Algebra Qualifying Exam Solutions

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Contents

Chapter 1

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Problem 1.1. Determine $\mathrm{Aut}(S_3)$.

Proof. $\sigma \in \mathrm{Aut}(S_3)$ is determined by where (12) and (123) are sent to. There are 6 options in total and all of them are homomorphisms (conjugation). It is easy to check that this group is not commutative, i.e.,

$$\mathrm{Aut}(S_3) \cong S_3$$

□

Problem 1.2. A group G is a semidirect product of subgroups $N, H \subset G$ if N is normal and every element of G has a unique presentation nh, $n \in N$, $h \in H$. Find all semidirect products (up to isomorphism) of $N = \mathbb{Z}/11\mathbb{Z}$, $H = \mathbb{Z}/5\mathbb{Z}$.

Proof. Let $G = N \rtimes_\theta H$, where

$$\theta : \mathbb{Z}/5\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/11\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$$

such that

$$5\theta(1) \equiv 0 \mod 10$$

Thus $\theta(1)$ could be $0, 2, 4, 6, 8$. When $\theta(1) = 0$, this gives the abelian group

$$G \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \times \frac{\mathbb{Z}}{11\mathbb{Z}}$$

We claim that all nontrivial θ give rise to the same semidirect product, namely, the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta'} & \mathbb{Z}/10\mathbb{Z} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta} & \mathbb{Z}/10\mathbb{Z} \end{array}$$

for $\theta : 1 \mapsto 2$ and any $\theta' : 1 \mapsto 4, 6, 8$, by taking m to be the multiplication map by $2, 3, 4$ respectively. Hence we see

$$\theta(h)(g) = g^{2^{2h}}$$

by observing

$$\mathbb{Z}/5\mathbb{Z} \xrightarrow{2} \mathbb{Z}/10\mathbb{Z} \xrightarrow{2^2} (\mathbb{Z}/11\mathbb{Z})^\times \xrightarrow[2^2\cdot(-)]{} \mathrm{Aut}(\mathbb{Z}/11\mathbb{Z})$$

In other words,

$$G = \left\langle g, h : g^1 1 = h^5 = 1, hgh^{-1} = g^{2^{2h}} \right\rangle$$

□

Problem 1.3. Let F be a finite field of order 2^n. Here $n > 0$. Determine all values of n such that the polynomial $x^2 - x + 1$ is irreducible in $F[x]$.

Proof. We know that $x^2 - x + 1$ is irreducible over $\mathbb{F}_2$, namely, it has no roots in $\mathbb{F}_2$. Since there is only one field of order 4, we must have

$$\mathbb{F}_4 \cong \frac{\mathbb{F}_2}{(x^2 - x + 1)}$$

Clearly $x^2 - x + 1$ is not irreducible over $\mathbb{F}_4$. For any $\mathbb{F}_{2^n}$, we know $(x^2 - x + 1)$ is irreducible if and only if $\mathbb{F}_4$ does not embed into $\mathbb{F}2^n$, i.e., $2 \nmid n$. This shows that when n is odd, the polynomial $x^2 - x + 1$ is irreducible over $\mathbb{F}_{2^n}$. □

Problem 1.4. (1) Determine the Galois group of $x^4 - 4x^2 - 2$ over $\mathbb{Q}$.

(2) Let G be a group of order 8 such that G is the Galois group of a polynomial of degree 4 over $\mathbb{Q}$. Show that G is isomorphic to the Galois group in part (1).

Proof. (1) The roots of this polynomial is $\pm\sqrt{2 \pm \sqrt{6}}$, and notice that

$$\sqrt{2}i = \sqrt{2 + \sqrt{6}}\sqrt{2 - \sqrt{6}}$$

This gives the splitting field (Galois extension) of this polynomila as

$$\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}, \sqrt{2}i\right)$$

We see that

$$\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}\right) \cap \mathbb{Q}(\sqrt{2}i) = \varnothing$$

because the first is contained in $\mathbb{R}$ and the second is not. We must have

$$\left[\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}, \sqrt{2}i\right)/\mathbb{Q}\right] = 8$$

By part b, we see Gal $\cong D_8$.

(2) Any Galois group of a polynomial with 4 roots in the splitting field embeds into S_4, and we notice that $|G| = 2^3, |S_4| = 2^3 \cdot 3$, i.e., G is a Sylow 2-subgroup of S_4, and all Sylow 2-subgroups are conjugate/isomorphic of one another, hence

$$\text{Gal} \cong D_8$$

□

Problem 1.5. Let A be a linear transformation of a finite dimensional vector space over a field of characteristic $\neq 2$.

(1) Define the wedge product linear transformation $\wedge^2 A = A \wedge A$.

(2) Prove that

$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2)).$$

Proof. (Recall we have analogous results for $A \otimes A$).

(1) The wedge product $A \wedge A$ is defined on the wedge product of vector spaces $V \wedge V$, so we first define the vector space: let $\{v_1, \dots, v_n\}$ be the basis of V, then $\{v_i \wedge v_j\}$ where $i < j$ forms a basis of $V \wedge V$, satisfying:

1. $v_i \wedge v_j = -v_j \wedge v_i$
2. $(a_i v_i + a_j v_j) \wedge (b_k v_k + b_l v_l) = (a_i b_k) v_i \wedge v_k + (a_i b_l) v_i \wedge v_l + (a_j b_k) v_j \wedge v_k + (a_j b_l) v_j \wedge v_l$

And $A \wedge A$ where $A : V \to V$ is defined as

$$A \wedge A(v_i \wedge v_j) = Av_i \wedge Av_j$$

(2) Consider the matrix representation of $A = (A_{ij})$, on the basis $\{v_i \wedge v_j : i < j\}$,

$$\begin{aligned} A \wedge A(v_i \wedge v_j) &= \sum_{k,l=1}^{n} A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki} A_{lj} (v_k \wedge v_l) + \sum_{l<k} A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki} A_{lj} (v_k \wedge v_l) - \sum_{l<k} A_{ki} A_{lj} (v_l \wedge v_k) \end{aligned}$$

Thus the diagonal term with respect to $v_i \wedge v_j$ is

$$A_{ii} A_{jj} - A_{ji} A_{ij}$$

Thus

$$\text{Tr}(A \wedge A) = \sum_{i<j} A_{ii} A_{jj} - A_{ji} A_{ij}$$

Now

$$\text{Tr}(A)^2 = \sum_{i=1}^{n} A_{ii}^2 + 2 \sum_{i<j} A_{ii} A_{jj}$$

and

$$\begin{aligned} \text{Tr}(A^2) &= \sum_{k,l=1}^{n} A_{lk} A_{kl} \\ &= \sum_{i=1}^{n} A_{ii}^2 + 2 \sum_{k<l} A_{lk} A_{kl} \end{aligned}$$

Thus we see that

$$tr(\wedge^2 A) = \frac{1}{2} (tr(A)^2 - tr(A^2))$$

□

Problem 1.6. Find a table of characters for the alternating group A_5.

Proof.

	1 Id	20 (1 2 3)	15 (1 2)(3 4)	12 (1 2 3 4 5)	12 (1 2 3 5 4)
χ_1	1	1	1	1	1
χ_2	3	0	-1	ϕ	$1-\phi$
χ_3	3	0	-1	$1-\phi$	ϕ
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

where $\phi = \frac{1+\sqrt{5}}{2}$. □