



# Functional Analysis

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# Chapter 1 Lecture 1

Here we go.

## 1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

**Textbook:** an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable number of seminorms).

**Remark** Let  $\mathcal{H}$  be a Hilbert space, then the dual space  $\mathcal{H}^*$  is itself.  $\mathcal{H} = \mathcal{H}^*$ . Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themselves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

## 1.0.2 Motivation

Let  $X$  be a compact Hausdorff space. Let  $C(X) = \{f : X \rightarrow \mathbb{R}, f \text{ continuous}\}$  be the algebra of continuous functions on  $X$  mapping in to  $\mathbb{R}$  or  $\mathbb{C}$ . Define the norm as the sup norm  $\|\cdot\|_{L^\infty}$ .

We will develop the spectral theorem of operators on the Hilbert space, i.e. self-adjoint operators can be diagonalized.

If  $T$  is a self-adjoint operator on a Hilbert space, then we take the product of  $T$  (polynomials of  $T$ ), let  $C^*(T, I_{\mathcal{H}})$  be the sub-algebra of operators generated by  $T$  and  $I$  the identity operator, then take the closure, i.e. making it closed in the operator norm.

**Remark** The  $*$  is to remind us,  $T$  is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

### Proposition 1.1

*We have the next two algebra isomorphic to each other.*

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \quad (1.1)$$

This is what we are aiming for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If  $T_1, \dots, T_n$  is a collection of self-adjoint operators on  $\mathcal{H}$ , and such all commute with each other, then we also have

$$C^*(T_1, \dots, T_n, I_{\mathcal{H}}) \cong C(X) \quad (1.2)$$

## 1.0.3 Groups

Let  $G$  be a group,  $B$  be a Banach space, for example, groups of automorphisms. Let

$$\text{Aut}(B) = \{T : T \text{ is isometric, onto, invertible on } B\}$$

### Definition 1.1

*Suppose that  $\alpha$  is a group homomorphism, and  $\alpha : G \rightarrow \text{Aut}(B)$ , is called a representation on  $B$  or an action of the group  $G$  on  $B$ .*

Then we can consider the subalgebra  $\mathcal{L}(B)$ , consisting of the bounded linear operators on  $B$ , generated by

$$\{\alpha_x : x \in G\}$$

**Remark** The identity on  $G$  should be mapped into the identity operator on  $B$ , hence no need to include it.

Elements of the form  $\sum_{z \in \Sigma} \alpha_x, z_x \in \mathbb{C}$ , (where  $\Sigma$  is a finite sum.)

Let's introduce,  $f \in C_c(G)$  are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x \in G} f(x) \alpha_x = \alpha_f$$

note for except finitely many  $x$ ,  $f(x) = 0$ .

Let  $f, g \in C_c(G)$ , then for

$$\alpha_f \alpha_g = \left( \sum f(x) \alpha_x \right) \left( \sum g(y) \alpha_y \right) = \sum_{x,y} f(x) g(y) \alpha_x \alpha_y = \sum_{x,y} f(x) g(y) \alpha_{xy}$$

The last inequality follows from  $\alpha$  being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f \alpha_g = \sum_x \sum_y f(x) g(x^{-1}y) \alpha_y = \sum (f * g)(y) \alpha_y$$

where we define  $f * g(y) = \sum f(x) g(x^{-1}y)$  as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f * g}$$

This is how we define convolution on  $C_c(G)$  Notice we have, by  $\|\alpha_x\| = 1$ ,

$$\|\alpha_f\| = \left\| \sum f(x) \alpha_x \right\| \leq \sum |f(x)| \|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We get  $l^1(G)$  is an algebra with ??

For  $G$  commutative, it is easily connected with the Fourier transform.

Consider  $l^2(G)$  with the counting measure on the group. For  $x \in G$ , let  $\xi \in l^2(G)$  define  $\alpha_x \xi(y) = \xi(x^{-1}y)$ ,  $\alpha_x$  being unitary.  $l^1(G)$  acts on operators in  $l^2(G)$  via  $\alpha$ .

If  $G$  is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where  $X$  is some compact space. Note that  $C_c(G)$  operators on  $l^2(G)$ , and  $\|\alpha_f\| \leq \|f\|_{l^1}$ .

## 1.1 Lecture 2

Let's do some math.

Let  $X$  be a Hausdorff compact space, and let  $C(X)$  denote the space of continuous functions defined on  $X$ . This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm  $\|\cdot\|_{L^\infty}$ . Note  $X$ , by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from  $C(X)$ .

### Definition 1.2 (Normed Algebra)

Let  $\mathcal{A}$  be an algebra on  $\mathbb{R}$  or  $\mathbb{C}$ , is a normed algebra if it has a norm  $\|\cdot\|$ , as a vector space, such that for  $a, b \in \mathcal{A}$ , we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity.

### Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm.

Given  $x \in X$ , define  $\varphi_x : C(X) \rightarrow \mathbb{C}$  the evaluation map such that

$$\varphi_x(f) = f(x)$$

$\varphi_x$  is an algebra homomorphisms between  $C(X) \rightarrow \mathbb{R}$  or  $C(X) \rightarrow \mathbb{C}$ . This simply implies

$$\varphi_x(f + g) = (f + g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that,  $C(X)$  has an identity element, which is the constant function 1, under multiplication. Hence  $C(X)$  is a unital algebra. Note that  $\varphi_x$  defined above is a unital homomorphism, meaning that it sends identity to identity.

Note  $\varphi_x$  is also a multiplicative linear functional, also unital.

### Proposition 1.2

Every multiplicative linear functional on  $C(X)$  is of the form  $\varphi_x$  for some  $x \in X$ .

**Proof** Main Claim: given a multiplicative linear functional  $\varphi$ , there exists a point  $x_0$  and if we have some  $f \in C(X)$ , we have  $\varphi(f) = 0$ , then we have  $f(x_0) = 0$ . To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each  $x \in X$ , there is an  $f_x \in C(X)$  such that  $f_x(x) \neq 0$ , but  $\varphi(f_x) = 0$ .

Set  $g_x = \overline{f_x} f_x$ , then we have  $g_x(x) > 0$ , but  $\varphi(g_x) = \varphi(f_x) \varphi(\overline{f_x}) = 0$ , then there is an open set  $O_x$  such that  $x \in O_x$ , and  $g_x(y) > 0$  for all  $y \in O_x$ . Now by compactness, there is  $x_1, \dots, x_n$  such that  $X = \bigcup_{j=1}^n O_{x_j}$ , let  $g = g_{x_1} + \dots + g_{x_n}$ , then we have  $g(y) > 0$  for all  $y \in X$ , and  $\varphi(g) = 0$ . Note that  $g$  is a continuous function, and  $g$  is invertible, and also  $re(\frac{1}{g}) \in C(X)$ , but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists  $x_0 \in X$  such that if  $\varphi(f) = 0$ , this means  $f(x_0) = 0$ . For any  $f$ , consider  $f - \varphi(f) \cdot 1$ , apply  $\varphi$ , we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies  $f(x_0) = \varphi(f)$  which implies  $\varphi(f) = \varphi_{x_0}(f)$ .

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.

For  $\mathcal{A} = C(X)$ , and  $\varphi \in \widehat{\mathcal{A}}$ .

### Definition 1.4

For any unital commutative algebra  $\mathcal{A}$  and let  $\widehat{\mathcal{A}}$  be the set of unital homomorphisms of  $\mathcal{A}$  into the field.

**Remark** We have  $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$ , since  $\varphi$  is unital, we have  $\|\varphi\| = 1$ .

This is not always true for normed algebra, Let

$$\mathcal{A} := \text{Poly} \subset C([0, 1])$$

We define  $\varphi(p) = p(2)$ ,  $p$  is a polynomial. This is not continuous, nor is the  $\|\varphi\| = 1$ .

### Proposition 1.3

If  $\mathcal{A}$  is a unital commutative Banach algebra, and if  $\phi \in \widehat{\mathcal{A}}$ , then we have  $\|\varphi\| = 1$ .

The word “unital” is key here.

### Proposition 1.4

Let  $\mathcal{A}$  be a unital Banach algebra (not necessarily commutative), then if  $a \in \mathcal{A}$ , and  $\|a\| < 1$ , then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

**Proof** For this, we use completeness.  $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ ,  $a^0 = 1_{\mathcal{A}}$ . You could look at the partial sums.  $S_m = \sum_{n=0}^m a^n$ , you want to show that  $\{S_m\}$  is a Cauchy sequence, and use completeness of Banach algebras.  $\lim_{m \rightarrow \infty} S_m = \frac{1}{1-a}$ .

To prove this is a Cauchy sequence:

$$\|S_n - S_m\| = \left\| \sum_{j=m+1}^n a^j \right\| \leq \sum_{j=m+1}^n \|a^j\| \leq \sum_{j=m+1}^n \|a\|^j$$

And the fact that  $\|a\| \leq 1$ , we have the sum bounded by  $\epsilon$ , hence  $\{S_n\}$  is a Cauchy sequence. Let  $b = \sum_{n=0}^{\infty} a^n$ , we want to show that  $b(1-a) = 1$ .

$$b(1-a) = \lim_{n \rightarrow \infty} S_n(1-a) = \lim_{n \rightarrow \infty} \left( \sum_{n=0}^{\infty} a^n \right) (1-a) = \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 0$$

The last inequality follows from  $\|a^{n+1}\| \leq \|a\|^{n+1} \rightarrow 0$ .

## 1.2 Lecture 3

We now begin.

Let  $\mathcal{A}$  be a unital Banach algebra, and if  $a \in \mathcal{A}$  and  $\|a\| < 1$ , then we have  $(1-a)$  has an inverse and if  $\mathcal{A} = \mathcal{B}(B)$ , where  $B$  is some Banach space, then  $T \in \mathcal{A}$ , and  $\|T\| < 1$ , then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Neumann series.

Now we have the following corollary.

### Corollary 1.1

If  $a \in \mathcal{A}$  and  $\|1-a\| < 1$ , then  $a$  is invertible.

**Proof**  $a = 1 - (1-a)$ .

### Proposition 1.5

The set of invertible elements of  $\mathcal{A}$  is an open subset of  $\mathcal{A}$ .

**Proof** The open ball about 1 consists of invertible elements. If  $d$  is any invertible element, then we define  $a \mapsto da$ . This map is continuous, i.e. it is the left representation  $L_b(a) = ab$  for all  $a \in \mathcal{A}$ . If  $d$  is invertible, then the inverse is also continuous, hence it is a homeomorphism of  $\mathcal{A}$  onto itself.

Denote the unit ball about 1 as  $B_1(1)$ , and let  $d$  be some invertible element, under  $L_d$ , homeomorphism,  $O \mapsto d \cdot O$ , this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.



□

**Proposition 1.6**

Let  $C(X)$  be the unital Banach algebra, and for  $f \in C(X)$ , we have  $\alpha \in \text{Range}(f)$  if and only if  $(f - \alpha \cdot 1)$  is not invertible.



**Proof** Let  $f \in C(X)$ , and if  $\alpha \in \text{range of } f$ , so  $\alpha = f(x_0)$  for some  $x_0$ . then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $(f - \alpha \cdot 1)$  is not invertible. Conversely, if we have  $f - \alpha \cdot 1$  is not invertible, then there exists  $x_0 \in X$  such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence  $f(x_0) = \alpha$ , i.e.,  $\alpha \in \text{range of } f$ .

□

**Definition 1.5 (spectrum of an element)**

For any unital algebra  $\mathcal{A}$  over some field  $\mathbb{F}$ , for any  $a \in \mathcal{A}$ , the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible} \}$$

is called the spectrum of  $a$ , denoted as  $\sigma(a)$ .



Interpret this in our familiar linear map:  $\lambda$  is called an eigenvalue, i.e. is in the spectrum of  $T$  if we have  $T - \lambda I$  is not invertible.

**Proposition 1.7**

Let  $\mathcal{A}$  be a unital Banach algebra, and let  $a \in \mathcal{A}$ , then if  $\lambda \in \sigma(a)$ , then

$$|\lambda| \leq \|a\|$$



**Proof** Suppose  $|\lambda| > \|a\|$ , then  $\lambda \neq 0$ , then

$$a - \lambda \cdot 1 = -\lambda \left(1 - \frac{a}{\lambda}\right)$$

And by assumption,  $\|a/\lambda\| \leq 1$ , hence  $(1 - a/\lambda)$  is invertible. Hence  $a - \lambda \cdot 1$  is invertible (product of two invertible elements), meaning  $\lambda \notin \sigma(a)$ .

□

**Proposition 1.8**

Let  $\varphi$  be a multiplicative linear functional on  $\mathcal{A}$ , i.e.  $\varphi \in \widehat{\mathcal{A}}$ , and then  $\varphi(a) \in \sigma(a)$ , and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$



**Proof**  $\varphi(a - \varphi(a) \cdot 1) = 0$ . Hence  $a - \varphi(a)1$  is not invertible.

□

**Proposition 1.9**

$\sigma(a)$  is a closed subset of  $\mathbb{R}, \mathbb{C}$ .



**Proof** Define the map  $\phi : \lambda \mapsto a - \lambda 1$ , the map  $\phi$  is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of  $\mathcal{A}$  is open, hence

$$\sigma(a) = \phi^{-1}(\text{noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{invertible})$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{invertible}))^c$$

Hence the spectrum of an element is closed.

□

Let  $\varphi \in \widehat{\mathcal{A}}$  then  $\|\varphi\| = 1$ . So  $\widehat{\mathcal{A}}$  is a subset of the unit ball of  $\mathcal{A}'$ , which denotes the dual vector space of continuous linear transformations.

On  $\mathcal{A}'$ , we can equip the weak-\* topology, i.e. the weakest topology, making the map  $\psi \mapsto \psi(a)$  continuous.

#### Proposition 1.10

$\widehat{\mathcal{A}}$  is closed for the weak-\* topology.



**Proof** let  $\{\varphi_\lambda\}$  be a net of elements of  $\widehat{\mathcal{A}}$ , that converges to some  $\psi \in \mathcal{A}'$  in the weak-\* topology, i.e., for every  $a \in \mathcal{A}$ ,  $\varphi_\lambda(a) \rightarrow \psi(a)$  for all  $a \in \mathcal{A}$ .

Then  $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$ .

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$ .

#### Theorem 1.1 (Alaoglu's theorem)

For any normed vector space  $V$ , the closed unit ball of  $V'$  is compact in the weak-\* topology.



As an immediate corollary, we have the following.

#### Corollary 1.2

$\widehat{\mathcal{A}}$  is compact with respect to the weak-\* topology.



**Proof**  $\widehat{\mathcal{A}}$  is a closed subset of a compact set, hence is also compact. □

Let  $\mathcal{A} = C(X)$ , and  $\widehat{\mathcal{A}}$ , we define  $x \mapsto \varphi_x$  is a bijection. The weak-\* topology in  $\widehat{\mathcal{A}}$  makes  $\varphi_x \mapsto \varphi_x(f) = f(x)$  continuous. Such  $x \mapsto \varphi_x$  is a homomorphism of  $X$  onto  $\mathcal{A}$ .

For  $\mathcal{A}$  unital Banach algebra, commutative, for any  $a \in \mathcal{A}$ , define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

#### Proposition 1.11

The map  $a \mapsto \widehat{a}$  is a unital algebra homomorphism from  $\mathcal{A}$  into  $C(\mathcal{A})$ .



**Proof** we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{ab})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$



## 1.3 Lecture 4

Today we talk about the structure of  $\widehat{l^1(S)}, \widehat{l^1(G)}$ , where  $S, G$  are semigroups and groups, and how they naturally identify with the unit disk  $\mathbb{D}$ , and the unit circle  $\mathbb{T}$ .

Let  $S$  be a commutative discrete semigroups, for example  $\mathbb{N} \cup \{0\}$ , and  $f \in C_c(S)$ , then we can write  $f = \sum_{x \in S} f(x)\delta_x$ , where we define  $\delta_x\delta_y = \delta_{xy}$ . Note that  $C_c(S)$  is dense in  $l^1(S)$ .

#### Definition 1.6 (Convolution)

Take any  $f, g \in C_c(S)$ , we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{y \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left( \sum_{xy=z} f(x)g(y) \right) \delta_z$$



where we define the convolution between two functions

$$f * g(z) = \sum_{x,y, xy=z} f(x)g(y)$$

And under this convolution operation, we have  $l^1(S), *$  as a Banach algebra.

**Example 1.1** If we consider polynomials of the form  $f(x) = \sum_{n=0}^{\infty} f(n)x^n$ , and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right) \left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f * g)(p)$$

And let  $f \in C_c(S)$ , where  $S = \mathbb{N}$ . we define  $\|f\|_{l^1} = \sum_{x \in S} |f(x)|$ .

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We let  $\mathcal{A} = l^1(S)$ , and  $\widehat{\mathcal{A}}$  denote the set of unital homomorphisms from  $\mathcal{A}$  to  $\mathbb{R}, \mathbb{C}$ . Note that  $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$ .

Note that we know  $(l^1(S))' = l^\infty(S)$ , hence  $\widehat{\mathcal{A}} \subset \mathcal{A}'$ . Note that we have  $\|\varphi\| = 1$ , hence if we  $\varphi \in l^\infty(S)$ , we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for  $z \in S, \|z\| \leq 1$ , we have  $|\varphi(z)| \leq 1$ .

#### Proposition 1.12

We naturally identify  $\widehat{l^1(S)}$  with  $\text{Hom}(S, \mathbb{D})$ , i.e.  $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$ .

**Proof** Given  $f \in \widehat{l^1(S)}$ , we know it's multiplicative, unital, hence all these transfer when viewing  $\varphi \in l^\infty(S)$ . This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here  $xy$  denotes the operation on  $S$  between  $x, y$ , for example, could be  $x + y$ . Hence naturally, if  $\varphi \in \widehat{l^1(S)}$ ,  $\varphi$  can also be viewed as  $\varphi : S \rightarrow \mathbb{D}$ , and thus is in  $l^\infty$ , with  $|\varphi(s)| \leq 1$ . □

Furthermore, we can identify elements in  $\widehat{l^1(S)}$  with the unit disk. Take  $S = \mathbb{N}$ .

#### Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where  $\mathbb{D}$  denotes the unit disk in  $\mathbb{C}$ .

**Proof** We motivate this by noticing  $\mathbb{N}$  is generated by 1, and thus viewing  $\varphi \in \widehat{l^1(\mathbb{N})}$  as  $\varphi \in l^\infty(\mathbb{N})$ , we have  $\varphi$  is determined by  $\varphi(1)$ . And denote  $\varphi(1) = z_0$ , then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for  $z \in \mathbb{D}$ ,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space. □

#### Proposition 1.14

The standard topology on  $\mathbb{D}$  coincides with the weak-\* topology on  $\widehat{l^1(\mathbb{N})}$ .

$$D_{std} \cong D_{weak-*}$$

**Proof** We just need to associate an element in  $\mathbb{D}$  with a function  $\varphi \in \widehat{l^1(\mathbb{N})}$ . And we do this by

$$z \mapsto \sum_{n \in \mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

### 1.3.1 On groups

We let  $G$  denote a discrete commutative group, and we see everything above follows, with one extra property.

#### Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where  $\mathbb{T}$  denotes the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ .

**Proof** For  $\varphi \in \widehat{l^1(G)}$ , we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because  $|\varphi(x)| \leq 1, \forall x$ , Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have  $\widehat{l^1(G)}$  naturally identifies with  $\mathbb{T}$ . Like what we described above, we have what is desired. □

**Remark** Take  $G = \mathbb{Z}$ , if we denote  $z \in \mathbb{T}$  as  $z = e^{2\pi it}$ , then we naturally identify with

$$\sum_{n \in \mathbb{Z}} f(n)e^{2\pi int}$$

we denote this mapping as  $\widehat{f}$ , i.e.

$$\widehat{f}(z) = \sum_{m \in \mathbb{Z}} f(m)e^{2\pi imt}$$

This is the Fourier transform.

## 1.4 Lecture 5

Last time, we talked about if we denote  $\mathcal{A} = l^1(G)$ , equipped with  $\|\cdot\|_{l^1}$ , under convolution, we have

$$\widehat{\mathcal{A}} \cong \text{Hom}(G, \mathbb{T})$$

If we take  $G = (\mathbb{Q}, +)$ , one can ask the question if  $\widehat{\mathcal{A}}$  is big enough. And we will see later in the course, the answer is yes.

For pointwise multiplication,  $\widehat{G}$  forms a group, and in fact  $\widehat{G}$  is a compact topological group.

For any compact commutative group  $G$ , for example  $\mathbb{R}^n$  under  $+$ . Define

$$\widehat{G} = \text{continuous homomorphisms into } \mathbb{T}$$

**Remark** We now require continuous with this general  $G$  (previously was not required for discrete group  $G$ ).

#### Proposition 1.16

Let  $G$  be a locally compact and commutative group, we have  $\widehat{G}$  as a locally compact, commutative group.

We define the pairing between  $G$  and  $\widehat{G}$  as follows:  $x \in G, \varphi \in \widehat{G}$ ,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the following map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let  $G, H$  denote locally compact groups, and  $\phi : G \rightarrow H$  be a continuous homomorphism. Note we have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \widehat{G} & \xleftarrow{\phi} & \widehat{H} \end{array}$$

If we take an element  $\psi \in \widehat{H}$ , we consider  $\psi \circ \phi$ . We get  $\psi \circ \phi \in \widehat{G}$ .

#### Definition 1.7 (category, functor)

A category is specified by

1. a set of objects
  2. morphisms between objects
- (a).  $X, Y, Z$  are objects, and if

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

- (b). For each object  $X$ , there is an identity morphism  $1_X$ .

And a functor is defined to be such a morphism between categories.



**Example 1.2** For category of finite vector spaces  $V$ , passing from vector space to its dual  $V'$  is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ V' & \xleftarrow{T^t} & W' \\ V'' & \xrightarrow{T^{tt}} & W'' \end{array}$$

The map going in the same directions  $V \rightarrow W$ , and  $V'' \rightarrow W''$  is called covariant, whereas  $V' \leftarrow W'$  is called contravariant.

**Example 1.3** For category of locally compact groups  $G, H$ , assigning the dual group is a functor:

$$\begin{array}{ccc} G & \rightarrow & H \\ \widehat{G} & \leftarrow & \widehat{H} \\ \widehat{\widehat{G}} & \rightarrow & \widehat{\widehat{H}} \end{array}$$

**Example 1.4** Now let  $X$  be a compact space. Given  $\Phi$  continuous map between  $X \rightarrow Y$ .

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ C(X) & \leftarrow C(\Phi) & C(Y) \end{array}$$

For  $f \in C(Y)$ , we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Z \\ C(X) & \xleftarrow{C(\varphi)} & C(Y) \xleftarrow{C(\phi)} C(Z) \end{array}$$

where for  $f \in C(Y)$ ,  $C(\varphi)(f) = f \circ \varphi$ , and  $g \in C(Z)$ ,  $C(\phi) = g \circ \phi$ . This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between  $C(X)$  and  $C(Y)$ , there exists a map from  $X$  to  $Y$ .

**Proposition 1.17**

Suppose  $X, Y$  are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism  $\check{F} : X \rightarrow Y$ .



**Proof** Define  $\varphi_x : C(X) \rightarrow \mathbb{C}$  as the evaluation map: take  $f \in C(X)$ ,

$$\varphi_x(f) = f(x)$$

Then  $\varphi_x \circ F \in \widehat{C(Y)}$ . And we know that any element in  $\widehat{C(Y)}$  is a point evaluation, i.e. there exists  $y \in Y$  such that

$$\varphi_y = \varphi_x \circ F$$

We thus define  $\check{F}(x) = y$  as such that it satisfies the above equation. We need to show  $\check{F}$  is continuous. Note that  $X, Y$  are compact Hausdorff spaces, and the topology on  $Y$  is the coarsest topology making all functions  $g \in C(Y)$  continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by  $F, g$  being continuous, we have  $\check{F}$  is also continuous.

□

There is a natural bijection between the continuous functions from  $X$  to  $Y$ , and the unital algebra homomorphism from  $C(X)$  to  $C(Y)$ .

A quick reminder:

**Remark** For  $X$  compact, the weak-\* topology coincides with the standard topology.

## 1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let  $\mathcal{A}$  be a unital Banach algebra.

We write  $GL_n(\mathcal{A})$  to denote the general linear group, the group formed by  $n \times n$  matrices with entries from  $\mathcal{A}$ .

The less standard notation is  $GL_I(\mathcal{A})$  is the group of invertible elements in  $\mathcal{A}$ . As we have shown previously, this is a closed subset of  $\mathcal{A}$ . This is the notation that we will use.

**Remark** It is easy to see that the product is jointly continuous.

**Proposition 1.18**

The following map is continuous.

$$a \mapsto a^{-1}$$



**Proof** Given  $\|a - b\| < \delta$ , we would like to show  $\|a^{-1} - b^{-1}\| < \epsilon$ . We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\| \|b - a\| \|b^{-1}\|$$

Take  $\delta = \epsilon / \|a^{-1}\| \|b^{-1}\|$  would suffice.

□

**Proposition 1.19**

Fix  $a \in GL(\mathcal{A})$ , there exists a neighborhood  $O$  of  $a$  and a constant  $K$  such that for all  $y \in O$ , we have

$$\|c^{-1}\| < K$$



**Proof** Let  $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$ , then  $d$  is invertible and

$$d^{-1} = \sum_{n=0}^{\infty} (1 - d)^n$$

We thus have

$$\|d^{-1}\| \leq \frac{1}{1 - \|1 - d\|} \leq \frac{1}{1 - 1/2} = 2$$

We then identify what our  $O$  should be. Let  $O = aV$ , then we want to show that every  $ad$  has an inverse with bounded norm. Because  $a, d$  are both invertible,  $ad$  is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \leq \|d^{-1}\| \|a^{-1}\| \leq 2\|a^{-1}\|$$

□

**Remark** For each invertible element, we can find a neighborhood of invertible elements around it, and using that  $(1 - d)$  is bounded, then  $d$  is invertible, we can bound  $\|d^{-1}\|$ .

**Definition 1.8**

Fix  $a \in \mathcal{A}$ , the resolvent set of  $\mathcal{A}$  is the complement of spectrum of  $\mathcal{A}$ , i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$$



Hence the resolvent set is an open, unbounded subset of  $\mathbb{C}$  or  $\mathbb{R}$ .

**Definition 1.9 (Resolvent function)**

On the resolvent set,  $\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$  is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that  $a$  is fixed, and  $\lambda$  is the variable here.



Now we note that this  $R_a(\lambda)$  function is nicely behaved.

**Proposition 1.20**

The resolvent function  $R_a(z)$  is analytic on the resolvent set, and vanishes as  $z \rightarrow \infty$ .



**Proof** We first define the notation of analyticity on an open subset of  $\mathbb{R}, \mathbb{C}$ : this means for every point in the open set  $O$ , we can find a power series expansion of the function such that its radius of convergence  $> 0$ .

Fix  $z_0$  in the resolvent set. We know  $z_0 1_{\mathcal{A}} - a$  is invertible. We consider  $(z 1_{\mathcal{A}} - a)$ , for  $z$  in the resolvent set. We will omit the  $1_{\mathcal{A}}$  for simplicity.

$$z 1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a) \left( 1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a} \right)$$

We know the latter term is invertible if  $\|\frac{z_0 - z}{z_0 - a}\| < 1$  has norm, hence we have

$$(z - a)^{-1} = \sum_{n=0}^{\infty} \left( \frac{z_0 - z}{z_0 - a} \right)^n (z_0 - a)^{-1}$$

What happens when we let  $z \rightarrow \infty$ , we consider  $R_a(1/z)$ , and let  $z \rightarrow 0$ . Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1 - az}{z}\right)^{-1} = z(1 - az)^{-1}$$

Let  $z \rightarrow 0$  makes  $R_a(1/z)$  go to zero.

□

Now given that  $R_a(z)$  is analytic and bounded at  $\infty$ , we can state the following important theorem.

**Theorem 1.2 (Nonemptiness of spectrum)**

Let  $\mathcal{A}$  be a unital Banach algebra over  $\mathbb{C}$ , then for any  $a \in \mathcal{A}$ , we have  $\sigma(a) \neq \emptyset$ .



**Proof** Assume there exists  $a \in \mathcal{A}$ , such that  $\sigma(a) = \emptyset$ . If  $\mathcal{A} = \mathbb{C}$ , then we would have  $R_a(\lambda)$  be a bounded entire, complex-valued function defined on all of  $\mathbb{C}$ . By Liouville's theorem, we must have  $R_a(z)$  a constant function, but we know  $z \rightarrow \infty$ ,  $R_a \rightarrow 0$ , hence  $R_a(z)$  is constantly 0, but this cannot be true.

If our  $\mathcal{A}$  is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where  $\phi$  is some nonzero element in  $\mathcal{A}'$ , guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at  $\infty$ . Again, the function is constantly 0.

With the nonemptiness of spectrum theorem, we now state the Gelfand-Mazur theorem.

**Theorem 1.3 (Gelfand-Mazur)**

Let  $\mathcal{A}$  be a unital Banach algebra over  $\mathbb{C}$ , if any nonzero element of  $\mathcal{A}$  is invertible, then  $\mathcal{A}$  is isomorphic to  $\mathbb{C}$ .



**Proof** For any  $a \in \mathcal{A}$ , we know  $\sigma(a) \neq \emptyset$ , hence there exists  $\lambda$  such that  $\lambda 1_{\mathcal{A}} - a$  is invertible, i.e.  $a = \lambda 1_{\mathcal{A}}$ , hence establishing an isomorphism between  $\mathcal{A}$  and  $\mathbb{C}$ . In other words,  $\mathcal{A} = \mathbb{C} 1_{\mathcal{A}}$ .

□

**1.5.1 Functional Calculus****Proposition 1.21**

Let  $a \in \mathcal{A}$ , then if  $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$  converges for  $|z| < r$ , where  $r > \|a\|$ , then  $\sum_{n=0}^{\infty} \alpha_n a^n$  converges as well.



We first start with proving the following statement.

**Lemma 1.1**

Let  $f$  be a polynomial,  $\mathcal{A}$  is a unital Banach algebra over  $\mathbb{C}$ ,  $f = \sum_{n=0}^k a_n x^n$ , then for  $a \in \mathcal{A}$ , we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of  $a$  under  $f$  is exactly the spectrum of  $f$  evaluated at  $a$ .



**Proof** ( $\Leftarrow$ ). We take  $\lambda \in \sigma(a)$ , and we would like to show  $f(\lambda)$  is in the spectrum of  $f(a)$ . We note that if  $\lambda \in \sigma(a)$ , then  $a = \lambda 1_{\mathcal{A}}$ , and  $f(\lambda 1_{\mathcal{A}}) = f(a)$ , hence by definition,  $f(a) - f(\lambda) 1_{\mathcal{A}}$  is not invertible implying  $f(\lambda)$  is in the spectrum of  $f(a)$ . Note that this also implies  $f(a) - f(\lambda) = (a - \lambda)Q(z)$  for some polynomial  $Q(z)$ .

( $\Rightarrow$ ). We take  $\lambda \in \sigma(f(a))$ , i.e.  $f(a) = \lambda 1_{\mathcal{A}}$ . we would like to show  $\lambda = f(y)$ , where  $y \in \sigma(a)$ . If  $f$  is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1) \dots (z - c_n)$$

Plugging in  $a$  we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}}) \dots (a - c_n 1_{\mathcal{A}})$$

If  $f(a) - \lambda$  is not invertible, then there exists  $j$  such that  $(a - c_j 1_{\mathcal{A}})$  is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show  $\lambda = f(y)$ , where  $y \in \sigma(a)$ . In fact, we have  $\lambda = f(c_j)$  by knowing  $f(c_j) - \lambda = 0$ .

□

Now let  $f(z) = z^n$ , and if  $\lambda \in \sigma(a)$ , then  $\lambda^n \in \sigma(a^n)$  by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$

This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

#### Definition 1.10

Fix  $a \in \mathcal{A}$ , we define the spectral radius of  $a$ , denoted by  $r(a)$ ,

$$r(a) = \sup_{\lambda} \{|\lambda| : \lambda \in \sigma(a)\}$$



#### Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$



**Proof** From the previous remark that  $|\lambda| \leq \|a^n\|^{1/n}$ , hence this follows.

## 1.6 Lecture 7

I have not typed up for this?

## 1.7 Lecture 8

Let  $\mathcal{A}$  be a unital Banach algebra. Then for  $a \in \mathcal{A}$ , and we look at the resolvent of  $a$ ,  $R_a(\lambda)$ , we've noted that as  $\lambda \rightarrow \infty$ , we have

$$\lim_{\lambda \rightarrow \infty} R_a(\lambda) = \lim_{\lambda \rightarrow \infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for  $|\lambda| \geq \|a\|$ .

Recall that we define the spectral radius,  $r(a)$ , as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

#### Proposition 1.22

$$r(a) = \lim \|a^n\|^{1/n}$$



**Proof**

If we let  $\lambda = 1/z$ , then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for  $|z| \leq \|a\|^{-1}$ , but maybe?? also for  $|z| < r(a)^{-1}$ ?

For  $r > r(a)$ , i.e.  $|z| \leq r^{-1}$ , we know  $\sum_n a^n r^n$  converges for  $r > r(a)$ .

know  $z \sum a^n z^n$  converges absolutely. In particular,

$$a^n z^n \rightarrow 0$$

Hence there exists  $M$  such that for  $n \geq M$ , we have

$$\|a^n r^{-n}\| \leq 1$$

This implies that

$$\|a^n\| \leq r^n \Rightarrow \|a^n\|^{1/n} \leq r$$



for all  $n \geq M$ .

This implies that

$$\limsup \|a^n\|^{1/n} \leq r$$

And note that  $r$  is arbitrary close to the spectral radius  $r(a)$ . Hence we have

$$\limsup \|a^n\|^{1/n} \leq r(a) \leq \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

□

For each  $\varphi \in \mathcal{A}'$ , consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n) \lambda^{-n}$$

This series converges for  $r > r(a)$ . We can apply the same process, to argue that there exists  $M_\varphi$  such that

$$\|\varphi(a^n) r^{-n}\| \leq M_\varphi$$

for all  $n \geq 0$ . Note that  $M_\varphi$  could be different for all  $\varphi$ .

Note that

$$\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}''$$

there is a natural injection of  $a \mapsto \hat{a} \in \mathcal{A}''$ .

For each  $n$ , define  $F_n \in \mathcal{A}''$ , by  $F_n(\varphi) = |\varphi(a^n r^{-n})| \leq M_\varphi$ . Applying the UBP, we have

$$|F_n(\varphi)| \leq M \Rightarrow |\varphi(a^n) r^{-n}| \leq M$$

This implies that

$$|\varphi(a^n)| \leq r^n M$$

Note that by Hahn-Banach, for any  $b \in \mathcal{A}$ , we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking  $n$ -th root of both sides, we get

$$\|a^n\| \leq r^n M \Rightarrow \|a^n\|^{1/n} \leq r M^{1/n} \rightarrow r$$

Hence we again obtain the same result.

□

Recall UBP.

#### Theorem 1.4 (Uniform Boundedness Principle)

Let  $X$  be Banach, and  $Y$  be normed, let  $T_n : X \rightarrow Y$  be a family of linear operators, and if for all  $x \in X$ , we have

$$\|T_n(x)\| < \infty$$

Then for all  $n$ , we have

$$\|T_n\| < \infty$$

♥

Note that if  $\mathcal{A}$  is unital, and if  $\mathcal{A} \subset \mathcal{B}$  with some unit. For  $a \in \mathcal{A}$ , if  $a$  is not invertible in  $\mathcal{A}$ , then it might be invertible in  $\mathcal{B}$ . Hence if we use  $\sigma_{\mathcal{A}}(a)$  to denote the spectrum of  $a$  in  $\mathcal{A}$ .

#### Proposition 1.23

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

♠

**Example 1.5** Let  $\mathcal{B} = l^1(\mathbb{Z})$ , and let  $\mathcal{A} = l^1(\mathbb{N})$ , equipped with convolution.

Clearly  $\mathcal{A} \subset \mathcal{B}$ . And note that the delta function at 1,  $\delta_1$  is not invertible in  $\mathcal{A}$  but it has an inverse  $\delta_{-1}$  in  $\mathcal{B}$ . Hence we see  $0 \in \sigma_{\mathcal{A}}(a)$ , but  $0 \notin \sigma_{\mathcal{B}}(a)$ .

**Proposition 1.24 (Spectral radius is preserved)**

For  $\mathcal{A} \subset \mathcal{B}$ , we have

$$r_{\mathcal{A}}(a) = \lim \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

**Proposition 1.25**

Let  $X$  be compact, and let  $\mathcal{A} = C(X)$ . Then for  $f \in C(X)$ , we have

$$\|f^2\|_{\infty} = \|f\|_{\infty}^2$$



**Proof** Look at where  $f$  takes  $\|f\|_{\infty}$ , and square it, since when  $X$  is compact, you can actually obtain the point where  $|f(x)| = \|f\|_{\infty}$ .

**Remark** The same property holds for  $f$  in any unital subalgebra of  $C(X)$ , for example, if  $X \subset \mathbb{C}$ , and let  $\mathcal{A}$ =functions that are holomorphic on an open subset of  $\mathbb{C}$  that are in  $X$ .

Let  $\mathcal{A}$  be a unital Banach algebra with the property such that for any  $a \in \mathcal{A}$ , we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any  $n$ , we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking  $1/2^n$ -root of both sides, we get that the spectral radius of  $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

Let  $\mathcal{H}$  be a Hilbert space, over  $\mathbb{C}$ , and let  $\mathcal{A} = B(\mathcal{H})$ , i.e. the bounded linear operators on  $\mathcal{H}$ , and equip with the operation of taking adjoint.  $T \mapsto T^*$ .

**Proposition 1.26**

For any  $T \in B(\mathcal{H})$ , we have

$$\|T^*T\| = \|T\|^2$$



**Proof** We know that  $\|T^*\| = \|T\|$ . And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let  $\xi \in \mathcal{H}$ , then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi \rangle = \langle \xi, T^*T\xi \rangle \leq \|T^*T\|\|\xi\|^2$$

where the last inequality follows from Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2}\|\xi\|$$

which by definition, gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result. □

**Corollary 1.4**

If  $T^* = T$ , then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements.

Note that for general  $T$ , we have  $T^*T$  is always self-adjoint,

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the \*-algebra structure.

## 1.8 Lecture 9

Let  $\mathcal{H}$  be a hilbert space over  $\mathbb{C}$ , and  $\mathcal{B}(\mathcal{H})$  with  $\|\cdot\|_\infty$ , and closed under taking involutions. If  $T \in \mathcal{B}(\mathcal{H})$ , then

$$\|T^*T\| = \|T\|^2$$

So if  $T^* = T$ , then we have

$$\sigma(T) = \|T\|$$

### Definition 1.11

A concrete  $C^*$ -algebra is a norm-closed sub-algebra  $\mathcal{A}$  of  $\mathcal{B}(\mathcal{H})$ , for some  $\mathcal{H}$  (could be finite or infinitely dimensional) such that is self-adjoint, i.e., if  $T \in \mathcal{A}$ , then  $T^* \in \mathcal{A}$ .  
Again, we call  $\mathcal{A}$  is unital if  $1_{\mathcal{H}} \in \mathcal{A}$ .

### Corollary 1.5

If  $\mathcal{A}$  is a  $C^*$  algebra, then for all  $a \in \mathcal{A}$ ,

$$\|a^*a\| = \|a\|^2$$

If  $\mathcal{A}$  is unital  $C^*$ -algebra, and if  $a^* = a$ , then  $r(a) = \|a\|$ .

Next we say a bit about the Gelfand transform.

### Definition 1.12

Let  $\mathcal{A}$  be a unital Banach algebra, and commutative, then we have the Gelfand transform  $\Gamma : \mathcal{A} \rightarrow C(\widehat{\mathcal{A}})$ :

$$\Gamma(a)(\varphi) = \varphi(a)$$

then he said something of homomor to the complex numbers or something

Note that if  $a \in \mathcal{A}$ , and  $\varphi \in \widehat{\mathcal{A}}$ , then  $\varphi(a) \in \sigma(a)$ , then we have

$$|\varphi(a)| \leq \|a\| = r(a)$$

then  $\|\widehat{a}\|_\infty \leq r(a)$ . Now we would like to show

$$\|\widehat{a}\|_\infty = r(a)$$

We will now dedicate the next 50 minutes of our life to proving this theorem.

### Theorem 1.5

If  $\lambda \in \sigma(a)$ , then there exists  $\varphi \in \widehat{\mathcal{A}}$  such that  $\varphi(a) = \lambda$ . This is equivalent to saying: if  $(a - \lambda 1)$  is not invertible, then there is a  $\varphi \in \widehat{\mathcal{A}}$ , such that

$$\varphi(a - \lambda 1) = 0$$

This means if  $a$  is not invertible, then there exists  $\varphi \in \widehat{\mathcal{A}}$  such that  $\varphi(a) = 0$ .

**Proof** Suppose  $a \in \mathcal{A}$ , and consider

$$a\mathcal{A} = \{ab : b \in \mathcal{A}\}$$

The set  $a\mathcal{A}$  does not contain the identity element i.e.  $1_{\mathcal{A}} \notin a\mathcal{A}$  (otherwise it would imply it has an inverse). And  $a\mathcal{A}$  is a two-sided proper ideal (by  $\mathcal{A}$  being commutative).

We now introduce a fact that we will use.

**Definition 1.13**

An ideal  $I$  is maximal if  $I$  is proper in  $R$ , and not contained in any bigger proper ideals.



**Lemma 1.2**

Let  $R$  be a unital ring commutative, every proper ideal is contained in a maximal ideal (by Zorn's lemma).



**Lemma 1.3**

For  $\mathcal{A}$  unital commutative Banach algebra, if  $I$  is a proper ideal, then its closure is a proper ideal.



**Proof** We have seen that  $GL(\mathcal{A})$ , the set of invertible elements, is open. Hence its complement is closed. Any proper ideal does not contain any elements in  $GL(\mathcal{A})$ , hence its closure is closed inside a closed set.

**Remark** Let  $X$  be locally compact, but compact, such as  $\mathbb{R}$ , then we have  $C_c(X) \subset C_\infty(X)$ . Note that  $C_c(X)$  is a proper ideal of  $C_\infty(X)$ , but it's dense in  $C_\infty(X)$ , hence its closure is the entire space, hence no longer proper. This tells us the closure of a proper ideal is not always proper, if  $\mathcal{A}$  is not unital.

**Theorem 1.6**

Every maximal ideal of  $\mathcal{A}$  is closed.



**Proof** The closure of any proper ideal is closed in unital algebras, hence its closure is itself.

First let  $V$  be a normed vector space. Let  $W$  be a closed subspace, and form the quotient space  $V/W$ . There is a natural way to equip  $V/W$  with a norm

$$\|v\| = \inf\{\|v - w\| : w \in W\}$$

i.e. the distance between  $v$  to  $W$ . This is a norm. Further if  $V$  is complete, so is  $V/W$ .

Let  $\mathcal{A}$  be a normed algebra, commutative, unital. Let  $I$  be a closed ideal.

**Proposition 1.27**

For  $a, b \in \mathcal{A}$ , we have

$$\|\dot{a}\dot{b}\| = \|\dot{a}b\| \leq \|\dot{a}\|\|\dot{b}\|$$

so that  $\mathcal{A}/I$  is a normed algebra.



**Proof** Let  $c, d \in I$ , and

$$(a - c)(b - d) = ab - (ad + cb - cd)$$

Note that  $(ad + cb - cd) \in I$ , hence

$$\|\dot{a}\dot{b}\| \leq \|ab - (ad + cb - cd)\| = \|(a - c)(b - d)\| \leq \|(a - c)\|\|(b - d)\|$$

Taking infimum over all  $c, d$ , we get

$$\|(\dot{a}\dot{b})\| \leq \|\dot{a}\|\|\dot{b}\|$$



**Proposition 1.28**

If  $\mathcal{A}$  is a Banach algebra and if  $I$  is a closed ideal, then  $\mathcal{A}/I$  is a Banach algebra for the norm  $\|\dot{a}\|$  defined above.



Let  $\mathcal{A}$  be a unital commutative Banach algebra over  $\mathbb{C}$ , let  $I$  be a maximal ideal of  $\mathcal{A}$ , then  $\mathcal{A}/I$  is a Banach algebra,

if  $\mathcal{A}/I$  has a proper ideal, then you can put this ideal back in  $\mathcal{A}$  such that it contains  $I$ . By  $I$  already being ideal, this implies that  $\mathcal{A}/I$  does not contain any proper ideals.

Now coming back. Let nonzero element in  $\mathcal{A}/I$  is invertible. The Gelfand Mazur theorem tells us

$$\mathcal{A}/I \cong \mathbb{C}$$

Moreover,  $\mathcal{A}/I = 1_{\mathcal{A}/I}\mathbb{C}$ . Then the quotient map

$$\mathcal{A} \rightarrow \mathcal{A}/I \cong \mathbb{C}$$

is an element of  $\mathcal{A}$ , i.e. an algebra homomorphism  $\varphi$ , with the property  $\varphi(I) = 0$ .

If  $a\mathcal{A} \subset I$ , then for  $y \in a\mathcal{A}$ , we have

$$\varphi(y) = 0$$

And we are therefore finally done. We thus have

$$\|\hat{a}\| = r(a)$$

□

## 1.9 Lecture 10

It's Friday.

Consider  $C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$ , and  $C_\infty(\mathbb{R})$  is an ideal of  $C_b(\mathbb{R})$ .

And the abstract  $C^*$ -algebra is a Banach algebra with involution  $*$ , such that

$$\|a^*a\| = \|a\|^2$$

Zorn's lemma states that  $C_\infty(\mathbb{R})$  is contained in a maximal ideal, in a commutative Banach algebras, maximal ideals give rise to bounded multiplicative linear functionals.

**Remark** There is ideals of  $C_b(\mathbb{R})$  that are bigger than  $C_\infty(\mathbb{R})$ .

There exists linear functionals that are 0 on  $C_\infty(\mathbb{R})$ , but nonzero on  $C_b(\mathbb{R})$ , but such functional is not “constructable.”

We also have  $c_0 \subset l^\infty(\mathbb{N})$ , where  $c_0$  are sequences that converge to 0 at infinity. Again, nonzero linear functionals exist on  $l^\infty(\mathbb{N})$ , and is identically zero on  $c_0$ , but such is also not constructable.

### Definition 1.14

$\prod_{j=1}^\infty \mathbb{Z}_5$  = the sequences of elements of  $\mathbb{Z}_5$ .  $\bigoplus \mathbb{Z}_5$  all sequences that are 0 except for finitely many number of entries.



Note that  $\bigoplus \mathbb{Z}_5$  is an ideal of  $\prod_{j=1}^\infty \mathbb{Z}_5$ .

### Proposition 1.29

$l^\infty(\mathbb{N})$  is not separable,  $\prod_{j=1}^\infty \mathbb{Z}_5$  is not separable, nor is it finitely generated.



### Proposition 1.30

If  $\mathcal{A}$  is a unital commutative Banach algebra over  $\mathbb{C}$ , which is separable, and if  $I$  is a closed ideal, then one can construct a maximal ideal containing  $I$  by countable *what*



**Proof** Let  $\{a_n\}$  be a countable subset of  $\mathcal{A}$ , whose linear span is dense.

### Lemma 1.4

Note that  $\mathcal{A}/I$  contains noninvertible elements if and only if  $I$  is not maximal.



If  $I$  is not maximal, then you can find the first  $a_n$  such that  $a_n \in \mathbb{C}1_{\mathcal{A}}$  such that  $a_n \notin \mathcal{A}/I$ , then

$$\overline{a_n \mathcal{A}/I} \text{ is a proper ideal of } \mathcal{A}$$

hence it generates a proper ideal in  $\mathcal{A}$ , we denote it as  $I_1$ . If  $I_1$  is not maximal, then repeat the process. By  $\{a_n\}$  being countable, and that they are dense, we have a countable addition, which gives a maximal ideal containing  $I$  by countable inclusions.  $\square$

**Remark**  $C_\infty(\mathbb{R})$  is separable, and  $C_b(\mathbb{R})$  is not separable.

Let's look at  $L^\infty([0, 1], m)$ , where  $m$  denotes the Lebesgue measure. This is a  $C^*$ -algebra, commutative, unital.

Could you exhibit any linear functionals on  $L^\infty$

Note that  $L^2([0, 1], m)$ , a Hilbert space, as an algebra on  $L^2$ ,  $L^\infty$  is closed for the strong operator topology on  $\mathcal{B}(\mathcal{H})$  by the seminorm,

$$T \in \mathcal{B}(\mathcal{H}), T \rightarrow \|t\xi\|, \text{ for } \xi \in \mathcal{H}$$

For example, figure 1. We say that  $L^\infty([0, 1])$  is a von Neumann algebra. Every commutative von Neumann algebra looks like some  $L^2([0, 1], \mu)$ . But note that noncommutative ones are quite interesting.

For a commutative Banach algebra over  $\mathbb{C}$ , the Gelfand transform

$$a \mapsto \hat{a}$$

We have

$$\|\hat{a}\|_\infty = r(a)$$

### Proposition 1.31

If  $\|a^2\| = \|a\|^2$ , then the Gelfand transform is isometric. Thus we have

$$\|\hat{a}\|_\infty = \|a\|$$

### Definition 1.15

For  $a_n$  involution on an algebra over  $\mathbb{C}$ , is a map  $*$  from  $\mathcal{A} \rightarrow \mathcal{A}$ , with the properties

1.  $(a^*)^* = a$
2.  $(a + b)^* = a^* + b^*$
3.  $(\alpha a)^* = \bar{\alpha} a^*$  for  $\alpha \in \mathbb{C}$ .
4.  $(ab)^* = b^* a^*$

### Definition 1.16 (Banach $*$ algebra)

If  $\mathcal{A}$  has a norm, then we say it is a  $*$  normed if

$$\|a^*\| = \|a\|$$

If  $\mathcal{A}$  is complete, then it is called a Banach  $*$  algebra.

Let  $G$  be a discrete group, let  $\mathcal{H}$  be a Hilbert space, then  $\text{Aut}(\mathcal{H}) = U(\mathcal{H})$  is the group of unitary operators on  $\mathcal{H}$  to itself. By a unitary representation of  $G$  on  $\mathcal{H}$ , we mean

### Definition 1.17 (Unitary representation)

A unitary representation of  $G$  on  $\mathcal{H}$  is a homomorphism  $\pi : G \rightarrow U(\mathcal{H})$ .

We note that  $C_c(G) \subset l^1(G)$ , and

$$\pi_f = \sum_{x \in G} f(x) \pi_x, \pi_f \pi_g = \pi_{f * g}$$

Now we ask what is  $(\pi_f)^*$ ?

$$(\pi_f)^* = \sum \overline{f(x)} \pi_x^* = \sum \overline{f(x)} \pi_{x^{-1}} = \sum \overline{f(x^{-1})} \pi_x$$

So we get

$$(f^*)(x) = \overline{f(x^{-1})}$$

This defines an involution on  $l^1(G)$ . And it is easy to check that

$$\|f^*\|_1 = \|f\|_1$$

**Remark** The same process would not work for semigroups without the presence of  $x^{-1}$  necessarily.

We would like to think of this involution as some sort of complex conjugation.

**Definition 1.18**

Let  $\mathcal{A}$  be a Banach  $*$ -algebra is symmetric if whenever  $a \in \mathcal{A}$ , and  $a^* = a$ , then

$$\sigma(a) \subset \mathbb{R}$$



If one looks at  $l^1$  over noncommutative groups, some are symmetric, some are not.

**Example 1.6** Let  $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ , with  $\|\cdot\|_\infty$ . We define an involution:

$$(\alpha, \beta)^* = (\bar{\beta}, \bar{\alpha})$$

This is a well-defined involution. However, this is not symmetric under this involution.

**Proposition 1.32**

If  $G$  is commutative, then  $l^1(G)$  is symmetric.



$$\widehat{\mathcal{A}} = \{ \text{set of homomorphisms } G \rightarrow \mathbb{T} \}$$

This is symmetric.

**Proposition 1.33**

Let  $\mathcal{A}$  be an abstract, unital  $C^*$ -algebra, then  $\mathcal{A}$  is symmetric.



This is quite strong! (Every  $C^*$ -algebra is symmetric).