

# Algebra Qualifying Exam Solutions

(Accuracy Not Guaranteed)

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April 28, 2025

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# Chapter 1

## Spring 2017

**Problem 1.1.** Let  $A$  be a commutative ring, and define the *nilradical*  $\sqrt{0}$  to be the set of nilpotent elements in  $A$ . Show that  $\sqrt{0}$  is equal to the intersection of all prime ideals in  $A$ . Show that if  $A$  is reduced, then  $A$  can be embedded into a product of fields.

*Proof.* Let  $\{P_i : i \in I\}$  be the collection of prime ideals in  $A$ . We first show that

$$\sqrt{0} = \bigcap_i P_i$$

Let  $a \in \sqrt{0}$ , then for some  $n \geq 0$ ,  $a^n = 0$ , this implies that for all  $i \in I$ ,

$$a^n \in P_i \Rightarrow a \in P_i \text{ or } a^{n-1} \in P_i$$

since  $P_i$  is prime. We claim that  $a \in P_i$ . If not, then  $a^{n-1} \in P_i$  which implies  $a^{n-2} \in P_i \dots$  which eventually implies  $a \in P_i$ , which is a contradiction. Hence  $\sqrt{0} \subset \bigcap_i P_i$ . Now for the reverse inclusion, we use the following lemma:

**Lemma 1.1.** Let  $S$  be a multiplicative set in  $A$  such that  $0 \notin S$ , then there exists a prime ideal  $P \subset A$  such that

$$S \cap P = \emptyset$$

Let  $a \in \bigcap_i P_i$ , then the set

$$S = \{a, a^2, \dots\}$$

is a multiplicative set, suppose that  $a$  is not nilpotent, i.e.,  $a \notin \sqrt{0}$ , then there exists a prime ideal that does not intersect  $S$ , which is a contradiction since  $a \in P_i$  for all  $i$ . Thus

$$\sqrt{0} = \bigcap_i P_i$$

Now we show that if  $A$  is reduced, then  $A$  can be embedded into a product of fields. If  $A$  is reduced, then  $\sqrt{0} = 0$ , i.e., if  $a \neq 0$ , then  $a$  cannot be in all the prime ideals. Suppose  $a \neq 0$ , then there exists some  $P_i$  such that  $a \notin P_i$ . Then we can consider the map

$$A \rightarrow \frac{A}{P_i} \rightarrow \text{Frac}\left(\frac{A}{P_i}\right)$$

where

$$a \mapsto a + P_i \mapsto \frac{a + P_i}{1}$$

Thus we claim that  $A$  embeds in

$$A \xrightarrow{\iota} \text{Frac}\left(\frac{A}{P_1}\right) \times \text{Frac}\left(\frac{A}{P_2}\right) \times \cdots = \prod_{i \in I} \text{Frac}\left(\frac{A}{P_i}\right)$$

where  $\text{Frac}$  denotes the field of fractions. If  $a = 0$ , then  $\iota(a) = (0, \dots, 0)$ , if  $a \neq 0$ , then  $a \notin P_j$  for some  $j$ , and

$$\iota(a) = \left(0, \dots, 0, \frac{a + P_j}{1}, 0, \dots, 0\right)$$

where only the  $j$ -th entry is nonzero. □

**Problem 1.2.** Write down the minimal polynomial for  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$  and prove that it is reducible over  $\mathbb{F}_p$  for every prime number  $p$ .

*Proof.* The minimal polynomial  $p_m$  is

$$p_m(t) = (t^2 - 5)^2 - 24 = t^4 - 10t^2 + 1$$

The roots are  $\pm\sqrt{2} \pm \sqrt{3}$ , thus this polynomial generates a field extension of  $\mathbb{Q}$ ,

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \frac{\mathbb{Q}[t]}{(p_m(t))}$$

We claim that it suffices to show that  $\sqrt{2}$  or  $\sqrt{3}$  or  $\sqrt{6}$  are in  $\mathbb{F}_p$  for any prime  $p$ . Take  $\sqrt{2} \in \mathbb{F}_p$  for example, we know  $p_m(t)$  is not irreducible over  $\mathbb{Q}(\sqrt{2})$ , because then it would mean the degree of field extension  $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$  is 8, which is a contradiction.

$$\begin{array}{c} \mathbb{Q}(\sqrt{2}, \sqrt{3}) \\ \uparrow \\ \mathbb{Q}(\sqrt{2}) \\ \uparrow \\ \mathbb{Q} \end{array}$$

Thus  $p_m(t)$  is reducible over  $\mathbb{Q}(\sqrt{2})$ . Now we show the following.

**Lemma 1.2.** For any prime  $p$ ,  $\sqrt{2}$  or  $\sqrt{3}$  or  $\sqrt{6}$  are in  $\mathbb{F}_p$  for any prime  $p$ .

There exists a homomorphism (Legendre symbol)  $\varphi : \mathbb{F}_p^\times \rightarrow \{\pm 1\}$ , such that

$$\varphi(g) = \begin{cases} 1, & \text{if } g \text{ is a square} \\ -1, & \text{otherwise} \end{cases}$$

Suppose that 2, 3 are not squares, i.e.,  $\sqrt{2}, \sqrt{3} \notin \mathbb{F}_p^\times$ , then

$$\varphi(2 \cdot 3) = 1$$

which implies  $\sqrt{6} \in \mathbb{F}_p^\times$ , concluding the proof. □

**Problem 1.3.** Let  $K/k$  be a finite separable field extension, and let  $L/k$  be any field extension. Show that  $K \otimes_k L$  is a product of fields.

*Proof.* We know  $K/k$  implies there exists  $\alpha \in K$  such that

$$K = k(\alpha)$$

moreover, for any  $t \in K$ , the minimal polynomial of  $t$  factors into distinct linear factors. Let  $p_\alpha$  be the minimal polynomial of  $\alpha$ ,

$$\begin{aligned} K \otimes_k L &= \frac{k[t]}{(p_\alpha(t))} \otimes_k L \\ &= \frac{L[t]}{(p_\alpha(t))} \\ &= \frac{L[t]}{(p_\alpha^1(t)) \cdots (p_\alpha^k(t))} \end{aligned}$$

where  $p_\alpha^i(t)$  are distinct irreducible factors over in  $L[t]$ . By Chinese Remainder Theorem, we must have

$$K \otimes_k L = \frac{L[t]}{(p_\alpha^1(t))} \cdots \frac{L[t]}{(p_\alpha^k(t))}$$

i.e., a product of fields. □

**Problem 1.4.** Let  $M$  be an invertible  $n \times n$  matrix with entries in an algebraically closed field  $k$  of characteristic not 2. Show that  $M$  has a square root, i.e. there exists  $N \in \text{Mat}_{n \times n}(k)$  such that  $N^2 = M$ .

*Proof.* It suffices to show that every Jordan block

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

where  $\lambda \neq 0$  is a square. We will proceed using inductino. When  $n = 2$ , the square root of

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^{\frac{1}{2}} & \frac{1}{2}\lambda^{-\frac{1}{2}} \\ 0 & \lambda^{\frac{1}{2}} \end{bmatrix}^2$$

Now assume that  $J_k$  is a square up to  $k = n - 1$ , we want to show  $J_n$  also has a square root. We claim  $J_n$  has the following square

$$J_n = \begin{bmatrix} B^2 & x \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} B & x \\ 0 & \lambda^{1/2} \end{bmatrix}^2$$

where  $B$  is a  $(n - 1) \times (n - 1)$  matrix and  $x = (x_1, \dots, x_{n-1}), 0 = (0, \dots, 0)$ . It suffices to find such an  $x$  exists. Let  $b_1, \dots, b_{n-1}$  denote the row vectors of  $B$ , we must satisfy

$$\begin{cases} b_1 \cdot x + x_1 \lambda^{\frac{1}{2}} = 0 \\ \cdots \\ b_{n-2} \cdot x + x_{n-2} \lambda^{\frac{1}{2}} = 0 \\ b_{n-1} \cdot x + x_{n-1} \lambda^{\frac{1}{2}} = 1 \end{cases}$$

Namely, we need to find  $x$  that satisfies

$$(B + \lambda^{\frac{1}{2}}I)x = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Since  $(B + \lambda^{1/2}I)$  is invertible, there exists a unique solution, hence such  $x$  exists,  $J_n$  has a square root!  $\square$

**Problem 1.5.** Prove directly from the definition of (left) semisimple ring that every such ring is (left) Noetherian and Artinian. (You may freely use facts about semisimple, Noetherian, and Artinian modules.)

*Proof.* If  $R$  is Artinian, then  $R$  can be decomposed into a finite sum of simple rings, let  $R_1, \dots, R_n$  be simple rings, we can write

$$R = \bigoplus_{i=1}^n R_i$$

where  $R_i$  contains only the trivial ideal and  $R_i$  as ideals. Now it is quite clear that every ascending and descending chain of ideals stabilizes because there are only finitely many distinct ideals.  $\square$

**Problem 1.6.** Let  $G$  be a finite group and  $H$  an abelian subgroup. Show that every irreducible representation of  $G$  over  $\mathbb{C}$  has dimension  $\leq [G : H]$ .

*Proof.* Any irreducible representation  $\rho : H \rightarrow \mathbb{C}^\times$  is one-dimensional, and we consider induced representation of  $\rho$ ,  $\text{Ind}_H^G \rho$ , we note that  $\text{Ind}_H^G \rho$  is not necessarily irreducible, hence for any irreducible representation  $\tilde{\rho} : G \rightarrow \text{GL}_n(\mathbb{C})$ , we have

$$\dim \tilde{\rho} \leq \dim(\text{Ind}_H^G \rho)$$

and

$$\text{Ind}_H^G \rho = \bigoplus_{i=1}^n g_i H$$

where  $g_i$  are the representatives of the coset and the sum consists of exactly one copy for each coset. Hence we see

$$\dim \tilde{\rho} \leq \dim(\text{Ind}_H^G \rho) = [G : H]$$

$\square$

# Chapter 2

## Fall 2016

**Problem 2.1.** Determine  $\text{Aut}(S_3)$ .

*Proof.*  $\sigma \in \text{Aut}(S_3)$  is determined by where  $(12)$  and  $(123)$  are sent to. There are 6 options in total and all of them are homomorphisms (conjugation). It is easy to check that this group is not commutative, i.e.,

$$\text{Aut}(S_3) \cong S_3$$

□

**Problem 2.2.** A group  $G$  is a semidirect product of subgroups  $N, H \subset G$  if  $N$  is normal and every element of  $G$  has a unique presentation  $nh, n \in N, h \in H$ . Find all semidirect products (up to isomorphism) of  $N = \mathbb{Z}/11\mathbb{Z}, H = \mathbb{Z}/5\mathbb{Z}$ .

*Proof.* Let  $G = N \rtimes_{\theta} H$ , where

$$\theta : \mathbb{Z}/5\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/11\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$$

such that

$$5\theta(1) \equiv 0 \pmod{10}$$

Thus  $\theta(1)$  could be 0, 2, 4, 6, 8. When  $\theta(1) = 0$ , this gives the abelian group

$$G \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \times \frac{\mathbb{Z}}{11\mathbb{Z}}$$

We claim that all nontrivial  $\theta$  give rise to the same semidirect product, namely, the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta'} & \mathbb{Z}/10\mathbb{Z} \\ m \downarrow & & \downarrow \text{id} \\ \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta} & \mathbb{Z}/10\mathbb{Z} \end{array}$$

for  $\theta : 1 \mapsto 2$  and any  $\theta' : 1 \mapsto 4, 6, 8$ , by taking  $m$  to be the multiplication map by 2, 3, 4 respectively. Hence we see

$$\theta(h)(g) = g^{2^{2h}}$$

by observing

$$\mathbb{Z}/5\mathbb{Z} \xrightarrow{2} \mathbb{Z}/10\mathbb{Z} \xrightarrow{2^2} (\mathbb{Z}/11\mathbb{Z})^{\times} \xrightarrow{2^2 \cdot (-)} \text{Aut}(\mathbb{Z}/11\mathbb{Z})$$

In other words,

$$G = \langle g, h : g^{11} = h^5 = 1, hgh^{-1} = g^4 \rangle$$

□

**Problem 2.3.** Let  $F$  be a finite field of order  $2^n$ . Here  $n > 0$ . Determine all values of  $n$  such that the polynomial  $x^2 - x + 1$  is irreducible in  $F[x]$ .

*Proof.* We know that  $x^2 - x + 1$  is irreducible over  $\mathbb{F}_2$ , namely, it has no roots in  $\mathbb{F}_2$ . Since there is only one field of order 4, we must have

$$\mathbb{F}_4 \cong \frac{\mathbb{F}_2}{(x^2 - x + 1)}$$

Clearly  $x^2 - x + 1$  is not irreducible over  $\mathbb{F}_4$ . For any  $\mathbb{F}_{2^n}$ , we know  $(x^2 - x + 1)$  is irreducible if and only if  $\mathbb{F}_4$  does not embed into  $\mathbb{F}_{2^n}$ , i.e.,  $2 \nmid n$ . This shows that when  $n$  is odd, the polynomial  $x^2 - x + 1$  is irreducible over  $\mathbb{F}_{2^n}$ .  $\square$

**Problem 2.4.** (1) Determine the Galois group of  $x^4 - 4x^2 - 2$  over  $\mathbb{Q}$ .

(2) Let  $G$  be a group of order 8 such that  $G$  is the Galois group of a polynomial of degree 4 over  $\mathbb{Q}$ . Show that  $G$  is isomorphic to the Galois group in part (1).

*Proof.* (1) The roots of this polynomial is  $\pm\sqrt{2 \pm \sqrt{6}}$ , and notice that

$$\sqrt{2}i = \sqrt{2 + \sqrt{6}}\sqrt{2 - \sqrt{6}}$$

This gives the splitting field (Galois extension) of this polynomial as

$$\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}, \sqrt{2}i\right)$$

We see that

$$\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}\right) \cap \mathbb{Q}(\sqrt{2}i) = \emptyset$$

because the first is contained in  $\mathbb{R}$  and the second is not. We must have

$$\left[\mathbb{Q}\left(\sqrt{2 + \sqrt{6}}, \sqrt{2}i\right) / \mathbb{Q}\right] = 8$$

By part b, we see  $\text{Gal} \cong D_8$ .

(2) Any Galois group of a polynomial with 4 roots in the splitting field embeds into  $S_4$ , and we notice that  $|G| = 2^3$ ,  $|S_4| = 2^3 \cdot 3$ , i.e.,  $G$  is a Sylow 2-subgroup of  $S_4$ , and all Sylow 2-subgroups are conjugate/isomorphic of one another, hence

$$\text{Gal} \cong D_8$$

$\square$

**Problem 2.5.** Let  $A$  be a linear transformation of a finite dimensional vector space over a field of characteristic  $\neq 2$ .

(1) Define the wedge product linear transformation  $\wedge^2 A = A \wedge A$ .

(2) Prove that

$$\text{tr}(\wedge^2 A) = \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2)).$$

*Proof.* (Recall we have analogous results for  $A \otimes A$ ).



- (1) The wedge product  $A \wedge A$  is defined on the wedge product of vector spaces  $V \wedge V$ , so we first define the vector space: let  $\{v_1, \dots, v_n\}$  be the basis of  $V$ , then  $\{v_i \wedge v_j\}$  where  $i < j$  forms a basis of  $V \wedge V$ , satisfying:

1.  $v_i \wedge v_j = -v_j \wedge v_i$
2.  $(a_i v_i + a_j v_j) \wedge (b_k v_k + b_l v_l) = (a_i b_k) v_i \wedge v_k + (a_i b_l) v_i \wedge v_l + (a_j b_k) v_j \wedge v_k + (a_j b_l) v_j \wedge v_l$

And  $A \wedge A$  where  $A : V \rightarrow V$  is defined as

$$A \wedge A(v_i \wedge v_j) = Av_i \wedge Av_j$$

- (2) Consider the matrix representation of  $A = (A_{ij})$ , on the basis  $\{v_i \wedge v_j : i < j\}$ ,

$$\begin{aligned} A \wedge A(v_i \wedge v_j) &= \sum_{k,l=1}^n A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k < l} A_{ki} A_{lj} (v_k \wedge v_l) + \sum_{l < k} A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k < l} A_{ki} A_{lj} (v_k \wedge v_l) - \sum_{l < k} A_{ki} A_{lj} (v_l \wedge v_k) \end{aligned}$$

Thus the diagonal term with respect to  $v_i \wedge v_j$  is

$$A_{ii} A_{jj} - A_{ji} A_{ij}$$

Thus

$$\text{Tr}(A \wedge A) = \sum_{i < j} A_{ii} A_{jj} - A_{ji} A_{ij}$$

Now

$$\text{Tr}(A)^2 = \sum_{i=1}^n A_{ii}^2 + 2 \sum_{i < j} A_{ii} A_{jj}$$

and

$$\begin{aligned} \text{Tr}(A^2) &= \sum_{k,l=1}^n A_{lk} A_{kl} \\ &= \sum_{i=1}^n A_{ii}^2 + 2 \sum_{k < l} A_{lk} A_{kl} \end{aligned}$$

Thus we see that

$$\text{tr}(\wedge^2 A) = \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2))$$

□

**Problem 2.6.** Find a table of characters for the alternating group  $A_5$ .

*Proof.*

	1	20	15	12	12
	Id	(1 2 3)	(1 2)(3 4)	(1 2 3 4 5)	(1 2 3 5 4)
$\chi_1$	1	1	1	1	1
$\chi_2$	3	0	-1	$\phi$	$1 - \phi$
$\chi_3$	3	0	-1	$1 - \phi$	$\phi$
$\chi_4$	4	1	0	-1	-1
$\chi_5$	5	-1	1	0	0

where  $\phi = \frac{1+\sqrt{5}}{2}$ .

□

# Chapter 3

## Spring 2016

I can't do 6

**Problem 3.1.** Classify all groups of order 66, up to isomorphism.

*Proof.* There are a total of 4 groups. We have  $n_{11} = 1$ , take any Sylow-3 subgroup, we can construct a subgroup of order 33 by taking the semidirect product.

**Lemma 3.1.** Let  $H$  be a subgroup of  $G$  such that  $[G : H] = p$  is the smallest prime dividing  $|G|$ , then  $H$  is normal.

Using the lemma, we know this subgroup  $N$  of order 33 must be normal and isomorphic to  $\mathbb{Z}/33\mathbb{Z}$ . Take any Sylow 2-subgroup of  $G$ , we know

$$G = \frac{\mathbb{Z}}{33\mathbb{Z}} \rtimes_{\theta} \frac{\mathbb{Z}}{2\mathbb{Z}}$$

where  $\theta : P_2 \rightarrow \text{Aut}(N) = (\mathbb{Z}/33\mathbb{Z})^{\times}$  satisfies

$$2\theta(1) \equiv 1 \pmod{33}$$

We see there are four numbers in  $(\mathbb{Z}/33\mathbb{Z})^{\times}$  that satisfy this:

$$\theta(1) \mapsto \{1, 10, 23, 32\}$$

When  $\theta(1) = 1$ ,

$$G_1 \cong \frac{\mathbb{Z}}{33\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

When  $\theta(1) = 10$ , then

$$G_2 = \langle g, h : g^{33} = h^2 = 1, hgh^{-1} = g^{10} \rangle$$

When  $\theta(1) = 23$ , we have

$$G_3 = \langle g, h : g^{33} = h^2 = 1, hgh^{-1} = g^{23} \rangle$$

When  $\theta(1) = 32$ , we have

$$G_4 = \langle g, h : g^{33} = h^2 = 1, hgh^{-1} = g^{32} \rangle$$

□

**Problem 3.2.** Let  $F \subset K$  be an algebraic extension of fields. Let  $F \subset R \subset K$  where  $R$  is a  $F$ -subspace of  $K$  with the property such that  $\forall a \in R, a^k \in R$  for all  $k \geq 2$ .

- (1) Assume that  $\text{char}(F) \neq 2$ . Show that  $R$  is a subfield of  $K$ .
- (2) Give an example such that  $R$  may not be a field if  $\text{char}(F) = 2$ .

*Proof.* I will only do (1), because I can't do (2). It suffices to show that  $R$  is closed under multiplication and taking inverses. Let  $a, b \in R$ , we know

$$(a + b)^2 \in R \Rightarrow a^2 + b^2 + 2ab \in R \Rightarrow ab \in R$$

Since  $F \subset K$  is algebraic, for any  $a \in R$ , there exists a minimal polynomial  $p_a(t)$  such that

$$p_a(a) = c_0 + c_1a + \cdots + c_na^n = 0$$

Multiplying both sides by  $a^{-n}$  and equating we get

$$c_0a^{-n} + \cdots + c_n = c_0 + c_1a + \cdots + c_na^n$$

i.e.,

$$c_0a^{-n} = c_0 + c_1a + \cdots + c_na^n - c_n - \cdots - c_1a^{-(n-1)}$$

multiplying both sides by  $a^{n-1}$ , we see that  $a^{-1} \in R$ , as desired.  $\square$

**Problem 3.3.** Determine the Galois group of  $x^6 - 10x^3 + 1$  over  $\mathbb{Q}$ .

*Proof.* Solving for the roots we see the splitting field for this polynomial is

$$\mathbb{Q}(\sqrt{5 + 2\sqrt{6}}, \sqrt{3}i)$$

which has degree 12, i.e., the order of the Galois group. Let

$$\begin{cases} \alpha_1 = \sqrt{5 + 2\sqrt{6}} \\ \beta_1 = \sqrt{5 - 2\sqrt{6}} \\ \alpha_2 = e^{\frac{2\pi i}{3}} \sqrt{5 + 2\sqrt{6}} \\ \beta_2 = e^{\frac{2\pi i}{3}} \sqrt{5 - 2\sqrt{6}} \\ \alpha_3 = e^{\frac{4\pi i}{3}} \sqrt{5 + 2\sqrt{6}} \\ \beta_3 = e^{\frac{4\pi i}{3}} \sqrt{5 - 2\sqrt{6}} \end{cases}$$

We see that there are two choices for  $\alpha_1 \mapsto \alpha_i, \beta_j$  for any  $i, j$ . This gives a group of order 12 and by drawing a hexagon (or by guessing), one can conclude that this is  $D_{12}$ .  $\square$

**Problem 3.4.** Let  $V$  and  $W$  be two finite dimensional vector spaces over a field  $K$ . Show that for any  $q > 0$ ,

$$\bigwedge^q(V \oplus W) \cong \sum_{i=0}^q \left( \bigwedge^i(V) \otimes_K \bigwedge^{q-i}(W) \right).$$

*Proof.* Any two finite dimensional vector spaces of the same dimension are isomorphic. Hence, it suffices to show that the dimensions are equal. We will convince ourselves it holds for  $q = 2$ . Let  $\{v_1, \dots, v_n\}$  be the basis of  $V$ , and  $\{w_1, \dots, w_k\}$  be the basis of  $W$ , then we begin with the LHS:

$$\bigwedge^2(V \oplus W)$$

We note that  $V \oplus W$  has basis

$$\{(v_i, w_j) : 1 \leq i \leq n, 1 \leq j \leq k\}$$

So we reenumerate the  $n + k$  basis as

$$\{e_1, \dots, e_{n+k}\}$$

Then  $\bigwedge^q(V \oplus W)$  has basis

$$\{e_i \wedge e_j : i < j\}$$

There are exactly  $1 + \cdots + (n + k - 1)$  basis vectors i.e.,

$$\dim \left( \bigwedge^2(V \oplus W) \right) = \frac{(n + k - 1)(n + k)}{2}$$

As for the RHS:

$$\dim \left( \sum_{i=0}^2 \left( \bigwedge^i(V) \otimes_K \bigwedge^{2-i}(W) \right) \right) = \frac{(k-1)k}{2} + nk + \frac{(n-1)n}{2}$$

And we observe that two two quantities are equal. Now we do the general case, just like above,

$$\dim \left( \bigwedge^q(V \oplus W) \right) = \binom{n+k}{q}$$

And the RHS:

$$\dim \left( \bigwedge^{q-1}(V \oplus W) \wedge (V \oplus W) \right) = \sum_{i=0}^q \binom{n}{i} \binom{k}{q-i}$$

and it is clear that these two quantities are equal. □

**Problem 3.5.** Prove that a finite dimensional algebra over a field is a division algebra if and only if it does not have zero divisors.

*Proof.* Recall a finite dimensional algebra is a ring with a field action, and it is a division algebra if every nonzero element  $a \in A$  has an  $a^{-1} \in A$ . We know  $A$  does not have a zero divisor if and only if for any  $a \in A$ , the multiplication map by  $a$  is injective. Since  $A$  is a finite dimensional vector space as well, an injective map is necessarily surjective, i.e., multiplication by  $a$  is surjective, this happens if and only if  $a$  is a unit, i.e.,  $A$  is a division algebra. □

**Problem 3.6.** Let  $A$  be a semi-simple finite dimensional algebra over  $\mathbb{C}$ , and let  $V$  be a direct sum of two isomorphic simple  $A$ -modules. Find the automorphism group of the  $A$ -module  $V$ .

# Chapter 4

## Fall 2011

2,3,4,5

**Problem 4.1.** (a) Let  $G$  be a group of order 5046. Show that  $G$  cannot be a simple group. You may not appeal to the classification of finite simple groups.

(b) Let  $p$  and  $q$  be prime numbers. Show that any group of order  $p^2q$  is solvable.

*Proof.* (a)  $5046 = 29^2 \cdot 2 \cdot 3$ , we must have  $n_{29} = 1$ . This shows  $G$  is not simple.

(b) There are two cases:  $p < q$  and  $p > q$ . When  $p < q$ , we if  $n_q = p^2$  then  $n_p = 1$ , thus when  $n_p = q$ , we have  $n_q = 1$ . In other words, you either have a normal group of order  $p^2$  or  $q$ , and both cases are solvable. When  $p > q$ , then  $n_q = 1$ , also solvable. □

**Problem 4.2.** Consider the special orthogonal group  $G = SO(3, \mathbb{R})$ , namely,

$$G = \{A \in GL(3, \mathbb{R}) : A^T A = I_3, \det(A) = 1\}$$

(a) Show that for any element  $A$  in  $G$ , there exists a real number  $\alpha$  with  $-1 \leq \alpha \leq 3$  such that

$$A^3 - \alpha A^2 + \alpha A - I_3 = 0.$$

(b) For which real numbers  $\alpha$  with  $-1 \leq \alpha \leq 3$  does there exist an element  $A$  in  $G$  whose minimal polynomial is  $x^3 - \alpha x^2 + \alpha x - 1$ ? Explain your answer.

*Proof.* This question is quite computational, and I will not list all the computations here.

(a) Writing  $A$  in a general form, we see that

$$\alpha = \text{tr}(A)$$

Thus it suffices to compute the eigenvalues of  $A$  (in the algebraic closure). □

**Problem 4.3.** Let  $G$  be a cyclic group of order 100. Let  $K = \mathbb{Q}$ , the field of rational numbers, or  $K = F_p$ , the finite field with  $p$  elements,  $p$  being a prime number. For each such  $K$ , construct a Galois extension  $L/K$  whose Galois group  $\text{Gal}(L/K)$  is isomorphic to  $G$ . Explain your construction in detail.

**Problem 4.4.** Let  $\rho : S_3 \rightarrow \mathbb{C}^2$  be a two-dimensional irreducible representation of the symmetric group  $S_3$ . Decompose  $\rho^{\otimes 2}$  and  $\rho^{\otimes 3}$  into a direct sum of irreducible representations of  $S_3$ .

**Problem 4.5.** Let  $A$  be a finite-dimensional semisimple algebra over  $\mathbb{C}$ , and  $V$  an  $A$ -module of finite type (i.e., finitely-generated as an  $A$ -module). Prove that  $V$  has only finitely many  $A$ -submodules if and only if  $V$  is a direct sum of pairwise non-isomorphic irreducible (i.e., simple)  $A$ -modules.

## Chapter 5

## Spring 2007

**Problem 5.1.** Prove that the integer orthogonal group  $O_n(\mathbb{Z})$  is a finite group. (By definition, an  $n \times n$  square matrix  $X$  over  $\mathbb{Z}$  is orthogonal if  $XX^t = I_n$ .)

**Problem 5.2.** Prove that no group of order 224 is simple.

**Problem 5.3.** Write down the irreducible polynomial for  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$  and prove that it is reducible modulo  $p$  for every prime  $p$ .

**Problem 5.4.** Find the invertible elements, the zero divisors and the nilpotent elements in the following rings:

- (a)  $\mathbb{Z}/p^n\mathbb{Z}$ , where  $n$  is a natural number,  $p$  is a prime one.
- (b) the upper triangular matrices over a field.

**Problem 5.5.** Prove that the group  $\text{GL}(2, \mathbb{C})$  does not contain a subgroup isomorphic to  $S_4$ .