Algebra Qualifying Exam Solutions (Accuracy Not Guaranteed)

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April 27, 2025

Contents

Chapter 1

Spring 2017

Problem 1.1. Let A be a commutative ring, and define the *nilradical* $\sqrt{0}$ to be the set of nilpotent elements in A. Show that $\sqrt{0}$ is equal to the intersection of all prime ideals in A. Show that if A is reduced, then A can be embedded into a product of fields.

Proof. Let $\{P_i : i \in I\}$ be the collection of prime ideals in A. We first show that

$$\sqrt{0} = \bigcap_i P_i$$

Let $a \in \sqrt{0}$, then for some $n \geq 0$, $a^n = 0$, this implies that for all $i \in I$,

$$a^n \in P_i \Rightarrow a \in P_i \text{ or } a^{n-1} \in P_i$$

since P_i is prime. We claim that $a \in P_i$. If not, then $a^{n-1} \in P_i$ which implie $a^{n-2} \in P_i \dots$ which eventually implies $a \in P_i$, which is a contradiction. Hence $\sqrt{0} \subset \bigcap_i P_i$. Now for the reverse inclusion, we use the following lemma:

Lemma 1.1. Let S be a multiplicative set in A such that $0 \notin S$, then there exists a prime ideal $P \subset A$ such that

$$S \cap P = \varnothing$$

Let $a \in \bigcap_i P_i$, then the set

$$S = \{a, a^2, \dots\}$$

is a multiplicative set, suppose that a is not nilpotent, i.e., $a \notin S$, then there exists a prime ideal that does not interserct S, which is a contradiction since $a \in P_i$ for all i. Thus

$$\sqrt{0} = \bigcap_i P_i$$

Now we show that if A is reduced, then A can be embedded into a product of fields. If A is reduced, then $\sqrt{0} = 0$, i.e., if $a \neq 0$, then a cannot be in all the prime ideals. Suppose $a \neq 0$, then there exists some P_i such that $a \notin P_i$. Then we can consider the map

$$A \to \frac{A}{P_i} \to \text{Frac}\left(\frac{A}{P_i}\right)$$

where

$$a \mapsto a + P_i \mapsto \frac{a + P_i}{1}$$

Thus we claim that A embeds in

$$A \xrightarrow{\iota} \mathrm{Frac}\left(\frac{A}{P_1}\right) \times \mathrm{Frac}\left(\frac{A}{P_2}\right) \times \cdots = \prod_{i \in I} \mathrm{Frac}\left(\frac{A}{P_i}\right)$$

where Frac denotes the field of fractions. If $a = 0$, then $\iota(a) = (0, \ldots, 0)$, if $a \neq 0$, then $a \notin P_j$ for some j, and

$$\iota(a) = \left(0, \ldots, 0, \frac{a + P_j}{1}, 0, \ldots, 0\right)$$

where only the j-th entry is nonzero. □

Problem 1.2. Write down the minimal polynomial for $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$ and prove that it is reducible over $\mathbb{F}_p$ for every prime number p.

Proof. The minimal polynomial p_m is

$$p_m(t) = (t^2 - 5)^2 - 24 = t^4 - 10t^2 + 1$$

The roots are $\pm\sqrt{2} \pm \sqrt{3}$, thus this polynomial generates a field extension of $\mathbb{Q}$,

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \frac{\mathbb{Q}[t]}{(p_m(t))}$$

We claim that it suffices to show that $\sqrt{2}$ or $\sqrt{3}$ or $\sqrt{6}$ are in $\mathbb{F}_p$ for any prime p. Take $\sqrt{2} \in \mathbb{F}_p$ for example, we know $p_m(t)$ is not irreducible over $\mathbb{Q}(\sqrt{2})$, because then it would mean the degree of field extension $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}]$ is 8, which is a contradiction.

$$\begin{array}{c} \mathbb{Q}(\sqrt{2}, \sqrt{3}) \\ \uparrow \\ \mathbb{Q}(\sqrt{2}) \\ \uparrow \\ \mathbb{Q} \end{array}$$

Thus $p_m(t)$ is reducible over $\mathbb{Q}(\sqrt{2})$. Now we show the following.

Lemma 1.2. For any prime p, $\sqrt{2}$ or $\sqrt{3}$ or $\sqrt{6}$ are in $\mathbb{F}_p$ for any prime p.

There exists a homomorphism (Legendre symbol) $\varphi : \mathbb{F}_p^\times \to \{\pm 1\}$, such that

$$\varphi(g) = \begin{cases} 1, & \text{if } g \text{ is a square} \\ -1, & \text{otherwise} \end{cases}$$

Suppose that $2, 3$ are not squares, i.e., $sqrt2, \sqrt{3} \notin \mathbb{F}_p^\times$, then

$$\varphi(2 \cdot 3) = 1$$

which implies $\sqrt{6} \in \mathbb{F}_p^\times$, concluding the proof. □

Problem 1.3. Let K/k be a finite separable field extension, and let L/k be any field extension. Show that $K \otimes_k L$ is a product of fields.

Proof. We know K/k implies there exists $\alpha \in K$ such that

$$K = k(\alpha)$$

moreover, for any $t \in K$, the minimal polynomial of t factors into distinct linear factors. Let p_α be the minimal polynomial of α,

$$\begin{aligned} K \otimes_k L &= \frac{k[t]}{(p_\alpha(t))} \otimes_k L \\ &= \frac{L[t]}{(p_\alpha(t))} \\ &= \frac{L[t]}{(p_\alpha^1(t)) \dots (p_\alpha^k(t))} \end{aligned}$$

where $p_\alpha^i(t)$ are distinct irreducible factors over in $L[t]$. By Chinese Remainder Theorem, we must have

$$K \otimes_k L = \frac{L[t]}{(p_\alpha^1(t))} \cdots \frac{L[t]}{(p_\alpha^k(t))}$$

i.e., a product of fields. $\square$

Problem 1.4. Let M be an invertible $n \times n$ matrix with entries in an algebraically closed field k of characteristic not 2. Show that M has a square root, i.e. there exists $N \in \mathrm{Mat}_{n\times n}(k)$ such that $N^2 = M$.

Proof. It suffices to show that every Jordan block

$$J_n(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

where $\lambda \neq 0$ is a square. We will proceed using inductino. When $n = 2$, the square root of

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^{\frac{1}{2}} & \frac{1}{2}\lambda^{-\frac{1}{2}} \\ 0 & \lambda^{\frac{1}{2}} \end{bmatrix}^2$$

Now assume that J_k is a square up to $k = n-1$, we want to show J_n also has a square root. We claim J_n has the following square

$$J_n = \begin{bmatrix} B^2 & x \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} B & x \\ 0 & \lambda^{1/2} \end{bmatrix}^2$$

where B is a $(n-1) \times (n-1)$ matrix and $x = (x_1, \dots, x_{n-1}), 0 = (0, \dots, 0)$. It suffices to find such an x exists. Let $b_1, \dots, b_{n-1}$ denote the row vectors of B, we must satisfy

$$\begin{cases} b_1 \cdot x + x_1 \lambda^{\frac{1}{2}} = 0 \\ \dots \\ b_{n-2} \cdot x + x_{n-2} \lambda^{\frac{1}{2}} = 0 \\ b_{n-1} \cdot x + x_{n-1} \lambda^{\frac{1}{2}} = 1 \end{cases}$$

Namely, we need to find x that satisfies

$$(B+\lambda^{\frac{1}{2}}I)x = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 1 \end{bmatrix}$$

Since $(B+\lambda^{1/2}I)$ is invertible, there exists a unique solution, hence such x exsits, J_n has a square root! $\square$

Problem 1.5. Prove directly from the definition of (left) semisimple ring that every such ring is (left) Noetherian and Artinian. (You may freely use facts about semisimple, Noetherian, and Artinian modules.)

Proof. If R is Artinian, then R can be decomposed into a finite sum of simple rings, let $R_1, \dots, R_n$ be simple rings, we can write

$$R = \bigoplus_{i=1}^{n} R_i$$

where R_i contains only the trivial ideal and R_i as ideals. Now it is quite clear that every ascending and descending chain of ideals stabilizes because there are only finitely many distinct ideals. $\square$

Problem 1.6. Let G be a finite group and H an abelian subgroup. Show that every irreducible representation of G over $\mathbb{C}$ has dimension $\leq [G:H]$.

Proof. Any irreduicble representation $\rho : H \to \mathbb{C}^{\times}$ is one-dimensional, and we consider induced representation of ρ, $\mathrm{Ind}_H^G \rho$, we note that $\mathrm{Ind}_H^G \rho$ is not necessarily irreducible, hence for any irreducible representation $\tilde{\rho} : G \to \mathrm{GL}_n(\mathbb{C})$, we have

$$\dim \tilde{\rho} \leq \dim(\mathrm{Ind}_H^G \rho)$$

and

$$\mathrm{Ind}_H^G \rho = \bigoplus_{i=1}^{n} g_i H$$

where g_i are the representatives of the coset and the sum consists of exactly one copy for each coset. Hence we see

$$\dim \tilde{\rho} \leq \dim(\mathrm{Ind}_H^G \rho) = [G:H]$$

$\square$

Chapter 2

Fall 2016

Problem 2.1. Determine $\text{Aut}(S_3)$.

Proof. $\sigma \in \text{Aut}(S_3)$ is determined by where (12) and (123) are sent to. There are 6 options in total and all of them are homomorphisms (conjugation). It is easy to check that this group is not commutative, i.e.,

$$\text{Aut}(S_3) \cong S_3$$

□

Problem 2.2. A group G is a semidirect product of subgroups $N, H \subset G$ if N is normal and every element of G has a unique presentation nh, $n \in N$, $h \in H$. Find all semidirect products (up to isomorphism) of $N = \mathbb{Z}/11\mathbb{Z}$, $H = \mathbb{Z}/5\mathbb{Z}$.

Proof. Let $G = N \rtimes_\theta H$, where

$$\theta : \mathbb{Z}/5\mathbb{Z} \to \text{Aut}(\mathbb{Z}/11\mathbb{Z}) \cong \mathbb{Z}/10\mathbb{Z}$$

such that

$$5\theta(1) \equiv 0 \mod 10$$

Thus $\theta(1)$ could be $0, 2, 4, 6, 8$. When $\theta(1) = 0$, this gives the abelian group

$$G \cong \frac{\mathbb{Z}}{5\mathbb{Z}} \times \frac{\mathbb{Z}}{11\mathbb{Z}}$$

We claim that all nontrivial θ give rise to the same semidirect product, namely, the following diagram commutes

$$\begin{array}{ccc} \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta'} & \mathbb{Z}/10\mathbb{Z} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \text{id}} \\ \mathbb{Z}/5\mathbb{Z} & \xrightarrow{\theta} & \mathbb{Z}/10\mathbb{Z} \end{array}$$

for $\theta : 1 \mapsto 2$ and any $\theta' : 1 \mapsto 4, 6, 8$, by taking m to be the multiplication map by $2, 3, 4$ respectively. Hence we see

$$\theta(h)(g) = g^{2^{2h}}$$

by observing

$$\mathbb{Z}/5\mathbb{Z} \xrightarrow{2} \mathbb{Z}/10\mathbb{Z} \xrightarrow{2^2} (\mathbb{Z}/11\mathbb{Z})^\times \xrightarrow[2^2\cdot(-)]{} \text{Aut}(\mathbb{Z}/11\mathbb{Z})$$

In other words,

$$G = \left\langle g, h : g^1 1 = h^5 = 1, hgh^{-1} = g^{2^{2h}} \right\rangle$$

□

Problem 2.3. Let F be a finite field of order 2^n. Here $n > 0$. Determine all values of n such that the polynomial $x^2 - x + 1$ is irreducible in $F[x]$.

Proof. We know that $x^2 - x + 1$ is irreducible over $\mathbb{F}_2$, namely, it has no roots in $\mathbb{F}_2$. Since there is only one field of order 4, we must have

$$\mathbb{F}_4 \cong \frac{\mathbb{F}_2}{(x^2 - x + 1)}$$

Clearly $x^2 - x + 1$ is not irreducible over $\mathbb{F}_4$. For any $\mathbb{F}_{2^n}$, we know $(x^2 - x + 1)$ is irreducible if and only if $\mathbb{F}_4$ does not embed into $\mathbb{F}2^n$, i.e., $2 \nmid n$. This shows that when n is odd, the polynomial $x^2 - x + 1$ is irreducible over $\mathbb{F}_{2^n}$. $\square$

Problem 2.4. (1) Determine the Galois group of $x^4 - 4x^2 - 2$ over $\mathbb{Q}$.

(2) Let G be a group of order 8 such that G is the Galois group of a polynomial of degree 4 over $\mathbb{Q}$. Show that G is isomorphic to the Galois group in part (1).

Proof. (1) The roots of this polynomial is $\pm\sqrt{2 \pm \sqrt{6}}$, and notice that

$$\sqrt{2}i = \sqrt{2+\sqrt{6}}\sqrt{2-\sqrt{6}}$$

This gives the splitting field (Galois extension) of this polynomila as

$$\mathbb{Q}\left(\sqrt{2+\sqrt{6}}, \sqrt{2}i\right)$$

We see that

$$\mathbb{Q}\left(\sqrt{2+\sqrt{6}}\right) \cap \mathbb{Q}(\sqrt{2}i) = \varnothing$$

because the first is contained in $\mathbb{R}$ and the second is not. We must have

$$\left[\mathbb{Q}\left(\sqrt{2+\sqrt{6}}, \sqrt{2}i\right)/\mathbb{Q}\right] = 8$$

By part b, we see Gal $\cong D_8$.

(2) Any Galois group of a polynomial with 4 roots in the splitting field embeds into S_4, and we notice that $|G| = 2^3, |S_4| = 2^3 \cdot 3$, i.e., G is a Sylow 2-subgroup of S_4, and all Sylow 2-subgroups are conjugate/isomorphic of one another, hence

$$\text{Gal} \cong D_8$$

$\square$

Problem 2.5. Let A be a linear transformation of a finite dimensional vector space over a field of characteristic $\neq 2$.

(1) Define the wedge product linear transformation $\wedge^2 A = A \wedge A$.

(2) Prove that

$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2)).$$

Proof. (Recall we have analogous results for $A \otimes A$).

(1) The wedge product $A \wedge A$ is defined on the wedge product of vector spaces $V \wedge V$, so we first define the vector space: let $\{v_1, \ldots, v_n\}$ be the basis of V, then $\{v_i \wedge v_j\}$ where $i < j$ forms a basis of $V \wedge V$, satisfying:

1. $v_i \wedge v_j = -v_j \wedge v_i$
2. $(a_i v_i + a_j v_j) \wedge (b_k v_k + b_l v_l) = (a_i b_k) v_i \wedge v_k + (a_i b_l) v_i \wedge v_l + (a_j b_k) v_j \wedge v_k + (a_j b_l) v_j \wedge v_l$

And $A \wedge A$ where $A : V \to V$ is defined as

$$A \wedge A(v_i \wedge v_j) = Av_i \wedge Av_j$$

(2) Consider the matrix representation of $A = (A_{ij})$, on the basis $\{v_i \wedge v_j : i < j\}$,

$$\begin{aligned} A \wedge A(v_i \wedge v_j) &= \sum_{k,l=1}^{n} A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki} A_{lj} (v_k \wedge v_l) + \sum_{l<k} A_{ki} A_{lj} (v_k \wedge v_l) \\ &= \sum_{k<l} A_{ki} A_{lj} (v_k \wedge v_l) - \sum_{l<k} A_{ki} A_{lj} (v_l \wedge v_k) \end{aligned}$$

Thus the diagonal term with respect to $v_i \wedge v_j$ is

$$A_{ii} A_{jj} - A_{ji} A_{ij}$$

Thus

$$\text{Tr}(A \wedge A) = \sum_{i<j} A_{ii} A_{jj} - A_{ji} A_{ij}$$

Now

$$\text{Tr}(A)^2 = \sum_{i=1}^{n} A_{ii}^2 + 2 \sum_{i<j} A_{ii} A_{jj}$$

and

$$\begin{aligned} \text{Tr}(A^2) &= \sum_{k,l=1}^{n} A_{lk} A_{kl} \\ &= \sum_{i=1}^{n} A_{ii}^2 + 2 \sum_{k<l} A_{lk} A_{kl} \end{aligned}$$

Thus we see that

$$tr(\wedge^2 A) = \frac{1}{2}(tr(A)^2 - tr(A^2))$$

□

Problem 2.6. Find a table of characters for the alternating group A_5.

Proof.

	1	20	15	12	12
	Id	$(1\,2\,3)$	$(1\,2)(3\,4)$	$(1\,2\,3\,4\,5)$	$(1\,2\,3\,5\,4)$
χ_1	1	1	1	1	1
χ_2	3	0	-1	ϕ	$1-\phi$
χ_3	3	0	-1	$1-\phi$	ϕ
χ_4	4	1	0	-1	-1
χ_5	5	-1	1	0	0

where $\phi = \frac{1+\sqrt{5}}{2}$. □