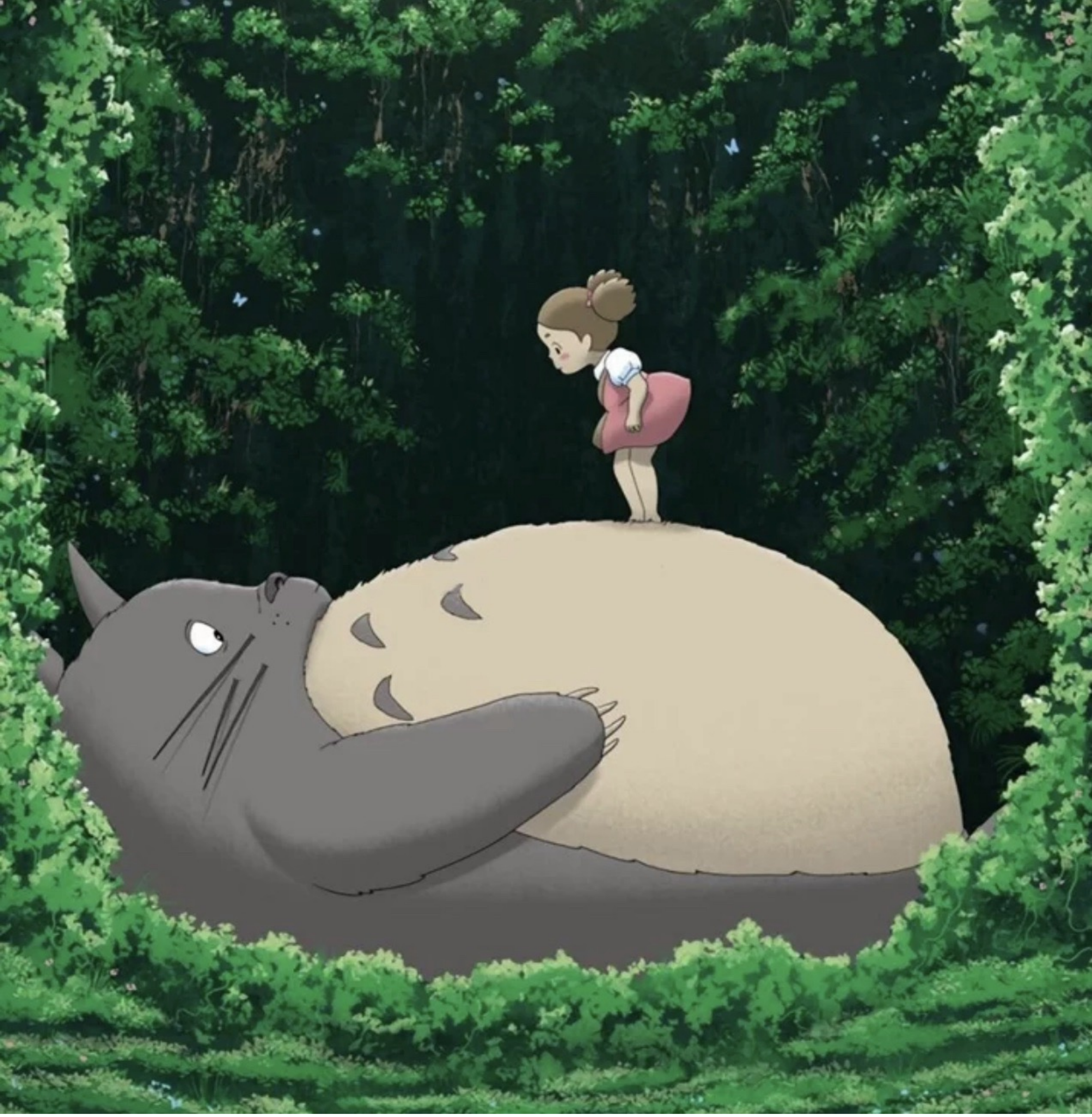

Functional Analysis

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Contents

Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

Remark Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \tag{1.1}$$

♠

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_{\mathcal{H}}) \cong C(X) \tag{1.2}$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on B or an action of the group G on B.

♣

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B, generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B, hence no need to include it.

Elements of the form $\Sigma_{z_x} \alpha_x, z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x\in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many x, $f(x)=0$.

Let $f,g\in C_c(G)$, then for

$$\alpha_f\alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f\alpha_g = \sum_x\sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f*g)(y)\alpha_y$$

where we define $f*g(y)=\sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f\alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\|=1$,

$$\|\alpha_f\| = \|\sum f(x)\alpha_x\| \le \sum |f(x)|\|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f*g\|_{l^1} \le \|f\|_{l^1}\|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x\in G$, let $\xi\in l^2(G)$ define $\alpha_x\xi(y)=\xi(x^{-1}y)$, α_x being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α.

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\|\le\|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X. This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X, by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let $\mathcal{A}$ be an algebra on $\mathbb{R}$ or $\mathbb{C}$, is a normed algebra if it has a norm $\|\|$, as a vector space, such that for for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity. ♣

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm. ♣

Given $x \in X$, define $\varphi_x : C(X) \to \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \to \mathbb{R}$ or $C(X) \to \mathbb{C}$. This simply implies

$$\varphi_x(f+g) = (f+g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$. ♠

Proof Main Claim: given a multiplicative linear functional φ, there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f(x) \neq 0$, but $\varphi(f) = 0$.

Set $g_x = \overline{f}_x f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f}_x) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is $x_1, ..., x_n$ such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + ...g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f, consider $f - \varphi(f) \cdot 1$, apply φ, we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field. ♣

Remark We have $|\varphi(f)| \leq \|\varphi\| \|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

Thss is not always true for normed algebra, Let

$$\mathcal{A} := Poly \subset C([0,1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If $\mathcal{A}$ is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word "unital" is key here.

Proposition 1.4

Let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} =? \sum_{n=0}^{\infty} a^n, a^0 = 1_{\mathcal{A}}$ You could look at the partial sums. $S_m = \sum_{n=0}^{m} a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m\to\infty} S_m = \frac{1}{1-a}$.

To prove this is a cauchy sequence:

$$\|S_n - S_m\| = \|\sum_{j=m+1}^{n} a^j\| \leq \sum_{m+1}^{n} \|a^j\| \leq \sum_{m+1}^{n} \|a\|^j$$

And the fact that $\|a\| \leq 1$, we have the sum bounded by ϵ, hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n\to\infty} S_n(1-a) = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = \lim_{n\to\infty}(1 - a^{n+1}) = 0$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$.

1.2 Lecture 3

We now begin.

Let $\mathcal{A}$ be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Newmann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1 - a\| < 1$, then a is invertible.

Proof $a = 1 - (1 - a)$.

Proposition 1.5

The set of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$.

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of $\mathcal{A}$ onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d, homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in Range(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.

Proof Let $f \in C(X)$, and if $\alpha \in$ range of f, so $\alpha = f(x_0)$ for some x_0. then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(-\alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in$ range of f.

□

Definition 1.5 (spectrum of an element)

For any unital algebra $\mathcal{A}$ over some field $\mathbb{F}$, for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible } \}$$

is called the spectrum of a, denoted as $\sigma(a)$.

Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$

Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda(1 - \frac{a}{\lambda})$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\alpha \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on $\mathcal{A}$, i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of $\mathbb{R}, \mathbb{C}$.

Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of $\mathcal{A}$ is open, hence

$$\sigma(a) = \phi^{-1}(\text{ noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{ invertible })$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of $\mathcal{A}'$, which denotes the dual vector space of continuous linear transformations.

On $\mathcal{A}'$, we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak- topology.*

Proof let $\{\varphi_\lambda\}$ be a net of elemnts of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \to \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V, the closed unit ball of V' is compact in the weak- topology.*

As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak- toplogy.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto $\mathcal{A}$.

For $\mathcal{A}$ unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ into $C(\mathcal{A})$.

Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{a}\widehat{b})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

□

1.3 Lecture 4

Today we talk about the structure of $\widehat{l^1(S)}, \widehat{l^1(G)}$, where S, G are semigroups and groups, and how they naturally identify with the unit disk $\mathbb{D}$, and the unit circle $\mathbb{T}$.

Let S be a commutative discrete semigroups, for example $\mathbb{N} \cup \{0\}$, and $f \in C_c(S)$, then we can write $f = \sum_{x \in S} f(x)\delta_x$, where we define $\delta_x\delta_y = \delta_{xy}$. Note that $C_c(S)$ is dense in $l^1(S)$.

Definition 1.6 (Convolution)

Take any $f, g \in C_c(S)$, we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{x \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left(\sum_{xy = z} f(x)g(y) \right) \delta_z$$

where we define the convolution between two functions

$$f * g(z) = \sum_{x,y,xy=z} f(x)g(y)$$

And under this convolution operation, we have $l^1(S), *$ as a Banach algebra.

Example 1.1 If we consider polynomials of the form $f(x) = \sum_{n=0}^{\infty} f(n)x^n$, and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right)\left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f*g)(p)$$

And let $f \in C_c(S)$, where $S = \mathbb{N}$. we define $\|f\|_{l^1} = \sum_{x\in S} |f(x)|$.

It is easy to check we have

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ denote the set of unital homomorphisms from $\mathcal{A}$ to $\mathbb{R}, \mathbb{C}$. Note that $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$. Note that we know $(l^1(S))' = l^\infty(S)$, hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. Note that we have $\|\varphi\| = 1$, hence if we $\varphi \in l^\infty(S)$, we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for $z \in S$, $\|z\| \leq 1$, we have $|\varphi(z)| \leq 1$.

Proposition 1.12

We naturally identify $\widehat{l^1(S)}$ with $Hom(S, \mathbb{D})$, i.e. $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$.

Proof Given $f \in \widehat{l^1(S)}$, we know it's multiplicative, unital, hence all these transfer when viewing $\varphi \in l^\infty(S)$. This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here xy denotes the operation on S between x, y, for example, could be $x + y$. Hence naturally, if $\varphi \in \widehat{l^1(S)}$, φ can also be viewed as $\varphi : S \to \mathbb{D}$, and thus is in l^∞, with $|\varphi(s)| \leq 1$.

□

Furthermore, we can identify elements in $\widehat{l^1(S)}$ with the unit disk. Take $S = \mathbb{N}$.

Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Proof We motivate this by noticing $\mathbb{N}$ is generated by 1, and thus viewing $\varphi \in \widehat{l^1(\mathbb{N})}$ as $\varphi \in l^\infty(\mathbb{N})$, we have φ is determined by $\varphi(1)$. And denote $\varphi(1) = z_0$, then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for $z \in \mathbb{D}$,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space.

□

Proposition 1.14

The standard topology on $\mathbb{D}$ coincides with the weak- topology on $\widehat{l^1(\mathbb{N})}$.*

$$D_{std} \cong D_{weak-*}$$

Proof We just need to associate an element in $\mathbb{D}$ with a function $\varphi \in \widehat{l^1(\mathbb{N})}$. And we do this by

$$z \mapsto \sum_{n \in \mathbb{N}} f(n)x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

1.3.1 On groups

We let G denote a discrete commutative group, and we see everything above follows, with one extra property.

Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where $\mathbb{T}$ *denotes the unit circle* $\{z \in \mathbb{C} : |z| = 1\}$.

Proof For $\varphi \in \widehat{l^1(G)}$, we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because $|\varphi(x)| \leq 1, \forall x$, Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have $\widehat{l^1(G)}$ naturally identifies with $\mathbb{T}$. Like what we described above, we have what is desired.

□

Remark Take $G = \mathbb{Z}$, if we denote $z \in \mathbb{T}$ as $z = e^{2\pi i t}$, then we naturally identify with

$$\sum_{n \in \mathbb{Z}} f(m)e^{2\pi i n t}$$

we denote this mapping as $\widehat{f}$, i.e.

$$\widehat{f}(z) = \sum_{m \in \mathbb{Z}} f(m)e^{2\pi i n t}$$

This is the Fourier transform.

1.4 Lecture 5

Last time, we talked about if we denote $\mathcal{A} = l^1(G)$, equipped with $\|\cdot\|_{l^1}$, under convolution, we have

$$\widehat{\mathcal{A}} \text{“=”} Hom(G, \mathbb{T})$$

If we take $G = (\mathbb{Q}, +)$, one can ask the question if $\widehat{\mathcal{A}}$ is big enough. And we will se later in the course, the answer is yes.

For pointwise multiplication, $\widehat{G}$ forms a group, and in fact $\widehat{G}$ is a compact topological group.

For any compact commutative group G, for exapmle $\mathbb{R}^n$ under $+$. Define

$$\widehat{G} = \text{ continuous homomorphisms into } \mathbb{T}$$

Remark We now require continuous with this general G (previously was not required for discrete group G).

Proposition 1.16

Let G *be a locally compact and commutative group, we have* $\widehat{G}$ *as a locally compact, commutative group.*

We define the pairing between G and $\widehat{G}$ as follows: $x \in G, \varphi \in \widehat{G}$,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the folliwng map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let G, H denote locally compact groups, and $\phi : G \to H$ bet a continuous homomorphism. Note we have the following diagram:

$$G \xrightarrow{\phi} H$$

$$\widehat{G} \xleftarrow{\phi} \widehat{H}$$

If we take an element $\psi \in \widehat{H}$, we consider $\psi \circ \phi$. We get $\psi \circ \phi \in \widehat{G}$.

Definition 1.7 (category, functor)

A category is specified by

1. *a set of objects*
2. *morphisms between objects*
   (a). *X, Y, Z are objects, and if*

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

   (b). *For each object X, there is an identity morphism 1_X.*

And a functor is defined to be such a morphism between categories. ♣

Example 1.2 For category of finite vector spaces V, passing from vector space to its dual V' is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$V \xrightarrow{T} W$$

$$V' \xLeftarrow{T^t} W'$$

$$V'' \xrightarrow{T^{tt}} W''$$

The map going in the same directions $V \to W$, and $V'' \to W''$ is called covariant, whereas $V' \leftarrow W'$ is called contravariant.

Example 1.3 For category of locally compact groups G, H, assigning the dual group is a functor:

$$G \to H$$

$$\widehat{G} \leftarrow \widehat{H}$$

$$\widehat{\widehat{G}} \to \widehat{\widehat{H}}$$

Example 1.4 Now let X be a compact space. Given Φ continuous map between $X \to Y$.

$$X \xrightarrow{\Phi} Y$$

$$C(X) \leftarrow C(\Phi) C(Y)$$

For $f \in C(Y)$, we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$X \xrightarrow{\varphi} \xrightarrow{\phi} Z$$

$$C(X) \xleftarrow{C(\varphi)} C(Y) \xleftarrow{C(\phi)} C(Z)$$

where for $f \in C(Y), C(\varphi)(f) = f \circ \varphi$, and $g \in C(Z), C(\phi) = g \circ \phi$. This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between $C(X)$ and $C(Y)$, there eixsts a map from X to Y.

Proposition 1.17

Suppose X, Y are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism $\check{F} : X \to Y$.

Proof Define $\varphi_x : C(X) \to \mathbb{C}$ as the evalutation map: take $f \in C(X)$,

$$\varphi_x(f) = f(x)$$

Then $\varphi_x \circ F \in \widehat{C(Y)}$. And we know that any element in $\widehat{C(Y)}$ is a point evaluation, i.e. there exists $y \in Y$ such that

$$\varphi_y = \varphi_x \circ F$$

We thus define $\check{F}(x) = y$ as such that it satisfies the above equation. We need to show $\check{F}$ is continuous. Note that X, Y are compact Hausdorff spaces, and the topology on Y is the coarest topology making all functions $g \in C(Y)$ continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by F, g being continuous, we have $\check{F}$ is also continuous.

□

There is a natural bijection between the continuous functiosn from X to Y, and the unital algebra homomorphism from $C(X)$ to $C(Y)$.

A quick reminder:

Remark For X compact, the weak-* topology coincides with the standard topology.

1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let $\mathcal{A}$ be a unital Banach algebra.

We write $GL_n(\mathcal{A})$ to denote the general linear group, the group formed by $n \times n$ matrices with entries from $\mathcal{A}$.

The less standard notation is $GL_I(\mathcal{A})$ is the group of invertible elements in $\mathcal{A}$. As we have shown previously, this is a closed subset of $\mathcal{A}$. This is the notation that we will use.

Remark It is easy to see that the product is jointly continuous.

Proposition 1.18

The following map is continuous.

$$a \mapsto a^{-1}$$

Proof Given $\|a - b\| < \delta$, we would like to show $\|a^{-1} - b^{-1}\| < \epsilon$. We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\| \|b - a\| \|b^{-1}\|$$

Take $\delta = \epsilon / \|a^{-1}\| \|b^{-1}\|$ would suffice.

□

Proposition 1.19

Fix $a \in GL(\mathcal{A})$, there exists a neighborhood O of a and a constant K such that for all $y \in O$, we have

$$\|c^{-1}\| < K$$

Proof Let $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$, then d is invertible and

$$d^{-1} = \sum_{n=0}^{\infty}(1-d)^n$$

We thus have

$$\|d^{-1}\| \le \frac{1}{1 - \|1-d\|} \le \frac{1}{1 - 1/2} = 2$$

We then identify what our O should be. Let $O = aV$, then we want to show that every ad has an inverse with bounded norm. Because a, d are both invertible, ad is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \le \|d^{-1}\|\|a^{-1}\| \le 2\|a^{-1}\|$$

□

Remark For each invertible element, we can find a neighborhood of invertible elements around it, and using that $(1-d)$ is bounded, then d is invertible, we can bound $\|d^{-1}\|$.

Definition 1.8

Fix $a \in \mathcal{A}$, the resolvent set of $\mathcal{A}$ is the complement of spectrum of $\mathcal{A}$, i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible }\}$$

Hence the resolvent set is an open, unbounded suset of $\mathbb{C}$ or $\mathbb{R}$.

Definition 1.9 (Resolvent function)

On the resolvent set, $\{\lambda \in \mathbb{F} : a - \lambda 1 \text{ is invertible }\}$ is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that a is fixed, and λ is the variable here.

Now we note that this $R_a(\lambda)$ function is nicely behaved.

Proposition 1.20

The resolvent function $R_a(z)$ is analytic on the resolvent set, and vanishes as $z \to \infty$.

Proof We first define the notation of analyticity on an open subset of $\mathbb{R}, \mathbb{C}$: this means for every point in the open set O, we can find a power series expansion of the function such that its radius of convergence > 0.

Fix z_0 in the resolvent set. We know $z_0 1_{\mathcal{A}} - a$ is invertible. We consider $(z1_{\mathcal{A}} - a)$, for z in the resolvent set. We will omit the $1_{\mathcal{A}}$ for simplicity.

$$z1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a)\left(1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a}\right)$$

We know the the latter term is invertible if $\|\frac{z_0-z}{z_0-a}\| < 1$ has norm, hence we have

$$(z-a)^{-1} = \sum_{n=0}^{\infty}\left(\frac{z_0 - z}{z_0 - a}^n\right)(z_0 - a)^{-1}$$

What happens when we let $z \to \infty$, we consider $R_a(1/z)$, and let $z \to 0$. Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1 - az}{z}\right)^{-1} = z(1 - az)^{-1}$$

Let $z \to 0$ makes $R_a(1/z)$ go to zero.

□

Now given that $R_a(z)$ is analytic and bounded at ∞, we can state the following important theorem.

Theorem 1.2 (Nonemptyness of spectrum)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, then for any $a \in \mathcal{A}$, we have $\sigma(a) \neq \emptyset$.

Proof Assume there exists $a \in \mathcal{A}$, such that $\sigma(a) = \emptyset$. If $\mathcal{A} = \mathcal{C}$, then we would have $R_a(\lambda)$ be a bounded entire, complex-valued function defined on all of $\mathbb{C}$. By Liouville's theorem, we must have $R_a(z)$ a constant function, but we know $z \to \infty$, $R_a \to 0$, hence $R_a(z)$ is constantly 0, but this cannot be true.

If our $\mathcal{A}$ is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where ϕ is some nonzero element in $\mathcal{A}'$, guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at ∞. Again, the function is constantly 0.

With the nonemptyness of spectrum theorem, we now state the Gelfand-Mazur theorem.

Theorem 1.3 (Gelfand-Mazur)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, if any nonzero element of $\mathcal{A}$ is invertible, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

Proof For any $a \in \mathcal{A}$, we know $\sigma(a) \neq \emptyset$, hence there exists λ such that $\lambda 1_{\mathcal{A}} - a$ is invertible, i.e. $a = \lambda 1_{\mathcal{A}}$, hence establishing an isomorphism between $\mathcal{A}$ and $\mathbb{C}$. In other words, $\mathcal{A} = \mathbb{C}1_{\mathcal{A}}$.

□

1.5.1 Functional Calculus

Proposition 1.21

Let $a \in \mathcal{A}$, then if $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges for $|z| < r$, where $r > \|a\|$, then $\sum_{n=0}^{\infty} \alpha_n a^n$ converges as well.

We first start with proving the following statement.

Lemma 1.1

Let f be a polynomial, $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, $f = \sum_{n=0}^{k} a_n x^n$, then for $a \in \mathcal{A}$, we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of a under f is exactly the spectrum of f evaluated at a.

Proof ($\Leftarrow$). We take $\lambda \in \sigma(a)$, and we would like to show $f(\lambda)$ is in the spectrum of $f(a)$. We note that if $\lambda \in \sigma(a)$, then $a = \lambda 1_{\mathcal{A}}$, and $f(\lambda 1_{\mathcal{A}}) = f(a)$, hence by definition, $f(a) - f(\lambda)1_{\mathcal{A}}$ is not invertible implying $f(\lambda)$ is in the spectrum of $f(a)$. Note that this also implies $f(a) - f(\lambda) = (a - \lambda)Q(z)$ for some polynomial $Q(z)$.

($\Rightarrow$). We take $\lambda \in \sigma(f(a))$, i.e. $f(a) = \lambda 1_{\mathcal{A}}$. we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. If f is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1)...(z - c_n)$$

Plugging in a we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}})...(a - c_n 1_{\mathcal{A}})$$

If $f(a) - \lambda$ is not invertible, then there exists j such that $(a - c_j 1_{\mathcal{A}})$ is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. In fact, we have $\lambda = f(c_j)$ by knowing $f(c_j) - \lambda = 0$.

□

Now let $f(z) = z^n$, and if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$

This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

Definition 1.10

Fix $a \in \mathcal{A}$, we define the spectral radius of a, denoted by $r(a)$,

$$r(a) = \sup_\lambda \{|\lambda| : \lambda \in \sigma(a)\}$$

♣

Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$

♡

Proof From the previous remark that $|\lambda| \leq \|a^n\|^{1/n}$, hence this follows.

1.6 Lecture 7

I have not typed up for this?

1.7 Lecture 8

Let $\mathcal{A}$ be a unital Banach algebra. Then for $a \in \mathcal{A}$, and we look at the resolvent of a, $R_a(\lambda)$, we've noted that as $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty} R_a(\lambda) = \lim_{\lambda\to\infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda\to\infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for $|\lambda| \geq \|a\|$.

Recall that we define the spectral raidus, $r(a)$, as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

Proposition 1.22

$$r(a) = \lim \|a^n\|^{1/n}$$

♠

Proof

If we let $\lambda = 1/z$, then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for $|z| \leq \|a\|^{-1}$, but maybe?? also for $|z| < r(a)^{-1}$?

For $r > r(a)$, i.e. $|z| \leq r^{-1}$, we know $\sum_n a^n r^n$ converges for $r > r(a)$. know $z \sum a^n z^n$ converges absolutely. In particular,

$$a^n z^n \to 0$$

Hence there exists M such that for $n \geq M$, we have

$$\|a^n r^{-n}\| \leq 1$$

This implies that

$$\|a^n\| \leq r^n \Rightarrow \|a^n\|^{1/n} \leq r$$

for all $n \geq M$.

This implies that

$$\limsup \|a^n\|^{1/n} \leq r$$

And note that r is arbitrary close to the spectral radius $r(a)$. Hence we have

$$\limsup \|a^n\|^{1/n} \leq r(a) \leq \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

$\square$

For each $\varphi \in \mathcal{A}'$, consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n)\lambda^{-n}$$

This series converges for $r > r(a)$. We can apply the same process, to argue that there exists M_φ such that

$$\|\varphi(a^n)r^{-n}\| \leq M_\varphi$$

for all $n \geq 0$. Note that M_φ could be different for all φ.

Note that

$$\mathcal{A} \to \mathcal{A}' \to \mathcal{A}''$$

there is a natural injection of $a \mapsto \widehat{a} \in \mathcal{A}''$.

For each n, definite $F_n \in \mathcal{A}''$, by $F_n(\varphi) = |\varphi(a^n r^{-n})| \leq M_\varphi$. Applying the UBP, we have

$$|F_n(\varphi)| \leq M \Rightarrow |\varphi(a^n)r^{-n}| \leq M$$

This implies that

$$|\varphi(a^n)| \leq r^n M$$

Note that by Hahn-Banach, for any $b \in \mathcal{A}$, we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking n-th root of both sides, we gets

$$\|a^n\| \leq r^n M \Rightarrow \|a^n\|^{1/n} \leq rM^{1/n} \to r$$

Hence we again obtain the same result.

$\square$

Recall UBP.

Theorem 1.4 (Uniform Boudnedness Principle)

Let X be Banach, and Y be normed, let $T_n : X \to Y$ be a family of linear operators, and if for all $x \in X$, we have

$$\|T_n(x)\| < \infty$$

Then for all n, we have

$$\|T_n\| < \infty$$

♡

Note that if $\mathcal{A}$ is unital, and if $\mathcal{A} \subset \mathcal{B}$ with some unit. For $a \in \mathcal{A}$, if a is not invertible in A, then it might be invertible in $\mathcal{B}$. Hence if we use $\sigma_{\mathcal{A}}(a)$ to denote the spectrum of a in $\mathcal{A}$.

Proposition 1.23

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

♠

Example 1.5 Let $\mathcal{B} = l^1(\mathbb{Z})$, and let $\mathcal{A} = l^1(\mathbb{N})$, equipped with convolution.

Clearly $\mathcal{A} \subset \mathcal{B}$. And note that the delta function at 1, δ_1 is not invertible in $\mathcal{A}$ but it has an inverse δ_{-1} in $\mathcal{B}$. Hence we see $0 \in \sigma_{\mathcal{A}}(a)$, but $0 \notin \sigma_{\mathcal{B}}(a)$.

Proposition 1.24 (Spectral radius is preserved)

For $\mathcal{A} \subset \mathcal{B}$, we have

$$r_{\mathcal{A}}(a) = \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

Proposition 1.25

Let X be compact, and let $\mathcal{A} = C(X)$. Then for $f \in C(X)$, we have

$$\|f^2\|_\infty = \|f\|_\infty^2$$

Proof Look at where f takes $\|f\|_\infty$, and sqaure it.

Remark The same property holds for f in any unitla subalgebra of $C(X)$, for example, if $X \subset \mathbb{C}$, and let $\mathcal{A}$=functions that are holomorphic on an open subset of $\mathbb{C}$ that are in X.

Let $\mathcal{A}$ be a unital Banach alagebra with the property such that for any $a \in \mathcal{A}$, we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any n, we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking $1/2^n$-root of both sides, we get that the spectral radius of $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

Let $\mathcal{H}$ be a Hilbert space, over $\mathbb{C}$, and let $\mathcal{A} = B(H)$, i.e. the bounded lineaer operators on $\mathcal{H}$., and equip with the operation of taking adjoint. $T \mapsto T^*$.

Proposition 1.26

For any $T \in B(\mathcal{H})$, we have

$$\|T^*T\| = \|T\|^2$$

Proof We know that $\|T^*\| = \|T\|$. And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let $\xi \in \mathcal{H}$, then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi \rangle = \langle \xi, T^*T\xi \rangle \leq \|T^*T\|\|\xi\|^2$$

where the last inequality follows form Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2}\|\xi\|$$

which by definition, gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result. □

Corollary 1.4

If $T^* = T$, then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements.

Note that for general T, we have

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Because T^*T is self-adjoint. Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the *-algebra structure.