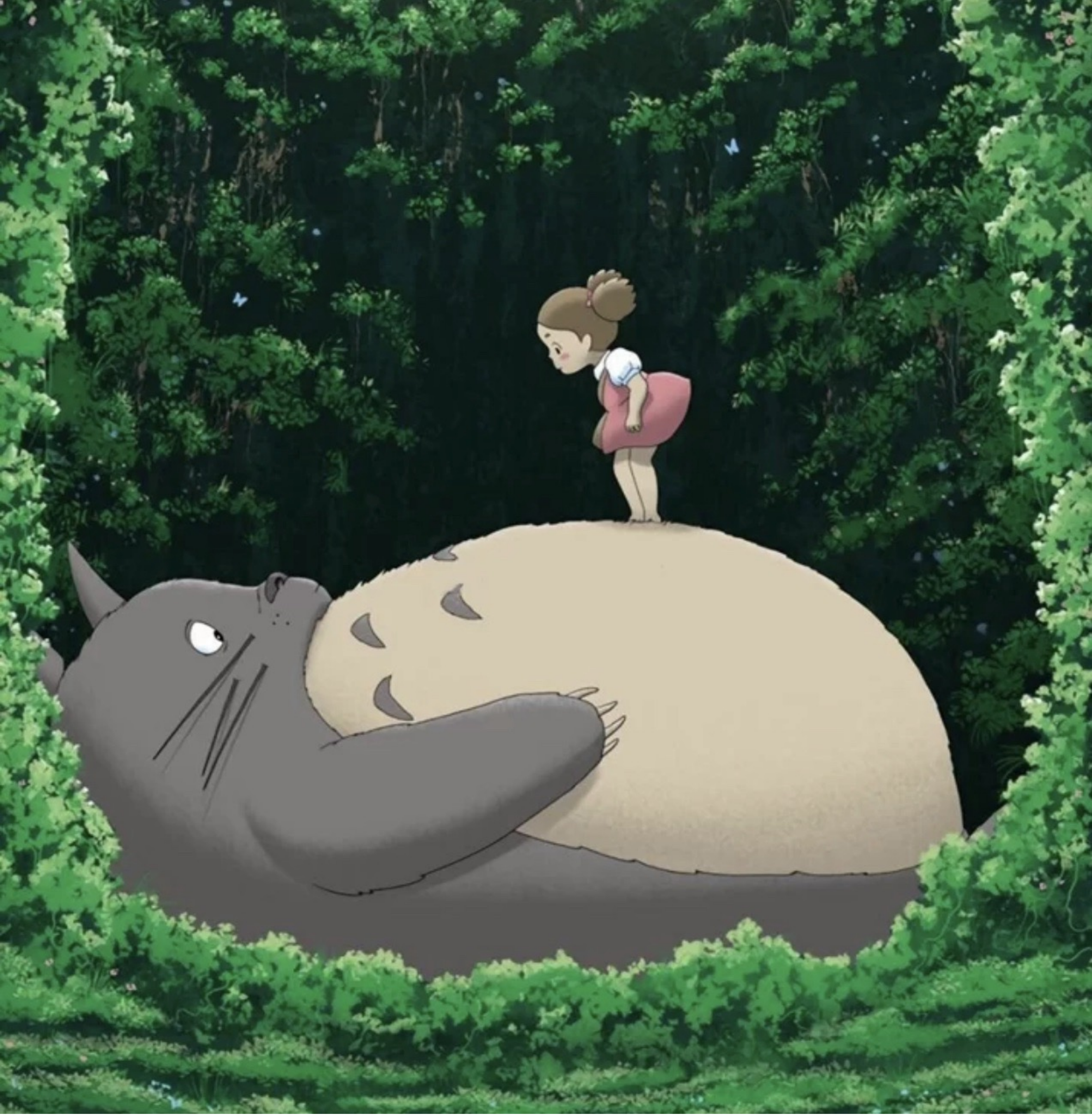

Functional Analysis

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Date: September 2, 2023

Contents

Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

Remark Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \tag{1.1}$$

♠

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_{\mathcal{H}}) \cong C(X) \tag{1.2}$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on B or an action of the group G on B.

♣

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B, generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B, hence no need to include it.

Elements of the form $\Sigma_{z_x} \alpha_x, z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x\in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many x, $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f \alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f\alpha_g = \sum_x \sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f * g)(y)\alpha_y$$

where we define $f * g(y) = \sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f \alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \|\sum f(x)\alpha_x\| \leq \sum |f(x)| \|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1} \|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x \xi(y) = \xi(x^{-1}y), \alpha_x$ being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α.

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X. This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X, by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let $\mathcal{A}$ be an algebra on $\mathbb{R}$ or $\mathbb{C}$, is a normed algebra if it has a norm $\|\|$, as a vector space, such that for for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity. ♣

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm. ♣

Given $x \in X$, define $\varphi_x : C(X) \to \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \to \mathbb{R}$ or $C(X) \to \mathbb{C}$. This simply implies

$$\varphi_x(f+g) = (f+g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$. ♠

Proof Main Claim: given a multiplicative linear functional φ, there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f(x) \neq 0$, but $\varphi(f) = 0$.

Set $g_x = \overline{f}_x f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f}_x) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is $x_1, ..., x_n$ such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + ...g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f, consider $f - \varphi(f) \cdot 1$, apply φ, we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field. ♣

Remark We have $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

Thss is not always true for normed algebra, Let

$$\mathcal{A} := Poly \subset C([0,1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If $\mathcal{A}$ is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word "unital" is key here.

Proposition 1.4

Let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} =? \sum_{n=0}^{\infty} a^n, a^0 = 1_{\mathcal{A}}$ You could look at the partial sums. $S_m = \sum_{n=0}^{m} a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m\to\infty} S_m = \frac{1}{1-a}$.

To prove this is a cauchy sequence:

$$\|S_n - S_m\| = \|\sum_{j=m+1}^{n} a^j\| \leq \sum_{m+1}^{n} \|a^j\| \leq \sum_{m+1}^{n} \|a\|^j$$

And the fact that $\|a\| \leq 1$, we have the sum bounded by ϵ, hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n\to\infty} S_n(1-a) = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = \lim_{n\to\infty}(1 - a^{n+1}) = 0$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$.

1.2 Lecture 3

We now begin.

Let $\mathcal{A}$ be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Newmann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1 - a\| < 1$, then a is invertible.

Proof $a = 1 - (1-a)$.

Proposition 1.5

The set of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$.

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of $\mathcal{A}$ onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d, homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in Range(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.

Proof Let $f \in C(X)$, and if $\alpha \in$ range of f, so $\alpha = f(x_0)$ for some x_0. then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(-\alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in$ range of f.

□

Definition 1.5 (spectrum of an element)

For any unital algebra $\mathcal{A}$ over some field $\mathbb{F}$, for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible } \}$$

is called the spectrum of a, denoted as $\sigma(a)$.

Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$

Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda(1 - \frac{a}{\lambda})$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\alpha \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on $\mathcal{A}$, i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of $\mathbb{R}, \mathbb{C}$.

Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of $\mathcal{A}$ is open, hence

$$\sigma(a) = \phi^{-1}(\text{ noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{ invertible })$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of $\mathcal{A}'$, which denotes the dual vector space of continuous linear transformations.

On $\mathcal{A}'$, we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak- topology.*

Proof let $\{\varphi_\lambda\}$ be a net of elemnts of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \to \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V, the closed unit ball of V' is compact in the weak- topology.*

As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak- toplogy.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact. □

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto $\mathcal{A}$.

For $\mathcal{A}$ unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ into $C(\mathcal{A})$.

Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{a}\widehat{b})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

□

1.3 Lecture 4

Today we talk about the structure of $\widehat{l^1(S)}, \widehat{l^1(G)}$, where S, G are semigroups and groups, and how they naturally identify with the unit disk $\mathbb{D}$, and the unit circle $\mathbb{T}$.

Let S be a commutative discrete semigroups, for example $\mathbb{N} \cup \{0\}$, and $f \in C_c(S)$, then we can write $f = \sum_{x \in S} f(x)\delta_x$, where we define $\delta_x\delta_y = \delta_{xy}$. Note that $C_c(S)$ is dense in $l^1(S)$.

Definition 1.6 (Convolution)

Take any $f, g \in C_c(S)$, we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{x \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left(\sum_{xy = z} f(x)g(y) \right) \delta_z$$

where we define the convolution between two functions

$$f * g(z) = \sum_{x,y,xy=z} f(x)g(y)$$

And under this convolution operation, we have $l^1(S), *$ as a Banach algebra.

Example 1.1 If we consider polynomials of the form $f(x) = \sum_{n=0}^{\infty} f(n)x^n$, and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right)\left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f*g)(p)$$

And let $f \in C_c(S)$, where $S = \mathbb{N}$. we define $\|f\|_{l^1} = \sum_{x\in S} |f(x)|$.

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ denote the set of unital homomorphisms from $\mathcal{A}$ to $\mathbb{R}, \mathbb{C}$. Note that $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$.

Note that we know $(l^1(S))' = l^\infty(S)$, hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. Note that we have $\|\varphi\| = 1$, hence if we $\varphi \in l^\infty(S)$, we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for $z \in S$, $\|z\| \leq 1$, we have $|\varphi(z)| \leq 1$.

Proposition 1.12

We naturally identify $\widehat{l^1(S)}$ with $Hom(S, \mathbb{D})$, i.e. $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$.

Proof Given $f \in \widehat{l^1(S)}$, we know it's multiplicative, unital, hence all these transfer when viewing $\varphi \in l^\infty(S)$. This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here xy denotes the operation on S between x, y, for example, could be $x + y$. Hence naturally, if $\varphi \in \widehat{l^1(S)}$, φ can also be viewed as $\varphi : S \to \mathbb{D}$, and thus is in l^∞, with $|\varphi(s)| \leq 1$.

□

Furthermore, we can identify elements in $\widehat{l^1(S)}$ with the unit disk. Take $S = \mathbb{N}$.

Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Proof We motivate this by noticing $\mathbb{N}$ is generated by 1, and thus viewing $\varphi \in \widehat{l^1(\mathbb{N})}$ as $\varphi \in l^\infty(\mathbb{N})$, we have φ is determined by $\varphi(1)$. And denote $\varphi(1) = z_0$, then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for $z \in \mathbb{D}$,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space.

□

Proposition 1.14

The standard topology on $\mathbb{D}$ coincides with the weak- topology on $\widehat{l^1(\mathbb{N})}$.*

$$D_{std} \cong D_{weak-*}$$

Proof We just need to associate an element in $\mathbb{D}$ with a function $\varphi \in \widehat{l^1(\mathbb{N})}$. And we do this by

$$z \mapsto \sum_{n\in\mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

1.3.1 On groups

We let G denote a discrete commutative group, and we see everything above follows, with one extra property.

Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Proof For $\varphi \in \widehat{l^1(G)}$, we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because $|\varphi(x)| \leq 1, \forall x$, Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have $\widehat{l^1(G)}$ naturally identifies with $\mathbb{T}$. Like what we described above, we have what is desired.

□

Remark Take $G = \mathbb{Z}$, if we denote $z \in \mathbb{T}$ as $z = e^{2\pi it}$, then we naturally identify with

$$\sum_{n\in\mathbb{Z}} f(m)e^{2\pi int}$$

we denote this mapping as $\widehat{f}$, i.e.

$$\widehat{f}(z) = \sum_{m\in\mathbb{Z}} f(m)e^{2\pi int}$$

This is the Fourier transform.

1.4 Lecture 5

Last time, we talked about if we denote $\mathcal{A} = l^1(G)$, equipped with $\|\cdot\|_{l^1}$, under convolution, we have

$$\widehat{\mathcal{A}} \text{ “=” } Hom(G, \mathbb{T})$$

If we take $G = (\mathbb{Q}, +)$, one can ask the question if $\widehat{\mathcal{A}}$ is big enough. And we will se later in the course, the answer is yes.

For pointwise multiplication, $\widehat{G}$ forms a group, and in fact $\widehat{G}$ is a compact topological group.

For any compact commutative group G, for exapmle $\mathbb{R}^n$ under $+$. Define

$$\widehat{G} = \text{ continuous homomorphisms into } \mathbb{T}$$

Remark We now require continuous with this general G (previously was not required for discrete group G).

Proposition 1.16

Let G be a locally compact and commutative group, we have $\widehat{G}$ as a locally compact, commutative group.

We define the pairing between G and $\widehat{G}$ as follows: $x \in G, \varphi \in \widehat{G}$,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the folliwng map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let G, H denote locally compact groups, and $\phi : G \to H$ bet a continuous homomorphism. Note we have the following diagram:

$$G \xrightarrow{\phi} H$$

$$\widehat{G} \xleftarrow{\phi} \widehat{H}$$

If we take an element $\psi \in \widehat{H}$, we consider $\psi \circ \phi$. We get $\psi \circ \phi \in \widehat{G}$.

Definition 1.7 (category, functor)

A category is specified by

1. *a set of objects*
2. *morphisms between objects*
   (a). *X, Y, Z are objects, and if*

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

   (b). *For each object X, there is an identity morphism 1_X.*

And a functor is defined to be such a morphism between categories. ♣

Example 1.2 For category of finite vector spaces V, passing from vector space to its dual V' is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$V \xrightarrow{T} W$$

$$V' \xLeftarrow{T^t} W'$$

$$V'' \xrightarrow{T^{tt}} W''$$

The map going in the same directions $V \to W$, and $V'' \to W''$ is called covariant, whereas $V' \leftarrow W'$ is called contravariant.

Example 1.3 For category of locally compact groups G, H, assigning the dual group is a functor:

$$G \to H$$

$$\widehat{G} \leftarrow \widehat{H}$$

$$\widehat{\widehat{G}} \to \widehat{\widehat{H}}$$

Example 1.4 Now let X be a compact space. Given Φ continuous map between $X \to Y$.

$$X \xrightarrow{\Phi} Y$$

$$C(X) \leftarrow C(\Phi) C(Y)$$

For $f \in C(Y)$, we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$X \xrightarrow{\varphi} \xrightarrow{\phi} Z$$

$$C(X) \xleftarrow{C(\varphi)} C(Y) \xleftarrow{C(\phi)} C(Z)$$

where for $f \in C(Y), C(\varphi)(f) = f \circ \varphi$, and $g \in C(Z), C(\phi) = g \circ \phi$. This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between $C(X)$ and $C(Y)$, there eixsts a map from X to Y.

Proposition 1.17

Suppose X, Y are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism $\check{F} : X \to Y$.

Proof Define $\varphi_x : C(X) \to \mathbb{C}$ as the evalutation map: take $f \in C(X)$,

$$\varphi_x(f) = f(x)$$

Then $\varphi_x \circ F \in \widehat{C(Y)}$. And we know that any element in $\widehat{C(Y)}$ is a point evaluation, i.e. there exists $y \in Y$ such that

$$\varphi_y = \varphi_x \circ F$$

We thus define $\check{F}(x) = y$ as such that it satisfies the above equation. We need to show $\check{F}$ is continuous. Note that X, Y are compact Hausdorff spaces, and the topology on Y is the coarest topology making all functions $g \in C(X)$ continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \\ &= F \circ g(x) \end{aligned}$$

Hence by F, g being continuous, we have $\check{F}$ is also continuous.

$\square$

There is a natural bijection between the continuous functiosn from X to Y, and the unital algebra homomorphism from $C(X)$ to $C(X)$.

A quick reminder:

Remark For X compact, the weak-* topology coincides with the standard topology.