

# Algebra Definition Theorem List

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# Chapter 1

## Category Theory

**Definition 1.1** (*initial, final*). Let  $\mathcal{C}$  be a category, then object  $I$  is initial if for every object  $A$ , there exists a unique morphism  $I \rightarrow A$ . We say  $F$  is final if for every  $A$ , there exists a unique morphism  $A \rightarrow F$ .

# Chapter 2

## Group Theory I

This corresponds to Aluffi Chapter II.

**Proposition 2.1.** Let  $G$  be a group, for all  $a, g, h \in G$ , if

$$ga = ha$$

then  $g = h$ .

**Proposition 2.2.** Let  $g \in G$  have order  $n$ , then

$$n \mid |G|$$

**Corollary 2.1.** If  $g$  is an element of finite order, and let  $N \in \mathbb{Z}$ , then

$$g^N = e \iff N \text{ is a multiple of } |g|$$

**Proposition 2.3.** Let  $g \in G$  be of finite order, then  $g^m$  also has finite order, for all  $m \geq 0$ , and

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\gcd(m, |g|)}$$

**Proposition 2.4.** If  $gh = hg$ , then  $|gh|$  divides  $\text{lcm}(|g|, |h|)$ .

**Definition 2.1 (Dihedral Group).** Let  $D_{2n}$  denote the group of symmetries of a  $n$ -sided polygon, consisting of  $n$  rotations and  $n$  reflections about lines through the origin and a vertex or a midpoint of a side.

**Proposition 2.5.** Let  $m \in \mathbb{Z}/n\mathbb{Z}$ , then

$$|m| = \frac{n}{\gcd(n, m)}$$

**Corollary 2.2.** The element  $m \in \mathbb{Z}/n\mathbb{Z}$  generates  $\mathbb{Z}/n\mathbb{Z}$  if and only if  $\gcd(m, n) = 1$ .

**Definition 2.2** (Multiplicative  $(\mathbb{Z}/n\mathbb{Z})^\times$ ). The multiplicative group of  $\mathbb{Z}/n\mathbb{Z}$  is

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{m \in \mathbb{Z}/n\mathbb{Z} : \gcd(m, n) = 1\}$$

**Proposition 2.6.** Let  $\varphi : G \rightarrow H$  be a homomorphism, and let  $g \in G$  be an element of finite order, then  $|\varphi(g)|$  divides  $|g|$ .

For example, there is no nontrivial homomorphism from  $\mathbb{Z}/n\mathbb{Z}$  to  $\mathbb{Z}$ .

**Proposition 2.7.** There is an isomorphism between  $D_6$  and  $S_3$ .

**Proposition 2.8.** Let  $\varphi : G \rightarrow H$  be an isomorphism, for all  $g \in G$ ,  $|\varphi(g)| = |g|$ , and  $G$  is commutative if and only if  $H$  is commutative.

**Proposition 2.9.** If  $H$  is commutative, then  $\text{Hom}(G, H)$  is a group.

**Definition 2.3.** Let  $A = \{1, \dots, n\}$ , then the free abelian group on  $A$  is

$$\mathbb{Z} \oplus \dots \oplus \mathbb{Z} = \mathbb{Z}^{\oplus n}$$

**Proposition 2.10.** Let  $\{H_\alpha\}$  be any family of subgroups of  $G$ , then

$$\bigcap_{\alpha} H_{\alpha}$$

is a subgroup of  $G$ .

**Proposition 2.11.** If  $\varphi : G_1 \rightarrow G_2$  is a group homomorphism, then if  $H_2 \subset G_2$  is a subgroup, then

$$\varphi^{-1}(H_2)$$

is a subgroup of  $G_1$ .

**Proposition 2.12.** Let  $H \subset \mathbb{Z}/n\mathbb{Z}$  be a subgroup, then  $H$  is generated by some  $m$  where  $m$  divides  $n$ .

**Proposition 2.13.** If  $\varphi : G_1 \rightarrow G_2$  is a homomorphism, then  $\ker(\varphi)$  is a normal subgroup.

**Theorem 2.1.** Let  $\varphi : G_1 \rightarrow G_2$  be a surjective homomorphism, then

$$G_2 \cong \frac{G_1}{\ker \varphi}$$

**Proposition 2.14.** Let  $H_1, H_2$  be normal subgroups of  $G_1, G_2$ , then  $H_1 \times H_2$  are normal subgroups of  $G_1 \times G_2$ , then

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}$$

For example,

$$\frac{\mathbb{Z}/6\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$$

**Proposition 2.15.** Let  $H$  be a normal subgroup of  $G$ , then every subgroup  $K$  containing  $H$ ,  $K/H$  can be identified with a subgroup of  $G/H$ .

**Proposition 2.16.** Let  $H$  be a normal subgroup of  $G$ , and  $N$  be a subgroup of  $G$  containing  $H$ , then  $N/H$  is normal in  $G/H$  if and only if  $N$  is normal in  $G$ , in this case

$$\frac{G/H}{N/H} = \frac{G}{N}$$

**Proposition 2.17.** Let  $H, K$  be subgroups of  $G$ , and if  $H$  is normal, then  $HK$  is a subgroup of  $G$  and  $H$  is normal in  $HK$ . Moreover,  $H \cap K$  is normal in  $K$ , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}$$

**Proposition 2.18.** Let  $H$  be a subgroup of  $G$ , then for all  $g \in G$ , the function  $H \rightarrow gH$  such that

$$h \mapsto gh$$

is a bijection.

**Theorem 2.2 (Lagrange).** If  $G$  is a finite group, and  $H \subset G$  is a subgroup, then

$$|G| = [G : H] \cdot |H|$$

In particular,  $|H|$  divides  $|G|$ .

**Theorem 2.3 (Fermat's Little Theorem).** Let  $p$  be a prime integer, and  $a$  be any integer, then

$$a^p \equiv a \pmod{p}$$

**Proposition 2.19.** Any group  $G$  acts on itself by left/right multiplications, and acts on the cosets  $G/H$ :

$$\varphi : g \mapsto (aH \mapsto gaH)$$

**Definition 2.4 (orbit).** The orbit of  $a \in A$  of a group action by  $G$  is

$$O(a) = \{g \cdot a : g \in G\}$$

The stabilizer of  $a$  is the following

$$\text{Stab}_G(a) = \{g \in G : g \cdot a = a\}$$

**Proposition 2.20.** The orbits of an action form a partition on the set  $A$ , and  $G$  acts transitively on each orbit.

**Definition 2.5 (transitive action, faithful action).** An action of  $G$  on  $A$  is transitive if for all  $a, b \in A$ , there exists  $g \in G$  such that

$$g \cdot a = b$$

In other words, the orbit of any element  $a \in A$  is the entire set.

An action is faithful if for any  $g \in G$ ,

$$g \cdot a = a \text{ for all } a$$

implies that  $g = e$ .

**Proposition 2.21.** Every transitive action of  $G$  on a set  $A$  is isomorphic to multiplication of  $G$  on  $G/H$ , where  $H = \text{Stab}(a)$  for any  $a \in A$ .

**Proposition 2.22.** If  $O(a)$  is an orbit of the action of a finite group  $G$ , then  $O(a)$  is a finite and  $|O(a)|$  divides  $|G|$ . Moreover,

$$|G| = |O(a)| \cdot |\text{Stab}_G(a)|$$

For example, there is no transitive action of  $S_3$  on the set of 5 elements.

## Chapter 3

# Group Theory II

This corresponds to Aluffi Chapter IV.

**Proposition 3.1.** Every **transitive** action of a group  $G$  on a set  $S$  is isomorphic to the left multiplication on the cosets  $G/H$ . Here,  $H$  can be taken to be the stabilizer of any element  $a \in S$ .

Moreover, suppose  $G$  is finite, then

$$|G| = |O_a| \cdot |\text{Stab}(a)|$$

for any  $a \in S$ . (The size of the orbit must divide  $|G|$ .)

**Proposition 3.2 (class formula).** Let  $S$  be a finite set, and  $G$  act on  $S$ , then

$$|S| = |Z| + \sum_{a \in A} [G : \text{Stab}(a)] = |Z| + \sum_{a \in A} |O_a|$$

where  $Z = \{a \in S : g \cdot a = a \text{ for all } g\}$ , i.e., the fixed elements, and  $A \subset S$  contains exactly one element from each nontrivial orbit of the action.

In other words,  $|S|$  is the sum of the number of trivial orbits and each nontrivial orbit.

**Proposition 3.3.** Let  $G$  be a  $p$ -group that acts on a finite set  $S$ , then let  $Z$  be fixed elements of this action, then

$$|S| \equiv |Z| \pmod{p}$$



**Warning 3.1.** The important takeaway is that each summand on the right,  $|O_a|$  divides  $|G|$ .

### 3.1 Conjugation Action

**Definition 3.1 (fixed points, centralizer, conjugacy class).** The fixed points under the conjugation action is the center of  $G$ . The centralizer  $Z_G(g)$  where  $g \in G$  is its stabilizer under conjugation:

$$Z_G(g) = \{h \in G : hgh^{-1} = g\}$$

The conjugacy class of  $g \in G$  is the orbit  $[g]$ . (In other words, centralizer is the set of elements that commute with  $g$ .)



For arbitrary  $a \in G$ , we have

$$Z(G) \subset Z_G(a)$$

Moreover,  $a$  is the only element in  $[a]$  iff  $a \in Z(G)$ .

**Proposition 3.4.** The center is the set of fixed points of  $G$  under the conjugation action, the conjugacy classes are the orbits.

**Theorem 3.2.** Let  $G$  be finite, and if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

*Proof.* One can show that every element  $a \in G$  can be written as

$$a = g^r z$$

for some  $z \in Z(G)$ , then compute  $ab = ba$ . □

**Proposition 3.5 (Class formula).** Let  $G$  be finite, then

$$\begin{aligned} |G| &= |Z(G)| + \sum_{[a] \in A} |[a]| \\ &= |Z(G)| + \sum_a [G : Z_G(a)] \end{aligned}$$

where  $A$  contains one representative for each nontrivial conjugacy class.



**Warning 3.3.** There are many consequences of the class formula, showing center is nontrivial, etc. Mainly using the summand divides  $|G|$ !

**Theorem 3.4.** Let  $G$  be a nontrivial  $p$ -group, then  $G$  has a nontrivial center.

**Proposition 3.6.** Let  $G$  be a group of  $p^2$  elements, where  $p$  is prime, then  $G$  is commutative.

**Proposition 3.7.** The only possibility for the class formula of a nonabelian group of order 6 is

$$6 = 1 + 2 + 3$$

The center must be trivial if  $G$  is nonabelian.

**Proposition 3.8.** Normal subgroups are unions of conjugacy classes. Thus, a noncommutative group of order 6 cannot have a normal subgroup of order 2.

It contains the identity, and there is no other conjugacy class of size 1.

**Definition 3.2 (normalizer).** Let  $A \subset G$  be a subset. The normalizer  $N_G(A)$  of  $A$  is

$$\text{Stab}_G(A) = \{g : gAg^{-1} = A\}$$

If  $H$  is subgroup of  $G$ , every conjugate  $gHg^{-1}$  is also a subgroup of  $G$ , and all conjugate groups have the same order.

The centralizer of  $A$  is the subgroup  $Z_G(A) \subset N_G(A)$  fixing each  $a \in A$ :

$$Z_G(A) = \{g : gag^{-1} = a \text{ for all } a \in A\}$$

**Proposition 3.9 (\*)**.  $H$  is a normal in  $G$  if and only if  $N_G(H) = G$ . More generally, the normalizer  $N_G(H)$  for any subgroup  $H$  is the largest subgroup such that  $H$  is normal in  $N_G(H)$ .

**Proposition 3.10 (\*)**. Let  $H \subset G$  be a subgroup, then the number of subgroups conjugate to  $H$  is the size of the orbit=index of the stabilizer, which is  $[G : N_G(H)]$ .

**Corollary 3.1**. If  $[G : H]$  is finite, then the number of subgroups conjugate to  $H$  is finite, and

$$[G : H] = [G : N_G(H)] \cdot [N_G(H) : H]$$

In other words, the number of subgroups conjugate to  $H$  divides the index  $[G : H]$ .

## 3.2 Sylow

**Theorem 3.5 (Cauchy's Theorem)**. Let  $G$  be a finite group, and let  $p$  be a prime divisor of  $|G|$ , then  $G$  contains an element of order  $p$ .

Moreover, let  $N$  be the number of cyclic subgroups of order  $p$ , then

$$N \equiv 1 \pmod{p}$$

**Definition 3.3 (simple)**. A group is simple if it is nontrivial and its only normal subgroups are  $\{e\}$  and  $G$  (has no nontrivial proper subgroup).

**Definition 3.4 ( $p$ -Sylow subgroups)**. Let  $p$  be prime, a  $p$ -Sylow subgroup of a finite group  $G$  is a subgroup of order  $p^r$ , where  $|G| = p^r m$ ,  $\gcd(p, m) = 1$ .

**Theorem 3.6 (Sylow I)**. Every finite group contains a  $p$ -Sylow subgroup for all prime  $p$ . If  $p^k$  divides  $|G|$ , then  $G$  has a subgroup of order  $p^k$ .

**Theorem 3.7 (Sylow II)**. Let  $G$  be finite, and  $P$  is a  $p$ -Sylow subgroup, let  $H \subset G$  be a  $p$ -group, then  $H$  is contained in a conjugate of  $P$ . If  $P_1, P_2$  are both  $p$ -Sylow subgroups, then they are conjugates to each other.

**Theorem 3.8 (Sylow III)**. Let  $|G| = p^r m$ , and  $\gcd(p, m) = 1$ , then the number of  $p$ -Sylow subgroups is

$$n_p \mid m$$

and

$$n_p \equiv 1 \pmod{p}$$

**Proposition 3.11.** Let  $G$  be a finite group, let  $P$  be a  $p$ -Sylow subgroup, the number of  $p$ -Sylow subgroup  $n_p$  is

$$n_p = [G : N_G(P)]$$

by definition.

**Proposition 3.12.** Let  $G$  be a group of order  $mp^r$ , where  $p$  is prime and  $1 < m < p$ , then  $G$  is not simple.

**Proposition 3.13 (\*)**. Let  $p < q$  be primes, let  $G$  has order  $pq$ , if  $p \nmid (q - 1)$ , then  $G$  is cyclic.

*Proof.* If  $G$  is abelian, use elements of orders  $p, q$ . If  $G$  not necessarily abelian, then use the conjugation action.  $\square$

**Proposition 3.14 (\*)**. Let  $q$  be an odd prime, and  $G$  be a noncommutative group of order  $2q$ , then

$$G \cong D_{2q}$$

### 3.3 Series and Solvability

**Definition 3.5 (composition series).** A comp series for  $G$  is a normal series

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that  $G_{i+1}/G_i$  is simple.

**Definition 3.6 (commutator subgroup).** Let  $G$  be a group, the commutator subgroup of  $G$  is the subgroup **generated** by all elements

$$ghg^{-1}h^{-1}$$

**Proposition 3.15.** Let  $[G, G]$  be the commutator subgroup of  $G$ , then  $[G, G]$  is normal in  $G$ , and the quotient, also called the abelianization of  $G$ ,

$$G^{\text{ab}} = \frac{G}{[G, G]}$$

is commutative.

If  $\varphi : G \rightarrow H$ , where  $H$  is commutative, then

$$[G, G] \subset \ker(\varphi)$$

**Definition 3.7.** A group  $G$  is solvable, if there exists a sequence such that

$$\{e\} = G_0 \subset \cdots \subset G_k = G$$

where  $G_i$  is normal in  $G_{i+1}$ , and  $G_{i+1}/G_i$  is abelian, or equivalently, cyclic.

**Proposition 3.16.** All  $p$ -groups are solvable!

**Proposition 3.17.** Let  $N$  be normal in  $G$ , then  $G$  is solvable if and only if  $N, G/N$  are solvable.

### 3.4 $S_n$ and $A_n$

**Proposition 3.18.** Disjoint cycles commute. For every  $\sigma \in S_n$ ,  $\sigma$  can be written as disjoint nontrivial cycles, unique up to rearranging.

**Proposition 3.19.** Two elements in  $S_n$  are conjugate in  $S_n$  if and only if they have the same type. Hence the number of conjugacy classes is the number of partitions of  $n$  as a sum.

**Proposition 3.20.** Let  $\sigma \in S_n$ , and  $(a_1 \dots a_n)$  is a cycle in  $S_n$ , then

$$\sigma(a_1 \dots a_n)\sigma^{-1} = (\sigma(a_1) \dots \sigma(a_n))$$

Proof: try  $\varphi(a_1)$  on the left hand side.



**Warning 3.9.** Very useful!

**Example 3.1.** In  $S_4$ , we have

$$(1234)(12)(1234)^{-1} = (23)$$

**Definition 3.8 (Even permutation).** Let  $\sigma \in S_n$ , then  $\sigma$  is even if

$$\prod_{i < j} (x_i - x_j) = \prod_{i < j} (x_{\sigma(i)} - x_{\sigma(j)})$$

**Proposition 3.21.**  $A_n$  is always normal in  $S_n$ , because it is the kernel of the  $\varepsilon : S_n \rightarrow \{\pm 1\}$  (determining parity).

**Proposition 3.22.** Let  $\sigma \in A_n$ , where  $n \geq 2$ , then the conjugacy class of  $\sigma$  in  $S_n$  splits into two conjugacy classes in  $A_n$  precisely if the type of  $\sigma$  consists of distinct odd numbers; or equivalently, the centralizer of  $\sigma$  is contained  $A_n$ . Otherwise, the conjugacy class stays the same.

**Example 3.2.**  $S_5$  has even permutations 5, 3, 2+2, 1, and only 5-cycle of  $S_5$  splits into 2 conjugacy classes in  $A_5$ .

**Proposition 3.23.** The group  $A_5$  is a simple noncommutative group of order 60.

**Proposition 3.24.** Every simple group of order  $< 60$  is commutative,  $A_5$  is the smallest simple group that is not commutative.

*Proof.* Any nontrivial normal subgroup consists of nontrivial conjugacy classes and  $\{e\}$ , the conjugacy classes of  $A_5$  has the following size:

$$1, 15, 20, 12, 12$$

Thus any subgroup of  $G$ , i.e., order that divides 60 cannot be written as a sum of the numbers above.  $\square$

**Proposition 3.25.** The alternating group is generated by 3-cycles.

**Proposition 3.26.** Let  $n \geq 5$ , if a normal subgroup of  $A_n$  contains a 3-cycle, then it contains all 3-cycles.

*Proof.* It suffices to note that the 3 cycles form a conjugacy class that doesn't split from  $S_n$  to  $A_n$ .  $\square$

**Proposition 3.27.** The alternating group  $A_n$  is simple for  $n \geq 5$ . As a result,  $S_n$  is not solvable for  $n \geq 5$ .

## 3.5 Product of Groups

**Proposition 3.28.** Let  $N, H$  be normal subgroups of  $G$ , let  $[N, H]$  be the commutator of  $N, H$ , then

$$[N, H] \subset N \cap H$$

Thus if  $N \cap H = \{e\}$ , then  $N, H$  commute with each other.

A stronger statement is the following:

**Theorem 3.10.** Let  $N, H$  be normal subgroups of  $G$ , such that  $N \cap H = \{e\}$ , then

$$NH \cong N \times H$$

**Definition 3.9 (Split Short exact sequence).** A short exact sequence of groups is a sequence:

$$1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

splits if  $H$  is identified with a subgroup of  $G$  such that

$$N \cap H = \{e\}$$

**Definition 3.10 (semidirect product).** Let  $N$  be a normal subgroup, and let  $\theta : H \rightarrow \text{Aut}(N)$ , then define an operator  $\cdot_\theta$  on  $N \times H$  as

$$(n_1, h_1) \cdot_\theta (n_2, h_2) = (n_1 \theta(h_1)(n_2), h_1 h_2)$$

The semidirect product of  $N \rtimes_\theta H$  is the group  $N \times H$  with operation  $\cdot_\theta$ .

**Proposition 3.29.** Let  $N, H$  be subgroups, and  $N$  is normal, suppose that  $N \cap H = \{e\}$ , and  $G = NH$ , then let  $\theta : H \rightarrow \text{Aut}(N)$  be  $\theta \mapsto \theta_h$ , and

$$\theta_h(n) = nhn^{-1}$$

Then

$$G \cong N \rtimes_{\theta} H$$

(Recall that the operation defined on  $N \rtimes_{\theta} H$  is  $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \theta_{h_1}(n_2), h_1 h_2)$ ).

**Proposition 3.30.** Let  $G$  be a noncommutative group of order  $pq$ , then there is exactly one group up to isomorphism.

### 3.6 Classification of Finite Abelian Groups

**Proposition 3.31.** Let  $G$  be abelian, let  $H, K$  be subgroups such that  $|H|, |K|$  are relatively prime, then

$$H + K \cong H \oplus K$$

*Proof.* Lagrange:  $N \cap H = \{e\}$ . □

**Proposition 3.32.** Every finite abelian group is a direct sum of its nontrivial Sylow subgroups.

**Theorem 3.11.** If  $G$  is finite and abelian, then  $G$  is a direct sum of cyclic  $p$ -groups.

**Theorem 3.12.** Let  $G$  be finite nontrivial abelian group, then there exists prime integers  $p_1, \dots, p_r$ , and positive integers  $n_{i(j)}$  such that

$$G = \bigoplus_{i,j} \frac{\mathbb{Z}}{p_i^{n_{i(j)}} \mathbb{Z}}$$

There exists positive integers  $1 < d_1 \mid \dots \mid d_s$  such that  $|G| = d_1 \dots d_s$ , and

$$G \cong \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{d_s \mathbb{Z}}$$

**Example 3.3.** Finite abelian group of order 360 has 6 isomorphism classes.

**Theorem 3.13.** Let  $F$  be a field, and  $G$  be a finite subgroup of the multiplicative group  $(F^{\times}, \cdot)$ , then  $G$  is cyclic.

*Proof.* Hard proof. Don't torture yourself. □

# Chapter 4

## Ring Theory

This corresponds to Aluffi Chapter III.

**Definition 4.1 (free action).** An action by  $G$  is free if there exists  $x \in X$  such that  $gx = x$  then  $g = e$ .

**Definition 4.2 (faithful action).** An action by  $G$  is faithful if  $gx = x$  for all  $x \in X$  implies that  $g = e$ .

**Definition 4.3 (zero-divisor).** An element  $a \in R$  is a (left) zero-divisor if there exists  $b \neq 0$  such that

$$ab = 0$$

**Proposition 4.1.** In a ring  $R$ ,  $a \in R$  is not a left zero-divisor if and only if the left multiplication by  $a$  is injective.

**Definition 4.4 (integral domain).** An ID is a nonzero commutative ring such that for all  $a, b \in R$ ,

$$ab = 0$$

implies  $a = 0$  or  $b = 0$ . In other words, IDs are commutative rings without zero divisors. Equivalently, if  $a, b \neq 0$ , then  $ab \neq 0$ .

**Proposition 4.2.** In a ring  $R$ :

1.  $u$  is left unit iff the left multiplication by  $u$  is surjective.
2. If  $u$  is a left unit, then the right multiplication by  $u$  is injective, i.e.,  $u$  is not a right zero-divisor.

Notice that in a commutative ring, this means  $u$  is a unit iff multiplication by  $u$  is bijective.

**Definition 4.5 (division ring, field).** A division ring is a ring in which every nonzero element is a unit. A field is a nonzero commutative ring in which every nonzero element is a unit.

**Proposition 4.3.** The group of units in  $\mathbb{Z}/n\mathbb{Z}$  is exactly the group  $(\mathbb{Z}/n\mathbb{Z})^*$ .

*Proof.*  $m$  is a unit iff multiplication by  $m$  is surjective, iff  $m$  generates  $\mathbb{Z}/n\mathbb{Z}$ , iff  $m \in (\mathbb{Z}/n\mathbb{Z})^*$ . □

**Definition 4.6 (Power Series Ring).** The power series ring

$$\sum_{i=0}^{\infty} a_i x^i$$

is denoted by  $R[[x]]$ .

**Definition 4.7 (Monoid Ring).** Given a monoid  $M$  and a ring  $R$ , the elements

$$\sum_{m \in M} a_m \cdot m$$

where  $a_m \in R$  and  $a_m \neq 0$  for finitely many terms, forms a ring denoted as  $R[M]$ .

**Proposition 4.4.** Assume  $R$  is a finite commutative ring, then  $R$  is an integral domain if and only if  $R$  is a field.

**Proposition 4.5.**  $\text{End}_{\text{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$ , where  $\text{End}_{\text{Ab}}(G) = \text{Hom}_{\text{Ab}}(G, G)$  where  $G$  is abelian.

*Proof.*  $\varphi \mapsto \varphi(1)$ . □

**Theorem 4.1.** Let  $I$  be a two-sided ideal of a ring  $R$ . Then for every ring homomorphism  $\varphi : R \rightarrow S$  such that  $I \subset \ker \varphi$  there exists a unique ring homomorphism  $\tilde{\varphi} : R/I \rightarrow S$  so that the diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \pi \downarrow & \nearrow \tilde{\varphi} & \\ R/I & & \end{array}$$

**Theorem 4.2.** Let  $\varphi : R \rightarrow S$  be a surjective ring homomorphism, then

$$S \cong \frac{R}{\ker(\varphi)}$$

**Proposition 4.6.** Let  $I$  be an ideal of a ring  $R$ , and let  $J$  be an ideal of  $R$  containing  $I$ , then  $J/I$  is an ideal of  $R/I$ , and

$$\frac{R/I}{J/I} = \frac{R}{J}$$

**Definition 4.8 (Noetherian).** A commutative ring  $R$  is Noetherian if every ideal of  $R$  is finitely generated. An ideal  $I$  is finitely generated if  $I = (a_1, \dots, a_n)$ , i.e., every element in  $I$  can be written as

$$r_1 a_1 + \dots + r_n a_n$$

for some  $r_1, \dots, r_n \in R$ .



**Proposition 4.7.** Let  $\bar{b}$  be the class of  $b$  in  $R/(a)$ , then

$$\frac{R/(a)}{(\bar{b})} \cong \frac{R}{(a, b)}$$

**Proposition 4.8.**  $\mathbb{Z}$  is a PID by taking the smallest positive element  $d$  in each ideal, obtaining  $(d)$ .

**Definition 4.9.**  $I$  is a prime ideal if  $R/I$  is an integral domain, and is a maximal ideal if  $R/I$  is a field.

**Definition 4.10.** Let  $I, J$  be ideals of  $R$ , then  $IJ$  is the ideal **generated** by elements  $ij, i \in I, j \in J$ .  
Note that  $IJ \subset I \cap J$ .

**Example 4.1.** In  $\mathbb{Z}$ :

$$(4) \cap (3) = (12)$$

and

$$(4) \cap (6) = (12)$$

**Definition 4.11 (Long division).** Let  $f(x) \in R[x]$  be monic, if  $g(x) \in R[x]$  be another polynomial, then there exists unique  $q, r \in R[x]$ , where  $\deg(r) < \deg(f)$ , such that

$$g(x) = f(x)q(x) + r(x)$$

Moreover,

$$g(x) + (f(x)) = r(x) + (f(x))$$

as cosets of  $(f(x))$ .

**Proposition 4.9.** Let  $I$  be an ideal of commutative  $R$ , if  $R/I$  is finite, then  $I$  is prime if and only if maximal.

**Proposition 4.10.** Let  $R$  be a PID, a nonzero ideal  $I$  is prime if and only if it is maximal.

*Proof.* Is simple proof, you just do it. □

**Theorem 4.3.** Let  $R$  be commutative, let  $f(x) \in R[x]$  be a monic polynomial of degree  $d$ , then

$$\varphi : R[x] \rightarrow R^{\oplus d}$$

where

$$\varphi : g(x) \mapsto r(x)$$

where  $r(x)$  is the remainder  $g(x) = f(x)q(x) + r(x)$  induces an isomorphism of **groups**:

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d}$$

**Ring Structure:** can be induced by the map  $\varphi$ .

**Example 4.2.** Let  $f(x) = x - a$  for some  $a \in R$ , then

$$\frac{R[x]}{(x - a)} \cong R$$

**Example 4.3.** Let  $f(x) = x^2 + 1$ , then there is isomorphism of groups:

$$R \oplus R \cong \frac{R[x]}{(x^2 + 1)}$$

note that elements on the right are of the form  $a_0 + a_1x$ . One can give a ring structure on  $R \oplus R$  by  $\varphi$ .

**Example 4.4.** The ideal  $(2, x)$  is maximal in  $\mathbb{Z}[x]$ .

**Example 4.5.** The maximal ideals in  $\mathbb{C}[x]$  are precisely

$$(x - a)$$

where  $a \in \mathbb{C}$ .

**Definition 4.12 (Krull dimension).** Let  $R$  be commutative, the Krull dimension is the length of the longest chain of prime ideals in  $R$ . For example, PIDs but not fields have Krull dimension 1.

$$(0) \subset (d)$$

has length 1.

Moreover,  $k[x_1, \dots, x_n]$  have Krull dimension  $n$ :

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \dots (x_1, \dots, x_n)$$

## 4.1 Modules

**Definition 4.13 (module).** A  $R$ -module  $M$  is an abelian group with a ring action, satisfying:

1.  $r(m + n) = rm + rn$
2.  $(r + s)m = rm + sm$
3.  $(rs)m = r(sm)$
4.  $1m = m$ .

A **submodule**  $N$  of  $M$  is an abelian group such that for all  $r \in R, n \in N$ ,

$$rn \in N$$

A **homomorphism** of  $R$ -modules  $\varphi : M \rightarrow M'$  is such that

$$\begin{cases} \varphi(m + n) = \varphi(m) + \varphi(n) \\ \varphi(rm) = r\varphi(m) \end{cases}$$

Let  $R = k$  be a field, then  $R$ -modules are called vector spaces over  $k$ .

**Definition 4.14.** Let  $r \in M$  be in the center of  $M$ , then

$$rM = \{rm : m \in M\}$$

is a submodule of  $M$ . If  $I$  is an ideal of  $R$ , then

$$IM = \left\{ \sum_i r_i m_i : r_i \in I, m_i \in M \right\}$$

i.e., generated by  $rm, r \in I$  is a submodule.

**Example 4.6.** If  $R$  is not commutative, then  $R/I$  is not a ring, where  $I$  is a left ideal, but is defined as a left-module. The multiplication given by  $r(a + I) = ra + I$ .

**Definition 4.15.** An  $R$ -algebra is a ring with a ring  $R$  action.

**Theorem 4.4.** Suppose  $\varphi : M \rightarrow M'$  be a surjective  $R$ -module homomorphism, then

$$M' \cong \frac{M}{\ker \varphi}$$

**Proposition 4.11.** Let  $N$  be a submodule of an  $R$ -module  $M$ , and let  $P$  be a submodule of  $M$  containing  $N$ . Then  $P/N$  is a submodule of  $M/N$ , and

$$\frac{M/N}{P/N} \cong \frac{M}{P}$$

**Proposition 4.12.** Let  $N, P$  be submodules, then  $N + P$  is a submodule of  $M$ , and  $N \cap P$  is a submodule of  $P$ , and

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

## 4.2 Free Modules

**Definition 4.16.** Let  $A$  be a set, then

$$F^R(A) \cong R^{\oplus A}$$

where  $F^R(A)$  denotes the free modules over  $A$ . Every element is written as

$$\sum_{a \in A} r_a a$$

(always a finite sum). We say a module  $M = \langle A \rangle$  is finitely generated if  $A$  is finite.

**Example 4.7.** Let  $R = \mathbb{Z}[x_1, \dots, x_n]$ , when  $R$  viewed as a  $R$ -module over itself, it is finitely generated (by 1), by the ideal

$$(x_1, x_2, \dots)$$

as an  $R$ -module, is not finitely generated.

**Definition 4.17 (Noetherian Modules).** An  $R$ -module is Noetherian if every submodule of  $M$  is finitely generated as an  $R$ -module.

**Proposition 4.13.** Let  $M$  be an  $R$ -module,  $N$  be a submodule, then  $M$  is Noetherian iff  $N, M/N$  are both Noetherian.

**Definition 4.18 (finite, finite-type  $R$ -algebra).** Let  $S$  be an  $R$ -algebra, it is called **finite** if it is finitely generated as an  $R$ -module; equivalently,

$$S \cong \frac{R^{\oplus n}}{M}$$

for some submodule  $M$ .

An  $R$ -algebra  $S$  is called **finite-type** if it is finitely generated as an  $R$ -algebra, i.e.,

$$S \cong \frac{R[x_1, \dots, x_n]}{I}$$

for some ideal  $I$ .

Elements in finite  $R$ -algebra is of the form:

$$\sum_{i=1}^n r_i s_i$$

where  $S = \langle s_1, \dots, s_n \rangle$ . Elements in finite-type  $R$ -algebra is of the form:

$$r_{11}s_1 + r_{12}s_1^2 + \dots + r_{21}s_2 + r_{22}s_2^2 + \dots + r_{nk}s_n^k$$

**Proposition 4.14.** The polynomial ring  $R[x]$  is finite-type, not finite.

**Proposition 4.15.** Let  $R$  be a PID, and  $F$  be a finitely generated free module over  $R$ , and let  $M \subset F$  be a submodule, then  $M$  is free.

**Definition 4.19 (???).** Let  $R$  be an integral domain, the rank of  $M$  is the maximal number of linearly independent elements of  $M$ .

**Definition 4.20** (SES, split). A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact iff  $f$  is injective,  $g$  is surjective, and

$$\ker(g) = \operatorname{im}(f)$$

A SES is said to **split** if it is isomorphic in a sense that the following diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong \\ 0 & \longrightarrow & A' & \longrightarrow & A \oplus C & \longrightarrow & C' \longrightarrow 0 \end{array}$$

# Chapter 5

## Ring Theory II

This corresponds to Aluffi Chapter V.

**Proposition 5.1.** Let  $N$  be a submodule of  $M$ , where  $M$  is finitely generated, let  $\langle m_1, \dots, m_k \rangle$  be the elements whose cosets generate  $M/N$ , then

$$M = N + \langle m_1, \dots, m_k \rangle$$

*Proof.* This is the same proof that if  $N, M/N$  are finitely generated, then  $M$  is. □

**Proposition 5.2.** Let  $R$  be commutative, and  $M$  be an  $R$ -module, then TFAE:

1.  $M$  is **Noetherian**.
2.  $M$  satisfies the **ascending chain condition**. (sequence of submodules.)
3. Every nonempty family of submodules has a maximal element with respect to inclusion.

*Proof.* Noetherian implies acc: given  $N_1 \subset N_2 \subset \dots$ , then  $N = \bigcup_i N_i$  is finitely generated. □

**Proposition 5.3 (Hilbert's basis theorem).** Let  $R$  be a Noetherian ring, then  $R[x_1, \dots, x_n]$  is Noetherian. This is the same as If  $R$  is Noetherian, then  $R[x]$  is also Noetherian.

**Proposition 5.4.** Let  $a, b \in R$ , then  $(a) = (b)$  iff  $a = ub$  for some unit  $u$ .

**Definition 5.1 (prime, irreducible elements).** Let  $R$  be commutative

1. Let  $R$  be an integral domain, an element  $a \in R$  is **prime** if the ideal  $(a)$  is prime.
2. An element  $a \in R$  is **irreducible** if  $a$  is not a unit and

$$a = bc$$

implies  $b$  is a unit or  $c$  is a unit. Equivalently,  $a$  is irreducible if  $(a) \subset (b)$  implies  $(b) = (a)$  or  $(b) = (1) = R$ , i.e.,  $(a)$  is maximal in principal ideals.

**Proposition 5.5.** Let  $R$  be an **integral domain**, then

$$\text{nonzero prime elements} \Rightarrow \text{irreducible}$$

**Definition 5.2 (factorization).**  $r \in R$  has a factorization if there exists **finite** irreducibles  $q_1, \dots, q_n$  such that

$$r = q_1 \dots q_n$$

**Proposition 5.6.** Let  $R$  be an integral domain, and let  $r$  be a nonzero, nonunit element of  $R$ . Assume that every ascending chain of principal ideals,

$$(r) \subset (r_1) \subset (r_2) \dots$$

stabilizes. Then  $r$  has a factorization into irreducibles.

Of course if a ring is ACC, then factorizations exist.

**Proposition 5.7.** Factorization exists in Noetherian rings.

**Example 5.1.** A non-Noetherian ring but factorization still exists:

$$\mathbb{Z}[x_1, \dots, x_n]$$

**Proposition 5.8.** Let  $R$  be Noetherian and  $I$  be an ideal, then  $R/I$  is also Noetherian.

## 5.1 UFD, PID, ED

**Definition 5.3 (gcd).** Let  $a, b \in R$ , then the gcd of  $a, b$  is  $d$  such that  $(d)$  is the smallest principal ideal such that

$$(a, b) \subset (d)$$

**Proposition 5.9.** Let  $R$  be a UFD, and  $a, b, c \in R$  be nonzero, then

$$(a) \subset (b) \iff m(b) \subset m(a)$$

where  $m(a)$  is the multiset of irreducible factors of  $a$ . Moreover, the irreducible factors of  $bc$  are the collection of irreducible factors of  $b$  and  $c$ .

**Proposition 5.10.** Let  $R$  be a UFD, then gcd of any  $a, b$  exists.

**Example 5.2.** There exists Noetherian rings that are not UFD.

$$\frac{\mathbb{C}[x, y, z, w]}{(xw - yz)}$$

since  $r = xw = yz$ .

**Proposition 5.11.** In UFD,  $a$  is irreducible implies  $a$  is prime.

*Proof.* Assume  $bc \in (a)$ , then  $(bc) \subset (a)$ , hence the multiset of irreducible factors of  $a$  is contained in the multiset of  $b, c$ , but  $a$  is irreducible implies that  $a$  must be among the factors of  $b$  or  $c$ .  $\square$

**Theorem 5.1.** An integral domain  $R$  is a UFD if and only if

1. The acc holds for principal ideals in  $R$ .
2. Every irreducible element of  $R$  is prime.

**Proposition 5.12.** If  $R$  is a PID, and  $a, b \in R$ , then  $d = \gcd(a, b)$  iff  $(a, b) = (d)$ . In other words, there exists  $r, s \in R$ , such that

$$d = ra + sb$$

**Example 5.3.** UFD but not PID:

$$\mathbb{Z}[x]$$

**Definition 5.4 (Euclidean domain).** A Euclidean valuation on an integral domain  $R$  is an valuation: for all  $a \in R$ , and all nonzero  $b \in R$ , there exists  $q, r$  such that

$$a = qb + r$$

with either  $r = 0$  or  $v(r) < v(b)$ . An integral domain is a ED if it admits a Euclidean valuation.

## 5.2 $R(x)$ and Field of Fractions

**Theorem 5.2.** Let  $R$  be a UFD, then  $R[x]$  is also a UFD.

**Example 5.4.**  $\mathbb{Z}[x], \mathbb{Z}[x_1, \dots, x_n]$  are UFD.

**Definition 5.5 (Field of fractions).** Let  $R$  be an integral domain, then the field of fractions is

$$\text{Frac}(R) = \left\{ \frac{a}{r} : a, r \in R, r \neq 0 \right\}$$

where  $\frac{a}{r}$  is the equivalence given by  $\frac{a}{r} \sim \frac{b}{s} \iff as = br$ .

**Definition 5.6.** The field of fractions  $R(x)$  is the field of rational functions with coefficients in  $R$ : elements are of the form

$$\frac{p(x)}{q(x)}, q(x) \neq 0$$

denoted as  $R(x)$ .



**Definition 5.7 (primitive).** Let  $R$  be a UFD,  $f$  is primitive if and only if  $\gcd(a_0, \dots, a_d) = 1$ .

**Proposition 5.13.** Let  $R$  be a UFD, and  $K$  be its field of fractions, let  $f \in R[x]$  be a nonconstant, irreducible polynomial, then  $f$  is irreducible in  $K[x]$ .

## 5.3 Irreducibility

**Proposition 5.14.** Let  $R$  be an ID, then  $f \in R[x]$  of degree  $d$  can have at most  $d$  roots.

This is not true for non-ID, for example,  $x^2 + 2$  over  $\mathbb{Z}/6\mathbb{Z}$ .

**Proposition 5.15.** Let  $k$  be a field, then  $f \in k[x]$  of degree 2 or 3 is irreducible iff it has no root in  $k$ .

**Example 5.5.**  $t^2 + t + 1$  is irreducible over  $\mathbb{F}_2$  (therefore over  $\mathbb{Q}$ ).

**Proposition 5.16 (rational root theorem).** Let  $R$  be a UFD, and  $K$  be its field of fractions, let

$$f(x) = a_0 + a_1x + \dots + a_nx^n \in R[x]$$

if  $\frac{p}{q} \in K$  is a root, ( $\gcd(p, q) = 1$ ), then

$$p \text{ divides } a_0, q \text{ divides } a_n$$

**Proposition 5.17.** Let  $k$  be a field, and  $f(t) \in k[t]$  be a nonzero irreducible polynomial. Then

$$F = \frac{k[t]}{(f(t))}$$

is a field, where  $k$  embeds into  $F$ . Moreover,  $f(x) \in k[x]$  has a root in  $F$ , which is

$$t + (f(t))$$

**Proposition 5.18.** A field is algebraically closed

- $k$  is algebraically closed  $\iff$  all irreducible polynomials in  $k[x]$  have degree 1
- $\iff$  every nonconstant polynomial  $f$  factors completely into linear factors
- $\iff$  every nonconstant  $f$  has a root in  $k$

**Proposition 5.19.** Finite fields are not algebraically closed. In other words, if a field  $k$  is algebraically closed, then it is infinite.

**Example 5.6.** The nonconstant irreducible polynomials of  $\mathbb{R}[x]$  are precisely those of degree 1 and quadratic  $f = ax^2 + bx + c$  where  $b^2 - 4ac < 0$ .

**Proposition 5.20.** Let  $f \in \mathbb{Z}[x]$  be such that  $\gcd(a_0, \dots, a_n) = 1$ , and let  $p$  be prime. If  $f \bmod p$  has the same degree as  $f$ , and is irreducible over  $\mathbb{F}_p$ , then  $f$  is irreducible over  $\mathbb{Z}$ .



**Warning 5.3.** This is important! We can show a polynomial is irreducible over  $\mathbb{Z}$  by showing it is irreducible over  $\mathbb{F}_p$  for some  $p$ .

**Example 5.7.** There exists reducible polynomial over  $\mathbb{Z}$  but irreducible over  $\mathbb{F}_p$  for every prime  $p$ :  $x^4 + 1$ . (Hint: Legendre symbol).

**Proposition 5.21 (Generalized Eisenstein).** Let  $R$  be a commutative ring, let  $p$  be a prime ideal in  $R$ , let  $f \in R[x]$ , assume that

1.  $a_n \notin p$ .
2.  $a_i \in p$ .
3.  $a_0 \notin p^2$ .

then  $f$  is not the product of polynomials with degree strictly less than  $\deg(f)$ .



**Warning 5.4.** Generalized Eisenstein works for commutative rings! Some examples:

$$\mathbb{C}[x, y], \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1x_2 - x_3x_4)}$$

**Example 5.8.** For all  $n$  and all primes  $p$ , the polynomial  $x^n - p$  is irreducible over  $\mathbb{Z}$ .

**Example 5.9.** Let  $p$  be a prime, then the cyclotomic polynomial  $\Phi_p(x)$  is irreducible.

$$1 + x + x^2 + \dots + x^{p-1}$$

*Proof.*

$$f(x) = \frac{x^p - 1}{x - 1} f(x + 1) = \frac{(x + 1)^p - 1}{x}$$

We see that coefficients are now

$$\binom{p}{k}, k = 1, \dots, p - 1$$

hence  $p$  divides all but leading coefficient. □

## 5.4 CRT

**Theorem 5.5 (CRT).** Let  $I_1, \dots, I_k$  be ideals of  $R$  such that  $I_i + I_j = (1)$  for all  $i \neq j$ . Then

$$\frac{R}{I_1 \cap \dots \cap I_k} = \frac{R}{I_1 I_2 \dots I_k} \cong \frac{R}{I_1} \times \dots \times \frac{R}{I_k}$$

(It uses if  $I_i + I_j = (1)$ , then  $I_1 \dots I_k = I_1 \cap \dots \cap I_k$ ).

**Proposition 5.22 (CRT in PID).** Let  $R$  be a PID, and let  $a_1, \dots, a_k$  be elements such that  $\gcd(a_i, a_j) = 1$ , let  $a = a_1 \dots a_k$ , then

$$\frac{R}{(a)} \cong \frac{R}{(a_1)} \times \dots \times \frac{R}{(a_k)}$$

## Chapter 6

# Linear Algebra I

This corresponds to Aluffi Chapter VI, excluding Section 4-5.

### 6.1 basis, free modules, IBN

**Proposition 6.1 (Zorn's).** Every module  $M$  has maximal linearly independent set. In other words, let  $S \subset M$  be a linearly independent subset. Then there exists a maximal linearly independent subset of  $M$  containing  $S$ .

**Definition 6.1 (basis).** A subset  $S \subset M$  is a basis if it is linearly independent and generates  $M$ . Every element in  $M$  can be written as

$$m = \sum_{s_i \in S} r_i s_i$$

where only finitely many terms are nonzero.

((2)  $\subset \mathbb{Z}$  is maximal but not a basis).

**Proposition 6.2.** Regarding basis,

1. An  $R$ -module  $M$  is free iff it admits a basis. (Any vector space is free as a  $k$ -module).
2. The converse holds when  $R = k$ : let  $B$  be a maximal linearly independent subset of  $M = V$ , then  $B$  is a basis.
3. When  $R = k$ , let  $S$  be a linearly independent subset, then there exists a basis  $B$  of  $V$  containing  $S$ . If  $B$  is a minimal generating set for  $V$ , then  $B$  is also a basis.

**Proposition 6.3.** Let  $R$  be an **integral domain**, and  $M$  a free  $R$ -module, let  $B$  be a maximal linearly independent subset of  $M$ . If  $S$  is any independent subset, then

$$|S| \leq |B|$$

**Example 6.1.** The basis  $\mathbb{C}[x]$  over  $\mathbb{C}$  is  $\{1, x, \dots\}$ , hence an uncountable subset of  $\mathbb{C}[x]$  is necessarily linearly dependent.

**Proposition 6.4.** Let  $R$  be an **integral domain**, let  $m, n$  be nonnegative integers,

$$R^m \cong R^n \iff m = n$$

If  $R$  satisfies the above, we say it satisfies the invariant basis number property. (All commutative rings satisfy this)!

**Definition 6.2 (rank of a module).** Let  $R$  be an integral domain, the rank of a free module  $M$  is the size of the maximal linearly independent subset of  $M$ .

**Proposition 6.5.** Let  $R$  be an integral domain, and let  $M$  be a free  $R$ -module, assume that  $M$  is generated by  $S$ :  $M = \langle S \rangle$ , then  $S$  contains a maximal linearly independent subset of  $M$ .

## 6.2 Homomorphisms $R^n \rightarrow R^m$

**Proposition 6.6.** Let  $\alpha : M \rightarrow N$  be a homomorphism of finitely generated modules, and let  $P$  be a matrix representing it wrt any basis of  $M, N$ , then with respect to any other choice of bases of  $M, N$ ,  $\alpha$  is of the form

$$N_1 \cdot P \cdot N_2$$

where  $N_1, N_2$  are invertible matrices.

**Proposition 6.7.** Two matrices  $P, Q \in M_n(R)$  are equivalent if they are the same up to elementary operations, i.e., iff the same up to multiplications by elementary matrices. In other words,  $M, N$  are equivalent if there exists invertible  $P_1, P_2$  such that

$$M = P_1 N P_2$$

**Example 6.2.** The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

interchanges the second and fourth row of a  $4 \times n$  matrix. Multiplying on the right by

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

adds to the third column of a  $m \times 3$  matrix the  $c$ -multiple of the first column.

**Proposition 6.8.** Let  $k$  be a field, then  $\text{GL}_n(k)$  is generated by elementary matrices!

**Proposition 6.9.** Over a field, every  $m \times n$  matrix is equivalent to a matrix of the form:

$$\left( \begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

In other words, up to multiplying some invertible matrix  $N_1, N_2$  on the left and right, every matrix is of the above form.

**Definition 6.3 (adjoint matrix).** Let  $M$  be an  $n \times n$  matrix, the adjoint matrix  $\text{adj}(A)$  is such that

$$A \cdot \text{adj}(A) = \text{adj}(A)A = \det(A)I_n$$

**Proposition 6.10 (Nakayama's lemma).** (Different versions of the same lemma).

1. Let  $R$  be a commutative ring,  $M$  and  $R$ -module, and let  $a \in R$  be a nilpotent element, then

$$M = 0 \iff aM = M$$

2. Let  $J$  be the Jacobson radical of  $R$ , where  $M$  is finitely generated  $R$ -module. If  $M = JM + N$ , then  $M = N$ . (A special case is when  $R$  is a local ring and  $\mathfrak{m} = J$ ).

# Chapter 7

## Field Theory

Aluffi Chapter VII.

**Proposition 7.1.** Any ring homomorphism from a field to a nonzero ring is injective.

**Definition 7.1 (finite field extension).** A field extension  $k \subset F$  is finite, of degree  $n$ , if  $F$  has finite dimension  $\dim F = n$  as a vector space over  $k$ .

**Definition 7.2 (simple extension).** A field extension  $k \subset F$  is simple if there exists an element  $\alpha \in F$  such that  $F = k(\alpha)$ .

For example, the extension  $\frac{K[t]}{(f(t))} = K(\alpha)$  for some  $f(\alpha) = 0$ .

**Proposition 7.2.** Let  $k \subset k(\alpha)$  be a simple extension, then consider the evaluation map

$$\varepsilon : f(t) \mapsto f(\alpha)$$

Then  $\varepsilon$  is not injective iff  $k(\alpha)$  is a finite extension, i.e., there exists a monic irreducible polynomial  $p$  such that

$$k(\alpha) = \frac{k[t]}{(p(t))}$$

**Definition 7.3.** Let  $k \subset F$  be an extension, then the group of automorphisms of this extension, denoted  $\text{Aut}_k(F)$  is the group of automorphisms  $\varphi : F \rightarrow F$  that fixes  $k$ .

**Corollary 7.1.** Let  $k \subset k(\alpha)$ , and  $p(x)$  be the minimal polynomial over  $k$ , then

$$|\text{Aut}_k(k(\alpha))| = \text{number of distinct roots of } p \text{ in } k(\alpha)$$

and

$$|\text{Aut}_k(k(\alpha))| \leq [k(\alpha) : k]$$

with equality if and only if  $p(x)$  factors over  $k(\alpha)$  as a product of distinct linear factors.

**Proposition 7.3.** Let  $k \subset F$  be finite, then it is also an algebraic extension, where for any  $\alpha \in F$ ,

$$[k(\alpha) : k] \leq [F : k]$$

**Proposition 7.4.** Let  $k \subset E \subset F$  be field extensions, then  $k \subset F$  is finite iff both  $E/k$  and  $F/E$  are finite, in this case

$$[F : k] = [F : E][E : k]$$

**Corollary 7.2.** Let  $k \subset F$  be finite, and  $E$  be an intermediate field, then both  $[E : k], [F : E]$  divide  $[F : k]$ .

**Definition 7.4.** A field ext  $k \subset F$  is finitely generated if there exists  $\{\alpha_i\} \subset F$  such that

$$F = k(\alpha_1) \dots (\alpha_n)$$

**Proposition 7.5.** Let  $k \subset k(\alpha_1, \dots, \alpha_n)$  be finitely generated, then  $k \subset F$  is algebraic implies that  $k \subset F$  is finite.

**Corollary 7.3.** Let  $k \subset F$  be a field extension, then

$$E = \{\alpha \in F : \alpha \text{ is algebraic over } k\}$$

is a field extension over  $k$ .

**Corollary 7.4.** Let  $k \subset E \subset F$ , then  $k \subset F$  is algebraic iff both  $k \subset E$  and  $E \subset F$  are algebraic.

**Definition 7.5.** Let  $f(x) \in k[x]$  be a polynomial of degree  $d$ , the splitting field of  $f$  over  $k$

$$F = k(\alpha_1) \dots (\alpha_d)$$

generated by all roots of  $f$ , i.e., such that  $f$  splits into linear factors over  $F$ .

**Proposition 7.6.** Splitting field of  $f$  is unique up to isomorphisms, and

$$[F : k] \leq (\deg(f))!$$

**Definition 7.6.** A field extension  $k \subset F$  is normal if every irred polynomial  $f$  has a root in  $F$  iff  $f$  splits into product of linear factors over  $F$ .

**Proposition 7.7 (normal).** A field extension  $k \subset F$  is **finite and normal** iff  $F$  is the splitting feild of some polynomial  $f \in k[x]$ .

**Definition 7.7.** Let  $k$  be a field,  $f \in k[x]$  is separable if it has no multiple factors over its splitting field.

**Proposition 7.8.** Let  $f \in k[x]$ , then  $f$  is separable iff  $f, f'$  are relatively prime. If it is inseparable, then  $f' = 0$ .



**Definition 7.8.** Let  $k$  be a field of characteristic  $p$ , the map from  $k \rightarrow k$  such that  $x \mapsto x^p$  is a homomorphism (Frobenius).

A field is perfect if  $\text{char}(k) = 0$  or the Frobenius map is surjective.

**Proposition 7.9.**  $k$  is perfect iff irred polynomial in  $k[x]$  are separable.

**Corollary 7.5.** Finite fields are perfect, i.e., irred polynomials are separable.

## 7.1 Finite fields

**Definition 7.9.** Let  $F$  be a finite field of characteristic  $p$ , then  $F$  is an extension of  $\mathbb{F}_p$ , i.e.,

$$F = \mathbb{F}_{p^d}$$

for some  $d \in \mathbb{Z}^+$ .

**Theorem 7.1.** The polynomial

$$x^{p^d} - x$$

is separable over  $\mathbb{F}_p$ , and the splitting field of  $x^{p^d} - x$  over  $\mathbb{F}_p$  is a field with  $p^d$  elements.

Conversely, let  $F$  be a field with  $p^d$  elements, then  $F$  is the splitting field of

$$x^{p^d} - x$$

over  $\mathbb{F}_p$ .

**Corollary 7.6.** For every  $p^d$  for some  $d$ , there exists only one finite field of order  $p^d$  up to isomorphisms. This is the Galois field of order  $p^d$ .

**Corollary 7.7.**  $\mathbb{F}_{p^d} \subset \mathbb{F}_{p^e}$  iff  $d \mid e$ .

**Corollary 7.8.** Let  $F = \mathbb{F}_{q^n}$ , then

$$x^{q^n} - x$$

factors over  $\mathbb{F}_q$  as irreducible polynomials of degree  $d$ , where  $d$  ranges over all divisors of  $n$ . These polynomials factor completely over  $\mathbb{F}_{q^n}$ .

**Theorem 7.2.**  $\text{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^d})$  is cyclic, generated by the Frobenius isomorphism.

## 7.2 Cyclotomic

**Definition 7.10.** Polynomial

$$\Phi_n(x) = \prod_{i=0}^{n-1} (x - \xi_n^i)$$

is called the  $n$ th cyclotomic polynomial.

**Proposition 7.10.** If  $n = p$  is prime, then

$$\Phi_p(x) = x^{p-1} + \cdots + x + 1 = \frac{x^p - 1}{x - 1}$$

For all positive integers  $n$ , we have

$$x^n - 1 = \prod_{1 \leq d|n} \Phi_d(x)$$

**Proposition 7.11.** For all positive  $n$ ,  $\Phi_n(x) \in \mathbb{Z}[x]$  is irreducible over  $\mathbb{Q}$ .

**Definition 7.11.** The splitting field  $\mathbb{Q}(\zeta_n)$  for  $x^n - 1 \in \mathbb{Q}[x]$  is the  $n$ th cyclotomic field.

**Proposition 7.12.**  $\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\zeta_n))$  is isomorphic to the group  $(\mathbb{Z}/n\mathbb{Z})^\times$

**Proposition 7.13.** An algebraic extension  $k \subset F$  is simple iff the number of distinct intermediate fields  $k \subset E \subset F$  is finite.

**Theorem 7.3.** Every finite separable is simple.

One should draw diagrams

$$k - E - F$$

and

$$\text{Aut}_k(F) - \text{Aut}_E(F) - \{e\}$$

each extension (reversely) corresponds to a subgroup that fixes that extension in the Galois group  $\text{Gal}(F/k)$ .

**Theorem 7.4.** Let  $k \subset F$  be Galois, then  $k \subset E \subset F$ ,  $k \subset E$  is Galois iff  $\text{Aut}_E(F)$  is normal in  $\text{Gal}(F/k)$ , in this case,

$$\text{Gal}(E/k) \cong \frac{\text{Gal}(F/k)}{\text{Gal}(F/E)}$$

**Definition 7.12 (discriminant).** The discriminant of  $f$ , separable, irreducible is

$$D(f) = \Delta^2 f = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$$

**Proposition 7.14.** Let  $k$  be field of char not equal to 2, and  $f$  is separable, with discriminant  $D$ . Then the Galois group of  $f$  is contained in  $A_n$  iff  $D$  is a square in  $k$ .  
(We note that  $\Delta$  is fixed by the Galois group  $G$  iff  $G \subset A_n$ )

**Proposition 7.15.** Let  $f \in \mathbb{Q}[x]$  be irred of degree  $p$ , assume that  $f$  has  $p - 2$  real roots and 2 complex roots, then the Galois group is  $S_p$ .

**Theorem 7.5.** Every finite abelian group is the Galois group of some extension  $F$  over  $\mathbb{Q}$ .

More specifically, every finite abelian group  $G$  is the group of some intermediate field of the extension  $\mathbb{Q} \subset \mathbb{Q}(\xi_n)$  in a cyclotomic field.

*Proof.* Classification:

$$G \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{n_r\mathbb{Z}}$$

Choose distinct  $p_i$  such that  $p_i \equiv 1 \pmod{n_i}$ . Let  $n = p_1 \cdots p_r$ , by CRT

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_r\mathbb{Z})^\times$$

Then  $(\mathbb{Z}/n\mathbb{Z})^\times$  has a subgroup  $H$  such that

$$G \cong \frac{(\mathbb{Z}/n\mathbb{Z})^\times}{H}$$

Since  $(\mathbb{Z}/n\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_n))$ ,  $H$  corresponds to an intermediate field  $F$ , where

$$\mathbb{Q} \subset F \subset \mathbb{Q}(\zeta_n)$$

$H$  is automatically normal, hence  $\mathbb{Q} \subset F$  is Galois and

$$\text{Gal}(F/\mathbb{Q}) = G$$

□

## Chapter 8

# Linear Algebra II

This corresponds to Aluffi Chapter VIII.

## Chapter 9

# Field Theory-Hilbert's Nullstellensatz

This corresponds to Aluffi Chapter VII 2.2-2.3.

**Proposition 9.1.** For a field  $K$ , TFAE:

1.  $K$  is algebraically closed.
2. There is no algebraic extension over  $K$  except for the trivial one.
3. If  $K \subset L$  is any extension, and  $\alpha \in L$  is algebraic over  $K$ , then  $\alpha \in K$ .

**Definition 9.1 (algebraic closure).** An algebraic closure of a field  $k$  is the algebraic extension such that  $\bar{k}$  is algebraically closed.

**Proposition 9.2 (Hilbert's Nullstellensatz).** Recall that if  $K$  is algebraically closed, then every maximal ideal in  $K[x]$  is of the form  $(x - \alpha)$ ,  $\alpha \in K$ .

**Proposition 9.3.** Let  $K$  be algebraically closed, and  $I \subset K[x_1, \dots, x_n]$  be an ideal, then  $I$  is maximal iff

$$I = (x_1 - c_1, \dots, x_n - c_n)$$

for some  $c_1, \dots, c_n \in K$ .

**Proposition 9.4 (normal basis theorem).** Let  $k \subset K$  be a Galois extension of degree  $n$ , let  $\{\sigma_1, \dots, \sigma_n\}$  be the elements of the Galois group, then there exists  $w \in K$  such that

$$\{\sigma_1(w), \dots, \sigma_n(w)\}$$

forms a basis of  $K$  over  $k$ .

## Chapter 10

# Representation Theory of Finite Groups

Let  $k$  be a field and  $G$  be a finite group, a representation  $\rho : G \rightarrow \text{GL}(V)$  is such that

$$\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$$

And  $V$  is a  $k[G]$ -module, i.e., elements in  $k[G]$  are of the form

$$\sum_{g \in G} a_g g$$

and they act on  $V$  by

$$\left( \sum_{g \in G} a_g g \right) \cdot v = \sum_{g \in G} a_g (\rho(g)(v))$$

## **Chapter 11**

# **Semisimple Algebra**