commutative semigroup.

Proposition 1.10

We have

$$\widehat{\mathcal{A}} \text{ “=” } Hom(S, \mathbb{D})$$

where $\mathbb{D}$ *denotes the unit disk in the complex plane.*

Proof In other words, a unital homomorphism acting on $l^1(S)$ can be viewed as a unital homomorphism that acts directly on the semigroup and mapping into $\mathbb{D}$.

We note that $\varphi \in \widehat{\mathcal{A}}$, then $\varphi \in l^\infty(S)$, and we also have $\|\varphi\|_{l^\infty} = 1$, hence $\|\varphi(s)\| \leq 1, s \in S$. For φ being multiplicative, we have $\varphi(\delta_{xy})\varphi(\delta_x\delta_y) = \varphi(\delta_x)\varphi(\delta_y)$, hence

$$\varphi(xy) = \varphi(x)\varphi(y)$$

We also have

$$\varphi(x + y) = \varphi(x) + \varphi(y)$$

And

$$\varphi(e) = 1$$

Remark We are simply using the fact that every $\varphi \in \widehat{\mathcal{A}}$ can be viewed as an element of $l^\infty(S)$.

□

Proposition 1.11

For $S = \mathbb{N}$*, There is a natural identification between* $\widehat{l^1(S)}$ *with the unit disk* $\mathbb{D}$*.*

Proof We note that $\mathbb{N}$ is generated by 1, so $l^1(S)$ is generated by δ_1, hence $\varphi \in \widehat{l^1(S)}$ is determined by $\varphi(\delta_1)$. Alternatively, if we view $\varphi \in l^\infty(S)$, then φ is determined by its value on $\varphi(1)$. Let $\varphi(1) = z_0$. Note $z_0 \in \mathbb{D}$, Then we we have, given φ is multiplicative,

$$\varphi(m) = z_0^m$$

Hence there is a natural identification from $\mathbb{D}$ to $\widehat{l^1(\mathbb{N})}$, taking an element in $z \in \mathbb{D}$, to a map $\varphi : \mathbb{N} \to \mathbb{C}, \varphi \in l^\infty(\mathbb{N})$, by the map

$$z \mapsto \varphi(n) = z^n$$

The map is bijective and continuous.

□

Proposition 1.12

The unit disk $\mathbb{D}$ *under the standard topology, coincides with the weak-* topology on* $\mathbb{D}$ *that is determined in the sense of* $\widehat{l^1(S)}$*. In other words,*

$$\mathbb{D}_{std} \cong \mathbb{D}_{weak-*}$$

Proof We would like to show the map

$$z \mapsto \varphi(f) = \sum_{n \in S} f(n)\varphi(n)$$

is continuous. We have noted the natural corespondence from $z \mapsto \varphi(n) = z^n$. And by definition of the pairing between $\varphi \in l^1(S), f \in l^1(S)$, we have

$$z \mapsto \varphi(z) = z^m \mapsto \sum_{n \in S} f(n)z^n$$

The first map is continuous, and the second is also continuous, hence we have a continuous, bijective map between $\mathbb{D}$, which is a compact space, to $\widehat{l^1(S)}$, a Hausdorff space, hence

$$\mathbb{D}_{std} \cong \mathbb{D}_{weak-*}$$

□

1.0.6 On groups

Let G be a discrete commutative group. Everything above applies, however, we note that in this case $\varphi \in \widehat{l^1(G)}$ implies $|\varphi(x)| = 1$ for all $x \in G$. This is because $\|x\| = 1, \forall x \in G$. This implies $|\varphi(x)| \leq 1$. Hence,

$$\|\varphi(e)\| = \|\varphi(x)\varphi(x^{-1})\| = 1$$

This means $|\varphi(x)| = 1, \forall x \in G$.

Previously, we had $\widehat{l^1(S)}$ "=" $\mathbb{D}$, since $|\varphi(s)| \leq 1$, and now we have

Proposition 1.13

For G a commutative discrete group, we have

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where $\mathbb{T} = \{x \in \mathbb{C} : |x| = 1\}$.

Just like $\mathbb{D}$, we have $\mathbb{T}$ as a compact topological group. Hence the standard topology on $\mathbb{T}$ coincides with the weak-* topology on $\mathbb{T}$, by the map $z \in \mathbb{T}$,

$$z \mapsto \sum_{n \in G} f(n) z^n$$

If we denote $z \in T$ as $z = e^{2\pi i t}$, then we would have

$$\sum_{n \in G} f(n) e^{2\pi i n t}$$

And this is the Fourier series!

Definition 1.7 (Self-adjoint Algebras)

A Banach algebra is called self-adjoint if for every $a \in \mathcal{A}$, we have $a^* \in \mathcal{A}$ as well.

Proposition 1.14

The Gelfand transformation is onto for $\mathcal{A}$ Banach algebras that are self-adjoint. It is also an isometry.

Our goal for the following few propositions is to establish the relationship between the spectral radius, Gelfand transform, and maximal ideals.

Let $\mathcal{A}$ be a commutative Banach algebra.

Proposition 1.15

There is a natural correspondance between the multiplication functionals φ on $\mathcal{A}$ and the set of maximal ideals in $\mathcal{A}$.

Namely, for every maximal ideal $\mathcal{M}$ in $\mathcal{A}$, we can find a $\varphi \in \widehat{\mathcal{A}}$ such that $\ker(\varphi) = \mathcal{M}$.

The proof uses algebra, and we did it in class, so we do not illustrate here. The important thing is the following result.

Corollary 1.6

$a \in \mathcal{A}$ is invertible if and only if $\widehat{a}$ is invertible, where $\widehat{a} = \Gamma(a)$ is the Gelfand transform.

Proof We know if a is invertible, then

$$\Gamma(aa^{-1}) = \Gamma(a)\Gamma(a^{-1}) = 1$$

Hence $\Gamma(a^{-1})$ is the inverse of $\Gamma(a) = \widehat{a}$, hence is invertible.

Now we want to show if $\widehat{a}$ is invertible, then a is invertible. Suppose a is not invertible, we show $\widehat{a}$ is not invertible. In other words, there exists φ such that

$$\widehat{a}(\varphi) = \varphi(a) = 0$$

Using the previous proposition, we notice that the set

$$\{ab : b \in \mathcal{A}\} \text{ is a proper ideal of } \mathcal{A}$$

This is due to a be not invertible, hence does not contain $1_{\mathcal{A}}$. And every proper ideal is contained in some maximal ideal $\mathcal{M}$, hence there exists $\varphi \in \widehat{\mathcal{A}}$ such that $\varphi(a) = 0$, and we are therefore done.

□

Now we connect the spectral radius with the Gelfand transform. Recall the definition of the spectral radius.

Definition 1.8 (spectral radius)

Let $a \in \mathcal{A}$, then we define

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$$

♣

We now have the following claim.

Proposition 1.16

We have

$$r(a) = \|\widehat{a}\|_\infty$$

♠

Proof We already know that for every $a \in \mathcal{A}$, $\widehat{a}(\varphi) = \varphi(a) \in \sigma(a)$, hence $\|\widehat{a}\|_\infty \leq r(a)$.

Lemma 1.2

For $a \in \mathcal{A}$, we have

$$\sigma(a) \subset Range(\widehat{a})$$

♡

Proof Suppose λ is not in the range of $\widehat{a}$, then $\widehat{a} - \lambda(\varphi) \neq 0$ for all φ, hence

$$\widehat{a} - \lambda \text{ is invertible} \Rightarrow a - \lambda \text{ is invertible}$$

Hence $\lambda \notin \sigma(a)$. Hence this implies $\lambda \in \sigma(a)$ implies $\lambda = \varphi(a)$ for some a.

□

Hence $r(a) \leq \|\widehat{a}\|_\infty$. Hence $r(a) = \|\widehat{a}\|$.

□

In class we saw if $\|a^2\| = \|a\|^2$, then

$$r(a) = \|a\|$$

Now we connect this with the Gelfand transform.

Proposition 1.17

The Gelfand transform is an isometry i.e. $\|\widehat{a}\| = \|a\|$ if and only if

$$\|a^2\| = \|a\|^2$$

♠

Proof We have $\|\widehat{a}\| = r(a)$, and by the previous remark, we already have one direction. Now we want to show if $r(a) = \|a\|$, then $\|a^2\| = \|a\|^2$.

Lemma 1.3 (Spectral mapping theorem)

For $a \in \mathcal{A}$, we have

$$\varphi(\sigma(a)) = \sigma(\varphi(a))$$

♡

what Hence we have $r(a^2) = (r(a))^2$, then we have

$$\|a^2\| = r(a^2) = (r(a))^2 = \|a\|^2$$

□

Now we enter the realm of Hilbert spaces.

Theorem 1.4

For $T \in \mathcal{B}(\mathcal{H})$ if $\langle T\xi, \xi\rangle = 0$ for all $\xi \in \mathcal{H}$, then we have $T = 0$

Remark This is proved by polarization.

Proposition 1.18

By the same reasoning, if $\langle T\xi, \xi\rangle$ is real for all ξ, then $T = T^*$.

Proof

$$\langle T\xi, \xi\rangle = \langle \xi, T^*\xi\rangle = \langle T^*\xi, \xi\rangle$$

By the previous theorem, we know $T = T^*$.

Proposition 1.19

If we have $\|T\xi\| \geq a\|\xi\|$, and similarly $\|T^*\xi\| \geq b\|\xi\|$, then we have T is invertible.

Proof For $T\xi = 0$, we have $\xi = 0$, hence T is injective. And similarly, T^* is injective, and we have

$$\ker T^* = (Range(T))^{\perp} = \{0\}$$

Hence we have $Range(T)$ is dense in $\mathcal{H}$. Thus, by T is injective, we can define T^{-1} on $Range(T)$. It now suffices to show that T^{-1} is bounded on $Range(T)$, then it will extend. Let $\xi \in Range(T)$, we have

$$\|\xi\| = \|TT^{-1}\xi\| \geq a\|T^{-1}\xi\|$$

Hence T^{-1} is bounded on a dense subset of $\mathcal{H}$.

We recall both big and small Gelfand-Naimark theorems.

Theorem 1.5 (Little Gelfand-Naimark theorem)

Let $\mathcal{A}$ be a unital commutative C^*-algebra, then it is isomorphic to $C(\widehat{\mathcal{A}})$, via the Gelfand transform $a \mapsto \widehat{a}$.

Theorem 1.6 (Big Gelfand-Naimark theorem)

*Let $\mathcal{A}$ be a commutative *-algebra, then there exists a *-representation π of $\mathcal{A}$ on such that*

$$\mathcal{A} \cong \{\pi(a) : a \in \mathcal{A}\}$$

This is to say every C^*-algebra is isomorphic to another C^*-algebra of bounded operators on a Hilbert space.

1.0.7 GNS construction

Let $\mathcal{A}$ be a unital commutative C^*-algebra, and we next define a correspondance between the cyclic *-representations of $\mathcal{A}$ and the states on $\mathcal{A}$.

Definition 1.9 (*-representation)

Let $\mathcal{A}$ be a unital commutative C^*-algebra, and let π be a non-degenerate representation on $\mathcal{A}$, such that π takes involution on $\mathcal{A}$ to involution of operators.

Note *For*

Definition 1.10 (Cyclic vector)

For $\xi \in \mathcal{H}$, and $\xi \neq 0$, it is called a cyclic vector if the set

$$\{\pi(a)\xi : a \in \mathcal{A}\}$$

is norm dense in $\mathcal{H}$. Then we call π a cyclic representation.

Not all Hilbert spaces have a cyclic representation of course, but they can be dissected into orthogonal subspaces that are π-cyclic.

Proposition 1.20

*Given a *-representation π of $\mathcal{A}$ on $\mathcal{H}$, and take $\xi \neq 0, \xi \in \mathcal{H}$, and $K = \overline{\{\pi(a)\xi : a \in \mathcal{A}\}}$, then K is called the π-cyclic subspace of $\mathcal{H}$. For any $\mathcal{H}$, there exists a family of orthogonal π-cyclic subspaces such that*

$$\mathcal{H} = \bigoplus_{\lambda} K_{\lambda}$$

Remark Any nonzero vector in an irreducible representation is cyclic.

Definition 1.11 (state)

*Let φ be a linear functional on $\mathcal{A}$, we say φ is **positive** if for all $a \in \mathcal{A}$, we have*

$$\varphi(aa^*) \geq 0$$

Moreover, if φ has norm 1, then we call them states.

Proposition 1.21

*Let ξ be a cyclic vector, and π a *-representation (or any $\xi \neq 0$), and the map $a \mapsto \langle \pi(a)\xi, \xi \rangle$ is a state on $\mathcal{A}$.*

Proof Denote this map as $\pi(a) = \langle \pi(a)\xi, \xi \rangle$.

$$\varphi(a^*a) = \langle \pi(a^*a)\xi, \xi \rangle = \langle \pi(a)\xi, \pi(a)\xi \rangle \geq 0$$

□

Is it true that $\varphi(a) = \overline{\varphi(a^*)}$ for φ a positive linear functional on $\mathcal{A}$. If $\varphi(a) = \varphi(a^*)$, but if $\lambda \in \mathbb{C}$, we have $\varphi(\lambda a) = \lambda(\varphi(a)) = \lambda\varphi(a^*)$, but $\varphi((\lambda a)^*) = \overline{\lambda}\varphi(a^*)$.