Functional Analysis

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Chapter 1

Preliminary

1.1 9/3 lecture

Definition 1.1 (orthonormal basis). Let S be an orthonormal set in the Hilbert space such that no other orthonormal set contains S as a proper subset. Then S is called an orthonormal basis.

Proposition 1.1. Every Hilbert space admits an orthonormal basis.

Proof. Zorn's lemma. $\square$

Remark: if H is separable, i.e., H has a countable dense subset, then the proof does not require Zorn's lemma. For example, L^2 is separable.

Proposition 1.2 (II.6, Parsevel's formula). Let $\mathcal{H}$ be a Hilbert space, and $S = \{x_n\}$ be an orthonormal basis, then for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha \in \mathcal{A}} (x_\alpha, y) x_\alpha, \quad \|y\|^2 = \sum |(x_n, y)|^2$$

where $\mathcal{A}$ is an index set.

Proof. Bessel's inequality states that for any $\mathcal{A}' \subset \mathcal{A}$ finite, we have

$$\sum_{\alpha \in \mathcal{A}'} |(x_\alpha, y)|^2 \leq \|y\|^2 < \infty$$

It follows that $|(x_\alpha, y)| > \frac{1}{n}$ for at most finitely many α's, and $|(x_\alpha, y)| \neq 0$ for at most countably many α's. Let $\{\alpha_i\}_{i=1}^{\infty}$ be an enumeration of such α's. Then

$$\sum_{i=1}^{N} |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$$

which implies

$$\sum_{i=1}^{\infty} |(x_{\alpha_i}, y)|^2 < \infty$$

Let

$$y_n = \sum_{i=1}^{n} (x_{\alpha_i}, y) x_{\alpha_i},$$

we would like to show that the sequence $\{y_n\}$ is cauchy,

$$\|y_n - y_m\|^2 = \left\| \sum_{i=m+1}^{n} (x_{\alpha_i}, y)x_{\alpha_i} \right\|^2 \to 0 \text{ as } m \to \infty$$

Thus $\{y_n\}$ is Cauchy. In other words,

$$y_n \to y = \sum_{i=1}^{\infty}(x_{\alpha_i}, y)x_{\alpha_i}$$

□

Definition 1.2. A metric space is separable if it has a countable dense subset.

Proposition 1.3 (II.7). Let $\mathcal{H}$ be a Hilbert space, then it is separable iff it has a countable orthonormal basis.

Proof. Suppose $\mathcal{H}$ is separable, let $\{x_n\}$ be a countable dense set, then we throw out terms in $\{x_n\}$ until we get a linearly indepedent dense subset $\{u_n\} \subset \{x_n\}$. Applying Gram-Schmidt, we can assume $\{u_n\}$ to be countable and orthonormal. Conversely, if $\{u_n\}$ is a countable orthonormal basis, then the set of linear combinations of $\{u_n\}$ with rational coefficients forms a countable dense subset of $\mathcal{H}$. □

Definition 1.3 (Fourier Coefficient). The nth Fourier coefficient of a 2π-periodic function f is

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x)dx$$

The Fourier series of f is

$$\tilde{f}(x) = \lim_{M\to\infty} \sum_{M=-N}^{N} \frac{1}{\sqrt{2\pi}} c_n e^{inx}$$

Proposition 1.4. The Fourier series $\sum_k c_k$ converges if $f \in L^2$. Moreover, the series converges uniformly to a continuous function if $\sum |c_k| < \infty$

I am too lazzy to type it up, but it uses the fun lemma below:

Lemma 1.1. Suppose f is 2π-periodic, and $(f, e^{inx}) = 0$ for all n, then $f \equiv 0$. (In other words, if all the Fourier coefficients are 0, then the function must be identically zero).

1.2 9/8 Lecture

Definition 1.4 (Banach space). A complete normed linear space is called a Banach space.

Example 1.1. 1. $L^\infty(\mathbb{R}) = \{f : f(x) \leq M \text{ a.e. }\}$, where $\|f\|_\infty$ is the smallest such M, is a Banach space.

2. Let $C(\mathbb{R})$ be the bounded continuous functions on $\mathbb{R}$. Let $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and equip it with the same norm. Moreover, $C(\mathbb{R})$ is also a Banach space (due to the uniform convergence of continuous functions is still continuous).

3. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support, and this is not a Banach space under $\|\cdot\|_\infty$.

4. L^p is complete for all $1 \leq p < \infty$.

5. Let $a = \{a_n\}$ be a sequence of complex numbers, ad

$$\|a\| = \sup_n |a_n| < \infty$$

let $c_0 = \{\lim_{n\to\infty} a_n = 0\}$, $s = \{\lim_{n\to\infty} n^N a_n = 0 \forall N\}$, and $l_p = \{\|a\|_p^p = \sum_{n=1}^\infty |a_n|^p < \infty\}$. Note that the space

$$f = \{a_n = 0 \text{ for al but finitely many } n\}$$

is not complete! However, it is a dense subset in l^p. Morever, the set of elements in f with rational coefficients, and the closure of f in s, l^p, c_0 are exactly the whole spaces, i.e., s, l^p, c_0 are separable.

6. Let $L(X,Y)$ be bounded linear operators from X, Y, with the operator norm, and $L(X,Y)$ is a Banach psace.

Proposition 1.5. Let $L^p(\mathbb{R})$, where $1 \leq p < \infty$ be the space of functions with the norm

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^p$$

then

1. (Minkowski's inequality) $\|f\|_p \leq \|f\|_p + \|g\|_p$.

2. (Riesz-Fischer) L^p is complete.

3. (Holder) Given $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have

$$\|fg\|_r \leq \|f\|_p \|g\|_q$$

if $f \in L^p, g \in L^q$.

Proposition 1.6. If Y is complete, then $L(X,Y)$ is a Banach space.

Proof. Suppose $\{A_n\}$ is Cauchy, now we construct the limit: for each x, $A_n x = y_n$ is a Cauchy sequence:

$$\|y_n - y_m\| \leq \|A_n - Am\| \cdot \|x\|$$

Now since Y is complete, we know that $A_n x \to y$. Let $Ax = y$. (This is our limit)! Now $\|A_n\| \leq C$ for all n, which implies $\|A\| \leq C$. Thus $L(X,Y)$ is complete! □

1.2.1 Duals

Definition 1.5 (dual space). The space of bounded linear functionals $L(X, \mathbb{C})$, where X is Banach, is called the dual space to X, denoted by X^*. Let $f \in X^*$, then define the norm

$$\|f\| = \sup_{x \in X, \|x\| \leq 1} |f(x)|$$

Example 1.2. 1. Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L^q$, then

$$G(f) = \int_{-\infty}^{\infty} \bar{g}(x) f(x) dx$$

Then G is in $(L^p)^*$. Moreover, any such linear functional on L^p can be written in this way for some $g \in L^q$. And

$$|G(f)| \leq \|f\|_p \|g\|_q$$

by Holder. Moreover,

$$L^q(\mathbb{R})^* = L^p, (L^q(\mathbb{R})^*)^* = L^q$$

because L^q is reflexive! In particular, L^2 is its own dual space.

2. Suppose $\{\lambda_k\} \subset l^q$, then

$$\Lambda(\{a_k\}) = \sum_k \lambda_k a_k$$

is a bounded linear functional on l^p. Thus

$$l_q \subset (l^p)^*$$

for $1 \leq p \leq \infty$. It turns out every linear functional on l^p can be written in this form.

Example 1.3. Let $p = 1$, we have

$$L^1(\mathbb{R})^* = L^\infty, \text{ but } L^\infty(\mathbb{R})^* \neq L^1(\mathbb{R})$$

in fact $L^\infty(\mathbb{R})^*$ is bigger.

1.3 9/10 Lecture

Proposition 1.7 (Geometric Hahn-Banach). Let V_1 be a subspace of V, $x \in V \setminus V_1$, then one can find a hyperplane (codim 1) V_2 such that $V_1 \subset V_2$, and $x \notin V_2$.

Proof. If V_1 has codim$V_1 = 1$, then we are done. Suppose that codim $V_1 > 1$, we would like to find V_2 such that $x \notin V_2$, where $V_2 \neq V_1$ such that $\dim(V/V_1) > 1$. Note that we define

$$\dim(V/V_1) = \{[z] : [z] = [w] \iff z + w \in V_1\}$$

(For any banach space, we can write $B = W \oplus (B/W)$). This implies that we can find $y = [y] \in V/V_1$ such that $y \not\emptyset$, and $y \neq x$ (by codim > 1). Set

$$V_2 = \{z + ty; z \in V_1, t \in \mathbb{R}\}$$

Then we can continue this process, and using Zorn's lemma, we can have V_2 to have codim 1. $\square$

Definition 1.6. A subset $A \subset V$ is called if for any $x, y \in V$, the line connecting them is contained in A. If the set is also open, then we call A convex linearly open.

Proposition 1.8 (Geometric HB for Convex sets)**.** Let $A \subset V$ be convex linearly open, and let V_1 be a linear subspace which does not intersect A. Then there is a hyperplane V_2 such that $V_1 \subset V_2$ and $V_2 \cap A = \varnothing$.

(Essentially proof by picture).

Proposition 1.9 (Hahn-Banach)**.** Let X be a real vector space, for all $x, y \in X$, and $\alpha \in [0, 1]$, with sublinear functional $p(x)$ satisfying

$$P(\alpha x + (1-\alpha)y) \leq \alpha p(x) + (1-\alpha)p(y)$$

Suppose that λ is a linear functional defined on a subspace on Y such that $\lambda(y) \leq p(y)$ for all $y \in Y$. Then there is a linear functional Λ on X such that $\Lambda = \lambda$ on Y, and

$$\Lambda(x) \leq p(x)$$

Proof. Let $x \in Y \setminus Y$, we will first show that we can extend λ to the subspace spanned by Y and z, following the same bound. Define

$$\tilde{\lambda}(az + y) = a\tilde{\lambda}(z) + \lambda(y)$$

Suppose that $y_1, y_2 \in Y$, and $\alpha, \beta > 0$, and

$$\begin{aligned} \beta\lambda(y_1) + \alpha\lambda(y_2) = \lambda(\alpha y_1 + \beta y_2) &= (\alpha+\beta)\lambda\left(\frac{\beta}{\alpha+\beta}y_1 + \frac{\alpha}{\alpha+\beta}y_2\right) \\ &\leq (\alpha+\beta)p(\dots) \\ &\leq \beta p(y_1 - \alpha z) + \alpha(y_2 + \beta z) \end{aligned}$$

deviding both sides by α, β, taking the sup over $\alpha > 0, y$, we see that

$$\tilde{\lambda}(x) \leq p(x)$$

for all x in this subspace. Using Zorn's lemma, we extend one subspace at a time, then we are done. □

1.4 9/15 Lecture

Proposition 1.10 (Geometric Hahn-Banach)**.** Let A be complex and linearly open, and $A \subset V$ be a vector space over $\mathbb{R}$. Let V_1 be a subspace of V, $V_1 \cap A = \varnothing$. Then there exists a hyperplane V_2 such that $V_2 \cap A = \varnothing, V_1 \subset V_2$.

Definition 1.7. A seminorm p is such that $p(x) \geq 0, p(x+y) = p(x) + p(y)$, and $p(\alpha x) = |\alpha| p(x)$.

And we have the following analytic version of Hahn-Banach.

Proposition 1.11 (Analytic Hahn-Banach)**.** Let W be a subspace of V, and f linear form on W such that $|f(x)| \leq p(x)$, for all $x \in W$. Then there is a linear form f_1 such that $f_1(x) = f(x)$ on W, and $|f_1(x)| \leq p(x)$ for all x.

I am too lazy to follow the proof.

1.5 9/17 Lecture

Proposition 1.12. Let $T : X \to Y$ be a linear map, T is bounded iff

$$T^{-1}(\{y : \|y\| \leq 1\})$$

has nonempty interior.

Proof. Suppose that it has a nonempty interior, then

$$\{x : \|x - x_0\| \leq \varepsilon\} \subset T^{-1}(\{y : \|y\| \leq 1\})$$

Then if $\|x\| \leq \varepsilon$, then

$$\|Tx\| \leq \|T(x - x_0)\| + \|Tx_0\| \leq 1 + \|Tx_0\|$$

then

$$T\left(\frac{\varepsilon x}{\|x\|}\right) = \frac{\varepsilon}{\|x\|}T(x) \Rightarrow \|T(x)\| \leq (1 + \|Tx_0\|)\frac{1}{\varepsilon}\|x\|$$

□

Definition 1.8 (nowhere dense)**.** A set S is called nowhere dense if $\overline{S}$ has empty interior.

Proposition 1.13 (Baire category theorem)**.** A complete metric space M is never a union of a countable number of nowhere dense sets.

Proof. Suppose $M = \bigcup_{n=1}^{\infty} A_n$, where A_n is nowhere dense. We will construct $\{x_n\}$ that stays away from each A_n so the limit x is in any A_n, which gives a contradiction. Take $x_1 \in \bar{A}_1$, and $B_1 \cap A_1 = \varnothing$, and $B_1 = B(x_1, r_1) = \{d(x, x_1) < \varepsilon\}$, let $x_2 \in B_1 \setminus A_1$, and let $B_2 = B(x_2, r_2)$, and so on, $\{x_n\}$ is a Cauchy sequence, $x_n \to x$ as $n \to \infty$. We have $x_n \in B_N, n \geq N$, but $\bar{B}_N \subset B_{N-1}$, thus $x \in B_{N-1}$ for any N, and $x \notin A_{N-1}$ for any N. This gives our contradiction. □

Proposition 1.14 (UBP)**.** Let X be a Banach space, let $\mathcal{F}$ be a family of bounded linear maps $T : X \to Y$. Suppose that for each $x \in X$,

$$\{\|Tx\| : T \in \mathcal{F}\}$$

is bounded, then

$$\sup_{T \in \mathcal{F}} \|T\| < \infty$$

Proof. Key ingredient: Baire category theorem. Let $B_n = \{x : \|Tx\| \leq n, T \in \mathcal{F}\}$, each x is in some B_n, and $X = \bigcup_{n=1}^{\infty} B_n$, where B_n is closed. Baire category theorem, some B_n has nonempty interior.

$$B_n\{x : \|Tx\| \leq n\} \Rightarrow \{|x : \|Tx\| \leq 1\}$$

And

$$\|Tx\| \leq (1 + \|Tx_0\|)\|x\| \Rightarrow \|T\| \leq 1 + \|T(x_0)\|$$

□