Calc III Section Notes with Answers

Spring 2026

Hui Sun

February 9, 2026

Contents

Chapter 1

The Geometry of Euclidean Spaces

Week 1 (1/19-23)

Logistics

- TA: Hui.
- Email: hsun95@jh.edu.
- Drop-in Hours: Tuesday 2-3 PM, 4-5 pm, Krieger 211.
- Biweekly Quizzes: 10%.
- Attendance: 5%. (If you can't make it, email me).

Definition 1.1 (standard basis of $\mathbb{R}^3$). The following vectors

$$i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$$

are called the **standard basis** vectors of $\mathbb{R}^3$, and for any vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, we can write

$$a = a_1 i + a_2 j + a_3 k$$

Definition 1.2 (dot product). Let $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$, the **dot product** $v \cdot w$ is given by

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Alternatively,

$$v \cdot w = \|v\| \|w\| \cos \theta$$

where

$$\theta = \arccos\left(\frac{v \cdot w}{\|v\| \|w\|}\right)$$

Definition 1.3 (length of vector). Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, the **length** or **norm** of v, denoted as $\|v\|$, is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v \dot{v}}$$

Definition 1.4 (linear combination)**.** Let $v, w \in \mathbb{R}^3$, a **linear combination** of v, w is a sum

$$av + bw$$

for some $a, b \in \mathbb{R}$. One can generalize this definition to n vectors: let $v_1, v_2 \ldots, v_n \in \mathbb{R}^3$, a linear combination of these vectors is a finite sum

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

for some $a_i \in \mathbb{R}, 1 \leq i \leq n$.

Proposition 1.1 (properties of the dot product)**.** Let $a, b, c \in \mathbb{R}^n$, then

(a) Nonnegativity: $a \cdot a \geq 0$, and $a \cdot a = 0$ if and only if $a = 0$.

(b) Scalar multiplication: let $\lambda \in \mathbb{R}$, then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry: $a \cdot b = b \cdot a$.

Problem 1.1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 1.2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 1.3. Can you express $w = (1,2)$ as a linear combination of v_1, v_2 for difference choices of v_1, v_2?

1. $v_1 = (1,1), v_2 = (-2,-2)$.
2. $v_1 = (2,1), v_2 = (-1,0)$.

Proof.

1. We first note that $(1,1), (-2,-2)$ lie on the same line through the origin. Hence, any linear combination of v_1, v_2 will stay in this line, i.e., of the form (a,a), for some $a \in \mathbb{R}$. Therefore, it is impossible to write $w = (1,2)$ as a linear combination of v_1, v_2.

2. Suppose $w = a_1 v_1 + a_2 v_2$ for some $a_1, a_2 \in \mathbb{R}$, then

$$\begin{cases} 2a_1 - a_2 = 1 \\ a_1 = 2 \end{cases} \Rightarrow \begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

Thus we can write w as a linear combination of v_1, v_2:

$$w = 2v_1 + 3v_2$$

□

Problem 1.4. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned} \|u+v\|^2 &= (u+v) \cdot (u+v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

given the assumption, we also have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u \cdot v = 0$$

□

Week 2 (1/26-30)

Topics: determinant, cross product.

Definition 1.5 (determinant). Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, the **determinant** of A is given by

$$\det(A) = ad - bc$$

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ be a 3×3 matrix, the **determinant** of A is given by

$$\det(A) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Definition 1.6 (cross product). Let $a, b \in \mathbb{R}^3$, write $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$, then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Proposition 1.2 (properties of the cross product). We have the following properties regarding the cross product: let $a, b \in \mathbb{R}^3$,

1. $a \times a = 0$.
2. $a \times b = -b \times a$.
3. $(a + b) \times c = a \times c + b \times c$, and $a \times (b + c) = a \times b + a \times c$.
4. $(\alpha a) \times b = \alpha(a \times b)$ for any $a \in \mathbb{R}$.
5. $a \times b$ is perpendicular to vectors a, b.
6. The length of the cross product is the area of the parallelogram spanned by a, b:

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$

   where $0 \leq \theta \leq \pi$ is the angle between them.

7. $a \times b = 0$ iff a, b are parallel or either a or b are 0.
8. The cross product is **not** associative! For example, compute

$$(i \times i) \times j, \quad i \times (i \times j)$$

Proposition 1.3 (determinant and linear combination)**.** Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ be a 3×3 matrix, let

$$a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$$

If any of $a, b,$ or c is a linear combination of the other two vectors, then $\det(A) = 0$. (Relevant topic: linear independence).

Problem 1.5. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 1.6. Compute the determinant of the following matrix A:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

Proof. Notice that the third row vector $(3, 1, 1)$ is the sum of the two row vectors above, hence by Proposition 1.3, we know we must have $\det(A) = 0$. □

Week 2 (Section Activity)

First, introduce yourself to one another:

1. Name.
2. Pronouns.
3. Year.
4. Major.
5. If you could have a noncat, nondog pet, what would it be?

Next, work on and **discuss** the following problems!

Problem 1.7. Let $u = (1, 2, -1)$, find a nonzero vector that is orthogonal to u.

Proof. There are many choices, for example, $v = (1, 0, 1)$. □

Problem 1.8. Let $u = (1, 0), v = (2, -1)$,

1. Using the dot product, what is the angle between u, v? (You do not need to simplify your answer).
2. Can you find a nonzero vector in $\mathbb{R}^2$ that is orthogonal to both of u, v?
3. Can you find nonzero vectors that are orthogonal to any basis of $\mathbb{R}^2$? (Hint: take $u = (1, 0)$, can you draw the set of vectors orthogonal to u? A basis of $\mathbb{R}^2$ contains two noncolinear vectors).

Proof.

1. We have
$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{2}{\sqrt{5}}$$
Hence
$$\theta = \arccos\left(\frac{2}{\sqrt{5}}\right)$$
2. Suppose there exists a nonzero vector $w = (w_1, w_2) \in \mathbb{R}^2$ such that $w \cdot u = w \cdot v = 0$, then this implies
$$w \cdot u = w_1 = 0, \quad w \cdot v = 2w_1 - w_2 = 0 \Rightarrow w_1 = w_2 = 0$$
Thus the only vector orthogonal to u, v is the zero vector, hence the answer is no.
3. Let $\{a, b\}$ be a basis of $\mathbb{R}^2$, where $a = (a_1, a_2), b = (b_1, b_2)$, we know they are noncolinear, hence
$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$$
Then suppose w is orthogonal to both a, b. Then (one can draw a picture), w should be of the form
$$w = (\lambda a_2, -\lambda a_1) = (\zeta b_2, -\zeta b_1)$$
for some $\lambda, \zeta \in \mathbb{R}$. But this implies
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{\zeta}{\lambda}$$
which is a contradiction to the above. This concludes the proof that no nonzero vector can be orthogonal to a basis in $\mathbb{R}^2$.

□

Problem 1.9. Let
$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
Compute the determinant of A.

Proof. We have $\det(A) = 10$. □

Problem 1.10. Construct a nonzero 3×3 matrix A such that $\det(A) = 0$.

Proof.
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$
□

Week 3 (2/2-2/6)

Topics: parametric equations, multivariable functions, and level sets.

Definition 1.7 (matrix addition, multiplication)**.** Let A, B be $m \times n$ matrices, C be $n \times k$ matrix as follows

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \dots & b_{mn} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{n1} & \dots & c_{nk} \end{pmatrix}$$

then **matrix addition** $A + B$ defined as

$$A + B := \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

and **matrix multiplication** is defined as

$$AC = \begin{pmatrix} \sum_{j=1}^n a_{1j}c_{j1} & \dots & \sum_{j=1}^n a_{1j}c_{jk} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_{mj}c_{j1} & \dots & \sum_{j=1}^n a_{mj}c_{jk} \end{pmatrix}$$

Remark. *Given matrices A, B, for the matrix multiplication AB to be well-defined,*

number of columns of A = number of rows of B

Definition 1.8 (matrix as linear transformation)**.** Let A be an $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

and let $x \in \mathbb{R}^n$, where $x = (x_1, \dots, x_n)$. Then A applied to x as a linear transformation is given by

$$Ax = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{pmatrix}$$

where

$$\sum_{j=1}^n a_{1j}x_j = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n, \quad \dots$$

Definition 1.9 (image, graph)**.** The **image** of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in U\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in U\}$$

Definition 1.10 (level set)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, and $c \in \mathbb{R}$ be some constant. Then the **level set** of f at c is the set $\mathcal{L}_c$

$$\mathcal{L}_c \coloneqq \{x \in U : f(x) = c\} \subset \mathbb{R}^n$$

Definition 1.11 (Equation of a line)**.** A **line** l in $\mathbb{R}^3$ through the tip of $a = (a_1, a_2, a_3)$ pointing in the direction of a vector $v = (v_1, v_2, v_3)$ is given by

$$l(t) = a + tv = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$$

where $t \in \mathbb{R}$. Alternatively, a line passing through two points $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$ is given by

$$l(t) = (x(t), y(t), z(t))$$

where

$$\begin{cases} x(t) = x_1 + (x_2 - x_1)t \\ y(t) = y_1 + (y_2 - y_1)t \\ z(t) = z_1 + (z_2 - z_1)t \end{cases}$$

Definition 1.12 (Plane in $\mathbb{R}^3$)**.** If a plane P passes through some point (x_0, y_0, z_0), and $n = (A, B, C)$ is a vector orthogonal to the plane, then the plane P is given by the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Remark. *A point P in the plane and a normal vector to the plane uniquely determine a plane in $\mathbb{R}^3$. Equivalently, three noncolinear points uniquely determine a plane.*

Problem 1.11. Compute the plane containing all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

□

Problem 1.12. (a) Find the equation of the line through $(1, 1, 0)$ in the direction of $2\mathbf{i} - \mathbf{k}$.

(b) Find the equation of the line passing through $(0, 1, 1)$ and $(0, 1, 0)$.

(c) Find an equation for the plane perpendicular to the vector $(-1, 1, -1)$ and passing through the point $(1, 1, 1)$.

Proof. (a) The equation is given by

$$l_1(t) = (1, 1, 0) + t(2, 0, -1), \quad t \in \mathbb{R}$$

(b) The equation is given by

$$l_2(t) = (0, 1, 1) + t(0, 0, -1), \quad t \in \mathbb{R}$$

(c) The equation of the plane is given by

$$(x - 1, y - 1, z - 1) \cdot (-1, 1, -1) = 0$$

simplifying we get

$$x - y + z = 1$$

□

Chapter 2

Differentiation

Week 4 (2/9-2/13)

Topic: limits.

Definition 2.1 (limit). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is open, let x_0 be in A or be a boundary point of A and N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0, f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that

$$\text{if } x \in U, \text{ then } f(x) \in N$$

If f is eventually in N for *any* neighborhood N around b, then the **limit** of f as $x \to x_0$ exists, denoted as

$$\lim_{x \to x_0} f(x) = b$$

Alternatively, if the limit exists, then $\lim_{x \to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x'_1, x'_2, \ldots, x'_n)$ from **all directions**, and $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 2.2 (continuity). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 2.1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 2.1. For the following functions, find their (1) image, (2) graph, (3) **draw their level sets**.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x, y) = x^2 + y^2$.

Proof. 1. Image$(f) = \{x^2 + 1 : x \in \mathbb{R}\}$, and Graph$(f) = \{(x, x^2 + 1) : x \in \mathbb{R}\}$.

2. Image$(g) = \{x^2 + y^2, (x, y) \in \mathbb{R}^2\}$, and Graph$(g) = \{(x, y, x^2 + y^2) : (x, y) \in \mathbb{R}^2\}$.

$\square$

Problem 2.2. Compute the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$$

   (Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$).

2. $$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}$$

3. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2 + y^2}$$

Proof. 1. Following the hint, we see

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = \lim_{x\to 0} x = 0$$

2. This one uses the exact same trick:

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{xy} \cdot y = 0$$

3. First letting $x \to 0$ along $y = 0$, we see the limit is 1; letting $x = y \to 0$, we see the limit is 0, thus the limit doesn't exist!

$\square$

Problem 2.3. Compute the limit of the following functions:

1. $$\lim_{(x,y)\to(0,0)} \frac{x}{x + y}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x + y}$$

   (Hint: try considering $y = x^2 - x$ and $y = x$)

3. $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x + y}$$

Proof. 1. First fix $x = 0$, let $y \to 0$, then the limit is 0; now fix $y = 0$, let $x \to 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \to 0, y \to 0$), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x + y} = \lim_{x\to 0} \frac{x^3 - x^2}{x^2} = \lim_{x\to 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

Warning 2.1. 2 does not follow from 1! A student suggests a proof: $\lim_{(x,y)\to(0,0)} = \frac{x}{x+y} \cdot y$, and by 1, the limit $\frac{x}{x+y}$ doesn't exist, this implies the limit of $\frac{xy}{x+y}$ also doesn't exist. This argument is not correct! Consider the following counterexample: $\lim_{y\to 0} \frac{1}{y}$ doesn't exist, but the limit

$$\lim_{y\to 0} \frac{1}{y} \cdot y = 1$$

exists! More concretely, if you multiply by any function that doesn't tend to 0, the argument follows, but it doesn't work when the function tends to 0!

3. We see that
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} \frac{xy}{x+y}$$
Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist!

□

Idea 2.2. How to find a a limit $\lim_{x\to x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.