



Functional Analysis

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Date: September 18, 2023

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Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable number of seminorms).

Remark Let \mathcal{H} be a Hilbert space, then the dual space \mathcal{H}^* is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themselves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \rightarrow \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to \mathbb{R} or \mathbb{C} . Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i.e. self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \quad (1.1)$$

This is what we are aiming for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If T_1, \dots, T_n is a collection of self-adjoint operators on \mathcal{H} , and such all commute with each other, then we also have

$$C^*(T_1, \dots, T_n, I_{\mathcal{H}}) \cong C(X) \quad (1.2)$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$\text{Aut}(B) = \{T : T \text{ is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphism, and $\alpha : G \rightarrow \text{Aut}(B)$, is called a representation on B or an action of the group G on B .

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B , generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B , hence no need to include it.

Elements of the form $\sum_{z \in \Sigma} \alpha_x z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x \in G} f(x) \alpha_x = \alpha_f$$

note for except finitely many x , $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f \alpha_g = \left(\sum f(x) \alpha_x \right) \left(\sum g(y) \alpha_y \right) = \sum_{x,y} f(x) g(y) \alpha_x \alpha_y = \sum_{x,y} f(x) g(y) \alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f \alpha_g = \sum_x \sum_y f(x) g(x^{-1}y) \alpha_y = \sum (f * g)(y) \alpha_y$$

where we define $f * g(y) = \sum f(x) g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f * g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \left\| \sum f(x) \alpha_x \right\| \leq \sum |f(x)| \|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x \xi(y) = \xi(x^{-1}y)$, α_x being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α .

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X . This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X , by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let \mathcal{A} be an algebra on \mathbb{R} or \mathbb{C} , is a normed algebra if it has a norm $\|\cdot\|$, as a vector space, such that for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity.

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm.

Given $x \in X$, define $\varphi_x : C(X) \rightarrow \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \rightarrow \mathbb{R}$ or $C(X) \rightarrow \mathbb{C}$. This simply implies

$$\varphi_x(f + g) = (f + g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$.

Proof Main Claim: given a multiplicative linear functional φ , there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f_x(x) \neq 0$, but $\varphi(f_x) = 0$.

Set $g_x = \overline{f_x} f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f_x}) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is x_1, \dots, x_n such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + \dots + g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f , consider $f - \varphi(f) \cdot 1$, apply φ , we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra \mathcal{A} and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of \mathcal{A} into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra \mathcal{A} and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of \mathcal{A} into the field.

Remark We have $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

This is not always true for normed algebra, Let

$$\mathcal{A} := \text{Poly} \subset C([0, 1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If \mathcal{A} is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word “unital” is key here.

Proposition 1.4

Let \mathcal{A} be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} = \sum_{n=0}^{\infty} a^n$, $a^0 = 1_{\mathcal{A}}$. You could look at the partial sums. $S_m = \sum_{n=0}^m a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m \rightarrow \infty} S_m = \frac{1}{1-a}$.

To prove this is a Cauchy sequence:

$$\|S_n - S_m\| = \left\| \sum_{j=m+1}^n a^j \right\| \leq \sum_{j=m+1}^n \|a^j\| \leq \sum_{j=m+1}^n \|a\|^j$$

And the fact that $\|a\| \leq 1$, we have the sum bounded by ϵ , hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n \rightarrow \infty} S_n(1-a) = \lim_{n \rightarrow \infty} \left(\sum_{n=0}^{\infty} a^n \right) (1-a) = \lim_{n \rightarrow \infty} (1 - a^{n+1}) = 0$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \rightarrow 0$.

1.2 Lecture 3

We now begin.

Let \mathcal{A} be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Neumann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1-a\| < 1$, then a is invertible.

Proof $a = 1 - (1-a)$.

Proposition 1.5

The set of invertible elements of \mathcal{A} is an open subset of \mathcal{A} .

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of \mathcal{A} onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d , homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in \text{Range}(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.



Proof Let $f \in C(X)$, and if $\alpha \in \text{range of } f$, so $\alpha = f(x_0)$ for some x_0 . then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $(f - \alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha \cdot 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in \text{range of } f$.

□

Definition 1.5 (spectrum of an element)

For any unital algebra \mathcal{A} over some field \mathbb{F} , for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible} \}$$

is called the spectrum of a , denoted as $\sigma(a)$.



Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let \mathcal{A} be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$



Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda \left(1 - \frac{a}{\lambda}\right)$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\lambda \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on \mathcal{A} , i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$



Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of \mathbb{R}, \mathbb{C} .



Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of \mathcal{A} is open, hence

$$\sigma(a) = \phi^{-1}(\text{noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{invertible})$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{invertible}))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of \mathcal{A}' , which denotes the dual vector space of continuous linear transformations.

On \mathcal{A}' , we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak-* topology.



Proof let $\{\varphi_\lambda\}$ be a net of elements of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \rightarrow \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V , the closed unit ball of V' is compact in the weak-* topology.



As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak-* topology.



Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact. □

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto \mathcal{A} .

For \mathcal{A} unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from \mathcal{A} into $C(\mathcal{A})$.



Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{ab})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$



1.3 Lecture 4

Today we talk about the structure of $\widehat{l^1(S)}, \widehat{l^1(G)}$, where S, G are semigroups and groups, and how they naturally identify with the unit disk \mathbb{D} , and the unit circle \mathbb{T} .

Let S be a commutative discrete semigroups, for example $\mathbb{N} \cup \{0\}$, and $f \in C_c(S)$, then we can write $f = \sum_{x \in S} f(x)\delta_x$, where we define $\delta_x\delta_y = \delta_{xy}$. Note that $C_c(S)$ is dense in $l^1(S)$.

Definition 1.6 (Convolution)

Take any $f, g \in C_c(S)$, we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{y \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left(\sum_{xy=z} f(x)g(y) \right) \delta_z$$

where we define the convolution between two functions

$$f * g(z) = \sum_{x,y, xy=z} f(x)g(y)$$

And under this convolution operation, we have $l^1(S), *$ as a Banach algebra.

Example 1.1 If we consider polynomials of the form $f(x) = \sum_{n=0}^{\infty} f(n)x^n$, and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right) \left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f * g)(p)$$

And let $f \in C_c(S)$, where $S = \mathbb{N}$. we define $\|f\|_{l^1} = \sum_{x \in S} |f(x)|$.

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ denote the set of unital homomorphisms from \mathcal{A} to \mathbb{R}, \mathbb{C} . Note that $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$.

Note that we know $(l^1(S))' = l^\infty(S)$, hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. Note that we have $\|\varphi\| = 1$, hence if we $\varphi \in l^\infty(S)$, we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for $z \in S, \|z\| \leq 1$, we have $|\varphi(z)| \leq 1$.

Proposition 1.12

We naturally identify $\widehat{l^1(S)}$ with $\text{Hom}(S, \mathbb{D})$, i.e. $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$.

Proof Given $f \in \widehat{l^1(S)}$, we know it's multiplicative, unital, hence all these transfer when viewing $\varphi \in l^\infty(S)$. This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here xy denotes the operation on S between x, y , for example, could be $x + y$. Hence naturally, if $\varphi \in \widehat{l^1(S)}$, φ can also be viewed as $\varphi : S \rightarrow \mathbb{D}$, and thus is in l^∞ , with $|\varphi(s)| \leq 1$. □

Furthermore, we can identify elements in $\widehat{l^1(S)}$ with the unit disk. Take $S = \mathbb{N}$.

Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where \mathbb{D} denotes the unit disk in \mathbb{C} .

Proof We motivate this by noticing \mathbb{N} is generated by 1, and thus viewing $\varphi \in \widehat{l^1(\mathbb{N})}$ as $\varphi \in l^\infty(\mathbb{N})$, we have φ is determined by $\varphi(1)$. And denote $\varphi(1) = z_0$, then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for $z \in \mathbb{D}$,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space. □

Proposition 1.14

The standard topology on \mathbb{D} coincides with the weak-* topology on $\widehat{l^1(\mathbb{N})}$.

$$D_{std} \cong D_{weak-*}$$

Proof We just need to associate an element in \mathbb{D} with a function $\varphi \in \widehat{l^1(\mathbb{N})}$. And we do this by

$$z \mapsto \sum_{n \in \mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

1.3.1 On groups

We let G denote a discrete commutative group, and we see everything above follows, with one extra property.

Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where \mathbb{T} denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Proof For $\varphi \in \widehat{l^1(G)}$, we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because $|\varphi(x)| \leq 1, \forall x$, Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have $\widehat{l^1(G)}$ naturally identifies with \mathbb{T} . Like what we described above, we have what is desired. □

Remark Take $G = \mathbb{Z}$, if we denote $z \in \mathbb{T}$ as $z = e^{2\pi it}$, then we naturally identify with

$$\sum_{n \in \mathbb{Z}} f(n)e^{2\pi int}$$

we denote this mapping as \widehat{f} , i.e.

$$\widehat{f}(z) = \sum_{m \in \mathbb{Z}} f(m)e^{2\pi imt}$$

This is the Fourier transform.

1.4 Lecture 5

Last time, we talked about if we denote $\mathcal{A} = l^1(G)$, equipped with $\|\cdot\|_{l^1}$, under convolution, we have

$$\widehat{\mathcal{A}} \cong \text{Hom}(G, \mathbb{T})$$

If we take $G = (\mathbb{Q}, +)$, one can ask the question if $\widehat{\mathcal{A}}$ is big enough. And we will see later in the course, the answer is yes.

For pointwise multiplication, \widehat{G} forms a group, and in fact \widehat{G} is a compact topological group.

For any compact commutative group G , for example \mathbb{R}^n under $+$. Define

$$\widehat{G} = \text{continuous homomorphisms into } \mathbb{T}$$

Remark We now require continuous with this general G (previously was not required for discrete group G).

Proposition 1.16

Let G be a locally compact and commutative group, we have \widehat{G} as a locally compact, commutative group.

We define the pairing between G and \widehat{G} as follows: $x \in G, \varphi \in \widehat{G}$,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the following map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let G, H denote locally compact groups, and $\phi : G \rightarrow H$ be a continuous homomorphism. Note we have the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ \widehat{G} & \xleftarrow{\phi} & \widehat{H} \end{array}$$

If we take an element $\psi \in \widehat{H}$, we consider $\psi \circ \phi$. We get $\psi \circ \phi \in \widehat{G}$.

Definition 1.7 (category, functor)

A category is specified by

1. a set of objects
 2. morphisms between objects
- (a). X, Y, Z are objects, and if

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

- (b). For each object X , there is an identity morphism 1_X .

And a functor is defined to be such a morphism between categories.



Example 1.2 For category of finite vector spaces V , passing from vector space to its dual V' is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ V' & \xleftarrow{T^t} & W' \\ V'' & \xrightarrow{T^{tt}} & W'' \end{array}$$

The map going in the same directions $V \rightarrow W$, and $V'' \rightarrow W''$ is called covariant, whereas $V' \leftarrow W'$ is called contravariant.

Example 1.3 For category of locally compact groups G, H , assigning the dual group is a functor:

$$\begin{array}{ccc} G & \rightarrow & H \\ \widehat{G} & \leftarrow & \widehat{H} \\ \widehat{\widehat{G}} & \rightarrow & \widehat{\widehat{H}} \end{array}$$

Example 1.4 Now let X be a compact space. Given Φ continuous map between $X \rightarrow Y$.

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & Y \\ C(X) & \leftarrow & C(\Phi)C(Y) \end{array}$$

For $f \in C(Y)$, we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Z \\ C(X) & \xleftarrow{C(\varphi)} & C(Y) \xleftarrow{C(\phi)} C(Z) \end{array}$$

where for $f \in C(Y)$, $C(\varphi)(f) = f \circ \varphi$, and $g \in C(Z)$, $C(\phi) = g \circ \phi$. This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between $C(X)$ and $C(Y)$, there exists a map from X to Y .

Proposition 1.17

Suppose X, Y are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism $\check{F} : X \rightarrow Y$.



Proof Define $\varphi_x : C(X) \rightarrow \mathbb{C}$ as the evaluation map: take $f \in C(X)$,

$$\varphi_x(f) = f(x)$$

Then $\varphi_x \circ F \in \widehat{C(Y)}$. And we know that any element in $\widehat{C(Y)}$ is a point evaluation, i.e. there exists $y \in Y$ such that

$$\varphi_y = \varphi_x \circ F$$

We thus define $\check{F}(x) = y$ as such that it satisfies the above equation. We need to show \check{F} is continuous. Note that X, Y are compact Hausdorff spaces, and the topology on Y is the coarsest topology making all functions $g \in C(Y)$ continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by F, g being continuous, we have \check{F} is also continuous.

□

There is a natural bijection between the continuous functions from X to Y , and the unital algebra homomorphism from $C(X)$ to $C(Y)$.

A quick reminder:

Remark For X compact, the weak-* topology coincides with the standard topology.

1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let \mathcal{A} be a unital Banach algebra.

We write $GL_n(\mathcal{A})$ to denote the general linear group, the group formed by $n \times n$ matrices with entries from \mathcal{A} .

The less standard notation is $GL_I(\mathcal{A})$ is the group of invertible elements in \mathcal{A} . As we have shown previously, this is a closed subset of \mathcal{A} . This is the notation that we will use.

Remark It is easy to see that the product is jointly continuous.

Proposition 1.18

The following map is continuous.

$$a \mapsto a^{-1}$$



Proof Given $\|a - b\| < \delta$, we would like to show $\|a^{-1} - b^{-1}\| < \epsilon$. We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\| \|b - a\| \|b^{-1}\|$$

Take $\delta = \epsilon / \|a^{-1}\| \|b^{-1}\|$ would suffice.

□

Proposition 1.19

Fix $a \in GL(\mathcal{A})$, there exists a neighborhood O of a and a constant K such that for all $y \in O$, we have

$$\|c^{-1}\| < K$$



Proof Let $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$, then d is invertible and

$$d^{-1} = \sum_{n=0}^{\infty} (1 - d)^n$$

We thus have

$$\|d^{-1}\| \leq \frac{1}{1 - \|1 - d\|} \leq \frac{1}{1 - 1/2} = 2$$

We then identify what our O should be. Let $O = aV$, then we want to show that every ad has an inverse with bounded norm. Because a, d are both invertible, ad is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \leq \|d^{-1}\| \|a^{-1}\| \leq 2\|a^{-1}\|$$

□

Remark For each invertible element, we can find a neighborhood of invertible elements around it, and using that $(1 - d)$ is bounded, then d is invertible, we can bound $\|d^{-1}\|$.

Definition 1.8

Fix $a \in \mathcal{A}$, the resolvent set of \mathcal{A} is the complement of spectrum of \mathcal{A} , i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$$



Hence the resolvent set is an open, unbounded subset of \mathbb{C} or \mathbb{R} .

Definition 1.9 (Resolvent function)

On the resolvent set, $\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible}\}$ is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that a is fixed, and λ is the variable here.



Now we note that this $R_a(\lambda)$ function is nicely behaved.

Proposition 1.20

The resolvent function $R_a(z)$ is analytic on the resolvent set, and vanishes as $z \rightarrow \infty$.



Proof We first define the notation of analyticity on an open subset of \mathbb{R}, \mathbb{C} : this means for every point in the open set O , we can find a power series expansion of the function such that its radius of convergence > 0 .

Fix z_0 in the resolvent set. We know $z_0 1_{\mathcal{A}} - a$ is invertible. We consider $(z 1_{\mathcal{A}} - a)$, for z in the resolvent set. We will omit the $1_{\mathcal{A}}$ for simplicity.

$$z 1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a) \left(1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a} \right)$$

We know the latter term is invertible if $\|\frac{z_0 - z}{z_0 - a}\| < 1$ has norm, hence we have

$$(z - a)^{-1} = \sum_{n=0}^{\infty} \left(\frac{z_0 - z}{z_0 - a} \right)^n (z_0 - a)^{-1}$$

What happens when we let $z \rightarrow \infty$, we consider $R_a(1/z)$, and let $z \rightarrow 0$. Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1 - az}{z}\right)^{-1} = z(1 - az)^{-1}$$

Let $z \rightarrow 0$ makes $R_a(1/z)$ go to zero.

□

Now given that $R_a(z)$ is analytic and bounded at ∞ , we can state the following important theorem.

Theorem 1.2 (Nonemptiness of spectrum)

Let \mathcal{A} be a unital Banach algebra over \mathbb{C} , then for any $a \in \mathcal{A}$, we have $\sigma(a) \neq \emptyset$.



Proof Assume there exists $a \in \mathcal{A}$, such that $\sigma(a) = \emptyset$. If $\mathcal{A} = \mathbb{C}$, then we would have $R_a(\lambda)$ be a bounded entire, complex-valued function defined on all of \mathbb{C} . By Liouville's theorem, we must have $R_a(z)$ a constant function, but we know $z \rightarrow \infty$, $R_a \rightarrow 0$, hence $R_a(z)$ is constantly 0, but this cannot be true.

If our \mathcal{A} is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where ϕ is some nonzero element in \mathcal{A}' , guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at ∞ . Again, the function is constantly 0.

With the nonemptiness of spectrum theorem, we now state the Gelfand-Mazur theorem.

Theorem 1.3 (Gelfand-Mazur)

Let \mathcal{A} be a unital Banach algebra over \mathbb{C} , if any nonzero element of \mathcal{A} is invertible, then \mathcal{A} is isomorphic to \mathbb{C} .



Proof For any $a \in \mathcal{A}$, we know $\sigma(a) \neq \emptyset$, hence there exists λ such that $\lambda 1_{\mathcal{A}} - a$ is invertible, i.e. $a = \lambda 1_{\mathcal{A}}$, hence establishing an isomorphism between \mathcal{A} and \mathbb{C} . In other words, $\mathcal{A} = \mathbb{C} 1_{\mathcal{A}}$.

□

1.5.1 Functional Calculus**Proposition 1.21**

Let $a \in \mathcal{A}$, then if $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges for $|z| < r$, where $r > \|a\|$, then $\sum_{n=0}^{\infty} \alpha_n a^n$ converges as well.



We first start with proving the following statement.

Lemma 1.1

Let f be a polynomial, \mathcal{A} is a unital Banach algebra over \mathbb{C} , $f = \sum_{n=0}^k a_n x^n$, then for $a \in \mathcal{A}$, we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of a under f is exactly the spectrum of f evaluated at a .



Proof (\Leftarrow). We take $\lambda \in \sigma(a)$, and we would like to show $f(\lambda)$ is in the spectrum of $f(a)$. We note that if $\lambda \in \sigma(a)$, then $a = \lambda 1_{\mathcal{A}}$, and $f(\lambda 1_{\mathcal{A}}) = f(a)$, hence by definition, $f(a) - f(\lambda) 1_{\mathcal{A}}$ is not invertible implying $f(\lambda)$ is in the spectrum of $f(a)$. Note that this also implies $f(a) - f(\lambda) = (a - \lambda)Q(z)$ for some polynomial $Q(z)$.

(\Rightarrow). We take $\lambda \in \sigma(f(a))$, i.e. $f(a) = \lambda 1_{\mathcal{A}}$. We would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. If f is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1) \dots (z - c_n)$$

Plugging in a we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}}) \dots (a - c_n 1_{\mathcal{A}})$$

If $f(a) - \lambda$ is not invertible, then there exists j such that $(a - c_j 1_{\mathcal{A}})$ is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. In fact, we have $\lambda = f(c_j)$ by knowing $f(c_j) - \lambda = 0$.

□

Now let $f(z) = z^n$, and if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$

This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

Definition 1.10

Fix $a \in \mathcal{A}$, we define the spectral radius of a , denoted by $r(a)$,

$$r(a) = \sup_{\lambda} \{|\lambda| : \lambda \in \sigma(a)\}$$



Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$



Proof From the previous remark that $|\lambda| \leq \|a^n\|^{1/n}$, hence this follows.

1.6 Lecture 7

I have not typed up for this?

1.7 Lecture 8

Let \mathcal{A} be a unital Banach algebra. Then for $a \in \mathcal{A}$, and we look at the resolvent of a , $R_a(\lambda)$, we've noted that as $\lambda \rightarrow \infty$, we have

$$\lim_{\lambda \rightarrow \infty} R_a(\lambda) = \lim_{\lambda \rightarrow \infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda \rightarrow \infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for $|\lambda| \geq \|a\|$.

Recall that we define the spectral radius, $r(a)$, as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

Proposition 1.22

$$r(a) = \lim \|a^n\|^{1/n}$$



Proof

If we let $\lambda = 1/z$, then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for $|z| \leq \|a\|^{-1}$, but maybe?? also for $|z| < r(a)^{-1}$?

For $r > r(a)$, i.e. $|z| \leq r^{-1}$, we know $\sum_n a^n r^n$ converges for $r > r(a)$.

know $z \sum a^n z^n$ converges absolutely. In particular,

$$a^n z^n \rightarrow 0$$

Hence there exists M such that for $n \geq M$, we have

$$\|a^n r^{-n}\| \leq 1$$

This implies that

$$\|a^n\| \leq r^n \Rightarrow \|a^n\|^{1/n} \leq r$$

for all $n \geq M$.

This implies that

$$\limsup \|a^n\|^{1/n} \leq r$$

And note that r is arbitrary close to the spectral radius $r(a)$. Hence we have

$$\limsup \|a^n\|^{1/n} \leq r(a) \leq \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

□

For each $\varphi \in \mathcal{A}'$, consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n) \lambda^{-n}$$

This series converges for $r > r(a)$. We can apply the same process, to argue that there exists M_φ such that

$$\|\varphi(a^n) r^{-n}\| \leq M_\varphi$$

for all $n \geq 0$. Note that M_φ could be different for all φ .

Note that

$$\mathcal{A} \rightarrow \mathcal{A}' \rightarrow \mathcal{A}''$$

there is a natural injection of $a \mapsto \hat{a} \in \mathcal{A}''$.

For each n , define $F_n \in \mathcal{A}''$, by $F_n(\varphi) = |\varphi(a^n r^{-n})| \leq M_\varphi$. Applying the UBP, we have

$$|F_n(\varphi)| \leq M \Rightarrow |\varphi(a^n) r^{-n}| \leq M$$

This implies that

$$|\varphi(a^n)| \leq r^n M$$

Note that by Hahn-Banach, for any $b \in \mathcal{A}$, we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking n -th root of both sides, we get

$$\|a^n\| \leq r^n M \Rightarrow \|a^n\|^{1/n} \leq r M^{1/n} \rightarrow r$$

Hence we again obtain the same result.

□

Recall UBP.

Theorem 1.4 (Uniform Boundedness Principle)

Let X be Banach, and Y be normed, let $T_n : X \rightarrow Y$ be a family of linear operators, and if for all $x \in X$, we have

$$\|T_n(x)\| < \infty$$

Then for all n , we have

$$\|T_n\| < \infty$$

♥

Note that if \mathcal{A} is unital, and if $\mathcal{A} \subset \mathcal{B}$ with some unit. For $a \in \mathcal{A}$, if a is not invertible in \mathcal{A} , then it might be invertible in \mathcal{B} . Hence if we use $\sigma_{\mathcal{A}}(a)$ to denote the spectrum of a in \mathcal{A} .

Proposition 1.23

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

♠

Example 1.5 Let $\mathcal{B} = l^1(\mathbb{Z})$, and let $\mathcal{A} = l^1(\mathbb{N})$, equipped with convolution.

Clearly $\mathcal{A} \subset \mathcal{B}$. And note that the delta function at 1, δ_1 is not invertible in \mathcal{A} but it has an inverse δ_{-1} in \mathcal{B} . Hence we see $0 \in \sigma_{\mathcal{A}}(a)$, but $0 \notin \sigma_{\mathcal{B}}(a)$.

Proposition 1.24 (Spectral radius is preserved)

For $\mathcal{A} \subset \mathcal{B}$, we have

$$r_{\mathcal{A}}(a) = \lim \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

**Proposition 1.25**

Let X be compact, and let $\mathcal{A} = C(X)$. Then for $f \in C(X)$, we have

$$\|f^2\|_{\infty} = \|f\|_{\infty}^2$$



Proof Look at where f takes $\|f\|_{\infty}$, and square it, since when X is compact, you can actually obtain the point where $|f(x)| = \|f\|_{\infty}$.

Remark The same property holds for f in any unital subalgebra of $C(X)$, for example, if $X \subset \mathbb{C}$, and let \mathcal{A} =functions that are holomorphic on an open subset of \mathbb{C} that are in X .

Let \mathcal{A} be a unital Banach algebra with the property such that for any $a \in \mathcal{A}$, we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any n , we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking $1/2^n$ -root of both sides, we get that the spectral radius of $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

Let \mathcal{H} be a Hilbert space, over \mathbb{C} , and let $\mathcal{A} = B(\mathcal{H})$, i.e. the bounded linear operators on \mathcal{H} , and equip with the operation of taking adjoint. $T \mapsto T^*$.

Proposition 1.26

For any $T \in B(\mathcal{H})$, we have

$$\|T^*T\| = \|T\|^2$$



Proof We know that $\|T^*\| = \|T\|$. And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let $\xi \in \mathcal{H}$, then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi \rangle = \langle \xi, T^*T\xi \rangle \leq \|T^*T\|\|\xi\|^2$$

where the last inequality follows from Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2}\|\xi\|$$

which by definition, gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result. □

Corollary 1.4

If $T^* = T$, then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements.

Note that for general T , we have T^*T is always self-adjoint,

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the $*$ -algebra structure.

1.8 Lecture 9

Let \mathcal{H} be a hilbert space over \mathbb{C} , and $\mathcal{B}(\mathcal{H})$ with $\|\cdot\|_\infty$, and closed under taking involutions. If $T \in \mathcal{B}(\mathcal{H})$, then

$$\|T^*T\| = \|T\|^2$$

So if $T^* = T$, then we have

$$\sigma(T) = \|T\|$$

Definition 1.11

A concrete C^* -algebra is a norm-closed sub-algebra \mathcal{A} of $\mathcal{B}(\mathcal{H})$, for some \mathcal{H} (could be finite or infinitely dimensional) such that is self-adjoint, i.e., if $T \in \mathcal{A}$, then $T^* \in \mathcal{A}$.
Again, we call \mathcal{A} is unital if $1_{\mathcal{H}} \in \mathcal{A}$.

Corollary 1.5

If \mathcal{A} is a C^* algebra, then for all $a \in \mathcal{A}$,

$$\|a^*a\| = \|a\|^2$$

If \mathcal{A} is unital C^* -algebra, and if $a^* = a$, then $r(a) = \|a\|$.

Next we say a bit about the Gelfand transform.

Definition 1.12

Let \mathcal{A} be a unital Banach algebra, and commutative, then we have the Gelfand transform $\Gamma : \mathcal{A} \rightarrow C(\widehat{\mathcal{A}})$:

$$\Gamma(a)(\varphi) = \varphi(a)$$

then he said something of homomor to the complex numbers or something

Note that if $a \in \mathcal{A}$, and $\varphi \in \widehat{\mathcal{A}}$, then $\varphi(a) \in \sigma(a)$, then we have

$$|\varphi(a)| \leq \|a\| = r(a)$$

then $\|\widehat{a}\|_\infty \leq r(a)$. Now we would like to show

$$\|\widehat{a}\|_\infty = r(a)$$

We will now dedicate the next 50 minutes of our life to proving this theorem.

Theorem 1.5

If $\lambda \in \sigma(a)$, then there exists $\varphi \in \widehat{\mathcal{A}}$ such that $\varphi(a) = \lambda$. This is equivalent to saying: if $(a - \lambda 1)$ is not invertible, then there is a $\varphi \in \widehat{\mathcal{A}}$, such that

$$\varphi(a - \lambda 1) = 0$$

This means if a is not invertible, then there exists $\varphi \in \widehat{\mathcal{A}}$ such that $\varphi(a) = 0$.

Proof Suppose $a \in \mathcal{A}$, and consider

$$a\mathcal{A} = \{ab : b \in \mathcal{A}\}$$

The set $a\mathcal{A}$ does not contain the identity element i.e. $1_{\mathcal{A}} \notin a\mathcal{A}$ (otherwise it would imply it has an inverse). And $a\mathcal{A}$ is a two-sided proper ideal (by \mathcal{A} being commutative).

We now introduce a fact that we will use.

Definition 1.13

An ideal I is maximal if I is proper in R , and not contained in any bigger proper ideals.



Lemma 1.2

Let R be a unital ring commutative, every proper ideal is contained in a maximal ideal (by Zorn's lemma).



Lemma 1.3

For \mathcal{A} unital commutative Banach algebra, if I is a proper ideal, then its closure is a proper ideal.



Proof We have seen that $GL(\mathcal{A})$, the set of invertible elements, is open. Hence its complement is closed. Any proper ideal does not contain any elements in $GL(\mathcal{A})$, hence its closure is closed inside a closed set.

Remark Let X be locally compact, but compact, such as \mathbb{R} , then we have $C_c(X) \subset C_\infty(X)$. Note that $C_c(X)$ is a proper ideal of $C_\infty(X)$, but it's dense in $C_\infty(X)$, hence its closure is the entire space, hence no longer proper. This tells us the closure of a proper ideal is not always proper, if \mathcal{A} is not unital.

Theorem 1.6

Every maximal ideal of \mathcal{A} is closed.



Proof The closure of any proper ideal is closed in unital algebras, hence its closure is itself.

First let V be a normed vector space. Let W be a closed subspace, and form the quotient space V/W . There is a natural way to equip V/W with a norm

$$\|v\| = \inf\{\|v - w\| : w \in W\}$$

i.e. the distance between v to W . This is a norm. Further if V is complete, so is V/W .

Let \mathcal{A} be a normed algebra, commutative, unital. Let I be a closed ideal.

Proposition 1.27

For $a, b \in \mathcal{A}$, we have

$$\|\dot{a}\dot{b}\| = \|\dot{a}b\| \leq \|\dot{a}\|\|\dot{b}\|$$

so that \mathcal{A}/I is a normed algebra.



Proof Let $c, d \in I$, and

$$(a - c)(b - d) = ab - (ad + cb - cd)$$

Note that $(ad + cb - cd) \in I$, hence

$$\|\dot{a}\dot{b}\| \leq \|ab - (ad + cb - cd)\| = \|(a - c)(b - d)\| \leq \|(a - c)\|\|(b - d)\|$$

Taking infimum over all c, d , we get

$$\|(\dot{a}\dot{b})\| \leq \|\dot{a}\|\|\dot{b}\|$$



Proposition 1.28

If \mathcal{A} is a Banach algebra and if I is a closed ideal, then \mathcal{A}/I is a Banach algebra for the norm $\|\dot{a}\|$ defined above.



Let \mathcal{A} be a unital commutative Banach algebra over \mathbb{C} , let I be a maximal ideal of \mathcal{A} , then \mathcal{A}/I is a Banach algebra,

if \mathcal{A}/I has a proper ideal, then you can put this ideal back in \mathcal{A} such that it contains I . By I already being ideal, this implies that \mathcal{A}/I does not contain any proper ideals.

Now coming back. Let nonzero element in \mathcal{A}/I is invertible. The Gelfand Mazur theorem tells us

$$\mathcal{A}/I \cong \mathbb{C}$$

Moreover, $\mathcal{A}/I = 1_{\mathcal{A}/I}\mathbb{C}$. Then the quotient map

$$\mathcal{A} \rightarrow \mathcal{A}/I \cong \mathbb{C}$$

is an element of \mathcal{A} , i.e. an algebra homomorphism φ , with the property $\varphi(I) = 0$.

If $a\mathcal{A} \subset I$, then for $y \in a\mathcal{A}$, we have

$$\varphi(y) = 0$$

And we are therefore finally done. We thus have

$$\|\hat{a}\| = r(a)$$

□

1.9 Lecture 10

It's Friday.

Consider $C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$, and $C_\infty(\mathbb{R})$ is an ideal of $C_b(\mathbb{R})$.

And the abstract C^* -algebra is a Banach algebra with involution $*$, such that

$$\|a^*a\| = \|a\|^2$$

Zorn's lemma states that $C_\infty(\mathbb{R})$ is contained in a maximal ideal, in a commutative Banach algebras, maximal ideals give rise to bounded multiplicative linear functionals.

Remark There is ideals of $C_b(\mathbb{R})$ that are bigger than $C_\infty(\mathbb{R})$.

There exists linear functionals that are 0 on $C_\infty(\mathbb{R})$, but nonzero on $C_b(\mathbb{R})$, but such functional is not “constructable.”

We also have $c_0 \subset l^\infty(\mathbb{N})$, where c_0 are sequences that converge to 0 at infinity. Again, nonzero linear functionals exist on $l^\infty(\mathbb{N})$, and is identically zero on c_0 , but such is also not constructable.

Definition 1.14

$\prod_{j=1}^\infty \mathbb{Z}_5$ = the sequences of elements of \mathbb{Z}_5 . $\bigoplus \mathbb{Z}_5$ all sequences that are 0 except for finitely many number of entries.



Note that $\bigoplus \mathbb{Z}_5$ is an ideal of $\prod_{j=1}^\infty \mathbb{Z}_5$.

Proposition 1.29

$l^\infty(\mathbb{N})$ is not separable, $\prod_{j=1}^\infty \mathbb{Z}_5$ is not separable, nor is it finitely generated.



Proposition 1.30

If \mathcal{A} is a unital commutative Banach algebra over \mathbb{C} , which is separable, and if I is a closed ideal, then one can construct a maximal ideal containing I by countable *what*



Proof Let $\{a_n\}$ be a countable subset of \mathcal{A} , whose linear span is dense.

Lemma 1.4

Note that \mathcal{A}/I contains noninvertible elements if and only if I is not maximal.



If I is not maximal, then you can find the first a_n such that $a_n \in \mathbb{C}1_{\mathcal{A}}$ such that $a_n \notin \mathcal{A}/I$, then

$$\overline{a_n \mathcal{A}/I} \text{ is a proper ideal of } \mathcal{A}$$

hence it generates a proper ideal in \mathcal{A} , we denote it as I_1 . If I_1 is not maximal, then repeat the process. By $\{a_n\}$ being countable, and that they are dense, we have a countable addition, which gives a maximal ideal containing I by countable inclusions. \square

Remark $C_\infty(\mathbb{R})$ is separable, and $C_b(\mathbb{R})$ is not separable.

Let's look at $L^\infty([0, 1], m)$, where m denotes the Lebesgue measure. This is a C^* -algebra, commutative, unital.

Could you exhibit any linear functionals on L^∞

Note that $L^2([0, 1], m)$, a Hilbert space, as an algebra on L^2 , L^∞ is closed for the strong operator topology on $\mathcal{B}(\mathcal{H})$ by the seminorm,

$$T \in \mathcal{B}(\mathcal{H}), T \rightarrow \|t\xi\|, \text{ for } \xi \in \mathcal{H}$$

For example, figure 1. We say that $L^\infty([0, 1])$ is a von Neumann algebra. Every commutative von Neumann algebra looks like some $L^\infty([0, 1], \mu)$. But note that noncommutative ones are quite interesting.

For a commutative Banach algebra over \mathbb{C} , the Gelfand transform

$$a \mapsto \hat{a}$$

We have

$$\|\hat{a}\|_\infty = r(a)$$

Proposition 1.31

If $\|a^2\| = \|a\|^2$, then the Gelfand transform is isometric. Thus we have

$$\|\hat{a}\|_\infty = \|a\|$$

Definition 1.15

For a_n involution on an algebra over \mathbb{C} , is a map $*$ from $\mathcal{A} \rightarrow \mathcal{A}$, with the properties

1. $(a^*)^* = a$
2. $(a + b)^* = a^* + b^*$
3. $(\alpha a)^* = \bar{\alpha}a^*$ for $\alpha \in \mathbb{C}$.
4. $(ab)^* = b^*a^*$

Definition 1.16 (Banach $*$ algebra)

If \mathcal{A} has a norm, then we say it is a $*$ normed if

$$\|a^*\| = \|a\|$$

If \mathcal{A} is complete, then it is called a Banach $*$ algebra.

Let G be a discrete group, let \mathcal{H} be a Hilbert space, then $\text{Aut}(\mathcal{H}) = U(\mathcal{H})$ is the group of unitary operators on \mathcal{H} to itself. By a unitary representation of G on \mathcal{H} , we mean

Definition 1.17 (Unitary representation)

A unitary representation of G on \mathcal{H} is a homomorphism $\pi : G \rightarrow U(\mathcal{H})$.

We note that $C_c(G) \subset l^1(G)$, and

$$\pi_f = \sum_{x \in G} f(x)\pi_x, \pi_f\pi_g = \pi_{f*g}$$

Now we ask what is $(\pi_f)^*$?

$$(\pi_f)^* = \sum \overline{f(x)}\pi_x^* = \sum \overline{f(x)}\pi_{x^{-1}} = \sum \overline{f(x^{-1})}\pi_x$$

So we get

$$(f^*)(x) = \overline{f(x^{-1})}$$

This defines an involution on $l^1(G)$. And it is easy to check that

$$\|f^*\|_1 = \|f\|_1$$

Remark The same process would not work for semigroups without the presence of x^{-1} necessarily.

We would like to think of this involution as some sort of complex conjugation.

Definition 1.18

Let \mathcal{A} be a Banach $*$ -algebra is symmetric if whenever $a \in \mathcal{A}$, and $a^* = a$, then

$$\sigma(a) \subset \mathbb{R}$$

If one looks at l^1 over noncommutative groups, some are symmetric, some are not.

Example 1.6 Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, with $\|\cdot\|_\infty$. We define an involution:

$$(\alpha, \beta)^* = (\bar{\beta}, \bar{\alpha})$$

This is a well-defined involution. However, this is not symmetric under this involution.

Proposition 1.32

If G is commutative, then $l^1(G)$ is symmetric.

$$\widehat{\mathcal{A}} = \{ \text{set of homomorphisms } G \rightarrow \mathbb{T} \}$$

This is symmetric.

Proposition 1.33

Let \mathcal{A} be an abstract, unital C^* -algebra, then \mathcal{A} is symmetric.

This is quite strong! (Every C^* -algebra is symmetric).

Lecture 11

Let \mathcal{A} be a $*$ -Banach algebra.

Definition 1.19

\mathcal{A} is symmetric if for any $a \in \mathcal{A}$, we have

$$a^* = a$$

we have $\sigma(a) \in \mathbb{R}$.

Note that a C^* algebra is necessarily a $*$ -algebra. Hence we have

$$\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|$$

And we apply it to a^* , hence we get

$$\|a^*\| \leq \|a\|$$

Hence the involution property is satisfied.

Proposition 1.34

Let \mathcal{A} be a unital C^* -algebra, i.e. we have

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

Then \mathcal{A} is symmetric.

In the Gelfand Noimark paper (1943),

Proof [Arens Truck, 1946] Given $a \in \mathcal{A}$ with $a^* = a$, for any $t \in \mathbb{R}$, let $b = a + it$, we look at $b^*b = (a - it)(a + it) = a^2 + t^2$. So we have

$$\|b^*b\| \leq \|a^2\| + t^2$$

Now let $\lambda \in \sigma(a)$, with $\lambda = r + is$, we would like to show $s = 0$. Note we have $\lambda + it \in \sigma(b)$, this gives

$$\lambda + it = r + i(s + t) \in \sigma(b)$$

Then we have

$$|r + i(s + t)| \leq \|b\|$$

Hence

$$r^2 + (s + t)^2 |r + i(s + t)|^2 \leq \|b^2\| = \|b\|^2 = \|b^*b\| \leq \|a^2\| + t^2$$

We thus have

$$r^2 + s^2 + 2st \leq \|a^2\|, \text{ for all } t$$

This gives $s = 0$. □

Let's step back. Let \mathcal{A} be a commutative symmetric Banach $*$ -algebra. Then if $a \in \mathcal{A}$, and if $a^* = a$, so $\sigma(a) \subset \mathbb{R}$. But $\sigma(a) = \text{range of } \widehat{a}$, so \mathcal{A} is an \mathbb{R} -valued function on \mathcal{A} .

For any $a \in \mathcal{A}$, we have

$$a = \frac{a + a^*}{2} + i \frac{a - a^*}{2i} = a_r + ia_i$$

then $\widehat{a} = \widehat{a}_r + i\widehat{a}_i$, note $a^* = a_r - ia_i$, then we have

$$\widehat{a^*} = \widehat{a}_r - i\widehat{a}_i = \overline{\widehat{a}}$$

Thus, we have

$$\widehat{a^*} = \overline{\widehat{a}}$$

So $a \mapsto \widehat{a}$ is a $*$ -algebra homomorphism of \mathcal{A} into $C(\widehat{\mathcal{A}})$.

For any unital commutative Banach algebra \mathcal{A} , then the image of the Gelfand transform Γ , separates the points of $\widehat{\mathcal{A}}$.

Proof If $\varphi, \psi \in \widehat{\mathcal{A}}$, and $\Gamma(a)(\varphi) = \Gamma(a)(\psi)$, then

$$\varphi(a) = \psi(a), \text{ for all } a$$

Hence $\varphi = \psi$. □

Proposition 1.35

If \mathcal{A} is a unital symmetric Banach $*$ -algebra, then the image of Γ is dense in $C(\widehat{\mathcal{A}})$. ♠

Proof [Key ingredient: Stone-Weierstrass] $\{\Gamma(\varphi) : \varphi \in \widehat{\mathcal{A}}\}$ is a unital subalgebra of $C(\widehat{\mathcal{A}})$ that separates the points of \mathcal{A} , and is closed under taking complex conjugates, so Stone-Weierstrass theorem applies (a compact space and a unital subalgebra of continuous functions that separates the points of the space, and closed under complex conjugation, then this algebra is dense for the $\|\cdot\|_\infty$ norm). □

Theorem 1.7 (Little Gelfand-Naimark theorem)

Let \mathcal{A} be a unital commutative C^* -algebra (abstract, which doesn't include hilbert space), then the Gelfand transform

$$\Gamma(a)(\varphi) = \varphi(a)$$

is an isometric $*$ -isomorphism of \mathcal{A} into $C(\widehat{\mathcal{A}})$. ♥

Proof Since \mathcal{A} is symmetric, the range of the Gelfand transform is dense. We also saw that $\|a^{2^n}\| = \|a\|^{2^n}$, so the

spectral radius of a , $r(a) = \|a\| = \|\Gamma(a)\|$.

Therefore $a \mapsto \Gamma(a)$ is isometric, and the range of $\Gamma(a)$ is norm-closed. □

Remark In a commutative Banach algebra, we have $r(a) = \|\widehat{a}\|_\infty$, and have $r(ab) \leq r(a)r(b)$, hence $\|a^*a\| = \|a\|^2$, $r(a^*a) \leq r(a)^2$, we have

$$\|a\|^2 \leq r(a)^2 \leq \|a\|^2$$

For $T \in \mathcal{B}(\mathcal{H})$, and $T = T^*$, then

$$C^*(T, I) \cong C(\sigma(T))$$

Proposition 1.36

Let G be a commutative group, then $l^1(G)$ with its $*$ is symmetric. ♠

Proof Let $\mathcal{A} = l^1(G)$, then $\widehat{\mathcal{A}}$ is isomorphic to the homomorphisms of G into \mathbb{T} .

If $\varphi \in \widehat{\mathcal{A}}$, then

$$\varphi(f) := \sum_{x \in G} f(x)\varphi(x), f \in l^1(G)$$

Then

$$\varphi(f^*) = \sum f^*(x)\varphi(x) = \sum \overline{f(x^{-1})}\varphi(x) = \sum \overline{f(x)}\varphi(x^{-1}) = \sum \overline{f(x)\varphi(x)} = \overline{\varphi(f)}$$

Note how we might have homomorphisms not mapping into \mathbb{T} if G is not commutative. □

Proposition 1.37

For G commutative, the range of the Gelfand transform Γ , which is the Fourier transform in this setting, this is dense in $C(\widehat{G})$. However, this is not isometric, and the range is not norm-closed unless G is finite. ♠