Aluffi Problems

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Problem 6.1 (6.10). Let F_1, F_2 be free R-modules of finite rank, and let α_1, resp., α_2, be linear transformations of F_1, resp., F_2. Let $F = F_1 \oplus F_2$, and let $\alpha = \alpha_1 \oplus \alpha_2$ be the linear transformation of F restricting to α_1 on F_1 and α_2 on F_2.

- Prove that $P_\alpha(t) = P_{\alpha_1}(t)P_{\alpha_2}(t)$. That is, the characteristic polynomial is multiplicative under direct sums.
- Find an example showing that the minimal polynomial is not multiplicative under direct sums.

Problem 6.2 (6.13). Let A be a square matrix with integer entries. Prove that if λ is a rational eigenvalue, then $\lambda \in \mathbb{Z}$.

Proof. Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be the characteristic polynomial of A, then $p(\lambda) = 0$, letting $\lambda = \frac{p}{q}$, then

$$p \mid a_0, \quad q \mid a_n$$

we know that p is monic, thus $a_n = 1$, hence $\lambda \in \mathbb{Z}$. □

Problem 6.3 (7.3). Prove that two linear transformations of a vector space of dimension ≤ 3 are similar if and only if they have the same characteristic and minimal polynomials. Is this true in dimension 4? [§6.2]

Problem 6.4 (7.4). Let k be a field, and let K be a field containing k. Two square matrices $A, B \in M_n(k)$ may be viewed as matrices with entries in the larger field K. Prove that A and B are similar over k if and only if they are similar over K.

Problem 6.5 (7.7). Let V be a k-vector space of dimension n, and let $\alpha \in \mathrm{End}_k(V)$. Prove that the minimal and characteristic polynomials of α coincide if and only if there is a vector $v \in V$ such that

$$v, \alpha(v), \ldots, \alpha^{n-1}(v)$$

is a basis of V.

Problem 6.6 (7.8). Let V be a k-vector space of dimension n, and let $\alpha \in \mathrm{End}_k(V)$. Prove that the characteristic polynomial $P_\alpha(t)$ divides a power of the minimal polynomial $m_\alpha(t)$.

Proof. Assume that k is algebraically closed, and polynomials factors, the minimal polynomial m_α contains all the $(t - \lambda_i)$ for distinct λ_i's by Lemma 7.12. Thus P_α divides $(m_\alpha)^n$. □

Problem 6.7 (7.12). Let V be a finite-dimensional k-vector space, and let $\alpha \in \mathrm{End}_k(V)$ be a diagonalizable linear transformation. Assume that $W \subseteq V$ is an invariant subspace, so that α induces a linear transformation $\alpha|_W \in \mathrm{End}_k(W)$. Prove that $\alpha|_W$ is also diagonalizable. (Use Proposition 7.18.)

Proof. Assume that characteristic polynomial factors completely over k, then α is diagonalizable iff minimal polynomial m_α has no repeated roots, thus $\alpha|_W$ also has no repeated roots as it divides m_α. □

Problem 6.8 (7.13). Let R be an integral domain. Assume that $A \in \mathcal{M}_n(R)$ is diagonalizable, with distinct eigenvalues. Let $B \in \mathcal{M}_n(R)$ be such that $AB = BA$. Prove that B is also diagonalizable, and in fact it is diagonal w.r.t. a basis of eigenvectors of A. (If P is such that PAP^{-1} is diagonal, note that PAP^{-1} and PBP^{-1} also commute.)

Proof. It suffices to see that if $v_1 \neq 0$ is such that $Av_1 = \lambda_1 v_1$, then

$$\begin{aligned} A(Bv_1) &= B(Av_1) \\ &= B\lambda_1 v_1 \\ &= \lambda_1(Bv_1) \end{aligned}$$

Thus Bv_1 is contained in the one-dimensional subspace generated by v_1. □

Problem 6.9 (7.14). Prove that "commuting transformations may be simultaneously diagonalized", in the following sense. Let V be a finite-dimensional vector space, and let $\alpha, \beta \in \mathrm{End}_k(V)$ be diagonalizable transformations. Assume that $\alpha\beta = \beta\alpha$. Prove that V has a basis consisting of eigenvectors of both α and β. (Argue as in Exercise 7.13 to reduce to the case in which V is an eigenspace for α; then use Exercise 7.12.)

Proof. Separate into eigenspaces: consider eigenspace E_1 of α, then diagonalize β in E_1 (by 7.12), note that E_1 is invariant under β. □

Problem 6.10 (7.15). A **complete flag** of subspaces of a vector space V of dimension n is a sequence of nested subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V$$

with $\dim V_i = i$. In other words, a complete flag is a composition series in the sense of Exercise 1.16.

Problem 6.11 (7.17). A matrix $M \in M_n(\mathbb{C})$ is **normal** if $MM^\dagger = M^\dagger M$. Note that unitary matrices ($UU^* = U^*U = I$) and Hermitian matrices ($U = U^*$) are both normal. Prove that a triangular normal matrix is diagonal. [7.18]

Chapter 7

Fields

Chapter 8

Linear Algebra II