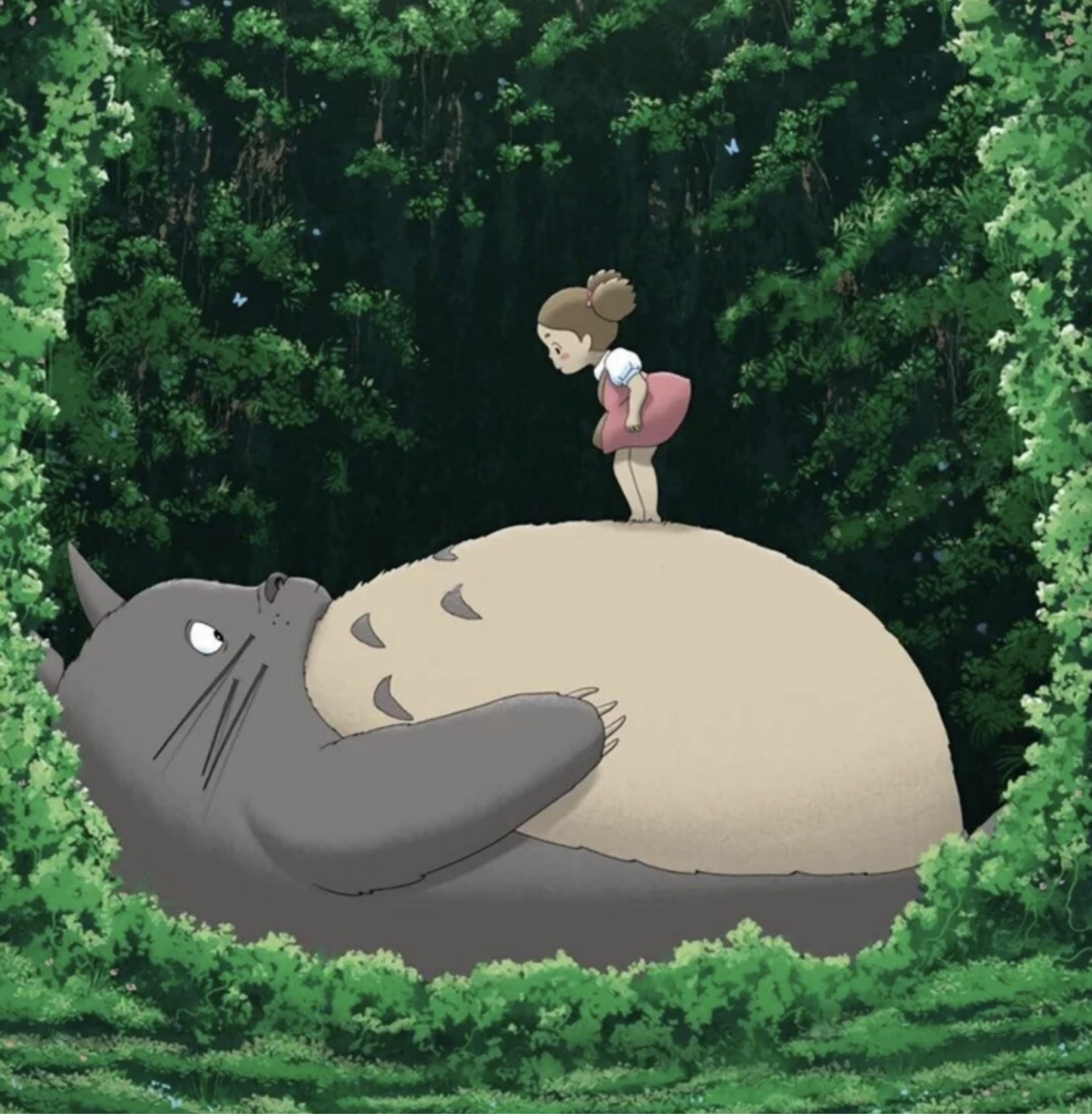

Functional Analysis

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Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

Remark Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_\mathcal{H})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_\mathcal{H}) \cong C(X) \tag{1.1}$$

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_\mathcal{H}) \cong C(X) \tag{1.2}$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on B or an action of the group G on B.

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B, generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B, hence no need to include it.

Elements of the form $\Sigma_{z_x} \alpha_x, z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x\in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many x, $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f\alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f\alpha_g = \sum_x \sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f*g)(y)\alpha_y$$

where we define $f * g(y) = \sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f\alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \|\sum f(x)\alpha_x\| \leq \sum |f(x)|\|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f*g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x\xi(y) = \xi(x^{-1}y), \alpha_x$ being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α.

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X. This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X, by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let $\mathcal{A}$ be an algebra on $\mathbb{R}$ or $\mathbb{C}$, is a normed algebra if it has a norm $\|\|$, as a vector space, such that for for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity. ♣

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm. ♣

Given $x \in X$, define $\varphi_x : C(X) \to \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \to \mathbb{R}$ or $C(X) \to \mathbb{C}$. This simply implies

$$\varphi_x(f+g) = (f+g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$. ♠

Proof Main Claim: given a multiplicative linear functional φ, there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f(x) \neq 0$, but $\varphi(f) = 0$.

Set $g_x = \overline{f}_x f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f}_x) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is $x_1, ..., x_n$ such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + ...g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f, consider $f - \varphi(f) \cdot 1$, apply φ, we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field. ♣

Remark We have $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

Thss is not always true for normed algebra, Let

$$\mathcal{A} := Poly \subset C([0, 1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If $\mathcal{A}$ is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word "unital" is key here.

Proposition 1.4

Let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} =? \sum_{n=0}^{\infty} a^n, a^0 = 1_{\mathcal{A}}$ You could look at the partial sums. $S_m = \sum_{n=0}^{m} a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m\to\infty} S_m = \frac{1}{1-a}$.

To prove this is a cauchy sequence:

$$\|S_n - S_m\| = \| \sum_{j=m+1}^{n} a^j \| \leq \sum_{m+1}^{n} \|a^j\| \leq \sum_{m+1}^{n} \|a\|^j$$

And the fact that $\|a\| \leq 1$, we have the sum bounded by ϵ, hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n\to\infty} S_n(1-a) = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = \lim_{n\to\infty}(1-a^{n+1}) = 0$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$.

1.2 Lecture 3

We now begin.

Let $\mathcal{A}$ be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Newmann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1 - a\| < 1$, then a is invertible.

Proof $a = 1 - (1-a)$.

Proposition 1.5

The set of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$.

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of $\mathcal{A}$ onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d, homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in Range(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.

Proof Let $f \in C(X)$, and if $\alpha \in$ range of f, so $\alpha = f(x_0)$ for some x_0. then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(-\alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in$ range of f.

□

Definition 1.5 (spectrum of an element)

For any unital algebra $\mathcal{A}$ over some field $\mathbb{F}$, for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible } \}$$

is called the spectrum of a, denoted as $\sigma(a)$.

Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$

Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda(1 - \frac{a}{\lambda})$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\alpha \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on $\mathcal{A}$, i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of $\mathbb{R}, \mathbb{C}$.

Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of $\mathcal{A}$ is open, hence

$$\sigma(a) = \phi^{-1}(\text{ noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{ invertible })$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of $\mathcal{A}'$, which denotes the dual vector space of continuous linear transformations.

On $\mathcal{A}'$, we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak- topology.*

Proof let $\{\varphi_\lambda\}$ be a net of elemnts of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \to \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V, the closed unit ball of V' is compact in the weak- topology.*

As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak- toplogy.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto $\mathcal{A}$.

For $\mathcal{A}$ unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ into $C(\mathcal{A})$.

Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{a}\widehat{b})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

□