

# Algebraic Topology

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# Chapter 1

## Category Theory

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### 1.1 Lecture 1 8/26

**Definition 1.1 (Category).** A category  $\mathcal{C}$  consists of the following data:

1. A collection of objects denoted as  $\text{Ob}(\mathcal{C})$
2. Given two objects  $X, Y \in \text{Ob}(\mathcal{C})$ , a collection of morphisms between  $X, Y$ ,  $f : X \rightarrow Y$ , denoted as  $\text{mor}_{\mathcal{C}}(X, Y)$ .
3. (Composition) We have  $\text{mor}_{\mathcal{C}}(X, Y) \times \text{mor}_{\mathcal{C}}(Y, Z) \rightarrow \text{mor}_{\mathcal{C}}(X, Z)$  that satisfies associativity

$$f \circ (g \circ h) = (f \circ g) \circ h$$

4. (Identity) There is a distinguished morphism for each  $X$ ,  $\text{id}_{\mathcal{C}}(X, X)$  such that given any  $f \in \text{mor}_{\mathcal{C}}(X, Y)$ , we have  $f \circ \text{id}_X = \text{id}_Y \circ f = f$ .

In this course, we will make the assumption that in all the categories that we work with,  $\text{Ob}(\mathcal{C})$  need not be a set, but given any  $X, Y \in \text{Ob}(\mathcal{C})$ ,  $\text{mor}(X, Y)$  will always be a set. Now we talk about some examples of categories.

**Example 1.1 (Sets).** Let  $\text{Ob}(\text{Sets})$  be all the sets in the universe. Given  $X, Y$  sets,  $\text{mor}(X, Y)$  be all the set maps from  $X$  to  $Y$ , and  $\text{id}_X$  is the identity map.

**Example 1.2 (Top).** Let  $\text{Ob}(\text{Top})$  be all the topological spaces, and  $\text{mor}(X, Y)$  be all the continuous maps from  $X$  to  $Y$ .

**Example 1.3 ( $\text{Vect}_{\mathbb{F}}$ ).** Let  $\mathbb{F}$  be a field, and let  $\text{Ob}$  be all the  $\mathbb{F}$ -vector spaces. Then  $\text{mor}(V, W)$  is all the  $\mathbb{F}$ -linear homomorphisms from  $V$  to  $W$ , where  $\text{id}_V$  is the identity homomorphism.

**Example 1.4 (Posets).** Fix a poset  $P$ , let  $\text{Ob}(P)$  be the collection of elements in  $P$ , and given  $p, q$  we define

$$\text{mor}(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \emptyset, & \text{otherwise} \end{cases}$$

**Problem 1.1. HW(Q1): check this is a category**

**Example 1.5 (Opposite category).** Given a category  $\mathcal{C}$ , there is another category called the opposite category, denoted as  $\mathcal{C}^{op}$ , where

1. The objects are the same as  $\mathcal{C}$
2. Given  $X, Y \in \text{Ob}(\mathcal{C}^{op})$ , we have  $\text{mor}_{op}(X, Y) := \text{mor}_{\mathcal{C}}(Y, X)$ .
3. Moreover, given  $f \in \text{mor}_{op}(X, Y), g \in \text{mor}_{op}(Y, Z)$ , then  $g \circ f$  in  $\mathcal{C}^{op}$  is  $f \circ g : Z \rightarrow X$ .

Naturally, we define isomorphisms now.

**Definition 1.2 (isomorphism).** Given a category  $\mathcal{C}$ , and a morphism  $f \in \text{mor}_{\mathcal{C}}(X, Y)$ , we say  $f$  is an isomorphism if there exists  $g \in \text{mor}_{\mathcal{C}}(Y, X)$  such that

$$f \circ g = \text{Id}_Y, g \circ f = \text{Id}_X$$

Now we introduce maps between categories.

**Definition 1.3 (functor).** Given categories  $\mathcal{C}, \mathcal{D}$ , a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is the following;

1. Given an object  $X$  in  $\mathcal{C}$ ,  $F(X)$  is an object in  $\mathcal{D}$ .
2. Given a morphism  $f : X \rightarrow Y$ ,  $F(f)$  is a functor  $F(f) : F(X) \rightarrow F(Y)$ . Moreover, it satisfies the following:
  - (a)  $F(\text{id}_X) = \text{id}_{F(X)}$
  - (b)  $F(f \circ g) = F(f) \circ F(g)$ . Alternatively, we can rewrite this condition as the following:

$$\begin{array}{ccc} \text{mor}(X, Y) \times \text{mor}(Y, Z) & \longrightarrow & \text{mor}(X, Z) \\ \downarrow \text{mor}(F) \times \text{mor}(F) & & \downarrow \text{mor}(F) \\ \text{mor}(F(X), F(Y)) \times \text{mor}(F(Y), F(Z)) & \longrightarrow & \text{mor}(F(X), F(Z)) \end{array}$$

such that this diagram commutes.

**Problem 1.2. HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

**Example 1.6.**  $F : \text{Top} \rightarrow \text{Set}$ , where  $X \mapsto X$ , where the latter is a set, and  $f \mapsto f$  as set maps.

**Example 1.7.** Let  $\mathbb{F}$  be a field, and  $F : \text{Sets} \rightarrow \text{Vect}_{\mathbb{F}}$ , where  $X \mapsto \mathbb{F}\langle X \rangle$ , where  $\mathbb{F}\langle X \rangle$  is the free vector space over  $\mathbb{F}$  on the set  $X$ .

**Problem 1.3. HW(Q3): extend this to a functor by defining  $\text{mor}(f)$  and show this is a functor.**

**Example 1.8.** Let  $\mathbb{F}$  be a field, then the following is a functor,  $F : \text{Sets}^{op} \rightarrow \text{Vect}_{\mathbb{F}}$ , where

$$hF : X \mapsto \text{Maps}(X, \mathbb{F})$$

**Problem 1.4. HW(Q4):** show this extends to a functor by defining  $F(f)$ , and show it is a functor.

## 1.2 Lecture 2 8/28

**Definition 1.4 (contravariant functor).** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a contravariant functor from  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ , (equivalently,  $\mathcal{C} \rightarrow \mathcal{D}^{op}$ ).

**Problem 1.5. HW(Q5):** Show that the following functor  $F$  from  $\text{Vect}_{\mathbb{F}}$  to  $\text{Vect}_{\mathbb{F}}$  extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = \text{Hom}(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let  $Cat$  be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

**Definition 1.5 (natural transformation).** Given functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ , a natural transformation  $T$  from  $F$  to  $G$  is the following:  $T : F \Rightarrow G$ :

1. given object  $X \in Ob(\mathcal{C})$ ,  $T(X) \in mor(F(X), G(X))$
2. Given  $f \in mor(X, Y)$ , the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ T(X) \downarrow & & \downarrow T(Y) \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

where  $mor_F, mor_G$  is the identification function on morphisms by functors  $F, G$

If for all  $X$ ,  $T(X)$  is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor  $F$  and turn it to another functor  $G$ . We will (in a homework) show there exists natural transformation between the following two functors.

**Example 1.9.** Consider  $F, G : \text{Vect}_{\mathbb{F}} \rightarrow \text{Vect}_{\mathbb{F}}$ , define

$$F(V) = V \otimes_{\mathbb{F}} V / \langle a \otimes b - b \otimes a \rangle = V \otimes_{\mathbb{F}} V / \Sigma_2, G(V) = (V \otimes_{\mathbb{F}} V)^{\Sigma_2} = \{ \alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha \}$$

Both are vector spaces are fixed under “swaps.” Then a natural transformation can be defined as follows  $T(V) :$

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

**Problem 1.6. HW(Q6):** For the above  $F, G$

1. Show that  $T$  defines a natural transformation from  $F$  to  $G$ .
2. Find conditions on  $\mathbb{F}$  for  $T$  being a natural isomorphism.

Next we define limits and colimits. Let  $\mathcal{C}, \mathcal{D}$  be categories,  $d$  be an object in  $\mathcal{D}$ , then we can define a functor  $F_d : \mathcal{C} \rightarrow \mathcal{D}$  such that for any object  $c$  in  $\mathcal{C}$ ,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the “constant functor” on  $\mathcal{D}$ , i.e., every object is sent to  $d$ , and every morphism is sent to  $id_d$ .

**Definition 1.6 (colimit).** Given any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the colimit of  $F$ , denoted as  $\text{colim}(F)$  is an object in  $\mathcal{D}$  endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object  $d$  in  $\mathcal{D}$  and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in  $\mathcal{D}$ ,  $f : \text{colim}(F) \rightarrow d$  making the following diagram commute: for any  $X, Y, g$ :

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(Y) \\ \searrow \varphi_F & & \swarrow \varphi_F \\ & \text{colim}(F) & \\ \searrow \varphi & \downarrow f & \swarrow \varphi \\ & d & \end{array}$$

Next we prove some facts about colimits and give an example, where  $\text{colim}(F)$  exists.

**Proposition 1.1.** If  $\text{colim} F$  exists, then  $\text{colim} F$  is unique up to isomorphisms.

*Proof.* Let  $\text{colim}(F), \text{colim}(F)'$  be two colimits that satisfy the criteria. They are both objects in  $\mathcal{D}$ , then we get a morphism  $f : \text{colim}(F) \rightarrow \text{colim}(F)'$ , and likewise  $g : \text{colim}(F) \rightarrow \text{colim}(F)'$ , then

$$f \circ g : \text{colim}(F)' \rightarrow \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for  $g \circ f$ . □

Next we demonstrate a fact via an example.

**Theorem 1.1.** Let  $\mathcal{C}$  be a category where  $Ob(\mathcal{C}), mor(X, Y)$  are all sets. Let  $F : \mathcal{C} \rightarrow \text{Top}$  be any functor, then  $\text{colim}(F)$  exists.

*Proof.* Define  $\text{colim}(F) := \bigsqcup_c F(c) / \sim$ , where  $\sim$  is induced by the equivalence relation given by

$$y \sim F(f)y$$

where  $y \in F(C_1), f : C_1 \rightarrow C_2, F(f)x \in F(C_2)$ . The natural transformation we endow on  $F$  as  $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$ :

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(\mathcal{C})} F(C) / \sim$$

□

**Problem 1.7. HW(Q7):** Show that  $\text{colim}(F)$ ,  $\varphi_F$  is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

**Definition 1.7 (limit).** Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the limit of  $F$ , denoted as  $\lim(F)$  is an object of  $\mathcal{D}$ , endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object  $d \in \text{Ob}(\mathcal{D})$  and a natural transformation

$$\varphi : F_d \rightarrow F$$

there exists a unique  $f : \lim F \rightarrow d$  such that the following diagram commutes:

$$\begin{array}{ccc}
 & \lim F & \\
 \varphi_F \swarrow & \downarrow f & \searrow \varphi_F \\
 & d & \\
 \swarrow \varphi & & \searrow \varphi \\
 F(X) & \xrightarrow{F(g)} & F(Y)
 \end{array}$$

Just like colimits, limits are unique up to isomorphisms.

**Problem 1.8. HW(Q8):** Given  $F : \mathcal{C} \rightarrow \mathcal{D}$ , consider  $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ , then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

## Chapter 2

# Homologies, Cohomologies

### 2.1 Lecture 3 9/4

Today we define (co)chain complexes: let  $R$  be a commutative ring, let  $Mod_R$  denote the category of  $R$ -modules and  $R$ -module maps.

**Definition 2.1 (chain complex).** A chain complex of  $R$ -modules is a collection of  $R$ -modules and  $R$ -modules maps

$$\cdots \rightarrow M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that  $\partial_i \circ \partial_{i+1} = 0$  for all  $i$ . In other words, the image of previous map is contained in the kernel of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by  $\{M.; \partial.^M\}$ .

Next we introduce morphisms between chain complexes.

**Definition 2.2 (morphism between complexes).** Let  $\{M.; \partial.^M\}, \{N.; \partial.^N\}$ , a morphism  $\{f.\}$  between chain complexes is a “ladder” such that the following commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{i+1} & \xrightarrow{\partial_{i+1}^M} & M_i & \xrightarrow{\partial_i^M} & M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots \\ & & \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} \\ \cdots & \longrightarrow & N_{i+1} & \xrightarrow{\partial_{i+1}^N} & N_i & \xrightarrow{\partial_i^N} & N_{i-1} \xrightarrow{\partial_{i-1}^N} \cdots \end{array}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where  $\{g.\} : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$ , and  $\{f.\} : \{N.; \partial.^N\} \rightarrow \{L.; \partial.^L\}$ , which is simply vertical stacking.

**Problem 2.1. HW(Q9):** Prove that chain complexes of  $R$ -modules form a category  $ch_R$ .

There are interesting functors  $F : ch_R \rightarrow Mod_R$ , and we begin with the following one:



**Definition 2.3** ( $H_n$ ,  $n$ th-homology). Given  $n \in \mathbb{Z}$ , there is a functor

$$H_n : \mathbf{ch}_R \rightarrow \mathbf{Mod}_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial_n^M / \text{Im} \partial_{n+1}^M$$

and for  $f : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$ , we define:  $H_n(f) : H_n(\{M.; \partial.^M\}) \rightarrow H_n(\{N.; \partial.^N\})$ ,

$$H_n(f)[x] := [f_n(x)]$$

where  $[x] \in H_n(\{M.; \partial.^M\})$ .

*Proof.* We need to show  $H_n$  is well-defined on objects and morphisms. We need to check that  $\text{Im} \partial_{n+1} \subset \ker \partial_n$ . This is a consequence of  $\partial^2 = 0$ .

On morphisms: for  $x \in \ker \partial_n^M$ , we have  $f_n(x) \in \ker \partial_n^N$ . This is we have

$$\partial_n^N(f_n(x)) = f_{n+1}(\partial_n^M(x)) = 0$$

Moreover, we need to check that this doesn't depend on the choice of representatives, i.e., we can check that

$$\text{Im} \partial_{n+1}^M \mapsto 0$$

Let  $x = \partial_{n+1}^M(y)$ , we have

$$f_n(x) = f_n(\partial_{n+1}^M(y)) = \partial_{n+1}^N(f_{n+1}(y)) = 0$$

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{\partial_{n+1}^M} & M_n \\ f_{n+1} \downarrow & & \downarrow f_n \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n \end{array}$$

□

Next we talk about homotopy between morphisms between chain complexes.

**Definition 2.4** (homotopy). Given two morphisms,  $f, g : M. \rightarrow N.$ , a chain homotopy  $h$ . between them is a collection of  $R$ -modules maps, for all  $n \in \mathbb{Z}$ ,

$$h_n : M_n \rightarrow N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as  $\partial h + h \partial = f - g$ .

$$\begin{array}{ccccc} M_{n+1} & \longrightarrow & M_n & \xrightarrow{\partial_n^M} & M_{n-1} \\ f_{n+1} \downarrow & \swarrow h_n & \downarrow f_n/g_n & \swarrow h_{n-1} & \downarrow f_{n-1} \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n & \longrightarrow & N_{n-1} \end{array}$$

**Problem 2.2. HW(Q10):** Show that homotopy is an equivalence relation between morphisms. Hint: replace  $h_n$  with  $-h_n : M_n \rightarrow N_{n+1}$ .

*Proof.* Reflexive is shown by defining  $h_n$  to be the zero map. For symmetry, we choose  $-h_n$ . Transitive is a ladder. □

**Proposition 2.1.** Let  $h.$  be a chain homotopy between  $f.$  and  $g.$ , then we have an equality

$$H_n(f.) = H_n(g.)$$

where  $H_n(f.), H_n(g.) : H_n(M.) \rightarrow H_n(N.)$ .

*Proof.* Given  $[x] \in H_n(M.)$ , we have

$$\begin{aligned} H_n(f)[x] &= [f_n(x)] \\ &= [g_n(x) + \partial h.(x) + h.\partial(x)] \\ &= [g_n(x) + \partial h.(x)] \\ &= [g_n(x)] \\ &= H_n(g)[x] \end{aligned}$$

□

Next we define a new category.

**Definition 2.5** ( $Hch_R$ ). Define the category  $Hch_R$  as follows:

1.  $Ob(Hch_R) = Ob(ch_R)$
2.  $mor_{Hch_R}(M., N.) = mor_{ch_R}(M., N.) / \sim$ , where  $\sim$  is the homotopy equivalence.

**Problem 2.3. HW(Q11):** Show that  $Hch_R$  is a category, admitting a functor

$$F : ch_R \rightarrow Hch_R$$

such that the following diagram commutes:

$$\begin{array}{ccc} ch_R & \xrightarrow{F} & Hch_R \\ & \searrow H_n \quad \swarrow H_n & \\ & mod_R & \end{array}$$

Next we introduce long and short exact sequences.

**Definition 2.6 (exactness).** Firstly, given a pair of  $R$ -module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at  $X_2$  if  $\ker(g) = \text{im}(f)$ . Hence given a sequence of  $R$ -module maps,

$$\cdots \rightarrow X_{i+1} \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots$$

this is called a long exact sequence if it is exact at all  $X_i$ . Finally, given a pair of  $R$ -module maps,

$$0 \rightarrow X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \rightarrow 0$$

This is a short exact sequence, and  $f$  is injective,  $g$  is surjective.

**Problem 2.4. HW(Q12):** Prove the following:

1. Given LES,

$$\cdots \rightarrow X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

show the following is a short exact sequence:

$$0 \rightarrow \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \rightarrow 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions  $X_i, Y_i$ , where  $i = 2, 3, 4$ , and assume the diagram commutes and if  $t_1, t_2, t_4, t_5$  are isomorphisms, show that  $t_3$  is also an isomorphism.

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ t_1 \downarrow & & t_2 \downarrow & & t_3 \downarrow & & t_4 \downarrow & & t_5 \downarrow \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & Y_5 \end{array}$$

Next we state the most important theorem in chain complexes:

**Theorem 2.1 (The snake lemma).** Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be a SES of chain complexes, i.e.,

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is a short exact sequence of all  $n$ . Then there exists a LES of homology groups.

$$\begin{array}{ccccc} & & & & H_{n+1}(C) \\ & & & \swarrow \delta_{n-1} & \\ H_n(A) & \xrightarrow{H_n(f)} & H_n(B) & \xrightarrow{H_n(g)} & H_n(C) \\ & & \searrow \delta_n & & \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) & \xrightarrow{H_{n-1}(g)} & H_{n-1}(C) \\ & & \swarrow \delta_{n-1} & & \\ H_{n-2}(A) & & & & \end{array}$$

## 2.2 Lecture 4 9/9

Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

*Proof.* First we define the map  $\delta_n : H_n(C) \rightarrow H_{n-1}(A)$ . Let  $[x] \in H_n(C)$ , then  $x \in \delta^C$ , where  $\delta^C : C_n \rightarrow C_{n-1}$ . We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for  $x \in C_n$ ,  $g_n : B_n \rightarrow C_n$  is surjective, hence there exists  $b \in B_n$  such that  $g_n(b) = x$ . Then consider  $d = \delta^B(b)$ , since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \text{im } f_{n-1}$$

Let  $y \in A_{n-1}$  be this unique  $y$  such that  $f_{n-1}(y) = d$ , where uniqueness is by  $f_{n-1}$  is injective. This is indicated in the below diagram:

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & b \in B_n & \xrightarrow{g_n} & x \in C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ y \in A_{n-1} & \xrightarrow{f_{n-1}} & d \in B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

We first need to check that  $[y]$  does not depend on the choice of  $b$ . Let  $g_n(b_1) = g_n(b_2) = x$ , then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let  $y_1, y_2$  be those determined by  $b_1, b_2$ , then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{array}{ccc} a \in A_n & \xrightarrow{f_n} & B_n \\ \delta^A \downarrow & & \downarrow \delta^B \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} \end{array}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e.,  $[y_1] = [y_2]$ , as they only differ by  $\text{im } \delta$ .

**Problem 2.5. HW(Q13):** Check that if  $x \in \text{im } \delta^C$ , then  $\delta_n[x] = 0$ .

the proof is not finished, too lazy to tex it up □

Next we review the tensor products of  $R$ -modules. We first review  $R$ -bilinear maps

**Definition 2.7 (bilinear maps).** Let  $M, N, P$  be  $R$ -modules, an  $R$ -bilinear map  $f : M \times N \rightarrow P$  is a map such that

1.  $f$  is linear in both coordinates, we have  $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$ , and similarly,  $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$ .
2. For all  $r \in R$ , we have  $f(rm, n) = f(m, rn) = rf(m, n)$ .

Next we define tensor products.

**Definition 2.8 (tensor product).** A tensor product of  $M \times N$  is an  $R$ -module denoted by  $M \otimes_R N$  such that

1.  $M \otimes_R N$  comes endowed with an  $R$ -bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other  $R$ -bilinear map  $f : M \times N \rightarrow P$ , there exists a unique  $R$ -module map  $\psi$  such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes_R N \\ f \downarrow & \swarrow \psi & \\ P & & \end{array}$$

It is not clear that  $M \otimes_R N$  exists or not. In fact, they exist!

**Theorem 2.2 ( $M \otimes_R N$  exists).** Define  $M \otimes_R N = R\langle M \times N \rangle / K$ , where  $R\langle M \times N \rangle$  is the free  $R$ -module on the set  $M \times N$ . We define  $K$  as the submodule generated by the following four relations:

1.  $\langle (m_1 + m_2, n) \rangle - \langle (m_1, n) \rangle - \langle (m_2, n) \rangle$
2.  $\langle (m, n_1 + n_2) \rangle - \langle (m, n_1) \rangle - \langle (m, n_2) \rangle$
3.  $r\langle (m, n) \rangle - \langle (rm, n) \rangle$
4.  $r\langle (m, n) \rangle - \langle (m, rn) \rangle$

Moreover, the map  $\varphi : M \times N \rightarrow M \otimes_R N$  given by

$$(m, n) \mapsto \langle (m, n) \rangle := m \otimes_R n$$

**Problem 2.6. HW(Q14):** show that  $M \otimes_R N$  is a tensor product.

## 2.3 Lecture 5 9/11

We continue with the tensors of  $R$ -modules. Let  $f : A \rightarrow B$  an  $R$ -module map, let  $N$  be some fixed  $R$ -module, then  $f$  induces maps:  $f \otimes id : A \otimes_R N \rightarrow B \otimes_R N$ ,

$$f \otimes id : a \otimes n \mapsto f(a) \otimes n$$

and  $id \otimes f : N \otimes_R A \rightarrow N \otimes_R B$  :

$$id \otimes f : n \otimes a \mapsto n \otimes f(a)$$

**Problem 2.7. HW(Q15(a)):** Show that the following maps induce functors:

1.  $- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$ , where

$$A \mapsto A \otimes_R N, f \mapsto f \otimes id$$

2.  $N \otimes_R - : \text{Mod}_R \rightarrow \text{Mod}_R$ , where

$$A \mapsto N \otimes_R A, f \mapsto id \otimes f$$

**Problem 2.8. HW(Q15(b)):** Show that one has the following natural isomorphisms:

1.  $0 \otimes_R M \cong 0$ , and  $0 \otimes_R - \cong F_0$  (recall the definition of  $F_0$  as a functor).
2.  $R \otimes_R M \cong M$ , and  $R \otimes_R - \cong id$ .
3.  $M \otimes_R N \cong N \otimes_R M$ , and  $M \otimes_R - \cong - \otimes_R M$ .
4.  $M \otimes_R (N \otimes_R K) \cong (M \otimes_R N) \otimes_R K$ .
5.  $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$ .

For convenience, we introduce the following definition:

**Definition 2.9 (positively graded chain complex).** A positively graded chain complex  $\{M_i; \partial^M\}$  is a chain complex so that  $M_i = 0$  for all  $i < 0$ . The category of positively graded chain complexes is denoted as  $ch_R^+$ .

We have our first important theorem for  $ch_R^+$ .

**Theorem 2.3.** There exists a functor  $\otimes_R$  and a natural transformation  $X$  such that the following diagram of functors commutes up to some  $X$ :

$$\begin{array}{ccc} ch_R^+ \times ch + R^+ & \xrightarrow{\otimes_R} & ch_R^+ \\ H_i \times H_j \downarrow & \nearrow X & \downarrow H_{i+j} \\ Mod_R \times Mod_R & \xrightarrow{\otimes_R} & Mod_R \end{array}$$

where  $X : \otimes_R \circ (H_i \times H_j) \Rightarrow H_{i+j} \circ \otimes_R$  is a natural transformation.

We note that the existence of  $X$  means this diagram doesn't commute "on the nose," but these two composition functors are the same up to some natural transformation. Before we given the proof, we recall that  $Ob(C \times D) = Ob(C) \times Ob(D)$ ,  $mor((X, Y), (X', Y')) = mor(X, Y) \times mor(X', Y')$ .

*Proof.* We define  $\otimes_R$  of positively graded chain complexes as follows: let  $\{M_i; \partial^M\}, \{N_i; \partial^N\}$  be two PGCC. Define  $\{M \otimes_R N; \partial^M\}$ :

$$(M \otimes_R N) = \bigoplus_{i+j=n} (M_i \otimes_R N_j)$$

note that the RHS is always a finite sum. Moreover,  $\partial^\otimes$  is defined as follows:

$$\partial^\otimes : (M \otimes_R N)_n \rightarrow (M \otimes_R N)_{n-1} \text{ is defined on the component } M_i \otimes_R N_j \text{ (from the RHS)}$$

and

$$\partial^\otimes(m_i \otimes n_j) := \partial^M(m_i) \otimes n_j + (-1)^i m_i \otimes \partial^N(n_j)$$

It is easy to check that  $\partial^\otimes \circ \partial^\otimes = 0$ .

Now we've show  $ch_R^+ \otimes_R ch_R^+$  is well-defined, it remains to define  $X$ , the natural transformation. We define

$$X : H_i(M_\bullet) \otimes_R H_j(N_\bullet) \rightarrow H_{i+j}(M_\bullet \otimes_R N_\bullet)$$

again, it suffices to define  $X$  on elementary tensors.

$$X : [\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$$

we need to check that

1.  $\partial^\otimes(\alpha \otimes \beta) = 0$  if  $\partial^M(\alpha) = 0$  and  $\partial^N(\beta) = 0$ .
2. If  $\alpha = \partial(r)i$ , then notice that  $\partial^\otimes(r \otimes \beta) = \alpha \otimes \beta$ , similarly for  $\beta$ . This would show that  $X$  is well-defined.

It is straightforward to check that  $X$  commutes with morphisms in  $ch_R^+ \times ch_R^+$ . □

Next we define cochain complexes and cohomologies.

**Definition 2.10 (cochain).** A cochain of  $R$ -modules  $(M^\bullet, \partial_M^\bullet)$  is a sequence of  $R$ -module maps:

$$\dots \longrightarrow M^i \xrightarrow{\partial^i} M^{i+1} \xrightarrow{\partial^{i+1}} M^{i+2} \longrightarrow \dots$$

such that  $\partial \circ \partial = 0$ .

Cochain complexes form a category, with morphisms  $\{f^\bullet\}$  form a ladder:

$$\begin{array}{ccccccc} \dots & \longrightarrow & M^i & \xrightarrow{\partial^i} & M^{i+1} & \xrightarrow{\partial^{i+1}} & M^{i+2} \longrightarrow \dots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \downarrow f^{i+2} \\ \dots & \longrightarrow & N^i & \xrightarrow{\partial^i} & N^{i+1} & \xrightarrow{\partial^{i+1}} & N^{i+2} \longrightarrow \dots \end{array}$$

The  $n$ -th cohomology of a cochain complex  $\{M^\bullet; \partial_M^\bullet\}$  is defined as:

$$H^n(M^\bullet; \partial_M^\bullet) := \frac{\ker \partial^i : M^i \rightarrow M^{i+1}}{\operatorname{im} \partial^{i-1} : M^{i-1} \rightarrow M^i}$$

We remark that there is nothing unexpected here from what we learned about chain complexes. Namely, if we reindex  $\{M^\bullet; \partial_M^\bullet\}$ , this defines a chain complex with  $M'_i = M^{-i}$ . This implies that the snake lemme holds! (with  $\partial^i : H^i(C) \rightarrow H^{i+1}(A)$ ).

**Theorem 2.4.** There is a functor  $D$  and a natural transformation  $\beta$  such that the following diagram of functors commute up to the natural transformation  $\beta$ :

$$\begin{array}{ccc} ch_R^{op} & \xrightarrow{D} & coch_R \\ H_n^{op} \downarrow & \nearrow \beta & \downarrow H^n \\ Mod_R^{op} & \xrightarrow{\bar{D}} & Mod_R \end{array}$$

where  $\bar{D}(M) = Hom_R(M, R)$ , and

$$D(\{M_\bullet; \partial_\bullet^M\}) \text{ is defined as } \{DM^\bullet; \partial^\bullet\}$$

where

$$DM^n := Hom_R(M_n, R), \partial^n : DM^n \rightarrow DM^{n+1} \text{ is the map induced by } \partial_{n+1} : M_{n+1} \rightarrow M_n$$

We observe that  $\partial^{n+1}\partial^n = 0$  since  $\partial_{n+2}\partial_{n+1} = 0$ .

**Problem 2.9. HW(Q16):** Show that  $D$  is a functor.

Next we define the natural transformation  $\beta$ . We have  $\beta : H^n(DM) \rightarrow \text{Hom}_R(H_n(M_\bullet), R)$ , such that

$$\beta : [\varphi] \mapsto \beta[\varphi]$$

let  $[x] \in H_n(M_\bullet)$ , where  $\beta[\varphi]([x]) = \varphi(x)$  (where  $\varphi \in \text{Hom}_R(M_n, \mathbb{R})$ ,  $x \in M_n$ ).

*Proof.* We first need to show that  $\beta$  is well-defined. If  $x = \partial_{n+1}(y)$ , then consider

$$\beta[\varphi][x] = \varphi(x) = \varphi(\partial_{n+1}(y)) = \partial^n(\varphi)(y) = 0, x \in \ker \varphi$$

Conversely, if  $\varphi = \delta^{n-1}(\psi)$ , we have

$$\beta[\varphi][x] = \varphi(x) = \delta^{n-1}\psi(x) = \psi(\partial_n(x)) = 0$$

It remains to check that  $\beta$  commutes with morphisms in  $ch_R^{op}$  (which we will do next time).  $\square$

## 2.4 Lecture 6 9/16

Today we continue our discussion of homological algebra. Let  $M$  be an  $R$ -module.

**Definition 2.11 (resolution).** A resolution of  $M$  is a positively graded chain complex  $\{P_\bullet, \partial_\bullet\}$  such that

1.  $H_n(P_\bullet) = 0$  for all  $n > 0$
2.  $H_0(P_\bullet) = \frac{P_0}{\text{im } \partial_1} \cong M$ , where  $\partial_1 : P_1 \rightarrow P_0$ .

We say  $\{P_\bullet, \partial_\bullet\}$  is a free resolution if  $P_i$  is a free  $R$ -module for each  $i$ .

For resolutions, we prove the following two things: first, free resolutions always exist; second, every  $R$ -module map can be extended to a map between their resolutions (with extra assumptions) and these extensions are unique up to homotopies.

**Proposition 2.2.** For any  $M$ , a free resolution for  $M$  exists.

*Proof.* We proceed this inductively. Define  $P_0$  to be  $R\langle M \rangle$ , where it is the free  $R$ -module defined on the set  $M$ . Note that

$$R\langle M \rangle \twoheadrightarrow M \text{ is surjective : } \langle m \rangle \mapsto m$$

Let  $K$  be the kernel of this map, we have an isomorphism:

$$\epsilon : P_0/K \cong M$$

Define  $P_1$  as  $R\langle K \rangle$ , note that  $P_1 \twoheadrightarrow K$ , then we define

$$\partial_1 : P_1 \rightarrow P_0$$

to be the composite

$$P_1 \twoheadrightarrow K \subset P_0$$

Now we consider  $P_2$ : let  $K_1 \subset P_1$  be the kernel of  $\delta_1$ , define  $P_2 = R\langle K_1 \rangle$ , then define  $\partial_2 : P_2 \rightarrow P_1$  to be the composite"

$$P_2 \twoheadrightarrow K_1 \subset P_1$$

note that  $\ker \delta_1 / \text{im } \delta_2 = K_1/K_1 = 0$ . Then we define  $K_2 = \ker \delta_2$ , define  $P_3 = R\langle K_2 \rangle, \dots$   $\square$

Just like the above proposition, the next theorem uses induction.



**Theorem 2.5 (extension theorem).** Let  $\{P_\bullet^M, \delta_\bullet^M, \epsilon_M\}$  be a free resolution on  $M$ , and let  $\{P_\bullet^N, \delta_\bullet^N, \epsilon_N\}$  be an arbitrary resolution of  $N$ . Then given a map of  $R$ -modules  $f : M \rightarrow N$ , we may extend it to a map of chain complexes:

$$f_\bullet : \{P_\bullet^M, \delta_\bullet^M\} \rightarrow \{P_\bullet^N, \delta_\bullet^N\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_0(P_\bullet^M) & \xrightarrow{H_0(f_\bullet)} & H_0(P_\bullet^N) \\ \downarrow \epsilon_M & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

Moreover, given any two extensions  $f_\bullet^1, f_\bullet^2$  of  $f$ , we have a chain homotopy  $h_\bullet$  between  $f_\bullet^1, f_\bullet^2$ .

Remark: if  $f_\bullet$  makes the diagram commute, and  $g_\bullet$  is homotopic to  $f_\bullet$ , then  $g_\bullet$  also makes the diagram commute: homotopy classes work the same on homologies (they are the same on the nose).

*Proof.* We will construct  $f_\bullet$  as follows. We construct  $f_i$  inductively on  $i$ . Consider the diagram:

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Since  $P_0^M$  is free, and  $\epsilon_N$  is surjective, we may lift  $f$  on generators of  $P_0^M$  by lifting the generators of  $P_0^M$  to elements in  $P_0^N$ . (Note: this lift may not be unique), but this lift extends uniquely to define  $f_0$ . We notice that the bottom square

$$\begin{array}{ccc} P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commutes on homologies ( $H_0(P_0^M), H_0(P_0^N)$ ). Now we construct  $f_1$ :

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & \xrightarrow{f_1} & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

We will follow the purple path above. Recall that  $\epsilon_M : H_0(P_0) = P_0 / \text{im}(\partial_1^M) \rightarrow M$  is an isomorphism. We

consider the composite:  $f_0 \circ \partial_1^M = g$ , we have

$$\begin{aligned}\epsilon_N \circ g &= \epsilon_N \circ f \circ \partial_1^M \\ &= f \circ \epsilon_M \circ \partial_1^M \\ &= 0\end{aligned}$$

This implies that  $\text{im}(g) \subset \ker(\partial_N) = \text{im}(\partial_1^N)$ . We can lift the generators of  $P_1^M$  to elements of  $P_1^N$ . (Once chosen a lift, one can extend this uniquely to define  $f_1$ ). Then we construct  $f_2, f_3, \dots$  the same way by considering  $f_n \circ \partial_{n+1}$  and show that it is in the kernel of  $\partial_n^N$  and lift it to define  $\partial_{n+1}$ .

Now we homotopy time. Assume  $f_\bullet^1, f_\bullet^2$  are two lifts of  $f$ , we construct  $h : P_\bullet^M \rightarrow P_{\bullet+1}^N$ . We define  $h_\bullet$  inductively, starting with  $h_0$  below:

$$\begin{array}{ccc} P_1^M & \xrightarrow{f_1} & P_1^N \\ \partial_1^M \downarrow & \nearrow h_0 & \downarrow \partial_1^N \\ P_0^M & \xrightarrow{f_0^1, f_0^2} & P_0^N \\ \epsilon_M \downarrow & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

We have  $\epsilon_N(f_0^1 - f_0^2) = 0$ , then

$$f_0^1 - f_0^2 \in \ker \epsilon_N = \text{im } \delta_1^N$$

we may lift  $f_0^1 - f_0^2$  on generators of  $P_0^M$ , where  $h_0 : P_0^M \rightarrow P_1^N$ . Hence

$$(h_{-1} \circ \delta_{-1}^N) + \delta_1^N \circ h_0 = f_0^1 - f_0^2$$

Inductively, we assume  $h_m$  exists for  $m \leq n$ , then

$$\begin{array}{ccc} P_{n+2}^M & \xrightarrow{f_1} & P_{n+2}^N \\ \partial_{n+2}^M \downarrow & \nearrow h_{n-1} & \downarrow \partial_{n+2}^N \\ P_{n+1}^M & \xrightarrow{f_{n+1}^1, f_{n+1}^2} & P_{n+1}^N \\ \partial_{n+1}^M \downarrow & \nearrow h_n & \downarrow \partial_{n+1}^N \\ P_n^M & \xrightarrow{f} & P_n^N \end{array}$$

consider the expressions

$$g_{n+1} := f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

we can check (by diagram chasing),  $\partial_{n+1}^N \circ g = 0$ . This implies that

$$\text{im}(g_{n+1}) \subset \text{im}(\partial_{n+2}^N)$$

we can construct  $h_{n+1}$  to get the map

$$\delta_{n+2}^N \circ h_{n+1} = g_{n+1} = f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

i.e.

$$\partial_{n+2}^N \circ h_{n+1} + h_n \circ \partial_{n+1}^M = f_{n+1}^1 - f_{n+1}^2$$

hence we are done! □

**Corollary 2.1.** Any two free resolutions of an  $R$ -module  $M$  are homotopy equivalent: given two free resolutions  $P_\bullet^M, Q_\bullet^M$ , there exists extension of  $\text{id} : M \rightarrow M$  and such that

$$f_\bullet : P_\bullet^M \rightarrow Q_\bullet^M, g_\bullet : Q_\bullet^M \rightarrow P_\bullet^M$$

such that

$$g_\bullet \circ f_\bullet = \text{id}, f_\bullet \circ g_\bullet = \text{id}$$

**Problem 2.10** (HW(2.1)). Prove this corollary.

Next we define Tor functors (pretty hard things).

**Definition 2.12** (tor functors). Let  $N$  be an  $R$ -module, recall the functor

$$- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$$

we define a collection of functors

$$\text{Tor}_R^i(-, N) : \text{Mod}_R \rightarrow \text{Mod}_R, i \in \mathbb{N}$$

given an object  $M$  in  $\text{Mod}_R$ , let  $\{P_\bullet^M, \partial_\bullet^M, \epsilon_M\}$  be a free resolution of  $M$ , define  $\text{Tor}^i(M, N)$  to be

$$\text{Tor}^i(M, N) = H_i(P_\bullet^M \otimes_R N, \partial_\bullet^M \otimes \text{id}_N)$$

where

$$\cdots \rightarrow P_i^M \otimes N \xrightarrow{\partial_i \otimes \text{id}} P_{i-1}^M \otimes N \rightarrow \cdots$$

We make the remark that there is a choice involved in picking the free resolution, but this is unique since homotopies are the same on homologies.

**Problem 2.11** (HW(2.2)). For all  $i$ , show that  $\text{Tor}_R^i(M, N)$  is a well-defined functor, and any other choice of free resolution of all objects yields an isomorphic functor. Hint: use the above corollary.

**Problem 2.12** (HW(2.3)). Show that

1.  $\text{Tor}_R^i(R, N) = 0$  for all  $i > 0$
2.  $\text{Tor}_R^i(M \oplus M', N) \cong \text{Tor}_R^i(M, N) \oplus \text{Tor}_R^i(M', N)$ , given by the natural isomorphism.

We claim that  $\epsilon_M : P_0^M \rightarrow M$  induces the following isomorphism

$$\text{Tor}_R^0(M, N) \cong M \otimes_R N$$

and  $\text{Tor}_R^i(M, N)$ 's are called the higher derived functors of  $- \otimes_R N$ .

## 2.5 Lecture 7 9/18

We continue with our discussion of tor functors. We claim that

**Proposition 2.3.** The natural isomorphism gives the following

$$\text{Tor}_R^0(-, N) \cong - \otimes_R N$$

i.e., for any  $M$ ,

$$\text{Tor}_R^0(M, N) \cong M \otimes_R N$$

*Proof.* By definition,  $\text{Tor}_R^0(M, N)$  is the 0-th homology of

$$\cdots \rightarrow P_1^M \otimes_R N \xrightarrow{\partial_1 \otimes \text{id}_N} P_0^M \otimes_R N \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

this implies that

$$\text{Tor}_R^0(M, N) = \frac{P_0^M \otimes_R N}{\text{im}(\partial_1 \otimes \text{id}_R)}$$

We complete the proof using the following lemma.

**Lemma 2.1.** We claim that the functor  $-\otimes_R N$  is right exact, meaning that given a sequence of  $R$ -modules,

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \rightarrow 0$$

that is exact at  $A_0$  and  $M$ , the following sequence:

$$A_1 \otimes_R N \xrightarrow{f \otimes \text{id}} A_0 \otimes_R N \xrightarrow{g \otimes \text{id}} M \otimes_R N \rightarrow 0$$

is also exact at  $A_0 \otimes_R N$  and  $M \otimes_R N$ .

If we assume the lemma for now, then applying it to

$$P_1 \xrightarrow{\partial_1^M} P_0 \xrightarrow{\epsilon} M \rightarrow 0$$

then we are done!

We prove the lemma now: exactness of  $M \otimes_R N$  implies that  $g \otimes \text{id}$  is surjective. Given that  $g: A_0 \rightarrow M$  is surjective, every generator of  $M \otimes n$  in  $M \otimes_R N$  is of the form  $g \otimes \text{id}(a \otimes n)$  for some  $a \in A_0$ . This implies that  $g \otimes \text{id}$  is surjective.

Next, we need to show that  $\ker(g \otimes \text{id}) = \text{im}(f \otimes \text{id})$ . It is clear that  $\supset$  holds, hence it suffices to show  $\subset$ . Let  $K = \ker g \otimes \text{id}$ , we need to show that

$$\frac{A_0 \otimes_R N}{K} \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

is surjective. It is enough to construct a map:

$$M \otimes_R N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

by the first isomorphism theorem in algebra and the fact that  $g \otimes \text{id}$  is surjective. To get such a map, we need to construct a bilinear map

$$M \times N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

defined as

$$(m, n) \mapsto (a, n)$$

and let  $a = g^{-1}(m)$  be a choice of the preimage. We remark that there could be many choices of  $a$ , but the difference  $a_1 - a_2$  comes from  $f$ , since  $A_0$  is exact. This implies that this map is well-defined. This implies that the above map is surjective. There for

$$M \times N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})} \xrightarrow{g \otimes \text{id}} M \otimes_R N$$

this composition is surjective. (Two surjective maps and maps to identity=isomorphism).  $\square$



**Warning 2.6.** We saw that tensor product preserves surjectivity, but it does not necessarily preserve injectivity. Namely, if we replace the statement of the claim with SES

$$0 \rightarrow A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \rightarrow 0$$

and consider

$$0 \rightarrow A_1 \otimes N \rightarrow \dots$$

$f$  need not to be injective.

Next we see the sufficient condition for  $\text{Tor}_R^i$  to vanish for all  $i \geq 2$ .

**Corollary 2.2.** If  $R$  is a PID, then  $\operatorname{Tor}_R^i(M, N) = 0$  for all  $i \geq 2$ .

*Proof.* Consider the free resolution of  $M$ :

$$0 \rightarrow K \rightarrow R\langle M \rangle \rightarrow 0 \rightarrow \dots$$

such that  $R\langle M \rangle/K \cong M$ . Recall that all submodules of a free module are free, so we can just take  $K = P_1$ , then we have

$$0 \rightarrow K \otimes_R N \rightarrow R\langle M \rangle \otimes_R N \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

so the only homologies are  $\operatorname{Tor}_R^0, \operatorname{Tor}_R^1$ . □

**Problem 2.13 (HW(2.4)).** Calculate  $\operatorname{Tor}_{\mathbb{Z}}^1$  and  $\operatorname{Tor}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$  for  $m, n > 0$ . (Note:  $m, n$  could be equal or not).