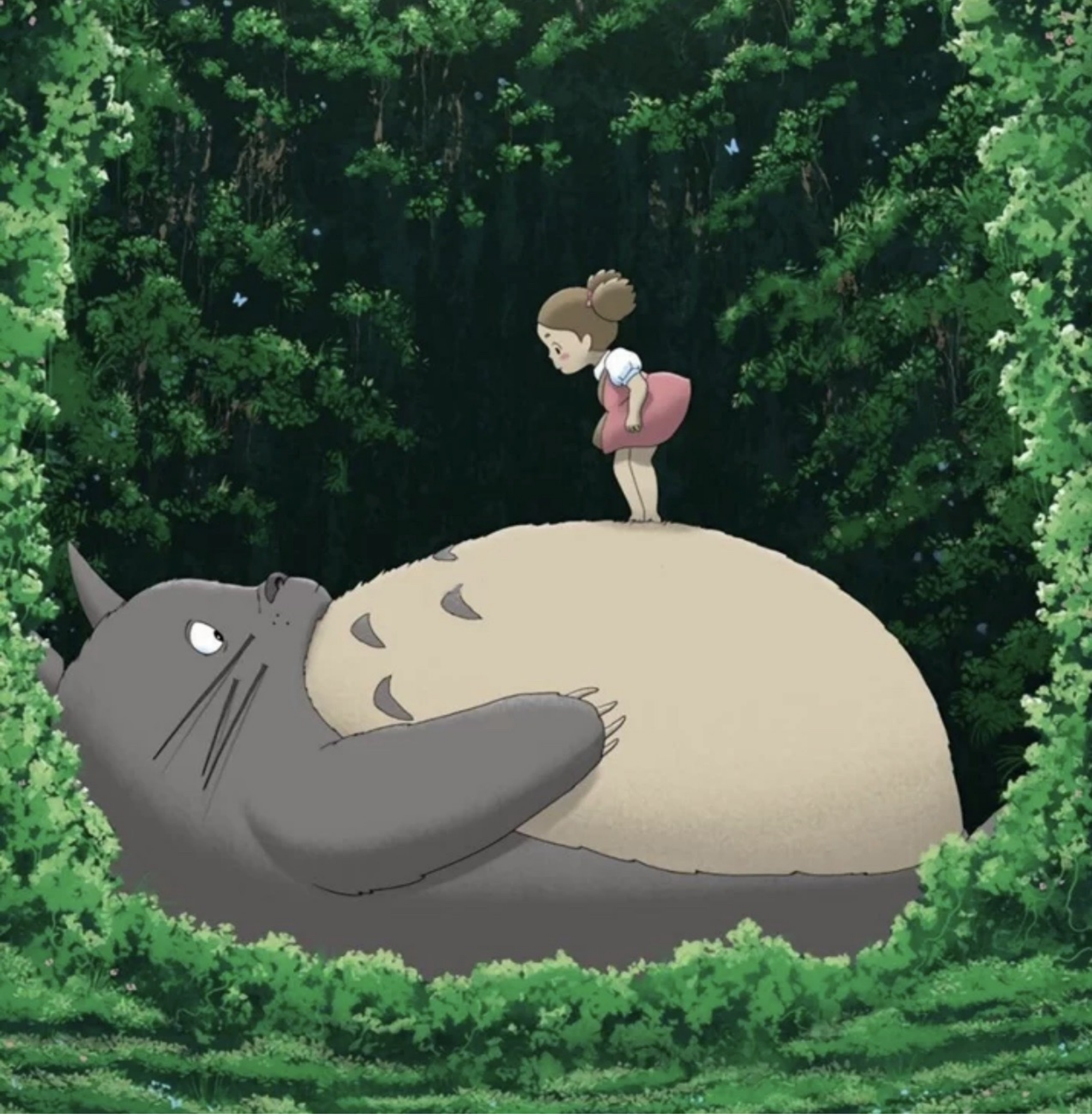

Fourier Analysis

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Chapter 1 Fourier Series and Integrals

We will go through the book's notes in this document. Chapter 1 is organized as follows:

1. Fourier coefficients and Series
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4. summability methods
5. The fourier transform of L^1 functions
6. Schwartz class and tempered distributions
7. Fourier transform on L^p, for $1 < p \leq 2$
8. Convergence and summability of Fourier Integrals
9. Further results

Some Notations The Lebesgue measure in $\mathbb{R}^n$ will be denoted using dx, and on the unit sphere S^{n-1} will be $d\sigma$. Let $a = (a_1, ..., a_n) \in \mathbb{N}^n$ be a multiindex, and $f : \mathbb{R}^n \to \mathbb{C}$, then

$$D^a f = \frac{\partial^{|a|} f}{\partial_{x_1}^{a_1} ... \partial_{x_n}^{a_n}}$$

where $|a| = a_1 + ... + a_n$.

Theorem 1.1 (Minkowski's integral inequality.)

Given $(X, \mu), (Y, \nu)$ as σ-finite measure spaces, we have the following inequality

$$\left(\int_X \left| \int_Y f(x,y) d\nu(y) \right|^p d\mu(x) \right)^{1/p} \leq \int_Y \left(\int_X |f(x,y)^p d\mu(x) \right)^{1/p} d\nu(y)$$

♡

Taking ν to be the counting measure over a two point set $S = 1, 2$ gives the usual Minkowski inequality

$$\|f_1 + f_2\|_{L^p} \leq \|f_!\|_{L^p} + \|f_2\|_{L^p}$$

We will use $\mathcal{D}$ to denote the space of test functions, i.e. C_c^∞, and $\mathcal{S}$ to denote the space of Schwartz functions. Recall the dual of $\mathcal{D}$, denoted as $\mathcal{D}'$ is the space of distrubitons, and $\mathcal{S}'$ is the space of temperate distributions.

Definition 1.1 (Convolution of distribution)

Let $T \in \mathcal{D}'$, and $f \in \mathcal{D}$, then we define

$$T * f(x) = \langle T, \tau_x \tilde{f} \rangle$$

where $\tilde{f}(y) = f(-y)$, and $\tau_x f(y) = f(x+y)$. Hence it can be read as $T * f(x) = \langle T, f(x-y) \rangle$

♣

1.0.1 1.1 Review of definitions

We now do some math. If f is a trigonometric series, of the form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}$$

Then we find c_n for any fixed n by multiplying $f(x)$ by $e^{-2\pi i n x}$, and integrate. Namely, we have

$$\int_0^1 \sum_n c_n e^{2\pi i n x} \cdot e^{-2\pi i n x} = \int_0^1 c_n = c_n$$

We denote the additive group of $\mathbb{R}/\mathbb{Z}$ by $\mathbb{T}$, which gives $[0, 1)$, and naturally identifies with S^1. Hence, saying a function f defined on $\mathbb{T}$ is the same as saying f is defined on $\mathbb{R}$ with period 1.

Definition 1.2 (Fourier coefficients)

Fix $f \in L^1(\mathbb{T})$, we associate the sequence $\{\hat{f}(n)\}$ of f defined by

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi inx}dx$$

And its Fourier series defined as

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}$$

♣

1.0.2 1.2 Critera for pointwise convergence

Define the N-th partial sum in the natural way. For pointwise convergence, the first result is due to Dirichlet, which states that if f is bounded and has a finite number of maxima and minima, and pointwise continuous, then the limit of $S_N(f)(x)$ exists and is equal to $\frac{1}{2}(f(x+) + f(x-))$. As we saw in class, one can express $S_N(f)(x)$ as a convolution

$$S_N(f)(x) = D_N(f) * f(x)$$

where $D_N(t) = \sum_{n=-N}^{N} e^{2\pi int}$. And for any $\delta > 0$, we have

$$\int_0^1 D_n(t)dt = 1, |D_N(t)| \leq \frac{1}{|\sin(\pi t)|}, \delta \leq |t| \leq \frac{1}{2}$$

To prove Dini's and Jordan's criterion, we first do some prep work.

Definition 1.3 (bounded variation)

The total variation of $f : I \to \mathbb{C}$ is defined by

$$V(f, I) = \sup_N \sup_{x_0 \leq \ldots \leq x_N} \sum_{j=1}^{N} |f(x_j) - f(x_{j-1})|$$

And f is of bounded variation if $V(f, I) < \infty$.

♣

Now we introduce the Riemann Localization principle, which states if two functions agree on a small neighborhood of a fixed point x, then their Fourier coefficients also agree at x. If we recall how Fourier coefficients are computed.

$$\hat{f}(n) = \int_0^1 f(x)e^{2\pi inx}$$

If we change f in other places, not in the neighborhood of x, it seems like the integral would change as well. But we will show now, this is not the case.

Theorem 1.2 (Riemann Localization Principle)

If $f = 0$ in $(x - \delta, x + \delta)$, then we have

$$\lim_{N \to \infty} S_N(f)(x) = 0$$

♡

Proof This is purely computational and it is done by writing out $S_N(f)(x)$ as a convolution and realizing $S_N(f)(x)$ is a sum of the Fourier coefficnets of two integral functions, hence its Fourier coefficients decay to 0, making $S_N(f)$ decay to 0 as well.

One ingredient in the proof was the Riemann Lebesgue lemma, which states that the Fourier coefficent of integrable functions decays to 0. This

Lemma 1.1 (Riemann-Lebesgue)

If $f \in L^1(\mathbb{T})$, then we have

$$\lim_{|n| \to \infty} \hat{f}(n) = 0$$

♡

Proof One could either approximate f with continuous functions or simple functions, either would work. If suffices to show $\hat{f}(n) \to 0$ for f continuous, and it's pure arithmatic manipulation, and for f simple functions, we can use integration by parts. Then we would recover that the Fourier transform decays as $|n| \to \infty$.

We now state two local pointwise results, using the theorems we proved above.

Theorem 1.3 (Dini's Criterion)

Let f be such that for a fixed x,

$$\int_{|t|\leq\delta} \left| \frac{f(x+t)-f(x)}{t} \right| dt < \infty$$

Then we have

$$\lim S_N(f)(x) = f(x)$$

Proof We would want to show that

$$S_N(f) - f(x) \to 0$$

This means

$$\begin{aligned} \int_{|t|\leq 1/2} (f(x-t)-f(x)) \frac{\sin((2N+1)\pi t)}{\sin(\pi t)} &= \int_{|t|\leq\delta} (f(x-t)-f(x)) \frac{\sin((2N+1)\pi t)}{\sin(\pi t)} \\ &+ \int_{\delta\leq t\leq 1/2} (f(x-t)-f(x)) \frac{\sin((2N+1)\pi t)}{\sin(\pi t)} \end{aligned}$$

Hence if we write $\sin((2N+1)\pi t)$ as $(e^{(2N+1)\pi t} - e^{-(2N+1)\pi t})/2$ again, we can write the above two integrals as the Fourier coefficient of two integrable functions, and by the Riemann Lebesgue lemma, we conclude as $N \to \infty$, $S_N(f)(x) \to f(x)$.

□

Remark In the proof of Dini's theorem and Riemann Localization principle, we both first write out the partial sum $S_N(f)$ as the convolution again $\frac{\sin((2N+1)\pi t)}{\sin(\pi t)}$, and to finish up, we transformed this back to a Fourier coefficient by expanding out $\sin((2N+1)\pi t)$, and used Riemann Lebesgue lemma.

Theorem 1.4 (Jordan's Criterion)

If f is of bounded variation in a neighborhood of x, then we have

$$\lim_{N\to\infty} S_N(f)(x) = \frac{1}{2}[f(x+) + f(x-)]$$

Proof If f is of bounded varation in $(x-\delta, x+\delta)$, then we have $f = f^+ - f^-$, where both f^+, f^- are monotonic. Hence we can assume f is monotonic in a nbhd of x. (Justification: if we can prove $S_N(f^+) = \frac{1}{2}(f^+(x+) + f^+(x-))$, and similarly for f^-, then we can add up and recover f.)

Assuming f is monotonic, we have

$$S_N(f)(x) = \int_{|t|\leq 1/2} f(x-t) D_N(t) dt = \int_0^{1/2} (f(x-t) + f(x+t)) D_N(t) dt$$

And thus assuming $x = 0$, it suffices to prove

$$\int_0^{1/2} f(t) D_N(t) dt \to \frac{1}{2} f(0^+)$$

Moving terms, and noting $D_n(t) = \frac{\sin((2N+1)\pi t)}{\sin(\pi t)}$ is an even function, we get $\frac{1}{2} f(0^+) = f(0^+) \int_0^{1/2} D_N(t)$. Hence it suffices to show the following integral tends to 0.

$$\int_0^{1/2} (f(t) - f(0^+)) D_N(t) dt = \int_0^{\delta} (f(t) - f(0^+)) D_N(t) dt + \int_{\delta}^{1/2} (f(t) - f(0^+)) D_N(t) dt$$

By the same arguemnt, the second integral tends to 0. To treat the first integral, we apply the second mean value theorem for integrals, which states for continuous φ, and monotonic h, there exists c, $a < c < b$,

$$\int_a^b h\varphi = h(a+) \int \varphi + h(b-) \int \varphi$$

This deals with the bad t around 0, and we are left with good, integrable terms,

$$\int_0^\delta (f(t) - f(0^+))D_N(t)dt = \int_c^\delta (f(t) - f(0^+))D_N(t)dt$$

Again, we have an integrable function, writing it as a Fourier coefficient, we have $\int_0^{1/2} f(t)D_N(t) \to \frac{1}{2}f(0^+)$.

□

1.0.3 1.3 Fourier series of continuous functions