Questions

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Problem 0.1. To see whether a polynomial is irreducible over $\mathbb{Q}$, is it sufficient to test whether $f \mod p$ is irreducible over any prime p?

For example, $x^5 - 5x^3 + 1$.

Problem 0.2. In the above example, how do we know that the Galois group contains an element of order 5? (It is clear why it contains a transposition because there exists complex roots).

Proof. This is because the Galois group G acts transitively on the set of roots, by the Orbit stabilizer theorem, we know

$$|G| = |\text{Orbit}(\alpha)| \cdot |\text{Stab}(\alpha)| = 5 \cdot |\text{Stab}(\alpha)|$$

i.e., 5 divides $|G|$. By Cauchy's theorem, there exists an element of order 5 in G, i.e., a 5-cycle. $\square$

Problem 0.3. The Galois action on the set of roots implies for any root r of the polynomial (where G is the splitting field of), we must have

$$\text{Orbit}(r) = \{ \text{ set of all roots}\}$$

Problem 0.4. Is it true that if I, J are ideals of a ring R, then

$$\frac{R}{I} \otimes_R \frac{R}{J} = \frac{R}{(I+J)}$$

in the case where $R = \mathbb{Q}[x]$, and I, J are irreducible polynomials, we have

$$\frac{R}{(f)} \otimes_R \frac{R}{(g)} = \frac{R}{(f_1) + (f_2)} \frac{R}{\gcd(f, g)}$$

Problem 0.5. Fall 2014 Q2, $\text{Hom}_R(M, N)$.

Problem 0.6. $\mathbb{Z}/55\mathbb{Z}$.