

Functional Analysis

Fall 2025

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September 3, 2025

Chapter 1

Preliminary

Definition 1.1 (orthonormal basis). Let S be an orthonormal set in the Hilbert space such that no other orthonormal set contains S as a proper subset. Then S is called an orthonormal basis.

Proposition 1.1. Every Hilbert space admits an orthonormal basis.

Proof. Zorn's lemma. □

Remark: if H is separable, i.e., H has a countable dense subset, then the proof does not require Zorn's lemma. For example, L^2 is separable.

Proposition 1.2 (II.6, Parseval's formula). Let \mathcal{H} be a Hilbert space, and $S = \{x_n\}$ be an orthonormal basis, then for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha \in \mathcal{A}} (x_\alpha, y) x_\alpha, \quad \|y\|^2 = \sum |(x_n, y)|^2$$

where \mathcal{A} is an index set.

Proof. Bessel's inequality states that for any $\mathcal{A}' \subset \mathcal{A}$ finite, we have

$$\sum_{\alpha \in \mathcal{A}'} |(x_\alpha, y)|^2 \leq \|y\|^2 < \infty$$

It follows that $|(x_\alpha, y)| > \frac{1}{n}$ for at most finitely many α 's, and $|(x_\alpha, y)| \neq 0$ for at most countably many α 's. Let $\{\alpha_i\}_{i=1}^\infty$ be an enumeration of such α 's. Then

$$\sum_{i=1}^N |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$$

which implies

$$\sum_{i=1}^\infty |(x_{\alpha_i}, y)|^2 < \infty$$

Let

$$y_n = \sum_{i=1}^n (x_{\alpha_i}, y) x_{\alpha_i},$$

we would like to show that the sequence $\{y_n\}$ is Cauchy,

$$\|y_n - y_m\|^2 = \left\| \sum_{i=m+1}^n (x_{\alpha_i}, y) x_{\alpha_i} \right\|^2 \rightarrow 0 \text{ as } m \rightarrow \infty$$

Thus $\{y_n\}$ is Cauchy. In other words,

$$y_n \rightarrow y = \sum_{i=1}^{\infty} (x_{\alpha_i}, y) x_{\alpha_i}$$

□

Definition 1.2. A metric space is separable if it has a countable dense subset.

Proposition 1.3 (II.7). Let \mathcal{H} be a Hilbert space, then it is separable iff it has a countable orthonormal basis.

Proof. Suppose \mathcal{H} is separable, let $\{x_n\}$ be a countable dense set, then we throw out terms in $\{x_n\}$ until we get a linearly independent dense subset $\{u_n\} \subset \{x_n\}$. Applying Gram-Schmidt, we can assume $\{u_n\}$ to be countable and orthonormal. □