Columbia HW Problems

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Contents

Chapter 1

HW2

Problem 1.1 (1). Determine whether the following statements are true or false. Justify your answers.

(a) Any subring of a field is an integral domain.

(b) The ring $\mathbb{Z}/49\mathbb{Z}$ is an integral domain.

(c) The direct product $F_1 \times F_2$ of two fields is a field.

(d) An element ab of a ring R is invertible if and only if both a and b are invertible.

(e) The ring $\mathbb{Z} \times \mathbb{Z}$ has exactly four idempotents.

Hint: First find all idempotents in the ring $\mathbb{Z}$. An idempotent is an element e such that $e^2 = e$.

Proof. (a) True (b) False (c) False, consider $(1,0) \cdot (0,1)$ (d) False Consider $5 + 5 = 1$ in $\mathbb{Z}/6\mathbb{Z}$ (e) True. $\square$

Problem 1.2 (6). 6. An element x of a ring R is called *nilpotent* if $x^n = 0$ for some $n > 0$. Note that $0 \in R$ is always nilpotent. (Remark: A nonzero nilpotent element is a zero divisor, while a zero divisor does not have to be nilpotent.)

(a) (5 points) Show that 0 is the only nilpotent element of an integral domain R.

(b) (5 points) Find all nilpotent elements in the following rings:

$$\mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{Z}/9\mathbb{Z}, \quad \mathbb{Z}/12\mathbb{Z}, \quad \mathbb{Q}[x].$$

(c) (optional, 10 points) Show that if x, y are nilpotent then $x+y$ is nilpotent (assume that $x^n = 0$, $y^m = 0$, use that the ring is commutative and apply the binomial theorem from lecture 1 to some large power of $x + y$).

Proposition 1.1. If $a \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent, then a is nilpotent in $\mathbb{Z}/d\mathbb{Z}$ for all divisors d of n. Moreover, units are not nilpotent.

(b) For $\mathbb{Z}/9\mathbb{Z}$: $\{0, 3, 6\}$ are nilpotents. For $\mathbb{Z}/12\mathbb{Z}$: $\{0, 6\}$ are nilpotents. (c) try $(x + y)^{m+n}$. $\square$

Proposition 1.2. An integral domain must have characteristic 0 or equal to some prime p; however, there exists char prime ring that is not an integral domain: $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Chapter 2

HW3

Problem 2.1 (Q7). Let $e \in R$ be an idempotent (i.e., $e^2 = e$) in a commutative ring R.

(a) Check that Re and $R(1-e)$ are ideals of R and that their intersection $Re \cap R(1-e) = 0$. Show that any element $a \in R$ has a unique presentation as a sum of an element in Re and an element in $R(1-e)$.

(b) Prove that Re is a ring, with identity e and addition and multiplication inherited from R. Likewise for $R(1-e)$. (Since $1-e$ is also an idempotent, you don't need to repeat your arguments twice.)

(c) Show that the map $\phi : Re \times R(1-e) \to R$ defined by $\phi(a, b) = a + b$ is an isomorphism of rings.

Proof. (a) Let $a = re = r'(1-e)$, then

$$a = re = r' - r'e = r'e - r'e = 0$$

Clearly we have $a = ae + a(1-e)$, and one can show this presentation is unique.

(b) True.

(c) It is surjective and injective both by part (a).

Proposition 2.1. Let $e \in R$ be a nontrivial idempotent, then viewing $Re, R(1-e)$ as rings with identities $e, (1-e)$, we can decompose R into a product of two rings:

$$R \cong Re \cong R(1-e)$$

□

Chapter 3

HW4

Problem 3.1. Compute the following sums and intersections of ideals in $\mathbb{Q}[x]$ (first recall the relation between sums and intersections of ideals in $F[x]$ to gcd and lcm of polynomials). Note that all the ideals of $\mathbb{Q}[x]$ are principal. For each ideal, list the monic polynomial which generates the ideal.

$$(x) + (x+2), \quad (x^2) + (2x), \quad (3x^2 + 2x) + (4x^2 + x),$$
$$(3x^2 + x + 5) + (0), \quad (x) \cap (2x+1), \quad (2x) \cap (3x^2).$$

Proof. $(1), (x), (x), (3x^2 + x + 5), (x(2x+1)), (6x^2) = (x^2)$. □

Proposition 3.1. For the sum and intersection of ideals, we have

$$(f) + (g) = (\gcd(f, g)), \quad (f) \cap (g) = (\text{lcm}(f, g))$$

Problem 3.2. Consider the ring $R = \mathbb{F}_p[x]/(f(x))$, where $f(x)$ is a polynomial of degree n. Show that R is a finite ring with p^n elements.

Proof. Every $g(x) \in R$ is identified with its remainder by $f(x)$, where $\deg(r) < n$, hence R consists of elements of the form

$$a_{n-1}x^n + \cdots + a_1 x + a_0$$

There are p choices for each a_i, therefore this ring has size p^n. □

Problem 3.3. Explain how to construct an infinite field of characteristic p.

Proof. Start with $\mathbb{F}_p[x]$ it is not a field, so

$$\mathbb{F}_p(x) = \left\{ \frac{p(x)}{q(x)} : q \neq 0 \right\}$$

is an infinite field of characteristic p. □