Functional Analysis

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Chapter 1

Preliminary

Definition 1.1 (orthonormal basis)**.** Let S be an orthonormal set in the Hilbert space such that no other orthonormal set contains S as a proper subset. Then S is called an orthonormal basis.

Proposition 1.1. Every Hilbert space admits an orthonormal basis.

Proof. Zorn's lemma. □

Remark: if H is separable, i.e., H has a countable dense subset, then the proof does not require Zorn's lemma. For example, L^2 is separable.

Proposition 1.2 (II.6, Parsevel's formula)**.** Let $\mathcal{H}$ be a Hilbert space, and $S = \{x_n\}$ be an orthonormal basis, then for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha \in \mathcal{A}} (x_\alpha, y) x_\alpha, \quad \|y\|^2 = \sum |(x_n, y)|^2$$

where $\mathcal{A}$ is an index set.

Proof. Bessel's inequality states that for any $\mathcal{A}' \subset \mathcal{A}$ finite, we have

$$\sum_{\alpha \in \mathcal{A}'} |(x_\alpha, y)|^2 \leq \|y\|^2 < \infty$$

It follows that $|(x_\alpha, y)| > \frac{1}{n}$ for at most finitely many α's, and $|(x_\alpha, y)| \neq 0$ for at most countably many α's. Let $\{\alpha_i\}_{i=1}^\infty$ be an enumeration of such α's. Then

$$\sum_{i=1}^N |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$$

which implies

$$\sum_{i=1}^\infty |(x_{\alpha_i}, y)|^2 < \infty$$

Let

$$y_n = \sum_{i=1}^n (x_{\alpha_i}, y) x_{\alpha_i},$$

we would like to show that the sequence $\{y_n\}$ is cauchy,

$$\|y_n - y_m\|^2 = \left\| \sum_{i=m+1}^{n} (x_{\alpha_i}, y) x_{\alpha_i} \right\|^2 \to 0 \text{ as } m \to \infty$$

Thus $\{y_n\}$ is Cauchy. In other words,

$$y_n \to y = \sum_{i=1}^{\infty} (x_{\alpha_i}, y) x_{\alpha_i}$$

$\square$

Definition 1.2. A metric space is separable if it has a countable dense subset.

Proposition 1.3 (II.7). Let $\mathcal{H}$ be a Hilbert space, then it is separable iff it has a countable orthonormal basis.

Proof. Suppose $\mathcal{H}$ is separable, let $\{x_n\}$ be a countable dense set, then we throw out terms in $\{x_n\}$ until we get a linearly indepedent dense subset $\{u_n\} \subset \{x_n\}$. Applying Gram-Schmidt, we can assume $\{u_n\}$ to be countable and orthonormal. Conversely, if $\{u_n\}$ is a countable orthonormal basis, then the set of linear combinations of $\{u_n\}$ with rational coefficients forms a countable dense subset of $\mathcal{H}$. $\square$

Definition 1.3 (Fourier Coefficient). The nth Fourier coefficient of a 2π-periodic function f is

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x) dx$$

The Fourier series of f is

$$\tilde{f}(x) = \lim_{M \to \infty} \sum_{M=-N}^{N} \frac{1}{\sqrt{2\pi}} c_n e^{inx}$$

Proposition 1.4. The Fourier series $\sum_k c_k$ converges if $f \in L^2$. Moreover, the series converges uniformly to a continuous function if $\sum |c_k| < \infty$

I am too lazzy to type it up, but it uses the fun lemma below:

Lemma 1.1. Suppose f is 2π-periodic, and $(f, e^{inx}) = 0$ for all n, then $f \equiv 0$. (In other words, if all the Fourier coefficients are 0, then the function must be identically zero).