Calc III Final Review

Fall 2025

(Please email hsun95@jh.edu if you see typos)

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Contents

Chapter 1

Definition Review

1.1 Before the midterm

Definition 1.1 (standard basis in $\mathbb{R}^3$)**.** The vectors

$$i = (1,0,0), j = (0,1,0), k = (0,0,1)$$

are called the **standard basis** vectors of $\mathbb{R}^3$, and for any vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, we can write

$$a = a_1 i + a_2 j + a_3 k$$

Definition 1.2 (Equation of a line)**.** A **line** l in $\mathbb{R}^3$ through the tip of $a = (a_1, a_2, a_3)$ pointing in the direction of a vector $v = (v_1, v_2, v_3)$ is given by

$$l(t) = a + tv = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$$

where $t \in \mathbb{R}$. Alternatively, a line passing through two points $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$ is given by

$$l(t) = (x(t), y(t), z(t))$$

where

$$\begin{cases} x(t) = x_1 + (x_2 - x_1)t \\ y(t) = y_1 + (y_2 - y_1)t \\ z(t) = z_1 + (z_2 - z_1)t \end{cases}$$

Definition 1.3 (inner product, dot product)**.** Let $a, b \in \mathbb{R}^3$, the **dot product**, also called the inner product, of a, b is

$$a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$$

where $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$. The **norm**, also called the length, of a is

$$\|a\| = (a \cdot a)^{\frac{1}{2}}$$

A vector of norm 1 is called a **unit vector**. Given any $u \in \mathbb{R}^3$, we can find the unit vector $\frac{u}{\|u\|}$ pointing in the same direction as u, this is called "normalizing" u.

Definition 1.4 (orthogonal projection)**.** The **orthogonal projection** of vector v onto another vector a is

$$\text{Proj}_a v = \frac{a \cdot v}{a \cdot a} a$$

For example, the orthogonal projection of $(1, 1, 0)$ onto $(1, 1, 1)$ is

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Definition 1.5 (orthogonal)**.** Let $a, b \in \mathbb{R}^n$, then a, b are called **orthogonal** or perpendicular iff

$$a \cdot b = 0$$

Definition 1.6 (determinant)**.** The **determinant** of a 2×2 matrix is given by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

and the determinant of a 3×3 matrix is given by

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Definition 1.7 (cross product)**.** Let $a, b \in \mathbb{R}^3$, write $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$, then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 1.8 (Plane in $\mathbb{R}^3$)**.** If a plane P passes through some point (x_0, y_0, z_0), and $n = (A, B, C)$ is a vector orthogonal to the plane, then the plane P is given by the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

(Notice that a point in P and a normal vector to P uniquely define a plane in $\mathbb{R}^3$.)

Definition 1.9 (image, graph)**.** The **image** of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in U\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in U\}$$

Definition 1.10 (level set)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, and $c \in \mathbb{R}$ be some constant. Then the **level set** of f at c is the set

$$\{x \in U : f(x) = c\} \subset \mathbb{R}^n$$

Definition 1.11 (open set, closed set, neighborhood, boundary)**.** Let $U \subset \mathbb{R}^n$, we say U is an **open set** if for every $x_0 \in U$, there exists some $r > 0$ such that $D_r(x_0) \subset U$, where $D_r(x_0)$ is the open disk of radius r centered at x_0:

$$D_r(x_0) = \{x \in \mathbb{R}^n : \|x - x_0\| < r\}$$

Some examples of open sets: $\mathbb{R}$, $D_1((0,0))$, $(1,2) \subset \mathbb{R}$. A **neighborhood** of $x_0 \in \mathbb{R}^n$ is an open set containing the point x_0. A point $x \in \mathbb{R}^n$ is called a **boundary point** of A if *every* neighborhood of x contains at least one point in A and at least one point not in A. A set is **closed** if it contains all its boundary points. Example of closed set: level sets of a continuous function f.

Definition 1.12 (limit)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is open, let x_0 be in A or be a boundary point of A and N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0, f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that

$$\text{if } x \in U, \text{ then } f(x) \in N$$

If f is eventually in N for *any* neighborhood N around b, then the **limit** of f as $x \to x_0$ exists, denoted as

$$\lim_{x \to x_0} f(x) = b$$

Definition 1.13 (continuous)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $x_0 \in A$, then f is **continuous at $\boldsymbol{x_0}$** if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Definition 1.14 (partial derivative)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, where U is open. Then the **partial derivative** with respect to x_i is defined by

$$\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}$$

where $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the ith coordinate.

Definition 1.15 (differentiability in two variables)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$, then f is **differentiable** at (x_0, y_0) if

(1) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0)

(2)
$$\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x - x_0) - \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y - y_0)}{\|(x,y) - (x_0,y_0)\|} = 0$$

The derivative of f at (x_0, y_0) is the 1×2 matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \end{bmatrix}$$

Moreover, the **tangent plane** of the graph of f at $(x_0, y_0, f(x_0, y_0))$ is given by

$$z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y - y_0)$$

Definition 1.16 (differentiability in the general setting). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$, then f is differentiable at $x_0 \in U$ if

(1) the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist for all $1 \leq i \leq m, 1 \leq j \leq n$.

(2)
$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$
where $T = Df(x_0)$ is the $m \times n$ matrix
$$Df(x_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \cdots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \cdots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}$$

The derivative of f at x_0 is the $m \times n$ matrix $Df(x_0)$.

Definition 1.17 (gradient). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, the **gradient** $\nabla f(x)$ is a special case of the general case above when $m = 1$, i.e., it is a $1 \times n$ matrix
$$Df(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Definition 1.18 (path and curve). A **path** in $\mathbb{R}^n$ is a map $c : [a, b] \to \mathbb{R}^n$, and the image of c is called a **curve**. We say the path c parametrizes the curve.
For example, $c(t) = (\cos t, \sin t)$ is a path, and the unit circle is a curve.

Definition 1.19 (velocity of a path). Let $c : [a, b] \to \mathbb{R}^n$ be a path, and we can write $c(t) = (c_1(t), \ldots, c_n(t))$. If c is differentiable, then we define the **velocity** of c at any $t_0 \in [a, b]$ as
$$c'(t_0) = (c_1'(t_0), \ldots, c_n'(t_0))$$
The velocity vector of c at t_0 is also a **tangent** vector to c at t_0. The **speed** of the path c at t_0 is the length of the velocity vector $\|c'(t_0)\|$.

Definition 1.20 (tangent line to a path). Let $c : [a, b] \to \mathbb{R}^n$ be a path, if $c'(t_0) \neq 0$, then the **tangent line** at x_0 is given by
$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

Definition 1.21 (directional derivative). Let $f : \mathbb{R}^3 \to \mathbb{R}$, be differentiable, then the **directional derivative** at $x_0 \in \mathbb{R}^3$ in the direction of a *unit vector* v is given by
$$\nabla f(x_0) \cdot v = \left[\frac{\partial f}{\partial x_1}(x_0)\right] v_1 + \left[\frac{\partial f}{\partial x_2}(x_0)\right] v_2 + \left[\frac{\partial f}{\partial x_3}(x_0)\right] v_3$$
where $v = (v_1, v_2, v_3)$.

Warning 1.1. Make sure you normalize any given direction v! This formula works for unit vectors.

Definition 1.22 (First order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + R_1(a, x)$$

where

$$\frac{R_1(a,x)}{\|x-a\|} \to 0 \text{ as } x \to a$$

Definition 1.23 (Second order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + R_2(a, x)$$

where

$$\frac{R_2(a,x)}{\|x-a\|} \to 0 \text{ as } x \to a$$

Definition 1.24 (critical point)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, a point $x_0 \in U$ is a **critical point** of f if either f is not differentiable at x_0, or $Df(x_0) = 0$. A critical point that is not a local extremum is called a saddle point.

Definition 1.25 (quadratic function)**.** A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic function** if it is given by

$$g(h_1, \dots, h_n) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

where (a_{ij}) is an $n \times n$ matrix. We can also write g as follows:

$$g(h_1, \dots, h_n) = [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Definition 1.26 (Hessian matrix)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, and suppose all the second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist, then the Hessian matrix of f is the $n \times n$ matrix given by

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

The Hessian as a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} \begin{bmatrix} h_1 & \dots & h_n \end{bmatrix} Hf(x) \begin{bmatrix} h_1 \\ \dots \\ h_n \end{bmatrix}$$

where $h = (h_1, \dots, h_n)$.

Definition 1.27 (degenerate/nondegenerate points)**.** Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, let (x_0, y_0) be a critical point. We define the **discriminant**, $\boldsymbol{\mathcal{D}}$,of the Hessian by

$$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $\mathcal{D} \neq 0$, the critical point (x_0, y_0) is called **nondegenerate**; if $\mathcal{D} = 0$, the point (x_0, y_0) is called **degenerate**.

Definition 1.28 (positive, negative-definite)**.** A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is called **positive-definite** if $g(h) \geq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. Similarly, g is **negative-definite** if $g(h) \leq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$.

Definition 1.29 (global extremum)**.** Let $f : A \to \mathbb{R}$ be a function defined on $A \subset \mathbb{R}^2$ or $A \subset \mathbb{R}^3$. A point $x_0 \in A$ is said to be an **absolute maximum** if $f(x_0) \geq f(x)$ for all $x \in A$. Similarly, x_0 is an **absolute minimum** if $f(x_0) \leq f(x)$ for all $x \in A$.

Definition 1.30 (bounded set)**.** A set $A \subset \mathbb{R}^n$ is said to be **bounded** if there is a number $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

1.2 After the midterm

Chapter 2

Theorem Review

2.1 Before the midterm

Proposition 2.1 (dot product)**.** Let $a, b \in \mathbb{R}^3$, and let θ be the angle between a, b, where $0 \leq \theta \leq \pi$, then

$$a \cdot b = \|a\|\|b\| \cos\theta$$

Proposition 2.2 (properties of the dot product)**.** Let $a, b, c \in \mathbb{R}^n$, then

(a) Nonnegativity: $a \cdot a \geq 0$, and $a \cdot a = 0$ if and only if $a = 0$.

(b) Scalar multiplication: let $\lambda \in \mathbb{R}$, then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry: $a \cdot b = b \cdot a$.

Proposition 2.3 (Cauchy-Schwarz)**.** Let $a, b \in \mathbb{R}^n$, then $a \cdot b \in \mathbb{R}$,

$$|a \cdot b| \leq \|a\|\|b\|$$

where the left hand side is the absolute value of $a \cdot b$, and the right hand side is multiplication of two nonnegative real numbers.

Proposition 2.4 (triangle inequality)**.** Let $a, b \in \mathbb{R}^n$, then

$$\|a + b\| \leq \|a\| + \|b\|$$

Proposition 2.5 (cross product). We have the following properties regarding the cross product: let $a, b \in \mathbb{R}^3$,

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:
$$\|a \times b\| = \|a\|\|b\| \sin\theta$$
where $0 \leq \theta \leq \pi$ is the angle between them.
3. $a \times b = -b \times a$, $(a+b) \times c = a \times c + b \times c$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. The cross product is **not** associative! For example, compute
$$(i \times i) \times j, \quad i \times (i \times j)$$

Proposition 2.6 (limits). Here are some properties of limits: let $f : U_1 \subset \mathbb{R}^n \to \mathbb{R}^m$, $g : U_2 \subset \mathbb{R}^n \to \mathbb{R}^m$,

(a) (Uniquessness):
$$\text{If } \lim_{x \to x_0} f(x) = b_1, \quad \lim_{x \to x_0} f(x) = b_2$$
then we must have
$$b_1 = b_2$$

(b) (Scalar multiplication): Let $c \in \mathbb{R}$, if $\lim_{x \to x_0} f(x) = b_1$, then
$$\lim_{x \to x_0} cf(x) = cb_1$$

(c) (Addition): Let f be as in (b), and $\lim_{x \to x_0} g(x) = b_2$, then
$$\lim_{x \to x_0} (f + g)(x) = b_1 + b_2$$

(d) (Component): Write $f(x) = (f_1(x), \ldots, f_n(x))$, if $\lim_{x \to x_0} f(x) = b = (b_1, \ldots, b_n)$, then
$$\lim_{x \to x_0} f_i(x) = b_i$$
for all $i = 1, \ldots, m$.

The same set of properties hold for continuity.

Proposition 2.7 (continuity of compositions). Let $g : A \subset \mathbb{R}^n \to \mathbb{R}^m$, and $f : B \subset \mathbb{R}^m \to \mathbb{R}^p$, and $g(A) \subset B$. If g is continuous at x_0, f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.

Proposition 2.8 (differentiability implies continuity). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. If f is differentiable at $x_0 \in U$, then f is continuous at x_0.

Proposition 2.9 (differentiability). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose $\partial f_i / \partial x_j$ exists for all i, j and are continuous in a neighborhood of $x_0 \in U$, then f is differentiable at x_0.

Proposition 2.10 (properties of derivatives). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, then the derivative of f at x_0 is an $m \times n$ matrix $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$. The derivative follows the same properties as derivative for single variable functions:

1. Let $c \in \mathbb{R}$, then
$$D(cf)(x_0) = cDf(x_0) \qquad \text{(multiplication of a matrix by constant } c\text{)}$$
2. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ also be differentiable at x_0, then
$$D(f+g)(x_0) = Df(x_0) + Dg(x_0) \qquad \text{(sum of two matrices)}$$
3. Let $h_1 : U \subset \mathbb{R}^n \to \mathbb{R}$, $h_2 : U \subset \mathbb{R}^n \to \mathbb{R}$,then
$$D(h_1h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \qquad \text{(product rule)}$$
and if $h_2 \neq 0$ on U.
$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \qquad \text{(quotient rule)}$$
4. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ such that $g(U) \subset V$, then
$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \qquad \text{(chain rule)}$$

Proposition 2.11 (fastest rate of change). Suppose that $\nabla f(x_0) \neq 0$, then the direction for which f increases the fastest at x_0 is along $\nabla f(x_0)$.

Proposition 2.12 (gradient is normal, tangent plane). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable, let S be a level surface of f, i.e., S is a surface described by
$$f(x, y, z) = k$$
were k is some constant. Let $(x_0, y_0, z_0) \in S$, then
$$\nabla f(x_0, y_0, z_0) \text{ is } \textbf{normal} \text{ to the level surface at } (x_0, y_0, z_0)$$
This means if $c(t)$ is a path in S, and $v(0) = (x_0, y_0, z_0)$, and if v is a tangent vector to $c(t)$ at $t = 0$, then
$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$
Moreover, if $\nabla f(x_0, y_0, z_0) \neq 0$, the **tangent plane** of S at (x_0, y_0, z_0) is given by
$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Proposition 2.13 (Equality of mixed partials). If $f(x, y)$ is twice continuously differentiable, then
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Proposition 2.14 (extremums are critical points). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, where U is open. If x_0 is a local extremum, then $Df(x_0) = 0$.

Proposition 2.15 (extremum). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be in C^3, and x_0 is a critical point of f. If the Hessian $Hf(x_0)$ is positive-definite, then x_0 is a local minimum of f; if $Hf(x_0)$ is negative-definite, then x_0 is a local maximum.

Proposition 2.16 (local minimum). Let $f(x, y)$ be of C^2, and U is open in $\mathbb{R}^2$. A point (x_0, y_0) is a strict local **minimum** of f if the following conditions hold:

1. $$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$
2. $$\mathcal{D}(x_0, y_0) > 0$$
   where $\mathcal{D}$ is the **discriminant** of the Hessian, defined by
   $$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
   where Hf is the 2×2 Hessian matrix.
3. $$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

Proposition 2.17 (local maximum). Let $f(x, y)$ be of C^2, and U is open in $\mathbb{R}^2$. A point (x_0, y_0) is a strict local **maximum** of f if the following conditions hold:

1. $$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$
2. $$\mathcal{D}(x_0, y_0) > 0$$
   where $\mathcal{D}$ is the discriminant of the Hessian, defined above.
3. $$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$$

Proposition 2.18 (saddle points). Let $f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, if $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, and $\mathcal{D}(x_0, y_0) < 0$, where $\mathcal{D}$ is the discriminant, then the critical point (x_0, y_0) is a saddle point, i.e., neither a maximum or a minimum.

Proposition 2.19 (continuous functions attain extremum on closed bounded sets). Let $f : D \to \mathbb{R}$ be continuous, where D is closed and bounded in $\mathbb{R}^n$. Then f assumes its absolute maximum and absolute minimum values at some point $x_0, x_1 \in D$.

2.2 After the midterm

Chapter 3

Practice Problems

3.1 Before the midterm

Problem 3.1. Find the equation of the line passing through $(1, 0, 2)$ in the direction $(2, -1, 3)$.

Problem 3.2. In which direction does the line

$$l(t) = (3 - 2t, 2 + 5t, 1 + t)$$

point?

Problem 3.3. Do the following two lines intersect?

$$l_1(t) = (1 + 2t, 2 + t, 3 - t), \quad l_2(s) = (3 - s, 4 - s, 2 + s)$$

Problem 3.4. Do the following points lie on the same line?

$$A = (1, 0, 1), \quad B = (2, 1, 1), \quad C = (0, -1, 1)$$

Problem 3.5. Find the angle between two vectors $(1, 2, 0), (3, 1, 1)$.

Problem 3.6. Let $b = (2, 1, 3)$ and P be the plane through the origin given by $x + y + 2z = 0$.

(a) Find two distinct vectors v_1, v_2 that are orthogonal in P.

(b) Find the projection of b onto the plane P, namely,

$$\text{Proj}_{v_1} b + \text{Proj}_{v_2} b$$

Problem 3.7. Find a unit vector orthognal to both vectors $a = (1, 2, -1), b = (2, 3, -1)$.

Problem 3.8. Find the equation of the plane containing all three points below:

$$P = (2, 1, -1), \quad Q = (1, 0, -2), \quad T = (3, 2, 1)$$

Problem 3.9. (a) Find an equation for the line that passes through the point $(1,1,0)$ and is perpendicular to the plane $3x+y-2z+1=0$.

(b) Find an equation for the plane that contains the line

$$l(t) = (-1+t, 2+2t, 1+3t)$$

and is perpendicular to the plane

$$2x+y-z+1=0$$

Problem 3.10. Compute the area of the parallelogram spanned by the vectors $(1,1,0), (0,2,1)$.

Problem 3.11. Use the triangle inequality 2.4 to show the reverse triangle inequality:

$$\Big| \|a\| - \|b\| \Big| \leq \|a-b\|$$

Problem 3.12. Compute the following limits if they exist; if the limits don't exist, please explain why.

1. $$\lim_{(x,y)\to(2,1)} \frac{x^2+y^2-2xy}{x-y}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{\cos x - 1}{x^2+y^2}$$

3. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{(x+y)^2}$$

4. $$\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x + y}{x^3+y}$$

5. $$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 y \cos z}{x^2+y^2}$$

6. $$\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2}$$

7. $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^6}{xy^3}$$

Problem 3.13. (a) Show that $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = (1-x)^8 + \cos(1+x^3)$$

is continuous.

(b) Show $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 e^x}{2 - \sin x}$$

is continuous.

Problem 3.14. Compute all the partial derivatives.

1. $w = e^{xy} \log(x^2 + y^2)$.
2. $w = \cos(ye^{xy}) \sin x$.

Problem 3.15. Compute the gradient of $h(x, y, z) = (x + z)e^{x-y}$ at $(1, 1, 0)$.

Problem 3.16. Determine the velocity vector of the given path:

$$c(t) = (\cos 2t, 3t^2 - t, -t)$$

Problem 3.17. Find the tangent line to the given path at $t = 0$

$$c(t) = (e^t \sin t, 2t, -t^3)$$

Problem 3.18. Compute the derivatives.

1. Let

$$f(u, v) = u^2 v + 2v, \quad u = -x^2 + y, v = x + y$$

Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

2. Let

$$g(u, v) = (e^u, u + \sin v), \quad f(x, y, z) = (x^2, yz)$$

Compute $D(g \circ f)$ at $(0, 1, 0)$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $c(t) = \mathbb{R} \to \mathbb{R}^3$. Suppose $c(0) = (1, 2, 0)$, and

$$\nabla f(1, 2, 0) = (0, 0, 1), \quad c'(0) = (2, 1, 1)$$

Compute $\frac{d(f \circ c)}{dt}$ at $t = 0$.

Problem 3.19. Determine the directional derivative of

$$f(x, y, z) = x^3 y - xyz$$

at $(1, 1, 0)$ along $v = (0, -1, 1)$.

Problem 3.20. Find a unit vector normal to the surface

$$xe^y + ye^z + ze^x = e + 1$$

at the point $(0, 1, 1)$.

Problem 3.21. Find the tangent plane of the level surface of $f(x, y, z) = \ln(x + y) - 2xz = \ln(3) + 2$ at $(1, 2, -1)$.

Problem 3.22. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is called an *even* function if $f(x) = f(-x)$ for every x in $\mathbb{R}^n$. If f is differentiable and even, find ∇f at the origin.

Problem 3.23. Consider the function

$$f(x, y) = \frac{1}{\log(x^2 + y)}.$$

Verify by hand that $f_{xy} = f_{yx}$.

Problem 3.24. Consider the function $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$. Show that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Problem 3.25. Find the second-order Taylor expansion for the function

$$f(x, y) = x^2 + 2xy$$

at $(1, 1)$.

Problem 3.26. Find and classify all critical points of the following function:

1. $$f(x, y) = e^x \cos y$$

2. $$g(x, y) = (2x^2 + x)(3y + 1)$$

Problem 3.27. Show that $(0, 0)$ is a critical point of

$$f(x, y) = x^2y - 2x^2 - y^2$$

and is it a local maximum, local minimum, or a saddle point?

3.2 After the midterm

Chapter 4

Answer Key

4.1 Before the midterm

Problem 4.1. Find the equation of the line passing through $(1, 0, 2)$ in the direction $(2, -1, 3)$.

Proof. By definition 1.2, the line is given by

$$l(t) = (1 + 2t, -t, 2 + 3t)$$

□

Problem 4.2. In which direction does the line

$$l(t) = (3 - 2t, 2 + 5t, 1 + t)$$

point?

Proof. In the direction of the vector $(-2, 5, 1)$. □

Problem 4.3. Do the following two lines intersect?

$$l_1(t) = (1 + 2t, 2 + t, 3 - t), \quad l_2(s) = (3 - s, 4 - s, 2 + s)$$

Proof. For them to intersect, we must have t, s such that

$$\begin{cases} 1 + 2t = 3 - s & (1) \\ 2 + t = 4 - s & (2) \\ 3 - t = 2 + s & (3) \end{cases}$$

$(2) - (1)$ gives $-t + 1 = 1$, which implies $t = 0, s = 2$, but this does not satisfy (3), hence these two lines do not intersect! □

Problem 4.4. Do the following points lie on the same line?

$$A = (1, 0, 1), \quad B = (2, 1, 1), \quad C = (0, -1, 1)$$

Proof. We can find the unique line passing through A, B by the equation given in1.2

$$l(t) = (1,0,1) + (1,1,0)t$$

then for C to lie on this line, there must exists some t such that

$$\begin{cases} 1 + t = 0 \\ t = -1 \\ 1 = 1 \end{cases}$$

and $t = -1$ satisfies. This means all three points lie on the same line! □

Problem 4.5. Find the angle between two vectors $(1, 2, 0), (3, 1, 1)$.

Proof. By Proposition 2.1

$$\cos\theta = \frac{a \cdot b}{\|a\|\|b\|} = \frac{5}{\sqrt{5}\sqrt{11}} = \sqrt{\frac{5}{11}}$$

hence

$$\theta = \arccos\left(\sqrt{\frac{5}{11}}\right)$$

□

Problem 4.6. Let $b = (2, 1, 3)$ and P be the plane through the origin given by $x + y + 2z = 0$.

(a) Find two distinct vectors v_1, v_2 that are orthogonal in P.

(b) Find the projection of b onto the plane P, namely,

$$\mathrm{Proj}_{v_1} b + \mathrm{Proj}_{v_2} b$$

Proof. (a) We can let $v_1 = (1, -1, 0), v_2 = (1, 1, -1)$. One can verify that $v_1, v_2 \in P$ and $v_1 \cdot v_2 = 0$.

(b) The projection is given by

$$\begin{aligned} \mathrm{Proj}_{v_1} b + \mathrm{Proj}_{v_2} b &= \frac{v_1 \cdot b}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot b}{v_2 \cdot v_2} v_2 \\ &= \frac{1}{2}(1, -1, 0) + 0 \\ &= \left(\frac{1}{2}, -\frac{1}{2}, 0\right) \end{aligned}$$

□

Problem 4.7. Find a unit vector orthognal to both vectors $a = (1, 2, -1), b = (2, 3, -1)$.

Proof. The cross product is orthognal to both of the vectors:

$$a \times b = \det \begin{bmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 3 & -1 \end{bmatrix} = (1, -1, -1)$$

Then we normalize it:

$$\frac{a \times b}{\|a \times b\|} = \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

□

Problem 4.8. Find the equation of the plane containing all three points below:

$$P = (2, 1, -1), \quad Q = (1, 0, -2), \quad T = (3, 2, 1)$$

Proof. We can find two vectors in this plane:

$$\mathrm{PQ} = Q - P = (-1, -1, -1), \quad \mathrm{PT} = T - P = (1, 1, 2)$$

then we can find a normal vector n to the plane by taking the cross product:

$$n = \mathrm{PQ} \times \mathrm{PT} = \det \begin{bmatrix} i & j & k \\ -1 & -1 & -1 \\ 1 & 1 & 2 \end{bmatrix} = (-1, 1, 0)$$

Then by Definition 1.8, using point Q, we see the plane can be written as

$$-1(x - 1) + y = 0$$

simplifying we get $x - y = 1$. □

Problem 4.9. (a) Find an equation for the line that passes through the point $(1, 1, 0)$ and is perpendicular to the plane $3x + y - 2z + 1 = 0$.

(b) Find an equation for the plane that contains the line

$$l(t) = (-1 + t, 2 + 2t, 1 + 3t)$$

and is perpendicular to the plane

$$2x + y - z + 1 = 0$$

Proof. (a) A normal vector to the plane $3x + y - 2z + 1$ is $(3, 1, -2)$, since the line is perpendicular to the plane, the line is parallel along the direction $(3, 1, -2)$. Now the line passes through $(1, 1, 0)$, thus we have the equation for the line

$$l(t) = (1, 1, 0) + t(3, 1, -2)$$

(b) A normal vector to $2x+y-z$ is $n = (2, 1, -1)$, and since our plane is perpendicular to this, it is parallel to the vector n. Thus a normal vector to our plane must be orthogonal to both n and $(1, 2, 3)$, where the latter is given by the line in the plane. Thus taking the cross product:

$$n_1 = n \times (1, 2, 3) = (5, -7, 3)$$

Hence the equation for the plane is given by:

$$5(x + 1) - 7(y - 2) + 3(z - 1) = 0$$

simplifying we get $5x - 7y + 3z + 16 = 0$.

□

Problem 4.10. Compute the area of the parallelogram spanned by the vectors $(1, 1, 0), (0, 2, 1)$.

Proof. Since we know

$$\|u \times v\| = \|u\|\|v\| \sin\theta$$

the length of the cross product is exactly the area of the parallelogram, thus computing

$$\|(1, 1, 0) \times (0, 2, 1)\| = \|(1, -1, 2)\| = \sqrt{6}$$

□

Problem 4.11. Use the triangle inequality 2.4 to show the reverse triangle inequality:

$$\Big| \|a\| - \|b\| \Big| \leq \|a - b\|$$

Proof. We know by traingle inequality

$$\begin{aligned} \|a\| &= \|(a-b) + b\| \\ &\leq \|a - b\| + \|b\| \end{aligned}$$

rearranging, we get $\|a\| - \|b\| \leq \|a - b\|$. Similarly

$$\|b\| - \|a\| \leq \|a - b\|$$

Together this implies

$$\Big| \|a\| - \|b\| \Big| \leq \|a - b\|$$

□

Problem 4.12. Compute the following limits if they exist; if the limits don't exist, please explain why.

1. $$\lim_{(x,y)\to(2,1)} \frac{x^2 + y^2 - 2xy}{x - y}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{\cos x - 1}{x^2 + y^2}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{(x+y)^2}$$
4. $$\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x + y}{x^3 + y}$$
5. $$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 y \cos z}{x^2 + y^2}$$
6. $$\lim_{(x,y)\to(2,1)} \frac{x^2 - 2xy}{x^2 - 4y^2}$$
7. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$$

Proof. 1.

$$\lim_{(x,y)\to(2,1)} \frac{x^2 + y^2 - 2xy}{x - y} = \lim_{(x,y)\to(2,1)} \frac{(x-y)^2}{x - y} = \lim_{(x,y)\to(2,1)} x - y = 1$$

2. The limit doesn't exist,
$$\lim_{(x,y)\to(0,0)} \frac{\cos x - 1}{x^2+y^2}$$
Consider the path $x=0, y\to 0$, we have
$$\lim_{x=0,y\to 0} \frac{0}{y^2} = 0$$
Consider the path $y=0, x\to 0$,
$$\lim_{y=0,x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{-\cos x}{2} = -\frac{1}{2}$$

3. The limit doesn't exist,
$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{(x+y)^2}$$
Consider the path $x=0, y\to 0$,
$$\lim_{x=0,y\to 0} \frac{y^2}{y^2} = 1$$
Consider the path $y=x\to 0$,
$$\lim_{x=y\to 0} \frac{0}{4x^2} = 0$$

4. The limit doesn't exist,
$$\lim_{(x,y)\to(0,0)} \frac{\sin 2x - 2x + y}{x^3+y}$$
Consider the path $x=0, y\to 0$,
$$\lim_{x=0,y\to 0} \frac{y}{y} = 1$$
Consider the path $y=0, x\to 0$,
$$\begin{aligned}\lim_{y=0,x\to 0} \frac{\sin 2x - 2x}{x^3} &= \lim_{x\to 0} \frac{2\cos 2x - 2}{3x^2} \\ &= \lim_{x\to 0} \frac{-4\sin 2x}{6x} \\ &= \lim_{x\to 0} \frac{-8\cos 2x}{6} \\ &= -\frac{4}{3}\end{aligned}$$

5.
$$\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2 y\cos z}{x^2+y^2}$$
Writing $x = r\cos\theta, y = r\sin\theta$ in polar coordinates, we can rewrite this as
$$\left|\frac{2r^3\cos^2\theta\sin\theta\cos z}{r^2}\right| = |2r\cos^2\theta\sin\theta\cos z| \le 2r \to 0$$
as $(x,y,z)\to(0,0,0)$. Thus the limit is 0.

6.
$$\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2}$$
We factor:
$$\lim_{(x,y)\to(2,1)} \frac{x^2-2xy}{x^2-4y^2} = \lim_{(x,y)\to(2,1)} \frac{(x-2y)x}{(x+2y)(x-2y)} = \lim_{(x,y)\to(2,1)} \frac{x}{x+2y} = \frac{1}{2}$$

7. The limit doesn't exist,

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^6}{xy^3}$$

Consider $x = y \to 0$, then

$$\lim_{x=y\to 0} \frac{x^2 - x^6}{x^4} = \lim_{x\to 0} \frac{1 - x^4}{x^2} = \infty$$

Consider $x = y^3 \to 0$, then

$$\lim_{x=y^3\to 0} \frac{0}{y^6} = 0$$

□

Problem 4.13. (a) Show that $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = (1 - x)^8 + \cos(1 + x^3)$$

is continuous.

(b) Show $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) = \frac{x^2 e^x}{2 - \sin x}$$

is continuous.

Proof. (a) $(1 - x)^8$ is a polynomial, thus continuous, and $\cos x, 1 + x^3$ are both continuous, thus the composition $\cos(1 + x^3)$ is also continuous. Thus adding continuous functions gives another continuous function.

(b) $x^2 e^x, 2 - \sin x$ are both continuous, and $\frac{x^2 e^x}{2 - \sin x}$ is continuous if $2 - \sin x \neq 0$ for all x. This is indeed true because $-1 \leq \sin x \leq 1$, thus $1 \leq 2 - \sin x \leq 3$.

□

Problem 4.14. Compute all the partial derivatives.

1. $w = e^{xy} \log(x^2 + y^2)$.
2. $w = \cos(ye^{xy}) \sin x$.

Proof. 1.

$$\frac{\partial w}{\partial x} = ye^{xy} \ln(x^2 + y^2) + e^{xy} \frac{2x}{x^2 + y^2}$$

and

$$\frac{\partial w}{\partial y} = xe^{xy} \ln(x^2 + y^2) + e^{xy} \frac{2y}{x^2 + y^2}$$

2.

$$\frac{\partial w}{\partial x} = -y^2 e^{xy} \sin(ye^{xy}) \sin x + \cos(ye^{xy}) \cos x$$

and

$$\frac{\partial w}{\partial y} = -(1 + xy)e^{xy} \sin(ye^{xy}) \sin x$$

□

Problem 4.15. Compute the gradient of $h(x, y, z) = (x + z)e^{x-y}$ at $(1, 1, 0)$.

Proof. The gradient is

$$\begin{aligned}\nabla h(x, y, z) &= \begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} e^{x-y}(1 + x + z) & -(x + z)e^{x-y} & e^{x-y} \end{bmatrix}\end{aligned}$$

Thus

$$\nabla h(1, 1, 0) = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}$$

□

Problem 4.16. Determine the velocity vector of the given path:

$$c(t) = (\cos 2t, 3t^2 - t, -t)$$

Proof. It is given by

$$c'(t) = (-2 \sin 2t, 6t - 1, -1)$$

□

Problem 4.17. Find the tangent line to the given path at $t = 0$

$$c(t) = (e^t \sin t, 2t, -t^3)$$

Proof. By the equation in Definition 1.20, we have

$$c'(t) = (e^t \sin t + e^t \cos t, 2, -3t^2)$$

and $c(0) = (0, 0, 0)$, $c'(0) = (1, 2, 0)$. Thus the tangent line is given by

$$l(t) = (t, 2t, 0)$$

□

Problem 4.18. Compute the derivatives.

1. Let
$$f(u, v) = u^2v + 2v, \quad u = -x^2 + y, v = x + y$$
Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

2. Let
$$g(u, v) = (e^u, u + \sin v), \quad f(x, y, z) = (x^2, yz)$$
Compute $D(g \circ f)$ at $(0, 1, 0)$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $c(t) = \mathbb{R} \to \mathbb{R}^3$. Suppose $c(0) = (1, 2, 0)$, and
$$\nabla f(1, 2, 0) = (0, 0, 1), \quad c'(0) = (2, 1, 1)$$
Compute $\frac{d(f \circ c)}{dt}$ at $t = 0$.

Proof. 1. We have

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = -4xuv + u^2 + 2$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 2uv + u^2 + 2$$

(You might want to replace u, v with x, y, but I am lazy).

2. We have

$$D(g \circ f)(0,1,0) = Dg(f(0,1,0))Df(0,1,0)$$

where $f(0,1,0) = (0,0)$

$$Dg(u,v) = \begin{bmatrix} e^u & 0 \\ 1 & \cos v \end{bmatrix}, \quad , Df(x,y,z) = \begin{bmatrix} 2x & 0 & 0 \\ 0 & z & y \end{bmatrix}$$

Thus

$$\begin{aligned} D(g \circ f)(0,1,0) &= Dg(0,0)Df(0,1,0) \\ &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

3. We have

$$\frac{d(f \circ c)}{dt}(0) = \nabla f(1,2,0)c'(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 1$$

□

Problem 4.19. Determine the directional derivative of

$$f(x,y,z) = x^3y - xyz$$

at $(1,1,0)$ along $v = (0,-1,1)$.

Proof. First we compute

$$\nabla f(x,y,z) = (3x^2y - yz, x^3 - xz, -xy)$$

Thus

$$\nabla f(1,1,0) = (3,1,-1)$$

Recall the directional derivative is given by

$$\nabla f(1,1,0) \cdot \frac{v}{\|v\|} = -\frac{2}{\sqrt{2}}$$

We need to make sure that the direction vector is a unit vector! □

Problem 4.20. Find a unit vector normal to the surface

$$xe^y + ye^z + ze^x = e + 1$$

at the point $(0,1,1)$.

Proof. This is a level set for the multivariate function $f(x, y, z) = xe^y + ye^z + ze^x$. We compute the gradient

$$\nabla f(x, y, z) = (e^y + ze^x, e^z + xe^y, e^x + ye^z).$$

hence $\nabla f(0, 1, 1) = (e + 1, e, e + 1)$, and this vector is normal to the surface. To make this a unit vector, we normalize to get

$$\frac{\nabla f(0,1,1)}{\|\nabla f(0,1,1)\|} = \frac{1}{\sqrt{3e^2 + 4e + 2}}(e + 1, e, e + 1),$$

□

Problem 4.21. Find the tangent plane of the level surface of $f(x, y, z) = \ln(x + y) - 2xz = \ln(3) + 2$ at $(1, 2, -1)$.

Proof. By the equation given in Proposition 2.12, we first compute a normal vector to the tangent plane, which is the gradient of f at $(1, 2, -1)$:

$$\nabla f(x, y, z) = \left(\frac{1}{x + y} - 2z, \frac{1}{x + y}, -2x\right)$$

and $\nabla f(1, 2, -1) = \left(\frac{7}{3}, \frac{1}{3}, -2\right)$, thus the tangent plane is given by

$$\frac{7}{3}(x - 1) + \frac{1}{3}(y - 2) - 2(z + 1) = 0$$

simplifying we get $7x + y - 6z - 15 = 0$. □

Problem 4.22. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called an *even* function if $f(x) = f(-x)$ for every x in $\mathbb{R}^n$. If f is differentiable and even, find ∇f at the origin.

Proof. We claim that $\nabla f(0, \ldots, 0) = 0$. It suffices to show that $\nabla f(0, \ldots, 0) \cdot v = \nabla f(0, \ldots, 0) \cdot (-v)$ for any vector $v \in \mathbb{R}^n$. Because this implies $2\nabla f(0, \ldots, 0) \cdot v = 0$ for every $v \in \mathbb{R}^n$, so $Df(0, \ldots, 0) = 0$. We know that

$$\nabla f(0, \ldots, 0) \cdot v = \frac{d}{dt} f(tv)\bigg|_{t=0}, \quad \nabla f(0, \ldots, 0)(-v) = \frac{d}{dt} f(-tv)\bigg|_{t=0}$$

But $f(tv) = f(-tv)$ since f is even, thus

$$\nabla f(0, \ldots, 0) \cdot v = \nabla f(0, \ldots, 0) \cdot (-v)$$

as desired. □

Problem 4.23. Consider the function

$$f(x, y) = \frac{1}{\log(x^2 + y)}.$$

Verify by hand that $f_{xy} = f_{yx}$.

Proof. We compute these separately.

$$f_x = \frac{2x}{x^2 + y}, \quad f_{xy} = -\frac{2x}{(x^2 + y)^2}$$

and

$$f_y = \frac{1}{x^2 + y}, \quad f_{yx} = -\frac{2x}{(x^2 + y)^2}$$

□

Problem 4.24. Consider the function $f(x,y,z) = \left(x^2+y^2+z^2\right)^{-1/2}$. Show that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Proof. Note

$$f_x = -\frac{1}{2} \cdot \frac{2x}{\left(x^2+y^2+z^2\right)^{3/2}} = -\frac{x}{\left(x^2+y^2+z^2\right)^{3/2}},$$

so

$$f_{xx} = -\frac{\left(x^2+y^2+z^2\right)^{3/2} - x \cdot \frac{3}{2}\left(x^2+y^2+z^2\right)^{1/2} \cdot 2x}{\left(x^2+y^2+z^2\right)^3},$$

which is

$$f_{xx} = -\frac{x^2+y^2+z^2-3x^2}{\left(x^2+y^2+z^2\right)^{5/2}},$$

or

$$f_{xx} = -\frac{-2x^2+y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}}.$$

By symmetry,

$$f_{yy} = -\frac{x^2-2y^2+z^2}{\left(x^2+y^2+z^2\right)^{5/2}},$$

and

$$f_{zz} = -\frac{x^2+y^2-2z^2}{\left(x^2+y^2+z^2\right)^{5/2}},$$

so we see that $f_{xx} + f_{yy} + f_{zz} = 0$. □

Problem 4.25. Find the second-order Taylor expansion for the function

$$f(x,y) = x^2 + 2xy$$

at $(1,1)$.

Proof. First $f(1,1) = 3$, then we find all first-order and second-order partial derivatives:

$$f_x = 2x+2y, f_y = 2x, f_{xx} = 2, f_{xy} = 2, f_{yy} = 0$$

Thus by formula in Definition 1.23, we have

$$\begin{aligned} f(x,y) &= 3 + 4(x-1) + 2(y-1) + \frac{1}{2}2(x-1)^2 + \frac{1}{2}2(x-1)(y-1) + \frac{1}{2}2(x-1)(y-1) + R_2((1,1),(x,y)) \\ &= 3 + 4(x-1) + 2(y-1) + (x-1)^2 + 2(x-1)(y-1) + R_2((1,1),(x,y)) \end{aligned}$$

where

$$\frac{R_2((1,1),(x,y))}{\|(x-1,y-1)\|} \to 0$$

as $(x,y) \to (1,1)$. □

Problem 4.26. Find and classify all critical points of the following function:

1. $$f(x,y) = e^x \cos y$$

2. $$g(x,y) = (2x^2 + x)(3y + 1)$$

Proof.

1. The critical point of f requires
$$f_x = f_y = 0$$
This gives
$$f_x = e^x \cos y = 0 \Rightarrow y = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$$
similarly,
$$f_y = -e^x \sin y = 0 \Rightarrow y = n\pi, n \in \mathbb{Z}$$
We see that there is no such y that makes $f_x = f_y = 0$ simultaneously. Hence there are no critical points.

2. We again compute x, y such that $g_x = g_y = 0$.
$$g_x = (4x+1)(3y+1) \Rightarrow x = -\frac{1}{4}, y = -\frac{1}{3}$$
and
$$g_y = (2x^2 + x)3 = 0 \Rightarrow x = 0 \text{ or } x = -\frac{1}{2}$$
Thus the points (x,y) such that $g_x = g_y = 0$ are
$$\left(0, -\frac{1}{3}\right), \quad \left(-\frac{1}{2}, -\frac{1}{3}\right)$$
Now we classify them by first computing their Hessians:
$$g_{xx} = 4(3y+1), \quad g_{xy} = 3(4x+1), \quad g_{yy} = 0$$
Thus
$$\mathcal{D} = \det(Hf) = g_{xx}g_{yy} - g_{xy}^2 = -9(4x+1)^2$$
Then we see that $x = 0, -\frac{1}{2}$ both result in $\mathcal{D} < 0$, which means
$$\left(0, -\frac{1}{3}\right), \quad \left(-\frac{1}{2}, -\frac{1}{3}\right)$$
are both saddle points.

□

Problem 4.27. Show that $(0,0)$ is a critical point of
$$f(x,y) = x^2 y - 2x^2 - y^2$$
and is it a local maximum, local minimum, or a saddle point?

Proof. We have

$$f_x = 2xy - 4x, \quad f_y = x^2 - 2y$$

and we see $f_x(0,0) = f_y(0,0) = 0$, thus $(0,0)$ is a critical point. Now we compute the discriminant:

$$f_{xx} = 2y - 4, \quad f_{xy} = 2x, \quad f_{yy} = -2$$

Then

$$\mathcal{D} = f_{xx}f_{yy} - f_{xy}^2 = -2(2y-4) - 4x^2$$

Hence $\mathcal{D}(0,0) = 8 > 0$, and $f_{xx}(0,0) = -4$ imply that $(0,0)$ is a local maximum. $\square$

4.2 After the midterm

Chapter 5

Tips

1. When asked to find the limit:

Step 1: Factor out common factor, for example,

$$\frac{x^2 - 2xy}{x^2 - 4y^2} = \frac{(x-2y)x}{(x-2y)(x+2y)} = \frac{x}{x+2y}$$

Step 2: Try the following four paths: take $(x, y) \to (0, 0)$ as an example,

i. $x = 0, y \to 0.$
ii. $y = 0, x \to 0.$
iii. $x = y \to 0.$
iv. $x = -y \to 0.$

Step 3: Try to put into expressions that you are familiar with, for example,

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{x} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} y$$

and use the fact that $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

If any two paths give different limits, then the limit doesn't exist. Step 2:

2. When asked to find a directional derivative of f along v : make sure you normalize v as $\frac{v}{\|v\|}$.

3. When asked to find an equation for a plane: identify a normal vector by

   (a) taking the cross product of two vectors in the plane 1.8.

   (b) computing the gradient if the plane is the tangent plane to a level surface 2.12.

4. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable, then Df is an $m \times n$ matrix. Let A be an $m \times n$ matrix and B be a $k \times p$ matrix, then the matrix multiplication AB only makes sense when $n = k$.