Calc III Sections

Fall 2025

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November 11, 2025

Calc III-Week 12 (11/10-14)

Definition 1 (path integral)**.** Let $c : [a,b] \to \mathbb{R}^3$ be a path of C^1 and $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $f \circ c$ is continuous on $[a,b]$, The **path integral** of $f(x,y,z)$ along the path c is given by

$$\begin{aligned}\int_c f ds &= \int_a^b f(c(t))\|c'(t)\|dt \\ &= \int_a^b f(x(t),y(t),z(t))\|c'(t)\|dt\end{aligned}$$

Definition 2 (line integral)**.** Let F be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$, where $c(t) = (x(t), y(t), z(t))$. We define $\int_c F \cdot ds$, the **line integral** of F along c by the following

$$\begin{aligned}\int_c F \cdot ds &= \int_a^b F(c(t)) \cdot c'(t)dt \\ &= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt \\ &:= \int_c F_1 dx + F_2 dy + F_3 dz\end{aligned}$$

the expression $F_1 dx + F_2 dy + F_3 dz$ is called the **differential form**.

Example 1 (work done along a path)**.** The force field F on a particle moving along a path $c : [a,b] \to \mathbb{R}^3$ is given by

$$\text{work done by } F = \int_a^b F(c(t)) \cdot c'(t)dt$$

Definition 3 (reparametrization)**.** Let $h : I \to I_1$ be a C^1 real-valued bijective function. Let $c : I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition

$$p = c \circ h : I \to \mathbb{R}^3$$

a **reparametrization** of c.

Example 2. Let $c : [0,1] \to \mathbb{R}^3$ be a C^1 path, then consider $h : [0,1] \to [0,1]$, where $h(t) = 1 - t$. Then the path

$$c_{\text{op}} = c \circ h(t) = c(1-t)$$

is the same path in the opposite direction.

Proposition 1 (reparametrization for path integrals)**.** Let c be a C^1 path and c' be any reparametrization of c, and let f be a continuous function on the image of c, then

$$\int_c f(x,y,z)ds = \int_{c'} f(x,y,z)ds$$

Proposition 2 (reparametrization for line integrals)**.** Let F be a vector feld continuous on the C^1 path $c : [a, b] \to \mathbb{R}^3$, and let $c' : [a', b'] \to \mathbb{R}^3$ be a reparametrization of c. If c' is orientation-preserving, then

$$\int_{c'} F \cdot ds = \int_c F \cdot ds$$

If c' is orientation-reversing, then

$$\int_{c'} F \cdot ds = -\int_c F \cdot ds$$

Proposition 3 (fundamental theorem of line integrals)**.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is of C^1 and that $c : [a, b] \to \mathbb{R}^3$ is piecewise C^1. Then

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Problem 1. Let $c : [a, b] \to \mathbb{R}^3$ be a C^1 path, find a reparametrization $\tilde{c} = c \circ h$, such that $\tilde{c} : [0, 1] \to \mathbb{R}^3$.

Proof. Define $h(t) = a + (b - a)t$, where $0 \le t \le 1$. □

Problem 2. Suppose that $\nabla f(x, y, z) = (2x^2, y, -z)$, and that $f(0, 1, -1) = 2$. What is the value of $f(2, 1, 1)$?

Proof. By the fundamental theorem of line integrals, we know

$$\begin{aligned} f(2,1,1) &= \int_\ell \nabla f \cdot ds + f(0,1,-1) \\ &= \int_0^1 \nabla f(2t, 1, 2t-1) \cdot (2,0,2) dt + f(0,1,-1) \\ &= \int_0^1 \left(16t^2 + 2 - 4t\right) dt + 2 \\ &= \frac{22}{3} \end{aligned}$$

□

Problem 3. Compute the path integral of $f(x, y, z) = 2x + y - z$ over the curve c, where c is the intersection of the two surfaces below:

$$y = x, \quad y^2 + z^2 = 4$$

Hint: first find a parametrization of c, then use the formula of path integral.

Proof. Using polar coordinates, we can write $y = 2\cos\theta$, $z = 2\sin\theta$, and since $y = x$, we know $x = 2\cos\theta$, thus the parametrization is given by

$$(2\cos\theta, 2\cos\theta, r\sin\theta), \quad 0 \le \theta \le 2\pi$$

Now

$$\int_0^{2\pi} 2(2\cos t) + 2\cos t - 2\sin t dt = 0$$

□

Problem 4. Suppose path c has length l, and F is a vector field such that $\|F\| \leq M$. Prove that

$$\left|\int_c F \cdot ds\right| \leq Ml$$

(Hint: Cauchy-Schwarz.)

Proof.

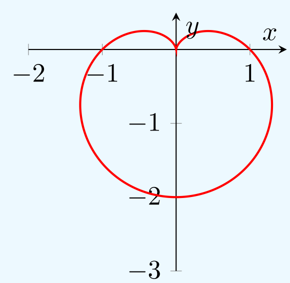
$$\begin{aligned}\left|\int_c F \cdot ds\right| &= \left|\int_a^b F(c(t)) \cdot c'(t)dt\right| \\ &\leq \int_a^b |F(c(t)) \cdot c'(t)|\, dt \\ &\leq \int_a^b \|F(c(t))\| \cdot \|c'(t)\| dt \\ &\leq Ml\end{aligned}$$

□

Problem 5 (Group work). Given vector field $F(x, y) = (1, 1)$, and the following heart is given by the parametrization:

$$c(t) = ((1 - \sin t)\cos t, (1 - \sin t)\sin t) \quad \text{where } 0 \leq t \leq 2\pi$$

Group 1: what is the work done by going up the top of the heart? ($0 \leq t \leq \pi$).
Group 2: what is the work done by going up the bottom of the heart? ($\pi \leq t \leq 2\pi$).



Proof. No need to actually compute $c'(t)$! For Group 1, we see that

$$\begin{aligned}\int_0^\pi F(c(t)) \cdot c'(t)dt &= \int_0^\pi x'(t) + y'(t)dt \qquad (c(t) = (x(t), y(t))) \\ &= x(\pi) + y(\pi) - x(0) - y(0) \\ &= -2\end{aligned}$$

Similarly,

$$\int_\pi^{2\pi} F(c(t)) \cdot c'(t)dt = 2$$

Thus combined

$$\int_0^{2\pi} F(c(t)) \cdot c'(t)dt = 0$$

□