

Real Analysis 605 MT Review

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Chapter 1

Definitions

Definition 1.1 (sequence of sets). Let $\{E_k\} \subset \mathbb{R}^n$ be a sequence of sets is said to increase to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ for all k , and decrease to $\bigcap_k E_k$ if $E_k \supset E_{k+1}$ for all k .

Definition 1.2 (limsup, liminf of sets). Let $\{E_k\}_{k=1}^\infty$ be a sequence of sets, we define

$$\limsup E_k = \bigcap_{j=1}^\infty \left(\bigcup_{k=j}^\infty E_k \right), \quad \liminf E_k = \bigcup_{j=1}^\infty \left(\bigcap_{k=j}^\infty E_k \right)$$

Definition 1.3 (metric). Let d be a metric on \mathbb{R}^n , let $x, y \in \mathbb{R}^n$, then

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 1.4 (limsup, liminf of sequences). Let $\{a_k\}$ be a sequence of points in \mathbb{R} , then

$$\limsup a_k := \lim_{j \rightarrow \infty} \left\{ \sup_{k \geq j} a_k \right\}$$

and

$$\liminf a_k := \lim_{j \rightarrow \infty} \left\{ \inf_{k \geq j} a_k \right\}$$

Definition 1.5 (distance between sets). Let $E_1, E_2 \subset \mathbb{R}^n$, then the distance between E_1 and E_2 is defined as

$$d(E_1, E_2) = \inf \{|x - y| : x \in E_1, y \in E_2\}$$

Definition 1.6 (open set). Let $E \subset \mathbb{R}^n$, then E is called open if for each $x \in E$, there exists δ such that $B_\delta(x) \subset E$.

A subset E_1 of E is said to be relatively open with respect to E if it can be written as $E_1 = E \cap G$ for some open set G .

Definition 1.7 (A_δ, A_σ sets). A set A is said to be of type A_δ if it can be written as a countable intersection of sets and to be of type A_σ if it can be written as a countable union of sets. Then G_δ implies a countable intersection of open sets, and F_σ implies the countable union of closed sets.

Definition 1.8 (perfect set). C is called a perfect set if it is a closed set such that every point in C is a limit point.

Definition 1.9 (compact set). A set E is compact if and only if every open cover of E has a finite sub-cover.

Definition 1.10 (monotone function). A function f defined on $I \subset \mathbb{R}$ is monotone increasing if $f(x) \leq f(y)$ whenever $x < y$. Similarly defined for monotonically decreasing.

Definition 1.11 (continuous). Let f be defined on a neighborhood of x_0 , then f is said to be continuous at x_0 if $f(x_0)$ is finite and $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

Definition 1.12 (continuous relative to a set). Let f be defined in only a set E containing x_0 , f is said to be continuous at x_0 relative to E if $f(x_0)$ is finite and either x_0 is an isolated point of E or x_0 is a limit point of E and for $x \in E$.

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

If $E_1 \subset E$, a function is continuous in E_1 relative to E if it is continuous relative to E at every point in E_1 .

Definition 1.13 (uniform convergence). A sequence $\{f_k\}$ defined on E is said to uniformly convergence on E to a finite f if given $\varepsilon > 0$, there exists K such that for all $k \geq K$, $x \in E$,

$$|f_k(x) - f(x)| < \varepsilon$$

Definition 1.14 (Riemann integral). Let f be bounded on an interval I , partition I into a finite collection Γ of nonoverlapping intervals, denote $|\Gamma| = \max_k \text{diam}(I_k)$, select points $\xi_k \in I_k$, let

$$R_\Gamma = \sum_{k=1}^N f(\xi_k) |I_k|$$

and

$$U_\Gamma = \sum_{k=1}^N \left(\sup_{x \in I_k} f(x) \right) |I_k|, \quad L_\Gamma = \sum_{k=1}^N \left(\inf_{x \in I_k} f(x) \right) |I_k|$$

The Riemann integral exists if $\lim_{|\Gamma| \rightarrow 0} R_\Gamma$ exists and the limit A is the Riemann integral. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\Gamma| < \delta$, we have $|A - R_\Gamma| < \varepsilon$ for any Γ and any chosen $\{\xi_k\}$.

This is equivalent to the statement:

$$\inf_{\Gamma} U_\Gamma = \sup_{\Gamma} L_\Gamma = A$$

We begin chapter 2.

Definition 1.15 (variation). Let f be defined on $[a, b]$, the variation of f over $[a, b]$ is

$$V(f) = \sup_{\Gamma} \sum_{i=1}^m |f(x_i) - f(x_{i-1})|$$

where Γ is any partition $\{x_0, x_1, \dots, x_m\}$ of $[a, b]$.

Definition 1.16 (Lipschitz). Let f be defined on $[a, b]$, then f is said to be Lipschitz if there exists an absolute constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in [a, b]$.

Definition 1.17 (splitting). For any $x \in \mathbb{R}$, we can write

$$x^+ = \begin{cases} x, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$x^- = \begin{cases} 0, & x > 0 \\ -x, & x \leq 0 \end{cases}$$

then $|x| = x^+ + x^-$, $x = x^+ - x^-$.

Definition 1.18 (P_{Γ}, N_{Γ}). For any f and any partition Γ , define

$$P_{\Gamma} = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^+$$

and

$$N_{\Gamma} = \sum_{i=1}^m [f(x_i) - f(x_{i-1})]^-$$

similarly, we define

$$P = \sup_{\Gamma} P_{\Gamma}, N = \sup_{\Gamma} N_{\Gamma}$$

Definition 1.19 (rectifiable curve). Let C be a curve, i.e.

$$C : \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

Let Γ be any partition, define

$$L = \sup_{\Gamma} \sum_{i=1}^m ((\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1}))^2)^{1/2}$$

then C is rectifiable if $L < +\infty$.

Definition 1.20 (Riemann-Stieltjes integral). Let f, ϕ be finite on an interval $[a, b]$, let $\Gamma = \{a = x_0 = \dots < x_m = b\}$ be any partition, define

$$R_\Gamma = \sum_{i=1}^m f(\xi_i) [\phi(x_i) - \phi(x_{i-1})]$$

If $\lim_{|\Gamma| \rightarrow 0} R_\Gamma$ exists, then we call this the Riemann-Stieltjes integral. That is, given any $\varepsilon > 0$, there is $\delta > 0$ such that when $|\Gamma| < \delta$ we have $|I - R_\Gamma| < \varepsilon$. We denote it as

$$I = \int_a^b f(x) d\phi(x) = \int_a^b f d\phi$$

Definition 1.21 (upper, lower R-S sum). Let f be bounded and ϕ be monotonically increasing. Let

$$m_i = \inf_{[x_{i-1}, x_i]} f(x), M_i = \sup_{[x_{i-1}, x_i]} f(x)$$

then we define the lower and upper Riemann-Stieltjes sums L_Γ, U_Γ as follows:

$$L_\Gamma = \sum_{i=1}^m m_i [\phi(x_i) - \phi(x_{i-1})], U_\Gamma = \sum_{i=1}^m M_i [\phi(x_i) - \phi(x_{i-1})]$$