Calc III Practice Exam

1. Let $\vec{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + xe^z\,\mathbf{k}$ and let S be the part of the cylinder $x^2+y^2=1$ underneath $z=2$ in the first octant. Suppose S is positively oriented. Calculate
$$\iint_S \vec{F}\cdot d\vec{S}.$$
Set $D=[0,\pi/2]\times[0,2]$. We parameterize S by $\Phi\colon D\to\mathbb{R}^3$ by $\Phi(\theta,v) = (\cos\theta,\sin\theta,v)$. Then $\Phi_u = (-\sin\theta,\cos\theta,0)$ and $\Phi_v=(0,0,1)$, so the normal is given by
$$\Phi_u\times\Phi_v = \det\begin{bmatrix} i & j & k\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix} = (\cos\theta,\sin\theta,0).$$
Note that this vector is already a unit normal vector. Thus, the flux integral is
$$\begin{aligned}\iint_S \vec{F}\cdot d\vec{S} &= \iint_D \vec{F}\cdot\vec{n}\,dA\\ &= \iint_D \vec{F}(\cos\theta,\sin\theta,v)\cdot(\cos\theta,\sin\theta,0)\,dA\\ &= \iint_D \left(\cos^2\theta+\sin^2\theta\right)\,dA\\ &= \iint_D da\\ &= \text{Area}(D).\end{aligned}$$
Thus, the flux integral evaluates to $\text{Area}(D)=\pi$.

2. Let D be the closed disk of radius 4 centered at $(0,0)$. Find the maximum and minimum of $f(x,y) = x^2+\frac{1}{4}(y+1)^2+1$ on D.
We find the critical points to bound the function on the interior of D and Lagrange multipliers to bound the function on the boundary of D.

   Because it is quick, we start by finding the critical points of f. Note that $f_x = 2x$ and $f_y = \frac{1}{2}(y+1)$, so the only critical point occurs at $(x,y)=(0,-1)$. Here, we have $f(0,-1)=1$.

   We now use Lagrange multipliers to maximize the function $f(x,y) = x^2+\frac{1}{4}(y+1)^2+1$ on the boundary of D, which occurs where the constraint $g(x,y)=x^2+y^2-16$ vanishes. Namely, the extrema will occur at pairs (x,y) where $\nabla f = \lambda\nabla g$ for some $\lambda\in\mathbb{R}$, which gives the equations
$$\begin{cases} 2\lambda x = 2x,\\ 2\lambda y = \frac{1}{2}(y+1),\\ x^2+y^2=16.\end{cases}$$

The first equation implies that $x = 0$ or $\lambda = 1$. If $x = 0$, then the last equation forces $y = \pm 4$, so

$$\begin{cases} f(0, +4) = 29/4, \\ f(0, -4) = 13/4. \end{cases}$$

Lastly, if $\lambda = 1$, then $2\lambda y = \frac{1}{2}(y+1)$ implies $4y = y + 1$, so $y = -1/3$. In this case, $x^2 = 16 - 1/9$, so $f(x, y) = 16 - \frac{1}{9} + \frac{4}{9} + 1 = 17 + \frac{1}{3} = \frac{52}{3}$.

Comparing all of our values, we see that the minimum is $f(0, -1) = 1$, and the maximum is $f(\sqrt{16 - 1/9}, -1/3) = 52/3$.

3. Let $E \subset \mathbb{R}^3$ be the solid region bounded by $x^2 + y^2 = 9$, $z = 0$ and $y + z = 9$.

   (a) Sketch the region E and express it as an elementary region as a set in terms of inequalities.

   (b) Find the volume of E using a triple integral.

   We do these calculations separately.

   (a) We do not provide a sketch, but we will give the inequalities. The region should satisfy

   $$\begin{cases} x^2 + y^2 \leq 9, \\ z \geq 0, \\ y + z \leq 9. \end{cases}$$

   To provide some visualization, we note that $x^2 + y^2 \leq 9$ produces a cylinder of radius 3 with axis along the z-axis. Then $z \geq 0$ cuts off the cylinder below, and $y + z \leq 9$ chops the cylinder off above diagonally.

   (b) We use cylindrical coordinates, where $(x, y, z) = (r\cos\theta, r\sin\theta, z)$. Then $dx\,dy\,dz = r\,dr\,d\theta\,dz$. Our cylinder is cut out by $r \leq 3$ and $z \in [0, 9 - y]$, so the volume integral is

   $$\begin{aligned} V &= \int_0^3 \int_0^{2\pi} \int_0^{9 - r\sin\theta} dz\,d\theta\,dr \\ &= \int_0^3 \int_0^{2\pi} (9 - r\sin\theta) r\,d\theta\,dr \\ &= \int_0^3 \left(9r\theta + r^2\cos\theta\right)\Big|_0^{2\pi} dr \\ &= \int_0^3 18\pi r\,dr \\ &= 81\pi. \end{aligned}$$

4. Let $\vec{F} = \langle ye^{xy} - zy, xe^{xy} - xz, -xy \rangle$ be a vector field in $\mathbb{R}^3$. Let C be the intersection of the paraboloid $x = y^2 + z^2$ and the cylinder $z^2 + y^2 = 9$. Calculate

$$\int_C \vec{F} \cdot d\vec{r}.$$

One can check that F is a conservative vector field, and the intersection is a circle $z^2 + y^2 = 9, x = 9$, and we know the line integral of a conservative vector field over any simply closed line is 0.

5. Find the line integral over C, the lines connecting $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$ and $(1, 0, 1)$, oriented clockwise, for the vector field

$$\vec{F} = (x \cos(x), xy - z, e^z + y).$$

This is an application of Stokes' theorem. We know

$$\int_C F \cdot ds = \int_S \nabla \times F \cdot dS$$

where S is the rectangle at $x = 1$ traced by the four points. And we can compute

$$\nabla \times F = (2, 0, y)$$

One can also compute the unit normal vector to the rectangle that agrees with the clockwise orientation to be $(-1, 0, 0)$. This gives

$$\begin{aligned}\int_C F \cdot ds &= \int_S \nabla \times F \cdot dS \\ &= \int_0^1 \int_0^1 (2, 0, y) \cdot (-1, 0, 0) dydz \\ &= -2.\end{aligned}$$

6. Find

$$\lim_{(x,y)\to(1,2)} \frac{y^3 - 4x}{x^3 + 4y^3}.$$

The function is continuous at $(1, 2)$, and we have

$$\lim_{(x,y)\to(1,2)} \frac{y^3 - 4x}{x^3 + 4y^3} = \frac{8 - 4}{1 + 32} = \frac{4}{33}.$$

7. Let $\vec{F} = (y^2, 2xy + x)$.

   (a) Is $\vec{F}$ conservative? If so, find a potential function for $\vec{F}$. If not, justify your answer.

(b) Let C be the positively oriented triangle connecting $(0,0)$, $(0,1)$ and $(-1,0)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(a) We can check
$$\frac{\partial Q}{\partial x} = 2y + 1 \neq \frac{\partial P}{\partial y} = 2y$$
thus F is not conservative.

(b) We apply Green's theorem:
$$\begin{aligned}\int_C F \cdot dr &= \int_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dA \\ &= \int_D dA \\ &= \frac{1}{2}\end{aligned}$$

8. Let $\vec{F}(x,y,z) = (xz - y^3\cos(z))\,\mathbf{i} + x^3e^{-z}\,\mathbf{j} + ze^{x^2+y^2+z^2}\,\mathbf{k}$. Find the flux of the curl of $\vec{F}$ across the upper hemisphere of $x^2+y^2+z^2 = 1$ oriented upwards.

This is an application of Stoke's theorem.
$$\int_S F \cdot dS = \int_{\partial S} F \cdot ds$$
where ∂S is the boundary parametrized by
$$\partial S = c(t) = (\cos t, \sin t, 0), 0 \leq t \leq 2\pi$$
Thus we have
$$\begin{aligned}\int_S F \cdot dS &= \int_{\partial S} F \cdot ds \\ &= \int_0^{2\pi} F(c(t)) \cdot c'(t)dt \\ &= \int_0^{2\pi} (-\sin^3(t), \cos^3(t), 0) \cdot (-\sin t, \cos t, 0)dt \\ &= \int_0^{2\pi} \sin^4 t + \cos^4 t dt \\ &= \int_0^{2\pi} \left(1 - \frac{1}{2}\sin^2(2t)\right) dt \\ &= \frac{3\pi}{2}\end{aligned}$$