Calc III Sections

Fall 2025

Hui Sun

September 9, 2025

Calc III-Week 3 (9/8-9/12)

Topics: (1) Graphing multivariable functions, (2) Introducing limits and continuity.

Definition 1 (graph). The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 2 (limit). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, ($A$ is open), let N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0 ($x_0 \in \bar{A}$,) f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that whenever $x \in U$, then $f(x) \in N$ as well.

The **limit** of f as $x \to x_0$, if it exists, is $\lim_{x \to x_0} f(x) := b \in \mathbb{R}^m$ such that f is eventually in N, for every neighborhood N of b.

Definition 3 (limit'). $\lim_{x \to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x_1', x_2', \ldots, x_n')$ from **all directions**, $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 4 (continuity). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 0.1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 1. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x) = x^2 + y^2$.

Problem 2. Compute the limit of the following functions:

1. $f(x, y) = \frac{x}{x+y}$ as $(x, y) \to (0, 0)$.
2. $g(x, y) = \frac{xy}{x+y}$ as $(x, y) \to (0, 0)$.
3. $h(x, y) = \frac{\sin(xy)}{x+y}$ as $(x, y) \to (0, 0)$. Recall that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$.