Real Analysis 605 MT Review

Hui Sun

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Contents

Chapter 1

Definitions

Definition 1.1 (sequence of sets)**.** Let $\{E_k\} \subset \mathbb{R}^n$ be a sequence of sets is said to increase to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ for all k, and decrease to $\bigcap_k E_k$ if $E_k \supset E_{k+1}$ for all k.

Definition 1.2 (limsup,liminf of sets)**.** Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets, we define

$$\limsup E_k = \bigcap_{j=1}^{\infty} \left(\bigcup_{k=j}^{\infty} E_k \right), \quad \liminf E_k = \bigcup_{j=1}^{\infty} \left(\bigcap_{k=j}^{\infty} E_k \right)$$

Definition 1.3 (metric)**.** Let d be a metric on $\mathbb{R}^n$, let $x, y \in \mathbb{R}^n$, then

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 1.4 (limsup, liminf of sequences)**.** Let $\{a_k\}$ be a sequence of points in $\mathbb{R}$, then

$$\limsup_{k \to \infty} a_k := \lim_{j \to \infty} \{\sup_{k \geq j} a_k\}$$

and

$$\liminf_{k \to \infty} a_k := \lim_{j \to \infty} \{\inf_{k \geq j} a_k\}$$

Definition 1.5 (distance between sets)**.** Let $E_1, E_2 \subset \mathbb{R}^n$, then the distance between E_1 and E_2 is defined as

$$d(E_1, E_2) = \inf\{|x - y| : x \in E_1, y \in E_2\}$$

Definition 1.6 (open set)**.** Let $E \subset \mathbb{R}^n$, then E is called open if for each $x \in E$, there exists δ such that $B_\delta(x) \subset E$.

A subset E_1 of E is said to be relatively open with respect to E if it can be written as $E_1 = E \cap G$ for some open set G.

Definition 1.7 (A_δ, A_σ sets)**.** A set A is said to be of type A_δ if it can be written as a countable intersection of sets and to be of type A_σ if it can be written as a countable union of sets. Then G_δ implies a countable intersection of open sets, and F_σ implies the countable union of closed sets.

Definition 1.8 (perfect set)**.** C is called a perfect set if it is a closed set such that every point in C is a limit point.

Definition 1.9 (compact set)**.** A set E is compact if and only if every open cover of E has a finite subcover.

Definition 1.10 (monotone function)**.** A fuction f defined on $I \subset \mathbb{R}$ is monotone increasing if $f(x) \leq f(y)$ whenever $x < y$. Similarly defined for monotonically decreasing.

Definition 1.11 (continuous)**.** Let f be defined on a neighorhood of x_0, then f is said to be continuous at x_0 if $f(x_0)$ is finite and $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition 1.12 (continuous relative to a set)**.** Let f be defined in only a set E containing x_0, f is said to be continuous at x_0 relative to E if $f(x_0)$ is finite and either x_0 is an isolated point of E or x_0 is a limit point of E and for $x \in E$.

$$\lim_{x\to x_0} f(x) = f(x_0)$$

If $E_1 \subset E$, a function is continuous in E_1 relative to E if it is continuous relative to E at every point in E_1.

Definition 1.13 (uniform convergence)**.** A sequence $\{f_k\}$ defined on E is said to uniformly convergence on E to a finite f if given $\varepsilon > 0$, there exists K such that for all $k \geq K$, $x \in E$,

$$|f_k(x) - f(x)| < \epsilon$$

Definition 1.14 (Riemann integral)**.** Let f be bounded on an interval I, partition I into a finite collection Γ of nonoverlapping intervals, denote $|\Gamma| = \max_k diam(I_k)$, select points $\xi_k \in I_k$, let

$$R_\Gamma = \sum_{k=1}^{N} f(\xi_k)|I_k|$$

and

$$U_\Gamma = \sum_{k=1}^{N}(\sup_{x\in I_k} f(x))|I_k|, \quad L_\Gamma = \sum_{k=1}^{N}(\inf_{x\in I_k} f(x))|I_k|$$

The Riemann integral exists if $\lim_{|\Gamma|\to 0} R_\Gamma$ exists and the limit A is the Riemann integral. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\Gamma| < \delta$, we have $|A - R_\Gamma| < \varepsilon$ for any Γ and any chosen $\{\xi_k\}$.

This is equivalent to the statement:

$$\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A$$

We begin chapter 2.

Definition 1.15 (variation). Let f be defined on $[a,b]$, the variation of f over $[a,b]$ is

$$V(f) = \sup_{\Gamma} \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$$

where Γ is any partition $\{x_0, x_1, \ldots, x_m\}$ of $[a,b]$.

Definition 1.16 (Lipschitz). Let f be defined on $[a,b]$, then f is said to be Lipschitz if there exists an absolute constant C such that

$$|f(x) - f(y)| \leq C|x-y|$$

for all $x, y \in [a,b]$.

Definition 1.17 (splitting). For any $x \in \mathbb{R}$, we can write

$$x^+ = \begin{cases} x, x > 0 \\ 0, x \leq 0 \end{cases}$$

$$x^- = \begin{cases} 0, x > 0 \\ -x, x \leq 0 \end{cases}$$

then $|x| = x^+ + x^-, x = x^+ - x^-$.

Definition 1.18 (P_Γ, N_Γ). For any f and any partition Γ, define

$$P_\Gamma = \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})]^+$$

and

$$N_\Gamma = \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})]^-$$

similarly, we define

$$P = \sup_{\Gamma} P_\Gamma, N = \sup_{\Gamma} N_\Gamma$$

Definition 1.19 (rectifiable curve). Let C be a curve, i.e.

$$C : \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

Let Γ be any partition, define

$$L = \sup_{\Gamma} \sum_{i=1}^{m} \left((\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1}))^2 \right)^{1/2}$$

then C is rectifiable if $L < +\infty$.

Definition 1.20 (Riemann-Stieltjes integral). Let f, ϕ be finite on an interval $[a, b]$, let $\Gamma = \{a = x_0 = \cdots < x_m = b\}$ be any partition, define

$$R_\Gamma = \sum_{i=1}^{m} f(\xi_i)\,[\phi(x_i) - \phi(x_{i-1})]$$

If $\lim_{|\Gamma|\to 0} R_\Gamma$ exists, then we call this the Riemann-Stieltjes integral. That is, given any $\varepsilon > 0$, there is $\delta > 0$ such that when $|\Gamma| < \delta$ we have $|I - R_\Gamma| < \varepsilon$. We denote it as

$$I = \int_a^b f(x)d\phi(x) = \int_a^b f d\phi$$

Definition 1.21 (upper, lower R-S sum). Let f be bounded and ϕ be monotonically increasing. Let

$$m_i = \inf_{[x_{i-1},x_i]} f(x), M_i = \sup_{[x_{i-1},x_i]} f(x)$$

then we define the lower and upper Riemann-Stieltjes sums L_Γ, U_Γ as follows:

$$L_\Gamma = \sum_{i=1}^{m} m_i[\phi(x_i) - \phi(x_{i-1})], U_\Gamma = \sum_{i=1}^{m} M_i[\phi(x_i) - \phi(x_{i-1})]$$

Definition 1.22 (Lebesgue outer measure). For let S be a collection of n-dimensional intervals that cover E, then the Lebesgue outer measure of E is given by

$$|E|_e = \inf \sigma(S)$$

where $\sigma(S) = \sum_{I_k \in S} |I_k|$.

Definition 1.23 (Lebesgue measurable). A subset E of $\mathbb{R}^n$ is called Lebesgue measurable if and only if given any $\varepsilon > 0$, there exists an open set G such that

$$E \subset G, |G - E|_e < \varepsilon$$

If E is measurable, then $|E| = |E|_e$.

Definition 1.24 (σ-algebra). A σ-algebra is a collection of sets that is closed under taking complement, countable union, and countable intersection.

The σ-algebra generated by containing all the open sets is called the Borel σ-algebra.

Definition 1.25 (Lebesgue measurable functions). Let E be a measurable set in $\mathbb{R}^n$, f is a mesaurable function if for all finite a, the set

$$\{x \in E : f(x) > a\}$$

is measuarble.

Definition 1.26 (upper,lower semicontinuous). Let f be defined on E, then f is usc at x_0 if for every $M > f(x_0)$, there exists $\delta > 0$ such that when $|x - x_0| < \delta$, we have $f(x) < M$.

f is called usc relative to E if it is usc at every limit point of E.

Definition 1.27 (convergence in measure). Let f, $\{f_k\}$ be defined and a.e. on E, then $f_k \to f$ in measure if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \varepsilon\} = 0$$