

# Calc III Sections

Fall 2025

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## Calc III-Week 10 (10/27-31)

Topics: (1) Divergence and Curl, (2) Double integrals.

**Definition 0.1 (divergence).** Let  $F$  be a vector field in  $\mathbb{R}^3$   $F = (F_1, F_2, F_3)$ , the divergence of  $F$  is the **scalar field** (assigns one number to an given point  $(x, y, z)$ ),

$$\operatorname{div} F := \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

More generally, if  $F = (F_1, \dots, F_n)$  is a vector field on  $\mathbb{R}^n$ , its divergence is

$$\operatorname{div} F = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n}$$

**Remark 1.** We write the divergence as  $\nabla \cdot F$  because

$$\nabla = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$$

and if  $F = (F_1, \dots, F_n)$ ,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \dots + \frac{\partial F_n}{\partial x_n} = \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right) \cdot (F_1, \dots, F_n) = \nabla \cdot F$$

**Definition 0.2 (curl).** Let  $F$  be a vector field in  $\mathbb{R}^3$ , writing  $F = (F_1, F_2, F_3)$ , the **curl** of  $F$  is the vector field

$$\operatorname{curl} F := \nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

If  $\operatorname{curl} F = 0$ , then we say the vector field is **irrotational**.

**Proposition 0.1 (gradient is irrotational).** Let  $f \in C^2$ , where  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , viewing  $\nabla f$  as a vector field, then

$$\nabla \times (\nabla f) = 0$$

**Proposition 0.2 (divergence of a curl vanishes).** For any  $C^2$  vector field  $F$ ,

$$\nabla \cdot (\nabla \times F) = 0$$

**Proposition 0.3 (Fubini's Theorem).** Let  $f$  be a continuous function on a rectangular domain  $R = [a, b] \times [c, d]$ , then

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

**Problem 0.1.** Show that the vector field  $V(x, y, z) = (x^2, -y, z)$  is not the curl of any vector field  $F$ . In other words, there is no vector field  $F$  such that

$$V = \operatorname{curl} F$$

**Problem 0.2** (Marsden-Tromba, IV.4, 21). Let  $F(x, y, z) = (x^2, x^2y, z + zx)$ . Can there exist a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  such that  $F = \nabla f$ ?

**Problem 0.3.** Change the order of integration to  $dydx$  for the following function:

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x} dx dy$$

**Problem 0.4.** Change the order of integrations for the following functions:

1.

$$\int_0^1 \int_x^1 f(x, y) dy dx$$

2.

$$\int_0^1 \int_{-y}^{y^3} f(x, y) dx dy$$

3.

$$\int_0^3 \int_{2x}^6 f(x, y) dy dx$$

4.

$$\int_0^1 \int_{-\sqrt{y}}^{y^2} f(x, y) dx dy$$

5.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 f(x, y) dx dy$$

6.

$$\int_0^1 \int_{\ln y}^1 f(x, y) dx dy$$