Calc III Sections

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Calc III-Week 10 (10/27-31)

Topics: (1) Divergence and Curl, (2) Double integrals.

Definition 0.1 (divergence)**.** Let F be a vector field in $\mathbb{R}^3$ $F = (F_1, F_2, F_3)$, the divergence of F is the **scalar field** (assigns one number to an given point (x, y, z)),

$$\operatorname{div} F := \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

More generally, if $F = (F_1, \ldots, F_n)$ is a vector field on $\mathbb{R}^n$, its divergence is

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Remark 1. *We write the divergence as $\nabla \cdot F$ because*

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

and if $F = (F_1, \ldots, F_n)$,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, \ldots, F_n) = \nabla \cdot F$$

Definition 0.2 (curl)**.** Let F be a vector field in $\mathbb{R}^3$, writing $F = (F_1, F_2, F_3)$, the **curl** of F is the vector field

$$\operatorname{curl} F := \nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

If $\operatorname{curl} F = 0$, then we say the vector field is **irrotational**.

Proposition 0.1 (gradient is irrotational)**.** Let $f \in C^2$, where $f : \mathbb{R}^3 \to \mathbb{R}$, viewing ∇f as a vector field, then

$$\nabla \times (\nabla f) = 0$$

Proposition 0.2 (divergence of a curl vanishes)**.** For any C^2 vector field F,

$$\nabla \cdot (\nabla \times F) = 0$$

Proposition 0.3 (Fubini's Theorem)**.** Let f be a continuous function on a rectangular domain $R = [a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Problem 0.1. Show that the vector field $V(x, y, z) = (x^2, -y, z)$ is not the curl of any vector field F. In other words, there is no vector field F such that

$$V = \operatorname{curl} F$$

Proof. By the above proposition, if there exists F such that

$$V = \operatorname{curl} F$$

then

$$\nabla \cdot V = 0$$

However,

$$\nabla \cdot V(x, y, z) = 2x \neq 0 \text{ if } x \neq 0$$

this is a contradiction, hence no such F exists. □

Problem 0.2 (Marsden-Tromba, IV.4, 21). Let $F(x, y, z) = (x^2, x^2 y, z + zx)$. Can there exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$?

Proof. No. Suppose there exists such f, then the gradient must be irrotational. In other words, we must have

$$\nabla \times F = 0$$

Thus we compute the curl of F:

$$\nabla \times F = (-2yz, z, 0)$$

and it is not identically 0. □

Problem 0.3. Change the order of integration to $dydx$ for the following function:

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} dx dy$$

Proof. It should be

$$\int_1^e \int_0^{\ln x} \frac{x}{\ln x} dy dx$$

□

Problem 0.4. Change the order of integrations for the following functions:

1. $$\int_0^1 \int_x^1 f(x,y)dydx$$
2. $$\int_0^1 \int_{-y}^{y^3} f(x,y)dxdy$$
3. $$\int_0^3 \int_{2x}^6 f(x,y)dydx$$
4. $$\int_0^1 \int_{-\sqrt{y}}^{y^2} f(x,y)dxdy$$
5. $$\int_0^8 \int_{\sqrt[3]{y}}^2 f(x,y)dxdy$$
6. $$\int_0^1 \int_{\ln y}^1 f(x,y)dxdy$$

Proof.

1. It should be
$$\int_0^1 \int_0^y f(x,y)dxdy$$
2. It should be
$$\int_{-1}^0 \int_{-x}^1 f(x,y)dydx + \int_0^1 \int_{\sqrt{x}}^1 f(x,y)dydx$$
3. It should be
$$\int_0^6 \int_0^{\frac{y}{2}} f(x,y)dxdy$$
4. It should be
$$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)dydx + \int_{-1}^0 \int_{x^2}^1 f(x,y)dydx$$
5. It should be
$$\int_0^2 \int_0^{x^3} f(x,y)dydx$$
6. It should be
$$\int_{-\infty}^0 \int_0^{e^x} f(x,y)dydx + \int_0^1 \int_0^1 f(x,y)dydx$$

□