# Algebraic Topology

Hui Sun

October 6, 2024

# Contents

# Chapter 1

# Category Theory

**Instructor:** Nitu Kitchro, **Office Hours**: Monday after class, **TA:** Anna Matsui

## 1.1 Lecture 1 8/26

**Definition 1.1 (Category).** A category $\mathcal{C}$ consists of the following data:

1. A collection of objects denoted as Ob($\mathcal{C}$)
2. Given two objects $X, Y \in$ Ob($\mathcal{C}$), a collection of morphisms between $X, Y$, $f : X \to Y$, denoted as mor$_{\mathcal{C}}(X, Y)$.
3. (Composition) We have mor$_{\mathcal{C}}(X, Y) \times mor_{\mathcal{C}}(Y, Z) \to mor_{\mathcal{C}}(X, Z)$ that satisfies associativity
$$f \circ (g \circ h) = (f \circ g) \circ h$$
4. (Identity) There is a distinguished morphism for each $X$, $Id_{\mathcal{C}}(X, X)$ such that given any $f \in mor(X, Y)$, we have $f \circ id_X = id_Y \circ f = f$.

In this course, we will make the assumption that in all the categories that we work with, Ob($\mathcal{C}$) need not be a set, but given any $X, Y \in Ob(\mathcal{C})$, mor$(X, Y)$ will always be a set. Now we talka bout some examples of categories.

**Example 1.1 (Sets).** Let Ob$(Sets)$ be all the sets in the universe. Given $X, Y$ sets, mor$(X, Y)$ be all the set maps from $X$ to $Y$, and $id_X$ is the identity map.

**Example 1.2 (Top).** Let Ob$(Top)$ be all the topological spaces, and mor$(X, Y)$ be all the continuous maps from $X$ to $Y$.

**Example 1.3 (Vect$_{\mathbb{F}}$).** Let $\mathbb{F}$ be a field, and let Ob be all the $\mathbb{F}$-vector spaces. Then mor$(V, W)$ is all the $\mathbb{F}$-linear homomorphisms from $V$ to $W$, where $id_V$ is the identity homomorphism.

**Example 1.4 (Posets).** Fix a poset $P$, let Ob$(P)$ be the collection of elements in $P$, and given $p, q$ we define
$$mor(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \varnothing, & \text{otherwise} \end{cases}$$

**Problem 1.1.** **HW(Q1): check this is a category**

**Example 1.5** (Opposite category). Given a category $\mathcal{C}$, there is another category called the opposite category, denoted as $\mathcal{C}^{op}$, where

1. The objects are the same as $\mathcal{C}$
2. Given $X, Y \in \text{Ob}(C^{op})$, we have $\text{mor}_{op}(X, Y) := \text{mor}_{\mathcal{C}}(Y, X)$.
3. Moreover, given $f \in mor_{op}(X, Y), g \in mor_{op}(Y, Z)$, then $g \circ f$ in $C^{op}$ is $f \circ g : Z \to X$.

Naturally, we define isomorphisms now.

**Definition 1.2** (isomorphism). Given a category $\mathcal{C}$, and a morphism $f \in mor_C(X, Y)$, we say $f$ is an isomorphism if there exists $g \in mor_C(Y, X)$ such that

$$f \circ g = Id_Y, g \circ f = Id_X$$

Now we introduce maps between categories.

**Definition 1.3** (functor). Given categories $\mathcal{C}, \mathcal{D}$, a functor $F : C \to D$ is the following;

1. Given an object $X$ in $\mathcal{C}$, $F(X)$ is an object in $D$.
2. Given a morphism $f : X \to Y$, $F(f)$ is a functor $F(f) : F(X) \to F(Y)$. Moreover, it satisfies the following:
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(f \circ g) = F(f) \circ F(g)$. Alternatively, we can rewrite this condition as the following:

$$\begin{tikzcd}
mor(X, Y) \times mor(Y, Z) \arrow[r] \arrow[d, "mor(F) \times mor(F)"] & mor(X, Z) \arrow[d, "mor(F)"] \\
mor(F(X), F(Y)) \times mor(F(Y), F(Z)) \arrow[r] & mor(F(X), F(Z))
\end{tikzcd}$$

   such that this diagram commutes.

**Problem 1.2.** **HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

**Example 1.6.** $F : Top \to Set$, where $X \mapsto X$, where the latter is a set, and $f \mapsto f$ as set maps.

**Example 1.7.** Let $\mathbb{F}$ be a field, and $F : Sets \to \text{Vect}_{\mathbb{F}}$, where $X \mapsto \mathbb{F}\langle X \rangle$, where $\mathbb{F}\langle X \rangle$ is the free vector space over $\mathbb{F}$ on the set $X$.

**Problem 1.3.** **HW(Q3): extend this to a functor by defining $mor(f)$ and show this is a functor.**

**Example 1.8.** Let $\mathbb{F}$ be a field, then the following is a functor, $F : Sets^{op} \to \text{Vect}_{\mathbb{F}}$, where

$$hF : X \mapsto Maps(X, \mathbb{F})$$

**Problem 1.4.** **HW(Q4)**: show this extends to a functor by defining $F(f)$, and show it is a functor.

## 1.2 Lecture 2 8/28

**Definition 1.4** (contravariant functor). Let $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor from $\mathcal{C}^{op} \to \mathcal{D}$, (equivalently, $\mathcal{C} \to \mathcal{D}^{op}$).

**Problem 1.5.** **HW(Q5):** Show that the following functor $F$ from $\text{Vect}_{\mathbb{F}}$ to $\text{Vect}_{\mathbb{F}}$ extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = Hom(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let $Cat$ be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

**Definition 1.5** (natural transformation). Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation $T$ from $F$ to $G$ is the following: $T : F \Rightarrow G$:

1. given object $X \in Ob(\mathcal{C})$, $T(X) \in mor(F(X), G(X))$
2. Given $f \in mor(X, Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T(X)"'] & F(Y) \arrow[d, "T(Y)"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

   where $mor_F, mor_G$ is the identification function on morphisms by functors $F, G$

If for all $X$, $T(X)$ is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor $F$ and turn it to another functor $G$. We will (in a homework) show there exists natural transofrmation between the following two functors.

**Example 1.9.** Consider $F, G : Vect_{\mathbb{F}} \to Vect_{\mathbb{F}}$, define

$$F(V) = V \otimes_{\mathbb{F}} V/_{\langle a\otimes b - b\otimes a\rangle} = V \otimes_{\mathbb{F}} V/\Sigma_2, G(V) = (V \otimes_F V)^{\Sigma_2} = \{\alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha\}$$

Both are vector spaces are fixed under "swaps." Then a natural transformation can be defined as follows $T(V)$ :

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

**Problem 1.6.** **HW(Q6):** For the above $F, G$

1. Show that $T$ defines a natural transformation from $F$ to $G$.
2. Find conditions on $\mathbb{F}$ for $T$ being a natural isomorphism.

Next we define limits and colimits. Let $\mathcal{C}, \mathcal{D}$ be categories, $d$ be an object in $\mathcal{D}$, then we can define a functor $F_d : \mathcal{C} \to \mathcal{D}$ such that for any object $c$ in $\mathcal{C}$,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the "constant functor" on $\mathcal{D}$, i.e., every object is sent to $d$, and every morphism is sent to $id_d$.

**Definition 1.6** (colimit)**.** Given any functor $F : \mathcal{C} \to \mathcal{D}$, the colimit of $F$, denoted as $\text{colim}(F)$ is an object in $\mathcal{D}$ endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object $d$ in $D$ and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in $\mathcal{D}$, $f : \text{colim}(F) \to d$ making the following diagram commute: for any $X, Y, g$:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(g)"] \arrow[dr, "\varphi_F"] \arrow[ddr, bend right, "\varphi"'] & & F(Y) \arrow[dl, "\varphi_F"'] \arrow[ddl, bend left, "\varphi"] \\
& \text{colim}(F) \arrow[d, "f"] & \\
& d &
\end{tikzcd}$$

Next we prove some facts about colimits and give an example, where $\text{colim}(F)$ exists.

**Proposition 1.1.** If $\text{colim}F$ exists, then $\text{colim}F$ is unique up to isomorphisms.

*Proof.* Let $\text{colim}(F), \text{colim}(F)'$ be two colimits that satisfy the criteria. They are both objects in $\mathcal{D}$, then we get a morphism $f : \text{colim}(F) \to \text{colim}(F)'$, and likewise $g : \text{colim}(F) \to \text{colim}(G)'$, then

$$f \circ g : \text{colim}(F)' \to \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for $g \circ f$. $\square$

Next we demonstrate a fact via an example.

**Theorem 1.1.** Let $\mathcal{C}$ be a category where $Ob(\mathcal{C}), mor(X, Y)$ are all sets. Let $F : \mathcal{C} \to \text{Top}$ be any functor, then $\text{colim}(F)$ exists.

*Proof.* Define $\text{colim}(F) := \bigsqcup_c F(c)/\sim$, where $\sim$ is induced by the equivalence relation given by

$$y \sim F(f)y$$

where $y \in F(C_1), f : C_1 \to C_2, F(f)x \in F(C_2)$. The natural transformation we endow on $F$ as $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$:

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(C)} F(C)/\sim$$

$\square$

**Problem 1.7.** **HW(Q7):** Show that colim$(F), \varphi_F$ is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

**Definition 1.7** (limit)**.** Given a functor $F : \mathcal{C} \to \mathcal{D}$, the limit of $F$, denoted as $\lim(F)$ is an object of $\mathcal{D}$, endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object $d \in Ob(\mathcal{D})$ and a natural transformation

$$\varphi : F_d \to F$$

there exists a unique $f : \lim F \to d$ such that the following diagram commutes:

$$\begin{tikzcd}
& \lim F \arrow[d, "f"] \arrow[ddl, bend right, "\varphi_F"'] \arrow[ddr, bend left, "\varphi_F"] & \\
& d \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
F(X) \arrow[rr, "F(g)"'] & & F(Y)
\end{tikzcd}$$

Just like colimits, limits are unique up to isomorphisms.

**Problem 1.8.** **HW(Q8):** Given $F : \mathcal{C} \to \mathcal{D}$, consider $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

# Chapter 2

# Homologies, Cohomologies

## 2.1 Lecture 3 9/4

Today we define (co)chain complexes: let $R$ be a commutative ring, let $Mod_R$ denote the category of $R$-modules and $R$-module maps.

**Definition 2.1** (chain complex). A chain complex of $R$-modules is a collection of $R$-modules and $R$-modules maps

$$\cdots \to M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \ldots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all $i$. In other words, the image of previous map is contained in the kernal of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by $\{M.; \partial.^M\}$.

Next we introduce morphisms between chain complexes.

**Definition 2.2** (morphism between complexes). Let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$, a morphism $\{f.\}$ between chain complexes is a "ladder" such that the following commutes:

$$\begin{tikzcd}
\ldots \arrow[r] & M_{i+1} \arrow[r, "\partial_{i+1}^M"] \arrow[d, "f_{i+1}"'] & M_i \arrow[r, "\partial_i^M"] \arrow[d, "f_i"'] & M_{i-1} \arrow[r, "\partial_{i-1}^M"] \arrow[d, "f_{i-1}"'] & \ldots \\
\ldots \arrow[r] & N_{i+1} \arrow[r, "\partial_{i+1}^N"] & N_i \arrow[r, "\partial_i^N"] & N_{i-1} \arrow[r, "\partial_{i-1}^N"] & \ldots
\end{tikzcd}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where $\{g.\} : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, and $\{f.\} : \{N.; \partial.^N\} \to \{L.; \partial.^L\}$, which is simply vertical stacking.

**Problem 2.1.** **HW(Q9):** Prove that chain complexes of $R$-modules form a category $\mathrm{ch}_R$.

There are interesting functors $F : \mathrm{ch}_R \to Mod_R$, and we begin with the following one:

**Definition 2.3** ($H_n$, $n$th-homology)**.** Given $n \in \mathbb{Z}$, there is a functor

$$H_n : \text{ch}_R \to Mod_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial_n^M / Im\partial_{n+1}^M$$

and for $f : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, we define: $H_n(f) : H_n(\{M.; \partial.^M\}) \to H_n(\{N.; \partial.^N\})$,

$$H_n(f)[x] := [f_n(x)]$$

where $[x] \in H_n(\{M.; \partial.^M\})$.

*Proof.* We need to show $H_n$ is well-defined on objects and morphisms. We need to check that $Im\partial_{n+1} \subset \ker \partial_n$. This is a consequence of $\partial^2 = 0$.

On morphisms: for $x \in \ker \partial_n^M$, we have $f_n(x) \in \ker \partial_n^N$. This is we have

$$\partial_n^N(f_n(x) = f_{n+1})(\partial_n^M(x)) = 0$$

Moreover, we need to check that this desn't depend on the choice of representatives, i.e., we can check that

$$Im\partial_{n+1}^M \mapsto 0$$

Let $x = \partial_{n+1}^M(y)$, we have

$$f_n(x) = f_n(\partial_{n+1}^M(y)) = \partial_{n+1}^N(f_{n+1}(y)) = 0$$

$$\begin{tikzcd} M_{n+1} \arrow[r, "\partial_{n+1}^M"] \arrow[d, "f_{n+1}"'] & M_n \arrow[d, "f_n"] \\ N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n \end{tikzcd}$$

□

Next we talk about homotopy between morphisms between chain complexes.

**Definition 2.4** (homotopy)**.** Given two morphisms, $f., g. : M. \to N.$, a chain homotopy $h.$ between them is a collection of $R$-modules maps, for all $n \in \mathbb{Z}$,

$$h_n : M_n \to N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as $\partial h + h\partial = f - g$.

$$\begin{tikzcd} M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & M_n \arrow[r, "\partial_n^M"] \arrow[d, "f_n/g_n"] \arrow[dl, "h_n"] & M_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] \\ N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n \arrow[r] & N_{n-1} \end{tikzcd}$$

**Problem 2.2.** **HW(Q10):** Show that homotopy is an equivalence relation between morphisms. Hint: replace $h_n$ with $-h_n : M_n \to N_{n+1}$.

*Proof.* Reflexive is shown by defining $h_n$ to be the zero map. For symmetry, we choose $-h_n$. Transitive is a ladder. □

**Proposition 2.1.** Let $h.$ be a chain homotopy between $f.$ and $g.$, then we have an equality

$$H_n(f.) = H_n(g.)$$

where $H_n(f.), H_n(g.) : H_n(M.) \to H_n(N.)$.

*Proof.* Given $[x] \in H_n(M.)$, we have

$$\begin{aligned} H_n(f)[x] &= [f_n(x)] \\ &= [g_n(x) + \partial h.(x) + h.\partial(x)] \\ &= [g_n(x) + \partial h.(x)] \\ &= [g_n(x)] \\ &= H_n(g)[x] \end{aligned}$$

$\square$

Next we define a new category.

**Definition 2.5** ($Hch_R$)**.** Define the category $\mathrm{Hch}_R$ as follows:

1. $Ob(\mathrm{Hch}_R) = Ob(\mathrm{ch}_R)$
2. $mor_{\mathrm{Hch}_R}(M., N.) = mor_{\mathrm{ch}_R}(M., N.)/\sim$, where $\sim$ is the homotopy equivalence.

**Problem 2.3.** **HW(Q11):** Show that $Hch_R$ is a category, admitting a functor

$$F : ch_R \to Hch_R$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{ch}_R \arrow[rr, "F"] \arrow[dr, "H_n"'] & & \mathrm{Hch}_R \arrow[dl, "H_n"] \\ & \mathrm{mod}_R & \end{tikzcd}$$

Next we introduce long and short exact sequences.

**Definition 2.6** (exactness)**.** Firstly, given a pair of $R$-module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at $X_2$ if $\ker(g) = \mathrm{im}(f)$. Hence given a sequence of $R$-module maps,

$$\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \dots$$

this is called a long exact sequence if it is exact at all $X_i$. Finally, given a pair of $R$-module maps,

$$0 \to X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$$

This is a short exact sequence, and $f$ is injective, $g$ is surjective.

**Problem 2.4.** **HW(Q12):** Prove the following:

1. Given LES,
$$\cdots \to X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$
show the following is a short exact sequence:
$$0 \to \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \to 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions $X_i, Y_i$, where $i = 2,3,4$, and assume the diagram commutes and if $t_1, t_2, t_4, t_5$ are isomorphisms, show that $t_3$ is also an isomorphism.
$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "t_1"'] & X_2 \arrow[r, "f_2"] \arrow[d, "t_2"'] & X_3 \arrow[r, "f_3"] \arrow[d, "t_3"'] & X_4 \arrow[r, "f_4"] \arrow[d, "t_4"'] & X_5 \arrow[d, "t_5"'] \\
Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & Y_4 \arrow[r, "g_4"] & Y_5
\end{tikzcd}$$

Next we state the most important theorem in chain complexes:

**Theorem 2.1** (The snake lemma). Let $A. \xrightarrow{f.} B. \xrightarrow{g.} C.$ be a SES of chain complexes, i.e.,
$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$
is a short exact sequence of all $n$. Then there exists a LES of homology groups.
$$\begin{tikzcd}
 & & H_{n+1}(C) \arrow[dll, "\delta_{n-1}"'] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta_n"'] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta_{n-1}"'] \\
H_{n-2}(A) & &
\end{tikzcd}$$

## 2.2 Lecture 4 9/9

Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & C_n \arrow[d, "\delta^C"'] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

*Proof.* First we define the map $\delta_n : H_n(C) \to H_{n-1}(A)$. Let $[x] \in H_n(C)$, then $x \in \delta^C$, where $\delta^C : C_n \to C_{n-1}$. We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for $x \in C_n$, $g_n : B_n \to C_n$ is surjective, hence there exists $b \in B_n$ such that $g_n(b) = x$. Then consider $d = \delta^B(b)$, since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \operatorname{im} f_{n-1}$$

Let $y \in A_{n-1}$ be this unique $y$ such that $f_{n-1}(y) = d$, where uniqueness is by $f_{n-1}$ is injective. This is indicated in the below diagram:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & {\color{red}b} \in B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & {\color{red}x} \in C_n \arrow[d, "\delta^C"'] \\
{\color{red}y} \in A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & {\color{red}d} \in B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

We first need to check that $[y]$ does not depend on the choice of $b$. Let $g_n(b_1) = g_n(b_2) = x$, then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let $y_1, y_2$ be those determined by $b_1, b_2$, then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{tikzcd}
{\color{red}a} \in A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[d, "\delta^B"] \\
A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1}
\end{tikzcd}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e., $[y_1] = [y_2]$, as they only differ by $\operatorname{im}\delta$.

**Problem 2.5.** **HW(Q13):** Check that if $x \in \operatorname{im}\delta^C$, then $\delta_n[x] = 0$.

the proof is not finished, too lazy to tex it up ☐

Next we review the tensor products of $R$-modules. We first review $R$-bilinear maps

**Definition 2.7** (bilinear maps). Let $M, N, P$ be $R$-modules, an $R$-bilinear map $f : M \times N \to P$ is a map such that

1. $f$ is linear in both coordinates, we have $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, and similarly, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.
2. For all $r \in R$, we have $f(rm, n) = f(m, rn) = rf(m, n)$.

Next we define tensor products.

**Definition 2.8 (tensor product).** A tensor product of $M \times N$ is an $R$-module denoted by $M \otimes_R N$ such that

1. $M \otimes_R N$ comes endowed with an $R$-bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other $R$-bilinear map $f : M \times N \to P$, there exists a unique $R$-module map $\psi$ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "f"'] & M \otimes_R N \arrow[dl, "\psi"] \\ P & \end{tikzcd}$$

It is not clear that $M \otimes_R N$ exists or not. In fact, they exist!

**Theorem 2.2 ($M \otimes_R N$ exists).** Define $M \otimes_R N = R\langle M \times N \rangle / K$, where $R\langle M \times N \rangle$ is the free $R$-module on the set $M \times N$. We define $K$ as the submodule generated by the following four relations:

1. $\langle (m_1 + m_2, n) \rangle - \langle (m_1, n) \rangle - \langle (m_2, n) \rangle$
2. $\langle (m, n_1 + n_2) \rangle - \langle (m, n_1) \rangle - \langle (m, n_2) \rangle$
3. $r\langle (m, n) \rangle - \langle (rm, n) \rangle$
4. $r\langle (m, n) \rangle - \langle (m, rn) \rangle$

Moreover, the map $\varphi : M \times N \to M \otimes_R N$ given by

$$(m, n) \mapsto \langle (m, n) \rangle := m \otimes_R n$$

**Problem 2.6. HW(Q14):** show that $M \otimes_R N$ is a tensor product.

## 2.3 Lecture 5 9/11

We continue with the tensors of $R$-modules. Let $f : A \to B$ an an $R$-module map, let $N$ be some fixed $R$-module, then $f$ induces maps: $f \otimes id : A \otimes_R N \to B \otimes_R N$,

$$f \otimes id : a \otimes n \mapsto f(a) \otimes n$$

and $id \otimes f : N \otimes f : N \otimes_R A \to N \otimes_R B :$

$$id \otimes f : n \otimes a \mapsto n \otimes f(a)$$

**Problem 2.7.** **HW(Q15(a)):** Show that the following maps induce functors:

1. $- \otimes_R N : Mod_R \to Mod_R$, where
$$A \mapsto A \otimes_R N, f \mapsto f \otimes id$$

2. $N \otimes_R - : Mod_R \to Mod_R$, where
$$A \mapsto N \otimes_R A, f \mapsto id \otimes f$$

**Problem 2.8.** **HW(Q15(b)):** Show that one has the following natural isomorphisms:

1. $0 \otimes_R M \cong 0$, and $0 \otimes_R - \cong F_0$ (recall the definition of $F_0$ as a functor).
2. $R \otimes_R M \cong M$, and $R \otimes_R - \cong id$.
3. $M \otimes_R N \cong N \otimes_R M$, and $M \otimes_R - \cong - \otimes_R M$.
4. $M \otimes_R (N \otimes_R K) \cong (M \otimes_R N) \otimes_R K$.
5. $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$.

For convenience, we introduce the following definition:

**Definition 2.9** (positively graded chain complex). A positively graded chain complex $\{M.; \partial.^M\}$ is a chain complex so that $M_i = 0$ for all $i < 0$. The category of positively graded chain complexed is denoted as $ch_R^+$.

We have our first important theorem for $ch_R^+$.

**Theorem 2.3.** There exists a functor $\otimes_R$ and a natural transformation $X$ such that the following diagram of functors commutes up to some $X$:

$$\begin{tikzcd}
ch_R^+ \times ch + R^+ \arrow[r, "\otimes_R"] \arrow[d, "H_i \times H_j"'] & ch_R^+ \arrow[d, "H_{i+j}"] \\
Mod_R \times Mod_R \arrow[r, "\otimes_R"'] \arrow[ur, Rightarrow, "X"] & Mod_R
\end{tikzcd}$$

where $X : \otimes_R \circ (H_i \times H_j) \Rightarrow H_{i+j} \circ \otimes_R$ is a natural transformation.

We note that the existence of $X$ means this diagram doesn't commute "on the nose," but these two composition functors are the same up to some natural transformation. Before we given the proof, we recall that $Ob(C \times D) = Ob(C) \times Ob(D)$, $mor((X,Y),(X',Y')) = mor(X,Y) \times mor(X',Y')$.

*Proof.* We define $\otimes_R$ of positively graded chain complexes as follows: let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$ be two PGCC. Define $\{M \otimes_R N.; \partial.^M\}$:
$$(M \otimes_R N) = \bigoplus_{i+j=n} (M_i \otimes_R N_j)$$

note that the RHS is always a finite sum. Moreover, $\partial^{\otimes}$ is defined as follows:

$$\partial^{\otimes} : (M \otimes_R N)_n \to (M \otimes_R N)_{n-1} \text{ is defined on the component } M_i \otimes_R N_j \text{ (from the RHS)}$$

and
$$\partial^{\otimes}(m_i \otimes n_j) := \partial^M(m_i) \otimes n_j + (-1)^i m_i \otimes \partial^N(n_j)$$

It is easy to check that $\partial^{\otimes} \circ \partial^{\otimes} = 0$.

Now we've show $ch_R^+ \otimes_R ch_R^+$ is well-defined, it remains to define $X$, the natural transformation. We define

$$X : H_i(M.) \otimes_R H_j(N.) \to H_{i+j}(M. \otimes_R N.)$$

again, it suffices to define $X$ on elementary tensors.

$$X : [\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$$

we need to check that

1. $\partial^{\otimes}(\alpha \otimes \beta) = 0$ if $\partial^M(\alpha) = 0$ and $\partial^N(\beta) = 0$.
2. If $\alpha = \partial(r)i$, then notice that $\partial^{\otimes}(r \otimes \beta) = \alpha \otimes \beta$, similarly for $\beta$. This would show that $X$ is well-defined.

It is straightforward to check that $X$ commutes with morphisms in $ch_R^+ \times ch_R^+$. □

Next we define cochain complexes and cohomologies.

**Definition 2.10 (cochain).** A cochain of $R$-modules $(M^\bullet, \partial_M^\bullet)$ is a sequence of $R$-module maps:

$$\dots \longrightarrow M^i \xrightarrow{\partial^i} M^{i+1} \xrightarrow{\partial^{i+1}} M^{i+2} \longrightarrow \dots$$

such that $\partial \circ \partial = 0$.

Cochain complexes form a category, with morphisms $\{f^\bullet\}$ form a ladder:

$$\begin{tikzcd}
\dots \arrow[r] & M^i \arrow[r, "\partial^i"] \arrow[d, "f^i"] & M^{i+1} \arrow[r, "\partial^{i+1}"] \arrow[d, "f^{i+1}"] & M^{i+2} \arrow[r] \arrow[d, "f^{i+2}"] & \dots \\
\dots \arrow[r] & N^i \arrow[r, "\partial^i"] & N^{i+1} \arrow[r, "\partial^{i+1}"] & N^{i+2} & \dots
\end{tikzcd}$$

The $n$-th cohomology of a cochain complex $\{M^\bullet; \partial_M^\bullet\}$ is defined as:

$$H^n(M^\bullet; \partial_M^\bullet) := \frac{\ker \partial^i : M^i \to M^{i+1}}{\operatorname{im} \partial^{i-1} : M^{i-1} \to M^i}$$

We remark that there is nothing unexpected here from what we learned about chain complexes. Namely, if we reindex $\{M^\bullet; \partial_M^\bullet\}$, this defines a chain complex with $M_i' = M^{-i}$. This implies that the snake lemme holds! (with $\partial^i : H^i(C) \to H^{i+1}(A)$).

**Theorem 2.4.** There is a functor $D$ and a natural transformation $\beta$ such that the following diagram of functors commute up to the natrual transformation $\beta$:

$$\begin{tikzcd}
ch_R^{op} \arrow[r, "D"] \arrow[d, "H_n^{op}"'] & coch_R \arrow[d, "H^n"] \\
Mod_R^{op} \arrow[r, "\overline{D}"'] \arrow[ur, "\beta"] & Mod_R
\end{tikzcd}$$

where $\overline{D}(M) = Hom_R(M, R)$, and

$$D(\{M_\bullet; \partial_\bullet^M\}) \text{ is defined as } \{DM^\bullet; \partial^\bullet\}$$

where

$$DM^n := Hom_R(M_n, R), \partial^n : DM^n \to DM^{n+1} \text{ is the map induced by } \partial_{n+1} : M_{n+1} \to M_n$$

We observe that $\partial^{n+1}\partial^n = 0$ since $\partial_{n+2}\partial_{n+1} = 0$.

**Problem 2.9.** **HW(Q16):** Show that $D$ is a functor.

Next we define the natural transformation $\beta$. We have $\beta : H^n(DM) \to Hom_R(H_n(M_\bullet), R)$, such that

$$\beta : [\varphi] \mapsto \beta[\varphi]$$

let $[x] \in H_n(M_\bullet)$, where $\beta[\varphi]([x]) = \varphi(x)$ (where $\varphi \in Hom_R(M_n, \mathbb{R}), x \in M_n$).

*Proof.* We first need to show that $\beta$ is well-defined. If $x = \partial_{n+1}(y)$, then consider

$$\beta[\varphi][x] = \varphi(x) = \varphi(\partial_{n+1}(y)) = \partial^n(\varphi)(y) = 0, x \in \ker \varphi$$

Conversely, if $\varphi = \delta^{n-1}(\psi)$, we have

$$\beta[\varphi][x] = \varphi(x) = \delta^{n-1}\psi(x) = \psi(\partial_n(x)) = 0$$

It remains to check that $\beta$ commutes with morphisms in $ch_R^{op}$ (which we will do next time). □

## 2.4 Lecture 6 9/16

Today we continue our discussion of homological algebra. Let $M$ be an $R$-module.

**Definition 2.11 (resolution).** A resolution of $M$ is a positively graded chain complex $\{P_\bullet, \partial_\bullet\}$ such that

1. $H_n(P_\bullet) = 0$ for all $n > 0$
2. $H_0(P_\bullet) = \frac{P_0}{\operatorname{im} \partial_1} \cong M$, where $\partial_1 : P_1 \to P_0$.

We say $\{P_\bullet, \partial_\bullet\}$ is a free resolution if $P_i$ is a free $R$-module for each $i$.

For resolutions, we prove the following two things: first, free resolutions always exist; second, every $R$-module map can be extended to a map between their resolutions (with extra assumptions) and these extensions are unique up to homotopies.

**Proposition 2.2.** For any $M$, a free resolution for $M$ exists.

*Proof.* We proceed this inductively. Defien $P_0$ to be $R\langle M\rangle$, where it is the free $R$-module defined on the set $M$. Note that

$$R\langle M\rangle \twoheadrightarrow M \text{ is surjective} : \langle m\rangle \mapsto m$$

Let $K$ be the kernel of this map, we have an isomorphism:

$$\epsilon : P_0/K \cong M$$

Define $P_1$ as $R\langle K\rangle$, note that $P_1 \twoheadrightarrow K$, then we define

$$\partial_1 : P_1 \to P_0$$

to be the composite

$$P_1 \twoheadrightarrow K \subset P_0$$

Now we consider $P_2$: let $K_1 \subset P_1$ be the kernel of $\delta_1$, define $P_2 = R\langle K_1\rangle$, then define $\partial_2; P_2 \to P_1$ to be the composite"

$$P_2 \twoheadrightarrow K_1 \subset P_1$$

note that $\ker \delta_1 / \operatorname{im} \delta_2 = K_1/K_1 = 0$. Then we define $K_2 = \ker \delta_2$, define $P_3 = R\langle K_2\rangle, \ldots$ □

Just like the above proposition, the next theorem uses induction.

**Theorem 2.5** (extension theorem). Let $\{P_\bullet^M, \delta_\bullet^M, \epsilon_M\}$ be a free resolution on $M$, and let $\{P_\bullet^N, \delta_\bullet^N, \epsilon^N\}$ be an arbitrary resolution of $N$. Then given a map of $R$-modules $f : M \to N$, we may extend it to a map of chain complexes:

$$f_\cdot : \{P_\bullet^M, \delta_\bullet^M\} \to \{P_\bullet^N, \delta_\bullet^N\}$$

such that the following diagram commutes:

$$\begin{tikzcd} H_0(P_\bullet^M) \arrow[r, "H_0(f_\bullet)"] \arrow[d, "\epsilon_M"] & H_0(P_\bullet^N) \arrow[d, "\epsilon_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Moreover, given any two extension $f_\bullet^1, f_\bullet^2$ of $f$, we have a chain homotopy $h_\bullet$ between $f_\bullet^1, f_\bullet^2$.

Remark: if $f_\bullet$ makes the diagram commute, and $g_\bullet$ is homotopic to $f_\bullet$, then $g_\bullet$ also makes the diagram commutes: homotopy classes work the same on homologies (they are the same on the nose).

*Proof.* We will constructu $f_\bullet$ as follows. We construct $f_i$ inductively on $i$. Consider the diagram:

$$\begin{tikzcd} \cdots \arrow[d] & \cdots \arrow[d] \\ P_1^M \arrow[d] & P_1^N \arrow[d] \\ P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Since $P_0^M$ is free, and $\epsilon_N$ is surjective, we may lift $f$ on generators of $P_0^M$ by lifting the geneartors of $P_0^M$ to elements in $P_0^N$. (Note: this lift may not be unique), but this lift extends uniquely to define $f_0$. We notice that the bottom square

$$\begin{tikzcd} P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes on homologies $(H_0(P_0^M), H_0(P_0^N))$. Now we construct $f_1$:

$$\begin{tikzcd} \cdots \arrow[d] & \cdots \arrow[d] \\ P_1^M \arrow[r, dashed, "f_1"] \arrow[d] & P_1^N \arrow[d] \\ P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We will follow the purple path above. Recall that $\epsilon_M : H_0(P_0) = P_0/\operatorname{im}(\partial_1^M) \to M$ is an isomorphism. We consider the composite: $f_0 \circ \partial_1^M = g$, we have

$$\begin{aligned} \epsilon_N \circ g &= \epsilon_N \circ f \circ \partial_1^M \\ &= f \circ \epsilon_M \circ \partial_1^M \\ &= 0 \end{aligned}$$

This implies that $\operatorname{im}(g) \subset \ker(\partial_N) = \operatorname{im}(\partial_1^N)$. We can lift the generators of $P_1^M$ to elements of $P_1^N$. (Once chosen a lift, one can extend this uniquely to define $f_1$). Then we construct $f_2, f_3, \dots$ the same way by considering $f_n \circ \partial_{n+1}$ and show that it is in the kernel of $\partial_n^N$ and lift it to define $\partial_{n+1}$.

Now we homotopy time. Assume $f_\bullet^1, f_\bullet^2$ are two lifts of $f$, we construct $h : P_\bullet^M \to P_{\bullet+1}^N$. We define $h_\bullet$ inductively, starting with $h_0$ below:

$$\begin{tikzcd} P_1^M \arrow[r, dashed, "f_1"] \arrow[d, "\partial_1^M"'] & P_1^N \arrow[d, "\partial_1^N"] \\ P_0^M \arrow[r, dashed, "f_0^1, f_0^2"'] \arrow[ur, dashed, "h_0"] \arrow[d, "\varepsilon_M"'] & P_0^N \arrow[d, "\varepsilon_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We have $\epsilon_N(f_0^1 - f_0^2) = 0$, then

$$f_0^1 - f_0^2 \in \ker \epsilon_N = \operatorname{im} \delta_1^N$$

we may lift $f_0^1 - f_0^2$ on generators of $P_0^M$, where $h_0 : P_0^M \to P_1^N$. Hence

$$(h_{-1} \circ \delta_{-1}^N) + \delta_1^N \circ h_0 = f_0^1 - f_0^2$$

Inductively, we assume $h_m$ exists for $m \leq n$, then

$$\begin{tikzcd} P_{n+2}^M \arrow[r, dashed, "f_1"] \arrow[d, "\partial_{n+2}^M"'] & P_{n+2}^N \arrow[d, "\partial_{n+2}^N"] \\ P_{n+1}^M \arrow[r, dashed, "f_{n+1}^1, f_{n+1}^2"] \arrow[ur, dashed, "h_{n-1}"] \arrow[d, "\partial_{n+1}^M"'] & P_{n+1}^N \arrow[d, "\partial_{n+1}^N"] \\ P_n^M \arrow[r, "f"] \arrow[ur, dashed, "h_n"] & P_n^N \end{tikzcd}$$

consier the expressions

$$g_{n+1} := f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

we can check (by diagram chasing), $\partial_{n+1}^N \circ g = 0$. This implies that

$$\operatorname{im}(g_{n+1}) \subset \operatorname{im}(\partial_{n+2}^N)$$

we can construct $h_{n+1}$ to get the map

$$\delta_{n+2}^N \circ h_{n+1} = g_{n+1} = f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

i.e.

$$\partial_{n+2}^N \circ h_{n+1} + h_n \circ \partial_{n+1}^M = f_{n+1}^1 - f_{n+1}^2$$

hence we are done! □

**Corollary 2.1.** Any two free resolutions of an $R$-module $M$ are homotopy equivalent: given two free resolutions $P_\bullet^M, Q_\bullet^M$, there exists extension of $\mathrm{id} : M \to M$ and such that

$$f_\bullet : P_\bullet^M \to Q_\bullet^M, g_\bullet : Q_\bullet^M \to P_\bullet^M$$

such that

$$g_\bullet \circ f_\bullet = \mathrm{id}, f_\bullet \circ g_\bullet = \mathrm{id}$$

**Problem 2.10** (HW(2.1)). Prove this corollary.

Next we deinfe Tor functors (pretty hard things).

**Definition 2.12** (tor functors). Let $N$ be an $R$-module, recall the functor

$$-\otimes_R N : Mod_R \to Mod_R$$

we define a collection of functors

$$\text{Tor}^i_R(-, N) : Mod_R \to Mod_R, i \in \mathbb{N}$$

given an object $M$ in $Mod_R$, let $\{P^M_\bullet, \partial^M_\bullet, \epsilon_M\}$ be a free resolution of $M$, define $\text{Tor}^i(M, N)$ to be

$$\text{Tor}^i(M, N) = H_i(P^M_\bullet \otimes_R N, \partial^M_\bullet \otimes \text{id}_N)$$

where

$$\cdots \to P^M_i \otimes N \xrightarrow{\partial_i \otimes \text{id}} P^M_{i-1} \otimes_R N \to \ldots$$

We make the remark that there is a choice involved in picking the free resolution, but this is unique since homotopies are the same on homologies.

**Problem 2.11** (HW(2.2)). For all $i$, show that $\text{Tor}^i_R(M, N)$ is a well-defined functor, and any other choice of free resolution of all objects yields an isomorphic functor. Hint: use the above corollary.

**Problem 2.12** (HW(2.3)). Show that

1. $\text{Tor}^i_R(R, N) = 0$ for all $i > 0$
2. $\text{Tor}^i_R(M \oplus M', N) \cong \text{Tor}^i_R(M) \oplus \text{Tor}^i_R(N)$, given by the natural isomorphism.

We claim that $\epsilon_M : P^M_0 \to M$ induces the following isomorphism

$$\text{Tor}^0_R(M, N) \cong M \otimes_R N$$

and $\text{Tor}^i_R(M, N)$'s are called the highest derived functors of $-\otimes_R N$.

## 2.5 Lecture 7 9/18

We continue with our discussion of tor functors. We claim that

**Proposition 2.3.** The natural isomorphism gives the following

$$\text{Tor}^0_R(-, N) \cong -\otimes_R N$$

i.e., for any $M$,

$$\text{Tor}^0_R(M, N) \cong M \otimes_R N$$

*Proof.* By definition, $\text{Tor}^0_R(M, N)$ is the 0-th hoology of

$$\cdots \to P^M_1 \otimes_R N \xrightarrow{\partial_1 \otimes \text{id}_N} P^M_0 \otimes_R N \to 0 \to 0 \to \ldots$$

this implies that

$$\text{Tor}^0_R(M, N) = \frac{P^M_0 \otimes_R N}{\text{im}(\partial_1 \otimes \text{id}_R)}$$

We complete the proof using the following lemma.

**Lemma 2.1.** We claim that the functor $-\otimes_R N$ is right exact, meaning that give a sequence of $R$-modules,

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

that is exact at $A_0$ and $M$, the following sequence:

$$A_1 \otimes_R N \xrightarrow{f \otimes \mathrm{id}} A_0 \otimes_R N \xrightarrow{g \otimes id} M \otimes_R N \to 0$$

is also exact at $A_0 \otimes_R N$ and $M \otimes_R N$.

If we assume the lemma for now, then applying it to

$$P_1 \xrightarrow{\partial_1^M} P_0 \xrightarrow{\epsilon} M \to 0$$

then we are done!

We prove the lemma now: exactness of $M \otimes_R N$ implies that $g \otimes \mathrm{id}$ is sujrective. givn that $g; A_0 \to M$ is sujective, every generator of $M \otimes n$ in $M \otimes_R N$ s of the form $g \otimes \mathrm{id}(a \otimes n)$ for some $a \in A_0$. This implies that $g \otimes \mathrm{id}$ is surjecctive.

Next, we need to show that $\ker(g \otimes \mathrm{id}) = \mathrm{im}(f \otimes \mathrm{id})$. It is clear that $\supset$ holds, hence it suffices to show $\subset$. Let $K = \ker g \otimes \mathrm{id}$, we need to show that

$$\frac{A_0 \otimes_R N}{K} \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

is surjective. It is enough to construct a map:

$$M \otimes_R N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

by the first isomorphism thoerem in algebra and the fact that $g \otimes \mathrm{id}$ is surjective. To get such a map, we need to construct a bilinear map

$$M \times N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

defined as

$$(m, n) \mapsto (a, n)$$

and let $a = g^{-1}(m)$ be a choice of the preimage. We remark that there could be many choices of $a$, but the difference $a_1 - a_2$ comes from $f$, since $A_0$ is exact. This implies that this map is well-defined. This implies that the above map is surjective. There for

$$M \times N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})} \xrightarrow{g \otimes \mathrm{id}} M \otimes_R N$$

this composition is surjective. (Two surjective maps and maps to identity=isomorphism). □

**Warning 2.6.** We saw that tensor product preserves surjectivity, but it does not necessarily preserve injectivity. Namely, if we replace the statement of the claim with SES

$$0 \to A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

and consider

$$0 \to A_1 \otimes N \to \dots$$

$f$ need not to be injective.

Next we see the sufficient condition for $\mathrm{Tor}_R^i$ to vanish for all $i \geq 2$.

**Corollary 2.2.** If $R$ is a PID, then $\mathrm{Tor}^i_R(M,N) = 0$ for all $i \geq 2$.

*Proof.* Consider the free resolution of $M$:

$$0 \to K \to R\langle M\rangle \to 0 \to \dots$$

such that $R\langle M\rangle/K \cong M$. Recall that all submodules of a free module are free, so we can just take $K = P_1$, then we have

$$0 \to K \otimes_R N \to R\langle M\rangle \otimes_R N \to 0 \to 0 \to \dots$$

so the only homologies are $\mathrm{Tor}^0_R, \mathrm{Tor}^1_R$. □

**Problem 2.13** (HW(2.4)). Calculate $\mathrm{Tor}^1_{\mathbb{Z}}$ and $\mathrm{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $m, n > 0$. (Note: $m, n$ could be equal or not).

**Definition 2.13** (ext functor). Fix an $R$-module $N$, consider the functor

$$\mathrm{Hom}_R(-;N) : Mod_R^{op} \to Mod_R$$

Define the functors $\mathrm{Ext}^i_R(-,N) : Mod_R^{op} \to Mod_R$ as follows:

$$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(P^M_\bullet, N))$$

where $P^M_\bullet$ is a free resoltuion of $M$.

We note that if $R$ is a PID, then $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \geq 2$.

**Proposition 2.4.** We have

$$\mathrm{Ext}^0_R(M,N) \cong \mathrm{Hom}(M,N)$$

*Proof.* This requires the following lemma:

**Lemma 2.2.** If

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

is right exact, then

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(A_0,N) \to 0$$

is exact at $\mathrm{Hom}_R(M,N)$ and $\mathrm{Hom}_R(A_0,N)$.

□

**Problem 2.14** (HW(2.5)). Prove the above lemma.

**Problem 2.15** (HW(2.6)). Prove the following statements about the Ext functor.

1. $$\mathrm{Ext}^i_R\left(\bigoplus_\alpha M_\alpha, N\right) \cong \prod_\alpha \mathrm{Ext}^i_R(M_\alpha, N)$$

2. $$\mathrm{Ext}^i_R(M, \prod_\alpha N_\alpha) \cong \prod_\alpha \mathrm{Ext}^i_R(M, N_\alpha)$$

3. Calculate $$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

Next we state and prove Algebraic Kunneth theorem.

**Theorem 2.7** (AKT). Let $R$ be a PID, and let $\{M_\bullet, \partial^M_\bullet\}, \{N_\bullet, \partial^N_\bullet\}$ be PGCC of $R$-modules such that $M_i$ is free for all $i$. Then there exists a SES:

$$0 \to \bigoplus_{i+j=n} H_i(M) \otimes_R H_j(N) \xrightarrow{X} H_n((M \otimes_R N)_\bullet) \to \bigoplus_{i+j=n-1} \mathrm{Tor}^1_R(H_i(M_\bullet), H_j(N_\bullet)) \to 0$$

where $X$ denotes the algebraic crossproduct.

*Proof.* too long, will type up later ☐

## 2.6 Lecture 8 9/23

**Corollary 2.3.** Let $R$ be a field, then the algebraic crossproduct induces an isomorphism:

$$0 \to \bigoplus_{i+j=n} H_i(M) \otimes_{\mathbb{F}} H_j(N) \cong H_n(M \otimes_{\mathbb{F}} N)$$

where the isomorphism is given by the algebraic crossproduct $X$.

**Corollary 2.4** (Universal Coefficient Theorem). Let $\{M_\bullet, \partial^M_\bullet\}$ be a chain complex of free $\mathbb{Z}$-modules, and let $R$ be any commutative ring, then there is a SES:

$$0 \to H_n(M_\bullet) \otimes_{\mathbb{Z}} R \xrightarrow{f} H_n(M \otimes_{\mathbb{Z}} R) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{n-1}(M), R) \to 0$$

where $f$ is injective but not necessarily surjective (the failure to be surjective is measured by $\mathrm{Tor}^1_R$).

*Proof.* Use AKT with

$$N_i = \begin{cases} R, i = 0 \\ 0, i > 0 \end{cases} \quad , \quad H_i(N) = \begin{cases} R, i = 0 \\ 0, i \neq 0 \end{cases}$$

Hence $(M \otimes_{\mathbb{Z}} N)_\bullet = M_\bullet \otimes_{\mathbb{Z}} R$. ☐

**Problem 2.16** (HW(2.7)). Prove the UCT in cohomology: let $\{M_\bullet, \partial_\bullet^M\}$ be a chain complex of free $\mathbb{Z}$-modules, let $R$ be any commutative ring, then there exists SES

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(M), R) \to H^n(\mathrm{Hom}_{\mathbb{Z}}(M_\bullet, R)) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(M), R) \to 0$$

Hint: use the same proof for AKT, instead of $\otimes$ with $N$, you take the Hom into $R$.

# Chapter 3

# Singular Cohomology

We begin with some basic definitions.

**Definition 3.1** ($n$-simplex)**.** The standard $n$-simplex $\Delta_n \subset \mathbb{R}^{n+1}$ is defined as

$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1} : x = \sum_{i=0}^{n} t_i e_i, t_i \geq 0, \sum_{t_i} = 1 \right\}$$

where $e_i, 0 \leq i \leq n$ are the standard basis vectors of $\mathbb{R}^{n+1}$.

**Definition 3.2** (face)**.** Let $0 \leq i \leq n$, then the $i$th face $F_i$ of $\Delta_n$ is the $(n-1)$-simplex

$$F_i = \{x \in \Delta_n : t_i = 0\}$$

**Definition 3.3** (singular chain complex)**.** Given a topological space $X$, the singular chain complex of $X$, with $\mathbb{Z}$ coefficients, denoted as $S_\bullet(X, Z)$ is defined as

$$S_i(X, Z) = \begin{cases} 0, i < 0 \\ \mathbb{Z}\langle \Delta_i(X) \rangle, i \geq 0 \end{cases}$$

where $\Delta_i(X)$ is the set of continuous maps from $\Delta_i \to X$. We define $\partial_n : S_n(X, \mathbb{Z}) \to S_{n-1}(X, \mathbb{Z})$ as follows:

$$\partial_n \langle f \rangle = \sum_{i=0}^{n} (-1)^i \langle f \circ F_i \rangle$$

where $\langle f \rangle$ is a generator of $\Delta_i(X)$, and $f : \Delta_X \to X$, where

$$f \circ F_i = \Delta_{n-1} \to \Delta_n \xrightarrow{f} X$$

Note to complete this definition, one needs to check that $\partial^2 = 0$, which we did in class. might include this later

## 3.1 Lecture 9 9/25

Recall that last time, we defined the singular chain complexes $S_\bullet(X,\mathbb{Z})$ with $\mathbb{Z}$-coefficients:

$$S_i(X,\mathbb{Z}) = \begin{cases} 0, i<0 \\ \mathbb{Z}\langle\Delta_i(X)\rangle, i\geq 0 \end{cases}$$

where $\Delta_i(X)$ is the set of continuous maps from $\Delta_i$ to $X$. Now we discuss some variations of this concept.

**Definition 3.4** (relative singular chain complex with $\mathbb{Z}$-coefficients)**.** Let $A \subset X$ be a subspace, define $S_\bullet(X, A\mathbb{Z})$ by

$$S_i(X,A,\mathbb{Z}) = \begin{cases} 0, i<0 \\ \frac{\mathbb{Z}\langle\Delta_i(X)\rangle}{\mathbb{Z}\langle\Delta_i(A)\rangle}, i\geq 0 \end{cases}$$

note that the quotient is still free.

We note that $S_\bullet(X,A,\mathbb{Z})$ is a chain complex with the following $\partial$ maps such that the following diagram commutes:

$$\begin{tikzcd} S_i(A,\mathbb{Z}) \arrow[r, "\partial_i"] \arrow[d] & S_{i-1}(A,\mathbb{Z}) \arrow[d] \\ S_i(X,\mathbb{Z}) \arrow[r, "\partial_i"] \arrow[d, two heads] & S_{i-1}(X,\mathbb{Z}) \arrow[d, two heads] \\ S_i(X,A,\mathbb{Z}) \arrow[r, "\partial_i"] & S_{i-1}(X,A,\mathbb{Z}) \end{tikzcd}$$

**Definition 3.5** ($S_\bullet(X,A,R)$)**.** We define $S_\bullet(X,A,R)$, where $R$ is any commutative ring, and

$$S_\bullet(X,A,R) = S_\bullet(X,A,\mathbb{Z}) \otimes_{\mathbb{Z}} R$$

it is a chain complex of $R$-modules with $\partial_i$ induced from $S_\bullet(X,A,\mathbb{Z})$.

The last variation is as follows:

**Definition 3.6** (singular cochain complex)**.** We define the singular cochain complex of $R$-modules $S^\bullet(X,A,R)$ as follows:

$$S^i(X,A,R) := \mathrm{Hom}_{\mathbb{Z}}(S_i(X,A,\mathbb{Z}),R) = \mathrm{Hom}_R(S_i(X,A,R),R)$$

where $\partial^i$ is induced from $\partial_i$ in $S_\bullet(X,A,R)$.