Calc III Section Notes with Answers

Fall 2025

Hui Sun

October 21, 2025

Contents

Chapter 1

The Geometry of Euclidean Spaces

Week 1 (8/25-29)

Logistics

- TA: Hui.
- Email: hsun95@jh.edu.
- Office Hours: Tuesday 4-6 PM, Krieger 211; Friday 1-2 PM Zoom.
- Biweekly Quizzes: 15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Icebreaking Activity

- In a group of three or four:
  1. Learn each other names, year, pronouns.
  2. Find something in common and different among you and share with the entire class.
  3. Play Buzz if you have time, with prime 7: say the number if it doens't contain or is not divisible by 7, say buzz otherwise.

Problem 1.1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 1.2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.
$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check
$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$
thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 1.3. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$
Bonus: is the converse true? (meaning assuming $\|u + v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have
$$\begin{aligned} \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$
because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$
given the assumption, we also have
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$
Thus equating them we get
$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u \cdot v = 0$$

□

Reminders

1. First HW due this Friday.
2. First Quiz next Tuesday.

Week 2 (9/1-5)

Topics: (1) cross product, (2) plane in $\mathbb{R}^3$.

Definition 1.1 (cross product)**.** Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$, the cross product of a, b is the vector $a \times b$,

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 1.2 (Plane in three dimensions)**.** A perpendicular vector and a normal vector uniquely define a plane in $\mathbb{R}^3$: given the plane $\mathcal{P}$ passing containing the point (x_0, y_0, z_0) that has a normal vector (A, B, C) is given by the equation:

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1.1. Here are some properties of the cross product:

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:

   $$\|a \times b\| = \|a\|\|b\| \sin \theta$$

   where θ is the angle between them. (Compare this with the dot product).
3. $a \times b = -b \times a$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. (HW) The cross product is **not** associative! For example, compute

   $$(i \times i) \times j, \quad i \times (i \times j)$$

Problem 1.4. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 1.5. Compute the plane containg all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

$\square$

Reminders HW is due Sunday 11:59PM.

Chapter 2

Differentiation

Week 3 (9/8-9/12)

Topics: (1) graphing multivariable functions, (2) limits and continuity.

Definition 2.1 (graph). The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 2.2 (limit). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is open, let x_0 be in A or be a boundary point of A and N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0, f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that

$$\text{if } x \in U, \text{ then } f(x) \in N$$

If f is eventually in N for *any* neighborhood N around b, then the **limit** of f as $x \to x_0$ exists, denoted as

$$\lim_{x \to x_0} f(x) = b$$

Definition 2.3 (limit'). If the limit exists, then $\lim_{x \to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x'_1, x'_2, \ldots, x'_n)$ from **all directions**, and $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 2.4 (continuity). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 2.1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 2.1. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x, y) = x^2 + y^2$.

Proof.

1. Image$(f) = \{x^2 + 1 : x \in \mathbb{R}\}$, and Graph$(f) = \{(x, x^2 + 1) : x \in \mathbb{R}\}$.
2. Image$(g) = \{x^2 + y^2, (x, y) \in \mathbb{R}^2\}$, and Graph$(g) = \{(x, y, x^2 + y^2) : (x, y) \in \mathbb{R}^2\}$.

$\square$

Problem 2.2. Compute the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$$
   (Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$).
2. $$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + y^2}$$

Proof.

1. Following the hint, we see
$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = \lim_{x\to 0} x = 0$$
2. This one uses the exact same trick:
$$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{xy} \cdot y = 0$$
3. First letting $x \to 0$ along $y = 0$, we see the limit is 1; letting $x = y \to 0$, we see the limit is 0, thus the limit doesn't exist!

$\square$

Problem 2.3. Compute the limit of the following functions:

1. $$\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

   (Hint: try considering $y = x^2 - x$ and $y = x$)

3. $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$$

Proof.

1. First fix $x = 0$, let $y \to 0$, then the limit is 0; now fix $y = 0$, let $x \to 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \to 0, y \to 0$), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3 - x^2}{x^2} = \lim_{x\to 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

Warning 2.1. 2 does not follow from 1! A student suggests a proof: $\lim_{(x,y)\to(0,0)} = \frac{x}{x+y} \cdot y$, and by 1, the limit $\frac{x}{x+y}$ doesn't exist, this implies the limit of $\frac{xy}{x+y}$ also doesn't exist. This argument is not correct! Consider the following counterexample: $\lim_{y\to 0} \frac{1}{y}$ doesn't exist, but the limit

$$\lim_{y\to 0} \frac{1}{y} \cdot y = 1$$

exists! More concretely, if you multiply by any function that doesn't tend to 0, the argument follows, but it doesn't work when the function tends to 0! (Sorry I wasn't able to give a concrete counterexample in class other than saying this gives "bad and untrue vibes"). Thank you (the student) who brought it up, your attempt still remains very very good.

3. We see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} \frac{xy}{x+y}$$

Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist!

$\square$

How to find a a limit $\lim_{x\to x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.

Week 4 (9/15-9/19)

Topics: (1) Partial derivatives. (2) Definition of total derivatives.

Problem 2.4. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the following functions:

1. $$x^3y^4 - xy^2$$
2. $$x^2\sin(2y) + 3$$
3. $$\ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{2}\right)$$

   You may use the following identities to simply the equation first:
   $$\ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln\left(\frac{1}{a}\right) = -\ln a$$

Proof. We have

1. $$\partial x : 3x^2y^4 - y^2, \quad \partial y : 4x^3y^3 - 2xy$$
2. $$\partial x : 2x\sin(2y), \quad \partial y : 2x^2\cos(2y)$$
3. $$\partial x : -\frac{2}{x} - \frac{1}{x+y}, \quad \partial y : \frac{1}{y} - \frac{1}{x+y}$$

□

Definition 2.5 (tangent plane)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0), then the **tangent plane** to the graph f in $\mathbb{R}^3$ is the given by

$$z = f(x_0, y_0) + \frac{\partial f}{\partial x}\bigg|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\bigg|_{(x_0,y_0)}(y - y_0)$$

Problem 2.5. Compute the plane tangent to the graph of $f(x, y) = x^2y + 2xy - y^2$ at $(1, 2)$.

Proof. We have

$$\frac{\partial f}{\partial x}(1,2) = 2xy + 2y|_{(1,2)} = 8, \quad \frac{\partial f}{\partial y}(1,2) = x^2 + 2x - 2y|_{(1,2)} = -1$$

and $f(1, 2) = 2$, thus the plane is given by

$$z = 2 + 8(x - 1) - (y - 2)$$

i.e., $z = 8x - y - 4$. □

Definition 2.6 (derivative for two variables)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$, then f is said to be **differentiable** at (x_0, y_0) if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{\|(x,y) - (x_0,y_0)\|} = 0$$

where $\mathcal{P}(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$ is the tangent plane to f at (x_0, y_0).

Definition 2.7 (derivative for n variables)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **gradient** of f, denoted as ∇f is given by

$$\nabla f = \left[\tfrac{\partial f}{\partial x_1}, \ldots, \tfrac{\partial f}{\partial x_n}\right]$$

is a $1 \times n$ matrix. And f is said to be **differentiable** at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)\|}{\|x - x_0\|} = 0$$

and the derivative of f is exactly the gradient ∇f at x_0.

Definition 2.8 (derivative for m outputs)**.** Let $f : \mathbb{R} \to \mathbb{R}^m$, where $f(x) = (f_1(x), \ldots, f_m(x))$, then let T denote the $n \times 1$ matrix

$$T = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \cdots \\ \frac{df_m}{dx}(x_0) \end{bmatrix}$$

Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - T(x - x_0)|}{|x - x_0|} = 0$$

and the matrix T is the derivative at x_0.

Example 2.2. Let $f(x) = (x^2, 2x, -x)$, then

$$T = Df(1) = \begin{bmatrix} 2x \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Definition 2.9 (derivative for general functions)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$, let T be the $m \times n$ matrix with entries $\partial f_i / \partial x_j$ evaluated at $x_0 \in \mathbb{R}^n$. Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

then f is differentiable at x_0, and the matrix T is the derivative at x_0. Note that T loosk like

$$T = Df(x_0), \quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 2.3. Let $f(x, y, z) = (ze^x, -ye^z)$, then

$$Df(x, y, z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$

Problem 2.6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = x^2y + y\sin(z) + ze^x.$$

Compute the gradient of f at $(1, 2, 0)$.

Proof. You can compute the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy + ze^x, \\ \frac{\partial f}{\partial y} &= x^2 + \sin(z), \\ \frac{\partial f}{\partial z} &= y\cos(z) + e^x. \end{aligned}$$

thus the gradient is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xy + ze^x, x^2 + \sin z, y\cos z + e^x)$$

Hence

$$\nabla f(1, 2, 0) = (4, 1, 2 + e)$$

□

Week 5 (9/22-9/26)

Topics: (1) properties of derivatives, (2) directional derivatives, (3) gradient.

Definition 2.10 (path). A **path** c is a map $c : [a, b] \to \mathbb{R}^n$. We can write $c(t) = (c_1(t), \ldots, c_n(t))$. If c is differentiable, then we can define the **velocity** of c at any $t_0 \in [a, b]$ as

$$c'(t_0) = (c_1'(t_0), \ldots, c_n'(t_0))$$

The velocity vector of c at t_0 is also a **tangent** vector to c at t_0. The **speed** of the path c at t_0 is the length of the velocity vector $\|c'(t_0)\|$.

Definition 2.11 (tangent line to a path). Let $c : [a, b] \to \mathbb{R}^n$ be a path, if $c'(t_0) \neq 0$, then the **tangent line** at x_0 is given by

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

Proposition 2.1. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, then the derivative of f at x_0 is an $m \times n$ matrix $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$. The derivative follows the same properties as derivative for single variable functions:

1. Let $c \in \mathbb{R}$, then
$$D(cf)(x_0) = cDf(x_0) \qquad \text{(multiplication of a matrix by constant } c\text{)}$$
2. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ also be differentiable at x_0, then
$$D(f + g)(x_0) = Df(x_0) + Dg(x_0) \qquad \text{(sum of two matrices)}$$
3. Let $h_1 : U \subset \mathbb{R}^n \to \mathbb{R}$, $h_2 : U \subset \mathbb{R}^n \to \mathbb{R}$,then
$$D(h_1 h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \qquad \text{(product rule)}$$
and if $h_2 \neq 0$ on U.
$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \qquad \text{(quotient rule)}$$
4. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ such that $g(U) \subset V$, then
$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \qquad \text{(chain rule)}$$

Definition 2.12 (directional derivative). Let $f : \mathbb{R}^3 \to \mathbb{R}$, be differentiable, then the directional directive at $x_0 \in \mathbb{R}^3$ in the direction of a **unit vector** v is given by

$$\nabla f(x_0) \cdot v = \left[\frac{\partial f}{\partial x_1}(x_0)\right] v_1 + \left[\frac{\partial f}{\partial x_2}(x_0)\right] v_2 + \left[\frac{\partial f}{\partial x_3}(x_0)\right] v_3$$

where $v = (v_1, v_2, v_3)$.

Proposition 2.2. Suppose that $\nabla f(x_0) \neq 0$, then the direction for which f increases the fastest at x_0 is along $\nabla f(x_0)$.

Proposition 2.3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable, let S be a level surface of f, i.e., S is a surface described by

$$f(x, y, z) = k$$

were k is some constant. Let $(x_0, y_0, z_0) \in S$, then

$$\nabla f(x_0, y_0, z_0) \text{ is normal to the level surface at } (x_0, y_0, z_0)$$

This means if $c(t)$ is a path in S, and $v(0) = (x_0, y_0, z_0)$, and if v is a tangent vector to $c(t)$ at $t = 0$, then

$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$

Moreover, if $\nabla f(x_0, y_0, z_0) \neq 0$, the **tangent plane** of S at (x_0, y_0, z_0) is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Problem 2.7. Consider the curve in $\mathbb{R}$: $c(t) = (2t, t^2, -t)$. Find the speed of the c at $t = 2$ and the tangent line at $t = 1$.

Proof. The velocity vector of c at $t = 2$ is

$$c'(t) = (2, 2t, -1)$$

evaluated at $t = 2$ is $c'(2) = (2, 4, -1)$. Thus the speed is the length of the velocity vector

$$\|c'(2)\| = \left(2^2 + 4^2 + (-1)^2\right)^{\frac{1}{2}} = \sqrt{21}$$

For the tangent line: the tangent vector is

$$c'(1) = (2, 2, -1)$$

and $c(1) = (2, 1, -1)$. Thus the tangent line l at $t = 1$ is given by

$$l(t) = (2, 1, -1) + t(2, 2, -1) = (2 + 2t, 1 + 2t, -1 - t)$$

□

Problem 2.8 (2.5, Q7). Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and

$$g(x, y) = (e^{x-y}, x - y).$$

Calculate $f \circ g$ and

$$D(f \circ g)(1, 1).$$

Proof. We have

$$f \circ g(x, y) = (\tan(e^{x-y} - 1) - e^{x-y}, e^{2(x-y)} - (x - y)^2)$$

and $g(1, 1) = (1, 0)$, thus using chain rule, we have

$$D(f \circ g)(1, 1) = Df(1, 0)Dg(1, 1)$$

where

$$Df(u, v) = \begin{bmatrix} \sec^2(u - 1) & -e^v \\ 2u & -2v \end{bmatrix}, \quad Dg(x, y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

Hence

$$D(f \circ g)(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$$

□

Problem 2.9 (2.5, Q8)**.** Let $f(u,v,w) = (e^{u-w}, \cos(v+u) + \sin(u+v+w))$ and $g(x,y) = (e^x, \cos(y-x), e^{-y})$. Calculate $f \circ g$ and $D(f \circ g)(0,0)$.

Proof. We have

$$f \circ g = (e^{e^x - \cos(y-x)}, \cos(e^x + \cos(y-x)), \sin(e^x + e^{-y} + \cos(y-x)))$$

and $g(0,0) = (1,1,1)$. Thus

$$D(f \circ g)(0,0) = Df(1,1,1)Dg(0,0)$$

where

$$Df(u,v,w) = \begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & -\sin(v+u) + \cos(u+v+w) & \cos(u+v+w) \end{bmatrix}$$

and

$$Dg(x,y) = \begin{bmatrix} e^x & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix}$$

Thus

$$D(f \circ g)(0,0) = \begin{bmatrix} 1 & 0 & -1 \\ -\sin 2 + \cos 3 & -\sin 2 + \cos 3 & \cos 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ -\sin 2 + \cos 3 & -\cos 3 \end{bmatrix}$$

□

Problem 2.10 (2.5, Q11)**.** Let $f(x,y,z) = (3y+2, x^2+y^2, x+z^2)$. Let

$$c(t) = (\cos(t), \sin(t), t).$$

(a) Find the path $p = f \circ c$ and the velocity vector

$$p'(\pi).$$

(b) Find $c(\pi), c'(\pi)$ and $Df(-1,0,\pi)$.
(c) Thinking of $Df(-1,0,\pi)$ as a linear map, find

$$Df(-1,0,\pi)\ (c'(\pi)).$$

Proof. (a) We have

$$p(t) = (3\sin t + 2, 1, \cos t + t^2)$$

and

$$p'(t) = (3\cos t + 0, -\sin t + 2t)$$

thus

$$p'(\pi) = (-3, 0, 2\pi)$$

(b) We have $c(\pi) = (-1, 0, \pi)$, and $c'(t) = (-\sin t, \cos t, 1)$, and $c'(\pi) = (0, -1, 1)$. And

$$Df(x, y, z) = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix}$$

Thus

$$Df(-1, 0, \pi) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}$$

(c) We have

$$Df(-1, 0, \pi)(c'(\pi)) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}$$

□

Problem 2.11 (2.6, Q3). Compute the directional derivatives of the following functions along unit vectors at the indicated points in directions parallel to the given vector:

(a)

$$f(x, y) = x^y, (x_0, y_0) = (e, e), \quad \mathbf{d} = 5\mathbf{i} + 12\mathbf{j}$$

(b)

$$f(x, y, z) = e^x + yz, \ (x_0, y_0, z_0) = (1, 1, 1), \quad \mathbf{d} = (1, -1, 1)$$

(c)

$$f(x, y, z) = xyz, \ (x_0, y_0, z_0) = (1, 0, 1), \quad \mathbf{d} = (1, 0, -1)$$

Proof. We find ∇f for all these functions and find the directional directive

$$\nabla f(x_0) \cdot \frac{d}{\|d\|}$$

(a) We have

$$\nabla f(x, y, z) = (yx^{y-1}, x^y \ln x)$$

hence

$$\nabla f(e, e) \cdot \frac{d}{\|d\|} = (e^e, e^e) \cdot \left(\frac{5}{13}, \frac{12}{13}\right) = \frac{17}{13} e^e$$

(b) We have

$$\nabla f(x, y, z) = (e^x, z, y)$$

hence

$$\nabla f(1, 1, 1) \cdot \frac{1}{\sqrt{3}}(1, -1, 1) = \frac{e}{\sqrt{3}}$$

(c) We have

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 0, 1) \cdot \frac{1}{\sqrt{2}}(1, 0, -1) = \frac{1}{\sqrt{2}}(0, 1, 0) \cdot (1, 0, -1) = 0$$

□

Problem 2.12 (2.6, Q6). Find a vector which is normal to the curve

$$x^3 + xy + y^3 = 11 \text{ at } (1, 2).$$

Proof. Consider the function $f(x, y) = x^3 + xy + y^3$, then the level set of $f(x, y) = 11$ coincides with the curve above. Thus it suffices to compute

$$\nabla f(x, y) = (3x^2 + y, x + 3y^2)$$

and

$$\nabla f(1, 2) = (5, 13)$$

is perpendicular to the the level curve. □

Problem 2.13 (2.6, Q7). Find the rate of change of $f(x, y, z) = xyz$ in the direction normal to the surface

$$yx^2 + xy^2 + yz^2 = 3 \text{ at } (1, 1, 1).$$

Proof. We first find a normal vector to the surface, consider the surface as a level set of the function

$$g(x, y, z) = yx^2 + xy^2 + yz^2$$

Thus

$$\nabla g(x, y, z) = (2xy + y^2, x^2 + 2xy + z^2, 2yz)$$

hence

$$u = \nabla g(1, 1, 1) = (3, 4, 2)$$

is a normal vector to the surface, and we normalize it to get a unit normal vector $n = \frac{u}{\|u\|} = \frac{1}{\sqrt{29}}(3, 4, 2)$. Now we find the directional derivative of $f(x, y, z) = xyz$ along $(3, 4, 2)$:

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 1, 1) = (1, 1, 1)$$

and the directional derivative is

$$\nabla f(1, 1, 1) \cdot n = \frac{9}{\sqrt{29}}$$

□

Summary

- If asked to find directional derivative/rate of change along a unit vector v at point x_0: find $\nabla f(x_0) \cdot v$.
- If asked to find along with direction f increases the fastest: find $\nabla f(x_0)$.
- If asked to find a normal vector to a surface at a point x_0: construct a function such that the surface is the level set of this function, then find $\nabla f(x_0)$.

Chapter 3

Higher-Order Derivatives: Maxima and Minima

Week 6 (9/29-10/3)

Topics: (1) Higher-order derivatives, (2) Taylor expansion.

Proposition 3.1. Let $f(x, y)$ be twice continuously differentiable, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Definition 3.1 (First order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + R_1(a, x)$$

where

$$\frac{R_1(a, x)}{\|x - a\|} \to 0 \text{ as } x \to a$$

Definition 3.2 (Alternative Definition (First-order))**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$. Then

$$f(a + h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + R_1(a, h)$$

where $R_1(a, h)/\|h\| \to 0$ as $h \to 0$.

Definition 3.3 (Second order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + R_2(a, x)$$

where

$$\frac{R_2(a, x)}{\|x - a\|} \to 0 \text{ as } x \to a$$

Definition 3.4 (Alternative Definition (Second-order)). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order. Then we can write

$$f(a+h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a) + R_2(a,h)$$

where $R_2(a,h)/\|h\|^2 \to 0$ as $h \to 0$.

Problem 3.1. Find all the second partial derivatives of $f(x,y) = xy + \ln(x-y)$.
(This includes $\partial^2 f/\partial x^2, \partial^2 f/\partial x\partial y, \partial^2 f/\partial y\partial x, \partial^2 f/\partial y^2$).

Proof. For first-order derivatives:

$$\frac{\partial f}{\partial x} = y + \frac{1}{x-y}, \quad \frac{\partial f}{\partial y} = x - \frac{1}{x-y}$$

Then

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{(x-y)^2}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 1 + \frac{1}{(x-y)^2}$$
$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{(x-y)^2}$$

□

Problem 3.2. Write the second-order Taylor expansion for the following function,

$$f(x,y) = e^{x+y}$$

centered at $(x,y) = (0,0)$.

Proof. We first compute all the partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} = e^{x+y}$$

then

$$f(x) = 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + R_2(x)$$

where $R_2(x)/\|x\| \to 0$ as $x \to 0$. □

Week 7 (10/6-10/10)

Topics: Extremum.

Definition 3.5 (quadratic function). A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic function** if it is given by

$$g(h_1, \dots, h_n) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

where (a_{ij}) is an $n \times n$ matrix. We can also write g as follows:

$$g(h_1, \dots, h_n) = [h_1, \dots, h_n] \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Definition 3.6 (Hessian matrix). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, and suppose all the second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist, then the Hessian matrix of f is the $n \times n$ matrix given by

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \dots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

The Hessian as a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} \begin{bmatrix} h_1 & \dots & h_n \end{bmatrix} Hf(x) \begin{bmatrix} h_1 \\ \dots \\ h_n \end{bmatrix}$$

where $h = (h_1, \dots, h_n)$.

Definition 3.7 (degenerate/nondegenerate points). Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, let (x_0, y_0) be a critical point. We define the **discriminant**, $\boldsymbol{\mathcal{D}}$,of the Hessian by

$$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $\mathcal{D} \neq 0$, the critical point (x_0, y_0) is called **nondegenerate**; if $\mathcal{D} = 0$, the point (x_0, y_0) is called **degenerate**.

Definition 3.8 (positive, negative-definite). A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is called **positive-definite** if $g(h) \geq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. Similarly, g is **negative-definite** if $g(h) \leq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. (The matrix is positive-definite iff it is symmetric $A^T = A$ and the eigenvalues are nonnegative).

Definition 3.9 (bounded set). A set $A \subset \mathbb{R}^n$ is said to be **bounded** if there is a number $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Proposition 3.2 (extremums are critical points). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, where U is open. If x_0 is a local extremum, then $Df(x_0) = 0$.

Proposition 3.3 (extremum)**.** Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be in C^3, and x_0 is a critical point of f. If the Hessian $Hf(x_0)$ is positive-definite, then x_0 is a local minimum of f; if $Hf(x_0)$ is negative-definite, then x_0 is a local maximum.

Proposition 3.4 (local minimum)**.** Let $f(x, y)$ be of C^2, and U is open in $\mathbb{R}^2$. A point (x_0, y_0) is a strict local **minimum** of f if the following conditions hold:

1. $$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

2. $$\mathcal{D}(x_0, y_0) > 0$$

   where $\mathcal{D}$ is the **discriminant** of the Hessian, defined by

   $$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

   where Hf is the 2×2 Hessian matrix.

3. $$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ in 3, then it becomes a local maximum.

Proposition 3.5 (saddle points)**.** Let $f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, if $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, and $\mathcal{D}(x_0, y_0) < 0$, where $\mathcal{D}$ is the discriminant, then the critical point (x_0, y_0) is a saddle point, i.e., neither a maximum or a minimum.

Proposition 3.6 (continuous functions attain extremum on closed bounded sets)**.** Let $f : D \to \mathbb{R}$ be continuous, where D is closed and bounded in $\mathbb{R}^n$. Then f assumes its absolute maximum and absolute minimum values at some point $x_0, x_1 \in D$.

Problem 3.3. Is the following matrix positive-definite?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof. It is not! Consider the vector $(0, 1)$, we have

$$[0, 1]\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0, 1]\begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1$$

□

Problem 3.4. Find the critical point of $f(x, y) = y + x \sin y$ and classify whether it is a local max/min or a saddle point.

Proof. We compute

$$f_x = \sin y, \quad f_y = 1 + x\cos y$$

Setting them both to 0 gives

$$(1, n\pi), \text{ when } n \text{ is odd}, \quad (-1, n\pi) \text{ when } n \text{ is even}$$

Now we compute the discriminant:

$$f_{xx} = 0, \quad f_{xy} = f_{yx} = \cos y, \quad f_{yy} = -x\sin y$$

Thus

$$\mathcal{D} = \det\begin{bmatrix} 0 & \cos y \\ \cos y & -x\sin y \end{bmatrix} = -\cos^2 y$$

Thus $\mathcal{D} < 0$ for all the critical points, hence they are saddle points! $\square$

Week 7 Additional content

Proposition 3.7. If f has a maximum or minimum at x_0 when constrained to a surface S, then $\nabla f(x)$ is perpendicular to S at x_0.

Consequence of Proposition 11:

Proposition 3.8. Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ and $\vec{x_0} \in U$ such that $g(x_0) = c$. Let $\mathcal{L}_c$ be the level set for g with value c and assume

$$\nabla g(\vec{x_0}) \neq \vec{0}.$$

If f restricted to x_c has a local minimum or maximum on $\mathcal{L}_c$ at x_0, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Problem 3.5 (Marsden-Tromba, III. 2)**.** Let $f(x, y, z) = x - y + z$, find the extremum of f subject to the constraint $x^2 + y^2 + z^2 = 2$.

Proof. We compute the gradient of f and $g(x, y, z) = x^2 + y^2 + z^2$:

$$\nabla f(x, y, z) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \quad \nabla g(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

By the proposition above, we need to have

$$\begin{aligned} \lambda &= 2x \\ -\lambda &= 2y \\ \lambda &= 2z \end{aligned}$$

Equating all λ gives

$$x = z = -y$$

Plugging in the constraint we get

$$3x^2 = 2$$

Hence two critical points

$$\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right), \quad \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$$

Plugging them in we find the extremums are $\sqrt{6}$ and $-\sqrt{6}$. □

Problem 3.6 (Marsden-Tromba, III. 16)**.** Use Lagrange multipliers to find the distance from the point $(2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$.

Proof. Let (x, y, z) be a point on the plane, then we would like the minimize the function

$$f(x, y, z) = (x - 2)^2 + y^2 + (z + 1)^2$$

with the constraint

$$3x - 2y + 8z + 1 = 0$$

Writing $g(x, y, z) = 3x - 2y + 8z + 1$, we do the exact same thing as we did above:

$$\nabla f = \begin{bmatrix} 2x - 4 & 2y & 2z + 2 \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 3 & -2 & 8 \end{bmatrix}$$

Then setting

$$\begin{aligned} 2x - 4 &= 3\lambda \\ 2y &= -2\lambda \\ 2z + 2 &= 8\lambda \end{aligned}$$

One can then replace x, y, z with expressions in λ in the constraint:

$$3x - 2y + 8z + 1 = 0$$

which gives

$$\lambda = 2/77$$

Then plugging in λ to solve for x, y, z, the $\sqrt{f(x, y, z)}$ is the final answer. (I am too lazy to do the computation). $\square$

Chapter 4

Vector-Valued Functions

Week 8 (Fall Break)

Topics: (1) Acceleration and Arc Length, (2) Vector Fields.

Proposition 4.1 (Newton's second law)**.** Let $c(t)$ be a path of a particle with mass m and $a(t) = c''(t)$ be the acceleration, then

$$F(c(t)) = ma(t)$$

where F is the force applying on the particle.

Definition 4.1 (arc length)**.** Let $c(t) = (x(t), y(t), z(t))$ be a path, then the length of the path in $\mathbb{R}^3$ from $t_0 \leq t \leq t_1$ is

$$\begin{aligned} L_{t_0 \to t_1}(c) &= \int_{t_0}^{t_1} \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right)^{\frac{1}{2}} dt \\ &= \int_{t_0}^{t_1} \|c'(t)\| dt \end{aligned}$$

More generally, if $c(t) = (x_1(t), \ldots, x_n(t))$ is a path in $\mathbb{R}^n$, then

$$L_{t_0 \to t_1}(c) = \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} x_i'(t)^2\right)^{\frac{1}{2}} dt$$

Definition 4.2 (vector field)**.** A vector field is a function $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns $x \in \mathbb{R}^n$ to another vector $F(x) \in \mathbb{R}^n$.

Problem 4.1. Find the velocity, speed, and acceleration of the following path at $t = 0$:

$$c(t) = (\cos t, 2t, -\sin t)$$

Proof. The velocity is

$$c'(t) = (-\sin t, 2, -\cos t), \quad c'(0) = (0, 2, -1)$$

And the speed is

$$\|c'(t)\| = \sqrt{5}$$

□

Problem 4.2. Find the length of the curve above from $t = 0$ to $t = 2$.

Proof.

$$\begin{aligned} L_{0\to 2}\|c'(t)\|dt &= \int_0^2 \left(\sin^2 t + 4 + \cos^2 t\right)^{\frac{1}{2}} dt \\ &= \int_0^2 \sqrt{5}dt \\ &= 2\sqrt{5} \end{aligned}$$

□