

Calc III Sections

Fall 2025

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Calc III-Week 3 (9/8-9/12)

Topics: (1) Graphing multivariable functions, (2) Introducing limits and continuity.

Definition 1 (graph). The **image** of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a subset of \mathbb{R}^m ,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of \mathbb{R}^{n+m} ,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 2 (limit). Let $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$, (A is open), let N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0 ($x_0 \in \bar{A}$), f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that whenever $x \in U$, then $f(x) \in N$ as well.

The **limit** of f as $x \rightarrow x_0$, if it exists, is $\lim_{x \rightarrow x_0} f(x) := b \in \mathbb{R}^m$ such that f is eventually in N , for every neighborhood N of b .

Definition 3 (limit'). $\lim_{x \rightarrow x'} f(x) = b$ is when $x = (x_1, x_2, \dots, x_n) \rightarrow x' = (x'_1, x'_2, \dots, x'_n)$ from **all directions**, $f(x)$ approaches $b = (b_1, \dots, b_m)$.

Definition 4 (continuity). Let $f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 1. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, and $f(x) = x^2 + 1$.

2. Let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$, and $g(x) = x^2 + y^2$.

Problem 2. Compute the following limits:

1.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{y}$$

(Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$).

2.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy} - 1}{y}$$

3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{(x-y)^2}{x^2 + y^2}$$

Proof. 1. Following the hint, we see

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{y} = \lim_{(x,y) \rightarrow (0,0)} \frac{\sin xy}{xy} x = \lim_{x \rightarrow 0} x = 0$$

2. This one uses the exact same trick:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{e^{xy} - 1}{xy} \cdot y = 0$$

3. First letting $x \rightarrow 0$ along $y = 0$, we see the limit is 1; letting $x = y \rightarrow 0$, we see the limit is 0, thus the limit doesn't exist!

□

Problem 3. Compute the limit of the following functions:

1.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x}{x+y}$$

2.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x+y}$$

(Hint: try considering $y = x^2 - x$ and $y = x$)

3.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{x+y}$$

Proof. 1. First fix $x = 0$, let $y \rightarrow 0$, then the limit is 0; now fix $y = 0$, let $x \rightarrow 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \rightarrow 0$, $y \rightarrow 0$), then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x+y} = \lim_{x \rightarrow 0} \frac{x^3 - x^2}{x^2} = \lim_{x \rightarrow 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

3. We see that

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(xy)}{xy} \frac{xy}{x+y}$$

Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist!

□

How to find a limit $\lim_{x \rightarrow x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.