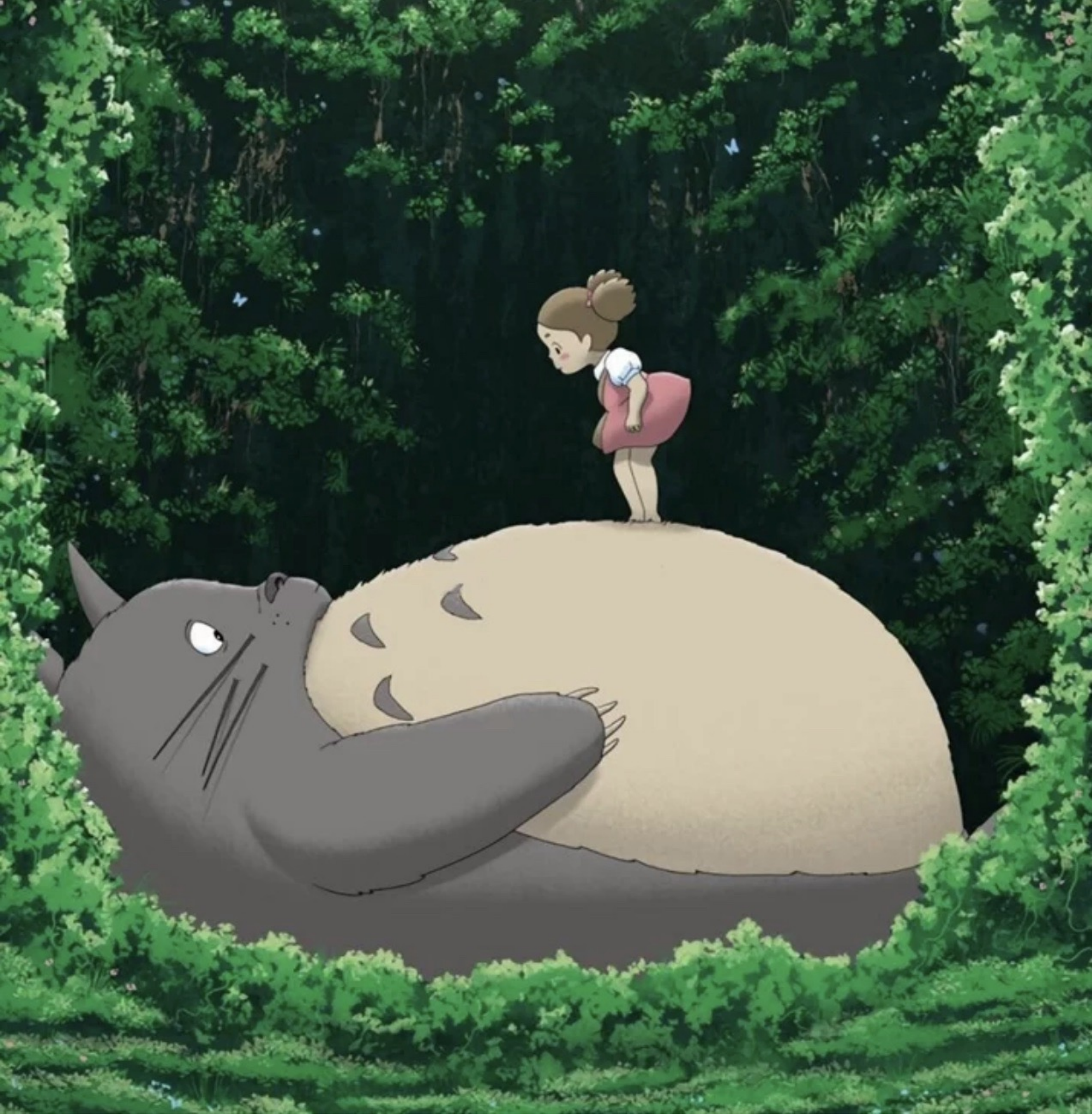

Functional Analysis

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Chapter 1 Prep work

We will start from the beginning and take baby steps. It's going to be okay.

An algebra is a vector space (with addition and scalar multiplication, usually over $\mathbb{R}, \mathbb{C}$), with an extra multiplication operation such that it is associative, and distributive. Then a normed algebra is an algebra with a sub-multiplicative norm, such that for all $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

A Banach algebra is a normed algebra that is complete under the metric induced by the norm. And we can form a Banach algebra by starting with a normed algebra and form its completion and by uniform continuity of addition and multiplicatoin extend to the completion of the algebra to form a Banach algebra.

We will begin with some important examples of Banach algebras. Let X be a compact topological space, and let $C(X)$ be the space of continuous functions, equip it with $\|\cdot\|_{L^\infty}$ norm, then $(C(X), \|\cdot\|_{L^\infty})$ is a Banach algebra. Similarly, if X is only locally compact, then $C_b(X)$, the space of bounded continuous functions under the $\|\cdot\|_{L^\infty}$ norm is also a Banach algebra.

1.0.1 Some Banach algebra examples

Another important example is that let X be a Banach space, and the space of all bounded/continuous operators on X, denoted by $\mathcal{B}(X)$ is a Banach algebra with the operator norm. Any closed subalgebra of $B(X)$ is also Banach.

If X is a Hilbert space, then we also have the operation of taking adjoints, namely $\|T\| = \|T^*\|$.

Definition 1.1

A C^* algebra is a closed subalgebra of the space of bounded (equivalently) functions defined on a Hilbert space, $\mathcal{B}(\mathcal{H})$. ♣

Remark The space of continuous/bdd operators on a Hilbert space, under the operator norm, then closed under the norm topology and taking adjoints of the operators. On wikipedia, C* algebra is defined to be a Banach algebra equipped with an involution that acts like a adjoint.

One of the goals of this course is to develop the following theorem.

Theorem 1.1

*Let $\mathcal{A}$ be a commutative C^*-algebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is isometrically and *-algebraically isomorphic to some $C(X)$, where X is some locally compact space.* ♡

Proposition 1.1

Multiplication is continuous in Banach algebras. ♠

Proof Multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, hence if we have x_n, y_n such that $x_n \to x, y_n \to y$, then we have

$$\|x_n y_n - xy\| \leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| < \epsilon$$

Hence multiplication is continuous.

Definition 1.2 (Unital Banach algebra and invertibility)

A Banach algebra (let's repeat, a complete vector space with addition, scalar multiplicatin, and multiplication such that the norm is sub-multiplicative) is called unital if there exists a multiplicative inverse.

An element $a \in \mathcal{A}$ is called invertible if there exists an element $a^{-1} \in \mathcal{A}$ such that

$$aa^{-1} = a^{-1}a = e$$

♣

Regarding invertibility, we can determine whether an element is invertible by knowing a related element's norm.

Proposition 1.2

Let $\mathcal{A}$ be a unital Banach algebra, and if $\|a\| < 1$, then $(1-a)$ is invertible.

Proof We would like to use the fact that every Cauchy sequence converges. Define

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$$

where $a^0 = 1$ by definition. We first show that this geometric series converges to an element in $\mathcal{A}$, and we will show that the quantity defined above is indeed the inverse of $(1-a)$.

Note that we define the partial sum $S_N = \sum_{n=0}^{N} a^n$, then

$$\|S_N - S_M\| \leq \sum_{M+1}^{N} \|a\|^n < \epsilon$$

Hence $\{S_N\}$ is a cauchy sequence, hence converges to some element which we denoted as $(1-a)^{-1} \in \mathcal{A}$. Now

$$(1-a) \cdot (1-a)^{-1} = (1-a) \cdot \lim_{N \to \infty} S_N = (1-a) \cdot \frac{1}{1-a} = 1$$

Likewise for the other side. Notice our $(1-a)^{-1}$ is a defined quantity, while $\frac{1}{1-a}$ is the sum of geometric series.

$\square$

Corollary 1.1

Let $\mathcal{A}$ be a unital Banach algebra, then if $\|(1-a)\| < 1$, then we have, a is invertible.

The implication of this corollary is interesting.

Corollary 1.2

The open ball of radius 1 around the identity element $1_{\mathcal{A}}$ consists of invertible elements.

$$\|1-a\| < 1 \Rightarrow a \in B_1(1_{\mathcal{A}})$$

And we know a is invertible.

Proposition 1.3

The set of invertible elements of a unital Banach algebra is an open subset.

Proof We use the fact that $B_1(1_{\mathcal{A}})$ is an open set. Note that for any invertible element d, we define the map, for all $a \in \mathcal{A}$,

$$L_d(a) = da$$

We observe this map is continuous, and by d be invertible, the inverse is also continuous, hence a homeomorphism. Bijectivity follows from $da = db \Rightarrow a = b$, and for every $c \in \mathcal{A}$, we can find $a = d^{-1}c$ such that $L_d(a) = c$.

Hence for every d invertible, we have $d \cdot O$ an open ball of invertible elements, and taking all union of these open balls give us the set of invertible elements, which is an open set.

$\square$

Proposition 1.4

For $f \in C(X)$, we have α is in the range of f if and only if $(f - 1 \cdot \alpha)$ is invertible.

Proof Refer to the lecture notes. In function spaces, the word **invertible** means having trivial kernel, i.e. $f(x) = 0$ implies $x = 0$.

$\square$

1.0.2 Algebra homomorphisms on $C(X)$

Definition 1.3 (Algebra homomorphism)

An algebra homomorphism is a homomorphism between two algebras. For example, consider X a compact space, and $C(X)$ the space of continuous functions, hence if we define the evalutation map as follows:

$$\varphi_x(f) = f(x)$$

This is an algebra homomorphism between $C(X)$ and $(\mathbb{C})$. Namely, the homomorphism property is justified as: (under both addition and multiplication)

$$\varphi_x(f+g) = f+g(x) = f(x)+g(x) = \varphi_x(f)+\varphi_x(g)$$
$$\varphi_x(fg) = (fg)(x) = f(x)g(x) = \varphi_x(f)\varphi_x(g)$$

And of course, same thing follows for scalar multiplication. ♣

Remark We need to check all three conditions to make sure such φ preserves the structures between the algebras.

An algebra homomorphism is called unital if if maps the (multiplicative identity) unity to unity. In the above example, a unital homomorphism would be $\varphi(1) = 1$, where the left 1 is the constant 1 function, and the right 1 is the number.

Now we will introduce the proposition that every multiplicative linear functional on $C(X)$. Note we can use algebra homomorphism and multiplicative linear functional synonomously on $C(X)$, hence they entail the same information.

Proposition 1.5

Let φ be a multiplicative linear functional on $C(X)$, i.e. a nontrivial algebra homomorphism, then $\varphi(f) = f(x_0)$ for some $x_0 \in X$. In other words, a multiplicative linear functional always takes this form. ♠

Proof It suffices to show the following lemma:

Lemma 1.1

There exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. ♡

We will first show how the lemma implies $\varphi(f) = f(x_0)$. Consider the function $f - \varphi(f)\cdot 1$, then we know

$$\varphi(f - \varphi(f)\cdot 1) = 0$$

Then there exists x_0 such that $f(x_0) - \varphi(f) = 0$, this gives $\varphi(f) = f(x_0)$.

Now we prove the lemma.

Proof Our claim is that there exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. Assume the contrary, which states for all x, there exists an f_x such that $\varphi(f_x) = 0$, but $f(x) \neq 0$. We define a nonnegative function $g_x = f_x\overline{f_x}$. And by multiplicativity, we have $\varphi(g_x) = 0$. We now note that because g is continuous, in a small nbd of x, denoted by O_x, we have $g(y) > 0$ for all $y \in O_x$.

Now using compactness, we can write X as a finite union of small neighborhoods $X = \bigcup_{j=1}^n O_{x_j}$, and define

$$g = g_{x_1} + ... + g_{x_n}$$

Then for each $y \in X$, $y \in O_{x_j}$ for some j, hence $g(y) > 0$ for all $y \in X$. This implies that g is invertible hence we have

$$\varphi(g \cdot 1/g) = 1$$

This contradicts with the fact that $\varphi(g) = 0$. And we are done.

□

Hence we have the following corollary.

Corollary 1.3

Let X be compact, and $C(X)$ the space of continuous functions, then φ is a multiplicative linear functional (i.e. a algebra homomorphism with $\mathbb{C}$) if and only if it is a point evaluation. ♡

Definition 1.4 ($\widehat{\mathcal{A}}$)

Given a unital commutative (or Banach) algebra, for example, $C(X)$ with $\|\cdot\|_{L^\infty}$, we define the set of unital homomorphisms, i.e., nonzero unital multiplicative linear functionals on $\mathcal{A}$ as $\widehat{\mathcal{A}}$. ♣

Proposition 1.6

If $\mathcal{A}$ is a unital algebra, then for $\varphi \in \widehat{\mathcal{A}}$, we have $\|\varphi\| = 1$ ♠

Proof We have

$$\|\varphi\| = \sup\{|\varphi(f)| : \|f\|_{L^\infty} = 1\}$$

Because $|\varphi(f)| = |f(x_0)|$ for some x_0, we always have $\|\varphi\| \leq 1$, but with the unity, we have $|\varphi(e)| = 1$, and taking the sup we have $\|\varphi\| = 1$.

□

1.0.3 Spectrum

We now define the spectrum of an element in a Banach algebra.

Definition 1.5 (spectrum)

Let $\mathcal{A}$ be a Banach algebra, fix $a \in \mathcal{A}$, we define the following set to be the spectrum of a, denoted by $\sigma(a)$.

$$\sigma(a) = \{\lambda \in \mathbb{F} : a - \lambda \cdot 1_{\mathcal{A}} \text{ is not invertible } \}$$ ♣

We have a bound on the size of λ given $\|a\|$.

Proposition 1.7

For $\lambda \in \sigma(a)$, we have

$$|\lambda| \leq \|a\|$$ ♠

Proof Assume the contrary, we have $|\lambda| > \|a\|$, then a/λ has norm $\|a/\lambda\| < 1$. Thus, $(1 - a/\lambda)$ is invertible.

$$a - \lambda \cdot 1 = -\lambda(1 - a/\lambda)$$

Because the product of two invertible elements is again, invertible, we get that $\lambda \notin \sigma(a)$. Hence a contradiction.

□

Proposition 1.8

Let $\mathcal{A}$ be a unital Banach algebra, and let $\varphi \in \widehat{\mathcal{A}}$, then we have

$$\varphi(a) \in \sigma(a)$$ ♠

Proof It suffices to show that $a - \varphi(a) \cdot 1$ is not invertible. Assuming that it is, denote its inverse by $(a - \varphi(a))^{-1}$, then

$$\varphi\left((a - \varphi(a)1)\frac{1}{a - \varphi(a)}\right) = 1$$

However, $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence a contradiction.

□

Remark To prove an element $a \in \mathcal{A}$ is not invertible, it suffices to prove $\varphi(a) = 0$.

Corollary 1.4

For the above, $|\varphi(a)| \leq \|a\|$, and again, $\|\varphi\| = 1$. ♡

Remark This is to say, every unital homomorphism $\varphi \in \mathcal{A}$ is continuous.

We now show that the spectrum of an element is always closed.

Proposition 1.9

Let $a \in \mathcal{A}$, then $\sigma(a)$ is closed.

Proof We define a map $\phi : \mathbb{F} \to \mathcal{A}$ as

$$\phi(\lambda) = a - \lambda \cdot 1$$

The map is continuous, and we notice that the $\sigma(a)$ is the complement of the preimage of invertible elements under ϕ, i.e.

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Using the fact that the set of invertible elements is open, we get $\sigma(a)$ is closed.

□

1.0.4 Weak-* topology

We now do some topology. Fix $\mathcal{A}$, Recall the weak-* topology is defined on $\mathcal{A}'$ and it is the weakest topology such that the map $\psi \in \mathcal{A}'$,

$$\psi \mapsto \psi(a) \text{ continuous}$$

We first note that if $\varphi \in \widehat{\mathcal{A}}$, then $\|\varphi\| = 1$. Hence $\widehat{\mathcal{A}}$ is a subset of the closed unit ball in $\mathcal{A}'$. Now with respect to the weak-* topology, we have some nice properties.

Theorem 1.2

$\widehat{\mathcal{A}}$ is closed with respect to the weak- topology.*

Proof Let $\{\varphi_\lambda\}$ be a net that converges to some φ in the weak-* topology, which is a linear functional, i.e. $\varphi \in \mathcal{A}'$. Weak-* convergence implies for all $a \in \mathcal{A}$, we have

$$\varphi_\lambda(a) \to \varphi(a)$$

We show that φ is multiplicative.

$$\varphi(ab) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a) \lim \varphi_\lambda(b) = \varphi(a)\varphi(b)$$

Now it remains to show that $\|\varphi\| = 1$ to show that it is closed. It suffices to show φ is unital.

$$\varphi(1) = \lim \varphi_\lambda(1) = 1$$

Hence $|\varphi(1)| \leq \|\varphi\|$, hence $\|\varphi\| = 1$.

□

Now we recall Alaoglu's theorem.

Theorem 1.3 (Alaoglu's)

The closed unit ball is compact in the weak- topology.*

Hence as an immediate corollary,

Corollary 1.5

$\widehat{\mathcal{A}}$ is compact with respect to the weak- topology.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□