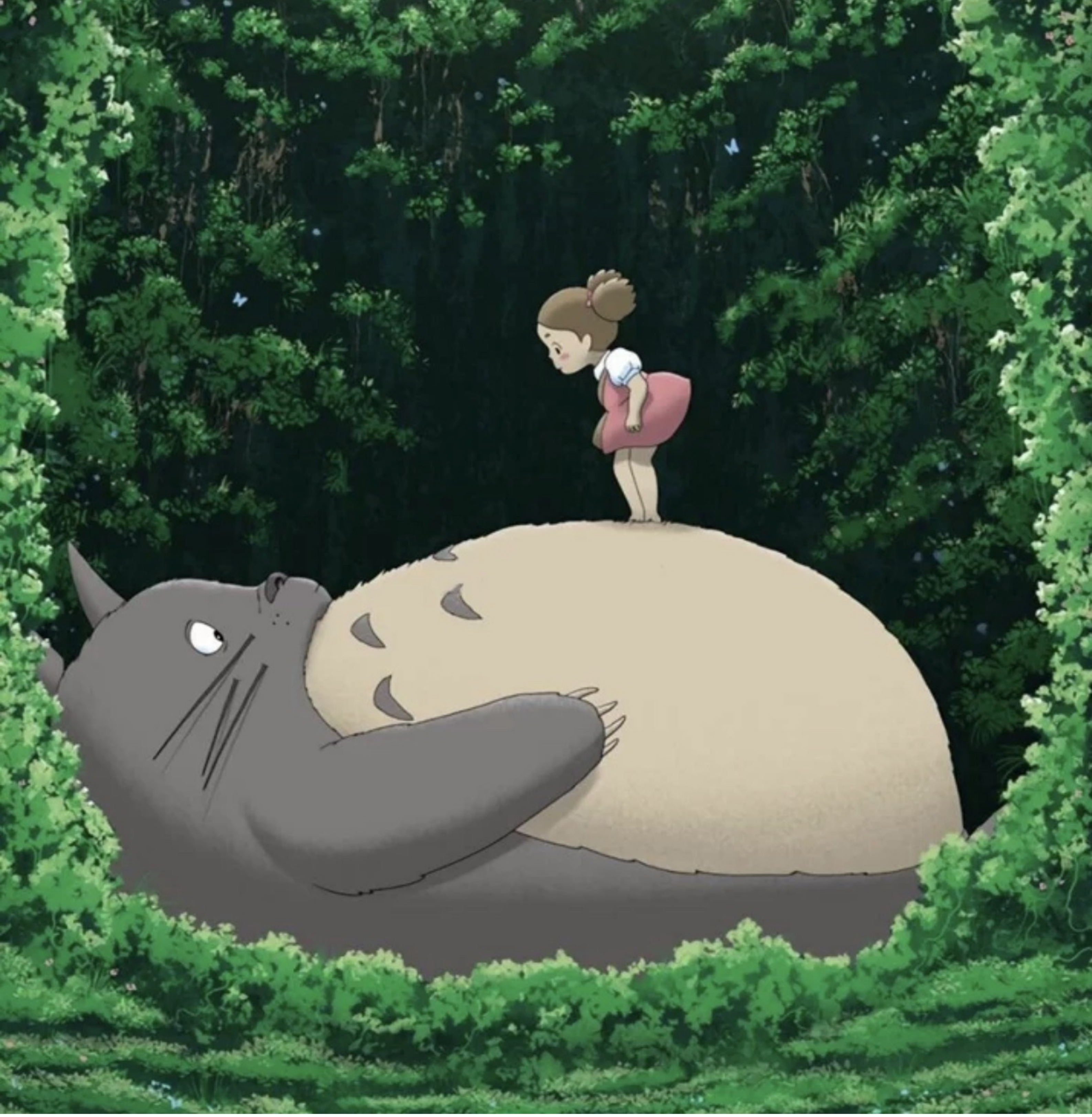

Functional Analysis

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Chapter 1 Prep work

We will start from the beginning and take baby steps. It's going to be okay.

An algebra is a vector space (with addition and scalar multiplication, usually over $\mathbb{R}, \mathbb{C}$), with an extra multiplication operation such that it is associative, and distributive. Then a normed algebra is an algebra with a sub-multiplicative norm, such that for all $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

A Banach algebra is a normed algebra that is complete under the metric induced by the norm. And we can form a Banach algebra by starting with a normed algebra and form its completion and by uniform continuity of addition and multiplicatoin extend to the completion of the algebra to form a Banach algebra.

We will begin with some important examples of Banach algebras. Let X be a compact topological space, and let $C(X)$ be the space of continuous functions, equip it with $\|\cdot\|_{L^\infty}$ norm, then $(C(X), \|\cdot\|_{L^\infty})$ is a Banach algebra. Similarly, if X is only locally compact, then $C_b(X)$, the space of bounded continuous functions under the $\|\cdot\|_{L^\infty}$ norm is also a Banach algebra.

Proposition 1.1

Multiplication is continuous in Banach algebras.

Proof Multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, hence if we have x_n, y_n such that $x_n \to x, y_n \to y$, then we have

$$\|x_n y_n - xy\| \leq \|x_n - x\|\|y_n\| + \|x\|\|y_n - y\| < \epsilon$$

Hence multiplication is continuous.

Definition 1.1 (Unital Banach algebra and invertibility)

A Banach algebra (let's repeat, a complete vector space with addition, scalar multiplicatin, and multiplication such that the norm is sub-multiplicative) is called unital if there exists a multiplicative inverse.
An element $a \in \mathcal{A}$ is called invertible if there exists an element $a^{-1} \in \mathcal{A}$ such that

$$aa^{-1} = a^{-1}a = e$$

Another important example is that let X be a Banach space, and the space of all bounded/continuous operators on X, denoted by $\mathcal{B}(X)$ is a Banach algebra with the operator norm. Any closed subalgebra of $B(X)$ is also Banach.

If X is a Hilbert space, then we also have the operation of taking adjoints, namely $\|T\| = \|T^*\|$.

Definition 1.2

A C^* algebra is a closed subalgebra of the space of bounded (equivalently) functions defined on a Hilbert space, $\mathcal{B}(\mathcal{H})$.

Remark The space of continuous/bdd operators on a Hilbert space, under the operator norm, then closed under the norm topology and taking adjoints of the operators. On wikipedia, C* algebra is defined to be a Banach algebra equipped with an involution that acts like a adjoint.

One of the goals of this course is to develop the following theorem.

Theorem 1.1

*Let $\mathcal{A}$ be a commutative C^*-algebra of $\mathcal{B}(\mathcal{H})$, then $\mathcal{A}$ is isometrically and *-algebraically isomorphic to some $C(X)$, where X is some locally compact space.*

We will mostly follow the lecture and the previous lecture notes.

Definition 1.3 (Algebra homomorphism)

An algebra homomorphism is a homomorphism between two algebras. For example, consider X a compact space, and $C(X)$ the space of continuous functions, hence if we define the evalutation map as follows:

$$\varphi_x(f) = f(x)$$

This is an algebra homomorphism between $C(X)$ and $(\mathbb{C})$. Namely, the homomorphism property is justified as: (under both addition and multiplication)

$$\varphi_x(f+g) = f + g(x) = f(x) + g(x) = \varphi_x(f) + \varphi_x(g)$$
$$\varphi_x(fg) = (fg)(x) = f(x)g(x) = \varphi_x(f)\varphi_x(g)$$

And of course, same thing follows for scalar multiplication. ♣

Remark We need to check all three conditions to make sure such φ preserves the structures between the algebras.

An algebra homomorphism is called unital if if maps the (multiplicative identity) unity to unity. In the above example, a unital homomorphism would be $\varphi(1) = 1$, where the left 1 is the constant 1 function, and the right 1 is the number.

Now we will introduce the proposition that every multiplicative linear functional on $C(X)$. Note we can use algebra homomorphism and multiplicative linear functional synonomously on $C(X)$, hence they entail the same information.

Proposition 1.2

Let φ be a multiplicative linear functional on $C(X)$, i.e. a nontrivial algebra homomorphism, then $\varphi(f) = f(x_0)$ for some $x_0 \in X$. In other words, a multiplicative linear functional always takes this form. ♠

Proof It suffices to show the following lemma:

Lemma 1.1

There exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. ♡

We will first show how the lemma implies $\varphi(f) = f(x_0)$. Consider the function $f - \varphi(f) \cdot 1$, then we know

$$\varphi(f - \varphi(f) \cdot 1) = 0$$

Then there exists x_0 such that $f(x_0) - \varphi(f) = 0$, this gives $\varphi(f) = f(x_0)$.

Now we prove the lemma.

Proof Our claim is that there exists x_0 such that if $\varphi(f) = 0$, then we have $f(x_0) = 0$. Assume the contrary, which states for all x, there exists an f_x such that $\varphi(f_x) = 0$, but $f(x) \neq 0$. We define a nonnegative function $g_x = f_x\overline{f_x}$. And by multiplicativity, we have $\varphi(g_x) = 0$. We now note that because g is continuous, in a small nbd of x, denoted by O_x, we have $g(y) > 0$ for all $y \in O_x$.

Now using compactness, we can write X as a finite union of small neighborhoods $X = \bigcup_{j=1}^{n} O_{x_j}$, and define

$$g = g_{x_1} + ... + g_{x_n}$$

Then for each $y \in X$, $y \in O_{x_j}$ for some j, hence $g(y) > 0$ for all $y \in X$. This implies that g is invertible hence we have

$$\varphi(g \cdot 1/g) = 1$$

This contradicts with the fact that $\varphi(g) = 0$. And we are done.

□

Hence we have the following corollary.

Corollary 1.1

Let X be compact, and $C(X)$ the space of continuous functions, then φ is a multiplicative linear functional (i.e. a algebra homomorphism with $\mathbb{C}$) if and only if it is a point evaluation. ♡

Definition 1.4 ($\widehat{\mathcal{A}}$)

Given a unital commutative (or Banach) algebra, for example, $C(X)$ with $\|\cdot\|_{L^\infty}$, we define the set of unital homomorphisms, i.e., nonzero unital multiplicative linear functionals on $\mathcal{A}$ as $\widehat{\mathcal{A}}$.

Proposition 1.3

If $\mathcal{A}$ is a unital algebra, then for $\varphi \in \widehat{\mathcal{A}}$, we have $\|\varphi\| = 1$

Proof We have

$$\|\varphi\| = \sup\{|\varphi(f)| : \|f\|_{L^\infty} = 1\}$$

Because $|\varphi(f)| = |f(x_0)|$ for some x_0, we always have $\|\varphi\| \leq 1$, but with the unity, we have $|\varphi(e)| = 1$, and taking the sup we have $\|\varphi\| = 1$.