Calc III Section Notes with Answers

Fall 2025

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Calc III-Week 1 (8/25-29)

Logistics

- TA: Hui.
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- Office Hours: Tuesday 4-6 PM, Krieger 211; Friday 1-2 PM Zoom.
- Biweekly Quizzes: 15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Icebreaking Activity

- In a group of three or four:
  1. Learn each other names, year, pronouns.
  2. Find something in common and different among you and share with the entire class.
  3. Play Buzz if you have time, with prime 7: say the number if it doens't contain or is not divisible by 7, say buzz otherwise.

Some Math

Problem 1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1,2), \quad v = (3,-2)$$

Compute $u+v, u-v$, and draw them in the plane.

Proof.

$$u+v = (4,0), \quad u-v = (-2,4)$$

□

Problem 2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1,2,3), \quad ,v = (-2,1,4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3,0,1)$. Note that this vector is **not** unique! For example, $(-1,-1,1)$ is another such vector.

□

Problem 3. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned}\|u+v\|^2 &= (u+v)\cdot(u+v)\\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v\\ &= \|u\|^2 + \|v\|^2\end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u\cdot v$$

given the assumption, we also have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u\cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u\cdot v = 0$$

□

Reminders

1. First HW due this Friday.
2. First Quiz next Tuesday.

Calc III-Week 2 (9/1-5)

Definition 1 (cross product). Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$, the cross product of a, b is the vector $a \times b$,

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 2 (Plane in three dimensions). A perpendicular vector and a normal vector uniquely define a plane in $\mathbb{R}^3$: given the plane $\mathcal{P}$ passing containing the point (x_0, y_0, z_0) that has a normal vector (A, B, C) is given by the equation:

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1. Here are some properties of the cross product:

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:

   $$\|a \times b\| = \|a\|\|b\| \sin\theta$$

   where θ is the angle between them. (Compare this with the dot product).
3. $a \times b = -b \times a$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. (HW) The cross product is **not** associative! For example, compute

   $$(i \times i) \times j, \quad i \times (i \times j)$$

Problem 4. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 5. Compute the plane containg all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

□

Reminders
HW is due Sunday 11:59PM.

Calc III-Week 3 (9/8-9/12)

Topics: (1) Graphing multivariable functions, (2) Introducing limits and continuity.

Definition 3 (graph). The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 4 (limit). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, ($A$ is open), let N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0 ($x_0 \in \bar{A}$,) f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that whenever $x \in U$, then $f(x) \in N$ as well.

The **limit** of f as $x \to x_0$, if it exists, is $\lim_{x \to x_0} f(x) := b \in \mathbb{R}^m$ such that f is eventually in N, for every neighborhood N of b.

Definition 5 (limit'). $\lim_{x \to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x'_1, x'_2, \ldots, x'_n)$ from **all directions**, $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 6 (continuity). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 0.1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 6. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x) = x^2 + y^2$.

Problem 7. Compute the limit of the following functions:

1. $f(x, y) = \frac{x}{x+y}$ as $(x, y) \to (0, 0)$.
2. $g(x, y) = \frac{xy}{x+y}$ as $(x, y) \to (0, 0)$.
3. $h(x, y) = \frac{\sin(xy)}{x+y}$ as $(x, y) \to (0, 0)$. Recall that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.