Calc III Sections

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Calc III-Week 7 (10/6-10/10)

Topics: (1) quadratic form, (2) constraint.

Definition 1 (quadratic function). A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic function** if it is given by

$$g(h_1, \dots, h_n) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

where (a_{ij}) is an $n \times n$ matrix. We can also write g as follows:

$$g(h_1, \dots, h_n) = [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

An example of a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} \begin{bmatrix} h_1 & \dots & h_n \end{bmatrix} Hf(x) \begin{bmatrix} h_1 \\ \dots \\ h_n \end{bmatrix}$$

where $h = (h_1, \dots, h_n)$, and $Hf(x)$ is the Hessian matrix of $f : \mathbb{R}^n \to \mathbb{R}^n$.

Definition 2 (positive, negative-definite). A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is called **positive-definite** if $g(h) \geq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. Similarly, g is **negative-definite** if $g(h) \leq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. (The matrix is positive-definite iff it is symmetric $A^T = A$ and the eigenvalues are nonnegative).

Definition 3 (bounded set). A set $A \subset \mathbb{R}^n$ is said to be **bounded** if there is a number $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Proposition 1 (continuous functions attain extremum on closed bounded sets). Let $f : D \to \mathbb{R}$ be continuous, where D is closed and bounded in $\mathbb{R}^n$. Then f assumes its absolute maximum and absolute minimum values at some point $x_0, x_1 \in D$.

Proposition 2. If f has a maximum or minimum at x_0 when constrained to a surface S, then $\nabla f(x)$ is perpendicular to S at x_0.

Consequence of Proposition 2:

Proposition 3. Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ and $\vec{x_0} \in U$ such that $g(x_0) = c$. Let $\mathcal{L}_c$ be the level set for g with value c and assume

$$\nabla g(\vec{x_0}) \neq \vec{0}.$$

If f restricted to x_c has a local minimum or maximum on $\mathcal{L}_c$ at x_0, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Problem 1. Is the following matrix positive-definite?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof. It is not! Consider the vector $(0,1)$, we have

$$[0,1]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0,1]\begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1$$

$\square$

Problem 2. Find the critical point of $f(x,y) = y + x\sin y$ and classify whether it is a local max/min or a saddle point.

Proof. We compute

$$f_x = \sin y, \quad f_y = 1 + x\cos y$$

Setting them both to 0 gives

$$(1, n\pi), \text{ when } n \text{ is odd}, \quad (-1, n\pi) \text{ when } n \text{ is even}$$

Now we compute the discriminant:

$$f_{xx} = 0, \quad f_{xy} = f_{yx} = \cos y, \quad f_{yy} = -x\sin y$$

Thus

$$\mathcal{D} = \det\begin{bmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{bmatrix} = -\cos^2 y$$

Thus $\mathcal{D} < 0$ for all the critical points, hence they are saddle points! $\square$

Problem 3 (Marsden-Tromba, III. 2). Let $f(x,y,z) = x - y + z$, find the extremum of f subject to the constraint $x^2 + y^2 + z^2 = 2$.

Proof. We compute the gradient of f and $g(x,y,z) = x^2 + y^2 + z^2$:

$$\nabla f(x,y,z) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \quad \nabla g(x,y,z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

By the proposition above, we need to have

$$\begin{aligned} \lambda &= 2x \\ -\lambda &= 2y \\ \lambda &= 2z \end{aligned}$$

Equating all λ gives

$$x = z = -y$$

Plugging in the constraint we get

$$3x^2 = 2$$

Hence two critical points

$$\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right), \quad \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$$

Plugging them in we find the extremums are $\sqrt{6}$ and $-\sqrt{6}$. $\square$

Problem 4 (Marsden-Tromba, III. 16)**.** Use Lagrange multipliers to find the distance from the point $(2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$.

Proof. Let (x, y, z) be a point on the plane, then we would like the minimize the function

$$f(x, y, z) = (x-2)^2 + y^2 + (z+1)^2$$

with the constraint

$$3x - 2y + 8z + 1 = 0$$

Writing $g(x, y, z) = 3x - 2y + 8z + 1$, we do the exact same thing as we did above:

$$\nabla f = \begin{bmatrix} 2x-4 & 2y & 2z+2 \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 3 & -2 & 8 \end{bmatrix}$$

Then setting

$$\begin{aligned} 2x - 4 &= 3\lambda \\ 2y &= -2\lambda \\ 2z + 2 &= 8\lambda \end{aligned}$$

One can then replace x, y, z with expressions in λ in the constraint:

$$3x - 2y + 8z + 1 = 0$$

which gives

$$\lambda = 2/77$$

Then plugging in λ to solve for x, y, z, the $\sqrt{f(x, y, z)}$ is the final answer. (I am too lazy to do the computation). ☐