Prelim Review

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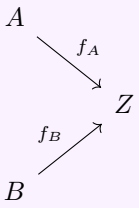
Chapter 1

Category Theory-Aluffi I.5

Definition 1.1 (initial, final)**.** Let C be a category. We say $I \in Obj(C)$ is **initial** if for every $A \in Obj(C)$, there exists exactly one morphism $f : I \to A$. (In other words, $Hom(I, A)$ is a singleton). We say $F \in Obj(C)$ is **final** if for every $B \in Obj(C)$, $Hom(B, F)$ is a singleton.

Definition 1.2 (coproduct)**.** Let A, B be objects of a category C, then the coproduct $A \amalg B$ is an object of C with two morphisms $i_A : A \to A \amalg B$, $i_b : B \to A \amalg B$ with the following universal property: for all objects $Z \in Obj(C)$ and for all morphisms f_A, f_B such that



there exists a unique $\sigma : A \amalg B \to Z$ such that the following diagram commutes:

Chapter 2

Aluffi II

Proposition 2.1. Let $|g|$ be the order of an element $g \in G$, then $|g| \leq |G|$.

Proof. It's trivial if $|G| = \infty$. For $|G| < \infty$, consider the following $|G| + 1$ terms,

$$g^0, g, g^2, \ldots, g^{|G|}$$

Note all can be distinct, hence there exists $i < j$ such that $g^i = g^j$, i.e., $g^{j-i} = e$. This implies that $|g| \leq |G|$. □

Example 2.1. There exists g, h with finite orders each, and gh has infinite order. For example, the group generated by relations:

$$\langle r, s : r^2 = s^2 = e \rangle$$

The term rs has infinite order. Note there exists geometric examples with matrix groups.

Proposition 2.2. The following is a list of statements/propositions that you should know.

1. If $g^N = e$, then $|g|$ divides N.
2. If $g^m = N$, then $|g| = \frac{lcm(|g|,m)}{m}$
3. If $gh = hg$, then $|gh|$ divides $lcm(|g|, |h|)$
4. We have $|g^m| = \frac{|g|}{\gcd(m,|g|)}$

Definition 2.1 (Dihedrayl group)**.** The group D_{2n} is the group of rotations and reflections of n-polygons. (There are $2n$ elements in this group).

We note that D_6 and S_3 are isomorphic. They are both generated by x, y such that $x^2 = e, y^3 = e$.
We first note that $\mathbb{Z}/n\mathbb{Z}$ is a cyclic group of order n,and $[1]$ is a generator.

Proposition 2.3. $[m]$ is a generator of $\mathbb{Z}/n\mathbb{Z}$ if and only if $\gcd(m, n) = 1$.

Proof. $[m]$ is a generator if and only if $[m]$ has order n, which by the above proposition 4, we have $[m] = m[1]$, hence $\gcd(m, n) = 1$. □

Proposition 2.4. $(\mathbb{Z}/n\mathbb{Z})^* = \{[m] : \gcd(m,n) = 1\}$ is a group under the multiplication defined as

$$[m] \cdot [n] = [mn]$$

Proof. We will only check the existence of inverse. For each $[m]$, we know $\gcd(m,n) = 1$, hence $[m]$ is a generator of the additive group $\mathbb{Z}/n\mathbb{Z}$, hence we know there exists some integer q such that $q[m] = [1]$, this implies that $[q][m] = [1]$, hence inverse. □

Example 2.2. $G \cong H \times G$ does not mean that H is the trivial group. For example, consider G as the infinite product $G = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \ldots$, and take $H = \mathbb{Z}$.

Note that if we take $G = \mathbb{Z}$, then H is the trivial group. Proof: consider where $\varphi(1)$ gets sent to. No matter where it is sent to, there are elements not mapped by φ.

Example 2.3. Let $\varphi : S_3 \to \mathbb{Z}/2\mathbb{Z}$ be a homomorphism, then φ sends elements of order 3 to 0. One can define the commutator subgroup. Let G be a group, then the commutator subgroup $[A, G]$ is generated by

$$\langle ghg^{-1}h^{1-}, g, h \in G \rangle$$

then

Proposition 2.5. Let $\varphi : G \to H$ be a homomorphism, then $\varphi([A, G]) \subset \ker(\varphi)$.

Moreover, one can show that any homomorphism $\varphi : S_3 \to \mathbb{Z}/2\mathbb{Z}$ should send all elements of order 2 to 0, or all to 1. One can either use the determinant map $\det : S_3 \to \{\pm 1\}$, or consider that

$$\varphi((12)(23)(12)) = \varphi(13) = \varphi(12)\varphi(23)\varphi(12)$$

then $\varphi(13) = \varphi(23)$.

Example 2.4. Any homomorphism $\varphi : \mathbb{Q} \to \mathbb{Q}$ is linear, i.e., $\varphi(p) = qp$ for some $q \in \mathbb{Q}$. We note that $\varphi(p) = p\varphi(1)$, and $\varphi\left(\frac{1}{p}\right) = \frac{1}{q}\varphi(1)$. Hence we have $\varphi\left(\frac{p}{q}\right) = \frac{p}{q}\varphi(1)$. By the same argument, any homomorphism from $\mathbb{Z} \to \mathbb{Z}$ is also linear.

We note that isomorphisms preserve the following things:

Proposition 2.6. If G, H are isomorphic, then

1. If G is abelian, then H is abelian.
2. The order of g=the order of $\varphi(g)$

If it's just a homomorphism, then the order of $\varphi(g)$ divides the order of g. For example, there is no nontrivial homomorphism from $\mathbb{Z}/4\mathbb{Z}$ to $\mathbb{Z}/7\mathbb{Z}$, because $\varphi(g)$'s order would have to divide 4 and 7.

Example 2.5. $(\mathbb{R}, \cdot)$ and $(\mathbb{C}, \cdot)$ are not isomorphic. There are no finite order elements in $\mathbb{R}$, but i has order 4 in $\mathbb{C}$.

Example 2.6. $Hom(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is a ring, with the gruop being under addition of maps. $Aut(\mathbb{Z}/n\mathbb{Z})$ is a group under composition of maps.

Proposition 2.7. Let H be a subset of a group G, then H is a subgroup if for all $a, b \in H$, $ab^{-1} \in H$.

Every homomorphism $\varphi : G \to G'$ determines two subgroups naturally, $\ker(G) \subset G, Im(G) \subset G'$. This is because if H' is a subgroup of G', then $\varphi^{-1}(H')$ is a subgroup of G. In fact, the image of any subgroup of G is also a subgroup of G'.

Definition 2.2 (cyclic group). A group is cyclic if it is either $\cong \mathbb{Z}$ or $\cong \mathbb{Z}/n\mathbb{Z}$.

Let's now classify all the subgroups of cyclic groups.

Proposition 2.8. If $H \subset \mathbb{Z}$ is a subgroup, then $H = d\mathbb{Z}$ for some $d \geq 0$. (Proof: let d be the smallest positive integer in H).

If $G \subset \mathbb{Z}/n\mathbb{Z}$ is a subgroup, then $G = \langle [d]_n \rangle$ for some d that divides n. Moreover, this exists a bijection between the subgroups of $\mathbb{Z}/n\mathbb{Z}$ with the divisors of n.

Idea 2.1. This means that all subgroups of cyclic groups are cyclic.

Example 2.7. Show that $\mathbb{Z}/12\mathbb{Z}$ has 6 subgroups. Proof: 12 has 6 divisors: 1,2,3,4,6,12.

Proposition 2.9. Let $\varphi : G \to H$ be a surjective homomorphism, then if G is cyclic, H is also cyclic.

This gives the result that the projection π_n allows us to go from a cyclic subgroup of $\mathbb{Z}$ to a cyclic subgroup of $\mathbb{Z}/n\mathbb{Z}$.

To understand $\varphi : G \to G'$ as a monic morphism means φ is injective. If we consider

$$\ker(\varphi) \underset{e}{\overset{i}{\rightrightarrows}} G \xrightarrow{\varphi} G'$$

Then $\varphi \circ i = \varphi \circ e$, where e is the trivial map and i is the inclusion map, this means that $\ker(\varphi) = \{e\}$.

Proposition 2.10. Let $S \subset G$ be a subset, S generates G if and only if $\pi : F(S) \to G$ is surjective.

Proof. Assume π is surjective, then for any $g \in G$, we have $\pi(w) = g$ for some $s \in F(S)$, and $w = s_1^{n_1} \dots s_n^{n_k}$, hence every g corresponds to that. □

Theorem 2.2. Let $\varphi : G \to G'$ be a homomorphism, then there exists a bijection $\tilde{\varphi}$

$$\tilde{\varphi} : G/\ker(\varphi) \to Im(\varphi)$$

Proof. Let $\tilde{\varphi}([a]) := \varphi(a)$. Then $Im(\tilde{\varphi})$ is a subgroup of G'. We just need to show that $\tilde{\varphi}$ is injective.

$$\tilde{\varphi}([a]) = \tilde{\varphi}([b]) \Rightarrow \varphi(a) = \varphi(b) \Rightarrow \varphi(ab^{-1}) = e \Rightarrow ab^{-1} \in \ker(\varphi) \Rightarrow [a] = [b]$$

Note that the last argument is by $H = \ker(\varphi)$ is normal, hence we want to show $Ha = Hb$, and $ab^{-1} \in H$ means $Hab^{-1} \subset H$, i.e., $Ha \subset Hb$, and vice versa. □

Example 2.8. The commutator subgroup of a group G is defined roughly to capture the group that is "not commutative with other elements." In other words, they should, in some sense, be complimentary to the center. The commutator subgroup $[G, G]$ of G is the subgroup **generated by**

$$\langle ghg^{-1}h^{-1}, g, h \in G \rangle$$

Proposition 2.11. The $G/[G, G]$ is abelian. In other words, the quotient group by the commutator subgroup is abelian. (This intuitively makes sense because if we view the "noncommutative elements" as the same, then the rest should just be abelian).

Proof. We would like to show that $g[G, G]h[G, G] = gh[G, G] = hg[G, G]$. In other words,

$$g^{-1}h^{-1}gh \in [G, G] \Rightarrow g^{-1}h^{-1}gh \subset [G, G] \Rightarrow g^{-1}h^{-1}gh \in [G, G] = [G, G]$$

This is because that any coset contained in one coset is equal to the entire coset. □

Lemma 2.1. we claim that for any subgroup H,

$$gH \subset H \Rightarrow gH = H$$

For any $h \in H$, we know that $g^{-1}gH \in H$, and because $gh \in H$, we have that $g^{-1} \in H$, hence we have that $h \in gH$ as well.

Warning 2.3. This fact should be memorized, i.e., if any coset gH is contained in an other coset $g'H$, then they are the same.

Example 2.9. Let H be a normal subgroup of G, and K be a subgroup of G, then $HK = \{hk : h \in H, k \in K\}$ is a normal subgroup.

To show that it is a subgroup, we note that

$$HK = \pi^{-1}(\pi(K))$$

where $\pi : G \to G/H$. In other words, $\pi^{-1}(gH) = gH$.

Lemma 2.2. For H normal in G, the usual projeciton $\pi : G \to G/H$, we have $\pi^{-1}(gH) = gH$.

Example 2.10. Cosets are disjoint. Let $g \in g_1H \cap g_2H$, then $g_1h_1 = g_2h_2$, then $g_1 = g_2h_2h_1^{-1}$, hence $g_1H \subset g_2H$.

Example 2.11 (8.13). Let $|G|$ be odd, show that every element is a square.

Method 1: it'd be nice if we can show that $f(g) = g^2$ is an injective homomrophism from $G \to G$, but this is not the case. It need not to be a homomorphism. In fact, it is a homomorphism only when G is abelian. Now consider $\langle g \rangle$, it is an abelian group, so we can restrict f to all the $\langle g \rangle$, then we can show that it is injective, hence surjective.

Method 2: observe that $g^{|G|} = 1, g^{|G|+1} = g$, then $(g^{|G|+1})^2 = g$.

Example 2.12 (8.25). An interesting homomorphism from $G/H \to Aut(G)$, when H is abelian and normal.

We know that if H is normal, then $f : g \mapsto \gamma_g$ satisfies $\gamma_g(H) \subset H$, and hence an homomorphism would be $g \mapsto \gamma_g|_H$. Now if H is abelian, we see that $H \subset \ker(f)$, i.e. there is the homomorphism from $G/H \to Aut(H)$ is well-defined.

Lemma 2.3. Let $f : G \to G'$, then the following are equivalent:

1. $\overline{f} : G/H \to G'$ is well-defined, i.e., $gH \mapsto f(g)$.
2. $H \subset \ker(f)$.

Definition 2.3 (action on a set). An action ρ of G on a set A is a function $\rho : G \times A \to A$ such that

$$\rho(e, a) = a, \forall a, \rho(gh, a) = \rho(g, \rho(h, a)), \forall g, h, a$$

We call an action is transitive if for all $a, b \in A$, there exists g such that

$$b = \rho(g, a)$$

We know that left multiplications are faithful and also transitive.

Theorem 2.4 (Cayley's theorem). Every group acts faithfully on some set.

Proof: Every group acts faithfully on itself by left multiplication.

Definition 2.4 (orbit, stabilizer). The orbit of an element $a \in A$ under the action of G is the set

$$O(a) = \{\rho(g, a) : g \in G\}$$

The stabilizer subgroup of G of $a \in A$ is

$$Stab(a) = \{g \in G : \rho(g, a) = a\}$$

Proposition 2.12. Orbits partition the set A.

Proof. We can show this directly: if $\rho(g_1, a) = \rho(g_2, b)$, then $\rho(g', a) \in O(b)$ for ant g'.

Alternatively, we can show that if $c = \rho(g, a) \in O(a)$, then $O(c) \subset O(a)$; then we can show if $c \in O(a)$, then $a \in O(c)$, so $O(a) \subset O(c)$, hence $O(a) = O(c)$, hence any overlap implies equality. □

From this we see that orbits partition A, and the induced action on each orbit is transitive. Hence it suffices to consider transitive actions if we want to understand any group actions on a set.

Proposition 2.13. If G acts transitively on a set A, then $|A|$ divides $|G|$.

Proof. One can define a bijection between $\varphi : G/H \to A$, where $H = Stab(a)$ for any $a \in A$, by $\varphi : gH \mapsto g \cdot a$. Then by Lagrange's, we know $|G| = |A| \cdot |H|$. □

Example 2.13 (9.11). Let G be a finite group, and p be the smallest prime such that it divides $|G|$. Let H be a subgroup of G of index p, then H is normal.

The proof relies on showing $\ker(\sigma) = H$, where $\sigma : G \to S_p$ corresponds to an action of G on G/H.

Chapter 3

Aluffi III: Rings and Modules

Definition 3.1. An integral domain is nonzero commutative ring R such that for all $a, b \in R$

$$ab = 0 \Rightarrow a = 0 \text{ or } b = 0$$

Equivalently, left and right multiplication by every nonzero element $u \in R$ is injective.

In other words, if we define a left zero divisor as $a \in R$ such that for some $b \neq 0$ we have $ab = 0$, then the integral domain requires NO nonzero zero divisors (0 is always going to be a zero divisor).

Proposition 3.1. Multiplication cancellation holds in integral domains, i.e., if $ab = ac$, then $b = c$, for $a \neq 0$.

Proof. We will know it holds if multiplication by a is an injective function, this is true if and only if a is not a zero divisor. □

Example 3.1. $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{Q}$ are integral domains, and $\mathbb{Z}/n\mathbb{Z}$ is not, for some n. However, $\mathbb{Z}/p\mathbb{Z}$ is a integral domain.

Definition 3.2. A left unit in R is u such that there exists v and $uv = 1$. In other words, a left unit has a right inverse.

We note that two-sided units have unique inverses, and if we call the two-sided units just units, the units of a ring form a group!

One warning: if u only has a right inverse, and no left inverse, then this u may have many right inverses.

Definition 3.3 (field)**.** A field is a commutative ring such that every nonzero element is a unit. (And of course, fields are integral domains, integral domains are not always fields, for example, $\mathbb{Z}$).

Proposition 3.2. In a field, left (and right) multiplication by any $u \neq 0$ is injective and surjective.

Proof. This uses the fact that $u \neq 0$ is a two-sided unit. □

Proposition 3.3. A finite integral domain is a field.

Proof. It suffices to show that the left and right multiplications by an element in this integral domain is surjective (this would imply this element is a unit). We know the multiplication is injective, and an injective map from a finite set to itself is also surjective. □

Example 3.2. $\mathbb{Z}/p\mathbb{Z}$ is a field, hence an integral domain. This is because the group of units in $\mathbb{Z}/n\mathbb{Z}$ is those m such that $\gcd(m, n) = 1$.

If m is a unit in $\mathbb{Z}/n\mathbb{Z}$, then $1 \equiv am$ for some m, then m is a generator of $\mathbb{Z}/n\mathbb{Z}$, hence $\gcd(m, n) = 1$. Proof: $am = 1 \mod n$, hence there exists some b such that $am + bn = 1$, hence $\gcd(m, n) = 1$.

Proposition 3.4. A polynomial ring $R[x]$ is an integral domain if R is an integral domain.

Chapter 4

Taylor 1