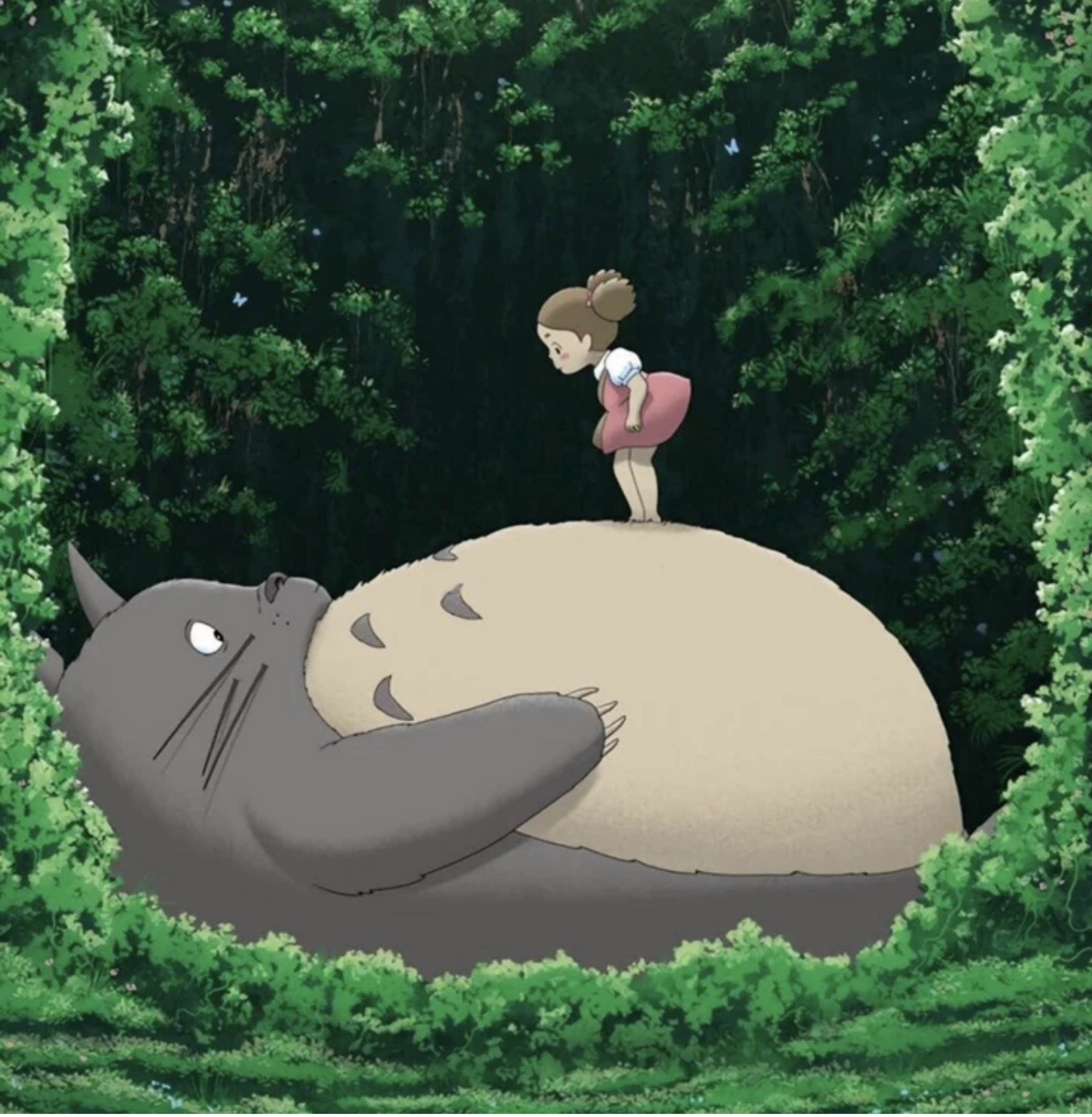

PDE Topics

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Chapter 1 Lecture 1

here we go.

Course overview

We will be discussing the nonlinear Schrodinger quations, which is a subcategory of nonlinear pde's, nonlinear dispersive equations, and infinite speed of propogation,

Let's start with the linear Schrondinger equation.

$$i\partial_t u + \Delta u = 0 \text{ in } \mathbb{R} \times \mathbb{R}^n, u(t=0) = u_0$$

The fundamental solution to a Schrondinger equation, is the $K(t,x)$ is such that $u_0 = \delta_0$.

Instead, one could look at other intial data, for exmaple, $\widehat{u}_0 = \delta_{\xi_0}$, or $u_0 = e^{ix\xi_0}$.

Remark If you localize the initial data in the physical space, then the fourier transform is constant and therefore cannot be localized in the Fourier space. The reverse is also true if you try to localize in the Foureir space.

If we have

$$u_0 = e^{-\frac{(x-x_0)^2}{2}} e^{i(x-x_0)\cdot\xi_0}$$

For this type of initial data, we call it the coherent state, localized at (x_0, ξ_0)

In non-coherent state, the solution spreads out immediately; in the coherent state, the solution remains nicely behaved and coherent for a period of time, then it spreads out evenetually.

Remark This is the idea of group velocity, waves with frequence ξ_0 move with velocity $2\xi_0$. This $2\xi_0$ is called the group velocity.

Dispersive equation: waves with different frequencies travel in different directions.

1.1 Nonlinear

We will start with the nonlinear boys now.

$$i\partial_t u + \Delta u = \lambda u \cdot |u|^p$$

We will ask the following standard pde questions.

1. existence
2. uniqueness
3. continuous dependent
4. global in time behavior, i.e. linear vs nonlinear effects

Remark If one just observes the RHS, then there is linear and nonlinear contributions, and one probalby would expect that one dominates over the other over time.

linear: scattering. nonlinear solution looks like the linear solution

nonlinear: solitons (solutions that remain concentrated for a very long time, such as a bump function), blow-ups.

We will comment on the dispersive aspect of the Schrondinger equation before the nonlinear aspect.

1.2 Dispersion

Here are ways to measure dispersion. Given nicely behaved initial data, $u(t=0) = u_0 \in H^s$.

1. dispersive estimates $\|u\|_{L^\infty} \leq t^{O(1)} \|u_0\|_{L^1}$
2. Strichartz estimates $\|u\|_{L^p_t L^q_x} \leq \|u_0\|_{L^2}$
3. Lateral Strichartz esiamtes, exchanging the role of t, x.

4. Improved function spaces (Bourgain spaces, U^p, V^p)
5. Local energy decay. If you have a dispersive solution, instead of measuring the solution everywhere, say, you measure it in the vertical cylinder.

Back to NLS.

$$i\partial_t u + \Delta u = \lambda u|u|^p$$

We will talk about the following:

1. local well-posedness
2. global well-posedness for small initial data
3. large initial data problem
4. energy critical problem $\int |\nabla u|^2$ and the mass critical problem $\int |u|^2$

Remark The exponenet p that we put on the RHS plays an important role int he above questions.

Some topics in the foreseeable future: Littlewood-Paley theory, Bessel's problem, etc

References: Tao's on nonlinear and dispersive pde.

Now we will talk about Schrondinger maps

$$u : \mathbb{R} \times \mathbb{R}^n \to (M, g)$$

Sasy we have $u_t = i\Delta u$, then $u_t \in TM$, where T stands for tangent, as we have rotated Δu 90 degrees hence should live in the tangent of the manifold.

$$u_t = P\Delta u, P \text{ projection on } TM$$

The RHS $P\Delta u$ is called the heat flow. Let M be a kahlan manifold.

Spherical case, $(M, g) = \mathbb{S}^2$. One can identify $\mathbb{S}^2$ as the comlex plane and compactified. Hence if we would like an object that is perpendicular to both u and Δu, and rotate by 90 degrees, then we look at the following equation

$$u_t = u \times \Delta u, u(t = 0) = u_0$$

Then we come to the next section of the class, Quasilinear Schrondinger equations.

$$iu_t + g^{jk}(u)\partial_j\partial_k(u) = N(u, \nabla u), u(t = 0) = u_0$$

Suppose g^{jk} is a positive definite matrix, and the N stands for nonlinear We will look at the local solvability.

If we start with a simple guess, $N = \partial_j u$, and this becomes a ill-posed linear problem due to exponential growth (by taking the Fourier transform). Then we can probably repalce $N = (\nabla u)^2, N = (\nabla)^3$.

Another difficulty is how waves propogate, and "trapping" refers to when waves are localized eternally and do not propogate (sit in the vertical cylinder for example). This leads to the discussion of local well-posed theory.

For the **last part of the course**, we wil look at global solutions for quasilinear Schrondinger equations for small initial data, if $n \geq 3$, then somehow you can use the dispersive estimates mentioned above, via Strichartz. In higher dimension, the decay is faster, than the estimate is stronger, and the linear component plays more role. In low dimension, the nonlinear interactiosn are more prominennt.

In $n = 1$, there exists a following conjecture.

Proposition 1.1 (Conjecture)

If one has $1 - d$ dispersive problem, that is cubic defocusing, then there exists a global solution for small data u_0.

In the case of QNLS, there is a proved theorem as above in 2023.

1.3 Lecture 2

We will now interpret the three quantities introduced above, mass, momentum and energy.

First interpretation The Hamiltonian interpretation,

denote $H(u) = E(u)$, and $w(u, v)$ antisymmetric in L^2, and

$$w(u, v) = \int Im(u\overline{v}), J = i$$

And $\partial_t u = JDH = -id\Delta u$, where D is the differential form.

Given two Hamiltonians, $\{H_1, H_2\} = 0$, can ask if they commute. This is to ask whether H_1, H_2 flows commute.

Theorem 1.1 (Noether)

Each sympletic symmetry of one Hamiltonian flow is generated by a commuting Hamiltonian. ♡

Remark Sympletic refers to the solution preserving the sympletic form.

E generate the linear Schrodinger equation.

If we look at

$$\frac{\delta P_j}{\delta u} - i\partial_j u, \partial)tu - i \cdot i\partial_j u = -\partial_j u$$

This gives

$$\partial_t u + \partial u = 0$$

This generates translations.

If we look at mass M, we have

$$\frac{\partial M}{\partial u} = 2u, \partial_\theta u = i2u$$

This generates phase rotations.

We note that although mass is a conserved quantity, the mass is moving around. Hence, we associate a flux to it, i.e. a mass density. We define the mass density as follows

$$m(u) = |u|^2$$

And we take the time derivative

$$\partial_t m = \partial_j f_j$$

where f_j is the mass transfer in the j-th direction.

$$\partial_t m = 2Re(\partial_t u \cdot \overline{u}) = 2Re(i\Delta u \cdot \overline{u})$$

Then by integration by parts, we have the above equal to

$$2Re(i\partial_j(\partial_j u \cdot \overline{u}) - i\partial_j u \cdot \partial_j \overline{u}) = \partial_j[2Im(\partial_j u \cdot \overline{u})] = -\partial_j p_j$$

We have shown that

$$\partial_j m(u) + \partial_j p_j = 0$$

We could do similar computations for momentum.

$$\partial_t p_j + \partial_k e_{jk} = 0$$

Where for the matrix e_{jk}, the trace of the matrix is equal to e, denoted as the energy density. The above two equations give rise to the "Energy-momentum tensor is divergence free."?

$\begin{bmatrix} m & P \\ P & e \end{bmatrix}$ The divergence of the above matrix is equal to 0.

Solutions for the (LS). We go back to the fundamental solution. For $u + 0 = \delta_0$. We have

$$\widehat{K} = e^{it\xi^2}\widehat{u}_0 = e^{it\xi^2}$$

We thus have

$$K(t,x) = \frac{1}{(2\pi i t)^{n/2}} e^{-ix^2/4t}$$

The dirac is Galilean invariant, and so is the fundamental solution, and $|K(t,x)|$ has to say constant due to Galilean invariance. Hence we have

$$|K(t,x)| = c_n \cdot t^{-n/2}$$

And we also have infinite speed of propogation, wave packets travel in all directions with the same speed, and do no discriminate different speeds.

Connection between propogation speed and frequency. For

$$\partial_t \widehat{u} = it\xi^2 \cdot u$$

Suppose u only has frequencies close to ξ_0. Let's approximate ξ, Given

$$\xi^2 = \xi_0^2 + 2\xi_0(\xi - \xi_0)$$

Approximate equation

$$\partial_t \widehat{u} = i[\xi_0^2 + 2\xi_0(\xi - \xi_0)]\widehat{u} = (2i\xi_0 \cdot \xi - i\xi_0^2)\widehat{u}$$

Coming back to the physical space, we have

$$\partial_t u = 2\xi_0 \cdot \partial_x u - i\xi_0^2 u$$

The first partial refers to the transport, and the second refers to the phase rotation.

This gives that $u(t,x) = u_0(x, x + 2t\xi_0)e^{-it\xi_0^2}$. This gives the conclusion that waves with frequency ξ_0 now move with velocity $2\xi_0$.

(We may have a sign error, but imagine we have $\tau + \xi_0^2$), and the velocity $2\xi_0$ is called the roup velocity. If we denote $\tau + \xi^2$ as $\tau + \alpha(\xi)$, then the group velocity is $\partial_\xi \alpha(\xi)$.

If the velocity of waves depends on frequency, we thus call this dispersive.

If I choose $u_0 = e^{ix\xi_0}$, then we get

$$u(t) = e^{i(x_2 t\xi_0)} \cdot e^{-it\xi_0^2}$$

The first part comes from the transport and the second part comes from the phase shift.

We have $\widehat{u}_0 = \delta_{\xi_0}$.

If we have $u_0 = e^{-x^2/2}$, and $\widehat{u}_0 = e^{-\xi^2/2}$, and $(x_0, \xi_0) = (0,0)$, and $(\delta x, \delta\xi) = (1,1)$.

Then $\widehat{u}(t,\xi) = e^{-it\xi^2}e^{-\xi^2/2}$.

This gives

$$u(t,x) = \frac{1}{((1/2_i t)2\pi)^{n/2}} e^{-x^2/(2-4it)}$$

Note that if $t \lesssim 1$, then it behaves nicely like a Gaussian, and if otherwise, we have waves spread out because the $4it$ term dominates.

Before a time threshold, we have the coherent state, where everything stays like a Gaussian, but becomes dispersive after some time.

Given Galilean invariance and translation invariance, we can move to ξ_0 and then to x_0. Hence now we have

$$u_0 = e^{-(x-x_0)^2/2} e^{i(x-x_0)\xi_0^2}$$

These translations do not commute, however, it only varies our solution by a constnat factor, say $e^{i\xi_0^2}$ or something. The form of above u_0 is called the coherent state.

And we call the coherent state solutions, moving with velocity $2\xi_0$ as wave packets. And still $\delta x = 1, \delta\xi = 1$. And the time of coherent is 1, $\delta t = 1$.

What if we want to change the scale, i.e. the sclaing symmetry. $(t,x) \mapsto (\lambda^2 t, \lambda x)$.

$$\delta t = T, \delta x = \sqrt{T}, \delta\xi = \frac{1}{\sqrt{T}}$$

By this scaling relation, We can adjust the time of the coherent state.

We can also think of LS solutions as superpositions of wave packets.

1.4 Lecture 3

We recall what we did last time.

$$(i\partial_t + \Delta)u = 0, u(0) = u_0$$

And we have

$$u_0(x) = e^{-(x-x_0)^2/2} e^{ix_0(x-x_0)}$$

This refers to the coherent state, and it suffices to study $x_0 = \xi_0 = 0$, and use translation invariance.

And note that we have (x_0, ξ_0) as the center and the scales are $\delta x = 1, \delta\xi = 1$.

We note this is not the only scale. We could instead have $\delta x = \sqrt{T}, \delta\xi = \frac{1}{\sqrt{T}}$, and for scale=1, we have coherent $T = 1$. And for the other scale, we have the coherent time T as T.

Now we ask the following questions: Question: can we think of arbitrary solutions as superpositions of wave packets?

We want u_0= linear combination of coherent states. In other words, we now replace our typical $e^{-x^2/2}$ with any Schwartz function φ, and we have

$$u_0(x) = \varphi(x - x_0)e^{i\xi_0(x-x_0)}$$

Moreover, another justification to use arbitrary schwartz function is that if we are trying to solve

$$i\partial_t u - A(D)u = 0, a \mapsto a(\xi)$$

Then there is no reason to use u_0 as the Gausssian $e^{-x^2/2}$ because the Gaussian does not stay Gaussian as it evolves.

We could use partition of unity. Let $\mathbb{Z} \subset \mathbb{R}^n$, we have

$$1 = \sum_{j\in\mathbb{Z}^n} \chi_j(x)$$

And $\chi_j(x) = \chi(x - x_j) \in \mathcal{S}$. Now we have each χ_j contained in a unit cube.

And we have

$$u_0 = \sum \chi_j(x)u_0 = \sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}^n} \chi_k(D)\chi_j(x) \cdot u_0$$

The above is called the spatial unit scale decomposition. However, this decomposition is not smooth compared to our initial form of the data.

Now we ask the question, how about a smooth wave packet decomposition.

$$u = \int f(x_0, \xi_0)u_{x_0,\xi_0}(x)dx_0 d\xi_0$$

Remark The representation is not unique, and it one wishes to make this unique, they would have to have some sort of restriction on $f(x_0, \xi_0)$

This is the Bargmann trnsform, or the Segal transform. If one includes the scaling, this is called FBi transform.

Definition 1.1

Let T be defined as follows: (using Gaussians)

$$f(x_0, \xi_0) = Tu(x_0, \xi_0) = \int e^{-(x-x_0)^2/2} e^{-i\xi_0(x-x_0)}u(x)dx$$

Note we cannot hope this to be surjective, hence we have twice the amount of variables of x_0, ξ_0. Note we start with x, and end with a function in two variables.

If we differentiate with respect to ξ_0, we get a $i(x - x_0)$ term, and if we differentiate with respect to x_0, we get a $(x - x_0)$ term, as well as $i\xi_0$ term, hence we have the following operator that kills the phase.

$$[\partial_{\xi_0} - i(\partial_{x_0} - \xi_0)]\, Tu = 0$$

Proposition 1.2

Such T defined above, as the Bargmann transform, is an L^2 isometry.

$$T * \circ T = I$$

We define a slightly different transform:

$$\tilde{T}u(z) = \int e^{-1/2(x-z)^2}u(x)dx, z = x_0 - i\xi_0$$

And now we have

$$\tilde{T} : L^2 \to L^2(e^{-\xi_0^2})$$

Definition 1.2

The FBi transform is simply a recaled version of the Bargmann transform.

$$T_\lambda u(x_0, \xi_0) = \int e^{-(x-x_0)^2/2\lambda} e^{-i\xi_0(x-x_0)} u(x) dx$$

♣

Now we have $\Delta x = \sqrt{\lambda}$, and $\xi = 1/\sqrt{\lambda}$.

We have

$$u_0 = \sum_{x_0, \xi_0 \in \mathbb{Z}^n} c_{x_0,\xi_0} u_0^{x_0,\xi_0}$$

Then we have

$$u = \sum c_{x_0,\xi_0} e^{-itD^2} u_0(x_0, \xi_0)$$

The above is the solution to the linear Schrondinger. This is useful up to time 1, and it is viewed as a superposition of wave packets.

We shall approximate wave packets, we have u_0 localized at (x_0, ξ_0). We approximate the solution as follows:

$$u(x, t) \sim u_0(x - 2t\xi_0) e^{it\xi_0^2}$$

If we start with $e^{i(x-x_0)\xi_0}$, then roughly it is $e^{i(x-2t\xi_0-x_0)}\xi_0$, where $\xi \sim \xi_0$, and $\tau \sim -\xi_0^2$.

Then u is a good approximate solution, you can verify this by

$$(i\partial_t + \Delta)u \sim O(1)$$

However, this error adds up and is acceptable if $t << 1$. Especially if you have nonhomogenous equation, the error adds up as time progresses.

We have

$$u(t, x) = \sum_{x_0,\xi_0} c_{x_0,\xi_0} u_0(x - 2t\xi_0) e^{it\xi_0^2}$$

And this is a good approxiamte solution up to time 1.

Remark We note that the composition here is almost orthogonal, hence we have different frequencies, just like we had agove when we had $u = \sum c_{x_0,\xi_0} e^{-itD^2} u_0(x_0, \xi_0)$

In the constant coefficient case, you start with (x_0, ξ_0), at $t = 0$, you end up at $(x_0 + 2\xi_0, \xi_0)$ But in variable coefficients, you have

$$(x_0, \xi_0) \to (x_t, \xi_t)$$

We no longer move in the linear fashion.

We now explore the case where we don't assume infinitely many derivatives.

Let's consider wave packets with less localization. Simply consider the case at $(0, 0)$ and use translation invariance and Galilean symmetry.

Take $u_0 \in L^2$, what does it mean for this to be localized at (0,0)?

One proposition could be

$$x \cdot u_0 \in L^2$$

This means as long as you move away from (0,0), u_0 decays. And we would also like to have decay in frequence, hence

$$\partial_x u_0(x, \xi) \in L^2$$

The above implies decay in ξ. Together they imply the solution u is localizated at $(0, 0)$ up to time $O(1)$.

If you wish it to localize at $(x_0 - 2t\xi_0, \xi_0)$, then we can have

$$(x - x_0 - 2t\xi_0)u \in L^2, [D_x - \xi_0]u_0 \in L^2$$

By energy estimates, we have if u solves the equation, then $D_x u$ also solves the equation, we have

$$\|D_x u\|_{L^2} = \|D_x u(0)\|_{L^2}$$

So it remains to consider xu, and how it interacts with the equation. We have

$$(i\partial_t + \Delta)x_j u = x_j(i\partial_t + \Delta)u + 2\partial_j u$$

And the first term is 0, hence we have

$$\frac{d}{dt}\|xu\|_{L^2}^2 = Re\int 2\overline{u}\partial_j u dx = O(1)$$

Then we have $\overline{u} \in O_{L^2}(1), \partial_j u \in O_{l^2}(1)$, hence the entire term is of $O(1)$ in L^2.

Recall the fundamental solution of linear Schrondinger equation.

$$K(t,x) = \frac{1}{(4\pi it)^{n/2}}e^{ix^2/4t}, |K| \lesssim t^{-n/2}$$

Hence we have $u(t) = K(t) * u_0$. And we have the dispersive estimate for linear Schrondinger

$$\|u(t)\|_{L^\infty} \leq \|K(t)\|_{L^\infty}\|u_0\|_{L^1} \lesssim t^{-n/2}\|u_0\|_{L^1}$$

Dispersive estimates via stationary phase.

Let's generalize to variable coefficents.

$$(i\partial_t + A(D)\Delta)u = 0$$

And our solution is of the form

$$u(t) = e^{itA(D)}u(0)$$

and

$$K(t) = \int_{\mathbb{R}^n} e^{ix\xi}e^{-ita(\xi)}d\xi$$

We have the oscillatory integral

$$I_\lambda = \int e^{i\lambda\varphi(\xi)}a(\xi)d\xi$$

Where we assume $a(\xi)$ having compact support.

And the value of the integral depends on the stationary points, defined by $D\varphi = 0$.

If there are no stationary points, then we get

$$|I_\lambda| \leq \lambda^{-N}$$

Replace φ with a quadratic expansion, and "replace" with a Gaussian. If we have nondegenerate points, $D\varphi \neq 0$. We have

$$|I_\lambda| \leq \lambda^{-n/2}$$

It's like putting φ in $n-1$ dimension and operate with separation of variables.

We now examine our solution to the Schrondinger equation, and find the critical points here. We differentiate with respect to x_j

$$x_j + t\partial_{\xi_j}a(\xi) = 0$$

We have

$$v \sim x/t = a_\xi$$

where a_ξ is our previous group velocity.

The nondegenerate points are characterized as nonvanishing Hessian, which is $\sim D^2a$.

Definition 1.3

The nondegenerate points are defined by the Hessian $\sim D^2a$, being a nondegenerate matrix. ♣

In LS, we have

$$a = \xi^2, D^2a = 2I$$

Now we introduce a third way of viewing dispersive estimates, which is via **wave packets**.

We first note that the dispersive estimate

$$\|u\|_{L^\infty} \lesssim t^{-n/2}\|u_0\|_{L^1}$$

is scale invariant. Hence it suffices to focus at $t = 1$.

Let's take

$$u_0 = \delta_0 = \sum_{x_0,\xi_0 \in \mathbb{Z}^n} c_{x_0,\xi_0}\varphi_{x_0,\xi_0}$$

we have

$$u_0 = \sum \chi_j(x)u_0 = u_0$$

And we want to localize frequency

$$\widehat{u_0} = \widehat{\delta_0} = 1 = \sum_{k\in\mathbb{Z}^n} \chi_k(\xi)$$

We have the $\|\chi_k\|_{L^2} = 1$, and this says the Fourier coefficents are of $O(1)$.

Then

$$u = \sum_{x_0=0,\xi_0\in\mathbb{Z}^n} cu_{x_0,\xi_0}$$

At time $t = 1$, we get one of the wave packet each, hence we have

$$|K(t = 1)| \leq \sup_{\xi_0} |u_{x_0,\xi_0}| = 1$$

Remark This no longe words for $t << 1$, hence there exists overlappings now. But to fix this, we rescale $\delta x = \sqrt{T}$, to get smaller and thinner tubes, nonoverlapping at time T.

Now we ask the question, what is the good way to measure dispersive decay for $u_0 \in L^2$.

A1: there is no uniform decay bound of the form

$$\|u(t)\|_{Sobolev} \leq C(t)\|u_0\|_{L^2}, \lim_{t\to\infty} C(t) = 0$$

This is because $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$, and taking t infinitely large, $C(t)$ tends to 0, and we would get

$$\|u\|_{sobolev} \leq 0, t \to \infty$$

Next time, we will talk about Strichartaz estimates, not in a uniform way, but in an average sense of $\|u\|_{L^p_t L^2_x}$.

1.5 Lecture 4

We will talk about Strichartz estimates this time. Suppose u is a solution to

$$iu_t + \Delta u = 0, u(0) = u_0$$

We ask if there exists a following bound

$$\|u\|_{L^p_{t,x}} \lesssim \|u_0\|_{L^2_x}$$

We cannot expect to use the same exponents for t, x hence we will attempt to put $L^p_t L^q_x$ instead.

Note we can also interchange the role of t, x, which are probably called maximal inequalities, but for now, we stick to this order for t, x. Sobolev embeddings in $\mathbb{R}^n$ also relates L^p norms with L^2, such as

$$\|u\|_{L^p} \lesssim \|u\|_{\dot{H}^s} = \||D|^s u\|_{L^2}$$

except having derivatives on the RHS. Hence in general, we look for inequalities of the form

$$\|u\|_{L^p_t L^q_x} \lesssim \||D|^s u_0\|_{L^2}$$

Before trying to figure out the indices p, q, s, we first note the symmetries of the Schrodinger equations. We use the scaling symmetry first. In other words, we want both spaces invariant under the transformation

$$u(t, x) \mapsto u(\lambda^2 t, \lambda x)$$

We end up with the following scaling relation.

$$\frac{2}{p}+\frac{n}{q}=\frac{n}{2}-s$$

The other piece of information can be derived from wave packets and the Galilean symmetry. What we saw last time is that we can produce wave packets $u_p \approx 1$ of size 1 in a rectangle $(T^{1/2})^{\times}T$, where $T^{1/2}$ is the spatial δx, and $\delta t \approx T$. The dual scale is $\delta\xi \approx T^{-1/2}$ due to the uncertainty principle. We can put $\delta\xi$ to localize at anywhere in the Fourier space, so if we restrict to $|\xi| \approx 1$, then we would want $\delta\xi \ll 1$, hence $T \gg 1$. This implies the following relationship:

$$T^{\frac{1}{p}}T^{\frac{n}{2q}} \lesssim T^{\frac{n}{4}}$$

for $T \gg 1$. Hence in fact we have

$$\frac{2}{p}+\frac{n}{q}\leq\frac{n}{2}$$

If you were to compare these two, then $s \geq 0$. We thus name them:

1. sharp Strichartz: $s = 0$
2. non-sharp Strichartz: $s > 0$

From the sharp to non-sharp, we will use Sobolev embeddings. Though it seems like the derivation with s included is redundant, if you take wave equations or the KDV equations instead, then there exists a number of derivatives s there for these kinds of Strichartz estimates.

1.5.1 Strichartz estimates, restriction theorems in Harmonic analysis

We will now focus on $\widehat{u} = \delta_{T=-|\xi|^2}\widehat{u_0}(\xi)$, which are L^2 measures on the paraboloids.

$$\widehat{u_0} \mapsto \mathcal{F}^{-1}\delta_{\tau=-|\xi|^2}\widehat{u_0} : L^2 \to L^p_t L^q_x$$

The adjoint operator is $f \mapsto (\mathcal{F}_{t,x}f)|_{\tau=-|\xi|^2} : L^{p'}L^{q'} \to L^2$, trace of this Fourier transform on the paraboloids, we want it to be from $L^{p'}L^{q'} \to L^2$.

Strichartz estiamtes $L^2 \to L^pL^q$, the second one is called the restriction theorem. In PDE's, we are looking for L^2 and harmonic analysts want to change L^2 to other exponents and look for the range of it, and don't care about different exponents p', q', hence choosing them to be equal.

Also, in harmonic analysis, one may not only consider the case for paraboloids.

1. sphere: Stein-Thomas theorem, 70s
2. cone: Strichartz, 80s
3. paraboloids, 80s

Remark All the above shapes considered have nonvanishing curavture, both sphere and hyperboloids have maximal number of nonvanishing curvature while the cone has one less.

1.5.2 Visualization, $p \geq 2$

Before we prove anything, we make one more observation. We look at the scaling relationship:

$$\frac{2}{p}+\frac{n}{q}=\frac{n}{2}$$

When $n = 1$, we draw a graph with $1/p - 1/q$ coordinates. The $L^\infty L^2$ endpoints is invariant regardless of the dimension, due to the L^2 norm being preserved. When $n \geq 3$, one of the endpoints is $\frac{4}{n}$, which might be less than 1.

What if we try to construct various wave packets u_j of size 1 inductively (where each is far away from the previous ones both in space and frequency)? If we measure $u = \sum c_j u_j$, then

$$\|u\|^2_{L^2} = \sum_j c_j^2$$

However, we also have

$$\|u\|^p_{L^pL^q} = \sum c_j^p l^2$$

which controls l^p, hence we would want $p \geq 2$.

For $n = 1$, we have the full range. For $n = 2$, we have one forbidden endpoint; for $n = 3$, we only care about the RHS of the shadow region, and we also have endpoint estimates.

1.5.3 T^*T-method

Last time, we proved the following dispersive estimates:

$$\|e^{itD^2}u_0\|_{L^\infty} \lesssim t^{-n/2}\|u_0\|_{L^1}$$

We want $u_0 \to e^{itD^2}u_0$ is $L^2 \to L^pL^q$. Combining with the trivial L^2 bound (isometry), we interpolate them to obtain the following

$$\|e^{itD^2}u_0\|_{L^p} \lesssim t^{-\frac{n}{2}(\frac{1}{p'}-\frac{1}{p})}\|u_0\|_{L^{p'}}$$

note that the exponent of t is determined by scaling. Note this holds for $(L^1)' = L^\infty$, and $(L^2)' = L^2$.

Let us denote the operator $T : u_0 \to e^{itD^2}u_0$. (The LHS has duality in n-dim, the RHS has duality in $n+1$-dim). Having this operator from $L^2 \to L^pL^q$ is the same as having the dual operator:

$$T^* : f \mapsto \int e^{-itD^2}f(t)dt, T^* : L^{p'}L^{q'} \to L^2$$

A fundamental result in functional analysis is that T is bounded if and only if T^* is bounded.

Definition 1.4 (TT^***-method)**

Given $T : L^2 \to L^pL^q$, we consider the following operator:

$$TT^* : L^{p'}L^{q'} \to L^pL^q$$

We have T is bounded if and only if T^* is bounded, if and only if TT^* is bounded.

$$\langle TT^*(f), g\rangle = \langle T^*(f), T^*(g)\rangle$$

where $g \in L^{p'}L^{q'}$, and taking $f = g$, we have

$$\|T^*(f)\|_{L^2}^2 \lesssim \|f\|_{L^pL^q}^2\|TT^*\|$$

This is called the TT^* argument. ♣

Remark The essence of this method is that one of the spaces is L^2.

We write

$$TT^*(f)(t) = e^{itD^2}T^*f = \int_{\mathbb{R}} e^{i(t-s)D^2}f(s)ds$$

where t, s are arbitrary instead of the actual fundamental solution for the Schrodinger equation. So

$$\|TT^*(f)\|_{L^q} \lesssim \int_{\mathbb{R}} |t-s|^{-\frac{n}{2}(\frac{1}{q'}-\frac{1}{q})}\|f(s)\|_{L^{q'}}ds$$

By Young's inequality, we have the exponents

$$\frac{n}{2}\left(\frac{1}{q'} - \frac{1}{q}\right) \in (0, 1)$$

hence $2 < p < \infty$.

Remark The convolution with $1/|t| : L^2 \to L^2$ is not true for Young's inequality, hence in Hilbert transform, we want to use p.v. instead of absolute value.

So far, we have looked at only homogeneous euqations. For inhomogeneous ones, i.e. $iu_t + \Delta u = f$, we can write the solution into two components,

$$u(t) = e^{itD^2}u_0 + \int_0^t e^{i(t-s)D^2}f(s)ds$$

which is the standard Duhamel's formula. For the first component, we know the mapping property from L^2. For the second one, if we want $u \in L_t^\infty L_x^2$, it is enough to fix t, then we look at the map

$$f \mapsto \int_0^t e^{i(t-s)D^2}f(s)ds$$

We want to compare this with the operator T^*. In fact, the time range 0 to t does nothing, so this maps from $L^{p'}L^{q'} \to L^2$. You might conjecture the correct form for the estimates are

$$\|u\|_s \leq \|u_0\|_{L^2} + \|f\|_{s'}$$

where s is our favorite Strichartz norm. We have proved the case for $f = 0$. Also, when $f \neq 0$, we also proved for $s = L^\infty L^2$. However, the nontrivial Strichartz norms can also be proved. In fact, we have more general forms:

$$\|u\|_{L^pL^q} \lesssim \|f\|_{L^{p_1'}L^{q_1'}}$$

For forward problems,

$$u(t) = \int_{-\infty}^{t} e^{i(t-s)D^2} f(s)ds$$

is a convolutional operator and the kernel for this is

$$1_{t\geq s}e^{i(t-s)D^2}$$

where restricting $1_{t\geq s}$ also means you are viewing the forward problem. Note that the difference between this and the kernel for TT^* operator is that the kernel for TT^* is $e^{i(t-s)D^2}$ without restriction.

1.5.4 Christ-Kiselev Lemma

We now state a classical lemma:

Lemma 1.1 (Christ-Kiselev)

Suppose the operators

$$L : f \mapsto \int K(t,s)f(s)ds, L^r : f \mapsto \int_{t\geq s} K(t,s)f(s)ds$$

which satisfy $L : L^{p_1} \to L^{p_2}$, then if $p_1 < p_2$, then we have

$$L^r : L^{p_1} \to L^{p_2}$$

♡

Proof For $1_{t\geq s}$, and $t, s \in [0, 1]$, a rectangle R is easy to manipulate since $1_R = 1_{t\in I} \times 1_{s\in J}$. However for $t \geq s$, it is not a rectangle.

The problem of choosing this: we cannot ensure the same amount of f being split inot different intervals. Instead of making a equipartition of the interval, we split the intervals containing the same amount of f. Assume $\|f\|_{L^{p'}} = 1$, and $[0, 1]_s = I_0 \cup I_1$ such that $\|f\|_{I_0} = \|f\|_{I_1} = \frac{1}{2^{p_1}}$. Then we split to $I_0 = I_{00} \cup I_{01}$.

Hence we write

$$I_{t\geq s} = \cup_k \cup_{R\in R_k} I_R \times J_R$$

where k tells us which layer we are at, and R_k are rectangles at level k. All disjoint, both horizontally and vertically. We want to write $L^r(f) = \sum_k \sum_{R\in R_k} 1_{I_R} L 1_{J_R} f$, where we want to estimate for each k,

$$\sum_k \sum_{R\in R_k} 1_{I_R} L 1_{J_R} f := \sum_k L_k^r f$$

In each layer, we gain something in size. Our claim is that

$$\|L_k^r\| \lesssim 2^{1/p_2 - 1/p_1} k$$

then this sums in a good way. We know the size $1_{J_R} f$ and mapping property of L. For $1_{I_R}'s$. they are disjoint, so the L^q norms add nicely. By composing these three, we get the desired result.

□

This will not cover the double endpoint case where $p = p' = 2$. Once you have bounds TT^*, you can consider T, T^*, to vary q, q_1' to have two different pair of admissible (p, q) and (p_1, q_1).

Note it is easy to handle the direction from $1_{t\geq s}e^{i(t-s)D^2}$ to $e^{i(t-s)D^2}$ just by time reversal. We just handle two parts and add them together but this is not the direction that we want.

By next time, this C-K will become obselete in the context of the Schrodinger equation. In $n \geq 3$, we need to discuss the endpoint case.

1.5.5 The forbidden endpoint in $n = 2$

We start with $n \geq 2$. The endpoint when $n \geq 3$ is resolved by Keel-Tao in 1996. When $n \geq 3$, the endpoint case is

$$\mathcal{S} = L_t^2 L_x^{\frac{2n}{n-2}}$$

There is not much difference from looking at the inhomogeneous or homogeneous operator. We consider

$$f(s) \mapsto \int e^{i(t-s)D^2} f(s)ds, L^2 L^{\frac{2n}{n+2}} \to L^2 L^{\frac{2n}{n-2}}$$

where the integral is over $\mathbb{R}$ or $t > s$ dependeing on inhomogeneous or homogeneous case. To avoid writing notations, we note $q_c = \frac{2n}{n-2}$, the difficulty is that

$$\|e^{itD^2}\|_{L^{q_c'} \to L_{q_c}} \lesssim \frac{1}{t}$$

where the convolution with $\frac{1}{t}$ does not have the mapping property $L^2 \to L^2$, so we cannot apply Young's inequality in this case directly. This is also the reason why the endpoint case fails is dimension 2 (forbidden endpoint) since the kernel $K(t, x = 0) = \frac{1}{t} 1_{t \geq 0}$.

If we focus on the integral over domain over $\mathbb{R}$, then it is just the TT^* operator, which is expected to have the mapping property $L^2 L^1 \to L^2 L^\infty$ in dimension 2.

Suppose by contradiction that it is, then one looks at $*K; L^2 L^1 \to L^2 L^\infty$, $f(t)\delta_{x=0} \to u(t, x = 0)$, where $\delta_{x=0}$ can be chosen to be a limit of L^1 functions.

$$u(t, x = 0) = \left(\frac{1}{t} 1_{t \geq 0}\right) * f(t)$$

where does not map $L_t^2 \to L_t^2$. Also, the first term, if one tries to make sense of it as a distribution, then its Fourier transform will not have a log component on it.

However, $*\frac{1}{t} : L^2 \to L^2$, which is called the Hilbert transform.

1.6 Lecture 5

1.6.1 Endpoint case when $n = 3$

We prove the case in $n \geq 3$ by restricting t to $t \approx 2^j$

$$\frac{1}{t} 1_{t \approx 2^j} \in L^1$$

To ge from fixed dyadic scale to various ones, we draw a picture. Suppose we focus on the case $t > s$, unlike the proof of CK lemma, we want to split inot equal pieces. We write

$$T = \sum_j T_j, T_j = \sum_{Q \in 2_j} 1_Q \cdot e^{i(t-s)D^2}$$

where we let the output restricted to Q. In Q_j, $t - s \approx 2^j$. Using notations from last time, we have

$$1_Q \cdot e^{i(t-s)D^2} = 1_{I_q}(t) 1_{J_Q}(s) e^{i(t-s)D^2}$$

so we can apply Young's inequality:

$$\|T_j\|_{L^2 L^{q_c'} \to L^2 L^{q_c}} \lesssim 1 (R)$$

which is called the restricted inequality. Now we want to add up different pieces but we cannot so far.

In (R), we cannot sum up. The idea of Keel-Tao is to expand the restricted inequality. We not only have this for T_j, instead of thinking of these as a linear operator $L^2 L^{q_c'} \to L^2 L^{q_c}$, we test the element in the dual space.

$$f \mapsto T_j f$$

changes to

$$\langle f, g \rangle \mapsto \langle T_j f, g \rangle$$

The restricted inequlaity (R) tells us

$$|\langle T_j f, g \rangle| \leq \|f\|_{L^2 L^{q_c'}} \|g\|_{L^2 L^{q_c'}}$$

and what we want is to have

$$\sum_j |\langle T_j f, g\rangle| \le \|f\|_{L^2 L^{q_c'}} \|g\|_{L^2 L^{q_c'}}$$

which is praphrased into a bilinear setting instead of a linear fashion. This idea is useful in nonlinear PDE's. You can match the properties of f with properties of g in some sense and to play with.

As an intermediate step, we are going to have (E).

$$|\langle T_j f, g\rangle| \le 2^{j\beta(q_1,q_2)} \|f\|_{L^2 L^{q_1}} \|g\|_{L^2 L^{q_2}} (E)$$

the scale of rectangle will be changed if you change the scaling, so this can hold if one puts a factor in front. The factor contains a function β, which is nontrivial and linear in $1/q_1$ and linear in $1/q_2$. β is also symmetric and is zero when $q_1 = q_2 = q_c$. So $\beta(q_1, q_2)$ is of the form $c(1/q_1 + 1/q_2 - 2/q_c)$.

Proof [E] We want to have enough points so that the convex hull contains the point $(1/q_c, 1/q_c)$ so that we can obtain the desired result by interpolation.

Since $|K(t,x)| \le 1/t^{n/2}$, it maps $L^2L^1 \to L^2L^\infty$ and mapping L^2 to L^2 since we are in a finite interval $(q_1' = 1, q_2 = \infty)$.

On the other hand, we want to use the nonendpoint estimates

$$\langle T_j f, g\rangle \le \|f\|_{L^{p_1'} L^{q_1'}} \|g\|_{L^{p_2'} L^{q_2'}}$$

Also we can restrict to intervals $t - s \approx 2^j$, and in such intervals, we can replace p_1', p_2' by 2 using Holder inequality.

$$\langle T_j f, g\rangle \le 2^j \|f\|_{L^2 L^{q_1'}} \|g\|_{L^2 L^{q_2'}}$$

which holds for all $2 \le q_1, q_2 \le< q_c$.

Now we know l_j^∞ estimates and we want l_1^j estimates, interpolation theory would allow to replace l^∞ by l^1, where one can find in Bergh-Loefstrom/ Triebel.

We can think of this in a concrete way. We look at spatial support of f, g:

$$f = f_k(t)1_{I(t)}, g = g_l(s)1_{J(s)}, |I(t)| = 2^k, |J(t)| = 2^l$$

We thus have

$$\|f\|_{L^2 L^{q_1'}} = \|f_k\|_{L^2} e^{k/q_1'}, \|g\|_{L^2 L^{q_2'}} = \|g_l\|_{L^2} e^{l/q_2'}$$

(E) implies

$$|\langle T_j f, g\rangle| \le 2^{j\beta(q_1,q_2)} \|f\|_{L^2 L^{q_1}} \|g\|_{L^2 L^{q_2}} \le 2^{j\beta(q_1,q_2)} \|f_k\|_{L^2} e^{k/q_1'} \|g_l\|_{L^2} e^{l/q_2'}$$

We write $f = \sum f_k(t) 1_{I_k(t)}, g = \sum g_l(s) 1_{J_l(s)}$, and apply the upgraded version of estimate above to obtain. Last task: to decompose

$$f = \sum f_k 1_{I_k}, |I_k| = 2^k, \|f\|_{L^{q'}}^{q'} \approx 2^k \|f_k\|_{L^\infty}^{q'}$$

suppose $f \in \mathbb{R}^+$ by replacing to $|f| f \to f^*$ a nondecreasing rearrangement of f, where the way to choose F^* is

$$|\{f^* \ge \lambda\}| = |\{|f| \ge \lambda\}|$$

Here, the dyadic decomposition is done on the base.

$$|\langle T_j f, g\rangle| \le \|(\|f_k 2^{k/q_c'}\|_{l_k^2})\|_{L_t^2} \|(\|g_l 2^{l/q_c'}\|_{l_l^2})\|_{L_t^2}$$

where we interchange l, L^2 and apply the estimate obtained from rearrangements and then we take advantage of $q_c' < 2$, so we have $l^{q_c'} \subset l^2$.

Strichartz estimates are sharp for wave packets, and no other things make these sharp.

1.7 Lecture 6

We look at the Schrodinger equation $iu_t + \Delta u = f, u(t=0) = u_0$. Last time, we looked at Strichartz estimates:

$$\|u\|_S \le \|u_0\|_{L^2} + \|f\|_{S'}$$

where $S = \cap_{(p,q)} L_t^p L_x^q$, is the Strichartz norm, where the exponents satisfy the scaling relation, $2/p + n/2 = n/2$.

Note that if $u \in C(L^2)$, then $u \in S$. The first thing we note that the Strichartz estiamtes are not specific to the Schrodinger equation, but rather, we can apply it to the more general setting:

$$i\partial_t u + A(D)u = f$$

we need $a(\xi)$ to have dispersive relation, and $\partial^2 a$, i.e. the Hessian, nondegenerate.

Back to linear Schrodinger. In $n = 1$, if we draw the dispersive relation of the Schrodinger (Figure 1). The idea now is to exchange the role of t and x. We get

$$\tau + \xi^2 = 0 \Rightarrow \xi = \pm\sqrt{-\tau}$$

Hence we now obtain two solutions u_R, u_L, and we take $u = u_L + u_R$.

$$(i\partial_x - \sqrt{-D_\tau})u^R = 0, (i\partial_x - \sqrt{-D_\tau})u^L = 0$$

Recall the (lateral) Strichartz estimates:

$$\||D_x|^S u\|_{L_x^p L_t^q} \leq \|u_0\|_{L^2}$$

Note that for the Schrodinger equation, we do not have any derivatives on the LHS, however, derivatives should arise when we change different p, q.

Recall the curvature relation: (we are still in $n = 1$)

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$$

Now the scaling relation, time counts as two space dimension, hence

$$\frac{1}{p} + \frac{2}{q} - s = \frac{1}{2}$$

If we take $(p, q) = (\infty, 2)$, then $s = 1/2$. This is one endpoint, and this is a gain. The other endpoint, $(p, q) = (4, \infty)$, we have $s = -1/4$, which is a loss.

Proposition 1.3

By lateral Strichartz estimates, we have the following:

$$\|D^{1/2}u\|_{L_x^\infty L_t^2} \leq \|u_0\|_{L^2}$$
$$\|D^{-1/4}u\|_{L_x^4 L_t^\infty} \leq \|u_0\|_{L^2}$$

What happens when $n > 1$? We gain have

$$\tau + \xi^2 = 0$$

Choose one distinct direction ξ_1, and x_1 as the evolutionary variable.

$$\tau + \xi_1^2 + (\xi')^2 = 0 \Rightarrow \xi_1 = \pm\sqrt{-\tau - (\xi')^2}$$

Note the RHS can vanish now, which creates problems. In other words, previously, ξ or ξ_1 could not be 0, since it would require $\tau = 0$ as well. Now if we are unlucky, and $\xi_1 = 0$ for some choice of τ, ξ', then there are waves that do not travel in the x_1 direction. (Figure 4)

In figure 5, we avoid this issue by restricting our solution to a cone, for example. Hence if we only look at solutiosn located in this region, we get

$$u \to P_1(D)u$$

we have a solution that is dispersive in the x_1 direction. Apply the lateral Strichartz estimates:

$$\||D_x|^s P_1 u\|_{L_{x_1}^p L_{x',t}^q} \leq \|u_0\|_{L^2}$$

Note that we have the same curvature relation, but for the scaling relation, we have (since time acts as two variable and x' has $n-1$, hence get $n+1$ for the second term)

$$\frac{1}{p} + \frac{n+1}{q} - s = \frac{n}{2}$$

Remark We make note that there exists gain and there exists loss, since one can always take $p = q$, which is a middle point. And this is like the Strichartz estimate that we had before, where we have no derivatives.

Proposition 1.4

We have the more refined lateral Strichartz estimates where we have gains and losses;

$$\|D_x^+ u\|_{L_x^\infty L_t^2} \leq \|u_0\|_{L^2}$$

Note that this one has nothing to do with dispersion, due to L^2 norm being perserved.

$$\|u\|_{L^p L^p} \leq \|u_0\|_{L^2}$$

This one does not see direction of travel, and it only sees curvature.

$$\|D_x^- u\|_{L_x^2 L_{x',t}^q} \leq \|u_0\|_{L^2}$$

This also depends on the curvature

Remark We say it depends on curvature when it does not depend on the angular decomposition like in Figure 5. And thus the middlepoint, the bottom endpoint does not depend on the angular decay.

Note there is something missing in the Strichartz estimates that need to be compensated for having something else.

Note that $L_x^p L_t^q$ are spaces that are invariant to translations and phase shifts.

$$\|u\|_{L^p L^q} = \|e^{ix\xi} u\|_{L^p L^q}$$

However, the phase shift generated by multiplying $e^{ix\xi}$ changes where the solution lies in the Fourier space. In other words, the Strichartz estimates do not see where the solution lies in the Fourier space.

$$i\partial_t u + \Delta u = u^2 = u \cdot u$$

where the first u localized in ξ_1, and second u localized in ξ_2. We multiply them together, hence get a convolution in the Fourier space, $(\xi_{11} - \xi_1^2), (\xi_{21} - \xi_2^2)$. Then we would add these to up due to convolution. And note that for paraboloids, which are convex, hence adding two points on the characteristic set would give you a point strictly inside the paraboloid.

Now Bourgain.

$$\Delta u = f, \dot{H}^s, |\xi|^s$$

And our characteristic set $\tau + \xi^2 = 0$, we use $|\tau + \xi^2|^b$, and $|\xi|^s$, where $|\tau + \xi^2|^b$ is the distance from the characteristic set in the verticle direction.

Definition 1.5 (Bourgain space)

We define a norm as follows:

$$\|\widehat{u} \cdot |\xi|^s |\tau + \xi^2|^b\|_{L^2} = \|u\|_{\dot{X}^{S,b}}$$

Remark s refers to the intial data regularity, $u_0 \in H^s$. and b refers to the Sobolev regularity away from the chraactersitic set.

Let's now interpret b. Fix $|\xi| = 1$, and set $s = 0$.

$$\widehat{u} \cdot |\tau + \xi^2|^n = f, \text{ where } f \in L^2$$

Does this definition make sense?

$$\widehat{u} = \frac{f}{|\tau + \xi^2|^b}$$

If we ignore ξ, i.e., L^2/τ^b, and L^2 is closed under multiplication of two L^2 functions, hence this makes sense as long as $1/\tau^b \in L^2$, i.e. $b < 1/2$. This gives an upperbound for b. Does there exist a lower bound?

If $b < 0$, then $\widehat{u}$ vanishes on $\tau + \xi^2 = 0$. Hence the worst f you could have is

$$f \approx |\tau + \xi^2|^{-\frac{1}{2}+}$$

If the exponent goes below $-1/2$, then $f \notin L^2$. Hence $b \geq -1/2$.

To sum up, we have two choices: in homogeneous spaces, we require $-1/2 < b < 1/2$; in inhomogeneous spaces, there is not restriction. We also consider scaling:

$$(i\partial_t + \Delta)u = 0, u(0) = u_0 \in \dot{H}^s, \Rightarrow \|u\|_{L^\infty \dot{H}^s}$$

Now we ask how does $X^{S,b}$ scale?

$$\frac{n+2}{2} - s - 2b = \frac{n}{2} - s$$

This gives $b = 1/2$.

Remark This is a major downside of Bourgain spaces, since scaling wouldnt' work since $b = 1/2$ is not allowed.

We connect them with the Strichartz spaces.

$$S \leftarrow X^{0,1/2}, S^s \leftarrow X^{s,1/2}$$

Hence the dual S' corresponds to the dual $X^{0,-1/2}$.

Remark One could work in Bourgain spaces if they care about resonant interactions, and Strichartz estimates if they don't...

Now we ask the question, can we find a middle ground between $X^{0,1/2}$ and Strichartz. Note that

$$X^{0,1/2} \subset S, X^{0,0} = L^2$$

We could interpolate in between

$$X^{0,b} \subset L^{p_1} L^{q_1}, \text{ where } \frac{1}{p_1} = (1-2b)\frac{1}{2} + 2b\frac{1}{p}$$

and the same relation holds for q. However, note that $\dot{X}^{0,1/2} \subset S$ is not well-defined. However, the above $X^{0,b}$ is true. Hence, we interpolate, again.

Theorem 1.2

$X^{s,b}$ to Strichartz embedding holds, where $b \in [0, 1/2)$.

We would like to prove something like $X^{0,b} \subset L^{p_1} L^{q_1}$, then we look at the map

$$u \to \frac{1}{|D_t + D_x^2|^b} u, L^2 \to L^{p_1} L^{q_1}$$

Proof We shall do a TT^* argument.

$$Q_b = T_b T_b^* : f \mapsto \frac{1}{|D_t + D_x^2|^{2b}} f$$

where $\frac{1}{|D_t + D_x^2|^{2b}}$ is a multiplier, and has a proper symbol in the dual space. And we would want

$$Q_b = TT^* : L^{p_1'} L^{q_1'} \to L^p L^q$$

Note that $T : X_0 \to Y_0$, and $T : X_1 \to Y_1$, then we can interpolate to have $X_\theta \to Y_\theta$. For our purposes, we consider a different interpolation, known as Stein's interpolation (for strips in the complex plane), shown in Figure 8.

We apply Stein's interpolation. $Re(b) = 0$, and $T_b : L^2 \to L^2$. Q_b, multipliers with symbol: (we would like to extend the range of b)

$$\Gamma(2b) = (\tau + \xi^2 + i0)^{2b}$$

This homogeneous expression does not have constraint on b. We can increase b until $q \to \infty$.

Note homogeneous functions act funny when the exponents are integers, hence we use the Gamma function Γ to kill the integers. Then $p_1 = q_1$ at endpoint, $p = q = \infty$. Now we just need to bound the Fourier transform of τ_ξ^2

$$Q_b = *(\tau + \xi^2)^{2b} : L^1 \to L^\infty$$

1.8 Lecture 7

$X^{s,b}$ Bourgain spaces and

$$u \in X^{S,b} if \widehat{u} \langle \xi \rangle^x \langle \tau + \xi^2 \rangle^b \in L^2$$

$$\frac{i\partial_t + \Delta}{-\tau - \xi^2} : X^{S,b} \to X^{S,b-1}$$

Bouragin solve in inhomogeneous spaces

$$(i\partial_t + \Delta)u - f, u(0) = u_0$$

$$(u_0, f) \to u$$

where $u_0 \in H^s$, and $f \in X^{S,b-1}$, where $u \in X^{S,b}$.

Now we ask the question: is the follwoing bound true?

$$\|u\|_{X^{S,b}} \leq \|u_0\|_{H^s} + \|f\|_{X^{S,b-1}}$$

Suppose $f = 0, s = 0$, then we have

$$\widehat{u} = \widehat{u_0}(\xi) \cdot \delta_{\tau=|\xi|^2} \notin X^{S,b}, \text{ for all } s, b$$

Hence, we only look at solutions that are local in time, i.e. we insert a cutoff funciton.

Hence we ask if the following bound holds:

$$\|\chi u\|_{X^{S,b}} \leq \|u_0\|_{H^s} + \|f\|_{X^{S,b-1}}$$

$$\widehat{\chi u} = \widehat{\chi} *_\tau \widehat{u} = \widehat{\chi} *_\tau \widehat{u_0}(\xi)\delta_{\tau+|\xi^2} = \widehat{u_0}(\xi)\widehat{\chi}(\tau + \xi^2)$$

Now we get

$$\|\chi u\|_{X^{0,b}} = \|\widehat{u_0}(\xi)\widehat{\chi}(\tau + \xi^2)\langle \tau + \xi^2\rangle^b\|_{L^2} \leq \|\widehat{u_0}\|_{L^2}$$

Note this is because $\widehat{\chi}(\tau + \xi^2)\langle \tau + \xi^2\rangle^b \in \mathcal{S}$, and $\mathcal{S} \subset L^2$. Hence, we lok at the contribution of f,

$$(i\partial_t + \Delta)(\chi u) = \chi(i\partial_t + \Delta)u_i\chi_t u$$

We would like to separate $\|\chi u\|_{X^{S,b}}$ into two parts into $\langle \tau + \xi^2\rangle > 1$ or not.

$$\|\chi u\|_{X^{S,b}} = \|(i\partial_t + \Delta)(\chi u)\|_{X^{S,b-1}} + \|\chi u\|_{X^{S,0}}$$

For $\langle \tau + \xi^2\rangle < 1$, we have the second term in L^2.

$$\|\chi u\|_{X^{S,b}} \leq \|f\|_{X^{S,b-1}} + \|u\|_{X^{S,0}}$$

Hence it remains to show

$$\|\chi u\|_{L^2 H^s} \leq \|f\|_{X^{S,b-1}} + \|u_0\|_{H^s}$$

We will replace $\|\chi u\|_{L^2 H^s}$ with $\|\chi u\|_{L^\infty H^s}$. Note that if we have $u_0 \in H^s$, we have $\chi u \in L^\infty H^s$. Now it remains to examine the contribution of f.

$$u(t) = \int_0^t e^{i(t-s)D^2} f(s) ds$$

Note that

$$\widehat{u}(t) = \int_0^t e^{i(t-s)\xi^2} \widehat{f}(s) ds$$

Now it's enough to consider fixed ξ. Hence we have

$$\widehat{u}(t, \xi) = e^{it\xi^2} \int_0^t e^{-is\xi^2} \widehat{f}(s, \xi) ds$$

Let

$$\widehat{g}(s, \xi) = e^{-is\xi^2} \widehat{f}(s, \xi)$$

Now we ask, where is $\widehat{g}$? Note $e^{-is\xi^2}$ is a translation in the Fourier time varialbe.

$$\widehat{g}(\tau, \xi) = \widehat{f}(\tau - \xi^2, \xi)$$

where $\widehat{f} \in L^2_{\langle\xi\rangle^s, \langle \tau+\xi^2\rangle^{b-1}}$, hence this puts $\widehat{g}$ in $L^2_{\langle\xi\rangle^s, \langle\tau\rangle^{b-1}}$. This gives $g \in H_t^{b-1} H_x^S$.

We have shown we are trying to bound the following in L^∞_{loc}.

$$\int_0^t g(s) ds$$

If $b = 1$, then this is L^2, hence is computable. If $b < 1$, consider

$$I = \langle g, 1_{[0,t]} \rangle$$

For $g \in H^{b-1}$, we would like to have $1_{[0,t]} \in H^{1-b}$.

$$\mathcal{F}_{1_{[0,t]}} = \frac{1}{\tau}(1 - e^{it\tau})$$

and this gives $\frac{1}{\tau^{1/2+}} \in L^2$.

Proposition 1.5

We have

$$1_{[0,t]} \in H^s, \text{ for } s < 1/2$$

Remark H^s embeds into C^0 the space of continuous functions, for $s > 1/2$, hence agrees with our computation.

Now we come back and consider

$$\begin{cases} iu_t + \Delta u = u^2 \\ u(0) = u_0 \in H^S \end{cases}$$

Goal: show there is a local solution in $X^{S,b}$. Note that

$$u = e^{itD^2} u_0 + \int_0^t e^{i(t-s)D^2} u^2 ds$$

where we think of $u = N(\chi u)$, and try to apply the contraction principle in $X^{S,b}$. We can make $\chi(0) = 1$ such that $\chi u(0) = u_0$, i.e. it agrees with our initial data. However, this is not the solution to our original problem anymore.

$$\begin{cases} (i\partial_t + \Delta)u = (\chi u)^2 \\ u(0) = u_0 \end{cases}$$

This coincides with the original solution as along as $\chi(t) = 1$. Now we consider

$$\chi u = \chi N(\chi u)$$

And we apply contraction principle for χu. However, this contraction defeats scaling.

Now we ask the question, is there a counterpart of the Bourgain space which works at $b = \pm\frac{1}{2}$. These spaces are important because scaling works and linear equations are taken into consideration.

For $u \in X^{S,b}$, we consider

$$v(t) = e^{-itD^2} u(t), u(t) = e^{itD^2} v(t)$$

This multiplier pushes u back to time 0 along the Schrodinger flow. For example, if u solves the homogeneous equation, then

$$u(t) = e^{itD^2} u_0 \Rightarrow v(t) = u_0$$

If we try to measure $\|u\|_{X^{S,b}}$, it is like measuring

$$\|u\|_{L^2_{\langle\xi\rangle^s, \langle\tau+\xi^2\rangle^{b-1}}} = \|v\|_{L^2_{\langle\xi\rangle^2 \langle\tau\rangle^b}}$$

This is to say

$$\|u\|_{X^{S,b}} = \|v\|_{H^b_t \dot{H}^s_x}$$

Hence we get,

$$\|v\|_{L^\infty H^s} = \|u\|_{L^\infty H^s} \leq \|u\|_{X^{S,b}}, b > 1/2$$

Writing this in terms of v, we get

$$\|v\|_{L^\infty H^s} \leq \|v\|_{H^b_t H^s_x}$$

And note $H^b \subset L^\infty$, when $b > 1/2$.

Note that if $b = 1/2$, we have

$$\dot{H}^{1/2} \not\subset L^\infty, \dot{H}^{1/2} \subset BMO$$

Now can we find a space where $b = 1/2$ and is scale invariant?

$$\begin{cases} L^\infty \subset L^\infty \\ \dot{W}^{1,1} \subset L^\infty \end{cases}$$

$W^{1,1}$ is the space of functions with continuous derivatives. We interpolate between L^∞ and $\dot{W}^{1,1}$. And note 2 is between 1 and ∞, we have

$$[L^\infty, \dot{W}^{1,1}]_{[1/2]}$$

However, there are spaces that come very close to it.

$$[]_{[1/2]} \sim V^2, U^2 \subset L^\infty$$

And $V^1 = BV$, the space of functions of boudned variation. We have the norm defined as follows:

$$\|u\|_{V^1} = \sup_{t_j} \sum |u(t_{j+1}) - u(t_j)|$$

Note that

$$W^{1,1} \subset BV$$

We would like to replace 1 with p,

$$\|u\|_{V^p}^p = \sup_{t_j} \sum |u(t_{j+1}) - u(t_j)|^p$$

As $p \to \infty$, we have that this resembles the L^∞ norm. Note that the pointwise values of functions in $u \in V^p$ is important, but we only care about functions that are defined almost everywhere.

Hence to avoid this issue, we know for any u of boudned variation, we could have a left and right limit, hence we can enforce u to be left or right continuous.

Thus, we have

$$BV = V^1 \subset V^p \subset V^q \subset L^\infty$$

for any p, q, and we do not care about the distance between p and q.

Note that

$$U^2 \subset [L^\infty, W^{1,1}]_{1/2} \subset V^2$$

And for the LHS inclusion, we have U^2, which is an atomic space.

Atoms $a \in \mathcal{A}$, are objects of size 1. and for every u in the space,

$$u = \sum c_j a_j, a_j \in \mathcal{A}$$

We can think of c_j's as basis of our space. We would like the sum to converge, hence we have

$$\|u\| = \inf \sum |c_j|$$

taking the infimimum to ensure this is unique.

Now it remains to specify the atoms.

$$a = \sum_j b_j 1_{I_j}, \text{ where } I_j \text{ are disjoint}$$

And we would like to have a to have size 1, hence we have

$$\sum_j |b_j|^2 = 1, i.e. l^2$$

Note I_j's can be as large and as small as you want.

Example 1.1 l^1 is an atomic space, with $a = \delta_{j=j_0}$. (can also heuristically think of L^1 as an atomic space, with atoms being the dirac delta functions)

Lemma 1.2

We have

$$U^2 \subset V^2$$

Proof Take $a \in U^2$, do we have $a \in V^2$? Note that

$$\sum_j |a(t_{j+1}) - a(t_j)^2 \leq \sum |b_{j_{k+1}} - b_{jk}|^2 \leq \sum b_j^2 = 1$$

Now consider $U^2(\mathbb{R}), V^2(\mathbb{R})$, and recall the transformation:

$$u \to v = e^{-itD^2} u$$

where $u \in X^{S,b}$, and $v \in H^b H^S$, and

$$u \in U^p_\Delta H^s \iff v \in U^p H^s$$

and

$$\|u\|^p_{V^p_\Delta H^s} = \|v\|^2_{V^p H^s} = \inf \sum \|v(t_{j+1}) - v(t_j)\|^p_{H^s} = \inf \sum \|u(t_{j+1}) - e^{i(t_{j+1}-t_j)D^2} u(t_j)\|^p_{H^s}$$

1.9 Lecture 8

We will continue our discussion of U^p, V^p spaces, and they are defined for functions defined on $\mathbb{R}$.

Definition 1.6 (U^p, V^p)

V^p is the bounded p variation, and we have the norm as follows:

$$\|u\|^p_{V^p} = \sup_{t_k} \sum |u(t_{k+1} - u(t_k))|^p$$

Note that we include left continuous in the definition for V^p since they have only countably many jump discontinuities and we define the value at that point to be the left limit.

U^p is the atomic space, and the atoms are define as

$$a = \sum c_k 1_{[t_k, t_{k+1}]}, \sum_k c_k^p = 1$$

And for $u \in U^p$, we have

$$u = \sum_k \lambda_k a_k, \|u\| = \inf \sum_k |\lambda_k|$$

♣

Last time we have

$$U^p \subset V^p$$

The reverse inclusion is not correct, but we have the following proposition if we change up the exponents a bit.

Proposition 1.6

We have

$$V^p \subset U^q, \text{ if } p > q$$

♠

Suppose that $\|v\|_{V^p} = 1$ and the range of v, $Range(v) \subset [0, 1]$, and our first guess is that u_1 is a step function so that $|v - u_1| \leq \frac{1}{2}$. And we look at the number of steps, in each step, we produce a difference of $1/2$, by the norm, $\left(\frac{1}{2}\right)^p$. If we denote the number of steps as N_1, we thus have

$$\left(\frac{1}{2}\right)^p N_1 \leq 1 \Rightarrow N_1 \leq 2^p$$

Every step if we divide the range in 2, we get a sequence v_n such that (we increase our accuracy by $1/2$ at each step)

$$\leq 2^{-n} v - (u_1 + ... u_n)$$

and u_n is a step function with steps $\leq 2^{-n+1}$. Then we count the number of steps N_n,

$$2^{-np} N_n \leq 1 \Rightarrow N_n \leq 2^{np}$$

These steps have preassigned dyadic size. Now we compute $\|u_n\|_{U^q}$, we have

$$\|u_n\|^q_{U^q} \leq 2^{(-n+1)q} \cdot N_n \approx 2^{n(p-q)}$$

Since we have $p < q$, hence we have the converges absolutely.

$$\|v\|_{U^q} \leq \sum 2^{n(p-q)} \lesssim 1$$

□

Remark Just like Christ-Kiselev lemma, we don't split the function based on equidistant intervals, but instead, we split the intervals based on the size of the function $|f(x)|$.

We discuss the connection with the Schrodinger.

$$U^p_\Delta L^2 = \{u \in L^\infty L^2 : e^{-itD^2}u \in U^p L^2\}$$

For $u \in U^p_\Delta$, we have

$$e^{-itD^2}(i\partial_t + \Delta)u = i\partial_t e^{-itD^2}u$$

Hence we have

$$(i\partial_t + \Delta), \text{ and } e^{itD^2}$$

as conjugate operators with respect to e^{-itD^2}.

Note that we have

$$i\partial_t + \Delta : \dot{X}^{S,b} \to \dot{X}^{S,b-1}, \partial_t : \dot{H}^b \to \dot{H}^{b-1}$$

And we have the following duality relation:

$$\left(X^{S,b}\right)^* = X^{-S,-b}$$

Similarly, we have the following duality relations:

$$\left(\dot{H}\right)^* = \dot{H}^{-b}$$

And

$$\partial_t : H^b \to H^{b-1}, \left(\dot{H}^{\frac{1}{2}}\right)^* = \dot{H}^{\frac{1}{2}}, \partial\dot{H}^{\frac{1}{2}} = \dot{H}^{\frac{-1}{2}}$$

Moreover, we would like

$$U^2, V^2 \approx \dot{H}^{\frac{1}{2}}$$

Proposition 1.7

We have

$$(\partial U^2)^* = V^2$$

First we observe that, for the Schrodinger equaiton, we have

$$\left[(i\partial_t + \Delta)U^2_\Delta L^2\right]^* = V^2_\Delta L^2$$

And

$$\begin{cases} (i\partial_t + \Delta)u = f \\ u(0) = u_0 \end{cases}$$

Hence we have

$$\|u\|_{U^2_\Delta L^2} \leq \|u_0\|_{L^2} + \|f\|_{DU_\Delta L^2}$$

Note that we have $U_\Delta L^2$ as our $X^{S,b}$, and S the Strichartz space, and $DU^2_\Delta L^2$ as S', and $X^{S,b-1}$.

We look at the Riemann-Stiegjer integral,

$$\langle \partial_t u, v\rangle = \int v du = \lim_{\Delta t_n \to 0} \sum v(t_j)\left(u(t_{j+1} - u(t_j))\right) = \lim_{\Delta t_n \to 0} \sum u(t_{j+1})(v(t_{j+1}) - v(t_j))$$

where $u \in U^2, v \in V^2$, we take a sequence of points t_n increasing, and $t_{j+1} - t_j \to 0$. (Exactly how we defined the Riemann integral).

We have

$$\|v\|_{(\partial U^2)^*} = \sup_{\|u\| \leq 1} \langle \partial_t u, v\rangle = \sup_{atoms} \langle \partial_t a, v\rangle$$

hence we have

$$\langle \partial_t a, v\rangle = \sum c_j(v(t_{j+1}) - v(t_j)), a = \sum c_j 1_{[t_j, t_{j+1})} \le \left(\sum (v(t_{j+1}) - v(t_j))^2\right)^{\frac{1}{2}}$$

due to $(l^2)^* = l^2$, hence it embeds into V^2, as this is exactly the V^2 norm. □

$$\begin{cases} U^p \subset V^p \\ (DU^p)^* = V^{p'} \end{cases}$$

These are called Besov norms. And we have $U^2, V^2 \approx \dot{H}^{\frac{1}{2}}$. Dyadic definition of $\dot{H}^{\frac{1}{2}}$, and $u \in \dot{H}^{\frac{1}{2}}$, this implies $\hat{u}|\xi|^{\frac{1}{2}} \in L^2$, we have

$$|\xi| \approx 2^j, 1 = \sum_j \chi_j(\xi)$$

and we have

$$\hat{u} = \sum \chi_j \hat{u} = \sum \hat{u}_j$$

Definition 1.7 (Besov norm)

$B^s_{p,q}$ is defined as

$$\|u\|_{B^s_{p,q}} = \sum \left(2^{\frac{j}{s}} \|u_j\|_{L^2}\right)^q$$

Hence we have

$$\|u\|_{B^{\frac{1}{2}}_{2p}} = \sum \left(2^{j/2}\|u_j\|_{L^2}\right)^p$$

♣

Note that we have

$$\dot{B}^{\frac{1}{2}}_{2,1} \subset \dot{H}^{\frac{1}{2}} = \dot{B}^{\frac{1}{2}}_{2,2} \subset \dot{B}^{1/2}_{2,\infty}$$

For the above, we used the following embedding:

$$l^1 \subset l^2 \subset l^\infty$$

and we have

$$\partial \dot{H}^{\frac{1}{2}} \to \dot{H}^{-1/2} = (H^{\frac{1}{2}})^*$$

And

Proposition 1.8

We have the following embedding:

$$\dot{B}^{\frac{1}{2}}_{2,1} \subset U^2 \subset V^2 \subset \dot{B}^{\frac{1}{2}}_{2,\infty}$$

♠

Proof It suffices to prove the first embedding, since if we have

$$B^{\frac{1}{2}}_{2,1} \subset U^2 \Rightarrow \subset DU^2 \Rightarrow V^2 \subset B^{\frac{1}{2}}_{2,\infty}$$

Recall we have

$$\|u\|_{B^{\frac{1}{2}}_{2,1}} = \sum_j 2^{j/2}\|u_j\|_{L^2}$$

We have

$$\|u\|_{U^2} \le \|u\|_{B^{\frac{1}{2}}_{2,1}}$$

by triangle inequality, this gets reduced to the following:

$$\|u_j\|_{U^2} \le 2^{j/2}\|u_j\|_{L^2}$$

By rescaling, we could take $j \to 0$. And we look at $u_0 = u \in L^2$, localized in

$$\{|\xi| = 1\} \Rightarrow u \in U^2$$

Enough to look at $u \in H^1 \Rightarrow u \in H^2$, and we decompose u based on unit intervals,

$$u = \sum_k u_k$$

where $supp(u_k) \subset [k-1, k]$. Hence if we take $u_k \in V^1$, and $\{u_k\} \in l_k^2 V^1 \subset U^2$, where we used the embedding $V^1 \subset U^2$. Hence we are done.

Local orthogonality gives us

$$\|u\|_{L^2}^2 \approx \sum_k \|u_k\|_{L^2}^2$$

□

Corollary 1.1

We have $U_\Delta^2 L^2 \approx V_\Delta^2 L^2 \approx \dot{X}^{0,\frac{1}{2}}$ for functions localized in $|\tau^2 + \xi^2| \approx 2^j$.

Proposition 1.9

Suppose $L^p L^q$ is a Strichartz esimate, then we have

$$U_\Delta^p L^2 \subset L^p L^q$$

Proof It is enough to show it for atoms, and the same p allows us to use Strichartz.

Corollary 1.2

For $p_1 < p$, we have

$$V_\Delta^{p_1} L^2 \subset L^p L^q$$

$$\|u\|_{L^{p_1} L^{q_1}} \le \|u_0\|_{L^2} + \|f\|_{L^{p_2'} L^{q_2'}} \Rightarrow \text{Christ-Kiselev} p_1 > p_2'$$

And note that we have

$$\|u\|_{U_\Delta^r L^2} \le \|u_0\|_{L^2} + \|f\|_{DU_\Delta^r L^2}$$

where $r \le p_1$ this implies

$$U_\Delta^r L^2 \subset L^{p_1} L^{q_1}$$

And for $r > p_2'$, this implies

$$L^{p_2'} L^{q_2'} \subset DU_\Delta^r L^2$$

And this gives

$$V_\Delta^{r_1} L^2 \subset L^{L^{p_2}} L^{q_2}$$

1.10 Lecture 10

Next we make the transition from linear to nonlinear Schrodinger equations (NLS).

$$\begin{cases} iu_t + \Delta u = \pm u|u|^p, p > 0, 1 \\ u(t=0) = u_0, u_0 \in H^s, \text{ or } \dot{H}^s \end{cases}$$

Next we ask the question, is this question well-posed? and is it globally well-posed?

We first begin with $u_0 \in L^2$, and try to find if we can solutions $u \in C(L^2)$.

$$\begin{cases} iu_t + \Delta u = f \\ u(0) = u_0 \end{cases}$$

And we have the Strichartz estimates:

$$\|u\|_{L^\infty L^2} + \|u\|_S \le \|u_0\|_{L^2} + \|f\|_{S'}$$

where $S = \cap L_t^p L_x^q$, and S' is the dual Strichartz space.

We will "replace the line with a number," namely, we will choose $p = q$, and

$$S = L_{t,x}^{\frac{2(n+2)}{n}}, S' = L_{t,x}^{\frac{2(n+2)}{(n+4)}}$$

where our $f = u \cdot |u|^p$.

$$u \to |u|^p \cdot u, L_{t,x}^{\frac{2(n+2)}{n}} \to L_{t,x}^{\frac{2(n+2)}{(n+4)}}$$

hence we obtain, $p = \frac{4}{n}$.

Using the contraction principle to solve the NLS.

We have the solution in the form:

$$u(t) = e^{itD^2}u_0 + \int_0^t e^{i(t-s)D^2} u \cdot |u|^p ds := N(u)$$

Note *Why do we pick out $f = u \cdot |u|^p$ instead of, say $|u|^{p+1}$, this is because this $u \cdot |u|^p$ preserves all the nice properties of solutions to the equation, such as the rotation symmetry, Galilean symmetry, etc. For example, $\overline{u}$ kills the phase rotation, Galilean symmetry.*

Proposition 1.10

$u \mapsto N(u)$ is a contraction, i.e. it converges to a fixed point.

First we would like to define the notion of "contraction" given a metric, we pick our Banach space to be the Strichartz estimate S.

By the linear Strichartz estimate, it suffices to look at the source term:

$$\begin{aligned}
\|N(u) - N(v)\|_S &\le \|u|u|^p - v|v|^p\|_{S'} \\
&\le \||u-v| \cdot (|u|^p + |v|^p)\|_{S'} \\
&\le \|u-v\|_{L^{\frac{2n+1}{n}}} \|(u,v)\|_{[L^{\frac{2n+2}{n}}]}
\end{aligned}$$

Instead of S'. we replace it with $L^{\frac{2(n+2)}{n+4}}$, where the last term is the Lipschitz solution, hence we look for small data solutions.

Assume $\|u_0\|_{L^2} < \epsilon$, then we look for solutions in $B_{L^{\frac{2(n+1)}{n}}}(0, 100\epsilon)$.

Theorem 1.3

*NLS with $p = \frac{4}{n}$ is **globally** well posed for initial data u_0 which is small in L^2.*

Note *Two caveats: we fixed p, do not have understanding of other p's, and we can only work with small initial data u_0.*

Remark When p is even, we are given the nice properties that u could be made analytic, hence $n = 1, 2$ cases are interesting.

Problem 1.1 How do we proceed with other p's?

n=1 $p = 4$. Can p go lower? $0 < p < 4$. Say, we pick $p = 2$. Hence we estimate: $u \cdot |u|^2$. The Strichartz space:

$$L^\infty L^2 \to L^4 L^\infty, S' : L^1 L^2 \to L^{\frac{4}{3}} L^1$$

For the following:

$$|u|^2 \cdot u = \overline{u} \cdot u \cdot u$$

where $\overline{u}, u \in L^4 L^\infty$, and the third $u \in L^\infty L^2$, hence taking the produt of both in time and space, we get our $|u|^2 \cdot u \in L^2 L^2$, and for fixed time interval T, we have

$$L^2 L^2 \subset L^1 L^2$$

Now going back to the contraction $N(u)$, we get

$$\begin{aligned} \|N(u) - N(v)\|_S &\leq \||u|^2 u - |v|^2 v\|_{L^1 L^2} \\ &\leq \sqrt{T} \| \cdot \|_{L^2 L^2} \\ &\leq \sqrt{T} \|(u,v)\|_S^2 \end{aligned}$$

Hence now, our Lipschitz constant is

$$L = \sqrt{T} \left(\|(u,v)\|_S^2\right)^2$$

Hence we can choose our T corresponding to kill the contribution of the rightmost term.

Corollary 1.3

For $\|u_0\|_{L^2} \leq M$, and $\|u\|_S \leq 10^5 M$, we can choose $T = 10^{-10} M^{-4}$, then u exists by the contraction principle. ♡

Hence we have the theorem.

Theorem 1.4

For the above NLS, with $0 < p < \frac{4}{n}$, it is locally (then globally) well-posed for initial data $u_0 \in L^2$. ♡

Remark We recall, if our ODE solution exists locally, and does not increase after that local time period, then the solution exists globally. Hence we check if solution u grows, we have $\frac{d}{dt}\|u\|_{L^2}^2 = 0$, as for the nonlinear portion, we have $\int Im(\pm u|u|^p \overline{u})$, hence picking u to be real, we have that NLS is well-posed globally.

Note that under scaling, the linear Schrodinger:

$$u(t,x) \to \mu u(\lambda^2 t, \lambda x)$$

and for the nonlinear equation it is the same, however, t, x are coupled:

$$(iu_t + \Delta u) = |u|^p \cdot u$$

Hence we have

$$u - \lambda^2 = \mu^{p+1} \Rightarrow \mu = \lambda^{\frac{2}{p}}$$

Then we ask the question: how about the L^2 norm of the data?

$$\|u_\lambda\|_{L^2}^2 = \int \mu^2 |u(\lambda x)|^2 dx = \mu^2 \lambda^{-n} \|u\|_{L^2}^2 = \lambda^{\frac{4}{p}} \lambda^{-n} \|u\|_{L^2}^2$$

Note that when $p = \frac{4}{n}$, the magical exponent for p!

Corollary 1.4 (L^2 critical)

When $p = \frac{n}{4}$, the L^2 problem is scale-invariant. (The L^2 norm does not change when you scale the solution). This is called the L^2 critical problem, and we get global well-posedness for small data. ♡

Now suppose $p < \frac{4}{n}$, this is called the **subcritical** problem, and $u(t) \in [0, T]$, and u_λ lives for $[0, \lambda^{-2} T]$. We could scale our data to preserve the size, we could have the data live longer by scaling the size at particular t to be smaller; or have the data live shorter by increasing the size.

Now suppose $p > \frac{4}{n}$, this is called the **supercritical** L^2 problem, and this problem is ill-posed. this is because the scaling is reversed as in the subcritical case. We obtain a "better" solution, larger and lives longer by scaling, which should be impossible.

Note *What do we mean by saying a problem is ill-posed? It means the nonexistence (where), or nonuniqueness. And maybe a third one is the continuous dependence on the initial data. This could happen with two initial data that start really close together, but the solution differs a lot.*

One could define well-posedness in S the Strichartz space, or one could talk about the unconditional uniqueness in $C(L^2)$, where it is continuous in L^2. One could surely do a combination of the two, where one compares S and one $C(L^2)$.

Recall we came up with $p = \frac{4}{n}$, since we look at $u_0 \in L^2$, (and we did this because the conservation law holds in L^2), and used the Strichartz estimate given that our intial data is in L^2.

Now we would like to move u_0 away from $L^2 \to \dot{H}^s$, and the reason for homogeneous Sobolev space is that we would like to do scaling. Now how does $\|u\|_{H^s}$ change under sclaing?

$$\|u_\lambda\|_{H^s}^2 \approx \|D^s u_\lambda\|_{L^2}^2 \approx \int u^2 \lambda^{2s} |D^s u(\lambda x)|^2 dx \approx \lambda^{\frac{4}{p}} \lambda^{-n} \lambda^{2s} \|u\|_{H^s}^2$$

Hence the dependence now also includes s. This gives us $\frac{4}{p} = n - 2s$, and $p_s = \frac{4}{n-2s}$. Now we again separate into three cases, and the only thing we no longer have the conservation in H^s.

1. $p < p_s$
2. $p = p_s$
3. $p > p_s$

Remark Subcritical means everything nice, and critical is when it is scale-invariant, and supercritical is ill-posed.

s=1 This gives rise to another conservation situation.

$$iu_t + \Delta u = u \cdot |u|^p$$

H^1 conservation law is the following: recall the mass $M = \frac{1}{2}\int |u|^2 dx$,

$$E = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{p+2}|u|^{p+2} dx$$

Hence for H^1 critical, we have $p_c = \frac{4}{n-2}$, and this only makes sense when $n \geq 3$.

Note *In $n = 2$, H^1 is almost like L^∞, and hence loses meaning.*

Note

$$0 < \frac{4}{n-2} < \frac{4}{n}$$

and the middle one is the energy critical point, and the right $4/n$ is the mass critical point.

1.11 Lecture 11

We've been discussing the semi-linear equation:

$$\begin{cases} iu_t + \Delta u = \pm |u|^p \cdot u \\ u(0) = u_0 \end{cases}$$

And we've seen the operator $P \to S_p$, where $p = \frac{4}{n-2}$, and we separate them into three cases, the $s > s_p, s = s_p, s < s_p$, where we have $s > s_p$ is LWP, the second being GWP for small dadta, and $s < s_p$ is ill-posed.

We also discussed the conservation of mass

$$M = \frac{1}{2}\int |u|^2 dx$$

And the conservation of energy

$$E = \int \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p+2}|u|^{p+2} dx$$

The conservation of mass lives in L^2, where $p = \frac{4}{n}$, and $n \geq 1$, and we call this the mass critical problem; and the conservation of energy, we want it to be $\dot{H}^1$, where $p\frac{4}{n-2}$, and $n \geq 3$, and we call this is the mass critical problem.

Next we discuss the Hamiltonian structure: we have

$$w(u, v) = \int Im(u \cdot \overline{v}) \Rightarrow J = i$$

where J denotes the sympletic form. If we have the Hamiltonian form $H = H(x)$, relaxed to the sympletic form J, and we should have

$$u_t = JDH(u)$$

We take the flow associated to the mass, $u_t = iu, u(t) = u(0)e^{it}$ where this acts as a phase rotation. Then if we take the energy:

$$u_t = i\left(-\Delta u \pm u|u|^{p-1}\right) \to (NLS)$$

Theorem 1.5 (Noether's theorem)

Phase rotation symmetry corresponds exactly to mass conservation.

We have another version of symmetry, the translation symmetry, where we have $u(t,x) \to u(t,x+he_j)$, and we thus have $u_t = -u_{x_j}$.

Proposition 1.11 (The mass critical case.)

If we start with $\|u_0\| \ll 1$, then we have $u \in S$.

we considered the Strichartz space, where $p = q$, and thus consider $L_{t,x}^{\frac{2(n+2)}{n}}$, and have $u \cdot |u|^p \in S'$.

Once we obtain a global solution. This gives rise to our discussion of scattering, which is how the solution behaves at infinity. **Remark** We would like to say this u to the NLS behaves just like the solution to the linear equation at ∞.

Theorem 1.6 (scattering)

If u is a small L^2 solution to mass critical problem. Then there exists $u^+ \in L^2$ such that

$$\lim_{t\to\infty} u(t) - e^{it\Delta}u^+ = 0 \text{ in } L^2$$

where we have $e^{it\Delta}u^+$ is our solution to the linear equation.

Proof We should have

$$u^+ = \lim_{t\to\infty} e^{-it\Delta}u(t)$$

We first have $u(t)$ the nonliear flow at time t, then we come back to u_0 for NLS, and also back to $t = 0$ for th linear flow, then we push it to $t \to \infty$.

$u(t) = e^{it\Delta}u(0) + \int_0^t e^{i(t-s)\Delta} f(s)ds$. Applying $e^{-it\Delta}$, we have

$$e^{-it\Delta}u(t) = u(0) + \int_0^t e^{-is\Delta} f(s)ds$$

where the RHS, we have $f \in S'$, and $e^{-is\Delta}$ send it back to dual Strichartz, and land in L^2.

Then $f \in L^{\frac{2(n+2)}{n}}([0,\infty)\times\mathbb{R}^n)$, hence

$$1_{[0,t]}f \to f \text{ as } t \to \infty$$

Hence we get our scattering property.

□

Note *If $p < \frac{4}{n}$, this is our subcritical problem, then local well-posedness implies global well-poseness, this gives us $u|u|^p \in S'$ on I.*

We are still a bit unhappy since for mass critical problem, we are only working with small intiial data $\|u_0\|_{L^2} \ll 1$.

Problem 1.2 What if our initial data is large?

Let's start witht he easier version: local well-posedness.

Theorem 1.7

If $u_0 \in L^2$, then there exists some time $T = T(u_0)$ such that the problem has a solution in $[0,T]$. (By having a solution, we again mean we can put the source term f in the dual Strichartz estimate).

Proof We formulate this as a fixed point problem.

$$u(t) = e^{it\Delta}u_0 + \int_0 0^t e^{i(t-s)\Delta}u \cdot |u|^{p-1}(s)ds$$

If $u = N(u)$, then you need some X such that $N : X \to X$, and such that N is a contraction. If you start with a small initial data, then there is no bad way to choose X.

Again we would like to look at the Lipschitz constant for $u \cdot |u|^{p-1}$, because we need u to be small in X. Note that we would like

$$L^\infty L^2 \not\subset X$$

We choose $X = B(0,\epsilon)$ in $L_{t,x}^{\frac{2(n+2)}{n}}$, to have a small Lipschitz constant for nonliearity.

Note that we have

$$\|e^{it\Delta}u_0\|_{L^2} \leq \|u_0\|_{L^2}$$

But note that our RHS could be large, then we could choose T such that

$$\|e^{it\Delta}u_0\|_{L^{\frac{2(n+2)}{n}}} \leq \epsilon$$

Note that we have $L^{\frac{2(n+2)}{n}}([0,T]\times\mathbb{R}^n)$. Wave packets can live on any time scale, and depending on how we choose the scale, we might stop at a very very small T, yet it is positive.

□

The scattering we had is $t \to \infty$, and we could also investigate $t \to -\infty$.

$$u_0 \to u^+(\infty), u_0 \to u^-(-\infty), L^2 \to L^2$$

Then we have a reverse problem (asymptotic completeness).

Problem 1.3 Given $u^+ \in L^2$, is there a solution u that matches our u^+ at infinity.

Theorem 1.8

Given $u^+ \in L^2$, then there exists T and a solution u in $[T,\infty)$.

Proof We apply the exact same argument, but just writing it from ∞.

$$u = e^{it\Delta}u^+ - \int_t^\infty e^{i(t-s)\Delta}u\cdot|u|^p(s)ds$$

Choose T such that

$$\|e^{it\Delta}u^+\|_{L^{\frac{2(n+2)}{n}}([T,\infty)\times\mathbb{R}^n)} \leq \epsilon$$

We choose T for as long as it keeps the norm small.

□

Note *The solution u is global if our u^+ is small.*

We know how to solve for u_0 is large from $t=0$, and from ∞. Now we ask if we can have **blow-up** if the initial data u_0 is large?

First we need to identify "blow-up." It is impossible to have $\|u(t)\|_{L^2} \to \infty$. Find maximal T such that $[0,T)$ maximal time interval of existence (not specified which space yet). Then

$$\lim_{t\to T} u(t) := u(T) \in L^2$$

If we above limit exists, then we could denote it as $u(T)$, and it leaves in L^2 since it is a limit of an L^2 sequence. Then we could continue to extend our solution for larger T.

Proposition 1.12

If the solution $u : [0,T) \to L^2$ is a maximal solution, then

$$\lim_{t\to T} u(T)$$

does not exist.

Note *We also mean the limit does not even exist for any subsequence. However, this is not a good enough criterion, since all continuous functions are by definition, cannot blow up.*

Next we introduce a better blow-up criterion.

We have $\|e^{it\Delta}u_0\|_{L^{\frac{2(n+2)}{n}}} = \epsilon$, and note now we use this as an initial data, and extend our solution, hence we have $T(u_0), T(u_1), \ldots$.

If we don't have a global solution, then we must have stopped somwhere, then we get a solution on $[0,T)$, and then we have

$$\|u\|_{L^{\frac{2(n+2)}{n}}([0,T)} = \infty$$

Theorem 1.9

The solution u blows up at time T if and only if

$$\|u\|_{L^{\frac{2(n+2)}{n}}([0,T)\times\mathbb{R}^n} = \infty$$

Same at ∞, where either $\|u\|_{L^{\frac{2(n+2)}{n}}([0,T)\times\mathbb{R}^n} < \infty$ with scattering or $\|u\|_{L^{\frac{2(n+2)}{n}}([0,T)\times\mathbb{R}^n} = \infty$ in which case we have no scattering.

The magical signs:

$$E(u) = \frac{1}{2}\int |\nabla u|^2 \pm \frac{1}{p+2}|u|^{p+2}dx$$

For $u \in L^2$, we have the GNS inequality:

$$\|u\|_{L^{2*}} \leq \|\nabla u\|_{L^2}$$

where

$$\frac{1}{2} > \frac{1}{p+2} = \frac{1}{2} - \frac{1}{n+2} > \frac{1}{2^*} = \frac{1}{2} - \frac{1}{n}$$

Note that we have the interpolation inequality.

$$\|u\|_{L^{p+2}}^{p+2} \leq \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^p$$

Let's think about global solutions, and those that scatter and those that don't.

The simplest solution that does not scatter is those that are constant in time, and we call them the **steady state**.

$$u_t = i \cdot \frac{\delta E(u)}{\delta u}$$

Steady state is if and only if critical points for E.

In the sign in the energy formula is +, then E is convex and there is no steady state; if the sign is -, then we have the mountain pass lemma. After we cross a critical convex point, then we pass the hill, and look for the smallest height that we can reach.

Lemma 1.3

There exists a smallest size steady state Q, the ground state, which is

$$\Delta Q = -|Q|^p \cdot Q$$

where Q is real-valued.

And we come to the conjecture that for the $+$ sign, we have the defocusing problem: we have well-posedness and scattering for large data; the $-$ sign, we have the focusing problem, and we have scattering and global solution for $\|u_0\|_{L^2} \leq \|Q\|_{L^2}$. (i.e. if you are stuck in the bowl) This is a theorem by Dodsin.

1.12 Lecture 10/3

We were looking at the mass critical problem:

$$\begin{cases} iu_t + \Delta u = |u|^p, p = \frac{4}{n} \end{cases}$$

where we have GWP fro small data, and LWP for large initial data, and blow up at $\|u\|_{[L^{\frac{2(n+2)}{n}}]} = \infty$.

We now talk about focusing and defocusing. The energy is

$$E(u) = \int \frac{1}{2}|\Delta u|^2 \pm \frac{1}{p+2}|u|^{p+2}dx$$

where we noted that if we have $+$, then it is defocusing and it is convex and coercive, and focusing when we have $-$.

$$\int |u|^{p+2} \lesssim \int |\Delta u|^2 \left(\int |u|^2\right)^{\frac{p}{2}}$$

Hence the energy is controlled by $|u|_{L^2}$ then E is coercive, and but not if $\|u\|_{L^2}$ gets large.

Now we ask the question how exactly is $\|u\|_{L^2}$ small, and what is the **threshold** for $\|u\|_{L^2}$ such that the energy stays positive definite? And for this, we need to find the sharp constant such that inequality in G-N

$$\int |u|^{p+2} \leq C \int |\Delta u|^2 \left(\int |\nabla u|^2\right)^{\frac{p}{2}}$$

And we look for the smallest constant such that the above holds, or equivalently,

$$\max \frac{\int \frac{1}{p+2}|u|^{p+2}}{\int \frac{1}{2}|\nabla u|^2} := H(u)$$

with $\|u\|_{L^2} = c$. This should remind you of the Lagrange multipliers, subject to some constraint.The existence of the maximizer will be delayed till later discussions.

Now we look at the following and would like to set this to 0.

$$\frac{d}{dh}H(u+hv) = \frac{d}{dh}\log H(i+hv) = \frac{\int |u|^p Re(u\overline{v})}{\int \frac{1}{p+2}|u|^{p+2}} - \frac{\int Re(\nabla u \nabla \overline{v})}{\int \frac{1}{2}|\nabla u|^2}$$

And $v \perp u$, hence we get

$$Re \int \left(\frac{|u|^p u}{\int \frac{1}{p+2}|u|^{p+2}dx} + \frac{\int \Delta x}{\int \frac{1}{2}|\nabla u|^2}\right) \cdot \overline{u} dx = 0$$

This gives that

$$\frac{|u|^p u}{\int \frac{1}{p+2}|u|^{p+2}dx} + \frac{\int \Delta x}{\int \frac{1}{2}|\nabla u|^2} = \lambda u$$

where λ is the Lagrange multiplier. As we move away from 0, i.e. $u \mapsto au$, as a increases, we get a positive power of a, a^p

We choose c such that the maximum is exactly 1. Then the denominators in the above equation are exactly equal, hence we move them to the RHS into λ, thus

$$|u|^p u + \Delta u = \lambda u, \lambda > 0$$

Because of the presence of λu, this is not a steady state problem. However, u

$$v(x,t) = u(x)e^{i\lambda t}$$

solves the NLS. This is called a soliton.

$$iv_t + \Delta v = (-\lambda u + \Delta u)e^{i\lambda t} = |u|^p u e^{i\lambda t} = v^p \cdot v$$

Hence we get the conclusion that there exists some c_0 such that at c_0, the max is 1, $E(v) \geq 0$, and the maximizer, $E(Q) = 0$, and $Qe^{i\lambda t}$ then solves the NLS.

Proposition 1.13

$c_0 = \|Q\|_{L^2}$ is the critical mass. And $\|u\|_{L^2} < c_0$ implies that E is coercive, and $\|u\|_{L^2} = c_0$ we have the extreme case where $u = Q_1$, and $E(Q) = 0$.

$$\max \frac{|u|^{p+2}}{\int |\nabla u|^2} dx$$

First by separating u into real and imaginary parts, we conclude that u is real. Then notice the quantity does not change if we replace u with $|u|$, then we can take $u \geq 0$. And we do the rearrangement: we take the decreasing rearrangement of λ, And we have $|\{u \geq \lambda\}| = |\{u^* \geq \lambda\}|$, and we want u^* in 1-d, and to generalize, we want u^* to be spherically symmetric in n-dim.

First we rewrite

$$-\Delta u + \lambda u = |u|^p \cdot u$$

And we take the Fourier transform: where the LHS is $\xi^2+\lambda$. This implies exponential decay for the fundamental solution. This implies that u is smooth and decays exponentially at ∞.

$$(\partial_r + \frac{r}{r-1}\partial_r + \lambda)u = u^{p+1}$$

This gives the solution in the form:

$$u \approx cr^{-\frac{n-1}{2}}e^{-\sqrt{\lambda}r}$$

Note *This is the shooting problem, and usually, the slope is not zero, and this ensures the uniqueness of solution.*

Theorem 1.10

There is a unique smooth nonnegative radial ground state.

Our first soliton is as follows: $u = e^{it}Q$, and we could do the following:

1. translate it
2. rescale it, making it wider and shorter
3. Galilean motion/transform, due to Galilean invariance.

To summerize, we can choose scale, position and velocity. These are the enemies of scattering.

There is a Schrodinger counterpart to Euclidean inversion: Euclidean:$x \mapsto \frac{1}{x}$, where the Schrodinger is

$$(t,x) \mapsto \left(\frac{1}{t}, \frac{x}{t}\right)$$

And

$$u(t,x) \mapsto \frac{1}{t^{-\frac{n}{2}}}e^{i(x^2+4)/(4t)}u(\frac{1}{t}, \frac{x}{t})$$

This is calle dhte pseudoconformal transformation. If $u = Q$, then $\tilde{Q}$ and note that $\tilde{Q}$ blows up at $t=0$, and

$$\|\tilde{u}\|_{L^2} = \|u|_{L^2}$$

We have the solitons, and solutions that blow up at finite times.

$$iu_t = -\Delta u \pm |u|^{p-1}u$$

On the RHS, $-\Delta u$ is the linear part, and it wants to disperse, the second term is $|u|^{p-1}u$, and wants to behave, together with the LHS, like an ODE.

And suppose $u_0 = A\cdot e^{\frac{(x-x_0)^2}{T}}e^{i(x-x_0)\cdot\xi_0}$, where A is some amplitude:

$$\delta_x \approx \sqrt{T}, \delta_\xi = \frac{1}{\sqrt{T}}$$

And linear flow stays coherent up to time T.

We note that $|u|^p$ is a conserved quantity, and we have

$$iu_t = \pm|u|^p\cdot u$$

and we have

$$u(t,x) = u_0(t,x)\cdot e^{\pm it|u_0(t,x)|^p}$$

We solved it using the fact it is now an ODE.

Now we ask the question, is the nonlinear effect slower of faster than the linear one?

We assumed $x_0 = 0$, and $\xi_0 = 0$. and note that the size of the phase is up to $T\cdot A^p$, and we have

$$T\cdot A^p \begin{cases} \ll 1, \text{ linear } wins \\ \approx 1, \text{ balance} \\ \gg 1, \text{ Nonlinear wins} \end{cases}$$

And

$$\|u_0\|_{L^2} \approx 1 \approx A(\sqrt{T})^{\frac{n}{2}} = AT^{\frac{n}{4}} \approx 1$$

If we have $p > \frac{4}{n}$, the linear part wins and you expect scattering, and $p = \frac{4}{n}$ (we assume $\|u_0\|_{L^2}\approx 1$), and if we drop the assumption, it depends on the size of the initial data, and $p < \frac{4}{n}$, and nonlinear wins, and expect no classical scattering.

You start with a bump funciton for initial data, and the nonlinear portion turns into quite oscillatory, then linear makes waves move along their group velocity.

1.13 Lecture 10/5

If we look at the energy critical problem, re call the equal of interest, for $p = \frac{4}{n-2}$, and $n \geq 3$:

$$iu_t + \Delta u = \pm |u|^p \cdot u$$

and $\dot{H}^1$ is critical , recall the energy is defined as

$$E(u) = \int \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p+2}|u|^{p+2}dx$$

and we noted that the energy is conserved and scale invariant for this choice of p.

Local well-posedness gives us $u_0 \in H^1$, and gives $u \in S'$. Where S is the strichartz space, and $S = \cap L^p L^q$, and $S' = \{u, \nabla_x u \in S\}$.

We also introduced the following theorem.

Theorem 1.11

If $\|u_0\|_{\dot{H}^1} \ll 1$, then there exists a global solution $u \in S'$.

Proof We gave this proof before, via the contraction principle, and with the same term in $N' = (S')^1$.

We would like $u \in S'$, and $|u|^p \cdot u \in N'$, and note

$$\||u|^p u\|_{N'} \leq \|u\|_S^{p+1}$$

And we would like to minimize this strichartz norm.

$$\||u|^p u\|_{N'} = \|\nabla(|u|^p \cdot u)\|_N = \|\nabla u u^p\|_N$$

where $\nabla u \in L^\infty L^2$, hence to put it in a good space, we would also like u^p to be in the mixed L^p norm. We have

$$\nabla u \in L^2 L^r, u \in L^q L^{r^*} \Rightarrow |u|^p \in L^{q/p} L^{r^*/p}$$

We make the choice that $q = r^*$, and

$$u_0 \in H^1, u \in L_t^q L_x^2$$

Hence by the scaling relation:

$$\frac{n}{2} - 1 = \frac{2}{q} + \frac{n}{q}$$

This gives that

$$\frac{r^*}{p} = \frac{2(n+2)}{n} \geq 2$$

where $u \in L^{\frac{2(n+2)}{2}}$, and is ≥ 2, when $n \geq 3$. Hence the solution does not blow up.

Theorem 1.12

For $u_0 \in \dot{H}^1$, there exists a unique local solution $u \in S^1[0,T]$, furthermore, this solution may be continued, for as long as $\|u\|_{L_{t,x}^{\frac{2(n+2)}{n-2}}}$ remains finite.

Theorem 1.13

If u is a global solution with $\|u\|_{L^{\frac{2(n+2)}{2}}}$, then the solution u is scattering, and there exists $u^+ \in \dot{H}^1$, such that

$$\lim_{t\to\infty} \|u(t) - e^{it\Delta^2}u^+\|_{\dot{H}^1} = 0$$

Now we investigate the main problem, namely what happens for large inital data? We have the following defocusing conjecture.

Proposition 1.14 (Conjecture)

There is well-posedness for large data (i.e. $u \in L^{\frac{2(n+1)}{2}}$).

For the focusing case, the energy is

$$E = \int \frac{1}{2}|\nabla u|^2 - \frac{1}{p+2}|u|^{p+2}dx$$

$u_t = i\frac{\delta E}{\delta t}$, which gives us steady state, i.e. critical points for E.

By scaling, via GNS, we have

$$\|u\|_{L^{p+2}} \leq c\|\nabla u\|_{L^2}$$

Note that the second term in the energy is dominated by the first term. For $\|u\|_{\dot{H}^1} \ll 1$, the energy is essentially quadratic and coercive. For large $\|u\|$, we might have our E being negative. Essentially by the mountain pass lemma, which states to cross a mountain, "you must obtain a point where it is the lowest."

Now we look at the best constant in GNS, i.e. the minimizer. And from the small data ball to $E(u) < 0$, we cross the highest point, and the minimzer which gives the best constant is a scalar multiple of the lowest moutain pass. This gives us

$$-\Delta u = \lambda|u|^p u$$

and if we scale u to cu, we could get $\lambda = 1$, which gives our critical point. Via the above equation, we found the ground state to be $Q = (1+r^2)^{\frac{-(n-2)}{2}}$. Now for the threshold conjecture.

Proposition 1.15 (Conjecture)

Global well-posedness holds for $E(u) < E(Q)$ in the good region, where the good region includes the small initial data region, and the bad region is outside, where $E(u) < 0$.
furthermore, we ave finite time blowup holds for $E(u) < E(Q)$, in the good region as above.

The L^2 case, the defocusing conjecture by Dodson in the book, and the threshold conjecture is also done by Dodson.

The $\dot{H}^1$ case , the defocusing conjecture, we will follow Visann notes, and the threshold conjecture is only known for $n \geq 5$, where $n = 3, 4$ still remain open.

Now we look at the in-between cases, where $\dot{H}^s$, and $s \in (0, 1)$, then $p = \frac{4}{n-2s}$. The defocusing conjecture remains largely open, but there is a note such that no type II blowup can occur(remain bounded on a finite time interval), but remains open for the type I blowup(where the solution grows). The threshold conjecture:

$$-\Delta Q + Q = |Q|^p Q$$

where Q is the critical point of E on the level set of M.

$$-\Delta Q - Q^{p+1} = Q$$

where the LHS is δE, and RHS from δM. If we are limmited to $\dot{H}^s$, we do not know where mass or energy is finite.

We ask the question for H^1, nonhomogenous initial data, $E < \infty, M < \infty$.

$$E^s M^{1-s}$$

and this is the scale invariant quantity. And the threshold conjecture asks

$$E^s(u)M^{1-s}(u) \leq E^s(Q)M^{1-s}(Q)$$

For $\|u_0\| \ll 1$, we have GWP. And

$$A = \{u_0 \in L^2 : \text{ we have GWP}\}$$

note that

$$B(0,\epsilon) \subset A$$

Proposition 1.16

A is open.

Proof Exactly because $B(0,\epsilon) \subset A$. If you have an initial data such that you have GWP, then all data that lives in a small neigbhorhood of it, you still should get GWP.

Now we try to move to the boundary of A, and $\|u\|_{L^p} \to \infty, p = \frac{2(n+2)}{n}$. Hence we take a point inside of A, very close to the boundary, and how can we make the u's norm large?

We first note that we must have wave packets, and if we suppose that the wave packets are spread out in more than one direction: this is because we can separate the initial energy into these two packets, and their energy is small, and separate, they are nice solutions, with controlled strichartz norms. And our operator is sublienar, hence the overall strichartz norm is nicely bounded. When they intersect, the norm is already small due to decay, hence the norm is again, well-controlled. This cannot happen. (Hence there can exist only one wave packet in one direction at any given time, and that packet could propogate over time, but when you dissect it, you should still obtain a wave packet.)

And to prove the above, there are two approaches: one by Bourgain (95), which is the induction on energy; and the other is by contradiction, where we deal with the "minimal enemy."