Aluffi Problems

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Category Theory

Chapter 2

Groups I

Problem 2.1 (1.8)**.** Let G be a finite abelian group with exactly one element f of order 2. Prove that $\prod_{g \in G} g = f$.

Proof. It suffices to see that $\prod_g g^2 = e$, which is true by every element has an inverse. $\square$

Problem 2.2 (1.13)**.** Give an example showing that $|gh|$ is not necessarily equal to $\mathrm{lcm}(|g|, |h|)$, even if g and h commute.

Proof. Let $g = h = 1 \in \mathbb{Z}/2\mathbb{Z}$. $\square$

Problem 2.3 (1.14)**.** If g and h commute and $\gcd(|g|, |h|) = 1$, then $|gh| = |g||h|$. (Hint: Let $N = |gh|$; then $g^N = (h^{-1})^N$. What can you say about this element?)

Proof. We know that $g^N = (h^{-1})^N = e$. $\square$

Problem 2.4 (6.7)**.** If $\mathrm{Aut}(G)$ is cyclic, then G is abelian.

Proof. This implies $\mathrm{Inn}(G)$ is cyclic, which is iff $\mathrm{Inn}(G)$ is trivial, iff G is abelian. $\square$

Problem 2.5 (6.9)**.** Prove that every finitely generated subgroup of $\mathbb{Q}$ is cyclic. Prove that $\mathbb{Q}$ is not finitely generated.

Proof. Suppose we just have $H = \left\langle \frac{p_1}{q_1}, \frac{p_2}{q_2} \right\rangle$, find $\mathrm{lcm}(q_1, q_2) = q$, then

$$H = \left\langle \frac{a_1}{q}, \frac{a_2}{q} \right\rangle$$

find $\gcd(a_1, a_2) = p$, we claim that

$$H = \left\langle \frac{p}{q} \right\rangle$$

If $\mathbb{Q}$ were to be finitely generated, then it is cyclic, $\mathbb{Q} = \langle \frac{p}{q} \rangle$, then try $(p+1)/q$. $\square$

Problem 2.6 (8.1). If a group H may be realized as a subgroup of two groups G_1 and G_2 and if

$$\frac{G_1}{H} \cong \frac{G_2}{H},$$

does it follow that $G_1 \cong G_2$? Give a counterexample.

Proof. Let $G_1 = S_3, G_2 = \mathbb{Z}/6\mathbb{Z}$, and $H = \mathbb{Z}/3\mathbb{Z}$. □

Problem 2.7 (8.2). Suppose G is a group and $H \subseteq G$ is a subgroup of index 2, that is, such that there are precisely two cosets of H in G. Prove that H is normal in G.

Proof. For any $g \notin H$, we have

$$G = H \sqcup gH = H \sqcup Hg$$

Thus $gH = Hg$. □

Problem 2.8 (8.13). Let G be a finite group, and assume $|G|$ is odd. Prove that every element of G is a square.

Proof. Consider the set function $\varphi : g \mapsto g^2$, this function is injective hence surjective. □

Problem 2.9 (8.18). Let G be an abelian group of order $2n$, where n is odd. Prove that G has exactly one element of order 2. (It has at least one, for example by Exercise [8.17]. Use Lagrange's theorem to establish that it cannot have more than one.) Does the same conclusion hold if G is not necessarily commutative?

Proof. There exists one element g of order 2, then take its quotient $G/\langle g \rangle$. □

Problem 2.10 (9.11). Let G be a finite group, and H be subgroup of index p, where p is the smallest prime dividing $|G|$, then H is normal in G.

Proof. (I will abuse the notatoin $\left|\frac{G}{H}\right| = [G : H]$). Let G act on the cosets G/H by left multiplication, this action $\sigma : G \to \mathrm{Aut}(G/H)$ is not trivial, hence

$$\left|\frac{G}{\ker(\sigma)}\right| \text{ divides } p!$$

Moreover, we notice that $\ker(\sigma) \subset H$, hence p divides $\left|\frac{G}{\ker(\sigma)}\right|$. Now we recall that p is the smallest prime dividing $|G|$, we must have $\left|\frac{G}{\ker(\sigma)}\right| = p$, hence $H = \ker(\sigma)$. □

Proposition 2.1 (1.12). There exists elements $g, h \in G$, such that $|g|, |h| < \infty$, but $|gh| = \infty$.

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

Proposition 2.2 (1.15)**.** Let G be a commutative group, and let $g \in G$ be an element of maximal finite order, that is, such that if $h \in G$ has finite order, then $|h| \leq |g|$. Then, if h has finite order in G, then $|h|$ divides $|g|$.

Proposition 2.3. When n is odd, the center of D_{2n} is trivial, when n is even, the center consists of $\{e, r^{\frac{n}{2}}\}$.

$$r^{\frac{n}{2}} s = sr^{-\frac{n}{2}} = sr^{\frac{n}{2}}$$

Proposition 2.4 (4.8)**.** The map $g \mapsto \left(r_g : a \mapsto gag^{-1}\right)$ defines a homomorphism from $G \to \mathrm{Aut}(G)$.

Proposition 2.5 (4.9)**.** Let m, n be positive integers such that $\gcd(m, n) = 1$, then

$$\frac{\mathbb{Z}}{mn\mathbb{Z}} \cong \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{n\mathbb{Z}}$$

Proposition 2.6 (4.14)**.** The order of the group of automorphisms of $\mathbb{Z}/n\mathbb{Z}$ is the the number of generators of $\mathbb{Z}/\mathbb{Z}$, i.e.,

$$|\mathrm{Aut}(\mathbb{Z}/n\mathbb{Z})| = \left|(\mathbb{Z}/n\mathbb{Z})^{\times}\right|$$

Proposition 2.7 (4.15)**.** Let p be a prime, then

$$\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong \frac{\mathbb{Z}}{(p-1)\mathbb{Z}}$$

Proposition 2.8 (6.3)**.** Every matrix in $\mathrm{SU}(2)$ may be written in the form

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} \gamma & \omega \\ -\bar{\omega} & \bar{\gamma} \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 + c^2 + d^2 = 1$.

Proposition 2.9 (6.10)**.** The set of 2×2 matrices with integer entries and determinant 1 is denoted $\mathrm{SL}_2(\mathbb{Z})$:

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{such that } a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

Note that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the matrices

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and } t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Proposition 2.10 (7.7)**.** Let G be a group and n a positive integer, let $H \subset G$ be the subgroup generated by all elements of order n in G, then H is normal.

Proposition 2.11 (7.14)**.** $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Proposition 2.12 (8.4)**.** The dihedral group D_{2n} can also be represented as

$$\langle a, b : a^2 = b^2 = (ab)^n = e \rangle$$

(a, b are two reflections, take $a = s, b = rs$).

Proposition 2.13 (8.8)**.** $\mathrm{SL}_n(\mathbb{R})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$, and

$$\frac{\mathrm{GL}_n(\mathbb{R})}{\mathrm{SL}_n(\mathbb{R})} = (\mathbb{R}^\times, \cdot)$$

as groups.

Chapter 3

Rings and Modules

Problem 3.1 (1.12). Just as complex numbers may be viewed as combinations $a+bi$, where $a, b \in \mathbb{R}$ and i satisfies the relation $i^2 = -1$ (and commutes with $\mathbb{R}$), we may construct a ring $\mathbb{H}$ by considering linear combinations $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$ and i, j, k commute with $\mathbb{R}$ and satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Addition in $\mathbb{H}$ is defined componentwise, while multiplication is defined by imposing distributivity and applying the relations. For example,

$$(1 + i + j) \cdot (2 + k) = 1 \cdot 2 + i \cdot 2 + j \cdot 2 + 1 \cdot k + i \cdot k + j \cdot k = 2 + 2i + 2j + k - j + i = 2 + 3i + j + k.$$

1. Verify that this prescription does indeed define a ring.
2. Compute $(a + bi + cj + dk)(a - bi - cj - dk)$, where $a, b, c, d \in \mathbb{R}$.
3. Prove that $\mathbb{H}$ is a division ring.
4. List all subgroups of $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$, and prove that they are all normal.
5. Prove that Q_8 and D_8 are not isomorphic.
6. Prove that Q_8 admits the presentation $\langle x, y \mid x^2 y^{-2}, y^4, xyx^{-1}y \rangle$.

Elements of $\mathbb{H}$ are called *quaternions*. Note that Q_8 forms a subgroup of the group of units of $\mathbb{H}$; it is a noncommutative group of order 8, called the *quaternionic group*.

Proof.

1. :)
2. $a^2 + b^2 + c^2 + d^2$.
3. follows from 2.
4. $\{\pm 1\}, \{\pm 1, \pm i\}, \{\pm 1, \pm j\}, \{\pm 1, \pm k\}$
5. Number of order 4 elements: 2 in D_8 and 6 in Q_8.
6. Take $x = i, y = j$, then

$$Q_8 = \{1, i, i^2, i^3, j, ij, i^2 j, i^3 j\}$$

□

Problem 3.2 (1.15)**.** Prove that $R[x]$ is an integral domain if and only if R is an integral domain.

Proof. For sufficiency: observe that if $f, g \neq 0 \in R[x]$, then $fg \neq 0$. □

Problem 3.3 (1.16)**.** Let R be a ring, and consider the ring of power series $R[[x]]$ (cf. {1.3}).

1. Prove that a power series $a_0 + a_1 x + a_2 x^2 + \cdots$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R. What is the inverse of $1 - x$ in $R[[x]]$?
2. Prove that $R[[x]]$ is an integral domain if and only if R is.

Proof. 1. For sufficiency: you do it term by term; the inverse of $(1-x)$ is $1 + x + x^2 + \cdots = \sum_{i=0}^{\infty} x^i$. □

Problem 3.4 (2.11)**.** Prove (by hand) that division ring R of p^2 elements where p is prime, is commutative.

Proof. Assume not commutative, then the center of R must contain p elements. Let $r \in R$ such that r is not in the center, then the centralizer of r must be the entire ring R, and this holds for all such r. □

Problem 3.5 (2.16)**.** Prove that there is (up to isomorphism) only one structure of ring with identity on the abelian group $(\mathbb{Z}, +)$. (Hint: Let R be a ring whose underlying group is $\mathbb{Z}$. By Proposition [2.7] there is an injective ring homomorphism $\lambda : R \to \mathrm{End}_{Ab}(R)$, and the latter is isomorphic to $\mathbb{Z}$. Prove that λ is surjective.)

Proof. There exists an injective map

$$\lambda : R \to \mathbb{Z}$$

note that this map is also surjective. □

Problem 3.6 (2.17)**.** Let R be a ring, and let $E = \mathrm{End}_{Ab}(R)$ be the ring of endomorphisms of the underlying abelian group $(R, +)$. Prove that the center of E is isomorphic to a subring of the center of R. (Prove that if $\alpha \in E$ commutes with all right-multiplications by elements of R, then α is left-multiplication by an element of R; then use Proposition [2.7])

Proof. If α commutes with all the right multiplications r_x, then

$$\alpha r_x(s) = \alpha(sx) = \alpha(s)x$$

letting $s = 1$, we see

$$\alpha(x) = \alpha(1)x$$

Thus α is a left multiplication. Let $\varphi : \alpha \mapsto \alpha(1)$, this is injective, surjective onto its image. □

Problem 3.7 (3.4). Let R be a ring such that every subgroup of $(R, +)$ is in fact an ideal of R. Prove that $R \cong \mathbb{Z}/n\mathbb{Z}$, where n is the characteristic of R.

Proof. It suffices to exhibit a surjective map from $\mathbb{Z}$ to R, consider the subgroup $\varphi(\mathbb{Z})$, where $\varphi : 1 \mapsto 1$. We know that $\varphi(\mathbb{Z})$ is an ideal, i.e., for every $r \in R$,

$$r \cdot 1 \in \varphi(\mathbb{Z})$$

since $1 \in \varphi(\mathbb{Z})$, thus this map is surjective. □

Problem 3.8 (4.5). Let I, J be ideals in a commutative ring R, such that $I+J = (1)$. Prove that $IJ = I \cap J$.

Proof. We know $IJ \subset I \cap J$, now let $r \in I \cap J$, then

$$r \cdot 1 = r(i + j) = ri + rj \in IJ$$

□

Problem 3.9 (4.6). Let I, J be ideals in a commutative ring R. Assume that $R/(IJ)$ is reduced (that is, it has no nonzero nilpotent elements). Prove that $IJ = I \cap J$.

Proof. Consider nonzero $r \in I \cap J$, then $r^2 \in IJ$, hence in R/IJ, $r = 0 + IJ$, i.e., $r \in IJ$. □

Problem 3.10 (4.11). Let R be a commutative ring, $a \in R$, and $f_1(x), \dots, f_r(x) \in R[x]$.

- Prove the equality of ideals

$$(f_1(x), \dots, f_r(x), x - a) = (f_1(a), \dots, f_r(a), x - a).$$

- Note the useful substitution trick

$$\frac{R[x]}{(f_1(x), \dots, f_r(x), x - a)} \cong \frac{R}{(f_1(a), \dots, f_r(a))}.$$

Proof. Use long division: $f_1(x) = q(x)(x - a) + f_1(a)$. □

Problem 3.11 (4.17). Let K be a compact topological space, and let R be the ring of continuous real-valued functions on K, with addition and multiplication defined pointwise.

(i) For $p \in K$, let $M_p = \{f \in R \mid f(p) = 0\}$. Prove that M_p is a maximal ideal in R.

(ii) Prove that if $f_1, \dots, f_r \in R$ have no common zeros, then $(f_1, \dots, f_r) = (1)$. (Hint: Consider $f_1^2 + \cdots + f_r^2$.)

(iii) Prove that every maximal ideal M in R is of the form M_p for some $p \in K$. (Hint: You will use the compactness of K and (ii).)

Proof. (i) Note that $\frac{R}{M_p} \cong \mathbb{R}$, given by evaluation at p.

(ii) Note that $g(p) = f_1^2 + \cdots + f_r^2(p) > 0$ for all $p \in K$, thus one can construct an inverse. Namely,

$$1 = h(f_1^2 + \cdots + f_r^2)$$

where $h = \frac{1}{g}$.

(iii) Let M be a maximal ideal, suppose M is not contained in M_p for any p. This implies that there exists $f \in M$ such that $f(p) \neq 0$ for every $p \in K$. Then we consider the set

$$\{f^{-1}(\mathbb{R} \setminus \{0\}) : f \in M\}$$

This is an open cover of K, hence there exists $f_1, \ldots, f_r$ such that

$$\{f_i(\mathbb{R} \setminus \{0\}) : 1 \leq i \leq r\}$$

is also a cover of K. We know that $f_1, \ldots, f_r$ have no common roots, thus

$$(f_1, \ldots, f_r) = R$$

which is a contradiction.

$\square$

Problem 3.12 (4.23). A ring R has Krull dimension 0 if every prime ideal in R is maximal. Prove that fields and Boolean rings have Krull dimension 0.

Proof. Let p be a prime ideal of a Boolean ring, then $R/p \cong \mathbb{Z}/2\mathbb{Z}$, which is a field, hence p is also a maxiaml ideal. $\square$

Problem 3.13 (6.3). Let R be a ring, M an R-module, and $p : M \to M$ an R-module homomorphism such that $p^2 = p$. (Such a map is called a projection.) Prove that $M \cong \ker p \oplus \operatorname{im} p$.

Proof. Let $m \in M$, then $m = (m - p(m)) + p(m)$. $\square$

Problem 3.14 (6.6). Let R be a ring, and let $F = R^{\oplus n}$ be a finitely generated free R-module. Prove that $\mathrm{Hom}_{R\text{-Mod}}(F, R) \cong F$. On the other hand, find an example of a ring R and a nonzero R-module M such that $\mathrm{Hom}_{R\text{-Mod}}(M, R) = 0$.

Proof. Define the map $F \to \mathrm{Hom}(F, R)$ as

$$(r_1, \ldots, r_n) \mapsto \left(\varphi : (a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i r_i\right)$$

Take $M = \mathbb{Z}/2\mathbb{Z}$, $R = \mathbb{Z}$ in the second question. $\square$

Problem 3.15 (6.16). Let R be a ring. A (left-)R-module M is *cyclic* if $M = \langle m \rangle$ for some $m \in M$.

(i) Prove that simple modules are cyclic.

(ii) Prove that an R-module M is cyclic if and only if $M \cong R/I$ for some (left-)ideal I.

(iii) Prove that every quotient of a cyclic module is cyclic.

Proof. (i) Take any nonzero $r \in R$, then $M = \langle r \rangle$.

(ii) For the forward directin, $M = \langle m \rangle$, consider the map $\varphi : m \mapsto 1$; for the backwards, $1+I$ is a generator of R/I, where R/I viewed as a R-module.

(iii) Follows from (ii) and the second isomorphism theorem.

□

Problem 3.16 (6.18). Let M be an R-module, and let N be a submodule of M. Prove that if N and M/N are both finitely generated, then M is finitely generated.

Proof. Suppose $N = \langle r_1, \ldots, r_k \rangle$, $M/N = \langle r_{k+1} + N, \ldots, r_{k+m} + N \rangle$, then we claim $M = \langle r_1, \ldots, r_{k+m} \rangle$. If $m \in M$ is such that $m \in N$, then done; if $m \notin N$, then $m \in r_i + N$ for some i, then

$$m = \sum a_i r_i \Rightarrow m - \sum a_i r_i \in N$$

thus again writting it as a finite sum, we are done. □

Proposition 3.1 (2.8). Every subring of a field is an integral domain.

Proposition 3.2 (2.9). The center of a division ring is a field.

Proposition 3.3 (3.9). A nonzero ring with ideals being only $\{0\}$ and R are called simple rings. The only simple commutative rings are fields. Moreover, $M_n(\mathbb{R})$ is also simple.

Proposition 3.4 (3.14). The characteristic of an integral domain is either 0 or a prime ideal p.

Proposition 3.5 (4.4). If k is a field, then $k[x]$ is a PID.

Proposition 3.6 (4.9). Let R be a commutative ring, and let $f(x)$ be a zero-divisor in $R[x]$. There exists $\exists b \in R, b \neq 0$, such that $f(x)b = 0$. (Let $fg = 0$, where $g = b_e x^e + \cdots + b_0$, set $b = b_e$.)

Proposition 3.7 (4.10). Let d be an integer that is not the square of an integer, and consider the subset of $\mathbb{C}$ defined by

$$\mathbb{Q}(\sqrt{d}) := \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}.$$

Then $\mathbb{Q}(\sqrt{d})$ is a field, and

$$\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}[t]/(t^2 - d)$$

Proposition 3.8 (4.19). Let R be a commutative ring, let P be a prime ideal in R, and let I_j be ideals of R.

(i) Assume that $I_1 \cdots I_r \subseteq P$, then that $I_j \subseteq P$ for some j.

(ii) By (i), if $P \supseteq \bigcap_{j=1}^{r} I_j$, then P contains one of the ideals I_j. The following is not true: $P \supseteq \bigcap_{j=1}^{\infty} I_j$, then P contains one of the ideals I_j. Consider $I_j = (p_j)$ then $\cap I_j = 0$.

Proposition 3.9 (4.20)**.** Let M be a two-sided ideal in a (not necessarily commutative) ring R. Then M is maximal if and only if R/M is a simple ring.

Proposition 3.10 (4.21)**.** Let k be an algebraically closed field, and let $I \subseteq k[x]$ be an ideal. Then I is maximal if and only if $I = (x - c)$ for some $c \in k$.

Proposition 3.11 (4.22)**.** $(x^2 + 1)$ is maximal in $\mathbb{R}[x]$.

Proposition 3.12 (5.4)**.** Let R be a ring. A nonzero R-module M is *simple* (or *irreducible*) if its only submodules are $\{0\}$ and M. Let M, N be simple modules, and let $\varphi : M \to N$ be a homomorphism of R-modules. Prove that either $\varphi = 0$ or φ is an isomorphism. (This rather innocent statement is known as Schur's lemma.)

Proposition 3.13 (5.5)**.** Let R be commutative, viewed as R-module over itself, let M be an R-module, then

$$\mathrm{Hom}(R, M) \cong M$$

as R-modules.

Proposition 3.14 (5.13)**.** Let R be an integral domain, let I be a nonzero principal ideal, then I is isomorphic to R as an R-module.

Proposition 3.15 (5.16)**.** Let R be commutative, $a \in R$ be nilpotent, consider the submodule aM of M. Then

$$M = 0 \iff aM = M$$

Proof. Multiplication by a is a surjective map, composition of surjective maps is still surjective. □

Proposition 3.16 (6.16)**.** Let M be an R-module, it is cyclic if $M = \langle m \rangle$, then M is cyclic if and only if $M \cong R/I$ for some ideal I.

Proposition 3.17 (6.18)**.** Let M be an R-module, and let N be a submodule of M. Prove that if N and M/N are both finitely generated, then M is finitely generated.

Chapter 4

Groups II

Chapter 5

Irreducibility of polynomials

Chapter 6

Linear Algebra I

Problem 6.1 (6.10)**.** Let F_1, F_2 be free R-modules of finite rank, and let α_1, resp., α_2, be linear transformations of F_1, resp., F_2. Let $F = F_1 \oplus F_2$, and let $\alpha = \alpha_1 \oplus \alpha_2$ be the linear transformation of F restricting to α_1 on F_1 and α_2 on F_2.

- Prove that $P_\alpha(t) = P_{\alpha_1}(t)P_{\alpha_2}(t)$. That is, the characteristic polynomial is multiplicative under direct sums.
- Find an example showing that the minimal polynomial is not multiplicative under direct sums.

here

Problem 6.2 (6.13)**.** Let A be a square matrix with integer entries. Prove that if λ is a rational eigenvalue, then $\lambda \in \mathbb{Z}$.

Proof. Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ be the characteristic polynomial of A, then $p(\lambda) = 0$, letting $\lambda = \frac{p}{q}$, then

$$p \mid a_0, \quad q \mid a_n$$

we know that p is monic, thus $a_n = 1$, hence $\lambda \in \mathbb{Z}$. □

Problem 6.3 (7.3)**.** Prove that two linear transformations of a vector space of dimension ≤ 3 are similar if and only if they have the same characteristic and minimal polynomials. Is this true in dimension 4? [§6.2]

here

Problem 6.4 (7.4)**.** Let k be a field, and let K be a field containing k. Two square matrices $A, B \in M_n(k)$ may be viewed as matrices with entries in the larger field K. Prove that A and B are similar over k if and only if they are similar over K.

here

Proof. For the interesting direction, if A, B are similar in K: □

Problem 6.5 (7.7)**.** Let V be a k-vector space of dimension n, and let $\alpha \in \mathrm{End}_k(V)$. Prove that the minimal and characteristic polynomials of α coincide if and only if there is a vector $v \in V$ such that

$$\{v, \alpha(v), \ldots, \alpha^{n-1}(v)\}$$

is a basis of V.

here

Problem 6.6 (7.8)**.** Let V be a k-vector space of dimension n, and let $\alpha \in \mathrm{End}_k(V)$. Prove that the characteristic polynomial $P_\alpha(t)$ divides a power of the minimal polynomial $m_\alpha(t)$.

Proof. Assume that k is algebraically closed, and polynomials factors, the minimal polynomial m_α contains all the $(t - \lambda_i)$ for distinct λ_i's by Lemma 7.12. Thus P_α divides $(m_\alpha)^n$. □

Problem 6.7 (7.12)**.** Let V be a finite-dimensional k-vector space, and let $\alpha \in \mathrm{End}_k(V)$ be a diagonalizable linear transformation. Assume that $W \subseteq V$ is an invariant subspace, so that α induces a linear transformation $\alpha|_W \in \mathrm{End}_k(W)$. Prove that $\alpha|_W$ is also diagonalizable. (Use Proposition 7.18.)

Proof. Assume that characteristic polynomial factors completely over k, then α is diagonalizable iff minimal polynomial m_α has no repeated roots, thus $\alpha|_W$ also has no repeated roots as it divides m_α. □

Problem 6.8 (7.13)**.** Let R be an integral domain. Assume that $A \in \mathcal{M}_n(R)$ is diagonalizable, with distinct eigenvalues. Let $B \in \mathcal{M}_n(R)$ be such that $AB = BA$. Prove that B is also diagonalizable, and in fact it is diagonal w.r.t. a basis of eigenvectors of A. (If P is such that PAP^{-1} is diagonal, note that PAP^{-1} and PBP^{-1} also commute.)

Proof. It suffices to see that if $v_1 \neq 0$ is such that $Av_1 = \lambda_1 v_1$, then

$$\begin{aligned} A(Bv_1) &= B(Av_1) \\ &= B\lambda_1 v_1 \\ &= \lambda_1 (Bv_1) \end{aligned}$$

Thus Bv_1 is contained in the one-dimensional subspace generated by v_1. □

Problem 6.9 (7.14)**.** Prove that "commuting transformations may be simultaneously diagonalized", in the following sense. Let V be a finite-dimensional vector space, and let $\alpha, \beta \in \mathrm{End}_k(V)$ be diagonalizable transformations. Assume that $\alpha\beta = \beta\alpha$. Prove that V has a basis consisting of eigenvectors of both α and β. (Argue as in Exercise 7.13 to reduce to the case in which V is an eigenspace for α; then use Exercise 7.12.)

Proof. Separate into eigenspaces: consider eigenspace E_1 of α, then diagonalize β in E_1 (by 7.12), note that E_1 is invariant under β. □

Problem 6.10 (7.15). A **complete flag** of subspaces of a vector space V of dimension n is a sequence of nested subspaces

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V_n = V$$

with $\dim V_i = i$. In other words, a complete flag is a composition series in the sense of Exercise 1.16. Let V be a finite-dim vector space over algebraically closed k. Prove that every linear transformation α of V preserves a complete flag: there is a complete flag as above and such that $\alpha(V_i) \subset V_i$.

Find a linear transformation of $\mathbb{R}^2$ that does not preserve a complete flag.

Proof. It suffices take V_i as the subspaces generated by eigenvectors. An example in $\mathbb{R}^2$:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

□

Chapter 7

Fields

Chapter 8

Linear Algebra II