Calc III Sections

Fall 2025

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October 7, 2025

Calc III-Week 7 (10/6-10/10)

Topics: Extremum.

Definition 1 (quadratic function). A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic function** if it is given by

$$g(h_1, \dots, h_n) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

where (a_{ij}) is an $n \times n$ matrix. We can also write g as follows:

$$g(h_1, \dots, h_n) = [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Definition 2 (Hessian matrix). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, and suppose all the second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist, then the Hessian matrix of f is the $n \times n$ matrix given by

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

The Hessian as a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} \begin{bmatrix} h_1 & \dots & h_n \end{bmatrix} Hf(x) \begin{bmatrix} h_1 \\ \dots \\ h_n \end{bmatrix}$$

where $h = (h_1, \dots, h_n)$.

Definition 3 (degenerate/nondegenerate points). Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, let (x_0, y_0) be a critical point. We define the **discriminant**, $\boldsymbol{\mathcal{D}}$,of the Hessian by

$$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $\mathcal{D} \neq 0$, the critical point (x_0, y_0) is called **nondegenerate**; if $\mathcal{D} = 0$, the point (x_0, y_0) is called **degenerate**.

Definition 4 (positive, negative-definite). A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is called **positive-definite** if $g(h) \geq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. Similarly, g is **negative-definite** if $g(h) \leq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. (The matrix is positive-definite iff it is symmetric $A^T = A$ and the eigenvalues are nonnegative).

Definition 5 (bounded set). A set $A \subset \mathbb{R}^n$ is said to be **bounded** if there is a number $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Proposition 1 (extremums are critical points). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, where U is open. If x_0 is a local extremum, then $Df(x_0) = 0$.

Proposition 2 (extremum). Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be in C^3, and x_0 is a critical point of f. If the Hessian $Hf(x_0)$ is positive-definite, then x_0 is a local minimum of f; if $Hf(x_0)$ is negative-definite, then x_0 is a local maximum.

Proposition 3 (local minimum). Let $f(x, y)$ be of C^2, and U is open in $\mathbb{R}^2$. A point (x_0, y_0) is a strict local **minimum** of f if the following conditions hold:

1. $$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

2. $$\mathcal{D}(x_0, y_0) > 0$$

   where $\mathcal{D}$ is the **discriminant** of the Hessian, defined by

   $$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

   where Hf is the 2×2 Hessian matrix.

3. $$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ in 3, then it becomes a local maximum.

Proposition 4 (saddle points). Let $f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, if $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, and $\mathcal{D}(x_0, y_0) < 0$, where $\mathcal{D}$ is the discriminant, then the critical point (x_0, y_0) is a saddle point, i.e., neither a maximum or a minimum.

Proposition 5 (continuous functions attain extremum on closed bounded sets). Let $f : D \to \mathbb{R}$ be continuous, where D is closed and bounded in $\mathbb{R}^n$. Then f assumes its absolute maximum and absolute minimum values at some point $x_0, x_1 \in D$.

Problem 1. Is the following matrix positive-definite?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Problem 2. Find the critical point of $f(x, y) = y + x \sin y$ and classify whether it is a local max/min or a saddle point.