Algebraic Topology

Hui Sun

October 9, 2024

Contents

Chapter 1

Category Theory

Instructor: Nitu Kitchro, **Office Hours**: Monday after class, **TA:** Anna Matsui

1.1 Lecture 1 8/26

Definition 1.1 (Category). A category $\mathcal{C}$ consists of the following data:

1. A collection of objects denoted as Ob($\mathcal{C}$)
2. Given two objects $X, Y \in$ Ob($\mathcal{C}$), a collection of morphisms between X, Y, $f : X \to Y$, denoted as mor$_{\mathcal{C}}(X, Y)$.
3. (Composition) We have mor$_{\mathcal{C}}(X, Y) \times mor_{\mathcal{C}}(Y, Z) \to mor_{\mathcal{C}}(X, Z)$ that satisfies associativity
$$f \circ (g \circ h) = (f \circ g) \circ h$$
4. (Identity) There is a distinguished morphism for each X, $Id_{\mathcal{C}}(X, X)$ such that given any $f \in mor(X, Y)$, we have $f \circ id_X = id_Y \circ f = f$.

In this course, we will make the assumption that in all the categories that we work with, Ob($\mathcal{C}$) need not be a set, but given any $X, Y \in Ob(\mathcal{C})$, mor(X, Y) will always be a set. Now we talka bout some examples of categories.

Example 1.1 (Sets). Let Ob($Sets$) be all the sets in the universe. Given X, Y sets, mor(X, Y) be all the set maps from X to Y, and id_X is the identity map.

Example 1.2 (Top). Let Ob(Top) be all the topological spaces, and mor(X, Y) be all the continuous maps from X to Y.

Example 1.3 (Vect$_{\mathbb{F}}$). Let $\mathbb{F}$ be a field, and let Ob be all the $\mathbb{F}$-vector spaces. Then mor(V, W) is all the $\mathbb{F}$-linear homomorphisms from V to W, where id_V is the identity homomorphism.

Example 1.4 (Posets). Fix a poset P, let Ob(P) be the collection of elements in P, and given p, q we define
$$mor(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \varnothing, & \text{otherwise} \end{cases}$$

Problem 1.1. **HW(Q1): check this is a category**

Example 1.5 (Opposite category). Given a category $\mathcal{C}$, there is another category called the opposite category, denoted as $\mathcal{C}^{op}$, where

1. The objects are the same as $\mathcal{C}$
2. Given $X, Y \in \text{Ob}(C^{op})$, we have $\text{mor}_{op}(X,Y) := \text{mor}_{\mathcal{C}}(Y,X)$.
3. Moreover, given $f \in mor_{op}(X,Y), g \in mor_{op}(Y,Z)$, then $g \circ f$ in C^{op} is $f \circ g : Z \to X$.

Naturally, we define isomorphisms now.

Definition 1.2 (isomorphism). Given a category $\mathcal{C}$, and a morphism $f \in mor_C(X,Y)$, we say f is an isomorphism if there exists $g \in mor_C(Y,X)$ such that

$$f \circ g = Id_Y, g \circ f = Id_X$$

Now we introduce maps between categories.

Definition 1.3 (functor). Given categories $\mathcal{C}, \mathcal{D}$, a functor $F : C \to D$ is the following;

1. Given an object X in $\mathcal{C}$, $F(X)$ is an object in D.
2. Given a morphism $f : X \to Y$, $F(f)$ is a functor $F(f) : F(X) \to F(Y)$. Moreover, it satisfies the following:
   (a) $F(id_X) = id_{F(X)}$
   (b) $F(f \circ g) = F(f) \circ F(g)$. Alternatively, we can rewrite this condition as the following:

$$\begin{tikzcd}
mor(X,Y) \times mor(Y,Z) \arrow[r] \arrow[d, "mor(F) \times mor(F)"] & mor(X,Z) \arrow[d, "mor(F)"] \\
mor(F(X),F(Y)) \times mor(F(Y),F(Z)) \arrow[r] & mor(F(X),F(Z))
\end{tikzcd}$$

   such that this diagram commutes.

Problem 1.2. **HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

Example 1.6. $F : Top \to Set$, where $X \mapsto X$, where the latter is a set, and $f \mapsto f$ as set maps.

Example 1.7. Let $\mathbb{F}$ be a field, and $F : Sets \to \text{Vect}_{\mathbb{F}}$, where $X \mapsto \mathbb{F}\langle X \rangle$, where $\mathbb{F}\langle X \rangle$ is the free vector space over $\mathbb{F}$ on the set X.

Problem 1.3. **HW(Q3): extend this to a functor by defining $mor(f)$ and show this is a functor.**

Example 1.8. Let $\mathbb{F}$ be a field, then the following is a functor, $F : Sets^{op} \to \text{Vect}_{\mathbb{F}}$, where

$$hF : X \mapsto Maps(X, \mathbb{F})$$

Problem 1.4. **HW(Q4)**: show this extends to a functor by defining $F(f)$, and show it is a functor.

1.2 Lecture 2 8/28

Definition 1.4 (contravariant functor). Let $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor from $\mathcal{C}^{op} \to \mathcal{D}$, (equivalently, $\mathcal{C} \to \mathcal{D}^{op}$).

Problem 1.5. **HW(Q5):** Show that the following functor F from $\text{Vect}_{\mathbb{F}}$ to $\text{Vect}_{\mathbb{F}}$ extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = Hom(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let Cat be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

Definition 1.5 (natural transformation). Given functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation T from F to G is the following: $T : F \Rightarrow G$:

1. given object $X \in Ob(\mathcal{C})$, $T(X) \in mor(F(X), G(X))$
2. Given $f \in mor(X, Y)$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T(X)"'] & F(Y) \arrow[d, "T(Y)"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

   where mor_F, mor_G is the identification function on morphisms by functors F, G

If for all X, $T(X)$ is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor F and turn it to another functor G. We will (in a homework) show there exists natural transofrmation between the following two functors.

Example 1.9. Consider $F, G : Vect_{\mathbb{F}} \to Vect_{\mathbb{F}}$, define

$$F(V) = V \otimes_{\mathbb{F}} V/_{\langle a\otimes b - b\otimes a\rangle} = V \otimes_{\mathbb{F}} V/\Sigma_2, G(V) = (V \otimes_F V)^{\Sigma_2} = \{\alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha\}$$

Both are vector spaces are fixed under "swaps." Then a natural transformation can be defined as follows $T(V)$:

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

Problem 1.6. **HW(Q6):** For the above F, G

1. Show that T defines a natural transformation from F to G.
2. Find conditions on $\mathbb{F}$ for T being a natural isomorphism.

Next we define limits and colimits. Let $\mathcal{C}, \mathcal{D}$ be categories, d be an object in $\mathcal{D}$, then we can define a functor $F_d : \mathcal{C} \to \mathcal{D}$ such that for any object c in $\mathcal{C}$,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the "constant functor" on $\mathcal{D}$, i.e., every object is sent to d, and every morphism is sent to id_d.

Definition 1.6 (colimit). Given any functor $F : \mathcal{C} \to \mathcal{D}$, the colimit of F, denoted as $\text{colim}(F)$ is an object in $\mathcal{D}$ endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object d in D and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in $\mathcal{D}$, $f : \text{colim}(F) \to d$ making the following diagram commute: for any X, Y, g:

$$\begin{tikzcd}
F(X) \arrow[rr, "F(g)"] \arrow[dr, "\varphi_F"] \arrow[ddr, bend right, "\varphi"'] & & F(Y) \arrow[dl, "\varphi_F"'] \arrow[ddl, bend left, "\varphi"] \\
& \text{colim}(F) \arrow[d, "f"] & \\
& d &
\end{tikzcd}$$

Next we prove some facts about colimits and give an example, where $\text{colim}(F)$ exists.

Proposition 1.1. If $\text{colim}F$ exists, then $\text{colim}F$ is unique up to isomorphisms.

Proof. Let $\text{colim}(F), \text{colim}(F)'$ be two colimits that satisfy the criteria. They are both objects in $\mathcal{D}$, then we get a morphism $f : \text{colim}(F) \to \text{colim}(F)'$, and likewise $g : \text{colim}(F) \to \text{colim}(G)'$, then

$$f \circ g : \text{colim}(F)' \to \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for $g \circ f$. $\square$

Next we demonstrate a fact via an example.

Theorem 1.1. Let $\mathcal{C}$ be a category where $Ob(\mathcal{C}), mor(X, Y)$ are all sets. Let $F : \mathcal{C} \to \text{Top}$ be any functor, then $\text{colim}(F)$ exists.

Proof. Define $\text{colim}(F) := \bigsqcup_c F(c)/\sim$, where $\sim$ is induced by the equivalence relation given by

$$y \sim F(f)y$$

where $y \in F(C_1), f : C_1 \to C_2, F(f)x \in F(C_2)$. The natural transformation we endow on F as $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$:

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(C)} F(C)/\sim$$

$\square$

Problem 1.7. **HW(Q7):** Show that colim$(F), \varphi_F$ is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

Definition 1.7 (limit). Given a functor $F : \mathcal{C} \to \mathcal{D}$, the limit of F, denoted as $\lim(F)$ is an object of $\mathcal{D}$, endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object $d \in Ob(\mathcal{D})$ and a natural transformation

$$\varphi : F_d \to F$$

there exists a unique $f : \lim F \to d$ such that the following diagram commutes:

$$\begin{tikzcd}
& \lim F \arrow[d, "f"] \arrow[ddl, bend right, "\varphi_F"'] \arrow[ddr, bend left, "\varphi_F"] & \\
& d \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
F(X) \arrow[rr, "F(g)"'] & & F(Y)
\end{tikzcd}$$

Just like colimits, limits are unique up to isomorphisms.

Problem 1.8. **HW(Q8):** Given $F : \mathcal{C} \to \mathcal{D}$, consider $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$, then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

Chapter 2

Homologies, Cohomologies

2.1 Lecture 3 9/4

Today we define (co)chain complexes: let R be a commutative ring, let Mod_R denote the category of R-modules and R-module maps.

Definition 2.1 (chain complex). A chain complex of R-modules is a collection of R-modules and R-modules maps

$$\cdots \to M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \ldots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all i. In other words, the image of previous map is contained in the kernal of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by $\{M.; \partial.^M\}$.

Next we introduce morphisms between chain complexes.

Definition 2.2 (morphism between complexes). Let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$, a morphism $\{f.\}$ between chain complexes is a "ladder" such that the following commutes:

$$\begin{tikzcd}
\ldots \arrow[r] & M_{i+1} \arrow[r, "\partial^M_{i+1}"] \arrow[d, "f_{i+1}"'] & M_i \arrow[r, "\partial^M_i"] \arrow[d, "f_i"'] & M_{i-1} \arrow[r, "\partial^M_{i-1}"] \arrow[d, "f_{i-1}"'] & \ldots \\
\ldots \arrow[r] & N_{i+1} \arrow[r, "\partial^N_{i+1}"] & N_i \arrow[r, "\partial^N_i"] & N_{i-1} \arrow[r, "\partial^N_{i-1}"] & \ldots
\end{tikzcd}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where $\{g.\} : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, and $\{f.\} : \{N.; \partial.^N\} \to \{L.; \partial.^L\}$, which is simply vertical stacking.

Problem 2.1. **HW(Q9):** Prove that chain complexes of R-modules form a category ch_R.

There are interesting functors $F : \mathrm{ch}_R \to Mod_R$, and we begin with the following one:

Definition 2.3 (H_n, nth-homology)**.** Given $n \in \mathbb{Z}$, there is a functor

$$H_n : \mathrm{ch}_R \to Mod_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial_n^M / Im\partial_{n+1}^M$$

and for $f : \{M.; \partial.^M\} \to \{N.; \partial.^N\}$, we define: $H_n(f) : H_n(\{M.; \partial.^M\}) \to H_n(\{N.; \partial.^N\})$,

$$H_n(f)[x] := [f_n(x)]$$

where $[x] \in H_n(\{M.; \partial.^M\})$.

Proof. We need to show H_n is well-defined on objects and morphisms. We need to check that $Im\partial_{n+1} \subset \ker \partial_n$. This is a consequence of $\partial^2 = 0$.

On morphisms: for $x \in \ker \partial_n^M$, we have $f_n(x) \in \ker \partial_n^N$. This is we have

$$\partial_n^N(f_n(x) = f_{n+1})(\partial_n^M(x)) = 0$$

Moreover, we need to check that this desn't depend on the choice of representatives, i.e., we can check that

$$Im\partial_{n+1}^M \mapsto 0$$

Let $x = \partial_{n+1}^M(y)$, we have

$$f_n(x) = f_n(\partial_{n+1}^M(y)) = \partial_{n+1}^N(f_{n+1}(y)) = 0$$

$$\begin{tikzcd}
M_{n+1} \arrow[r, "\partial_{n+1}^M"] \arrow[d, "f_{n+1}"'] & M_n \arrow[d, "f_n"] \\
N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n
\end{tikzcd}$$

□

Next we talk about homotopy between morphisms between chain complexes.

Definition 2.4 (homotopy)**.** Given two morphisms, $f., g. : M. \to N.$, a chain homotopy $h.$ between them is a collection of R-modules maps, for all $n \in \mathbb{Z}$,

$$h_n : M_n \to N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as $\partial h + h\partial = f - g$.

$$\begin{tikzcd}
M_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & M_n \arrow[r, "\partial_n^M"] \arrow[d, "f_n/g_n"] \arrow[dl, "h_n"] & M_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "h_{n-1}"] \\
N_{n+1} \arrow[r, "\partial_{n+1}^N"'] & N_n \arrow[r] & N_{n-1}
\end{tikzcd}$$

Problem 2.2. **HW(Q10):** Show that homotopy is an equivalence relation between morphisms. Hint: replace h_n with $-h_n : M_n \to N_{n+1}$.

Proof. Reflexive is shown by defining h_n to be the zero map. For symmetry, we choose $-h_n$. Transitive is a ladder. □

Proposition 2.1. Let $h.$ be a chain homotopy between $f.$ and $g.$, then we have an equality

$$H_n(f.) = H_n(g.)$$

where $H_n(f.), H_n(g.) : H_n(M.) \to H_n(N.)$.

Proof. Given $[x] \in H_n(M.)$, we have

$$\begin{aligned} H_n(f)[x] &= [f_n(x)] \\ &= [g_n(x) + \partial h.(x) + h.\partial(x)] \\ &= [g_n(x) + \partial h.(x)] \\ &= [g_n(x)] \\ &= H_n(g)[x] \end{aligned}$$

□

Next we define a new category.

Definition 2.5 (Hch_R)**.** Define the category Hch_R as follows:

1. $Ob(\mathrm{Hch}_R) = Ob(\mathrm{ch}_R)$
2. $mor_{\mathrm{Hch}_R}(M., N.) = mor_{\mathrm{ch}_R}(M., N.)/\sim$, where $\sim$ is the homotopy equivalence.

Problem 2.3. HW(Q11): Show that Hch_R is a category, admitting a functor

$$F : ch_R \to Hch_R$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{ch}_R \arrow[rr, "F"] \arrow[dr, "H_n"'] & & \mathrm{Hch}_R \arrow[dl, "H_n"] \\ & \mathrm{mod}_R & \end{tikzcd}$$

Next we introduce long and short exact sequences.

Definition 2.6 (exactness)**.** Firstly, given a pair of R-module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at X_2 if $\ker(g) = \mathrm{im}(f)$. Hence given a sequence of R-module maps,

$$\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \ldots$$

this is called a long exact sequence if it is exact at all X_i. Finally, given a pair of R-module maps,

$$0 \to X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \to 0$$

This is a short exact sequence, and f is injective, g is surjective.

Problem 2.4. **HW(Q12):** Prove the following:

1. Given LES,
$$\cdots \to X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$
show the following is a short exact sequence:
$$0 \to \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \to 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions X_i, Y_i, where $i = 2, 3, 4$, and assume the diagram commutes and if t_1, t_2, t_4, t_5 are isomorphisms, show that t_3 is also an isomorphism.
$$\begin{tikzcd}
X_1 \arrow[r, "f_1"] \arrow[d, "t_1"'] & X_2 \arrow[r, "f_2"] \arrow[d, "t_2"'] & X_3 \arrow[r, "f_3"] \arrow[d, "t_3"'] & X_4 \arrow[r, "f_4"] \arrow[d, "t_4"'] & X_5 \arrow[d, "t_5"'] \\
Y_1 \arrow[r, "g_1"] & Y_2 \arrow[r, "g_2"] & Y_3 \arrow[r, "g_3"] & Y_4 \arrow[r, "g_4"] & Y_5
\end{tikzcd}$$

Next we state the most important theorem in chain complexes:

Theorem 2.1 (The snake lemma). Let $A. \xrightarrow{f.} B. \xrightarrow{g.} C.$ be a SES of chain complexes, i.e.,
$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$
is a short exact sequence of all n. Then there exists a LES of homology groups.
$$\begin{tikzcd}
 & & H_{n+1}(C) \arrow[dll, "\delta_{n-1}"'] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta_n"'] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta_{n-1}"'] \\
H_{n-2}(A) & &
\end{tikzcd}$$

2.2 Lecture 4 9/9

Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & C_n \arrow[d, "\delta^C"'] \\
A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

Proof. First we define the map $\delta_n : H_n(C) \to H_{n-1}(A)$. Let $[x] \in H_n(C)$, then $x \in \delta^C$, where $\delta^C : C_n \to C_{n-1}$. We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for $x \in C_n$, $g_n : B_n \to C_n$ is surjective, hence there exists $b \in B_n$ such that $g_n(b) = x$. Then consider $d = \delta^B(b)$, since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \operatorname{im} f_{n-1}$$

Let $y \in A_{n-1}$ be this unique y such that $f_{n-1}(y) = d$, where uniqueness is by f_{n-1} is injective. This is indicated in the below diagram:

$$\begin{tikzcd}
A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\delta^A"'] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta^B"'] & C_{n+1} \arrow[d, "\delta^C"'] \\
A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & b \in B_n \arrow[r, "g_n"] \arrow[d, "\delta^B"'] & x \in C_n \arrow[d, "\delta^C"'] \\
y \in A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\delta^A"'] & d \in B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\delta^B"'] & C_{n-1} \arrow[d, "\delta^C"'] \\
A_{n-2} \arrow[r, "f_{n-2}"] & B_{n-2} \arrow[r, "g_{n-2}"] & C_{n-2}
\end{tikzcd}$$

We first need to check that $[y]$ does not depend on the choice of b. Let $g_n(b_1) = g_n(b_2) = x$, then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let y_1, y_2 be those determined by b_1, b_2, then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{tikzcd}
a \in A_n \arrow[r, "f_n"] \arrow[d, "\delta^A"'] & B_n \arrow[d, "\delta^B"] \\
A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1}
\end{tikzcd}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e., $[y_1] = [y_2]$, as they only differ by $\operatorname{im} \delta$.

Problem 2.5. **HW(Q13):** Check that if $x \in \operatorname{im} \delta^C$, then $\delta_n[x] = 0$.

the proof is not finished, too lazy to tex it up □

Next we review the tensor products of R-modules. We first review R-bilinear maps

Definition 2.7 (bilinear maps)**.** Let M, N, P be R-modules, an R-bilinear map $f : M \times N \to P$ is a map such that

1. f is linear in both coordinates, we have $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, and similarly, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.
2. For all $r \in R$, we have $f(rm, n) = f(m, rn) = rf(m, n)$.

Next we define tensor products.

Definition 2.8 (tensor product). A tensor product of $M \times N$ is an R-module denoted by $M \otimes_R N$ such that

1. $M \otimes_R N$ comes endowed with an R-bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other R-bilinear map $f : M \times N \to P$, there exists a unique R-module map ψ such that the following diagram commutes:

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[d, "f"'] & M \otimes_R N \arrow[dl, "\psi"] \\ P & \end{tikzcd}$$

It is not clear that $M \otimes_R N$ exists or not. In fact, they exist!

Theorem 2.2 ($M \otimes_R N$ exists). Define $M \otimes_R N = R\langle M \times N\rangle / K$, where $R\langle M \times N\rangle$ is the free R-module on the set $M \times N$. We define K as the submodule generated by the following four relations:

1. $\langle (m_1 + m_2, n)\rangle - \langle (m_1, n)\rangle - \langle (m_2, n)\rangle$
2. $\langle (m, n_1 + n_2)\rangle - \langle (m, n_1)\rangle - \langle (m, n_2)\rangle$
3. $r\langle (m, n)\rangle - \langle (rm, n)\rangle$
4. $r\langle (m, n)\rangle - \langle (m, rn)\rangle$

Moreover, the map $\varphi : M \times N \to M \otimes_R N$ given by

$$(m, n) \mapsto \langle (m, n)\rangle := m \otimes_R n$$

Problem 2.6. **HW(Q14):** show that $M \otimes_R N$ is a tensor product.

2.3 Lecture 5 9/11

We continue with the tensors of R-modules. Let $f : A \to B$ an an R-module map, let N be some fixed R-module, then f induces maps: $f \otimes id : A \otimes_R N \to B \otimes_R N$,

$$f \otimes id : a \otimes n \mapsto f(a) \otimes n$$

and $id \otimes f : N \otimes f : N \otimes_R A \to N \otimes_R B :$

$$id \otimes f : n \otimes a \mapsto n \otimes f(a)$$

Problem 2.7. **HW(Q15(a)):** Show that the following maps induce functors:

1. $-\otimes_R N : Mod_R \to Mod_R$, where
$$A \mapsto A \otimes_R N, f \mapsto f \otimes id$$
2. $N \otimes_R - : Mod_R \to Mod_R$, where
$$A \mapsto N \otimes_R A, f \mapsto id \otimes f$$

Problem 2.8. **HW(Q15(b)):** Show that one has the following natural isomorphisms:

1. $0 \otimes_R M \cong 0$, and $0 \otimes_R - \cong F_0$ (recall the definition of F_0 as a functor).
2. $R \otimes_R M \cong M$, and $R \otimes_R - \cong id$.
3. $M \otimes_R N \cong N \otimes_R M$, and $M \otimes_R - \cong - \otimes_R M$.
4. $M \otimes_R (N \otimes_R K) \cong (M \otimes_R N) \otimes_R K$.
5. $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$.

For convenience, we introduce the following definition:

Definition 2.9 (positively graded chain complex)**.** A positively graded chain complex $\{M.; \partial.^M\}$ is a chain complex so that $M_i = 0$ for all $i < 0$. The category of positively graded chain complexed is denoted as ch_R^+.

We have our first important theorem for ch_R^+.

Theorem 2.3. There exists a functor $\otimes_R$ and a natural transformation X such that the following diagram of functors commutes up to some X:

$$\begin{tikzcd}
ch_R^+ \times ch + R^+ \arrow[r, "\otimes_R"] \arrow[d, "H_i \times H_j"'] & ch_R^+ \arrow[d, "H_{i+j}"] \\
Mod_R \times Mod_R \arrow[r, "\otimes_R"'] \arrow[ur, Rightarrow, "X"] & Mod_R
\end{tikzcd}$$

where $X : \otimes_R \circ (H_i \times H_j) \Rightarrow H_{i+j} \circ \otimes_R$ is a natural transformation.

We note that the existence of X means this diagram doesn't commute "on the nose," but these two composition functors are the same up to some natural transformation. Before we given the proof, we recall that $Ob(C \times D) = Ob(C) \times Ob(D), mor((X,Y),(X',Y')) = mor(X,Y) \times mor(X',Y')$.

Proof. We define $\otimes_R$ of positively graded chain complexes as follows: let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$ be two PGCC. Define $\{M \otimes_R N.; \partial.^M\}$:
$$(M \otimes_R N) = \bigoplus_{i+j=n} (M_i \otimes_R N_j)$$
note that the RHS is always a finite sum. Moreover, $\partial^{\otimes}$ is defined as follows:
$$\partial^{\otimes} : (M \otimes_R N)_n \to (M \otimes_R N)_{n-1} \text{ is defined on the component } M_i \otimes_R N_j \text{ (from the RHS)}$$
and
$$\partial^{\otimes}(m_i \otimes n_j) := \partial^M(m_i) \otimes n_j + (-1)^i m_i \otimes \partial^N(n_j)$$

It is easy to check that $\partial^{\otimes} \circ \partial^{\otimes} = 0$.

Now we've show $ch_R^+ \otimes_R ch_R^+$ is well-defined, it remains to define X, the natural transformation. We define

$$X : H_i(M.) \otimes_R H_j(N.) \to H_{i+j}(M. \otimes_R N.)$$

again, it suffices to define X on elementary tensors.

$$X : [\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$$

we need to check that

1. $\partial^{\otimes}(\alpha \otimes \beta) = 0$ if $\partial^M(\alpha) = 0$ and $\partial^N(\beta) = 0$.
2. If $\alpha = \partial(r)i$, then notice that $\partial^{\otimes}(r \otimes \beta) = \alpha \otimes \beta$, similarly for β. This would show that X is well-defined.

It is straightforward to check that X commutes with morphisms in $ch_R^+ \times ch_R^+$. □

Next we define cochain complexes and cohomologies.

Definition 2.10 (cochain). A cochain of R-modules $(M^\bullet, \partial_M^\bullet)$ is a sequence of R-module maps:

$$\dots \longrightarrow M^i \xrightarrow{\partial^i} M^{i+1} \xrightarrow{\partial^{i+1}} M^{i+2} \longrightarrow \dots$$

such that $\partial \circ \partial = 0$.

Cochain complexes form a category, with morphisms $\{f^\bullet\}$ form a ladder:

$$\begin{tikzcd}
\dots \arrow[r] & M^i \arrow[r, "\partial^i"] \arrow[d, "f^i"] & M^{i+1} \arrow[r, "\partial^{i+1}"] \arrow[d, "f^{i+1}"] & M^{i+2} \arrow[r] \arrow[d, "f^{i+2}"] & \dots \\
\dots \arrow[r] & N^i \arrow[r, "\partial^i"] & N^{i+1} \arrow[r, "\partial^{i+1}"] & N^{i+2} & \dots
\end{tikzcd}$$

The n-th cohomology of a cochain complex $\{M^\bullet; \partial_M^\bullet\}$ is defined as:

$$H^n(M^\bullet; \partial_M^\bullet) := \frac{\ker \partial^i : M^i \to M^{i+1}}{\operatorname{im} \partial^{i-1} : M^{i-1} \to M^i}$$

We remark that there is nothing unexpected here from what we learned about chain complexes. Namely, if we reindex $\{M^\bullet; \partial_M^\bullet\}$, this defines a chain complex with $M_i' = M^{-i}$. This implies that the snake lemme holds! (with $\partial^i : H^i(C) \to H^{i+1}(A)$).

Theorem 2.4. There is a functor D and a natural transformation β such that the following diagram of functors commute up to the natrual transformation β:

$$\begin{tikzcd}
ch_R^{op} \arrow[r, "D"] \arrow[d, "H_n^{op}"'] & coch_R \arrow[d, "H^n"] \\
Mod_R^{op} \arrow[r, "\overline{D}"'] \arrow[ur, "\beta"] & Mod_R
\end{tikzcd}$$

where $\overline{D}(M) = Hom_R(M, R)$, and

$$D(\{M_\bullet; \partial_\bullet^M\}) \text{ is defined as } \{DM^\bullet; \partial^\bullet\}$$

where

$$DM^n := Hom_R(M_n, R), \partial^n : DM^n \to DM^{n+1} \text{ is the map induced by } \partial_{n+1} : M_{n+1} \to M_n$$

We observe that $\partial^{n+1}\partial^n = 0$ since $\partial_{n+2}\partial_{n+1} = 0$.

Problem 2.9. **HW(Q16):** Show that D is a functor.

Next we define the natural transformation β. We have $\beta : H^n(DM) \to Hom_R(H_n(M_\bullet), R)$, such that

$$\beta : [\varphi] \mapsto \beta[\varphi]$$

let $[x] \in H_n(M_\bullet)$, where $\beta[\varphi]([x]) = \varphi(x)$ (where $\varphi \in Hom_R(M_n, \mathbb{R}), x \in M_n$).

Proof. We first need to show that β is well-defined. If $x = \partial_{n+1}(y)$, then consider

$$\beta[\varphi][x] = \varphi(x) = \varphi(\partial_{n+1}(y)) = \partial^n(\varphi)(y) = 0, x \in \ker \varphi$$

Conversely, if $\varphi = \delta^{n-1}(\psi)$, we have

$$\beta[\varphi][x] = \varphi(x) = \delta^{n-1}\psi(x) = \psi(\partial_n(x)) = 0$$

It remains to check that β commutes with morphisms in ch_R^{op} (which we will do next time). □

2.4 Lecture 6 9/16

Today we continue our discussion of homological algebra. Let M be an R-module.

Definition 2.11 (resolution). A resolution of M is a positively graded chain complex $\{P_\bullet, \partial_\bullet\}$ such that

1. $H_n(P_\bullet) = 0$ for all $n > 0$
2. $H_0(P_\bullet) = \frac{P_0}{\operatorname{im} \partial_1} \cong M$, where $\partial_1 : P_1 \to P_0$.

We say $\{P_\bullet, \partial_\bullet\}$ is a free resolution if P_i is a free R-module for each i.

For resolutions, we prove the following two things: first, free resolutions always exist; second, every R-module map can be extended to a map between their resolutions (with extra assumptions) and these extensions are unique up to homotopies.

Proposition 2.2. For any M, a free resolution for M exists.

Proof. We proceed this inductively. Defien P_0 to be $R\langle M\rangle$, where it is the free R-module defined on the set M. Note that

$$R\langle M\rangle \twoheadrightarrow M \text{ is surjective} : \langle m\rangle \mapsto m$$

Let K be the kernel of this map, we have an isomorphism:

$$\epsilon : P_0/K \cong M$$

Define P_1 as $R\langle K\rangle$, note that $P_1 \twoheadrightarrow K$, then we define

$$\partial_1 : P_1 \to P_0$$

to be the composite

$$P_1 \twoheadrightarrow K \subset P_0$$

Now we consider P_2: let $K_1 \subset P_1$ be the kernel of ∂_1, define $P_2 = R\langle K_1\rangle$, then define $\partial_2; P_2 \to P_1$ to be the composite"

$$P_2 \twoheadrightarrow K_1 \subset P_1$$

note that $\ker \partial_1 / \operatorname{im} \partial_2 = K_1/K_1 = 0$. Then we define $K_2 = \ker \partial_2$, define $P_3 = R\langle K_2\rangle, \dots$ □

Just like the above proposition, the next theorem uses induction.

Theorem 2.5 (extension theorem)**.** Let $\{P_\bullet^M, \partial_\bullet^M, \epsilon_M\}$ be a free resolution on M, and let $\{P_\bullet^N, \partial_\bullet^N, \epsilon^N\}$ be an arbitrary resolution of N. Then given a map of R-modules $f : M \to N$, we may extend it to a map of chain complexes:

$$f_\cdot : \{P_\bullet^M, \partial_\bullet^M\} \to \{P_\bullet^N, \partial_\bullet^N\}$$

such that the following diagram commutes:

$$\begin{tikzcd} H_0(P_\bullet^M) \arrow[r, "H_0(f_\bullet)"] \arrow[d, "\epsilon_M"] & H_0(P_\bullet^N) \arrow[d, "\epsilon_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

Moreover, given any two extension $f_\bullet^1, f_\bullet^2$ of f, we have a chain homotopy $h_\bullet$ between $f_\bullet^1, f_\bullet^2$.

Remark: if $f_\bullet$ makes the diagram commute, and $g_\bullet$ is homotopic to $f_\bullet$, then $g_\bullet$ also makes the diagram commutes: homotopy classes work the same on homologies (they are the same on the nose).

Proof. We will constructu $f_\bullet$ as follows. We construct f_i inductively on i. Consider the diagram:

$$\begin{tikzcd} \cdots \arrow[d] & \cdots \arrow[d] \\ P_1^M \arrow[d] & P_1^N \arrow[d] \\ P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Since P_0^M is free, and ϵ_N is surjective, we may lift f on generators of P_0^M by lifting the geneartors of P_0^M to elements in P_0^N. (Note: this lift may not be unique), but this lift extends uniquely to define f_0. We notice that the bottom square

$$\begin{tikzcd} P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

commutes on homologies $(H_0(P_0^M), H_0(P_0^N))$. Now we construct f_1:

$$\begin{tikzcd} \cdots \arrow[d] & \cdots \arrow[d] \\ P_1^M \arrow[r, dashed, "f_1"] \arrow[d] & P_1^N \arrow[d] \\ P_0^M \arrow[r, dashed, "f_0"] \arrow[d] & P_0^N \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

We will follow the purple path above. Recall that $\epsilon_M : H_0(P_0) = P_0/\operatorname{im}(\partial_1^M) \to M$ is an isomorphism. We consider the composite: $f_0 \circ \partial_1^M = g$, we have

$$\begin{aligned} \epsilon_N \circ g &= \epsilon_N \circ f \circ \partial_1^M \\ &= f \circ \epsilon_M \circ \partial_1^M \\ &= 0 \end{aligned}$$

This implies that $\mathrm{im}(g) \subset \ker(\partial_N) = \mathrm{im}(\partial_1^N)$. We can lift the generators of P_1^M to elements of P_1^N. (Once chosen a lift, one can extend this uniquely to define f_1). Then we construct $f_2, f_3, \ldots$ the same way by considering $f_n \circ \partial_{n+1}$ and show that it is in the kernel of ∂_n^N and lift it to define ∂_{n+1}.

Now we homotopy time. Assume $f_\bullet^1, f_\bullet^2$ are two lifts of f, we construct $h : P_\bullet^M \to P_{\bullet+1}^N$. We define $h_\bullet$ inductively, starting with h_0 below:

$$\begin{tikzcd}
P_1^M \arrow[r, dashed, "f_1"] \arrow[d, "\partial_1^M"'] & P_1^N \arrow[d, "\partial_1^N"] \\
P_0^M \arrow[r, dashed, "{f_0^1, f_0^2}"'] \arrow[ur, dashed, "h_0"] \arrow[d, "\varepsilon_M"'] & P_0^N \arrow[d, "\varepsilon_N"] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

We have $\epsilon_N(f_0^1 - f_0^2) = 0$, then

$$f_0^1 - f_0^2 \in \ker \epsilon_N = \mathrm{im}\, \delta_1^N$$

we may lift $f_0^1 - f_0^2$ on generators of P_0^M, where $h_0 : P_0^M \to P_1^N$. Hence

$$(h_{-1} \circ \delta_{-1}^N) + \delta_1^N \circ h_0 = f_0^1 - f_0^2$$

Inductively, we assume h_m exists for $m \leq n$, then

$$\begin{tikzcd}
P_{n+2}^M \arrow[r, dashed, "f_1"] \arrow[d, "\partial_{n+2}^M"'] & P_{n+2}^N \arrow[d, "\partial_{n+2}^N"] \\
P_{n+1}^M \arrow[r, dashed, "{f_{n+1}^1, f_{n+1}^2}"] \arrow[ur, dashed, "h_{n-1}"] \arrow[d, "\partial_{n+1}^M"'] & P_{n+1}^N \arrow[d, "\partial_{n+1}^N"] \\
P_n^M \arrow[r, "f"] \arrow[ur, dashed, "h_n"] & P_n^N
\end{tikzcd}$$

consier the expressions

$$g_{n+1} := f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

we can check (by diagram chasing), $\partial_{n+1}^N \circ g = 0$. This implies that

$$\mathrm{im}(g_{n+1}) \subset \mathrm{im}(\partial_{n+2}^N)$$

we can construct h_{n+1} to get the map

$$\delta_{n+2}^N \circ h_{n+1} = g_{n+1} = f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

i.e.

$$\partial_{n+2}^N \circ h_{n+1} + h_n \circ \partial_{n+1}^M = f_{n+1}^1 - f_{n+1}^2$$

hence we are done! □

Corollary 2.1. Any two free resolutions of an R-module M are homotopy equivalent: given two free resolutions $P_\bullet^M, Q_\bullet^M$, there exists extension of $\mathrm{id} : M \to M$ and such that

$$f_\bullet : P_\bullet^M \to Q_\bullet^M, g_\bullet : Q_\bullet^M \to P_\bullet^M$$

such that

$$g_\bullet \circ f_\bullet = \mathrm{id}, f_\bullet \circ g_\bullet = \mathrm{id}$$

Problem 2.10 (HW(2.1)). Prove this corollary.

Next we deinfe Tor functors (pretty hard things).

Definition 2.12 (tor functors). Let N be an R-module, recall the functor

$$- \otimes_R N : Mod_R \to Mod_R$$

we define a collection of functors

$$\text{Tor}^i_R(-, N) : Mod_R \to Mod_R, i \in \mathbb{N}$$

given an object M in Mod_R, let $\{P^M_\bullet, \partial^M_\bullet, \epsilon_M\}$ be a free resolution of M, define $\text{Tor}^i(M, N)$ to be

$$\text{Tor}^i(M, N) = H_i(P^M_\bullet \otimes_R N, \partial^M_\bullet \otimes \text{id}_N)$$

where

$$\cdots \to P^M_i \otimes N \xrightarrow{\partial_i \otimes \text{id}} P^M_{i-1} \otimes_R N \to \ldots$$

We make the remark that there is a choice involved in picking the free resolution, but this is unique since homotopies are the same on homologies.

Problem 2.11 (HW(2.2)). For all i, show that $\text{Tor}^i_R(M, N)$ is a well-defined functor, and any other choice of free resolution of all objects yields an isomorphic functor. Hint: use the above corollary.

Problem 2.12 (HW(2.3)). Show that

1. $\text{Tor}^i_R(R, N) = 0$ for all $i > 0$
2. $\text{Tor}^i_R(M \oplus M', N) \cong \text{Tor}^i_R(M) \oplus \text{Tor}^i_R(N)$, given by the natural isomorphism.

We claim that $\epsilon_M : P^M_0 \to M$ induces the following isomorphism

$$\text{Tor}^0_R(M, N) \cong M \otimes_R N$$

and $\text{Tor}^i_R(M, N)$'s are called the highest derived functors of $- \otimes_R N$.

2.5 Lecture 7 9/18

We continue with our discussion of tor functors. We claim that

Proposition 2.3. The natural isomorphism gives the following

$$\text{Tor}^0_R(-, N) \cong - \otimes_R N$$

i.e., for any M,

$$\text{Tor}^0_R(M, N) \cong M \otimes_R N$$

Proof. By definition, $\text{Tor}^0_R(M, N)$ is the 0-th hoology of

$$\cdots \to P^M_1 \otimes_R N \xrightarrow{\partial_1 \otimes \text{id}_N} P^M_0 \otimes_R N \to 0 \to 0 \to \ldots$$

this implies that

$$\text{Tor}^0_R(M, N) = \frac{P^M_0 \otimes_R N}{\text{im}(\partial_1 \otimes \text{id}_R)}$$

We complete the proof using the following lemma.

Lemma 2.1. We claim that the functor $-\otimes_R N$ is right exact, meaning that give a sequence of R-modules,

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

that is exact at A_0 and M, the following sequence:

$$A_1 \otimes_R N \xrightarrow{f\otimes \mathrm{id}} A_0 \otimes_R N \xrightarrow{g\otimes id} M \otimes_R N \to 0$$

is also exact at $A_0 \otimes_R N$ and $M \otimes_R N$.

If we assume the lemma for now, then applying it to

$$P_1 \xrightarrow{\partial_1^M} P_0 \xrightarrow{\epsilon} M \to 0$$

then we are done!

We prove the lemma now: exactness of $M \otimes_R N$ implies that $g \otimes \mathrm{id}$ is sujrective. givn that $g; A_0 \to M$ is sujective, every generator of $M \otimes n$ in $M \otimes_R N$ s of the form $g \otimes \mathrm{id}(a \otimes n)$ for some $a \in A_0$. This implies that $g \otimes \mathrm{id}$ is surjecctive.

Next, we need to show that $\ker(g \otimes \mathrm{id}) = \mathrm{im}(f \otimes \mathrm{id})$. It is clear that $\supset$ holds, hence it suffices to show $\subset$. Let $K = \ker g \otimes \mathrm{id}$, we need to show that

$$\frac{A_0 \otimes_R N}{K} \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

is surjective. It is enough to construct a map:

$$M \otimes_R N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

by the first isomorphism thoerem in algebra and the fact that $g \otimes \mathrm{id}$ is surjective. To get such a map, we need to construct a bilinear map

$$M \times N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})}$$

defined as

$$(m, n) \mapsto (a, n)$$

and let $a = g^{-1}(m)$ be a choice of the preimage. We remark that there could be many choices of a, but the difference $a_1 - a_2$ comes from f, since A_0 is exact. This implies that this map is well-defined. This implies that the above map is surjective. There for

$$M \times N \to \frac{A_0 \otimes_R N}{\mathrm{im}(f \circ \mathrm{id})} \xrightarrow{g\otimes \mathrm{id}} M \otimes_R N$$

this composition is surjective. (Two surjective maps and maps to identity=isomorphism). □

Warning 2.6. We saw that tensor product preserves surjectivity, but it does not necessarily preserve injectivity. Namely, if we replace the statement of the claim with SES

$$0 \to A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

and consider

$$0 \to A_1 \otimes N \to \dots$$

f need not to be injective.

Next we see the sufficient condition for Tor_R^i to vanish for all $i \geq 2$.

Corollary 2.2. If R is a PID, then $\text{Tor}^i_R(M,N) = 0$ for all $i \geq 2$.

Proof. Consider the free resolution of M:

$$0 \to K \to R\langle M\rangle \to 0 \to \dots$$

such that $R\langle M\rangle/K \cong M$. Recall that all submodules of a free module are free, so we can just take $K = P_1$, then we have

$$0 \to K \otimes_R N \to R\langle M\rangle \otimes_R N \to 0 \to 0 \to \dots$$

so the only homologies are $\text{Tor}^0_R, \text{Tor}^1_R$. □

Problem 2.13 (HW(2.4)). Calculate $\text{Tor}^1_{\mathbb{Z}}$ and $\text{Tor}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ for $m, n > 0$. (Note: m, n could be equal or not).

Definition 2.13 (ext functor). Fix an R-module N, consider the functor

$$\text{Hom}_R(-;N) : Mod_R^{op} \to Mod_R$$

Define the functors $\text{Ext}^i_R(-,N) : Mod_R^{op} \to Mod_R$ as follows:

$$\text{Ext}^i_R(M,N) = H^i(\text{Hom}_R(P^M_\bullet, N))$$

where $P^M_\bullet$ is a free resoltuion of M.

We note that if R is a PID, then $\text{Ext}^i_R(M,N) = 0$ for all $i \geq 2$.

Proposition 2.4. We have

$$\text{Ext}^0_R(M,N) \cong \text{Hom}(M,N)$$

Proof. This requires the following lemma:

Lemma 2.2. If

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \to 0$$

is right exact, then

$$0 \to \text{Hom}_R(M,N) \to \text{Hom}_R(A_0,N) \to 0$$

is exact at $\text{Hom}_R(M,N)$ and $\text{Hom}_R(A_0,N)$.

□

Problem 2.14 (HW(2.5)). Prove the above lemma.

Problem 2.15 (HW(2.6)). Prove the following statements about the Ext functor.

1. $$\mathrm{Ext}^i_R\left(\bigoplus_\alpha M_\alpha, N\right) \cong \prod_\alpha \mathrm{Ext}^i_R(M_\alpha, N)$$

2. $$\mathrm{Ext}^i_R(M, \prod_\alpha N_\alpha) \cong \prod_\alpha \mathrm{Ext}^i_R(M, N_\alpha)$$

3. Calculate
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

Next we state and prove Algebraic Kunneth theorem.

Theorem 2.7 (AKT). Let R be a PID, and let $\{M_\bullet, \partial^M_\bullet\}, \{N_\bullet, \partial^N_\bullet\}$ be PGCC of R-modules such that M_i is free for all i. Then there exists a SES:

$$0 \to \bigoplus_{i+j=n} H_i(M) \otimes_R H_j(N) \xrightarrow{X} H_n((M \otimes_R N)_\bullet) \to \bigoplus_{i+j=n-1} \mathrm{Tor}^1_R(H_i(M_\bullet), H_j(N_\bullet)) \to 0$$

where X denotes the algebraic crossproduct.

Proof. too long, will type up later □

2.6 Lecture 8 9/23

Corollary 2.3. Let R be a field, then the algebraic crossproduct induces an isomorphism:

$$0 \to \bigoplus_{i+j=n} H_i(M) \otimes_{\mathbb{F}} H_j(N) \cong H_n(M \otimes_{\mathbb{F}} N)$$

where the isomorphism is given by the algebraic crossproduct X.

Corollary 2.4 (Universal Coefficient Theorem). Let $\{M_\bullet, \partial^M_\bullet\}$ be a chain complex of free $\mathbb{Z}$-modules, and let R be any commutative ring, then there is a SES:

$$0 \to H_n(M_\bullet) \otimes_{\mathbb{Z}} R \xrightarrow{f} H_n(M \otimes_{\mathbb{Z}} R) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{n-1}(M), R) \to 0$$

where f is injective but not necessarily surjective (the failure to be surjective is measured by Tor^1_R).

Proof. Use AKT with

$$N_i = \begin{cases} R, i = 0 \\ 0, i > 0 \end{cases} \quad , \quad H_i(N) = \begin{cases} R, i = 0 \\ 0, i \neq 0 \end{cases}$$

Hence $(M \otimes_{\mathbb{Z}} N)_\bullet = M_\bullet \otimes_{\mathbb{Z}} R$. □

Problem 2.16 (HW(2.7)). Prove the UCT in cohomology: let $\{M_\bullet, \partial_\bullet^M\}$ be a chain complex of free $\mathbb{Z}$-modules, let R be any commutative ring, then there exists SES

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(M), R) \to H^n(\mathrm{Hom}_{\mathbb{Z}}(M_\bullet, R)) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(M), R) \to 0$$

Hint: use the same proof for AKT, instead of $\otimes$ with N, you take the Hom into R.

Chapter 3

Singular Cohomology

We begin with some basic definitions.

Definition 3.1 (n-simplex)**.** The standard n-simplex $\Delta_n \subset \mathbb{R}^{n+1}$ is defined as

$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1} : x = \sum_{i=0}^{n} t_i e_i, t_i \geq 0, \sum_{t_i} = 1 \right\}$$

where $e_i, 0 \leq i \leq n$ are the standard basis vectors of $\mathbb{R}^{n+1}$.

Definition 3.2 (face)**.** Let $0 \leq i \leq n$, then the ith face F_i of Δ_n is the $(n-1)$-simplex

$$F_i = \{x \in \Delta_n : t_i = 0\}$$

Definition 3.3 (singular chain complex)**.** Given a topological space X, the singular chain complex of X, with $\mathbb{Z}$ coefficients, denoted as $S_\bullet(X, Z)$ is defined as

$$S_i(X, Z) = \begin{cases} 0, i < 0 \\ \mathbb{Z}\langle \Delta_i(X) \rangle, i \geq 0 \end{cases}$$

where $\Delta_i(X)$ is the set of continuous maps from $\Delta_i \to X$. We define $\partial_n : S_n(X, \mathbb{Z}) \to S_{n-1}(X, \mathbb{Z})$ as follows:

$$\partial_n \langle f \rangle = \sum_{i=0}^{n} (-1)^i \langle f \circ F_i \rangle$$

where $\langle f \rangle$ is a generator of $\Delta_i(X)$, and $f : \Delta_X \to X$, where

$$f \circ F_i = \Delta_{n-1} \to \Delta_n \xrightarrow{f} X$$

Note to complete this definition, one needs to check that $\partial^2 = 0$, which we did in class. might include this later

3.1 Lecture 9 9/25

Recall that last time, we defined the singular chain complexes $S_\bullet(X, \mathbb{Z})$ with $\mathbb{Z}$-coefficients:

$$S_i(X, \mathbb{Z}) = \begin{cases} 0, i < 0 \\ \mathbb{Z}\langle \Delta_i(X) \rangle, i \geq 0 \end{cases}$$

where $\Delta_i(X)$ is the set of continuous maps from Δ_i to X. Now we discuss some variations of this concept.

Definition 3.4 (relative singular chain complex with $\mathbb{Z}$-coefficients)**.** Let $A \subset X$ be a subspace, define $S_\bullet(X, A\mathbb{Z})$ by

$$S_i(X, A, \mathbb{Z}) = \begin{cases} 0, i < 0 \\ \frac{\mathbb{Z}\langle \Delta_i(X) \rangle}{\mathbb{Z}\langle \Delta_i(A) \rangle}, i \geq 0 \end{cases}$$

note that the quotient is still free.

We note that $S_\bullet(X, A, \mathbb{Z})$ is a chain complex with the following ∂ maps such that the following diagram commutes:

$$\begin{tikzcd} S_i(A, \mathbb{Z}) \arrow[r, "\partial_i"] \arrow[d] & S_{i-1}(A, \mathbb{Z}) \arrow[d] \\ S_i(X, \mathbb{Z}) \arrow[r, "\partial_i"] \arrow[d, two heads] & S_{i-1}(X, \mathbb{Z}) \arrow[d, two heads] \\ S_i(X, A, \mathbb{Z}) \arrow[r, "\partial_i"] & S_{i-1}(X, A, \mathbb{Z}) \end{tikzcd}$$

Definition 3.5 ($S_\bullet(X, A, R)$)**.** We define $S_\bullet(X, A, R)$, where R is any commutative ring, and

$$S_\bullet(X, A, R) = S_\bullet(X, A, \mathbb{Z}) \otimes_{\mathbb{Z}} R$$

it is a chain complex of R-modules with ∂_i induced from $S_\bullet(X, A, \mathbb{Z})$.

The last variation is as follows:

Definition 3.6 (singular cochain complex)**.** Define the singular cochain complex of R-modules $S^\bullet(X, A, R)$ as follows:

$$S^i(X, A, R) := \mathrm{Hom}_{\mathbb{Z}}(S_i(X, A, \mathbb{Z}), R) = \mathrm{Hom}_R(S_i(X, A, R), R)$$

where ∂^i is induced from ∂_i in $S_\bullet(X, A, R)$.

We make the following remark: if $A = \varnothing$, then $S_\bullet(X, A, \mathbb{Z}) = S_\bullet(X, \mathbb{Z})$. Previously, we did UCT for chain complexes of free $\mathbb{Z}$-modules, here we state the universal coefficient theorem for singular chain complexes:

Theorem 3.1 (Universal Coefficient Theorems)**.** We have some SES's:

1. There exists a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X, A, \mathbb{Z}) \otimes H_j(Y, B, \mathbb{Z}) \to H_n(S_\bullet(X, A, \mathbb{Z}) \otimes S_\bullet(Y, B, \mathbb{Z})) \to$$
$$\bigoplus_{i+j=n-1} \mathrm{Tor}^1_{\mathbb{Z}}(H_i(X, A, \mathbb{Z}), H_j(Y, B, \mathbb{Z})) \to 0$$

   where $H_i(X, A, Z) = H_i(S_\bullet(X, A, \mathbb{Z}))$

2. There exists a SES:

$$0 \to H_i(X, A, \mathbb{Z}) \otimes_{\mathbb{Z}} R \xrightarrow{f} H_i(X, A, R) \to \mathrm{Tor}^1_{\mathbb{Z}}(H_{i-1}(X, A, \mathbb{Z}), R) \to 0$$

   again f is injective, and the failure to be surjective is measured by $\mathrm{Tor}^1_{\mathbb{Z}}$.

3. There exists a SES:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X, A, \mathbb{Z}), R) \to H^n(X, A, R) \xrightarrow{\beta} \mathrm{Hom}_{\mathbb{Z}}(H_n(X, A, \mathbb{Z}), R) \to 0$$

note all the above assumes R is a PID.

We next introduce the category PTop.

Definition 3.7 (PTop)**.** The category PTop has objects pairs (X, A) where $A \subset X$ is a subspace of a toplogical space X. where

$$mor_{\text{PTop}}((X, A), (Y, B)) = \text{ set of continuous maps from } X \to Y \text{ that sends } A \text{ to } B$$

i.e., the image of f in A is contained in B.

Theorem 3.2. $S_\bullet(X, A, R)$ is a functor from PTop $\to ch^+_R$, and $S^\bullet(X, A, R)$ is the contravariant functor from PTop to $coch^+_R$.

Proof. To show that it is a functor, we know it's defined on objects, we now define it on morphisms. Given $f : (X, A) \to (Y, B)$, we define

$$f_* : S_i(X, A, R) \to S_i(Y, B, R)$$

as follows:

$$(f_*)_i\,[\langle g : \Delta_i \to X\rangle] := [\langle f \circ g : \Delta_i \to Y\rangle]$$

∎

We need to check that it commutes in a ladder as follows:

$$\begin{array}{c} F_j \\ \big\downarrow \\ \Delta_i \xrightarrow{g} X \xrightarrow{f} Y \end{array}$$

we have

$$\begin{aligned}\partial_i \circ (f_*)_i[\langle g : \Delta_i \to X\rangle] &= \partial_i[\langle f \circ g : \Delta_i \to Y]\\ &= \sum_{j=0}^{i}(-1)^j (f_*)_{i-1}[\langle g : F_j \to X\rangle]\\ &= (f_*)_{i-1}\sum_{j=0}^{i}(-1)^j[\langle g : F_i \to X\rangle]\\ &= (f_*)_{i-1} \circ \partial_i\end{aligned}$$

Moreover, it is easy to see that

$$(f \circ g)_* = f_* \circ g_*$$

Definition 3.8 (singular homology). The nth singular homology with coefficients in R is the composite functor:

$$\text{PTop} \xrightarrow{S_\bullet(X,A,R)} ch_R^+ \xrightarrow{H_n} Mod_R$$

and similarly for cohomologies.

Example 3.1. We consider the following simple example $X = \text{pt}$, and $A = \varnothing$, $S_\bullet(\text{pt}, R)$ since the set $\Delta_i(\text{pt})$ is a singleton $i \geq 0$. So $S_\bullet(\text{pt}, R)$ looks like

$$\cdots \to R \to R \xrightarrow{\partial_2} R \xrightarrow{\partial_1} R \to 0 \to \dots$$

where

$$H_i(\text{pt}, R) = \begin{cases} 0, i \neq 0 \\ R, i = 0 \end{cases}$$

Definition 3.9 (path-connected). A space X is path-connected if given any $a, b \in X$, there exists a continuous path $\gamma : [0, 1] \to X$ such that

$$\gamma(0) = \alpha, \quad \gamma(1) = b$$

Proposition 3.1. If X is path-connected, then

$$H_0(X, R) \cong R$$

(implying that H_0 a homology group, could be tiny!)

Proof. Recall that by definition, we have

$$H_0(X, R) = \frac{R\langle\Delta_0(X)\rangle}{\text{im}(\partial_1)}$$

where $\partial_1 : R\langle\Delta_1(X)\rangle \to R\langle\Delta_0(X)\rangle$. We consider the homomorphism:

$$\varepsilon : R\langle\Delta_0(X)\rangle \to R$$

such that $\varepsilon\langle x\rangle = 1$, for generator $x \in X = \Delta_0(X)$. Notice that

$$\partial_1\langle\gamma\rangle = \langle\gamma(1)\rangle - \langle\gamma(0)\rangle$$

Hence

$$\varepsilon\partial_1\langle\gamma\rangle = \varepsilon\langle\gamma(1)\rangle - \varepsilon\langle\gamma(0)\rangle = 1 - 1 = 0$$

Hence ε extends to a surjective map.

Problem 3.1 (HW(2.8)). Show that ε is also injective.

□

Next we stated Eilenberg-Steenrod Axioms. will fill in later

3.2 Lecture 10 9/30

we restated ES axioms at the beginning of class.

Definition 3.10 (contractible). A space X is said to be contractible if the identity map $i : X \to X$ is homotopic to the map that sends all X to some $x_0 \in X$, where x_0 is any point. This means, there exists $h : X \times [0, 1] \to X$ such that

$$h(x, 0) = x, h(x, 1) = x_0$$

Problem 3.2. Use ES axioms to show that if X is contractible, then

$$H_n(X) = \begin{cases} 0, n \neq 0 \\ R, n = 0 \end{cases}$$

then we proved this fact without using ES axiom, will fill in later

Corollary 3.1. Let B be an open ball in $\mathbb{R}^n$, then

$$H_n(B) = \begin{cases} R, n = 0 \\ 0, n > 0 \end{cases}$$

sinc the ball is contractible.

Then we consider some spheres and hemispheres.

Definition 3.11 (n-sphere, hemisphere). Let $S^n \subset \mathbb{R}^{n+1}$ be the n-sphere

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\}$$

define the hemispheres:

$$D^n_+ = \{(x_0, \dots, x_n) \in S^n : x_0 \geq 0\}, \quad D^n_- = \{(x_0, \dots, x_n) \in S^n : x_0 \leq 0\}$$

notice that

$$D^n_+ \cap D^n_- = S^{n-1}$$

We make the following observations.

Proposition 3.2. We have the following isomorphism.

1. $$H_i(S^n, D^n_-) \cong H_i(S^n, \{s\}) := \tilde{H}_i(S^n)$$

   where we can choose $\{s\} \in D^n_-$ to be the south pole.

2. $$H_i(S^n, D^n_-) \cong H_i(D^n_+, S^{n-1})$$

Problem 3.3 (HW(2.9)). Give a proof of 1 in the above proposition using the 5-lemma.

we proved these using ES axioms A2A3, insert later

Theorem 3.3. There are the following isomorphisms:

1. $$H_i(D^{n+1}_+, S^n) = \begin{cases} R, i = n+1 \\ 0, \text{else} \end{cases}$$

   for $n \geq 0$.

2. $$H_i(S^n) = \begin{cases} R, i = 0, n \\ 0, \text{else} \end{cases}$$

   where $n > 0$.

Proof. We will conduct a simultaneous induction. We will do the base case now: for $n = 0$, we prove 1. Consider the LES:

$$\cdots \to H_{i+1}(D^1, S^0) \to H_i(S^0) \to H_i(D^1) \to H_i(D^1, S^0) \to \ldots$$

For $i > 0$, we have

$$H_i(S^0) = H_i(D^1) = 0$$

this implies that for all $i > 1$,

$$H_i(D^1, S^0) = 0$$

For $i = 0, 1$, consider

$$0 \to H_1(D^1, S^0) \to H_0(S^0) \to H_0(S^1) \to H_0(D^1, S^0) \to 0$$

this implies that

$$H_0(D^1, S^0) = 0, \quad H_1(D^1, S^0) = R$$

Now we begin the induction step, assume 1,2 holds untill n, we consider $n+1$. Consider the LES:

$$\cdots \to H_i(D^{n+1}_-) \to H_i(S^{n+1}) \to H_i(S^{n+1}, D^{n+1}_-) \to H_{i-1}(D^{n+1}_-) \to \ldots$$

where

$$H_i(S^{n+1}, D^{n+1}_-) \cong H_i(D^{n+1}_+, S^n)$$

since $H_i(D^{n+1}_-) = 0$ for all $i > 0$, we have

$$H_i(S^{n+1}) \cong H_i(D^{n+1}_+, S^n) \cong \begin{cases} R, i = n+1 \\ 0, 1 \leq i < n+1 \end{cases}$$

We only need to understand $H_1(S^{n+1})$ to fully prove (b). Notice that

$$H_1(D^{n+1}) \to H_1(S^{n+1}) \to H_1(S^{n+1}, D^{n+1}) \to H_0(D^{n+1}) \cong H_0(S^{n+1})$$

and it is cclear that $H_1(S^{n+1}) \cong H_1(D^{n+1}, S^n) = 0$.

Problem 3.4 (HW(2.10)). Prove (a) for $n+1$ using (b) and the following LES:

$$\cdots \to H_{i+1}(D_+^{n+2}, S^{n+1}) \to H_i(S^{n+1}) \to H_i(D_+^{n+2}) \to \ldots$$

□

3.3 Lecture 11 10/02

We begin by showing S^n is not a retraction of D^{n+1} for $n > 0$.

Corollary 3.2. There is no map $\gamma : D^{n+1} \to S^n$ such that

$$\gamma \circ i = \mathrm{id}$$

where $i : S^n \to D_{n+1}$ is the inclusion map.

fill in theorem later

Theorem 3.4 (Brouwer Fixed point theorem). Let f be any continuous map $f : D^{n+1} \to D^{n+1}$, for $n > 0$. Then there exists $x \in D^{n+1}$ such that

$$f(x) = x$$

Proof. Assume f existed without a fixed point, then we use this f to construct a retraction $g : D^{n+1} \to S^n$. We define g to be the point on S^n the line segment $(x, f(x))$ intersects it at. incomplete □

Problem 3.5. Show the following:

1. Show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic, if $m \neq n$.
2. Show that $\mathbb{S}^n$ is not a retraction of S^m if $n < m$.

Definition 3.12 (augmented CC)**.** A augmented chain complex of R-modules is a postively graded chain complex of R-modules endowed with an R-module map:

$$\varepsilon : C_0 \to R$$

such that

$$\varepsilon \circ \partial_1 = 0$$

i.e., ε factors through $H_0(C_\bullet)$.

$$\begin{tikzcd} C_1 \arrow[r, "\partial_1"] & C_0 \arrow[r, "\varepsilon"] \arrow[d] & R \\ & H_0(C_\bullet) \arrow[ur, dashed] & \end{tikzcd}$$

An augmented chain complex is called

1. acyclic. If $H_i(C_\bullet) = 0$ for all $i > 0$, and
$$H_0(C_\bullet) \cong R$$
via the map ε.
2. free. If C_i is a free-module for all i.

A morphism of two augmented chain complex $\{C_\bullet, \partial_\bullet^C, \varepsilon^C\}, \{D_\bullet, \partial_\bullet^D, \varepsilon^D\}$ is a morphism $\{f_\bullet\}$ of chain complexes between $\{C_\bullet, \partial_\bullet^C\}$ and $\{D_\bullet, \partial_\bullet^D\}$ such that the following diagram commutes:

$$\begin{tikzcd} C_0 \arrow[r, "f_0"] \arrow[dr, "\varepsilon^C"'] & D_0 \arrow[d, "\varepsilon^D"] \\ & R \end{tikzcd}$$

Definition 3.13 (Ach_R)**.** Let Ach_R be the category of augmented chain complexes of R-modules.

Problem 3.6 (HW(2.12))**.** Show that if $\{C_\bullet, \partial_\bullet^C, \varepsilon^C\}$ and $\{D_\bullet, \partial_\bullet^D, \varepsilon^D\}$ are objects in Ach_R, then

$$\{(C \otimes D)_\bullet, \partial_\bullet^{C\otimes D}, \varepsilon^C \otimes \varepsilon^C\}$$

is also an object in Ach_R.

Proof. It suffices to check that

$$\varepsilon^C \otimes \varepsilon^D : C_0 \otimes D_0 \to R$$

such that

$$\varepsilon^C \otimes \varepsilon^D \circ \partial_1^{C\otimes D} = 0$$

□