Calc III Section Notes with Answers

Spring 2026

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Contents

Chapter 1

The Geometry of Euclidean Spaces

Week 1 (1/19-23)

Logistics

- TA: Hui.
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- Office Hours: Tuesday 4-6 PM, Krieger 211; Friday 1-2 PM Zoom.
- Biweekly Quizzes: 15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Definition 1 (standard basis of $\mathbb{R}^3$)**.** The following vectors

$$i = (1,0,0), j = (0,1,0), k = (0,0,1)$$

are called the **standard basis** vectors of $\mathbb{R}^3$, and for any vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, we can write

$$a = a_1 i + a_2 j + a_3 k$$

Definition 2 (dot product)**.** Let $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$, the **dot product** $v \cdot w$ is given by

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Alternatively,

$$v \cdot w = \|v\|\|w\| \cos\theta$$

where

$$\theta = \arccos\left(\frac{v \cdot w}{\|v\|\|w\|}\right)$$

Definition 3 (length of vector)**.** Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, the **length** or **norm** of v, denoted as $\|v\|$, is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v\dot{v}}$$

Definition 4 (linear combination)**.** Let $v, w \in \mathbb{R}^3$, a **linear combination** of v, w is a sum

$$av + bw$$

for some $a, b \in \mathbb{R}$. One can generalize this definition to n vectors: let $v_1, v_2 \dots, v_n \in \mathbb{R}^3$, a linear combination of these vectors is a finite sum

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

for some $a_i \in \mathbb{R}, 1 \le i \le n$.

Proposition 1 (properties of the dot product)**.** Let $a, b, c \in \mathbb{R}^n$, then

(a) Nonnegativity: $a \cdot a \geq 0$, and $a \cdot a = 0$ if and only if $a = 0$.

(b) Scalar multiplication: let $\lambda \in \mathbb{R}$, then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry: $a \cdot b = b \cdot a$.

Problem 1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 3. Can you express $w = (1, 0)$ as a linear combination of v_1, v_2 for difference choices of v_1, v_2?

1. $v_1 = (1, 1), v_2 = (-2, -2)$.
2. $v_1 = (2, 1), v_2 = (-1, 0)$.

Week 2 (1/26-30)

Definition 5 (cross product)**.** Let $a, b \in \mathbb{R}^3$, write $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$, then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Proposition 2 (properties of the cross product)**.** We have the following properties regarding the cross product: let $a, b \in \mathbb{R}^3$,

1. $a \times a = 0$.
2. $a \times b = -b \times a$.
3. $(a + b) \times c = a \times c + b \times c$, and $a \times (b + c) = a \times b + a \times c$.
4. $(\alpha a) \times b = \alpha(a \times b)$ for any $a \in \mathbb{R}$.
5. $a \times b$ is perpendicular to vectors a, b.
6. The length of the cross product is the area of the parallelogram:

   $$\|a \times b\| = \|a\|\|b\| \sin\theta$$

   where $0 \leq \theta \leq \pi$ is the angle between them.
7. $a \times b = 0$ iff a, b are parallel or either a or b are 0.
8. The cross product is **not** associative! For example, compute

   $$(i \times i) \times j, \quad i \times (i \times j)$$