Calc III Sections

Fall 2025

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Calc III-Week 11 (11/3-7)

Topic: theorem of change of variables.

Definition 1 (injective, surjective)**.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map, then T is **injective or one-to-one** if for $x, y \in \mathbb{R}^2$

$$Tx = Ty$$

then

$$x = y.$$

We say T is **surjective or onto** if for all $y \in \mathbb{R}^2$, there exists $x \in \mathbb{R}^2$ such that

$$Tx = y$$

If T is both injective and bijective, then we say T is **bijective**.

Definition 2 (linear map)**.** Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$, then T is linear if and only if for all $x, y \in \mathbb{R}^2, \lambda \in \mathbb{R}$.

$$\begin{cases} T(x+y) = Tx + Ty \\ T(\lambda x) = \lambda Tx \end{cases}$$

Definition 3 (Jacobian Determinant)**.** Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be of C^1 defined by

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$$

The **Jacobian determinant** of T, denoted as $\frac{\partial(x,y)}{\partial(u,v)}$ is the determinant of the matrix $DT(u,v)$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Example 1. The polar coordinate transformation is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

The Jacobian matrix is given by

$$\det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r$$

Proposition 1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, then there exists a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that

$$Tx = Ax$$

Proposition 2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, then T is injective if and only if it is also surjective if and only if $\det(A) \neq 0$, where A is the matrix associated to T.

Theorem 1 (change of variables formula)**.** Let D, D^* be elementary regions in $\mathbb{R}^2$, suppose $T : D^* \to D$ is bijective. Then for any integral function $f : D \to \mathbb{R}$, the **change of variable formula** states

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\det(J)\right| dudv$$

where

$$\det(J) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

is the determinant of the Jacobian matrix.

Corollary 1. An immediate corollary of the above theorem is the change of variables into polar coorid-nates:

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Corollary 2. Let W, W^* be elementary regions in $\mathbb{R}^3$, and suppose $T : W^* \to W$ is bijective. Then the change of varialbes formula for triple integrals states:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \mid \det(J) \mid dudvdw$$

where

$$\det(J) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Problem 1. Compute the following integral:

$$\iint_D (x^2+y^2)^{1/2} dxdy$$

where D is the disk $D = \{(x,y) : x^2+y^2 \le 9\}$.

Problem 2 (Marsden-Tromba, 6.2, Exercise 19)**.** Compute $\iint_R (x+y)^2 e^{x-y} dxdy$, where R is the region bounded by $x+y=1, x+y=4, x-y=-1, x-y=1$.