

Additive Combinatorics

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Chapter 1

Probabilistic Methods

1.1 First Moment Method

The first moment method upperbounds the probability of events occurring using expected value.



Warning 1.1. The first moment method does not give a lower bound.

If we want to show A contains a subset B that satisfies property \mathcal{P} , then it suffices to show that a randomly chosen subset B satisfies \mathcal{P} with positive probability.

Definition 1.1 (expected value, variance). Let X be a real-valued random variable with discrete support, then its expected value is

$$\mathbb{E}(X) = \sum_x xP(X)$$

And its variance is given by

$$\text{Var}(X) = \mathbb{E}(|X - \mathbb{E}(X)|^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

If we let $I(E)$ denote the indicator function on E , $I(E) = \begin{cases} 1, E \text{ happens} \\ 0, E \text{ doesn't happen} \end{cases}$ Then $\mathbb{E}(I(E)) = P(E)$.

Let F be an event with nonzero probability.

Definition 1.2 (conditional probability and expected value). The conditional probability of event E with respect to F is

$$P(E|F) = \frac{P(E \wedge F)}{P(F)}$$

We define conditional expected value as follows

$$\mathbb{E}(X|F) = \frac{\mathbb{E}(XI(F))}{\mathbb{E}(I(F))} = \sum_x xP(X = x|F)$$

A random variable is called Boolean if it only takes values $\{0, 1\}$, equivalently, it is the indicator function of some event E .

Proposition 1.1 (Borel-Cantellilemma). Let E_1, \dots, E_n be a sequence of events (possibly infinite), such that $\sum_n P(E_n) < \infty$, then

$$P(\text{fewer than } M \text{ events happen}) \geq 1 - \frac{\sum_n P(E_n)}{M}$$

Proof. By moving terms, we show the probability of more than M events happen is $\leq \frac{\sum_n P(E_n)}{M}$. By Markov's inequality,

$$P\left(\sum_n I(E_n) \geq M\right) \leq \frac{\mathbb{E}(\sum_n I(E_n))}{M} = \frac{\sum_n P(E_n)}{M}$$

Note that in the last equality, we need the assumption there are finitely many events, which we can do by monotone convergence $\sum_n P(E_n) < \infty$. \square

1.2 Second Moment Method

The second moment method tells us that X cannot deviate from the expected value $\mathbb{E}(X)$ too much, and these tools are known as large deviation inequalities.

Theorem 1.2 (Chebyshev's inequality). Let X be a random variable, then for any $\lambda > 0$, we have

$$P(|X - \mathbb{E}(X)| > \lambda \text{Var}(X)^{1/2}) \leq \frac{1}{\lambda^2}$$

Proof. If $\text{Var}(X) = 0$, then $X = \mathbb{E}(X)$, hence the inequality satisfies. Let $\text{Var}(X) > 0$, we note

$$P(|X - \mathbb{E}(X)| > \lambda \text{Var}(X)^{1/2}) = P(|X - \mathbb{E}(X)|^2 > \lambda^2 \text{Var}(X))$$

And by Markov's inequality, we have

$$P(|X - \mathbb{E}(X)|^2 > \lambda^2 \text{Var}(X)) \leq \frac{\mathbb{E}(|X - \mathbb{E}(X)|^2)}{\lambda^2 \text{Var}(X)} = \frac{1}{\lambda^2}$$

\square

From this, we know that $X = \mathbb{E}(X) + O(\text{Var}(X)^{1/2})$ with high probability. And $\text{Var}(X) = \mathbb{E}(|X - \mathbb{E}(X)|^2)$, then we know $|X - \mathbb{E}(X)|^2 \geq \text{Var}(X)$ with positive probability. The tools using variance is the second moment method.



Warning 1.3. The second moment method gives two-sided bounds on how X distributes, i.e., the two tails decay at $1/\lambda^2$.

Recall the linearity of expected value is as follows:

$$\mathbb{E}\left(\sum_n X_n\right) = \sum_n \mathbb{E}(X_n)$$

But this is not true for variance in general! If $\{X_j\}$ are pairwise independent random variables, then we have

$$\text{Var}\left(\sum_n X_n\right) = \sum_n \text{Var}(X_n)$$

Lemma 1.1. Let X_j be random variables, then we have

$$\text{Var} \left(\sum_n X_n \right) = \sum_n \text{Var}(X_n) + \sum_{i,j, i \neq j} \text{Cov}(X_i, X_j)$$

where $\text{Cov}(X_i, X_j) = \mathbb{E}(X_i X_j) - \mathbb{E}(X_i)\mathbb{E}(X_j)$.

Let's do an example.

Example 1.1. Let B be a randomly chosen set in A , then we have

$$\text{Var}(|B|) = \sum_a P(a \in B) - P(a \in B)^2$$