

# Algebra Qualifying Exam

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# Chapter 1

## Group Theory

### 1.1 Sylow Theorems

We first talk about semidirect products. Let  $G$  be any group, and  $N, H$  be subgroups of  $G$ .

**Definition 1.1.** For  $\varphi : H \rightarrow \text{Aut}(N)$ , define  $N \rtimes_{\varphi} H$  by

- (1)  $N \rtimes_{\varphi} H = N \times H$  as a set.
- (b) Equipped with the group structure

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)$$

The structure  $(N \rtimes_{\varphi} H, \cdot)$  forms a group.

**Example 1.1.** If  $N$  is a normal subgroup of  $G$ , and  $N \cap H = \{e\}$ , and  $\varphi : H \rightarrow \text{Aut}(N)$  where

$$\varphi : h \mapsto (n \mapsto hnh^{-1})$$

(acting by conjugation), and  $G = NH$ . Then

$$N \rtimes_{\varphi} H \rightarrow G$$

where

$$(n, h) \mapsto nh$$

is a bijective homomorphism. Hence

$$G \cong N \rtimes_{\varphi} H$$

Next we present some divisibility results.

**Proposition 1.1 (Lagrange, Orbit-Stabilizer).** We have the following divisibility results:

- Let  $H$  be a subgroup of  $G$ , let  $[G : H]$  denote the number of cosets of  $H$  in  $G$ , then

$$|G| = |H| [G : H]$$

- Let  $G$  be a finite group acting transitively on a finite set  $A$ , then for any  $a \in A$ , we have

$$|\text{Stab}_G(a)| \cdot |O_G(a)| = |G|$$

The class formula is when  $G$  acts on itself by conjugation:

**Proposition 1.2 (class formula).** Let  $G$  act on a finite set  $S$ , and let  $Z$  denote fixed points of this action, then

$$|S| = |Z| + \sum_{a \in A} |O_G(a)|$$

where  $A$  includes exactly one element from each nontrivial orbit.

If  $G$  acts on itself by conjugation, then

$$|G| = |Z(G)| + \sum_g |[g]| = |Z(G)| + \sum_g \frac{|G|}{|C_G(g)|}$$

where  $[g]$  denote the conjugacy class of  $g$ , and the sum includes exactly one from each nontrivial conjugacy class in  $G$ .

**Problem 1.1 (F2019-Q2).** 2. Let  $p, q$  be two prime numbers such that  $p \mid q - 1$ . Prove that

- (a) there exists an integer  $r \not\equiv 1 \pmod{q}$  such that  $r^p \equiv 1 \pmod{q}$ ;
- (b) there exists (up to an isomorphism) only one noncommutative group of order  $pq$ .

*Proof.* (a) We want to show that there exists an element  $r \in (\mathbb{Z}/q\mathbb{Z})^\times$  such that

$$r^p \equiv 1 \pmod{q}$$

We can do this because  $(\mathbb{Z}/q\mathbb{Z})^\times$  has order  $(q - 1)$  and  $p \mid (q - 1)$ . Therefore by Cauchy's theorem, there exists an element of order  $p$  in  $(\mathbb{Z}/q\mathbb{Z})^\times$ .

- (b) Let  $n_p, n_q$  denote the number of  $p, q$ -Sylow subgroups. We see that  $n_q \mid p$  and  $n_q \equiv 1 \pmod{q}$ , since  $p < q$ , we must have  $n_q = 1$ . Now  $n_p = 1$  or  $q$  by the same reasoning. Suppose  $n_q = 1$ , let  $P, Q$  denote the normal subgroups of order  $p, q$ , then

$$G \cong P \times Q$$

by a standard argument (included in the lemma below). Then  $G$  is commutative. Since  $G$  is noncommutative, we have  $n_p = q$ . Choose any  $p$ -Sylow subgroup  $P$ , we know that

$$G \cong Q \rtimes_\theta P$$

where  $Q$  is the normal subgroup of order  $q$  and  $\theta : P \rightarrow \text{Aut}(Q) = (\mathbb{Z}/q\mathbb{Z})^\times$ . We know either  $\theta : 1 \mapsto 1$ , is the trivial map which produces a commutative group; or  $\theta : 1 \mapsto r$ , where  $r \in (\mathbb{Z}/q\mathbb{Z})^\times$  is some element of order  $p$ . □

**Lemma 1.1.** Let  $p, q$  be two primes such that  $q \nmid (p - 1)$ , and  $N, H$  has order  $p, q$  respectively, suppose that  $N$  is normal in  $G$ , and  $N \cap H = \{e\}$ , then

$$G \cong N \times H$$

*Proof.* We consider the map

$$\psi : N \times H \rightarrow G$$

such that

$$(n, h) \mapsto nh$$

We want to show that  $\psi$  is a homomorphism and  $\psi$  is injective (hence bijective by size argument). It is clearly injective:

$$nh = e \Rightarrow n, h \in N \cap H = \{e\}$$

It suffices to show that  $\psi$  is a homomorphism. We see that this implies

$$n_1 n_2 h_1 h_2 = n_1 h_1 n_2 h_2$$

Therefore it suffices to for any  $n \in N, h \in H$ , one has

$$nh = hn$$

Consider the conjugation action

$$\varphi : H \rightarrow \text{Aut}(N)$$

where

$$h \mapsto (n \mapsto hnh^{-1})$$

Then we claim that  $\varphi$  is trivial. This is because  $\ker(\varphi)$  has size either 1 or  $q$ . If it has size  $q$ , then the map is trivial; if it has size 1, then  $H$  embeds in  $\text{Aut}(N)$ , however,  $|H| = q$ ,  $\text{Aut}(N) = p - 1$ , and  $q \nmid (p - 1)$ , hence impossible. This shows that the map is trivial, i.e., for  $n \in N, h \in H$ ,

$$hn = nh$$

as desired. □

**Problem 1.2 (F2015-Q1).** Prove every group of order 15 is cyclic.

*Proof.* We will show that any group  $G$  of order 15 is isomorphic to

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

For this, using the above lemma, it suffices to show that there is one normal subgroup of order 3 and one normal subgroup of order 5. We repeat the argument above,  $n_5 \mid 3$  and  $n_5 \equiv 1 \pmod{5}$ , hence  $n_5 = 1$ . Moreover,  $n_3 \mid 5$  and  $n_3 \equiv 1 \pmod{3}$ , hence  $n_3 = 1$  as well. By the lemma above, we know that

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

hence cyclic as desired. □

**Problem 1.3 (S2013-Q2).** Let  $p$  and  $q$  be primes with  $p < q$ . Let  $G$  be a group of order  $pq$ . Prove the following statements:

- (a) If  $p$  does not divide  $q - 1$  (i.e.,  $p \nmid q - 1$ ), then  $G$  is cyclic.
- (b) If  $p$  divides  $q - 1$  (i.e.,  $p \mid q - 1$ ), then  $G$  is either cyclic or isomorphic to a non-abelian group on two generators. Give the presentation of this non-abelian group.

*Proof.* This question is exactly the same as F19-Q2, we will only outline here.

- (a) We have  $n_q = 1$ , and  $n_p \mid q$ , hence  $n_p = 1$  or  $q$ , moreover  $n_p \equiv 1 \pmod{p}$ . If  $n_p = q$ , this implies that  $p \mid (q - 1)$ , hence  $n_p = 1$ . Therefore by the above argument

$$G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$$

- (b) If  $p \mid (q - 1)$ , then  $n_p = 1$  or  $q$ . Hence  $G$  is either of the form above or isomorphic to the non-abelian group

$$G = Q \rtimes_{\theta} P$$

We know from F2019-Q2, the trivial  $\theta$  defines the abelian, hence cyclic group  $G = \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$ . And  $\theta : 1 \mapsto r$ , for some  $r \in (\mathbb{Z}/q\mathbb{Z})^{\times}$  of order  $p$  defines a non-abelian group. So we have

$$G = \langle g, h : g^q = h^p = e, hgh^{-1} = g^r \rangle$$

□

**Problem 1.4 (F2007-Q1).** Prove that no group of order 148 is simple.

*Proof.* We note the prime factorization of 148 is

$$148 = 2^2 \cdot 37$$

We see that  $n_{37} \mid 4$  and  $n_{37} \equiv 1 \pmod{37}$ , therefore  $n_{37} = 1$ . This shows that there exists a normal subgroup of order 37, i.e., the group is not simple. □

**Problem 1.5 (F2017-Q1).** Show that there is no simple group of order 30.

*Proof.* This is slightly more complicated, and we will use a counting argument. Same reasoning as the above. The prime factorization of 30 is as below:

$$30 = 2 \cdot 3 \cdot 5$$

We see  $n_5 \mid 6$ , and  $n_5 \equiv 1 \pmod{5}$ . Unfortunately,  $n_5$  could either be 1 or 6. Now  $n_3 \mid 10$ , and  $n_3 \equiv 1 \pmod{3}$ , unfortunately again  $n_3$  could be 10. However, we argue that  $n_3 = 10$  and  $n_5 = 6$  cannot happen at the same time. Suppose this is the case, then there are 20 elements of order 2 and 24 elements of order 5, but this is too many! Hence either  $n_3 = 1$  or  $n_5 = 1$ , as desired. □

**Problem 1.6 (F2011-Q1).**

- (a) Let  $G$  be a group of order 5046. Show that  $G$  cannot be a simple group. You may not appeal to the classification of finite simple groups.
- (b) Let  $p$  and  $q$  be prime numbers. Show that any group of order  $p^2q$  is solvable.

*Proof.* The proof is very similar like above.

- (a) The prime factorization of 5046 is as follows:

$$5046 = 2 \cdot 3 \cdot 29^2$$

Hence we see  $n_{29} = 1$ , i.e., there is a normal subgroup of order 29, therefore not simple.

- (b) We will do discussion by cases.

- (1)  $p > q$ . Then  $n_p = 1$  or  $q$  and  $n_p \equiv 1 \pmod{p}$ , therefore  $n_p = 1$ . Let  $P$  be the normal subgroup of  $G$  of order  $p^2$ , we thus have

$$\{e\} \subset P \subset G$$

It is clear that  $|G/P| = q$ , thus abelian, and  $|P| = p^2$  also abelian as well (by the lemma below). This shows that  $G$  is solvable.

- (2)  $p < q$ . Then  $n_p = 1$  or  $q$ , and  $n_q = 1$  or  $p^2$ . Suppose that  $n_q = 1$ , let  $Q$  denote the normal subgroup of order  $q$ , then

$$\{e\} \subset Q \subset G$$

It is clear that  $Q$  and  $G/Q$  are both abelian. Suppose that  $n_q = p^2$  instead, then there are only  $p^2q - p^2(q-1) = p^2$  elements of order  $\neq q$ . Since any  $p$ -Sylow subgroup has  $p^2$  elements with order  $\neq q$ , we must have  $n_p = 1$ . Hence we are in case (1) again. This shows that  $G$  is solvable in either case  $n_q = 1, p^2$ . □

**Lemma 1.2 ( $p^2$  abelian).** Fix prime  $p$ , any group of order  $p^2$  is abelian.

*Proof.* For any nontrivial  $p$  group, by the class formula, the center  $Z(G)$  is nontrivial, thus the center has order either  $p$  or  $p^2$ . If it has order  $p^2$ , then the group is abelian. If it has order  $p$ , then

$$|G/Z(G)| = p$$

is also cyclic, therefore  $G$  is abelian (strictly speaking is a contradiction that  $|Z(G)| = p$ ). In either case, we see that  $G$  is abelian. □

**Problem 1.7.** Any  $p$ -group is solvable, for any prime  $p$ .

*Proof.* Suppose  $|G| = p^r$  for some  $r \geq 0$ , we will use induction on  $r$ . If  $r = 0$ , then the trivial group is trivially solvable.

- Base case: if  $r = 1$ ,  $|G| = p$ , then  $G$  is cyclic, hence solvable.
- Induction step: suppose that  $G$  is solvable for all  $|G| = p^k$ , where  $0 \leq k \leq r-1$ . Now we want to show that  $G$  of order  $p^r$  is solvable. We know  $G$  has a nontrivial center, suppose that  $|Z(G)| = p^k$ , where  $1 \leq k \leq r$ , then

$$|G/Z(G)| = p^{r-k}, 0 \leq r-k \leq r-1$$

We know any group  $G$  is solvable if and only if there exists a sequence of subgroups  $G_0, \dots, G_k$

$$\{e\} = G_0 \subset \dots \subset G_k = G$$

such that  $G_{i-1}$  is normal in  $G_i$  and  $G_i/G_{i-1}$  is solvable. Therefore we see when  $|G| = p^r$ ,

$$\{e\} \subset Z(G) \subset G$$

has  $Z(G)$  solvable, and  $G/Z(G)$  also solvable by the induction hypothesis, so we close the induction. □

**Problem 1.8 (S2016-Q1).** Classify all groups of order 66, up to isomorphism.

*Proof.* By  $66 = 2 \cdot 3 \cdot 11$ , we know  $n_{11} = 1$ . We claim that there is a normal subgroup isomorphic to  $\mathbb{Z}/33\mathbb{Z}$ .

1. First we show that there is a subgroup of order 33. Let  $P_{11}$  denote the normal subgroup of order 11 and let  $P_3$  denote a 3-Sylow subgroup of  $G$ . Then we claim that the following

$$H = \{gh : g \in P_{11}, h \in P_3\}$$

forms a subgroup and is isomorphic to  $\mathbb{Z}/33\mathbb{Z}$ . By the Lemma 1.1, we see that

$$H \cong \frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} = \frac{\mathbb{Z}}{33\mathbb{Z}}$$

2. Now we show that it is normal. This follows from the following general lemma:

**Lemma 1.3.** Let  $p$  be the smallest prime factor of  $|G|$ , and let  $H$  be a subgroup with index  $p$ , then  $H$  is normal.

*Proof.* We will only prove in the case that  $H$  is a subgroup of index 2, i.e.,  $G = H \sqcup (G \setminus H)$ . We see for all  $g \in G$ ,

$$gH = Hg$$

since if  $g \in H$ , then the equality holds; if  $g \notin H$ , then  $gH = G \setminus H$ , so is  $Hg$ .  $\square$

Now since there is a subgroup of order 2, we can write  $G$  as a semidirect product

$$G = \frac{\mathbb{Z}}{33\mathbb{Z}} \rtimes_{\theta} \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The number of nonisomorphic groups will depend on the choice of  $\theta$ . There are four different choices for  $\theta : H \rightarrow \text{Aut}\left(\frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}\right) = \frac{\mathbb{Z}}{10\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

$$\begin{cases} \theta_1 : 1 \mapsto (0, 0) \\ \theta_2 : 1 \mapsto (0, 1) \\ \theta_3 : 1 \mapsto (5, 0) \\ \theta_4 : 1 \mapsto (5, 1) \end{cases}$$

There are 4 different groups and one can write them in cyclic notation using the  $\theta$  above.  $\square$

**Problem 1.9 (S2007-Q2).** Prove that no group of order 224 is simple.

*Proof.* The prime factorization is

$$224 = 2^5 \cdot 7$$

If  $n_2 = 1$  or  $n_7 = 1$ , then we are done; assume that  $n_2 = 7$  instead, then we recall  $G$  has a nontrivial transitive action on the set of 2-Sylow subgroups, i.e., there is a homomorphism  $\varphi : G \rightarrow S_7$ . We know  $\ker(\varphi)$  is a normal subgroup of  $G$ . Since the action is nontrivial transitive, we know  $\ker(\varphi) \neq G$ . If  $\ker(\varphi) = \{e\}$ , then  $\varphi$  produces an embedding of  $G$  into  $S_7$ . However,  $|G| = 224 \nmid |S_7|$ . This shows that  $\ker(\varphi)$  is a nontrivial proper normal subgroup of  $G$ , concluding that  $G$  is not simple.  $\square$

**Problem 1.10 (F2008-Q1).** Show that no group of order 36 is simple.

*Proof.*

$$36 = 2^2 \cdot 3^2$$

We know  $n_2 \mid 9$ ,  $n_2 \equiv 1 \pmod{2}$ , and  $n_3 \mid 4$ ,  $n_3 \equiv 1 \pmod{3}$ . We know  $n_3 = 1$  or 4, suppose that  $n_3 = 4$ , then there is a nontrivial action of  $G$  on the set of 3-Sylow subgroups, i.e.,

$$\varphi : G \rightarrow S_4$$

Suppose that  $G$  is simple, we know  $\ker(\varphi) \neq G$  since the action is nontrivial, by assumption  $\ker(\varphi) = \{e\}$ , which implies that  $\varphi$  is an embedding, but  $|G| = 36 \nmid |S_4|$ , which is a contradiction. This implies that  $G$  is not simple.  $\square$



**Problem 1.11 (S2014-Q2).** All groups of order less than 60 are solvable, i.e., there exists a sequence of subgroups of  $G$ ,  $G_0, \dots, G_k$  such that  $G_i$  is normal in  $G_{i+1}$  and  $G_{i+1}/G_i$  is abelian, and

$$1 = G_0 \subset \dots \subset G_k = G$$

*Proof.* Groups of order  $p, pq, p^2, p^2q$  are solvable.

$$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, \\ 31, 33, 34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59\}$$

And any  $p$ -group is also solvable.

$$\{8, 16, 27, 32\}$$

The remaining groups are

$$\{24, 36, 40, 42, 48, 54, 56\}$$

24: If  $n_2 = 1$  or  $n_3 = 1$ , then we are done. We see  $n_2 = 1$  or 3, consider the action  $\varphi : G \rightarrow S_3$ . We see  $\ker(\varphi)$  is a proper normal subgroup of  $G$ , this implies that

$$\{e\} \subset \ker(\varphi) \subset G$$

where  $|\ker(\varphi)|$  is a known solvable group, hence we are done.

36: Exactly same as above, we assume  $n_3 \neq 1$ , therefore  $n_3 = 4$ , the action  $\varphi : G \rightarrow S_4$  is not injective, hence  $\ker(\varphi)$  is again a proper normal subgroup of  $G$  that is solvable.

40: We see  $n_5 = 1$ , therefore

$$\{e\} \subset \mathbb{Z}/5\mathbb{Z} \subset G$$

42: We see  $n_7 = 1$ .

48: We see  $n_2 = 1$  or 3, the the action  $\varphi : G \rightarrow S_3$  is not injective, hence  $\ker(\varphi)$  is a proper normal subgroup of  $G$  that is solvable.

54: We see  $n_3 = 1$ .

56: We know  $n_7 = 1$  or 8 and  $n_2 = 1$  or 7. The group action argument does not work. We assume  $n_7 = 8$ , then there can be at most  $56 - 8(7 - 1) = 8$  elements of order  $\neq 7$ . This shows that  $n_2 = 1$ . Hence

$$\{e\} \subset P_2 \subset G$$

□

**Problem 1.12 (S2012-Q1).** Let  $G$  be a group of order  $p^3q^2$ , where  $p$  and  $q$  are prime integers. Show that for  $p$  sufficiently large and  $q$  fixed,  $G$  contains a normal subgroup other than  $\{1\}$  and  $G$ .

*Proof.* We want to show that there exists a normal group of size  $p^3$ , i.e.,  $n_p = 1$ . We know  $n_p \mid q^2, n_p \equiv 1 \pmod p$ . Let  $p$  be large enough such that  $p > (q^2 - 1)$ , then this forces  $n_p = 1$ , as desired. □

**Problem 1.13 (F2014-Q4).**

- (a) Let  $G$  be a group of order  $p^2q^2$ , where  $p$  and  $q$  are distinct odd primes, with  $p > q$ . Show that  $G$  has a normal subgroup of order  $p^2$ .
- (b) Can a solvable group contain a non-solvable subgroup? Explain.

*Proof.* (a) We know  $n_p = 1$  or  $q$  or  $q^2$ , and  $n_p \equiv 1 \pmod{p}$ . Since  $p > q$ , we know  $n_p \neq q$ . It suffices to show that  $n_p \neq q^2$ : suppose that  $n_p = q^2$ , then

$$p \mid (q^2 - 1) = (q + 1)(q - 1)$$

Since  $p$  is prime,  $p \mid (q + 1)$  or  $p \mid (q - 1)$ . The latter impossible since  $q < p$ .  $p \mid (q + 1)$  is also impossible because this implies that  $q = p + 1$ , which implies that  $q$  is even, a contradiction.

(b) It is not possible. Suppose  $G$  is a solvable group, let  $H$  be a subgroup of  $G$ , then we know there exists sequence

$$\{e\} = G_0 \subset \cdots \subset G_k = G$$

such that  $G_i$  is normal in  $G_{i+1}$  and  $\frac{G_{i+1}}{G_i}$  is abelian. We define  $H_i = G_i \cap H$ , then we see  $H$  is solvable with sequence  $H_0 \subset \cdots \subset H_k$ . □

**Problem 1.14 (F2018-Q2).** Let  $G$  be a group of order 24. Assume that no Sylow subgroup of  $G$  is normal in  $G$ . Show that  $G$  is isomorphic to the symmetric group  $S_4$ .

*Proof.* By Sylow, we have  $n_3 = 4, n_2 = 3$ . Denote  $\text{Syw}_3(G) = \{P_1, P_2, P_3, P_4\}$  and consider the transitive action by of  $G$  by conjugation on this set, which embeds in  $S_4$ , i.e.,  $\varphi : G \rightarrow S_4$ . By a size argument, it suffices to show that  $\varphi$  is injective. We see that

$$\ker(\varphi) = \{g \in G : gP_i g^{-1} = P_i \text{ for each } i\} = \bigcap_{i=1}^4 N_G(P_i)$$

By the orbit-stabilizer theorem,  $|N_G(P_i)| = 6$  for all  $i$ . However, for any  $i \neq j$ , 3 does not divide  $|N_G(P_i) \cap N_G(P_j)|$ : if not, the intersection would include a 3-Sylow subgroup but  $P_i$  is the only 3-Sylow subgroup in  $N_G(P_i)$ , thus this is impossible. It remains to see that  $|\ker(\varphi)| \neq 2$ . Suppose that it is, then  $\text{im}(\varphi)$  is an index 2 subgroup of  $S_4$ , hence

$$\frac{G}{\ker \varphi} \cong A_4$$

and  $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  is normal in  $A_4$ , hence so is  $\varphi^{-1}(K)$  (it has size 8) in  $G$ . This is a contradiction because this implies there is a normal 2-Sylow subgroup. □

**Problem 1.15 (F2001-Q1).** Let  $G$  be a finite group and let  $N$  be a normal subgroup of  $G$  such that  $N$  and  $G/N$  have relatively prime orders.

1. Assume that there exists a subgroup  $H$  of  $G$  having the same order as  $G/N$ . Show that  $G = HN$ . (Here  $HN$  denotes the set  $\{xy \mid x \in H, y \in N\}$ .)
2. Show that  $\phi(N) = N$ , for all automorphisms  $\phi$  of  $G$ .

*Proof.* 1. Since  $N, H$  have relatively prime orders,  $N \cap H = \{e\}$ , thus we can write

$$G = N \rtimes_{\theta} H$$

where  $\theta(h)n = hn h^{-1}$ . One can show that the map  $\varphi : N \rtimes_{\theta} H \rightarrow G$  as

$$\varphi : (n, h) \mapsto nh$$

It is clear that  $\varphi$  is a homomorphism and injective, thus by a size argument we have  $\varphi$  is an isomorphism. This shows  $G = NH$  and similarly  $G = HN$ .

2. Any automorphism  $\phi$  of  $G$  permutes the  $p$ -Sylow subgroups. Suppose that  $|G| = p_1^{i_1} \dots p_k^{i_k}$ , then after rearranging,

$$|N| = p_1^{i_1} \dots p_j^{i_j}$$

because  $N$  and  $G/N$  have relatively prime orders. Hence  $N$  contains all the Sylow  $p_i$ -subgroups, hence  $\phi(N) = N$  for all automorphisms  $\phi$  of  $G$ . □

**Problem 1.16 (S2001-Q1).** Let  $G$  be a finite group and  $p$  the smallest prime number dividing the order  $|G|$  of  $G$ . Let  $H$  be a subgroup of  $G$  of index  $p$  in  $G$ . Show that  $H$  is necessarily a normal subgroup of  $G$ .

*Proof.*  $G$  has an action on  $G/H$  by left multiplication:  $\varphi : G \rightarrow \text{Aut}(G/H)$  such that

$$\varphi(g)(\bar{g}H) = g\bar{g}H$$

We will show that  $H = \ker(\varphi)$ . First we see that  $\ker(\varphi) \subset H$ :

$$\ker(\varphi) = \{g \in G : g\bar{g}H = \bar{g}H : \text{for all } \bar{g} \in G\}$$

letting  $\bar{g} \in H$  we see  $g \in \ker(\varphi)$  implies  $g \in H$ , i.e.,  $\ker(\varphi) \subset H$ .

Now we use a size argument to show  $|H| \leq |\ker(\varphi)|$ . We note that  $\text{im}(\varphi)$  is a subgroup of  $\text{Aut}(G/H) = S_p$ , thus

$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p!$$

because  $\frac{|G|}{|\ker(\varphi)|}$  also divides  $|G|$  and  $p$  is the smallest prime that divides  $p$ , we must have

$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p$$

Note that  $\frac{|G|}{|H|} = p$ , this gives

$$|H| \leq |\ker(\varphi)|$$

which shows  $H \subset \ker(\varphi)$ , hence  $H = \ker(\varphi)$ . □

(End of Page 5)

## 1.2 Class Formula, Classification of $p$ -groups

**Definition 1.2 (nilpotent group).** Let  $G$  be a group. Define inductively an increasing sequence  $\{e\} = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \dots$  of subgroups of  $G$  as follows: for  $i \geq 1$ ,  $Z_i$  is the subgroup of  $G$  corresponding to the center of  $G/Z_{i-1}$ . One can show that  $Z_i$  is normal in  $G$ . A group is *nilpotent* if  $Z_m = G$  for some  $m$ .

**Example 1.2.**

- $p$ -groups are nilpotent.
- Nilpotent groups are solvable.

**Proposition 1.3.** We have the following classification of groups of order  $p, p^2, p^3$ , for prime  $p$ .

- $|G| = p$  implies  $G \cong \mathbb{Z}/p\mathbb{Z}$ .

- $|G| = p^2$  implies

$$G \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}} \quad \text{or} \quad G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}$$

- $|G| = p^3$  implies that

$$G \cong \frac{\mathbb{Z}}{p^3\mathbb{Z}} \quad \text{or} \quad G/Z(G) \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \quad \text{or} \quad [G, G] = Z(G)$$

**Problem 1.17 (S2010-Q1).** Let  $G$  be a non-abelian group of order  $p^3$ , where  $p$  is prime. Determine the number of distinct conjugacy classes in  $G$ .

*Proof.* We know  $G$  has a nontrivial center, and if  $|Z(G)| = p^2$  or  $p^3$ , then  $G$  is abelian, this shows that  $|Z(G)| = p$ , now let  $g \in G \setminus Z(G)$ , then

$$Z(G) \subsetneq Z_g(G) \subsetneq G$$

where  $Z(G) \subsetneq Z_g(G)$  because  $g \in Z_g(G)$ , and  $Z_g(G) \subsetneq G$  since  $g \notin Z(G)$ . This shows that  $Z_g(G)$  is a subgroup of order  $p^2$ , in other words, the size of the conjugacy class of any  $g \in G \setminus Z(G)$  is

$$|[g]| = \left| \frac{G}{Z_g(G)} \right| = p$$

By the class formula,

$$|G| = |Z(G)| + \sum_{a \in A} |[a]|$$

where  $A$  contains one  $a$  from each nontrivial conjugacy class  $[a]$ . Thus we have

$$p^3 = p + Np \Rightarrow N = p^2 - 1$$

Every element in  $Z(G)$  is its own conjugacy class, thus the total number of conjugacy classes is

$$p^2 + p - 1$$

□

**Problem 1.18 (F2013-Q1).** Let  $p > 2$  be a prime. Classify groups of order  $p^3$  up to isomorphism. The two nonabelian groups of order  $p^3$  (for  $p \neq 2$ ), up to isomorphism, are:

$$\text{Heis}(\mathbb{Z}/(p)) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z}/(p) \right\}$$

and

$$\begin{aligned} G_p &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}/(p^2), a \equiv 1 \pmod{p} \right\} \\ &= \left\{ \begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix} \middle| m, b \in \mathbb{Z}/(p^2) \right\} \end{aligned}$$

**Problem 1.19** (F2014-Q5).

- (a) Prove that every group of order  $p^2$  (with  $p$  prime) is abelian. Then classify such groups up to isomorphism.
- (b) Give an example of a non-abelian group of order  $p^3$  for  $p = 3$ .  
*Suggestion: Represent the group as a group of matrices.*

*Proof.* (a) See Lemma 1.2. There are two abelian groups:  $\frac{\mathbb{Z}}{p^2\mathbb{Z}}, \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$ .

(b) See Problem 1.18. □

**Problem 1.20** (F2019-Q4, S2015-Q3). Find all irreducible representations of a finite  $p$ -group over a field of characteristic  $p$ .

*Proof.* Let  $G$  any finite  $p$ -group. Let  $V$  be an irreducible representation over  $\mathbb{F}_p$ , consider the  $[\mathbb{F}_p G]$ -module  $W$  generated by any  $v \in V \setminus \{0\}$ . We see  $W$  is a finite-dimensional vector space over  $\mathbb{F}_p$ , i.e.,

$$|W| = p^d$$

for some  $d \geq 1$ . We consider the action of  $G$  on  $W$ , all the orbits of this action either has size 1 or is a power of  $p$ , since  $G$  is a  $p$ -group, by the class formula, let  $N$  be the number of nontrivial orbits of size 1,

$$|W| \equiv 1 + N \pmod{p} \Rightarrow 1 + N \equiv 0 \pmod{p}$$

Hence there exists at least one nontrivial orbit  $\{v\}$  of size 1. We consider the vector space  $\bar{W}$  generated by  $v$  over  $\mathbb{F}_p$ : it is one-dimensional vector space contained in  $V$ , invariant under  $G$ , since  $V$  is irreducible, we must have  $V = \bar{W}$ . The action of  $G$  on  $\bar{W}$  is the trivial action, thus all irreducible representations of a finite  $p$ -group over  $\mathbb{F}_p$  are trivial. □

## 1.3 Random Problems

**Problem 1.21** (F2010-Q1). Let  $G$  be a group. Let  $H$  be a subset of  $G$  that is closed under group multiplication. Assume that  $g^2 \in H$  for all  $g \in G$ . Show that:

- $H$  is a normal subgroup of  $G$
- $G/H$  is abelian

*Proof.* • We first show that  $H$  is subgroup. It remains to show that if  $h \in H$ , then  $h^{-1} \in H$ , we know  $(h^{-1})^2 \in H$ , thus

$$h(h^{-1})^2 = h^{-1} \in H$$

as desired. Now we show that  $H$  is normal: for any  $h \in H, g \in G$ , we want to show  $ghg^{-1} \in H$ .

$$\begin{aligned} ghg^{-1} &= (gh)^2(gh)^{-1}hg^{-1} \\ &= (gh)^2h^{-1}g^{-1}hg^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2(g^{-1}h)^{-1}g^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2h^{-1} \in H \end{aligned}$$

as desired.

- It suffices to show that for any  $g_1, g_2 \in G$ , we have

$$g_1 g_2 H \subset g_2 g_1 H$$

Take any  $h \in H$ , we want to show  $(g_2 g_1)^{-1} g_1 g_2 h \in H$ ,

$$\begin{aligned} (g_2 g_1)^{-1} g_1 g_2 h &= (g_2 g_1)^{-2} g_2 g_1^2 g_2 h \\ &= (g_2 g_1)^{-2} (g_2 g_1^2)^2 (g_2 g_1^2)^{-1} g_2 h \\ &= (g_2 g_1)^{-2} (g_2 g_1^2)^2 g_1^{-2} h \in H \end{aligned}$$

as desired. □

**Problem 1.22 (S2014-Q1).** Find the number of colorings of the faces of a cube using 3 colors, where two colorings are considered equal if they can be transformed into each other by a rotation of the cube.

[Hint: Use Burnside's formula:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where a group  $G$  acts on a set  $X$ ,  $X/G$  is the set of orbits, and for every  $g \in G$ ,  $X^g$  is the fixed subset of  $g$  in  $X$ .]

*Proof.* Let  $X$  be the set of all possible colorings of the cube (equal cubes allowed), we have  $|X| = 3^6$ . We notice two things:

1. The group of rotations of a cube is  $S_4$ .
2. For  $\sigma_1, \sigma_2 \in S_4$  that are conjugates of each other,  $|X^{\sigma_1}| = |X^{\sigma_2}|$ . Therefore for the Burnside's formula becomes

$$|X/S_4| = \frac{1}{|S_4|} \sum_{[\sigma] \text{ conj classes}} |[ \sigma ]| \cdot |X^\sigma|$$

Now we analyze for each conjugacy class  $[\sigma]$ , what is  $|X^\sigma|$ .

- $(1 + 1 + 1 + 1)$ ,  $|[e]| = 1$  and  $|X^e| = 3^6$ .
- $(1 + 1 + 2)$ ,  $|[\sigma_1]| = 6$  and  $|X^{\sigma_1}| = 3^3$ .
- $(1 + 3)$ ,  $|[\sigma_2]| = 8$ , and  $|X^{\sigma_2}| = 3^2$ .
- $(2 + 2)$ ,  $|[\sigma_3]| = 6$ , and  $|X^{\sigma_3}| = 3^4$ .
- $(4)$ ,  $|[\sigma_4]| = 6$ , and  $|X^{\sigma_4}| = 3^3$ .

Thus combining we get

$$|X/S_4| = \frac{1}{24} (3^6 + 6 \cdot 3^3 + 8 \cdot 3^2 + 6 \cdot 3^4 + 6 \cdot 3^3) = 57$$

□

**Problem 1.23 (S2019-Q4).** Let  $f$  be a polynomial with  $n$  variables and define

$$\text{Sym}(f) = \{\sigma \in S_n \mid f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = f(x_1, x_2, \dots, x_n)\}.$$

1. Prove that  $\text{Sym}(f)$  is a subgroup of  $S_n$ .
2. Prove that the dihedral group  $D_4$  (the group of symmetries of the square) is isomorphic to  $\text{Sym}(x_1x_2 + x_3x_4)$ .

*Proof.* 1. The group  $S_n$  acts on the polynomial ring  $k[x_1, \dots, x_n]$ , by permuting the  $x_i$  to  $x_{\sigma(i)}$ , and we see that  $\text{Sym}(f)$  is the centralizer of a fixed element  $f \in k[x_1, \dots, x_n]$ , hence is a subgroup.

2. We have a total of 8 elements in  $\text{Sym}(x_1x_2 + x_3x_4)$ :

$$\{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$$

and we can be drawing a square that this corresponds to the group  $D_4$ . □

**Problem 1.24 (S2011-Q1, F2004-Q1).**

- (a) Let  $H$  be a proper nontrivial subgroup of a finite group  $G$  (i.e.,  $H \neq \{1\}$  and  $H \neq G$ ). Prove that  $G$  is not the union of all conjugates of  $H$  in  $G$ .
- (b) Give an example of an infinite group  $G$  for which the assertion in part (a) fails.

*Proof.* (a) If  $H$  is normal, then all conjugations of  $H$  is equal to  $H$ , but  $H \subsetneq G$ , this  $G$  is not the union of all conjugates of  $H$  in  $G$ . Now suppose  $H$  is not normal, assume the contrary that  $G$  is the union of all conjugates of  $H$ , then the number of distinct conjugates of  $H$  is  $[G : N_G(H)]$ , hence

$$|G| = [G : N_G(H)] \cdot |H| \iff [G : H] = [G : N_G(H)] \iff [N_G(H) : H] = 1$$

this is a contradiction since  $H$  is not normal. Thus  $G$  is not the union of all conjugates of  $H$  in  $G$ .

- (b) Consider

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2(\mathbb{C})$$

It is clear that conjugation of matrices in  $B$  do not give matrices with nonzero left bottom entry. □

**Problem 1.25 (S2009-Q1).** Let  $H$  and  $K$  be two solvable subgroups of a group  $G$  such that  $G = HK$ .

1. Show that if either  $H$  or  $K$  is normal in  $G$ , then  $G$  is solvable.
2. Give an example where  $G$  may not be solvable without the assumption in (a).

*Proof.* 1. WLOG suppose that  $H$  is normal, then the composite map  $\varphi = \pi \circ \iota$ :

$$K \xrightarrow{\iota} G \xrightarrow{\pi} G/H$$

is surjective, therefore

$$\{e\} \subset H \subset G$$

$G/H \cong K/\ker(\varphi)$  is solvable, hence  $G$  is solvable.

2. The smallest nonsolvable group is  $A_5$ , we have

$$A_5 = HK$$

where  $H = \langle (12345) \rangle$ ,  $K = A_4 = \{\sigma \in A_5 : \sigma(5) = 5\}$ . Now  $H, K$  are both solvable, but  $G$  is not.  $\square$

**Problem 1.26 (F2003-Q1).** In a group  $G$ , let 1 denote the identity element and let  $[x, y] = xyx^{-1}y^{-1}$  denote the commutator of elements  $x, y \in G$ .

1. Express  $[z, xy]x$  in terms of  $x$ ,  $[z, x]$ , and  $[z, y]$ .
2. Prove that if the identity  $[[x, y], z] = 1$  holds in  $G$ , then the following identities hold in  $G$ :

$$[x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z].$$

*Proof.* 1. We have

$$\begin{aligned} [z, xy]x &= zxyx^{-1}y^{-1}x^{-1}x \\ &= zxx^{-1}x^{-1}xyz^{-1}y^{-1} \\ &= [z, x]x[z, y] \end{aligned}$$

2. The identity  $[[x, y], z] = 1$  implies

$$[x, y]z = z[x, y]$$

Therefore using the identity in 1, we have

$$\begin{aligned} [x, yz] &= [x, y]y[x, z]y^{-1} \\ &= [x, y]yy^{-1}[x, z] \\ &= [x, y][x, z] \end{aligned}$$

Similarly

$$\begin{aligned} [xy, z] &= xyz^{-1}y^{-1}x^{-1}z^{-1} \\ &= xyz^{-1}z^{-1}zx^{-1}z^{-1} \\ &= x[y, z]x^{-1}[x, z] \\ &= [y, z][x, z] \\ &= [x, z][y, z] \end{aligned}$$

$\square$

**Problem 1.27 (S2005-Q1).** Let  $k$  be a field. Let  $G = \text{GL}_n(k)$  be the general linear group, where  $n > 0$ . Let  $D$  be the subgroup of diagonal matrices, and let  $N = N_G(D)$  be the normalizer of  $D$  in  $G$ . Determine the quotient group  $N/D$ .

**Problem 1.28 (F2009-Q1).** Let  $G$  be a finite group, and let  $\text{Aut}(G)$  be its automorphism group. Consider the group action  $\phi: \text{Aut}(G) \times G \rightarrow G$  defined by  $\phi(\sigma, g) = \sigma(g)$ . Assume  $G$  has exactly two orbits under this action.

1. Determine all such groups  $G$  up to isomorphism.
2. For each case from (a), determine when  $\text{Aut}(G)$  is solvable.



**Problem 1.29** (F2016-Q1). Determine  $\text{Aut}(S_3)$ .

*Proof.* Every element  $\sigma \in \text{Aut}(S_3)$  must send order 2 elements  $\{(12), (23), (13)\}$  to one another, and order 3 elements  $\{(123), (132)\}$  to each other. However,  $\sigma$  is determined by how it permutes

$$\{(12), (23), (13)\}$$

Thus every  $\sigma$  is an inner automorphism of the form  $\sigma_g(h) = ghg^{-1}$  for  $g, h \in S_3$  and  $g$  is some transposition. Hence

$$\text{Aut}(S_3) \cong S_3$$

□

## Chapter 2

# Representation Theory

**Proposition 2.1.** One should probably know the character table for  $S_3, S_4, A_5, S_5$ .

**Theorem 2.1 (Compilation of theorems).** Schur's lemma:

1. If  $\varphi : V \rightarrow W$  is a  $G$ -invariant map, i.e.,

$$\varphi(\rho(g)(v)) = \rho(g)\varphi(v)$$

where  $V, W$  are irreducible representations, then  $\varphi = 0$  or an isomorphism. This is true for any field  $k$  that  $V, W$  are over.

2. If  $\varphi : V \rightarrow V$  and everything as above, then

$$\varphi(v) = \lambda v$$

for some  $\lambda \in k^\times$ . This is only true when  $k$  is algebraically closed.

3.  $\text{Hom}_G(V, W) = \begin{cases} k & \text{if } V \cong W \\ 0 & \text{if not} \end{cases}$ , where  $V, W$  are irreducible. This is true for  $k$  algebraically closed.

4. Maschke's theorem: any finite dimensional representation  $V$  of a finite group  $G$  can be decomposed into a direct sum of irreducible representations.

$$V = V_1^{r_1} \oplus \cdots \oplus V_k^{r_k}$$

where  $V_i$ 's are irreducible. This is true when the characteristic  $k$  does not divide  $|G|$ , notably this always holds for characteristic 0 fields.

5. Do not mix them up.

**Proposition 2.2.**  $G$  is abelian if and only if every irreducible representation  $\rho$  is one-dimensional.

*Proof.* If  $G$  is abelian, take any irreducible representation  $\rho$ ,

$$\{\rho(g) : g \in G\}$$

can be simultaneously diagonalized (minimal polynomial has no repeated factor), i.e., there exists an eigenbasis  $\{e_1, \dots, e_n\}$  such that  $\rho(g)$  is a diagonal matrix for all  $g$ . This implies that the vector space generated by  $\{e_i\}$  for each  $i$  is a  $\rho$ -invariant subspace, since  $\rho$  is irreducible,  $\rho$  must be one-dimensional.

Conversely, let  $|G| = n$ , if every irreducible  $\rho$  is one-dimensional, then there are  $n$  irreducible representations, i.e.,  $n$  conjugacy classes, i.e.,  $G$  is abelian.  $\square$

## 2.1 Characters

**Problem 2.1 (S2008-Q4).** Let  $V \cong \mathbb{C}^n$  be an  $n$ -dimensional complex vector space with standard basis  $e_1, \dots, e_n$ . Consider the permutation action  $S_n \times V \rightarrow V$  defined by:

$$\sigma \cdot e_i = e_{\sigma(i)} \quad \text{for } \sigma \in S_n$$

Decompose  $V$  into irreducible  $\mathbb{C}[S_n]$ -modules.

*Proof.* We denote this action as  $\chi_v$ , we see that

$$\langle \chi_v, \chi_v \rangle = 2$$

Hence it is a sum of 2 irreducible representations, and because

$$\langle \chi_v, \chi_{\text{triv}} \rangle = 1$$

The computation is as follows:

$$\begin{aligned} \langle \chi_v, \chi_v \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} (\text{number of fixed points of } \sigma) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{number of } \{(i, j) : \sigma \text{ fixes } i, j\} \\ &= \frac{1}{n!} \sum_{1 \leq i, j \leq n} \text{number of } \{\sigma : \sigma \text{ fixes } i, j\} \\ &= 2 \end{aligned}$$

and similarly for  $\langle \chi_v, \chi_{\text{triv}} \rangle = 1$ . Thus,

$$\chi_v = \chi_{\text{triv}} \oplus \chi_{\text{std}}$$

where

$$\chi_{\text{std}} = \left\{ (v_1, \dots, v_n) : \sum_{i=1}^n v_i = 0 \right\}$$

$\square$

**Problem 2.2 (S2014-Q5).** Find the table of characters for  $S_4$ .

*Proof.*

Class Size	$[e]$ 1	$[(12)]$ 6	$[(12)(34)]$ 3	$[(123)]$ 8	$[(1234)]$ 6
$\chi_{\text{triv}}$	1	1	1	1	1
$\chi_{\text{sgn}}$	1	-1	1	1	-1
$\chi_2$	2	0	2	-1	0
$\chi_{\text{perm}} - \chi_{\text{triv}}$	3	1	-1	0	-1
$\chi_3 \otimes \chi_{\text{sgn}}$	3	-1	-1	0	1

$\square$

**Problem 2.3 (F2016-Q6).** Find a table of characters for the alternating group  $A_5$ .

**Problem 2.4 (F2015-Q3).** Let  $G = S_4$  (the symmetric group on four letters).

- Prove that  $G$  has two non-equivalent irreducible complex representations of dimension 3; call them  $\rho_1$  and  $\rho_2$ .
- Decompose the tensor product representation  $\rho_1 \otimes \rho_2$  into a direct sum of irreducible representations of  $G$ .

*Proof.* (a) We do this by the following formula: let  $d_i$  be the dimension of each irreducible representation of  $S_4$ , then

$$|S_4| = 24 = \sum_{i=1}^5 d_i^2$$

We notice that  $d_i \leq 3$  and there are two  $d_i = d_j = 3$ . We can write down the character table of  $S_4$ , and their character does not agree on all  $\sigma \in S_4$ , hence non-equivalent.

(b) We have

$$\chi_1 \otimes \chi_2(g) = \begin{cases} 9, & g = e \\ -1, & g = (12) \\ 0, & g = (123) \\ -1, & g = (1234) \\ 1, & g = (12)(34) \end{cases}$$

Hence we see

$$\rho_1 \otimes \rho_2 = \rho_{\text{sgn}} \oplus \rho_{\text{perm-triv}} \oplus \chi_{3 \otimes \text{sgn}} \oplus \chi_2$$

In other words, it is a direct sum of four irreducible representations of  $S_4$  except for the trivial one.  $\square$

**Problem 2.5 (F2011-Q4).** Let  $\rho: S_3 \rightarrow \text{GL}(2, \mathbb{C})$  be a two-dimensional irreducible representation of the symmetric group  $S_3$ .

- Decompose the tensor square  $\rho^{\otimes 2}$  into irreducible representations of  $S_3$ .
- Decompose the tensor cube  $\rho^{\otimes 3}$  into irreducible representations of  $S_3$ .

*Proof.* Using the character table of  $S_3$ :

Class Size	$[e]$ 1	$[(12)]$ 3	$[(123)]$ 2
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

(a) Let  $\chi \otimes \chi$  denote the corresponding character, we have

$$\chi \otimes \chi(g) = \begin{cases} 4, & g = e \\ 0, & g = (12) \\ 1, & g = (123) \end{cases}$$

Thus we see

$$\rho \otimes \rho = \rho_{\text{triv}} \oplus \rho_{\text{sgn}} \oplus \rho$$

(b) Similarly, we see

$$\chi^{\otimes 3}(g) = \begin{cases} 8, & g = e \\ 0, & g = (12) \\ -1, & g = (123) \end{cases}$$

Thus

$$\rho^{\otimes 3} = \rho_{\text{triv}} \oplus \rho_{\text{sgn}} \oplus \rho \oplus \rho \oplus \rho$$

□

**Problem 2.6 (F2014-Q3).** Let  $G = S_3$  be the symmetric group on three elements.

- (a) Prove that  $G$  has an irreducible complex representation of dimension 2 (call it  $\rho$ ), but none of higher dimension.
- (b) Decompose the triple tensor product  $\rho \otimes \rho \otimes \rho$  into a direct sum of irreducible representations of  $G$ .

*Proof.* (a) Notice that  $|S_3| = 6 = d_1^2 + d_2^2 + d_3^2$ .

(b) Same as the above question.

□

**Problem 2.7 (S2006-Q6).** Let  $S_4$  be the symmetric group on four elements.

- (a) Give an example of a non-trivial 8-dimensional complex representation of  $S_4$ .
- (b) Show that every 8-dimensional complex representation of  $S_4$  contains a 2-dimensional invariant subspace.

*Proof.* (a) There exists a nontrivial 2-dimensional irreducible representation of  $S_4$ , if we denote it as  $\rho$ , then

$$\rho \otimes \rho \otimes \rho$$

is an 8-dimensional representation of  $S_4$ .

- (b) We notice that 8 has the sum of a multiple of 3 and 1, thus it must contain another 1 or 2 in the sum, proving there exists a 2-dimensional invariant subspace. Warning: this subspace is not necessarily irreducible.

□

**Problem 2.8 (F2007-Q5).** Prove that every 5-dimensional complex representation of the alternating group  $A_4$  (the alternating group of degree 4) contains a 1-dimensional invariant subspace.

*Proof.* The character table is as follows:

Class	$[e]$	$[(123)]$	$[(12)(34)]$	$[(132)]$
Size	1	4	3	4
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	$\omega$	1	$\omega^2$
$\chi^{(3)}$	1	$\omega^2$	1	$\omega$
$\chi^{(4)}$	3	0	-1	0

where  $\omega = e^{\frac{2\pi i}{3}}$ . Since 5 cannot be written as a multiple of 3, it must contain a 1-dimensional invariant subspace (also 2, 3, 4). □

**Problem 2.9 (S2004-Q6).** Consider complex representations of a finite group  $G$ . Let  $\sigma_1, \dots, \sigma_s$  be representatives of the conjugacy classes of  $G$ , and let  $\chi_1, \dots, \chi_s$  be the irreducible characters of  $G$ .

- (a) Define an inner product on the  $\mathbb{C}$ -vector space of class functions on  $G$  such that  $\{\chi_1, \dots, \chi_s\}$  forms an orthonormal basis.
- (b) Let  $A = (a_{ij})$  be the character table matrix of  $G$ , where  $a_{ij} = \chi_i(\sigma_j)$  for  $1 \leq i, j \leq s$ . Prove that  $A$  is invertible.

*Proof.* (a) As expected, take two class functions  $f_1, f_2$ , we define

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

(b) **not finished**

□

**Problem 2.10 (S2018-Q4, S2007-Q5).** Is  $S_4$  isomorphic to a subgroup of  $\mathrm{GL}_2(\mathbb{C})$ ?

*Proof.* There is no injective homomorphism  $\varphi : S_4 \rightarrow \mathrm{GL}_2(\mathbb{C})$ . Note any such  $\varphi$  is called a 2-dim representation, it is either irreducible or not, we know that the 2-dimensional irreducible representation is not injective, and a direct sum of 1-dimensional representations is also not injective. □

**Problem 2.11 (S2010-Q6).** Let  $G$  be a group of order 24. Using representation theory, prove that  $G \neq [G, G]$ , where  $[G, G]$  denotes the commutator subgroup of  $G$ .

*Proof.* Suppose  $G = [G, G]$ , then we claim the only 1-dimensional representation  $\rho : G \rightarrow \mathbb{C}^\times$  is the trivial one. This is because if  $\rho$  is one-dim, then

$$[G, G] \subset \ker(\rho)$$

i.e.,  $\rho$  is trivial. However, there is no way to write

$$|G| = 24 = 1 + d_1^2 + \dots + d_k^2$$

where  $d_i \geq 2$ . Thus  $G \neq [G, G]$ . □

**Problem 2.12 (F2017-Q6).** Let  $G$  be a finite group with center  $Z(G)$ . Show that if  $G$  admits a faithful irreducible representation  $\rho : G \rightarrow \mathrm{GL}_n(k)$  for some positive integer  $n \in \mathbb{Z}^+$  and some field  $k$ , then the center  $Z(G)$  is cyclic.

*Proof.* (We will only do the case where  $k$  is algebraically closed). For any  $z \in Z(G)$ ,  $\rho(z) : V \rightarrow V$  is a  $G$ -map, i.e.,

$$\rho(z)(\rho(g)v) = \rho(g)(\rho(z)v)$$

We know by Schur's lemma that  $\rho(z)$  is a scalar multiplication:

$$\rho(z) \in k^\times$$

Because  $\rho$  is faithful,  $Z$  embeds into  $k^\times$  via  $\rho$ .

**Lemma 2.1 (Fact).** Any finite subgroup of  $k^\times$  for field  $k$  is cyclic.

Hence  $Z$  is cyclic. □

**Problem 2.13 (S2005-Q6).** Let  $V$  be a finite-dimensional vector space over a field  $k$ , and let  $G$  be a finite group with an irreducible representation  $\varphi: G \rightarrow \text{GL}(V)$ . Suppose  $H$  is a finite abelian subgroup of  $\text{GL}(V)$  contained in the centralizer of  $\varphi(G)$ . Prove that  $H$  must be cyclic.

*Proof.* Just like above, we embed  $H$  into  $k^\times$ . Let any  $h \in H$ , we note that  $h$  is a  $G$ -map, i.e., for any  $g \in G$ ,

$$h(\varphi(g)v) = \varphi(g)hv$$

this is because  $h$  is contained in the centralizer of  $\varphi(G)$ , i.e., commutes with all  $\varphi(g)$ . By Schur's Lemma, we have

$$h = \lambda I, \text{ where } \lambda \in k^\times$$

One can define a homomorphism  $\psi: H \rightarrow k^\times$  such that

$$\psi(\lambda I) = \lambda$$

This map embeds  $H$  into  $k^\times$ , and we are done by again observing any finite subgroup of  $k^\times$  is cyclic. □

**Problem 2.14 (F2010-Q6).** Let  $G$  be a non-abelian group of order  $p^3$ , where  $p$  is prime.

1. Determine the number of isomorphism classes of irreducible complex representations of  $G$ , and find their dimensions.
2. Which of these irreducible complex representations are faithful? Justify your answer.

*Proof.* 1. In S2010-Q1, we showed there are  $p^2 - 1 + p$  conjugacy classes in a non-abelian group  $G$  of order  $p^3$ . There are  $p^2$  one-dimensional irreducible representations because one dimensional representations of  $G$  are equivalent to one-dimensional representations of  $G/[G, G]$  which has size  $p^2$ , thus abelian and all irreducible representations are one-dimensional.

**Lemma 2.2 (Fact).** Let  $V$  be an irreducible representation, then  $\dim V$  divides  $|G|$ . (This is true when  $k$  is algebraically closed and characteristic 0).

Thus it is clear that there are  $p - 1$  representations of dimension  $p$ . (Sanity check:  $|G| = p^3 = p^2 + (p - 1)p^2$ ).

2. We claim that all the one-dimensional representations are not faithful and all the  $p$ -dimensional representations are. Recall  $\rho$  is irreducible if and only if  $\ker(\rho) = \{g : \rho(g)v = v \text{ for all } v\} = \{e\}$ .

**Lemma 2.3 (Fact).** Let  $\rho: G \rightarrow \mathbb{C}^\times$  be a one-dimensional irreducible representation, then

$$[G, G] \subset \ker(\rho)$$

Thus if  $\rho$  is one-dimensional, then  $\rho$  is not faithful. Now for the higher dimensional case:

**Lemma 2.4 (Fact).** If  $\rho : G \rightarrow \mathrm{GL}_p(\mathbb{C})$  is an irreducible representation, then  $\bar{\rho} : \frac{G}{\ker \rho} \rightarrow \mathrm{GL}_p(\mathbb{C})$  is also irreducible.

If  $\ker \rho$  is nontrivial, then it must divide the size of  $|G|$ , hence  $\frac{G}{\ker \rho}$  is abelian, i.e., all irreducible representations are one-dimensional. This is a contradiction since  $\rho$  is  $p$ -dimensional, thus  $\ker(\rho) = \{e\}$ , as desired.  $\square$

**Problem 2.15 (S2011-Q5).** Let  $K$  be a field, and let  $\Phi : G \rightarrow \mathrm{GL}_n(K)$  be an  $n$ -dimensional matrix representation of a group  $G$ . Define a  $G$ -action on the matrix ring  $M_n(K)$  by:

$$(g, A) \mapsto \Phi(g) \cdot A \quad (\text{matrix multiplication})$$

for  $g \in G$  and  $A \in M_n(K)$ . This action induces a group homomorphism  $\Psi : G \rightarrow \mathrm{GL}(M_n(K))$ . Express the character  $\chi_\Psi$  of  $\Psi$  in terms of  $\chi_\Phi$  (the character of  $\Phi$ ).

*Proof.* We compute the trace of the multiplication map by  $\Phi(g)$ , we consider a basis of  $M_n(K)$

$$\{M_{ij} : 1 \leq i, j \leq n\}$$

where  $M_{ij}$  is the matrix with only nonzero entry 1 at the  $ij$ th position. Then we see

$$\Phi(g)M_{ij} = (\Phi(g))_{ii}$$

Thus

$$\chi_\Psi = n \sum_{i=1}^n (\Phi(g))_{ii} = n \mathrm{tr}(\Phi(g))$$

$\square$

**Problem 2.16 (S2015-Q5).** Prove that a tensor product of irreducible representations over an algebraically closed field is irreducible.

*Proof.* Let  $\rho_1$  be irreducible of  $G_1$ ,  $\rho_2$  of  $G_2$ , then over an algebraically closed field, we know  $\rho_1 \otimes \rho_2$  is an irreducible representation of  $G_1 \times G_2$ , and we define

$$\rho_1 \otimes \rho_2(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$$

where  $g_1 \in G_1, g_2 \in G_2$ . And define  $\chi_1 \otimes \chi_2$  similarly, we have

$$\chi_1 \otimes \chi_2(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$$

One can use this to show that the tensor product  $\rho_1 \otimes \rho_2$  of two irreducible representations  $\rho_1$  and  $\rho_2$  is irreducible on  $G_1 \times G_2$ .  $\square$

**Problem 2.17 (S2001-Q3).** Calculate the complete character table for  $\mathbb{Z}/3\mathbb{Z} \times S_3$ , where  $S_3$  is the symmetric group in 3 letters.

*Proof.* Using the question above, it suffices to find all the irreducible characters of  $\mathbb{Z}/3\mathbb{Z}$  and  $S_3$ . There are 3 irreducible representations for each, hence there are 9 irreducible characters on  $\mathbb{Z}/3\mathbb{Z} \times S_3$  in total. We will skip the character table here but the exact filling of the table should follow the above

$$\chi_1 \otimes \chi_2(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$$

One thing to note is that let  $\rho$  be an irreducible representation of  $\mathbb{Z}/3\mathbb{Z}$ , then

$$\rho(g)^3 = 1$$

hence  $\rho(g) = e^{\frac{2\pi i}{3}}$ , where  $i = 0, 1, 2$ .  $\square$



## 2.2 Induced representations, Frobenius Reciprocity

**Problem 2.18 (S2009-Q6).** Let  $G = S_4$  and consider the subgroup  $H = \langle (1\ 2), (3\ 4) \rangle$ .

- Determine the number of irreducible complex characters of  $H$ .
- Choose a non-trivial irreducible character  $\psi$  of  $H$  over  $\mathbb{C}$  satisfying  $\psi((1\ 2)(3\ 4)) = -1$ . Compute the values of the induced character  $\text{Ind}_H^G(\psi)$  on all conjugacy classes of  $G$ , and express it as a sum of irreducible characters of  $G$ .

**Problem 2.19 (S2017-Q6).** Let  $G$  be a finite group and  $H$  an abelian subgroup. Show that every irreducible representation of  $G$  over  $\mathbb{C}$  has dimension  $\leq [G : H]$ .

*Proof.* We know that if  $A$  is commutative, then all the irreducible representations  $\rho$  of  $A$  are one-dimensional. Now we induce  $\rho$  to a representation on  $G$ :

$$\bar{\rho} : G \rightarrow \text{GL}(\mathbb{C})$$

We have

$$\text{Ind}_A^G = \bigoplus_{i=1}^n g_i V$$

where  $g_i$  is the representative for each coset  $G/A$ , and  $n = [G : A]$ . Therefore all representations of  $G$  has dimension  $[G : A]$ . Since not all induced representations are irreducible, any irreducible representation of  $G$  has dimension  $\leq [G : A]$ , as desired.  $\square$

**Problem 2.20 (S2008-Q6).** Give an example of non-isomorphic finite groups with same character table. Construct the character table in detail.

**Problem 2.21 (S2012-Q4).** Let  $Q$  be the quaternion group with presentation:

$$Q = \langle t, s_i, s_j, s_k \mid t^2 = 1, s_i^2 = s_j^2 = s_k^2 = s_i s_j s_k = t \rangle.$$

- Find four non-isomorphic 1-dimensional real representations of  $Q$ .
- Prove that the natural embedding  $\rho : Q \rightarrow \mathbb{H}$  given by:

$$\rho(t) = -1, \quad \rho(s_i) = i, \quad \rho(s_j) = j, \quad \rho(s_k) = k$$

defines an irreducible 4-dimensional real representation of  $Q$ , where  $\mathbb{H}$  is the algebra of real quaternions.

- Classify all irreducible complex representations of  $Q$  up to isomorphism.

**Problem 2.22 (F2004-Q6).** Let  $D_8$  be the dihedral group of order 8, with presentation:

$$D_8 = \langle r, s \mid r^4 = 1 = s^2, rs = sr^{-1} \rangle.$$

- Determine all conjugacy classes of  $D_8$ .
- Find the commutator subgroup  $D'_8$  of  $D_8$  and determine the number of distinct degree-1 (linear) characters of  $D_8$ .
- Construct the complete complex character table of  $D_8$ .

**Problem 2.23 (F2000-Q7).** Let  $D_{10}$  be the dihedral group of order 10, with presentation:

$$D_{10} = \langle r, s \mid r^5 = 1 = s^2, rs = sr^{-1} \rangle.$$

1. Determine all conjugacy classes of  $D_{10}$ .
2. Compute the commutator subgroup  $D'_{10}$  of  $D_{10}$ .
3. Prove that  $D_{10}/D'_{10} \cong \mathbb{Z}/2\mathbb{Z}$  and deduce that  $D_{10}$  has exactly two distinct degree-1 characters.
4. Construct the complete complex character table of  $D_{10}$ .

## Chapter 3

# Semisimple Algebra

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The most recent semisimple question appeared in 2020.

**Problem 3.1 (F2019-Q5).** Determine the number of two-sided ideals in the group algebra  $\mathbb{C}[S_3]$ , where  $S_3$  is the symmetric group of permutations of  $\{1, 2, 3\}$ .

**Problem 3.2 (F2009-Q6, F2001-Q5).** Let  $\rho: G \rightarrow \text{GL}_n(\mathbb{C})$  be an irreducible complex representation of a finite group  $G$ , with character  $\chi$ , and let  $C$  be the center of  $G$ .

1. Prove that for every  $s \in C$ , the matrix  $\rho(s)$  is a scalar multiple of the identity matrix  $I_n$ .
2. Using part (a), show that  $|\chi(s)| = n$  for all  $s \in C$ .
3. Establish the inequality  $n^2 \leq [G : C]$ , where  $[G : C]$  is the index of  $C$  in  $G$ .
4. Prove that if  $\rho$  is faithful (i.e., injective), then  $C$  must be cyclic.

*Proof.* 1.  $\mathbb{C}$  is algebraically closed therefore Schur's lemma applies (see F2017-Q6)

2. We know that

$$\rho(z) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & & & \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

We also know that  $C$  is finite and  $\rho(z^r) = I$ , which implies  $|r| = 1$ . This gives  $|\chi(s)| = n$  for all  $s \in C$ .

3. Y

4. If  $\rho$  is faithful, then  $C$  embeds into  $k^\times$ , and any finite subgroup of  $k^\times$  is cyclic.

□

**Problem 3.3 (S2017-Q5).** Prove directly from the definition of (left) semisimple ring that every such ring is (left) Noetherian and Artinian. (You may freely use facts about semisimple, Noetherian, and Artinian modules.)

**Problem 3.4 (S2005-Q4).** Let  $R$  be a ring and  $L$  a minimal left ideal of  $R$  (i.e.,  $L$  contains no non-zero proper left ideals of  $R$ ). Assuming  $L^2 \neq 0$ , prove that  $L = Re$  for some non-zero idempotent element  $e \in R$ .

**Problem 3.5** (S2016-Q6, F2006-Q6, F2008-Q6). Let  $A$  be a finite-dimensional semisimple algebra over  $\mathbb{C}$ , and let  $V$  be an  $A$ -module that decomposes as  $V \cong S \oplus S$ , where  $S$  is a simple  $A$ -module. Determine the automorphism group  $\text{Aut}_A(V)$  of  $V$  as an  $A$ -module.

**Problem 3.6** (S2010-Q5). Classify all non-commutative semi-simple rings with 512 elements. (You can use the fact that finite division rings are fields.)

**Problem 3.7** (F2011-Q5). Let  $A$  be a finite-dimensional semisimple algebra over  $\mathbb{C}$ , and let  $V$  be a finitely-generated  $A$ -module. Prove that  $V$  has only finitely many  $A$ -submodules if and only if  $V$  decomposes into a direct sum of pairwise irreducible non-isomorphic (i.e., simple)  $A$ -modules.



**Warning 3.1.** For one-dimensional irreducible representation,  $\rho : G \rightarrow \text{GL}_n(\mathbb{C})$ , they are equivalent to  $\rho : G^{ab} \rightarrow \text{GL}_n(\mathbb{C})$ .

# Chapter 4

## Linear Algebra I

Topics: finitely generated modules/PID, triangularization, diagonalization, Jordan canonical form.  
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**Problem 4.1 (F2018-Q1).** Let  $V$  be an  $n$ -dimensional vector space over a field  $k$  and let  $\alpha : V \rightarrow V$  be a linear endomorphism. Prove that the minimal and characteristic polynomials of  $\alpha$  coincide if and only if there is a vector  $v \in V$  so that:

$$\{v, \alpha(v), \dots, \alpha^{n-1}(v)\}$$

is a basis for  $V$ .

*Proof.* We will show there is not vector  $v \in V$  such that  $\{v, \alpha(v), \dots, \alpha^{n-1}(v)\}$  is linearly independent if and only if the minimal and characteristic polynomials aren't the same.

no  $v$  s.t.  $\{v, \dots, \alpha^{n-1}(v)\}$  is linearly independent  $\iff$  for all  $v \in V$ ,  $\{v, \dots, \alpha^{n-1}(v)\}$  is linearly dependent  
 $\iff \alpha^{n-1}v = c_0v + \dots + c_{n-2}\alpha^{n-2}v$  for all  $v$

We then claim that  $\alpha^{n-1}v = c_0v + \dots + c_{n-2}\alpha^{n-2}v$  for all  $v$  if and only if the minimal polynomial  $p_m$  has degree less than the characteristic polynomial  $p_n$ . Note that  $p_n$  has degree  $n$ , and we can write

$$p_n(t) = a_nt^n + \dots + a_1t + a_0$$

and  $p_n(\alpha)v = 0$  for all  $v \in V$ , therefore  $\alpha^{n-1}v = c_0v + \dots + c_{n-2}\alpha^{n-2}v$  for all  $v$  if and only if

$$p_n(\alpha)v = a_n\alpha^{n-1}v + \dots + a_1\alpha v + a_0 = p_{n-1}(\alpha)v = 0$$

where  $p_{n-1}$  is a polynomial of degree at most  $n-1$  and is such that  $p_{n-1}(\alpha) = 0$ . Thus no  $v$  such that  $\{v, \dots, \alpha^{n-1}v\}$  is linearly independent if and only if minimal and characteristic polynomial have different degrees.  $\square$

**Problem 4.2 (F2018-Q3).**

- (a) Fix a positive integer  $n$  and classify all finite modules over the ring  $\mathbb{Z}/n$ .
- (b) Prove, either using (a) or from first principles, for a fixed prime  $p$  that all finite modules over  $\mathbb{Z}/p$  are free.

**Problem 4.3 (F2017-Q2).** Let  $\Lambda$  be a free abelian group of finite rank  $n$ , and let  $\Lambda' \subset \Lambda$  be a subgroup of the same rank. Let  $x_1, \dots, x_n$  be a  $\mathbb{Z}$ -basis for  $\Lambda$ , and let  $x'_1, \dots, x'_n$  be a  $\mathbb{Z}$ -basis for  $\Lambda'$ . For each  $i$ , write  $x'_i = \sum_{j=1}^n a_{ij}x_j$ , and let  $A := (a_{ij}) \in \text{Mat}_{n \times n}(\mathbb{Z})$ . Show that the index  $[\Lambda : \Lambda']$  equals  $|\det A|$ .

**Problem 4.4 (S2001-Q5).**

- (a) Prove that an  $n \times n$  matrix  $A$  with entries in the field  $\mathbb{C}$  of complex numbers, satisfying  $A^3 = A$ , can be diagonalized over  $\mathbb{C}$ .
- (b) Does the statement in (a) remain true if one replaces  $\mathbb{C}$  by an arbitrary algebraically closed field  $F$ ? Why or why not?

*Proof.* (a)  $A$  is diagonalizable if and only if the minimal polynomial splits into distinct linear factors. The characteristic polynomial is  $p(t) = t(t+1)(t-1)$  and the minimal polynomial  $p_m \mid p$  thus  $A$  is diagonalizable.

- (b) This is not true. Take  $k$  to be a field of characteristic 2, then

$$p(t) = t(t^2 - 1) = t(t-1)^2$$

Thus the minimal polynomial could be  $(t-1)^2$ , i.e.,  $A$  is not necessarily diagonalizable. □

**Problem 4.5 (F2001-Q3).** Let  $A$  be an  $n \times n$  complex matrix with  $A^m = 0$  for some integer  $m > 0$ .

1. Show that if  $\lambda$  is an eigenvalue of  $A$ , then  $\lambda = 0$ .
2. Determine the characteristic polynomial of  $A$ .
3. Prove that  $A^n = 0$ .
4. Construct a  $5 \times 5$  matrix  $B$  satisfying  $B^3 = 0$  but  $B^2 \neq 0$ .
5. For any  $5 \times 5$  complex matrix  $M$  with  $M^3 = 0$  and  $M^2 \neq 0$ , is  $M$  necessarily similar to your matrix  $B$  from part (d)? Justify your answer.

1. Suppose  $\lambda$  is an eigenvalue, then there exists  $v \neq 0$ , such that

$$A^m v = \lambda^m v = 0 \Rightarrow \lambda = 0$$

2. The characteristic polynomial is  $p(t) = t^n$ .

3. Cayley-Hamilton theorem.

4. Can have

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The important is that the top left  $3 \times 3$  matrix  $A$  satisfies  $A^3 = 0, A^2 \neq 0$ . This is constructed by building  $B$  using the Jordan form.

5. No, the lower  $2 \times 2$  matrix could be

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

**Problem 4.6 (F2018-Q4).** In this question all modules are left modules.

Let  $k$  be a field of characteristic different from 2 and let  $G = \{e, g\}$  be the multiplicative group with two elements. Consider the group ring  $A = k[G]$ .

- (a) Show that the  $A$ -module  $A$  is a direct sum of two ideals of  $A$ .
  - List all proper ideals of  $A$ .
  - Is  $A$  a principal ideal domain?
- (b) Show that every  $A$ -module decomposes into a direct sum of simple  $A$ -modules.
- (c) Assume now that the characteristic of  $k$  is 2. Give an example of an  $A$ -module that cannot be decomposed into a direct sum of two simple  $A$ -modules.

**Problem 4.7 (S2003-Q3).** Prove that if a linear operator on a complex vector space is diagonal in some basis, then its restriction to any invariant subspace  $L$  is also diagonal in some basis of  $L$ .

*Proof.*

□

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**Problem 4.8 (S2017-Q4).** Let  $M$  be an invertible  $n \times n$  matrix with entries in an algebraically closed field  $k$  of characteristic not 2. Show that  $M$  has a square root, i.e. there exists  $N \in \text{Mat}_{n \times n}(k)$  such that  $N^2 = M$ .

**Problem 4.9 (S2008-Q1).** Let  $k$  be a field. Consider the subgroup  $B \subset \text{GL}_2(k)$  where

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in k, ad \neq 0 \right\}.$$

- (a) Let  $Z$  be the center of  $\text{GL}_2(k)$ . Show that

$$\bigcap_{x \in \text{GL}_2(k)} x^{-1} B x = Z.$$

- (b) Assume  $k$  is algebraically closed. Show that

$$\bigcup_{x \in \text{GL}_2(k)} x^{-1} B x = \text{GL}_2(k).$$

- (c) Assume  $k$  is a finite field. Can the statement in (b) still be true?

*Proof.* (a) Let  $y \in \bigcap_{x \in \text{GL}_2(k)} x^{-1} B x$ , then for all  $x \in \text{GL}_2(k)$ , we have  $xyx^{-1} \in B$ . This shows that

$$\begin{aligned} xyx^{-1} \in B \text{ for all } x &\iff xyx^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \\ &\iff x^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is a subspace for } y \text{ for all } x \\ &\iff \text{the whole vector space is the eigenspace of } y \\ &\iff y \text{ is a scalar} \\ &\iff y \in Z \end{aligned}$$

- (b) If  $k$  is algebraically closed, then any matrix can be written as a triangular matrix up to some basis change.
- (c) (If  $k$  is  $\mathbb{R}$ , the statement of (b) is also wrong). It's the same idea for finite groups, one can take  $g \in \overline{\mathbb{F}_p} \setminus \mathbb{F}_p$ , then the characteristic polynomial for the map of multiplication by  $g : \overline{\mathbb{F}_p} \rightarrow \overline{\mathbb{F}_p}$  where  $\overline{\mathbb{F}_p} = \mathbb{F}_{p^2}$  is a vector space over  $\mathbb{F}$  the minimal polynomial is  $(t - g)^2$  which is irreducible over  $\mathbb{F}_p$ .  $\square$

**Problem 4.10 (S2009-Q4).** Let  $E$  be a finite-dimensional vector space over an algebraically closed field  $k$ . Let  $A, B$  be  $k$ -endomorphisms of  $E$ . Assume  $AB = BA$ . Show that  $A$  and  $B$  have a common eigenvector.

*Proof.* Since  $k$  is algebraically closed, we know there exists at least one eigenvector of  $A$ , i.e., there exists  $\lambda$  such that  $Av = \lambda v$  for some  $v \neq 0$ . We denote this eigenspace by  $E_\lambda$ , and we note that  $E_\lambda$  is invariant under  $B$ : let  $v \in E_\lambda$

$$A(Bv) = \lambda(Bv)$$

thus  $Bv \in E_\lambda$  as well. Then it suffices to find an eigenvector of  $B$  living inside  $E_\lambda$ , this is done by noting  $B|_{E_\lambda}$  has an eigenvector in  $E_\lambda$ , as desired.  $\square$

**Problem 4.11 (F2005-Q6).** Let  $E$  be a finite-dimensional vector space over a field  $k$ . Assume  $S, T \in \text{End}_k(E)$ . Assume  $ST = TS$  and both of them are diagonalizable. Show that there exists a basis of  $E$  consisting of eigenvectors for both  $S$  and  $T$ .

*Proof.* It is the same proof as above except now we do this for all  $E_{\lambda_1}, \dots, E_{\lambda_k}$ .  $\square$

**Problem 4.12 (S2015-Q2).** Let  $A, B$  be two commuting operators on a finite dimensional space  $V$  over  $\mathbb{C}$  such that  $A^n = B^m$  is the identity operator on  $V$  for some positive integers  $n, m$ . Prove that  $V$  is a direct sum of 1-dimensional invariant subspaces with respect to  $A$  and  $B$  simultaneously.

*Proof.*  $\square$



## Chapter 5

# Linear Algebra II

Topics: exterior power, tensor algebras, trances, determinants  
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**Problem 5.1 (F2016-Q5).** Let  $A$  be a linear transformation of a finite dimensional vector space over a field of characteristic  $\neq 2$ .

- (1) Define the wedge product linear transformation  $\wedge^2 A = A \wedge A$ .
- (2) Prove that

$$\text{tr}(\wedge^2 A) = \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2)).$$

**Problem 5.2 (S2006-Q5).** Let  $V$  be a finite-dimensional vector space over a field  $k$ . Let  $T \in \text{End}_k(V)$ . Show that  $\text{tr}(T \otimes T) = (\text{tr}(T))^2$ . Here  $\text{tr}(T)$  is the trace of  $T$ .

**Problem 5.3 (S2016-Q4).** Let  $V$  and  $W$  be two finite dimensional vector spaces over a field  $K$ . Show that for any  $q > 0$ ,

$$\bigwedge^q (V \oplus W) \cong \sum_{i=0}^q \left( \bigwedge^i (V) \otimes_K \bigwedge^{q-i} (W) \right).$$

**Problem 5.4 (S2011-Q4).** Let  $F$  be a field, and  $V$  a finite-dimensional vector space over  $F$ , with  $\dim_F V = n$ .

- (a) Prove that if  $n > 2$ , the spaces  $\bigwedge^2(\bigwedge^2(V))$  and  $\bigwedge^4(V)$  are not isomorphic.
- (b) Let  $k$  be a positive integer. Prove that when  $v \in \bigwedge^k(V)$  and  $0 \neq x \in V$ ,  $v \wedge x = 0$  holds if and only if  $v = x \wedge y$  for some  $y \in \bigwedge^{k-1}(V)$ .

**Problem 5.5 (S2010-Q4).** Let  $V$  be a  $n$ -dimensional vector space over a field  $k$ . Let  $T \in \text{End}_k(V)$ .

- (a) Show that  $\text{tr}(T \otimes T \otimes T) = (\text{tr}(T))^3$ . Here  $\text{tr}(T)$  is the trace of  $T$ .
- (b) Find a similar formula for the determinant  $\det(T \otimes T \otimes T)$ .

## Chapter 6

# Linear Algebra III

Topics: random linear algebra problems

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**Proposition 6.1.** Let  $V$  be a  $m$  dimensional vector space, and  $W$  be  $n$  dimensional. Show that  $A : V \rightarrow V$  and  $B : W \rightarrow W$  has

$$\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$$

**Problem 6.1 (S2013-Q5).** Let  $A$  and  $B$  be  $n \times n$  matrices with complex coefficients. Assume that  $(A - I)^n = 0$  and  $A^k B = B A^k$  for some natural number  $k$ . Prove that  $AB = BA$  (Hint: Prove that  $A$  can be expressed as a function of  $A^k$ ).

**Problem 6.2 (F2011-Q2).** Consider the special orthogonal group  $G = SO(3, \mathbb{R})$ , namely,

$$G = \{A \in GL(3, \mathbb{R}) : A^T A = I_3, \det(A) = 1\}$$

(a) Show that for any element  $A$  in  $G$ , there exists a real number  $\alpha$  with  $-1 \leq \alpha \leq 3$  such that

$$A^3 - \alpha A^2 + \alpha A - I_3 = 0.$$

(b) For which real numbers  $\alpha$  with  $-1 \leq \alpha \leq 3$  does there exist an element  $A$  in  $G$  whose minimal polynomial is  $x^3 - \alpha x^2 + \alpha x - 1$ ? Explain your answer.

**Problem 6.3 (F2007-Q3).** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a real matrix such that  $a, b, c, d > 0$ .

- (1) Prove that  $A$  has two distinct real eigenvalues,  $\lambda > \mu$ .
- (2) Prove that  $\lambda$  has an eigenvector in the first quadrant and  $\mu$  has an eigenvector in the second quadrant.

**Problem 6.4 (S2007-Q1).** Prove that the integer orthogonal group  $O_n(\mathbb{Z})$  is a finite group. (By definition, an  $n \times n$  square matrix  $X$  over  $\mathbb{Z}$  is orthogonal if  $XX^t = I_n$ .)

**Problem 6.5 (F2008-Q4).** A differentiation of a ring  $R$  is a mapping  $D : R \rightarrow R$  such that, for all  $x, y \in R$ ,

- (1)  $D(x + y) = D(x) + D(y)$ ; and
- (2)  $D(xy) = D(x)y + xD(y)$ .

If  $K$  is a field and  $R$  is a  $K$ -algebra, then its differentiation are supposed to be over  $K$ , that is,

- (3)  $D(x) = 0$  for any  $x \in K$ .

Let  $D$  be a differentiation of the  $K$ -algebra  $M_n(K)$  of  $n \times n$ -matrices. Prove that there exists a matrix  $A \in M_n(K)$  such that  $D(X) = AX - XA$  for all  $X \in M_n(K)$ .

**Problem 6.6 (F2006-Q1).** Let  $SL_n(k)$  be the special linear group over a field  $k$ , i.e.  $n \times n$  matrices with determinant 1. Let  $I$  be the identity matrix, and  $E_{ij}$  be the elementary matrix that has 1 at  $(i, j)$ -entry and 0 elsewhere. Here  $1 \leq i \neq j \leq n$ .

- (1) Let  $C_{ij}$  be the centralizer of the matrix  $I + E_{ij}$ . Find explicit generators of  $C_{ij}$ .
- (2) Find the intersection

$$\bigcap_{1 \leq i \neq j \leq n} C_{ij}.$$

- (3) Determine all the elements in the conjugacy class of  $I + E_{ij}$ .

**Problem 6.7 (S2018-Q1).** Let  $F$  be a field of characteristic not equal to 2. Let  $D$  be the non-commutative algebra over  $F$  generated by elements  $i, j$  that satisfy the relations

$$i^2 = j^2 = 1, \quad ij = -ji.$$

Define  $k = ij$ .

- (a) Verify that  $D$  is isomorphic to the algebra  $M_2(F)$  of  $2 \times 2$  matrices in such a way that

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- (b) Write  $q = x + yi + zj + uk$  for  $x, y, z, u \in F$ . Verify that the norm

$$N(q) = x^2 - y^2 - z^2 + u^2$$

corresponds to the determinant under the isomorphism of part (a).

- (c) What does the involution  $q \mapsto \bar{q} = x - yi - zj - uk$  on  $D$  correspond to on the matrix side?

**Problem 6.8 (S2006-Q3).** Let  $V$  be a  $n$ -dimensional vector space over a field  $k$ , with a basis  $\{e_1, \dots, e_n\}$ . Let  $A$  be the ring of all  $n \times n$  diagonal matrices over  $k$ .  $V$  is a  $A$ -module under the action:

$$\text{diag}(\lambda_1, \dots, \lambda_n) \cdot (a_1 e_1 + \dots + a_n e_n) = (\lambda_1 a_1 e_1 + \dots + \lambda_n a_n e_n).$$

Find all  $A$ -submodules of  $V$ .

**Problem 6.9 (S2006-Q1).** Let  $\mathbb{F}_p$  be the field with  $p$  elements, here  $p$  is prime. Let  $\mathrm{SL}_2(\mathbb{F}_p)$  be the group of  $2 \times 2$  matrices over  $\mathbb{F}_p$  with determinant 1.

- (1) Find the order of  $\mathrm{SL}_2(\mathbb{F}_p)$ . Deduce that

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}$$

is a Sylow-subgroup of  $\mathrm{SL}_2(\mathbb{F}_p)$ .

- (2) Determine the normalizer of  $H$  in  $\mathrm{SL}_2(\mathbb{F}_p)$  and find its order.

**Problem 6.10 (S2004-Q1).** Let  $\mathbb{F}_2$  be the finite field with 2 elements.

- (a) What is the order of  $\mathrm{GL}_3(\mathbb{F}_2)$ , the group of  $3 \times 3$  invertible matrices over  $\mathbb{F}_2$ ?
- (b) Assuming the fact that  $\mathrm{GL}_3(\mathbb{F}_2)$  is a simple group, find the number of elements of order 7 in  $\mathrm{GL}_3(\mathbb{F}_2)$ .

**Problem 6.11 (S2002-Q4).** For a field  $K$ , let  $\mathrm{SL}_2(K)$  be the special linear group over  $K$ , i.e. the group of  $2 \times 2$ -matrices over  $K$  with determinant 1, and let  $\mathrm{PSL}_2(K)$  be the quotient of  $\mathrm{SL}_2(K)$  by its center, i.e. the projective special linear group. Find the order of  $\mathrm{PSL}_2(F_7)$  where  $F_7$  denotes the finite field of 7 elements.

**Problem 6.12 (S2007-Q4).** Find the invertible elements, the zero divisors and the nilpotent elements in the following rings:

- (a)  $\mathbb{Z}/p^n\mathbb{Z}$ , where  $n$  is a natural number,  $p$  is a prime one.
- (b) the upper triangular matrices over a field.

## Chapter 7

# Homological Algebra

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### Problem 7.1 (S2012-Q2).

- (a) Prove that if  $M$  is an abelian group and  $n$  is a positive integer, the tensor product  $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$  can be naturally identified with  $M/nM$ .
- (b) Compute the tensor product over  $\mathbb{Z}$  of  $\mathbb{Z}/n\mathbb{Z}$  with each of  $\mathbb{Z}/m\mathbb{Z}$ ,  $\mathbb{Q}$  and  $\mathbb{Q}/\mathbb{Z}$ . Also compute the tensor products  $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ ,  $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$ , and  $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$ .
- (c) Let  $\mathbb{Z}^{\mathbb{N}}$  denote the (abelian) group of sequences  $(a_i)_{i \in \mathbb{N}}$  in  $\mathbb{Z}$  under termwise addition, and  $\mathbb{Z}^{(\mathbb{N})}$  the subgroup of sequences for which  $a_i = 0$  for all but finitely many  $i$ . Define  $\mathbb{Q}^{\mathbb{N}}$  and  $\mathbb{Q}^{(\mathbb{N})}$  analogously. Compare  $\mathbb{Z}^{(\mathbb{N})} \otimes_{\mathbb{Z}} \mathbb{Q}$  to  $\mathbb{Q}^{(\mathbb{N})}$ , and  $\mathbb{Z}^{\mathbb{N}} \otimes_{\mathbb{Z}} \mathbb{Q}$  to  $\mathbb{Q}^{\mathbb{N}}$ .

### Problem 7.2 (F2006-Q4). Let $R$ be a commutative ring. Let $M$ be an $R$ -module.

- (1) Write down the definition of  $\mathcal{T}(M)$ , the tensor algebra of  $M$ .
- (2) Assume  $R = \mathbb{Z}$  and  $M = \mathbb{Q}/\mathbb{Z}$ . Compute  $\mathcal{T}(M)$ .
- (3) If  $M$  is a vector space over a field  $R$ , show that  $\mathcal{T}(M)$  contains no zero divisors.

### Problem 7.3 (S2009-Q5). Consider the $\mathbb{Z}$ -modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$ for all positive integers $i$ . Let $M = \prod_{i=1}^{\infty} M_i$ . Let $S = \mathbb{Z} - \{0\}$ .

- (a) Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} M \cong S^{-1}M.$$

Here  $S^{-1}M$  is the localization of  $M$ .

- (b) Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i=1}^{\infty} M_i \neq \prod_{i=1}^{\infty} (\mathbb{Q} \otimes_{\mathbb{Z}} M_i).$$

### Problem 7.4 (S2013-Q1). Prove that, as a $\mathbb{Z}$ -module, $\mathbb{Q}$ is flat but not projective.

**Problem 7.5 (F2008-Q5).** For each  $n \in \mathbb{Z}$ , define the ring homomorphism

$$\phi_n : \mathbb{Z}[x] \rightarrow \mathbb{Z} \text{ by } \phi_n(f) = f(n).$$

This gives a  $\mathbb{Z}[x]$ -module structure on  $\mathbb{Z}$ , i.e.,

$$f \circ a = f(n) \cdot a \text{ for all } f \in \mathbb{Z}[x] \text{ and } a \in \mathbb{Z}.$$

Now given two integers  $m, n \in \mathbb{Z}$ , compute the tensor product  $\mathbb{Z} \otimes_{\mathbb{Z}[x]} \mathbb{Z}$  where the left-hand copy of  $\mathbb{Z}$  uses the module structure from  $\phi_n$  and the right-hand copy of  $\mathbb{Z}$  uses the module structure from  $\phi_m$ . (Note: The answer depends on the numbers  $n$  and  $m$ .)

**Problem 7.6 (F2014-Q2).** Let  $R = \mathbb{Q}[X]$ ,  $I$  and  $J$  the principal ideals generated by  $X^2 - 1$  and  $X^3 - 1$  respectively. Let  $M = R/I$  and  $N = R/J$ . Express in simplest terms [the isomorphism type of] the  $R$ -modules  $M \otimes_R N$  and  $\text{Hom}_R(M, N)$ . **Explain.**

**Problem 7.7 (F2004-Q5).** Consider the ideal  $I = (2, x)$  in  $R = \mathbb{Z}[x]$ .

- (a) Construct a non-trivial  $R$ -module homomorphism  $I \otimes_R I \rightarrow R/I$ , and use that to show that  $2 \otimes x - x \otimes 2$  is a non-zero element in  $I \otimes_R I$ .
- (b) Determine the annihilator of  $2 \otimes x - x \otimes 2$ .

**Problem 7.8 (S2018-Q5).** Let  $n$  be a positive integer and  $A$  an abelian group. Prove that

$$\text{Ext}^1(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA.$$

**Problem 7.9 (F2002-Q3).** Working over the integers, calculate (and show your work in a readable fashion)  $\text{Tor}(\mathbb{Z}/(p), \mathbb{Z}/(p))$ .

**Problem 7.10 (F2002-Q4).** Working over the integers, calculate (and show your work in a readable fashion)  $\text{Ext}(\mathbb{Z}/(p), \mathbb{Z}/(p))$ .

**Problem 7.11 (S2018-Q2).** Let  $R$  be a commutative ring. An  $R$ -module  $M$  is said to be finitely presented if there exists a right-exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some non-negative integers  $m, n$ . Prove that any finitely generated projective  $R$ -module  $P$  is finitely presented.

**Problem 7.12 (F2013-Q3).** Let  $R$  be a commutative ring with unity. Given an  $R$ -module  $A$  and an ideal  $I \subset R$ , there is a natural  $R$ -module homomorphism  $A \otimes_R I \rightarrow A \otimes_R R \cong A$  induced by the inclusion  $I \subset R$ . In the following three steps you shall prove the flatness criterion:  *$A$  is flat if and only if for every finitely generated ideal  $I \subset R$  the natural map  $A \otimes_R I \rightarrow A \otimes_R R$  is injective.*

- (a) Prove that if  $A$  is flat and  $I \subset R$  is a finitely generated ideal then  $A \otimes_R I \rightarrow A \otimes_R R$  is injective.
- (b) If  $A \otimes_R I \rightarrow A \otimes_R R$  is injective for every finitely generated ideal  $I$ , prove that  $A \otimes_R I \rightarrow A \otimes_R R$  is injective for every ideal  $I$ . Show that if  $K$  is any submodule of a free module  $F$  then the natural map  $A \otimes_R K \rightarrow A \otimes_R F \cong A$  induced by the inclusion  $K \subset F$  is injective (*Hint: the general case reduces to the case when  $F$  has finite rank*).
- (c) Let  $\psi : L \rightarrow M$  be an injective homomorphism of  $R$ -modules. Prove that the induced map  $1 \otimes \psi : A \otimes_R L \rightarrow A \otimes_R M$  is injective (*Hint: Write  $M$  as a quotient  $f : F \rightarrow M$  of a free module  $F$ , giving a short exact sequence  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  and consider the commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & J & \longrightarrow & L \longrightarrow 0 \\ & & \downarrow \text{id} & & & & \downarrow \varphi \\ 0 & \longrightarrow & K & \longrightarrow & F & \xrightarrow{f} & M \longrightarrow 0 \end{array}$$

where  $J = f^{-1}(\psi(L))$ .

**Problem 7.13 (F2013-Q4).**

- (a) Let  $R$  be a P.I.D. Prove that a finitely generated  $R$ -module  $M$  is flat if and only if  $M$  is torsion-free (hence, free by the structure theorem).
- (b) Give an example of an integral domain  $R$  and a torsion-free  $R$ -module  $M$  such that  $M$  is not free.

**Problem 7.14 (F2000-Q6).** Let  $R$  be the ring  $\mathbb{Q}[X]/(X^7 - 1)$ , where  $(X^7 - 1)$  is the ideal generated by  $X^7 - 1$  in  $\mathbb{Q}[X]$ . Give an example of a finitely generated projective  $R$ -module which is not  $R$ -free. (We remind you that an  $R$ -module is called projective if it is a direct summand of a free  $R$ -module.)

## Chapter 8

# Commutative Algebra

Topics: basic properties, Nakayama's lemma, integrality.

**Problem 8.1 (S2017-Q1).** Let  $A$  be a commutative ring, and define the *nilradical*  $\sqrt{0}$  to be the set of nilpotent elements in  $A$ . Show that  $\sqrt{0}$  is equal to the intersection of all prime ideals in  $A$ . Show that if  $A$  is reduced, then  $A$  can be embedded into a product of fields.

This one is complicated manipulation, so we omit.

**Problem 8.2 (F2004-Q2).** Let  $\mathfrak{N}$  be the set of all nilpotent elements in a ring  $R$ . Assume first that  $R$  is commutative.

- (a) Show that  $\mathfrak{N}$  is an ideal in  $R$ , and  $R/\mathfrak{N}$  contains no non-zero nilpotent elements.
- (b) Show that  $\mathfrak{N}$  is the intersection of all the prime ideals of  $R$ .
- (c) Give an example with  $R$  **non**-commutative where  $\mathfrak{N}$  is not an ideal in  $R$ .

**Problem 8.3 (S2014-Q4).** Let  $L/K$  be a Galois extension of degree  $p$  with  $\text{char} K = p$ . Show that  $L = K(\theta)$ , where  $\theta$  is a root of  $x^p - x - a$ ,  $a \in K$ , and, conversely, any such extension is Galois of degree 1 or  $p$ .

**Problem 8.4 (S2009-Q2).** Consider  $\mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\}$  where  $\omega$  is a non-trivial cube root of 1. Show that  $\mathbb{Z}[\omega]$  is an Euclidean domain.

**Problem 8.5 (F2006-Q3).** Let  $A$  be a principal integral domain and  $K$  be its field of fractions. Assume that  $R$  is a ring such that  $A \subset R \subset K$ . Show that  $R$  is also a principal integral domain.

**Problem 8.6 (F2001-Q2).** Let  $S$  denote the ring  $\mathbb{Z}[X]/X^2\mathbb{Z}[X]$ , where  $X$  is a variable.

- (a) Show that  $S$  is a free  $\mathbb{Z}$ -module and find a  $\mathbb{Z}$ -basis for  $S$ .
- (b) Which elements of  $S$  are units (i.e. invertible with respect to multiplication)?
- (c) List all the ideals of  $S$ .
- (d) Find all the nontrivial ring morphisms defined on  $S$  and taking values in the ring of Gaussian integers  $\mathbb{Z}[i]$ .



**Problem 8.7 (S2001-Q6).** Let  $R$  be the ring  $\mathbb{Z}[X, Y]/(YX^2 - Y)$ , where  $X$  and  $Y$  are two algebraically independent variables, and  $(YX^2 - Y)$  is the ideal generated by  $YX^2 - Y$  in  $\mathbb{Z}[X, Y]$ .

- Show that the ideal  $I$  generated by  $Y - 4$  in  $R$  is not prime.
- Provide the complete list of prime ideals in  $R$  containing the ideal  $I$  described in question (a).
- Which of the ideals found in (b) are maximal?

**Problem 8.8 (F2017-Q3).** In this problem all rings are commutative.

- Let  $R$  be a local ring with maximal ideal  $\mathfrak{m}$ , let  $N$  and  $M$  be finitely generated  $R$ -modules, and let  $f: N \rightarrow M$  be an  $R$ -linear map. Show that  $f$  is surjective if and only if the induced map  $N/\mathfrak{m}N \rightarrow M/\mathfrak{m}M$  is.
- Recall that a module  $M$  over a ring  $R$  is *projective* if the functor  $\text{Hom}_R(M, -)$  is exact. Show that if  $R$  is local and  $M$  is finitely generated projective, then  $M$  is free.

**Problem 8.9 (F2010-Q4).** Let  $A$  be a commutative Noetherian local ring with maximal ideal  $\mathfrak{m}$ . Assume  $\mathfrak{m}^n = \mathfrak{m}^{n+1}$  for some  $n > 0$ . Show that  $A$  is Artinian.

**Problem 8.10 (F2009-Q5).** Let  $A, B$  be two Noetherian local rings with maximal ideals  $\mathfrak{m}_A, \mathfrak{m}_B$ , respectively. Let  $f: A \rightarrow B$  be a ring homomorphism such that  $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ . Assume that:

- $A/\mathfrak{m}_A \rightarrow B/\mathfrak{m}_B$  is an isomorphism.
- $\mathfrak{m}_A \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$  is surjective.
- $B$  is a finitely generated  $A$ -module (via  $f$ ). Show that  $f$  is surjective.

**Problem 8.11 (F2015-Q6).** Let  $K$  be a finite algebraic extension of  $\mathbb{Q}$ .

- Give the definition of an integral element of  $K$ .
- Show that the set of integral elements in  $K$  form a sub-ring of  $K$ .
- Determine the ring of integers in each of the following two fields. No credit for memorized answers:  $\mathbb{Q}(\sqrt{13})$ , and  $\mathbb{Q}(\sqrt[3]{2})$ .

**Problem 8.12 (F2009-Q2).** Consider  $\mathbb{Q}[\sqrt{5}] = \{a + b\sqrt{5} | a, b \in \mathbb{Q}\}$ . Determine the integral closure of  $\mathbb{Z}$  in  $\mathbb{Q}[\sqrt{5}]$ .

**Problem 8.13 (S2012-Q5).**

- Give the definition of a Dedekind domain.
- Give an example of a Dedekind domain that is not a principal ideal domain. Verify from the definition that it is a Dedekind domain, and also that it isn't a principal ideal domain.

**Problem 8.14 (S2005-Q5).** Let  $A$  be an integral domain and let  $K$  be its field of fractions. Let  $A'$  be the integral closure of  $A$  in  $K$ . Let  $P \subset A$  be a prime ideal and let  $S = A - P$ . (Note that  $A_P = S^{-1}A$  is contained in  $K$ .) Show that  $A_P$  is integrally closed in  $K$  if and only if  $(A'A) \otimes_A A_P = 0$ .

**Problem 8.15** (F2013-Q2). Let  $a$  be an integral algebraic number such that its norm is 1 for any imbedding into  $\mathbb{C}$ , the field of complex numbers. Prove that  $a$  is a root of unity.

**Problem 8.16** (F2004-Q4). Let  $\lambda_1, \dots, \lambda_n$  be roots of unity, with  $n \geq 2$ . Assume that  $\frac{1}{n} \sum_{i=1}^n \lambda_i$  is integral over  $\mathbb{Z}$ . Show that either  $\sum_{i=1}^n \lambda_i = 0$  or  $\lambda_1 = \lambda_2 = \dots = \lambda_n$ .

## Chapter 9

# Ring Theory Random

**Proposition 9.1.** Let  $I \subset R$  be an ideal, then the following are equivalent:

1.  $I$  is a prime ideal.
2. There exists a field  $K$  and  $\varphi : R \rightarrow K$  such that  $I = \ker(\varphi)$ .

*Proof.* (1) $\Rightarrow$ (2). Let  $K$  be the field of fractions of  $R/I$ , which is an integral domain. (2) $\Rightarrow$  (1) is obvious given  $K$  is a field.  $\square$

**Problem 9.1 (S2010-Q2).** Let  $R$  be a ring such that  $r^3 = r$  for all  $r \in R$ . Show that  $R$  is commutative. (Hint: First show that  $r^2$  is central for all  $r \in R$ .)

*Proof.* This question is not so constructive and is purely computational (as far as I am aware) so I will skip it here.  $\square$

**Problem 9.2 (S2006-Q2).** Let  $R$  be a ring with identity 1. Let  $x, y \in R$  such that  $xy = 1$ .

- (1) Assume  $R$  has no zero-divisor. Show that  $yx = 1$ .
- (2) Assume  $R$  is finite. Show that  $yx = 1$ .

*Proof.* (1) We know  $x, y \neq 0$ , therefore consider

$$x(yx - 1) = 0$$

Since  $R$  has no zero-divisor, we must have  $yx - 1 = 0$ , as desired.

(2) We note the right multiplication map  $m_x : R \rightarrow R$  by  $x$  is injective: suppose  $r_1, r_2 \in R$  and

$$r_1x = xr_2x$$

multiplying both sides by  $y$  we see  $r_1 = r_2$ . Since  $R$  is finite, this map is also surjective, i.e., there exists  $s \in R$  such that

$$sx = 1$$

Now we see

$$yx - 1 = sx(yx - 1) = sx - sx = 0$$

as desired.  $\square$

# Chapter 10

## Tensor Products over Fields

Page 42-43

**Proposition 10.1.** If  $L/k$  is finite separate extension, then there exists  $\alpha \in L$  such that

$$L = k(\alpha)$$

**Example 10.1.** Write  $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$  as a product of fields:

$$\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \frac{\mathbb{Q}[x]}{(x^2 - 3)} \cong \frac{\mathbb{Q}(\sqrt{2}[x])}{(x^3 - 2)}$$

and  $(x^3 - 2)$  does not have a root in  $\mathbb{Q}(\sqrt{2})$ , thus

$$\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{2})\sqrt{3}$$

**Example 10.2.** Similarly, write the following as a product of fields

$$\begin{aligned} \mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{2}) &= \mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \frac{\mathbb{Q}[x]}{(x^4 - 2)} \\ &= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x^4 - 2)} \\ &= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x - \sqrt[4]{2})(x + \sqrt[4]{2})(x^2 + \sqrt{2})} \\ &= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x - \sqrt[4]{2})} \times \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x + \sqrt[4]{2})} \times \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x^2 + \sqrt{2})} \end{aligned}$$

By the Chinese Remainder theorem

**Lemma 10.1 (CRT).** Let  $R$  be a PID, and  $I + J = (1)$ , then

$$\frac{R}{IJ} = \frac{R}{I} \times \frac{R}{J}$$

We have

$$\mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{2}) \cong \mathbb{Q}(\sqrt[4]{2}) \times \mathbb{Q}(\sqrt[4]{2}) \times \mathbb{Q}(\sqrt[4]{2})(i)$$

**Example 10.3.** The field extension generated  $(x^p - t)$  of field  $\mathbb{F}_p(t)$  is not separable, i.e.,

$$\frac{\mathbb{F}_p(t)[x]}{(x^p - t)}$$

is not separable. Consider the element  $x$ , then the minimal polynomial  $m(s) = s^p - t$  can be written as

$$s^p - t = s^p - x^p = (s - x)^p$$

**Problem 10.1 (S2017-Q3).** Let  $K/k$  be a finite separable field extension, and let  $L/k$  be any field extension. Show that  $K \otimes_k L$  is a product of fields.

**Problem 10.2 (F2019-Q3).** Let  $F, L$  be extensions of a field  $K$ . Suppose that  $F/K$  is finite. Show that there exists an extension  $E/K$  such that there are monomorphisms of  $F$  into  $E$  and of  $L$  into  $E$  which are identical on  $K$ .

**Problem 10.3 (F2009-Q4).** Let  $E$  and  $F$  be finite field extensions of a field  $k$  such that  $E \cap F = k$ , and that  $E$  and  $F$  are both contained in a larger field  $L$ . Assume that  $E$  is Galois over  $k$ . Show that  $E \otimes_k F \cong EF$ .

**Problem 10.4 (S2008-Q5).** Let  $k$  be a field of characteristic zero. Assume that  $E$  and  $F$  are algebraic extensions of  $k$  and both contained in a larger field  $L$ . Show that the  $k$ -algebra  $E \otimes_k F$  has no nonzero nilpotent elements.

**Problem 10.5 (S2004-Q5).** Show that there is a  $\mathbb{C}$ -algebra isomorphism between  $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$  and  $\mathbb{C} \times \mathbb{C}$ .

**Problem 10.6 (F2005-Q5).** Let  $\mathbb{C}$  and  $\mathbb{R}$  be complex and real number fields. Let  $\mathbb{C}(x)$  and  $\mathbb{C}(y)$  be function fields of one variable. Consider  $\mathbb{C}(x) \otimes_{\mathbb{R}} \mathbb{C}(y)$  and  $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$ .

- (1) Determine if they are integral domains.
- (2) Determine if they are fields.

**Problem 10.7 (F2003-Q4).** Verify the isomorphism of algebras over a field  $K$ :

$$\mathbb{M}_n(K) \otimes_K \mathbb{M}_m(K) \simeq \mathbb{M}_{mn}(K).$$

[Note:  $\mathbb{M}_n(K)$  denotes the algebra of  $n \times n$  matrices over  $K$ .]

# Chapter 11

## Irreducibility of Polynomials

Page 44-45

**Proposition 11.1.** Fix any prime  $p$ , the polynomial

$$f(x) = x^{p-1} + \cdots + x + 1$$

is irreducible over  $\mathbb{Q}$ . Similarly

$$g(x) = x^{p-1} - x^{p-2} + \cdots - x + 1$$

is irreducible over  $\mathbb{Q}$ .

*Proof.* This is an application of Eisenstein. Write

$$f(x) = \frac{x^p - 1}{x - 1}$$

and replace  $x$  with  $x + 1$  we get

$$\begin{aligned} f(x) &= \frac{(x+1)^p - 1}{x} \\ &= \frac{\sum_{k=1}^n \binom{p}{k} x^k}{x} \\ &= \sum_{k=1}^n \binom{p}{k} x^{k-1} \end{aligned}$$

We apply Eisenstein with prime  $p$  to see  $f$  is irreducible. □

**Proposition 11.2.** For any prime  $p$ , either  $\sqrt{2} \in \mathbb{F}_p$  or  $\sqrt{3} \in \mathbb{F}_p$  or  $\sqrt{6} \in \mathbb{F}_p$ .

*Proof.* We know there exists a legendre symbol (a character)  $\chi : \mathbb{F}_p^\times \rightarrow \{\pm 1\}$  such that for  $g \in \mathbb{F}_p$ ,

$$\chi(g) = \begin{cases} 1, & \text{if } g \text{ is a square} \\ -1, & \text{if } g \text{ is not a square} \end{cases}$$

Suppose that  $\sqrt{2}$  and  $\sqrt{3}$  are not in  $\mathbb{F}_p$ , then

$$\chi(2) = \chi(3) = -1$$

i.e., 2, 3 are not squares. However,

$$\chi(2 \cdot 3) = \chi(6) = 1$$

This implies that 6 is a square and  $\sqrt{6} \in \mathbb{F}_p$ , as desired.  $\square$

**Corollary 11.1.** The following polynomial

$$f(x) = (x^2 - 1)(x^3 - 1)(x^6 - 1)$$

has a linear factor.

**Problem 11.1 (S2018-Q3).** Let  $R$  be the ring  $\mathbb{Z}[\zeta_p]$ , where  $p$  is a prime number and  $\zeta_p$  denotes a primitive  $p$ th root of unity in  $\mathbb{C}$ . Prove that if an integer  $n \in \mathbb{Z}$  is divisible by  $1 - \zeta_p$  in  $R$ , then  $p$  divides  $n$ .

**Problem 11.2 (F2008-Q2).** Show that the polynomial  $x^5 - 5x^4 - 6x - 2$  is irreducible in  $\mathbb{Q}[x]$ .

**Problem 11.3 (F2003-Q3).** Obtain a factorization into irreducible factors in  $\mathbb{Z}[x]$  of the polynomial  $x^{10} - 1$ .

**Problem 11.4 (S2004-Q3).** Let  $k$  be a field with characteristic 0. Let  $m \geq 2$  be an integer. Show that  $f(x, y) = x^m + y^m + 1$  is irreducible in  $k[x, y]$ .

**Problem 11.5 (S2017-Q2, S2007-Q3).** Write down the minimal polynomial for  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$  and prove that it is reducible over  $\mathbb{F}_p$  for every prime number  $p$ .

*Proof.* The minimal polynomial of  $\sqrt{2} + \sqrt{3}$  is

$$f(x) = x^4 - 10x^2 + 1 = 0$$

By the corollary, we know in any  $\mathbb{F}_p$  for any prime  $p$ , either  $\sqrt{2}, \sqrt{3}, \sqrt{6}$  is in  $\mathbb{F}_p$ . We claim that if  $\sqrt{2} \in \mathbb{F}_p$ , then  $f$  is factors over  $\mathbb{Q}(\sqrt{2})$ . Suppose that  $f$  does not factor over  $\mathbb{Q}(\sqrt{2})$ , i.e.,  $f$  is irreducible over  $\mathbb{Q}(\sqrt{2})$ , then the degree of extension

$$[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 8$$

which is a contradiction. Hence  $f$  factors over  $\mathbb{Q}(\sqrt{2})$ . Similar arguments work if  $\sqrt{3}$  or  $\sqrt{6}$  are in  $\mathbb{F}_p$ .  $\square$

**Problem 11.6 (S2015-Q4).** Prove that the polynomial  $x^4 + 1$  is not irreducible over any field of positive characteristic.

*Proof.* The idea is the same as above, and it suffices to note that the field extension generated by  $x^4 + 1$  is  $\mathbb{Q}(\sqrt{2}, i)$ . Using the Legendre symbol, the proof is similar to the above.  $\square$

**Problem 11.7 (F2010-Q2).**

- Find the complete factorization of the polynomial  $f(x) = x^6 - 17x^4 + 80x^2 - 100$  in  $\mathbb{Z}[x]$ .
- For which prime numbers  $p$  does  $f(x)$  have a root in  $\mathbb{Z}/p\mathbb{Z}$  (i.e.,  $f(x)$  has a root modulo  $p$ )? Explain your answer.

*Proof.* (a) Letting  $y = x^2$ , we need to factorize

$$f(y) = y^3 - 17y + 80y - 100$$

Now  $f$  is cubic, we need to find the roots of  $f$ : 5 is a root,

$$f(y) = (y - 5)(y - 2)(y - 10)$$

i.e.,

$$f(x) = (x^2 - 2)(x^2 - 5)(x^2 - 10)$$

which consists of only irreducible factors over  $\mathbb{Z}$ .

(b)  $f$  has a root in  $\mathbb{F}_p$  for all prime  $p$ , by the above corollary. □

## 11.1 Quick finite field review

If  $p$  is prime, then  $\mathbb{F}_p$  is a field of  $p$  elements, isomorphic to  $\mathbb{Z}/p\mathbb{Z}$ .

**Proposition 11.3 (Fact).** For every prime power  $p^n$ , there is exactly one finite field of  $p^n$  elements, namely  $\mathbb{F}_{p^n}$ , up to isomorphisms.

**Theorem 11.1 (Galois theory of finite fields).** We have

(1)  $\mathbb{F}_{p^n}/\mathbb{F}$  is a Galois extension, and

$$\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}) \text{ is cyclic}$$

where the generator is the Frobenius automorphism  $\sigma : \mathbb{F}_{p^n} \rightarrow \mathbb{F}_{p^n}$  where

$$\sigma : x \mapsto x^p$$

(2) We also have

$$\mathbb{F}_{p^n} = \{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^n} - \alpha = 0\}$$

This statement implies that  $\mathbb{F}_{p^n}$  is the splitting field of  $x^{p^n} - 1$ .

*Proof.* We note that  $\mathbb{F}_{p^n}$  is the splitting field of  $x^{p^n} - x$  over  $\mathbb{F}_p$ .

$$\mathbb{F}_{p^n} = \{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^n} - \alpha = 0\}$$

If  $\alpha \in \mathbb{F}_{p^n}$ , then we want to show that  $\alpha^{p^n} = \alpha$ : if  $\alpha = 0$ , then done; if  $\alpha \in \mathbb{F}_p^\times$ , then using the fact that any finite field is cyclic, we know

$$\mathbb{F}_{p^n} \cong \mathbb{Z}/(p^n - 1)\mathbb{Z} \Rightarrow \alpha^{p^n - 1} = 1$$

and we are done. Now we observe that  $\{\alpha \in \overline{\mathbb{F}_p} : \alpha^{p^n} - \alpha = 0\}$  has  $p^n$  elements, and is also a field, thus we are done.

This fact can be used to show (1) and the above proposition. □

**Proposition 11.4.**  $\mathbb{F}_{p^n}$  embeds into  $\mathbb{F}_{p^m}$  iff  $n \mid m$ .



*Proof.* If  $n \mid m$ , then  $m = nk$  for some integer  $k$ . We then notice that

$$\alpha^{p^n} = \alpha \Rightarrow \alpha^{p^{kn}} = \alpha^{p^m} = \alpha$$

Thus  $\mathbb{F}_{p^n}$  embeds into  $\mathbb{F}_{p^m}$ . Conversely, consider the Galois field extensions

$$\mathbb{F}_p \subset \mathbb{F}_{p^n} \subset \mathbb{F}_{p^m}$$

Then by degree of field extensions, we know  $n \mid m$ . □

**Problem 11.8 (F2016-Q3).** If field  $|F| = 2^n$ , find all  $n$  such that  $x^2 - x + 1$  is irreducible over  $F$ .

*Proof.* We know that  $x^2 - x + 1$  is irreducible over  $\mathbb{F}_2$ , namely, it has no roots in  $\mathbb{F}_2$ . Since there is only one field of order 4, we must have

$$\mathbb{F}_4 \cong \frac{\mathbb{F}_2}{(x^2 - x + 1)}$$

Clearly  $x^2 - x + 1$  is not irreducible over  $\mathbb{F}_4$ . For any  $\mathbb{F}_{2^n}$ , we know  $(x^2 - x + 1)$  is irreducible if and only if  $\mathbb{F}_4$  does not embed into  $\mathbb{F}_{2^n}$ , i.e.,  $2 \nmid n$ . This shows that when  $n$  is odd, the polynomial  $x^2 - x + 1$  is irreducible over  $\mathbb{F}_{2^n}$ . □

# Chapter 12

## Galois Theory

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Quick reminder whether a polynomial has a rational root:

**Proposition 12.1.** Let  $f(t) = a_n t^n + \cdots + a_1 t + a_0$ , and if a rational (expressed in lowest terms)  $\frac{p}{q}$  is a root of  $f$ , then  $p \mid a_0, q \mid a_0$ .

**Definition 12.1 (Galois extension).** A field extension  $k \subset L$  is Galois if for all  $x \in L$ , the minimal polynomial  $f(x) \in k[x]$  splits into a linear factor without repeated roots.

**Definition 12.2 (normal extension).** An extension  $k \subset K$  is normal if  $f$  has a root in  $K$  if and only if  $f$  splits completely into linear factors over  $K$ . An extension that is normal and separable is Galois.

**Theorem 12.1.** Suppose  $k \subset L$  is Galois,

$$\{k \subset M \subset L\} \xleftrightarrow{\text{one-to-one}} \{\text{Subgroups of } \text{Gal}(L/k)\}$$

Moreover, the order of the Galois group is the degree of the field extension.

$$|\text{Gal}(L/k)| = [L : k]$$

**Proposition 12.2.** Let  $G$  be a Galois group of a polynomial  $f$  of degree 4, and  $|G| = 8$ , then

$$G \cong D_8$$

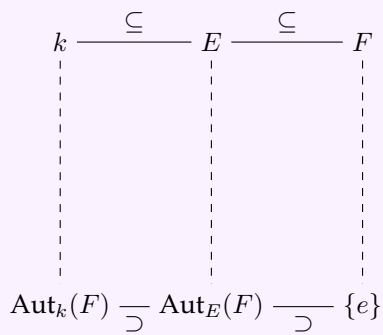
*Proof.* We know that  $G$  permutes the four roots of  $f$ , i.e.,  $G$  embeds into  $S_4$ . Since  $|G| = 8$ , we know  $G$  is a Sylow-2 subgroup of  $S_4$ , and all Sylow-2 subgroups are conjugates (isomorphic to one another), i.e.,

$$G \cong D_8$$

as desired. □

**Proposition 12.3.** Let  $k \subset K$  be a Galois extension, then the intermediate field extensions  $k \subset E \subset K$  is determined by the subgroups of  $\text{Gal}(K/k)$ . Namely, let  $E$  be an intermediate extension, there exists a subgroup  $H$  of  $\text{Gal}(K/k)$  that fixes  $E$ . This extension is normal if and only if  $H$  is normal. And  $E/k$  is Galois if and only if  $H$  is normal.

**Proposition 12.4.** We can draw the lattice of subgroups of  $\text{Gal}(K/k)$  and lattice of subfields:



**Problem 12.1 (S2009-Q3).** Consider the field  $K = \mathbb{Q}(\sqrt{a})$  where  $a \in \mathbb{Z}, a < 0$ . Show that  $K$  cannot be embedded in a cyclic extension whose degree over  $\mathbb{Q}$  is divisible by 4.

*Proof.* Suppose  $K$  embeds into a degree  $4n$  extension  $L$ , and

$$\text{Gal}(L/\mathbb{Q}) = \frac{\mathbb{Z}}{4n\mathbb{Z}}$$

Since  $K$  is a degree 2 extension of  $\mathbb{Q}$ , thus  $L/K$  is a degree  $4n/2$  Galois extension, with Galois group

$$\text{Gal}(L/K) = \frac{2\mathbb{Z}}{4n\mathbb{Z}}$$

We notice that  $\sqrt{a}$  is complex, hence the complex conjugation  $\tau$  is in  $\text{Gal}(L/\mathbb{Q})$ , i.e., it is an order 2 element in  $\frac{\mathbb{Z}}{4n\mathbb{Z}}$ , it is therefore  $[2n]$  i.e.,

$$\tau \in \frac{2\mathbb{Z}}{4n\mathbb{Z}} = \text{Gal}(L/\mathbb{Q})$$

This implies  $\tau$  fixed  $K$ , however  $\tau(\sqrt{a}) \neq \sqrt{a}$ , hence a contradiction.  $\square$

**Problem 12.2 (F2000-Q4).** Let  $G$  be a finite group. Show that there exists a Galois field extension  $K/k$  whose Galois group is isomorphic to  $G$ .

**Problem 12.3 (S2001-Q2).** Let  $K$  be the splitting field of  $f(X) = X^3 - 2$  over  $\mathbb{Q}$ .

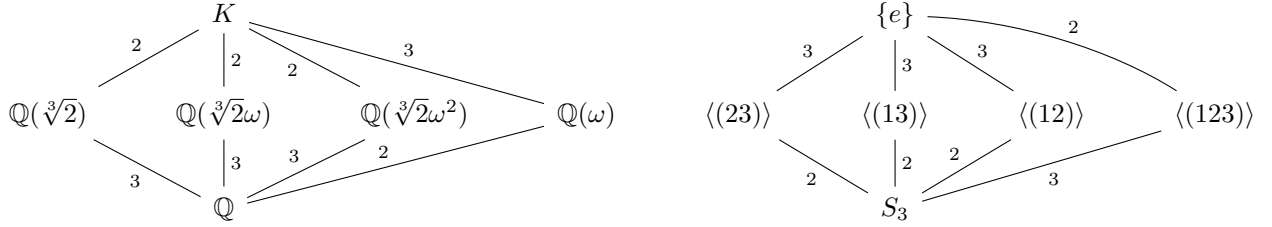
- Determine an explicit set of generators for  $K$  over  $\mathbb{Q}$ .
- Show that the Galois group  $G(K/\mathbb{Q})$  of  $K$  over  $\mathbb{Q}$  is isomorphic to the symmetric group  $S_3$ .
- Provide the complete list of intermediate fields  $k$ ,  $\mathbb{Q} \subseteq k \subseteq K$ , satisfying  $[k : \mathbb{Q}] = 3$ .
- Which of the fields determined in (c) are normal extensions of  $\mathbb{Q}$ ?

*Proof.* (a) The set of generators is

$$\left\{ \sqrt[3]{2}, e^{\frac{2\pi i}{3}} \right\}$$

- The minimal polynomial of  $e^{\frac{2\pi i}{3}}$  is  $x^2 + x + 1$ . This shows that the Galois group  $G$  has order 6, and because of the complex root, there exists an element of order 2, a transposition, that only swaps the two complex roots, and  $G$  also has an element of order 3 because 3 divides  $|G|$ , this shows that  $G$  must be  $S_3$ .

(c) The following is the subgroup lattice of  $S_3$  and subfield lattice:



Thus all the  $\mathbb{Q} \subset k$  such that  $[k : \mathbb{Q}] = 3$  are

$$\{\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2})e^{\frac{2\pi i}{3}}, \mathbb{Q}(\sqrt[3]{2})e^{\frac{4\pi i}{3}}\}$$

(d) None of the above are normal because the subgroups

$$\{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\}$$

are all Sylow 2-subgroups of  $S_3$ , hence all conjugates to one another, i.e., not normal.  $\square$

**Problem 12.4 (F2001-Q4).** Let  $K := \mathbb{Q}(\sqrt{3} + \sqrt{5})$ .

- (a) Show that  $K$  is the splitting field of  $X^4 - 6X^2 + 4$ .
- (b) Find the structure of the Galois group of  $K/\mathbb{Q}$ .
- (c) List all the fields  $k$ , satisfying  $\mathbb{Q} \subseteq k \subseteq K$ .

*Proof.* (a) I believe there is typo in (a) where the polynomial should be  $f(X) = X^4 - 16X^2 + 4$ . This is the minimal polynomial of  $\sqrt{3} + \sqrt{5}$ . We see that  $\mathbb{Q}(\sqrt{3} + \sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$ , hence it contains all the roots of  $f$ .

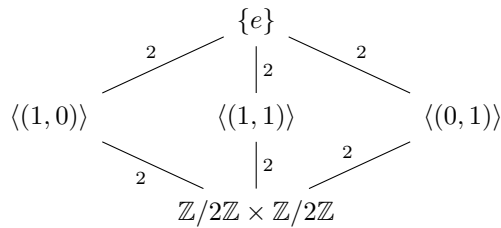
(b) We let  $\alpha = \sqrt{3} + \sqrt{5}$ , and  $\beta = \sqrt{3} - \sqrt{5}$ , then we see Galois group permutes

$$\{\alpha, -\alpha, \beta, -\beta\}$$

and we have  $\alpha\beta \in \mathbb{Q}$ . Thus just like the above, we have

$$\text{Gal}(K/\mathbb{Q}) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

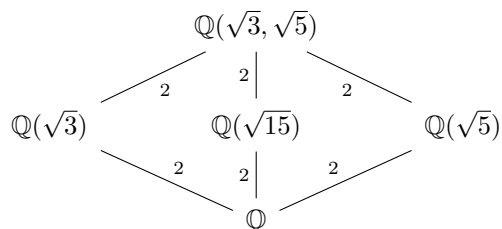
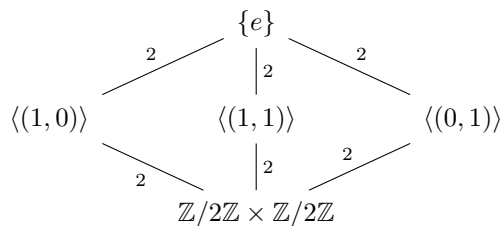
(c) We know the intermediate fields are determined by the subgroup of  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .



and let  $(1,0)$  be the element such that

$$(1,0) \cdot (\sqrt{3} + \sqrt{5}) = \sqrt{3} - \sqrt{5}$$

then we have the corresponding lattice of subfields



So all intermediate fields are

$$\{\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{15}), \mathbb{Q}(\sqrt{5})\}$$

□

**Problem 12.5 (F2013-Q5).** Compute the Galois group of  $f(x) = x^4 + 1$  over  $\mathbb{Q}$ .

*Proof.* The splitting field for  $f$  is  $\mathbb{Q}(\xi_8)$  where  $\xi_8 = e^{\frac{2\pi i}{8}}$ , and the Galois group

$$\text{Gal}(\mathbb{Q}(\xi_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$$

thus

$$(\mathbb{Z}/8\mathbb{Z})^\times \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

□

**Problem 12.6 (F2016-Q4).**

- (1) Determine the Galois group of  $x^4 - 4x^2 - 2$  over  $\mathbb{Q}$ .
- (2) Let  $G$  be a group of order 8 such that  $G$  is the Galois group of a polynomial of degree 4 over  $\mathbb{Q}$ . Show that  $G$  is isomorphic to the Galois group in part (1).

*Proof.* (a) There are four roots of this polynomial

$$\{\alpha, -\alpha, \beta, -\beta\}$$

where

$$\alpha = \sqrt{2 + \sqrt{6}}, \beta = \sqrt{2 - \sqrt{6}}$$

Thus the Galois group embeds into  $S_4$ . Notice that

$$\alpha\beta = \sqrt{2}i$$

Thus we see the Galois extension has degree 8:

$$\begin{array}{c} \mathbb{Q}(\sqrt{2+\sqrt{6}}, \sqrt{2}i) \\ | \quad 2 \\ \mathbb{Q}(\sqrt{2+\sqrt{6}}) \\ | \quad 4 \\ \mathbb{Q} \end{array}$$

Notice that the Galois group  $G$  is an order 8 subgroup of  $S_4$ , which implies that  $G$  is a Sylow 2 subgroup, and all Sylow 2 subgroups are isomorphic:

$$G \cong D_8$$

(b) The argument is given in (a). □

**Problem 12.7 (S2008-Q3).** Let  $K$  be the splitting field of the polynomial  $X^4 - 6X^2 - 1$  over  $\mathbb{Q}$ .

- (a) Compute  $\text{Gal}(K/\mathbb{Q})$ .
- (b) Determine all intermediate fields that are Galois over  $\mathbb{Q}$ .

*Proof.* (a) This computation is exactly same as above, as we have the four roots

$$\left\{ \pm\sqrt{3+\sqrt{10}}, \pm\sqrt{3-\sqrt{10}} \right\}$$

and we see that  $\alpha\beta = i$ , thus the Galois group  $\text{Gal}(K/\mathbb{Q})$  has order 8, and embeds into  $S_4$ , thus

$$\text{Gal}(K/\mathbb{Q}) \cong D_8$$

(b) **not finished** □

**Problem 12.8 (S2010-Q3).** Compute Galois groups of the following polynomials.

- (a)  $x^3 + t^2x - t^3$  over  $k$ , where  $k = \mathbb{C}(t)$  is the field of rational functions in one variable over complex numbers  $\mathbb{C}$ .
- (b)  $x^4 - 14x^2 + 9$  over  $\mathbb{Q}$ .

*Proof.* (a) The polynomial completely factors over  $\mathbb{C}(t)$ , so the Galois group is  $\{e\}$ .

(b) The roots are

$$\left\{ \pm\sqrt{7 \pm 2\sqrt{10}} \right\}$$

and  $\alpha\beta \in \mathbb{Q}$  again, hence the Galois group is  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ . □

**Problem 12.9 (S2013-Q6).** Let  $K$  be the splitting field of  $x^6 - 5$  over  $\mathbb{Q}$ .

- (a) Prove that  $x^6 - 5$  is irreducible over  $\mathbb{Q}$ .
- (b) Compute the Galois group of  $K$  over  $\mathbb{Q}$ .
- (c) Describe an intermediate field  $F$  such that  $F$  is not  $\mathbb{Q}$  or  $K$  and  $F/\mathbb{Q}$  is Galois.

*Proof.* (a) By Eisenstein.

- (b) We know  $K = \mathbb{Q}(\sqrt[6]{5}, \zeta_6)$ , where  $\zeta_6$  is the 6th root of unity. The roots are

$$\{\sqrt[6]{5}, \sqrt[6]{5}\zeta_6, \dots, \sqrt[6]{5}\zeta_6^5\}$$

Note that the minimal polynomial for  $\zeta_6$  is  $x^2 - x + 1$ , so the size of  $\text{Gal}(K/\mathbb{Q})$  is 12. We see that any  $\sigma \in \text{Gal}(K/\mathbb{Q})$  is determined by where it sends  $\sqrt[6]{5}$  and  $\zeta_6$ , so we only need to compute the possibilities of them. The Galois action is transitive implies that there  $\sqrt[6]{5}$  can be sent to any  $\sqrt[6]{5}\zeta_6^k$ , where  $k = 0, 1, 2, 3, 4, 5$ , and since  $\zeta_6$  has minimal polynomial

$$x^2 - x + 1$$

Then there are two possibilities for  $\zeta_6 \mapsto \zeta_6, \bar{\zeta}_6$ , where  $\bar{\zeta}_6 = \zeta_6^5$ . Now we see that

$$\text{Gal}(K/\mathbb{Q}) = D_{12}$$

as it is generated by

$$\sigma : \sqrt[6]{5} \mapsto \zeta_6 \sqrt[6]{5}, \zeta_6 \mapsto \zeta_6, \quad \tau : \sqrt[6]{5} \mapsto \sqrt[6]{5}, \zeta_6 \mapsto \zeta_6^5$$

satisfying  $\tau\sigma = \tau\sigma^{-1}$ . (One can draw a hexagon)

- (c)  $F/\mathbb{Q}$  corresponds to a normal subgroup of  $D_{12}$ . Any subgroup of 6 is normal, i.e., the subgroup

$$\{e, \sigma, \dots, \sigma^5\}$$

This subgroup fixes the field  $\mathbb{Q}(\zeta_6)$ . Hence it corresponds to

$$F = \mathbb{Q}(\zeta_6)$$

□

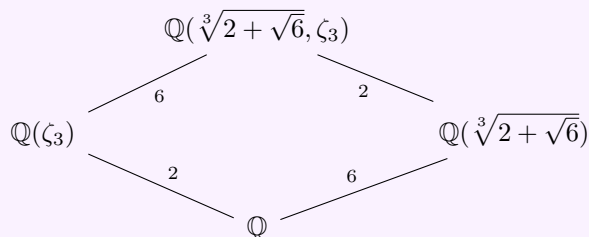
**Problem 12.10 (S2016-Q3).** Determine the Galois group of  $x^6 - 10x^3 + 1$  over  $\mathbb{Q}$ .

*Proof.* This is the same process as above, the roots are

$$\left\{ \zeta_3^i \sqrt[3]{5 \pm 2\sqrt{6}} : i = 0, 1, 2 \right\}$$

The order of the Galois group  $G$  is 12, but now we need another trick.

**Lemma 12.1.** Transitive subgroup of  $S_6$  of order 12 can only be  $D_{12}$  or  $A_4$ . However,  $A_4$  has no index 2 subgroups, i.e., this Galois extension cannot have a subfield extension of degree 2 over  $\mathbb{Q}$ , this gives that  $G$  must be  $D_{12}$ :



□

**Problem 12.11 (F2010-Q3).** Let  $K = \mathbb{Q}(\sqrt[8]{2}, \sqrt{-1})$  and  $F = \mathbb{Q}(\sqrt{-2})$ . Show that  $K$  is Galois over  $F$  and determine the Galois group  $\text{Gal}(K/F)$ .

*Proof.* Since  $\sqrt{2} = \zeta_8^4$ , we see  $F$  is a subfield such that

$$\mathbb{Q} \subset F \subset K$$

Since  $K$  is Galois over  $\mathbb{Q}$  (splitting field of  $x^8 - 2$ ), we know  $K/F$  is also Galois. Now The Galois group  $\text{Gal}(K/F)$  corresponds to the subgroup of  $\text{Gal}(K/\mathbb{Q})$  which is  $D_{10}$  of index 5, i.e., a subgroup of order 2, hence

$$\text{Gal}(K/F) = \frac{\mathbb{Z}}{2\mathbb{Z}}$$

□

**Problem 12.12 (F2015-Q2).** The dihedral group  $D_{2n}$  is the group on two generators  $r$  and  $s$ , with respective orders  $o(r) = n$  and  $o(s) = 2$ , subject to the relation  $rsr = s$ .

- (a) Calculate the order of  $D_{2n}$ .
- (b) Let  $K$  be the splitting field of the polynomial  $x^8 - 2$ . Determine whether the Galois group  $\text{Gal}(K/\mathbb{Q})$  is dihedral (i.e., isomorphic to  $D_{2n}$  for some  $n$ ).

*Proof.* (a) Because of the relation  $sr s = r^{-1}$ , we can express all the terms in  $D_{2n}$  as

$$r^k s^m$$

where  $0 \leq k \leq n-1, m = 0, 1$ . Hence there are  $2n$  elements.

- (b) The Galois group is indeed diahedral, note that

$$K = \mathbb{Q}(\zeta_8, \sqrt[8]{2})$$

Thus  $\text{Gal}(K/\mathbb{Q})$  has order 16 and the only transitive subgroup of  $S_8$  of order 16 is  $D_{16}$ .

□

**Problem 12.13 (S2019-Q1).** Show that any transitive subgroup of  $A_5$  is isomorphic to one of the following groups:

- (a) the cyclic group  $\mathbb{Z}/5\mathbb{Z}$ ,
- (b) the dihedral group  $D_5$ ,
- (c)  $A_5$ .



**Problem 12.14 (S2019-Q2).** Let  $f(x) = x^5 - 5x + 12$ . Verify that  $f(x)$  is irreducible in  $\mathbb{Q}[x]$  and its discriminant is  $d(f) = (2^6 5^3)^2$ . If  $r_1, \dots, r_5$  are the roots of  $f$ , let

$$P(x) = \prod_{1 \leq i < j \leq 5} (x - (r_i + r_j)).$$

Show that  $P(x)$  is a product of two monic irreducible polynomials in  $\mathbb{Q}[x]$ :

$$P(x) = (x^5 - 5x^3 - 10x^2 + 30x - 36)(x^5 + 5x^3 + 10x^2 + 10x + 4).$$

Use this information, Problem 1 and properties of  $f_3 \in \mathbb{F}_3[x]$ , the reduction of  $f$  modulo 3, to show that the the Galois group  $G_f$  of  $f$  is isomorphic to  $D_5$ .

**Problem 12.15 (F2018-Q6).** Determine the Galois group over  $\mathbb{Q}$  of the polynomial

$$X^6 + 22X^5 - 9X^4 + 12X^3 - 37X^2 - 29X - 15.$$

**Problem 12.16 (F2017-Q4).** Compute the Galois group of  $x^5 - 10x + 5$  over  $\mathbb{Q}$ .

**Problem 12.17 (F2004-Q3).** Let  $f(x) = x^5 - 9x + 3$ . Determine the Galois group of  $f$  over  $\mathbb{Q}$ .

**Problem 12.18 (F2006-Q2).** Let  $f$  be a polynomial in  $\mathbb{Q}[x]$ . Let  $E$  be a splitting field of  $f$  over  $\mathbb{Q}$ . For the following cases, determine whether  $E$  is solvable by radicals.

- (1)  $f(x) = x^4 - 4x + 2$ .
- (2)  $f(x) = x^5 - 4x + 2$ .

**Problem 12.19 (S2011-Q3).** Determine the Galois group [up to isomorphism] of the splitting field of each of the following polynomials over  $\mathbb{Q}$ :

- (a)  $f(x) = x^4 - 9x^3 + 9x + 4$ ,
- (b)  $g(x) = x^5 - 6x^2 + 2$ .

**Problem 12.20 (F2014-Q1).**

- (a) Let  $S_n$  be the symmetric group (permutation group) on  $n$  objects. Prove that if  $\sigma \in S_n$  is an  $n$ -cycle and  $\tau \in S_n$  is a transposition (i.e., a 2-cycle), then  $\sigma$  and  $\tau$  generate  $S_n$ .
  - (b) Let  $f_a(x)$  be the polynomial  $x^5 - 5x^3 + a$ . Determine an integer  $a$  with  $-4 \leq a \leq 4$  for which  $f_a$  is irreducible over  $\mathbb{Q}$ , and the Galois group of [the splitting field of]  $f_a$  over  $\mathbb{Q}$  is  $S_5$ . Then explain why the equation  $f_a(x) = 0$  is not solvable in radicals.
- (a) It suffices to assume that the  $n$  cycle is  $(1 \dots n)$  (up to rearranging the terms), and the transposition is  $(12)$ . One can show that conjugation gives all the transpositions, hence generate  $S_n$ .
- (b) Take  $a = 1$ , then  $f_a(x)$  is irreducible: it doesn't have a root by the Rational Root Theorem and cannot be factored into lower degree polynomial by term matching. Moreover, we see that  $f'_a(x)$  has 3 roots, by Rolle's theorem, there are at most 4 real roots, this implies that there exists a complex root  $r_1$ , and

since this has odd degree, it must also exist a real root  $r_2$ . This shows that there exists an element in the Galois group that has order 5 and a transposition (sending conjugate complex roots to each other). Thus by (a), since the Galois group is a subgroup of  $S_5$ , we must have it equal to  $S_5$ .

**Problem 12.21 (F2009-Q3).** Determine the Galois group of  $x^4 - 4x^2 + 7x - 3$  over  $\mathbb{Q}$ .

**Problem 12.22 (S2012-Q3).** In this problem,  $G$  denotes the group  $S_5 \times C_2$ , where  $S_5$  is the symmetric group on five letters and  $C_2$  is the cyclic group of order 2.

- (a) Determine all normal subgroups of  $G$ .
- (b) Give an example of a polynomial with rational coefficients whose Galois group is  $G$ , deducing that from basic principles.

**Problem 12.23 (F2015-Q4).** Let  $H = S_3 \times S_5$ .

- (a) Determine all normal subgroups of  $H$ . Make sure you have them all! What would be different if  $H$  were replaced by  $S_2 \times S_5$ ?
- (b) Describe, in full detail, the construction of a polynomial with rational coefficients, whose Galois group is isomorphic to  $H$ .