Calc III Sections

Fall 2025

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Calc III-Week 6 (9/29-10/3)

Topics: (1) Higher-order derivatives, (2) Taylor expansion.

Proposition 1 (symmetry of second partials). Let $f(x,y)$ be twice continuously differentiable, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Definition 1 (First order Taylor expansion). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + R_1(a,x)$$

where

$$\frac{R_1(a,x)}{\|x-a\|} \to 0 \text{ as } x \to a$$

Definition 2 (Alternative Definition (First-order)). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$. Then

$$f(a+h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + R_1(a,h)$$

where $R_1(a,h)/\|h\| \to 0$ as $h \to 0$.

Definition 3 (Second order Taylor expansion). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + R_2(a,x)$$

where

$$\frac{R_2(a,x)}{\|x-a\|} \to 0 \text{ as } x \to a$$

Problem 1. Find all the second partial derivatives of $f(x,y) = xy + \ln(x-y)$.
(This includes $\partial^2 f/\partial x^2, \partial^2 f/\partial x \partial y, \partial^2 f/\partial y \partial x, \partial^2 f/\partial y^2$).

Problem 2. Write the second-order Taylor expansion for the following function,

$$f(x,y) = e^{x+y}$$

centered at $(x,y) = (0,0)$.