Harmonic Analysis

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Contents

Chapter 1

Some things to remember

Convergence of Fourier series and Fourier transforms. We define the partial sum operator for Fourier transforms as follows:

$$(S_R f)\widehat{\ } = \chi_{B_R}\widehat{f}$$

We first talk about the L^p convergence of the Fourier transform. For $n = 1$, Riesz showed that $\|S_R f - f\|_p = 0$ as $R \to \infty$. For $n > 1$, C. Fefferman then showed that S_R is not bounded unless $p = 2$.

Theorem 1.1. $S_R f$ converges to f in L^p if and only if S_R is bounded.

Proof. ($\Rightarrow$) By Uniform Boundedness Principle, either S_R is bounded or there exists $f \in L^p$ such that $\sup_R \|S_R f\|_p = \infty$. This contradicts that $\|S_R f - f\|_p \to 0$ as $R \to \infty$.

($\Leftarrow$) One can find smooth, compactly supported g such that $S_R g = g$, and $\|f - g\|_p < \epsilon$. And the result follows. $\square$

For pointwise convergence, it is the theorem by Carleson-Hunt. The Fourier series of a L^p function for $1 < p < \infty$ converges pointwise. However, the pointwise convergence of Fourier transforms is such that it converges pointwise for $1 < p \leq 2$.

Chapter 2

Chapter 1

In this chapter, we will introduce two useful covering lemmas, and prove that the maximal function is weak type (1,1) and strong (p, p). Then we will prove Calderon-Zygmund decomposition, a scheme where we can "cut" functions using maximal functions. Then we will prove a general result about Calderon-Zygmund operators, i.e., they are weak (1,1) and hence strong (p, p). Then we will do some examples and discuss some further results.

Now we prove the following theorem about operators of a certain form.

Theorem 2.1. Let T be an operator that is $\|Tf\|_q \le A\|f\|_q$, for all $f \in L^q$, and satisfies

$$\int_{B^c(y,c\delta)} |K(x,y) - K(x,\overline{y})|dx \le A, \overline{y} \in B(y,\delta)$$

Then T is bounded on $L^p \cap L^q$, for $1 < p < q$.

Proof. We would like to show that the operator is weak (1,1), and by a standard argument, we can show that it is strong (p, p), In other words, we show that

$$\mu\left(\{x : |Tf| > \alpha\}\right) \lesssim \frac{\|f\|_{L^1}}{\alpha}$$

We apply the Calderon-Zygmund decomposition, and $f = g + h$, where $f = g$ on $B^c = (\bigcup B_k^*)^c$. For $|Tf| > \alpha$, we either have $|Tg| > \alpha/2$ or $|Tb| > \alpha_2$. Hence

$$\mu\left(\{x : |Tf| > \alpha\}\right) \le \mu(\{x : |Tg| > \alpha/2\}) + \mu(\{x : |Tb| > \alpha/2\})$$

Now it suffices to show each term above is bounded by $\frac{\|f\|_{L^1}}{\alpha}$. For the g term, we have that on B^c, $|g| \lesssim \alpha$, hence

$$\int |g|^q \lesssim \alpha^{q-1}\|f\|_{L^1}$$

And on B, we have

$$\int |g|^q \lesssim \alpha^{q-1}\mu(B) \lesssim \alpha^{p-1}\|f\|_{L^1}$$

Hence we have

$$\int |g|^q \lesssim \alpha^{p-1}\|f\|_{L^1}$$

Thus

$$\mu(\{x : |Tg| > \alpha/2\}) \lesssim \alpha^{-q}\|Tg\|_{L^q}^q \lesssim \alpha^{-1}\|f\|_{L^1}$$

Hence we are done with the g part. Now for b, it suffices to show that

$$\mu(\{x : |Tb| > \alpha/2\} \cap B_1^c) \lesssim \alpha^{-1}\|f\|_{L^1}$$

where $B_1 = \bigcup B_k^{**}$. Because $\mu(B)_1 \lesssim \|f\|_{L^1}/\alpha$. This is to show that $\int_{B_1^c} |Tb| \lesssim \|f\|_{L^1}$ unfinished $\square$

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