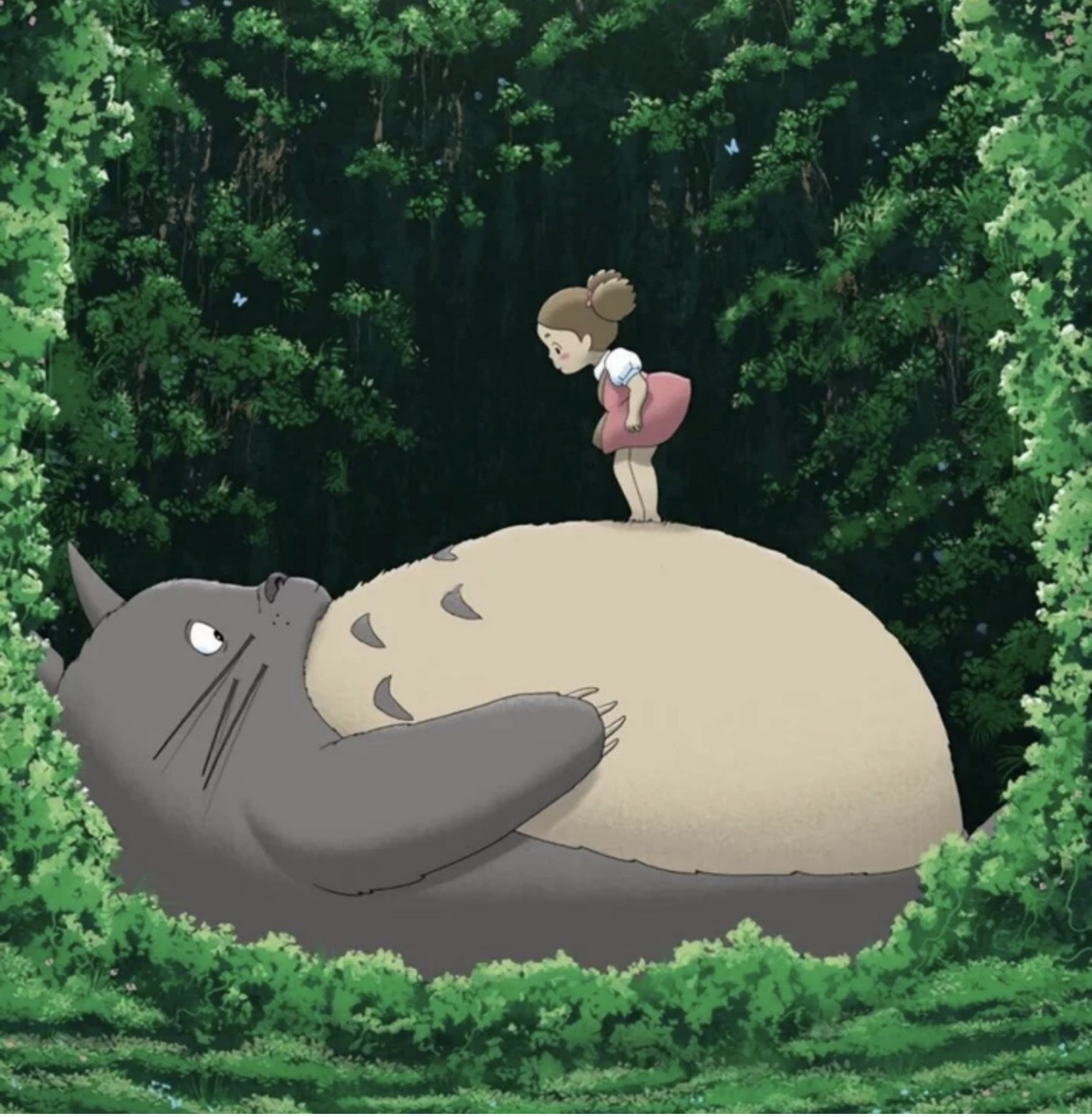

Geometric measure theory

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Chapter 1 Introduction

We will first introduce three questions in incidence geometry: the projection problem, the distance set problem, and the discrete Kakeya problem in $\mathbb{R}^2$. Let P be a discrete subset of $\mathbb{R}^2$.

Problem 1.1 (Projection) Let $e \in S^1$, and π_e be the projection onto the line l_e. We ask the upper bound on the number of e such that $\pi_e(P) \leq \frac{n}{8}$, given that P is a discrete set with $|P| = n$.

Problem 1.2 (Distance set) What is the lower bound the distance set $\Delta(P)$

$$\Delta(P) = \{|p - p'| : p, p' \in P\}$$

Problem 1.3 (Discrete Kakeya/Joints problem) Given a set of m lines $\mathcal{L}$, such that each line $l \in \mathcal{L}$ is m-rich, i.e.

$$|P \cap l| \geq m \text{ for each } l$$

Can we put a lower bound on the size of P.

We remind ourselves of a sharp bound regarding how the lines and points intersect. Let $I(P, \mathcal{L}) = \{(p, l) \in P \times \mathcal{L} : p \in l\}$

Theorem 1.1 (Szemeredi-Trotter theorem)

For any $P \subset \mathbb{R}^2$, and a finite set of lines, then we have

$$|I(P, \mathcal{L})| \lesssim (|P||\mathcal{L}|)^{\frac{2}{3}} + |\mathcal{L}| + |P|$$

We will prove a weaker result for some intuition, and gain some insight into the projection problem and the discrete Kakeya problem.

Proposition 1.1 (Weaker S-T)

In $\mathbb{R}^2$, we have that

$$|I(P, \mathcal{L})| \lesssim 4 \min\{|P|^{\frac{1}{2}}|\mathcal{L}| + |P|, |\mathcal{L}|^{\frac{1}{2}}|P| + |\mathcal{L}|\} \tag{1.1}$$

Using Proposition 1.1, we get the following lower bound on the discrete Kakeya problem in $\mathbb{R}^2$.

Corollary 1.1

we get that for a set of m lines such that each line intersects the point set P at least m times, we get that

$$|P| \gtrsim m^2$$

Note *The distance set problem can be realized as intersections between points and circles, instead of points and lines.*

We make a similar conjecture in $\mathbb{R}^n$, for m^{n-1} lines such that each line intersects the point set P at least m times, then we should have

$$|P| \gtrsim m^n$$

This statement fails for $\mathbb{R}^3$. Yet we could enforce some assumption to push to a nicer result.

Theorem 1.2 (G-N, Joints Problem)

For a set of m^2 lines such that no more than m lines lie in the same plane, and each line intersects the point set P at at least m points, then we have

$$|P| \gtrsim m^3$$

(This is in fact a conjecture by Bourgain and a corollary to the Joints problem in $\mathbb{R}^3$).

We now prove Proposition 1.1. unfinished here, the key idea is to use cauchy schwartz to get an l^2 norm to interpret as two points.

We now give some general bounds on the size of $\Delta(P)$ given that $|P| = n$.

✍ **Exercise 1.1** For a given $n \in \mathbb{N}$, there exists a set P such that $|\Delta(P)| \lesssim n$, for example, the set of n points arranged on a straight line.

✍ **Exercise 1.2** We now get some general lower bound on $\Delta(P)$. We can show $|\Delta(P)| \gtrsim n^{\frac{1}{2}}$. Consider two distinct points p_1, p_2, if we show that either

$$|\{|p_1 - p| : q \in P\}| \gtrsim n^{\frac{1}{2}} \text{ or } |\{|p_2 - q| : q \in P\}| \gtrsim n^{\frac{1}{2}}$$

WLOG, assume p_1 has that

$$|\{|p_1 - q| : q \in P\}| \lesssim n^{\frac{1}{2}} \tag{1.2}$$

Then we would like to show that

$$|\{|p_2 - q| : q \in P\}| \gtrsim n^{\frac{1}{2}}$$

If the equation 1.2 is true, then there exists a distance r such that

$$|Q| = |\{q \in P : |p_1 - q| = r|\}| \gtrsim n^{\frac{1}{2}}$$

And for $p_1 \neq p_2$, we have

$$|\{|p_2 - q| : q \in Q\}| \gtrsim n^{\frac{1}{2}}$$

Chapter 2 Dimensions

We now discuss some ways of measuring size of fractal sets.

Definition 2.1

Given a bounded set E, we define its δ-covering number $|E|_\delta$ as the smallest number of δ-balls needed to cover E. ♣

We note that as $\delta \to 0$, $|E|_\delta \to \infty$, so does $\frac{1}{\delta}$, hence comparing the rate of increase between the two gives us the Minkowski dimension (box counting dimension).

Example 2.1 Let $f : (X, d) \to (Y, d')$ is biLipschitz, if there eixts a constant C such that

$$C^{-1}d'(f(x), f(y)) \leq d(x, y) \leq Cd'(f(x), f(y))$$

Let $f : [0, 1]^n \to \mathbb{R}^n$ be biLipschitz, where $E = f([0, 1]^n)$, then we have

$$C^{-1}E \leq |[0, 1]^n| \leq CE$$

Hence $[0, 1] \sim E$, and $|E|_\delta \sim \delta^{-n}$.

Definition 2.2 (Upper and Lower Minkowski's dimension)

Let E be a bounded set in $\mathbb{R}^n$, and $|E|_\delta$ be the δ-covering number, then we define the upper and lower Minkowski dimension as follows:

$$\overline{\dim_B}(E) = \limsup_{\delta\to 0} \frac{\log(|E|_\delta)}{\log(1/\delta)}, \underline{\dim_B}(E) = \liminf_{\delta\to 0} \frac{\log(|E|_\delta)}{\log(1/\delta)}$$

♣

Example 2.2 The countable set $E = \mathbb{Q} \cap [0, 1]$, has Lebesgue measure 0, and has Minkowski dimension:

$$\dim_B(E) = \lim_{\delta\to 0} \frac{\log(\delta^{-1})}{\log(\delta^{-1})} = 1$$

Example 2.3 The set $E = \{\frac{1}{n} : n \in \mathbb{N}\}$ has Minkowski dimension: for every $\frac{1}{n}$, it could be covered by a $\delta = n^{-2}$-length disjoint interval, hence

$$\dim_B(E) = \lim_{\delta\to 0} \frac{\log(n)}{\log(n^2)} = \frac{1}{2}$$

Example 2.4 The set $E = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ is "too sparse" of a fractal so its box counting dimension is the same as the topological dimension.

$$\dim_B(E) = \lim_{\delta\to 0} \frac{\log(n)}{\log(2^n)} = \lim_{n\to\infty} \frac{\log(n)}{n\log(2)} = 0$$

One could generalize this to get any set $E = \{a^{-n} : n \in \mathbb{N}\}$ has Minkowski dimension 0.

Example 2.5 The Cantor set, splits into 2^n intervals of length $\frac{1}{3^n}$.

$$\dim_B(E) = \lim_{\delta\to 0} \frac{\log(2^n)}{\log(3^n)} = \frac{\log(2)}{\log(3)}$$

Note *Minkowski dimension does not always exist if the upper or lower Minkowski dimensions don't agree, and it does not work with unbounded sets E.*

Note *The example 2.2 has Minkowski dimension 1, but it is a countable set, hence we would like to assign it measure 0.*

$$\dim \cup_i E_i = \sup_i \dim E_i$$

To address the above two concerns, we introduce the Hausdorff dimension. We do it in three steps: introduce an up-to-δ-cover $\{U_j\}$, construct Hausdorff δ-measure, and letting $\delta \to 0$.

2.0.1 Hausdorff measure

Definition 2.3 (s-dim Hausdorff measure)

Fix $s \geq 0$, and $\delta \in (0, \infty]$, given a set $E \in \mathbb{R}^n$, an "up-to-δ"-cover of E is a ***countable*** *family of sets $\{U_j\}_{j \in \mathbb{N}}$ such that*

$$E \subset \cup_j U_j, diam(U_j) \leq \delta, \text{ for all } j$$

And an s-dimensional Hausdorff δ-meausre of the set E is

$$H^s_\delta(E) = \inf \left\{ \sum_j diam(U_j)^s, \{U_j\}_j \text{ is an up-to-}\delta\text{-cover of } E \right\}$$

Finally, the s-dimensional Hausdorff measure of E is

$$H^s(E) = \lim_{\delta \to 0} H^s_\delta(E)$$

♣

Remark The limit is well justified since as $\delta \to 0$, $H^s_\delta(E)$ is an increasing function.

There are many nice properties regarding the Hausdorff measure, for example, n-dim Hausdorff measure agrees with the n-dim Lebesgue measure, and there is a unique number such that the Hausdorff measure stops being ∞, and equivalently drops to zero. Hence based on this observation, we introduce the Hausdorff dimension of a set E.

Definition 2.4 (Hausdorff dimension)

For a set $E \subset \mathbb{R}^n$, we have

$$\dim_H(E) = \sup\{s : H^s(E) = \infty\} = \inf\{s : H^s(E) = 0\}$$

♣

Before anything, we first check that the s-dimensional Hausdorff measure defined above is indeed a measure.

Proposition 2.1

For $s \geq 0$, the s-dimensional measure is indeed a measure.

♠

Proof We have that $\mu(\emptyset) = 0$, and $\mu(E) \geq 0$ for all E. Finally we check the measure is countably additive. For $\{E_j\}_{j \in \mathbb{N}}$ disjoint sets, we consider $E = \cup_j E_j$, as $\delta \to 0$, (or for δ sufficiently small, given E_j's are disjoint), all the up-to-δ-covers are disjoint, hence

$$H^s_\delta(\cup_j E_j) = \sum_j H^s_\delta(E_j)$$

And letting $\delta \to 0$, we get

$$H^s(\cup_j E_j) = \sum_j H^s(E_j)$$

□

Proposition 2.2

The following are basic facts about the Hausdorff measure:

1. *for $n \in \mathbb{N}$, let m be the n-dim Lebesgue measure, there exists a constant C such that*
$$C^{-1} H^n(E) \leq m(E) \leq C H^n(E)$$
2. *$H^s(E)$ is a nonincreasing function of s.*
3. *For $0 \leq s_1 < s_2 < \infty$*
$$\text{either } H^{s_1}(E) = \infty \text{ or } H^{s_2}(E) = 0$$
4. *For $s > n$, and $E \subset \mathbb{R}^n$, we have that*
$$H^s(E) = 0$$

5. *For $E \subset \mathbb{R}^n$, and $s \geq 0$, we have that*

$$H^s(E) = 0 \iff H^s_\infty(E) = 0$$

Example 2.6 For a set $E \subset \mathbb{R}^n$, we have that the n-dimensional Hausdorff measure should agree with the standard Lebesgue measure on $\mathbb{R}^n$. For if E is unbounded, then $m(E) = \infty$, and

Exercise 2.1 We have that for $f : A \to R^m$, $A \subset R^n$, for a fixed $s \geq 0$, and f is Lipschitz with Lipschitz constant L, we have that

$$H^s(f(A)) \lesssim_L H^s(A)$$

Proof For any up-to-δ cover $\{E_j\}$ of A, we have $\{f(E_j)\}_j$ is an (up-to-some constant)-δ cover of $f(A)$, hence

Proposition 2.3

The Hausdorff measure is monotone: for $E_1 \subset E_2$, we have that

$$H^s(E_1) \leq H^s(E_2)$$

Proof For $E_1 \subset E_2$, for each δ, an up-to-δ-cover of E_2 is also an up-to-δ cover of E_1, and hence taking the infimimum, we get that $H^s(E_1) \leq H^s(E_2)$.

$\square$

Proposition 2.4

The Hausdorff dimension satisfies that the dimension is a local property:

$$\dim(\cup_j E_j) = \sup_j \dim(E_j)$$

Proof We would like to show that $H^s(\cup_j E_j) = \infty$ if and only if $\sup_j H^s(E_j) = \infty$, and similarly, $H^s(\cup_j E_j) = 0$ if and only if $\sup_j H^s(E_j) = 0$.

This is a total of 4 directions. By monotonicity, two directions are shown:

$$\sup_j H^s(E_j) = \infty \Rightarrow H^s(\cup_j E_j) = \infty$$

Moreover,

$$H^s(\cup_j E_j) = 0 \Rightarrow \sup_j H^s(E_j) = 0$$

Moreover, by H^s being a measure, if we have $\sup_j H^s(E_j) = 0$, then all $H^s(E_j) = 0$ for all j, thus

$$H^s(\cup_j E_j) \leq \sum_j H^s(E_j) = 0$$

Now it remains to show that what

Now we justify the usage of H^s, instead of just working H^s_δ.

Exercise 2.2 For $0 \leq s \leq 1, n \geq 2$, we have

$$H^s_2(B_1) = H^s_2(\overline{B_1}) = H^s_2(\partial(B_1))$$

We see that

$$H^s_2(B) = H^s_2(\overline{B}) = 2$$

Then $H^s_2(\partial B) = 0$ if $\overline{B}$ was indeed measurable. But for $0 \leq s \leq$, it is more reasonable to cover $\overline{\partial B}$ with bigger covers.

Hence we work with H^s to get a Borel regular measure. Recall the following definitions.

Definition 2.5

A measure μ is a Borel measure if all Borel sets are μ-measurable. Moreover, μ is called Borel regular if for any Borel set A, there exists another Borel set B such that $B \subset A$, and $\mu(A) = \mu(B)$.

With our construction, we claim that the Hausdorff measure H^s for any $s > 0$ is a Borel regular measure.

Proposition 2.5

H^s_δ is a Borel regular measure.

Proof We first accept the fact that every Borel set is H^s-measurable. We show that H^s is Borel-regular. For a Borel set A, we would like to approximate it by "fattening up" the covers. For each n, let $B_n := \cup_j E_{n,j}$ be a cover of A, and such that $\sum_j (diam(E_{n,j}))^s \le H^s_{\frac{1}{n}}(A) + \frac{1}{n}$. Then if we take $B = \cap_n B_n$, we have that $A \subset B$, and $H^s(A) = \cap_n H^s_{\frac{1}{n}}(A) \ge \sum_j (diam(E_{n,j}))^s - \frac{1}{n} \ge \cap_n H^s_{\frac{1}{n}}(B_n) - \frac{1}{n}$, which by our construction, is $H^s(B)$. Then by monotonicity of H^s, we have that

$$H^s(A) = H^s(B)$$

Note *The countably additivity of H^s comes from the fact that all Borel sets are H^s-measurable, and any meausure if countably additive on its measurable sets.*

Section 3 This is part to be typed up. We did Mass distribution principle, which states that if E has that a r_0 Frostman measure μ, then $H_{r_0}(E) \ge \mu(E)/C$, and if further we have that $\mu(E) > 0$, then $\dim_H E \ge s$.

1. Frostman implies positive Hausdorff dimension
2. definition of support of a measure
3. push-forward measure

This is page 12 on weak convergence of measures.

Definition 2.6 (Weak convergence of measures)

Let $\{\mu_j\}$ be a sequence of locally finite measures (they automatically assign finite measures to all compact sets), and we say $\{\mu_j\}$ converges to μ weakly if for all $\varphi \in C_c(X)$, we have

$$\lim_{j\to\infty} \int \varphi d\mu_j = \int \varphi d\mu$$

Our goal for tonight is to understand the proof of the Frostman Lemma.

Lemma 2.1 (Frostman Lemma)

Assume $E \subset \mathbb{R}^n$ is a compact set with $H^s(E) > 0$, then there exists a compactly supported Borel measure μ with $supp(\mu) \subset E$ and $\mu(E) \gtrsim H^s_\infty(E)$, and such that for all $x \in \mathbb{R}^n, r > 0$, we have

$$\mu(B(x,r)) \le r^s$$

Proof

There are some things we need to establish before the we prove the Frostman lemma. For a set $E \in \mathbb{R}^n$, we use $\mathcal{M}(E)$ to denote the set of finite Borel measures whose support is contained in E, i.e. if $\mu \in \mathcal{M}(E)$, we have

$$supp(\mu) \subset E, \text{ and } 0 < \mu(E) < \infty$$

Next we state the "Bolzano-Weierstrass" theorem for measures.

Lemma 2.2

Let $\{\mu_j\}$ be a sequence of locally finite Borel measures on $\mathbb{R}^n$, i.e., for all K compact subset in $\mathbb{R}^n$, we have

$$\sup_{j\in\mathbb{N}} \mu_j(K) < \infty$$

Then there exists a subsequence μ_{j_k} such that as $k \to \infty$, the subsequence converges to μ.

2.1 Hausdorff dimension of product sets

Theorem 2.1 (Hausdorff dimension of product sets)

Let A, B be Borel sets, and $s, t \geq 0$, then we have

$$H^{s+t}_{\infty}(A \times B) \gtrsim_{d_1,d_2} H^s_{\infty}(A) H^t_{\infty}(B)$$

Proof We use the theorem that positive Hausdorff meausre if and only if there exists a Frostman measure.

Assume $H^s_{\infty}(A) > 0, H^t_{\infty}(B) > 0$, then there exists μ_1, μ_2 such that

$$\mu_1(A) \gtrsim H^s_{\infty}(A), \mu_2(B) \gtrsim H^t_{\infty}(B)$$

Then we consider any ball $B((x_1, x_2), r)$, we have that

$$B((x_1, x_2), r) \subset B(x_1, r) \times B(x_2, r)$$

Hence we have

$$\mu_1 \times \mu_2(A \times B) \gtrsim r^{s+t}$$

Hence $\mu_1 \times \mu_2$ is a Frostman measure on $A \times B$, hence

$$H^{s+t}_{\infty}(E) \geq r^{s+t} \gtrsim H^s_{\infty}(A) H^t_{\infty}(B)$$

Corollary 2.1

For A, B Borel sets, we have

$$\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B)$$

Proof Once we have $H^{s+t}_{\infty}(A \times B) \gtrsim_{d_1,d_2} H^s_{\infty}(A) H^t_{\infty}(B)$, it is easy to see, if $\dim_H(A) > s$, and $\dim_H(B) > t$, then we have that $H^s_{\infty}(A) \geq 0$, and $H^t_{\infty}(B) \geq 0$, thus $H^{s+t}_{\infty}(A \times B) \geq 0$, hence we have

$$\dim_H(A \times B) \geq s + t$$

□

Corollary 2.2

We also have that

$$\dim_H(A \times B) \leq \dim_H A + \overline{\dim}_M B$$

Proof If we assume that $\dim_H(A) < s, \dim_H(B) \leq \overline{\dim}_M(B) < t$, then we have

$$\dim_H(B) < t, i.e. H^t_{\infty}(B) = 0$$

$$\dim_{\infty}(A \times B)$$

2.2 Riesz Energies

We revisit the question regarding projections. Assume E is compact,do we always have

$$\dim_H(\pi_e(E)) = \min\{\dim_H(E), 1\}$$

for some $e \in S^1$ or for all $e \in S^1$.

The important implication is we would like to transfer an s-dimensional Frostman measure on E to an s-dimensional Frostman measure(the push-forward measure) on $\pi_e(E)$, i.e. we would like to have if $\mu(B(x, r)) \lesssim r^s$ for $supp(\mu) \subset E$,

$$\mu_{\pi_e}(B(x, r)) \lesssim r^s$$

where $supp(\mu_{\pi_e}) \subset \pi_e(E)$.

Definition 2.7 (Riesz potential and energy of measures)

Let $0 \leq s \leq d$, and let μ be a Borel measure on $\mathbb{R}^d$. The s-dimensional Riesz potential of the measure μ at a particular point is

$$V_s(\mu)(x) = \mu * k_s(x) = \int \frac{1}{|x-y|^s} d\mu(y)$$

where k_s is the s-dimensional Riesz kernel

$$k_s(x) = \frac{1}{|x|^s}$$

And the s-dimensional Riesz energy of the measure μ is given by integrating the potential:

$$I_s(\mu) = \int V_s(\mu) d\mu(x) = \int \int \frac{1}{|x-y|^s} d\mu(x) d\mu(y)$$

♣

The reason for introducing the Riesz energies and potentials is because they carry the same information as a measure being s-Frostman.

Proposition 2.6

Let $\mu \in M(\mathbb{R}^d)$ be s-Frostman, then for all $0 \leq t < s$, we have $\|V_t(\mu)\|_\infty < \infty$, and

$$I_t(\mu) < \infty$$

Conversely, if we have finite s-Riesz energy, $I_s(\mu) < \infty$, then there exists a subset $B \subset \mathbb{R}^d$, that we have

$$\mu(B) > 0, \mu|_B \text{ is } s\text{-Frostman}$$

♠