Calc III Final Review

Fall 2025

(This document only contains materials after the midterm;
please email hsun95@jh.edu if you see typos)

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Contents

Chapter 1

Definition Review

Definition 1.1 (acceleration)**.** Let $c(t)$ be a path, the **acceleration** $a(t)$ of $c(t)$ is

$$a(t) = c''(t)$$

Definition 1.2 (arc length)**.** Let $c(t) = (x(t), y(t), z(t))$ be a path, then the length of the path in $\mathbb{R}^3$ from $t_0 \leq t \leq t_1$ is

$$\begin{aligned} L_{t_0 \to t_1}(c) &= \int_{t_0}^{t_1} \left(x'(t)^2 + y'(t)^2 + z'(t)^2 \right)^{\frac{1}{2}} dt \\ &= \int_{t_0}^{t_1} \|c'(t)\| dt \end{aligned}$$

More generally, if $c(t) = (x_1(t), \ldots, x_n(t))$ is a path in $\mathbb{R}^n$, then

$$L_{t_0 \to t_1}(c) = \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} x_i'(t)^2 \right)^{\frac{1}{2}} dt$$

Definition 1.3 (vector field)**.** A vector field is a function $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns $x \in \mathbb{R}^n$ to another vector $F(x) \in \mathbb{R}^n$.

Definition 1.4 (flow line)**.** If F is a vector field, a **flow line** for F is a path $c(t)$ such that

$$c'(t) = F(c(t))$$

Intuitively speaking, flow lines are the “streamlines” threading through vector fields.

Definition 1.5 (divergence)**.** Let F be a vector field in $\mathbb{R}^3$ $F = (F_1, F_2, F_3)$, the divergence of F is the **scalar field** (assigns one number to an given point (x, y, z)),

$$\operatorname{div} F \coloneqq \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

More generally, if $F = (F_1, \dots, F_n)$ is a vector field on $\mathbb{R}^n$, its divergence is

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Definition 1.6 (curl)**.** Let F be a vector field in $\mathbb{R}^3$, writing $F = (F_1, F_2, F_3)$, the **curl** of F is the vector field

$$\operatorname{curl} F \coloneqq \nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

If $\operatorname{curl} F = 0$, then we say the vector field is **irrotational**.

Definition 1.7. We say a region $D \subset \mathbb{R}^2$ is ***y*-simple** if there are continuous functions ϕ_1, ϕ_2 such that D is the set of pionts (x, y) satisfying

$$x \in [a, b], \quad \phi_1(x) \le y \le \phi_2(x)$$

Similarly, we define D to be ***x*-simple**if there are continuous ψ_1, ψ_2 such that D is the set of points (x, y) satisfying

$$y \in [c, d], \quad \psi_1(y) \le x \le \psi_2(y)$$

A **simple** region is one that is both x- and y-simple.

Definition 1.8 (injective, surjective)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map, we say T is **injective** or **one-to-one** on D^*, if for $x, y \in D^*$

$$Tx = Ty$$

implies

$$x = y.$$

We say T is **surjective**, or **onto** D, if for all $y \in D$, there exists x in the domain of T such that

$$Tx = y$$

If T is both injective and surjective, then we say T is **bijective**.

Definition 1.9 (Jacobian Determinant)**.** Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be of C^1 defined by

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$$

The **Jacobian determinant** of T, denoted as $\frac{\partial(x,y)}{\partial(u,v)}$ is the determinant of the matrix $DT(u, v)$:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Definition 1.10 (path integral)**.** Let $c : [a, b] \to \mathbb{R}^3$ be a path of C^1 and $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $f \circ c$ is continuous on $[a, b]$, The **path integral** of $f(x, y, z)$ along the path c is given by

$$\begin{aligned}\int_c f ds &= \int_a^b f(c(t))\|c'(t)\|dt \\ &= \int_a^b f(x(t), y(t), z(t))\|c'(t)\|dt\end{aligned}$$

Definition 1.11 (line integral)**.** Let F be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $c : [a, b] \to \mathbb{R}^3$, where $c(t) = (x(t), y(t), z(t))$. We define $\int_c F \cdot ds$, the **line integral** of F along c by the following

$$\begin{aligned}\int_c F \cdot ds &= \int_a^b F(c(t)) \cdot c'(t)dt \\ &= \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt \\ &:= \int_c F_1 dx + F_2 dy + F_3 dz\end{aligned}$$

the expression $F_1dx + F_2dy + F_3dz$ is called the **differential form**.

For example, the work done by a force field F on a particle moving along a path c is given by

$$\text{work done by } F = \int_a^b F(c(t)) \cdot c'(t)dt$$

Definition 1.12 (reparametrization)**.** Let $h : I \to I_1$ be a C^1 real-valued bijective function. Let $c : I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition

$$p = c \circ h : I \to \mathbb{R}^3$$

a **reparametrization** of c.

For example, let $c : [0, 1] \to \mathbb{R}^3$ be a C^1 path, then consider $h : [0, 1] \to [0, 1]$, where $h(t) = 1 - t$. Then the path

$$c_{\text{op}} = c \circ h(t) = c(1 - t)$$

is the same path in the opposite direction.

Definition 1.13 (parametrization of surface)**.** Let S be a surface in $\mathbb{R}^3$, a **surface parametrization** is a map $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

Definition 1.14 (regular surface, tangent plane)**.** Let $\Phi(u, v)$ be a parametrization of a surface $S \subset \mathbb{R}^3$. We say S is **regular** at $\Phi(u_0, v_0)$ if

$$T_u \times T_v \neq 0 \text{ at } (u_0, v_0)$$

where

$$T_u = \frac{\partial \Phi}{\partial u}, \quad T_v = \frac{\partial \Phi}{\partial v}$$

If S is regular at $\Phi(u_0, v_0)$, then we can find the tangent plane by first finding a normal vector to the surface at this point: $n = T_u \times T_v$, then the tangent plane at $(x_0, y_0, z_0) = \Phi(u_0, v_0)$ is given by

$$(x - x_0, y - y_0, z - z_0) \cdot n = 0$$

Definition 1.15 (surface area)**.** Let $S \subset \mathbb{R}^3$ be a parametrized surface, then the **surface area** $A(S)$ of S is given by

$$\begin{aligned} A(S) &= \iint_D \|T_u \times T_v\| dudv \\ &= \iint_D \left(\left[\frac{\partial(x, y)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)} \right]^2 + \left[\frac{\partial(x, z)}{\partial(u, v)} \right]^2 \right)^{1/2} dudv \end{aligned}$$

where $\|T_u \times T_v\|$ is the norm of $T_u \times T_v$, and

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \quad \frac{\partial(y, z)}{\partial(u, v)} = \det \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}, \quad \frac{\partial(x, z)}{\partial(u, v)} = \det \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

Definition 1.16 (integral over a surface)**.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous, i.e., f is a scalar-valued continuous function defined on a parametrized surface S by $\Phi : D \to S \subset \mathbb{R}^3$, we define the integral of f over S as

$$\iint_S f dS = \iint_D f(\Phi(u, v)) \|T_u \times T_v\| dudv$$

A special case is when we take S as the graph of some function $g(x, y)$. Then we have

$$\iint f dS = \iint_D \frac{f(x, y, g(x, y))}{\cos \theta} dxdy$$

where θ is the angle between the unit vector k at $(x, y, g(x, y))$ and the normal vector to the surface. (Recall that the normal vector of a graph is given by $n = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} j + k$).

Definition 1.17 (surface integral of vector fields)**.** Let F be a vector field defined on S, parametrized by Φ. The surface integral of F over $\Phi : D \to \mathbb{R}^3$, denoted by

$$\iint_\Phi F \cdot dS$$

is defined by

$$\iint_\Phi F \cdot dS = \iint_D F \cdot (T_u \times T_v) dudv$$

Definition 1.18 (oriented surface)**.** An oriented surface is a two-sided surface with one side as the **outside (positive)** and one side as the **inside (negative)**. Let $\Phi : D \to \mathbb{R}^3$ be a parametrization of an oriented surface S, then the parametrization Φ is said to be orientation-preserving if

$$\frac{T_u \times T_v}{\|T_u \times T_v\|} = n(\Phi(u, v))$$

at all $(u, v) \in D$ for which S is smooth at $\Phi(u, v)$, where $n(\Phi(u, v))$ is the unit normal vector to S at (u, v) pointing away from the positive side of S (n is given).

Chapter 2

Theorem Review

Proposition 2.1. Let f be constrained to a surface S, if f has a max or a min at x_0, then $\nabla f(x_0)$ is perpendicular to S at x_0.

Proposition 2.2 (Lagrange). Suppose that $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ are C^1 functions. Let $x_0 \in U$ and $g(x_0) = c$, and let S be the level set for g at c, i.e., $S = \{x : g(x) = c\}$. Assume $\nabla g(x_0) \neq 0$, then if f has a local maximum or minimum on S at x_0, then there exists some real number λ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0) \tag{2.1}$$

Proposition 2.3 (Bordered Hessian). Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ and $g : U \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth functions. Let $x_0 \in U$, $g(x_0) = c$, and let S be the level curve of g with value c. Assume that $\nabla g(x_0) \neq 0$ and that there exists a real number λ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0)$$

Let $h = f - \lambda g$ and the bordered Hessian determinant is defined by

$$|\bar{H}| = \det \begin{vmatrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ -\frac{\partial g}{\partial y} & \frac{\partial^2 h}{\partial x \partial y} & \frac{\partial^2 h}{\partial y^2} \end{vmatrix}$$

For f restricted to the curve S,

1. If $|\bar{H}| > 0$, then x_0 is a local max.
2. If $|\bar{H}| < 0$, then x_0 is a local min.
3. If $|\bar{H}| = 0$, then it is inconclusive.

Proposition 2.4 (Newton's Second Law). Let F be the force acting on a particle of mass m, then

$$F = ma$$

where a is the acceleration.

Proposition 2.5 (gradient is irrotational). Let $f \in C^2$, viewing ∇f as a vector field, then

$$\nabla \times (\nabla f) = 0$$

Proposition 2.6 (divergence of a curl vanishes)**.** For any C^2 vector field F,

$$\nabla \cdot (\nabla \times F) = 0$$

Proposition 2.7 (Fubini's Theorem for rectangles)**.** Let f be a continuous function on a rectangular domain $R = [a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Proposition 2.8 (Fubini's Theorem for general regions)**.** Suppose D is a set of points (x, y) such that $y \in [c, d]$ and $\psi_1(y) \leq x \leq \psi_2(y)$, and similarly for $x \in [a, b]$, $\varphi_1(x) \leq y \leq \varphi_2(x)$. If f is continuous on D, then

$$\iint_D f(x, y) dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y) dy dx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx dy$$

Proposition 2.9. We have the following identities regarding divergence and curl:

1. $\nabla(f + g) = \nabla f + \nabla g$.
2. $\nabla(cf) = c\nabla f$, for constant c.
3. $\nabla(fg) = f\nabla g + g\nabla f$.
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at points x where $g(x) \neq 0$.
5. $\operatorname{div}(F + G) = \operatorname{div} F + \operatorname{div} G$.
6. $\operatorname{curl}(F + G) = \operatorname{curl} F + \operatorname{curl} G$.
7. $\operatorname{div}(fF) = f \operatorname{div} F + F \cdot \nabla f$.
8. $\operatorname{div}(F \times G) = G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G$.
9. $\operatorname{div} \operatorname{curl} F = 0$.
10. $\operatorname{curl}(fF) = f \operatorname{curl} F + \nabla f \times F$.
11. $\operatorname{curl} \nabla f = 0$.
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$.
13. $\operatorname{div}(\nabla f \times \nabla g) = 0$.
14. $\operatorname{div}(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$.

Proposition 2.10 (integrability)**.** For different assumptions on f, we have the following integrability results:

1. Let f be continuous and defined on a closed rectangle R, then f is integrable over R.
2. Let $f : R \to \mathbb{R}$ be a bounded function on R and suppose the set of points where f is discontinuous lies on a finite union of graphs of continuous functions, then f is integrable over R.

Proposition 2.11 (Fubini's Theorem for rectangles). For different assumptions on f, we have the following Fubini's theorem results:

1. Let f be a continuous function on a rectangular domain $R = [a,b] \times [c,d]$, then

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy = \iint_R f(x,y)dA$$

2. Let f be bounded with domain $R = [a,b] \times [c,d]$ and the discontinuities of f lie on a finite union of graphs of continuous functions. If the integral $\int_a^b f dy$ exists for each $x \in [a,b]$, then

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx$$

exists and

$$\int_a^b \int_c^d f(x,y)dydx = \iint_R f(x,y)dA$$

Similar results hold if $\int_a^b f dx$ exists for each $y \in [c,d]$. If both hold simultaneously, then

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy = \iint_R f(x,y)dA$$

Proposition 2.12 (Fubini's Theorem for general regions). Suppose D is a set of points (x,y) such that $y \in [c,d]$ and $\psi_1(y) \le x \le \psi_2(y)$, and similarly for $x \in [a,b]$, $\varphi_1(x) \le y \le \varphi_2(x)$. If f is continuous on D, then

$$\iint_D f(x,y)dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dxdy$$

Proposition 2.13 (simple-regions). If D is a x-simple region with $y \in [c,d], \psi_1(y) \le x \le \psi_2(y)$, and if f is continuous on D, then

$$\iint_D f(x,y)dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)dx \right) dy$$

Similarly, if D is y-simple, then

$$\iint_D f(x,y)dA = \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)dydx$$

If D is simple, then the two expressions above are equal.

For example, the area of a x-simple region D can be computed as

$$\iint_D dA = \int_c^d \psi_2(y) - \psi_1(y)dy$$

Proposition 2.14. Let A be 2×2 matrix with $\det(A) \neq 0$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map $Tx = Ax$. Then T transforms parallelograms into parallelograms and vertices into vertices.

Proposition 2.15. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, i.e., there exists $n \times n$ matrix A such that $Tx = Ax$, then T is injective iff surjective iff $\det(A) \neq 0$.

Theorem 2.1 (change of variables formula)**.** Let D, D^* be elementary regions in $\mathbb{R}^2$, suppose $T : D^* \to D$ is both one-to-one and onto. Then for any integral function $f : D \to \mathbb{R}$, the **change of variable formula** states

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\det(J)\right| dudv$$

where

$$\det(J) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

is the Jacobian determinant.

Proposition 2.16 (change of variables-polar coordinates)**.** As a corollary to the theorem above, we have the following change of variables formula for polar coordinates:

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Proposition 2.17 (change of variables-triple)**.** Let W, W^* be elementary regions in $\mathbb{R}^3$, and suppose $T : W^* \to W$ is bijective. Then the change of varialbes formula for triple integrals states:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \mid \det(J) \mid dudvdw$$

where

$$\det(J) = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

is the Jacobian determinant.

Proposition 2.18 (change of variables-triple cylindrical)**.** As a corollary to the above, we have the following change of variables formula for cylindrical coordinates:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z) r dr d\theta dz$$

Recall cylinder cooridnates is setting up the following

$$x = r\cos\theta, y = r\sin\theta, z = z$$

Proposition 2.19 (change of variables-triple spherical)**.** As a corollary to the above, we have the following change of variables formula for spherical coordinates:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi d\rho d\theta d\phi$$

Recall the spherical cooridnates is setting up the following

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

Proposition 2.20 (reparametrization for path integrals)**.** Let c be a C^1 path and c' be any reparametrization of c, and let f be a continuous function on the image of c, then

$$\int_c f(x,y,z)ds = \int_{c'} f(x,y,z)ds$$

Proposition 2.21 (reparametrization for line integrals)**.** Let F be a vector feld continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$, and let $c' : [a',b'] \to \mathbb{R}^3$ be a reparametrization of c. If the reparametrization c' is orientation-preserving, then

$$\int_{c'} F \cdot ds = \int_c F \cdot ds$$

If c' is orientation-reversing, then

$$\int_{c'} F \cdot ds = -\int_c F \cdot ds$$

Proposition 2.22 (fundamental theorem of line integrals)**.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is of C^1 and that $c : [a,b] \to \mathbb{R}^3$ is piecewise C^1. Then

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Proposition 2.23 (surface integral of vector fields and orientations)**.** Let S be an oriented surface and let Φ_1, Φ_2 be two regular orientation-preserving parametrizations, with F a continuous vector field defined on S. Then

$$\iint_{\Phi_1} F \cdot dS = \iint_{\Phi_2} F \cdot dS$$

If Φ_1 is orientation-preserving and Φ_2 is orientation-reversing, then

$$\iint_{\Phi_1} F \cdot dS = -\iint_{\Phi_2} F \cdot dS$$

If f is a real-valued continuous function defined on S, and Φ_1, Φ_2 are parametrizations of S, then

$$\iint_{\Phi_1} f dS = \iint_{\Phi_2} f dS$$

Proposition 2.24. The surface integral of F over a surface S is equal to the integral of the normal component of F over S: let S be an oriented smooth surface S and an orientation-preserving parametrization Φ of, then we denote $\iint_S F \cdot dS = \iint_\Phi F \cdot dS$, and

$$\iint_S F \cdot dS = \iint_S (F \cdot n)\, dS$$

Proposition 2.25. Let S be the graph of a function $g(x,y)$, then

$$\iint_S F \cdot dS = \iint_D F \cdot (T_x \times T_y) dxdy = \iint_D \left(F_1 \left(-\frac{\partial g}{\partial x}\right) + F_2 \left(-\frac{\partial g}{\partial y}\right) + F_3\right) dxdy$$

Theorem 2.2 (Green's theorem). Let $F(x, y) = (P(x, y), Q(x, y))$ be a continuously differentiable vector field. For a simple region $D \subset \mathbb{R}^2$ with $\partial D = C$ as its positively oriented boundary, we have

$$\int_{\partial D} F \cdot ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

Theorem 2.3 (Green's theorem (curl form)). Let $D \subset \mathbb{R}^2$ be a region to which Green's theorem applies, let ∂D be its positively oriented boundary and let $F(P, Q)$ be a C^1 vector field on D. Then

$$\int_{\partial D} F \cdot ds = \iint_D \operatorname{curl} F \cdot k dA = \iint_D (\nabla \times F) \cdot k dA$$

Proposition 2.26. Let $D \subset \mathbb{R}^2$ be a region where Green's theorem applies and let ∂D be its boundary. Let n denote the outward unit normal to ∂D. If $c : [a, b] \to \mathbb{R}^2$, $t \mapsto c(t) = (x(t), y(t))$ is a positively oriented parametrization of ∂D, n is given by

$$n = \frac{(y'(t), -x'(t))}{\sqrt{x'(t)^2 + y'(t)^2}}$$

Let $F = (P, Q)$ be a C^1 vector field on D. Then

$$\int_{\partial D} F \cdot n ds = \iint_D \operatorname{div} F dA$$

Proposition 2.27 (area of a region). If C is a simple closed curve that bounds a region to which Green's theorem applies, then the area of the region D bounded by $C = \partial D$ is

$$A = \frac{1}{2} \int_{\partial D} x dy - y dx$$

Theorem 2.4 (Stokes' theorem). Let S be the oriented surface defined by a C^2 function $z = f(x, y)$, where $(x, y) \in D$, a region to which Green's theorem applies, and let F be a C^1 vector field on S. Then if ∂S denotes the oriented boundary curve of S, then

$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$$

More generally, let S be an oriented surface defined by a one-to-one parametrization $\Phi : D \subset \mathbb{R}^2 \to S$, where D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary of S and let F be a C^1 vector field on S. Then

$$\iint_S (\nabla \times F) \cdot dS = \int_{\partial S} F \cdot ds$$