Functional Analysis

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Chapter 1

Preliminary

1.1 9/3 lecture

Definition 1.1 (orthonormal basis). Let S be an orthonormal set in the Hilbert space such that no other orthonormal set contains S as a proper subset. Then S is called an orthonormal basis.

Proposition 1.1. Every Hilbert space admits an orthonormal basis.

Proof. Zorn's lemma. □

Remark: if H is separable, i.e., H has a countable dense subset, then the proof does not require Zorn's lemma. For example, L^2 is separable.

Proposition 1.2 (II.6, Parsevel's formula). Let $\mathcal{H}$ be a Hilbert space, and $S = \{x_n\}$ be an orthonormal basis, then for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha \in \mathcal{A}} (x_\alpha, y) x_\alpha, \quad \|y\|^2 = \sum |(x_n, y)|^2$$

where $\mathcal{A}$ is an index set.

Proof. Bessel's inequality states that for any $\mathcal{A}' \subset \mathcal{A}$ finite, we have

$$\sum_{\alpha \in \mathcal{A}'} |(x_\alpha, y)|^2 \leq \|y\|^2 < \infty$$

It follows that $|(x_\alpha, y)| > \frac{1}{n}$ for at most finitely many α's, and $|(x_\alpha, y)| \neq 0$ for at most countably many α's. Let $\{\alpha_i\}_{i=1}^\infty$ be an enumeration of such α's. Then

$$\sum_{i=1}^N |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$$

which implies

$$\sum_{i=1}^\infty |(x_{\alpha_i}, y)|^2 < \infty$$

Let

$$y_n = \sum_{i=1}^n (x_{\alpha_i}, y) x_{\alpha_i},$$

we would like to show that the sequence $\{y_n\}$ is cauchy,

$$\|y_n - y_m\|^2 = \left\| \sum_{i=m+1}^{n} (x_{\alpha_i}, y) x_{\alpha_i} \right\|^2 \to 0 \text{ as } m \to \infty$$

Thus $\{y_n\}$ is Cauchy. In other words,

$$y_n \to y = \sum_{i=1}^{\infty} (x_{\alpha_i}, y) x_{\alpha_i}$$

□

Definition 1.2. A metric space is separable if it has a countable dense subset.

Proposition 1.3 (II.7). Let $\mathcal{H}$ be a Hilbert space, then it is separable iff it has a countable orthonormal basis.

Proof. Suppose $\mathcal{H}$ is separable, let $\{x_n\}$ be a countable dense set, then we throw out terms in $\{x_n\}$ until we get a linearly indepedent dense subset $\{u_n\} \subset \{x_n\}$. Applying Gram-Schmidt, we can assume $\{u_n\}$ to be countable and orthonormal. Conversely, if $\{u_n\}$ is a countable orthonormal basis, then the set of linear combinations of $\{u_n\}$ with rational coefficients forms a countable dense subset of $\mathcal{H}$. □

Definition 1.3 (Fourier Coefficient). The nth Fourier coefficient of a 2π-periodic function f is

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x) dx$$

The Fourier series of f is

$$\tilde{f}(x) = \lim_{M \to \infty} \sum_{M=-N}^{N} \frac{1}{\sqrt{2\pi}} c_n e^{inx}$$

Proposition 1.4. The Fourier series $\sum_k c_k$ converges if $f \in L^2$. Moreover, the series converges uniformly to a continuous function if $\sum |c_k| < \infty$

I am too lazzy to type it up, but it uses the fun lemma below:

Lemma 1.1. Suppose f is 2π-periodic, and $(f, e^{inx}) = 0$ for all n, then $f \equiv 0$. (In other words, if all the Fourier coefficients are 0, then the function must be identically zero).

1.2 9/8 Lecture

Definition 1.4 (Banach space). A complete normed linear space is called a Banach space.

Example 1.1. 1. $L^\infty(\mathbb{R}) = \{f : f(x) \leq M \text{ a.e. }\}$, where $\|f\|_\infty$ is the smallest such M, is a Banach space.

2. Let $C(\mathbb{R})$ be the bounded continuous functions on $\mathbb{R}$. Let $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and equip it with the same norm. Moreover, $C(\mathbb{R})$ is also a Banach space (due to the uniform convergence of continuous functions is still continuous).

3. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support, and this is not a Banach space under $\|\cdot\|_\infty$.

4. L^p is complete for all $1 \leq p < \infty$.

5. Let $a = \{a_n\}$ be a sequence of complex numbers, ad

$$\|a\| = \sup_n |a_n| < \infty$$

let $c_0 = \{\lim_{n\to\infty} a_n = 0\}$, $s = \{\lim_{n\to\infty} n^N a_n = 0 \forall N\}$, and $l_p = \{\|a\|_p^p = \sum_{n=1}^\infty |a_n|^p < \infty\}$. Note that the space

$$f = \{a_n = 0 \text{ for al but finitely many } n\}$$

is not complete! However, it is a dense subset in l^p. Morever, the set of elements in f with rational coefficients, and the closure of f in s, l^p, c_0 are exactly the whole spaces, i.e., s, l^p, c_0 are separable.

6. Let $L(X,Y)$ be bounded linear operators from X, Y, with the operator norm, and $L(X,Y)$ is a Banach psace.

Proposition 1.5. Let $L^p(\mathbb{R})$, where $1 \leq p < \infty$ be the space of functions with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^p$$

then

1. (Minkowski's inequality) $\|f\|_p \leq \|f\|_p + \|g\|_p$.
2. (Riesz-Fischer) L^p is complete.
3. (Holder) Given $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

if $f \in L^p, g \in L^q$.

Proposition 1.6. If Y is complete, then $L(X,Y)$ is a Banach space.

Proof. Suppose $\{A_n\}$ is Cauchy, now we construct the limit: for each x, $A_n x = y_n$ is a Cauchy sequence:

$$\|y_n - y_m\| \leq \|A_n - Am\| \cdot \|x\|$$

Now since Y is complete, we know that $A_n x \to y$. Let $Ax = y$. (This is our limit)! Now $\|A_n\| \leq C$ for all n, which implies $\|A\| \leq C$. Thus $L(X,Y)$ is complete! □

1.2.1 Duals

Definition 1.5 (dual space). The space of bounded linear functionals $L(X, \mathbb{C})$, where X is Banach, is called the dual space to X, denoted by X^*. Let $f \in X^*$, then define the norm

$$\|f\| = \sup_{x \in X, \|x\| \leq 1} |f(x)|$$

Example 1.2. 1. Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L^q$, then

$$G(f) = \int_{-\infty}^{\infty} \bar{g}(x) f(x) dx$$

Then G is in $(L^p)^*$. Moreover, any such linear functional on L^p can be written in this way for some $g \in L^q$. And

$$|G(f)| \leq \|f\|_p \|g\|_q$$

by Holder. Moreover,

$$L^q(\mathbb{R})^* = L^p, (L^q(\mathbb{R})^*)^* = L^q$$

because L^q is reflexive! In particular, L^2 is its own dual space.

2. Suppose $\{\lambda_k\} \subset l^q$, then

$$\Lambda(\{a_k\}) = \sum_k \lambda_k a_k$$

is a bounded linear functional on l^p. Thus

$$l_q \subset (l^p)^*$$

for $1 \leq p \leq \infty$. It turns out every linear functional on l^p can be written in this form.

Example 1.3. Let $p = 1$, we have

$$L^1(\mathbb{R})^* = L^\infty, \text{ but } L^\infty(\mathbb{R})^* \neq L^1(\mathbb{R})$$

in fact $L^\infty(\mathbb{R})^*$ is bigger.

1.3 9/10 Lecture

Proposition 1.7 (Geometric Hahn-Banach). Let V_1 be a subspace of V, $x \in V \setminus V_1$, then one can find a hyperplane (codim 1) V_2 such that $V_1 \subset V_2$, and $x \notin V_2$.

Proof. If V_1 has codim$V_1 = 1$, then we are done. Suppose that codim $V_1 > 1$, we would like to find V_2 such that $x \notin V_2$, where $V_2 \neq V_1$ such that $\dim(V/V_1) > 1$. Note that we define

$$\dim(V/V_1) = \{[z] : [z] = [w] \iff z + w \in V_1\}$$

(For any banach space, we can write $B = W \oplus (B/W)$). This implies that we can find $y = [y] \in V/V_1$ such that $y \not\emptyset$, and $y \neq x$ (by codim > 1). Set

$$V_2 = \{z + ty; z \in V_1, t \in \mathbb{R}\}$$

Then we can continue this process, and using Zorn's lemma, we can have V_2 to have codim 1. $\square$

Definition 1.6. A subset $A \subset V$ is called if for any $x, y \in V$, the line connecting them is contained in A. If the set is also open, then we call A convex linearly open.

Proposition 1.8 (Geometric HB for Convex sets)**.** Let $A \subset V$ be convex linearly open, and let V_1 be a linear subspace which does not intersect A. Then there is a hyperplane V_2 such that $V_1 \subset V_2$ and $V_2 \cap A = \varnothing$.

(Essentially proof by picture).

Proposition 1.9 (Hahn-Banach)**.** Let X be a real vector space, for all $x, y \in X$, and $\alpha \in [0, 1]$, with sublinear functional $p(x)$ satisfying

$$P(\alpha x + (1 - \alpha)y) \leq \alpha p(x) + (1 - \alpha)p(y)$$

Suppose that λ is a linear functional defined on a subspace on Y such that $\lambda(y) \leq p(y)$ for all $y \in Y$. Then there is a linear functional Λ on X such that $\Lambda = \lambda$ on Y, and

$$\Lambda(x) \leq p(x)$$

Proof. Let $x \in Y \setminus Y$, we will first show that we can extend λ to the subspace spanned by Y and z, following the same bound. Define

$$\tilde{\lambda}(az + y) = a\tilde{\lambda}(z) + \lambda(y)$$

Suppose that $y_1, y_2 \in Y$, and $\alpha, \beta > 0$, and

$$\begin{aligned} \beta\lambda(y_1) + \alpha\lambda(y_2) = \lambda(\alpha y_1 + \beta y_2) &= (\alpha + \beta)\lambda\left(\frac{\beta}{\alpha + \beta}y_1 + \frac{\alpha}{\alpha + \beta}y_2\right) \\ &\leq (\alpha + \beta)p(\dots) \\ &\leq \beta p(y_1 - \alpha z) + \alpha(y_2 + \beta z) \end{aligned}$$

deviding both sides by α, β, taking the sup over $\alpha > 0, y$, we see that

$$\tilde{\lambda}(x) \leq p(x)$$

for all x in this subspace. Using Zorn's lemma, we extend one subspace at a time, then we are done. □

1.4 9/15 Lecture

Proposition 1.10 (Geometric Hahn-Banach)**.** Let A be complex and linearly open, and $A \subset V$ be a vector space over $\mathbb{R}$. Let V_1 be a subspace of V, $V_1 \cap A = \varnothing$. Then there exists a hyperplane V_2 such that $V_2 \cap A = \varnothing, V_1 \subset V_2$.

Definition 1.7. A seminorm p is such that $p(x) \geq 0, p(x + y) = p(x) + p(y)$, and $p(\alpha x) = |\alpha|p(x)$.

And we have the following analytic version of Hahn-Banach.

Proposition 1.11 (Analytic Hahn-Banach)**.** Let W be a subspace of V, and f linear form on W such that $|f(x)| \leq p(x)$, for all $x \in W$. Then there is a linear form f_1 such that $f_1(x) = f(x)$ on W, and $|f_1(x)| \leq p(x)$ for all x.

I am too lazy to follow the proof.

1.5 9/17 Lecture

Proposition 1.12. Let $T : X \to Y$ be a linear map, T is bounded iff

$$T^{-1}(\{y : \|y\| \leq 1\})$$

has nonempty interior.

Proof. Suppose that it has a nonempty interior, then

$$\{x : \|x - x_0\| \leq \varepsilon\} \subset T^{-1}(\{y : \|y\| \leq 1\})$$

Then if $\|x\| \leq \varepsilon$, then

$$\|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| \leq 1 + \|Tx_0\|$$

then

$$T\left(\frac{\varepsilon x}{\|x\|}\right) = \frac{\varepsilon}{\|x\|}T(x) \Rightarrow \|T(x)\| \leq (1 + \|Tx_0\|)\frac{1}{\varepsilon}\|x\|$$

□

Definition 1.8 (nowhere dense). A set S is called nowhere dense if $\overline{S}$ has empty interior.

Proposition 1.13 (Baire category theorem). A complete metric space M is never a union of a countable number of nowhere dense sets.

Proof. Suppose $M = \bigcup_{n=1}^{\infty} A_n$, where A_n is nowhere dense. We will construct $\{x_n\}$ that stays away from each A_n so the limit x is in any A_n, which gives a contradiction. Take $x_1 \in \bar{A}_1$, and $B_1 \cap A_1 = \varnothing$, and $B_1 = B(x_1, r_1) = \{d(x, x_1) < \varepsilon\}$, let $x_2 \in B_1 \setminus A_1$, and let $B_2 = B(x_2, r_2)$, and so on, $\{x_n\}$ is a Cauchy sequence, $x_n \to x$ as $n \to \infty$. We have $x_n \in B_N, n \geq N$, but $\bar{B}_N \subset B_{N-1}$, thus $x \in B_{N-1}$ for any N, and $x \notin A_{N-1}$ for any N. This gives our contradiction. □

Proposition 1.14 (UBP). Let X be a Banach space, let $\mathcal{F}$ be a family of bounded linear maps $T : X \to Y$. Suppose that for each $x \in X$,

$$\{\|Tx\| : T \in \mathcal{F}\}$$

is bounded, then

$$\sup_{T \in \mathcal{F}} \|T\| < \infty$$

Proof. Key ingredient: Baire category theorem. Let $B_n = \{x : \|Tx\| \leq n, T \in \mathcal{F}\}$, each x is in some B_n, and $X = \bigcup_{n=1}^{\infty} B_n$, where B_n is closed. Baire category theorem, some B_n has nonempty interior.

$$B_n\{x : \|Tx\| \leq n\} \Rightarrow \{|x : \|Tx\| \leq 1\}$$

And

$$\|Tx\| \leq (1 + \|Tx_0\|)\|x\| \Rightarrow \|T\| \leq 1 + \|T(x_0)\|$$

□

1.6 9/22 Lecture

Definition 1.9 (homeomorphism). Let $f : S \to T$, be a map between two topological spaces, then f is called a homeomorphism if it is a continuous bijection with a continuous inverse.

Definition 1.10. Let K be a family of functions $f : S \to (\tau, \tau_U)$, then X is the weak topology on S if it is the weakest topology making all $f \in K$ continuous: for any $U \in \tau_U$, $f^{-1}(U)$ is open in X.

Example 1.4. Let $\mathcal{H}$ be a Hilbert space. Then the weak topology on $\mathcal{H}$ is the weakest topology that makes

$$\varphi \mapsto \langle \varphi, \psi \rangle$$

continuous for each $\psi \in \mathcal{H}$.

Definition 1.11 (Hausdorff). A topological space is called Hausdorff iff for each $x \neq y$, there exists open sets U_1, U_2 such that $x \in U_1, y \in U_2$ and $U_1 \cap U_2 = \varnothing/$. A topological space is called

1. separable if it contains a countable dense subset.
2. first countable if each point has a countable neighborhood base.
3. second countable iff it has a countable base.

We note that every metric space is first countable, and is second countable if it is separable. Finally, any second countable space is separable.

Now we talk about some nets.

Definition 1.12 (directed system, net). A directed system is an index set I together with an ordering $\prec$ such that

1. $\alpha, \beta \in I$, there is $\gamma \succ \alpha, \gamma \succ \beta$.
2. there is a partial ordering.

A net on a set S is a mapping $I : A \mapsto x_A \in S$ such that $\{x_\alpha\}_{\alpha \in I}$. If I is positive integers then the nets are just sequences $\{x_i\}_{i=1}^{\infty}$.

A net $\{x_\alpha\}_{\alpha \in I}$ is said to converge to $x \in S$ if for any neighborhood N of x, there is $\beta \in I$ so $x_\alpha \in N$ if $\alpha \succ \beta$.

Proposition 1.15. A function $f : S \to T$ is continuous iff for eery convergent net $\{x_\alpha\}$, the net $\{f(x_\alpha)\}$ converges too. (If S, T are first countable, then the nets can be taken as sequences).

Then we defined compact and FIP for topological spaces, and claimed that they are equal.

Example 1.5. The unit ball in l_2 is not compact.

Proposition 1.16 (Tychnoff). An arbitrary product of compact topological spaces is compact.

Proposition 1.17 (Stone-Weierstrass). Let B be a subalgebra of $C_{\mathbb{R}}(X)$, where X is Hausdorff, compact that separates points, i.e., if $x \neq y$, this implies $f(x) \neq f(y)$ for some $f \in B$ and $L \in B$, then $B = C_{\mathbb{R}}(X)$.

Example 1.6. Let B be the closure of the set $\{e^{ikx}\}_{k=-\infty}^{\infty}$, on $[-\pi, \pi]$. Then by Stone Weierstrass, we must have $B = C[-\pi, \pi]$.

1.7 Lecture 9/24

Proposition 1.18 (Riesz-Markov). Let X be compact and Hausdorff, then for any positive linear functional l on $C(X)$, there is a unique Baire measure μ such that

$$l(f) = \int f d\mu$$

For example, let $X = [-1, 1]$, let $l(f) = f(0)$, then $\mu = \delta_0$. This means that you can continuously extend l so that it is defined on characteristic functions on Baire sets.

Definition 1.13 (weak topology). The weak topology on X is the weakest topology that make each $l \in X^k$ continuous. A sequence converges weakly $x_n \xrightarrow{w} x$ iff $l(x_n) \to l(x)$ for all $l \in X^*$.

Proposition 1.19. We have the following properties:

1. The weak topology is weaker than the norm topology.
2. For weakly convergent s sequence is norm bounded.
3. The weak topology is Hausdorff.

Example 1.7. Let $\mathcal{H}$ be a Hilbert space, and $\{\varphi_\alpha\}$ an orthonormal base. Given ψ_n a sequence in $\mathcal{H}$. And

$$\varphi_n^{(a)} = (\psi_n, \varphi_\alpha)$$

where $\psi_n = \sum \psi_n^{(a)} \varphi_\alpha$, then

$$\psi_n x \xrightarrow{w} \psi$$

iff $\psi_n^{(a)} \to \psi^{(a)}$ and $\{\|\psi_n\|\}$ is bounded.

Proposition 1.20. A linear functional l on a Banach space is weakly continuous iff it is norm continuous.