

Calc III Practice Exam

1. Let $\vec{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + xe^z\mathbf{k}$ and let S be the part of the cylinder $x^2 + y^2 = 1$ underneath $z = 2$ in the first octant. Suppose S is positively oriented. Calculate

$$\iint_S \vec{F} \cdot d\vec{S}.$$

Set $D = [0, \pi/2] \times [0, 2]$. We parameterize S by $\Phi: D \rightarrow \mathbb{R}^3$ by $\Phi(\theta, z) = (\cos \theta, \sin \theta, z)$. Then $\Phi_\theta = (-\sin \theta, \cos \theta, 0)$ and $\Phi_z = (0, 0, 1)$, so the normal is given by

$$\Phi_\theta \times \Phi_z = \det \begin{bmatrix} i & j & k \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\cos \theta, \sin \theta, 0).$$

Note that this vector is already a unit normal vector. Thus, the flux integral is

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_D \vec{F} \cdot \vec{n} dA \\ &= \iint_D \vec{F}(\cos \theta, \sin \theta, v) \cdot (\cos \theta, \sin \theta, 0) dA \\ &= \iint_D (\cos^2 \theta + \sin^2 \theta) dA \\ &= \iint_D da \\ &= \text{Area}(D). \end{aligned}$$

Thus, the flux integral evaluates to $\text{Area}(D) = \pi$.

2. Let D be the closed disk of radius 4 centered at $(0, 0)$. Find the maximum and minimum of $f(x, y) = x^2 + \frac{1}{4}(y+1)^2 + 1$ on D .

We find the critical points to bound the function on the interior of D and Lagrange multipliers to bound the function on the boundary of D .

Because it is quick, we start by finding the critical points of f . Note that $f_x = 2x$ and $f_y = \frac{1}{2}(y+1)$, so the only critical point occurs at $(x, y) = (0, -1)$. Here, we have $f(0, -1) = 1$.

We now use Lagrange multipliers to maximize the function $f(x, y) = x^2 + \frac{1}{4}(y+1)^2 + 1$ on the boundary of D , which occurs where the constraint $g(x, y) = x^2 + y^2 - 16$ vanishes. Namely, the extrema will occur at pairs (x, y) where $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$, which gives the equations

$$\begin{cases} 2\lambda x = 2x, \\ 2\lambda y = \frac{1}{2}(y+1), \\ x^2 + y^2 = 16. \end{cases}$$

The first equation implies that $x = 0$ or $\lambda = 1$. If $x = 0$, then the last equation forces $y = \pm 4$, so

$$\begin{cases} f(0, +4) = 29/4, \\ f(0, -4) = 13/4. \end{cases}$$

Lastly, if $\lambda = 1$, then $2\lambda y = \frac{1}{2}(y+1)$ implies $4y = y+1$, so $y = 1/3$. In this case, $x^2 = 16 - 1/9$, so $f(x, y) = 16 - \frac{1}{9} + \frac{4}{9} + 1 = 17 + \frac{1}{3} = \frac{52}{3}$.

Comparing all of our values, we see that the minimum is $f(0, -1) = 1$, and the maximum is $f(\sqrt{16 - 1/9}, -1/3) = 52/3$.

3. Let $E \subset \mathbb{R}^3$ be the solid region bounded by $x^2 + y^2 = 9$, $z = 0$ and $y + z = 9$.
 - (a) Sketch the region E and express it as an elementary region as a set in terms of inequalities.
 - (b) Find the volume of E using a triple integral.

We do these calculations separately.

- (a) We do not provide a sketch, but we will give the inequalities. The region should satisfy

$$\begin{cases} x^2 + y^2 \leq 9, \\ z \geq 0, \\ y + z \leq 9. \end{cases}$$

To provide some visualization, we note that $x^2 + y^2 \leq 9$ produces a cylinder of radius 3 with axis along the z -axis. Then $z \geq 0$ cuts off the cylinder below, and $y + z \leq 9$ chops the cylinder off above diagonally.

- (b) We use cylindrical coordinates, where $(x, y, z) = (r \cos \theta, r \sin \theta, z)$. Then $dx dy dz = r dr d\theta dz$. Our cylinder is cut out by $r \leq 3$ and $z \in [0, 9 - y]$, so the volume integral is

$$\begin{aligned} V &= \int_0^3 \int_0^{2\pi} \int_0^{9-r \sin \theta} dz d\theta dr \\ &= \int_0^3 \int_0^{2\pi} (9 - r \sin \theta)r d\theta dr \\ &= \int_0^3 (9r\theta + r^2 \cos \theta) \Big|_0^{2\pi} dr \\ &= \int_0^3 18\pi r dr \\ &= 81\pi. \end{aligned}$$

4. Let $\vec{F} = \langle ye^{xy} - zy, xe^{xy} - xz, -xy \rangle$ be a vector field in \mathbb{R}^3 . Let C be the intersection of the paraboloid $x = y^2 + z^2$ and the cylinder $z^2 + y^2 = 9$. Calculate

$$\int_C \vec{F} \cdot d\vec{r}.$$

One can check that F is a conservative vector field, and the intersection is a circle $z^2 + y^2 = 9, x = 9$, and we know the line integral of a conservative vector field over any simply closed line is 0.

5. Find the line integral over C , the lines connecting $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$ and $(1, 0, 1)$, oriented clockwise, for the vector field

$$\vec{F} = (x \cos(x), xy - z, e^z + y).$$

This is an application of Stokes' theorem. We know

$$\int_C F \cdot ds = \int_S \nabla \times F \cdot dS$$

where S is the rectangle at $x = 1$ traced by the four points. And we can compute

$$\nabla \times F = (2, 0, y)$$

One can also compute the unit normal vector to the rectangle that agrees with the clockwise orientation to be $(-1, 0, 0)$. This gives

$$\begin{aligned} \int_C F \cdot ds &= \int_S \nabla \times F \cdot dS \\ &= \int_0^1 \int_0^1 (2, 0, y) \cdot (-1, 0, 0) dy dz \\ &= -2. \end{aligned}$$

6. Find

$$\lim_{(x,y) \rightarrow (1,2)} \frac{y^3 - 4x}{x^3 + 4y^3}.$$

The function is continuous at $(1, 2)$, and we have

$$\lim_{(x,y) \rightarrow (1,2)} \frac{y^3 - 4x}{x^3 + 4y^3} = \frac{8 - 4}{1 + 32} = \frac{4}{33}.$$

7. Let $\vec{F} = (y^2, 2xy + x)$.

- (a) Is \vec{F} conservative? If so, find a potential function for \vec{F} . If not, justify your answer.

- (b) Let C be the positively oriented triangle connecting $(0, 0)$, $(0, 1)$ and $(-1, 0)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

(a) We can check

$$\frac{\partial Q}{\partial x} = 2y + 1 \neq \frac{\partial P}{\partial y} = 2y$$

thus F is not conservative.

- (b) We apply Green's theorem:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) dA \\ &= \int_D dA \\ &= \frac{1}{2}\end{aligned}$$

8. Let $\vec{F}(x, y, z) = (xz - y^3 \cos(z)) \mathbf{i} + x^3 e^{-z} \mathbf{j} + z e^{x^2 + y^2 + z^2} \mathbf{k}$. Find the flux of the curl of \vec{F} across the upper hemisphere of $x^2 + y^2 + z^2 = 1$ oriented upwards.

This is an application of Stoke's theorem.

$$\int_S \vec{F} \cdot dS = \int_{\partial S} \vec{F} \cdot ds$$

where ∂S is the boundary parametrized by

$$\partial S = c(t) = (\cos t, \sin t, 0), 0 \leq t \leq 2\pi$$

Thus we have

$$\begin{aligned}\int_S \vec{F} \cdot dS &= \int_{\partial S} \vec{F} \cdot ds \\ &= \int_0^{2\pi} F(c(t)) \cdot c'(t) dt \\ &= \int_0^{2\pi} (-\sin^3(t), \cos^3(t), 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} \sin^4 t + \cos^4 t dt \\ &= \int_0^{2\pi} \left(1 - \frac{1}{2} \sin^2(2t) \right) dt \\ &= \frac{3\pi}{2}\end{aligned}$$