Calc III Section Notes with Answers

Fall 2025

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Calc III-Week 1 (8/25-29)

Logistics

- TA: Hui.
- Email: hsun95@jh.edu.
- Office Hour (tentative): Tuesday 4-6 PM, Krieger 211.
- Biweekly Quizzes: 10-15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Icebreaking Activity

- In a group of three or four:
  1. Learn each other names, year, pronouns.
  2. Find something in common and different among you and share with the entire class.
  3. Play Buzz if you have time, with prime 7: say the number if it doens't contain or is not divisible by 7, say buzz otherwise.

Some Math

Problem 1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 3. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned}
\|u+v\|^2 &= (u+v)\cdot(u+v) \\
&= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\
&= \|u\|^2 + \|v\|^2
\end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u\cdot v$$

given the assumption, we also have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u\cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u\cdot v = 0$$

$\square$

Reminders

1. First HW due this Friday.
2. First Quiz next Tuesday.

Calc III-Week 2 (9/1-5)

Definition 1 (cross product). Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$, the cross product of a, b is the vector $a \times b$,

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 2 (Plane in three dimensions). A perpendicular vector and a normal vector uniquely define a plane in $\mathbb{R}^3$: given the plane $\mathcal{P}$ passing containing the point (x_0, y_0, z_0) that has a normal vector (A, B, C) is given by the equation:

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1. Here are some properties of the cross product:

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:
$$\|a \times b\| = \|a\|\|b\| \sin \theta$$
where θ is the angle between them. (Compare this with the dot product).
3. $a \times b = -b \times a$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. (HW) The cross product is **not** associative! For example, compute
$$(i \times i) \times j, \quad i \times (i \times j)$$

Problem 4. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 5. Compute the plane containg all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

$\square$

Reminders
HW is due Sunday 11:59PM.