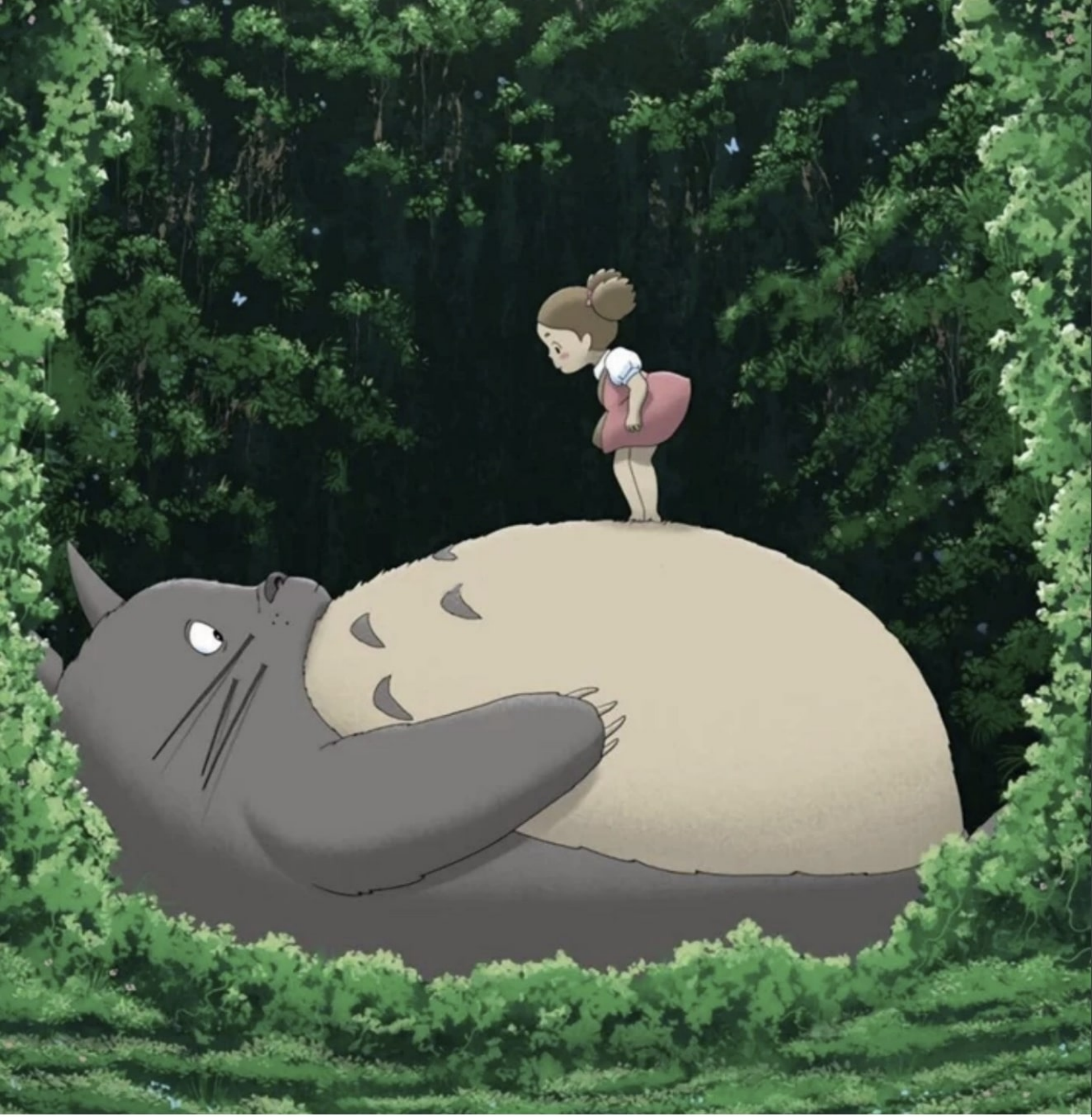

Functional Analysis

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Contents

Chapter 1 Lecture 1

Here we go.

1.0.1 Course Overview and Logistics

Some administrative things. OH are Monday, Fridays 1:45 to 2:45, Wednesdays 12:45-1:45 in Evans 811.

Textbook: an introduction to functional analysis by Conway. We will be talking about operators on Hilbert spaces, and more generally, Banach spaces, and Frechet spaces (defined by a countable numer of seminomrs).

Remark Let $\mathcal{H}$ be a Hilbert space, then the dual space $\mathcal{H}^*$ is itself. $\mathcal{H} = \mathcal{H}^*$. Hilbert spaces are the best spaces to work with. They are self-dual, and identified with themslves.

Then in the next section, we will look at groups, motivated by their actions on Banach spaces, connected with Fourier transforms.

1.0.2 Motivation

Let X be a compact Hausdorff space. Let $C(X) = \{f : X \to \mathbb{R}, f \text{ continuous}\}$ be the algebra of continuous functions on X mapping in to $\mathbb{R}$ or $\mathbb{C}$. Define the norm as the sup norm $\|\cdot\|_{L^\infty}$.

We will develop the spectral theorem of operators on the Hilbert space, i..e self-adjoint operators can be diagonalized.

If T is a self-adjoint operator on a Hilbert space, then we take the product of T (polynomials of T), let $C^*(T, I_{\mathcal{H}})$ be the sub-algebra of operators generated by T and I the identity operator, then take the closure, i.e. making it closed in the operator norm.

Remark The $*$ is to remind us, T is self-adjoint and when you take the adjoint and generate with it, it gets back into the same space.

Proposition 1.1

We have the next two algebra isomorphic to each other.

$$C^*(T, I_{\mathcal{H}}) \cong C(X) \tag{1.1}$$

♠

This is what we are aiminig for. We can generalize this even further to finitely many self-adjoint operators, in some sense, we are diagonalizing finitely many operators at the same time. If $T_1, ..., T_n$ is a collection of self-adjoint operators on $\mathcal{H}$, and such all commute with each other, then we also have

$$C^*(T_1, ..., T_n, I_{\mathcal{H}}) \cong C(X) \tag{1.2}$$

1.0.3 Groups

Let G be a group, B be a Banach space, for example, groups of automorphisms. Let

$$Aut(B) = \{T : T \text{is isometric, onto, invertible on } B\}$$

Definition 1.1

Suppose that α is a group homomorphisms, and $\alpha : G \to Aut(B)$, is called a representation on B or an action of the group G on B.

♣

Then we can consider the subalgebra $\mathcal{L}(B)$, consisting of the bounded linear operators on B, generated by

$$\{\alpha_x : x \in G\}$$

Remark The identity on G should be mapped into the identity operator on B, hence no need to include it.

Elements of the form $\Sigma_{z_x}\alpha_x, z_x \in \mathbb{C}$, (where Σ is a finite sum.)

Let's introduce, $f \in C_c(G)$ are functions with compact support and in discrete groups, imply they are of finite support.

$$\sum_{x\in G} f(x)\alpha_x = \alpha_f$$

note for except finitely many x, $f(x) = 0$.

Let $f, g \in C_c(G)$, then for

$$\alpha_f\alpha_g = (\sum f(x)\alpha_x)(\sum g(y)\alpha_y) = \sum_{x,y} f(x)g(y)\alpha_x\alpha_y = \sum_{x,y} f(x)g(y)\alpha_{xy}$$

The last inequality follows from α being a group homomorphism. And the sums are finite hence are able to exchange the orders. We further have,

$$\alpha_f\alpha_g = \sum_x \sum_y f(x)g(x^{-1}y)\alpha_y = \sum (f * g)(y)\alpha_y$$

where we define $f * g(y) = \sum f(x)g(x^{-1}y)$ as the convolution operator.

We get

$$\alpha_f\alpha_g = \alpha_{f*g}$$

This is how we define convolution on $C_c(G)$ Notice we have, by $\|\alpha_x\| = 1$,

$$\|\alpha_f\| = \|\sum f(x)\alpha_x\| \leq \sum |f(x)|\|\alpha_x\| = \sum |f(x)| = l^1(f) = \|f\|_{l^1}$$

It is therefore, easy to check

$$\|f * g\|_{l^1} \leq \|f\|_{l^1}\|g\|_{l^1}$$

We get $l^1(G)$ is an algebra with ??

For G commutative, it is easily connected with the Fourier transform.

Consider $l^2(G)$ with the counting measure on the group. For $x \in G$, let $\xi \in l^2(G)$ define $\alpha_x\xi(y) = \xi(x^{-1}y), \alpha_x$ being unitary. $l^1(G)$ acts on operators in $l^2(G)$ via α.

If G is commutative, then we have

$$\overline{\alpha_{l^1(G)}} \cong C(X)$$

where X is some compact space. Note that $C_c(G)$ operators on $l^2(G)$, and $\|\alpha_f\| \leq \|f\|_{l^1}$.

1.1 Lecture 2

Let's do some math.

Let X be a Hausdorff compact space, and let $C(X)$ denote the space of continuous functions defined on X. This is an algebra. You can multiply them, associatively and commutatively. We equip it with a norm $\|\cdot\|_{L^\infty}$. Note X, by assumption, is a normal space, you could have continuous functions mapped to 1 on one subset, 0 to the other subset. Hence there are many elements from $C(X)$.

Definition 1.2 (Normed Algebra)

Let $\mathcal{A}$ be an algebra on $\mathbb{R}$ or $\mathbb{C}$, is a normed algebra if it has a norm $\|\|$, as a vector space, such that for for $a, b \in \mathcal{A}$, we have

$$\|ab\| \leq \|a\|\|b\|$$

The above is called submultiplicity. ♣

Definition 1.3 (Banach Algebra)

A Banach Algebra is a normed algebra that is complete in the metric space from the norm. ♣

Given $x \in X$, define $\varphi_x : C(X) \to \mathbb{C}$ the evaluation map such that

$$\varphi_x(f) = f(x)$$

φ_x is an algebra homomorphisms between $C(X) \to \mathbb{R}$ or $C(X) \to \mathbb{C}$. This simply implies

$$\varphi_x(f+g) = (f+g)(x) = f(x) + g(x), \varphi_x(fg) = (fg)(x) = f(x)g(x)$$

We now make the note that, $C(X)$ has an identity element, which is the constant function 1, under multiplication. Hence $C(X)$ is a unital algebra. Note that φ_x defined above is a unital homomorphism, meaning that it sends identity to identity.

Note φ_x is also a multiplicative linear functional, also unital.

Proposition 1.2

Every multiplicative linear functional on $C(X)$ is of the form φ_x for some $x \in X$. ♠

Proof Main Claim: given a multiplicative linear functional φ, there exists a point x_0 and if we have some $f \in C(X)$, we have $\varphi(f) = 0$, then we have $f(x_0) = 0$. To prove this claim, we need compactness. Suppose the contrary of the claim. Suppose that for each $x \in X$, there is an $f_x \in C(X)$ such that $f(x) \neq 0$, but $\varphi(f) = 0$.

Set $g_x = \overline{f}_x f_x$, then we have $g_x(x) > 0$, but $\varphi(g_x) = \varphi(f_x)\varphi(\overline{f}_x) = 0$, then there is an open set O_x such that $x \in O_x$, and $g_x(y) > 0$ for all $y \in O_x$. Now by compactness, there is $x_1, ..., x_n$ such that $X = \bigcup_{j=1}^n O_{x_j}$, let $g = g_{x_1} + ...g_{x_n}$, then we have $g(y) > 0$ for all $y \in X$, and $\varphi(g) = 0$. Note that g is a continuous function, and g is invertible, and also $re(\frac{1}{g}) \in C(X)$, but we also have

$$\varphi\left(g \cdot \frac{1}{g}\right) = 1$$

Hence we've reached a contradiction. Then there exists $x_0 \in X$ such that if $\varphi(f) = 0$, this means $f(x_0) = 0$. For any f, consider $f - \varphi(f) \cdot 1$, apply φ, we have

$$\varphi(f - \varphi(f) \cdot 1) = 0, \text{ this implies there exists } x_0, \text{ such that } (f - \varphi(f)1)(x_0) = 0$$

This implies $f(x_0) = \varphi(f)$ which implies $\varphi(f) = \varphi_{x_0}(f)$.

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field.

For $\mathcal{A} = C(X)$, and $\varphi \in \widehat{\mathcal{A}}$.

Definition 1.4

For any unital commutative algebra $\mathcal{A}$ and let $\widehat{\mathcal{A}}$ be the set of unital homomorphisms of $\mathcal{A}$ into the field. ♣

Remark We have $|\varphi(f)| \leq \|\varphi\|\|f\|_{L^\infty}$, since φ is unital, we have $\|\varphi\| = 1$.

Thss is not always true for normed algebra, Let

$$\mathcal{A} := Poly \subset C([0,1])$$

We define $\varphi(p) = p(2)$, p is a polynomial. This is not continuous, nor is the $\|\varphi\| = 1$.

Proposition 1.3

If $\mathcal{A}$ is a unital commutative Banach algebra, and if $\phi \in \widehat{\mathcal{A}}$, then we have $\|\varphi\| = 1$.

The word "unital" is key here.

Proposition 1.4

Let $\mathcal{A}$ be a unital Banach algebra (not necessarily commutative), then if $a \in \mathcal{A}$, and $\|a\| < 1$, then we have

$$1_{\mathcal{A}} - a \text{ is invertible in } \mathcal{A}$$

Proof For this, we use completeness. $\frac{1}{1-a} =? \sum_{n=0}^{\infty} a^n, a^0 = 1_{\mathcal{A}}$ You could look at the partial sums. $S_m = \sum_{n=0}^{m} a^n$, you want to show that $\{S_m\}$ is a Cauchy sequence, and use completeness of Banach algebras. $\lim_{m\to\infty} S_m = \frac{1}{1-a}$.

To prove this is a cauchy sequence:

$$\|S_n - S_m\| = \|\sum_{j=m+1}^{n} a^j\| \leq \sum_{m+1}^{n} \|a^j\| \leq \sum_{m+1}^{n} \|a\|^j$$

And the fact that $\|a\| < 1$, we have the sum bounded by ϵ, hence $\{S_n\}$ is a Cauchy sequence. Let $b = \sum_{n=0}^{\infty} a^n$, we want to show that $b(1-a) = 1$.

$$b(1-a) = \lim_{n\to\infty} S_n(1-a) = \lim_{n\to\infty} \left(\sum_{n=0}^{\infty} a^n\right)(1-a) = \lim_{n\to\infty} (1 - a^{n+1}) = 1$$

The last inequality follows from $\|a^{n+1}\| \leq \|a\|^{n+1} \to 0$.

1.2 Lecture 3

We now begin.

Let $\mathcal{A}$ be a unital Banach algebra, and if $a \in \mathcal{A}$ and $\|a\| < 1$, then we have $(1-a)$ has an inverse and if $\mathcal{A} = \mathcal{B}(B)$, where B is some Banach space, then $T \in \mathcal{A}$, and $\|T\| < 1$, then we have

$$(1-T)^{-1} = \sum T^n$$

The above is called the Newmann series.

Now we have the following corollary.

Corollary 1.1

If $a \in \mathcal{A}$ and $\|1-a\| < 1$, then a is invertible.

Proof $a = 1 - (1-a)$.

Proposition 1.5

The set of invertible elements of $\mathcal{A}$ is an open subset of $\mathcal{A}$.

Proof The open ball about 1 consists of invertible elements. If d is any invertible element, then we define $a \mapsto da$. This map is continuous, i.e. it is the left representation $L_b(a) = ab$ for all $a \in \mathcal{A}$. If d is invertible, then the inverse is also continuous, hence it is a homeomorphism of $\mathcal{A}$ onto itself.

Denote the unit ball about 1 as $B_1(1)$, and let d be some invertible element, under L_d, homeomorphism, $O \mapsto d \cdot O$, this set is open, and consists of invertible elements. We take the union of all these elements, which give us an open set including every invertible elements.

□

Proposition 1.6

Let $C(X)$ be the unital Banach algebra, and for $f \in C(X)$, we have $\alpha \in Range(f)$ if and only if $(f - \alpha \cdot 1)$ is not invertible.

Proof Let $f \in C(X)$, and if $\alpha \in$ range of f, so $\alpha = f(x_0)$ for some x_0. then

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(-\alpha \cdot 1)$ is not invertible. Conversely, if we have $f - \alpha 1$ is not invertible, then there exists $x_0 \in X$ such that

$$(f - \alpha \cdot 1)(x_0) = 0$$

Hence $f(x_0) = \alpha$, i.e., $\alpha \in$ range of f.

□

Definition 1.5 (spectrum of an element)

For any unital algebra $\mathcal{A}$ over some field $\mathbb{F}$, for any $a \in \mathcal{A}$, the set

$$\{\lambda \in \mathbb{F} : a - \lambda 1_{\mathcal{A}} \text{ is not invertible } \}$$

is called the spectrum of a, denoted as $\sigma(a)$.

Interpret this in our familiar linear map: λ is called an eigenvalue, i.e. is in the spectrum of T if we have $T - \lambda I$ is not invertible.

Proposition 1.7

Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$, then if $\lambda \in \sigma(a)$, then

$$|\lambda| \leq \|a\|$$

Proof Suppose $|\lambda| > \|a\|$, then $\lambda \neq 0$, then

$$a - \lambda \cdot 1 = -\lambda(1 - \frac{a}{\lambda})$$

And by assumption, $\|a/\lambda\| \leq 1$, hence $(1 - a/\lambda)$ is invertible. Hence $a - \lambda \cdot 1$ is invertible (product of two invertible elements), meaning $\alpha \notin \sigma(a)$.

□

Proposition 1.8

Let φ be a multiplicative linear functional on $\mathcal{A}$, i.e. $\varphi \in \widehat{\mathcal{A}}$, and then $\varphi(a) \in \sigma(a)$, and we have

$$|\varphi(a)| \leq \|a\|, \|\varphi\| = 1$$

Proof $\varphi(a - \varphi(a) \cdot 1) = 0$. Hence $a - \varphi(a)1$ is not invertible.

□

Proposition 1.9

$\sigma(a)$ is a closed subset of $\mathbb{R}, \mathbb{C}$.

Proof Define the map $\phi : \lambda \mapsto a - \lambda 1$, the map ϕ is continuous (multiplication and subtraction are both continuous). We know the set of invertible elements of $\mathcal{A}$ is open, hence

$$\sigma(a) = \phi^{-1}(\text{ noninvertible}) = \phi^{-1}(\mathcal{A} \setminus \text{ invertible })$$

Or simply,

$$\sigma(a) = (\phi^{-1}(\text{ invertible }))^c$$

Hence the spectrum of an element is closed.

□

Let $\varphi \in \widehat{\mathcal{A}}$ then $\|\varphi\| = 1$. So $\widehat{\mathcal{A}}$ is a subset of the unit ball of $\mathcal{A}'$, which denotes the dual vector space of continuous linear transformations.

On $\mathcal{A}'$, we can equip the weak-* topology, i.e. the weakest topology, making the map $\psi \mapsto \psi(a)$ continuous.

Proposition 1.10

$\widehat{\mathcal{A}}$ is closed for the weak- topology.*

Proof let $\{\varphi_\lambda\}$ be a net of elemnts of $\widehat{\mathcal{A}}$, that converges to some $\psi \in \mathcal{A}'$ in the weak-* topology, i.e., for every $a \in \mathcal{A}$, $\varphi_\lambda(a) \to \psi(a)$ for all $a \in \mathcal{A}$.

Then $\varphi(a, b) = \lim \varphi_\lambda(ab) = \lim \varphi_\lambda(a)\varphi_\lambda(b) = \varphi(a)\varphi(b)$.

$\varphi(1) = \lim(\varphi_\lambda(1)) = \lim 1 = 1$.

Theorem 1.1 (Alaoglu's theorem)

For any normed vector space V, the closed unit ball of V' is compact in the weak- topology.*

As an immediate corollary, we have the following.

Corollary 1.2

$\widehat{\mathcal{A}}$ is compact with respect to the weak- toplogy.*

Proof $\widehat{\mathcal{A}}$ is a closed subset of a compact set, hence is also compact.

□

Let $\mathcal{A} = C(X)$, and $\widehat{\mathcal{A}}$, we define $x \mapsto \varphi_x$ is a bijection. The weak-* topology in $\widehat{\mathcal{A}}$ makes $\varphi_x \mapsto \varphi_x(f) = f(x)$ continuous. Such $x \mapsto \varphi_x$ is a homomorphism of X onto $\mathcal{A}$.

For $\mathcal{A}$ unital Banach algebra, commutative, for any $a \in \mathcal{A}$, define

$$\widehat{a} \in C(\widehat{\mathcal{A}}), \widehat{a}(\varphi) = \varphi(a)$$

Proposition 1.11

The map $a \mapsto \widehat{a}$ is a unital algebra homomorphism from $\mathcal{A}$ into $C(\mathcal{A})$.

Proof we have

$$\widehat{ab}(\varphi) = \varphi(ab) = \varphi(a)\varphi(b) = \widehat{a}(\varphi)\widehat{b}(\varphi) = (\widehat{a}\widehat{b})(\varphi)$$

Hence

$$(\widehat{ab}) = \widehat{a}\widehat{b}, \widehat{(a+b)} = \widehat{a} + \widehat{b}, \widehat{1_a} = 1$$

□

1.3 Lecture 4

Today we talk about the structure of $\widehat{l^1(S)}, \widehat{l^1(G)}$, where S, G are semigroups and groups, and how they naturally identify with the unit disk $\mathbb{D}$, and the unit circle $\mathbb{T}$.

Let S be a commutative discrete semigroups, for example $\mathbb{N} \cup \{0\}$, and $f \in C_c(S)$, then we can write $f = \sum_{x \in S} f(x)\delta_x$, where we define $\delta_x\delta_y = \delta_{xy}$. Note that $C_c(S)$ is dense in $l^1(S)$.

Definition 1.6 (Convolution)

Take any $f, g \in C_c(S)$, we consider the following:

$$\sum_{x \in S} f(x)\delta_x \sum_{x \in S} g(y)\delta_y = \sum_{x \cdot y} \delta_{xy} = \sum_{z \in S} \left(\sum_{xy = z} f(x)g(y) \right) \delta_z$$

where we define the convolution between two functions

$$f * g(z) = \sum_{x,y,xy=z} f(x)g(y)$$

And under this convolution operation, we have $l^1(S), *$ as a Banach algebra.

Example 1.1 If we consider polynomials of the form $f(x) = \sum_{n=0}^{\infty} f(n)x^n$, and consider the operation between two polynomials

$$\left(\sum f(m)x^m\right)\left(\sum g(n)x^n\right) = \sum_p \left(\sum_{m+n=p} f(m)g(n)x^p\right) = \sum_p (f*g)(p)$$

And let $f \in C_c(S)$, where $S = \mathbb{N}$. we define $\|f\|_{l^1} = \sum_{x\in S} |f(x)|$.
It is easy to check we have

$$\|f * g\|_{l^1} \le \|f\|_{l^1}\|g\|_{l^1}$$

We let $\mathcal{A} = l^1(S)$, and $\widehat{\mathcal{A}}$ denote the set of unital homomorphisms from $\mathcal{A}$ to $\mathbb{R}, \mathbb{C}$. Note that $\|\varphi\| = 1, \varphi \in \widehat{\mathcal{A}}$.
Note that we know $(l^1(S))' = l^\infty(S)$, hence $\widehat{\mathcal{A}} \subset \mathcal{A}'$. Note that we have $\|\varphi\| = 1$, hence if we $\varphi \in l^\infty(S)$, we have

$$\|\varphi\|_{l^\infty} = 1$$

Then for $z \in S$, $\|z\| \le 1$, we have $|\varphi(z)| \le 1$.

Proposition 1.12

We naturally identify $\widehat{l^1(S)}$ with $Hom(S, \mathbb{D})$, i.e. $\{\varphi \in l^\infty(S) : \|\varphi\|_{l^\infty} = 1\}$.

Proof Given $f \in \widehat{l^1(S)}$, we know it's multiplicative, unital, hence all these transfer when viewing $\varphi \in l^\infty(S)$. This implies

$$\varphi(\delta_x)\varphi(\delta_y) = \varphi(\delta_{xy}) \Rightarrow \varphi(x)\varphi(y) = \varphi(xy)$$

Note here xy denotes the operation on S between x, y, for example, could be $x + y$. Hence naturally, if $\varphi \in \widehat{l^1(S)}$, φ can also be viewed as $\varphi : S \to \mathbb{D}$, and thus is in l^∞, with $|\varphi(s)| \le 1$.

□

Furthermore, we can identify elements in $\widehat{l^1(S)}$ with the unit disk. Take $S = \mathbb{N}$.

Proposition 1.13

$$\widehat{l^1(\mathbb{N})} \cong \mathbb{D}$$

where $\mathbb{D}$ denotes the unit disk in $\mathbb{C}$.

Proof We motivate this by noticing $\mathbb{N}$ is generated by 1, and thus viewing $\varphi \in \widehat{l^1(\mathbb{N})}$ as $\varphi \in l^\infty(\mathbb{N})$, we have φ is determined by $\varphi(1)$. And denote $\varphi(1) = z_0$, then we have

$$\varphi(n) = z_0^n$$

We thus define a map as follows, for $z \in \mathbb{D}$,

$$z \mapsto \varphi(n) = z^n$$

The map is continuous, bijective, and thus a homeomorphism between compact and Hausdorff space.

□

Proposition 1.14

The standard topology on $\mathbb{D}$ coincides with the weak- topology on $\widehat{l^1(\mathbb{N})}$.*

$$D_{std} \cong D_{weak-*}$$

Proof We just need to associate an element in $\mathbb{D}$ with a function $\varphi \in \widehat{l^1(\mathbb{N})}$. And we do this by

$$z \mapsto \sum_{n \in \mathbb{N}} f(n) x^n$$

Both maps are continuous, bijective, and between compact and Hausdorff space, hence is a homeomorphism.

1.3.1 On groups

We let G denote a discrete commutative group, and we see everything above follows, with one extra property.

Proposition 1.15

We have the following:

$$\widehat{l^1(G)} \cong \mathbb{T}$$

where $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Proof For $\varphi \in \widehat{l^1(G)}$, we have

$$|\varphi(x \cdot x^{-1})| = |\varphi(e)| = 1$$

Because $|\varphi(x)| \leq 1, \forall x$, Hence we have

$$|\varphi(x)| = 1, \forall x$$

Hence we have $\widehat{l^1(G)}$ naturally identifies with $\mathbb{T}$. Like what we described above, we have what is desired.

□

Remark Take $G = \mathbb{Z}$, if we denote $z \in \mathbb{T}$ as $z = e^{2\pi i t}$, then we naturally identify with

$$\sum_{n \in \mathbb{Z}} f(m) e^{2\pi i n t}$$

we denote this mapping as $\widehat{f}$, i.e.

$$\widehat{f}(z) = \sum_{m \in \mathbb{Z}} f(m) e^{2\pi i n t}$$

This is the Fourier transform.

1.4 Lecture 5

Last time, we talked about if we denote $\mathcal{A} = l^1(G)$, equipped with $\|\cdot\|_{l^1}$, under convolution, we have

$$\widehat{\mathcal{A}} \text{ “=” } Hom(G, \mathbb{T})$$

If we take $G = (\mathbb{Q}, +)$, one can ask the question if $\widehat{\mathcal{A}}$ is big enough. And we will se later in the course, the answer is yes.

For pointwise multiplication, $\widehat{G}$ forms a group, and in fact $\widehat{G}$ is a compact topological group.

For any compact commutative group G, for exapmle $\mathbb{R}^n$ under $+$. Define

$$\widehat{G} = \text{ continuous homomorphisms into } \mathbb{T}$$

Remark We now require continuous with this general G (previously was not required for discrete group G).

Proposition 1.16

Let G be a locally compact and commutative group, we have $\widehat{G}$ as a locally compact, commutative group.

We define the pairing between G and $\widehat{G}$ as follows: $x \in G, \varphi \in \widehat{G}$,

$$\varphi(x) = \langle x, \varphi \rangle$$

And we have the folliwng map is a homeomorphism.

$$G \mapsto \widehat{\widehat{G}}$$

Now let G, H denote locally compact groups, and $\phi : G \to H$ bet a continuous homomorphism. Note we have the following diagram:

$$G \xrightarrow{\phi} H$$

$$\widehat{G} \xleftarrow{\phi} \widehat{H}$$

If we take an element $\psi \in \widehat{H}$, we consider $\psi \circ \phi$. We get $\psi \circ \phi \in \widehat{G}$.

Definition 1.7 (category, functor)

A category is specified by

1. *a set of objects*
2. *morphisms between objects*
   (a). *X, Y, Z are objects, and if*

$$X \xrightarrow{\Phi} Y \xrightarrow{\Psi} Z$$

   (b). *For each object X, there is an identity morphism 1_X.*

And a functor is defined to be such a morphism between categories. ♣

Example 1.2 For category of finite vector spaces V, passing from vector space to its dual V' is a functor.

Note that we have the following diagram, assuming they are vector spaces over the reals,

$$V \xrightarrow{T} W$$

$$V' \xLeftarrow{T^t} W'$$

$$V'' \xrightarrow{T^{tt}} W''$$

The map going in the same directions $V \to W$, and $V'' \to W''$ is called covariant, whereas $V' \leftarrow W'$ is called contravariant.

Example 1.3 For category of locally compact groups G, H, assigning the dual group is a functor:

$$G \to H$$

$$\widehat{G} \leftarrow \widehat{H}$$

$$\widehat{\widehat{G}} \to \widehat{\widehat{H}}$$

Example 1.4 Now let X be a compact space. Given Φ continuous map between $X \to Y$.

$$X \xrightarrow{\Phi} Y$$

$$C(X) \leftarrow C(\Phi) C(Y)$$

For $f \in C(Y)$, we define

$$C(\Phi)(f) = f \circ \Phi$$

Similarly, we take

$$X \xrightarrow{\varphi}\xrightarrow{\phi} Z$$

$$C(X) \xleftarrow{C(\varphi)} C(Y) \xleftarrow{C(\phi)} C(Z)$$

where for $f \in C(Y), C(\varphi)(f) = f \circ \varphi$, and $g \in C(Z), C(\phi) = g \circ \phi$. This is a contravariant functor from the category of compact Hausdorff space into the category of unital commutative Banach algebra.

Now we build an important intuition that given a unital algebra homomorphism map between $C(X)$ and $C(Y)$, there eixsts a map from X to Y.

Proposition 1.17

Suppose X, Y are compact, there exists a unital algebra homomorphism

$$C(X) \xleftarrow{F} C(Y)$$

Then there exists a continuous homomorphism $\check{F} : X \to Y$.

Proof Define $\varphi_x : C(X) \to \mathbb{C}$ as the evalutation map: take $f \in C(X)$,

$$\varphi_x(f) = f(x)$$

Then $\varphi_x \circ F \in \widehat{C(Y)}$. And we know that any element in $\widehat{C(Y)}$ is a point evaluation, i.e. there exists $y \in Y$ such that

$$\varphi_y = \varphi_x \circ F$$

We thus define $\check{F}(x) = y$ as such that it satisfies the above equation. We need to show $\check{F}$ is continuous. Note that X, Y are compact Hausdorff spaces, and the topology on Y is the coarest topology making all functions $g \in C(Y)$ continuous.

$$\begin{aligned} g \circ \check{F}(x) &= g(\check{F}(x)) \\ &= g(y : \varphi_y = \varphi_x \circ F) \\ &= \varphi_y(g : \varphi_y = \varphi_x \circ F) \\ &= \varphi_x \circ F(g) \\ &= F(g)(x) \end{aligned}$$

Hence by F, g being continuous, we have $\check{F}$ is also continuous.

□

There is a natural bijection between the continuous functiosn from X to Y, and the unital algebra homomorphism from $C(X)$ to $C(Y)$.

A quick reminder:

Remark For X compact, the weak-* topology coincides with the standard topology.

1.5 Lecture 6

Now we begin. From Aren "not talking to you is torture."

Let $\mathcal{A}$ be a unital Banach algebra.

We write $GL_n(\mathcal{A})$ to denote the general linear group, the group formed by $n \times n$ matrices with entries from $\mathcal{A}$.

The less standard notation is $GL_I(\mathcal{A})$ is the group of invertible elements in $\mathcal{A}$. As we have shown previously, this is a closed subset of $\mathcal{A}$. This is the notation that we will use.

Remark It is easy to see that the product is jointly continuous.

Proposition 1.18

The following map is continuous.

$$a \mapsto a^{-1}$$

Proof Given $\|a - b\| < \delta$, we would like to show $\|a^{-1} - b^{-1}\| < \epsilon$. We first rewrite

$$a^{-1} - b^{-1} = a^{-1}(b - a)b^{-1}$$

Hence we have

$$\|a^{-1} - b^{-1}\| \leq \|a^{-1}\|\|b - a\|\|b^{-1}\|$$

Take $\delta = \epsilon / \|a^{-1}\|\|b^{-1}\|$ would suffice.

□

Proposition 1.19

Fix $a \in GL(\mathcal{A})$, there exists a neighborhood O of a and a constant K such that for all $y \in O$, we have

$$\|c^{-1}\| < K$$

Proof Let $V = \{d \in \mathcal{A} : \|1 - d\| < 1/2\}$, then d is invertible and

$$d^{-1} = \sum_{n=0}^{\infty}(1-d)^n$$

We thus have

$$\|d^{-1}\| \leq \frac{1}{1 - \|1-d\|} \leq \frac{1}{1-1/2} = 2$$

We then identify what our O should be. Let $O = aV$, then we want to show that every ad has an inverse with bounded norm. Because a, d are both invertible, ad is also invertible.

$$\|(ad^{-1})\| = \|d^{-1}a^{-1}\| \leq \|d^{-1}\|\|a^{-1}\| \leq 2\|a^{-1}\|$$

□

Remark For each invertible element, we can find a neighborhood of invertible elements around it, and using that $(1-d)$ is bounded, then d is invertible, we can bound $\|d^{-1}\|$.

Definition 1.8

Fix $a \in \mathcal{A}$, the resolvent set of $\mathcal{A}$ is the complement of spectrum of $\mathcal{A}$, i.e. it is the set

$$\{\lambda \in \mathbb{F} : a - \lambda I \text{ is invertible }\}$$

Hence the resolvent set is an open, unbounded suset of $\mathbb{C}$ or $\mathbb{R}$.

Definition 1.9 (Resolvent function)

On the resolvent set, $\{\lambda \in \mathbb{F} : a - \lambda 1 \text{ is invertible }\}$ is as follows:

$$R(a, \lambda) = (\lambda 1_{\mathcal{A}} - a)^{-1}$$

note that a is fixed, and λ is the variable here.

Now we note that this $R_a(\lambda)$ function is nicely behaved.

Proposition 1.20

The resolvent function $R_a(z)$ is analytic on the resolvent set, and vanishes as $z \to \infty$.

Proof We first define the notation of analyticity on an open subset of $\mathbb{R}, \mathbb{C}$: this means for every point in the open set O, we can find a power series expansion of the function such that its radius of convergence > 0.

Fix z_0 in the resolvent set. We know $z_0 1_{\mathcal{A}} - a$ is invertible. We consider $(z1_{\mathcal{A}} - a)$, for z in the resolvent set. We will omit the $1_{\mathcal{A}}$ for simplicity.

$$z1_{\mathcal{A}} - a = (z_0 - a) - (z_0 - z) = (z_0 - a)\left(1_{\mathcal{A}} - \frac{z_0 - z}{z_0 - a}\right)$$

We know the the latter term is invertible if $\|\frac{z_0-z}{z_0-a}\| < 1$ has norm, hence we have

$$(z-a)^{-1} = \sum_{n=0}^{\infty}\left(\frac{z_0 - z}{z_0 - a}^n\right)(z_0 - a)^{-1}$$

What happens when we let $z \to \infty$, we consider $R_a(1/z)$, and let $z \to 0$. Note that we have the following:

$$R_a\left(\frac{1}{z}\right) = \left(\frac{1}{z} - a\right)^{-1} = \left(\frac{1-az}{z}\right)^{-1} = z(1-az)^{-1}$$

Let $z \to 0$ makes $R_a(1/z)$ go to zero.

□

Now given that $R_a(z)$ is analytic and bounded at ∞, we can state the following important theorem.

Theorem 1.2 (Nonemptyness of spectrum)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, then for any $a \in \mathcal{A}$, we have $\sigma(a) \neq \emptyset$.

Proof Assume there exists $a \in \mathcal{A}$, such that $\sigma(a) = \emptyset$. If $\mathcal{A} = \mathcal{C}$, then we would have $R_a(\lambda)$ be a bounded entire, complex-valued function defined on all of $\mathbb{C}$. By Liouville's theorem, we must have $R_a(z)$ a constant function, but we know $z \to \infty$, $R_a \to 0$, hence $R_a(z)$ is constantly 0, but this cannot be true.

If our $\mathcal{A}$ is a more general Banach algebra, then we take a slight detour of creating an entire bounded function, via the following map

$$z \mapsto \phi(R_a(z))$$

where ϕ is some nonzero element in $\mathcal{A}'$, guaranteed by Hahn-Banach theorem. Then we have the above map is complex-valued, entire, bounded at ∞. Again, the function is constantly 0.

With the nonemptyness of spectrum theorem, we now state the Gelfand-Mazur theorem.

Theorem 1.3 (Gelfand-Mazur)

Let $\mathcal{A}$ be a unital Banach algebra over $\mathbb{C}$, if any nonzero element of $\mathcal{A}$ is invertible, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.

Proof For any $a \in \mathcal{A}$, we know $\sigma(a) \neq \emptyset$, hence there exists λ such that $\lambda 1_{\mathcal{A}} - a$ is invertible, i.e. $a = \lambda 1_{\mathcal{A}}$, hence establishing an isomorphism between $\mathcal{A}$ and $\mathbb{C}$. In other words, $\mathcal{A} = \mathbb{C}1_{\mathcal{A}}$.

□

1.5.1 Functional Calculus

Proposition 1.21

Let $a \in \mathcal{A}$, then if $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ converges for $|z| < r$, where $r > \|a\|$, then $\sum_{n=0}^{\infty} \alpha_n a^n$ converges as well.

We first start with proving the following statement.

Lemma 1.1

Let f be a polynomial, $\mathcal{A}$ is a unital Banach algebra over $\mathbb{C}$, $f = \sum_{n=0}^{k} a_n x^n$, then for $a \in \mathcal{A}$, we have

$$\sigma(f(a)) = f(\sigma(a))$$

This states the spectrum of a under f is exactly the spectrum of f evaluated at a.

Proof $(\Leftarrow)$. We take $\lambda \in \sigma(a)$, and we would like to show $f(\lambda)$ is in the spectrum of $f(a)$. We note that if $\lambda \in \sigma(a)$, then $a = \lambda 1_{\mathcal{A}}$, and $f(\lambda 1_{\mathcal{A}}) = f(a)$, hence by definition, $f(a) - f(\lambda)1_{\mathcal{A}}$ is not invertible implying $f(\lambda)$ is in the spectrum of $f(a)$. Note that this also implies $f(a) - f(\lambda) = (a - \lambda)Q(z)$ for some polynomial $Q(z)$.

$(\Rightarrow)$. We take $\lambda \in \sigma(f(a))$, i.e. $f(a) = \lambda 1_{\mathcal{A}}$. we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. If f is some polynomial, then we can rewrite as follows:

$$f(z) - \lambda = d(z - c_1)...(z - c_n)$$

Plugging in a we get

$$f(a) - \lambda = d(a - c_1 1_{\mathcal{A}})...(a - c_n 1_{\mathcal{A}})$$

If $f(a) - \lambda$ is not invertible, then there exists j such that $(a - c_j 1_{\mathcal{A}})$ is not invertible. This implies,

$$c_j \in \sigma(a)$$

Recall we would like to show $\lambda = f(y)$, where $y \in \sigma(a)$. In fact, we have $\lambda = f(c_j)$ by knowing $f(c_j) - \lambda = 0$.

□

Now let $f(z) = z^n$, and if $\lambda \in \sigma(a)$, then $\lambda^n \in \sigma(a^n)$ by the previous lemma. Then we know that

$$|\lambda^n| = |\lambda|^n \leq \|a^n\|$$

This implies

$$|\lambda| \leq \|a^n\|^{1/n}, \forall n$$

Hence we have

$$|\lambda| \leq \liminf_n \{\|a^n\|^{1/n}\}$$

Definition 1.10 (spectral radius)

Fix $a \in \mathcal{A}$, we define the spectral radius of a, denoted by $r(a)$,

$$r(a) = \sup_\lambda \{|\lambda| : \lambda \in \sigma(a)\}$$

♣

Next we introduce an equivalent definition of the spectral radius which connects to the Gelfand transform.

Proposition 1.22

For $\mathcal{A}$ a Banach algebra, we have the following relationship:

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|\Gamma(a)\|_\infty$$

♠

Example 1.5 Note it we have a self-adjoint operator T, then the spectral radius of T would be the absolute value of the largest eigenvalue, $|\lambda|$.

Corollary 1.3

$$r(a) \leq \limsup_n \{\|a^n\|^{1/n}\}$$

♡

Proof From the previous remark that $|\lambda| \leq \|a^n\|^{1/n}$, hence this follows.

1.6 Lecture 7

I have not typed up for this?

1.7 Lecture 8

Let $\mathcal{A}$ be a unital Banach algebra. Then for $a \in \mathcal{A}$, and we look at the resolvent of a, $R_a(\lambda)$, we've noted that as $\lambda \to \infty$, we have

$$\lim_{\lambda\to\infty} R_a(\lambda) = \lim_{\lambda\to\infty} (\lambda 1_{\mathcal{A}} - a)^{-1} = \lim_{\lambda\to\infty} \lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$$

And the above Laurent series converges for $|\lambda| \geq \|a\|$.

Recall that we define the spectral raidus, $r(a)$, as

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} \leq \|a\|$$

Now we would like to prove the following proposition.

Proposition 1.23 (Gelfand-Beurling)

$$r(a) = \lim \|a^n\|^{1/n}$$

♠

Proof

If we let $\lambda = 1/z$, then

$$R(a, z) = z \sum_{n=0}^{\infty} a^n z^n$$

This converges for $|z| \le \|a\|^{-1}$, but maybe?? also for $|z| < r(a)^{-1}$?

For $r > r(a)$, i.e. $|z| \le r^{-1}$, we know $\sum_n a^n r^n$ converges for $r > r(a)$.

know $z \sum a^n z^n$ converges absolutely. In particular,

$$a^n z^n \to 0$$

Hence there exists M such that for $n \ge M$, we have

$$\|a^n r^{-n}\| \le 1$$

This implies that

$$\|a^n\| \le r^n \Rightarrow \|a^n\|^{1/n} \le r$$

for all $n \ge M$.

This implies that

$$\limsup \|a^n\|^{1/n} \le r$$

And note that r is arbitrary close to the spectral radius $r(a)$. Hence we have

$$\limsup \|a^n\|^{1/n} \le r(a) \le \liminf \|a^n\|^{1/n}$$

We've derived the second inequality from last class. Hence all inequalities become equalities. This gives us

$$r(a) = \lim \|a^n\|^{1/n}$$

□

For each $\varphi \in \mathcal{A}'$, consider the map

$$\lambda \mapsto \lambda^{-1} \sum \varphi(a^n)\lambda^{-n}$$

This series converges for $r > r(a)$. We can apply the same process, to argue that there exists M_φ such that

$$\|\varphi(a^n)r^{-n}\| \le M_\varphi$$

for all $n \ge 0$. Note that M_φ could be different for all φ.

Note that

$$\mathcal{A} \to \mathcal{A}' \to \mathcal{A}''$$

there is a natural injection of $a \mapsto \widehat{a} \in \mathcal{A}''$.

For each n, definite $F_n \in \mathcal{A}''$, by $F_n(\varphi) = |\varphi(a^n r^{-n})| \le M_\varphi$. Applying the UBP, we have

$$|F_n(\varphi)| \le M \Rightarrow |\varphi(a^n)r^{-n}| \le M$$

This implies that

$$|\varphi(a^n)| \le r^n M$$

Note that by Hahn-Banach, for any $b \in \mathcal{A}$, we have

$$\|b\| = \sup\{|\varphi(b)| : \|\varphi\| = 1\}$$

Taking n-th root of both sides, we gets

$$\|a^n\| \le r^n M \Rightarrow \|a^n\|^{1/n} \le r M^{1/n} \to r$$

Hence we again obtain the same result.

□

Recall UBP.

Theorem 1.4 (Uniform Boudnedness Principle)

Let X be Banach, and Y be normed, let $T_n : X \to Y$ be a family of linear operators, and if for all $x \in X$, we have

$$\|T_n(x)\| < \infty$$

Then for all n, we have

$$\|T_n\| < \infty$$

♡

Note that if $\mathcal{A}$ is unital, and if $\mathcal{A} \subset \mathcal{B}$ with some unit. For $a \in \mathcal{A}$, if a is not invertible in A, then it might be invertible in $\mathcal{B}$. Hence if we use $\sigma_{\mathcal{A}}(a)$ to denote the spectrum of a in $\mathcal{A}$.

Proposition 1.24

$$\sigma_{\mathcal{B}}(a) \subset \sigma_{\mathcal{A}}(a)$$

Example 1.6 Let $\mathcal{B} = l^1(\mathbb{Z})$, and let $\mathcal{A} = l^1(\mathbb{N})$, equipped with convolution.

Clearly $\mathcal{A} \subset \mathcal{B}$. And note that the delta function at 1, δ_1 is not invertible in $\mathcal{A}$ but it has an inverse δ_{-1} in $\mathcal{B}$. Hence we see $0 \in \sigma_{\mathcal{A}}(a)$, but $0 \notin \sigma_{\mathcal{B}}(a)$.

Proposition 1.25 (Spectral radius is preserved)

For $\mathcal{A} \subset \mathcal{B}$, we have

$$r_{\mathcal{A}}(a) = \lim \|a^n\|^{1/n} = r_{\mathcal{B}}(a)$$

This proposition tells us that the spectral radius of an element $a \in \mathcal{A}$ is independent of the Banach algebra it is considered in, but rather only depends on itself.

Proposition 1.26

Let X be compact, and let $\mathcal{A} = C(X)$. Then for $f \in C(X)$, we have

$$\|f^2\|_\infty = \|f\|_\infty^2$$

Proof Look at where f takes $\|f\|_\infty$, and sqaure it, since when X is compact, you can actually obtain the point where $|f(x)| = \|f\|_\infty$.

Remark The same property holds for f in any unital subalgebra of $C(X)$, for example, if $X \subset \mathbb{C}$, and let $\mathcal{A}$=functions that are holomorphic on an open subset of $\mathbb{C}$ that are in X.

Proposition 1.27

Let $\mathcal{A}$ be a unital Banach algebra such that for $a \in \mathcal{A}$, we have

$$\|a^2\| = \|a\|^2$$

Then we have

$$r(a) = \|a\|$$

Proof If we have

$$\|a^2\| = \|a\|^2$$

This implies that

$$\|a^4\| = \|a\|^4$$

By induction, for any n, we have

$$\|a^{2^n}\| = \|a\|^{2^n}$$

Hence by taking $1/2^n$-root of both sides, we get that the spectral radius of $r(a)$

$$r(a) = \|a^{2^n}\|^{1/2^n} = \|a\|$$

□

Let $\mathcal{H}$ be a Hilbert space, over $\mathbb{C}$, and let $\mathcal{A} = B(H)$, i.e. the bounded linear operators on $\mathcal{H}$, and equip with the operation of taking adjoint. $T \mapsto T^*$.

Proposition 1.28

For any $T \in B(\mathcal{H})$, we have

$$\|T^*T\| = \|T\|^2$$

Proof We know that $\|T^*\| = \|T\|$. And thus

$$\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$$

For the reverse direction, let $\xi \in \mathcal{H}$, then

$$\|T(\xi)\|^2 = \langle T\xi, T\xi\rangle = \langle \xi, T^*T\xi\rangle \leq \|T^*T\|\|\xi\|^2$$

where the last inequality follows form Cauchy-Schwartz. This implies that

$$\|T(\xi)\| \leq \|T^*T\|^{1/2}\|\xi\|$$

which by definition ($\|T\|$ is the smallest constant for the inequality), gives

$$\|T\| \leq \|T^*T\|^{1/2}$$

Taking squares we get the desired result.

□

Note *We used the inner product to justify $\|T\|^2 \leq \|T^*T\|$, which we cannot necessarily do in a non-Hilbert space.*

Corollary 1.4

If $T^* = T$, then

$$\|T^2\| = \|T\|^2$$

And we have

$$r(T) = \|T\|$$

where the spectral radius is determined by the algebra elements.

Note that for general T, we have T^*T is always self-adjoint,

$$\|T\|^2 = \|T^*T\| = r(T^*T)$$

Then we have

$$\|T\| = (r(T^*T))^{1/2}$$

where the spectral radius is determined by the *-algebra structure.

1.8 Lecture 9

Let $\mathcal{H}$ be a hilbert space over $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ with $\|\cdot\|_\infty$, and closed under taking involutions. If $T \in \mathcal{B}(\mathcal{H})$, then

$$\|T^*T\| = \|T\|^2$$

So if $T^* = T$, then we have

$$\sigma(T) = \|T\|$$

Definition 1.11 (Concrete C^*-algebra)

*A concrete C^*-algebra is a norm-closed sub-algebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, for some $\mathcal{H}$ such that is **self-adjoint**, i.e., if $T \in \mathcal{A}$, then $T^* \in \mathcal{A}$. We call $\mathcal{A}$ is unital if $1_{\mathcal{H}} \in \mathcal{A}$.*

Corollary 1.5

If $\mathcal{A}$ is a C^* algebra, then for all $a \in \mathcal{A}$,

$$\|a^*a\| = \|a\|^2$$

If $\mathcal{A}$ is unital C^*-algebra, and if $a^* = a$, then $r(a) = \|a\|$.

This follows from our discussio above. Next we say a bit about the Gelfand transform.

Definition 1.12

Let $\mathcal{A}$ be a unital Banach algebra, and commutative, then we have the Gelfand transform $\Gamma : \mathcal{A} \to C(\widehat{A})$:

$$\Gamma(a)(\varphi) = \varphi(a)$$

then he said something of homomor to the complex numbers or something

Note that if $a \in \mathcal{A}$, and $\varphi \in \widehat{A}$, then $\varphi(a) \in \sigma(a)$, then we have

$$|\varphi(a)| \le \|a\| = r(a)$$

then $\|\widehat{a}\|_\infty \le r(a)$. Now we would like to show

$$\|\widehat{a}\|_\infty = r(a)$$

Theorem 1.5

$r(a)$ is the spectral radius of a, which is defined as $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}$, and we have

$$r(a) = \|\widehat{a}\|_\infty$$

Note *This gives us a correspondance between φ and non-invertible elements of $\mathcal{A}$. This says if a is not invertible, then we can find some φ such that $\varphi(a) = 0$, i.e. a φ kills the non-invertible element.*

We will now dedicate the next 50 minutes of our life to proving this theorem.

Theorem 1.6 (φ and maximal ideal)

If $\lambda \in \sigma(a)$, then there exists $\varphi \in \widehat{A}$ such that $\varphi(a) = \lambda$. This is equiavalent to saying: if $(a - \lambda 1)$ is not invertible, then there is a $\varphi \in \widehat{\mathcal{A}}$, such that

$$\varphi(a - \lambda 1) = 0$$

This means if a is not invertible, then there exists $\varphi \in \widehat{\mathcal{A}}$ such that $\varphi(a) = 0$.

Proof Suppose $a \in \mathcal{A}$, and consider

$$a\mathcal{A} = \{ab : b \in \mathcal{A}\}$$

The set $a\mathcal{A}$ does not contain the identity element i.e. $1_{\mathcal{A}} \notin a\mathcal{A}$ (otherwise it would imply it has an inverse). And $a\mathcal{A}$ is a two-sided proper ideal (by $\mathcal{A}$ being commutative).

We now introduce a fact that we will use.

Definition 1.13

An ideal I is maximal if I is proper in R, and not contained in any bigger proper ideals.

Lemma 1.2

Let R be a unital ring commutatitve, every proper ideal is contained in a maximal ideal (by Zorn's lemma).

Lemma 1.3

For $\mathcal{A}$ unital commutative Banach algebra, if I is a proper ideal, then its closure is a proper ideal.

Proof We have seen that $GL(\mathcal{A})$, the set of invertible elements, is open. Hence its complement is closed. Any proper ideal does not contain any elements in $GL(\mathcal{A})$, hence its closure is closed inside a closed set.

Remark Let X be locally compact, but compact, such as $\mathbb{R}$, then we have $C_c(X) \subset C_\infty(X)$. Note that $C_c(X)$ is a proper ideal of $C_\infty(X)$, but it's dense in $C_\infty(X)$, hence its closure is the entire space, hence no longer proper. This tells us the closure of a proper ideal is not always proper, if $\mathcal{A}$ is not unital.

Theorem 1.7

Every maximal ideal of $\mathcal{A}$ is closed.

Proof The closure of any proper ideal is closed in unital algebras, hence its closure is itself.

First let V be a normed vector space. Let W be a closed subspace, and form the quotient space V/W. There is a natrual way to equip V/W with a norm

$$\|\dot{v}\| = \inf\{\|v - w\| : w \in W\}$$

i.e. the distance between v to W. This is a norm. Further if V is complete, so is V/W.

Let $\mathcal{A}$ be a normed algebra, commutative, unital. Let I be a closed ideal.

Proposition 1.29

For $a, b \in \mathcal{A}$, we have

$$\|\dot{a}\dot{b}\| = \|\dot{ab}\| \leq \|\dot{a}\|\|\dot{b}\|$$

so that $\mathcal{A}/I$ is a normed algebra.

Proof Let $c, d \in I$, and

$$(a - c)(b - d) = ab - (ad + cb - cd)$$

Note that $(ad + cb - cd) \in I$, hence

$$\|\dot{ab}\| \leq \|ab - (ad + cb - cd)\| = \|(a - c)(b - d)\| \leq \|(a - c)\|\|(b - d)\|$$

Taking infimum over all c, d, we get

$$\|(\dot{ab})\| \leq \|\dot{a}\|\|\dot{b}\|$$

□

Proposition 1.30

If $\mathcal{A}$ is a Banach algebra and if I is a closed ideal, then $\mathcal{A}/I$ is a Banach algebra for the norm $\|\dot{a}\|$ defined above.

Let $\mathcal{A}$ be a unital commutative Banach algebra over $\mathbb{C}$, let I be a maximal ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is a Banach algebra, if $\mathcal{A}/I$ has a proper ideal, then you can put this ideal back in $\mathcal{A}$ such that it contains I. By I already being ideal, this implies that $\mathcal{A}/I$ does not contain any proper ideals.

Now coming back. Let nonzero element in $\mathcal{A}/I$ is invertible. The Gelfand Mazur theorem tells us

$$\mathcal{A}/I \cong \mathbb{C}$$

Moreover, $\mathcal{A}/I = 1_{\mathcal{A}/I}\mathbb{C}$. Then the quotient map

$$\mathcal{A} \to \mathcal{A}/I \cong \mathbb{C}$$

is an element of $\mathcal{A}$, i.e. an algebra homomorphism φ, with the property $\varphi(I) = 0$.

If $a\mathcal{A} \subset I$, then for $y \in a\mathcal{A}$, we have

$$\varphi(y) = 0$$

And we are therefore finally done. We thus have

$$\|\widehat{a}\| = r(a)$$

□

Corollary 1.6

We have

$$\sigma(a) = Range(\widehat{a})$$

1.9 Lecture 10

Consider $C_\infty(\mathbb{R}) \subset C_b(\mathbb{R})$, and $C_\infty(\mathbb{R})$ is an ideal of $C_b(\mathbb{R})$.

Definition 1.14 (Abstract C^*-algebra)

An abstract C^*-algebra is a Banach algebra with an involution, such that

$$\|a^*a\| = \|a\|^2$$

Remark Zorn's lemma states that $C_\infty(\mathbb{R})$ is contained in a maximal ideal, in a commutative Banach algebras, maximal ideals give rise to bounded multiplcative linear functionals.

Remark There is ideals of $C_b(\mathbb{R})$ that are bigger than $C_\infty(\mathbb{R})$.

There exists linear functionals that are 0 on $C_\infty(\mathbb{R})$, but nonzero on $C_b(\mathbb{R})$, but such functional is not "constructable."

We also have $c_0 \subset l^\infty(\mathbb{N})$, where c_0 are sequences that converge to 0 at infinity. Again, nonzero linear functionals exist on $l^\infty(\mathbb{N})$, and is identically zero on c_0, but such is also not constructable.

Definition 1.15

$\prod_{j=1}^\infty \mathbb{Z}_5$= *the sequences of elements of* $\mathbb{Z}_5$. $\bigoplus \mathbb{Z}_5$ *all sequences that are 0 except for finitely many number of entries.*

Note that $\bigoplus \mathbb{Z}_5$ is an ideal of $\prod^\infty \mathbb{Z}_5$.

Proposition 1.31

$l^\infty(\mathbb{N})$ is not separable, $\prod^\infty \mathbb{Z}_5$ is not separable, nor is it finitely generated.

Proposition 1.32

If $\mathcal{A}$ is a unital commutative Banach algebra over $\mathbb{C}$, which is separable, and if I is a closed ideal, then one can construct a maximal ideal containing I by countable what

Proof Let $\{a_n\}$ be a countable subset of $\mathcal{A}$, whose linear span is dense.

Lemma 1.4

Note that $\mathcal{A}/I$ contains noninvertible elements if and only if I is not maximal.

If I is not maximal, then you can find the first a_n such that $a_n \in \mathbb{C}1_{\mathcal{A}}$ such that $a_n \notin \mathcal{A}/I$, then

$$\overline{a_n\mathcal{A}/I} \text{ is a proper ideal of } \mathcal{A}$$

hence it generates a proper ideal in $\mathcal{A}$, we denote it as I_1. If I_1 is not maximal, then repeat the process. By $\{a_n\}$ being countable, and that they are dense, we have a countable addition, which gives a maximal ideal containing I by countable inclusions.

□

Remark $C_\infty(\mathbb{R})$ is separable, and $C_b(\mathbb{R})$ is not separable.

Let's look at $L^\infty([0,1], m)$, where m denotes the Lebesgue measure. This is a C^*-algebra, commutaitve, unital.

Could you exhibit any linear functionals on L^∞

Note that $L^2([0,1], m)$, a Hilbert space, as an algebra on L^2, L^∞ is clsoed for the strong operator topology on $\mathcal{B}(\mathcal{H})$ by the seminorm,

$$T \in \mathcal{B}(\mathcal{H}), T \to \|t\xi\|, \text{ for } \xi \in \mathcal{H}$$

For example, figure 1. We say that $L^\infty([0,1])$ is a von Neumann algebra. Every commutative von Neumann algebra looks like some $L^2([0,1], \mu)$. But note that noncommutative ones are quite interesting.

For a commutative Banach algebra over $\mathbb{C}$, the Gelfand transform

$$a \mapsto \widehat{a}$$

We have

$$\|\widehat{a}\|_\infty = r(a)$$

Proposition 1.33 (Gelfand isometric condition)

If $\|a^2\| = \|a\|^2$, then the Gelfand transform is isometric. Thus we have

$$\|\widehat{a}\|_\infty = \|a\|$$

Definition 1.16 (Involution $*$)

For an involution on an algebra over $\mathbb{C}$, is a map $*$ from $\mathcal{A} \to \mathcal{A}$, with the properties

1. $(a^*)^* = a$
2. $(a+b)^* = a^* + b^*$
3. $(\alpha a)^* = \overline{\alpha} a^*$ *for* $\alpha \in \mathbb{C}$.
4. $(ab)^* = b^* a^*$

Definition 1.17 (Banach $*$ algebra)

If $\mathcal{A}$ has a norm, then we say it is a $*$ normed if

$$\|a^*\| = \|a\|$$

If $\mathcal{A}$ is complete, then it is called a Banach $*$ algebra.

Let G be a discrete group, let $\mathcal{H}$ be a Hilbert space, then $Aut(\mathcal{H}) = U(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$ to itself. By a unitary representation of G on $\mathcal{H}$, we mean

Definition 1.18 (Unitary representation)

A unitary representation of G on $\mathcal{H}$ is a homomorphism $\pi : G \to U(\mathcal{H})$.

We note that $C_c(G) \subset l^1(G)$, and

$$\pi_f = \sum_{x \in G} f(x)\pi_x, \pi_f \pi_g = \pi_{f*g}$$

Now we ask what is $(\pi_f)^*$?

$$(\pi_f)^* = \sum \overline{f(x)}\pi_x^* = \sum_{x \in G} \overline{f(x)}\pi_{x^{-1}} = \sum \overline{f(x^{-1})}\pi_x$$

So we gets

$$(f^*)(x) = \overline{f(x^{-1})}$$

This defines an involution on $l^1(G)$. And it is easy to check that

$$\|f^*\|_1 = \|f\|_1$$

Remark The same process would not work for semingroups without the presence of x^{-1} necessarily.

We would like to think of this involution as some sort of complex conjugation.

Definition 1.19

Let $\mathcal{A}$ be a Banach $*$-algebra is symmetric if whenever $a \in \mathcal{A}$, and $a^* = a$, then

$$\sigma(a) \subset \mathbb{R}$$

If one looks at l^1 over noncommutative groups, some are symmetric, some are not.

Example 1.7 Let $\mathcal{A} = \mathbb{C} \bigoplus \mathbb{C}$, with $\|\cdot\|_\infty$. We define an involution:

$$(\alpha, \beta)^* = (\overline{\beta}, \overline{a})$$

This is a well-defined involution. However, this is not symmetric under this involution.

Proposition 1.34

If G is commutative, then $l^1(G)$ is symmetric.

$$\widehat{\mathcal{A}} = \{ \text{ set of homomorphisms } G \to \mathbb{T} \}$$

This is symmetric.

Proposition 1.35

Let $\mathcal{A}$ be an abstract, unital C^*-algebra, then $\mathcal{A}$ is symmetric.

This is quite strong! (Every C^*-algebra is symmetric).

Lecture 11

Let $\mathcal{A}$ be a *-Banach algebra.

Definition 1.20 (symmetric $*$-algebra)

$\mathcal{A}$ is symmetric if for any $a \in \mathcal{A}$, we have

$$a^* = a$$

we have $\sigma(a) \in \mathbb{R}$.

Note that a C^* algebra is necessarily a *-algebra. Hence we have

$$\|a\|^2 = \|a^*a\| \leq \|a^*\|\|a\|$$

And we apply it to a^*, hence we get

$$\|a^*\| \leq \|a\|$$

Hence the involution property is satisfied.

Proposition 1.36 (C^*-algebras are symmetric)

Let $\mathcal{A}$ be a unital C^*-algebra, i.e. we have

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

Then $\mathcal{A}$ is symmetric, i.e. $a^* = a$.

In the Gelfand Noimark paper (1943),

Proof [Arens Truck, 1946] Given $a \in \mathcal{A}$ with $a^* = a$, for any $t \in \mathbb{R}$, let $b = a + it$, we look at $b^*b = (a - it)(a + it) = a^2 + t^2$. So we have

$$\|b^*b\| \leq \|a^2\| + t^2$$

Now let $\lambda \in \sigma(a)$, with $\lambda = r + is$, we would like to show $s = 0$. Note we have $\lambda + it \in \sigma(b)$, this gives

$$\lambda + it = r + i(s + t) \in \sigma(b)$$

Then we have

$$|r + i(s + t)| \leq \|b\|$$

Hence

$$r^2 + (s + t)^2 |r + i(s + t)|^2 \leq \|b^2\| = \|b\|^2 = \|b^*b\| \leq \|a^2\| + t^2$$

We thus have

$$r^2 + s^2 + 2st \leq \|a^2\|, \text{ for all } t$$

This gives $s = 0$.

□

Let's step back. Let $\mathcal{A}$ be a commutative symmetric Banach *-algebra. Then if $a \in \mathcal{A}$, and if $a^* = a$, so $\sigma(a) \subset \mathbb{R}$. But $\sigma(a) = Range(\widehat{a})$, so $\mathcal{A}$ is an $\mathbb{R}$-valued function on $\mathcal{A}$.

For any $a \in \mathcal{A}$, we have

$$a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i} = a_r + ia_i$$

then $\widehat{a} = \widehat{a_r} + i\widehat{a_i}$, note $a^* = a_r - ia_i$, then we have

$$\widehat{a^*} = \widehat{a_r} - i\widehat{a_i} = \overline{\widehat{a}}$$

Thus, we have

$$\widehat{a^*} = \overline{\widehat{a}}$$

So $a \mapsto \widehat{a}$ is a *-algebra homomorphism of $\mathcal{A}$ into $C(\widehat{\mathcal{A}})$.

Definition 1.21 (separation of points by functions)

A collection of functions $\{f\}_j$ defined on X is said to separate points if for all $x, y \in X$, such that $x \neq y$, there exists f such that we have

$$f(x) \neq f(y)$$

♣

Proposition 1.37

For any unital commutative Banach algebra $\mathcal{A}$, then Gelfand transform $a \mapsto \widehat{a}$, separates the points of $\widehat{\mathcal{A}}$. ♠

Proof We prove the contrapositive, if we assume for all $\widehat{a}$, we have $\widehat{a}(\varphi) = \widehat{a}(\psi)$, then we would like to show $\varphi = \psi$. If $\varphi, \psi \in \widehat{\mathcal{A}}$, and $\widehat{a}(\varphi) = \widehat{a}(\psi)$, then

$$\varphi(a) = \psi(a), \text{ for all } a$$

Hence $\varphi = \psi$.

□

Proposition 1.38

*If $\mathcal{A}$ is a unital symmetric Banach *-algebra, then the image of Γ is dense in $C(\widehat{\mathcal{A}})$.* ♠

Proof [Key ingredient: Stone-Weierstrass] $\{\Gamma(\varphi) : \varphi \in \widehat{\mathcal{A}}\}$ is a unital subalgebra of $C(\widehat{\mathcal{A}})$ that separates the points of $\mathcal{A}$, and is closed under taking complex conjugates, so Stone-Weierstrass theorem applies (a compact space and a unital subalgebra of continuous functions that separates the points of the space, and closed under complex conjugation, then this algebra is dense for the $\|\cdot\|_\infty$ norm).

□

Theorem 1.8 (Little Galfand-Naimark theorem)

Let $\mathcal{A}$ be a unital commutative C^*-algebra (abstract, which doesn't have to include hilbert space), then the Gelfand transform

$$\widehat{a}(\varphi) = \varphi(a)$$

*is an isometric *-isomorphism of $\mathcal{A}$ into $C(\widehat{\mathcal{A}})$, i.e. $\|\widehat{a}\| = \|a\|$.* ♡

Proof Since $\mathcal{A}$ is symmetric, the range of the Gelfand transform is dense. We also saw that $\|a^{2^n}\| = \|a\|^{2^n}$, so the spectral radius of a, $r(a) = \|a\| = \|\Gamma(a)\|$.

Therefore $a \mapsto \Gamma(a)$ is isometric, and the range of $\Gamma(a)$ is norm-closed.

□

Remark In a commutative Banach algebra, we have $r(a) = \|\widehat{a}\|_\infty$, and have $r(ab) \leq r(a)r(b)$, hence $\|a^*a\| = \|a\|^2, r(a^*a) \leq r(a)^2$, we have

$$\|a\|^2 \leq r(a)^2 \leq \|a\|^2$$

For $T \in \mathcal{B}(\mathcal{H})$, and $T = T^*$, then

$$C^*(T, I) = \cong C(\sigma(T))$$

Proposition 1.39

Let G be a commutative group, then $l^1(G)$ with its $*$ is symmetric.

Proof Let $\mathcal{A} = l^1(G)$, then $\widehat{\mathcal{A}}$ is isomorphic to the homomorphisms of G into T.

If $\varphi \in \widehat{\mathcal{A}}$, then

$$\varphi(f) := \sum_{x\in G} f(x)\varphi(x), f \in l^1(G)$$

Then

$$\varphi(f^*) = \sum f^*(x)\varphi(x) = \sum \overline{f(x^{-1})}\varphi(x) = \sum \overline{f(x)}\varphi(x^{-1}) = \sum \overline{f(x)\varphi(x)} = \overline{\varphi(f)}$$

Note how we might have homomorphisms not mapping into $\mathbb{T}$ if G is not commutative.

□

Proposition 1.40

For G commutative, the range of the Gelfand transform Γ, which is the Fourier transform (for $f \in l^1(\mathbb{Z})$, we have $\widehat{f}(e^{i\theta}) = \sum f(n)e^{i\theta^n}$, note $\widehat{\mathbb{Z}} = \mathbb{T}$) in this setting, this is dense in $C(\widehat{G})$. However, this is not isometric, and the range is not norm-closed unless G is finite.

1.10 Lecture 12

Recall last time, we proved the Little-Galfand-Naimark theorem.

Theorem 1.9

Let $\mathcal{A}$ be a unital commutative C^*-algebra then we have

$$\mathcal{A} \cong C(\widehat{\mathcal{A}})$$

And $\widehat{\mathcal{A}}$ is comapct.

Definition 1.22

Let C be a category, we think of objects as categories X, Y, with morphisms in between X, Y, the abstract dual of C, which is another category, has the same objects, but you reverse all the morphisms (arrows).

Next we introduce the important summary of the things we've been doing.

Theorem 1.10

The category of unitial commutative C^*-algebras, with unital $*$-homomorphisms is a concrete realization of dual of the category of compact Hausdorff spaces.

$$X \to y$$

$$C(X) \leftarrow C(Y)$$

X is normal? if $C(X)$ contains no proper projects, i.e. elemnets like P such that

$$P^2 = P = P^*$$

We now look at the following.

Proposition 1.41

Let $\mathcal{A}$ be a unital Banach algebra, and let $a_0 \in \mathcal{A}$, suppose $\mathcal{A}$ is generated by a_0, i.e.e the norm closure of all the polynomials in a_0, with the identity $1_{\mathcal{A}}$. Then the $\widehat{A}$ is homeomorphic to $\sigma(a_0)$, via $\varphi \in \widehat{\mathcal{A}}$,

$$\varphi \mapsto \varphi(a_0) \in \sigma(a_0)$$

Proof φ is entirely determined by $\varphi(a_0)$, hence the map above is one-to-one. note that for every element in $\lambda \in \sigma(a_0)$, there exist φ such that $\varphi(a_0) = \lambda$. Hence the map is also surjective.

Let $\mathcal{A}$ be a unital C^*-algebra, and let $a \in \mathcal{A}$ and $a^* = a$, then $C^*(a, 1_{\mathcal{A}})$ is commutative, and unital, and generated by a so that

$$\widehat{B} = \sigma_B(a)$$

such that $\mathcal{B} = C(\widehat{\mathcal{B}}) = C(\sigma(a))$, hence we have

$$C^*(a, 1) \cong C(\sigma(a)), a \mapsto \widehat{a}$$

The continuous functional calculus (dealt with operators on a Hilbert space). Still we assume $a^* = a$.

Given $f \in C(\sigma(a))$, then there exists a $b \in C^*(a, 1)$, such that

$$\widehat{b} = f$$

we denote b as $f(a)$, then

$$f \mapsto f(a)$$

is a *-homomorphism of $C(\sigma(a))$ onto $C^*(a, 1)$, and so into $\mathcal{A}$.

Corollary 1.7

If $T \in \mathcal{B}(\mathcal{H})$, and if $T^* = T$, then for any function $f \in C(\sigma(T))$, we can form $f(T)$. This is part of the spectral theorem.

Definition 1.23

In any unital C^*-algebra, and $a \in \mathcal{A}$, we say that a is normal if $a^* \in C^*(a, 1)$, and a^* and a commute.

We introduce the spectral permanents)

Theorem 1.11

Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{B}$ be a unital C^*-subalgebra of $\mathcal{A}$. Then for any $b \in \mathcal{B}$, we have

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

Note that we are not requiring commutativity for $\mathcal{A}$ or $\mathcal{B}$.

Proof Note that we already have $\sigma_{\mathcal{A}}(b) \subset \sigma_{\mathcal{B}}(b)$ (an element not invertible in $\mathcal{B}$ may be invertible in $\mathcal{A}$). Then to show the other inclusion. We will first assume that $b^* = b$. If b has an inverse in $\mathcal{A}$, then it has a inverse in $\mathcal{B}$. So suppose c is an inverse in $\mathcal{A}$ for b, then

$$cb = 1_{\mathcal{A}} = bc$$

Now it suffices to consider $(b - \lambda 1)^{-1}$. Assume b is not invertible in $\mathcal{B}$ Consider the C^* algebra generated by $b, 1$, then $C^*(b, 1) := \mathcal{B}_1 \subset \mathcal{B}$. If b is not invertible in $\mathcal{B}$, then b is not invertible in $\mathcal{B}_1$, hence $\widehat{b}$ has no inverse in $C(\sigma_{\mathcal{B}}(\mathcal{H}))$: which has a criterion that it takes nonzero elements to 0.

Thus $\widehat{b}$ takes value 0 at some $\lambda \in \sigma_{\mathcal{B}}(b)$, and note that $\widehat{b}$ is continuous, there is an open neighborhood of λ_0 such that

$$|\widehat{b}(\lambda)| \leq \frac{1}{2\|c\|}$$

So by Urysohn's lemma, there is a $g \in C(\sigma(b))$ such that $supp(g) \subset O$, and $g = 0$ outside of O, with $\|g\|_\infty = 1$.

Then for $d := g(b)$, so for the Gelfand transform,

$$\widehat{g(b)} = g$$

we have $\|\widehat{b}g\|_\infty \leq \frac{1}{2\|c\|}$, then

$$\|db\| \leq \frac{1}{\|c\|}$$

then

$$1 = \|g\|_\infty = \|d\| = \|(cb)d\| \leq \|c\|\|bd\| \leq \|c\|\frac{1}{2\|c\|} = \frac{1}{2}$$

Hence we've reached a contradiction. Hence the self-adjoint case is done.

□

1.11 Lecture 13

Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{B}$ be a unital C^*-subalgebra (implying the same unit), and in previous lecture, we saw if $b \in \mathcal{B}$, and $b^* = b$, and if b has an inverse in $\mathcal{A}$, then it has an inverse in $\mathcal{B}$. This implies that

$$\sigma_{\mathcal{B}}(b) = \sigma_{\mathcal{A}}(b)$$

Proposition 1.42

Let $\mathcal{A}$ be a C^*-algebra, and $\mathcal{B}$ subalgebra, if b is invertible in $\mathcal{A}$, then b is invertible in $\mathcal{B}$. ♠

Proof We already know that this holds for $b^* = b$. For general $b \in \mathcal{B}$, (we no longer require $b^* = b$), then if b has an inverse in $\mathcal{A}$, then so does b^*, then b^*b has an inverse in $\mathcal{A}$, hence b^*b has an inverse in $\mathcal{B}$. Hence b has a left inverse $a(b^*b) = (b^*b)a = 1$, and bb^* is also invertible in $\mathcal{A}$, hence invertible in $\mathcal{B}$, hence b has a right inverse.

□

if we look at the shift operators on $l^2(\mathbb{N})$,

$$Se_n = e_{n+1} \tag{1.3}$$

Then the adjoint of this would be

$$S^*(e_n) = \begin{cases} e_{n-1}, n \geq 2 \\ 0, n = 1 \end{cases}$$

Then we have

$$S^*S = I_{\mathcal{H}}, SS^* = I - P_{e_1}$$

where P_{e_1} is the projection onto e_1.

Let $\mathcal{A} = l^1(\mathbb{N}_{\geq 0})$, and let $\mathcal{D} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Theorem 1.12

If $f \in l^1(\mathbb{N})$, and if $\hat{f} \in C(D)$, then nowhere takes value 0, then $\hat{f}$ is invertible as a function in $C(D)$, and $\hat{f}$ has absolutely convergence power series, then $\frac{1}{f}$ also has an absolutely convergence power series. ♡

Note *If we take $\mathcal{A} = l^1(\mathbb{Z})$, so $\widehat{\mathcal{A}} = \mathbb{T}$, and if $\hat{f}$, which is the Fourier series, nowhere takes value 0 on $\mathbb{T}$, then the function $\frac{1}{f}$ has absolutely convergent Fourier series. (This is much harder to prove).*

Let G be a group and let $(\mathcal{H}, U)$ be a unitary representation of G on $\mathcal{H}$, and let $U : G \to U(\mathcal{H})$, and $l^1(G)$ with convolution, for $f \in l^1(G)$, and for

$$U_f = \sum f(x)U_x \in \mathcal{B}(\mathcal{H})$$

with $\|U_f\| \leq \|f\|_{L^1}$. We then define

$$f^*(x) := \overline{f(x^{-1})}, U_f^* = (U^*)_f$$

We take $G = SL(n, \mathbb{Z})$, the $n \times n$ matrices with $\det(T) = 1, T \in SL(n, \mathbb{Z})$.

Problem 1.1 What are the unitary representations of G on $l^1(G)$

G acts on G by left translations α (actions), and acts on G/H (sets of cosets) for H any subgroups. Note that G/H are often called homogenous spaces. If G acts on a set M, consider $l^p(M)$, then G acts on isometries on $l^p(M)$, by

$$(\alpha)x\xi(y) = \xi(\alpha_x^{-1(y)})$$

$$X \to Y, C(X) \leftarrow C(Y)$$

In particular, this action on $l^2(M)$ is unitary.

Definition 1.24 (Left regular representation)

The representaiton U of G on $l^1(G)$ is called the left regular representation of G if we define, for $x, y \in G$, $\xi \in l^2(G)$, we have

$$(U_x\xi)(y) = \xi(x^{-1}y)$$

♣

We now, naturally define the integrated form of the representation of G on $l^1(G)$:

Definition 1.25 (integrated form of a representation)

We define $U_f \in End(l^1(G))$, we define U_f naturally as follows: let $\xi \in l^1(G)$, and

$$U_f(\xi) = \sum_{g \in G} f(g)U_x(\xi)$$

i.e. $U_f = \sum_{x \in G} f(x)U_x$, where $U_x \in End(V)$ by

$$x \xrightarrow{U} U_x$$

where U is the map : $G \to End(l^1(G))$.

♣

Definition 1.26 (Reduced C^*-algebra)

The operator norm closure of $\{U_f : f \in l^1(G)\} \in \mathcal{B}(l^1(G))$ is called the reduced C^*-algebra of G, denoted as $C_r^*(G)$.

♣

Note *Again, the defining property of a C^*-algebra is $\|a^*a\| = \|a\|^2$*

In 1975, we see that $C_r^*(F_2)$ is simple, and has no proper ideals, note F_2 can be thought of the group generated by a, b, a^{-1}, b^{-1} with unit. Note that the trivial representation is not continuous for $\|\|_{C_r^*}$.

Definition 1.27 (amenable groups)

G is amenable if the integrated form of trivial representation if continuous for $\|\|_{C_r^*}$.

♣

Remark This implies that the integrated form of all unitary representations of G are continuous for $\|\|_{C_r^*(G)}$. There are many equivalent properies of amenable using the geometric properties of G.

Note *All commutative groups are amenable.*

Definition 1.28 (Non-degenerate representation)

Let π be a representation of $\mathcal{A}$ on V, $\pi : \mathcal{A} \to V$, then π is non-degenerate if

the linear span of $\{\pi(a)v : a \in \mathcal{A}, v \in V\}$ is dense in V

♣

Definition 1.29 (faithful representations)

(In short, it's a representation that is injective.) The left representation is faithful if whenever $U_f = 0$, then we have

$$f = 0, f \in l^1(G)$$

♣

Proof $l^1(G)$, with convolution, with identity δ_e. note that $\delta_e \in l^2(G)$. If we consider $\delta_e \in l^2(G)$, and we look at

$$U_f\delta_e = f \in l^2(G)$$

Because we have the embedding $l^1 \subset l^2$.

Now assume that G is commutative, then we consider $l^1(G)$ acts on $l^2(G)$, and

$$C_r^*(G) = C^*(G) = C(\widehat{G})$$

note that we still have $\|f\|_{C_r^*(G)} \leq \|f\|_{L^1}$.

1.12 Lecture 13

Let G be commutative, and $(l^2(G), U)$ be the left regular representation. We have the interal form $U_f, hf \in l^1(G)$, and

$$U : l^1(G) \to \mathcal{B}(l^2(G))$$

and we let

$$C_r^*(G) = \{U_f : f \in l^1(G)\}^-$$

where we take the closure with respect to the operator norm.

For G commutative, we have $C_r^*(G)$ is a commutative C^*-algebra hence

$$\mathcal{A} := C_r^*(G) \cong C(\widehat{\mathcal{A}})$$

We've shown last time, that U is injective from $l^1(G)$ to $C_r^*(G)$. Each $\varphi \in \widehat{\mathcal{A}}$ a multiplicative linear functional, we have

$$f \mapsto \varphi(\pi_f) \in (l^1(G))^\frown$$

For any $f \in l^1(G)$, and $f \neq 0$, we have $U_f = 0$, and the map is injective, hence $\pi_f = 0$, we have, there exists φ such that $\varphi(\pi_f) \neq 0$.

Corollary 1.8 (largeness of the dual group)

$\widehat{G}$ is big enough given that $f, g \in l^1(G)$, if $f \neq g$, then there exists $\varphi \in \widehat{G}$ succh that

$$\widehat{f}(\varphi) \neq \widehat{g}(\varphi)$$

In other words $\varphi(f) \neq \varphi(g)$. ♡

We now state the big Galfand-Naimark theorem.

Theorem 1.13 (Big Gelfand-Naimark)

*Let $\mathcal{A}$ be an abstract C^*-algebra, e.g. a Banach *-algebra such that*

$$\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$$

*Then there exists a *-representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ which is isometric, i.e.*

$$\|\pi(a)\| = \|a\|$$

And

$$\mathcal{A} \cong \{\pi(a) : a \in \mathcal{A}\}$$ ♡

Note *In Little Galfand-Naimark, $C(\widehat{\mathcal{A}})$ is explicitly determined, but for big, it is not determined.*

For X a compact space, $C(X)$ is a C^*-algebra.

Consider X_d, which is discrete, and take $l^1(X_d)$ with the counting measure, is called the atomic representation.

Take any Borel measure μ on X, we have

$$L^2(X, \mu)$$

If μ has full support, then map of $C(X)$ is isometric.

Now to introduce positive linear functionals on $\mathcal{A}$. Let $\mathcal{A}$ be a *-algebra, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a *-representation. Let's take any $\xi \in \mathcal{H}$, and $\xi \neq 0$, define

$$\varphi_\xi(\mathcal{A}) = \langle \pi(a)\xi, \xi \rangle$$

then we look at

$$\varphi(a^*a) = \langle \pi(a^*a)\xi, \xi \rangle = \langle \pi(a)\xi, \pi(a)\xi \rangle \geq 0$$

Note *For $f \in C(X)$, we always have $f^*f \geq 0$.*

Definition 1.30 (Positive linear functionals)

If φ is a linear functional on $C(X)$, such that for all f,

$$\varphi(f^* f) \geq 0$$

Remark They are always continuous. And they give rise to a measure μ_φ on X.

You think of a^*a as being "positive."

Proposition 1.43

If $\mathcal{A}$ is a C^*-algebra, if $a, b \in \mathcal{A}$, there exists c such that

$$a^*a + b^*b = c^*c$$

Definition 1.31 (positive linear functionals)

*It $\mathcal{A}$ is a *-algebra, and if φ is a linear functional on $\mathcal{A}$, if for all $a \in \mathcal{A}$, we have*

$$\varphi(a^*a) \geq 0$$

then we call φ ***positive****.*

Proposition 1.44

If φ, ψ are positive, then $r\varphi + s\psi$, with $r, s \in \mathbb{R}^+$, then the positive linear functionals form a ***cone****.*

Let $\mathcal{A}$ be a *-algebra, and let φ be a positive linear functional on $\mathcal{A}$, then define a pre-inner product on $\mathcal{A}$ by

$$\langle a, b \rangle_\varphi = \varphi(b^*a)$$

Then we have

$$\langle a, a \rangle_\varphi = \varphi(a^*a) \geq 0$$

Proposition 1.45

For $\langle a, b \rangle_\varphi = \varphi(b^*a)$ defines a pre-inner product on $\mathcal{A}$. (A pre-inner product does not require $\langle \xi, \xi \rangle = 0$ implies $\xi = 0$.)

Proof We have $\overline{\langle a, b \rangle_\varphi} = \langle b, a \rangle_\varphi$.

1.12.1 GNS construction

Now for a positive linear functional, we attempt to associate a cyclic representation with it: we will define a pair $(\pi, \mathcal{H})$ such that it satisfies:

$$\varphi(a) = \langle \pi(a)\xi, \xi \rangle$$

Let $\eta_\varphi = \{a \in \mathcal{A} : \langle a, a \rangle_\varphi = 0\}$, then if $b \in \mathcal{A}, a \in \eta_\varphi$, then we have

$$|\langle ba, ba \rangle| = |\langle b^*ba, a \rangle| \leq \langle a, a \rangle^{1/2} = 0$$

So we get that η_φ is an ideal of $\mathcal{A}$, so form $\mathcal{A}/\eta_\varphi$, then

$$\langle , \rangle_\varphi$$

drops to an inner product on $\mathcal{A}/\eta_\varphi$, denote its complement by $L^2(a, \varphi)$, if we are given $c \in \mathcal{A}$, let π be the left regular representation of $\mathcal{A}$ on $\mathcal{A}$ via

$$\pi_c a = ca$$

then we have

$$\langle \pi_c a, b \rangle_\varphi = \langle ca, b \rangle = \varphi(b^*ca) = \varphi((c^*b)^*a) = \langle a, c^*b \rangle = \langle a, \pi(c^*)b \rangle$$

Then the left regular representation "is" a *-representation on $\mathcal{A}$, this drops to a *-representation on $\mathcal{A}/\eta_\varphi$.

Next time: we show π_c is not continuous, and we will use polynomials. Now if you assume $\mathcal{A}$ is Banach *-algebra, then this π_c are always bounded. **GNS representation**, where S is Siegel.

1.13 Lecture 15

Recall the GNS construction. Given a *-normed unital algebra $\mathcal{A}$, and a positive linear functional μ on $\mathcal{A}$, we mean

$$\mu(a^*a) \geq 0, \forall a \in \mathcal{A}$$

Now we define a pre-innner product on $\mathcal{A}$ by

$$\langle a, b\rangle_\mu = \mu(b^*a)$$

And one could check this is indeed a pre-inner product.

1. $\overline{\langle a, b\rangle_\mu} = \langle b, a\rangle_\mu$
2. $\langle a+b, a+b\rangle_\mu \geq 0$

If we let $\eta_\mu = \{a \in \mathcal{A} : \langle a, a\rangle = 0\}$, then by Cauchy-Schwarz inequality, we would show that η_μ is a left ideal. Form $\mathcal{A}/\eta_\mu$, then $\langle , \rangle$ becomes an inner product on $\mathcal{A}/\eta_\mu$.

One could complete $\mathcal{A}/\eta_\mu$ to get a Hilbert space $L^2(\mathcal{A}, \mu)$. Then we have

$$\langle f, g\rangle = \int f\bar{g}d\mu$$

Definition 1.32 (left regular representation)

For $c \in \mathcal{A}$, define the left regular representation π on $\mathcal{A}$ and on $\mathcal{A}/\eta_\mu$ via

$$\pi_c a = ca, a \in \mathcal{A}$$

Then we have

$$\langle \pi_c a, b\rangle_\mu = \langle ca, b\rangle_\mu = \langle a, \pi_c^* b\rangle$$

*Hence π is a *-representation.* ♣

Now let P be the algebra of polynomials with complex coefficients, viewed as functions on $\mathbb{R}$. Define μ on P as follow:

$$\mu(p) = \int_{-\infty}^{\infty} p(t)e^{-t^2}dt$$

Then our μ is a positive linear functional. Then we identify this as $L^2(\mathbb{R}, e^{-t^2}dt)$.

Proposition 1.46

Let $\mathcal{A}, \mu$, assume $\mathcal{A}$ is complete, let 1 denote the unit of the algebra. And μ is continuous with $\|\mu\| = \mu(1)$. ♠

Remark By completeness, you immediately get continuity in some cases.

Proof Now we introduce the main lemma.

unfinished

1.14 Lecture 16

We now discuss classical physics and quantum physics.

All possible "states" for the system. Then you have observables. $\mathbb{R}$-valued for the phase space P.

If the configuration space is $\mathbb{R}$, the velocities are given as $\mathbb{R}$ numbers, then the phase space is the cartesian product $\mathbb{R} \times \mathbb{R}$, and the Poisson bracket is given by

$$\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)(p)$$

In quantum physics, there is no configuration space that we work with. The phase space becomes a C^*-algebra, usually non-commutative. And the observables are the self-adjoint elements of $\mathcal{A}$, where $\mathcal{A} = \mathcal{B}(\mathcal{H})$, and such choice of Hilbert space is not unique. Possibles values for an observalbe $a = a^*$ form $\sigma(a)$.

When you observe a system in a certain state, and you measure and observable a several times, you can get different values (all in $\sigma(a)$).

For the system in a given state, there is a porobablity measure on $\sigma(a)$ that gives you the probability of getting a particular value of a. As the system evolves, this probably changes.

The **state** (this is why we call them states, i.e. the set of linear transformations that have norm 1) space is the set of all positive linear transformations on $\mathcal{A}$ of norm 1, μ, then for any observable a, form

$$C^*(a, 1) \cong C(\sigma(a))$$

Then the restriction of μ to $C^*(a, 1)$ give syou a probability measure on $\sigma(a)$.

Experiments show that for certain pairs of observables a, b, it is impossible to measure both simultaneously at high accuracy.

The way this is modeled is that it happens exactly if $ab \neq ba$, i.e. non-commutativity makes us impossible to measure them simultaneously.

Definition 1.33 (Poisson bracket)

For a, b observables such that they are noncommutative, we define the Poisson bracket as follows:

$$i[a, b] = ab - ba$$

♣

For the evolution of a quantum system given by $\mathbb{R} \mapsto Aut(\mathcal{A})$, and $t \mapsto \alpha_t$, and states evolve by $\mu \mapsto \mu \circ \alpha_t$.

If we let this Planck's constant h go to 0, then we have

$$ih[a, b] \to \text{ Poisson bracket}$$

i.e. getting back from quantum back to the classical system.

Let $\mathcal{A}$ be a *-normed unital algebra, and μ is a continuous positive linear functional, then $L^2(\mathcal{A}, \mu)$ and $\pi^{\mathcal{H}} : \mathcal{A} \to \mathcal{B}(L^2(\mathcal{A}, \mu))$.

Let $\xi_\mu = 1_{\mathcal{A}}$, viewed as an element of $L^2(\mathcal{A}, \mu)$, and

$$\{\pi_\mu(a)\xi_\mu : a \in \mathcal{A}\} = \mathcal{A}/\pi_\mu$$

viewed in $L^2(\mathcal{A}, \mu)$ is dense in $L^2(\mathcal{A}, \mu)$.

Definition 1.34

If we have $\mathcal{A}$ some representation $(\mathcal{H}, \pi)$, and a vector $\xi \in \mathcal{H}$ is said to be a cyclic vector of $\{\pi_\mu \xi : a \in \mathcal{A}\}$ is dense in $\mathcal{H}$.

♣

1.15 Lecture 17

If you look at $\mathcal{A}$ a *-normed unital algebra, and μ a continuous linear functional, and the GHS representation: $(\mathcal{H}_\mu, \pi_\mu, \xi_\mu)$, where $\overline{\mathcal{A}/\eta_\mu}$, where ξ_μ is the unital element $1_{\mathcal{A}}$ in the Hilbert space.

Given any *-representation of $\mathcal{A}$ on K, and any vector $\xi \in K$, let

$$\mu_\xi(a) = \langle \pi(a)\xi, \xi \rangle$$

Now we ask what positive linear functional comes from the cyclic vector ξ_μ, and we look at

$$\langle \pi_m u(a)\xi_\mu, \xi_\mu \rangle_\mu$$

where it is essentially

$$\langle a 1_{\mathcal{A}}, 1 \rangle = \mu(a)$$

Every state is a vector state, for some representation, namely, the GNS representation.

Consider the set $\{\pi(a) : a \in \mathcal{A}\}$, or for a unitary representation U of a group G, consider the set

$$\{U_x : x \in G\}$$

More importantly, given a Hilbert space $\mathcal{H}$, let S be a subset of the bounded operators on $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ such that if $a \in S$, then $a^* \in S$, and $I_H \in S$.

Definition 1.35

A subspace $K \subset H$ is said to be S-invariant (where $S \subset \mathcal{B}(\mathcal{H})$), if whenever $\xi \in K$, and $a \in S$, and $a\xi \in K$. You could write $aK \subset K$.

Proposition 1.47

If K is S-invariant, then so is $K^\perp$.

Proof Let $\xi \in K^\perp$, $a \in S$, want to show that $a\xi \in K^\perp$.

Let $\eta \in K$, then $\langle a\xi, \eta\rangle = \langle \xi, a^*\eta\rangle = 0$. This is true for all η, hence $a\xi \in K^\perp$. □

Let $\mathcal{A}$ be *-normed unital algebra, and let $(\mathcal{H}, \pi)$ be a *-representation of $\mathcal{A}$ on $\mathcal{H}$. Choose $\xi \in \mathcal{H}$, and let

$$K_\xi = \overline{\{\pi(a)\xi : a \in \mathcal{A}\}} \subset \mathcal{H}$$

This is a $\{\pi(a) : a \in \mathcal{A}\}$-invariant subspace. If we look at $K_\xi^\perp$, this is also $\{\pi(a) : a \in \mathcal{A}\}$-invariant.

Definition 1.36

If $(\mathcal{H}, \pi)$ a representation of $\mathcal{A}$, and H contains no proper nonempty π-invariant closed subspaces, we say then $(\mathcal{H}, \pi)$ is irreducible. (This gives the name "simple module.")

If $K_\xi^\perp$ is not the zero subspace, then you can choose $\xi_1 \in K_\xi^\perp, \xi_1 \neq 0$.

Set K_{ξ_1} to be the cyclic subspace generated by ξ_1, and π-invariant, if $K_{\xi_1}^\perp$ is π-invariant. And if it's not the zero subspace, then you could repeat this process.

If $\mathcal{H}$ is finite-dimensional, then can what we said above to get

$$\mathcal{H} = K_1 \bigoplus K_2 \bigoplus ... \bigoplus K_n$$

And for each K_j, we have π on these is irreducible. You could decompose any $\mathcal{H}$ into a direct sum of irreducible representations.

What happens when the space is infinite-dimensional? Group G has a representation on $l^2(G)$. Given any $(\mathcal{H}, \pi)$ with $\mathcal{H}$ infinite-dimensional, you can apply the above process, and may not get any irreducible representations, but you could keep getting **cyclic representations**.

For any Hilbert space, pick unit vector, look at orthogonal subspace, pick a unit vector in that, etc... What happens when you keep on going. Note that **when you keep going, you use Zorn's lemma**.

Via Zorn's lemma, you get the following.

Proposition 1.48

Every representation is a possibly infinite direct sum of cyclic representations.

We need to define infinite sum of representations.

Let $\{\mathcal{H}_\lambda\}_{\lambda\in\Lambda}$ be a collection of Hilbert spaces. Set the direct sum of these Hilbert spaces $\bigoplus \mathcal{H}$, and functions on these to be $\xi : \Lambda \to \cup\mathcal{H}_\lambda : \xi(\lambda) \in \mathcal{H}_\lambda$ for all λ and $\sum_{\lambda\in\Lambda} \|\xi_\lambda\|^2 < \infty$, hence we have $\xi_\lambda = 0$ except for a finite number of λ's.

And define $\langle \xi, \eta\rangle = \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \eta_\lambda\rangle$, such that the sum $\sum \|\xi_n\|^2 < \infty$, and $\langle \xi, \eta\rangle = \sum \xi_\lambda \overline{\eta}_\lambda$.

You know how to show $\mathcal{H}$ is complete under $\langle \xi, \eta\rangle = \sum \xi_\lambda \overline{\eta}_\lambda$, then you know how to show $\bigoplus \mathcal{H}_\lambda$ is complete under $\langle \xi, \eta\rangle = \sum_{\lambda\in\Lambda} \langle \xi_\lambda, \eta_\lambda\rangle$.

If $\{T_\lambda : T_\lambda \in \mathcal{B}(\mathcal{H}_\lambda)\}$, and $\bigoplus_{\lambda\in\Lambda} T_\lambda$, and if we define

$$(\bigoplus T_\lambda)(\xi) := \{T_\lambda \xi_\lambda\}_{\lambda\in\Lambda}$$

and we would like to put these in $\bigoplus \mathcal{H}_\lambda$, there is a constant c such that $\|T_\lambda\| \leq c$ for all λ, i.e. T_λ is bounded in norm.

Hence

$$\|T_\lambda \xi_\lambda\| \leq \|T_\lambda\| \cdot \|\xi_\lambda\| \leq c\|\xi_\lambda\|$$

If $\mathcal{A}$ is a *-normed unital algebra, and $(\mathcal{H}_\lambda, \pi_\lambda)$ is a collection of *-representation of $\mathcal{A}$, continuous, then for $\bigoplus_{\lambda\in\Lambda} \mathcal{H}_\lambda$, and for $a \in \mathcal{A}$, set

$$\bigoplus(\pi_\lambda)(a)\xi = \{\pi_\lambda(a)\xi_\lambda : \|\pi_\lambda(a)\| \leq \|a\|\}$$

and $\{\xi_\lambda\} \in \bigoplus_\lambda \mathcal{H}_\lambda$.

By Zorn's lemma, every Hilbert space is a direct sum of cyclic representations.

Next: spectral theorem for self-adjoint operators.

1.16 Lecture 18

Definition 1.37 (Equivalent representations)

*Let $\mathcal{A}$ be a (group) a *-algebra, and let $(\mathcal{H}, \pi)$, and (K, ρ) are two *-representations of $\mathcal{A}$, then these representations are equivalent, if there is a unitary operator $U : H \to K$, $(U^{-1} : K \to H)$, such that*

$$U\pi(a) = \rho(a)U$$

U is the representation that equivalates π and ρ. "U intertwines π and ρ"

♣

Proposition 1.49

*Let $\mathcal{A}$ be a unital *-normed algebra, and let $(\mathcal{H}, \pi, \xi)$ and (K, ρ, η), where ξ, η are cyclic vectors, and they each determine a positive linear functional on $\mathcal{A}$.*

Let μ_ξ, and μ_η be the correspondonig positive linear functionals, and if $\mu_\xi = \mu_\eta$, then $(\mathcal{H}, \pi)$ and (K, ρ) are unitarily equiavalent via a unitary operator such that $U\xi = \eta$.

♠

Proof Since we want that $U\xi = \eta$, then we want

$$U(\pi(a)\xi) = U\pi(a)(\xi) = \rho(a)U\xi = \rho(a)\eta$$

Start by try and define U by

$$U(\pi(a)\xi) = \rho(a)\xi$$

However, this raises the question, is this well-defined? If $\pi(a)\xi = \pi(b)\xi$, for some a, b, then do we have $\rho(a)\eta = \rho(b)\eta$? (This would imply U is well-defined).

Remark In linear situations like this, if $\pi(a - b)\xi = 0$, is it true that $\rho(a - b)\eta = 0$?

In other words, **it suffices to show that** if $\pi(\xi) = 0$, we have $\rho(c) = 0$?.

$$\|\rho(c)\eta\|^2 = \langle\rho(c)\eta, \rho(c)\eta\rangle = \langle\rho(c^*c)\eta, \eta\rangle = \mu_\eta(c^*c) = \mu_\xi(c^*c) = \langle\pi(c)\xi, \pi(c)\xi\rangle = 0$$

We note that

$$\|\rho(c)\eta\|^2 = \|\pi(c)\xi\|^2$$

The operator preserves norm on a dense subspace, we further have U is unitary.

We now see that U intertwines π, ρ, we have

$$U\pi(a)(\pi(c)\xi) = U\pi(ac)\xi = \rho(ac)\eta = \rho(a)\rho(c)\eta = \rho(a)U(\pi(c)\xi)$$

From this we see

$$U\pi(a) = \rho(a)U$$

□

Corollary 1.9

There is a bijection between the positive linear functionals on $\mathcal{A}$, and the pointed cyclic representations on $\mathcal{A}$. ♡

Let $\mathcal{A}$ be a commutative unital *-normed algebra and let $(\mathcal{H}, \mu)$ be a cyclic *-representation with cyclic vector ξ. Then let $\mathcal{B} = \overline{\pi(a)}$ be the norm closure is a C^*-algebra. So $\mathcal{B} = C(X)$.

Then μ_ξ is a positive linear functional on $\mathcal{B} = C(X)$. Hence μ_ξ gives a finite regular Borel measure on X, $\tilde{\mu}_\xi$, so you can form the $L^2(X, \tilde{\mu}_\xi)$ with $\mathcal{B}$ acting on $L^2(X, \tilde{\mu}_\xi)$ by pointwise multiplication. **This is basically the GNS representation.** Then you have the cyclic vector as the constant function 1. And find that $\mu_1 = \mu_\xi$, so

$$(\mathcal{H}, \pi, \xi) \cong (L^2(X, \tilde{\mu}_\xi), 1)$$

via the unitary U such that $U\xi = 1$.

Given $C(X)$ and any *-representation of it, (H, π) and you decompose $(\mathcal{H}, \pi)$ into direct sum of cyclic representations, $\bigoplus(H_\lambda, \pi_\lambda, \xi_\mu)$ where each ξ_μ is cyclic.

Each one $(H_\lambda, \pi_\lambda, \xi_\mu)$ is isomorphic to some $L^2(X, \mu_\lambda, \xi_\lambda)$, with $C(X)$ acts pointwise on each $L^2(X, \mu_\lambda, \xi_\lambda)$.

Example 1.8 Let $X = [0, 1]$, and $L^2(X, m)$, where m is the Lebesgue measure. And let

$$\nu = \delta_{\frac{1}{4}} + \delta_{\frac{1}{2}} + \delta_{\frac{1}{5}}$$

All the measures above and mutually exclusive to m, and $L^2(X, m + \nu)$.

This gives the Borel functional calculus. For any bounded Borel function on X (Borel functions are those that are measurable with respect to all possible Borel measures), it acts on each $L^2(X, \mu_\lambda)$ by pointwise multiplication, hence acts on $\mathcal{H}$. If you have a given Hilbert space $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$, and $T^* = T$, and

$$C^*(T, I_\mathcal{H}) = C(\sigma(T))$$

so $C(\sigma(T))$ acts on $\mathcal{H}$, hence every Borel function acts on $\mathcal{H}$.

Again we have $C(X)$ acting on $\mathcal{H}$ and the bounded Borel functions acting on $\mathcal{H}$, let E be any Borel subset of X, so χ_E acts on $\mathcal{H}$ and is a projection. So we have $E \mapsto \chi_E \in \mathcal{B}(\mathcal{H})$ via ν, and this is a projection-valued measure on X.

If E, F are disjoint, then

$$\chi_E \chi_F = \chi_{E \cap F}, \chi_{E \bigoplus F} = \chi_E + \chi_F$$

Remark This is somewhat countably additive in a weak sense.

1.17 Lecture 19

Let $\mathcal{A}$ be a unital commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and let $(\mathcal{H}, \pi, \xi)$ be a cyclic representation (note this is not unique at all, like choosing an orthonormal basis of $\mathcal{H}$). And $\mathcal{A} \cong C(X)$, and we've said

$$\mathcal{H} \cong \bigoplus_{\lambda \in \Lambda} L^2(X, \mu_\lambda)$$

If f is a $\mathbb{C}$-valued bounded Borel function on X, where X is a compact space, then f determines an opeartor T on each $L^2(X, \mu)$ by pointwise multiplication. with $\|T\| \leq \|f\|_\infty$. Then this gives a bounded operator on $\mathcal{H}$. Let $\mathcal{B}$ be the C^*-algebra of bounded Borel functions, then you get a representations of $\mathcal{B}$ on $\mathcal{H}$.

For each Borel subset E of X, χ_E goes to a projection operator on $\mathcal{H}$.

Definition 1.38

Let Σ be a σ-algebra of subsets of a set X (don't quite need a topology), and let $\mathcal{H}$ be a Hilbert space by a projection-valued measure on Σ, we mean an assignment

$$E \xrightarrow{\mu} \mu(E)$$

where $\mu(E)$ is a projection operator on Σ that satisfies

1. $\mu(\emptyset) = 0, \mu(X) = I_\mathcal{H}$ *the identity operator*
2. *If $E, F \in \Sigma$, then $\mu(E \cap F) = \mu(E)\mu(F)$.*

3. *If $\{E_j\}$ is a countable collection of disjoint elements in Σ, then $\mu(\bigoplus_j E_) = \sum_{j=1}^{\infty}\mu(E_j)$, for either the strong or the weak operator topology.*

Note *Every here looks like a mesaure, and we have to be careful about how the sum is well-defined.*

Let X be a compact set, and $B(X)$ denote the set of Borel sets, let $E_j \subset B(X)$ be disjoint. These viewed in $L^2(X,\mu)$, we have

$$\sum_{j=1}^{\infty}\chi_{E_j} = \lim_{n\to\infty}\sum_{j=1}^{n}\chi_{E_j} = \lim_{j=1}^{n}\chi_{\bigoplus_j E_j} = \chi_{\bigoplus_{j=1}^{\infty}E_j}$$

And note as $j\to\infty$, this is an increasing function, this is not converging to the uniform form. But these do converge, for the L^1 or L^2 norm by Dominated Converge theorem.

If $\xi \in L^2(X,\mu)$, and we look at

$$\chi_{\cup_1^{\infty}E_j}\xi \to \chi_{\cup_1^{\infty}}\xi \text{ for the } L^2 - norm$$

If any $\eta \in L^2(X,\mu)$,

$$\langle\chi_{\cup_1^{n}E_j}\xi,\eta\rangle = \int \chi_{\cup_1^{\infty}E_j}\xi\overline{\eta} \to \langle\chi_{\cup_1^{\infty}E_j}\xi,\eta\rangle$$

where the L^2 case follows from that $\xi\overline{\eta}\in L^2(X,\mu)$.

Definition 1.39

The strong operator topology on $\mathcal{B}(\mathcal{H})$ is the topology defined by the seminorms $T\to\|T\xi\|$ for all the ? $\xi\in\mathcal{H}$
If $\{T_\lambda\}$ is a bounded net of operators on $\mathcal{B}(\mathcal{H})$, then it converges to T in the strong operator norm if for all ξ,

$$\|T_\lambda\xi - T\xi\|| \xrightarrow{\lambda} 0$$

Then the weak operator topology on $\mathcal{B}(\mathcal{H})$ is defined by the seminorms, for all ξ,η

$$T\mapsto|\langle T\xi,\eta\rangle$$

If $\{T_\lambda\}$ is a bounded net, then we say it converges to T if

$$|\langle T_\lambda\xi,\eta\rangle - \langle T\xi,\eta\rangle| \to 0$$

Remark Strong operator topology implies the weak operator topology, not the reverse.

Remark If you look at the group of unitary operators, then the strong and weak operator topologies are equivalent.

Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, and $\mathcal{H} = \bigoplus(X,\mu_\lambda)$, and for $E\in Borel(X)$, χ_E actives on each one of (X,μ_λ), and

$$E\mapsto\chi_E$$

acting on $\mathcal{H}$ is a projection-valued measure, and is called the spectral measure for action of $\mathcal{A}$ on $\mathcal{H}$.

For $T\in\mathcal{B}(\mathcal{H})$, $T^*=T$, we denote $C^*(T,I)=\mathcal{A}$, and $\widehat{\mathcal{A}}=\sigma(\mathcal{A})$, then $C(\widehat{A})=\mathcal{A}$. So far each $E\in Borel(\sigma(T))$, you get an operator.

Let P_r be the projection $\chi_{(-\infty,r)}$, and for $r>\|T\|$, we have $P_r = I_{\mathcal{H}}$. These are called the "resolution of the identity" for T.

You could make sense of the operator T as follows:

$$T=\int_{-\infty}^{\infty} t dP_t$$

Let G be a locally compact group. And the product is jointly continuous $G\times G\to G,(x,y)\to xy$, and taking inverses is also continuous.

Example 1.9 $\mathbb{R},\mathbb{R}^n,\mathbb{T},\mathbb{T}^n,GL(n,\mathbb{R})$

We know need is a projection measure on G that is invariant for left translations.

1.18 Lecture 20

We will discuss the Haar measure today.

Let G be a group, and M a topological space, and action of G on M is:

$$\alpha : G \to Hom(G)$$

And if f is a $\mathbb{C}$-valued function on M, then the set

$$\alpha_x(f)m) := f(L_x^{-1}(m))$$

So $\alpha : G \to Aut(FunctionsonM)$. If M is locally compact, $f \in C_c(M)$. For example, $G = \mathbb{R}$.

Now let G be a compact group, then G acts on itself by left translations: λ:

$$\lambda_x(y) = xy$$

We can get an action on functions:

$$\lambda_x(f)(y) = f(x^{-1}y)$$

We then talk about a measure on a group, which is also a topological space in a sense, and would like to assign finite measure on comapct sets. The goal is to construct such a measure that is translation invariant with respect to the left trasnlations.

Definition 1.40 (Translation-invariant measures)

For M locally compact, a measure μ on M is α-invariant, if

$$\mu(\alpha_x(E)) = \mu(E), \text{ for all } x \in G$$

where E is a Borel subset of M. ♣

Remark On discrete groups, the counting measures are invariant under translations.

In 1933, Haar proved that every locally compact, second countable group (i.e. a countable base for the topology) has a left-(translatioin)invariant (nonzero) Borel measure, which we call the Haar measure. This is unique up to positive scalar multiplicaton.

Note *The Lebesgue measure in a Haar measure on the real numbers $\mathbb{R}$.*

Weil: worked instead with positive linear functionals φ on $C_c(G)$, and he proved that we always hav ea φ that is left invariant, i.e.

$$\varphi(\lambda_x f) = \varphi(f), \forall x \in G$$

Sketch: choose $f_0 \in C_c^+(G)$, so that $\varphi(f_0) = 1$, and . For a small nbhd O of the identity element e, choose a function $g_O = g$ supported in O, a smooth bump function.

Let $(f : g_O) = \inf\{\sum_{j=1}^n c_j, c_j > 0 : \text{ there exists } x_1, ..., x_j \in G \text{ with } f \leq \sum_j c_j(\lambda_{x_j} g)\}$, This expression here $f(: g_O)$ is left translation-invariant, now we consider

$$\frac{(f : g_O)}{(f_0 : g_O)}$$

You show that this converges to a linear functional on $C_c(G)$, which necessarily gives rise to a linear functional that is left translation-invariant.

If for $f \in C_c(G)$, define $\tilde{f}(x) = f(x^{-1})$, then the function

$$f \mapsto \varphi(\tilde{f})$$

is a right invariant positive linear functional on $C_c(G)$.

Sometimes $\varphi(f) = \int_G f(x)dx$, given x_0, if we look at

$$\int f(xx_0)dx = \int \rho_{x_0}(f)dx$$

This is also left-invariant, hence there exists $\delta(x_0)$ such that

$$\int f(xx_0)dx = \Delta(x_0) \int f(x)dx$$

Proposition 1.50

$\Delta : G \to \mathbb{R}^+$, a group under multiplication, is and a group homomorphism, and later we will show this is continuous. And this Δ is called the "modular funtion" of G.

Definition 1.41

G is said to be unimodular if $\delta \equiv 1$, i.e. the left Haar measure is right-invariant.

Example 1.10 The discrete group, commutative groups, compact groups, and nilpotent Lie groups, semi-simple lie groups.

But many are lie groups are not unimodular.

$$\int f(x^{-1})\Delta(x^{-1})dx = \int f(x)dx$$

Let H be a closed subgroup of G, let G/H be the corresponding homogenous space, and G acts on G/H, we now ask if there is an invariant measure that satisfies $\Delta_H = \Delta_G|_H$.

Definition 1.42

For a topological group G, and an action which we call a representation $\pi : G \to Aut(V)$ of G on a Banach space V.

Example 1.11 $\mathbb{R}$ acts on $L^2(\mathbb{R})$ by translations U_t. One could ask if this is continuous, i.e. as $t \to 0$, do we get

$$\|U_t \to U_0\| \to 0$$

You could have a little bump around 0 as your L^2 function, and when you translate it, the supports are disjoint and hence the L^2 norm gets preserved.

But there eixsts a continuous one for the strong operator topology. (The rep via translation itself does not converge, but when you apply it to something, it is indeed convergent).

Example 1.12 If $f \in C_c(X)$, and α, if you have $\alpha_x(f) \to f$, and $x \to e$.

1.19 Lecture 21

Let G be a locally compact group, and let M be a locally compact space, and let α be an action of G on M, and

$$\alpha : G \to Hom(M)$$

This is with α jointly continuous, i.e. $G \times M$ equipped with the product topology, and the mapping $G \times M \to M$

$$(x, m) \mapsto \alpha_x(m)$$

is continuous.

One classic example is G acting by left translation on G.

Theorem 1.14

The action of G on $C_\infty(M)$ via

$$\alpha_x(f)(m) = f(\alpha_x^{-1}m)$$

is strongly continuous, for the strong operator topology.

Proposition 1.51

If α is an ction of a topology G, on a Banach space, by isometries, (i.e. each α_x is an isometry on the Banach space), and if there is a dense subspace $V_0 \subset V$,(or the vectors whose span is a dense subspace) such that on V_0, *this action is strongly continuous, i.e. for all $v \in V_0$, then function*

$$x \mapsto \alpha_x(v) \text{ is continuous}$$

Then α is strongly continuous on V.

Proof You do the standard approximation argument by $\frac{\epsilon}{3}$-argument.

Now back to the theorem, it suffices to prove this for $C_c(M)$, since it is a dense subset of $C_\infty(M)$ with $\|\cdot\|_\infty$ norm. And it suffices to prove continuity at the identity element at e_G (by uniformity).

$$\|\alpha_x(v) - \alpha_y(v)\| = \|\alpha_{y^{-1}}(\alpha_x(v) - \alpha_y(v))\| = \|\alpha_{y^{-1}x}(v) - v\|$$

Proof Let $f \in C_c(G)$, and want to show that the map

$$x \mapsto \alpha_x(f) \text{ is strongly continuous}$$

suffices to show at $x = e_G$. Let $\epsilon > 0$ be given, let $K = supp(f)$, and choose a comapct neighborhood O of e_G. By joint continuity of $G \times M \xrightarrow{\alpha} M$, then we have

$$O \times K \mapsto \alpha_O(K) := OK \in G \text{ is compact}$$

Now because f is continuous, for each $m \in OK$, there is an open neighborhood U_m of m such that if $n \in U_m$, then

$$\|f(n) - f(m)\| < \epsilon/2$$

So there is an open neighborhood W_m of the identity element, such that $O_m W_m \subset U_m$, you can even required $O_m \subset O$ (what we started with, (O_m's are in G, and W_m's are in M)).The W_m's form an open cover of OK so there is a finite subcover $W_{m_1}, ..., W_{m_n}$ and let $O^* = \cap_{j=1}^n O_j$,

Now we claim that for $x \in O^*$, we have that

$$\|\alpha_x(f) - f\| < \epsilon$$

For every m, if $\alpha_x(f)(m) - f(m) \neq 0$, then

$$f(\alpha_x^{-1} m) - f(n) \neq 0$$

So each $m \in K$, $m \in O^*K$, and have $m \in O^*K \subset OK$, so there exists j such that $m \in W_{m_j}$, $x \in O^* \subset O_j$, and so

$$\alpha_x(m) \in U_{m_j}$$

This gives

$$\|f(\alpha_x(m_j)) - f(m_j)\| < \epsilon/2, \|f(m) - f(m_j)\| < \epsilon/2$$

Let α be an action of G on M, and suppose there is an α-invariant Borel (or even Radon measure) measure on M. This gives an action of G on $L^p(M, \nu)$, and

$$\alpha_x(f)(m) = f(\alpha_x^{-1} m)$$

where $\|\alpha_x(f)\|_{L^p} = \|f\|_{L^p}$ by α-invariant.

Proposition 1.52

This action is strongly continuous.

Proof Check this is true on $C_c(M) \subset L^p(M, \nu)$, where $C_c(M)$ is a dense subspace.

1.20 Lecture 21

G locally compact group, and x locally compact space, and α a jointly continuous actionof G on X, then the corresponding action of G on $C_\infty(X)$ is strongly continuous.

For any $f \in C_\alpha(X)$, and

$$x \mapsto \alpha_x f \text{ is continuous for } \|\cdot\|_\infty$$

Assume that μ is an α-invariant measure on X, i.e.

$$\mu(\alpha_x f) = \mu(f), \text{ for all } x, f$$

We then can form $L^p(X, \mu)$, with the $\|\cdot\|_{L^p}$ norm:

$$\|f\|_{L^p} = (\mu(|f|^p))^{\frac{1}{p}}$$

And $L^p(X)$ contains $C_c(X)$, and what we want is the action α on the L^p is again strongly continuous. (One could take f such that $\|f\|_\infty$ is small, but the L^p norm is large.)

The action α of G on $C_c(X)$ is continuous for the "inductive limit topology." We shall it define it now.

Definition 1.43 (Inductive limit topology)

If $O \subset X$, and $\overline{O}$ is compact, then you can view $C_\infty(O)$, equipped with $\|\cdot\|_\infty$, as a subset of $C_c(X)$ (simply by extending by 0), then the inductive limit topology is the strongest topology on $C_c(X)$ making the inclusion continuous. ♣

Definition 1.44 (Convergence in ILT)

If a net $\{f_\lambda\}$ converges to f uniformly, and eventually, and there is a compact set K such that eventually all f_λ are supported in K, then this $\{f_\lambda\}$ converges to f in the inductive limit topology. ♣

If μ is a Radon measure on $C_c(X)$, then the net $\{f_\lambda\}$ converges to f for $\|\cdot\|_{L^p}$ of $L^p(X,\mu)$. We can find f_λ converges uniformly to f in the $\|\cdot\|_\infty$ sense.

$$(\|f - f_\lambda\|_{L^p})^p = \int_X |f - f_\lambda|^p d\mu \leq \|f - f_\lambda\|_\infty^p \mu(K)$$

And $\mu(K) < \infty$, and $\|f - f_\lambda\|_\infty < \epsilon$. Everything is of finite measure, and uniform convergence gives you convergence in the L^p norm.

To show that $x \mapsto \alpha_x(F)$ is continuous for $\|\cdot\|_{L^p}$, it suffices to show that it is continuous at e, let O be a compact neighborhood of E, and let $K = supp(f)$, then for $x \in O$, then $\alpha_x(f)$ is compactly supported in $\alpha_O(K)$, and $\|f - \alpha_x(f)\|_\infty \to 0$, hence is strongly continuous for the $\|\cdot\|_{L^p}$ norm.

Let G be a locally compact and let α be an action by isometries on a Banach space V, we define the integrated form of α, $f \in C_c(G)$, and $v \in V$:

$$\alpha_f(v) = \int_G f(x)\alpha_x(v)dx, \alpha_x(v) \in V$$

where dx is a choice of left-invariant Haar measure.

Given $\varphi \in V^*$,

$$\int f(x)\langle \alpha_x(v), \varphi\rangle dx$$

the above form is called the "weak integral." and $range(f)$ is a normed-closed compact subset of V.

Consider, $C_c(X, \mathbb{R})$, and V is over $\mathbb{R}$. and

$$(X, \mathbb{R}) \bigotimes V = \{\sum_{i=1}^n f_i \bigotimes v_i, f_i \in C_c(X, \mathbb{R}), v \in V\}$$

and viewed as an element of $C_c(X, V)$, and $(x) = \sum_i f_i(x)v_i \in V$. Given a Radon measure μ, we have

$$\int \sum_{j=1}^n f_j \bigotimes v_j d\mu = \sum \mu(f_j)v_j$$

The above is well-defined, and apply to V^*, if we denote $F = \sum_{j=1}^n f_j \bigotimes v_j$, then

$$\varphi(\int F(x)d\mu(x)) = \int \varphi(F(x))d\mu(x)$$

One should think of these F as simple functions $f = \sum_j \chi_{E_j} v_j$.

Note that

$$\{\sum_{j=1}^{n} f_j \bigotimes v_j\} \text{ is dense for the inductive limit topology}$$

Given $F \in C_c(X,V)$, and some $\epsilon > 0$, let $K = supp(F)$, and C be a compact neighborhood of K. Then, for each $x \in C$, there is an open neighborhood O_x with

$$\|f(y) - f(x)\| < \epsilon \text{ if } g \in O_x$$

And by C being compact, there exists a finite subcover $O_{x_1}, ..., O_{x_n}$. Choose a partition of unity, subordinate to the $\{O_{x_j}\}_{j=1}^n$. This means $F_j \in C_c(X,\mathbb{R})$, and $0 \leq f_j \leq 1$, and $supp(f_j) \subset O_j$, such that $\sum \varphi_j = 1$ on K. Then the set

$$F_\epsilon = \sum f_j \bigotimes F(x_j)$$

and $\|F - F_\epsilon\|_\infty < \epsilon$.

Let α be an action on V, if we write $\alpha_f(v) = \int_G f\alpha_x(v)dx$, and $\alpha_f\alpha_g = \alpha_{f*g}$, then

$$(f*g)(x) = \int f(y)g(y^{-1}x)dy$$

Fubini's theorem for Radon measures, use $C_c(X) \bigotimes C_c(Y)$, dense in $C_c(X \times Y)$

$$\sum_{j=1}^{n} f_j \bigotimes g_j$$

for the inductive limit topology. This is the Stone-Weierstrass theorem. Now we ask the question, what is $(\pi_f)^*$.

1.21 Lecture 22

Let G be locally compact, and if $(\mathcal{H}, \pi)$ is a unitary representation of G, with the integrated form:

$$f \mapsto \pi_f, f \in C_c(G) \subset L^1(G)$$

we have

$$(\pi_f)^* = \left(\int f(x)\pi_x dx\right)^* = \int \overline{f(x)}\pi_x^* dx = \int \overline{f(x)}\pi_{x^{-1}}dx = \int \overline{f(x^{-1})}\Delta(x)\pi_{x^{-1}}dx$$

Proposition 1.53

If G is locally compact, but not discrete, then $L^1(G)$ does not have an identity.

Example 1.13 $C_\infty(X)$.

Note *Many Banach algebras have an "approximate" identity elements.*

Example 1.14 Let's look at $C_\infty(\mathbb{R})$. You could take a function that is a smooth characteristic function, whose support is contained in $[-N, N]$, and pushing $N \to \infty$. If you take N large enough, you eventually approximate the function itself.

Definition 1.45 (Approximate identities)

For a normed algebra $\mathcal{A}$, let a be a left approximate identity , we mean a net e_λ such that for any $a \in \mathcal{A}$, we have that $e_\lambda a \to a$ in norm. A right approximate identity is one such that $ae_\lambda \to a$ in norm. A two-sided approximate is one such that it satisfies both. And a bounded approximate identity is one such that $\|e_\lambda\| \leq c$ for all λ and such that $e_\lambda a \to a$ in norm. (There is therefore, left bounded approximate identity, and right, and two-sided). A norm 1 approximate identity is such that $\|e_\lambda\| = 1$.

If $\mathcal{A}$ has a * operation, then you can define a self-adjoint approximate identity, if

$$e_\lambda^* = e_\lambda \text{ for all } \lambda$$

If $\mathcal{A}$ is a C^*-algebra, a positive approximate identity is a net with $e_\lambda > 0$ for all λ.

Proposition 1.54

For any locally compact group, $L^1(G)$ has a approixmate identity of norm 1.

Note *The picture should look like a tall bump function around the identity e_G.*

Proof Let Γ be a collection of comapct neighborhoods of e_G, and for each O, choose $e_O \in C_c(G)$ with $supp(e_O) \subset O$, and $\int e_O(x)dx = 1$.

$f * g(x) = \int f(y)g(y^{-1}x)dy$ with integrated form of the action of G on $L^1(G)$ by translation $g(x) \to g(y^{-1}x)$. In fact, for any represntation of a Banach space V of G by $\pi_{e_\lambda} v \to v$ in norm. We would like the following to go to zero.

$$\| \int e_O(x)\pi_x v - v\| = \| \int e_O(x)\pi_x v - \int e_O(x)dxv\| = \| \int e_O(x)(\pi_x v - v)\|$$

For any $\epsilon > 0$, choose O_ϵ and that for $x \in O_\epsilon$, we have $\|\pi_x v - v\| < \epsilon$ (by SOT). For any neighborhood smaller than O_ϵ, we still have this property, $O \subset O_\epsilon$. So for the above equation, we get that

$$\| \int e_O(x)\pi_x(v) - \int e_O(x)dxv\| \leq \int Oe_O(x)\|\pi_x v - v\| \leq \epsilon$$

□

Proposition 1.55

For any c^*-algebra $\mathcal{A}$ has a positive approximate identity of norm 1.

Proof Let Γ be the collection of finite subsets of $\mathcal{A}$, and for $\lambda \in \Gamma$, let the finite sum be as follows:

$$b_\lambda = \sum \{b^* b : b \in \lambda\}$$

let

$$e_\lambda = \frac{b_\lambda}{\frac{1}{n} + b_\lambda} < 1, n = |\lambda| = \text{ the number of elements in } \lambda$$

□

For any Banach normed algebra $\mathcal{A}$, we can get an identity element by

$$\tilde{A} = \{(a, z) : a + z1_{\tilde{A}}\}$$

If $\mathcal{A}$ has an approximate identity of norm 1, and if μ is continuous positive $\mu(a^* a) \geq 0$ linear functional, on $\mathcal{A}$, but no identity element, but does have a norm $\|\mu\|$. Now if you have an approximate identity, then μ extends to a positive linear functional on $\tilde{\mathcal{A}}$, where $\|\mu\| = 1$. Then you could apply GNS to this μ.

Note *Any C^*-algebra $\mathcal{A}$ has a positive approximate identity of norm 1. For example G, $L^2(G)$, and π*

$$\overline{\{\pi_f : f \in \mathcal{A}\}} = D_r^*(f)$$

1.22 Lecture 23

Proposition 1.56

Let G be a locally compact group, the left-regular representation of $L^1(G)$ on $L^2(G)$ is faithful.

Proof let $f \in L^1(G)$, and $f \neq 0$, with $L^1(G)$ has a norm-1 approximate identity of functions in $C_c(G)$. Then there exists $g \in C_c(G)$ such that $f * g \neq 0$, and

$$f * g = \int_G f(x)\lambda_x g dx \in C_\infty(G) \cap L^1(G) \cap L^2(G)$$

so now g is an element of $L^2(G)$,

$$\lambda_f(g) = f * g \in L^2(G)$$

we have that

$$\|f * g\|_{L^2}^2 = \int |f * g|^2 d\mu > 0$$

So $\lambda_f \neq 0$. For G commutative, the norm closure

$$\overline{\{\lambda_f : f \in L^1(G)\}} = C_r^*(G)$$

Proposition 1.57

Let $\mathcal{A}$ be a C^*-algebra, without an identity element. Let $\tilde{\mathcal{A}}$ be $\mathcal{A}$ with 1 adjoint. (Side note: $\|a+\alpha 1\| = \|a\| + |\alpha|$, and note such norm is not good for $\mathcal{A} = C_\infty(X)$.) For the left regular representation on $\mathcal{A}$, i.e. if $b \in \tilde{\mathcal{A}}$, set $\|b\| := \sup\{\|ba\| : \|a\| \leq 1\}$. With this norm $\|b\|$, we have that $\tilde{\mathcal{A}}$ is a C^*-algebra.

This is to check that $\|b^*b\| = \|b\|^2$. We already have $\|b^*b\| \leq \|b\|^2$. And $\|ba\|^2$, we have that

$$\|ba\|^2 = \|(ba)^*(ba)\| \leq \|a^*\|\|b^*ba\| \leq \|b^*b\|\|a\| = \|a\|^2\|b^*b\|$$

And it is easy to see that if $\mathcal{A}$ is complete, then $\tilde{\mathcal{A}}$ is also complete.

If $\mathcal{A}$ is a commutative C^*-algebra without identity, and $\tilde{\mathcal{A}} = C(X)$, and this is the nonzero homomorphisms of $\mathcal{A}$ onto $\mathbb{C}$, with $\{\infty\}$. So $\tilde{\mathcal{A}} = X \setminus \{\infty\}$, and $\mathcal{A} \cong C_\infty(X \setminus \{\infty\})$, where the space $X \setminus \{\infty\}$ is locally compact.

Let G be a commutative, discrete group, where

$$C_r^*(G) \cong C_\infty(X)$$

where each point of X gives a nonzero homeomorphism of $L^1(G)$ into $\mathbb{C}$. Then get for any $f \in L^1(G)$, there is a continuous $*$-homomorphism of $L^1(G)$, then there exists a $\varphi \in \widehat{L^1(G)}$ such that $\varphi(f) \neq 0$. We want to describe all $\varphi \in \widehat{L^1(G)}$, let $f, g \in L^1(G), g \neq 0$. We have that

$$\varphi(f)\varphi(g) = \varphi(f * g) = \varphi(\int f(x)\lambda_x g dx)$$

So

$$\varphi(f) = \int f(x)\varphi(\lambda_x g)/\varphi(g) dx$$

Define $\omega\varphi(x) = \varphi(\lambda_x g)/\varphi(g)$, is continuous. hence we have that

$$\varphi_f\varphi_g = f(x)\varphi(\lambda_x g)dx$$

We then have

$$\varphi(g)\omega_\varphi(x) = \varphi(\lambda_x g)$$

If we look at

$$\omega_\varphi(xy)\varphi(g) = \varphi(\lambda_{xy}g) = \varphi(\lambda_x(\lambda_y g)) = \varphi(\lambda_y g)\omega_\varphi(y)\omega_\varphi(x)$$

Because we have $\varphi(g) \neq 0$, we divide both sides, then we have

$$\omega_\varphi(xy) = \omega_\varphi(x)\omega_\varphi(y)$$

So $\omega_\varphi \in \widehat{G}$ into T.

For $f \in L^1(G)$, define

$$\widehat{f}(\sigma) = \int f(x)\sigma(x)dx \in \widehat{G} \in C_\infty(\widehat{G})$$

where we equip $\widehat{G}$ with the weak-* topology, and $\sigma \in \widehat{G}$. And the above defines the Fourier transform.

Corollary 1.10

The Fourier trasnform defined above is injective.

For any $f \in L^1(G)$, if $f \neq 0$, then $\widehat{f} \neq 0$. And we have

$$\widehat{f^*} = \overline{\widehat{f}}$$

Next we state a fact:

Proposition 1.58

The weak- topology on $\widehat{G}$ agrees with the topology of uniform convergence on compact subsets of G. Moreover, under this uniform topology on compact subsets of G, we can see that $\widehat{G}$ is a topological group.*

Next time: unbounded operators on Hilbert spaces! (And the spectral theorem for those).

1.23 Lecture 24

Definition 1.46 (Power series of a bounded operator)

If V is a Banach space, and A is a bounded linear operator on V, then have the operator:

$$T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

Then $T_t T_s = T_{t+s}$, and $s, t \in \mathbb{R}$, where $T_0 = I$. For $v \in V$. For $v \in V$, and

$$t \mapsto T_t v \text{ is norm continuous}$$

If $V = C_\infty(\mathbb{R}^n, L^p(\mathbb{R}^n))$, and

$$Af = \Delta f, \text{ only defined for smooth } f$$

And

$$\frac{dI}{dt} f = \Delta f$$

Let V be a Banach space, and let $\{T_t\}$ be a semigroup of bounded operators on V, and $T_s T_t = T_{s+t}$, and $T_0 = I_V$, with each strongly continuous, so

$$t \mapsto T_t v \text{ is norm continuous for each } v$$

Definition 1.47

v is differentiable if

$$Av = \lim_{h \to 0} \frac{T_{t+h} v - T_t v}{h}$$

has a limit as $h \to 0$.

Let $\mathcal{D} = \{v \in V : v \text{ is differentiable }\}$

Definition 1.48

If $v, w \in \mathcal{D}$, then $u + v \in \mathcal{D}$, and $\alpha v \in \mathcal{D}$ if $\alpha \in \mathbb{C}$. And A is a linear operator from $\mathcal{D}$ to V.

For each $\{T_t v : t \in [0,1]\}$ is compact, so bounded, so there exists k_v such that $\|T_t v\| \le k_v$ for all $t \in [0,1]$. So by uniform boundedness principle, there is a k such that

$$\|T_t\| \le k, \text{ for } t \in [0,1]$$

Then any $t \ge 0$, is of the norm $n + s$ with $s \in [0,1]$. So

$$T_t = T_n T_s = (T_1)^n T_s, \|T_t\| \le k^{n+1} = k k^n = k e^{n \ln(k)} \le k e^{t\beta}, \beta = \ln(k)$$

Let $\tilde{T}_t = k^{-1} e^{-t\beta} T_t$, then $\|\tilde{T}_t\| \le 1$, this gives that $t \|\tilde{T}_t\|$ is a strongly continuous operator.

$$\frac{T_{t+h} - T_t}{h} = T_t \frac{T_h - T_0}{h}$$

And we also get

$$T_t A v = A T_t$$

Proposition 1.59

If $v \in \mathcal{D}(A)$, then so does $T_t v$ for all t, and we have

$$T_t A v = A T_t$$

Given f continuous on $[0, \infty)$, and $v \in V$. We can define

$$\int_0^t f(s) T_s(v) ds$$

is a norm continuous function on $\mathbb{R}^+$. This integral is defined by the Riemann integral (because is norm-continuous!). One should think of this integrated form of T for $f|_{[0,t]}$.

Proposition 1.60

We have

$$\int_0^t T_s v ds \in \mathcal{D}(A)$$

And this is to show

$$\int_0^t T_s v ds \in \mathcal{D}(A)$$

. We will prove this next time. And note that

$$\frac{1}{t} \int_0^t T_s v dt \to v, \text{ as } t \to 0$$

And this converges to the approximation of norm 1.

1.24 Lecture 25

Let V be a Banach space, and $\{T_t\}$ a strongly continuous semigroup.

Proposition 1.61

For any $t > 0$, we have that

$$\int_0^t T_s v ds \in D(A)$$

where $D(A)$ is the domain of A.

Proof

$$\begin{aligned}
\frac{T_h - T_0(I)}{h} \left(\int_0^t T_s v ds \right) &= \frac{1}{h} \left(T_h \left(int_0^t T_s v ds \right) - \int_0^t T_s v ds \right) \\
&= \frac{1}{h} \left(\int_0^t T_{s+h} v ds \right) - \frac{1}{h} \int_0^t T_s v ds \\
&= \frac{1}{h} \left(\int_h^{t+h} T_s v ds - \int_0^s T_s v ds \right) \\
&= \frac{1}{h} \left(\int_t^{t+h} T_s v ds - \int_0^t T s v ds \right) = T_t A v - v
\end{aligned}$$

In order words,

$$A \left(\int_0^t T_s v ds \right) = T_t A v - v$$

$\square$

Hence, to reiterate, we have

$$\int_0^t T_s v ds \in D(A), \text{ so } \frac{1}{t} \int_0^t T_s v ds \in D(A)$$

And as

$$\lim_{t\to 0}\left(\frac{1}{t}\int_0^t\right) = v$$

Then

Proposition 1.62

The domain of A, $D(A)$ is dense in V.

For each operator S on a domain D, with $D \subset V$, the graph of S,

$$\Gamma(S) = \{(v, Sv) : v \in D(S)\} \subset V \times V$$

And there is no unique norm that you put on here, but there exists some equivalent norms to put on here.

Example 1.15 We could have the following norms on the graph of S,

$$\|(v,w)\| = \begin{cases} \|v\|_V \|w\|_\infty \\ \|v\| + \|w\| \\ (\|v\|^2 + \|w\|^2)^{\frac{1}{2}} \end{cases}$$

Definition 1.49 (Closed and closable graph)

You call S is closed if $\Gamma(S)$ is a closed subset of $V \times V$.
We call that S is closable, if we have the closure of $\Gamma(S)$ is the graph of an operator.

Note *The issue here is that if $(v, w_1), (v, w_2)$ both belong to $\overline{\Gamma(S)}$. Then, you need $w_1 = w_2$. In other words, you need that if $(0, w) \in \overline{\Gamma(S)}$, then $w = 0$. Note that this is also a sufficient condition. If closable, then closure of S is the operator from $\overline{\Gamma(S)}$.*

Remark This does not mean that the closure is the whole space.

From an unbounded operator, how to create a one-parameter semigroup, such that is closable.

Proposition 1.63

This is the aim: for A from $\{T_t\}$, A is a closed operator.

$$A\left(\int_0^t T_s v dx\right) = T_t v - v$$

Proposition 1.64

For $v \in D(A)$, we have

$$\int_0^t A(T_s v) ds = \int_0^t T_s(Av) = T_t v - v$$

Proof Let V^* be the dual vector space, and let $\varphi \in V^*$, and let

$$f(t) = \varphi\left(\int_0^t AT_s v ds\right) = \int_0^t \varphi(AT_s v) ds, g(t) = \varphi(T_t v - v)$$

We have $f(0) = g(0) = 0$. Then we consider the derivatives:

$$f'(t) = \varphi(AT_t v), g'(t) = \varphi(AT_t v)$$

Now by the unique solutions of ordinary differential equations, we get that

$$f(t) = g(t), \forall t$$

□

Now we prove the proposition that A on T_t is a closed operator.

Proof Suppose we have a sequence of $\{v_n\} \subset D(A)$, with $v_n \to v \in V$, and $Av_n \to w \in V$. We thus have

$$(v_n, Av_n) \to (v, w)$$

Then for any t, we have (given T_t is bounded)

$$\int_0^t AT_s v_n ds = T_t v_n - v_n \to T_t v - v$$

Note that this also converges to

$$\int_0^t T_s(Av_n)ds \to \int_0^t T_s w ds$$

So we have that

$$\int_0^t T_s w ds = T_t v - v$$

$$\frac{1}{t}\int_0^t T_s w ds = \frac{T_t v - v}{t}$$

Let $t \to 0$, then we have (the LHS is approximate of identity)

$$w = Av$$

Hence we have that $v \in D(A)$, i.e. the pair

$$(v_n, Av_n) \to (v, w) = (v, Av) \in \Gamma(A)$$

A is closed.

□

Note *what semigroup/*

Let $\mathcal{H}$ be a Hilbert space, and let $A \in \mathcal{B}(\mathcal{H})$, and let

$$T_t = e^{tA}$$

We ask the question: when is T_t a group of **unitary** operators, such that $T_t T_t^* = T_t T_T^{-1} = I$

1.25 Lecture 26

Let $A \in \mathcal{B}(\mathcal{H})$, $T_t = e^{tA}$, unitary, and

$$T_t T_t^* = I$$

and

$$\frac{d}{dt}|_{t=0} = A + A^* = 0$$

And we have

$$e^{tA}e^{tA^*} = I$$

i.e. $A^* = -A$.

Let $B = -iA$, and B is self-adjoint, and

$$A = iB, e^{itB} \text{ is self-adjoint}$$

We now take

$$U_t = e^{itA}$$

where A is unbounded, we ask if it is self-adjoint? For an unbounded operator A on $\mathcal{H}$, $D(A)$, dense. Ho do we define A^*?

$$\langle A\xi, eta\rangle = \langle \xi, A^*\eta\rangle, \xi \in D(A)$$

where

$$D(A^*) = \{h \in \mathcal{H} : \xi \to A\xi \text{ is continuous }\}$$

$$\Gamma(A^*) = \{(\eta, \zeta) : \langle A\xi, \eta\rangle = \xi, \zeta, \forall \xi \in D(A)\}$$

In other words,

$$\langle A\xi, \eta\rangle - \langle \xi, \zeta\rangle = 0$$

which implies

$$\langle (A\xi - \xi), (\eta, \zeta)\rangle = 0$$

And we have

$$\langle V(\xi, A\xi), (\eta, \zeta)\rangle = 0$$

$$\Gamma(A^*) = (V(\Gamma(A)))^{\perp} = V(\Gamma(A)^{\perp})$$

We have that $\Gamma(A^*)$ is closed, and hence $V(\overline{\Gamma(A)})$ is closed.

Hence if A is closed, then

$$\Gamma(A^*) = (V\Gamma(A))^{\perp} = V((\Gamma(A))^{\perp})$$

Then

$$\mathcal{H} \bigoplus \mathcal{H} = \Gamma(A^*) \bigoplus V\Gamma(A)$$

Proposition 1.65

We have $D(A^*)$ is dense.

Proof If $D(A^*)$ is not dense, then there exists $\xi \in \mathcal{H}$, such that

$$\langle \xi, D(A^*)\rangle = 0$$

$$0 = \langle (0, \xi), (A^*\eta, -\eta)\rangle = \langle (0, \xi), V((\eta, A^*\eta))\rangle$$

for $\eta \in \Gamma(A^*)$. So $(0, \xi)$ is perpendicular to $V(\Gamma(A^*))$, so

$$(0, \xi) \in \Gamma(A), A0 = \xi \Rightarrow \xi = 0$$

□

$$\mathcal{H} \bigoplus \mathcal{H} = \Gamma(A^*) \bigoplus V\Gamma(A) = \Gamma(A) \bigoplus V(\Gamma(A^*))$$

For an $\xi \in \mathcal{H}$, and

$$(\xi, 0) = (\eta, A\eta) + (A^*S, -S)$$

and

$$\zeta = \eta + A^*\zeta = \eta + A^*A\eta = (I + A^*A)\eta, 0 = A\eta - \zeta$$

and then

$$(I + A^*A) : D(A) \to \mathcal{H}$$

For any $\zeta \in D(A)$, note that we have

$$\langle (I + A^*A)\eta, \eta\rangle = \langle \eta, \eta\rangle + \langle A^*A\eta, \eta\rangle = \langle A\eta, A\eta\rangle = 0, \text{ if } \eta \neq 0$$

Corollary 1.11

$(I + A^*A)$ is injective and onto $\mathcal{H}$.

If S, T are unbounded operators on $\mathcal{H}$, and $D(S), D(T)$, then

$$D(ST) = \{\xi : \xi \in D(T), \text{ and } T\xi \in D(S)\}$$

$$D(S + T) = D(S) \cap D(T)$$

We have that

$$\|(I + A^*A)\xi\| \geq \|\xi\|$$

And again we have $(I+A^*A)^{-1}$ exists on $\mathcal{H}$ and mapping onto $D(A)$. So

$$\|(I+A^*A)^{-1}\| \leq 1$$

1.26 Lecture 27

Proposition 1.66

For A closed, we have $I+A^*A$ is injective on $D(A^*A)$, and $D(A^*A)$ is dense in $\mathcal{H}$.

Proof We would like to show for a given $\xi \in \mathcal{H}$, we have that

$$\langle \xi, \eta\rangle = \langle (I+A^*A)\eta, \eta\rangle = \langle \eta, \eta\rangle = \langle A\eta, A\eta\rangle$$

So that if the above is equal to 0, we get that $\eta = 0$, which we get $\xi = 0$, hence the density claim.

Let $\eta \in D(A^*A)$, and let $\xi = (I+A^*A)\xi$, if $\xi = 0$, then we must have our $\eta = 0$. So $(I+A^*A) = I_{\mathcal{H}}$. And we get that $(I+A^*A)$ is one-to-one from $D(A^*A)$ on $\mathcal{H}$, hence

$$Range(S) = D(A^*A)$$

□

For any ξ, we get

$$\langle S^*\xi, \xi\rangle - \langle S^*(I+A^*A)S\xi, \xi\rangle = \langle (I+A^*A)S\xi, S\xi\rangle = \langle S\xi, S\xi\rangle^{\perp}\langle AS\xi, AS\xi\rangle \geq 0$$

Being positive implies that $S^* = S$, and S is positive and adjoint. S is also one-to-one, and

$$Range(S) = D(A^*A)$$

This gives that $(I+A^*A)S = I_{\mathcal{H}}$, then

$$S = (I+A^*A)^{-1}$$

Definition 1.50

For a closed operator A, we call $(A, D(A))$ is self-adjoint, if

$$(A, D(A)) = (A^*, D(A^*))$$

We will show that $(I+A^*A, D(A^*A))$ is self-adjoint.

Definition 1.51

For $(A, D(A))$ is symmetric if for all $\xi \in D(A)$, we have

$$\langle A\xi, \xi\rangle = \langle \xi, A\xi\rangle$$

We will show some operators are self-adjoint next time.

1.27 Lecture 28

For A closed on $D(A) \subset \mathcal{H}$, and

$$I + A^*A : D(A^*A) \to \mathcal{H}$$

And $S : \mathcal{H} \to D(A^*A)$, and

$$(I+A^*A)S = I_{\mathcal{H}}$$

And last time we saw that $I+A^*A$ is injective. Then we also have that S is onto $D(A^*A)$, and then

$$S = (I+A^*A)^{-1}$$

from $\mathcal{H}$ to $D(A^*A)$.

Note

If $T : D(T) \to D'$ is bijective, thus T^{-1} exists, then

$$\Gamma(T^{-1}) = W\Gamma(T), W((\xi, \eta)) = (\eta, \xi)$$

Hence $\Gamma(T^{-1})$ is closed if and only if $\Gamma(T)$ is closed.

This implies that S is closed.

Proposition 1.67

$I + A^*A$ is closed.

Proof Above.

Definition 1.52

$(A, D(A))$, let A be densely defined, is self-adjoint, if $D(A^*) = D(A)$ (A^* is also densely defined), and on this domain,

$$A = A^*$$

Proposition 1.68

If $A, D(A)$ is closed, then $D(A^*)$ is dense.

Proof If $D(A^*)$ is not dense, then there exists a nonzero ξ such that $\xi \perp D(A^*)$, and we have

$$(0, \xi) \perp (A^*\eta, -\eta)$$

for all $\eta \in D(A^*)$.

$$(0, \xi) \perp V(\eta, A^*\eta), \text{ for all } \eta$$

This gives that

$$(0, \xi) \in \Gamma(A)$$

Hence $\xi = 0$.

$\square$

Proposition 1.69

If A is closed, then $A = A^{**}$.

Proof Note that we have

$$\Gamma(A^*) = (V\Gamma(A))^{\perp}$$

$\Gamma(A^*A) = (V(\Gamma(A^*))^{\perp})$, and note that $V^2 = -I$, hence

$$\Gamma(A^{**}) = (V(\Gamma(A^*))^{\perp}) = V(\Gamma(A^*)^{\perp}) = V(V\Gamma(\overline{A}))$$

Thus,

$$\Gamma(A^{**}) = \Gamma(A)$$

$\square$

Proposition 1.70

If A is intertible and closed, and $A : D(A) \to D'$, then

$$\Gamma((A^{-1})^*) = \Gamma((A^*)^{-1})$$

Proof We have that

$$WV\Gamma(A^{-1})^* = W\Gamma(A^{-1})^{\perp} = (W\Gamma(A^{-1}))^{\perp} = \Gamma(A)^{\perp}$$

And note that we also have

$$VW\Gamma((A^*)^{-1}) = V(A^*) = \Gamma(A^{**})^{\perp} = \Gamma(A)^{\perp}$$

Note that we have

$$VW = -VW$$

$$(WV)(\xi, \eta) = W(\eta, -\xi) = (-\xi, \eta)$$

$$(VW(\xi, \eta)) = V(\eta, \xi) = (\xi, -\eta)$$

Note *The negative sign does not matter when describing graphs.*

□

Theorem 1.15
$I + A^*A$ is self-adjoint.

Proof

$$I + A^*A = S^{-1}, (I + A^*A)^{-1} = S$$

so

$$((I + A^*A)^{-1})^* = S^* = S = (I + A^*A)^{-1}$$

Hence $(I + A^*A)^{-1}$ is self-adjoint, hence $I + A^*A$ is also self-adjoint.

□

Theorem 1.16
If A is self-adjoint, S, $AS = R$, then we have $SR = RS$.

If A is self-adjoint, S, $AS = R$, then we want $SR = RS$ Let $\xi \in D(A)$, then

$$\xi = (I + A^*A)S\xi = S\xi + A^*AS\xi$$

We know that $S\xi \in D(A^*A) \subset D(A)$, so

$$A^*AS\xi \in D(A)$$

Then we apply A to everything

$$A\xi = AS\xi + AA^*AS\xi = (I + AA^*)AS\xi$$

Notice the * is in the wrong place now.

$$(I + AA^*)^{-1}A\xi = AS\xi$$

we call

$$\tilde{S} = (I + AA^*)^{-1}$$

Note *It looks like S, but just the * is in the wrong place.*

Note that

$$\tilde{S}A\xi = AS\xi$$

If A is self-adjoint, so $A^* = A$,

Definition 1.53 (Normal operators)
We say A is normal if

$$A^*A = AA^*, D(A^*A) = D(AA^*)$$

If you have A as normal, then you could say $S = \tilde{S}$.
So if A self-adjoint, then for $\xi \in D(A)$,

$$SA\xi = AS\xi$$

For any $\eta \in \mathcal{H}$, let $\xi = S\eta \in D(A^*A) \subset D(A)$,

$$S(AS)\xi = (AS)(S\xi)$$

Hence for A self-adjoint,

$$SR\xi = RS\xi, \text{ for all } \xi \in \mathcal{H}$$

1.28 Lecture Nov 3

We have

$$l^1(\mathbb{N}) \subset l^1(\mathbb{Z}), \mathcal{A} = L^1(\mathbb{R}^+) \subset L^1(\mathbb{R})$$

We note that

$$\widehat{\mathcal{A}} = \{ \text{ continuous homomorphisms of } \mathbb{R}^+ \text{ into the unit disk } \}$$

This is

$$\{\varphi_\lambda : \varphi_\lambda(t) = e^{-\lambda t}, Re(\lambda) \geq 0\}$$

This looks like the right half plane of $\mathbb{R}^2$.

The Gelfand transform sends $f \in L^1(\mathbb{R}^+)$ is

$$\widehat{f}(\lambda) = \int_0^\infty f(t)e^{-\lambda t} dt$$

Note *This is the Laplace transform.*

Note *Notice this if we don't assume f is integrable, but instead just f is bounded, and the above integral is well-defined for $Re(\lambda) > 0$. (Would not make sense if $Re(\lambda) = 0$).*

Let V be a Banach space, and let $\{T_t\}$ be a strongly continuous 1-parameter semigroup on $\mathbb{R}^+$ on V, continuous, i.e.

$$\|T_t\|| \leq 1, \forall t$$

For $Re(\lambda) > 0$, set

$$R_\lambda \xi = \int_0^\infty e^{-\lambda s} T_s \lambda ds$$

This will convergge given $Re(\lambda) > 0$. Let B be the "generator" of $\{T_t\}$. Then we have

$$T_h(R_\lambda \xi) = \int_0^\infty e^{-\lambda s} T_{s+h} \xi ds = \int_h^\infty e^{\lambda(s-h)} T_s \xi ds$$

This has

$$\begin{aligned}
\frac{T_h - I}{h} &= \frac{1}{h} \int_h^\infty e^{\lambda(s-h)} T_s \xi dt - \int_0^\infty e^{-\lambda s} T_s \xi ds \\
&= \frac{1}{h} \int_0^\infty e^{\lambda(s-h)} T_s \xi ds - \frac{1}{h} \int_0^\infty e^{-\lambda s} T_s \xi ds - \frac{1}{h} \int_0^h e^{-\lambda(s-h)} T_s \xi ds \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda s} T_s \xi ds - \frac{1}{h} \int_0^h e^{-\lambda(s-h)} T_s \xi \\
&= \lambda R_\lambda \xi - \xi
\end{aligned}$$

Thus for all $\xi \in V$, and $R_\lambda \xi \in D(B)$, and

$$B(R_\lambda \xi) = \lambda R_\lambda \xi - \xi$$

Then we have

$$\xi = (\lambda R_\lambda - BR_\lambda)\xi = (\lambda - B)R_\lambda \xi$$

All of the above leads to the following theorem.

Theorem 1.17

For $R_\lambda \xi \in D(B)$ for all $\xi \in V$, $Re(\lambda) > 0$, and

$$(\lambda - B)R_\lambda \xi = \xi$$

Thus $Range(\lambda - B) = V$. And

$$R_\lambda = \frac{1}{\lambda - B} \text{ “ $=$'' the resolvent of } B$$

Let $\mathcal{H}$, and a semigroup of unitary operators, $\{U_t\}$, A, we have

$$\langle U_t\xi, U_t\eta\rangle = \langle \xi, \eta\rangle$$

And

$$\langle A\xi, \eta\rangle + \langle \xi, A\eta\rangle = 0, \xi, \eta \in D(A)$$

So $-A \subset A^*$. Hence A is skew-symmetric. We apply to $\{U_t\}_{t=0}$, for $\lambda > 0$, $Range(A - \lambda) = \mathcal{H}$.

And let $\tilde{T}_t = T_{-t}$, and $Range(A + \lambda) = \mathcal{H}$. Then range

$$range(iA \pm i\lambda) = \mathcal{H}$$

Proposition 1.71

C is a symmetric operator on $\mathcal{H}$, and that $Range(C \pm i) = \mathcal{H}$, then C is a self-adjoint in the technical sense.

Proof We have $D(C) \subset D(C^*)$, and on $D(C)$, we have $C^*|_{D(C)} = C$, then we want: if $\xi \in D(C^*)$, then $\xi \in D(C)$. Now let $\xi \in D(C^*)$ be given, then because there exists $\eta \in D(C)$, so

$$(C^* - i)\eta = (C^* - i)\xi$$

But $D(C) \subset D(C^*)$,

$$(C^* - i)\eta$$

This implies that $\xi - \eta \in Ker(C^* - i) = Ker(C + i)^* = (Range(C + i))^\perp$. And note that $Range(C + i) = \mathcal{H}$, hence this is 0.

$$\xi = \eta, \xi \in D(C)$$

Hence we are done.

□

We will talk about compact operators next time.

1.29 Lecture Nov 6

We will discuss compact operators now.

Definition 1.54

Let V, W be normed vector spaces, let $T : V \to W$ be linear, then T is said to be compact if

$$T(V_1) \text{ is totally bounded in } W$$

where $V_1 = \overline{B}(0, 1)$ in V, the closed unit ball. Recall that totally bounded means for every $\epsilon > 0$, the set can be covered with finite number of balls of radius at most ϵ. If W is complete, then T is compact if

$$\overline{T(V_1)} \text{ is compact}$$

Note *This implies that T is a bounded operator.*

A bit of notation: let $B_0(V, W)$ be the set of compact operators from V, W.

Proposition 1.72

If $S, T \in B_0(V, W)$, then $S + T \in B_0(V, W)$. (Less nontrivially, so is αT.)

Proof Let $\epsilon > 0$, then there exists $w_1, ..., w_m \in W$ so that $Ball(w_j, \epsilon/2)$ cover $S(V_1)$. There exists $w'_1, ..., w'_n \in W$ such that $Ball(w'_j, \epsilon/2)$ cover $T(V_1)$.

So for any $v \in V_1$, there is w_j and such that $Sv \in Ball(w_j, \epsilon/2)$, and a w'_R such that $Tv \in Ball(w'_R, \epsilon/2)$. This gives that

$$\|(S+T)v - (w_j + w'_R)\| \leq \|Sv - w_j\| + \|Tv - w'_R\| \leq \epsilon/2 + \epsilon/2 = \epsilon$$

□

Proposition 1.73

If $T \in B_0(V,W)$, and $S \in B(U,V)$ (just a bounded operator), then $TS : U \to W$ is in $B_0(U,W)$. If $R : W \to Y$, and $R \in B(W,Y)$, then $RT : V \to Y$ is in $B_0(V,Y)$. This implies that

$$\{T : T \in B_0(V,W)\} \text{ is a two-sided ideal in } B(V,w)$$

Note *R is uniformly coninuous, hence maps totally bounded sets to totally bounded sets.*

Note *$B_0(V)$ for $B_0(V,V)$, then $B_0(V)$ is a 2 sided ideal in $B(V)$.*

Proposition 1.74

$B_0(V,W)$ is norm-closed in $B(V,W)$.

Proof Let $\{T_n\}$ be a sequence in $B_0(V,W)$, and $T_n \to T$ in norm in $B(V,W)$. We now want to show T is compact. Let $\epsilon > 0$ be given, choose N such that

$$\|T - T_N\| < \epsilon/2$$

Find $w_1, ..., w_n \in W$ so that $Ball(w_j, \epsilon/2)$ cover $T_N(V_1)$. Then for any $v \in V_1$, there exists w_j such that $Tv \in Ball(w_j, \epsilon/2)$, and $\|v\| \leq 1$, hence

$$\|Tv - w_j\| \leq \|Tv - T_N v\| + \|T_N v - w_j\| < \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $T \in B_0(V,W)$. Now we have shown that $B_0(V)$ is a closed two-sided ideal.

□

Next we talk about some important examples of compact operators.

Example 1.16 Let $T : V \to W$, bounded, has finite rank, i.e. $Range(T)$ is finite dimensional.

Example 1.17 Let $V = l^p(X), 1 \leq p < \infty$, and let $f \in C_0(X) = C_\infty(X)$. And let $T = M_p$ by pointwise multiplication by f (if X is finite, then it follows from the above example that T is finite rank), such T is compact.

The next example is slightly interesting.

Example 1.18 Integral operators: $A = \{\alpha_{j,k}\}^n$ act on $\mathbb{C}^n$

$$(Av)_j = \sum a_{jk} v_k = \sum (\alpha_{j,k}) v(k)$$

Let X, Y be measure spaces, and K measurable on $X \times Y$, and for a function ξ measurable on Y, define

$$(T_k \xi)(x) = \int K(x,y)\xi(y)dy$$

The above $K(x,y)$ is called the kernel.

Example 1.19 If $K \in L^\infty(X,Y)$, then for almost every x,

$$y \mapsto K(x,y) \in L^\infty(Y)$$

If $\xi \in L^1(Y)$, then

$$|(T_K \xi)(x)| = \int |K(x,y)||\xi(y)|dy < \infty$$

Hence $T_k \xi \in L^\infty$. (Such operator is probably not compact).

If $K \in L^1(X,Y)$, then for almost every x, $K(x,\cdot)$ (viewed as a function of y), if $K(x,\cdot) \in L^1$, and $\xi \in L^\infty(Y)$, then

$$T_K \xi(x) = \int K(x-y)\xi(y)dy$$

and

$$\|T_K \xi\|_{L^1} \leq \|K\|_{L^1} \|\xi\|_{L^\infty}$$

Proposition 1.75

$T_K : L^\infty(Y) \to L^1(X)$ is compact.

Proof If $K(x,y) = \chi_E(x)\chi_F(y)$, and $Range(T_k) = \mathbb{R}\chi_E$. This is of rank 1, hence compact. If

$$K = \sum_{j=1}^{n} \alpha_j \chi_{E_j} \chi_{F_j}$$

For $K \in L^1(X,Y)$, approximate by simple functions above, in the L^1 norm. This implies that

$$T_{K_n} \to T_K$$

in operator norm, and by the previous proposition, T_K is compact. □

Let $K \in L^2(X \times Y)$, so

$$\int |K|^2 dxdy < \infty$$

For almost every x, $K(x,\cdot) \in L^2(Y)$. Hence if $\xi \in L^2(Y)$,

$$(T_K\xi)(x) = \int K(x,y)\xi(y)dy$$

This integral makes sense for almost every x.

$$|T_k\xi(x)| = \left|\int K(x,y)\xi(y)\right| \leq \int |K(x,y)||\xi(y)|dy \leq \|K(x,y)\|_{L^2(Y)}\|\xi\|_{L^2} < \infty$$

Now

$$\|T_K\xi\|^2_{L^2(X\times Y)} \leq \|K(x,y)\|^2_{L^2(X\times Y)}\|\xi\|^2_{L^2}$$

This gives us

$$\|T_K\| \leq \|K\|_{L^2}$$

Hence we again approximate with simple functions, and by the same proof that we gave for L^1.

Corollary 1.12

T_K with $K \in L^2(X \times Y)$ defined by

$$T_K f(x) = \int K(x,y)f(y)dy$$

is compact.

1.30 Lecture Nov 8

We talked about self-adjoint compact operators last time. Let $T \in B(\mathcal{H})$, and $T^* = T$.

Proposition 1.76

For T bounded and self-adjoint, then we have

$$\|T\| = \sup\{|\langle T\xi, \xi\rangle| : \|\xi\| \leq 1\}$$

Proof

$$\langle \xi, \eta\rangle = \frac{1}{4}\sum_{n=0}^{3} i^n \langle \xi + i^n\eta, \xi + i^n\xi\rangle$$

Apply $T\xi$ to the first coordinate, and the fact that $T^* = T$, take the real part.

We present another proof.

Proof $T \geq 0$, such that $T = S^2$, for some $S \geq 0$, then

$$\langle T\xi, \xi\rangle = \langle S\xi, S\xi\rangle = \|S\xi\|^2$$

The taking the sup over $\|\xi\| \le 1$, we just get $\|S\xi\|^2 = \|S\|^2 = \|T\|$.

□

Proposition 1.77

We prove T self-adjoint has an approximate eigvenvector.

Fix $T, \sigma(T)$, we consider the projection to $[0, \infty)$, denote the projection operator as P^+, and projection to $(-\infty, 0]$ as P^-. Note that

$$P^+ + P^- = I$$

Hence for a Hilbert space $\mathcal{H}$, we can decompose it,

$$\mathcal{H} = P^+\mathcal{H} + P^-\mathcal{H}$$

Then there exists $\{\xi_n\}, \|\xi_n\| = 1$, such that

$$|\langle T\xi_n, \xi_n\rangle| \to \|T\| \neq 0$$

And $\|T\xi_n\| \to \|T\|$. So one can assume that

$$\langle T\xi_n, \xi_n\rangle \to \|T\|$$

Let $\lambda = \|T\|$, and now look at

$$\|(T - \lambda I)\xi_n\|^2 = \langle (T - \lambda)\xi_n, (T - \lambda)\xi_n\rangle = \langle T\xi_n, T\xi_n\rangle - 2\lambda\langle T\xi_n, \xi_n\rangle + \lambda^2\langle \xi_n, \xi_n\rangle$$

Now we take the limit, $n \to \infty$, we get

$$\|T\|^2 - 2\|T\|^2 + \|T\|^2 = 0$$

Definition 1.55 (approximate eigenvector)

For any $T \in B(\mathcal{H})$, if we have λ, and $\{\xi_n\}$, with $\|\xi_n\| = 1$, and

$$\|(T - \lambda)\xi_n\| \to 0$$

We then say that $\{\xi_n\}$ is an approximate eigenvector for $\lambda \in \sigma(T)$.

Corollary 1.13

We see from above, that for T self-adjoint, either $\|T\|$ or $-\|T\|$, one can show for any $\lambda \in \sigma(T)$, T has an approximate eigevnvector.

Proof Above.

Proposition 1.78

Compact operators have an eigenvector.

Let $T \in B_0(\mathcal{H})$, a compact operator, and $T^* = T$. Assume $\|T\|$ that has an approximate eigenvector $\{\xi_n\}$, with $\|\xi_n\| = 1$. Then $\{\xi_n\} \subset \mathcal{H}$, $\{T\xi_n\}$ is totally bounded.

And as a subsequence converges to some vector η, so assume that $T\xi_n \to \eta$, then

$$\|(T - \lambda)\xi_n\| \to 0$$

This implies that

$$(T - \lambda)\xi_n \to 0, T\xi_n - \lambda\xi_n \to 0$$

But we know $T\xi_n \to \eta$,

$$(T - \lambda)\eta$$

Let $T\lambda\xi_n - \lambda\lambda\xi_n = \lambda(T\xi_n - \lambda\xi_n) \to 0$, for $\lambda \neq 0$.

For T compact, let λ not be any eigenvalue, then the eigenspace $\mathcal{H}_\lambda$ must be finite dimensional. For any $r > 0$, let H_r be the direct sum of all the eigen-subspaces for eigenvalues λ with $|\lambda| \ge r$. Then T on $\mathcal{H}_r$ is commutative, with norm $\le \frac{1}{r}$. This implies that H_r is finite dimensional. This has bounded inverse.

Then $T|_{\mathcal{H}_r}S = I_{\mathcal{H}}$ (the identity opeartor is compact). T acts on H_r into itself, and $T^* = T$. Note that $T|_{H_r^\perp}$ is compact, if $\|T|_{H_R^\perp}\| \geq r$, then H^+ contains an eigenvector for an eigenvalue λ with $\lambda \geq r$.

This gives that $\|T|_{H_r^\perp}\| < r$. And we take a sequence $r_n \to 0$, note that all H_{r_n} are finite dimensional, so get a sequence $\{\lambda_n\}$, and $|\lambda_n| \to 0$. And the eigenspaces are finite dimensional, so $\bigoplus \mathcal{H}_{\lambda_n}$ is all of $\mathcal{H}$ except for elements in the kernel of T, which may be infinite-dimensional, or can just be $\{0\}$.

From this, we get there is an orthonormal basis for $\mathcal{H}$ consisting of eigenvalues.

Note *Any compact operator can be diagonalized, with a countable number of eigenspaces.*

1.31 Lecture Nov 15

We say $T \in B(\mathcal{H})$ is compact if $T(Ball)$ is totally bounded. The finite rank operator form a dense ideal in $B_0(\mathcal{H})$.

Fact: every self-adjoint compact operator can be approximated in norm by finite rank (self-adjoint) operators. Given T, we have

$$T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$$

So if we have T is compact, then T^* is also compact.

Definition 1.56

By a partial isometry on $\mathcal{H}$, we mean an operator W such that W is an isometry on $(kerW)^\perp$, and onto some other closed subspace of $\mathcal{H}$. We have W^*W is the projection on $(kerW)^\perp$, and WW^* is the projection on the range of W.

For $T \in B(\mathcal{H})$, define $|T| = (T^*T)^{\frac{1}{2}}$.

Theorem 1.18 (Polar Decomposition)

For any given $T \in B(\mathcal{H})$, then there is a partial isometry W, such that

$$T = W|T|, |T| = \sqrt{T^*T}$$

Proof For $\xi \in \mathcal{H}$,

$$\|T\xi\|^2 = \langle T\xi, T\xi\rangle = \langle T^*T\xi, \xi\rangle = \langle |T|^2\xi, \xi\rangle = \langle |T|\xi, |T|\xi\rangle = \||T|\xi\|^2$$

We have

$$ker(T) = (rangeT^*)^\perp$$

Hence

$$kerT = ker|T| = (range|T|)^\perp$$

Set for any ξ,

$$W(|T|\xi) = T\xi$$

And $W|T|\xi = T\xi, W|T| = T$. This extends to an isometry from the $\overline{range(|T|)}$, and onto the $\overline{range(T)}$.

Define W such that on $ker(T)$, $W = 0$. If $T \in B_0(\mathcal{H})$, $T^*T \in B(\mathcal{H})$ since it is an ideal. $|T| \in B_0(\mathcal{H})$ as well. If we take $T = W|T|, T^* = |T|W^*$.

From last calss,$\mathcal{A}$ is a C^*-algebra, and ω is a weight on $\mathcal{A}$. We have m_ω, n_ω, and n_ω is a left ideal in $\mathcal{A}$, and m_ω is a right ideal in $\mathcal{A}$. m_a is a two-sided ideal of $\mathcal{A}$, and

$$m_a = n_a^* n_a$$

Definition 1.57

A weight ω is tracial if for all $a \in \mathcal{A}$,

$$\omega(a^*a) = \omega(aa^*)$$

Proposition 1.79

*If $a \in N_\omega$, then $a^* \in n_\omega$. So n_ω is a 2-sided * ideal in $\mathcal{A}$, then m_ω is a 2-sided * ideal in $\mathcal{A}$.*
If $a, b \in n_\omega$, then

$$\omega(ab) = \omega(ba)$$

♠

Proposition 1.80

If $b \in m_\omega^+$, and $a \in \mathcal{A}$, and let $c = \sqrt{b} \in n_\omega$, then

$$\omega(ab) = \omega(acc) = \omega(cac \leq \omega(\|a\|c^2)) = \|a\|\omega(b)$$

Thus $b \in m_\omega^\perp$ determines a continuous linear functional on $\mathcal{A}$ of nomr $\omega(b)$, such that $a \mapsto \omega(ab)$.

♠

Let $\mathcal{A} = B(\mathcal{H})$, and $\omega = \tau$ for $T \in B(\mathcal{H})$, and we define

$$\tau(T) := \sum \langle T\xi_j, \xi_j\rangle$$

for some orthonormal basis, and $\tau(T^*T) = \tau(TT^*)$. We have

$$\tau(T^*T) = \sum \langle T^*T\xi, \xi\rangle = \sum \|T\xi_j\|^2$$

We claim the following:

Lemma 1.5

If $\{b_j\}$ is any other o.n. basis, then

$$\sum \|T\xi_j\|^2 = \sum \|T^* h_j\|^2$$

♡

1.32 Lecture Nov 17

For $B(\mathcal{H})$, and $\{\xi_j\}$ is an o.n.b, and for $T \geq 0$,

$$\tau(T) = \sum \langle T\xi_j, \xi_j\rangle$$

and we would like to know if we have

$$\tau(T^*T) = \tau(TT^*)$$

And let $\{\eta_k\}$ be another set of o.n.b, then for any j, we have

$$T\xi_j = \sum \langle T\xi_j, \eta_k\rangle \eta_k$$

And Parsevel's identity states that

$$\|T\xi_j\|^2 = \sum_k |\langle T\xi_j, \eta_k\rangle|^2$$

And

$$\begin{aligned}
\tau(T^*T) &= \sum \langle T^*T\xi_j, \xi_j\rangle \\
&= \sum \|T\xi_j\|^2 = \sum_j \sum_k |\langle T\xi_j, \eta_k\rangle|^2 \\
&= \sum_k \left(\sum_j |\langle \xi_j, T^*\eta_k\rangle| \right)^2 \\
&= \sum \|T^*\eta_k\|^2 \\
&= \tau(TT^*)
\end{aligned}$$

Corollary 1.14

τ defined above is indeed a trace.

♡

And if we define

$$\eta_\tau = \{T : \tau(T^*T) < \infty\}$$

And $T^*T = |T|^2$, hence equivalently, we have

$$\eta_\tau = \{T : \tau(|T|^2) < \infty\}$$

And thus defines a norm via this inner product by

$$\|T\|_\tau = (\tau(T^*t))^{1/2}$$

Definition 1.58

The opeartors T such that $\tau(T^*T) < \infty$ is called Hilbert-Schmidt operators, denoted as $B_2(\mathcal{H})$, where

$$B_\infty(\mathcal{H}) = B(\mathcal{H}) \text{ with operator norm } \|\cdot\|_\infty$$

If $S \in B(\mathcal{H})$, and $S \geq 0$, then for any $\epsilon > 0$, there is an $\xi \in \mathcal{H}$ such that $\|\xi\| = 1$, and $\|S\xi\| \geq \|S\|_\infty - \epsilon$. So if use ξ so part of an o.n. basis, such that

$$\tau(S) \geq \|S\|_\infty$$

So far any $T \in \mathcal{H}$,

$$\tau(T^*T) \geq \|T^*T\|_\infty = \|T\|_\infty^2$$

-evaluatioIf $T \in B_2(\mathcal{H})$, then

$$\|T\|_{L^2} \geq \|T\|_\infty$$

Proposition 1.81

If $T \in B_2(\mathcal{H})$, i.e. it is a Hilbert-Schmidt operator, then T is compact.

Proof Let $T \in B_2(\mathcal{H})$, and $\tau(T^*T) < \infty$.

$$\tau(T^*T) = \sum_j \|T\xi_j\|^2 < \infty$$

Let $\epsilon > 0$ be given, and there exists N such that if $n \geq N$, then

$$\sum_{j=n}^{\infty} \|T_j\|^2 < \epsilon$$

For any orthogonal projection P of finite rank,

$$\|T - TP\|_\infty \leq \|T - TP\|_2$$

Note that ξ_j is an o.n.b, If we let P be the projection on $\xi_1, ..., \xi_{n-1}$,

$$\sum_{j=1}^{\infty} \|T(1-P)\xi_j\| = \sum_{j=n}^{\infty} \|T\xi_j\| < \epsilon$$

□

Theorem 1.19

The space of Hilbert-Schmidt operators, $B_2(\mathcal{H})$, is complete for the $\|\cdot\|_2$, so $B_2(\mathcal{H})$ is a Hilbert space.

Proof Let $\{T_n\} \subset B_2(\mathcal{H})$ be a Cauchy sequence for $\|\cdot\|_2$. Thus $\{T_n\}$ is Cauchy for the operator norm, and in the operator norm, we have $B(\mathcal{H})$ is complete. So there is a $T \in B(\mathcal{H})$ such that $T_n \to T$ for $\|\cdot\|_\infty$.

Now we want is that it converges for the $\|\cdot\|_2$ norm. Let $\epsilon > 0$ be given, let P be any finite rank projection. And let N be such that if $m, n \geq N$, then $\|T_m - T_n\|_2 < \epsilon$. For $m \geq N$,

$$\|(T_m - T)P\|_2 \leq \|(T_n - T)P\|_2 + \|(T_m - T_n)P\|_2$$

So the second term is bounded by $\epsilon/2$. Now we can choose n such that $\|T - T_n\|_\infty < \epsilon/2$, and $n \geq N$, claim:

$$\|SP\|_2^2 = \sum \|S\xi_n\|^2 \lesssim \|S\|_\infty$$

Hence by choosing n this way, we get the first term is also bounded by $\epsilon/2$.

Now we prove the claim: if $S \in B(\mathcal{H})$, such that $\|SP\|2 < \epsilon$ for all projections.

$$\tau(S^*S) = \sum_{j=1}^{\infty} \|S\xi_j\|^2$$

And you take the partial sum

$$\sum_{j=1}^{k} \|S\xi_j\|^2 = \|SP\|2 < \epsilon$$

All the partial sums are bounded by ϵ, hence

$$\|S\|_2^2 < \epsilon$$

□

Next thing on Monday: every Hilbert Schmidt operator can be expressed by the kernels $K \in L^2(X \times X)$.

1.33 Lecture Nov 20

Proposition 1.82

Let $B_f(\mathcal{H})$ be the space of finite rank operators (range is finite dimension, and can be written as a sum of rank 1 operators), $B_f(\mathcal{H}) \subset B_2(\mathcal{H})$, and $B_f(\mathcal{H})$ is dense in $B_2(\mathcal{H})$ for the $\|\cdot\|_2$ norm.

Proof If $T \in B_2(\mathcal{H})$, then for an o.b. basis $\{\xi_j\}$, we have

$$\sum_{j=1}^{\infty} \|T\xi_j\|^2 < \infty$$

So given $\epsilon > 0$, we can find N such that

$$\sum_{j=N}^{\infty} \|T\xi_j\|^2 < \epsilon$$

Then let P be the orthogonal projection on $\xi_1, ..., \xi_{N-1}$ (P is finite rank), then

$$\|T - TP\|_2^2 = \sum_{j=1}^{\infty} \|(T - TP)\xi_j\|^2$$

for the first $N-1$ temrs, we get 0 contribution, hence

$$\|T - TP\|_2^2 = \sum_{j=N}^{\infty} \|(T - TP)\xi_j\|^2 = \sum_{j=N}^{\infty} \|T\xi_j\|^2 < \epsilon$$

□

Let $\mathcal{H} = L^2(X, \mu)$, and let $K \in L^2(X \times X, \mu \times \mu)$, and let

$$(T_k\xi)(x) = \int_X K(x, y)\xi(y)dy$$

Let $\{\xi_j\}$ be an o.n.b. for $L^2(X, \mu)$, and we would like to show these T_k are Hilbert-Schmidt operators. Then

$$\|T_K\xi_j\|^2 = \sum_{k=1}^{\infty} |\langle T\xi_j, \xi_k\rangle|^2 = \sum_{k=1}^{\infty} |\langle \int K(x, y)\xi_j(y)dy, \xi_k\rangle|^2$$

Summing over j, we get

$$\sum_j \sum_k |\int K(x, y)\xi_j(y)\xi_k(x)dydx|^2 = \sum_{j,k} |\langle K, \xi_j(y)\xi_k(x)\rangle|^2 = \|K\|_{L^2} < \infty$$

And by Parsevel's identity, since $\{\xi_j(y)\xi_k(x)\}$ form an o.n.b for $L^2(X \times X, \mu \times \mu)$, we get the last equality.

Note *You can run the argument backwards, and cna show that every Hilbert-Schimidt opeartor on $L^2(X, \mu)$ is of the form T_k for some $K \in L^2(X \times X)$.*

Recall that m_ω, where ω is tracial, (on n_ω, we showed that there exists an inner product-induced norm), and now we discuss m_ω. Let $\mathcal{A}$ be an algebra, and if $a \in \mathcal{A}$, and $b \in m_\omega$, $b \geq 0$, then

$$|\omega(ab)| \leq \|a\| |\omega(b)\|$$

For a trace τon $B(\mathcal{H})$, we have polar decomposition for $T \in B(\mathcal{H})$. If $T \in m_\tau$, we have $T = V|T|$, where V is a partial isometry, and $|T| = \sqrt{T^*T}$., and

$$V^*T = V^*V|T| = |T|$$

Then for $A \in B(\mathcal{H})$,

$$|\tau(AT)| = |\tau(aV|T|)| \leq \|AV\|_\infty \tau(|T|) \leq \|A\| \tau(|T|)$$

Proposition 1.83

For $A \in B(\mathcal{H})$, $T \in m_\tau$we have

$$|\tau(AT)| \leq \|A\| \tau(|T|)$$

If $T \in B(\mathcal{H})$, and $|T| \in m_\tau$, then $T = V|T|$, and $|T| = V^*T$, so $T \in m_\tau$.

Proposition 1.84

We have

$$m_\tau = \{T \in B(\mathcal{H}) : \tau(|T|) < \infty\}$$

is a two-sided ideal in $B(\mathcal{H})$.

So on m_τ, define

$$\|T\|_1 = \tau(|T|)$$

Proposition 1.85

If $S, T \in m_\tau$, then

$$\|S + T\|_1 \leq \|S\|_1 + \|T\|_1$$

The space $B_1(\mathcal{H})$ which is finite for the 1-norm, is called the trace-class operators.

Proof This would like the 1-norm defined above is indeed a norm. For $S, T \in m_\tau$, we have

$$S + T = W|S + T|$$

Hence $|S + T| = W^*(S + T)$, so

$$\tau(|S + T|) = \tau(W^*(S + T)) = \tau(W^*S) + \tau(W^*T)$$

but note that each term on the right does not need to be nonnegative, even though the sum is nonnegative. Hence

$$\|S + T\|_1 = \tau(|S + T|) \leq |\tau(W^*S)| + |\tau(W^*T)| \leq \|W^*\| |\tau(S)| + \|W^*\| |\tau(T)| = \|S\|_1 + \|T\|_1$$

□

It is therefore natural to move from 1 to p.

Definition 1.59

For $1 \leq p < \infty$, we have

$$B_p(\mathcal{H}) = \{T \in B_0(\mathcal{H}) : \tau(|T|^p) < \infty\}$$

and we define

$$\|T\|_p = (\tau(|T|^p))^{1/p} \text{ is a norm}$$

The space $B_p(\mathcal{H})$ is also called the Shatter ideals. And we define the dual of B_p as B_q via the following relation $\frac{1}{p} + \frac{1}{q} = 1$

Given $\mathcal{H}$, and an o.n.b $\{\xi_j\}$, then you can write an operator as an infinite matrix, and look at the ones that are diagonal.

Then $B_p(\mathcal{H}) \cap$ diagonal matrices $= l^p$.

We first observe that

$$\|T\|_1 \geq \|T\|_\infty$$

Proposition 1.86

$B_1(\mathcal{H})$ is complete for the $\|\cdot\|_1$ norm.

Proof This is quite analogous to the proof $B_2(\mathcal{H})$ is complete for $\|\cdot\|_2$. Let $\{T_n\} \subset B_1(\mathcal{H})$ be a Cauchy sequence for $\|\cdot\|_1$, then it is Cauchy for $\|\cdot\|_\infty$ norm, and so there is $T \in B(\mathcal{H})$ such that

$$\|T - T_n\|_\infty \to 0$$

This implies that $T \in B_0(\mathcal{H})$, and $B_0(\mathcal{H}) \cap$ diagonal $\cong c_0$. Let $\epsilon > 0$ be given, then we can find N such that for $m, n \geq N$,

$$\|T_m - T_n\|_1 < \epsilon$$

For any projection P of finite rank, for a fixed m, we look at the following finite rank operator,

$$\|P(T - T_m)\|_1 = \|PW^*|T - T_m|\|_1 \leq \|PW^*(T - T_n)\|_1 + \|PW^*(T_n - T_m)\|_1$$

(unfinished, the class ended here).

1.34 Lecture Nov 27

We introduced $B(\mathcal{H}), B_2(\mathcal{H}), B_1(\mathcal{H})$. For $T \in B_1(\mathcal{H})$, let $\|T\|_1 = \tau(|T|)$, and we've shown this is a norm. As a theorem, we were in the middle of showing that $B(\mathcal{H})$ is complete for the $\|\cdot\|_1$ norm.

If $S = 0$, then

$$\tau(S) = \sum_{j=1}^{\infty} \langle S\xi_j, \xi_j \rangle$$

If we let P_k be the orthogonal projection onto ξ_k.

$$\tau(PS) = \sum \langle PS\xi_j, \xi_j \rangle = \sum \langle S\xi_j, P\xi_j \rangle = \sum^{k} \langle SP\xi_j, P\xi_j \rangle = \sum^{k} \langle PSP\xi_j, \xi_j \rangle$$

Let $\{T_n\} \subset B(\mathcal{H})$ be a Cauchy sequence for $\|\|_1$, then $\{T_n\}$ is a Cauchy sequence for $\|\|_\infty$. So there is $T \in B(\mathcal{H})$ such that $T_n \to T$ in $\|\|_\infty$. Let $\epsilon > 0$ be given, for any finite dimensional projection P, we choose N so that for $m, n \geq N$, $\|T_m - T_n\|_1 < \epsilon/2$. Choose $m \geq N$. Then we have

$$\tau(P|T - T_m|P) = \tau(PV^*(T - T_m)P) \leq |\tau(PV^*(T - T_n)P)| + |\tau(PV^*(T_n - T_m)P)|$$

We can choose n such that

$$\tau(PV^*(T - T_n)P) < \epsilon/2$$

For $S, A \in B(\mathcal{H})$, then

$$|\tau(AS)| \leq \|A\|_\infty \|S\|_1, |\tau(SA)| \leq \|S\|_\infty \|A\|_1$$

And notice that we have

$$|\tau(PV^*(T_n - T_m)P)| \leq \|OV^*\|_\infty \|T_n - T_m\|_1 \leq \epsilon/2$$

Then we have

$$\tau(P_k|T - T_m|P_k) < \epsilon, \forall P$$

This gives that $\|T - T_m\|_1 < \infty$ so $T - T_m \in B(\mathcal{H})$, $T_m \in B_1(\mathcal{H})$, so $T \in B_1(\mathcal{H})$.

Theorem 1.20

The dual Banach space to $B_0(\mathcal{H})$ is, via the map $|\tau(AT)\| \leq \|A\|\|T\|_1$, $B_1(\mathcal{H})$. This implies that for all

$T \in B_1(\mathcal{H})$, define

$$\varphi_T(A) = \tau(AT)$$

for all $A \in B_0(\mathcal{H})$, with $\|\|_\infty$.
In other words, every element is dual of $B_0(\mathcal{H})$ is of the form φ_T for $T \in B_1(\mathcal{H})$.

Remark The dual of c_0 is l^1.
Proof Let $B_0(\mathcal{H}) \supset B_2(\mathcal{H})$, and if $A \in B_2(\mathcal{H})$, we have

$$\varphi(A) \leq \|\varphi\|\|A\|_\infty \leq \|\varphi\|\|A\|_2$$

So there is a $T \in B_2(\mathcal{H})$, such that

$$\varphi(A) = \langle A, T\rangle_2 = \tau(AT^*)$$

To show that $T \in B_1(\mathcal{H})$. Note that we have

$$|\varphi(P|T|)| = |\varphi(PV^*T)| = |\varphi(PV^*)| \leq \|\varphi\|\|PV^*\| = \|\varphi\|$$

Then we have

$$|\tau(P|T|)| = |\tau(PV^*T)| = |\varphi(PV^*)| \leq \|\varphi\|\|PV^*\|_\infty \leq \|\varphi\|$$

for all P. Hence $T \in B_1(\mathcal{H})$, and $\|T\|_1 \leq \|\varphi\|$.

□

Theorem 1.21

The dual of $B_1(\mathcal{H})$ is $B(\mathcal{H})$ (note that the dual of l^1 is l^∞).

Corollary 1.15

The unit ball of $B(\mathcal{H})$ is compact for the weak- topology. By definition, a von Neumann algebra is a unital *-algebra of $B(\mathcal{H})$ that is closed for the weak-* topology. Then we look at the set*

$$\{\varphi \in B_1(\mathcal{H}) : \varphi(N) = 0\} = N^\perp$$

And

$$\left(B_1(\mathcal{H})/N^\perp\right)^* = N$$

Hence every von Neumann algebra has a pre-dual, hence the closed unit ball in a von Neumann algebra is compact in the weak-* topology.

Proposition 1.87

The above is the ultra-weak opeartor topology on $B(\mathcal{H})$. If we view that $B_f(\mathcal{H}) \subset B_1(\mathcal{H})$, and use elements of $B_f(\mathcal{H})$ to get linear functionals on $\mathcal{B}(\mathcal{H})$, that is the weak operator topology.

1.35 Last Lecture

One can show that

$$\tau(\langle \eta, \eta\rangle_0) = \langle \eta, \xi\rangle_{\mathcal{H}}$$

Definition 1.60

Let V, W be vector spaces, T is said to be Fredholm if $\dim(ker(T)) < \infty$, and $\dim(coker(T)) < \infty$, where $coker = W/range(T)$.

Definition 1.61

The index of T is the dimension of $Ker(T) - coKer(T)$.

Proposition 1.88

If for $i = 1, 2$, $T_i : V_i \to W_i$, *and*

$$T_1 \bigoplus T_2 : V_1 \bigoplus V_2 \to W_1 \bigoplus W_2$$

then we have $T_1 \bigoplus T_2$ *is Fredholm and* $index(T_1 \bigoplus T_2) = index(T_1) + index(T_2)$.

Notation: $Fred(V, W) =$ set of F such that Fredholm from V to W.

Proposition 1.89

If $T \in F(V, W)$, *and if* $S \in F(W, Z)$, *then* $ST \in Fred(V, Z)$, *and* $index(ST) = index(S) + index(T)$.

Proof Exercise.

Proposition 1.90

If $T \in End(V, W)$, *and if* $index(T) = 0$, *then either (1)* T *is invertible, or (2)* $kernel(T) \neq \{0\}$, *and* $range(T) \neq V$.

In (1), the equation $Tx = w$ has a solution, and it is unique. In (2), the equation $Tx = w$ won't have a solution for some w. If it has a solution to some w, then the solution is not unique (you can add something in the kernel). There exists $w_1, ..., w_j$, such that $Tx_j = w_j$ has a solution, for $j = 1, ..., n$.

If V, W are Banach spaces, let $Fred(V, W)$ be the bounded Fredholm operators.

Theorem 1.22

If $T \in Fred(V, W)$, *and if* $K \in B_0(V, W)$, *then* $T + K \in Fred(V, W)$. *And we have* $index(T + K) = index(T)$.

Corollary 1.16

Let $K \in B_0(V)$, *then* $\lambda \neq 0$, *then if we look at* $\lambda I - K$ *is Fredholm, and* $index(\lambda I - K) = 0$.
So either $(\lambda I - K)$ *is invertible, so* $ker(\lambda I - K) \neq \{0\}$, *i.e.* λ *is an eigenvalue for* K, *and* $\lambda \in \sigma(K)$.

Theorem 1.23

For $K \in B_0(V)$, *and* λ *an eigenvalue, is* $(\lambda I - K)$ *is not invertible, there is an* $N < \infty$ *such that* $ker(\lambda I - K)^{N+n} = ker(\lambda I - K)^N$.

Definition 1.62

Psedodifferential operators: On $\mathbb{R}^n$, *given by a symbol on* $\mathcal{S}(\mathbb{R}^{2n})$, *define* T_a *on* $\mathcal{S}(\mathbb{R}^n)$, *and for* $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{T_a f(x)} = a(x, t)\hat{f}(t)$$

On $l^2(\mathbb{N})$, let T be the left shift operator, $Te_n = e_{n+1}$, the kernel is 0, but the cokernel is $\{e_1\}$. This is Fredholm of index -1.

Remark $B_(\mathcal{H})/B_0(\mathcal{H})$, and this is called the cackelim algebra, is a C^*-algebra. $T \in F(\mathcal{H})$ if and only if the image of T is invertible. Index exactly consists of $GL(calkim)$.

Remark The following isomorphisms: $B(\mathcal{H}) \sim l^\infty$, and $B_0(\mathcal{H}) \sim c_0$, and $B(\mathcal{H})/B_0(\mathcal{H}) \sim l^\infty/c_0$.