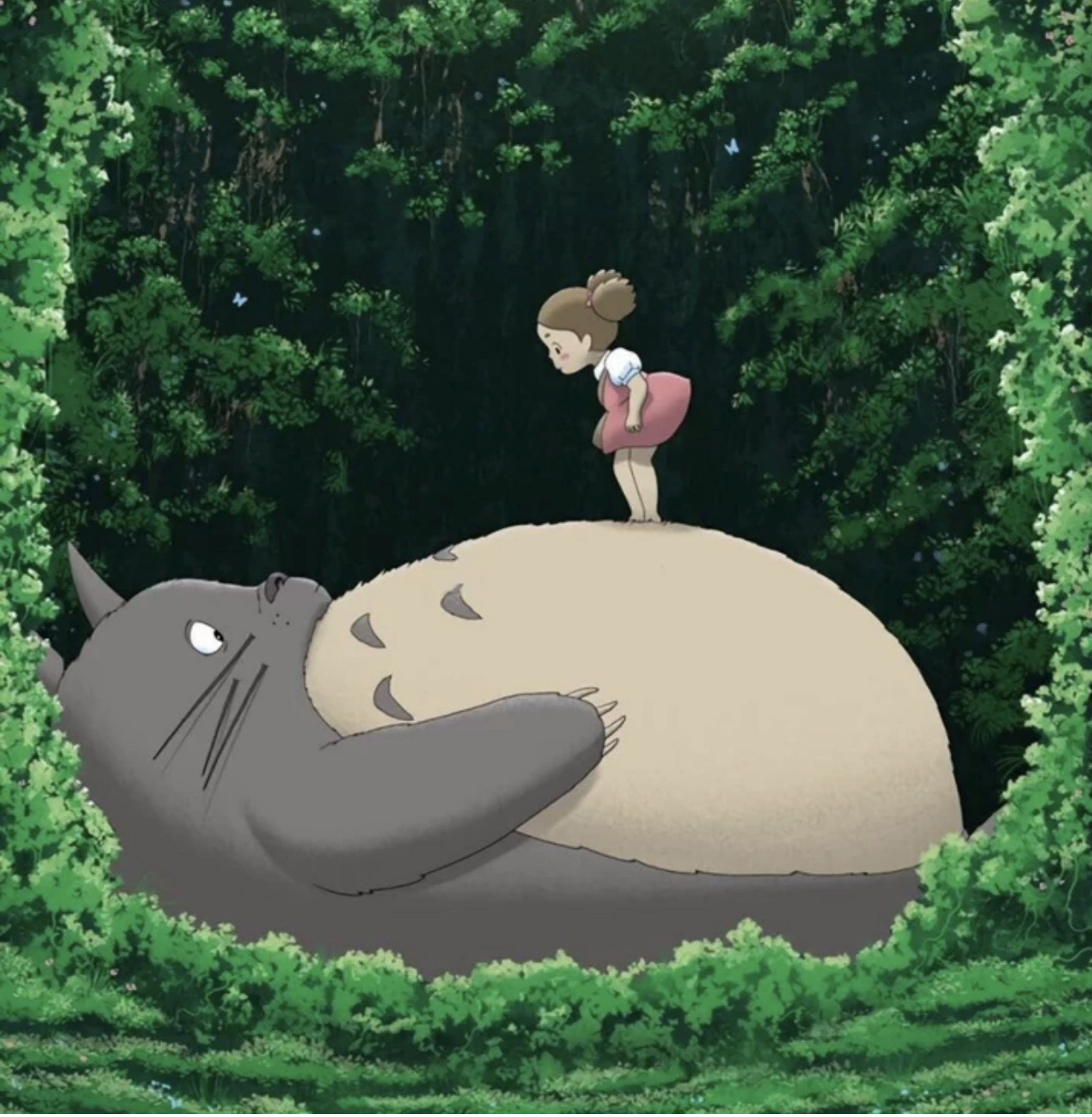

PDE Topics

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Chapter 1 Lecture 1

here we go.

Course overview

We will be discussing the nonlinear Schrodinger quations, which is a subcategory of nonlinear pde's, nonlinear dispersive equations, and infinite speed of propogation,

Let's start with the linear Schrondinger equation.

$$i\partial_t u + \Delta u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^n, u(t = 0) = u_0$$

The fundamental solution to a Schrondinger equation, is the $K(t, x)$ is such that $u_0 = \delta_0$.

Instead, one could look at other intial data, for exmaple, $\widehat{u}_0 = \delta_{\xi_0}$, or $u_0 = e^{ix\xi_0}$.

Remark If you localize the initial data in the physical space, then the fourier transform is constant and therefore cannot be localized in the Fourier space. The reverse is also true if you try to localize in the Foureir space.

If we have

$$u_0 = e^{-\frac{(x-x_0)^2}{2}} e^{i(x-x_0)\cdot\xi_0}$$

For this type of initial data, we call it the coherent state, localized at (x_0, ξ_0)

In non-coherent state, the solution spreads out immediately; in the coherent state, the solution remains nicely behaved and coherent for a period of time, then it spreads out evenetually.

Remark This is the idea of group velocity, waves with frequence ξ_0 move with velocity $2\xi_0$. This $2\xi_0$ is called the group velocity.

Dispersive equation: waves with different frequencies travel in different directions.

1.1 Nonlinear

We will start with the nonlinear boys now.

$$i\partial_t u + \Delta u = \lambda u \cdot |u|^p$$

We will ask the following standard pde questions.

1. existence
2. uniqueness
3. continuous dependent
4. global in time behavior, i.e. linear vs nonlinear effects

Remark If one just observes the RHS, then there is linear and nonlinear contributions, and one probalby would expect that one dominates over the other over time.

linear: scattering. nonlinear solution looks like the linear solution

nonlinear: solitons (solutions that remain concentrated for a very long time, such as a bump function), blow-ups.

We will comment on the dispersive aspect of the Schrondinger equation before the nonlinear aspect.

1.2 Dispersion

Here are ways to measure dispersion. Given nicely behaved initial data, $u(t = 0) = u_0 \in H^s$.

1. dispersive estimates $\|u\|_{L^\infty} \leq t^{O(1)} \|u_0\|_{L^1}$
2. Strichartz estimates $\|u\|_{L_t^p L_x^q} \leq \|u_0\|_{L^2}$
3. Lateral Strichartz esiamtes, exchanging the role of t, x.

4. Improved function spaces (Bourgain spaces, U^p, V^p)
5. Local energy decay. If you have a dispersive solution, instead of measuring the solution everywhere, say, you measure it in the vertical cylinder.

Back to NLS.

$$i\partial_t u + \Delta u = \lambda u|u|^p$$

We will talk about the following:

1. local well-posedness
2. global well-posedness for small initial data
3. large initial data problem
4. energy critical problem $\int |\nabla u|^2$ and the mass critical problem $\int |u|^2$

Remark The exponenet p that we put on the RHS plays an important role int he above questions.

Some topics in the foreseeable future: Littlewood-Paley theory, Bessel's problem, etc

References: Tao's on nonlinear and dispersive pde.

Now we will talk about Schrondinger maps

$$u : \mathbb{R} \times \mathbb{R}^n \to (M, g)$$

Sasy we have $u_t = i\Delta u$, then $u_t \in TM$, where T stands for tangent, as we have rotated Δu 90 degrees hence should live in the tangent of the manifold.

$$u_t = P\Delta u, P \text{ projection on } TM$$

The RHS $P\Delta u$ is called the heat flow. Let M be a kahlan manifold.

Spherical case, $(M, g) = \mathbb{S}^2$. One can identify $\mathbb{S}^2$ as the comlex plane and compactified. Hence if we would like an object that is perpendicular to both u and Δu, and rotate by 90 degrees, then we look at the following equation

$$u_t = u \times \Delta u, u(t=0) = u_0$$

Then we come to the next section of the class, Quasilinear Schrondinger equations.

$$iu_t + g^{jk}(u)\partial_j\partial_k(u) = N(u, \nabla u), u(t=0) = u_0$$

Suppose g^{jk} is a positive definite matrix, and the N stands for nonlinear We will look at the local solvability.

If we start with a simple guess, $N = \partial_j u$, and this becomes a ill-posed linear problem due to exponential growth (by taking the Fourier transform). Then we can probably repalce $N = (\nabla u)^2, N = (\nabla)^3$.

Another difficulty is how waves propogate, and "trapping" refers to when waves are localized eternally and do not propogate (sit in the vertical cylinder for example). This leads to the discussion of local well-posed theory.

For the **last part of the course**, we wil look at global solutions for quasilinear Schrondinger equations for small initial data, if $n \geq 3$, then somehow you can use the dispersive estimates mentioned above, via Strichartz. In higher dimension, the decay is faster, than the estimate is stronger, and the linear component plays more role. In low dimension, the nonlinear interactiosn are more prominennt.

In $n = 1$, there exists a following conjecture.

Proposition 1.1 (Conjecture)

If one has $1-d$ dispersive problem, that is cubic defocusing, then there exists a global solution for small data u_0.

In the case of QNLS, there is a proved theorem as above in 2023.

1.3 Lecture 2

We will now interpret the three quantities introduced above, mass, momentum and energy.

First interpretation The Hamiltonian interpretation,

denote $H(u) = E(u)$, and $w(u, v)$ antisymmetric in L^2, and

$$w(u, v) = \int Im(u\overline{v}), J = i$$

And $\partial_t u = JDH = -id\Delta u$, where D is the differential form.

Given two Hamiltonians, $\{H_1, H_2\} = 0$, can ask if they commute. This is to ask whether H_1, H_2 flows commute.

Theorem 1.1 (Noether)

Each sympletic symmetry of one Hamiltonian flow is generated by a commuting Hamiltonian.

Remark Sympletic refers to the solution preserving the sympletic form.

E generate the linear Schrodinger equation.

If we look at

$$\frac{\delta P_j}{\delta u} - i\partial_j u, \partial)tu - i \cdot i\partial_j u = -\partial_j u$$

This gives

$$\partial_t u + \partial u = 0$$

This generates translations.

If we look at mass M, we have

$$\frac{\partial M}{\partial u} = 2u, \partial_\theta u = i2u$$

This generates phase rotations.

We note that although mass is a conserved quantity, the mass is moving around. Hence, we associate a flux to it, i.e. a mass density. We define the mass density as follows

$$m(u) = |u|^2$$

And we take the time derivative

$$\partial_t m = \partial_j f_j$$

where f_j is the mass transfer in the j-th direction.

$$\partial_t m = 2Re(\partial_t u \cdot \overline{u}) = 2Re(i\Delta u \cdot \overline{u})$$

Then by integration by parts, we have the above equal to

$$2Re(i\partial_j(\partial_j u \cdot \overline{u}) - i\partial_j u \cdot \partial_j \overline{u}) = \partial_j[2Im(\partial_j u \cdot \overline{u})] = -\partial_j p_j$$

We have shown that

$$\partial_j m(u) + \partial_j p_j = 0$$

We could do similar computations for momentum.

$$\partial_t p_j + \partial_k e_{jk} = 0$$

Where for the matrix e_{jk}, the trace of the matrix is equal to e, denoted as the energy density. The above two equations give rise to the ”Energy-momentum tensor is divergence free.”?

$\begin{bmatrix} m & P \\ P & e \end{bmatrix}$ The divergence of the above matrix is equal to 0.

Solutions for the (LS). We go back to the fundamental solution. For $u + 0 = \delta_0$. We have

$$\widehat{K} = e^{it\xi^2}\widehat{u}_0 = e^{it\xi^2}$$

We thus have

$$K(t, x) = \frac{1}{(2\pi it)^{n/2}} e^{-ix^2/4t}$$

The dirac is Galilean invariant, and so is the fundamental solution, and $|K(t, x)|$ has to say constant due to Galilean invariance. Hence we have

$$|K(t, x)| = c_n \cdot t^{-n/2}$$

And we also have infinite speed of propogation, wave packets travel in all directions with the same speed, and do no discriminate different speeds.

Connection between propogation speed and frequency. For

$$\partial_t \widehat{u} = it\xi^2 \cdot u$$

Suppose u only has frequencies close to ξ_0. Let's approximate ξ, Given

$$\xi^2 = \xi_0^2 + 2\xi_0(\xi - \xi_0)$$

Approximate equation

$$\partial_t \widehat{u} = i[\xi_0^2 + 2\xi_0(\xi - \xi_0)]\widehat{u} = (2i\xi_0 \cdot \xi - i\xi_0^2)\widehat{u}$$

Coming back to the physical space, we have

$$\partial_t u = 2\xi_0 \cdot \partial_x u - i\xi_0^2 u$$

The first partial refers to the transport, and the second refers to the phase rotation.

This gives that $u(t,x) = u_0(x, x + 2t\xi_0)e^{-it\xi_0^2}$. This gives the conclusion that waves with frequency ξ_0 now move with velocity $2\xi_0$.

(We may have a sign error, but imagine we have $\tau + \xi_0^2$), and the velocity $2\xi_0$ is called the roup velocity. If we denote $\tau + \xi^2$ as $\tau + \alpha(\xi)$, then the group velocity is $\partial_\xi \alpha(\xi)$.

If the velocity of waves depends on frequency, we thus call this dispersive.

If I choose $u_0 = e^{ix\xi_0}$, then we get

$$u(t) = e^{i(x_2 t\xi_0)} \cdot e^{-it\xi_0^2}$$

The first part comes from the transport and the second part comes from the phase shift.

We have $\widehat{u}_0 = \delta_{\xi_0}$.

If we have $u_0 = e^{-x^2/2}$, and $\widehat{u}_0 = e^{-\xi^2/2}$, and $(x_0, \xi_0) = (0,0)$, and $(\delta x, \delta\xi) = (1,1)$.

Then $\widehat{u}(t,\xi) = e^{-it\xi^2}e^{-\xi^2/2}$.

This gives

$$u(t,x) = \frac{1}{((1/2_it)2\pi)^{n/2}} e^{-x^2/(2-4it)}$$

Note that if $t \lesssim 1$, then it behaves nicely like a Gaussian, and if otherwise, we have waves spread out because the $4it$ term dominates.

Before a time threshold, we have the coherent state, where everything stays like a Gaussian, but becomes dispersive after some time.

Given Galilean invariance and translation invariance, we can move to ξ_0 and then to x_0. Hence now we have

$$u_0 = e^{-(x-x_0)^2/2} e^{i(x-x_0)\xi_0^2}$$

These translations do not commute, however, it only varies our solution by a constnat factor, say $e^{i\xi_0^2}$ or something. The form of above u_0 is called the coherent state.

And we call the coherent state solutions, moving with velocity $2\xi_0$ as wave packets. And still $\delta x = 1, \delta\xi = 1$. And the time of coherent is 1, $\delta t = 1$.

What if we want to change the scale, i.e. the sclaing symmetry. $(t,x) \mapsto (\lambda^2 t, \lambda x)$.

$$\delta t = T, \delta x = \sqrt{T}, \delta\xi = \frac{1}{\sqrt{T}}$$

By this scaling relation, We can adjust the time of the coherent state.

We can also think of LS solutions as superpositions of wave packets.