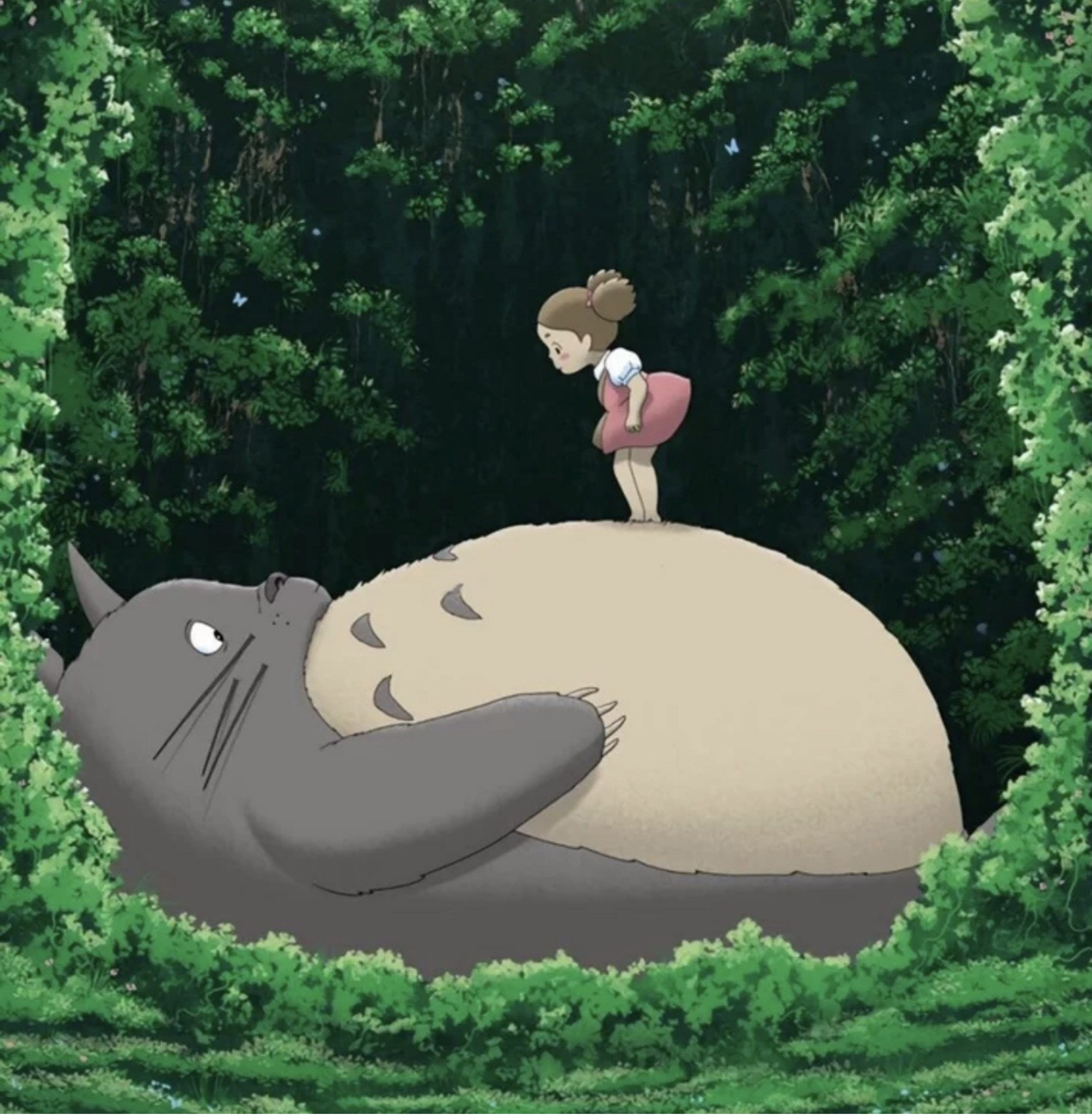

Fourier Analysis

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Chapter 1 Fourier series and Integrals

This chapter is an introduction to the Fourier series and Fourier integrals.

We begin with two ways to check pointwise convergence of partial sums of Fourier series

Theorem 1.1 (Dini's criterion)

If for some x, there exists $\delta > 0$ such that

$$\int_{|t|<\delta} \left| \frac{f(x+t) - f(x)}{t} \right| dt < \infty$$

then we have

$$\lim_{N\to\infty} S_N f(x) = f(x)$$

♡

Theorem 1.2 (Jordan's criterion)

If f is a function of bounded variation in a neighborhood of x, then

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}[f(x+) + f(x-)]$$

♡

Remark If f is continuous at x, then $f(x+) = f(x-) = f(x)$, we actually have $\lim_{N\to\infty} S_N f(x) = f(x)$.

Note *All these convergence results of the partial sums are local. And it is made clear using the following lemma.*

Theorem 1.3 (Riemann-Lebesgue localization principle)

If f is zero in a neighborhood of x, then

$$\lim_{N\to\infty} S_N f(x) = 0$$

♡

To show that, using the following lemma.

Lemma 1.1 (Riemann-Lebesgue lemma)

If $f \in L^1(\mathbb{T})$, then we have

$$\lim_{|n|\to\infty} \hat{f}(n) = 0$$

♡

Then we discuss the Fourier series of continuous functions.

Theorem 1.4

Ther exists a continuous function f whose Fourier series diverges at some point x, i.e.

$$\lim_{N\to\infty} S_N f(x) = +\infty$$

♡

Next, we move away from pointwise convergence, and instead we talked about convergence in the L^p norm.

Lemma 1.2

$S_N f$ converges to f in L^p, for $1 \le p < \infty$, if and only if $S_N : L^p \to L^p$ has $\|S_N\| < \infty$, i.e. the following holds:

$$\|S_N f\|_{L^p} \lesssim \|f\|_{L^p}$$

♡

next we note that take the Fourier series, is just like taking the fouier trasform, where it is a isometry from L^2 to $l^2(\mathbb{N})$.

Theorem 1.5

The mapping $f \mapsto \{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is an isometry from L^2 to l^2, i.e.

$$\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$$

Now we discuss some better summability methods, such as the Cesaro and Abel sum.

Theorem 1.6

If $f \in L^p$, where $1 \leq p < \infty$, then we have

$$\lim_{N\to\infty} \|\sigma_N f - f\|_{L^p} = 0$$

If $p = \infty$, and if f is continuous, then the above also holds.

Corollary 1.1

The trigonometric polynomials are dense in L^p, for $1 \leq p < \infty$.
And if f is integrable, where $\hat{f}(n) = 0$ for all n, then f is identically 0.

Note *The above all hold if we replace σ_N with P_r, and letting r go to $1-$.*

Theorem 1.7

The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$, such that for $f, g \in \mathcal{S}$, we have

$$\int f\hat{g} = \int \hat{f}g$$

and we also have

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)e^{2\pi i n\cdot\xi}d\xi$$

Corollary 1.2

For $f \in \mathcal{S}$, we have

$$(\hat{f})\hat{} = f(-x)$$

Hence the Fourier transform has period 4.

Theorem 1.8

The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded.

Since we've defined the Fourier transform on $\mathcal{S}$, we have it defined on L^p.

Theorem 1.9 (Plancherel)

The Fourier transform is an isometry on L^2, that is $\hat{f} \in L^2$, and we have

$$\|f\|_{L^2} = \|\hat{f}\|_{L^2}$$

Next we like to have some general bounds on Fourier transforms of L^p functions, namely, the following two.

Theorem 1.10

If $f \in L^p$ and $1 \leq p \leq 2$, then we have $f \in L^{p'}$ where p' is the dual exponent of p. And we have

$$\|\hat{f}\|_{L^{p'}} \leq \|f\|_{L^p}$$

Theorem 1.11 (Young's inequality)

For $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$, and $f \in L^p$, $g \in L^q$, we have

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$$

♡

We will state the Riesz-Thorin interpolation theorem here.

Theorem 1.12 (Riesz-Thorin)

Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and for $0<\theta<1$ define p, q by

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}, \frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}$$

If T is a linear operator from $L^{p_0}+L^{p_1}$, to $L^{q_0}+L^{q_1}$ such that

$$\|Tf\|_{q_0} \leq A\|f\|_{p_0}$$
$$\|Tf\|_{q_1} \leq B\|f\|_{p_1}$$

Then we have

$$\|Tf\|_{L^q} \leq A^{1-\theta}B^{\theta}\|f\|_{L^p}$$

♡

Proposition 1.1

We state some useful here. For any $f \in L^p$, whre $1<p<2$, we can disect $f=f_1+f_2$ where $f_1 \in L^1, f_2 \in L^2$. ♠

Chapter 2 The Hardy-Littlewood Maximal function

Proposition 2.1 (Pointwise convergence of ϕ_t)

For $\phi \in L^1(\mathbb{R}^n)$ and $\int \phi = 1$, we define $\phi_t = t^{-n}\phi(t^{-1}x)$, we have for $g \in \mathcal{S}$, we have

$$\lim_{t\to 0} \phi_t * g(x) = g(x)$$

♠

Now we address the L^p convergence of $\phi_t * f$. We have

Theorem 2.1

Let $\{\phi_t : t > 0\}$ be an approxiation of the identity, and $f \in L^p, 1 \leq p < \infty$, then we have

$$\lim_{t\to 0} \|\phi_t * f - f\|_{L^p} = 0$$

♡

Corollary 2.1

There exists a sequence $\{t_k\}$, depending on f, such that as $t_k \to 0$, we have

$$\lim_{k\to\infty} \phi_{t_k} * f(x) = f(x) \text{ a.e.}$$

♡

Theorem 2.2

Let $\{T_t\}$ be a family of linear and sublinear operators on L^p and define the maximal function as

$$T^* f(x) = \sup_t |T_t f(x)|$$

And if this maximal function T^* is weak (p,q), then the set

$$\{f \in L^p : \lim_{t\to t_0} T_t f(x) = f(x) a.e.\}$$

is closed in L^p.

♡

Hence if we show that pointwise limit of any linear or sublinear operator $Tf(x) = f(x)$ a.e. holds for $f \in \mathcal{S}$, then we know this is true for all $f \in L^p$.

Proposition 2.2

We have an alternative form of the L^p norm:

$$\|f\|_{L^p}^p = \int_0^\infty \lambda^{p-1} a_f(\lambda) d\lambda$$

More specifically, if we have $p = 1$, we then have

$$\|f\|_{L^1} = \int_0^\infty a_f(\lambda) d\lambda$$

♠

Next we introduced the Hardy-Littlewood maximal function:

$$Mf(x) = \sup_r \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy$$

Theorem 2.3

M is weak (1,1) and strong (p,p), for $1 < p \leq \infty$.

♡

Proposition 2.3

For ϕ a radial function, i.e. $\phi(t) = \phi(|t|)$, such that it is positive, decreasing (from $(0, \infty)$), then we have

$$\sup_t |\phi_t * f(x)| \leq \|\phi\|_{L^1} Mf(x)$$

♠

Corollary 2.2

If $|\phi(x)| \leq \psi(x)$ *a.e., where* ψ *is positive, radial, and decreasing, then we have note that* $|\phi(x) * f(x)| \leq \psi(x)$*, and from the previous proposition, we have*

$$\sup_t |\phi_t * f(x)| \leq \sup_t |\psi_t * f(x)| \leq \|\psi\|_{L^1} Mf(x)$$

Hence $\sup_t |\phi_t * f(x)|$ *is weak (1,1), and strong (p,p), for* $1 < p \leq \infty$*.* ♡

Corollary 2.3

All assumptions remain the same, and if $f \in L^p$, $1 \leq p < \infty$*, we have*

$$\lim_{t \to 0} \phi_t * f(x) = \left(\int \phi \right) f(x) \text{ a.e.}$$

♡