

# Calc III Section Notes with Answers

Spring 2026

Hui Sun

January 26, 2026

# Contents

1	The Geometry of Euclidean Spaces	3
---	----------------------------------	---

# Chapter 1

## The Geometry of Euclidean Spaces

### Week 1 (1/19-23)

#### Logistics

- TA: Hui.
- Email: [hsun95@jh.edu](mailto:hsun95@jh.edu).
- Office Hours: Tuesday 2-3 PM, 4-5 pm, Krieger 211.
- Biweekly Quizzes: 10%.
- Attendance: 5%. (If you can't make it, email me).

**Definition 1** (standard basis of  $\mathbb{R}^3$ ). The following vectors

$$i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$$

are called the **standard basis** vectors of  $\mathbb{R}^3$ , and for any vector  $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ , we can write

$$a = a_1 i + a_2 j + a_3 k$$

**Definition 2** (dot product). Let  $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$ , the **dot product**  $v \cdot w$  is given by

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Alternatively,

$$v \cdot w = \|v\| \|w\| \cos \theta$$

where

$$\theta = \arccos \left( \frac{v \cdot w}{\|v\| \|w\|} \right)$$

**Definition 3** (length of vector). Let  $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ , the **length** or **norm** of  $v$ , denoted as  $\|v\|$ , is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v \cdot v}$$

**Definition 4** (linear combination). Let  $v, w \in \mathbb{R}^3$ , a **linear combination** of  $v, w$  is a sum

$$av + bw$$

for some  $a, b \in \mathbb{R}$ . One can generalize this definition to  $n$  vectors: let  $v_1, v_2, \dots, v_n \in \mathbb{R}^3$ , a linear combination of these vectors is a finite sum

$$a_1v_1 + a_2v_2 + \dots + a_nv_n$$

for some  $a_i \in \mathbb{R}, 1 \leq i \leq n$ .

**Proposition 1** (properties of the dot product). Let  $a, b, c \in \mathbb{R}^n$ , then

(a) Nonnegativity:  $a \cdot a \geq 0$ , and  $a \cdot a = 0$  if and only if  $a = 0$ .

(b) Scalar multiplication: let  $\lambda \in \mathbb{R}$ , then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry:  $a \cdot b = b \cdot a$ .

**Problem 1.** Draw the following vectors in  $\mathbb{R}^2$ :

$$u = (1, 2), \quad v = (3, -2)$$

Compute  $u + v, u - v$ , and draw them in the plane.

*Proof.*

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

**Problem 2.** Consider the following vectors in  $\mathbb{R}^3$ :

$$u = (1, 2, 3), \quad v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors  $a, b \in \mathbb{R}^3$  are called **orthogonal** if  $a \cdot b = 0$ . Are  $u, v$  orthogonal? If not, find a nonzero vector orthogonal to  $u$ .

*Proof.* 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthogonal to  $u$ :  $(-3, 0, 1)$ . Note that this vector is **not** unique! For example,  $(-1, -1, 1)$  is another such vector.

□

**Problem 3.** Can you express  $w = (1, 0)$  as a linear combination of  $v_1, v_2$  for different choices of  $v_1, v_2$ ?

1.  $v_1 = (1, 1), v_2 = (-2, -2)$ .
2.  $v_1 = (2, 1), v_2 = (-1, 0)$ .

## Week 2 (1/26-30)

**Definition 5 (determinant).** Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a  $2 \times 2$  matrix, the **determinant** of  $A$  is given by

$$\det(A) = ad - bc$$

Let  $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$  be a  $3 \times 3$  matrix, the **determinant** of  $A$  is given by

$$\det(A) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

**Definition 6 (cross product).** Let  $a, b \in \mathbb{R}^3$ , write  $a = (a_1, a_2, a_3)$ ,  $b = (b_1, b_2, b_3)$ , then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where  $i, j, k$  are the standard vectors in  $\mathbb{R}^3$ .

**Proposition 2 (properties of the cross product).** We have the following properties regarding the cross product: let  $a, b \in \mathbb{R}^3$ ,

1.  $a \times a = 0$ .
2.  $a \times b = -b \times a$ .
3.  $(a + b) \times c = a \times c + b \times c$ , and  $a \times (b + c) = a \times b + a \times c$ .
4.  $(\alpha a) \times b = \alpha(a \times b)$  for any  $\alpha \in \mathbb{R}$ .
5.  $a \times b$  is perpendicular to vectors  $a, b$ .
6. The length of the cross product is the area of the parallelogram spanned by  $a, b$ :

$$\|a \times b\| = \|a\|\|b\| \sin \theta$$

where  $0 \leq \theta \leq \pi$  is the angle between them.

7.  $a \times b = 0$  iff  $a, b$  are parallel or either  $a$  or  $b$  are 0.
8. The cross product is **not** associative! For example, compute

$$(i \times i) \times j, \quad i \times (i \times j)$$

**Proposition 3 (determinant and linear combination).** Let  $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$  be a  $3 \times 3$  matrix, let

$$a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$$

If any of  $a, b$ , or  $c$  is a linear combination of the other two vectors, then  $\det(A) = 0$ . (Relevant topic: linear independence).