Calc III Sections

Fall 2025

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September 20, 2025

Calc III-Week 5 (9/22-9/26)

Topics: (1) properties of derivatives, (2) directional derivatives, (3) gradient.

Definition 1 (path). A **path** c is a map $c : [a, b] \to \mathbb{R}^n$. We can write $c(t) = (c_1(t), \dots, c_n(t))$. If c is differentiable, then we can define the **velocity** of c at any $t_0 \in [a, b]$ as

$$c'(t_0) = (c_1'(t_0), \dots, c_n'(t_0))$$

The velocity vector of c at t_0 is also a **tangent** vector to c at t_0. The **speed** of the path c at t_0 is the length of the velocity vector $\|c'(t_0)\|$.

Definition 2 (tangent line to a path). Let $c : [a, b] \to \mathbb{R}^n$ be a path, if $c'(t_0) \neq 0$, then the **tangent line** at x_0 is given by

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

Proposition 1. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, then the derivative of f at x_0 is an $m \times n$ matrix $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$. The derivative follows the same properties as derivative for single variable functions:

1. Let $c \in \mathbb{R}$, then
$$D(cf)(x_0) = cDf(x_0) \qquad \text{(multiplication of a matrix by constant } c\text{)}$$
2. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ also be differentiable at x_0, then
$$D(f + g)(x_0) = Df(x_0) + Dg(x_0) \qquad \text{(sum of two matrices)}$$
3. Let $h_1 : U \subset \mathbb{R}^n \to \mathbb{R}$, $h_2 : U \subset \mathbb{R}^n \to \mathbb{R}$,then
$$D(h_1 h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \qquad \text{(product rule)}$$
and if $h_2 \neq 0$ on U.
$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \qquad \text{(quotient rule)}$$
4. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$, $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ such that $g(U) \subset V$, then
$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \qquad \text{(chain rule)}$$

Definition 3 (directiona derivative). Let $f : \mathbb{R}^3 \to \mathbb{R}$, be differentiable, then the directional directive at $x_0 \in \mathbb{R}^3$ in the direction of a **unit vector** v is given by

$$\nabla f(x_0) \cdot v = \left[\frac{\partial f}{\partial x_1}(x_0)\right] v_1 + \left[\frac{\partial f}{\partial x_2}(x_0)\right] v_2 + \left[\frac{\partial f}{\partial x_3}(x_0)\right] v_3$$

where $v = (v_1, v_2, v_3)$.

Proposition 2. Suppose that $\nabla f(x_0) \neq 0$, then the direction for which f increases the fastest at x_0 is along $\nabla f(x_0)$.

Proposition 3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable, let S be a level surface of f, i.e., S is a surface described by

$$f(x, y, z) = k$$

were k is some constant. Let $(x_0, y_0, z_0) \in S$, then

$$\nabla f(x_0, y_0, z_0) \text{ is normal to the level surface at } (x_0, y_0, z_0)$$

This means if $c(t)$ is a path in S, and $v(0) = (x_0, y_0, z_0)$, and if v is a tangent vector to $c(t)$ at $t = 0$, then

$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$

Moreover, if $\nabla f(x_0, y_0, z_0) \neq 0$, the **tangent plane** of S at (x_0, y_0, z_0) is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Problem 1. Consider the curve in $\mathbb{R}$: $c(t) = (2t, t^2, -t)$. Find the speed of the c at $t = 2$ and the tangent line at $t = 1$.

Problem 2 (2.5, Q7). Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and

$$g(x, y) = (e^{x-y}, x - y).$$

Calculate $f \circ g$ and

$$D(f \circ g)(1, 1).$$

Problem 3 (2.5, Q8). Let $f(u, v, w) = (e^{u-w}, \cos(v+u)+\sin(u+v+w))$ and $g(x, y) = (e^x, \cos(y-x), e^{-y})$. Calculate $f \circ g$ and $D(f \circ g)(0, 0)$.

Problem 4 (2.5, Q11). Let $f(x, y, z) = (3y + 2, x^2 + y^2, x + z^2)$. Let

$$c(t) = (\cos(t), \sin(t), t).$$

(a) Find the path $p = f \circ c$ and the velocity vector

$$p'(\pi).$$

(b) Find $c(\pi), c'(\pi)$ and $Df(-1, 0, \pi)$.
(c) Thinking of $Df(-1, 0, \pi)$ as a linear map, find

$$Df(-1, 0, \pi)\,(c'(\pi)).$$

Problem 5 (2.6, Q3). Compute the directional derivatives of the following functions along unit vectors at the indicated points in directions parallel to the given vector:

(a)
$$f(x, y) = x^y, (x_0, y_0) = (e, e), \quad \mathbf{d} = 5\mathbf{i} + 12\mathbf{j}$$

(b)
$$f(x, y, z) = e^x + yz,\ (x_0, y_0, z_0) = (1, 1, 1), \quad \mathbf{d} = (1, -1, 1)$$

(c)
$$f(x, y, z) = xyz,\ (x_0, y_0, z_0) = (1, 0, 1), \quad \mathbf{d} = (1, 0, -1)$$

Problem 6 (2.6, Q6). Find a vector which is normal to the curve

$$x^3 + xy + y^3 = 11 \text{ at } (1, 2).$$

Problem 7 (2.6, Q7). Find the rate of change of $f(x, y, z) = xyz$ in the direction normal to the surface

$$yx^2 + xy^2 + yz^2 = 3 \text{ at } (1, 1, 1).$$

Summary

- If asked to find directional derivative/rate of change along a unit vector v at point x_0: find $\nabla f(x_0) \cdot v$.
- If asked to find along with direction f increases the fastest: find $\nabla f(x_0)$.
- If asked to find a normal vector to a surface at a point x_0: construct a function such that the surface is the level set of this function, then find $\nabla f(x_0)$.