

# Algebra Definition Theorem List

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# Chapter 1

## Category Theory

**Definition 1.1** (*initial, final*). Let  $\mathcal{C}$  be a category, then object  $I$  is initial if for every object  $A$ , there exists a unique morphism  $I \rightarrow A$ . We say  $F$  is final if for every  $A$ , there exists a unique morphism  $A \rightarrow F$ .

## Chapter 2

# Group Theory I

This corresponds to Aluffi Chapter II.

**Proposition 2.1.** Let  $G$  be a group, for all  $a, g, h \in G$ , if

$$ga = ha$$

then  $g = h$ .

**Corollary 2.1.** If  $g$  is an element of finite order, and let  $N \in \mathbb{Z}$ , then

$$g^N = e \iff N \text{ is a multiple of } |g|$$

**Proposition 2.2.** Let  $g \in G$  be of finite order, then  $g^m$  also has finite order, for all  $m \geq 0$ , and

$$|g^m| = \frac{\text{lcm}(m, |g|)}{m} = \frac{|g|}{\text{gcd}(m, |g|)}$$

**Proposition 2.3.** If  $gh = hg$ , then  $|gh|$  divides  $\text{lcm}(|g|, |h|)$ .

**Definition 2.1 (Dihedral Group).** Let  $D_{2n}$  denote the group of symmetries of a  $n$ -sided polygon, consisting of  $n$  rotations and  $n$  reflections about lines through the origin and a vertex or a midpoint of a side.

**Proposition 2.4.** Let  $m \in \mathbb{Z}/n\mathbb{Z}$ , then

$$|m| = \frac{n}{\text{gcd}(n, m)}$$

**Corollary 2.2.** The element  $m \in \mathbb{Z}/n\mathbb{Z}$  generates  $\mathbb{Z}/n\mathbb{Z}$  if and only if  $\text{gcd}(m, n) = 1$ .

**Definition 2.2 (Multiplicative  $(\mathbb{Z}/n\mathbb{Z})^\times$ ).** The multiplicative group of  $\mathbb{Z}/n\mathbb{Z}$  is

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{m \in \mathbb{Z}/n\mathbb{Z} : \text{gcd}(m, n) = 1\}$$

**Proposition 2.5.** Let  $\varphi : G \rightarrow H$  be a homomorphism, and let  $g \in G$  be an element of finite order, then  $|\varphi(g)|$  divides  $|g|$ .

For example, there is no nontrivial homomorphism from  $\mathbb{Z}/n\mathbb{Z}$  to  $\mathbb{Z}$ .

**Proposition 2.6.** Let  $\varphi : G \rightarrow H$  be an isomorphism, for all  $g \in G$ ,  $|\varphi(g)| = |g|$ , and  $G$  is commutative if and only if  $H$  is commutative.

**Proposition 2.7.** If  $H$  is commutative, then  $\text{Hom}(G, H)$  is a group.

**Definition 2.3.** Let  $A = \{1, \dots, n\}$ , then the free abelian group on  $A$  is

$$\mathbb{Z} \oplus \dots \oplus \mathbb{Z} = \mathbb{Z}^{\oplus n}$$

**Proposition 2.8.** Let  $\{H_\alpha\}$  be any family of subgroups of  $G$ , then

$$\bigcap_{\alpha} H_{\alpha}$$

is a subgroup of  $G$ .

**Proposition 2.9.** If  $\varphi : G_1 \rightarrow G_2$  is a group homomorphism, then if  $H_2 \subset G_2$  is a subgroup, then

$$\varphi^{-1}(H_2)$$

is a subgroup of  $G_1$ .

**Proposition 2.10.** Let  $H \subset \mathbb{Z}/n\mathbb{Z}$  be a subgroup, then  $H$  is generated by some  $m$  where  $m$  divides  $n$ .

**Proposition 2.11.** If  $\varphi : G_1 \rightarrow G_2$  is a homomorphism, then  $\ker(\varphi)$  is a normal subgroup.

**Proposition 2.12.** Let  $\varphi : G_1 \rightarrow G_2$  be a surjective homomorphism, then

$$G_2 \cong \frac{G_1}{\ker \varphi}$$

**Proposition 2.13.** Let  $H_1, H_2$  be normal subgroups of  $G_1, G_2$ , then  $H_1 \times H_2$  are normal subgroups of  $G_1 \times G_2$ , then

$$\frac{G_1 \times G_2}{H_1 \times H_2} \cong \frac{G_1}{H_1} \times \frac{G_2}{H_2}$$

For example,

$$\frac{\mathbb{Z}/6\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z}$$

**Proposition 2.14.** Let  $H$  be a normal subgroup of  $G$ , then every subgroup  $K/H$  of  $G/H$  can be identified with a subgroup of  $G$  containing  $H$ .

**Proposition 2.15.** Let  $H$  be a normal subgroup of  $G$ , and  $N$  be a subgroup of  $G$  containing  $H$ , then  $N/H$  is normal in  $G/H$  if and only if  $N$  is normal in  $G$ , in this case

$$\frac{G/H}{N/H} = \frac{G}{N}$$

**Proposition 2.16.** Let  $H, K$  be subgroups of  $G$ , and if  $H$  is normal, then  $HK$  is a subgroup of  $G$  and  $H$  is normal in  $HK$ . Moreover,  $H \cap K$  is normal in  $K$ , and

$$\frac{HK}{H} \cong \frac{K}{H \cap K}$$

**Proposition 2.17.** Let  $H$  be a subgroup of  $G$ , then for all  $g \in G$ , the function  $H \rightarrow gH$  such that

$$h \mapsto gh$$

is a bijection.

**Theorem 2.1 (Lagrange).** If  $G$  is a finite group, and  $H \subset G$  is a subgroup, then

$$|G| = [G : H] \cdot |H|$$

In particular,  $|H|$  divides  $|G|$ .

**Proposition 2.18 (Fermat's Little Theorem).** Let  $a \in \mathbb{Z}/p\mathbb{Z}$ , then

$$a^p \equiv a$$

**Proposition 2.19.** Any group  $G$  acts on itself by left/right multiplications, and acts on the cosets  $G/H$ :

$$\varphi : g \mapsto (aH \mapsto gaH)$$

**Definition 2.4 (orbit).** The orbit of  $a \in A$  of a group action by  $G$  is

$$O(a) = \{g \cdot a : g \in G\}$$

The stabilizer of  $a$  is the following

$$\text{Stab}_G(a) = \{g \in G : g \cdot a = a\}$$

**Proposition 2.20.** The orbits of an action form a partition on the set  $A$ , and  $G$  acts transitively on each orbit.

**Definition 2.5** (transitive action, faithful action). An action of  $G$  on  $A$  is transitive if for all  $a, b \in A$ , there exists  $g \in G$  such that

$$g \cdot a = b$$

In other words, the orbit of any element  $a \in A$  is the entire set.

An action is faithful if for any  $g \in G$ ,

$$g \cdot a = a \text{ for all } a$$

implies that  $g = e$ .

**Proposition 2.21.** Every transitive action of  $G$  on a set  $A$  is isomorphic to multiplication of  $G$  on  $G/H$ , where  $H = \text{Stab}(a)$  for any  $a \in A$ .

**Proposition 2.22.** If  $O(a)$  is an orbit of the action of a finite group  $G$ , then  $O(a)$  is a finite and  $|O|$  divides  $|G|$ . Moreover,

$$|G| = |O(a)| \cdot |\text{Stab}_G(a)|$$

For example, there is no transitive action of  $S_3$  on the set of 5 elements.

**Proposition 2.23.** Let  $G$  act on itself by conjugation, let  $g \in G$ , then

$$|[g]| \mid |G|$$

and

$$|Z_G(g)| \mid |G|$$

the size of the conjugacy class and the centralizer both divide  $|G|$ .



# Chapter 3

## Group Theory II

This corresponds to Aluffi Chapter IV.

**Proposition 3.1.** Every **transitive** action of a group  $G$  on a set  $S$  is isomorphic to the left multiplication on the cosets  $G/H$ . Here,  $H$  can be taken to be the stabilizer of any element  $a \in S$ .

Moreover, suppose  $G$  is finite, then

$$|G| = |O_a| \cdot |\text{Stab}(a)|$$

for any  $a \in S$ . (The size of the orbit must divide  $|G|$ .)

**Proposition 3.2 (class formula).** Let  $S$  be a finite set, and  $G$  act on  $S$ , then

$$|S| = |Z| + \sum_{a \in A} [G : \text{Stab}(a)] = |Z| + \sum_{a \in A} |O_a|$$

where  $Z = \{a \in S : g \cdot a = a \text{ for all } g\}$ , i.e., the fixed elements, and  $A \subset S$  contains exactly one element from each nontrivial orbit of the action.

In other words,  $|S|$  is the sum of the number of trivial orbits and each nontrivial orbit.

**Proposition 3.3.** Let  $G$  be a  $p$ -group that acts on a finite set  $S$ , then let  $Z$  be fixed elements of this action, then

$$|S| \equiv |Z| \pmod{p}$$



**Warning 3.1.** The important takeaway is that each summand on the right,  $|O_a|$  divides  $|G|$ .

### 3.1 Conjugation Action

**Definition 3.1 (fixed points, centralizer, conjugacy class).** The fixed points under the conjugation action is the center of  $G$ . The centralizer  $Z_G(g)$  where  $g \in G$  is its stabilizer under conjugation:

$$Z_G(g) = \{h \in G : hgh^{-1} = g\}$$

The conjugacy class of  $g \in G$  is the orbit  $[g]$ . (In other words, centralizer is the set of elements that commute with  $g$ .)

For arbitrary  $a \in G$ , we have

$$Z(G) \subset Z_G(a)$$

Moreover,  $a$  is the only element in  $[a]$  iff  $a \in Z(G)$ .

**Proposition 3.4.** The center is the set of fixed points of  $G$  under the conjugation action, the conjugacy classes are the orbits.

**Theorem 3.2.** Let  $G$  be finite, and if  $G/Z(G)$  is cyclic, then  $G$  is abelian.

*Proof.* One can show that every element  $a \in G$  can be written as

$$a = g^r z$$

for some  $z \in Z(G)$ , then compute  $ab = ba$ . □

**Proposition 3.5 (Class formula).** Let  $G$  be finite, then

$$\begin{aligned} |G| &= |Z(G)| + \sum_{[a] \in A} |[a]| \\ &= |Z(G)| + \sum_a [G : Z_G(a)] \end{aligned}$$

where  $A$  contains one representative for each nontrivial conjugacy class.



**Warning 3.3.** There are many consequences of the class formula, showing center is nontrivial, etc. Mainly using the summand divides  $|G|$ !

**Theorem 3.4.** Let  $G$  be a nontrivial  $p$ -group, then  $G$  has a nontrivial center.

**Proposition 3.6.** Let  $G$  be a group of  $p^2$  elements, where  $p$  is prime, then  $G$  is commutative.

**Proposition 3.7.** The only possibility for the class formula of a nonabelian group of order 6 is

$$6 = 1 + 2 + 3$$

The center must be trivial if  $G$  is nonabelian.

**Proposition 3.8.** Normal subgroups are unions of conjugacy classes. Thus, a noncommutative group of order 6 cannot have a normal subgroup of order 2.

It contains the identity, and there is no other conjugacy class of size 1.

**Definition 3.2 (normalizer).** Let  $A \subset G$  be a subset. The normalizer  $N_G(A)$  of  $A$  is

$$\text{Stab}_G(A) = \{g : gAg^{-1} = A\}$$

If  $H$  is subgroup of  $G$ , every conjugate  $gHg^{-1}$  is also a subgroup of  $G$ , and all conjugate groups have the same order.

The centralizer of  $A$  is the subgroup  $Z_G(A) \subset N_G(A)$  fixing each  $a \in A$ :

$$Z_G(A) = \{g : gag^{-1} = a \text{ for all } a \in A\}$$

**Proposition 3.9.**  $H$  is a normal in  $G$  if and only if  $N_G(H) = G$ . More generally, the normalizer  $N_G(H)$  for any subgroup  $H$  is the largest subgroup such that  $H$  is normal in  $N_G(H)$ .

**Theorem 3.5.** Let  $H \subset G$  be a subgroup, then the number of subgroups conjugate to  $H$  is the size of the orbit=index of the stabilizer, which is  $[G : N_G(H)]$ .

**Corollary 3.1.** If  $[G : H]$  is finite, then the number of subgroups conjugate to  $H$  is finite, and

$$[G : H] = [G : N_G(H)] \cdot [N_G(H) : H]$$

In other words, the number of subgroups conjugate to  $H$  divides the index  $[G : H]$ .

## 3.2 Sylow

**Theorem 3.6 (Cauchy's Theorem).** Let  $G$  be a finite group, and let  $p$  be a prime divisor of  $|G|$ , then  $G$  contains an element of order  $p$ .

Moreover, let  $N$  be the number of cyclic subgroups of order  $p$ , then

$$N \equiv 1 \pmod{p}$$

**Definition 3.3 (simple).** A group is simple if it is nontrivial and its only normal subgroups are  $\{e\}$  and  $G$  (has no nontrivial proper subgroup).

**Definition 3.4 ( $p$ -Sylow subgroups).** Let  $p$  be prime, a  $p$ -Sylow subgroup of a finite group  $G$  is a subgroup of order  $p^r$ , where  $|G| = p^r m$ ,  $\gcd(p, m) = 1$ .

**Theorem 3.7 (Sylow I).** Every finite group contains a  $p$ -Sylow subgroup for all prime  $p$ . If  $p^k$  divides  $|G|$ , then  $G$  has a subgroup of order  $p^k$ .

**Theorem 3.8 (Sylow II).** Let  $G$  be finite, and  $P$  is a  $p$ -Sylow subgroup, let  $H \subset G$  be a  $p$ -group, then  $H$  is contained in a conjugate of  $P$ . If  $P_1, P_2$  are both  $p$ -Sylow subgroups, then they are conjugates to each other.

**Theorem 3.9 (Sylow III).** Let  $|G| = p^r m$ , and  $\gcd(p, m) = 1$ , then the number of  $p$ -Sylow subgroups is

$$n_p \mid m$$

and

$$n_p \equiv 1 \pmod{p}$$

**Theorem 3.10.** Let  $G$  be a finite group, let  $P$  be a  $p$ -Sylow subgroup, the number of  $p$ -Sylow subgroup  $n_p$  is

$$n_p = [G : N_G(P)]$$

by definition.

**Theorem 3.11.** Let  $G$  be a group of order  $mp^r$ , where  $p$  is prime and  $1 < m < p$ , then  $G$  is not simple.

**Theorem 3.12.** Let  $p < q$  be primes, let  $G$  has order  $pq$ , if  $p \nmid (q - 1)$ , then  $G$  is cyclic.

*Proof.* If  $G$  is abelian, use elements of orders  $p, q$ . If  $G$  not necessarily abelian, then use the conjugation action.  $\square$

**Theorem 3.13.** Let  $p$  be a prime  $\neq 2$ , and  $G$  be a noncommutative group of order  $2p$ , then

$$G \cong D_{2p}$$

### 3.3 Series and Solvability

**Definition 3.5 (composition series).** A comp series for  $G$  is a normal series

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

such that  $G_{i+1}/G_i$  is simple.

**Definition 3.6 (commutator subgroup).** Let  $G$  be a group, the commutator subgroup of  $G$  is the subgroup **generated** by all elements

$$ghg^{-1}h^{-1}$$

**Proposition 3.10.** Let  $[G, G]$  be the commutator subgroup of  $G$ , then  $[G, G]$  is normal in  $G$ , and the quotient, also called the abelianization of  $G$ ,

$$G^{\text{ab}} = \frac{G}{[G, G]}$$

is commutative.

**Theorem 3.14.** If  $\varphi : G \rightarrow H$ , where  $H$  is commutative, then

$$[G, G] \subset \ker(\varphi)$$

**Definition 3.7.** A group  $G$  is solvable, if there exists a sequence such that

$$\{e\} = G_0 \subset \cdots \subset G_k = G$$

where  $G_i$  is normal in  $G_{i+1}$ , and  $G_{i+1}/G_i$  is abelian, or equivalently, cyclic.

**Proposition 3.11.** All  $p$ -groups are solvable!

**Proposition 3.12.** Let  $N$  be normal in  $G$ , then  $G$  is solvable if and only if  $N, G/N$  are solvable.

### 3.4 $S_n$ and $A_n$

**Proposition 3.13.** Disjoint cycles commute. For every  $\sigma \in S_n$ ,  $\sigma$  can be written as disjoint nontrivial cycles, unique up to rearranging.

**Proposition 3.14.** Two elements in  $S_n$  are conjugate in  $S_n$  if and only if they have the same type. Hence the number of conjugacy classes is the number of partitions of  $n$  as a sum.

**Proposition 3.15.** Let  $\sigma \in S_n$ , and  $(a_1 \dots a_n)$  is a cycle in  $S_n$ , then

$$\sigma(a_1 \dots a_n)\sigma^{-1} = (\sigma(a_1) \dots \sigma(a_n))$$

Proof: try  $\varphi(a_1)$  on the left hand side.



**Warning 3.15.** Very useful!

**Example 3.1.** In  $S_4$ , we have

$$(1234)(12)(1234)^{-1} = (23)$$

**Definition 3.8 (Even permutation).** Let  $\sigma \in S_n$ , then  $\sigma$  is even if

$$\prod_{i < j} (x_i - x_j) = \prod_{i < j} (x_{\sigma(i)} - x_{\sigma(j)})$$

**Proposition 3.16.**  $A_n$  is always normal in  $S_n$ , because it is the kernel of the  $\varepsilon : S_n \rightarrow \{\pm 1\}$  (determining parity).

**Proposition 3.17.** Let  $\sigma \in A_n$ , where  $n \geq 2$ , then the conjugacy class of  $\sigma$  in  $S_n$  splits into two conjugacy classes in  $A_n$  precisely if the type of  $\sigma$  consists of distinct odd numbers; or equivalently, the centralizer of  $\sigma$  is contained  $A_n$ . Otherwise, the conjugacy class stays the same.

**Example 3.2.**  $S_5$  has even permutations 5, 3, 2+2, 1, and only 5-cycle of  $S_5$  splits into 2 conjugacy classes in  $A_5$ .

**Proposition 3.18.** The group  $A_5$  is a simple noncommutative group of order 60.

**Proposition 3.19.** Every simple group of order  $< 60$  is commutative,  $A_5$  is the smallest simple group that is not commutative.

*Proof.* Any nontrivial normal subgroup consists of nontrivial conjugacy classes and  $\{e\}$ , the conjugacy classes of  $A_5$  has the following size:

$$1, 15, 20, 12, 12$$

Thus any subgroup of  $G$ , i.e., order that divides 60 cannot be written as a sum of the numbers above.  $\square$

**Proposition 3.20.** The alternating group is generated by 3-cycles.

**Proposition 3.21.** Let  $n \geq 5$ , if a normal subgroup of  $A_n$  contains a 3-cycle, then it contains all 3-cycles.

*Proof.* It suffices to note that the 3 cycles form a conjugacy class that doesn't split from  $S_n$  to  $A_n$ .  $\square$

**Proposition 3.22.** The alternating group  $A_n$  is simple for  $n \geq 5$ . As a result,  $S_n$  is not solvable for  $n \geq 5$ .

### 3.5 Product of Groups

**Proposition 3.23.** Let  $N, H$  be normal subgroups of  $G$ , let  $[N, H]$  be the commutator of  $N, H$ , then

$$[N, H] \subset N \cap H$$

Thus if  $N \cap H = \{e\}$ , then  $N, H$  commute with each other.

A stronger statement is the following:

**Theorem 3.16.** Let  $N, H$  be normal subgroups of  $G$ , such that  $N \cap H = \{e\}$ , then

$$NH \cong N \times H$$

**Definition 3.9 (Split Short exact sequence).** A short exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\varphi} G \xrightarrow{\psi} H \longrightarrow 1$$

**splits** if  $H$  is identified with a subgroup of  $G$  such that

$$N \cap H = \{e\}$$

**Definition 3.10 (semidirect product).** Let  $N$  be a normal subgroup, and let  $\theta : H \rightarrow \text{Aut}(N)$ , then define an operator  $\cdot_\theta$  on  $N \times H$  as

$$(n_1, h_1) \cdot_\theta (n_2, h_2) = (n_1 \theta(h_1)(n_2), h_1 h_2)$$

The semidirect product of  $N \rtimes_\theta H$  is the group  $N \times H$  with operation  $\cdot_\theta$ .

**Proposition 3.24.** Let  $N, H$  be subgroups, and  $N$  is normal, suppose that  $N \cap H = \{e\}$ , and  $G = NH$ , then let  $\theta : H \rightarrow \text{Aut}(N)$  be  $\theta \mapsto \theta_h$ , and

$$\theta_h(n) = nhn^{-1}$$

Then

$$G \cong N \rtimes_{\theta} H$$

(Recall that the operation defined on  $N \rtimes_{\theta} H$  is  $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \theta_{h_1}(n_2), h_1 h_2)$ ).

**Proposition 3.25.** Let  $G$  be a noncommutative group of order  $pq$ , then there is exactly one group up to isomorphism.

## 3.6 Classification of Finite Abelian Groups

**Proposition 3.26.** Let  $G$  be abelian, let  $H, K$  be subgroups such that  $|H|, |K|$  are relatively prime, then

$$H + K \cong H \oplus K$$

*Proof.* Lagrange:  $N \cap H = \{e\}$ . □

**Proposition 3.27.** Every finite abelian group is a direct sum of its nontrivial Sylow subgroups.

**Theorem 3.17.** If  $G$  is finite and abelian, then  $G$  is a direct sum of cyclic  $p$ -groups.

**Theorem 3.18.** Let  $G$  be finite nontrivial abelian group, then there exists prime integers  $p_1, \dots, p_r$ , and positive integers  $n_{i(j)}$  such that

$$G = \bigoplus_{i,j} \frac{\mathbb{Z}}{p_i^{n_{i(j)}} \mathbb{Z}}$$

There exists positive integers  $1 < d_1 \mid \dots \mid d_s$  such that  $|G| = d_1 \dots d_s$ , and

$$G \cong \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \oplus \dots \oplus \frac{\mathbb{Z}}{d_s \mathbb{Z}}$$

**Example 3.3.** Finite abelian group of order 360 has 6 isomorphism classes.

**Theorem 3.19.** Let  $F$  be a field, and  $G$  be a finite subgroup of the multiplicative group  $(F^{\times}, \cdot)$ , then  $G$  is cyclic.

*Proof.* Hard proof. Don't torture yourself. □



**Warning 3.20.** Next one is important.

**Proposition 3.28.** Let  $G$  be a finite group of order  $n$ , then  $G$  can be embedded into  $S_n$ .

*Proof.*  $G$  acts on itself by left multiplication. □

**Proposition 3.29.** The number of conjugacy classes of  $D_n = \langle r, s : r^n = s^2 = e, rs = sr^{-1} \rangle$ :

1.  $n = \text{odd}$ , then  $\{e\}$  is its own conjugacy class, the pairs of rotations  $\{r^k, r^{-k}\}$  are conjugacy classes, the reflections form ONE conjugacy class:

$$1 + \frac{n-1}{2} + 1 = \frac{n+3}{2}$$

conjugacy classes.

2.  $n = \text{even}$ , then  $[e], [r^{n/2}]$  forms their own conjugacy classes, the remaining rotations  $[r^k, r^{-k}]$ , and there are TWO conjugacy classes of reflection:

$$1 + 1 + \frac{n-2}{2} + 1 + 1 = \frac{n+6}{2}$$

conjugacy classes.



# Chapter 4

## Ring Theory

This corresponds to Aluffi Chapter III.

**Definition 4.1 (free action).** An action by  $G$  is free if there exists  $x \in X$  such that  $gx = x$  then  $g = e$ .

**Definition 4.2 (faithful action).** An action by  $G$  is faithful if  $gx = x$  for all  $x \in X$  implies that  $g = e$ .

**Definition 4.3 (zero-divisor).** An element  $a \in R$  is a (left) zero-divisor if there exists  $b \neq 0$  such that

$$ab = 0$$

**Proposition 4.1.** In a ring  $R$ ,  $a \in R$  is not a left zero-divisor if and only if the left multiplication by  $a$  is injective.

**Definition 4.4 (integral domain).** An ID is a nonzero commutative ring such that for all  $a, b \in R$ ,

$$ab = 0$$

implies  $a = 0$  or  $b = 0$ . In other words, IDs are commutative rings without zero divisors. Equivalently, if  $a, b \neq 0$ , then  $ab \neq 0$ .

**Proposition 4.2.** In a ring  $R$ :

1.  $u$  is left unit iff the left multiplication by  $u$  is surjective.
2. If  $u$  is a left unit, then the right multiplication by  $u$  is injective, i.e.,  $u$  is not a right zero-divisor.

Notice that in a commutative ring, this means  $u$  is a unit iff multiplication by  $u$  is bijective.

**Definition 4.5 (division ring, field).** A division ring is a ring in which every nonzero element is a unit. A field is a nonzero commutative ring in which every nonzero element is a unit.

**Proposition 4.3.** The group of units in  $\mathbb{Z}/n\mathbb{Z}$  is exactly the group  $(\mathbb{Z}/n\mathbb{Z})^*$ .

*Proof.*  $m$  is a unit iff multiplication by  $m$  is surjective, iff  $m$  generates  $\mathbb{Z}/n\mathbb{Z}$ , iff  $m \in (\mathbb{Z}/n\mathbb{Z})^*$ . □

**Definition 4.6 (Power Series Ring).** The power series ring

$$\sum_{i=0}^{\infty} a_i x^i$$

is denoted by  $R[[x]]$ .

**Definition 4.7 (Monoid Ring).** Given a monoid  $M$  and a ring  $R$ , the elements

$$\sum_{m \in M} a_m \cdot m$$

where  $a_m \in R$  and  $a_m \neq 0$  for finitely many terms, forms a ring denoted as  $R[M]$ .

**Proposition 4.4.** Assume  $R$  is a finite commutative ring, then  $R$  is an integral domain if and only if  $R$  is a field.

**Proposition 4.5.**  $\text{End}_{\text{Ab}}(\mathbb{Z}) \cong \mathbb{Z}$ , where  $\text{End}_{\text{Ab}}(G) = \text{Hom}_{\text{Ab}}(G, G)$  where  $G$  is abelian.

*Proof.*  $\varphi \mapsto \varphi(1)$ . □

**Theorem 4.1.** Let  $I$  be a two-sided ideal of a ring  $R$ . Then for every ring homomorphism  $\varphi : R \rightarrow S$  such that  $I \subset \ker \varphi$  there exists a unique ring homomorphism  $\tilde{\varphi} : R/I \rightarrow S$  so that the diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \pi \downarrow & \nearrow \tilde{\varphi} & \\ R/I & & \end{array}$$

**Theorem 4.2.** Let  $\varphi : R \rightarrow S$  be a surjective ring homomorphism, then

$$S \cong \frac{R}{\ker(\varphi)}$$

**Proposition 4.6.** Let  $I$  be an ideal of a ring  $R$ , and let  $J$  be an ideal of  $R$  containing  $I$ , then  $J/I$  is an ideal of  $R/I$ , and

$$\frac{R/I}{J/I} = \frac{R}{J}$$

**Definition 4.8 (Noetherian).** A commutative ring  $R$  is Noetherian if every ideal of  $R$  is finitely generated. An ideal  $I$  is finitely generated if  $I = (a_1, \dots, a_n)$ , i.e., every element in  $I$  can be written as

$$r_1 a_1 + \dots + r_n a_n$$

for some  $r_1, \dots, r_n \in R$ .

**Proposition 4.7.** Let  $\bar{b}$  be the class of  $b$  in  $R/(a)$ , then

$$\frac{R/(a)}{(\bar{b})} \cong \frac{R}{(a, b)}$$

**Proposition 4.8.**  $\mathbb{Z}$  is a PID by taking the smallest positive element  $d$  in each ideal, obtaining  $(d)$ .

**Definition 4.9.**  $I$  is a prime ideal if  $R/I$  is an integral domain, and is a maximal ideal if  $R/I$  is a field.

**Definition 4.10.** Let  $I, J$  be ideals of  $R$ , then  $IJ$  is the ideal **generated** by elements  $ij, i \in I, j \in J$ .  
Note that  $IJ \subset I \cap J$ .

**Example 4.1.** In  $\mathbb{Z}$ :

$$(4) \cap (3) = (12)$$

and

$$(4) \cap (6) = (12)$$

**Definition 4.11 (Long division).** Let  $f(x) \in R[x]$  be monic, if  $g(x) \in R[x]$  be another polynomial, then there exists unique  $q, r \in R[x]$ , where  $\deg(r) < \deg(f)$ , such that

$$g(x) = f(x)q(x) + r(x)$$

Moreover,

$$g(x) + (f(x)) = r(x) + (f(x))$$

as cosets of  $(f(x))$ .

**Proposition 4.9.** Let  $I$  be an ideal of commutative  $R$ , if  $R/I$  is finite, then  $I$  is prime if and only if maximal.

**Proposition 4.10.** Let  $R$  be a PID, a nonzero ideal  $I$  is prime if and only if it is maximal.

*Proof.* Is simple proof, you just do it. □

**Theorem 4.3.** Let  $R$  be commutative, let  $f(x) \in R[x]$  be a monic polynomial of degree  $d$ , then

$$\varphi : R[x] \rightarrow R^{\oplus d}$$

where

$$\varphi : g(x) \mapsto r(x)$$

where  $r(x)$  is the remainder  $g(x) = f(x)q(x) + r(x)$  induces an isomorphism of **groups**:

$$\frac{R[x]}{(f(x))} \cong R^{\oplus d}$$

**Ring Structure:** can be induced by the map  $\varphi$ .

**Example 4.2.** Let  $f(x) = x - a$  for some  $a \in R$ , then

$$\frac{R[x]}{(x - a)} \cong R$$

**Example 4.3.** Let  $f(x) = x^2 + 1$ , then there is isomorphism of groups:

$$R \oplus R \cong \frac{R[x]}{(x^2 + 1)}$$

note that elements on the right are of the form  $a_0 + a_1x$ . One can give a ring structure on  $R \oplus R$  by  $\varphi$ .

**Example 4.4.** The ideal  $(2, x)$  is maximal in  $\mathbb{Z}[x]$ .

**Example 4.5.** The maximal ideals in  $\mathbb{C}[x]$  are precisely

$$(x - a)$$

where  $a \in \mathbb{C}$ .

**Definition 4.12 (Krull dimension).** Let  $R$  be commutative, the Krull dimension is the length of the longest chain of prime ideals in  $R$ . For example, PIDs but not fields have Krull dimension 1.

$$(0) \subset (d)$$

has length 1.

Moreover,  $k[x_1, \dots, x_n]$  have Krull dimension  $n$ :

$$(0) \subset (x_1) \subset (x_1, x_2) \subset \dots (x_1, \dots, x_n)$$

**Proposition 4.11.** The product of two fields need not to be a field:

$$\mathbb{F}_2 \times \mathbb{F}_2$$

has zero divisors!

## 4.1 Modules

**Definition 4.13 (module).** A  $R$ -module  $M$  is an abelian group with a ring action, satisfying:

1.  $r(m + n) = rm + rn$
2.  $(r + s)m = rm + sm$
3.  $(rs)m = r(sm)$
4.  $1m = m$ .

A **submodule**  $N$  of  $M$  is an abelian group such that for all  $r \in R, n \in N$ ,

$$rn \in N$$

A **homomorphism** of  $R$ -modules  $\varphi : M \rightarrow M'$  is such that

$$\begin{cases} \varphi(m + n) = \varphi(m) + \varphi(n) \\ \varphi(rm) = r\varphi(m) \end{cases}$$

Let  $R = k$  be a field, then  $R$ -modules are called vector spaces over  $k$ .

**Definition 4.14.** Let  $r \in M$  be in the center of  $M$ , then

$$rM = \{rm : m \in M\}$$

is a submodule of  $M$ . If  $I$  is an ideal of  $R$ , then

$$IM = \left\{ \sum_i r_i m_i : r_i \in I, m_i \in M \right\}$$

i.e., generated by  $rm, r \in I$  is a submodule.

**Example 4.6.** If  $R$  is not commutative, then  $R/I$  is not a ring, where  $I$  is a left ideal, but is defined as a left-module. The multiplication given by  $r(a + I) = ra + I$ .

**Definition 4.15.** An  $R$ -algebra is a ring with a ring  $R$  action.

**Theorem 4.4.** Suppose  $\varphi : M \rightarrow M'$  be a surjective  $R$ -module homomorphism, then

$$M' \cong \frac{M}{\ker \varphi}$$

**Proposition 4.12.** Let  $N$  be a submodule of an  $R$ -module  $M$ , and let  $P$  be a submodule of  $M$  containing  $N$ . Then  $P/N$  is a submodule of  $M/N$ , and

$$\frac{M/N}{P/N} \cong \frac{M}{P}$$

**Proposition 4.13.** Let  $N, P$  be submodules, then  $N + P$  is a submodule of  $M$ , and  $N \cap P$  is a submodule of  $P$ , and

$$\frac{N + P}{N} \cong \frac{P}{N \cap P}$$

## 4.2 Free Modules

**Definition 4.16.** Let  $A$  be a set, then

$$F^R(A) \cong R^{\oplus A}$$

where  $F^R(A)$  denotes the free modules over  $A$ . Every element is written as

$$\sum_{a \in A} r_a a$$

(always a finite sum). We say a module  $M = \langle A \rangle$  is finitely generated if  $A$  is finite.

**Example 4.7.** Let  $R = \mathbb{Z}[x_1, x_2, \dots]$ , viewed as a  $R$ -module over itself, it is finitely generated (by 1). But the ideal

$$(x_1, x_2, \dots) \subset R$$

as an  $R$ -module, is not finitely generated.

**Definition 4.17 (Noetherian Modules).** An  $R$ -module is Noetherian if every submodule of  $M$  is finitely generated as an  $R$ -module. The above example is not Noetherian.

**Proposition 4.14.** Let  $M$  be an  $R$ -module,  $N$  be a submodule, then  $M$  is Noetherian iff  $N, M/N$  are both Noetherian.

**Definition 4.18 (finite, finite-type  $R$ -algebra).** Let  $S$  be an  $R$ -algebra, it is called **finite** if it is finitely generated as an  $R$ -module; equivalently,

$$S \cong \frac{R^{\oplus n}}{M}$$

for some submodule  $M$ .

An  $R$ -algebra  $S$  is called **finite-type** if it is finitely generated as an  $R$ -algebra, i.e.,

$$S \cong \frac{R[x_1, \dots, x_n]}{I}$$

for some ideal  $I$ .

Elements in finite  $R$ -algebra is of the form:

$$\sum_{i=1}^n r_i s_i$$

where  $S = \langle s_1, \dots, s_n \rangle$ . Elements in finite-type  $R$ -algebra is of the form:

$$r_{11}s_1 + r_{12}s_1^2 + \dots + r_{21}s_2 + r_{22}s_2^2 + \dots + r_{nk}s_n^k$$

**Proposition 4.15.** The polynomial ring  $R[x]$  is finite-type, not finite.

**Proposition 4.16.** Let  $R$  be a PID, and  $F$  be a finitely generated free module over  $R$ , and let  $M \subset F$  be a submodule, then  $M$  is free.

**Definition 4.19 (rank of a module).** Let  $R$  be an integral domain, the rank of  $M$  is the maximal number of linearly independent elements of  $M$ .

**Definition 4.20 (SES, split).** A sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is short exact iff  $f$  is injective,  $g$  is surjective, and

$$\ker(g) = \operatorname{im}(f)$$

A SES is said to **split** if it is isomorphic in a sense that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \uparrow \cong & & \uparrow \cong & & \uparrow \cong & & \\ 0 & \longrightarrow & A' & \longrightarrow & A \oplus C & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

# Chapter 5

## Ring Theory II

This corresponds to Aluffi Chapter V.

**Proposition 5.1.** Let  $N$  be a submodule of  $M$ , where  $M$  is finitely generated, let  $\langle m_1, \dots, m_k \rangle$  be the elements whose cosets generate  $M/N$ , then

$$M = N + \langle m_1, \dots, m_k \rangle$$

*Proof.* This is the same proof that if  $N, M/N$  are finitely generated, then  $M$  is. □

**Proposition 5.2.** Let  $R$  be commutative, and  $M$  be an  $R$ -module, then TFAE:

1.  $M$  is **Noetherian**.
2.  $M$  satisfies the **ascending chain condition**. (sequence of submodules.)
3. Every nonempty family of submodules has a maximal element with respect to inclusion.

*Proof.* Noetherian implies acc: given  $N_1 \subset N_2 \subset \dots$ , then  $N = \bigcup_i N_i$  is finitely generated. □

**Proposition 5.3 (Hilbert's basis theorem).** Let  $R$  be a Noetherian ring, then  $R[x_1, \dots, x_n]$  is Noetherian. This is the same as If  $R$  is Noetherian, then  $R[x]$  is also Noetherian.

**Proposition 5.4.** Let  $a, b \in R$ , then  $(a) = (b)$  iff  $a = ub$  for some unit  $u$ .

**Definition 5.1 (prime, irreducible elements).** Let  $R$  be commutative

1. Let  $R$  be an integral domain, an element  $a \in R$  is **prime** if the ideal  $(a)$  is prime.
2. An element  $a \in R$  is **irreducible** if  $a$  is not a unit and

$$a = bc$$

implies  $b$  is a unit or  $c$  is a unit. Equivalently,  $a$  is irreducible if  $(a) \subset (b)$  implies  $(b) = (a)$  or  $(b) = (1) = R$ , i.e.,  $(a)$  is maximal in principal ideals.



**Proposition 5.5.** Let  $R$  be an **integral domain**, then

$$\text{nonzero prime elements} \Rightarrow \text{irreducible}$$

**Definition 5.2 (factorization).**  $r \in R$  has a factorization if there exists **finite** irreducibles  $q_1, \dots, q_n$  such that

$$r = q_1 \dots q_n$$

**Proposition 5.6.** Let  $R$  be an integral domain, and let  $r$  be a nonzero, nonunit element of  $R$ . Assume that every ascending chain of principal ideals,

$$(r) \subset (r_1) \subset (r_2) \dots$$

stabilizes. Then  $r$  has a factorization into irreducibles.

Of course if a ring is ACC, then factorizations exist.

**Proposition 5.7.** Factorization exists in Noetherian rings.

**Example 5.1.** A non-Noetherian ring but factorization still exists:

$$\mathbb{Z}[x_1, \dots, x_n]$$

**Proposition 5.8.** Let  $R$  be Noetherian and  $I$  be an ideal, then  $R/I$  is also Noetherian.

## 5.1 UFD, PID, ED

**Definition 5.3 (gcd).** Let  $a, b \in R$ , then the gcd of  $a, b$  is  $d$  such that  $(d)$  is the smallest principal ideal such that

$$(a, b) \subset (d)$$

**Proposition 5.9.** Let  $R$  be a UFD, and  $a, b, c \in R$  be nonzero, then

$$(a) \subset (b) \iff m(b) \subset m(a)$$

where  $m(a)$  is the multiset of irreducible factors of  $a$ . Moreover, the irreducible factors of  $bc$  are the collection of irreducible factors of  $b$  and  $c$ .

**Proposition 5.10.** Let  $R$  be a UFD, then gcd of any  $a, b$  exists.

**Example 5.2.** There exists Noetherian rings that are not UFD.

$$\frac{\mathbb{C}[x, y, z, w]}{(xw - yz)}$$

since  $r = xw = yz$ .

**Proposition 5.11.** In UFD,  $a$  is irreducible implies  $a$  is prime.

*Proof.* Assume  $bc \in (a)$ , then  $(bc) \subset (a)$ , hence the multiset of irreducible factors of  $a$  is contained in the multiset of  $b, c$ , but  $a$  is irreducible implies that  $a$  must be among the factors of  $b$  or  $c$ .  $\square$

**Theorem 5.1.** An integral domain  $R$  is a UFD if and only if

1. The acc holds for principal ideals in  $R$ .
2. Every irreducible element of  $R$  is prime.

**Theorem 5.2.** If  $R$  is a PID, and  $a, b \in R$ , then  $d = \gcd(a, b)$  iff  $(a, b) = (d)$ . In other words, there exists  $r, s \in R$ , such that

$$d = ra + sb$$

**Example 5.3.** UFD but not PID:

$$\mathbb{Z}[x]$$

**Definition 5.4 (Euclidean domain).** A Euclidean valuation on an integral domain  $R$  is an valuation: for all  $a \in R$ , and all nonzero  $b \in R$ , there exists  $q, r$  such that

$$a = qb + r$$

with either  $r = 0$  or  $v(r) < v(b)$ . An integral domain is a ED if it admits a Euclidean valuation.

## 5.2 $R(x)$ and Field of Fractions

**Theorem 5.3.** Let  $R$  be a UFD, then  $R[x]$  is also a UFD.

**Example 5.4.**  $\mathbb{Z}[x], \mathbb{Z}[x_1, \dots, x_n]$  are UFD.

**Definition 5.5 (Field of fractions).** Let  $R$  be an integral domain, then the field of fractions is

$$\text{Frac}(R) = \left\{ \frac{a}{r} : a, r \in R, r \neq 0 \right\}$$

where  $\frac{a}{r}$  is the equivalence given by  $\frac{a}{r} \sim \frac{b}{s} \iff as = br$ .

**Definition 5.6.** The field of fractions  $R[x]$  is the field of rational functions with coefficients in  $R$ : elements are of the form

$$\frac{p(x)}{q(x)}, q(x) \neq 0$$

denoted as  $R(x)$ .

**Definition 5.7 (primitive).** Let  $R$  be a UFD,  $f$  is primitive if and only if  $\gcd(a_0, \dots, a_d) = 1$ .

**Proposition 5.12.** Let  $R$  be a UFD, and  $K$  be its field of fractions, let  $f \in R[x]$  be a nonconstant, irreducible polynomial, then  $f$  is irreducible in  $K[x]$ .

## 5.3 Irreducibility

**Proposition 5.13.** Let  $R$  be an ID, then  $f \in R[x]$  of degree  $d$  can have at most  $d$  roots.

This is not true for non-ID, for example,  $x^2 + 2$  over  $\mathbb{Z}/6\mathbb{Z}$ .

**Proposition 5.14.** Let  $k$  be a field, then  $f \in k[x]$  of degree 2 or 3 is irreducible iff it has no root in  $k$ .

**Example 5.5.**  $t^2 + t + 1$  is irreducible over  $\mathbb{F}_2$  (therefore over  $\mathbb{Q}$ ).

**Proposition 5.15 (rational root theorem).** Let  $R$  be a UFD, and  $K$  be its field of fractions, let

$$f(x) = a_0 + a_1x + \dots + a_nx^n \in R[x]$$

if  $\frac{p}{q} \in K$  is a root, ( $\gcd(p, q) = 1$ ), then

$$p \text{ divides } a_0, q \text{ divides } a_n$$

**Proposition 5.16.** Let  $k$  be a field, and  $f(t) \in k[t]$  be a nonzero irreducible polynomial. Then

$$F = \frac{k[t]}{(f(t))}$$

is a field, where  $k$  embeds into  $F$ . Moreover,  $f(x) \in k[x]$  has a root in  $F$ , which is

$$t + (f(t))$$

**Proposition 5.17.** A field is algebraically closed

- $k$  is algebraically closed  $\iff$  all irreducible polynomials in  $k[x]$  have degree 1
- $\iff$  every nonconstant polynomial  $f$  factors completely into linear factors
- $\iff$  every nonconstant  $f$  has a root in  $k$

**Theorem 5.4.** Finite fields  $\mathbb{F}_{p^n}$  are not algebraically closed.

**Example 5.6.** The nonconstant irreducible polynomials of  $\mathbb{R}[x]$  are precisely those of degree 1 and quadratic  $f = ax^2 + bx + c$  where  $b^2 - 4ac < 0$ .

**Proposition 5.18.** Let  $f \in \mathbb{Z}[x]$  be such that  $\gcd(a_0, \dots, a_n) = 1$ , and let  $p$  be prime. If  $f \pmod p$  has the same degree as  $f$ , and is irreducible over  $\mathbb{F}_p$ , then  $f$  is irreducible over  $\mathbb{Z}$ .



**Warning 5.5.** This is important! We can show a polynomial is irreducible over  $\mathbb{Z}$  by showing it is irreducible over  $\mathbb{F}_p$  for some  $p$ .

**Example 5.7.** There exists reducible polynomial over  $\mathbb{Z}$  but irreducible over  $\mathbb{F}_p$  for every prime  $p: x^4 + 1$ . (Hint: Legendre symbol).

**Proposition 5.19 (Generalized Eisenstein).** Let  $R$  be a commutative ring, let  $p$  be a prime ideal in  $R$ , let  $f \in R[x]$ , assume that

1.  $a_n \notin p$ .
2.  $a_i \in p$ .
3.  $a_0 \notin p^2$ .

then  $f$  is not the product of polynomials with degree strictly less than  $\deg(f)$ .



**Warning 5.6.** Generalized Eisenstein works for commutative rings! Some examples:

$$\mathbb{C}[x, y], \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{(x_1x_2 - x_3x_4)}$$

**Example 5.8.** For all  $n$  and all primes  $p$ , the polynomial  $x^n - p$  is irreducible over  $\mathbb{Z}$ .

**Example 5.9.** Let  $p$  be a prime, then the cyclotomic polynomial  $\Phi_p(x)$  is irreducible.

$$1 + x + x^2 + \cdots + x^{p-1}$$

*Proof.*

$$f(x) = \frac{x^p - 1}{x - 1}$$

$$f(x+1) = \frac{(x+1)^p - 1}{x}$$

We see that coefficients are now

$$\binom{p}{k}, k = 1, \dots, p-1$$

hence  $p$  divides all but leading coefficient. □

## 5.4 CRT

**Theorem 5.7 (CRT).** Let  $I_1, \dots, I_k$  be ideals of  $R$  such that  $I_i + I_j = (1)$  for all  $i \neq j$ . Then

$$\frac{R}{I_1 \cap \cdots \cap I_k} = \frac{R}{I_1 I_2 \cdots I_k} \cong \frac{R}{I_1} \times \cdots \times \frac{R}{I_k}$$

(It uses if  $I_i + I_j = (1)$ , then  $I_1 \cdots I_k = I_1 \cap \cdots \cap I_k$ ).

**Proposition 5.20 (CRT in PID).** Let  $R$  be a PID, and let  $a_1, \dots, a_k$  be elements such that  $\gcd(a_i, a_j) = 1$ , let  $a = a_1 \dots a_k$ , then

$$\frac{R}{(a)} \cong \frac{R}{(a_1)} \times \dots \times \frac{R}{(a_k)}$$

## Chapter 6

# Linear Algebra I

This corresponds to Aluffi Chapter VI, excluding Section 4-5.

### 6.1 basis, free modules, IBN

**Proposition 6.1 (Zorn's).** Every module  $M$  has maximal linearly independent set. In other words, let  $S \subset M$  be a linearly independent subset. Then there exists a maximal linearly independent subset of  $M$  containing  $S$ .

**Definition 6.1 (basis).** A subset  $S \subset M$  is a basis if it is linearly independent and generates  $M$ . Every element in  $M$  can be written as

$$m = \sum_{s_i \in S} r_i s_i$$

where only finitely many terms are nonzero.

((2)  $\subset \mathbb{Z}$  is maximal but not a basis).

**Proposition 6.2.** Regarding basis,

1. An  $R$ -module  $M$  is free iff it admits a basis. (Any vector space is free as a  $k$ -module).
2. The converse holds when  $R = k$ : let  $B$  be a maximal linearly independent subset of  $M = V$ , then  $B$  is a basis.
3. When  $R = k$ , let  $S$  be a linearly independent subset, then there exists a basis  $B$  of  $V$  containing  $S$ . If  $B$  is a minimal generating set for  $V$ , then  $B$  is also a basis.

**Proposition 6.3.** Let  $R$  be an **integral domain**, and  $M$  a free  $R$ -module, let  $B$  be a maximal linearly independent subset of  $M$ . Then the rank of  $M$  is  $|B|$ . If  $S$  is any independent subset, then

$$|S| \leq |B|$$

**Example 6.1.** The basis  $\mathbb{C}[x]$  over  $\mathbb{C}$  is  $\{1, x, \dots\}$ , hence an uncountable subset of  $\mathbb{C}[x]$  is necessarily linearly dependent.

**Proposition 6.4.** Let  $R$  be an **integral domain**, let  $m, n$  be nonnegative integers,

$$R^m \cong R^n \iff m = n$$

If  $R$  satisfies the above, we say it satisfies the invariant basis number property. (All commutative rings satisfy this)!

**Definition 6.2 (rank of a module).** Let  $R$  be an integral domain, the rank of a free module  $M$  is the size of the maximal linearly independent subset of  $M$ .

**Proposition 6.5.** Let  $R$  be an integral domain, and let  $M$  be a free  $R$ -module, assume that  $M$  is generated by  $S$ :  $M = \langle S \rangle$ , then  $S$  contains a maximal linearly independent subset of  $M$ .

## 6.2 Homomorphisms $R^n \rightarrow R^m$

**Proposition 6.6.** Let  $\alpha : M \rightarrow N$  be a homomorphism of finitely generated modules, and let  $P$  be a matrix representing it wrt any basis of  $M, N$ , then with respect to any other choice of bases of  $M, N$ ,  $\alpha$  is of the form

$$N_1 \cdot P \cdot N_2$$

where  $N_1, N_2$  are invertible matrices.

**Proposition 6.7.** Two matrices  $P, Q \in M_n(R)$  are equivalent if they are the same up to elementary operations, i.e., iff the same up to multiplications by elementary matrices. In other words,  $M, N$  are equivalent if there exists invertible  $P_1, P_2$  such that

$$M = P_1 N P_2$$

**Example 6.2.** The matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

interchanges the second and fourth row of a  $4 \times 4$  matrix. Multiplying on the right by

$$\begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

adds to the third column of a  $m \times 3$  matrix the  $c$ -multiple of the first column.

**Proposition 6.8.** Let  $k$  be a field, then  $\text{GL}_n(k)$  is generated by elementary matrices!

**Proposition 6.9.** Over a field, every  $m \times n$  matrix is equivalent to a matrix is equivalent to a matrix of the form:

$$\left( \begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

In other words, up to multiplying some invertible matrix  $N_1, N_2$  on the left and right, every matrix is of the above form.

**Proposition 6.10.** Let  $R$  be commutative, a square matrix  $A$  is invertible iff  $\det(A)$  is a unit in  $R$ ; The determinant is a homomorphism  $\det : \text{GL}_n(R) \rightarrow (R^*, \cdot)$ , and for  $A, B \in M_n(R)$ ,

$$\det(AB) = \det(BA)$$

**Proposition 6.11.** The row rank of a matrix over a field is equal to its column rank, recall that every matrix is equivalent to a matrix of the form

$$\left( \begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

From this we also know that

$$\dim V = \text{rank of } \alpha + \text{nullity of } \alpha$$

**Definition 6.3 (adjoint matrix).** Let  $M$  be an  $n \times n$  matrix, the adjoint matrix  $\text{adj}(A)$  is such that

$$A \cdot \text{adj}(A) = \text{adj}(A)A = \det(A)I_n$$

**Proposition 6.12 (Nakayama's lemma).** (Different versions of the same lemma).

1. Let  $R$  be a commutative ring,  $M$  and  $R$ -module, and let  $a \in R$  be a nilpotent element, then

$$M = 0 \iff aM = M$$

2. Let  $J$  be the Jacobson radical of  $R$ , where  $M$  is finitely generated  $R$ -module. If  $M = JM + N$ , then  $M = N$ . (A special case is when  $R$  is a local ring and  $\mathfrak{m} = J$ ).

### 6.3 Invariants in Linear Transformations

**Definition 6.4 (similar matrix).** Two matrices  $A, B$  are similar iff there exists invertible  $P$  such that

$$A = PBP^{-1}$$

For example,  $A$  is similar to  $A^t$ .

**Proposition 6.13.** Similar implies equivalent, but equivalent does not imply similar.

**Proposition 6.14.** Let  $\alpha$  be a linear transformation of  $R^n$ , a free  $R$ -module, then

$$\det(\alpha) \neq 0 \iff \alpha \text{ is injective}$$



**Proposition 6.15.** For  $A, B \in M_n(R)$ ,

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

If  $A, B$  are similar, then

$$\operatorname{tr}(A) = \operatorname{tr}(B)$$

**Definition 6.5 (characteristic polynomial).** Let  $\alpha \in \operatorname{End}(F)$ , where  $F = R^n$ , then the characteristic polynomial of  $\alpha$  is

$$P_\alpha(t) = \det(tI - \alpha)$$

**Theorem 6.1.** Let  $\alpha \in \operatorname{End}(F)$ , and  $F = R^n$ , let  $P_\alpha(t) = t^n + a_{n-1}t^{n-1} + \dots + a_0$  be characteristic polynomial,

1.  $P_\alpha(t)$  is of degree  $n$ .
2.  $a_{n-1}t^{n-1}$  is such that  $a_{n-1} = -\operatorname{tr}(\alpha)$ .
3.  $a_0 = (-1)^n \det(\alpha)$ .
4. If  $\alpha, \beta$  are similar, then  $\det(\alpha) = \det(\beta)$ .
5. We have

$$P_\alpha(t) = t^n - \operatorname{tr}(\alpha)t^{n-1} + \dots + (-1)^n \det(\alpha)$$

**Theorem 6.2.** Let  $\alpha : V \rightarrow V$  be an endomorphism over  $k$ , then

$$\operatorname{tr}(\alpha) = \sum_{i=1}^n \lambda_i$$

where  $\lambda_i$  is the eigenvalues of  $V$  over the algebraic closure  $\bar{k}$ .

**Example 6.3.** Having the same characteristic polynomial does not guarantee they are similar:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

are not similar.

**Definition 6.6 (annihilator ideal).** Given  $\alpha \in \operatorname{End}(F)$ , and  $f(x) \in R[x]$ , the annihilator ideal of  $\alpha$  is

$$\mathcal{A}(\alpha) = \{f \in R[x] : f(\alpha) = 0\}$$

**Definition 6.7.** Let  $k$  be a field, the minimal polynomial of  $\alpha$  is the monic generator  $m_\alpha(t)$  of  $\mathcal{A}(\alpha) = ((m_\alpha(t)))$ .

**Proposition 6.16.** If  $\alpha, \beta$  are similar, then

$$\mathcal{A}(\alpha) = \mathcal{A}(\beta)$$

**Proposition 6.17 (Cayley-Hamilton).** Let  $P_\alpha(t)$  be the characteristic polynomial of  $\alpha$ , then

$$P_\alpha(\alpha) = 0$$

**Proposition 6.18.** If  $\alpha, \beta$  are similar, then they have the same eigenvalues. Moreover,  $\lambda \in R$  is an eigenvalue of  $\alpha$  iff it is a root of the characteristic polynomial of  $\alpha$ .

**Proposition 6.19.** If  $R$  is algebraically closed, then  $\alpha$  has exactly  $n$  eigenvalues; more generally, it has at most  $n$  eigenvalues.

**Proposition 6.20.** The dimension of the eigenspace wrt  $\lambda$  is always less than or equal to its algebraic multiplicity  $(t - \lambda)^k$ . If for each  $\lambda$ , they are equal, then  $\alpha$  is diagonalizable with respect to an eigenbasis.

## 6.4 The canonical form

**Proposition 6.21.** Let  $R$  be a PID, then submodules of a free  $R$ -module  $M$  is also free.

**Proposition 6.22.** If  $M$  is finitely generated as an  $R$ -module, then

$$M \cong \frac{R^{\oplus n}}{N}$$

for some submodule of  $R^{\oplus n}$ .

**Proposition 6.23.** Recall: every finitely generated  $R$ -module over a PID (meaning when  $R$  is a PID) is a direct sum of cyclic modules. Let  $R = k[t]$ , then

$$\frac{k[t]}{(f(t))}$$

is cyclic viewed as a  $k[t]$ -module. (The generator is 1.).

**Proposition 6.24.** We call a module finitely generated over a PID if  $R$  is a PID and  $M$  is finitely generated as an  $R$ -module,

**Proposition 6.25.** There is a one-to-one correspondence

$$\{(V, \alpha) : \alpha : V \rightarrow V\} \leftrightarrow \{k[t] - \text{modules of } V\}$$

The isomorphism  $(\rightarrow)$  is given by

$$(V, \alpha) \mapsto (k[t] \rightarrow \text{End}(V) : t \mapsto \alpha)$$

and  $(\leftarrow)$  is given by

$$(\varphi : k[t] \rightarrow \text{End}(V)) \mapsto (V, \varphi(t))$$

**Proposition 6.26.** Let  $k$  be a field, and  $V$  finite dimensional vector space, let  $\alpha$  be a linear transformation, endow  $V$  with the  $k[t]$ -structure, there exists distinct monic irreducible polynomials  $p_i(t) \in k[t]$  such that

$$V \cong \bigoplus_{i,j} \frac{k[t]}{(p_i(t)^{r_{ij}})}$$

as  $k[t]$ -modules. Moreover, there exists monic  $f_1, \dots, f_m$  such that

$$V \cong \frac{k[t]}{(f_1(t))} \oplus \dots \oplus \frac{k[t]}{(f_m(t))}$$

as  $k[t]$ -modules, where  $f_1(t) \mid \dots \mid f_m(t)$ . The characteristic and minimal polynomials are such that

$$P_\alpha(t) = f_1(t) \dots f_m(t) = \prod_{i,j} p_i(t)^{r_{ij}}$$

and

$$m_\alpha(t) = f_m(t)$$

**Proposition 6.27.** If  $P_\alpha(t)$  factors completely over  $k$ , i.e.,

$$P_\alpha(t) = \prod_{i=1}^s (t - \lambda_i)^{m_i}$$

where  $\lambda_i$  are distinct eigenvalues of  $\alpha$ , then

$$V \cong \bigoplus_{i=1}^s \frac{k[t]}{(t - \lambda_i)^{m_i}}$$

where  $m_i = \sum_j r_{ij}$  in the above expression. Moreover,

$$m_\alpha(t) = \prod_{i=1}^s (t - \lambda_i)^{\max_j \{r_{ij}\}}$$

**Theorem 6.3.** The above translated into Jordan form is:

$$m_\alpha(t) = \prod_{i=1}^s (t - \lambda_i)^{\text{biggest Jordan block of } \lambda_i}$$

where  $\lambda_i$  are distinct eigenvalues.

**Example 6.4.** One use of the Jordan canonical form is the enumeration of all possible similarity classes of transformations with given eigenvalues. For example, there are 5 similarity classes of linear transformations with a single eigenvalue  $\lambda$  with algebraic multiplicity 4, over a 4-dimensional vector space: indeed, there are 5 different ways to stack together Jordan blocks corresponding to the same eigenvalue, within a  $4 \times 4$  square matrix:

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

**Proposition 6.28.** Two matrices are similar if and only if they have the same Jordan form.

**Proposition 6.29.** The dimension of the eigenspace with respect to  $\lambda$  is the **number** of the Jordan blocks with respect to  $\lambda$ .

**Proposition 6.30.** Assume  $P_\alpha(t)$  factors completely over  $k$ , then  $\alpha$  is diagonalizable iff either of following :

1. The dimension of eigenspace=algebraic multiplicity of  $\lambda$  for all eigenvalues  $\lambda$  of  $\alpha$ .
2. Minimal polynomial  $m_\alpha(t)$  has no repeated roots.



**Warning 6.4.** This is important.

**Proposition 6.31.** Let  $k$  be algebraically closed, the minimal polynomial coincide with the characteristic iff the Jordan form has a single Jordan block for each distinct eigenvalue.

# Chapter 7

## Linear Algebra II

This corresponds to Aluffi Chapter VIII. ( Section 2.1, 2.2 Section 3 Section 4)

### 7.1 Tensor

**Definition 7.1 (bilinear).** Let  $M, N, P$  be  $R$ -modules. A function  $\varphi : M \times N \rightarrow P$  is  $R$ -bilinear if

1. For all  $m \in M, n \mapsto (m, n)$  is an  $R$ -module homomorphism  $N \rightarrow P$ .
2. For all  $n \in N, m \mapsto (m, n)$  is an  $R$ -module homomorphism  $M \rightarrow P$ .

In other words,

$$\varphi(m, r_1 n_1 + r_2 n_2) = r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2)$$

similarly for  $M \rightarrow P$ .

**Proposition 7.1 (Tensor product).** The tensor product can be constructed as follows:

1. Take the **free  $R$ -module** generated by symbols  $\{m \otimes n \mid m \in M, n \in N\}$ .
2. **Quotient** by the submodule generated by the relations (to enforce bilinearity):
  - $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n,$
  - $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2,$
  - $(r \cdot m) \otimes n = m \otimes (r \cdot n) = r \cdot (m \otimes n)$  for  $r \in R$ .

Thus, elements of  $M \otimes_R N$  are finite sums of the form  $\sum_i m_i \otimes n_i$ , subject to the above rules.

#### Key Properties of Tensor Products

1. **Bilinearity:** The map  $\otimes : M \times N \rightarrow M \otimes_R N$  is  $R$ -bilinear.
2. **Functoriality:** If  $f : M \rightarrow M'$  and  $g : N \rightarrow N'$  are  $R$ -linear, there is an induced map:

$$f \otimes g : M \otimes_R N \rightarrow M' \otimes_R N', \quad (f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

3. **Associativity:**  $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$ .
4. **Commutativity:**  $M \otimes_R N \cong N \otimes_R M$  (if  $R$  is commutative).
5. **Base Change:** If  $S$  is an  $R$ -algebra, then  $M \otimes_R S$  is an  $S$ -module.

**Proposition 7.2 (universal property).** Every  $R$ -bilinear map  $\varphi : M \times N \rightarrow P$  factors uniquely through the tensor product  $M \otimes_R N$ ,

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & P \\ \otimes \downarrow & \nearrow \exists! \bar{\varphi} & \\ M \otimes_R N & & \end{array}$$

in such a way that the map  $\bar{\varphi}$  is unique.

**Example 7.1.** For all  $R$ -modules,

1.  $R \otimes_R N \cong N$ .
2.  $M \otimes_R N \cong N \otimes_R M$ .

**Proposition 7.3.** Let  $\alpha, \beta : M \otimes N \rightarrow P$ , if

$$\alpha(m \otimes n) = \beta(m \otimes n)$$

for all  $m \in M, n \in N$ , then  $\alpha = \beta$ . (This means it suffices to check on pure tensors).

## 7.2 Hom and Tensor

**Proposition 7.4.** Let  $\alpha : M_1 \rightarrow M_2$  be an  $R$ -module homomorphism, let  $N$  be an  $R$ -module, there is an induced  $R$ -linear map

$$\alpha \otimes N : M_1 \otimes_R N \rightarrow M_2 \otimes_R N$$

On pure tensors, this map is given by

$$m \otimes n \mapsto \alpha(m) \otimes n$$

**Theorem 7.1.** For all  $R$ -modules  $M, N, P$ , there is an isomorphism of  $R$ -modules

$$\text{Hom}_R(M, \text{Hom}(N, P)) \cong \text{Hom}_R(M \otimes_R N, P)$$

*Proof.* This says any bilinear map from  $M \times N$  comes from  $M \otimes_R N$ . □

**Proposition 7.5.** For all  $R$ -modules,  $M_1, M_2, N$ , we have

$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N)$$

The same statement is true for  $\sum_{\alpha} M_{\alpha} \otimes N$ . This also implies that

$$R^{\oplus n} \otimes_R R^{\oplus m} \cong R^{\oplus nm}$$

**Proposition 7.6.** Let  $M, N$  be free  $R$ -modules of rank  $m, n$ , then  $M \otimes_R N$  has rank  $mn$ . (Let  $e_1, \dots, e_m$  generate  $M$ ,  $v_1, \dots, v_n$  generate  $N$ , where  $M, N$  are free  $R$ -modules, then  $M \otimes_R N$  is generated by  $e_i \otimes v_j$ , and these  $mn$  elements are the basis for  $M \otimes_R N$ .)



**Warning 7.2.** The next examples are important.

**Proposition 7.7.** Let  $N$  be an  $R$ -module, and  $I$  be an ideal of  $R$ , then

$$\frac{R}{I} \otimes_R N \cong \frac{N}{IN}$$

( $R \otimes_R N \cong N$ ). Moreover, let  $J \subset R$  also be an ideal, then

$$\frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}$$

**Proposition 7.8.** Let  $S$  be a ring with an  $R$  action, and  $R$  is a ring, then

$$R[x] \otimes_R S \cong S[x]$$

**Proposition 7.9.** Let  $R, S$  be the above,

$$\frac{R[x]}{(f(x))} \otimes_R S \cong \frac{S[x]}{(f(x))}$$

For example,

$$\mathbb{C} \otimes_R \mathbb{C} \cong \frac{\mathbb{R}[x]}{(x^2+1)} \otimes_R \mathbb{C} \cong \frac{\mathbb{C}[x]}{(x^2+1)}$$

**Example 7.2.** We have

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{\gcd(m, n)}$$

(Recall that  $(m) + (n) = \gcd(m, n)$  in  $\mathbb{Z}$ , and  $(m) \cap (n) = (\text{lcm}(m, n))$ ). For example,

$$\frac{\mathbb{Z}}{2\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{3\mathbb{Z}} = 0$$

So if  $\gcd(m, n) = 0$ , then

$$\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}} = 0$$

**Definition 7.2 (reduced ring).** Let  $R$  be a ring, it is **reduced** if there are no nonzero nilpotent elements.

## 7.3 Multilinear Algebra: Wedge and Symmetric Product

Every  $\varphi : M_1 \times \dots \times M_k \rightarrow P$  factors unique through  $M_1 \otimes \dots \otimes M_k$ , and we will denote

$$M^{\otimes k} := M \otimes_R \dots \otimes_R M \text{ } k \text{ times}$$

**Definition 7.3 (symmetric and alternating map).** Let  $\varphi : M^k \rightarrow P$ , then it is called **symmetric** if for all  $\sigma \in S_k$ , and all  $m_1, \dots, m_k$ , we have

$$\varphi(m_{\sigma(1)}, \dots, m_{\sigma(k)}) = \varphi(m_1, \dots, m_k)$$

And  $\varphi : M^k \rightarrow P$  is called **alternating** if

$$\varphi(m_1, \dots, m_k) = 0 \text{ whenever } m_i = m_j \text{ for some } i \neq j$$

**Proposition 7.10.** Let  $\varphi : M^k \rightarrow P$  be  $R$ -multilinear, then

1. If  $\varphi$  is alternating, then for all  $\sigma \in S_k$ ,

$$\varphi(m_{\sigma(1)}, \dots, m_{\sigma(k)}) = (-1)^\sigma \varphi(m_1, \dots, m_k)$$

2. If 2 is a unit in  $R$ , and for all  $\sigma \in S_k$ ,  $\varphi(m_{\sigma(1)}, \dots, m_{\sigma(k)}) = (-1)^\sigma \varphi(m_1, \dots, m_k)$ , then  $\varphi$  is alternating.

It suffices to reduce to the case where  $k = 2$ .

**Definition 7.4 (Wedge product).** The module  $\bigwedge^k(M)$  is generated by pure alternating tensors:

$$e_{i_1} \wedge \dots \wedge e_{i_k}$$

where  $1 \leq i_1 < \dots < i_l \leq r$ . For example, suppose  $M = V$  is a 3-dimensional vector space, then  $V \wedge V$  has a basis

$$v_1 \wedge v_2, v_1 \wedge v_3, v_2 \wedge v_3$$

The dimension of  $V \wedge V$  is  $\frac{n(n-1)}{2}$  if  $\dim(V) = n$ .

Next we generalize it.

**Proposition 7.11.** Let  $R$  be commutative, and  $M$  a free  $R$ -module of rank  $n$ , then

$$\bigwedge^k(M) \text{ is a free } R\text{-module of rank } \binom{n}{k}$$

**Example 7.3.** If  $M$  is a free module of rank  $n$ , then

$$\bigwedge^n(M) \cong R$$

where the isomorphism  $\varphi : (e_{i_1}, \dots, e_{i_n}) \mapsto \begin{cases} \pm 1 & \text{if } i_1, \dots, i_n \text{ are distinct} \\ 0 & \end{cases}$

**Proposition 7.12.** Let  $\text{Sym}^n(V)$  be the **symmetric product**. A basis for  $\text{Sym}^n(V)$  is given by the **monomials**:

$$\left\{ e_1^{k_1} e_2^{k_2} \dots e_d^{k_d} \mid k_1 + \dots + k_d = n, k_i \geq 0 \right\},$$

where  $e_i^{k_i}$  denotes the symmetric product  $e_i \dots e_i$  ( $k_i$  times).



**Proposition 7.13.** Let  $V$  have dimension  $n$  with basis  $\{e_1, \dots, e_n\}$ , then  $\text{Sym}^k(V)$  is spanned by basis:

$$\{e_{i_1} \dots e_{i_k}, 1 \leq i_1 \leq \dots \leq i_k \leq n\}$$

(It contains the equality case compared to the wedge product). Moreover, the dimension of  $\text{Sym}^k V$  is

$$\binom{n+k-1}{k}$$

**Example 7.4.**  $\text{Sym}^2(V)$  for  $\dim V = 2$  Let  $V$  have basis  $\{e_1, e_2, e_3\}$ . Then:

$$\text{Sym}^2(V) = \text{span}\{e_1 e_1, e_1 e_2, e_2 e_2\},$$

where:

$$e_1 \odot e_1 = e_1 \otimes e_1,$$

$$e_1 \odot e_2 = \frac{1}{2}(e_1 \otimes e_2 + e_2 \otimes e_1),$$

$$e_2 \odot e_2 = e_2 \otimes e_2.$$

**Dimension:**  $\binom{2+2-1}{2} = 3$ .

**Definition 7.5 (determinant).** Let  $F$  be a free  $R$ -module of rank  $n$ , then

$$\bigwedge^n F$$

is called the determinant of  $F$ ,  $\det(F)$ . (In other words, it is the top exterior power). Recall that

$$\bigwedge^n F \cong R$$

since it is one-dimensional and spanned by  $\{e_1 \wedge \dots \wedge e_n\}$ .



**Warning 7.3.** Again, two matrices are similar if and only if they have the same jordan normal form!



**Warning 7.4.** Let  $M$  be an  $n \times n$  matrix over  $k$ , then factor its characteristic polynomial  $p(t)$  over its algebraic closure

$$p(t) = (t - \lambda_1) \dots (t - \lambda_d)$$

where  $\lambda_i$  are its eigenvalues in  $\bar{k}$ . Then

$$\text{tr}(M) = \sum_i \lambda_i, \quad \det(M) = \prod_i \lambda_i$$

**Proposition 7.14.** Let  $M \in \text{GL}_n(k)$ , then  $M$  is triangularizable iff the characteristic polynomial factors completely into linear factors. It is diagonalizable iff the characteristic polynomial completely factors into linear factors, and the minimal polynomial has no repeated roots.

# Chapter 8

## Field Theory

Aluffi Chapter VII.

**Definition 8.1 (radical).** The **radical** of an ideal  $I \subset R$  is

$$\text{rad}(I) = \sqrt{I} = \{a \in R : a^n \in I \text{ for some } n\}$$

An ideal is called radical if for any  $a \in R$ ,  $a^n \in I$  for some  $n$ , then  $a \in I$ .

**Proposition 8.1.** The radical  $\sqrt{I}$  of an ideal  $I$  in  $R$  is an ideal. Moreover,  $\sqrt{I}$  is radical.

**Example 8.1.** The nilradical of  $R$  is  $\sqrt{(0)}$ , i.e., the radical of the zero ideal.

**Proposition 8.2.** Any ring homomorphism from a field to a nonzero ring is injective.

**Proposition 8.3.** The characteristic of a field is either 0 or a prime number. (This is also true for integral domains). Moreover, let  $k \subset E$  be an extension, then  $\text{char}(k) = \text{char}(E)$ . Moreover, for such extension,  $E$  is a vector space over  $k$ .

**Definition 8.2 (finite field extension).** A field extension  $k \subset F$  is finite of degree  $n$ , if  $F$  has is a dimension  $n$  vector space over  $k$ . We denote

$$[F : k] = \dim_k(F)$$

**Example 8.2.** Let  $k$  be a field, and  $f$  is an irreducible polynomial over  $k$ , then

$$K = \frac{k[t]}{(f(t))}$$

is an extension in which  $f$  has a root. (To see this is a field, we see  $f(t)$  is irreducible, which is prime, which is maximal in  $k[t]$ ).

**Definition 8.3 (simple extension).** A field extension  $k \subset F$  is simple if there exists  $\alpha \in F$  such that  $F = k(\alpha)$ , where  $k(\alpha)$  is the smallest field containing  $\alpha$  and  $k$ . If  $k(\alpha)/k$  is a finite extension, then  $\alpha$  is algebraic, if infinite, then  $\alpha$  is called transcendental.

**Example 8.3.** The extension  $k \subset \frac{k[t]}{(f(t))}$  is simple because

$$\frac{k[t]}{(f(t))} \cong k(\alpha)$$

for some  $\alpha$  such that  $f(\alpha) = 0$ .

**Proposition 8.4.** Let  $k \subset k(\alpha)$  be a simple extension, then consider the evaluation map

$$\varepsilon : f(t) \mapsto f(\alpha)$$

Then  $\varepsilon$  is not injective iff  $k(\alpha)$  is a finite extension, i.e.,  $\alpha$  is algebraic, thus there exists a monic irreducible polynomial  $p$  such that

$$k(\alpha) = \frac{k[t]}{(p(t))}$$

And  $\varepsilon$  is injective iff  $\alpha$  is transcendental.

**Proposition 8.5 (lifting).** Let  $k_1 \subset k_1(\alpha_1), k_2 \subset k_2(\alpha_2)$  be two simple finite extensions, then let  $p_1, p_2$  be the minimal polynomials of  $\alpha_1, \alpha_2$ , let  $i : k_1 \rightarrow k_2$  be an isomorphism such that

$$i(p_1(t)) = p_2(t)$$

Then there exists a unique isomorphism  $j : k_1(\alpha_1) \rightarrow k_2(\alpha_2)$  such that  $j = i$  on  $k_1$  and

$$j(\alpha_1) = \alpha_2$$

This says that we can extend isomorphisms between fields into their simple extensions provided that this isomorphism agrees with the structure of the extensions.

**Definition 8.4 (Aut group).** Let  $k \subset F$  be an extension, then the group of automorphisms of this extension, denoted  $\text{Aut}_k(F)$  is the group of automorphisms  $\varphi : F \rightarrow F$  that fixes  $k$ ,  $\varphi(x) = x$  for all  $x \in k$ ,  $\varphi \in \text{Aut}_k(F)$ .

**Proposition 8.6.** Let  $k \subset k(\alpha)$ , and  $p(x)$  be the minimal polynomial over  $k$ , then

$$|\text{Aut}_k(k(\alpha))| = \text{number of distinct roots of } p \text{ in } k(\alpha)$$

and

$$|\text{Aut}_k(k(\alpha))| \leq [k(\alpha) : k] = \deg(p)$$

with equality if and only if  $p(x)$  factors over  $k(\alpha)$  as a product of distinct linear factors.

**Proposition 8.7.** Let  $k \subset k(\alpha) = F$ , then  $\text{Aut}_k F$  acts faithfully and transitively on the set of roots of  $p(t)$  in  $F$ .

**Definition 8.5 (algebraic extension).** Let  $k \subset F$ , and  $\alpha \in F$ , then  $\alpha$  is algebraic over  $k$  iff  $k(\alpha)$  is a finite extension; this is equivalent to saying there exists a nonzero  $f(x) \in k[x]$  such that  $f(\alpha) = 0$ . If  $k(\alpha)/k$  is not finite, then  $\alpha$  is transcendental.

If  $\alpha$  is algebraic over  $k$ , then every element in  $k(\alpha)$  can be written as a polynomial in  $\alpha$ .

**Proposition 8.8.** Finite extensions are algebraic.

*Proof.* Let  $k \subset F$  be finite, then consider  $\alpha \in F$ , we have  $k \subset k(\alpha) \subset F$ , hence  $k(\alpha)$  is also finite.  $\square$

**Proposition 8.9.** Let  $k \subset E \subset F$ , then  $k \subset F$  is finite iff both  $E/k$  and  $F/E$  are finite, in this case

$$[F : k] = [F : E][E : k]$$

This implies: let  $k \subset F$  be finite, and  $E$  be an intermediate field, then both  $[E : k], [F : E]$  divide  $[F : k]$ .

**Example 8.4.** Let  $k \subset F$ , let  $\alpha \in F$  be an algebraic element of odd degree over  $k$ . Then  $\alpha$  can be written as a polynomial in  $\alpha^2$ . It suffices to show that  $k(\alpha^2) = k(\alpha)$ . We consider

$$k \subset k(\alpha^2) \subset k(\alpha)$$

We see that  $k(\alpha)/k(\alpha^2)$  has degree at most 2 because  $t^2 - \alpha^2$ , and the degree must divide  $[k(\alpha) : k]$ , thus it must be 1.

**Definition 8.6 (finitely generated field extensions).** A field extension  $k \subset F$  is finitely generated if there exists  $\{\alpha_i\} \subset F$  such that

$$F = k(\alpha_1) \dots (\alpha_n)$$

**Proposition 8.10.** Let  $k \subset k(\alpha_1, \dots, \alpha_n)$  be finitely generated, then  $F/k$  is algebraic iff  $F/k$  is finite iff all  $\alpha_i$  are algebraic over  $k$ . (Thus given a finitely generated extension, to show that it is finite, it suffices to show each  $\alpha_i$  is algebraic).

**Proposition 8.11.** If  $\alpha, \beta$  are algebraic over  $k$ , then

$$\alpha + \beta, \alpha\beta, \alpha\beta^{-1}$$

are all algebraic over  $k$ . (For example,  $k(\alpha + \beta) \subset k(\alpha, \beta)$ ). This implies that given  $k \subset F$ ,

$$E = \{\alpha \in F : \alpha \text{ algebraic over } k\}$$

is a field.

**Proposition 8.12.** (Composite algebraic extensions are algebraic). Let  $k \subset E \subset F$ , then  $k \subset F$  is algebraic iff both  $k \subset E$  and  $E \subset F$  are algebraic.

**Example 8.5.**  $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ .

## 8.1 Algebraic Closure

**Proposition 8.13.** Recall that  $k$  is algebraically closed iff all irreducible polynomials in  $k[x]$  have degree 1, iff every polynomial factors into linear factors, iff every maximal ideal is of the form  $(x - c)$  for some  $c \in k$ .

**Proposition 8.14.** Field  $k$  is algebraically closed iff  $k$  has no nontrivial algebraic extensions, iff if  $k \subset F$ , and  $\alpha \in F$  is algebraic over  $k$ , then  $\alpha \in k$ .

**Definition 8.7 (algebraic closure).** The  $\bar{k}$  of  $k$  is such that  $\bar{k}$  is an algebraic extension and  $\bar{k}$  is algebraically closed. (The requirement that  $\bar{k}/k$  is algebraic is to ensure there is no intermediate field that is also algebraically closed). Equivalently,  $\bar{k}$  is the smallest field that is algebraically closed containing  $k$ .

## 8.2 splitting, normal, separable

**Definition 8.8 (splitting field).** Let  $f(x) \in k[x]$  be a polynomial of degree  $d$ , the splitting field of  $f$  over  $k$  is an extension  $F/k$  such that  $f$  factors into linear factors over  $F$ . In other words,

$$F = k(\alpha_1, \dots, \alpha_d)$$

where  $\alpha_i$  are roots of  $f$ .

**Proposition 8.15.** Small fact: let  $n$  = even, then the  $n$ th root of unity such that  $\omega_n^n = 1$  also satisfies

$$\omega_n^{\frac{n}{2}} + 1 = 0$$

For example, the 8th root of unity  $\omega_8 = e^{\frac{2\pi i}{8}}$  is also a root of

$$f(x) = x^4 + 1$$

which is irreducible over  $\mathbb{Q}$ .

**Proposition 8.16.** Splitting field of  $f$  is unique up to isomorphisms, and

$$[F : k] \leq (\deg(f))!$$

Recall that if  $k \subset k(\alpha)$  is simple extension, the  $|\text{Aut}_k k(\alpha)| =$  the number of distinct roots of  $m_\alpha$ , and  $|\text{Aut}_k k(\alpha)| \leq \deg(m_\alpha)$ .

**Proposition 8.17.** The polynomial  $x^n - 1$  is not irreducible over  $\mathbb{Q}$  for any  $n \geq 2$ . The polynomial  $x^n + 1$  is only irreducible if  $n = 2^k$  for some integer  $k$ .

**Proposition 8.18.** Let  $f(x) \in k[x]$ , the splitting field  $F$  of  $f$  is such that

$$[F : k] \leq (\deg f)!$$

**Example 8.6.** The splitting field of  $f(x) = x^3 - 2$  is  $\mathbb{Q}(\sqrt[3]{2}, \omega_3)$ , which is a degree  $6 = (\deg f)! = 3!$  extension.

**Example 8.7.** The splitting field of  $x^8 - 1$  is  $\mathbb{Q}(\omega_8)$ , and the minimal polynomial for  $\omega_8 = e^{\frac{2\pi i}{8}}$  is

$$f(x) = x^4 + 1$$

Thus  $[\mathbb{Q}(\omega_8) : \mathbb{Q}] = 4$ . (The splitting field for  $x^8 - 1$  is the same as  $x^4 + 1$ ). Moreover, the Galois group is

$$\mathbb{Q}(\omega_8) = \mathbb{Q}(i, \sqrt{2})$$

is

$$\text{Gal}(\mathbb{Q}(\omega_8)/\mathbb{Q}) = \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

**Example 8.8.** The splitting field for  $x^4 + 2$  is

$$\mathbb{Q}(\sqrt[4]{2}, i)$$

and the Galois group is

$$\text{Gal}(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q}) \cong D_4$$

because it is an order 8 subgroup of  $S_4$ .



**Warning 8.1.** Galois group doesn't act transitively on the roots of a random polynomial! They act transitively on the roots of the irreducible factors.

**Definition 8.9 (normal).** A field extension  $k \subset F$  is normal if every **irreducible** polynomial  $f$  has a root in  $F$  iff  $f$  splits into product of linear factors over  $F$ . (If it contains one root, then it contains all the other roots).

**Proposition 8.19.** A field extension  $k \subset F$  is **finite and normal** iff  $F$  is the **splitting field** of some polynomial  $f \in k[x]$ .

**Example 8.9.** If a complex root of an irreducible polynomial can be expressed as a polynomial in  $i, \sqrt[4]{2}$ , then all the roots can be expressed as a polynomial in them.

Proof:  $\mathbb{Q}(i, \sqrt[4]{2})$  is the splitting field of  $x^4 + 2$ , thus is normal.

**Definition 8.10 (separable).** Let  $k$  be a field,  $f \in k[x]$  is separable if it has no multiple factors over its splitting field.



**Warning 8.2.** This following is a classic example. All the bad examples arise from this one (inseparable).

**Example 8.10 (irreducible but not separable).** Let  $\mathbb{F}_p(t)$  be the field of rational functions in  $t$  over  $\mathbb{F}_p$ . Then the polynomial

$$f(x) = x^p - t \in \mathbb{F}_p(t)[x]$$

is irreducible, but not separable (cannot be factored into distinct linear factors in its algebraic closure).

Let  $u = t^{1/p}$  be a root of  $f$  in its algebraic closure, then we see

$$f(x) = x^p - t = (x - u)^p$$

**Proposition 8.20.** Let  $f \in k[x]$ , then  $f$  is separable iff  $f, f'$  are relatively prime. If it is inseparable, then  $f' = 0$ .

**Proposition 8.21.** Irreducible polynomials are separable in characteristic 0.

**Proposition 8.22.** Let  $k$  be a field of characteristic  $p$ , then the Frobenius homomorphism

$$x \mapsto x^p$$

is a surjective map over finite fields. (It is the identity over  $\mathbb{F}_p$ ).

**Proposition 8.23.** Every irreducible polynomial in  $k[x]$  is separable  $\iff \text{char}(k) = 0$  or the Frobenius homomorphism is surjective. This implies that every irreducible polynomial is separable in finite fields or  $\mathbb{Q}$ .

Remark: the reason the example  $x^p - t \in \mathbb{F}_p(t)[x]$  works is because  $\mathbb{F}_p(t)$  is not a finite field.

**Definition 8.11 (separable element).** Let  $k \subset F, \alpha \in F$ , then  $\alpha$  is called separable iff the minimal polynomial of  $\alpha$  is separable over  $k$ . An extension is called separable if every  $\alpha \in F$  is separable. For example, all extensions of  $\mathbb{Q}$  or finite fields are separable.

**Definition 8.12 (separable degree).** Let  $k \subset F \subset \bar{k}$ , the separable degree of  $F/k$  is the number of different homomorphisms/embeddings  $\varphi : F \rightarrow \bar{k}$  (such that  $\varphi|_k = \text{id}_k$ ), denoted  $[F : k]_s$ .

**Proposition 8.24.** Let  $k \subset k(\alpha)$  be a simple algebraic extension, then  $[k(\alpha) : k]_s = \text{number of distinct roots in } \bar{k} \text{ of the minimal polynomial } p_\alpha \text{ of } \alpha$ . Moreover,

$$[k(\alpha) : k]_s \leq \deg(p_\alpha)$$

the equality holds iff  $\alpha$  i.e.,  $p_\alpha$  is separable over  $k$ .

**Example 8.11.** For separability degree:

1. Every separability degree is equal to the degree of extension over  $\mathbb{Q}$  and finite fields.
2.  $x^p - t \in \mathbb{F}_p(t)[x]$  has separability degree 1 but degree of extension  $p$ .

**Proposition 8.25.** Let  $k \subset F$  be a finite extension, then  $F$  is finitely generated: you can just take a basis of  $F/k$ .

## 8.3 Finite fields

**Proposition 8.26.** For every prime power  $q$ , there is a unique field  $\mathbb{F}_q$  of size  $q$ , up to isomorphisms. (This is the splitting field of  $x^q - x$  over  $\mathbb{F}_p$ .)

*Proof.* Existence: we claim that

$$F = \{x \in \bar{\mathbb{F}}_p : x^q = x\}$$

is a field of  $q$  elements. Note that  $f(x) = x^q - x \in \mathbb{F}_p[x]$  is separable over  $\mathbb{F}_p$ , hence it splits into distinct linear factors over  $\bar{\mathbb{F}}_p$ , i.e., it has  $p$  distinct roots.

(One can show this is a subring, hence an integral domain, hence is a field because it is finite.)

Uniqueness: for every  $\mathbb{F}_p \subset F$ , this is a finite, hence algebraic extension, thus it can be embedded into  $\bar{\mathbb{F}}_p$ . And we claim that there's exactly one subfield of  $\bar{\mathbb{F}}_p$  with  $q$  elements:

$$F \subset 0 \cup \{x \in \bar{\mathbb{F}}_p : x^q = x\}$$

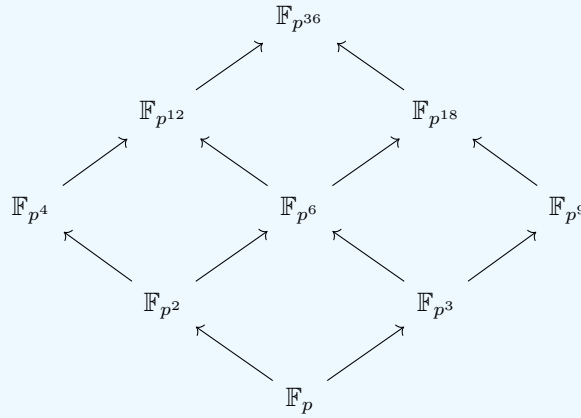
This implies  $F = RHS$ . □

**Proposition 8.27.** Fix a prime  $p$ , then

$$F = \mathbb{F}_{p^d} \hookrightarrow \mathbb{F}_{p^e}$$

iff  $d \mid e$ . Moreover,  $\mathbb{F}_{p^e}$  contains a unique copy of  $\mathbb{F}_{p^d}$ . (Nothing like  $\mathbb{Q}(\sqrt[3]{2})$ ,  $\mathbb{Q}(\sqrt[3]{2}\omega_3)$  will happen).

**Example 8.12.** An extension embedding tree looks like



**Proposition 8.28.** There exist a surjective map  $\varphi : \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{Z}/e\mathbb{Z}$  iff  $e \mid d$ .

**Proposition 8.29.** The Galois group  $\text{Gal}(\mathbb{F}_{p^d}/\mathbb{F}_p)$  is cyclic of order  $d$ , and it is generated by the Frobenius transformation

$$\sigma : x \mapsto x^p$$

More generally, let  $q$  be a prime power, then

$$\text{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q) = \langle \sigma : x \mapsto x^q \rangle$$

is cyclic of order  $d$ . In other words,

$$|\text{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^d})| = d$$

In particular,  $\mathbb{F}_q$  is fixed by  $\sigma$ : for all  $x \in \mathbb{F}_q$ , we must have

$$x^q = x$$

(also by Fermat). This completely describes  $\mathbb{F}_q$ :

$$\mathbb{F}_q = \{x \in \bar{\mathbb{F}}_q : x^q = x\}$$



**Proposition 8.30.** Let  $n$  be a positive integer, then over  $\mathbb{F}_q$ , we have

$$x^{q^n} - x = \prod_{f: \deg(f) | n} f(x)$$

where  $f$  is irreducible and monic. (All you need to do is to find all the irreducible polynomials of degree dividing  $n$ ).

These polynomials factor completely over  $\mathbb{F}_{q^n}$ .

**Example 8.13.** We will do a few examples:

1. Over  $\mathbb{F}_2$ ,

$$x^2 - x = x(x + 1)$$

consisting of all irreducible polynomials of degree 1 over  $\mathbb{F}_2$ .

2. Over  $\mathbb{F}_2$ , we have

$$x^4 - x = x(x + 1)(x^2 + x + 1)$$

consisting of all irreducible polynomials of degree 1, 2. ( $n = 2$ ).

3. Over  $\mathbb{F}_2$ , when  $n = 3$ , we factor

$$x^8 - x = x(x + 1)(x^3 + x^2 + 1)(x^3 + x + 1)$$

consisting of irreducible polynomials of degree 1, 3.

4. over  $\mathbb{F}_2$ , when  $n = 6$ , we factor

$$x^{64} - x$$

into two polynomials of degree 1, one degree 2, two degree 3, and 9 degree 6 polynomials.

## 8.4 Cyclotomic

**Definition 8.13** ( *$n$ th cyclotomic polynomial*). Let  $\omega_n = e^{2\pi i/n}$  be the primitive  $n$ th root, then  $\omega_n^m$  is also a primitive  $n$ th root iff  $\gcd(m, n) = 1$ . This implies that there are  $\phi(n)$  primitive  $n$ th roots of unity, and the polynomial

$$\Phi_n(x) = \prod_{1 \leq m \leq n, \gcd(m, n)=1} (x - \omega_n^m)$$

taking over all the primitive  $n$ th roots of unity,  $\Phi_n$  is called the  $n$ th cyclotomic polynomial. For example,

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + 1$$

**Proposition 8.31.** The cyclotomic polynomials  $\Phi_n(x)$  satisfy:

1. When  $n = p$  for some prime, then

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + \cdots + 1$$

2. When  $n$  not prime,

$$x^n - 1 = \prod_{1 \leq d, d|n} \Phi_d(x)$$

In other words,

$$\Phi_n(x) = \frac{x^n - 1}{\prod_{d|n, d < n} \Phi_d(x)}$$

*Proof.* Note that the proof involves

$$x^n - 1 = \prod_{1 \leq d, d|n} \prod_{d \text{th root of unity}} (x - \omega) = \prod_{1 \leq d, d|n} \Phi_d(x)$$

□

**Proposition 8.32.** For all  $n$ ,  $\Phi_n(x)$  is irreducible over  $\mathbb{Q}$ . (And they have integer coefficients).

The proof is hard.

**Definition 8.14** ( *$n$ th cyclotomic field*). Let  $\omega_n = e^{\frac{2\pi i}{n}}$ , then the splitting field  $\mathbb{Q}(\omega_n)$  of  $x^n - 1$  is called the  $n$ th cyclotomic field.

**Proposition 8.33.** We have

- 1.

$$[\mathbb{Q}(\omega_n) : \mathbb{Q}] = \phi(n)$$

2.  $\Phi_n(x)$  is the minimal polynomial of  $\omega_n$ .

- 3.

$$\text{Aut}_{\mathbb{Q}}(\mathbb{Q}(\omega_n)) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$$

where  $(\mathbb{Z}/n\mathbb{Z})^{\times}$  is the group of units under multiplication.

**Example 8.14.**  $\mathbb{Q}(\omega_8)/\mathbb{Q}$  is a degree 4 extension, with Galois group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .

$$\Phi_8(x) = x^4 + 1.$$



**Warning 8.3.** Important theorem.

**Proposition 8.34.** Finite + separable  $\Rightarrow$  simple extension.

**Proposition 8.35.** Transitive subgroups of  $S_4$  are as follows:

$$S_4, A_4, D_8, \frac{\mathbb{Z}}{4\mathbb{Z}}, \frac{\mathbb{Z}}{2\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$$

**Proposition 8.36.** Every finite extension of  $\mathbb{Q}$  or of finite fields  $\mathbb{F}_q$  is simple.

*Proof.* Finite+separable  $\Rightarrow$  simple. □

## 8.5 Galois I

**Definition 8.15 (Galois Correspondence).** Let  $k \subset F$ , and  $G = \text{Aut}_k(F)$ , let  $H \subset G$  be a subgroup, then  $H$  corresponds to an intermediate field extension

$$k \subset F^H \subset F$$

where

$$F^H = \{x \in F : h \cdot x = x \text{ for all } h \in H\}$$

the fixed elements of  $F$  by  $H$ . Conversely, given  $k \subset E \subset F$ , we can identify a subgroup of  $G$

$$H := \text{Aut}_E(F) \subset \text{Aut}_k(F)$$

This establishes a one-to-one correspondence.

**Proposition 8.37.** We have the following trivial inclusions:

$$E \subset F^{\text{Aut}_E(F)}, \quad H \subset \text{Aut}_{F^H}(F)$$

Let  $E_1 E_2$  denote the smallest subfield of  $F$  containing  $E_1, E_2$ , and  $\langle H_1, H_2 \rangle$  denote the smallest subgroup containing  $H_1, H_2 \subset \text{Aut}_k(F)$ , then

$$\text{Aut}_{E_1 E_2}(F) = \text{Aut}_{E_1}(F) \cap \text{Aut}_{E_2}(F)$$

and

$$F^{\langle G_1, G_2 \rangle} = F^{G_1} \cap F^{G_2}$$

**Proposition 8.38.** Let  $k \subset F$  be finite. The degree of the field extension  $[F : E]$  is the index of the subgroup  $\text{Aut}_E(F)$  in  $\text{Aut}_k(F)$ . Equivalently, the size of  $H \subset \text{Aut}_k(F)$  is the degree  $[F : F^H]$ , and

$$H = \text{Aut}_{F^H}(F)$$

**Proposition 8.39.** Let  $k \subset F$  be finite, and let  $H \subset \text{Aut}_k(F)$  be a subgroup, then  $F^H \subset F$  is finite and separable, hence simple, and also normal.

**Definition 8.16** (Galois extension).  $k \subset F$  is Galois iff any of the following holds:

1.  $F$  is the splitting field of a separable polynomial  $f \in k[t]$ .
2.  $k \subset F$  is normal and separable.
3.  $|\text{Aut}_k(F)| = [F : k]$ .
4.  $k = F^{\text{Aut}_k(F)}$ .
5. The Galois correspondence  $H \mapsto F^H$ ,  $E \mapsto \text{Aut}_E(F)$  is a bijection: the map

$$H \mapsto F^H \mapsto \text{Aut}_{F^H} F$$

is the identity.

6.  $k \subset F$  is separable, and if  $k \subset F \subset K$  is an algebraic extension and  $\sigma \in \text{Aut}_k(K)$ , then  $\sigma(F) = F$ .

**Proposition 8.40.** If  $k \subset F$  is finite and separable, if it is not Galois, then it can be embedded into some larger field  $k \subset K$  in many possible ways, i.e., the last criterion of Galois:  $\sigma_1(F) \neq \sigma_2(F)$ . If  $k \subset F$  is Galois, then all the images of embeddings coincide.

**Example 8.15.** The extension  $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$  is not Galois because there are three embeddings into  $\mathbb{C}$  and they do not coincide.

**Example 8.16.** Some examples of Galois extensions:

1.  $\mathbb{Q}(i)/\mathbb{Q}$ ,  $\mathbb{Z}/2\mathbb{Z}$ .
2.  $\mathbb{Q}(\omega_8)/\mathbb{Q}$ ,  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ .
3. Splitting field of  $x^4 + 2$ ,  $D_4$ .
4. Galois fields such as

$$\frac{\mathbb{F}_2[x]}{(x^3 + x^2 + 1)} / \mathbb{F}_2$$

with  $\mathbb{Z}/3\mathbb{Z}$ .

5.  $\mathbb{F}_p \subset \mathbb{F}_{p^d}$ , cyclic  $\mathbb{Z}/d\mathbb{Z}$ .
6.  $\mathbb{Q}(\omega_n)$  as splitting field of  $x^n - 1$ , which is separable, with  $(\mathbb{Z}/n\mathbb{Z})^\times$  under multiplication.

**Proposition 8.41.** Let  $k \subset F$  be Galois, and  $k \subset E \subset F$ , then

$$[F : E] = |\text{Aut}_E(F)|$$

and  $E \subset F$  is also Galois. Moreover,

$$[E : k] = [\text{Aut}_k(F) : \text{Aut}_E(F)]$$

**Proposition 8.42.** The Galois correspondence is inclusion-reversing:

$$\begin{array}{ccc}
 F & & \text{Aut}_F(F) = \{e\} \\
 | & & | \\
 E & & \text{Aut}_E(F) \\
 | & & | \\
 k & & \text{Aut}_k(F)
 \end{array}$$

Note that

$$|\text{Aut}_E(F)| = [F : E]$$

**Example 8.17.** The extension

$$\mathbb{F}_2 \subset \mathbb{F}_8 \subset \mathbb{F}_{64}$$

corresponds to subgroup

$$\text{Aut}_{\mathbb{F}_8} \mathbb{F}_{64} = [\mathbb{F}_{64} : \mathbb{F}_8] \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

**Proposition 8.43.** Let  $k \subset F$  be Galois, then  $k \subset E \subset F$ ,  $k \subset E$  is Galois iff  $\text{Aut}_E(F)$  is normal in  $\text{Aut}_k(F)$ . In this case,

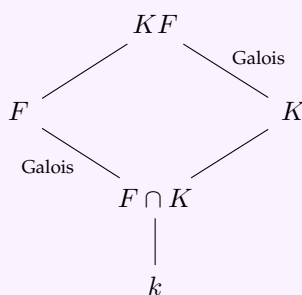
$$\text{Aut}_k(E) \cong \frac{\text{Aut}_k(F)}{\text{Aut}_E(F)}$$

Recall that

$$[E : k] = [\text{Aut}_k(F) : \text{Aut}_E(F)]$$

**Proposition 8.44.** Consider  $k \subset F$  Galois, and  $k \subset K$  any finite extension, then  $K \subset KF$  is Galois, and

$$\text{Aut}_K(KF) \cong \text{Aut}_{K \cap F}(F)$$



**Definition 8.17 (symmetric functions).** Consider  $P_n(x) = (x - t_1) \dots (x - t_n) \in \mathbb{Z}[t_1, \dots, t_n]$ , and consider its coefficients

$$P_n(x) = x^n - s_1(t_1, \dots, t_n)x^{n-1} + \dots + (-1)^n s_n(t_1, \dots, t_n)$$

are elementary symmetric functions of  $t_1, \dots, t_n$ . For example, when  $n = 3$ , the symmetric functions are

$$s_1 = t_1 + t_2 + t_3, s_2 = t_1 t_2 + t_1 t_3 + t_2 t_3, s_3 = t_1 t_2 t_3$$

**Proposition 8.45.** Let  $k$  be a field, and  $\varphi \in k(t_1, \dots, t_n)$ , then  $\varphi$  is symmetric iff  $\varphi$  is a rational function of elementary symmetric functions  $s_1, \dots, s_n$ .

*Proof.* Let  $k' = k(s_1, \dots, s_n)$  and  $F = k(t_1, \dots, t_n)$ , then  $k' \subset F$  is Galois because  $F$  is the splitting field of the separable polynomial  $P_n(x)$  over  $k'$ . The symmetric group  $S_n$  acts on  $F$  by permuting  $t_1, \dots, t_n$ , and  $S_n$  is the identity on  $k'$ . Thus  $S_n$  is a subgroup of  $\text{Aut}_{k'}(F)$ , i.e.,  $n! \leq [F : k'] \leq n!$ , where the second inequality follows from degree of  $P_n(x)$ . This implies that

$$\text{Aut}_{k'}(F) = S_n, k' = k(s_1, \dots, s_n) = F^{S_n}$$

We reiterate:

$$k(s_1, \dots, s_n) \subset k(t_1, \dots, t_n)$$

is Galois, with Galois group  $S_n$ . □

**Proposition 8.46.** As a corollary, every finite group  $G$  can be realized as the Galois group of some extension.

**Proposition 8.47.** Next we construct Galois extension with Galois group  $A_n$ : recall

$$\Delta = \prod_{1 \leq i < j \leq n} (t_i - t_j)$$

Then

$$k(s_1, \dots, s_n)(\Delta) \subset k(t_1, \dots, t_n)$$

is Galois, with Galois group  $A_n$ .

*Proof.*  $\Delta$  has degree 2 over  $k(s_1, \dots, s_n)$  because

$$\Delta^2 = D = \prod_{i < j} (t_i - t_j)^2 \in k(s_1, \dots, s_n)$$

and the only transitive subgroup of index 2 is  $A_n$ . □

**Proposition 8.48.** Let  $k$  be a field of characteristic 0, and  $f(x) \in k[x]$  be an irreducible polynomial. Then  $f(x)$  is solvable by radicals iff its Galois group is solvable.

**Proposition 8.49.** (This may seem repetitive). Let  $f(x) \in k[x]$  be an irreducible separable polynomial of degree  $n$ , then  $\text{Gal}_k(f(x))$  acts transitively on the set of roots of  $f$  in  $\bar{k}$ , i.e.,  $\text{Gal}_k(f(x))$  can be identified as a subgroup of  $S_n$ .

**Example 8.18.** The only transitive subgroups of  $S_3$ :  $S_3$  and  $A_3$ . This implies that an irreducible cubic separable polynomial can only have Galois group  $A_3$  or  $S_3$ .

**Example 8.19.** The only transitive subgroups of  $S_4$ :

$$S_4, A_4, D_8, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$$

**Definition 8.18 (discriminant).** Let  $f \in k[x]$  be separable, and  $\alpha_1, \dots, \alpha_n$  be its roots in some splitting field, then the **discriminant** of  $f$  is

$$D = \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2 = \Delta^2$$

We notice that  $D$  is always fixed by Galois group  $G$ , whereas  $\Delta$  is fixed by  $G$  iff  $G \subset A_n$ .

**Proposition 8.50.** Let  $k$  be field of characteristic  $\neq 2$ , let  $f(x) \in k[x]$  be a separable polynomial, with discriminant  $D$ . Then the Galois group of  $f(x)$  is contained in  $A_n$  iff  $D$  is a square in  $k$ , i.e.,  $\Delta \in k$ .

**Example 8.20.** For  $S_3$ : the Galois group is  $A_3$  if  $D$  is a square, and  $S_3$  if it is not a square.

**Proposition 8.51.** Let  $f \in \mathbb{Q}[x]$  be an irreducible polynomial of degree  $p$ , where  $p$  is prime. Assume that  $f$  has exactly 2 complex roots, then the Galois group is  $S_p$ .

*Proof.*  $G$  is a subgroup of  $S_p$ : complex conjugation gives an order 2 element of  $G$ ; we also know  $p$  divides  $|G|$ , thus there exists an element of order  $p$ , which is a  $p$ -cycle, i.e., they generate  $S_p$ .  $\square$

**Proposition 8.52.** Every finite abelian group is the Galois group of some extension  $F$  over  $\mathbb{Q}$ .

More specifically, every finite abelian group  $G$  is the group of some intermediate field of the extension  $\mathbb{Q} \subset \mathbb{Q}(\zeta_n)$  in a cyclotomic field.

*Proof.* Classification:

$$G \cong \frac{\mathbb{Z}}{n_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{n_r\mathbb{Z}}$$

Choose distinct  $p_i$  such that  $p_i \equiv 1 \pmod{n_i}$ . Let  $n = p_1 \cdots p_r$ , by CRT

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong (\mathbb{Z}/p_1\mathbb{Z})^\times \times \cdots \times (\mathbb{Z}/p_r\mathbb{Z})^\times$$

Then  $(\mathbb{Z}/n\mathbb{Z})^\times$  has a subgroup  $H$  such that

$$G \cong \frac{(\mathbb{Z}/n\mathbb{Z})^\times}{H}$$

Since  $(\mathbb{Z}/n\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta_n))$ ,  $H$  corresponds to an intermediate field  $F$ , where

$$\mathbb{Q} \subset F \subset \mathbb{Q}(\zeta_n)$$

$H$  is automatically normal, hence  $\mathbb{Q} \subset F$  is Galois and

$$\text{Gal}(F/\mathbb{Q}) = G$$

$\square$



**Warning 8.4.** If you are just given irreducible  $f$ , asked to compute the Galois group of  $f$ : remember the degree of extension could be as large as  $\deg(f)!$ .



**Warning 8.5.** Sometimes we get asked to find  $\text{Gal}(f)$ , if it's irreducible, then the Galois group acts transitively on the set of roots, i.e., can be identified with a subgroup of

$$S_{\deg(f)}$$

**Proposition 8.53.** Let  $k \subset F$  be Galois, fix  $H \subset \text{Gal}(F/k)$ , for any element  $\alpha \in L$ , then

$$\sum_{h \in H} h(\alpha), \quad \prod_{h \in H} h(\alpha)$$

are in  $L^H$ .

This is  $\text{tr}_{L/L^H}(m_\alpha)$ , where  $m_\alpha$  is the multiplication by  $\alpha$ , and similarly the determinant.

**Proposition 8.54 (normal basis theorem).** Let  $k \subset K$  be a Galois extension of degree  $n$ , let  $\{\sigma_1, \dots, \sigma_n\}$  be the elements of the Galois group, then there exists  $w \in K$  such that

$$\{\sigma_1(w), \dots, \sigma_n(w)\}$$

forms a basis of  $K$  over  $k$ .

**Proposition 8.55.** Let  $k \subset k(\alpha)$  be a simple extension, then for  $f(\alpha) = 0$ ,

$$[k(\alpha) : k] \leq (\deg(f))$$

and the splitting field  $F$

$$[F : k] \leq (\deg(f))!$$

**Proposition 8.56 (Artin-Schreier).** Let  $k$  be  $\text{char}(k) = p$ , then  $f(x) = x^p - x - a$  is either irreducible over  $k$ , or  $f$  splits into linear factors over  $k$ . If it is irreducible, let  $x_0$  be a root of  $f$ , then

$$k \subset k(x_0)$$

is a Galois extension of degree  $p$ , generated by

$$\sigma : x_0 \mapsto x_0 + 1$$



## Chapter 9

# Field Theory-Hilbert's Nullstellensatz

This corresponds to Aluffi Chapter VII 2.2-2.3.

**Proposition 9.1.** For a field  $K$ , TFAE:

1.  $K$  is algebraically closed.
2. There is no algebraic extension over  $K$  except for the trivial one.
3. If  $K \subset L$  is any extension, and  $\alpha \in L$  is algebraic over  $K$ , then  $\alpha \in K$ .

**Definition 9.1 (algebraic closure).** An algebraic closure of a field  $k$  is the algebraic extension such that  $\bar{k}$  is algebraically closed.

**Proposition 9.2 (Hilbert's Nullstellensatz).** Recall that if  $K$  is algebraically closed, then every maximal ideal in  $K[x]$  is of the form  $(x - \alpha)$ ,  $\alpha \in K$ .

**Proposition 9.3.** Let  $K$  be algebraically closed, and  $I \subset K[x_1, \dots, x_n]$  be an ideal, then  $I$  is maximal iff

$$I = (x_1 - c_1, \dots, x_n - c_n)$$

for some  $c_1, \dots, c_n \in K$ .

## Chapter 10

# Representation Theory of Finite Groups

Let  $k$  be a field and  $G$  be a finite group, a representation  $\rho : G \rightarrow \text{GL}(V)$  is such that

$$\rho(g_1 g_2) = \rho(g_1) \circ \rho(g_2)$$

And  $V$  is a  $k[G]$ -module, i.e., elements in  $k[G]$  are of the form

$$\sum_{g \in G} a_g g$$

and they act on  $V$  by

$$\left( \sum_{g \in G} a_g g \right) \cdot v = \sum_{g \in G} a_g (\rho(g)(v))$$

**Proposition 10.1.** The only two-sided ideals of  $M_2(\mathbb{R})$  is  $\{0\}$  and  $M_2(\mathbb{R})$ .

**Proposition 10.2.** Let  $k$  be a field, then every finite-dimensional  $k$ -algebra is left and right Noetherian and Artinian

**Proposition 10.3.** Let  $R$  be a commutative ring, then if  $R$  is Artinian, then  $R$  is Noetherian.

Hard proof.

**Definition 10.1 (representation).** Representation of  $G$  over  $k$  is a homomorphism  $\rho : G \rightarrow \text{GL}(V)$  for some vector space  $V$  over  $k$ , a representation can also be defined as a  $kG$ -module, where elements in the ring  $kG$  are of the form

$$\sum_{g \in G} a_g g, a_g \in k$$

i.e.,  $k$ -linear combinations of group elements.

An invariant subspace of  $W \subset V$  is such that for all  $w \in W, g \in G$ , we have

$$\rho(g)w \in W$$

i.e., an  $kG$ -submodule.

**Definition 10.2 (homomorphism between  $kG$ -modules).** Let  $f : V \rightarrow W$  be  $kG$ -modules, then  $f$  is a homomorphism iff

$$f(\rho(g)v) = \rho(g)f(v)$$

**Proposition 10.4 (Mascheke's theorem).** Let  $V$  be a representation of a finite group  $G$  over  $k$ , and let  $|G|$  be invertible (nonzero in characteristic 0, and not divisible by  $p$  in char  $p$ ). Let  $W$  be an invariant subspace of  $V$ , then there exists an invariant subspace  $W'$  such that

$$V = W \oplus W'$$

In other words, if  $V$  is not irreducible, then it can be decomposed into irreducible representations.

**Definition 10.3 (semisimple).** A  $R$ -module  $M$  is semisimple iff  $M$  can be written as a finite direct sum of simple modules.

$$M \cong M_1 \oplus \cdots \oplus M_k$$

where  $M_i$ 's are simple  $R$ -modules.

**Proposition 10.5.** Let  $k$  be a field and  $|G|$  is invertible, then every finite-dimensional  $kG$ -module is semisimple: it can be decomposed into a finite direct sum of simple  $kG$ -submodules.

**Proposition 10.6.** Let  $\rho$  be a finite-dimensional representation of  $G$  over  $\mathbb{C}$ , then for every  $g \in G$ , the matrix  $\rho(g)$  is diagonalizable.

**Proposition 10.7 (Schur's lemma).** Let  $S_1, S_2$  be simple  $R$ -modules, where  $R$  is a finite-dimensional algebra over  $k$  (finite-dimensional vector space but you can multiply things), then

$$\text{Hom}_R(S_1, S_2) = \begin{cases} 0, & \text{if } S_1 \not\cong S_2 \\ k, & \text{if } S_1 \cong S_2 \end{cases}$$

Equivalently, let  $V, W$  be irreducible representations of a finite group  $G$ , then  $V, W$  are  $kG$ -modules, where  $kG$  is a finite-dimensional  $k$ -algebra. Thus,

$$\text{Hom}_k(V, W) = \begin{cases} k, & V \cong W \\ 0, & \text{if not} \end{cases}$$

any map  $\varphi : V \rightarrow V$  is a multiplication by scalar  $\lambda \in k$ .



**Warning 10.1.** This assumes  $k$  is algebraically closed!

**Definition 10.4 (semisimple ring).** A ring  $R$  is semisimple iff

1. Every  $R$ -module is semisimple.
2.  $R$  can be written as

$$R \cong I_1 \oplus \cdots \oplus I_k$$

where  $I_i$  are simple ideals.

**Proposition 10.8 (Artin-Wedderburn).** A finite-dimensional  $k$ -algebra  $R$  is semisimple iff

$$R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

where  $M_{n_i}(D_i)$  is the matrix rings over division rings  $D_i$ . If  $k$  is algebraically closed, then

$$R \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_k}(k) \cong V_1 \oplus \cdots \oplus V_k$$

The example for finite-dimensional  $k$ -algebra is  $kG$ , where  $G$  is finite.

**Proposition 10.9.** Let  $k$  be algebraically closed, and  $G$  is finite, then

$$kG \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_k}(k)$$

Thus

$$|G| = \sum_{i=1}^k n_i^2$$

where  $n_i = \dim(V_i)$ .

## 10.1 Characters

**Proposition 10.10.** Some basic properties of  $\chi$ , let  $\rho$  be a representation of  $G$  over  $\mathbb{C}$ , and  $\chi(g) = \text{tr}(\rho(g))$ , then

1.  $\chi(1) = \dim \rho$ .
2. Let  $g \in G$  have order  $n$ , then  $\chi(g)$  is a sum of  $n$ th roots of unity.
3.  $|\chi(g)| \leq \chi(1)$ , with equality iff  $\rho(g)$  is a multiplication by scalar.
4.  $\chi(g) = \chi(1)$  iff  $\rho(g) = \text{id}$ .
5.  $\chi(g^{-1}) = \overline{\chi(g)}$ .
6.  $\chi(g) = \chi(h)$  if  $g, h$  are conjugates.
7. If  $V, W$  are isomorphic  $\mathbb{C}G$ -modules, then  $\chi_V = \chi_W$ .

We can extend characters onto larger representations:

**Proposition 10.11.** Let  $V, W$  be finite dimensional representations of  $G$  over  $k$  of char 0, then

1.  $V \oplus W$  has character  $\chi_V \oplus \chi_W$ .
2.  $V \otimes W$  has character  $\chi_V \cdot \chi_W$ .
3.  $V^*$  has character  $\chi_{V^*}(g) = \overline{\chi_V(g)}$ .
4.  $\text{Hom}_k(V, W)$  has character  $\chi_{V^*} \cdot \chi_W$ .

**Proposition 10.12 (Row Orthogonality).** Let  $G$  be finite, and  $V, W$  be irreducible complex representations of  $G$  with characters  $\chi_V, \chi_W$ , then

$$\langle \chi_V, \chi_W \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \begin{cases} 1, & V \cong W \\ 0, & \text{otherwise} \end{cases}$$

**Proposition 10.13.** Let  $V$  be a  $\mathbb{C}G$ -module, then in any expression

$$V \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$$

where  $S_i$  are non-isomorphic irreducible modules, we have

$$n_i = \langle \chi_V, \chi_{S_i} \rangle$$

where  $\chi_V, \chi_{S_i}$  are characters of  $V, S_i$ .



**Warning 10.2.** We can use the above to decompose any representation into irreducible ones, and the powers  $n_i$  can be determined by taking the inner product.

**Example 10.1.** Let  $V$  be the 2-dimensional representation of  $S_3$ , then we can consider  $V \otimes V$ , then  $\chi_2 \otimes \chi_2$  is a character of  $V \otimes V$ , now we decompose  $V \otimes V$  into irreducible representations:

$$\langle \chi_2 \otimes \chi_2, \chi_{\text{triv}} \rangle = 1$$

and similarly

$$\langle \chi_2 \otimes \chi_2, \chi_{\text{sgn}} \rangle = 1, \langle \chi_2 \otimes \chi_2, \chi_2 \rangle = 1$$

Thus we see

$$V \otimes V \cong \mathbb{C} \oplus S \oplus V$$

where  $V$  is the trivial and  $S$  is the sign representation.

**Proposition 10.14.** For finite dimensional complex representations  $V, W$ , then

$$V \cong W \iff \chi_V(g) = \chi_W(g), \text{ for all } g \in G$$

Moreover,

$$\langle \chi, \chi \rangle = 1$$

iff  $\chi$  is irreducible.

*Proof.* We write  $V \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$ , then the inner product being 1 implies

$$\sum_{i=1}^k n_i^2 = 1$$

which implies there exists only one nonzero  $n_i = 1$ , i.e.,  $V \cong S_i$ . □

**Proposition 10.15 (orthogonality).** Let  $\chi_1, \dots, \chi_k$  be the irreducible complex characters of  $G$ , with dimensions  $d_1, \dots, d_k$ , then

1.  $|G| = \sum_{i=1}^k d_i^2$ .
2.  $\sum_{i=1}^k d_i \chi_i(g) = 0$ , if  $g \neq 1$ .

**Proposition 10.16.** For any commutative ring  $R$ , the center

$$Z(M_n(R)) = \{\lambda I : \lambda \in R\} \cong R$$

**Proposition 10.17 (column orthogonality).** Let  $\chi_i$  be the irreducible representations of  $G$ , then

$$\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |Z_G(g)|, & \text{if } g, h \text{ are conjugates} \\ 0, & \text{if not} \end{cases}$$

where  $Z_G(g)$  is the centralizer of  $g$ .

**Proposition 10.18 (all orthogonality relations).** You have

1. Row orthogonality:  $\sum_{g \in G} \chi_V(g) \overline{\chi_W(g)} = \begin{cases} |G|, & \text{if } V \cong W \\ 0, & \text{if not} \end{cases}$ .
2. Dimension orthogonality:  $\sum_{i=1}^k d_i \chi_i(g) = 0$ .
3. Column orthogonality:  $\sum_{i=1}^k \chi_i(g) \overline{\chi_i(h)} = \begin{cases} |Z_G(g)|, & \text{if } g, h \text{ are conjugates} \\ 0, & \text{if not} \end{cases}$ .

**Proposition 10.19.** The dimension of an irreducible complex representation divides  $|G|$ .

**Proposition 10.20 (characters of cyclic groups).** Let  $G = \langle g : g^n = 1 \rangle$  be cyclic, and  $\omega_n$  be the primitive  $n$ th root of unity, then the irreducible complex characters of  $G$  are

$$\chi_i : g \mapsto \omega_n^i$$

where  $0 \leq i \leq n-1$ . These are exactly the  $n$  irreducible characters of  $G$ .

**Example 10.2.** Consider the character table for  $\mathbb{Z}/4\mathbb{Z}$ ,  $1 \mapsto 1, i, -1, -i$  defines 4 one-dimensional irreducible representations.

**Proposition 10.21.** Let  $V_1, \dots, V_n$  and  $W_1, \dots, W_m$  be the irreducible representations of  $G_1, G_2$ , then

$$V_i \otimes W_j$$

is the complete list of irreducible representations of  $G_1 \times G_2$ . As a consequence, the character table of  $G_1 \times G_2$  is the tensor product of the character tables of  $G_1$  and  $G_2$ .

**Proposition 10.22.**  $G$  is abelian iff all complex irreducible representations of  $G$  are one-dimensional.

*Proof.* We know cyclic groups have one-dimensional irreducible representations, and every finite abelian group is a direct product of cyclic groups, hence by the above, we know they are again one-dimensional. Conversely, consider

$$|G| = \sum_{i=1}^r d_i^2$$

This implies  $r = |G|$ , i.e., every conjugacy class has size 1, i.e.,  $G$  is abelian.  $\square$

**Proposition 10.23.** The one-dimensional irreducible representations of  $G$  exactly correspond to the one-dimensional representations of  $G^{\text{ab}} = \frac{G}{[G:G]}$ .  
(You put 1's at  $[G, G]$ , think about  $D_8$ .)

**Proposition 10.24.** Let  $\chi_1, \dots, \chi_n$  be pairwise distinct one-dimensional characters over  $k$ , then  $\chi_1, \dots, \chi_n$  are linearly independent over  $k$ .

## 10.2 Induction and Restriction

**Definition 10.5 (Induced Representation).** Let  $H$  be a subgroup of  $G$ , let  $V$  be a  $kH$ -module. Let  $g_1H, \dots, g_{[G:H]}H$  be the list of left cosets of  $G/H$ . Then

$$\text{Ind}_H^G(V) = \bigoplus_{i=1}^{[G:H]} g_i \otimes V$$

where

$$g_i \otimes V = \{g_i \otimes v : v \in V\}$$

note that if  $g \in G$ , then

$$g(g_i \otimes V) = g_j \otimes V$$

where  $gg_i = g_jh$  for some  $h \in H$ . Note that  $g_i \otimes V \cong V$  for each  $i$ .

The  $g_i \otimes V$  submodules of  $\text{Ind}_H^G(V)$  are permuted under the action of  $G$ , which is transitive.

**Definition 10.6 (Induced character).** Let  $H$  be a subgroup of  $G$ , let  $V$  be a  $\mathbb{C}H$ -module with character  $\chi$ . Then the induced character on  $G$  is

$$\text{Ind}_H^G(\chi)(g) = \frac{1}{|H|} \sum_{t \in G: t^{-1}gt \in H} \chi(t^{-1}gt)$$

**Example 10.3.** Consider  $G = S_3$ ,  $H = \langle (123) \rangle$ , and let  $\chi$  be the trivial representation on  $H$ , then

$$\text{Ind}_H^G\chi(g) = \begin{cases} 2, & g = e \\ 0, & g = (12) \\ 2, & g = (123) \end{cases}$$

**Proposition 10.25 (Frobenius reciprocity).** Let  $\text{Res}_H^G(V)$  be the restriction of representation of  $V$  on  $G$  to  $H$ . Let  $H \subset K \subset G$  be subgroups of  $G$  and  $V$  be a  $kH$ -module and  $W$  be a  $kG$ -module.

$$\text{Hom}_{kG}(\text{Ind}_H^G V, W) \cong \text{Hom}_{kG}(V, \text{Res}_H^G W)$$

and similarly,

$$\text{Hom}_{kG}(W, \text{Ind}_H^G V) \cong \text{Hom}_{kG}(\text{Res}_H^G W, V)$$

**Proposition 10.26.** Some more properties of induction and restriction:

1.  $\text{Ind}_K^G \text{Ind}_H^K V \cong \text{Ind}_H^G V$ .
2. Similarly for restrictions.
3.  $\text{Ind}_H^G V \cong \text{Hom}_{kH}(kG, V)$  as  $kG$ -modules.

**Proposition 10.27 (Frobenius reciprocity for characters).** Let  $\psi, \chi$  be complex characters of  $G$ .

$$\langle \text{Ind}_H^G \chi, \psi \rangle_G = \langle \chi, \text{Res}_H^G \psi \rangle_H$$

**Proposition 10.28.** Let  $V$  be a representation of  $G$  over  $k$  whose characteristic is not 2, then

$$V \otimes V \cong S^2(V) \oplus \wedge^2(V)$$

Moreover,

$$S^2 \chi(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$$

and

$$\wedge^2(g) = \frac{1}{2}(\chi(g)^2 - \chi(g^2))$$



**Warning 10.3.** So how to construct character tables.

**Proposition 10.29.** 1. Find the number of conjugacy classes of  $G$  (to identify the size of the table).

2. Take the abelianization  $G/[G, G]$  to find all the one-dimensional characters.
3. For characters of dimension greater than 1:

- (a) Find the natural representations.
- (b) Find representations lifted from quotient groups.
- (c) Representations induced from subgroups.
- (d) Tensor products of other representations.
- (e) Symmetric and alternating powers of representations.

Keep in mind of **Frobenius Reciprocity** and **Orthogonality Relations**.

**Proposition 10.30.** Suppose an abelian group  $G$  admits a faithful representation, then show  $G$  is cyclic.