Calc III Sections

Fall 2025

Hui Sun

September 16, 2025

Calc III-Week 4 (9/15-9/19)

Topics: (1) Partial derivatives. (2)

Definition 1 (graph)**.** The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 2 (limit)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, ($A$ is open), let N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0 ($x_0 \in \bar{A}$,) f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that whenever $x \in U$, then $f(x) \in N$ as well.

The **limit** of f as $x \to x_0$, if it exists, is $\lim_{x\to x_0} f(x) := b \in \mathbb{R}^m$ such that f is eventually in N, for every neighborhood N of b.

Definition 3 (limit')**.** $\lim_{x\to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x_1', x_2', \ldots, x_n')$ from **all directions**, $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 4 (continuity)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 1. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x) = x^2 + y^2$.

Problem 2. Compute the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$$
   (Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$).
2. $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

Proof. 1. Following the hint, we see

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = \lim_{x\to 0} x = 0$$

2. This one uses the exact same trick:

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} \cdot y = 0$$

3. First letting $x \to 0$ along $y = 0$, we see the limit is 1; letting $x = y \to 0$, we see the limit is 0, thus the limit doesn't exist!

□

Problem 3. Compute the limit of the following functions:

1. $$\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$
   (Hint: try considering $y = x^2 - x$ and $y = x$)
3. $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$$

Proof. 1. First fix $x = 0$, let $y \to 0$, then the limit is 0; now fix $y = 0$, let $x \to 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \to 0, y \to 0$), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3 - x^2}{x^2} = \lim_{x\to 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

3. We see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}\frac{xy}{x+y}$$

Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist!

$\square$

How to find a a limit $\lim_{x\to x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.