Algebra Qualifying Exam Solutions

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Contents

Chapter 1

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Problem 1.1. Determine Aut(S_3).

Problem 1.2. A group G is a semidirect product of subgroups $N, H \subset G$ if N is normal and every element of G has a uni $h \in H$. Find all semidirect products (up to isomorphism) of $N = \mathbb{Z}/11\mathbb{Z}$, $H = \mathbb{Z}/5\mathbb{Z}$.

Problem 1.3. Let F be a finite field of order 2^n. Here $n > 0$. Determine all values of n such that the polynomial $x^2 - x + 1$ is irreducible in $F[x]$.

Problem 1.4. (1) Determine the Galois group of $x^4 - 4x^2 - 2$ over $\mathbb{Q}$.

(2) Let G be a group of order 8 such that G is the Galois group of a polynomial of degree 4 over $\mathbb{Q}$. Show that G is isomorphic to the Galois group in part (1).

Problem 1.5. Let A be a linear transformation of a finite dimensional vector space over a field of characteristic $\neq 2$.

(1) Define the wedge product linear transformation $\wedge^2 A = A \wedge A$.

(2) Prove that

$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2)).$$

Problem 1.6. Find a table of characters for the alternating group A_5.