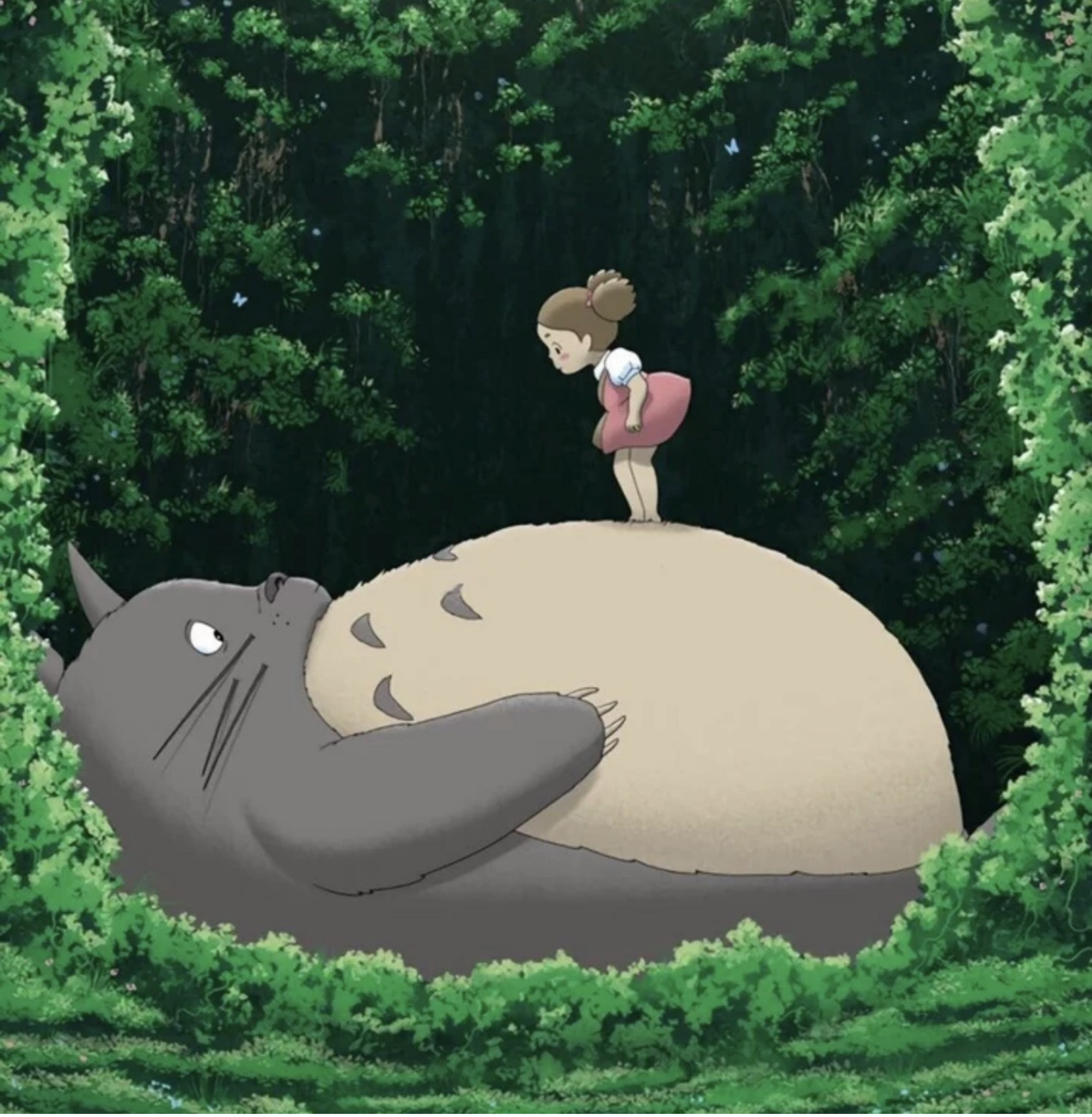

# Functional Analysis

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# Contents

This is for myself.

## 0.1 Chapter 1: Hilbert Spaces

-orthogonality

-riesz representation

We will state the Riesz representation theorem here, if $T : \mathcal{H} \to \mathbb{F}$ is a bounded linear functional, then there is a unique vector $h_0$ in $\mathcal{H}$ such that

$$T(h) = \langle h, h_0 \rangle, \forall h \in \mathcal{H}$$

Moreover, we have

$$\|T\| = \|h_0\|$$

-orthonormal sets of vectors and bases -isomorphic hilbert spaces and the fourier transform for the circle

We define an isomorphism between two Hilbert spaces as follows:

**Definition 0.1**

*Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, then an isomorophism $U$ is a surjective isometry, i.e.*

$$\langle Uh, Ug \rangle = \langle h, g \rangle$$

*Here we use $U$ to highlight such isomorphism is also called the unitary operator.* ♣

We conclude this section by a few theorems regarding the Fourier transform.

**Proposition 0.1**

*$\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is a basis for $L^2_{\mathbb{C}}[0, 2\pi]$.* ♠

And we also have

**Theorem 0.1**

*The Fourier transform is a linear isometry from $L^2_{\mathbb{C}}[0, 2\pi]$ to $l^2(\mathbb{Z})$.* ♡

Then we talk a little bit about the direct sum of hilbert spaces

## 0.2 Chapter 2: Operators on Hilbert Space

-We start with some properties of operators on the Hilbert space, such as the Schur's test.

**Proposition 0.2 (Schur's test)**

*If we have a kernel $k(x, y)$, and such that*

$$\int |k(x, y)|d\mu(x) \le c_1$$

$$\int |k(x, y)|d\mu(y) \le c_2$$

*Then if we define an operator $K : L^2(\mu) \to L^2(\mu)$ as*

$$T(f)(x) = \int k(x, y)f(y)d\mu(y)$$

*Then we have $L^2$ boundedness of $T$, and that*

$$\|T\| \le (c_1c_2)^{1/2}$$

♠

# Chapter 1 Lecture 1

Here we go.

### 1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

**Textbook**: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

**Remark** Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

### 1.0.2 Motivation

Let $X$ be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functons on $X$ mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If $T$ is a self-adjoint operator on a Hilbert space, then we take the product of $T$ (polynomials of $T$), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by $T$ and $I$ the identity operator, then take the closure, i.e. making it closed in the operator norm.

**Remark** The $*$ is to remind us, $T$ is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

**Proposition 1.1**

*We have the next two algebra isomorphic to each other.*

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \tag{1.1}$$

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_{\mathcal{H}}) \cong C(X) \tag{1.2}$$

### 1.0.3 Groups

Let $G$ be a group, $B$ be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

**Definition 1.1**

*Suppose that $\alpha$ is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on $B$ or an action of the group $G$ on $B$.*

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on $B$, generated by

$$\{\alpha_x : x \in G\}$$

**Remark** The identity on $G$ should be mapped into the identity operator on $B$, hence no need to include it.

Elements of the form $\Sigma_{z_x} \alpha_x, z_x \in \mathbb{C}$, (where $\Sigma$ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x \in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many $x$, $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f \alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from $\alpha$ being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f \alpha_g = \sum_x \sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f * g)(y)\alpha_y$$

where we define $f * g(y) = \sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \|\sum f(x)\alpha_x\| \leq \sum |f(x)|\|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For $G$ commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x\xi(y) = \xi(x^{-1}y), \alpha_x$ being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via $\alpha$.

If $G$ is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where $X$ is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.