

# Calc III Midterm Essay Review

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# Chapter 1

## Definition review

**Definition 1.1** (standard basis in  $\mathbb{R}^3$ ). The vectors

$$i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$$

are called the **standard basis** vectors of  $\mathbb{R}^3$ , and for any vector  $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ , we can write

$$a = a_1i + a_2j + a_3k$$

**Definition 1.2** (Equation of a line). A **line**  $l$  in  $\mathbb{R}^3$  through the tip of  $a = (a_1, a_2, a_3)$  pointing in the direction of a vector  $v = (v_1, v_2, v_3)$  is given by

$$l(t) = a + tv = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$$

where  $t \in \mathbb{R}$ . Alternatively, a line passing through two points  $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$  is given by

$$l(t) = (x(t), y(t), z(t))$$

where

$$\begin{cases} x(t) = x_1 + (x_2 - x_1)t \\ y(t) = y_1 + (y_2 - y_1)t \\ z(t) = z_1 + (z_2 - z_1)t \end{cases}$$

**Definition 1.3** (inner product, dot product). Let  $a, b \in \mathbb{R}^3$ , the **dot product**, also called the inner product, of  $a, b$  is

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

where  $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ . The **norm**, also called the length, of  $a$  is

$$\|a\| = (a \cdot a)^{\frac{1}{2}}$$

A vector of norm 1 is called a **unit vector**. Given any  $u \in \mathbb{R}^3$ , we can find the unit vector  $\frac{u}{\|u\|}$  pointing in the same direction as  $u$ , this is called “normalizing”  $u$ .

**Definition 1.4** (orthogonal projection). The **orthogonal projection** of vector  $v$  onto another vector  $a$  is

$$\text{Proj}_a v = \frac{a \cdot v}{a \cdot a} a$$

For example, the orthogonal projection of  $(1, 1, 0)$  onto  $(1, 1, 1)$  is

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

**Definition 1.5** (orthogonal). Let  $a, b \in \mathbb{R}^n$ , then  $a, b$  are called **orthogonal** or perpendicular iff

$$a \cdot b = 0$$

**Definition 1.6** (determinant). The **determinant** of a  $2 \times 2$  matrix is given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and the determinant of a  $3 \times 3$  matrix is given by

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

**Definition 1.7** (cross product). Let  $a, b \in \mathbb{R}^3$ , write  $a = (a_1, a_2, a_3)$ ,  $b = (b_1, b_2, b_3)$ , then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where  $i, j, k$  are the standard vectors in  $\mathbb{R}^3$ .

**Definition 1.8** (Plane in  $\mathbb{R}^3$ ). If a plane  $P$  passes through some point  $(x_0, y_0, z_0)$ , and  $n = (A, B, C)$  is a vector orthogonal to the plane, then the plane  $P$  is given by the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

(Notice that a point in  $P$  and a normal vector to  $P$  uniquely define a plane in  $\mathbb{R}^3$ .)

**Definition 1.9** (image, graph). The **image** of a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a subset of  $\mathbb{R}^m$ ,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in U\}$$

and the **graph** of  $f$  is a subset of  $\mathbb{R}^{n+m}$ ,

$$\text{Graph}(f) = \{(x, f(x)) : x \in U\}$$

**Definition 1.10** (level set). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $c \in \mathbb{R}$  be some constant. Then the **level set** of  $f$  at  $c$  is the set

$$\{x \in U : f(x) = c\} \subset \mathbb{R}^n$$

**Definition 1.11** (open set, closed set, neighborhood, boundary). Let  $U \subset \mathbb{R}^n$ , we say  $U$  is an **open set** if for every  $x_0 \in U$ , there exists some  $r > 0$  such that  $D_r(x_0) \subset U$ , where  $D_r(x_0)$  is the open disk of radius  $r$  centered at  $x_0$ :

$$D_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$$

Some examples of open sets:  $\mathbb{R}$ ,  $D_1((0,0))$ ,  $(1,2) \subset \mathbb{R}$ . A **neighborhood** of  $x_0 \in \mathbb{R}^n$  is an open set containing the point  $x_0$ . A point  $x \in \mathbb{R}^n$  is called a **boundary point** of  $A$  if *every* neighborhood of  $x$  contains at least one point in  $A$  and at least one point not in  $A$ . A set is **closed** if it contains all its boundary points. Example of closed set: level sets of a continuous function  $f$ .

**Definition 1.12** (limit). Let  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , where  $A$  is open, let  $x_0$  be in  $A$  or be a boundary point of  $A$  and  $N$  be a neighborhood of a point  $b \in \mathbb{R}^m$ . Now let  $x$  approach  $x_0$ ,  $f$  is said to be **eventually in**  $N$  if there exists a neighborhood  $U$  of  $x_0$  such that

$$\text{if } x \in U, \text{ then } f(x) \in N$$

If  $f$  is eventually in  $N$  for *any* neighborhood  $N$  around  $b$ , then the **limit** of  $f$  as  $x \rightarrow x_0$  exists, denoted as

$$\lim_{x \rightarrow x_0} f(x) = b$$

**Definition 1.13** (continuous). Let  $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $x_0 \in A$ , then  $f$  is **continuous at**  $x_0$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

**Definition 1.14** (partial derivative). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $U$  is open. Then the **partial derivative** with respect to  $x_i$  is defined by

$$\frac{\partial f}{\partial x_i}(x_1, \dots, x_n) = \lim_{h \rightarrow 0} \frac{f(x + he_i) - f(x)}{h}$$

where  $e_i = (0, \dots, 1, \dots, 0)$  with 1 in the  $i$ th coordinate.

**Definition 1.15** (differentiability in two variables). Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , then  $f$  is **differentiable** at  $(x_0, y_0)$  if

(1)  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$  exist at  $(x_0, y_0)$

(2)

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[ \frac{\partial f}{\partial x}(x_0,y_0) \right] (x - x_0) - \left[ \frac{\partial f}{\partial y}(x_0,y_0) \right] (y - y_0)}{\|(x,y) - (x_0,y_0)\|} = 0$$

The derivative of  $f$  at  $(x_0, y_0)$  is the  $1 \times 2$  matrix

$$\left[ \frac{\partial f}{\partial x}(x_0, y_0) \quad \frac{\partial f}{\partial y}(x_0, y_0) \right]$$

Moreover, the **tangent plane** of the graph of  $f$  at  $(x_0, y_0, f(x_0, y_0))$  is given by

$$z = f(x_0, y_0) + \left[ \frac{\partial f}{\partial x}(x_0, y_0) \right] (x - x_0) + \left[ \frac{\partial f}{\partial y}(x_0, y_0) \right] (y - y_0)$$

**Definition 1.16** (differentiability in the general setting). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , then  $f$  is differentiable at  $x_0 \in U$  if

- (1) the partial derivatives  $\frac{\partial f_i}{\partial x_j}$  exist for all  $1 \leq i \leq m, 1 \leq j \leq n$ .  
 (2)

$$\lim_{x \rightarrow x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

where  $T = Df(x_0)$  is the  $m \times n$  matrix

$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

The derivative of  $f$  at  $x_0$  is the  $m \times n$  matrix  $Df(x_0)$ .

**Definition 1.17** (gradient). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , the **gradient**  $\nabla f(x)$  is a special case of the general case above when  $m = 1$ , i.e., it is a  $1 \times n$  matrix

$$Df(x) = \left[ \frac{\partial f}{\partial x_1} \quad \cdots \quad \frac{\partial f}{\partial x_n} \right]$$

**Definition 1.18** (path and curve). A **path** in  $\mathbb{R}^n$  is a map  $c : [a, b] \rightarrow \mathbb{R}^n$ , and the image of  $c$  is called a **curve**. We say the path  $c$  parametrizes the curve.

For example,  $c(t) = (\cos t, \sin t)$  is a path, and the unit circle is a curve.

**Definition 1.19** (velocity of a path). Let  $c : [a, b] \rightarrow \mathbb{R}^n$  be a path, and we can write  $c(t) = (c_1(t), \dots, c_n(t))$ . If  $c$  is differentiable, then we define the **velocity** of  $c$  at any  $t_0 \in [a, b]$  as

$$c'(t_0) = (c'_1(t_0), \dots, c'_n(t_0))$$

The velocity vector of  $c$  at  $t_0$  is also a **tangent** vector to  $c$  at  $t_0$ . The **speed** of the path  $c$  at  $t_0$  is the length of the velocity vector  $\|c'(t_0)\|$ .

**Definition 1.20** (tangent line to a path). Let  $c : [a, b] \rightarrow \mathbb{R}^n$  be a path, if  $c'(t_0) \neq 0$ , then the **tangent line** at  $x_0$  is given by

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

**Definition 1.21** (directional derivative). Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , be differentiable, then the **directional derivative** at  $x_0 \in \mathbb{R}^3$  in the direction of a *unit vector*  $v$  is given by

$$\nabla f(x_0) \cdot v = \left[ \frac{\partial f}{\partial x_1}(x_0) \right] v_1 + \left[ \frac{\partial f}{\partial x_2}(x_0) \right] v_2 + \left[ \frac{\partial f}{\partial x_3}(x_0) \right] v_3$$

where  $v = (v_1, v_2, v_3)$ .



**Warning 1.1.** Make sure you normalize any given direction  $v$ ! This formula works for unit vectors.

**Definition 1.22** (First order Taylor expansion). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable at  $a \in U$ , then

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + R_1(a, x)$$

where

$$\frac{R_1(a, x)}{\|x - a\|} \rightarrow 0 \text{ as } x \rightarrow a$$

**Definition 1.23** (Second order Taylor expansion). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be twice continuously differentiable at  $a \in U$ , then

$$f(x) = f(a) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + R_2(a, x)$$

where

$$\frac{R_2(a, x)}{\|x - a\|} \rightarrow 0 \text{ as } x \rightarrow a$$

**Definition 1.24** (critical point). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , a point  $x_0 \in U$  is a **critical point** of  $f$  if either  $f$  is not differentiable at  $x_0$ , or  $Df(x_0) = 0$ . A critical point that is not a local extremum is called a saddle point.

**Definition 1.25** (quadratic function). A function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is called a **quadratic function** if it is given by

$$g(h_1, \dots, h_n) = \sum_{i,j=1}^n a_{ij} h_i h_j$$

where  $(a_{ij})$  is an  $n \times n$  matrix. We can also write  $g$  as follows:

$$g(h_1, \dots, h_n) = [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

**Definition 1.26** (Hessian). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , and suppose all the second-order partial derivatives  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  exist, then the Hessian matrix of  $f$  is the  $n \times n$  matrix given by

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

The Hessian as a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} [h_1 \quad \dots \quad h_n] Hf(x) \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

where  $h = (h_1, \dots, h_n)$ .

**Definition 1.27** (degenerate/nondegenerate points). Let  $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be of  $C^2$ , let  $(x_0, y_0)$  be a critical point. We define the **discriminant**, **disc  $f$** , of the Hessian by

$$\text{disc } f = \det(Hf) = \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$$

If  $\text{disc } f \neq 0$ , the critical point  $(x_0, y_0)$  is called **nondegenerate**; if  $\text{disc } f = 0$ , the point  $(x_0, y_0)$  is called **degenerate**.

**Definition 1.28** (positive, negative-definite). A quadratic function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is called **positive-definite** if  $g(h) \geq 0$  for all  $h \in \mathbb{R}^n$  and  $g(h) = 0$  implies  $h = 0$ . Similarly,  $g$  is **negative-definite** if  $g(h) \leq 0$  for all  $h \in \mathbb{R}^n$  and  $g(h) = 0$  implies  $h = 0$ .

**Definition 1.29** (global extremum). Let  $f : A \rightarrow \mathbb{R}$  be a function defined on  $A \subset \mathbb{R}^2$  or  $A \subset \mathbb{R}^3$ . A point  $x_0 \in A$  is said to be an **absolute maximum** if  $f(x_0) \geq f(x)$  for all  $x \in A$ . Similarly,  $x_0$  is an **absolute minimum** if  $f(x_0) \leq f(x)$  for all  $x \in A$ .

**Definition 1.30** (bounded set). A set  $D \subset \mathbb{R}^n$  is said to be **bounded** if there is a number  $M > 0$  such that  $\|x\| \leq M$  for all  $x \in D$ .

**Proposition 1.1** (extremum). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be in  $C^3$ , and  $x_0$  is a critical point of  $f$ . If the Hessian  $Hf(x_0)$  is positive-definite, then  $x_0$  is a local minimum of  $f$ ; if  $Hf(x_0)$  is negative-definite, then  $x_0$  is a local maximum.



## Chapter 2

# Theorem Review

**Proposition 2.1 (dot product).** Let  $a, b \in \mathbb{R}^3$ , and let  $\theta$  be the angle between  $a, b$ , where  $0 \leq \theta \leq \pi$ , then

$$a \cdot b = \|a\| \|b\| \cos \theta$$

**Proposition 2.2 (properties of the dot product).** Let  $a, b, c \in \mathbb{R}^n$ , then

(a) Nonnegativity:  $a \cdot a \geq 0$ , and  $a \cdot a = 0$  if and only if  $a = 0$ .

(b) Scalar multiplication: let  $\lambda \in \mathbb{R}$ , then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry:  $a \cdot b = b \cdot a$ .

**Proposition 2.3 (Cauchy-Schwarz).** Let  $a, b \in \mathbb{R}^n$ , then  $a \cdot b \in \mathbb{R}$ ,

$$|a \cdot b| \leq \|a\| \|b\|$$

where the left hand side is the absolute value of  $a \cdot b$ , and the right hand side is multiplication of two nonnegative real numbers.

**Proposition 2.4 (triangle inequality).** Let  $a, b \in \mathbb{R}^n$ , then

$$\|a + b\| \leq \|a\| + \|b\|$$

**Proposition 2.5 (cross product).** We have the following properties regarding the cross product: let  $a, b \in \mathbb{R}^3$ ,

1.  $a \times b$  is perpendicular to vectors  $a, b$ .
2. The length of the cross product is the area of the parallelogram:

$$\|a \times b\| = \|a\|\|b\|\sin \theta$$

where  $0 \leq \theta \leq \pi$  is the angle between them.

3.  $a \times b = -b \times a$ ,  $(a + b) \times c = a \times c + b \times c$ , and  $a \times (b + c) = a \times b + a \times c$ . Moreover,  $a \times b = 0$  iff  $a, b$  are parallel or either  $a$  or  $b$  are 0.

4. The cross product is **not** associative! For example, compute

$$(i \times i) \times j, \quad i \times (i \times j)$$

**Proposition 2.6 (limits).** Here are some properties of limits: let  $f : U_1 \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, g : U_2 \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,

- (a) (Uniqueness):

$$\text{If } \lim_{x \rightarrow x_0} f(x) = b_1, \quad \lim_{x \rightarrow x_0} f(x) = b_2$$

then we must have

$$b_1 = b_2$$

- (b) (Scalar mutliplication): Let  $c \in \mathbb{R}$ , if  $\lim_{x \rightarrow x_0} f(x) = b_1$ , then

$$\lim_{x \rightarrow x_0} cf(x) = cb_1$$

- (c) (Addition): Let  $f$  be as in (b), and  $\lim_{x \rightarrow x_0} g(x) = b_2$ , then

$$\lim_{x \rightarrow x_0} (f + g)(x) = b_1 + b_2$$

- (d) (Component): Write  $f(x) = (f_1(x), \dots, f_n(x))$ , if  $\lim_{x \rightarrow x_0} f(x) = b = (b_1, \dots, b_n)$ , then

$$\lim_{x \rightarrow x_0} f_i(x) = b_i$$

for all  $i = 1, \dots, m$ .

The same set of properties hold for continuity.

**Proposition 2.7 (continuity of compositions).** Let  $g : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $f : B \subset \mathbb{R}^m \rightarrow \mathbb{R}^p$ , and  $g(A) \subset B$ . If  $g$  is continuous at  $x_0$ ,  $f$  is continuous at  $g(x_0)$ , then  $f \circ g$  is continuous at  $x_0$ .

**Proposition 2.8 (differentiability implies continuity).** Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . If  $f$  is differentiable at  $x_0 \in U$ , then  $f$  is continuous at  $x_0$ .

**Proposition 2.9 (differentiability).** Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Suppose  $\partial f_i / \partial x_j$  exists for all  $i, j$  and are continuous in a neighborhood of  $x_0 \in U$ , then  $f$  is differentiable at  $x_0$ .

**Proposition 2.10 (properties of derivatives).** Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable at  $x_0$ , then the derivative of  $f$  at  $x_0$  is an  $m \times n$  matrix  $Df(x_0) = \left( \frac{\partial f_i}{\partial x_j} \right)_{ij}$ . The derivative follows the same properties as derivative for single variable functions:

1. Let  $c \in \mathbb{R}$ , then

$$D(cf)(x_0) = cDf(x_0) \quad (\text{multiplication of a matrix by constant } c)$$

2. Let  $g : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  also be differentiable at  $x_0$ , then

$$D(f+g)(x_0) = Df(x_0) + Dg(x_0) \quad (\text{sum of two matrices})$$

3. Let  $h_1 : U \subset \mathbb{R}^n \rightarrow \mathbb{R}, h_2 : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ , then

$$D(h_1 h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \quad (\text{product rule})$$

and if  $h_2 \neq 0$  on  $U$ .

$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \quad (\text{quotient rule})$$

4. Let  $g : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, f : V \subset \mathbb{R}^m \rightarrow \mathbb{R}^p$  such that  $g(U) \subset V$ , then

$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \quad (\text{chain rule})$$

**Proposition 2.11 (fastest rate of change).** Suppose that  $\nabla f(x_0) \neq 0$ , then the direction for which  $f$  increases the fastest at  $x_0$  is along  $\nabla f(x_0)$ .

**Proposition 2.12 (gradient is normal, tangent plane).** Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be differentiable, let  $S$  be a level surface of  $f$ , i.e.,  $S$  is a surface described by

$$f(x, y, z) = k$$

where  $k$  is some constant. Let  $(x_0, y_0, z_0) \in S$ , then

$$\nabla f(x_0, y_0, z_0) \text{ is } \mathbf{normal} \text{ to the level surface at } (x_0, y_0, z_0)$$

This means if  $c(t)$  is a path in  $S$ , and  $v(0) = (x_0, y_0, z_0)$ , and if  $v$  is a tangent vector to  $c(t)$  at  $t = 0$ , then

$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$

Moreover, if  $\nabla f(x_0, y_0, z_0) \neq 0$ , the **tangent plane** of  $S$  at  $(x_0, y_0, z_0)$  is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

**Proposition 2.13 (Equality of mixed partials).** If  $f(x, y)$  be twice continuously differentiable, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

**Proposition 2.14 (extremums are critical points).** Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be differentiable, where  $U$  is open. If  $x_0$  is a local extremum, then  $Df(x_0) = 0$ .

**Proposition 2.15 (local minimum).** Let  $f(x, y)$  be of  $C^2$ , and  $U$  is open in  $\mathbb{R}^2$ . A point  $(x_0, y_0)$  is a strict local **minimum** of  $f$  if the following conditions hold:

1.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

2.

$$\text{disc } f(x_0, y_0) > 0$$

where  $\text{disc } f$  is the **discriminant** of the Hessian, defined by

$$\text{disc } f = \det(Hf) = \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2$$

where  $Hf$  is the  $2 \times 2$  Hessian matrix.

3.

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

**Proposition 2.16 (local maximum).** Let  $f(x, y)$  be of  $C^2$ , and  $U$  is open in  $\mathbb{R}^2$ . A point  $(x_0, y_0)$  is a strict local **maximum** of  $f$  if the following conditions hold:

1.

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

2.

$$\text{disc } f(x_0, y_0) > 0$$

where  $\text{disc } f$  is the discriminant of the Hessian, defined above.

3.

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$$

**Proposition 2.17 (saddle points).** Let  $f(x, y) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be of  $C^2$ , if  $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$ , and  $\text{disc } f(x_0, y_0) < 0$ , where  $\text{disc } f$  is the discriminant, then the critical point  $(x_0, y_0)$  is a saddle point, i.e., neither a maximum or a minimum.

**Proposition 2.18 (continuous functions attain extremum on closed bounded sets).** Let  $f : D \rightarrow \mathbb{R}$  be continuous, where  $D$  is closed and bounded in  $\mathbb{R}^n$ . Then  $f$  assumes its absolute maximum and absolute minimum values at some point  $x_0, x_1 \in D$ .

## Chapter 3

# Practice Problems

**Problem 3.1.** Find the equation of the line passing through  $(1, 0, 2)$  in the direction  $(2, -1, 3)$ .

**Problem 3.2.** In which direction does the line

$$l(t) = (3 - 2t, 2 + 5t, 1 + t)$$

point?

**Problem 3.3.** Do the following two lines intersect?

$$l_1(t) = (1 + 2t, 2 + t, 3 - t), \quad l_2(s) = (3 - s, 4 - s, 2 + s)$$

**Problem 3.4.** Do the following points lie on the same line?

$$A = (1, 0, 1), \quad B = (2, 1, 1), \quad C = (0, -1, 1)$$

**Problem 3.5.** Find the angle between two vectors  $(1, 2, 0), (3, 1, 1)$ .

**Problem 3.6.** Let  $b = (2, 1, 3)$  and  $P$  be the plane through the origin given by  $x + y + 2z = 0$ .

- (a) Find two distinct vectors  $v_1, v_2$  that are orthogonal in  $P$ .
- (b) Find the projection of  $b$  onto the plane  $P$ , namely,

$$\text{Proj}_{v_1} b + \text{Proj}_{v_2} b$$

**Problem 3.7.** Find a unit vector orthogonal to both vectors  $a = (1, 2, -1), b = (2, 3, -1)$ .

**Problem 3.8.** Find the equation of the plane containing all three points below:

$$P = (2, 1, -1), \quad Q = (1, 0, -2), \quad T = (3, 2, 1)$$

**Problem 3.9.** (a) Find an equation for the line that passes through the point  $(1, 1, 0)$  and is perpendicular to the plane  $3x + y - 2z + 1 = 0$ .

(b) Find an equation for the plane that contains the line

$$l(t) = (-1 + t, 2 + 2t, 1 + 3t)$$

and is perpendicular to the plane

$$2x + y - z + 1 = 0$$

**Problem 3.10.** Compute the area of the parallelogram spanned by the vectors  $(1, 1, 0)$ ,  $(0, 2, 1)$ .

**Problem 3.11.** Use the triangle inequality 2.4 to show the reverse triangle inequality:

$$\left| \|a\| - \|b\| \right| \leq \|a - b\|$$

**Problem 3.12.** Compute the following limits if they exist; if the limits don't exist, please explain why.

1.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 + y^2 - 2xy}{x - y}$$

2.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\cos x - 1}{x^2 + y^2}$$

3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x - y)^2}{(x + y)^2}$$

4.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin 2x - 2x + y}{x^3 + y}$$

5.

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2x^2y \cos z}{x^2 + y^2}$$

6.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

7.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^6}{xy^3}$$

**Problem 3.13.** (a) Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = (1 - x)^8 + \cos(1 + x^3)$$

is continuous.

(b) Show  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \frac{x^2 e^x}{2 - \sin x}$$

is continuous.

**Problem 3.14.** Compute all the partial derivatives.

1.  $w = e^{xy} \log(x^2 + y^2)$ .
2.  $w = \cos(ye^{xy}) \sin x$ .

**Problem 3.15.** Compute the gradient of  $h(x, y, z) = (x + z)e^{x-y}$  at  $(1, 1, 0)$ .

**Problem 3.16.** Determine the velocity vector of the given path:

$$c(t) = (\cos 2t, 3t^2 - t, -t)$$

**Problem 3.17.** Find the tangent line to the given path at  $t = 0$

$$c(t) = (e^t \sin t, 2t, -t^3)$$

**Problem 3.18.** Compute the derivatives.

1. Let

$$f(u, v) = u^2 v + 2v, \quad u = -x^2 + y, v = x + y$$

Compute  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ .

2. Let

$$g(u, v) = (e^u, u + \sin v), \quad f(x, y, z) = (x^2, yz)$$

Compute  $D(g \circ f)$  at  $(0, 1, 0)$ .

3. Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $c(t) : \mathbb{R} \rightarrow \mathbb{R}^3$ . Suppose  $c(0) = (1, 2, 0)$ , and

$$\nabla f(1, 2, 0) = (0, 0, 1), \quad c'(0) = (2, 1, 1)$$

Compute  $\frac{d(f \circ c)}{dt}$  at  $t = 0$ .

**Problem 3.19.** Determine the directional derivative of

$$f(x, y, z) = x^3 y - xyz$$

at  $(1, 1, 0)$  along  $v = (0, -1, 1)$ .

**Problem 3.20.** Find a unit vector normal to the surface

$$xe^y + ye^z + ze^x = e + 1$$

at the point  $(0, 1, 1)$ .

**Problem 3.21.** Find the tangent plane of  $f(x, y, z) = \ln(x + y) - 2xz$  at  $(1, 2, -1)$ .

**Problem 3.22.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is called an *even* function if  $f(x) = f(-x)$  for every  $x$  in  $\mathbb{R}^n$ . If  $f$  is differentiable and even, find  $\nabla f$  at the origin.

**Problem 3.23.** Consider the function

$$f(x, y) = \frac{1}{\log(x^2 + y)}.$$

Verify by hand that  $f_{xy} = f_{yx}$ .

**Problem 3.24.** Consider the function  $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ . Show that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

**Problem 3.25.** Find the second-order Taylor expansion for the function

$$f(x, y) = x^2 + 2xy$$

at  $(1, 1)$ .

**Problem 3.26.** Find and classify all critical points of the following function:

1.

$$f(x, y) = e^x \cos y$$

2.

$$g(x, y) = (2x^2 + x)(3y + 1)$$

**Problem 3.27.** Show that  $(0, 0)$  is a critical point of

$$f(x, y) = x^2y - 2x^2 - y^2$$

and is it a local maximum, local minimum, or a saddle point?



## Chapter 4

# Answer Key

**Problem 4.1.** Find the equation of the line passing through  $(1, 0, 2)$  in the direction  $(2, -1, 3)$ .

*Proof.* By definition 1.2, the line is given by

$$l(t) = (1 + 2t, -t, 2 + 3t)$$

□

**Problem 4.2.** In which direction does the line

$$l(t) = (3 - 2t, 2 + 5t, 1 + t)$$

point?

*Proof.* In the direction of the vector  $(-2, 5, 1)$ .

□

**Problem 4.3.** Do the following two lines intersect?

$$l_1(t) = (1 + 2t, 2 + t, 3 - t), \quad l_2(s) = (3 - s, 4 - s, 2 + s)$$

*Proof.* For them to intersect, we must have  $t, s$  such that

$$\begin{cases} 1 + 2t = 3 - s & (1) \\ 2 + t = 4 - s & (2) \\ 3 - t = 2 + s & (3) \end{cases}$$

$(2) - (1)$  gives  $-t + 1 = 1$ , which implies  $t = 0, s = 2$ , but this does not satisfy (3), hence these two lines do not intersect! □

**Problem 4.4.** Do the following points lie on the same line?

$$A = (1, 0, 1), \quad B = (2, 1, 1), \quad C = (0, -1, 1)$$

*Proof.* We can find the unique line passing through  $A, B$  by the equation given in 1.2

$$l(t) = (1, 0, 1) + (1, 1, 0)t$$

then for  $C$  to lie on this line, there must exist some  $t$  such that

$$\begin{cases} 1 + t = 0 \\ t = -1 \\ 1 = 1 \end{cases}$$

and  $t = -1$  satisfies. This means all three points lie on the same line! □

**Problem 4.5.** Find the angle between two vectors  $(1, 2, 0), (3, 1, 1)$ .

*Proof.* By Proposition 2.1

$$\cos \theta = \frac{a \cdot b}{\|a\| \|b\|} = \frac{5}{\sqrt{5}\sqrt{11}} = \sqrt{\frac{5}{11}}$$

hence

$$\theta = \arccos\left(\sqrt{\frac{5}{11}}\right)$$
□

**Problem 4.6.** Let  $b = (2, 1, 3)$  and  $P$  be the plane through the origin given by  $x + y + 2z = 0$ .

- (a) Find two distinct vectors  $v_1, v_2$  that are orthogonal in  $P$ .
- (b) Find the projection of  $b$  onto the plane  $P$ , namely,

$$\text{Proj}_{v_1} b + \text{Proj}_{v_2} b$$

*Proof.* (a) We can let  $v_1 = (1, -1, 0), v_2 = (1, 1, -1)$ . One can verify that  $v_1, v_2 \in P$  and  $v_1 \cdot v_2 = 0$ .

(b) The projection is given by

$$\begin{aligned} \text{Proj}_{v_1} b + \text{Proj}_{v_2} b &= \frac{v_1 \cdot b}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot b}{v_2 \cdot v_2} v_2 \\ &= \frac{1}{2}(1, -1, 0) + 0 \\ &= \left(\frac{1}{2}, -\frac{1}{2}, 0\right) \end{aligned}$$
□

**Problem 4.7.** Find a unit vector orthogonal to both vectors  $a = (1, 2, -1), b = (2, 3, -1)$ .

*Proof.* The cross product is orthogonal to both of the vectors:

$$a \times b = \det \begin{bmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{bmatrix} = (1, -1, -1)$$

Then we normalize it:

$$\frac{a \times b}{\|a \times b\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$
□

**Problem 4.8.** Find the equation of the plane containing all three points below:

$$P = (2, 1, -1), \quad Q = (1, 0, -2), \quad T = (3, 2, 1)$$

*Proof.* We can find two vectors in this plane:

$$\mathbf{PQ} = Q - P = (-1, -1, -1), \quad \mathbf{PT} = T - Q = (1, 1, 2)$$

then we can find a normal vector  $n$  to the plane by taking the cross product:

$$n = \mathbf{PQ} \times \mathbf{PT} = \det \begin{bmatrix} i & j & k \\ -1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} = (-1, 1, 0)$$

Then by Definition 1.8, using point  $Q$ , we see the plane can be written as

$$-1(x - 1) + y = 0$$

simplifying we get  $x - y = 1$ . □

**Problem 4.9.** (a) Find an equation for the line that passes through the point  $(1, 1, 0)$  and is perpendicular to the plane  $3x + y - 2z + 1 = 0$ .

(b) Find an equation for the plane that contains the line

$$l(t) = (-1 + t, 2 + 2t, 1 + 3t)$$

and is perpendicular to the plane

$$2x + y - z + 1 = 0$$

*Proof.* (a) A normal vector to the plane  $3x + y - 2z + 1$  is  $(3, 1, -2)$ , since the line is perpendicular to the plane, the line is parallel along the direction  $(3, 1, -2)$ . Now the line passes through  $(1, 1, 0)$ , thus we have the equation for the line

$$l(t) = (1, 1, 0) + t(3, 1, -2)$$

(b) A normal vector to  $2x + y - z$  is  $n = (2, 1, -1)$ , and since our plane is perpendicular to this, it is parallel to the vector  $n$ . Thus a normal vector to our plane must be orthogonal to both  $n$  and  $(1, 2, 3)$ , where the latter is given by the line in the plane. Thus taking the cross product:

$$n_1 = n \times (1, 2, 3) = (5, -7, 3)$$

Hence the equation for the plane is given by:

$$5(x + 1) - 7(y - 2) + 3(z - 1) = 0$$

simplifying we get  $5x - 7y + 3z + 16 = 0$ . □

**Problem 4.10.** Compute the area of the parallelogram spanned by the vectors  $(1, 1, 0)$ ,  $(0, 2, 1)$ .

*Proof.* Since we know

$$\|u \times v\| = \|u\| \|v\| \sin \theta$$

the length of the cross product is exactly the area of the parallelogram, thus computing

$$\|(1, 1, 0) \times (0, 2, 1)\| = \|(1, -1, 2)\| = \sqrt{6}$$

□

**Problem 4.11.** Use the triangle inequality 2.4 to show the reverse triangle inequality:

$$\left| \|a\| - \|b\| \right| \leq \|a - b\|$$

*Proof.* We know by triangle inequality

$$\begin{aligned} \|a\| &= \|(a - b) + b\| \\ &\leq \|a - b\| + \|b\| \end{aligned}$$

rearranging, we get  $\|a\| - \|b\| \leq \|a - b\|$ . Similarly

$$\|b\| - \|a\| \leq \|a - b\|$$

Together this implies

$$\left| \|a\| - \|b\| \right| \leq \|a - b\|$$

□

**Problem 4.12.** Compute the following limits if they exist; if the limits don't exist, please explain why.

1.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 + y^2 - 2xy}{x - y}$$

2.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\cos x - 1}{x^2 + y^2}$$

3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x - y)^2}{(x + y)^2}$$

4.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin 2x - 2x + y}{x^3 + y}$$

5.

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2x^2y \cos z}{x^2 + y^2}$$

6.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

7.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^6}{xy^3}$$

*Proof.* 1.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 + y^2 - 2xy}{x - y} = \lim_{(x,y) \rightarrow (2,1)} \frac{(x - y)^2}{x - y} = \lim_{(x,y) \rightarrow (2,1)} x - y = 1$$

2. The limit doesn't exist,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\cos x - 1}{x^2 + y^2}$$

Consider the path  $x = 0, y \rightarrow 0$ , we have

$$\lim_{x=0, y \rightarrow 0} \frac{0}{y^2} = 0$$

Consider the path  $y = 0, x \rightarrow 0$ ,

$$\lim_{y=0, x \rightarrow 0} \frac{\cos x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{-\sin x}{2x} = \lim_{x \rightarrow 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

3. The limit doesn't exist,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-y)^2}{(x+y)^2}$$

Consider the path  $x = 0, y \rightarrow 0$ ,

$$\lim_{x=0, y \rightarrow 0} \frac{y^2}{y^2} = 1$$

Consider the path  $y = x \rightarrow 0$ ,

$$\lim_{x=y \rightarrow 0} \frac{0}{4x^2} = 0$$

4. The limit doesn't exist,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin 2x - 2x + y}{x^3 + y}$$

Consider the path  $x = 0, y \rightarrow 0$ ,

$$\lim_{x=0, y \rightarrow 0} \frac{y}{y} = 1$$

Consider the path  $y = 0, x \rightarrow 0$ ,

$$\begin{aligned} \lim_{y=0, x \rightarrow 0} \frac{\sin 2x - 2x}{x^3} &= \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2}{3x^2} \\ &= \lim_{x \rightarrow 0} \frac{-4 \sin 2x}{6x} \\ &= \lim_{x \rightarrow 0} \frac{-8 \cos 2x}{6} \\ &= -\frac{4}{3} \end{aligned}$$

5.

$$\lim_{(x,y,z) \rightarrow (0,0,0)} \frac{2x^2y \cos z}{x^2 + y^2}$$

Writing  $x = r \cos \theta, y = r \sin \theta$  in polar coordinates, we can rewrite this as

$$\left| \frac{2r^3 \cos^2 \theta \sin \theta \cos z}{r^2} \right| = |2r \cos^2 \theta \sin \theta \cos z| \leq 2r \rightarrow 0$$

as  $(x, y, z) \rightarrow (0, 0, 0)$ . Thus the limit is 0.

6.

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$

We factor:

$$\lim_{(x,y) \rightarrow (2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2} = \lim_{(x,y) \rightarrow (2,1)} \frac{(x-2y)x}{(x+2y)(x-2y)} = \lim_{(x,y) \rightarrow (2,1)} \frac{x}{x+2y} = \frac{1}{2}$$

7. The limit doesn't exist,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^6}{xy^3}$$

Consider  $x = y \rightarrow 0$ , then

$$\lim_{x=y \rightarrow 0} \frac{x^2 - x^6}{x^4} = \lim_{x \rightarrow 0} \frac{1 - x^4}{x^2} = \infty$$

Consider  $x = y^3 \rightarrow 0$ , then

$$\lim_{x=y^3 \rightarrow 0} \frac{0}{y^6} = 0$$

□

**Problem 4.13.** (a) Show that  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = (1 - x)^8 + \cos(1 + x^3)$$

is continuous.

(b) Show  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,

$$f(x) = \frac{x^2 e^x}{2 - \sin x}$$

is continuous.

*Proof.* (a)  $(1 - x)^8$  is a polynomial, thus continuous, and  $\cos x, 1 + x^3$  are both continuous, thus the composition  $\cos(1 + x^3)$  is also continuous. Thus adding continuous functions gives another continuous function.

(b)  $x^2 e^x, 2 - \sin x$  are both continuous, and  $\frac{x^2 e^x}{2 - \sin x}$  is continuous if  $2 - \sin x \neq 0$  for all  $x$ . This is indeed true because  $-1 \leq \sin x \leq 1$ , thus  $1 \leq 2 - \sin x \leq 3$ .

□

**Problem 4.14.** Compute all the partial derivatives.

1.  $w = e^{xy} \log(x^2 + y^2)$ .

2.  $w = \cos(ye^{xy}) \sin x$ .

*Proof.* 1.

$$\frac{\partial w}{\partial x} = ye^{xy} \ln(x^2 + y^2) + e^{xy} \frac{2x}{x^2 + y^2}$$

and

$$\frac{\partial w}{\partial y} = xe^{xy} \ln(x^2 + y^2) + e^{xy} \frac{2y}{x^2 + y^2}$$

2.

$$\frac{\partial w}{\partial x} = -y^2 e^{xy} \sin(ye^{xy}) \sin x + \cos(ye^{xy}) \cos x$$

and

$$\frac{\partial w}{\partial y} = -(1 + xy)e^{xy} \sin(ye^{xy}) \sin x$$

□

**Problem 4.15.** Compute the gradient of  $h(x, y, z) = (x + z)e^{x-y}$  at  $(1, 1, 0)$ .

*Proof.* The gradient is

$$\begin{aligned}\nabla h(x, y, z) &= \left[ \frac{\partial h}{\partial x} \quad \frac{\partial h}{\partial y} \quad \frac{\partial h}{\partial z} \right] \\ &= [e^{x-y}(1 + x + z) \quad -(x + z)e^{x-y} \quad e^{x-y}]\end{aligned}$$

Thus

$$\nabla h(1, 1, 0) = [2 \quad -1 \quad 1]$$

□

**Problem 4.16.** Determine the velocity vector of the given path:

$$c(t) = (\cos 2t, 3t^2 - t, -t)$$

*Proof.* It is given by

$$c'(t) = (-2 \sin 2t, 6t - 1, -1)$$

□

**Problem 4.17.** Find the tangent line to the given path at  $t = 0$

$$c(t) = (e^t \sin t, 2t, -t^3)$$

*Proof.* By the equation in Definition 1.20, we have

$$c'(t) = (e^t \sin t + e^t \cos t, 2, -3t^2)$$

and  $c(0) = (0, 0, 0)$ ,  $c'(0) = (1, 2, 0)$ . Thus the tangent line is given by

$$l(t) = (t, 2t, 0)$$

□

**Problem 4.18.** Compute the derivatives.

1. Let

$$f(u, v) = u^2v + 2v, \quad u = -x^2 + y, v = x + y$$

Compute  $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ .

2. Let

$$g(u, v) = (e^u, u + \sin v), \quad f(x, y, z) = (x^2, yz)$$

Compute  $D(g \circ f)$  at  $(0, 1, 0)$ .

3. Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  and  $c(t) = \mathbb{R} \rightarrow \mathbb{R}^3$ . Suppose  $c(0) = (1, 2, 0)$ , and

$$\nabla f(1, 2, 0) = (0, 0, 1), \quad c'(0) = (2, 1, 1)$$

Compute  $\frac{d(f \circ c)}{dt}$  at  $t = 0$ .

*Proof.* 1. We have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = -4xuv + u^2 + 2$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = 2uv + u^2 + 2$$

(You might want to replace  $u, v$  with  $x, y$ , but I am lazy).

2. We have

$$D(g \circ f)(0, 1, 0) = Dg(f(0, 1, 0))Df(0, 1, 0)$$

where  $f(0, 1, 0) = (0, 0)$

$$Dg(u, v) = \begin{bmatrix} e^u & 0 \\ 1 & \cos v \end{bmatrix}, \quad Df(x, y, z) = \begin{bmatrix} 2x & 0 & 0 \\ 0 & z & y \end{bmatrix}$$

Thus

$$\begin{aligned} D(g \circ f)(0, 1, 0) &= Dg(0, 0)Df(0, 1, 0) \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3. We have

$$\frac{d(f \circ c)}{dt}(0) = \nabla f(1, 2, 0)c'(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 1$$

□

**Problem 4.19.** Determine the directional derivative of

$$f(x, y, z) = x^3y - xyz$$

at  $(1, 1, 0)$  along  $v = (0, -1, 1)$ .

*Proof.* First we compute

$$\nabla f(x, y, z) = (3x^2y - yz, x^3 - xz, -xy)$$

Thus

$$\nabla f(1, 1, 0) = (3, 1, -1)$$

Recall the directional derivative is given by

$$\nabla f(1, 1, 0) \cdot \frac{v}{\|v\|} = -\frac{2}{\sqrt{2}}$$

We need to make sure that the direction vector is a unit vector!

□

**Problem 4.20.** Find a unit vector normal to the surface

$$xe^y + ye^z + ze^x = e + 1$$

at the point  $(0, 1, 1)$ .



*Proof.* This is a level set for the multivariate function  $f(x, y, z) = xe^y + ye^z + ze^x$ . We compute the gradient

$$\nabla f(x, y, z) = (e^y + ze^x, e^z + xe^y, e^x + ye^z).$$

hence  $\nabla f(0, 1, 1) = (e + 1, e, e + 1)$ , and this vector is normal to the surface. To make this a unit vector, we normalize to get

$$\frac{\nabla f(0, 1, 1)}{\|\nabla f(0, 1, 1)\|} = \frac{1}{\sqrt{3e^2 + 4e + 2}}(e + 1, e, e + 1),$$

□

**Problem 4.21.** Find the tangent plane of  $f(x, y, z) = \ln(x + y) - 2xz$  at  $(1, 2, -1)$ .

*Proof.* By the equation given in Proposition 2.12, the point  $(1, 2)$  lies on the level set

$$f(x, y) = 2 - \ln 3$$

In order to find a normal vector to the tangent plane, we compute the gradient of  $f$  at  $(1, 2, -1)$ :

$$\nabla f(x, y) = \left( \frac{1}{x+y} - 2z, \frac{1}{x+y}, -2x \right)$$

and  $\nabla f(1, 2, -1) = \left( \frac{7}{3}, \frac{1}{3}, -2 \right)$ , thus the tangent plane is given by

$$\frac{7}{3}(x - 1) + \frac{1}{3}(y - 2) - 2(z + 1) = 0$$

simplifying we get  $7x + y - 6z - 15 = 0$ .

□

**Problem 4.22.** A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is called an *even* function if  $f(x) = f(-x)$  for every  $x$  in  $\mathbb{R}^n$ . If  $f$  is differentiable and even, find  $\nabla f$  at the origin.

*Proof.* We claim that  $\nabla f(0, \dots, 0) = 0$ . It suffices to show that  $\nabla f(0, \dots, 0) \cdot v = \nabla f(0, \dots, 0) \cdot (-v)$  for any vector  $v \in \mathbb{R}^n$ . Because this implies  $2\nabla f(0, \dots, 0) \cdot v = 0$  for every  $v \in \mathbb{R}^n$ , so  $Df(0, \dots, 0) = 0$ . We know that

$$\nabla f(0, \dots, 0) \cdot v = \left. \frac{d}{dt} f(tv) \right|_{t=0}, \quad \nabla f(0, \dots, 0) \cdot (-v) = \left. \frac{d}{dt} f(-tv) \right|_{t=0}$$

But  $f(tv) = f(-tv)$  since  $f$  is even, thus

$$\nabla f(0, \dots, 0) \cdot v = \nabla f(0, \dots, 0) \cdot (-v)$$

as desired.

□

**Problem 4.23.** Consider the function

$$f(x, y) = \frac{1}{\log(x^2 + y)}.$$

Verify by hand that  $f_{xy} = f_{yx}$ .

*Proof.* We compute these separately.

$$f_x = \frac{2x}{x^2 + y}, \quad f_{xy} = -\frac{2x}{(x^2 + y)^2}$$

and

$$f_y = \frac{1}{x^2 + y}, \quad f_{yx} = -\frac{2x}{(x^2 + y)^2}$$

□

**Problem 4.24.** Consider the function  $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$ . Show that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

*Proof.* Note

$$f_x = -\frac{1}{2} \cdot \frac{2x}{(x^2 + y^2 + z^2)^{3/2}} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}},$$

so

$$f_{xx} = -\frac{(x^2 + y^2 + z^2)^{3/2} - x \cdot \frac{3}{2}(x^2 + y^2 + z^2)^{1/2} \cdot 2x}{(x^2 + y^2 + z^2)^3},$$

which is

$$f_{xx} = -\frac{x^2 + y^2 + z^2 - 3x^2}{(x^2 + y^2 + z^2)^{5/2}},$$

or

$$f_{xx} = -\frac{-2x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

By symmetry,

$$f_{yy} = -\frac{x^2 - 2y^2 + z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

and

$$f_{zz} = -\frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

so we see that  $f_{xx} + f_{yy} + f_{zz} = 0$ .

□

**Problem 4.25.** Find the second-order Taylor expansion for the function

$$f(x, y) = x^2 + 2xy$$

at  $(1, 1)$ .

*Proof.* First  $f(1, 1) = 3$ , then we find all first-order and second-order partial derivatives:

$$f_x = 2x + 2y, f_y = 2x, f_{xx} = 2, f_{xy} = 2, f_{yy} = 0$$

Thus by formula in Definition 1.23, we have

$$\begin{aligned} f(x, y) &= 3 + 4(x - 1) + 2(y - 1) + \frac{1}{2}2(x - 1)^2 + \frac{1}{2}2(x - 1)(y - 1) + \frac{1}{2}2(x - 1)(y - 1) + R_2((1, 1), (x, y)) \\ &= 3 + 4(x - 1) + 2(y - 1) + (x - 1)^2 + 2(x - 1)(y - 1) + R_2((1, 1), (x, y)) \end{aligned}$$

where

$$\frac{R_2((1,1), (x,y))}{\|(x-1, y-1)\|} \rightarrow 0$$

as  $(x, y) \rightarrow (1, 1)$ . □

**Problem 4.26.** Find and classify all critical points of the following function:

1.

$$f(x, y) = e^x \cos y$$

2.

$$g(x, y) = (2x^2 + x)(3y + 1)$$

*Proof.* 1. The critical point of  $f$  requires

$$f_x = f_y = 0$$

This gives

$$f_x = e^x \cos y = 0 \Rightarrow y = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$

similarly,

$$f_y = -e^x \sin y = 0 \Rightarrow y = \pi + n \in \mathbb{Z}$$

We see that there is no such  $y$  that makes  $f_x = f_y = 0$  simultaneously. Hence there are no critical points.

2. We again compute  $x, y$  such that  $g_x = g_y = 0$ .

$$g_x = (4x + 1)(3y + 1) \Rightarrow x = -\frac{1}{4}, y = -\frac{1}{3}$$

and

$$g_y = (2x^2 + x)3 = 0 \Rightarrow x = 0 \text{ or } x = -\frac{1}{2}$$

Thus the points  $(x, y)$  such that  $g_x = g_y = 0$  are

$$\left(0, -\frac{1}{3}\right), \quad \left(-\frac{1}{2}, -\frac{1}{3}\right)$$

Now we classify them by first computing their Hessians:

$$g_{xx} = 4(3y + 1), \quad g_{xy} = 3(4x + 1), \quad g_{yy} = 0$$

Thus

$$\text{disc } Hf = \det(Hf) = g_{xx}g_{yy} - g_{xy}^2 = -9(4x + 1)^2$$

Then we see that  $x = 0, -\frac{1}{2}$  both result in  $\text{disc } Hf < 0$ , which means

$$\left(0, -\frac{1}{3}\right), \quad \left(-\frac{1}{2}, -\frac{1}{3}\right)$$

are both saddle points. □

**Problem 4.27.** Show that  $(0, 0)$  is a critical point of

$$f(x, y) = x^2y - 2x^2 - y^2$$

and is it a local maximum, local minimum, or a saddle point?

*Proof.* We have

$$f_x = 2xy - 4x, \quad f_y = x^2 - 2y$$

and we see  $f_x(0, 0) = f_y(0, 0) = 0$ , thus  $(0, 0)$  is a critical point. Now we compute the discriminant:

$$f_{xx} = 2y - 4, \quad f_{xy} = 2x, \quad f_{yy} = -2$$

Then

$$\text{disc } Hf = f_{xx}f_{yy} - f_{xy}^2 = -2(2y - 4) - 4x^2$$

Hence  $\text{disc } Hf(0, 0) = 8 > 0$ , and  $f_{xx}(0, 0) = -4$  imply that  $(0, 0)$  is a local maximum. □

# Chapter 5

## Tips

1. When asked to find the limit:

Step 1: Factor out common factor, for example,

$$\frac{x^2 - 2xy}{x^2 - 4y^2} = \frac{(x - 2y)x}{(x - 2y)(x + 2y)} = \frac{x}{x + 2y}$$

Step 2: Try the following four paths: take  $(x, y) \rightarrow (0, 0)$  as an example,

- i.  $x = 0, y \rightarrow 0$ .
- ii.  $y = 0, x \rightarrow 0$ .
- iii.  $x = y \rightarrow 0$ .
- iv.  $x = -y \rightarrow 0$ .

Step 3: Try to put into expressions that you are familiar with, for example,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{x} = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{xy} y$$

and use the fact that  $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$ .

If any two paths give different limits, then the limit doesn't exist. Step 2:

2. When asked to find a directional derivative of  $f$  along  $v$ : make sure you normalize  $v$  as  $\frac{v}{\|v\|}$ .
3. When asked to find an equation for a plane: identify a normal vector by
  - (a) taking the cross product of two vectors in the plane [1.8](#).
  - (b) computing the gradient if the plane is the tangent plane to a level surface [2.12](#).
4. Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  be differentiable, then  $Df$  is an  $m \times n$  matrix. Let  $A$  be an  $m \times n$  matrix and  $B$  be a  $k \times p$  matrix, then the matrix multiplication  $AB$  only makes sense when  $n = k$ .