Calc III Sections

Fall 2025

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Calc III-Week 4 (9/15-9/19)

Topics: (1) Partial derivatives. (2) Definition of total derivatives.

Problem 1. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the following functions:

1. $$x^3y^4 - xy^2$$

2. $$x^2\sin(2y) + 3$$

3. $$\ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{2}\right)$$

   You may use the following identities to simply the equation first:

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln\left(\frac{1}{a}\right) = -\ln a$$

Definition 1 (tangent plane). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0), then the **tangent plane** to the graph f in $\mathbb{R}^3$ is the given by

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)$$

Problem 2. Compute the plane tangent to the graph of $f(x, y) = x^2y + 2xy - y^2$ at $(1, 2)$.

Definition 2 (derivative for two variables). Let $f : \mathbb{R}^2 \to \mathbb{R}$, then f is said to be **differentiable** at (x_0, y_0) if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \mathcal{P}(x, y)}{\|(x, y) - (x_0, y_0)\|} = 0$$

where $\mathcal{P}(x, y) = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)$ is the tangent plane to f at (x_0, y_0).

Definition 3 (derivative for n variables). Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **gradient** of f, denoted as ∇f is given by

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right]$$

is a $1 \times n$ matrix. And f is said to be **differentiable** at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)|}{\|x - x_0\|} = 0$$

and the derivative of f is exactly the gradient ∇f at x_0.

Definition 4 (derivative for m outputs). Let $f : \mathbb{R} \to \mathbb{R}^m$, where $f(x) = (f_1(x), \ldots, f_m(x))$, then let T denote the $n \times 1$ matrix

$$T = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \cdots \\ \frac{df_m}{dx}(x_0) \end{bmatrix}$$

Then f is said to be **differentiable** at x_0 if

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - T(x - x_0)|}{|x - x_0|} = 0$$

and the matrix T is the derivative at x_0.

Example 1. Let $f(x) = (x^2, 2x, -x)$, then

$$T = Df(1) = \begin{bmatrix} 2x \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Definition 5 (derivative for general functions). Let $f : \mathbb{R}^n \to \mathbb{R}^m$, let T be the $m \times n$ matrix with entries $\partial f_i / \partial x_j$ evaluated at $x_0 \in \mathbb{R}^n$. Then f is said to be **differentiable** at x_0 if

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

then f is differentiable at x_0, and the matrix T is the derivative at x_0. Note that T loosk like

$$T = Df(x_0), \quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 2. Let $f(x, y, z) = (ze^x, -ye^z)$, then

$$Df(x, y, z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$

Problem 3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = x^2 y + y \sin(z) + ze^x.$$

Compute the gradient of f at $(1, 2, 0)$.