Algebraic Topology

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Chapter 1

Category Theory

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Definition 1.1 (Category). A category $\mathcal{C}$ consists of the following data:

1. A collection of objects denoted as Ob($\mathcal{C}$)
2. Given two objects $X, Y \in$ Ob($\mathcal{C}$), a collection of morphisms between X, Y, $f : X \to Y$, denoted as mor$_{\mathcal{C}}(X, Y)$.
3. (Composition) We have mor$_{\mathcal{C}}(X, Y) \times mor_{\mathcal{C}}(Y, Z) \to mor_{\mathcal{C}}(X, Z)$ that satisfies associativity
$$f \circ (g \circ h) = (f \circ g) \circ h$$
4. (Identity) There is a distinguished morphism for each X, $Id_{\mathcal{C}}(X, X)$ such that given any $f \in mor(X, Y)$, we have $f \circ id_X = id_Y \circ f = f$.

In this course, we will make the assumption that in all the categories that we work with, Ob($\mathcal{C}$) need not be a set, but given any $X, Y \in Ob(\mathcal{C})$, mor(X, Y) will always be a set. Now we talka bout some examples of categories.

Example 1.1 (Sets). Let Ob$(Sets)$ be all the sets in the universe. Given X, Y sets, mor(X, Y) be all the set maps from X to Y, and id_X is the identity map.

Example 1.2 (Top). Let Ob(Top) be all the topological spaces, and mor(X, Y) be all the continuous maps from X to Y.

Example 1.3 (Vect$_{\mathbb{F}}$). Let $\mathbb{F}$ be a field, and let Ob be all the $\mathbb{F}$-vector spaces. Then mor(V, W) is all the $\mathbb{F}$-linear homomorphisms from V to W, where id_V is the identity homomorphism.

Example 1.4 (Posets). Fix a poset P, let Ob(P) be the collection of elements in P, and given p, q we define
$$mor(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \varnothing, & \text{otherwise} \end{cases}$$

Problem 1.1. **HW(Q1): check this is a category**

Example 1.5 (Opposite category). Given a category $\mathcal{C}$, there is another category called the opposite category, denoted as $\mathcal{C}^{op}$, where

1. The objects are the same as $\mathcal{C}$
2. Given $X, Y \in \mathrm{Ob}(C^{op})$, we have $\mathrm{mor}_{op}(X, Y) := \mathrm{mor}_{\mathcal{C}}(Y, X)$.
3. Moreover, given $f \in mor_{op}(X, Y), g \in mor_{op}(Y, Z)$, then $g \circ f$ in C^{op} is $f \circ g : Z \to X$.

Naturally, we define isomorphisms now.

Definition 1.2 (isomorphism). Given a category $\mathcal{C}$, and a morphism $f \in mor_C(X, Y)$, we say f is an isomorphism if there exists $g \in mor_C(Y, X)$ such that

$$f \circ g = Id_Y, g \circ f = Id_X$$

Now we introduce maps between categories.

Definition 1.3 (functor). Given categories $\mathcal{C}, \mathcal{D}$, a functor $F : C \to D$ is the following;

1. Given an object X in $\mathcal{C}$, $F(X)$ is an object in D.
2. Given a morphism $f : X \to Y$, $F(f)$ is a functor $F(f) : F(X) \to F(Y)$. Moreover, it satisfies the following:
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(f \circ g) = F(f) \circ F(g)$. Alternatively, we can rewrite this condition as the following:

$$\begin{tikzcd} mor(X,Y) \times mor(Y,Z) \arrow[r] \arrow[d, "mor(F)\times mor(F)"] & mor(X,Z) \arrow[d, "mor(F)"] \\ mor(F(X),F(Y)) \times mor(F(Y),F(Z)) \arrow[r] & mor(F(X),F(Z)) \end{tikzcd}$$

   such that this diagram commutes.

Problem 1.2. **HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

Example 1.6. $F : Top \to Set$, where $X \mapsto X$, where the latter is a set, and $f \mapsto f$ as set maps.

Example 1.7. Let $\mathbb{F}$ be a field, and $F : Sets \to \mathrm{Vect}_{\mathbb{F}}$, where $X \mapsto \mathbb{F}\langle X \rangle$, where $\mathbb{F}\langle X \rangle$ is the free vector space over $\mathbb{F}$ on the set X.

Problem 1.3. **HW(Q3): extend this to a functor by defining $mor(f)$ and show this is a functor.**

Example 1.8. Let $\mathbb{F}$ be a field, then the following is a functor, $F : Sets^{op} \to \mathrm{Vect}_{\mathbb{F}}$, where

$$hF : X \mapsto Maps(X, \mathbb{F})$$

Problem 1.4. **HW(Q4)**: show this extends to a functor by defining $F(f)$, and show it is a functor.

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Definition 1.4 (contravariant functor). Let $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor from $\mathcal{C}^{op} \to \mathcal{D}$, (equivalently, $\mathcal{C} \to \mathcal{D}^{op}$).

Problem 1.5. **HW(Q5):** Show that the following functor F from $\text{Vect}_{\mathbb{F}}$ to $\text{Vect}_{\mathbb{F}}$ extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = Hom(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let Cat be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

Definition 1.5 (natural transformation). Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation T from F to G is the following: $T : F \Rightarrow G$:

1. given object $X \in Ob(\mathcal{C})$, $T(X) \in mor(F(X), G(X))$
2. Given $f \in mor(X, Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T(X)"'] & F(Y) \arrow[d, "T(Y)"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

   where mor_F, mor_G is the identification function on morphisms by functors F, G

If for all X, $T(X)$ is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor F and turn it to another functor G. We will (in a homework) show there exists natural transofrmation between the following two functors.

Example 1.9. Consider $F, G : Vect_{\mathbb{F}} \to Vect_{\mathbb{F}}$, define

$$F(V) = V \otimes_{\mathbb{F}} V/_{\langle a\otimes b - b\otimes a\rangle} = V \otimes_{\mathbb{F}} V/\Sigma_2, G(V) = (V \otimes_F V)^{\Sigma_2} = \{\alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha\}$$

Both are vector spaces are fixed under "swaps." Then a natural transformation can be defined as follows $T(V)$:

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

Problem 1.6. **HW(Q6):** For the above F, G

1. Show that T defines a natural transformation from F to G.
2. Find conditions on $\mathbb{F}$ for T being a natural isomorphism.

Next we define limits and colimits. Let $\mathcal{C}, \mathcal{D}$ be categories, d be an object in $\mathcal{D}$, then we can define a functor $F_d : \mathcal{C} \to \mathcal{D}$ such that for any object c in $\mathcal{C}$,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the "constant functor" on $\mathcal{D}$, i.e., every object is sent to d, and every morphism is sent to id_d.

Definition 1.6 (colimit). Given any functor $F : \mathcal{C} \to \mathcal{D}$, the colimit of F, denoted as $\text{colim}(F)$ is an object in $\mathcal{D}$ endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object d in D and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in $\mathcal{D}$, $f : \text{colim}(F) \to d$ making the following diagram commute: for any X, Y, g:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(g)"] \arrow[dr, "\varphi_F"] \arrow[ddr, bend right, "\varphi"'] & & F(Y) \arrow[dl, "\varphi_F"'] \arrow[ddl, bend left, "\varphi"] \\
& \text{colim}(F) \arrow[d, "f"] & \\
& d &
\end{tikzcd}$$

Next we prove some facts about colimits and give an example, where $\text{colim}(F)$ exists.

Proposition 1.1. If $\text{colim}F$ exists, then $\text{colim}F$ is unique up to isomorphisms.

Proof. Let $\text{colim}(F), \text{colim}(F)'$ be two colimits that satisfy the criteria. They are both objects in $\mathcal{D}$, then we get a morphism $f : \text{colim}(F) \to \text{colim}(F)'$, and likewise $g : \text{colim}(F) \to \text{colim}(G)'$, then

$$f \circ g : \text{colim}(F)' \to \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for $g \circ f$. □

Next we demonstrate a fact via an example.

Theorem 1.1. Let $\mathcal{C}$ be a category where $Ob(\mathcal{C}), mor(X,Y)$ are all sets. Let $F : \mathcal{C} \to \text{Top}$ be any functor, then $\text{colim}(F)$ exists.

Proof. Define $\text{colim}(F) := \bigsqcup_c F(c)/\sim$, where $\sim$ is induced by the equivalence relation given by

$$y \sim F(f)y$$

where $y \in F(C_1), f : C_1 \to C_2, F(f)x \in F(C_2)$. The natural transformation we endow on F as $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$:

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(C)} F(C)/\sim$$

□

Problem 1.7. HW(Q7): Show that colim$(F), \varphi_F$ is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

Definition 1.7 (limit). Given a functor $F : \mathcal{C} \to \mathcal{D}$, the limit of F, denoted as $\lim(F)$ is an object of $\mathcal{D}$, endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object $d \in Ob(\mathcal{D})$ and a natural transformation

$$\varphi : F_d \to F$$

there exists a unique $f : \lim F \to d$ such that the following diagram commutes:

$$\begin{tikzcd}
& \lim F \arrow[d, "f"] \arrow[ddl, bend right, "\varphi_F"'] \arrow[ddr, bend left, "\varphi_F"] & \\
& d \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
F(X) \arrow[rr, "F(g)"'] & & F(Y)
\end{tikzcd}$$

Just like colimits, limits are unique up to isomorphisms.

Problem 1.8. HW(Q8): Given $F : \mathcal{C} \to \mathcal{D}$, consider $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

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Today we define (co)chain complexes: let R be a commutative ring, let Mod_R denote the category of R-modules and R-module maps.

Definition 1.8 (chain complex). A chain complex of R-modules is a collection of R-modules and R-modules maps

$$\cdots \to M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \dots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all i. In other words, the image of previous map is contained in the kernal of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by $\{M.; \partial.^M\}$.

Next we introduce morphisms between chain complexes.

Definition 1.9 (morphism between complexes)**.** Let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$, a morphism $\{f.\}$ between chain complexes is a "ladder" such that the following commutes:

$$\begin{tikzcd} \dots \arrow[r] & M_{i+1} \arrow[r, "\partial^M_{i+1}"] \arrow[d, "f_{i+1}"'] & M_i \arrow[r, "\partial^M_i"] \arrow[d, "f_i"'] & M_{i-1} \arrow[r, "\partial^M_{i-1}"] \arrow[d, "f_{i-1}"'] & \dots \\ \dots \arrow[r] & N_{i+1} \arrow[r, "\partial^N_{i+1}"] & N_i \arrow[r, "\partial^N_i"] & N_{i-1} \arrow[r, "\partial^N_{i-1}"] & \dots \end{tikzcd}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where $\{g.\} : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, and $\{f.\} : \{N.; \partial.^N\} \to \{L.; \partial.^L\}$, which is simply vertical stacking.

Problem 1.9. **HW(Q9):** Prove that chain complexes of R-modules form a category ch_R.

There are interesting functors $F : \mathrm{ch}_R \to Mod_R$, and we begin with the following one:

Definition 1.10 (H_n, nth-homology)**.** Given $n \in \mathbb{Z}$, there is a functor

$$H_n : \mathrm{ch}_R \to Mod_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial^M_n / Im \partial^M_{n+1}$$

and for $f : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, we define: $H_n(f) : H_n(\{M.; \partial.^M\}) \to H_n(\{N.; \partial.^N\})$,

$$H_n(f)[x] := [f_n(x)]$$

where $[x] \in H_n(\{M.; \partial.^M\})$.

Proof. We need to show H_n is well-defined on objects and morphisms. We need to check that $Im\partial_{n+1} \subset \ker \partial_n$. This is a consequence of $\partial^2 = 0$.

On morphisms: for $x \in \ker \partial^M_n$, we have $f_n(x) \in \ker \partial^N_n$. This is we have

$$\partial^N_n(f_n(x) = f_{n+1})(\partial^M_n(x)) = 0$$

Moreover, we need to check that this desn't depend on the choice of representatives, i.e., we can check that

$$Im\partial^M_{n+1} \mapsto 0$$

Let $x = \partial^M_{n+1}(y)$, we have

$$f_n(x) = f_n(\partial^M_{n+1}(y)) = \partial^N_{n+1}(f_{n+1}(y)) = 0$$

$$\begin{tikzcd} M_{n+1} \arrow[r, "\partial^M_{n+1}"] \arrow[d, "f_{n+1}"'] & M_n \arrow[d, "f_n"] \\ N_{n+1} \arrow[r, "\partial^N_{n+1}"'] & N_n \end{tikzcd}$$

□

Next we talk about homotopy between morphisms between chain complexes.

Definition 1.11 (homotopy)**.** Given two morphisms, $f., g. : M. \to N.$, a chain homotopy $h.$ between them is a collection of R-modules maps, for all $n \in \mathbb{Z}$,

$$h_n : M_n \to N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as $\partial h + h\partial = f - g$.

$$\begin{tikzcd}
M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & M_n \arrow[r, "\partial_n^M"] \arrow[d, "f_n/g_n"] \arrow[dl, "h_n"] & M_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] \\
N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n \arrow[r] & N_{n-1}
\end{tikzcd}$$

Problem 1.10. **HW(Q10):** Show that homotopy is an equivalence relation between morphisms. Hint: replace h_n with $-h_n : M_n \to N_{n+1}$.

Proof. Reflexive is shown by defining h_n to be the zero map. For symmetry, we choose $-h_n$. Transitive is a ladder. $\square$

Proposition 1.2. Let $h.$ be a chain homotopy between $f.$ and $g.$, then we have an equality

$$H_n(f.) = H_n(g.)$$

where $H_n(f.), H_n(g.) : H_n(M.) \to H_n(N.)$.

Proof. Given $[x] \in H_n(M.)$, we have

$$\begin{aligned}
H_n(f)[x] &= [f_n(x)] \\
&= [g_n(x) + \partial h.(x) + h.\partial(x)] \\
&= [g_n(x) + \partial h.(x)] \\
&= [g_n(x)] \\
&= H_n(g)[x]
\end{aligned}$$

$\square$

Next we define a new category.

Definition 1.12 (Hch_R)**.** Define the category Hch_R as follows:

1. $Ob(\mathrm{Hch}_R) = Ob(\mathrm{ch}_R)$
2. $mor_{\mathrm{Hch}_R}(M., N.) = mor_{\mathrm{ch}_R}(M., N.)/\sim$, where $\sim$ is the homotopy equivalence.

Problem 1.11. **HW(Q11):** Show that Hch_R is a category, admitting a functor

$$F : ch_R \to Hch_R$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{ch}_R \arrow[rr, "F"] \arrow[dr, "H_n"'] & & \mathrm{Hch}_R \arrow[dl, "H_n"] \\ & \mathrm{mod}_R & \end{tikzcd}$$

Next we introduce long and short exact sequences.

Definition 1.13 (exactness)**.** Firstly, given a pair of R-module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at X_2 if $\ker(g) = \mathrm{im}(f)$. Hence given a sequence of R-module maps,

$$\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \ldots$$

this is called a long exact sequence if it is exact at all X_i. Finally, given a pair of R-module maps,

$$0 \to X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$$

This is a short exact sequence, and f is injective, g is surjective.

Problem 1.12. **HW(Q12):** Prove the following:

1. Given LES,

$$\cdots \to X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

show the following is a short exact sequence:

$$0 \to \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \to 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions X_i, Y_i, where $i = 2, 3, 4$, and assume the diagram commutes and if t_1, t_2, t_4, t_5 are isomorphisms, show that t_3 is also an isomorphism.

$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d, "t_1"'] & X_2 \arrow[r, "f_2"] \arrow[d, "t_2"'] & X_3 \arrow[r, "f_3"] \arrow[d, "t_3"'] & X_4 \arrow[r, "f_4"] \arrow[d, "t_4"'] & X_5 \arrow[d, "t_5"'] \\ Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & Y_4 \arrow[r, "g_4"] & Y_5 \end{tikzcd}$$

Next we state the most important theorem in chain complexes:

Theorem 1.2 (The snake lemma)**.** Let $A. \xrightarrow{f.} B. \xrightarrow{g.} C.$ be a SES of chain complexes, i.e.,

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is a short exact sequence of all n. Then there exists a LES of homology groups.

$$\begin{tikzcd}
 & & H_{n+1}(C) \arrow[dll, bend right, "\delta_{n-1}"'] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, bend right, "\delta_n"'] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, bend right, "\delta_{n-1}"'] \\
H_{n-2}(A) & &
\end{tikzcd}$$

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Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & C_n \arrow[d, "\delta^C"'] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

Proof. First we define the map $\delta_n : H_n(C) \to H_{n-1}(A)$. Let $[x] \in H_n(C)$, then $x \in \delta^C$, where $\delta^C : C_n \to C_{n-1}$. We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for $x \in C_n$, $g_n : B_n \to C_n$ is surjective, hence there exists $b \in B_n$ such that $g_n(b) = x$. Then consider $d = \delta^B(b)$, since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \operatorname{im} f_{n-1}$$

Let $y \in A_{n-1}$ be this unique y such that $f_{n-1}(y) = d$, where uniqueness is by f_{n-1} is injective. This is indicated in the below diagram:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & b \in B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & x \in C_n \arrow[d, "\delta^C"'] \\
y \in A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & d \in B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

We first need to check that $[y]$ does not depend on the choice of b. Let $g_n(b_1) = g_n(b_2) = x$, then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let y_1, y_2 be those determined by b_1, b_2, then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{tikzcd} a \in A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[d, "\delta^B"] \\ A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1} \end{tikzcd}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e., $[y_1] = [y_2]$, as they only differ by $\operatorname{im} \delta$.

Problem 1.13. **HW(Q13):** Check that if $x \in \operatorname{im} \delta^C$, then $\delta_n[x] = 0$.

the proof is not finished □

Next we review the tensor products of R-modules. We first review R-bilinear maps

Definition 1.14 (bilinear maps)**.** Let M, N, P be R-modules, an R-bilinear map $f : M \times N \to P$ is a map such that

1. f is linear in both coordinates, we have $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, and similarly, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.
2. For all $r \in R$, we have $f(rm, n) = f(m, rn) = rf(m, n)$.

Next we define tensor products.

Definition 1.15 (tensor product)**.** A tensor product of $M \times N$ is an R-module denoted by $M \otimes_R N$ such that

1. $M \otimes_R N$ comes endowed with an R-bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other R-bilinear map $f : M \times N \to P$, there exists a unique R-module map ψ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "f"'] & M \otimes_R N \arrow[dl, "\psi"] \\ P & \end{tikzcd}$$

It is not clear that $M \otimes_R N$ exists or not. In fact, they exist!

Theorem 1.3 ($M \otimes_R N$ **exists).** Define $M \otimes_R N = R\langle M \times N\rangle / K$, where $R\langle M \times N\rangle$ is the free R-module on the set $M \times N$. We define K as the submodule generated by the following four relations:

1. $\langle (m_1 + m_2, n)\rangle - \langle (m_1, n)\rangle - \langle (m_2, n)\rangle$
2. $\langle (m, n_1 + n_2)\rangle - \langle (m, n_1)\rangle - \langle (m, n_2)\rangle$
3. $r\langle (m, n)\rangle - \langle (rm, n)\rangle$
4. $r\langle (m, n)\rangle - \langle (m, rn)\rangle$

Moreover, the map $\varphi : M \times N \to M \otimes_R N$ given by

$$(m, n) \mapsto \langle (m, n)\rangle := m \otimes_R n$$