Algebraic Topology

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Chapter 1

Category Theory

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Definition 1.1 (Category). A category $\mathcal{C}$ consists of the following data:

1. A collection of objects denoted as Ob($\mathcal{C}$)
2. Given two objects $X, Y \in$ Ob($\mathcal{C}$), a collection of morphisms between X, Y, $f : X \to Y$, denoted as mor$_{\mathcal{C}}(X, Y)$.
3. (Composition) We have mor$_{\mathcal{C}}(X, Y) \times mor_{\mathcal{C}}(Y, Z) \to mor_{\mathcal{C}}(X, Z)$ that satisfies associativity
$$f \circ (g \circ h) = (f \circ g) \circ h$$
4. (Identity) There is a distinguished morphism for each X, $Id_{\mathcal{C}}(X, X)$ such that given any $f \in mor(X, Y)$, we have $f \circ id_X = id_Y \circ f = f$.

In this course, we will make the assumption that in all the categories that we work with, Ob($\mathcal{C}$) need not be a set, but given any $X, Y \in Ob(\mathcal{C})$, mor(X, Y) will always be a set. Now we talka bout some examples of categories.

Example 1.1 (Sets). Let Ob($Sets$) be all the sets in the universe. Given X, Y sets, mor(X, Y) be all the set maps from X to Y, and id_X is the identity map.

Example 1.2 (Top). Let Ob(Top) be all the topological spaces, and mor(X, Y) be all the continuous maps from X to Y.

Example 1.3 (Vect$_{\mathbb{F}}$). Let $\mathbb{F}$ be a field, and let Ob be all the $\mathbb{F}$-vector spaces. Then mor(V, W) is all the $\mathbb{F}$-linear homomorphisms from V to W, where id_V is the identity homomorphism.

Example 1.4 (Posets). Fix a poset P, let Ob(P) be the collection of elements in P, and given p, q we define
$$mor(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \varnothing, & \text{otherwise} \end{cases}$$

Problem 1.1. **HW(Q1): check this is a category**

Example 1.5 (Opposite category). Given a category $\mathcal{C}$, there is another category called the opposite category, denoted as $\mathcal{C}^{op}$, where

1. The objects are the same as $\mathcal{C}$
2. Given $X, Y \in \text{Ob}(C^{op})$, we have $\text{mor}_{op}(X, Y) := \text{mor}_{\mathcal{C}}(Y, X)$.
3. Moreover, given $f \in mor_{op}(X, Y), g \in mor_{op}(Y, Z)$, then $g \circ f$ in C^{op} is $f \circ g : Z \to X$.

Naturally, we define isomorphisms now.

Definition 1.2 (isomorphism). Given a category $\mathcal{C}$, and a morphism $f \in mor_C(X, Y)$, we say f is an isomorphism if there exists $g \in mor_C(Y, X)$ such that

$$f \circ g = Id_Y, g \circ f = Id_X$$

Now we introduce maps between categories.

Definition 1.3 (functor). Given categories $\mathcal{C}, \mathcal{D}$, a functor $F : C \to D$ is the following;

1. Given an object X in $\mathcal{C}$, $F(X)$ is an object in D.
2. Given a morphism $f : X \to Y$, $F(f)$ is a functor $F(f) : F(X) \to F(Y)$. Moreover, it satisfies the following:
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(f \circ g) = F(f) \circ F(g)$. Alternatively, we can rewrite this condition as the following:

$$\begin{tikzcd}
mor(X,Y) \times mor(Y,Z) \arrow[r] \arrow[d, "mor(F)\times mor(F)"] & mor(X,Z) \arrow[d, "mor(F)"] \\
mor(F(X),F(Y)) \times mor(F(Y),F(Z)) \arrow[r] & mor(F(X),F(Z))
\end{tikzcd}$$

   such that this diagram commutes.

Problem 1.2. **HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

Example 1.6. $F : Top \to Set$, where $X \mapsto X$, where the latter is a set, and $f \mapsto f$ as set maps.

Example 1.7. Let $\mathbb{F}$ be a field, and $F : Sets \to \text{Vect}_{\mathbb{F}}$, where $X \mapsto \mathbb{F}\langle X \rangle$, where $\mathbb{F}\langle X \rangle$ is the free vector space over $\mathbb{F}$ on the set X.

Problem 1.3. **HW(Q3): extend this to a functor by defining $mor(f)$ and show this is a functor.**

Example 1.8. Let $\mathbb{F}$ be a field, then the following is a functor, $F : Sets^{op} \to \text{Vect}_{\mathbb{F}}$, where

$$hF : X \mapsto Maps(X, \mathbb{F})$$

Problem 1.4. **HW(Q4)**: show this extends to a functor by defining $F(f)$, and show it is a functor.

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Definition 1.4 (contravariant functor). Let $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor from $\mathcal{C}^{op} \to \mathcal{D}$, (equivalently, $\mathcal{C} \to \mathcal{D}^{op}$).

Problem 1.5. **HW(Q5):** Show that the following functor F from $\text{Vect}_{\mathbb{F}}$ to $\text{Vect}_{\mathbb{F}}$ extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = Hom(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let Cat be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

Definition 1.5 (natural transformation). Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation T from F to G is the following: $T : F \Rightarrow G$:

1. given object $X \in Ob(\mathcal{C})$, $T(X) \in mor(F(X), G(X))$
2. Given $f \in mor(X, Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T(X)"'] & F(Y) \arrow[d, "T(Y)"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

   where mor_F, mor_G is the identification function on morphisms by functors F, G

If for all X, $T(X)$ is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor F and turn it to another functor G. We will (in a homework) show there exists natural transofrmation between the following two functors.

Example 1.9. Consider $F, G : Vect_{\mathbb{F}} \to Vect_{\mathbb{F}}$, define

$$F(V) = V \otimes_{\mathbb{F}} V/_{\langle a \otimes b - b \otimes a \rangle} = V \otimes_{\mathbb{F}} V/\Sigma_2, G(V) = (V \otimes_F V)^{\Sigma_2} = \{\alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha\}$$

Both are vector spaces are fixed under "swaps." Then a natural transformation can be defined as follows $T(V)$:

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

Problem 1.6. **HW(Q6):** For the above F, G

1. Show that T defines a natural transformation from F to G.
2. Find conditions on $\mathbb{F}$ for T being a natural isomorphism.

Next we define limits and colimits. Let $\mathcal{C}, \mathcal{D}$ be categories, d be an object in $\mathcal{D}$, then we can define a functor $F_d : \mathcal{C} \to \mathcal{D}$ such that for any object c in $\mathcal{C}$,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the "constant functor" on $\mathcal{D}$, i.e., every object is sent to d, and every morphism is sent to id_d.

Definition 1.6 (colimit). Given any functor $F : \mathcal{C} \to \mathcal{D}$, the colimit of F, denoted as $\text{colim}(F)$ is an object in $\mathcal{D}$ endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object d in D and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in $\mathcal{D}$, $f : \text{colim}(F) \to d$ making the following diagram commute: for any X, Y, g:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(g)"] \arrow[dr, "\varphi_F"] \arrow[ddr, bend right, "\varphi"'] & & F(Y) \arrow[dl, "\varphi_F"'] \arrow[ddl, bend left, "\varphi"] \\
& \text{colim}(F) \arrow[d, "f"] & \\
& d &
\end{tikzcd}$$

Next we prove some facts about colimits and give an example, where $\text{colim}(F)$ exists.

Proposition 1.1. If $\text{colim}F$ exists, then $\text{colim}F$ is unique up to isomorphisms.

Proof. Let $\text{colim}(F), \text{colim}(F)'$ be two colimits that satisfy the criteria. They are both objects in $\mathcal{D}$, then we get a morphism $f : \text{colim}(F) \to \text{colim}(F)'$, and likewise $g : \text{colim}(F) \to \text{colim}(G)'$, then

$$f \circ g : \text{colim}(F)' \to \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for $g \circ f$. □

Next we demonstrate a fact via an example.

Theorem 1.1. Let $\mathcal{C}$ be a category where $Ob(\mathcal{C}), mor(X, Y)$ are all sets. Let $F : \mathcal{C} \to \text{Top}$ be any functor, then $\text{colim}(F)$ exists.

Proof. Define $\text{colim}(F) := \bigsqcup_c F(c)/\sim$, where $\sim$ is induced by the equivalence relation given by

$$y \sim F(f)y$$

where $y \in F(C_1), f : C_1 \to C_2, F(f)x \in F(C_2)$. The natural transformation we endow on F as $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$:

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(C)} F(C)/\sim$$

□

Problem 1.7. **HW(Q7):** Show that colim$(F), \varphi_F$ is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

Definition 1.7 (limit). Given a functor $F : \mathcal{C} \to \mathcal{D}$, the limit of F, denoted as $\lim(F)$ is an object of $\mathcal{D}$, endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object $d \in Ob(\mathcal{D})$ and a natural transformation

$$\varphi : F_d \to F$$

there exists a unique $f : \lim F \to d$ such that the following diagram commutes:

$$\begin{tikzcd}
& \lim F \arrow[d, "f"] \arrow[ddl, bend right, "\varphi_F"'] \arrow[ddr, bend left, "\varphi_F"] & \\
& d \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
F(X) \arrow[rr, "F(g)"'] & & F(Y)
\end{tikzcd}$$

Just like colimits, limits are unique up to isomorphisms.

Problem 1.8. **HW(Q8):** Given $F : \mathcal{C} \to \mathcal{D}$, consider $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

1.3 Lecture 3 9/4

Today we define (co)chain complexes: let R be a commutative ring, let Mod_R denote the category of R-modules and R-module maps.

Definition 1.8 (chain complex). A chain complex of R-modules is a collection of R-modules and R-modules maps

$$\cdots \to M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all i. In other words, the image of previous map is contained in the kernal of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by $\{M.; \partial.^M\}$.

Next we introduce morphisms between chain complexes.

Definition 1.9 (morphism between complexes). Let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$, a morphism $\{f.\}$ between chain complexes is a "ladder" such that the following commutes:

$$\ldots \longrightarrow M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \longrightarrow \ldots$$

$$\ldots \longrightarrow N_{i+1} \xrightarrow{\partial^N_{i+1}} N_i \xrightarrow{\partial^N_i} N_{i-1} \longrightarrow \ldots$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where $\{g.\} : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, and $\{f.\} : \{N.; \partial.^N\} \to \{L.; \partial.^L\}$, which is simply vertical stacking.

Problem 1.9. **HW(Q9):** Prove that chain complexes of R-modules form a category ch_R.

There are interesting functors $F : \text{ch}_R \to Mod_R$, and we begin with the following one:

Definition 1.10 (H_n, nth-homology). Given $n \in \mathbb{Z}$, there is a functor

$$H_n : \text{ch}_R \to Mod_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial^M_n / Im \partial^M_{n+1}$$

and for $f : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, we define: $H_n(f) : H_n(\{M.; \partial.^M\}) \to H_n(\{N.; \partial.^N\})$,

$$H_n(f)[x] := [f_n(x)]$$

where $[x] \in H_n(\{M.; \partial.^M\})$.