Algebra Qualifying Exam

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Contents

Chapter 1

Group Theory

1.1 Sylow Theorems

We first talk bout semidirect products. Let G be any group, and N, H be subgroups of G.

Definition 1.1. For $\varphi : H \to \mathrm{Aut}(N)$, define $N \rtimes H$ by

(1) $N \rtimes_\varphi H = N \times H$ as a set.

(b) Equipped with the group structure

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1\varphi(h_1)n_2, h_1h_2)$$

The structure $(N \rtimes_\varphi H, \cdot)$ forms a group.

Example 1.1. If N is a normal subgroup of G, and $N \cap H = \{e\}$, and $\varphi : H \to \mathrm{Aut}(N)$ where

$$\varphi : h \mapsto (n \mapsto hnh^{-1})$$

(acting by conjugation), and $G = NH$. Then

$$N \rtimes_\varphi H \to G$$

where

$$(n, h) \mapsto nh$$

is a bijective homomorphism homomorphism. Hence

$$G \cong N \rtimes_\varphi H$$

Next we present some divisibility results.

Proposition 1.1 (Lagrange, Orbit-Stabilizer)**.** We have the following divisibility results:

- Let H be a subgroup of G, let $[G : H]$ denote the number of cosets of H in G, then

$$|G| = |H|[G : H]$$

- Let G be a finite group acting transitively on a finite set A, then for any $a \in A$, we have

$$|\mathrm{Stab}_G(a)| \cdot |O_G(a)| = |G|$$

The class formula is when G acts on itself by conjugation:

Proposition 1.2 (class formula)**.** Let G act on a finite set S, and let Z denote fixed points of this action, then

$$|S| = |Z| + \sum_{a \in A} |O_G(a)|$$

where A includes exactly one element from each nontrivial orbit.

If G acts on itself by conjugation, then

$$|G| = |Z(G)| + \sum_{g} |[g]| = |Z(G)| + \sum_{g} \frac{|G|}{|C_G(g)|}$$

where $[g]$ denote the conjugacy class of g, and the sum includes exactly one from each nontrivial conjugacy class in G.

Problem 1.1 (F2019-Q2)**.** 2. Let p, q be two prime numbers such that $p \mid q - 1$. Prove that

(a) there exists an integer $r \neq 1 \mod q$ such that $r^p \equiv 1 \mod q$;

(b) there exists (up to an isomorphism) only one noncommutative group of order pq.

Proof. (a) We want to show that there exists an element $r \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that

$$r^p \equiv 1 \mod q$$

We can do this because $(\mathbb{Z}/q\mathbb{Z})^\times$ has order $(q-1)$ and $p|(q-1)$. Therefore by Cauchy's theorem, there exists an element of order p in $(\mathbb{Z}/p\mathbb{Z})^\times$.

(b) Let n_p, n_q denote the number of p, q-Sylow subgroups. We see that $n_q|p$ and $n_q \equiv 1 \mod q$, since $p < q$, we must have $n_q = 1$. Now $n_p = 1$ or q by the same reasoning. Suppose $n_q = 1$, let P, Q denote the normal subgroups of order p, q, then

$$G \cong P \times Q$$

by a standard argument (included in the lemma below). Then G is commutative. Since G is noncommutative, we have $n_p = q$. Choose any p-Sylow subgroup P, we know that

$$G \cong Q \rtimes_\theta P$$

where Q is the normal subgroup of order q and $\theta : P \to \mathrm{Aut}(Q) = (\mathbb{Z}/q\mathbb{Z})^\times$. We know either $\theta : 1 \mapsto 1$, is the trivial map which produces a commutative group; or $\theta : 1 \mapsto r$, where $r \in (\mathbb{Z}/q\mathbb{Z})^\times$ is some element of order p.

Warning 1.1. For completeness, we show that

□

Lemma 1.1. Let p, q be two primes such that $q \nmid (p-1)$, and N, H has order p, q respectively, suppose that N is normal in G, and $N \cap H = \{e\}$, then

$$G \cong N \times H$$

Proof. We consider the map

$$\psi : N \times H \to G$$

such that

$$(n,h) \mapsto nh$$

We want to show that ψ is a homomorphism and ψ is injective (hence bijective by size argument). It is clearly injective:

$$nh = e \Rightarrow n, h \in N \cap H = \{e\}$$

It suffices to show that ψ is a homomorphism. We see that this implies

$$n_1 n_2 h_1 h_2 = n_1 h_1 n_2 h_2$$

Therefore it suffices to for any $n \in N, h \in H$, one has

$$nh = hn$$

Consider the conjugation action

$$\varphi : H \to \text{Aut}(N)$$

where

$$h \mapsto \left(n \mapsto hnh^{-1}\right)$$

Then we claim that φ is trivial. This is because $\ker(\varphi)$ has size either 1 or q. If it has size q, then the map is trivial; if it has size 1, then H embeds in $\text{Aut}(N)$, however, $|H| = q$, $\text{Aut}(N) = p - 1$, and $q \nmid (p-1)$, hence impossible. This shows that the map is trivial, i.e., for $n \in N, h \in H$,

$$hn = nh$$

as desired. □

Problem 1.2 (F2015-Q1). Prove every group of order 15 is cyclic.

Proof. We will show that any group G of order 15 is isomorphic to

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

For this, using the above lemma, it suffices to show that there is one normal subgroup of order 3 and one normal subgroup of order 5. We repeat the argument above, $n_5 \mid 3$ and $n_5 \equiv 1 \mod 5$, hence $n_5 = 1$. Moreover, $n_3 \mid 5$ and $n_3 \equiv 1 \mod 3$, hence $n_3 = 1$ as well. By the lemma above, we know that

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

hence cyclic as desired. □

Problem 1.3 (S2013-Q2). Let p and q be primes with $p < q$. Let G be a group of order pq. Prove the following statements:

(a) If p does not divide $q-1$ (i.e., $p \nmid q-1$), then G is cyclic.

(b) If p divides $q-1$ (i.e., $p \mid q-1$), then G is either cyclic or isomorphic to a non-abelian group on two generators. Give the presentation of this non-abelian group.

Proof. This question is exactly the same as F19-Q2, we will only outline here.

(a) We have $n_q = 1$, and $n_p \mid q$, hence $n_p = 1$ or q, moreover $n_p \equiv 1 \mod p$. If $n_p = q$, this implies that $p \mid (q-1)$, hence $n_p = 1$. Therefore by the above argument

$$G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$$

(b) If $p \mid (q-1)$, then $n_p = 1$ or q. Hence G is either of the form above or isomorphic to the non-abelian group

$$G = Q \rtimes_\theta P$$

We know from F2019-Q2, the trivial θ defines the abelian, hence cyclic group $G = \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$. And $\theta : 1 \mapsto r$, for some $r \in (\mathbb{Z}/q\mathbb{Z})^\times$ of order p defines a non-abelian group. So we have

$$G = \langle g, h : g^q = h^p = e, hgh^{-1} = g^r \rangle$$

$\square$

Problem 1.4 (F2007-Q1). Prove that no group of order 148 is simple.

Proof. We note the prime factorization of 148 is

$$148 = 2^2 \cdot 37$$

We see that $n_{37} \mid 4$ and $n_{37} \equiv 1 \mod 37$, therefore $n_{37} = 1$. This shows that there exists a normal subgroup of order 37, i.e., the group is not simple. $\square$

Problem 1.5 (F2017-Q1). Show that there is no simple group of order 30.

Proof. This is slightly more complicated, and we will use a counting argument. Same reasoning as the above. The prime factorization of 30 is as below:

$$30 = 2 \cdot 3 \cdot 5$$

We see $n_5 \mid 6$, and $n_5 \equiv 1 \mod 5$. Unfortunately, n_5 could either be 1 or 6. Now $n_3 \mid 10$, and $n_3 \equiv 1 \mod 3$, unfortunately again n_3 could be 10. However, we argue that $n_3 = 10$ and $n_5 = 6$ cannot happen at the same time. Suppose this is the case, then there are 20 elements of order 2 and 24 elements of order 5, but this is too many! Hence either $n_3 = 1$ or $n_5 = 1$, as desired. $\square$

Problem 1.6 (F2011-Q1).

(a) Let G be a group of order 5046. Show that G cannot be a simple group. You may not appeal to the classification of finite simple groups.

(b) Let p and q be prime numbers. Show that any group of order p^2q is solvable.

Proof. The proof is very similar like above.

(a) The prime factorization of 5049 is as follows:

$$5049 = 2 \cdot 3 \cdot 29^2$$

Hence we see $n_{29} = 1$, i.e., there is a normal subgroup of order 29, therefore not simple.

(b) We will do discussion by cases.

(1) $p > q$. Then $n_p = 1$ or q and $n_p \equiv 1 \mod p$, therefore $n_p = 1$. Let P be the normal subgroup of G of order p^2, we thus have

$$\{e\} \subset P \subset G$$

It is clear that $|G/P| = q$, thus abelian, and $|P| = p^2$ also abelian as well (by the lemma below). This shows that G is solvable.

(2) $p < q$. Then $n_p = 1$ or q, and $n_q = 1$ or p^2. Suppose that $n_q = 1$, let Q denote the normal subgroup of order q, then

$$\{e\} \subset Q \subset G$$

It is clear that Q and G/Q are both abelian. Suppose that $n_q = p^2$ instead, then there are only $p^2q - p^2(q-1) = p^2$ elements of order $\neq q$. Since any p-Sylow subgroup has p^2 elements with order $\neq q$, we must have $n_p = 1$. Hence we are in case (1) again. This shows that G is solvable in either case $n_q = 1, p^2$.

□

Lemma 1.2 (p^2 abelian)**.** Fix prime p, any group of order p^2 is abelian.

Proof. For any nontrivial p group, by the class formula, the center $Z(G)$ is nontrivial, thus the center has order either p or p^2. If it has order p^2, then the group is abelian. If it has order p, then

$$|G/Z(G)| = p$$

is also cyclic, therefore G is abelian (strictly speaking is a contradiction that $|Z(G)| = p$). In either case, we see that G is abelian. □

Problem 1.7. Any p-group is solvable, for any prime p.

Proof. Suppose $|G| = p^r$ for some $r \geq 0$, we will use induction on r. If $r = 0$, then the trivial group is trivally solvable.

- Base case: if $r = 1$, $|G| = p$, then G is cyclic, hence solvable.
- Induction step: suppose that G is solvable for all $|G| = p^k$, where $0 \leq k \leq r-1$. Now we want to show that G of order p^r is solvable. We know G has a nontrivial center, suppose that $|Z(G)| = p^k$, where $1 \leq k \leq r$, then

  $$|G/Z(G)| = p^{r-k}, 0 \leq r-k \leq r-1$$

  We know any group G is solvable if and only if there exists a sequence of subgroups $G_0, \ldots, G_k$

  $$\{e\} = G_0 \subset \cdots \subset G_k = G$$

  such that G_{i-1} is normal in G_i and G_i/G_{i-1} is solvable. Therefore we see when $|G| = p^r$,

  $$\{e\} \subset Z(G) \subset G$$

  has $Z(G)$ solvable, and $G/Z(G)$ also solvable by the induction hypothesis, so we close the induction.

□

Problem 1.8 (S2016-Q1). Classify all groups of order 66, up to isomorphism.

Proof. By $66 = 2 \cdot 3 \cdot 11$, we know $n_{11} = 1$. We claim that there is a normal subgroup isomorphic to $\mathbb{Z}/33\mathbb{Z}$.

1. First we show that there is a subgroup of order 33. Let P_{11} denote the normal subgroup of order 11 and let P_3 denote a 3-Sylow subgroup of G. Then we claim that the following

$$H = \{gh : g \in P_{11}, h \in P_3\}$$

forms a subgroup and is isomorphic to $\mathbb{Z}/33\mathbb{Z}$. By the Lemma 1.1, we see that

$$H \cong \frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} = \frac{\mathbb{Z}}{33\mathbb{Z}}$$

2. Now we show that it is normal. This follows from the following general lemma:

Lemma 1.3. Let p be the smallest prime factor of $|G|$, and let H be a subgroup with index p, then H is normal.

Proof. We will only prove in the case that H is a subgroup of index 2, i.e., $G = H \sqcup (G \setminus H)$. We see for all $g \in G$,

$$gH = Hg$$

since if $g \in H$, then the equality holds; if $g \notin H$, then $gH = G \setminus H$, so is Hg. $\square$

Now since there is a subgroup of order 2, we can write G as a semidirect product

$$G = \frac{\mathbb{Z}}{33\mathbb{Z}} \rtimes_\theta \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The number of nonisomorphic groups will depend on the choice of θ. There are four different choices for $\theta : H \to \mathrm{Aut}\left(\frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}\right) = \frac{\mathbb{Z}}{10\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

$$\begin{cases} \theta_1 : 1 \mapsto (0,0) \\ \theta_2 : 1 \mapsto (0,1) \\ \theta_3 : 1 \mapsto (5,0) \\ \theta_4 : 1 \mapsto (5,1) \end{cases}$$

There are 4 different groups and one can write them in cyclic notation using the θ above. $\square$

Problem 1.9 (S2007-Q2). Prove that no group of order 224 is simple.

Proof. The prime factorization is

$$224 = 2^5 \cdot 7$$

If $n_2 = 1$ or $n_7 = 1$, then we are done; assume that $n_2 = 7$ instead, then we recall G has a nontrivial transitive action on the set of 2-Sylow subgroups, i.e., there is a homomorphism $\varphi : G \to S_7$. We know $\ker(\varphi)$ is a normal subgroup of G. Since the action is nontrivial transitive, we know $\ker(\varphi) \neq G$. If $\ker(\varphi) = \{e\}$, then φ produces an embedding of G into S_7. However, $|G| = 224 \nmid |S_7|$. This shows that $\ker(\varphi)$ is a nontrivial proper normal subgroup of G, concluding that G is not simple. $\square$

Problem 1.10 (F2008-Q1). Show that no group of order 36 is simple.

Proof.

$$36 = 2^2 \cdot 3^3$$

We know $n_2 \mid 9, n_2 \equiv 1 \mod 2$, and $n_3 \mid 4, n_3 \equiv 1 \mod 3$. We know $n_3 = 1$ or 4, suppose that $n_3 = 4$, then there is a nontrivial action of G on the set of 3-Sylow subgroups, i.e.,

$$\varphi : G \to S_4$$

Suppose that G is simple, we know $\ker(\varphi) \neq G$ since the action is nontrivial, by assumption $\ker(\varphi) = \{e\}$, which implies that φ is an embedding, but $|G| = 32 \nmid |S_4|$, which is a contradiction. This implies that G is not simple. □

Problem 1.11 (S2014-Q2). All groups of order less than 60 are solvable, i.e., there exists a sequence of subgroups of G, $G_0, \ldots, G_k$ such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, and

$$1 = G_0 \subset \cdots \subset G_k = G$$

Proof. Groups of order p, pq, p^2, p^2q are solvable.

$$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30,$$
$$31, 33, 34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59\}$$

And any p-group is also solvable.

$$\{8, 16, 27, 32\}$$

The remaining groups are

$$\{24, 36, 40, 42, 48, 54, 56\}$$

24: If $n_2 = 1$ or $n_3 = 1$, then we are done. We see $n_2 = 1$ or 3, consider the action $\varphi : G \to S_3$. We see $\ker(\varphi)$ is a proper normal subgroup of G, this implies that

$$\{e\} \subset \ker(\varphi) \subset G$$

where $|\ker(\varphi)|$ is a known solvable group, hence we are done.

36: Exactly same as above, we assume $n_3 \neq 1$, therefore $n_3 = 4$, the action $\varphi : G \to S_4$ is not injective, hence $\ker(\varphi)$ is again a proper normal subgroup of G that is solvable.

40: We see $n_5 = 1$, therefore

$$\{e\} \subset \mathbb{Z}/5\mathbb{Z} \subset G$$

42: We see $n_7 = 1$.

48: We see $n_2 = 1$ or 3, the the action $\varphi : G \to S_3$ is not injective, hence $\ker(\varphi)$ is a proper normal subgroup of G that is solvable.

54: We see $n_3 = 1$.

56: We know $n_7 = 1$ or 8 and $n_2 = 1$ or 7. The group action argument does not work. We assume $n_7 = 8$, then there can be at most $56 - 8(7 - 1) = 8$ elements of order $\neq 7$. This shows that $n_2 = 1$. Hence

$$\{e\} \subset P_2 \subset G$$

□

Problem 1.12 (S2012-Q1). Let G be a group of order p^3q^2, where p and q are prime integers. Show that for p sufficiently large and q fixed, G contains a normal subgroup other than $\{1\}$ and G.

Proof. We want to show that there exists a normal group of size p^3, i.e., $n_p = 1$. We know $n_p \mid q^2, n_p \equiv 1 \mod p$. Let p be large enough such that $p > (q^2 - 1)$, then ths forces $n_p = 1$, as desired. $\square$

Problem 1.13 (F2014-Q4).

(a) Let G be a group of order p^2q^2, where p and q are distinct odd primes, with $p > q$. Show that G has a normal subgroup of order p^2.

(b) Can a solvable group contain a non-solvable subgroup? Explain.

Proof. (a) We know $n_p = 1$ or q or q^2, and $n_p \equiv 1 \mod p$. Since $p > q$, we know $n_p \neq q$. It suffices to show that $n_p \neq q^2$: suppose that $n_p = q^2$, then

$$p \mid (q^2 - 1) = (q+1)(q-1)$$

Since p is prime, $p \mid (q+1)$ or $p \mid (q-1)$. The latter impossible since $q < p$. $p \mid (q+1)$ is also impossible because this implies that $q = p + 1$, which implies that q is even, a contradiction.

(b) It is not possible. Suppose G is a solvable group, let H be a subgroup of G, then we know there exists sequence

$$\{e\} = G_0 \subset \cdots \subset G_k = G$$

such that G_i is normal in G_{i+1} and $\frac{G_{i+1}}{G_i}$ is abelian. We define $H_i = G_i \cap H$, then we see H is solvable with sequence $H_0 \subset \ldots H_k$.

$\square$

Problem 1.14 (F2018-Q2). Let G be a group of order 24. Assume that no Sylow subgroup of G is normal in G. Show that G is isomorphic to the symmetric group S_4.

Proof. By Sylow, we have $n_3 = 4, n_2 = 3$. Denote $\text{Syw}_3(G) = \{P_1, P_2, P_3, P_4\}$ and consider the transitive action by of G by conjugation on this set, which embeds in S_4, i.e., $\varphi : G \to S_4$. By a size argument, it suffices to show that φ is injective. We see that

$$\ker(\varphi) = \{g \in G : gP_ig^{-1} = P_i \text{ for each } i\} = \bigcap_{i=1}^{4} N_G(P_i)$$

By the orbit-stabilizer theorem, $|N_G(P_i)| = 6$ for all i. However, for any $i \neq j$, 3 does not divide $|N_G(P_i) \cap N_G(P_j)|$: if not, the intersection would include a 3-Sylow subgroup but P_i is the only 3-Sylow subgroup in $N_G(P_i)$, thus this is impossible. It remains to see that $|\ker(\varphi)| \neq 2$. Suppose that it is, then $\text{im}(\varphi)$ is an index 2 subgroup of S_4, hence

$$\frac{G}{\ker \varphi} \cong A_4$$

and $K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is normal in A_4, hence so is $\varphi^{-1}(K)$ (it has size 8) in G. This is a contradiction because this implies there is a normal 2-Sylow subgroup. $\square$

Problem 1.15 (F2001-Q1). Let G be a finite group and let N be a normal subgroup of G such that N and G/N have relatively prime orders.

1. Assume that there exists a subgroup H of G having the same order as G/N. Show that $G = HN$. (Here HN denotes the set $\{xy \mid x \in H, y \in N\}$.)
2. Show that $\phi(N) = N$, for all automorphisms ϕ of G.

Proof.

1. Since N, H have relatively prime orders, $N \cap H = \{e\}$, thus we can write
$$G = N \rtimes_\theta H$$
where $\theta(h)n = hnh^{-1}$. One can show that the map $\varphi : N \rtimes_\theta H \to G$ as
$$\varphi : (n, h) \mapsto nh$$
It is clear that φ is a homomorphism and injective, thus by a size argument we have φ is an isomorphism. This shows $G = NH$ and similarly $G = HN$.
2. Any automorphism ϕ of G permutes the p-Sylow subgroups. Suppose that $|G| = p_1^{i_1} \dots p_k^{i_k}$, then after rearranging,
$$|N| = p_1^{i_1} \dots p_j^{i_j}$$
because N and G/N have relatively prime orders. Hence N contains all the Sylow p_i-subgroups, hence $\phi(N) = N$ for all automorphisms ϕ of G.

□

Problem 1.16 (S2001-Q1). Let G be a finite group and p the smallest prime number dividing the order $|G|$ of G. Let H be a subgroup of G of index p in G. Show that H is necessarily a normal subgroup of G.

Proof. G has an action on G/H by left multiplication: $\varphi : G \to \mathrm{Aut}(G/H)$ such that
$$\varphi(g)(\bar{g}H) = g\bar{g}H$$
We will show that $H = \ker(\varphi)$. First we see that $\ker(\varphi) \subset H$:
$$\ker(\varphi) = \{g \in G : g\bar{g}H = \bar{g}H : \text{ for all } \bar{g} \in G\}$$
letting $\bar{g} \in H$ we see $g \in \ker(\varphi)$ implies $g \in H$, i.e., $\ker(\varphi) \subset H$.

Now we use a size argument to show $|H| \leq |\ker\varphi|$. We note that $\mathrm{im}(\varphi)$ is a subgroup of $\mathrm{Aut}(G/H) = S_p$, thus
$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p!$$
because $\frac{|G|}{|\ker(\varphi)|}$ also divides $|G|$ and p is the smallest prime that divides p, we must have
$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p$$
Note that $\frac{|G|}{|H|} = p$, this gives
$$|H| \leq |\ker(\varphi)|$$
which shows $H \subset \ker(\varphi)$, hence $H = \ker(\varphi)$. □

(End of Page 5)

1.2 Class Formula, Classification of p-groups

Definition 1.2 (nilpotent group). Let G be a group. Define inductively an increasing sequence $\{e\} = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$ of subgroups of G as follows: for $i \geq 1$, Z_i is the subgroup of G corresponding to the center of G/Z_{i-1}. One can show that Z_i is normal in G. A group is *nilpotent* if $Z_m = G$ for some m.

Example 1.2.

- p-groups are nilpotent.
- Nilpotent groups are solvable.

Proposition 1.3. We have the following classification of groups of order p, p^2, p^3, for prime p.

- $|G| = p$ implies $G \cong \mathbb{Z}/p\mathbb{Z}$.
- $|G| = p^2$ implies
$$G \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}} \quad \text{or} \quad G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}$$
- $|G| = p^3$ implies that
$$G \cong \frac{\mathbb{Z}}{p^3\mathbb{Z}} \quad \text{or} \quad G/Z(G) \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \quad \text{or} \quad [G,G] = Z(G)$$

Problem 1.17 (S2010-Q1). Let G be a non-abelian group of order p^3, where p is prime. Determine the number of distinct conjugacy classes in G.

Proof. We know G has a nontrivial center, and if $|Z(G)| = p^2$ or p^3, then G is abelian, this shows that $|Z(G)| = p$, now let $g \in G \setminus Z(G)$, then
$$Z(G) \subsetneq Z_g(G) \subsetneq G$$
where $Z(G) \subsetneq Z_g(G)$ because $g \in Z_g(G)$, and $Z_g(G) \subsetneq G$ since $g \notin Z(G)$. This shows that $Z_g(G)$ is a subgroup of order p^2, in other words, the size of the conjugacy class of any $g \in G \setminus Z(G)$ is
$$|[g]| = \left|\frac{G}{Z_g(G)}\right| = p$$
By the class formula,
$$|G| = |Z(G)| + \sum_{a \in A} |[a]|$$
where A contains one a from each nontrivial conjugacy class $[a]$. Thus we have
$$p^3 = p + Np \Rightarrow N = p^2 - 1$$
Every element in $Z(G)$ is its own conjugacy class, thus the total number of conjugacy classes is
$$p^2 + p - 1$$
□

Problem 1.18 (F2013-Q1). Let $p > 2$ be a prime. Classify groups of order p^3 up to isomorphism. The two nonabelian groups of order p^3 (for $p \neq 2$), up to isomorphism, are:

$$\operatorname{Heis}(\mathbb{Z}/(p)) = \left\{ \left(\begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{array} \right) \middle| a, b, c \in \mathbb{Z}/(p) \right\}$$

and

$$G_p = \left\{ \left(\begin{array}{cc} a & b \\ 0 & 1 \end{array} \right) \middle| a, b \in \mathbb{Z}/(p^2), a \equiv 1 \bmod p \right\}$$
$$= \left\{ \left(\begin{array}{cc} 1+pm & b \\ 0 & 1 \end{array} \right) \middle| m, b \in \mathbb{Z}/(p^2) \right\}$$

Problem 1.19 (F2014-Q5).

(a) Prove that every group of order p^2 (with p prime) is abelian. Then classify such groups up to isomorphism.

(b) Give an example of a non-abelian group of order p^3 for $p = 3$.
Suggestion: Represent the group as a group of matrices.

Proof. (a) See Lemma 1.2. There are two abeliean groups: $\frac{\mathbb{Z}}{p^2\mathbb{Z}}, \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$.

(b) See Problem 1.18.

□

Problem 1.20 (F2019-Q4, S2015-Q3). Find all irreducible representations of a finite p-group over a field of characteristic p.

Proof. Let G any finite p-group. Let V be an irreducible representation over $\mathbb{F}_p$, which is a $[\mathbb{F}_p G]$-module. Thus $|V| = p^d$, since it is a finite-dimensional vector space over $\mathbb{F}_p$, i.e.,

$$|V| = p^d$$

for some $d \geq 1$. We consider the action of G on V, all the orbits of this action either has size 1 or is a power of p, since G is a p-group, by the class formula, let N be the number of nontrivial orbits of size 1,

$$|W| \equiv 1 + N \mod p \Rightarrow 1 + N \equiv 0 \mod p$$

Hence there exists at least one nontrivial orbit $\{v\}$ of size 1. We consider the vector space W generated by v over $\mathbb{F}_p$: it is one-dimensional vector space contained in V, invariant under G, since V is irreducible, we must have $V = W$.Thus all irreducible representations of a finite p-group over $\mathbb{F}_p$ are trivial. □

1.3 Random Problems

Problem 1.21 (F2010-Q1). Let G be a group. Let H be a subset of G that is closed under group multiplication. Assume that $g^2 \in H$ for all $g \in G$. Show that:

- H is a normal subgroup of G
- G/H is abelian

Proof.

- We first show that H is subgroup. It remains to show that if $h \in H$, then $h^{-1} \in H$, we know $(h^{-1})^2 \in H$, thus
$$h(h^{-1})^2 = h^{-1} \in H$$
as desired. Now we show that H is normal: for any $h \in H$, $g \in G$, we want to show $ghg^{-1} \in H$.
$$\begin{aligned} ghg^{-1} &= (gh)^2(gh)^{-1}hg^{-1} \\ &= (gh)^2h^{-1}g^{-1}hg^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2(g^{-1}h)^{-1}g^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2h^{-1} \in H \end{aligned}$$
as desired.

- It suffices to show that for any $g_1, g_2 \in G$, we have
$$g_1g_2H \subset g_2g_1H$$
Take any $h \in H$, we want to show $(g_2g_1)^{-1}g_1g_2h \in H$,
$$\begin{aligned} (g_2g_1)^{-1}g_1g_2h &= (g_2g_1)^{-2}g_2g_1^2g_2h \\ &= (g_2g_1)^{-2}(g_2g_1^2)^2(g_2g_1^2)^{-1}g_2h \\ &= (g_2g_1)^{-2}(g_2g_1^2)^2g_1^{-2}h \in H \end{aligned}$$
as desired.

□

Problem 1.22 (S2014-Q1). Find the number of colorings of the faces of a cube using 3 colors, where two colorings are considered equal if they can be transformed into each other by a rotation of the cube.

[*Hint*: Use Burnside's formula:
$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|,$$
where a group G acts on a set X, X/G is the set of orbits, and for every $g \in G$, X^g is the fixed subset of g in X.]

Proof. Let X be the set of all possible colorings of the cube (equal cubes allowed), we have $|X| = 3^6$. We notice two things:

1. The group of rotations of a cube is S_4.
2. For $\sigma_1, \sigma_2 \in S_4$ that are conjugates of each other, $|X^{\sigma_1}| = |X^{\sigma_2}$. Therefore for the Burnside's formula becomes
$$|X/S_4| = \frac{1}{|S_4|}\sum_{[\sigma] \text{ conj classes}} |[\sigma]| \cdot |X^\sigma|$$

Now we analyze for each conjugacy class $[\sigma]$, what is $|X^\sigma|$.

- $(1+1+1+1)$, $|[e]| = 1$ and $|X^e| = 3^6$.
- $(1+1+2)$, $|[\sigma_1]| = 6$ and $|X^{\sigma_1}| = 3^3$.
- $(1+3)$, $|[\sigma_2]| = 8$, and $|X^{\sigma_2}| = 3^2$.
- $(2+2)$, $|[\sigma_3]| = 6$, and $|X^{\sigma_3}| = 3^4$.

- (4), $|[\sigma_4]| = 6$, and and $|X^{\sigma_4}| = 3^3$.

Thus combining we get

$$|X/S_4| = \frac{1}{24}\left(3^6 + 6\cdot 3^3 + 8\cdot 3^2 + 6\cdot 3^4 + 6\cdot 3^3\right) = 57$$

□

Problem 1.23 (S2019-Q4). Let f be a polynomial with n variables and define

$$\mathrm{Sym}(f) = \{\sigma \in S_n \mid f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}) = f(x_1, x_2, \ldots, x_n)\}.$$

1. Prove that $\mathrm{Sym}(f)$ is a subgroup of S_n.
2. Prove that the dihedral group D_4 (the group of symmetries of the square) is isomorphic to $\mathrm{Sym}(x_1x_2 + x_3x_4)$.

Proof. 1. The group S_n acts on the polynomial ring $k[x_1, \ldots, x_n]$, by permuting the x_i to $x_{\sigma(i)}$, and we see that Sym(f) is the centralizer of a fixed element $f \in k[x_1, \ldots, x_n]$, hence is a subgroup.

2. We have a total of 8 elements in Sym$(x_1x_2 + x_3x_4)$:

$$\{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$$

and we can by drawing a square tha this corresponds to the group D_4.

□

Problem 1.24 (S2011-Q1, F2004-Q1).

(a) Let H be a proper nontrivial subgroup of a finite group G (i.e., $H \neq \{1\}$ and $H \neq G$). Prove that G is not the union of all conjugates of H in G.

(b) Give an example of an infinite group G for which the assertion in part (a) fails.

Proof. (a) If H is normal, then all conjugations of H is equal to H, but $H \subsetneq G$, this G is not not the union of all conjugates of H in G. Now suppose the contrary that G is the union of all conjugates of H, then the number of distinct conjugates of H is $[G : N_G(H)]$, hence

$$|G| = [G : N_G(H)]\cdot|H| \iff [G : H] = [G : N_G(H)] \iff [N_G(H) : H] = 1$$

this is a contradiction since H is not normal. Thus G not the union of all conjugates of H in G.

(b) Consider the upper triangular matrices over $\mathbb{C}$:

$$B = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} \subset \mathrm{GL}_2(\mathbb{C})$$

It is clear that conjugation of matrices in B do not give matrices with nonzero left bottom entry.

□

Problem 1.25 (S2009-Q1). Let H and K be two solvable subgroups of a group G such that $G = HK$.

1. Show that if either H or K is normal in G, then G is solvable.
2. Give an example where G may not be solvable without the assumption in (a).

Proof. 1. WLOG suppose that H is normal, then the composite map $\varphi = \pi \circ \iota$:

$$K \xrightarrow{\iota} G \xrightarrow{\pi} G/H$$

is surjective, therefore

$$\{e\} \subset H \subset G$$

$G/H \cong K/\ker(\varphi)$ is solvable, hence G is solvable.

2. The smallest nonsolvable group is A_5, we have

$$A_5 = HK$$

where $H = \langle (12345) \rangle$, $K = A_4 = \{\sigma \in A_5 : \sigma(5) = 5\}$. Now H, K are both solvable, but G is not.

$\square$

Problem 1.26 (F2003-Q1). In a group G, let 1 denote the identity element and let $[x, y] = xyx^{-1}y^{-1}$ denote the commutator of elements $x, y \in G$.

1. Express $[z, xy]x$ in terms of x, $[z, x]$, and $[z, y]$.
2. Prove that if the identity $[[x, y], z] = 1$ holds in G, then the following identities hold in G:

$$[x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z].$$

Proof. 1. We have

$$\begin{aligned} [z, xy]x &= zxyz^{-1}y^{-1}x^{-1}x \\ &= zxz^{-1}x^{-1}xzyz^{-1}y^{-1} \\ &= [z, x]x[z, y] \end{aligned}$$

2. The identity $[[x, y], z] = 1$ implies

$$[x, y]z = z[x, y]$$

Therefore using the identity in 1, we have

$$\begin{aligned} [x, yz] &= [x, y]y[x, z]y^{-1} \\ &= [x, y]yy^{-1}[x, z] \\ &= [x, y][x, z] \end{aligned}$$

Similarly

$$\begin{aligned} [xy, z] &= xyzy^{-1}x^{-1}z^{-1} \\ &= xyzy^{-1}z^{-1}zx^{-1}z^{-1} \\ &= x[y, z]x^{-1}[x, z] \\ &= [y, z][x, z] \\ &= [x, z][y, z] \end{aligned}$$

$\square$

Problem 1.27 (S2005-Q1). Let k be a field. Let $G = \mathrm{GL}_n(k)$ be the general linear group, where $n > 0$. Let D be the subgroup of diagonal matrices, and let $N = N_G(D)$ be the normalizer of D in G. Determine the quotient group N/D.

Proof. Consider the normalizers:

$$N = \{g \in G : gDg^{-1} = D\}$$

g basically permutes the n eigenvectors, i.e.,

$$N/D \cong S_n$$

□

Problem 1.28 (F2009-Q1). Let G be a finite group, and let $\mathrm{Aut}(G)$ be its automorphism group. Consider the group action $\phi\colon \mathrm{Aut}(G) \times G \to G$ defined by $\phi(\sigma, g) = \sigma(g)$. Assume G has exactly two orbits under this action.

1. Determine all such groups G up to isomorphism.
2. For each case from (a), determine when $\mathrm{Aut}(G)$ is solvable.

Problem 1.29 (F2016-Q1). Determine Aut(S_3).

Proof. Every element $\sigma \in$ Aut(S_3) must send order 2 elements $\{(12), (23), (13)\}$ to one another, and order 3 elements $\{(123), (132)\}$ to each other. However, σ is determined by how it permutes

$$\{(12), (23), (13)\}$$

Thus every σ is an inner automorphism of the form $\sigma_g(h) = ghg^{-1}$ for $g, h \in S_3$ and g is some transposition. Hence

$$\text{Aut}(S_3) \cong S_3$$

□

Chapter 2

Representation Theory

Proposition 2.1. One should probably know the character table for S_3, S_4, A_5, S_5.

Theorem 2.1 (Compilation of theorems). Schur's lemma:

1. If $\varphi : V \to W$ is a G-invariant map, i.e.,

$$\varphi(\rho(g)(v)) = \rho(g)\varphi(v)$$

   where V, W are irreducible representations, then $\varphi = 0$ or an isomorphism. This is true for any field k that V, W are over.

2. If $\varphi : V \to V$ and everything as above, then

$$\varphi(v) = \lambda v$$

   for some $\lambda \in k^\times$. This is only true when k is algebraically clsoed.

3. $\mathrm{Hom}_G(V, W) \begin{cases} k \text{ if } V \cong W \\ 0 \text{ if not} \end{cases}$, where V, W are irreducible. This is true for k algebraically closed.

4. Mascheke's theorem: any finite dimensional representation V of a finite group G can be decomposed into a direct sum of irreducible representations.

$$V = V_1^{r_1} \oplus \cdots \oplus V_k^{r_k}$$

   where V_i's are irreducible. This is true when the characteristic k does not divide $|G|$, notably this always holds for characteristic 0 fields.

5. Do not mix them up.

Proposition 2.2. G is abelian if and only if every irreducible representation ρ is one-dimensional.

Proof. If G is abelian, take any irreducible representation ρ,

$$\{\rho(g) : g \in G\}$$

can be simultaneously diagonalized (minimal polynomial has no repeated factor), i.e., there exists an eigenbasis $\{e_1, \ldots, e_n\}$ such that $\rho(g)$ is a diagonal matrix for all g. This implies that the vector space generated by $\{e_i\}$ for each i is a ρ-invariant subspace, since ρ is irreducible, ρ must be one-dimensional.

Conversely, let $|G| = n$, if every irreducible ρ is one-dimensional, then there are n irreducible representations, i.e., n conjugacy classes, i.e., G is abelian. □

2.1 Problems

Problem 2.1 (S2008-Q4). Let $V \cong \mathbb{C}^n$ be an n-dimensional complex vector space with standard basis $e_1, \ldots, e_n$. Consider the permutation action $S_n \times V \to V$ defined by:

$$\sigma \cdot e_i = e_{\sigma(i)} \quad \text{for } \sigma \in S_n$$

Decompose V into irreducible $\mathbb{C}[S_n]$-modules.

Proof. We claim that

$$V = \text{Span}\{e_1 + \cdots + e_n\} \bigoplus \text{Std}$$

where Std stands for the standard representation

$$\text{Std} = \text{Span}\{e_1, \ldots, e_n : e_1 + \cdots + e_n = 0\}$$

We verify these are the only irreducible components. Denote the given character as χ_v, we see that

$$\langle \chi_v, \chi_v \rangle = 2$$

Hence it is a sum of 2 irreducible representations, and because

$$\langle \chi_v, \chi_{\text{triv}} \rangle = 1$$

The computation is as follows:

$$\begin{aligned}
\langle \chi_v, \chi_v \rangle &= \frac{1}{n!} \sum_{\sigma \in S_n} (\text{ number of fixed points of } \sigma) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \text{number of } \{(i, j) : \sigma \text{ fixes } i, j\} \\
&= \frac{1}{n!} \sum_{1 \leq i, j \leq n} \text{number of } \{\sigma : \sigma \text{ fixes } i, j\} \\
&= 2
\end{aligned}$$

and similarly for $\langle \chi_v, \chi_{\text{triv}} \rangle = 1$. Thus,

$$\chi_v = \chi_{\text{triv}} \oplus \chi_{\text{std}}$$

where

$$\chi_{\text{std}} = \left\{ (v_1, \ldots, v_n) : \sum_{i=1}^{n} v_i = 0 \right\}$$

□

Problem 2.2 (S2014-Q5). Find the table of characters for S_4.

Proof.

Class Size	$[e]$ 1	$[(12)]$ 6	$[(12)(34)]$ 3	$[(123)]$ 8	$[(1234)]$ 6
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ_2	2	0	2	-1	0
$\chi_{\text{perm}} - \chi_{\text{triv}}$	3	1	-1	0	-1
$\chi_3 \otimes \chi_{\text{sgn}}$	3	-1	-1	0	1

□

Problem 2.3 (F2016-Q6)**.** Find a table of characters for the alternating group A_5.

Problem 2.4 (F2015-Q3)**.** Let $G = S_4$ (the symmetric group on four letters).

(a) Prove that G has two non-equivalent irreducible complex representations of dimension 3; call them ρ_1 and ρ_2.

(b) Decompose the tensor product representation $\rho_1 \otimes \rho_2$ into a direct sum of irreducible representations of G.

Proof. (a) We do this by the following formula: let d_i be the dimension of each irreducible representation of S_4, then

$$|S_4| = 25 = \sum_{i=1}^{5} d_i^2$$

We notice that $d_i \leq 3$ and there are two $d_i = d_j = 3$. We can write down the character table of S_4, and their character does not agree on all $\sigma \in S_4$, hence non-equivalent.

(b) We have

$$\chi_1 \otimes \chi_2(g) = \begin{cases} 9, g = e \\ -1, g = (12) \\ 0, g = (123) \\ -1, g = (1234) \\ 1, g = (12)(34) \end{cases}$$

Hence we see

$$\rho_1 \otimes \rho_2 = \rho_{\text{sgn}} \oplus \rho_{\text{perm}-\text{triv}} \oplus \chi_{3\otimes\text{sgn}} \oplus \chi_2$$

In other words, it is a direct sum of four irreducible representations of S_4 except for the trivial one.

□

Problem 2.5 (F2011-Q4)**.** Let $\rho\colon S_3 \to \mathrm{GL}(2, \mathbb{C})$ be a two-dimensional irreducible representation of the symmetric group S_3.

1. Decompose the tensor square $\rho^{\otimes 2}$ into irreducible representations of S_3.

2. Decompose the tensor cube $\rho^{\otimes 3}$ into irreducible representations of S_3.

Proof. Using the character table of S_3:

Class	$[e]$	$[(12)]$	$[(123)]$
Size	1	3	2
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	-1	1
$\chi^{(3)}$	2	0	-1

(a) Let $\chi \otimes \chi$ denote the corresponding character, we have

$$\chi \otimes \chi(g) = \begin{cases} 4, g = e \\ 0, g = (12) \\ 1, g = (123) \end{cases}$$

Thus we see

$$\rho \otimes \rho = \rho_{\text{triv}} \oplus \rho_{\text{sgn}} \oplus \rho$$

(b) Similarly, we see

$$\chi^{\otimes 3}(g) = \begin{cases} 8, g = e \\ 0, g = (12) \\ -1, g = (123) \end{cases}$$

Thus

$$\rho^{\otimes 3} = \rho_{\text{triv}} \oplus \rho_{\text{sgn}} \oplus \rho \oplus \rho \oplus \rho$$

□

Problem 2.6 (F2014-Q3)**.** Let $G = S_3$ be the symmetric group on three elements.

(a) Prove that G has an irreducible complex representation of dimension 2 (call it ρ), but none of higher dimension.

(b) Decompose the triple tensor product $\rho \otimes \rho \otimes \rho$ into a direct sum of irreducible representations of G.

Proof. (a) Notice that $|S_3| = 6 = d_1^2 + d_2^2 + d_3^2$.

(b) Same as the above question.

□

Problem 2.7 (S2006-Q6)**.** Let S_4 be the symmetric group on four elements.

(a) Give an example of a non-trivial 8-dimensional complex representation of S_4.

(b) Show that every 8-dimensional complex representation of S_4 contains a 2-dimensional invariant subspace.

Proof. (a) There exists a nontrivial 2-dimensional irreducible representation of S_4, if we denote it as ρ, then

$$\rho \otimes \rho \otimes \rho$$

is an 8-dimensional representation of S_4.

(b) We notice that it is impossible to write 8 has the sum of a multiple of 3 and 1, thus it must contain another 1 or 2 in the sum, proving there exists a 2-dimensional invaraint subspace. Warning: this subspace is not necessarily irreducible.

□

Problem 2.8 (F2007-Q5). Prove that every 5-dimensional complex representation of the alternating group A_4 (the alternating group of degree 4) contains a 1-dimensional invariant subspace.

Proof. The character table is as follows:

Class	$[e]$	$[(123)]$	$[(12)(34)]$	$[(132)]$
Size	1	4	3	4
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	ω	1	ω^2
$\chi^{(3)}$	1	ω^2	1	ω
$\chi^{(4)}$	3	0	-1	0

where $\omega = e^{\frac{2\pi i}{3}}$. Since 5 cannot be written as a multiple of 3, it must contain a 1-dimensional invariant subspace (also $2, 3, 4$). ☐

Problem 2.9 (S2004-Q6). Consider complex representations of a finite group G. Let $\sigma_1, \ldots, \sigma_s$ be representatives of the conjugacy classes of G, and let $\chi_1, \ldots, \chi_s$ be the irreducible characters of G.

(a) Define an inner product on the $\mathbb{C}$-vector space of class functions on G such that $\{\chi_1, \ldots, \chi_s\}$ forms an orthonormal basis.

(b) Let $A = (a_{ij})$ be the character table matrix of G, where $a_{ij} = \chi_i(\sigma_j)$ for $1 \leq i, j \leq s$. Prove that A is invertible.

Proof. (a) As expected, take two class functions f_1, f_2 , we define

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

(b) It suffices to see the rows of this matrix are all nonzero and orthogonal to each other, hence linearly independent, i.e., the sqaure matrix is invertible.

☐

Problem 2.10 (S2018-Q4, S2007-Q5). Is S_4 isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{C})$?

Proof. There is no injective homomorphism $\varphi : S_4 \to \mathrm{GL}_2(\mathbb{C})$. Note any such φ is called a 2-dim representation, it is either irreducible or not, we know that the 2-dimensional irreducible representation is not injective, and a direct sum of 1-dimensional representations is also not injective. ☐

Problem 2.11 (S2010-Q6). Let G be a group of order 24. Using representation theory, prove that $G \neq [G, G]$, where $[G, G]$ denotes the commutator subgroup of G.

Proof. Suppose $G = [G, G]$, then we claim the only 1-dimensional representation $\rho : G \to \mathbb{C}^\times$ is the trivial one. This is because if ρ is one-dim, then

$$[G, G] \subset \ker(\rho)$$

i.e., ρ is trivial. However, there is no way to write

$$|G| = 24 = 1 + d_1^2 + \cdots + d_k^2$$

where $d_i \geq 2$. Thus $G \neq [G, G]$. ☐

Problem 2.12 (F2017-Q6). Let G be a finite group with center $Z(G)$. Show that if G admits a faithful irreducible representation $\rho\colon G \to \mathrm{GL}_n(k)$ for some positive integer $n \in \mathbb{Z}^+$ and some field k, then the center $Z(G)$ is cyclic.

Proof. (We will only do the case where k is algebraically closed). For any $z \in Z(G)$, $\rho(z) : V \to V$ is a G-map, i.e.,

$$\rho(z)(\rho(g)v) = \rho(g)(\rho(z)v)$$

We know by Schur's lemma that $\rho(z)$ is a scalar multiplication:

$$\rho(z) \in k^\times$$

Because ρ is faithful, Z embeds into $k^\times$ via ρ.

Lemma 2.1 (Fact). Any finite subgroup of $k^\times$ for field k is cyclic.

Hence Z is cyclic. □

Problem 2.13 (S2005-Q6). Let V be a finite-dimensional vector space over a field k, and let G be a finite group with an irreducible representation $\varphi\colon G \to \mathrm{GL}(V)$. Suppose H is a finite abelian subgroup of $\mathrm{GL}(V)$ contained in the centralizer of $\varphi(G)$. Prove that H must be cyclic.

Proof. Just like above, we embed H into $k^\times$. Let any $h \in H$, we note that h is a G-map, i.e., for any $g \in G$,

$$h(\varphi(g)v) = \varphi(g)hv$$

this is because h is contained in the centralizer of $\varphi(G)$, i.e., commutes with all $\varphi(g)$. By Schur's Lemma, we have

$$h = \lambda I, \text{ where } \lambda \in k^\times$$

One can define a homomorphism $\psi : H \to k^\times$ such that

$$\psi(\lambda I) = \lambda$$

This map embeds H into $k^\times$, and we are done by again observing any finite subgroup of $k^\times$ is cyclic, □

Problem 2.14 (F2010-Q6). Let G be a non-abelian group of order p^3, where p is prime.

1. Determine the number of isomorphism classes of irreducible complex representations of G, and find their dimensions.

2. Which of these irreducible complex representations are faithful? Justify your answer.

Proof.

1. In S2010-Q1, we showed there are $p^2 - 1 + p$ conjugacy classes in a non-abelian group G of order p^3. There are p^2 one-dimensional irreducible representations because one dimensional representations of G are equivalent to one-dimensional representations of $G/[G,G]$ which has size p^2, thus abelian and all irreducible representations are one-dimensional.

   Lemma 2.2 (Fact). Let V be an irreducible representation, then $\dim V$ divides $|G|$. (This is true when k is algebraically closed and characteristic 0).

   Thus it is clear that there are $p - 1$ representations of dimension p. (Sanity check: $|G| = p^3 = p^2 + (p - 1)p^2$).

2. We claim that all the one-dimensional representations are not faithful and all the p-dimensional representations are. Recall ρ is irreducible if and only if $\ker(\rho) = \{g : \rho(g)v = v \text{ for all } v\} = \{e\}$.

Lemma 2.3 (Fact). Let $\rho : G \to \mathbb{C}^\times$ be a one-dimensional irreducible representation, then

$$[G, G] \subset \ker(\rho)$$

Thus if ρ is one-dimensional, then ρ is not faithful. Now for the higher dimensional case:

Lemma 2.4 (Fact). If $\rho : G \to \mathrm{GL}_p(\mathbb{C})$ is an irreducible representation, then $\bar{\rho} : \frac{G}{\ker \rho} \to \mathrm{GL}_p(\mathbb{C})$ is also irreducible.

If $\ker \rho$ is nontrivial, then it must divide the size of $|G|$, hence $\frac{G}{\ker \rho}$ is abelian, i.e., all irreducible representations are one-dimensional. This is a contradiction since ρ is p-dimensional, thus $\ker(\rho) = \{e\}$, as desired.

□

Problem 2.15 (S2011-Q5). Let K be a field, and let $\Phi\colon G \to \mathrm{GL}_n(K)$ be an n-dimensional matrix representation of a group G. Define a G-action on the matrix ring $M_n(K)$ by:

$$(g, A) \mapsto \Phi(g) \cdot A \quad \text{(matrix multiplication)}$$

for $g \in G$ and $A \in M_n(K)$. This action induces a group homomorphism $\Psi\colon G \to \mathrm{GL}(M_n(K))$. Express the character χ_Ψ of Ψ in terms of χ_Φ (the character of Φ).

Proof. We compute the trace of the multiplication map by $\Phi(g)$, we consider a basis of $M_n(K)$

$$\{M_{ij} : 1 \leq i, j \leq n\}$$

where M_{ij} is the matrix with only nonzero entry 1 at the ijth position. Then we see

$$\Phi(g)M_{ij} = (\Phi(g))_{ii}$$

Thus

$$\chi_\Psi = n\sum_{i=1}^{n}(\Phi(g))_{ii} = n\mathrm{tr}(\Phi(g))$$

□

Problem 2.16 (S2015-Q5). Prove that a tensor product of irreducible representations over an algebraically closed field is irreducible.

Proof. Let ρ_1 be irreducible of G_1, ρ_2 of G_2, then over an algebraically closed field, we know $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \times G_2$, and we define

$$\rho_1 \otimes \rho_2(g_1, g_2) = \rho_1(g_1) \otimes \rho_2(g_2)$$

where $g_1 \in G_1, g_2 \in G_2$. And define $\chi_1 \otimes \chi_2$ similarly, we have

$$\chi_1 \otimes \chi_2(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$$

One can use this to show that the tensor product $\rho_1 \otimes \rho_2$ of two irreducible representations ρ_1 and ρ_2 is irreducible on $G_1 \times G_2$. □

Problem 2.17 (S2001-Q3). Calculate the complete character table for $\mathbb{Z}/3\mathbb{Z} \times S_3$, where S_3 is the symmetric group in 3 letters.

Proof. Using the question above, it suffices to find all the irreducible characters of $\mathbb{Z}/3\mathbb{Z}$ and S_3. There are 3 irreducible representations for each, hence there are 9 irreducible characters on $\mathbb{Z}/3\mathbb{Z} \times S_3$ in total. We will skip the character table here but the exact filling of the table should follow the above

$$\chi_1 \otimes \chi_2(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$$

One thing to note is that let ρ be an irreducible representation of $\mathbb{Z}/3\mathbb{Z}$, then

$$\rho(g)^3 = 1$$

hence $\rho(g) = e^{\frac{2\pi i}{3}}$, where $i = 0, 1, 2$. $\square$

Warning 2.2. For one-dimensional irreducible representation, $\rho : G \to \mathrm{GL}_n(\mathbb{C})$, they are equivalent to $\rho : G^{ab} = \frac{G}{[G,G]} \to \mathrm{GL}_n(\mathbb{C})$.

Moreover, let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation, then $\bar{\rho} : \frac{G}{\ker \rho} \to \mathrm{GL}_n(\mathbb{C})$ is also irreducible.

Problem 2.18. Every irreducible representation of a finite cyclic group G over $\mathbb{R}$ has dimension ≤ 2.

Proof. Consider $\rho(g) : \mathbb{R}^n \to \mathbb{R}^n$, then

$$\mathbb{R}^n \cong \bigoplus_{i=1}^{d} \frac{\mathbb{R}[x]}{(f_i(x))}$$

where $f_1 \mid f_2 \mid \cdots \mid f_d$, because ρ is irreducible, there can only be one summand. Namely, f needs to be irreducible and

$$\mathbb{R}^n \cong \frac{\mathbb{R}[x]}{(f(x))}$$

since f has degree at most 2, we know $n \leq 2$. $\square$

Warning 2.3. The above problem is linear algebra and rep theeory!

Proposition 2.3. Let $\rho : G \to \mathrm{GL}_n(\bar{k})$, if $\rho = \sigma(\rho)$ for all $\sigma \in \mathrm{Gal}(\bar{k}/k)$, then $\rho : G \to \mathrm{GL}_n(k)$. In other words, it is a representation over k.

For example, if ρ is a complex representation, and $\rho = \bar{\rho}$, then ρ is a real representation.

Now we give an alternative proof of the above problem:

Proof. $\rho : G \to \mathrm{GL}_n(\mathbb{R})$ can be viewed as a representation over $\mathbb{C}$, then

$$\rho = \rho_1 \oplus \cdots \oplus \rho_k$$

If ρ_i is real for any i, then we are done; if not, we note that

$$\rho = \bar{\rho} = \bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_k$$

then $\rho_i = \bar{\rho}_1$ for some i, then we can consider

$$\rho' = \rho_1 \oplus \bar{\rho}_1$$

This is a real representation because $\rho' = \bar{\rho}'$, by Galois descent, ρ' is a real representation, i.e., ρ is at most two-dimensional. $\square$

2.2 Induced representations, Frobenius Reciprocity

Problem 2.19 (S2009-Q6). Let $G = S_4$ and consider the subgroup $H = \langle (1\,2), (3\,4) \rangle$.

(a) Determine the number of irreducible complex characters of H.

(b) Choose a non-trivial irreducible character ψ of H over $\mathbb{C}$ satisfying $\psi((1\,2)(3\,4)) = -1$. Compute the values of the induced character $\mathrm{ind}_H^G(\psi)$ on all conjugacy classes of G, and express it as a sum of irreducible characters of G.

Problem 2.20 (S2017-Q6). Let G be a finite group and H an abelian subgroup. Show that every irreducible representation of G over $\mathbb{C}$ has dimension $\leq [G : H]$.

Proof. We know that if A is commutative, then all the irreducible representations ρ of A are one-dimensional. Now we induce ρ to a representation on G:

$$\bar{\rho} : G \to \mathrm{GL}(\mathbb{C})$$

We have

$$\mathrm{Ind}_A^G = \bigoplus_{i=1}^{n} g_i V$$

where g_i is the representative for each coset G/A, and $n = [G : A]$. Therefore all representations of G has dimension $[G : A]$. Since not all induced representataions are irreducible, any irreducible representation of G has dimension $\leq [G : A]$, as desired. ☐

Problem 2.21 (S2008-Q6). Give an example of non-isomorphic finite groups with same character table. Construct the character table in detail.

Proof. D_8 and Q_8. They both have the trivial representation; subgroup $\mathbb{Z}/2\mathbb{Z}$ gives $D_8/\mathbb{Z}/2\mathbb{Z}$ a Klein 4 group, thus ☐

Problem 2.22. Decompose the permutation representation of S_n into irreducible representataions.

Proof. Recall that S_n acts on an n-dimensional vector space by permuting the basis elements $\{e_1, \ldots, e_n\}$. We claim that

$$V = V_{\text{triv}} + V_{\text{std}}$$

where

$$V_{\text{triv}} = \mathrm{Span}\{e_1 + e_2 + \cdots + e_n\}, \quad V_{\text{std}} = \left\{ \sum_i a_i e_i : \sum_i a_i = 0 \right\}$$

☐

Problem 2.23 (S2012-Q4). Let Q be the quaternion group with presentation:

$$Q = \langle t, s_i, s_j, s_k \mid t^2 = 1,\ s_i^2 = s_j^2 = s_k^2 = s_i s_j s_k = t \rangle.$$

(a) Find four non-isomorphic 1-dimensional real representations of Q.

(b) Prove that the natural embedding $\rho\colon Q \to \mathbb{H}$ given by:

$$\rho(t) = -1, \quad \rho(s_i) = i, \quad \rho(s_j) = j, \quad \rho(s_k) = k$$

defines an irreducible 4-dimensional real representation of Q, where $\mathbb{H}$ is the algebra of real quaternions.

(c) Classify all irreducible complex representations of Q up to isomorphism.

Proof. □

Problem 2.24 (F2004-Q6). Let D_8 be the dihedral group of order 8, with presentation:

$$D_8 = \langle r, s \mid r^4 = 1 = s^2,\ rs = sr^{-1} \rangle.$$

1. Determine all conjugacy classes of D_8.
2. Find the commutator subgroup D_8' of D_8 and determine the number of distinct degree-1 (linear) characters of D_8.
3. Construct the complete complex character table of D_8.

Proof. D_4 has $\frac{4+6}{2} = 5$ conjugacy classes. The commutator subgroup $[D_4, D_4] = \{e, r^2\}$, thus D_4^{ab} gives 4 one-dimensional representations of D_4. □

Problem 2.25 (F2000-Q7). Let D_{10} be the dihedral group of order 10, with presentation:

$$D_{10} = \langle r, s \mid r^5 = 1 = s^2,\ rs = sr^{-1} \rangle.$$

1. Determine all conjugacy classes of D_{10}.
2. Compute the commutator subgroup D_{10}' of D_{10}.
3. Prove that $D_{10}/D_{10}' \cong \mathbb{Z}/2\mathbb{Z}$ and deduce that D_{10} has exactly two distinct degree-1 characters.
4. Construct the complete complex character table of D_{10}.

Proposition 2.4. The character table for D_n, when n =odd. There are $\frac{n+3}{2}$ conjugacy classes, for example, D_5 has 4 conjugacy classes:

$$e, \{r, r^4\}, \{r^2, r^3\}, s$$

And there are two one-dimensional irreducible representataions: trivial and sign: sending reflection s to -1, and rotations to 1. The rest are two-dimensional irreducible representations: one example is

$$r \mapsto 2\cos(2\pi i/n), s \mapsto 0$$

The character table for D_n, when n =even. There are $\frac{n+6}{2}$ conjugacy classes: for example, D_4 has 5 conjugacy classes. And the character is more complicated, know that of D_4. Remember that it has at most dimension 2 irreducible representations.

Chapter 3

Semisimple Algebra

Definition 3.1 (Division ring)**.** Any nonzero element in a unit.

Proposition 3.1. Let A be a semisimple finite-dimensional algebra over F, then A can be decomposed into a direct sum of matrix algebras over a division ring:

$$A = M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

where D_i's are division rings, M_{n_i} is the algebra of $n_i \times n_i$ matrices with entries in D_i. This decomposition is unique up to permutation.

For example, let G be a finite, group, then the group algebra $\mathbb{C}(G)$ can be decomposed to

$$\mathbb{C}(G) = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$$

where $|G| = \sum_i d_i^2$, and k is the number of conjugacy classes of G. Hence it suffices to compute the irreducible representations of G.

Proposition 3.2. Any semisimple ring R can be decomposed into a finite direct sum of simple ideals J_i:

$$R = \bigoplus_{i=1}^{n} J_i.$$

Problem 3.1 (F2019-Q5)**.** Determine the number of two-sided ideals in the group algebra $\mathbb{C}[S_3]$, where S_3 is the symmetric group of permutations of $\{1, 2, 3\}$.

Proof. Using the Proposition above, we know that

$$\mathbb{C}(S_3) = M_1(\mathbb{C}) + M_1(\mathbb{C}) + M_2(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$$

□

Problem 3.2 (F2009-Q6, F2001-Q5). Let $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible complex representation of a finite group G, with character χ, and let C be the center of G.

1. Prove that for every $s \in C$, the matrix $\rho(s)$ is a scalar multiple of the identity matrix I_n.
2. Using part (a), show that $|\chi(s)| = n$ for all $s \in C$.
3. Establish the inequality $n^2 \leq [G : C]$, where $[G : C]$ is the index of C in G.
4. Prove that if ρ is faithful (i.e., injective), then C must be cyclic.

Proof. 1. $\mathbb{C}$ is algebraically closed therefore Schur's lemma applies (see F2017-Q6)

2. We know that

$$\rho(z) = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & & & \\ 0 & \dots & 0 & \lambda \end{pmatrix}$$

We also know that C is finite and $\rho(z^r) = I$, which implies $|r| = 1$. This gives $|\chi(s)| = n$ for all $s \in C$.

3. We know that ρ is irreducible, hence the corresponding character χ satisfies

$$\frac{1}{|G|}\sum_{g\in G}|\chi(g)|^2 = \frac{|C|}{|G|}n^2 + \frac{1}{|G|}\sum_{g\notin C}|\chi(g)|^2 = 1$$

This implies that

$$\frac{|C|}{|G|}n^2 \leq 1 \Rightarrow n^2 \leq [G : C]$$

4. If ρ is faithful, then C embeds into $k^\times$, and any finite subgroup of $k^\times$ is cyclic.

□

Problem 3.3 (S2017-Q5). Prove directly from the definition of (left) semisimple ring that every such ring is (left) Noetherian and Artinian. (You may freely use facts about semisimple, Noetherian, and Artinian modules.)

Proof. Any semisimple ring R can be decomposed into a finite direct sum of simple ideals I_i:

$$R = \bigoplus_{i=1}^{n} I_i$$

This directly implies that the ascending and descending chain condition: there aren't infinitely sequence of ideals of strict inclusions. □

Problem 3.4 (S2005-Q4). Let R be a ring and L a minimal left ideal of R (i.e., L contains no non-zero proper left ideals of R). Assuming $L^2 \neq 0$, prove that $L = Re$ for some non-zero idempotent element $e \in R$.

Proof. We recall that a ring element $e \in R$ is an idempotent if and only if

$$e^2 = e$$

It suffices to show that there exists a nonzero idempotent element $e \in L$ since Re is an ideal contained in L, thus $Re = L$. Take any $x \neq 0$ in L, such that there exists $g \in L$ such that $gx \neq 0$ (this is guaranteed by $L^2 \neq 0$). The ideal Lx is contained in L, since L is simple, we must have

$$L = Lx$$

Hence $x \in L$ can be written as

$$x = ex$$

for some $e \in L$, multiplying both sides by e and moving terms, we get

$$(e^2 - e)x = 0$$

It suffices to show that

$$\{g \in L : gx = 0\} = \{0\}$$

This is because $\{g \in L : gx = 0\}$ is again an ideal contained in L, since we assumed that there exists some $g \in L$ such that $gx \neq 0$,

$$\{g \in L : gx = 0\} = \{0\}$$

and we are done! □

Problem 3.5 (S2016-Q6, F2006-Q6, F2008-Q6)**.** Let A be a finite-dimensional semisimple algebra over $\mathbb{C}$, and let V be an A-module that decomposes as $V \cong S \oplus S$, where S is a simple A-module. Determine the automorphism group $\mathrm{Aut}_A(V)$ of V as an A-module.

Proof. By Schur's lemma, since S is a simple A-module, we know

$$\mathrm{End}_A(S) \cong \mathbb{C}$$

Thus

$$\mathrm{End}(V) \cong M_2(\mathbb{C})$$

hence

$$\mathrm{Aut}_A(V) \cong \mathrm{GL}_2(\mathbb{C})$$

□

Problem 3.6 (S2010-Q5)**.** Classify all non-commutative semi-simple rings with 512 elements. (You can use the fact that finite division rings are fields.)

Proof. By Artin-Wedderburn, we know that this finite semisimple ring can be decomposed into a finite direct sum of matrix rings:

$$R \cong M_{n_1}(F_1) \oplus \cdots \oplus M_{n_k}(F_k)$$

where F_i are finite fields. Further more we can assume $n_1 \geq n_2 \geq \cdots \geq n_k$.The total number of elements is

$$F_1^{n_1^2} \ldots F_k^{n_k^2} = 512 = 2^9$$

Thus we see all the components are powers of 2. Since R is noncommutative, we may assume that $n_1 \geq 2$. If $n_1 = 3$, then $F_1 = \mathbb{F}_2$, we have

$$R \cong M_3(\mathbb{F}_2)$$

If $n_1 = 2$, we can have $n_2 = 2$, then

$$R \cong M_2(\mathbb{F}_2) \oplus M_2(\mathbb{F}_2) \oplus \mathbb{F}_2$$

or $n_2 = n_3 = \cdots = n_k = 1$, then we have (different ways of adding to 5):

$$R \cong M_2(\mathbb{F}_2) \oplus \begin{cases} \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus F_2 \\ \mathbb{F}_4 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \oplus F_2 \\ \mathbb{F}_8 \oplus \mathbb{F}_2 \oplus \mathbb{F}_2 \\ \mathbb{F}_{16} \oplus \mathbb{F}_2 \\ \mathbb{F}_8 \oplus \mathbb{F}_4 \\ \mathbb{F}_{16} \oplus F_2 \\ \mathbb{F}_{32} \end{cases}$$

□

Problem 3.7 (F2011-Q5). Let A be a finite-dimensional semisimple algebra over $\mathbb{C}$, and let V be a finitely-generated A-module. Prove that V has only finitely many A-submodules if and only if V decomposes into a direct sum of pairwise irreducible non-isomorphic (i.e., simple) A-modules.

Proof. Suppose that V is a direct sum of distinct irreducible A-modules, then

$$V = S_1 \oplus \cdots \oplus S_n$$

where S_i's are nonisomorphic and simple. Hence the only submodules of S_i is $\{0\}$ and S_i, i.e., there are only finitely many submodules of V.

Conversely, we suppose that there are finitely many A-submodules of V, because V is semisimple, we know

$$V = \bigoplus_{i=1}^{n} S_i^{n_i}$$

where S_i's are semisimple. It suffices to show that $n_i = 1$ for all i. Suppose that

$$V = S_i \oplus S_i$$

By Schur's lemma, we have

$$\mathrm{Hom}(S_i, S_i) \cong \mathbb{C}$$

there are infinitely many distinct $\phi : S_i \to S_i$, and we note that

$$\{(s, \phi(s)) : \phi \in \mathrm{Hom}(S_i, S_i)\}$$

is a submodule of V, thus there are infinitely many submodules, which is a contradiction. □

Chapter 4

Linear Algebra I

Topics: finitely generated modules/PID, triangularization, diagonalization, Jordan canonical form.

Proposition 4.1. Assume that characteristic a linear operator $T : V \to V$ factors completely over k, then T is diagonalizable if and only if the minimal polynomial splits into distinct linear factors (has no repeated roots).

Problem 4.1 (F2018-Q1)**.** Let V be an n-dimensional vector space over a field k and let $\alpha : V \to V$ be a linear endomorphism. Prove that the minimal and characteristic polynomials of α coincide if and only if there is a vector $v \in V$ so that:

$$\{v, \alpha(v), \ldots, \alpha^{n-1}(v)\}$$

is a basis for V.

Proof. □

Problem 4.2 (F2018-Q3)**.**

(a) Fix a positive integer n and classify all finite modules over the ring $\mathbb{Z}/n\mathbb{Z}$.

(b) Prove, either using (a) or from first principles, for a fixed prime p that all finite modules over $\mathbb{Z}/p\mathbb{Z}$ are free.

Proof. By the classification of finite abelian groups

(a) $G \cong \bigoplus_{i,j} \frac{\mathbb{Z}}{p_i^{ij}\mathbb{Z}}$, but G cannot be a $\mathbb{Z}/n\mathbb{Z}$-module unless p_i^{ij} divides n for all i:

$$reasoning$$

Thus for a fixed n, let $n = p_1^{a_1} \ldots p_k^{a_k}$ be its prime factorization, then

$$G \cong \bigoplus_{p|n} \bigoplus_{i} \frac{\mathbb{Z}}{p^i\mathbb{Z}}$$

where $i \leq a_i$ for each i.

(b) By (a).

□

Problem 4.3 (F2017-Q2). Let Λ be a free abelian group of finite rank n, and let $\Lambda' \subset \Lambda$ be a subgroup of the same rank. Let $x_1, \ldots, x_n$ be a $\mathbb{Z}$-basis for Λ, and let $x_1', \ldots, x_n'$ be a $\mathbb{Z}$-basis for Λ'. For each i, write $x_i' = \sum_{j=1}^n a_{ij}x_j$, and let $A := (a_{ij}) \in \mathrm{Mat}_{n\times n}(\mathbb{Z})$. Show that the index $[\Lambda : \Lambda']$ equals $|\det A|$.

Proof. Up to some basis change, we can write

$$\Gamma' = d_1\mathbb{Z} \oplus \cdots \oplus d_k\mathbb{Z}$$

given $\Gamma = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$. Then Since we are taking the determinant, it is invariant under change of basis. One can compute the matrix using the standard basis for Γ and Γ', and it is clear that $[\Gamma : \Gamma'] = \prod_{i=1}^k d_i = |\det(A)|$. ∎

Problem 4.4 (S2001-Q5).

(a) Prove that an $n \times n$ matrix A with entries in the field $\mathbb{C}$ of complex numbers, satisfying $A^3 = A$, can be diagonalized over $\mathbb{C}$.

(b) Does the statement in (a) remain true if one replaces $\mathbb{C}$ by an arbitrary algebraically closed field F? Why or why not?

Proof. (a) A is diagonalizable if and only if the minimal polynomial splits into distinct linear factors. The characteristic polynomial is $p(t) = t(t+1)(t-1)$ and the minimal polynomial $p_m \mid p$ thus A is diagonalizable.

(b) This is not true. Take k to be a field of characteristic 2, then

$$p(t) = t(t^2-1) = t(t-1)^2$$

Thus the minimal polynomial could be $(t-1)^2$, i.e., A is not necessarily diagonalizable. ∎

Problem 4.5 (F2001-Q3). Let A be an $n \times n$ complex matrix with $A^m = 0$ for some integer $m > 0$.

1. Show that if λ is an eigenvalue of A, then $\lambda = 0$.
2. Determine the characteristic polynomial of A.
3. Prove that $A^n = 0$.
4. Construct a 5×5 matrix B satisfying $B^3 = 0$ but $B^2 \neq 0$.
5. For any 5×5 complex matrix M with $M^3 = 0$ and $M^2 \neq 0$, is M necessarily similar to your matrix B from part (d)? Justify your answer.

1. Suppose λ is an eigenvalue, then there exists $v \neq 0$, such that

$$A^m v = \lambda^m v = 0 \Rightarrow \lambda = 0$$

2. The characteristic polynomial is $p(t) = t^n$.
3. Cayley-Hamilton theorem.

4. Can have
$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
The important is that the top left 3×3 matrix A satisfies $A^3 = 0, A^3 \neq 0$. This is constructed by building B using the Jordan form.

5. No, the lower 2×2 matrix could be
$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Problem 4.6 (F2018-Q4). In this question all modules are left modules.

Let k be a field of characteristic different from 2 and let $G = \{e, g\}$ be the multiplicative group with two elements. Consider the group ring $A = k[G]$.

(a) Show that the A-module A is a direct sum of two ideals of A.
- List all proper ideals of A.
- Is A a principal ideal domain?

(b) Show that every A-module decomposes into a direct sum of simple A-modules.

(c) Assume now that the characteristic of k is 2. Give an example of an A-module that cannot be decomposed into a direct sum of two simple A-modules.

Proof. not finished □

Problem 4.7 (S2003-Q3). Prove that if a linear operator on a complex vector space is diagonal in some basis, then its restriction to any invariant subspace L is also diagonal in some basis of L.

Proof. The linear operator T is diagonalizable if and only if the minimal polynomial has no repeated factors, i.e.,
$$f_m(x) = (x - \lambda_1) \dots (x - \lambda_k)$$
And $T|_L$ has minimal polynomial dividng f_m, hence it also has no repeated factors, thus $T|_L$ is also diagonalizable. □

Problem 4.8 (S2017-Q4). Let M be an invertible $n \times n$ matrix with entries in an algebraically closed field k of characteristic not 2. Show that M has a square root, i.e. there exists $N \in \text{Mat}_{n\times n}(k)$ such that $N^2 = M$.

Proof. It suffices to show that every Jordan block
$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

where $\lambda \neq 0$ is a square. We will proceed using induction. When $n = 2$, the square root of

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^{\frac{1}{2}} & \frac{1}{2}\lambda^{-\frac{1}{2}} \\ 0 & \lambda^{\frac{1}{2}} \end{bmatrix}^2$$

Now assume that J_k is a square up to $k = n-1$, we want to show J_n also has a square root. We claim J_n has the following square

$$J_n = \begin{bmatrix} B^2 & x \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} B & x \\ 0 & \lambda^{1/2} \end{bmatrix}^2$$

where B is a $(n-1) \times (n-1)$ matrix and $x = (x_1, \dots, x_{n-1}), 0 = (0, \dots, 0)$. It suffices to find such an x exists. Let $b_1, \dots, b_{n-1}$ denote the row vectors of B, we must satisfy

$$\begin{cases} b_1 \cdot x + x_1 \lambda^{\frac{1}{2}} = 0 \\ \dots \\ b_{n-2} \cdot x + x_{n-2} \lambda^{\frac{1}{2}} = 0 \\ b_{n-1} \cdot x + x_{n-1} \lambda^{\frac{1}{2}} = 1 \end{cases}$$

Namely, we need to find x that satisfies

$$(B + \lambda^{\frac{1}{2}} I)x = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

Since $(B + \lambda^{1/2} I)$ is invertible, there exists a unique solution, hence such x exsits, J_n has a square root! $\square$

Problem 4.9 (S2008-Q1)**.** Let k be a field. Consider the subgroup $B \subset \mathrm{GL}_2(k)$ where

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in k, ad \neq 0 \right\}.$$

(a) Let Z be the center of $\mathrm{GL}_2(k)$. Show that

$$\bigcap_{x \in \mathrm{GL}_2(k)} x^{-1} B x = Z.$$

(b) Assume k is algebraically closed. Show that

$$\bigcup_{x \in \mathrm{GL}_2(k)} x^{-1} B x = \mathrm{GL}_2(k).$$

(c) Assume k is a finite field. Is the equation in (b) still true?

Proof. (a) Let $y \in \bigcap_{x \in \mathrm{GL}_2(k)}$, then for all $x \in \mathrm{GL}_2(k)$, we have $xyx^{-1} \in B$. This shows that

$$\begin{aligned} xyx^{-1} \in B \text{ for all } x &\iff xyx^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle \\ &\iff x^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is a subspace for } y \text{ for all } x \\ &\iff \text{the whole vector space is the eigenspace of } y \\ &\iff y \text{ is a scalar} \\ &\iff y \in Z \end{aligned}$$

(b) If k is algebraically closed, then any matrix can be written as a triangular matrix up to some basis change.

(c) It's false for finite fields. (b) is true when only when the characteristic polynomial can be factored completely over k. Take $k = \mathbb{F}_2$, then we know $x^2 + x + 1$ is irreducible over $\mathbb{F}_2$, then we notice the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

which has characteristic polynomial exactly this. This is a counterexample.

(One can take $g \in \overline{\mathbb{F}_p} \setminus \mathbb{F}_p$, then the characteristic polynomial for the map of multiplication by $g : \overline{\mathbb{F}_p} \to \overline{\mathbb{F}_p}$ where $\overline{\mathbb{F}_p} = \mathbb{F}_{p^2}$ is a vector space over $\mathbb{F}$ the minimal polynomial is $(t-g)^2$ which is irreducible over $\mathbb{F}_p$.)

$\square$

Problem 4.10 (S2009-Q4)**.** Let E be a finite-dimensional vector space over an algebraically closed field k. Let A, B be k-endomorphisms of E. Assume $AB = BA$. Show that A and B have a common eigenvector.

Proof. Since k is algebraically closed, we know there exists at least one eigenvector of A, i.e., there exists λ such that $Av = \lambda v$ for some $v \neq 0$. We denote this eigenspace by E_λ, and we note that E_λ is invariant under B: let $v \in E_\lambda$

$$A(Bv) = \lambda(Bv)$$

thus $Bv \in E_\lambda$ as well. Then it suffices to find an eigenvector of B living inside E_λ, this is done by noting $B|_{E_\lambda}$ has an eigenvector in E_λ, as desired. $\square$

Problem 4.11 (F2005-Q6)**.** Let E be a finite-dimensional vector space over a field k. Assume $S, T \in \mathrm{End}_k(E)$. Assume $ST = TS$ and both of them are diagonalizable. Show that there exists a basis of E consisting of eigenvectors for both S and T.

Proof. It is the same proof as above except now we do this for all $E_{\lambda_1}, \ldots, E_{\lambda_k}$. $\square$

Problem 4.12 (S2015-Q2)**.** Let A, B be two commuting operators on a finite dimensional space V over $\mathbb{C}$ such that $A^n = B^m$ is the identity operator on V for some positive integers n, m. Prove that V is a direct sum of 1-dimensional invariant subspaces with respect to A and B simultaneously.

Proof. Because

$$A^n = B^m = I$$

We know that the minimal polynomial of A, B both have no repeated roots, because $(t^n - 1), (t^m - 1)$ factor completely over $\mathbb{C}$. This shows that A, B are commuting diagonalizable matrices, thus they can be simultaneously diagonalized. $\square$

Chapter 5

Linear Algebra II

Topics: exterior power, tensor algebras, traces, determinants

Problem 5.1 (F2016-Q5). Let A be a linear transformation of a finite dimensional vector space over a field of characteristic $\neq 2$.

(1) Define the wedge product linear transformation $\wedge^2 A = A \wedge A$.

(2) Prove that
$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2)).$$

Proof. (a) We recall the wedge product of vector space $V \wedge V$ is given by the basis
$$\{v_i \wedge v_j : i < j\}$$
satisfying
$$v_i \wedge v_j = -v_j \wedge v_i$$
where $\{v_1, \ldots, v_n\}$ is a basis for V. And we define
$$A \wedge A(v_i \wedge v_j) = A(v_i) \wedge A(v_j)$$

(b) Consider the matrix representation of $A = (A_{ij})$, on the basis $\{v_i \wedge v_j : i < j\}$,
$$\begin{aligned} A \wedge A(v_i \wedge v_j) &= \sum_{k,l=1}^{n} A_{ki}A_{lj}(v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki}A_{lj}(v_k \wedge v_l) + \sum_{l<k} A_{ki}A_{lj}(v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki}A_{lj}(v_k \wedge v_l) - \sum_{l<k} A_{ki}A_{lj}(v_l \wedge v_k) \end{aligned}$$

Thus the diagonal term with respect to $v_i \wedge v_j$ is
$$A_{ii}A_{jj} - A_{ji}A_{ij}$$

Thus
$$\text{Tr}(A \wedge A) = \sum_{i<j} A_{ii}A_{jj} - A_{ji}A_{ij}$$

Now
$$\text{Tr}(A)^2 = \sum_{i=1}^{n} A_{ii}^2 + 2\sum_{i<j} A_{ii}A_{jj}$$

and

$$\begin{aligned} \mathrm{Tr}(A^2) &= \sum_{k,l=1}^{n} A_{lk}A_{kl} \\ &= \sum_{i=1}^{n} A_{ii}^2 + 2\sum_{k<l} A_{lk}A_{kl} \end{aligned}$$

Thus we see that

$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2))$$

□

Problem 5.2 (S2006-Q5). Let V be a finite-dimensional vector space over a field k. Let $T \in \mathrm{End}_k(V)$. Show that $\mathrm{tr}(T \otimes T) = (\mathrm{tr}(T))^2$. Here $\mathrm{tr}(T)$ is the trace of T.

Proof. We will show that $\mathrm{tr}(T \otimes T) = (\mathrm{tr}T)^2$, and the $T \otimes T \otimes$ is done similarly. We will use matrix representation to do an explicit computation. Let $\{v_1, \ldots, v_n\}$ be a basis of$V$, then $V \otimes V$ has basis

$$\{v_i \otimes v_j : 1 \le i, j \le n\}$$

and

$$T \otimes T(v_i \otimes v_j) = Tv_i \otimes Tv_j$$

Let $T = (a_{ij})$, then we know

$$(\mathrm{tr}(T))^2 = \left(\sum_{i=1}^{n} a_{ii}\right)^2$$

And we have

$$T \otimes T(v_i \otimes v_j) = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{ki}a_{lj} v_k \otimes v_l$$

Therefore computing the trace we see

$$\mathrm{tr}(T \otimes T) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ii}a_{jj} = \mathrm{tr}(T)^2$$

as desired! □

Problem 5.3 (S2016-Q4). Let V and W be two finite dimensional vector spaces over a field K. Show that for any $q > 0$,

$$\bigwedge^q (V \oplus W) \cong \sum_{i=0}^{q} (\bigwedge^i (V) \otimes_K \bigwedge^{q-i} (W)).$$

Proof. Any two finite dimensional vector spaces of the same dimension are isomorphic. Hence, it suffices to show that the dimensions are equal. We will convince ourselves it holds for $q = 2$. Let $\{v_1, \ldots, v_n\}$ be the basis of V, and $\{w_1, \ldots, w_k\}$ be the basis of W, then we begin with the LHS:

$$\bigwedge^2 (V \oplus W)$$

We note that $V \oplus W$ has basis
$$\{(v_i, w_j) : 1 \leq i \leq n, 1 \leq j \leq k\}$$
So we reenumerate the $n + k$ basis as
$$\{e_1, \dots, e_{n+k}\}$$
Then $\bigwedge^q(V \oplus W)$ has basis
$$\{e_i \wedge e_j : i < j\}$$
There are exactly $1 + \cdots + (n + k - 1)$ basis vectors i.e.,
$$\dim\left(\bigwedge^2(V \oplus W)\right) = \frac{(n + k - 1)(n + k)}{2}$$
As for the RHS:
$$\dim\left(\sum_{i=0}^{2}(\bigwedge^i(V) \otimes_K \bigwedge^{2-i}(W))\right) \frac{(k-1)k}{2} + nk + \frac{(n-1)n}{2}$$
And we observe that two two quantities are equal. Now we do the general case, just like above,
$$\dim\left(\bigwedge^q(V \oplus W)\right) = \binom{n+k}{q}$$
And the RHS:
$$\dim\left(\bigwedge^{q-1}(V \oplus W) \wedge (V \oplus W)\right) = \sum_{i=0}^{q}\binom{n}{i}\binom{k}{q-i}$$
and it is clear that these two quantities are equal. $\square$

Problem 5.4 (S2011-Q4). Let F be a field, and V a finite-dimensional vector space over F, with $\dim_F V = n$.

(a) Prove that if $n > 2$, the spaces $\bigwedge^2(\bigwedge^2(V))$ and $\bigwedge^4(V)$ are not isomorphic.

(b) Let k be a positive integer. Prove that when $v \in \bigwedge^k(V)$ and $0 \neq x \in V$, $v \wedge x = 0$ holds if and only if $v = x \wedge y$ for some $y \in \bigwedge^{k-1}(V)$.

Proof. (a) This is by a dimension argument:
$$\dim\left(\bigwedge^2(\bigwedge^2(V))\right) = \binom{\binom{n}{2}}{2} = \frac{n(n-1)(n-2)(n+1)}{2}$$
whereas
$$\dim\left(\bigwedge^4(V)\right) = \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4}$$
Thus not equal if $n > 2$.

(b) If there exists such y where $v = x \wedge y$, then
$$v \wedge x = (x \wedge y) \wedge x = (-y \wedge x) \wedge x = 0$$
Conversely, if $v = 0$, then it is immediate that $v = x \wedge x$. It suffices to assume that $v \neq 0$, thus if we write
$$v = v_1 \wedge \cdots \wedge v_k$$

where v_i's are distinct. Then

$$v \wedge x = 0$$

If $v_i = \pm x$ for any i, we are done. If not, then we derive a contradiction: $v_1 \neq x$, thus

$$v_1 \wedge (v_2 \wedge \cdots \wedge v_k \wedge x) = 0$$

i.e., $v_2 \wedge \cdots \wedge v_k \wedge x = 0$, now $v_2 \neq x$, and we keep going, eventually $v_k \wedge x = 0$ which implies $x = \pm v_k$. □

Problem 5.5 (S2010-Q4). Let V be a n-dimensional vector space over a field k. Let $T \in \mathrm{End}_k(V)$.

(a) Show that $tr(T \otimes T \otimes T) = (tr(T))^3$. Here $tr(T)$ is the trace of T.

(b) Find a similar formula for the determinant $\det(T \otimes T \otimes T)$.

Proof. (a) The trace computation is exactly the same as the one above.

(b) We can compute via some combinatorics:

$$\det(T \otimes T) = (\det T)^{2n}, \det(T \otimes T \otimes T) = (\det T)^{3n^2}$$

□

Chapter 6

Linear Algebra III

Topics: random linear algebra problems

Proposition 6.1. Let V be a m dimensional vector space, and W be n dimensional. Show that $A : V \to V$ and $B : W \to W$ has
$$\text{Tr}(A \otimes B) = \text{Tr}(A)\text{Tr}(B)$$

Proof. Use matrix representations. □

Problem 6.1 (S2013-Q5). Let A and B be $n \times n$ matrices with complex coefficients. Assume that $(A - I)^n = 0$ and $A^k B = BA^k$ for some natural number k. Prove that $AB = BA$ (*Hint*: Prove that A can be expressed as a function of A^k).

Proof. □

Problem 6.2 (F2011-Q2). Consider the special orthogonal group $G = SO(3, \mathbb{R})$, namely,
$$G = \{A \in GL(3, \mathbb{R}) : A^T A = \mathrm{I}_3, \det(A) = 1\}$$

(a) Show that for any element A in G, there exists a real number α with $-1 \leq \alpha \leq 3$ such that
$$A^3 - \alpha A^2 + \alpha A - \mathrm{I}_3 = 0.$$

(b) For which real numbers α with $-1 \leq \alpha \leq 3$ does there exist an element A in G whose minimal polynomial is $x^3 - \alpha x^2 + \alpha x - 1$? Explain your answer.

Proof. (a) The determinant forces the eigenvalues (over $\mathbb{C}$) to have norm 1. The form is done by explicit computations.

(b) It has the minimal polynomial equal to the characteristic polynomial if the polynomial splits into three distinct roots, we know $x = 1$ has a root,
$$(x - 1)(x^2 + (1 - \alpha)x + 1)$$
Hence as long as $\alpha \neq -1, 3$, the minimal polynomial and the characteristic polynomial coincide.

□

Problem 6.3 (F2007-Q3). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix such that $a, b, c, d > 0$.

(1) Prove that A has two distinct real eigenvalues, $\lambda > \mu$.

(2) Prove that λ has an eigenvector in the first quadrant and μ has an eigenvector in the second quadrant.

Problem 6.4 (S2007-Q1). Prove that the integer orthogonal group $O_n(\mathbb{Z})$ is a finite group. (By definition, an $n \times n$ square matrix X over $\mathbb{Z}$ is orthogonal if $XX^t = I_n$.)

Problem 6.5 (F2008-Q4). A differentiation of a ring R is a mapping $D : R \to R$ such that, for all $x, y \in R$,

(1) $D(x + y) = D(x) + D(y)$; and

(2) $D(xy) = D(x)y + xD(y)$.

If K is a field and R is a K-algebra, then its differentiation are supposed to be over K, that is,

(3) $D(x) = 0$ for any $x \in K$.

Let D be a differentiation of the K-algebra $M_n(K)$ of $n \times n$-matrices. Prove that there exists a matrix $A \in M_n(K)$ such that $D(X) = AX - XA$ for all $X \in M_n(K)$.

Problem 6.6 (F2006-Q1). Let $\mathrm{SL}_n(k)$ be the special linear group over a field k, i.e, $n \times n$ matrices with determinant 1. Let I be the identity matrix, and E_{ij} be the elementary matrix that has 1 at (i, j)-entry and 0 elsewhere. Here $1 \leq i \neq j \leq n$.

(1) Let C_{ij} be the centralizer of the matrix $I + E_{ij}$. Find explicit generators of C_{ij}.

(2) Find the intersection

$$\bigcap_{1 \leq i \neq j \leq n} C_{ij}.$$

(3) Determine all the elements in the conjugacy class of $I + E_{ij}$.

Problem 6.7 (S2018-Q1). Let F be a field of characteristic not equal to 2. Let D be the non-commutative algebra over F generated by elements i, j that satisfy the relations

$$i^2 = j^2 = 1, \quad ij = -ji.$$

Define $k = ij$.

(a) Verify that D is isomorphic to the algebra $M_2(F)$ of 2×2 matrices in such a way that

$$1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, j \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, k \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(b) Write $q = x + yi + zj + uk$ for $x, y, z, u \in F$. Verify that the norm

$$N(q) = x^2 - y^2 - z^2 + u^2$$

corresponds to the determinant under the isomorphism of part (a).

(c) What does the involution $q \mapsto \bar{q} = x - yi - zj - uk$ on D correspond to on the matrix side?

Problem 6.8 (S2006-Q3). Let V be a n-dimensional vector space over a field k, with a basis $\{e_1, \ldots, e_n\}$. Let A be the ring of all $n \times n$ diagonal matrices over k. V is a A-module under the action:

$$\operatorname{diag}(\lambda_1, \ldots, \lambda_n) \cdot (a_1 e_1 + \cdots + a_n e_n) = (\lambda_1 a_1 e_1 + \cdots + \lambda_n a_n e_n).$$

Find all A-submodules of V.

Problem 6.9 (S2006-Q1). Let $\mathbb{F}_p$ be the field with p elements, here p is prime. Let $\mathrm{SL}_2(\mathbb{F}_p)$ be the group of 2×2 matrices over $\mathbb{F}_p$ with determinant 1.

(1) Find the order of $\mathrm{SL}_2(\mathbb{F}_p)$. Deduce that

$$H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}$$

is a Sylow-subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$.

(2) Determine the normalizer of H in $\mathrm{SL}_2(\mathbb{F}_p)$ and find its order.

Problem 6.10 (S2004-Q1). Let $\mathbb{F}_2$ be the finite field with 2 elements.

(a) What is the order of $\mathrm{GL}_3(\mathbb{F}_2)$, the group of 3×3 invertible matrices over $\mathbb{F}_2$?

(b) Assuming the fact that $\mathrm{GL}_3(\mathbb{F}_2)$ is a simple group, find the number of elements of order 7 in $\mathrm{GL}_3(\mathbb{F}_2)$.

Problem 6.11 (S2002-Q4). For a field K, let $\mathrm{SL}_2(K)$ be the special linear group over K, i.e. the group of 2×2-matrices over K with determinant 1, and let $\mathrm{PSL}_2(K)$ be the quotient of $\mathrm{SL}_2(K)$ by its center, i.e. the projective special linear group. Find the order of $\mathrm{PSL}_2(F_7)$ where F_7 denotes the finite field of 7 elements.

Problem 6.12 (S2007-Q4). Find the invertible elements, the zero divisors and the nilpotent elements in the following rings:

(a) $\mathbb{Z}/p^n\mathbb{Z}$, where n is a natural number, p is a prime one.

(b) the upper triangular matrices over a field.

Chapter 7

Homological Algebra

Problem 7.1 (S2012-Q2).

(a) Prove that if M is an abelian group and n is a positive integer, the tensor product $M \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ can be naturally identified with M/nM.

(b) Compute the tensor product over $\mathbb{Z}$ of $\mathbb{Z}/n\mathbb{Z}$ with each of $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Also compute the tensor products $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$, and $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})$.

(c) Let $\mathbb{Z}^{\mathbb{N}}$ denote the (abelian) group of sequences $(a_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}$ under termwise addition, and $\mathbb{Z}^{(\mathbb{N})}$ the subgroup of sequences for which $a_i = 0$ for all but finitely many i. Define $\mathbb{Q}^{\mathbb{N}}$ and $\mathbb{Q}^{(\mathbb{N})}$ analogously. Compare $\mathbb{Z}^{(\mathbb{N})} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{(\mathbb{N})}$, and $\mathbb{Z}^{\mathbb{N}} \otimes_{\mathbb{Z}} \mathbb{Q}$ to $\mathbb{Q}^{\mathbb{N}}$.

Problem 7.2 (F2006-Q4). Let R be a commutative ring. Let M be an R-module.

(1) Write down the definition of $\mathcal{T}(M)$, the tensor algebra of M.

(2) Assume $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$. Compute $\mathcal{T}(M)$.

(3) If M is a vector space over a field R, show that $\mathcal{T}(M)$ contains no zero divisors.

Problem 7.3 (S2009-Q5). Consider the $\mathbb{Z}$-modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$ for all positive integers i. Let $M = \prod_{i=1}^{\infty} M_i$. Let $S = \mathbb{Z} - \{0\}$.

(a) Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} M \cong S^{-1}M.$$

Here $S^{-1}M$ is the localization of M.

(b) Show that

$$\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i=1}^{\infty} M_i \neq \prod_{i=1}^{\infty} (\mathbb{Q} \otimes_{\mathbb{Z}} M_i).$$

Problem 7.4 (F2008-Q5). For each $n \in \mathbb{Z}$, define the ring homomorphism

$$\phi_n : \mathbb{Z}[x] \to \mathbb{Z} \text{ by } \phi_n(f) = f(n).$$

This gives a $\mathbb{Z}[x]$-module structure on $\mathbb{Z}$, i.e,

$$f \circ a = f(n) \cdot a \text{ for all } f \in \mathbb{Z}[x] \text{ and } a \in \mathbb{Z}.$$

Now given two integers $m, n \in \mathbb{Z}$, compute the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}[x]} \mathbb{Z}$ where the left-hand copy of $\mathbb{Z}$ uses the module structure from ϕ_n and the right-hand copy of $\mathbb{Z}$ uses the module structure from ϕ_m. (Note: The answer depends on the numbers n and m.)

Problem 7.5 (F2014-Q2). Let $R = \mathbb{Q}[X]$, I and J the principal ideals generated by $X^2 - 1$ and $X^3 - 1$ respectively. Let $M = R/I$ and $N = R/J$. Express in simplest terms [the isomorphism type of] the R-modules $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$. **Explain.**

Problem 7.6 (F2004-Q5). Consider the ideal $I = (2, x)$ in $R = \mathbb{Z}[x]$.

(a) Construct a non-trivial R-module homomorphism $I \otimes_R I \to R/I$, and use that to show that $2 \otimes x - x \otimes 2$ is a non-zero element in $I \otimes_R I$.

(b) Determine the annihilator of $2 \otimes x - x \otimes 2$.

Problem 7.7 (S2018-Q2). Let R be a commutative ring. An R-module M is said to be finitely presented if there exists a right-exact sequence

$$R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$$

for some non-negative integers m, n. Prove that any finitely generated projective R-module P is finitely presented.

Problem 7.8 (F2013-Q3). Let R be a commutative ring with unity. Given an R-module A and an ideal $I \subset R$, there is a natural R-module homomorphism $A \otimes_R I \to A \otimes_R R \cong A$ induced by the inclusion $I \subset R$. In the following three steps you shall prove the flatness criterion: *A is flat if and only if for every finitely generated ideal $I \subset R$ the natural map $A \otimes_R I \to A \otimes_R R$ is injective.*

(a) Prove that if A is flat and $I \subset R$ is a finitely generated ideal then $A \otimes_R I \to A \otimes_R R$ is injective.

(b) If $A \otimes_R I \to A \otimes_R R$ is injective for every finitely generated ideal I, prove that $A \otimes_R I \to A \otimes_R R$ is injective for every ideal I. Show that if K is any submodule of a free module F then the natural map $A \otimes_R K \to A \otimes_R F \cong A$ induced by the inclusion $K \subset F$ is injective (*Hint*: the general case reduces to the case when F has finite rank).

(c) Let $\psi : L \to M$ be an injective homomorphism of R-modules. Prove that the induced map $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective (*Hint*: Write M as a quotient $f : F \to M$ of a free module F, giving a short exact sequence $0 \to K \to F \to M \to 0$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d, "\mathrm{id}"] & J \arrow[r] & L \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & K \arrow[r] & F \arrow[r, "f"] & M \arrow[r] & 0
\end{tikzcd}$$

where $J = f^{-1}(\psi(L))$).

Chapter 8

Ring Theory Random

Proposition 8.1. Let $I \subset R$ be an ideal, then the following are equivalent:

1. I is a prime ideal.
2. There exists a field K and $\varphi : R \to K$ such that $I = \ker(\varphi)$.

Proof. (1)⇒(2). Let K be the field of fractions of R/I, which is an integral domain. (2)⇒ (1) is obvious given K is a field. □

Problem 8.1 (S2010-Q2). Let R be a ring such that $r^3 = r$ for all $r \in R$. Show that R is commutative. (Hint: First show that r^2 is central for all $r \in R$.)

Proof. This question is not so constructive and is purely computational (as far as I am aware) so I will skip it here. □

Problem 8.2 (S2006-Q2). Let R be a ring with identity 1. Let $x, y \in R$ such that $xy = 1$.

(1) Assume R has no zero-divisor. Show that $yx = 1$.

(2) Assume R is finite. Show that $yx = 1$.

Proof. (1) We know $x, y \neq 0$, therefore consider

$$x(yx - 1) = 0$$

Since R has no zero-divisor, we must have $yx - 1 = 0$, as desired.

(2) We note the right multiplication map $m_x : R \to R$ by x is injective: suppose $r_1, r_2 \in R$ and

$$r_1 x = x r_2 x$$

multiplying both sides by y we see $r_1 = r_2$. Since R is finite, this map is also surjective, i.e., there exists $s \in R$ such that

$$sx = 1$$

Now we see

$$yx - 1 = sx(yx - 1) = sx - sx = 0$$

as desired.

□

Chapter 9

Tensor Products over Fields

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Proposition 9.1. If L/k is finite separate extension, then there exists $\alpha \in L$ such that

$$L = k(\alpha)$$

Example 9.1. Write $\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})$ as a product of fields:

$$\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \frac{\mathbb{Q}[x]}{(x^2-3)} \cong \frac{\mathbb{Q}(\sqrt{2})[x]}{(x^3-2)}$$

and $(x^3 - 2)$ does not have a root in $\mathbb{Q}(\sqrt{2})$, thus

$$\mathbb{Q}(\sqrt{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3}) \cong \mathbb{Q}(\sqrt{2})\sqrt{3}$$

Example 9.2. Similarly, write the following as a product of fields

$$\begin{aligned}
\mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{2}) &= \mathbb{Q}(\sqrt[4]{2}) \otimes_{Q} \frac{\mathbb{Q}[x]}{(x^4-2)} \\
&= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x^4-2)} \\
&= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x-\sqrt[4]{2})(x+\sqrt[4]{2})(x^2+\sqrt{2})} \\
&= \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x-\sqrt[4]{2})} \times \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x+\sqrt[4]{2})} \times \frac{\mathbb{Q}(\sqrt[4]{2})[x]}{(x^2+\sqrt{2})}
\end{aligned}$$

By the Chinese Remainder theorem

Lemma 9.1 (CRT). Let R be a PID, and $I + J = (1)$, then

$$\frac{R}{IJ} = \frac{R}{I} \times \frac{R}{J}$$

We have

$$\mathbb{Q}(\sqrt[4]{2}) \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt[4]{2}) \cong \mathbb{Q}(\sqrt[4]{2}) \times \mathbb{Q}(\sqrt[4]{2}) \times \mathbb{Q}(\sqrt[4]{2})(i)$$

Example 9.3. The field extension generated $(x^p - t)$ of field $\mathbb{F}_p(t)$ is not separable, i.e.,

$$\frac{\mathbb{F}_p(t)[x]}{(x^p - t)}$$

is not separable. Consider the element x, then the minimal polynomial $m(s) = s^p - t$ can be written as

$$s^p - t = s^p - x^p = (s - x)^p$$

Proposition 9.2. Recall that a finite separable extension implies algebraic.

Problem 9.1 (S2017-Q3)**.** Let K/k be a finite separable field extension, and let L/k be any field extension. Show that $K \otimes_k L$ is a product of fields.

Proof. Finite separable implies simple. There exists $\alpha \in K$ such that

$$K = k(\alpha)$$

Let p_α be the minimal polynomial of α, then

$$\begin{aligned} K \otimes_k L &= \frac{k[x]}{(p_\alpha(x))} \otimes_k L \\ &= \frac{L[x]}{(p_\alpha(x))} \end{aligned}$$

We note $p_\alpha(x)$ factors into irreducible linear factors over K. Hence

$$\begin{aligned} K \otimes_k L &= \frac{L[x]}{(p_\alpha^1(x)) \dots (p_\alpha^k(x))} \\ &= \frac{L[x]}{(p_\alpha^1(x))} \times \dots \times \frac{L[x]}{(p_\alpha^k(x))} \end{aligned}$$

□

Problem 9.2 (F2019-Q3)**.** Let F, L be extensions of a field K. Suppose that F/K is finite. Show that there exists an extension E/K such that there are monomorphisms of F into E and of L into E which are identical on K.

Proof. Consider the ring $F \otimes_k L$, and taking a maximal ideal

$$E = \frac{F \otimes_K L}{(m)}$$

Then one can show that the morphisms of F, L into E are injective. □

Problem 9.3 (F2009-Q4)**.** Let E and F be finite field extensions of a field k such that $E \cap F = k$, and that E and F are both contained in a larger field L. Assume that E is Galois over k. Show that $E \otimes_k F \cong EF$.

Proof. □

Problem 9.4 (S2008-Q5). Let k be a field of characteristic zero. Assume that E and F are algebraic extensions of k and both contained in a larger field L. Show that the k-algebra $E \otimes_k F$ has no nonzero nilpotent elements.

Problem 9.5 (S2004-Q5). Show that there is a $\mathbb{C}$-algebra isomorphism between $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$.

Problem 9.6 (F2005-Q5). Let $\mathbb{C}$ and $\mathbb{R}$ be complex and real number fields. Let $\mathbb{C}(x)$ and $\mathbb{C}(y)$ be function fields of one variable. Consider $\mathbb{C}(x) \otimes_{\mathbb{R}} \mathbb{C}(y)$ and $\mathbb{C}(x) \otimes_{\mathbb{C}} \mathbb{C}(y)$.

(1) Determine if they are integral domains.

(2) Determine if they are fields.

Problem 9.7 (F2003-Q4). Verify the isomorphism of algebras over a field K:

$$\mathbb{M}_n(K) \otimes_K \mathbb{M}_m(K) \simeq \mathbb{M}_{mn}(K).$$

[Note: $\mathbb{M}_n(K)$ denotes the algebra of $n \times n$ matrices over K.]

Chapter 10

Irreducibility of Polynomials

Reminder:

Proposition 10.1. Let K be a finite field, then $K^\times$ is cyclic.

Proposition 10.2 (Artin-Schreier). $x^p - x - 1 \in \mathbb{Q}[x]$ is irreducible.

Proof. It suffices to check irreducibility mod p.
$x^p - x - a$ is either irreducible or factors completely into linear factors. □

Proposition 10.3. For $x \in \mathbb{F}_p$, $x^p = x$.

A fact that I keep forgetting.

Proposition 10.4. Fix any prime p, the polynomial

$$f(x) = x^{p-1} + \cdots + x + 1$$

is irreducible over $\mathbb{Q}$. Similarly

$$g(x) = x^{p-1} - x^{p-2} + \cdots - x + 1$$

is irreducible over $\mathbb{Q}$.

Proof. This is an application of Eisenstein. Write

$$f(x) = \frac{x^p - 1}{x - 1}$$

and replace x with $x + 1$ we get

$$\begin{aligned} f(x) &= \frac{(x+1)^p - 1}{x} \\ &= \frac{\sum_{k=1}^n \binom{p}{k} x^k}{x} \\ &= \sum_{k=1}^n \binom{p}{k} x^{k-1} \end{aligned}$$

We apply Eisenstein with prime p to see f is irreducible. □

Proposition 10.5. For any prime p, either $\sqrt{2} \in \mathbb{F}_p$ or $\sqrt{3} \in \mathbb{F}_p$ or $\sqrt{6} \in \mathbb{F}_p$.

Proof. We know there exists a legendre symbol (a character) $\chi : \mathbb{F}_p^\times \to \{\pm 1\}$ such that for $g \in \mathbb{F}_p$,

$$\chi(g) = \begin{cases} 1, & \text{if } g \text{ is a square} \\ -1, & \text{if } g \text{ is not a square} \end{cases}$$

Suppose that $\sqrt{2}$ and $\sqrt{3}$ are not in $\mathbb{F}_p$, then

$$\chi(2) = \chi(3) = -1$$

i.e., $2, 3$ are not squares. However,

$$\chi(2 \cdot 3) = \chi(6) = 1$$

This implies that 6 is a square and $\sqrt{6} \in \mathbb{F}_p$, as desired. □

Corollary 10.1. The following polynomial

$$f(x) = (x^2 - 1)(x^3 - 1)(x^6 - 1)$$

has a linear factor.

Proposition 10.6. The polynomial

$$f(x) = (x - 1)(x - 2)(x - 3)(x - 4) + 1$$

is irreducible.

Problem 10.1 (S2018-Q3)**.** Let R be the ring $\mathbb{Z}[\zeta_p]$, where p is a prime number and ζ_p denotes a primitive pth root of unity in $\mathbb{C}$. Prove that if an integer $n \in \mathbb{Z}$ is divisible by $1 - \zeta_p$ in R, then p divides n.

Proof. We know the polynomial

$$x^p - 1 = (x - 1)(x^{p-1} - \cdots - x + 1)$$

And ζ_p is a roots of $(x^{p-1} - \cdots - x + 1)$, hence we are write ζ_p^{p-1} as

$$\zeta_p^{p-1} = -\zeta_p^{p-2} - \cdots - 1$$

Hence

$$n = (1 - \zeta_p)(a_0 + \cdots + a_{p-2}\zeta_p^{p-2})$$

We see that p divides the constant term, hence $p \mid n$. □

Problem 10.2 (F2008-Q2)**.** Show that the polynomial $x^5 - 5x^4 - 6x - 2$ is irreducible in $\mathbb{Q}[x]$.

Proof. It suffices to see that it is irreducible $\mod 5$. □

Problem 10.3 (F2003-Q3). Obtain a factorization into irreducible factors in $\mathbb{Z}[x]$ of the polynomial $x^{10} - 1$.

Proof. There are four irreducible factors, one linear, two cyclotomic. □

Problem 10.4 (S2004-Q3). Let k be a field with characteristic 0. Let $m \geq 2$ be an integer. Show that $f(x, y) = x^m + y^m + 1$ is irreducible in $k[x, y]$.

Proof. Take an irreducible factor of $y^m + 1$, and $y^m + 1$ is separable, hence there exists one irreducible factor whose square doesn't divide $y^m + 1$. By generalized Eisenstein, we know

$$f(x, y) \in k[y][x]$$

is irreducible, and done by $k[y][x] = k[x, y]$. □

Problem 10.5 (S2017-Q2, S2007-Q3). Write down the minimal polynomial for $\sqrt{2}+\sqrt{3}$ over $\mathbb{Q}$ and prove that it is reducible over $\mathbb{F}_p$ for every prime number p.

Proof. The minimal polynomial of $\sqrt{2} + \sqrt{3}$ is

$$f(x) = x^4 - 10x^2 + 1 = 0$$

By the corollary, we know in any $\mathbb{F}_p$ for any prime p, either $\sqrt{2}$, $\sqrt{3}$, $\sqrt{6}$ is in $\mathbb{F}_p$. We claim that if $\sqrt{2} \in \mathbb{F}_p$, then f is factors over $\mathbb{Q}(\sqrt{2})$. Suppose that f does not factor over $\mathbb{Q}(\sqrt{2})$, i.e., f is irreducible over $\mathbb{Q}(\sqrt{2})$, then the degree of extension

$$[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}(\sqrt{2})][\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 8$$

which is a contradiction. Hence f factors over $\mathbb{Q}(\sqrt{2})$. Similar arguments work if $\sqrt{3}$ or $\sqrt{6}$ are in $\mathbb{F}_p$. □

Problem 10.6 (S2015-Q4). Prove that the polynomial $x^4 + 1$ is not irreducible over any field of positive characteristic.

Proof. The idea is the same as above, and it suffices to note that the field extension generated by $x^4 + 1$ is $\mathbb{Q}(\sqrt{2}, i)$. Using the Legendre symbol, the proof is similar to the above. □

Problem 10.7 (F2010-Q2).

(a) Find the complete factorization of the polynomial $f(x) = x^6 - 17x^4 + 80x^2 - 100$ in $\mathbb{Z}[x]$.

(b) For which prime numbers p does $f(x)$ have a root in $\mathbb{Z}/p\mathbb{Z}$ (i.e, $f(x)$ has a root modulo p)? Explain your answer.

Proof. (a) Letting $y = x^2$, we need to factorize

$$f(y) = y^3 - 17y + 80y - 100$$

Now f is cubic, we need to find the roots of f: 5 is a root,

$$f(y) = (y - 5)(y - 2)(y - 10)$$

i.e.,

$$f(x) = (x^2 - 2)(x^2 - 5)(x^2 - 10)$$

which consists of only irreducible factors over $\mathbb{Z}$.

(b) f has a root in $\mathbb{F}_p$ for all prime p, by the above corollary.

$\square$

Chapter 11

Finite Fields

If p is prime, then $\mathbb{F}_p$ is a field of p elements, isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Proposition 11.1 (Fact)**.** For every prime power p^n, there is exactly one finite field of p^n elements, namely $\mathbb{F}_{p^n}$, up to isomorphisms.

Theorem 11.1 (Galois theory of finite fields)**.** We have

(1) $\mathbb{F}_{p^n}/\mathbb{F}$ is a Galois extension, and

$$\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}) \text{ is cyclic}$$

where the generator is the Forbenius automorphism $\sigma : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ where

$$\sigma : x \mapsto x^p$$

(2) We also have

$$\mathbb{F}_{p^n} = \left\{\alpha \in \overline{\mathbb{F}}_p : \alpha^{p^n} - \alpha = 0\right\}$$

This statement implies that $\mathbb{F}_{p^n}$ is the splitting field of $x^{p^n} - x$.

Proof. We note that $\mathbb{F}_{p^n}$ is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$.

$$\mathbb{F}_{p^n} = \left\{\alpha \in \overline{\mathbb{F}}_p : \alpha^{p^n} - \alpha = 0\right\}$$

If $\alpha \in \mathbb{F}_{p^n}$, then we want to show that $\alpha^{p^n} = \alpha$: if $\alpha = 0$, then done; if $\alpha \in \mathbb{F}_p^\times$, then using the fact that any finite field is cyclic, we know

$$\mathbb{F}_{p^n} \cong \mathbb{Z}/(p^n - 1)\mathbb{Z} \Rightarrow \alpha^{p^n - 1} = 1$$

and we are done. Now we observe that $\left\{\alpha \in \overline{\mathbb{F}}_p : \alpha^{p^n} - \alpha = 0\right\}$ has p^n elements, and is also a field, thus we are done.

This fact can be used to show (1) and the above proposition. $\square$

Proposition 11.2. $\mathbb{F}_{p^n}$ embeds into $\mathbb{F}_{p^m}$ iff $n \mid m$.

Proof. If $n \mid m$, then $m = nk$ for some integer k. We then notice that

$$\alpha^{p^n} = \alpha \Rightarrow \alpha^{p^{kn}} = \alpha^{p^m} = \alpha$$

Thus $\mathbb{F}_{p^n}$ embeds into $\mathbb{F}_{p^m}$. Conversely, consider the Galois field extensions

$$\mathbb{F}_p \subset \mathbb{F}_{p^n} \subset \mathbb{F}_{p^m}$$

Then by degree of field extensions, we know $n \mid m$. □

Problem 11.1 (F2016-Q3). If field $|F| = 2^n$, find all n such that $x^2 - x + 1$ is irreducible over F.

Proof. We know that $x^2 - x + 1$ is irreducible over $\mathbb{F}_2$, namely, it has no roots in $\mathbb{F}_2$. Since there is only one field of order 4, we must have

$$\mathbb{F}_4 \cong \frac{\mathbb{F}_2}{(x^2 - x + 1)}$$

Clearly $x^2 - x + 1$ is not irreducible over $\mathbb{F}_4$. For any $\mathbb{F}_{2^n}$, we know $(x^2 - x + 1)$ is irreducible if and only if $\mathbb{F}_4$ does not embed into $\mathbb{F}2^n$, i.e., $2 \nmid n$. This shows that when n is odd, the polynomial $x^2 - x + 1$ is irreducible over $\mathbb{F}_{2^n}$. □

Problem 11.2 (F2015-Q5). Let L be a finite field. Let a and b be elements of $L^\times$ (the multiplicative group of L) and $c \in L$. Show that there exist x and y in L such that $ax^2 + by^2 = c$.

Problem 11.3 (F2013-Q6). Let p be a prime and let F be a field of characteristic p.

(a) Prove that the map $\varphi : F \to F, \varphi(a) = a^p$ is a field homomorphism.

(b) F is said to be *perfect* if the above homomorphism φ is an automorphism. Prove that every finite field is perfect.

(c) If x is an indeterminate and F is any field of characteristic p, prove that the field $F(x)$ is not perfect.

Proof. (a) You just do it, the field has character p.

(b) Observe that it is surjective.

(c) x is not in the image of φ.

□

Problem 11.4 (F2017-Q5). Let K/k be an extension of finite fields with $\#k = q$, let $\Phi\colon x \mapsto x^q$ denote the qth power Frobenius map on K, and let $G := \mathrm{Gal}(K/k)$.

(a) Compute the minimal polynomial of Φ as a k-linear endomorphism of K.

(b) Use (a) to prove the *normal basis theorem* in the case of the extension K/k: there exists $x \in K$ such that the set $\{\sigma x \mid \sigma \in G\}$ is a k-basis for K.

Proof. (a) Same as above.

(b)

□

Problem 11.5 (F2010-Q5). Let $\mathbb{F}_q$ be a finite field with $q = p^n$ elements. Here p is a prime number. Let $\varphi : \mathbb{F}_q \to \mathbb{F}_q$ be given by $\varphi(x) = x^p$.

(a) Show that φ is a linear transformation on $\mathbb{F}_q$ (as vector space over $\mathbb{F}_p$), then determine its minimal polynomial.

(b) Supposed that φ is diagonalizable over $\mathbb{F}_p$. Show that n divides $p - 1$.

Problem 11.6 (S2011-Q2). Let p be a prime, F a finite field with p elements and K a finite extension of F. Denote by $F^\times$ and $K^\times$ the multiplicative groups of nonzero elements of fields F and K, respectively. Prove that the norm homomorphism $N : K^\times \to F^\times$ is surjective.

Proof. □

Problem 11.7 (F2008-Q3). Let k be a finite field and K be a finite extension of k. Let $\mathfrak{Tr} = \mathrm{Tr}_k^K$ be the trace function from K to k. Determine the image of $\mathfrak{Tr}$ and prove your answer.

Proof. Step 1: show that there exists an element α such that □

Problem 11.8 (S2014-Q3). Let L/K be a Galois extension of degree p with $\mathrm{char} K = p$. Show that $L = K(\theta)$, where θ is a root of $x^p - x - a, a \in K$, and, conversely, any such extension is Galois of degree 1 or p.

Warning 11.2. The $f = gh$, and $g = \prod_{i \in S}(x - \alpha_i)$ trick.

Proof. Artin-Schreier. □

Problem 11.9 (S2015-Q1). Let K be a field of characteristic $p > 0$. Prove that a polynomial $f(x) = x^p - x - a \in K[x]$ either irreducible, or is a product of linear factors. Find this factorization if f has a root $x_0 \in K$.

Proof. Artin-Schreier! If it has a root x_0, then all the roots $x_0 + k$ for any $k \in \mathbb{F}_p$ is a root. □

Problem 11.10 (S2002-Q5). Let $\zeta = e^{\frac{2\pi i}{5}}$ and $K = \mathbb{Q}(\zeta)$ the field generated by ζ over the field of rational numbers. Prove that K contains $\sqrt{5}$.

Problem 11.11 (S2008-Q2). Let ξ be a primitive 9-th root of unity. Find the minimal polynomial of $\xi + \xi^{-1}$ over $\mathbb{Q}$.

Proof. Draw the picture, a degree 3 polynomial works. □

Problem 11.12 (F2007-Q1). Let G be a cyclic group of order 12. Construct a Galois extension K over $\mathbb{Q}$ so that the Galois group is isomorphic to G.

Proof. The Galois extension $\mathbb{Q}(\zeta_{13})$. □

Problem 11.13 (F2011-Q3). Let G be a cyclic group of order 100. Let $K = \mathbb{Q}$, the field of rational numbers, or $K = F_p$, the finite field with p elements, p being a prime number. For each such K, construct a Galois extension L/K whose Galois group $\mathrm{Gal}(L/K)$ is isomorphic to G. Explain your construction in detail.

Proof. If $K = \mathbb{Q}$, then take $\mathbb{Q}(\zeta_{101})$. If $K = \mathbb{F}_p$, then take $\mathbb{F}_{p^{100}}$, we know it is the splitting field of

$$\mathbb{F}_{p^{100}} = \{x \in \overline{\mathbb{F}}_p : x^{p^{100}} - x = 0\}$$

the Galois group has the Frobenius generator $x \mapsto x^p$. □

Proposition 11.3. The polynomial $x^p - px - 1$ is irreducible over $\mathbb{Q}$.

Proof. Eisenstein. □

Proposition 11.4. Let ω_n be the nth root of unity, then the minimal polynomial is Φ_n and it has degree $|(\mathbb{Z}/n\mathbb{Z})^\times|$.

Problem 11.14 (S2006-Q4). Let k be a field, and p be a prime, let $a \in k$, show that $x^p - a$ either has a root in k or is irreducible over k.

Warning 11.3. The $f = gh$, and $g = \prod_{i \in S}(x - \alpha_i)$ trick.

Proof. We will show that if f does not have a root, then it is irreducible. Suppose that it is not irreducible, then

$$f(x) = g(x)h(x)$$

where $\deg(g) < p$, and we know

$$g(x) = \prod_{i \in S}(x - \alpha_i)$$

in the algebraic closure of k, and

$$\sum_{i \in S} \alpha_i \in k, \prod_{i \in S} \alpha_i \in k$$

We will now show that $a^{\frac{1}{p}} \in k$. We note that

$$c_0^p = \prod_{i \in S} \alpha_i^p = a^{|S|} \in k$$

We know that

$$c_0 = a^{\frac{|S|}{p}} \in k$$

Since $a \in k$, we can know find k, m such that $k|S| - pm = 1$, and

$$a^{\frac{k|S|}{p}} \cdot a^{-m} = a^{\frac{k|S| - pm}{p}} \in k$$

i.e., $a^{\frac{1}{p}} \in k$. Thus a contradiction. □

Problem 11.15 (S2005-Q2). Let $\mathbb{F}_p$ be the field with p elements, where p is a prime number. Let $f_{n,p}(x) = x^{p^n} - x + 1$, and suppose that $f_{n,p}(x)$ is irreducible in $\mathbb{F}_p[x]$. Let α be a root of $f_{n,p}(x)$.

(a) Show that $\mathbb{F}_{p^n} \subset \mathbb{F}_p(\alpha)$ and $[\mathbb{F}_p(\alpha) : \mathbb{F}_{p^n}] = p$.

(b) Determine all pairs (n, p) for which $f_{n,p}(x)$ is irreducible.

Proof. 1. Let $x \in \mathbb{F}_{p^n}$, one can show that $(x + \alpha)$ is also a root of f, i.e., $x + \alpha \in \mathbb{F}_p(\alpha)$, because $\mathbb{F}_p(\alpha)$ is Galois over $\mathbb{F}_{p^n}$, thus containing all the roots.

For $[\mathbb{F}_p(\alpha) : \mathbb{F}_{p^n}]$, we want to show that Galois group has order p, i.e., the Frobenius

$$x \mapsto x^{p^n}$$

has order p. This is true because

$$x \mapsto x^{p^n} = x - 1$$

Hence it clearly has order p.

(b) Uses part (a), not irreducible unless $n = 1$.

□

Proposition 11.5. Any finite subgroup of the multiplicative group of a field is cyclic. For example, any finite field $\mathbb{F}_{p^n}^\times$ is generated by some g, such that for all $x \in \mathbb{F}_{p^n}^\times$,

$$x = g^k$$

for some k.

Problem 11.16 (F2005-Q1). Let k be a finite field, with p^n elements, let d be a positive integer, compute

$$\sum_{x \in k} x^d$$

Proof. We know $\mathbb{F}_{p^n}^\times$ is generated by some g, then

$$\sum_{x \in k} x^d = \sum_{i=0}^{p^n - 2} g^{id} = \frac{g^{d(p^n-1)} - 1}{g^d - 1}$$

□

Chapter 12

Galois Theory

Quick reminder whether a polynomial has a rational root:

Proposition 12.1. Let $f(t) = a_n t^n + \cdots + a_1 t + a_0$, and if a rational (expressed in lowest terms) $\frac{p}{q}$ is a root of f, then $p \mid a_0, q \mid a_0$.

Definition 12.1 (Galois extension). A field extension $k \subset L$ is Galois if for all $x \in L$, the minimal polynomial $f(x) \in k[x]$ splits into a linear factor without repeated roots.

Definition 12.2 (normal extension). An extension $k \subset K$ is normal if f has a root in K if and only if f splits completely into linear factors over K. An extension that is normal and separable is Galois.

Theorem 12.1. Suppose $k \subset L$ is Galois,

$$\{k \subset M \subset L\} \xleftrightarrow{\text{one-to-one}} \{\text{Subgroups of } \mathrm{Gal}(L/k)\}$$

Moreover, the order of the Galois group is the degree of the field extension.

$$|\mathrm{Gal}(L/k)| = [L : k]$$

Proposition 12.2. Let G be a Galois group of a polynomial f of degree 4, and $|G| = 8$, then

$$G \cong D_8$$

Proof. We know that G permutes the four roots of f, i.e., G embeds into S_4. Since $|G| = 8$, we know G is a Sylow-2 subgroup of S_4, and all Sylow-2 subgroups are conjugates (isomorphic to one another), i.e.,

$$G \cong D_8$$

as desired. □

Proposition 12.3. Let $k \subset K$ be a Galois extension, then the intermediate field extensions $k \subset E \subset K$ is determined by the subgroups of $\mathrm{Gal}(K/k)$. Namely, let E be an intermediate extension, there exists a subgroup H of $\mathrm{Gal}(K/k)$ that fixes E. This extension is normal if and only if H is normal. And E/k is Galois if and only if H is normal.

Problem 12.1 (S2009-Q3). Consider the field $K = \mathbb{Q}(\sqrt{a})$ where $a \in \mathbb{Z}, a < 0$. Show that K cannot be embedded in a cyclic extension whose degree over $\mathbb{Q}$ is divisible by 4.

Proof. Suppose K embedes into a degree $4n$ extension L, and

$$\mathrm{Gal}(L/\mathbb{Q}) = \frac{\mathbb{Z}}{4n\mathbb{Z}}$$

Since K is a degree 2 extension of $\mathbb{Q}$, thus L/K is a degree $4n/2$ Galois extension, with Galois group

$$\mathrm{Gal}(L/K) = \frac{2\mathbb{Z}}{4n\mathbb{Z}}$$

We notice that $\sqrt{a}$ is complex, hence the complex conjugation τ is in $\mathrm{Gal}(L/\mathbb{Q})$, i.e., it is an order 2 element in $\frac{\mathbb{Z}}{4n\mathbb{Z}}$, it is therefore $[2n]$ i.e.,

$$\tau \in \frac{2\mathbb{Z}}{4n\mathbb{Z}} = \mathrm{Gal}(L/\mathbb{Q})$$

This implies τ fixed K, however $\tau(\sqrt{a}) \neq \sqrt{a}$, hence a contradiction. □

Problem 12.2 (F2000-Q4). Let G be a finite group. Show that there exists a Galois field extension K/k whose Galois group is isomorphic to G.

Proof. Embed any group into S_n, and S_n embeds into S_p for p large enough. □

12.1 Problems

Problem 12.3 (S2001-Q2). Let K be the splitting field of $f(X) = X^3 - 2$ over $\mathbb{Q}$.

(a) Determine an explicit set of generators for K over $\mathbb{Q}$.

(b) Show that the Galois group $G(K/\mathbb{Q})$ of K over $\mathbb{Q}$ is isomorphic to the symmetric group S_3.

(c) Provide the complete list of intermediate fields k, $\mathbb{Q} \subseteq k \subseteq K$, satisfying $[k : \mathbb{Q}] = 3$.

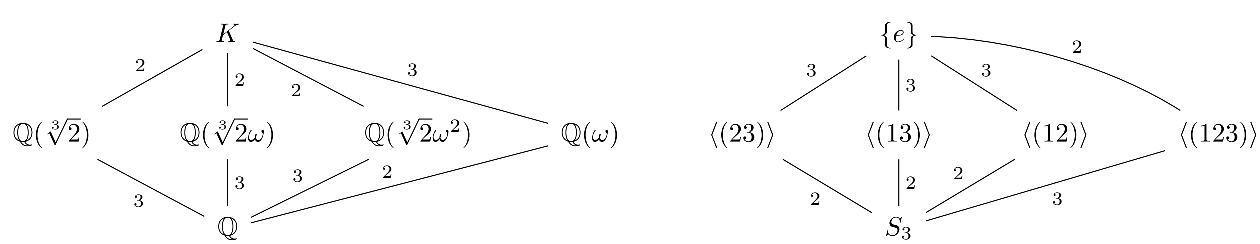
(d) Which of the fields determined in (c) are normal extensions of $\mathbb{Q}$?

Proof. (a) The set of generators is

$$\left\{\sqrt[3]{2}, e^{\frac{2\pi i}{3}}\right\}$$

(b) The Galois group is a subgroup of S_3, hence it suffices to show G has order 6, i.e., the extension is of degree 6.

(c) The following is the **complete** subgroup lattice of S_3 and subfield lattice:



Thus all the $\mathbb{Q} \subset k$ such that $[k : \mathbb{Q}] = 3$ are

$$\{\mathbb{Q}(\sqrt[3]{2}), \mathbb{Q}(\sqrt[3]{2}\omega_3), \mathbb{Q}(\sqrt[3]{2}\omega_3^2)\}$$

(d) None of the above are normal because the subgroups

$$\{\langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle\}$$

are all Sylow 2-subgroups of S_3, hence all conjugates to one another, i.e., not normal.

$\square$

Problem 12.4 (F2001-Q4). Let $K := \mathbb{Q}(\sqrt{3}+\sqrt{5})$.

(a) Show that K is the splitting field of $X^4 - 6X^2 + 4$.

(b) Find the structure of the Galois group of $K/\mathbb{Q}$.

(c) List all the fields k, satisfying $\mathbb{Q} \subseteq k \subseteq K$.

Proof. (a) I belive there is typo in (a) where the polynomial should be $f(X) = X^4 - 16X^2 + 4$. This is the minimal polynomial of $\sqrt{3}+\sqrt{5}$. We see that $\mathbb{Q}(\sqrt{3}+\sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$, hence it contains all the roots of f.

(b) We let $\alpha = \sqrt{3}+\sqrt{5}$, and $\beta = \sqrt{3}-\sqrt{5}$, then we see Galois group permutes

$$\{\alpha, -\alpha, \beta, -\beta\}$$

and we have $\alpha\beta \in \mathbb{Q}$. Thus just like the above, we have

$$\mathrm{Gal}(K/\mathbb{Q}) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

(c) We know the intermediate fields are determined by the subgroup of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd} & \{e\} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\ \langle(1,0)\rangle \arrow[dr, dash, "2"'] & \langle(1,1)\rangle \arrow[d, dash, "2"] & \langle(0,1)\rangle \arrow[dl, dash, "2"] \\ & \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

and let $(1,0)$ be the element such that

$$(1,0)\cdot(\sqrt{3}+\sqrt{5}) = \sqrt{3}-\sqrt{5}$$

then we have the corresponding lattice of subfields

$$\begin{tikzcd} & \{e\} \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\ \langle(1,0)\rangle \arrow[dr, dash, "2"'] & \langle(1,1)\rangle \arrow[d, dash, "2"] & \langle(0,1)\rangle \arrow[dl, dash, "2"] \\ & \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z} & \end{tikzcd}$$

$$\begin{tikzcd} & \mathbb{Q}(\sqrt{3},\sqrt{5}) \arrow[dl, dash, "2"] \arrow[d, dash, "2"'] \arrow[dr, dash, "2"'] & \\ \mathbb{Q}(\sqrt{3}) \arrow[dr, dash, "2"] & \mathbb{Q}(\sqrt{15}) \arrow[d, dash, "2"'] & \mathbb{Q}(\sqrt{5}) \arrow[dl, dash, "2"'] \\ & \mathbb{Q} & \end{tikzcd}$$

So all intermediate fields are
$$\left\{\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{15}), \mathbb{Q}(\sqrt{5})\right\}$$
□

Problem 12.5 (F2013-Q5). Compute the Galois group of $f(x) = x^4 + 1$ over $\mathbb{Q}$.

Proof. The splitting field for f is $\mathbb{Q}(\xi_8)$ where $\xi_8 = e^{\frac{2\pi i}{8}}$, and the Galois group
$$\mathrm{Gal}(\mathbb{Q}(\xi_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$$
thus
$$(\mathbb{Z}/8\mathbb{Z})^\times \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$
Alternatively, we can find $K = \mathbb{Q}(i, \sqrt{2})$, then $\mathrm{Gal}(F/\mathbb{Q}) \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$. □

Problem 12.6 (F2016-Q4).

(1) Determine the Galois group of $x^4 - 4x^2 - 2$ over $\mathbb{Q}$.

(2) Let G be a group of order 8 such that G is the Galois group of a polynomial of degree 4 over $\mathbb{Q}$. Show that G is isomorphic to the Galois group in part (1).

Proof. (a) There are four roots of this polynomial
$$\{\alpha, -\alpha, \beta, -\beta\}$$
where
$$\alpha = \sqrt{2+\sqrt{6}}, \beta = \sqrt{2-\sqrt{6}}$$
Thus the Galois group embeds into S_4. Notice that
$$\alpha\beta = \sqrt{2}i$$
Thus we see the Galois extension has degree 8:
$$\begin{array}{c} \mathbb{Q}(\sqrt{2+\sqrt{6}}, \sqrt{2}i) \\ \Big| \, {\scriptstyle 2} \\ \mathbb{Q}(\sqrt{2+\sqrt{6}}) \\ \Big| \, {\scriptstyle 4} \\ \mathbb{Q} \end{array}$$
Notice that the Galois grop G is an order 8 subgruop of S_4, which implies that G is a Sylow 2 subgroup, and all Sylow 2 subgruops are isomorphic:
$$G \cong D_8$$

(b) Notice that we need to check that f is irreducible, then we can embed Gal into S_4. Suppose that it is not irreducible, then either $f = g_1 g_2, g_i$ is quadratic, or $f = g(x)(x-a)$, for some $a \in \mathbb{Q}$. In former case, we see Gal embeds in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so cannot be of order 8; similarly for cubic+linear, S_3 does not have subgroup of order 8. Hence a degree 4 polynomial with Galois group of order 8 must be irreducible.

□

Problem 12.7 (S2008-Q3). Let K be the splitting field of the polynomial $X^4 - 6X^2 - 1$ over $\mathbb{Q}$.

(a) Compute $\mathrm{Gal}(K/\mathbb{Q})$.

(b) Determine all intermediate fields that are Galois over $\mathbb{Q}$.

Proof. (a) This computation is exactly same as above, as we have the four roots

$$\left\{\pm\sqrt{3+\sqrt{10}}, \pm\sqrt{3-\sqrt{10}}\right\}$$

and we see that $\alpha\beta = i$, thus the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ has order 8, and embeds into S_4, thus

$$\mathrm{Gal}(K/\mathbb{Q}) \cong D_8$$

(b) There are 10 subgroups of D_8, and 6 of them are normal. Let

$$r : \alpha \mapsto \beta, s : i \mapsto i$$

Then we see, for example, r^2 fixes i and $\sqrt{10}$, thus we must have the lattice

$$\begin{tikzcd}
& K \arrow[d, dash] & \\
& \mathbb{Q}(i,\sqrt{10}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(i) \arrow[dr, dash] & \mathbb{Q}(\sqrt{10}i) \arrow[d, dash] & \mathbb{Q}(\sqrt{10}) \arrow[dl, dash] \\
& \mathbb{Q} &
\end{tikzcd}$$

$\square$

Problem 12.8 (S2010-Q3). Compute Galois groups of the following polynomials.

(a) $x^3 + t^2x - t^3$ over k, where $k = \mathbb{C}(t)$ is the field of rational functions in one variable over complex numbers $\mathbb{C}$.

(b) $x^4 - 14x^2 + 9$ over $\mathbb{Q}$.

Proof. (a) The polynomial completely factors over $\mathbb{C}(t)$, so the Galois group is $\{e\}$. Try taking $x = \lambda t$, then solving for λ, which splits into linear factors because $\mathbb{C}$ is algebraically closed.

(b) The roots are

$$\left\{\pm\sqrt{7 \pm 2\sqrt{10}}\right\}$$

and $\alpha\beta \in \mathbb{Q}$ again, hence the Galois group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

$\square$

Problem 12.9 (S2013-Q6). Let K be the splitting field of $x^6 - 5$ over $\mathbb{Q}$.

(a) Prove that $x^6 - 5$ is irreducible over $\mathbb{Q}$.

(b) Compute the Galois group of K over $\mathbb{Q}$.

(c) Describe an intermediate field F such that F is not $\mathbb{Q}$ or K and $F/\mathbb{Q}$ is Galois.

Proof. (a) By Eisenstein.

(b) We know $K = \mathbb{Q}(\sqrt[6]{5}, \zeta_6)$, where ζ_6 is the 6th root of unity. The roots are

$$\left\{\sqrt[6]{5}, \sqrt[6]{5}\zeta_6, \dots, \sqrt[6]{5}\zeta_6^5\right\}$$

Note that the minimal polynomial for ζ_6 is $x^2 - x + 1$, so the size of $\mathrm{Gal}(K/\mathbb{Q})$ is 12. We see that any $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ is determined by where it sends $\sqrt[6]{5}$ and ζ_6, so we only need to compute the possibilities of them. The Galois action is transitive implies that there $\sqrt[6]{5}$ can be sent to any $\sqrt[6]{5}\zeta_6^k$, where $k = 0, 1, 2, 3, 4, 5$, and since ζ_6 has minimal polynomial

$$x^2 - x + 1$$

Then there are two possibilities for $\zeta_6 \mapsto \zeta_6, \bar{\zeta}_6$, where $\bar{\zeta}_6 = \zeta_6^5$. Now we see that

$$\mathrm{Gal}(K/Q) = D_{12}$$

as it is generated by

$$\sigma : \sqrt[6]{5} \mapsto \zeta_6 \sqrt[6]{5}, \zeta_6 \mapsto \zeta_6, \quad \tau : \sqrt[6]{5} \mapsto \sqrt[6]{5}, \zeta_6 \mapsto \zeta_6^5$$

satisfying $\tau\sigma = \tau\sigma^{-1}$. (One can draw a hexagon)

(c) $F/\mathbb{Q}$ corresponds to a normal subgroup of D_{12}. Any subgroup of 6 is normal, i.e., the subgroup

$$\{e, \sigma, \dots, \sigma^5\}$$

This subgroup fixes the field $\mathbb{Q}(\zeta_6)$. Hence it corresponds to

$$F = \mathbb{Q}(\zeta_6)$$

□

Problem 12.10 (S2016-Q3). Determine the Galois group of $x^6 - 10x^3 + 1$ over $\mathbb{Q}$.

Proof. This is the same process as above, the roots are

$$\left\{\zeta_3^i \sqrt[3]{5 \pm 2\sqrt{6}} : i = 0, 1, 2\right\}$$

The order of the Galois group G is 12, but now we need another trick.

Lemma 12.1. Transitive subgroup of S_6 of order 12 can only be D_{12} or A_4. However, A_4 has no index 2 subgroups, i.e., this Galois extension cannot have a subfield extension of degree 2 over $\mathbb{Q}$, this gives that G must be D_{12}:

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt[3]{2+\sqrt{6}}, \zeta_3) \arrow[dl, dash, "6"'] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\zeta_3) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{2+\sqrt{6}}) \arrow[dl, dash, "6"] \\
& \mathbb{Q} &
\end{tikzcd}$$

□

Problem 12.11 (F2010-Q3)**.** Let $K = \mathbb{Q}(\sqrt[8]{2}, \sqrt{-1})$ and $F = \mathbb{Q}(\sqrt{-2})$. Show that K is Galois over F and determine the Galois group $\mathrm{Gal}(K/F)$.

Proof. Since $\sqrt{2} = \zeta_8^4$, we see F is a subfield such that

$$\mathbb{Q} \subset F \subset K$$

The Galois group can be computed to be Q_8.

□

Problem 12.12 (F2015-Q2)**.** The dihedral group D_{2n} is the group on two generators r and s, with respective orders $o(r) = n$ and $o(s) = 2$, subject to the relation $rsr = s$.

(a) Calculate the order of D_{2n}.

(b) Let K be the splitting field of the polynomial $x^8 - 2$. Determine whether the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is dihedral (i.e., isomorphic to D_{2n} for some n).

Proof. (a) Because of the relation $srs = r^{-1}$, we can express all the terms in D_{2n} as

$$r^k s^m$$

where $0 \leq k \leq n-1, m = 0, 1$. Hence there are $2n$ elements.

(b) It is not D_{16}, you can compute the number of elements of each order.

□

Proposition 12.4 (S2019-Q1)**.** Any transitive subgroup of A_5 is isomorphic to one of the following groups:

(a) the cyclic group $\mathbb{Z}/5\mathbb{Z}$,

(b) the dihedral group D_5,

(c) A_5.

Problem 12.13 (F2017-Q4)**.** Compute the Galois group of $x^5 - 10x + 5$ over $\mathbb{Q}$.

Proof. S_5. □

Problem 12.14 (F2004-Q3)**.** Let $f(x) = x^5 - 9x + 3$. Determine the Galois group of f over $\mathbb{Q}$.

Proof. S_5. □

Problem 12.15 (F2006-Q2). Let f be a polynomial in $\mathbb{Q}[x]$. Let E be a splitting field of f over $\mathbb{Q}$. For the following cases, determine whether E is solvable by radicals. (i.e., whether the Galois group is solvable or not).

(1) $f(x) = x^4 - 4x + 2$.

(2) $f(x) = x^5 - 4x + 2$.

Proof. (1) It is a subgroup of S_4, so solvable.

(2) The Galois group is S_5, so not solvable.

□

Proposition 12.5. Any group of order < 60 is solvable.

Problem 12.16 (S2011-Q3). Determine the Galois group of the splitting field of each of the following polynomials over $\mathbb{Q}$:

(a) $f(x) = x^4 - 9x^3 + 9x + 4$,

(b) $g(x) = x^5 - 6x^2 + 2$.

Proof. For (a): do the modulo thing to find different cycle types. (b) is S_5 as usual. □

Problem 12.17 (F2014-Q1).

(a) Let S_n be the symmetric group (permutation group) on n objects. Prove that if $\sigma \in S_n$ is an n-cycle and $\tau \in S_n$ is a transposition (i.e., a 2-cycle), then σ and τ generate S_n.

(b) Let $f_a(x)$ be the polynomial $x^5 - 5x^3 + a$. Determine an integer a with $-4 \leq a \leq 4$ for which f_a is irreducible over $\mathbb{Q}$, and the Galois group of [the splitting field of] f_a over $\mathbb{Q}$ is S_5. Then explain why the equation $f_a(x) = 0$ is not solvable in radicals.

(a) It suffices to assume that the n cycle is $(1 \ldots n)$ (up to rearranging the terms), and the transposition is (12). One can show that conjugation gives all the transpositions, hence generate S_n.

(b) Take $a = 1$, then $f_a(x)$ is irreducible: it doesn't have a root by the Rational Root Theorem and cannot be factored into lower degree polynomial by term matching. Moreover, we see that $f_a'(x)$ has 3 roots, by Rolle's theorem, there are at most 4 real roots, this implies that there exists a complex root r_1, and since this has odd degree, it must also exist a real root r_2. This shows that there exists an element in the Galois group that has order 5 and a transposition (sending conjugate complex roots to each other). Thus by (a), since the Galois group is a subgroup of S_5, we must have it equal to S_5.

Problem 12.18 (F2009-Q3). Determine the Galois group of $x^4 - 4x^2 + 7x - 3$ over $\mathbb{Q}$.

Proof. $f \mod 2$ is irreducible of degree 4, hence there is a 4-cycle. And $f \mod 3$ gives a 3-cycle. This implies the galois group has order at least 12, inside of S_4, this means A_4 or S_4, but it cannot be A_4 because it contains no 4-cycle. □

Problem 12.19 (S2012-Q3). In this problem, G denotes the group $S_5 \times C_2$, where S_5 is the symmetric group on five letters and C_2 is the cyclic group of order 2.

(a) Determine all normal subgroups of G.

(b) Give an example of a polynomial with rational coefficients whose Galois group is G, deducing that from basic principles.

Proof. Consider $(x^5 - 4x - 2)(x^2 - 3)$. □

Problem 12.20 (F2015-Q4). Let $H = S_3 \times S_5$.

(a) Determine all normal subgroups of H. Make sure you have them all! What would be different if H were replaced by $S_2 \times S_5$?

(b) Describe, in full detail, the construction of a polynomial with rational coefficients, whose Galois group is isomorphic to H.

Proof. Consider $(x^5 - 4x - 2)(x^3 - 2)$. □