

# Calc III Section Notes with Answers

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# Chapter 1

## The Geometry of Euclidean Spaces

### Week 1 (1/19-23)

#### Logistics

- TA: Hui.
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- Drop-in Hours: Tuesday 2-3 PM, 4-5 pm, Krieger 211.
- Biweekly Quizzes: 10%.
- Attendance: 5%. (If you can't make it, email me).

**Definition 1.1** (standard basis of  $\mathbb{R}^3$ ). The following vectors

$$i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$$

are called the **standard basis** vectors of  $\mathbb{R}^3$ , and for any vector  $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ , we can write

$$a = a_1 i + a_2 j + a_3 k$$

**Definition 1.2** (dot product). Let  $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$ , the **dot product**  $v \cdot w$  is given by

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Alternatively,

$$v \cdot w = \|v\| \|w\| \cos \theta$$

where

$$\theta = \arccos \left( \frac{v \cdot w}{\|v\| \|w\|} \right)$$

**Definition 1.3** (length of vector). Let  $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ , the **length** or **norm** of  $v$ , denoted as  $\|v\|$ , is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v \cdot v}$$

**Definition 1.4** (linear combination). Let  $v, w \in \mathbb{R}^3$ , a **linear combination** of  $v, w$  is a sum

$$av + bw$$

for some  $a, b \in \mathbb{R}$ . One can generalize this definition to  $n$  vectors: let  $v_1, v_2, \dots, v_n \in \mathbb{R}^3$ , a linear combination of these vectors is a finite sum

$$a_1v_1 + a_2v_2 + \dots + a_nv_n$$

for some  $a_i \in \mathbb{R}, 1 \leq i \leq n$ .

**Proposition 1.1** (properties of the dot product). Let  $a, b, c \in \mathbb{R}^n$ , then

(a) Nonnegativity:  $a \cdot a \geq 0$ , and  $a \cdot a = 0$  if and only if  $a = 0$ .

(b) Scalar multiplication: let  $\lambda \in \mathbb{R}$ , then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry:  $a \cdot b = b \cdot a$ .

**Problem 1.1.** Draw the following vectors in  $\mathbb{R}^2$ :

$$u = (1, 2), \quad v = (3, -2)$$

Compute  $u + v, u - v$ , and draw them in the plane.

*Proof.*

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

**Problem 1.2.** Consider the following vectors in  $\mathbb{R}^3$ :

$$u = (1, 2, 3), \quad v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors  $a, b \in \mathbb{R}^3$  are called **orthogonal** if  $a \cdot b = 0$ . Are  $u, v$  orthogonal? If not, find a nonzero vector orthogonal to  $u$ .

*Proof.* 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthogonal to  $u$ :  $(-3, 0, 1)$ . Note that this vector is **not** unique! For example,  $(-1, -1, 1)$  is another such vector.

□

**Problem 1.3.** Can you express  $w = (1, 2)$  as a linear combination of  $v_1, v_2$  for different choices of  $v_1, v_2$ ?

1.  $v_1 = (1, 1), v_2 = (-2, -2)$ .
2.  $v_1 = (2, 1), v_2 = (-1, 0)$ .

*Proof.* 1. We first note that  $(1, 1), (-2, -2)$  lie on the same line through the origin. Hence, any linear combination of  $v_1, v_2$  will stay in this line, i.e., of the form  $(a, a)$ , for some  $a \in \mathbb{R}$ . Therefore, it is impossible to write  $w = (1, 2)$  as a linear combination of  $v_1, v_2$ .

2. Suppose  $w = a_1 v_1 + a_2 v_2$  for some  $a_1, a_2 \in \mathbb{R}$ , then

$$\begin{cases} 2a_1 - a_2 = 1 \\ a_1 = 2 \end{cases} \Rightarrow \begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

Thus we can write  $w$  as a linear combination of  $v_1, v_2$ :

$$w = 2v_1 + 3v_2$$

□

**Problem 1.4.** Let  $u, v \in \mathbb{R}^3$ , suppose that  $u, v$  are orthogonal, show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming  $\|u + v\|^2 = \|u\|^2 + \|v\|^2$ , is it true that  $u \cdot v = 0$ ?)

*Proof.* We have

$$\begin{aligned} \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

because  $u \cdot v = v \cdot u = 0$ . The converse is also true: we know by definition that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

given the assumption, we also have

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u \cdot v = 0$$

□

## Week 2 (1/26-30)

Topics: determinant, cross product.

**Definition 1.5** (determinant). Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a  $2 \times 2$  matrix, the **determinant** of  $A$  is given by

$$\det(A) = ad - bc$$

Let  $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$  be a  $3 \times 3$  matrix, the **determinant** of  $A$  is given by

$$\det(A) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

**Definition 1.6** (cross product). Let  $a, b \in \mathbb{R}^3$ , write  $a = (a_1, a_2, a_3)$ ,  $b = (b_1, b_2, b_3)$ , then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where  $i, j, k$  are the standard vectors in  $\mathbb{R}^3$ .

**Proposition 1.2** (properties of the cross product). We have the following properties regarding the cross product: let  $a, b \in \mathbb{R}^3$ ,

1.  $a \times a = 0$ .
2.  $a \times b = -b \times a$ .
3.  $(a + b) \times c = a \times c + b \times c$ , and  $a \times (b + c) = a \times b + a \times c$ .
4.  $(\alpha a) \times b = \alpha(a \times b)$  for any  $a \in \mathbb{R}$ .
5.  $a \times b$  is perpendicular to vectors  $a, b$ .
6. The length of the cross product is the area of the parallelogram spanned by  $a, b$ :

$$\|a \times b\| = \|a\|\|b\|\sin \theta$$

where  $0 \leq \theta \leq \pi$  is the angle between them.

7.  $a \times b = 0$  iff  $a, b$  are parallel or either  $a$  or  $b$  are 0.
8. The cross product is **not** associative! For example, compute

$$(i \times i) \times j, \quad i \times (i \times j)$$

**Proposition 1.3 (determinant and linear combination).** Let  $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$  be a  $3 \times 3$  matrix, let

$$a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$$

If any of  $a, b$ , or  $c$  is a linear combination of the other two vectors, then  $\det(A) = 0$ . (Relevant topic: linear independence).

**Problem 1.5.** Let  $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$  be vectors in  $\mathbb{R}^3$ , compute the area of the parallelogram spanned by these two vectors.

*Proof.*

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

**Problem 1.6.** Compute the determinant of the following matrix  $A$ :

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

*Proof.* Notice that the third row vector  $(3, 1, 1)$  is the sum of the two row vectors above, hence by Proposition 1.3, we know we must have  $\det(A) = 0$ . □

## Week 2 (Section Activity)

First, introduce yourself to one another:

1. Name.
2. Pronouns.
3. Year.
4. Major.
5. If you could have a noncat, nondog pet, what would it be?

Next, work on and **discuss** the following problems!

**Problem 1.7.** Let  $u = (1, 2, -1)$ , find a nonzero vector that is orthogonal to  $u$ .

*Proof.* There are many choices, for example,  $v = (1, 0, 1)$ . □

**Problem 1.8.** Let  $u = (1, 0)$ ,  $v = (2, -1)$ ,

1. Using the dot product, what is the angle between  $u, v$ ? (You do not need to simplify your answer).
2. Can you find a nonzero vector in  $\mathbb{R}^2$  that is orthogonal to both of  $u, v$ ?
3. Can you find nonzero vectors that are orthogonal to any basis of  $\mathbb{R}^2$ ? (Hint: take  $u = (1, 0)$ , can you draw the set of vectors orthogonal to  $u$ ? A basis of  $\mathbb{R}^2$  contains two noncolinear vectors).

*Proof.* 1. We have

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|} = \frac{2}{\sqrt{5}}$$

Hence

$$\theta = \arccos\left(\frac{2}{\sqrt{5}}\right)$$

2. Suppose there exists a nonzero vector  $w = (w_1, w_2) \in \mathbb{R}^2$  such that  $w \cdot u = w \cdot v = 0$ , then this implies

$$w \cdot u = w_1 = 0, \quad w \cdot v = 2w_1 - w_2 = 0 \Rightarrow w_1 = w_2 = 0$$

Thus the only vector orthogonal to  $u, v$  is the zero vector, hence the answer is no.

3. Let  $\{a, b\}$  be a basis of  $\mathbb{R}^2$ , where  $a = (a_1, a_2)$ ,  $b = (b_1, b_2)$ , we know they are noncolinear, hence

$$\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$$

Then suppose  $w$  is orthogonal to both  $a, b$ . Then (one can draw a picture),  $w$  should be of the form

$$w = (\lambda a_2, -\lambda a_1) = (\zeta b_2, -\zeta b_1)$$

for some  $\lambda, \zeta \in \mathbb{R}$ . But this implies

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{\zeta}{\lambda}$$

which is a contradiction to the above. This concludes the proof that no nonzero vector can be orthogonal to a basis in  $\mathbb{R}^2$ . □



**Problem 1.9.** Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Compute the determinant of  $A$ .

*Proof.* We have  $\det(A) = 10$ . □

**Problem 1.10.** Construct a nonzero  $3 \times 3$  matrix  $A$  such that  $\det(A) = 0$ .

*Proof.*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$$

□

### Week 3 (2/2-2/6)

Topics: parametric equations, multivariable functions, and level sets.

**Definition 1.7** (matrix addition, multiplication). Let  $A, B$  be  $m \times n$  matrices,  $C$  be  $n \times k$  matrix as follows

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$$

$$C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nk} \end{pmatrix}$$

then **matrix addition**  $A + B$  defined as

$$A + B := \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & \cdots & a_{mn} + b_{mn} \end{pmatrix}$$

and **matrix multiplication** is defined as

$$AC = \begin{pmatrix} \sum_{j=1}^n a_{1j}c_{j1} & \cdots & \sum_{j=1}^n a_{1j}c_{jk} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^n a_{mj}c_{j1} & \cdots & \sum_{j=1}^n a_{mj}c_{jk} \end{pmatrix}$$

**Remark.** Given matrices  $A, B$ , for the matrix multiplication  $AB$  to be well-defined,  
number of columns of  $A$  = number of rows of  $B$

**Definition 1.8** (matrix as linear transformation). Let  $A$  be an  $m \times n$  matrix

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

and let  $x \in \mathbb{R}^n$ , where  $x = (x_1, \dots, x_n)$ . Then  $A$  applied to  $x$  as a linear transformation is given by

$$Ax = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{pmatrix}$$

where

$$\sum_{j=1}^n a_{1j}x_j = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \quad \dots$$

**Definition 1.9** (image, graph). The **image** of a function  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a subset of  $\mathbb{R}^m$ ,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in U\}$$

and the **graph** of  $f$  is a subset of  $\mathbb{R}^{n+m}$ ,

$$\text{Graph}(f) = \{(x, f(x)) : x \in U\}$$

**Definition 1.10** (level set). Let  $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ , and  $c \in \mathbb{R}$  be some constant. Then the **level set** of  $f$  at  $c$  is the set  $\mathcal{L}_c$

$$\mathcal{L}_c := \{x \in U : f(x) = c\} \subset \mathbb{R}^n$$

**Definition 1.11** (Equation of a line). A **line**  $l$  in  $\mathbb{R}^3$  through the tip of  $a = (a_1, a_2, a_3)$  pointing in the direction of a vector  $v = (v_1, v_2, v_3)$  is given by

$$l(t) = a + tv = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3)$$

where  $t \in \mathbb{R}$ . Alternatively, a line passing through two points  $P = (x_1, y_1, z_1), Q = (x_2, y_2, z_2)$  is given by

$$l(t) = (x(t), y(t), z(t))$$

where

$$\begin{cases} x(t) = x_1 + (x_2 - x_1)t \\ y(t) = y_1 + (y_2 - y_1)t \\ z(t) = z_1 + (z_2 - z_1)t \end{cases}$$

**Definition 1.12** (Plane in  $\mathbb{R}^3$ ). If a plane  $P$  passes through some point  $(x_0, y_0, z_0)$ , and  $n = (A, B, C)$  is a vector orthogonal to the plane, then the plane  $P$  is given by the equation:

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

**Remark.** A point  $P$  in the plane and a normal vector to the plane uniquely determine a plane in  $\mathbb{R}^3$ . Equivalently, three noncolinear points uniquely determine a plane.

**Problem 1.11.** Compute the plane containing all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

*Proof.* Let  $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$ , then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point  $A$ , we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

□

- Problem 1.12.** (a) Find the equation of the line through  $(1, 1, 0)$  in the direction of  $2\mathbf{i} - \mathbf{k}$ .  
 (b) Find the equation of the line passing through  $(0, 1, 1)$  and  $(0, 1, 0)$ .  
 (c) Find an equation for the plane perpendicular to the vector  $(-1, 1, -1)$  and passing through the point  $(1, 1, 1)$ .

*Proof.* (a) The equation is given by

$$l_1(t) = (1, 1, 0) + t(2, 0, -1), \quad t \in \mathbb{R}$$

(b) The equation is given by

$$l_2(t) = (0, 1, 1) + t(0, 0, -1), \quad t \in \mathbb{R}$$

(c) The equation of the plane is given by

$$(x - 1, y - 1, z - 1) \cdot (-1, 1, -1) = 0$$

simplifying we get

$$x - y + z = 1$$

□