

Calc III Sections

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Calc III-Week 11 (11/3-7)

Topic: theorem of change of variables.

Definition 1 (injective, surjective). Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a map, then T is **injective or one-to-one** if for $x, y \in \mathbb{R}^2$

$$Tx = Ty$$

then

$$x = y.$$

We say T is **surjective or onto** if for all $y \in \mathbb{R}^2$, there exists $x \in \mathbb{R}^2$ such that

$$Tx = y$$

If T is both injective and bijective, then we say T is **bijective**.

Definition 2 (linear map). Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, then T is linear if and only if for all $x, y \in \mathbb{R}^2, \lambda \in \mathbb{R}$.

$$\begin{cases} T(x + y) = Tx + Ty \\ T(\lambda x) = \lambda Tx \end{cases}$$

Definition 3 (Jacobian Determinant). Let $T : D^* \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be of C^1 defined by

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}$$

The **Jacobian determinant** of T , denoted as $\frac{\partial(x, y)}{\partial(u, v)}$ is the determinant of the matrix $DT(u, v)$:

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Example 1. The polar coordinate transformation is given by:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

The Jacobian matrix is given by

$$\det \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix} = r$$

Proposition 1. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map, then there exists a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that

$$Tx = Ax$$

Proposition 2. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear map, then T is injective if and only if it is also surjective if and only if $\det(A) \neq 0$, where A is the matrix associated to T .

Theorem 1 (change of variables formula). Let D, D^* be elementary regions in \mathbb{R}^2 , suppose $T : D^* \rightarrow D$ is bijective. Then for any integral function $f : D \rightarrow \mathbb{R}$, the **change of variable formula** states

$$\iint_D f(x, y) dx dy = \iint_{D^*} f(x(u, v), y(u, v)) |\det(J)| du dv$$

where

$$\det(J) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

is the determinant of the Jacobian matrix.

Corollary 1. An immediate corollary of the above theorem is the change of variables into polar coordinates:

$$\iint_D f(x, y) dx dy = \iint_{D^*} f(r \cos \theta, r \sin \theta) r dr d\theta$$

Corollary 2. Let W, W^* be elementary regions in \mathbb{R}^3 , and suppose $T : W^* \rightarrow W$ is bijective. Then the change of variables formula for triple integrals states:

$$\iiint_W f(x, y, z) dx dy dz = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) |\det(J)| du dv dw$$

where

$$\det(J) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Problem 1. Compute the following integral:

$$\iint_D (x^2 + y^2)^{1/2} dx dy$$

where D is the disk $D = \{(x, y) : x^2 + y^2 \leq 9\}$.

Proof. Writing this in polar coordinates and using the change of variables formula, we have

$$\begin{aligned} \iint_D (x^2 + y^2)^{1/2} dx dy &= \int_0^{2\pi} \int_0^3 (r^2)^{1/2} r dr d\theta \\ &= \int_0^{2\pi} \int_0^3 r^2 dr d\theta \\ &= 18\pi \end{aligned}$$

□

Problem 2 (Marsden-Tromba, 6.2, Exercise 19). Compute $\iint_R (x + y)^2 e^{x-y} dx dy$, where R is the region bounded by $x + y = 1, x + y = 4, x - y = -1, x - y = 1$.

Problem 3. Compute $\iint_R xy dA$ where R is the region bounded by $xy = 1, xy = 3, y = 2, y = 6$, using the transformation $x = \frac{v}{6u}, y = 2u$.