

Algebraic Topology

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Chapter 1

Category Theory

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1.1 Lecture 1 8/26

Definition 1.1 (Category). A category \mathcal{C} consists of the following data:

1. A collection of objects denoted as $\text{Ob}(\mathcal{C})$
2. Given two objects $X, Y \in \text{Ob}(\mathcal{C})$, a collection of morphisms between X, Y , $f : X \rightarrow Y$, denoted as $\text{mor}_{\mathcal{C}}(X, Y)$.
3. (Composition) We have $\text{mor}_{\mathcal{C}}(X, Y) \times \text{mor}_{\mathcal{C}}(Y, Z) \rightarrow \text{mor}_{\mathcal{C}}(X, Z)$ that satisfies associativity

$$f \circ (g \circ h) = (f \circ g) \circ h$$

4. (Identity) There is a distinguished morphism for each X , $\text{id}_{\mathcal{C}}(X, X)$ such that given any $f \in \text{mor}_{\mathcal{C}}(X, Y)$, we have $f \circ \text{id}_X = \text{id}_Y \circ f = f$.

In this course, we will make the assumption that in all the categories that we work with, $\text{Ob}(\mathcal{C})$ need not be a set, but given any $X, Y \in \text{Ob}(\mathcal{C})$, $\text{mor}(X, Y)$ will always be a set. Now we talk about some examples of categories.

Example 1.1 (Sets). Let $\text{Ob}(\text{Sets})$ be all the sets in the universe. Given X, Y sets, $\text{mor}(X, Y)$ be all the set maps from X to Y , and id_X is the identity map.

Example 1.2 (Top). Let $\text{Ob}(\text{Top})$ be all the topological spaces, and $\text{mor}(X, Y)$ be all the continuous maps from X to Y .

Example 1.3 ($\text{Vect}_{\mathbb{F}}$). Let \mathbb{F} be a field, and let Ob be all the \mathbb{F} -vector spaces. Then $\text{mor}(V, W)$ is all the \mathbb{F} -linear homomorphisms from V to W , where id_V is the identity homomorphism.

Example 1.4 (Posets). Fix a poset P , let $\text{Ob}(P)$ be the collection of elements in P , and given p, q we define

$$\text{mor}(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \emptyset, & \text{otherwise} \end{cases}$$

Problem 1.1. HW(Q1): check this is a category

Example 1.5 (Opposite category). Given a category \mathcal{C} , there is another category called the opposite category, denoted as \mathcal{C}^{op} , where

1. The objects are the same as \mathcal{C}
2. Given $X, Y \in \text{Ob}(\mathcal{C}^{op})$, we have $\text{mor}_{op}(X, Y) := \text{mor}_{\mathcal{C}}(Y, X)$.
3. Moreover, given $f \in \text{mor}_{op}(X, Y), g \in \text{mor}_{op}(Y, Z)$, then $g \circ f$ in \mathcal{C}^{op} is $f \circ g : Z \rightarrow X$.

Naturally, we define isomorphisms now.

Definition 1.2 (isomorphism). Given a category \mathcal{C} , and a morphism $f \in \text{mor}_{\mathcal{C}}(X, Y)$, we say f is an isomorphism if there exists $g \in \text{mor}_{\mathcal{C}}(Y, X)$ such that

$$f \circ g = \text{Id}_Y, g \circ f = \text{Id}_X$$

Now we introduce maps between categories.

Definition 1.3 (functor). Given categories \mathcal{C}, \mathcal{D} , a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is the following;

1. Given an object X in \mathcal{C} , $F(X)$ is an object in \mathcal{D} .
2. Given a morphism $f : X \rightarrow Y$, $F(f)$ is a morphism $F(f) : F(X) \rightarrow F(Y)$. Moreover, it satisfies the following:
 - (a) $F(\text{id}_X) = \text{id}_{F(X)}$
 - (b) $F(f \circ g) = F(f) \circ F(g)$. Alternatively, we can rewrite this condition as the following:

$$\begin{array}{ccc} \text{mor}(X, Y) \times \text{mor}(Y, Z) & \longrightarrow & \text{mor}(X, Z) \\ \downarrow \text{mor}(F) \times \text{mor}(F) & & \downarrow \text{mor}(F) \\ \text{mor}(F(X), F(Y)) \times \text{mor}(F(Y), F(Z)) & \longrightarrow & \text{mor}(F(X), F(Z)) \end{array}$$

such that this diagram commutes.

Problem 1.2. HW(Q2): functors take isomorphisms to isomorphisms.

Now we talk about some examples of functors.

Example 1.6. $F : \text{Top} \rightarrow \text{Set}$, where $X \mapsto X$, where the latter is a set, and $f \mapsto f$ as set maps.

Example 1.7. Let \mathbb{F} be a field, and $F : \text{Sets} \rightarrow \text{Vect}_{\mathbb{F}}$, where $X \mapsto \mathbb{F}\langle X \rangle$, where $\mathbb{F}\langle X \rangle$ is the free vector space over \mathbb{F} on the set X .

Problem 1.3. HW(Q3): extend this to a functor by defining $\text{mor}(f)$ and show this is a functor.

Example 1.8. Let \mathbb{F} be a field, then the following is a functor, $F : \text{Sets}^{op} \rightarrow \text{Vect}_{\mathbb{F}}$, where

$$hF : X \mapsto \text{Maps}(X, \mathbb{F})$$

Problem 1.4. HW(Q4): show this extends to a functor by defining $F(f)$, and show it is a functor.

1.2 Lecture 2 8/28

Definition 1.4 (contravariant functor). Let $F : \mathcal{C} \rightarrow \mathcal{D}$ is a contravariant functor from $\mathcal{C}^{op} \rightarrow \mathcal{D}$, (equivalently, $\mathcal{C} \rightarrow \mathcal{D}^{op}$).

Problem 1.5. HW(Q5): Show that the following functor F from $\text{Vect}_{\mathbb{F}}$ to $\text{Vect}_{\mathbb{F}}$ extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = \text{Hom}(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let Cat be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

Definition 1.5 (natural transformation). Given functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, a natural transformation T from F to G is the following: $T : F \Rightarrow G$:

1. given object $X \in Ob(\mathcal{C})$, $T(X) \in mor(F(X), G(X))$
2. Given $f \in mor(X, Y)$, the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ T(X) \downarrow & & \downarrow T(Y) \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

where mor_F, mor_G is the identification function on morphisms by functors F, G

If for all X , $T(X)$ is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor F and turn it to another functor G . We will (in a homework) show there exists natural transformation between the following two functors.

Example 1.9. Consider $F, G : \text{Vect}_{\mathbb{F}} \rightarrow \text{Vect}_{\mathbb{F}}$, define

$$F(V) = V \otimes_{\mathbb{F}} V / \langle a \otimes b - b \otimes a \rangle = V \otimes_{\mathbb{F}} V / \Sigma_2, G(V) = (V \otimes_{\mathbb{F}} V)^{\Sigma_2} = \{ \alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha \}$$

Both are vector spaces are fixed under “swaps.” Then a natural transformation can be defined as follows $T(V) :$

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

Problem 1.6. HW(Q6): For the above F, G

1. Show that T defines a natural transformation from F to G .
2. Find conditions on \mathbb{F} for T being a natural isomorphism.

Next we define limits and colimits. Let \mathcal{C}, \mathcal{D} be categories, d be an object in \mathcal{D} , then we can define a functor $F_d : \mathcal{C} \rightarrow \mathcal{D}$ such that for any object c in \mathcal{C} ,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the “constant functor” on \mathcal{D} , i.e., every object is sent to d , and every morphism is sent to id_d .

Definition 1.6 (colimit). Given any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the colimit of F , denoted as $\text{colim}(F)$ is an object in \mathcal{D} endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object d in \mathcal{D} and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in \mathcal{D} , $f : \text{colim}(F) \rightarrow d$ making the following diagram commute: for any X, Y, g :

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(Y) \\ \searrow \varphi_F & & \swarrow \varphi_F \\ & \text{colim}(F) & \\ \searrow \varphi & \downarrow f & \swarrow \varphi \\ & d & \end{array}$$

Next we prove some facts about colimits and give an example, where $\text{colim}(F)$ exists.

Proposition 1.1. If $\text{colim} F$ exists, then $\text{colim} F$ is unique up to isomorphisms.

Proof. Let $\text{colim}(F), \text{colim}(F)'$ be two colimits that satisfy the criteria. They are both objects in \mathcal{D} , then we get a morphism $f : \text{colim}(F) \rightarrow \text{colim}(F)'$, and likewise $g : \text{colim}(F) \rightarrow \text{colim}(F)'$, then

$$f \circ g : \text{colim}(F)' \rightarrow \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for $g \circ f$. □

Next we demonstrate a fact via an example.

Theorem 1.1. Let \mathcal{C} be a category where $Ob(\mathcal{C}), mor(X, Y)$ are all sets. Let $F : \mathcal{C} \rightarrow \text{Top}$ be any functor, then $\text{colim}(F)$ exists.

Proof. Define $\text{colim}(F) := \bigsqcup_c F(c) / \sim$, where \sim is induced by the equivalence relation given by

$$y \sim F(f)y$$

where $y \in F(C_1), f : C_1 \rightarrow C_2, F(f)x \in F(C_2)$. The natural transformation we endow on F as $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$:

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(\mathcal{C})} F(C) / \sim$$

□

Problem 1.7. HW(Q7): Show that $\text{colim}(F)$, φ_F is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

Definition 1.7 (limit). Given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the limit of F , denoted as $\lim(F)$ is an object of \mathcal{D} , endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object $d \in \text{Ob}(\mathcal{D})$ and a natural transformation

$$\varphi : F_d \rightarrow F$$

there exists a unique $f : \lim F \rightarrow d$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & \lim F & \\
 \varphi_F \swarrow & \downarrow f & \searrow \varphi_F \\
 & d & \\
 \swarrow \varphi & & \searrow \varphi \\
 F(X) & \xrightarrow{F(g)} & F(Y)
 \end{array}$$

Just like colimits, limits are unique up to isomorphisms.

Problem 1.8. HW(Q8): Given $F : \mathcal{C} \rightarrow \mathcal{D}$, consider $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$, then

$$\lim F = \text{colim } F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

Chapter 2

Homologies, Cohomologies

2.1 Lecture 3 9/4

Today we define (co)chain complexes: let R be a commutative ring, let Mod_R denote the category of R -modules and R -module maps.

Definition 2.1 (chain complex). A chain complex of R -modules is a collection of R -modules and R -modules maps

$$\cdots \rightarrow M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that $\partial_i \circ \partial_{i+1} = 0$ for all i . In other words, the image of previous map is contained in the kernel of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by $\{M.; \partial.^M\}$.

Next we introduce morphisms between chain complexes.

Definition 2.2 (morphism between complexes). Let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$, a morphism $\{f.\}$ between chain complexes is a “ladder” such that the following commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{i+1} & \xrightarrow{\partial_{i+1}^M} & M_i & \xrightarrow{\partial_i^M} & M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots \\ & & \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} \\ \cdots & \longrightarrow & N_{i+1} & \xrightarrow{\partial_{i+1}^N} & N_i & \xrightarrow{\partial_i^N} & N_{i-1} \xrightarrow{\partial_{i-1}^N} \cdots \end{array}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where $\{g.\} : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$, and $\{f.\} : \{N.; \partial.^N\} \rightarrow \{L.; \partial.^L\}$, which is simply vertical stacking.

Problem 2.1. HW(Q9): Prove that chain complexes of R -modules form a category ch_R .

There are interesting functors $F : ch_R \rightarrow Mod_R$, and we begin with the following one:

Definition 2.3 (H_n , n th-homology). Given $n \in \mathbb{Z}$, there is a functor

$$H_n : \mathbf{ch}_R \rightarrow \mathbf{Mod}_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial_n^M / \text{Im} \partial_{n+1}^M$$

and for $f : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$, we define: $H_n(f) : H_n(\{M.; \partial.^M\}) \rightarrow H_n(\{N.; \partial.^N\})$,

$$H_n(f)[x] := [f_n(x)]$$

where $[x] \in H_n(\{M.; \partial.^M\})$.

Proof. We need to show H_n is well-defined on objects and morphisms. We need to check that $\text{Im} \partial_{n+1} \subset \ker \partial_n$. This is a consequence of $\partial^2 = 0$.

On morphisms: for $x \in \ker \partial_n^M$, we have $f_n(x) \in \ker \partial_n^N$. This is we have

$$\partial_n^N(f_n(x)) = f_{n+1}(\partial_n^M(x)) = 0$$

Moreover, we need to check that this doesn't depend on the choice of representatives, i.e., we can check that

$$\text{Im} \partial_{n+1}^M \mapsto 0$$

Let $x = \partial_{n+1}^M(y)$, we have

$$f_n(x) = f_n(\partial_{n+1}^M(y)) = \partial_{n+1}^N(f_{n+1}(y)) = 0$$

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{\partial_{n+1}^M} & M_n \\ f_{n+1} \downarrow & & \downarrow f_n \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n \end{array}$$

□

Next we talk about homotopy between morphisms between chain complexes.

Definition 2.4 (homotopy). Given two morphisms, $f, g : M. \rightarrow N.$, a chain homotopy h . between them is a collection of R -modules maps, for all $n \in \mathbb{Z}$,

$$h_n : M_n \rightarrow N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as $\partial h + h \partial = f - g$.

$$\begin{array}{ccccc} M_{n+1} & \longrightarrow & M_n & \xrightarrow{\partial_n^M} & M_{n-1} \\ f_{n+1} \downarrow & \swarrow h_n & \downarrow f_n/g_n & \swarrow h_{n-1} & \downarrow f_{n-1} \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n & \longrightarrow & N_{n-1} \end{array}$$

Problem 2.2. HW(Q10): Show that homotopy is an equivalence relation between morphisms. Hint: replace h_n with $-h_n : M_n \rightarrow N_{n+1}$.

Proof. Reflexive is shown by defining h_n to be the zero map. For symmetry, we choose $-h_n$. Transitive is a ladder. □

Proposition 2.1. Let $h.$ be a chain homotopy between $f.$ and $g.$, then we have an equality

$$H_n(f.) = H_n(g.)$$

where $H_n(f.), H_n(g.) : H_n(M.) \rightarrow H_n(N.)$.

Proof. Given $[x] \in H_n(M.)$, we have

$$\begin{aligned} H_n(f)[x] &= [f_n(x)] \\ &= [g_n(x) + \partial h.(x) + h.\partial(x)] \\ &= [g_n(x) + \partial h.(x)] \\ &= [g_n(x)] \\ &= H_n(g)[x] \end{aligned}$$

□

Next we define a new category.

Definition 2.5 (Hch_R). Define the category Hch_R as follows:

1. $Ob(Hch_R) = Ob(ch_R)$
2. $mor_{Hch_R}(M., N.) = mor_{ch_R}(M., N.) / \sim$, where \sim is the homotopy equivalence.

Problem 2.3. HW(Q11): Show that Hch_R is a category, admitting a functor

$$F : ch_R \rightarrow Hch_R$$

such that the following diagram commutes:

$$\begin{array}{ccc} ch_R & \xrightarrow{F} & Hch_R \\ & \searrow H_n \quad \swarrow H_n & \\ & mod_R & \end{array}$$

Next we introduce long and short exact sequences.

Definition 2.6 (exactness). Firstly, given a pair of R -module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at X_2 if $\ker(g) = \text{im}(f)$. Hence given a sequence of R -module maps,

$$\cdots \rightarrow X_{i+1} \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots$$

this is called a long exact sequence if it is exact at all X_i . Finally, given a pair of R -module maps,

$$0 \rightarrow X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \rightarrow 0$$

This is a short exact sequence, and f is injective, g is surjective.

Problem 2.4. HW(Q12): Prove the following:

1. Given LES,

$$\cdots \rightarrow X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

show the following is a short exact sequence:

$$0 \rightarrow \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \rightarrow 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions X_i, Y_i , where $i = 2, 3, 4$, and assume the diagram commutes and if t_1, t_2, t_4, t_5 are isomorphisms, show that t_3 is also an isomorphism.

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ t_1 \downarrow & & t_2 \downarrow & & t_3 \downarrow & & t_4 \downarrow & & t_5 \downarrow \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & Y_5 \end{array}$$

Next we state the most important theorem in chain complexes:

Theorem 2.1 (The snake lemma). Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a SES of chain complexes, i.e.,

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is a short exact sequence of all n . Then there exists a LES of homology groups.

$$\begin{array}{ccccc} & & & & H_{n+1}(C) \\ & & & \swarrow \delta_{n-1} & \\ H_n(A) & \xrightarrow{H_n(f)} & H_n(B) & \xrightarrow{H_n(g)} & H_n(C) \\ & & \searrow \delta_n & & \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) & \xrightarrow{H_{n-1}(g)} & H_{n-1}(C) \\ & & \swarrow \delta_{n-1} & & \\ & & H_{n-2}(A) & & \end{array}$$

2.2 Lecture 4 9/9

Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

Proof. First we define the map $\delta_n : H_n(C) \rightarrow H_{n-1}(A)$. Let $[x] \in H_n(C)$, then $x \in \delta^C$, where $\delta^C : C_n \rightarrow C_{n-1}$. We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for $x \in C_n$, $g_n : B_n \rightarrow C_n$ is surjective, hence there exists $b \in B_n$ such that $g_n(b) = x$. Then consider $d = \delta^B(b)$, since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \text{im } f_{n-1}$$

Let $y \in A_{n-1}$ be this unique y such that $f_{n-1}(y) = d$, where uniqueness is by f_{n-1} is injective. This is indicated in the below diagram:

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & b \in B_n & \xrightarrow{g_n} & x \in C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ y \in A_{n-1} & \xrightarrow{f_{n-1}} & d \in B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

We first need to check that $[y]$ does not depend on the choice of b . Let $g_n(b_1) = g_n(b_2) = x$, then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let y_1, y_2 be those determined by b_1, b_2 , then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{array}{ccc} a \in A_n & \xrightarrow{f_n} & B_n \\ \delta^A \downarrow & & \downarrow \delta^B \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} \end{array}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e., $[y_1] = [y_2]$, as they only differ by $\text{im } \delta$.

Problem 2.5. HW(Q13): Check that if $x \in \text{im } \delta^C$, then $\delta_n[x] = 0$.

the proof is not finished, too lazy to tex it up □

Next we review the tensor products of R -modules. We first review R -bilinear maps

Definition 2.7 (bilinear maps). Let M, N, P be R -modules, an R -bilinear map $f : M \times N \rightarrow P$ is a map such that

1. f is linear in both coordinates, we have $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$, and similarly, $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.
2. For all $r \in R$, we have $f(rm, n) = f(m, rn) = rf(m, n)$.

Next we define tensor products.

Definition 2.8 (tensor product). A tensor product of $M \times N$ is an R -module denoted by $M \otimes_R N$ such that

1. $M \otimes_R N$ comes endowed with an R -bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other R -bilinear map $f : M \times N \rightarrow P$, there exists a unique R -module map ψ such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes_R N \\ f \downarrow & \swarrow \psi & \\ P & & \end{array}$$

It is not clear that $M \otimes_R N$ exists or not. In fact, they exist!

Theorem 2.2 ($M \otimes_R N$ exists). Define $M \otimes_R N = R\langle M \times N \rangle / K$, where $R\langle M \times N \rangle$ is the free R -module on the set $M \times N$. We define K as the submodule generated by the following four relations:

1. $\langle (m_1 + m_2, n) \rangle - \langle (m_1, n) \rangle - \langle (m_2, n) \rangle$
2. $\langle (m, n_1 + n_2) \rangle - \langle (m, n_1) \rangle - \langle (m, n_2) \rangle$
3. $r\langle (m, n) \rangle - \langle (rm, n) \rangle$
4. $r\langle (m, n) \rangle - \langle (m, rn) \rangle$

Moreover, the map $\varphi : M \times N \rightarrow M \otimes_R N$ given by

$$(m, n) \mapsto \langle (m, n) \rangle := m \otimes_R n$$

Problem 2.6. HW(Q14): show that $M \otimes_R N$ is a tensor product.

2.3 Lecture 5 9/11

We continue with the tensors of R -modules. Let $f : A \rightarrow B$ an R -module map, let N be some fixed R -module, then f induces maps: $f \otimes id : A \otimes_R N \rightarrow B \otimes_R N$,

$$f \otimes id : a \otimes n \mapsto f(a) \otimes n$$

and $id \otimes f : N \otimes_R A \rightarrow N \otimes_R B$:

$$id \otimes f : n \otimes a \mapsto n \otimes f(a)$$

Problem 2.7. HW(Q15(a)): Show that the following maps induce functors:

1. $- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$, where

$$A \mapsto A \otimes_R N, f \mapsto f \otimes id$$

2. $N \otimes_R - : \text{Mod}_R \rightarrow \text{Mod}_R$, where

$$A \mapsto N \otimes_R A, f \mapsto id \otimes f$$

Problem 2.8. HW(Q15(b)): Show that one has the following natural isomorphisms:

1. $0 \otimes_R M \cong 0$, and $0 \otimes_R - \cong F_0$ (recall the definition of F_0 as a functor).
2. $R \otimes_R M \cong M$, and $R \otimes_R - \cong id$.
3. $M \otimes_R N \cong N \otimes_R M$, and $M \otimes_R - \cong - \otimes_R M$.
4. $M \otimes_R (N \otimes_R K) \cong (M \otimes_R N) \otimes_R K$.
5. $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$.

For convenience, we introduce the following definition:

Definition 2.9 (positively graded chain complex). A positively graded chain complex $\{M.; \partial.^M\}$ is a chain complex so that $M_i = 0$ for all $i < 0$. The category of positively graded chain complexes is denoted as ch_R^+ .

We have our first important theorem for ch_R^+ .

Theorem 2.3. There exists a functor \otimes_R and a natural transformation X such that the following diagram of functors commutes up to some X :

$$\begin{array}{ccc} ch_R^+ \times ch + R^+ & \xrightarrow{\otimes_R} & ch_R^+ \\ H_i \times H_j \downarrow & \nearrow X & \downarrow H_{i+j} \\ Mod_R \times Mod_R & \xrightarrow{\otimes_R} & Mod_R \end{array}$$

where $X : \otimes_R \circ (H_i \times H_j) \Rightarrow H_{i+j} \circ \otimes_R$ is a natural transformation.

We note that the existence of X means this diagram doesn't commute "on the nose," but these two composition functors are the same up to some natural transformation. Before we given the proof, we recall that $Ob(C \times D) = Ob(C) \times Ob(D)$, $mor((X, Y), (X', Y')) = mor(X, Y) \times mor(X', Y')$.

Proof. We define \otimes_R of positively graded chain complexes as follows: let $\{M.; \partial.^M\}, \{N.; \partial.^N\}$ be two PGCC. Define $\{M \otimes_R N.; \partial.^{M \otimes_R N}\}$:

$$(M \otimes_R N) = \bigoplus_{i+j=n} (M_i \otimes_R N_j)$$

note that the RHS is always a finite sum. Moreover, ∂^\otimes is defined as follows:

$$\partial^\otimes : (M \otimes_R N)_n \rightarrow (M \otimes_R N)_{n-1} \text{ is defined on the component } M_i \otimes_R N_j \text{ (from the RHS)}$$

and

$$\partial^\otimes(m_i \otimes n_j) := \partial^M(m_i) \otimes n_j + (-1)^i m_i \otimes \partial^N(n_j)$$

It is easy to check that $\partial^\otimes \circ \partial^\otimes = 0$.

Now we've show $ch_R^+ \otimes_R ch_R^+$ is well-defined, it remains to define X , the natural transformation. We define

$$X : H_i(M_\bullet) \otimes_R H_j(N_\bullet) \rightarrow H_{i+j}(M_\bullet \otimes_R N_\bullet)$$

again, it suffices to define X on elementary tensors.

$$X : [\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$$

we need to check that

1. $\partial^\otimes(\alpha \otimes \beta) = 0$ if $\partial^M(\alpha) = 0$ and $\partial^N(\beta) = 0$.
2. If $\alpha = \partial(r)i$, then notice that $\partial^\otimes(r \otimes \beta) = \alpha \otimes \beta$, similarly for β . This would show that X is well-defined.

It is straightforward to check that X commutes with morphisms in $ch_R^+ \times ch_R^+$. □

Next we define cochain complexes and cohomologies.

Definition 2.10 (cochain). A cochain of R -modules $(M^\bullet, \partial_M^\bullet)$ is a sequence of R -module maps:

$$\dots \longrightarrow M^i \xrightarrow{\partial^i} M^{i+1} \xrightarrow{\partial^{i+1}} M^{i+2} \longrightarrow \dots$$

such that $\partial \circ \partial = 0$.

Cochain complexes form a category, with morphisms $\{f^\bullet\}$ form a ladder:

$$\begin{array}{ccccccc} \dots & \longrightarrow & M^i & \xrightarrow{\partial^i} & M^{i+1} & \xrightarrow{\partial^{i+1}} & M^{i+2} \longrightarrow \dots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \downarrow f^{i+2} \\ \dots & \longrightarrow & N^i & \xrightarrow{\partial^i} & N^{i+1} & \xrightarrow{\partial^{i+1}} & N^{i+2} \longrightarrow \dots \end{array}$$

The n -th cohomology of a cochain complex $\{M^\bullet; \partial_M^\bullet\}$ is defined as:

$$H^n(M^\bullet; \partial_M^\bullet) := \frac{\ker \partial^i : M^i \rightarrow M^{i+1}}{\text{im } \partial^{i-1} : M^{i-1} \rightarrow M^i}$$

We remark that there is nothing unexpected here from what we learned about chain complexes. Namely, if we reindex $\{M^\bullet; \partial_M^\bullet\}$, this defines a chain complex with $M'_i = M^{-i}$. This implies that the snake lemme holds! (with $\partial^i : H^i(C) \rightarrow H^{i+1}(A)$).

Theorem 2.4. There is a functor D and a natural transformation β such that the following diagram of functors commute up to the natural transformation β :

$$\begin{array}{ccc} ch_R^{op} & \xrightarrow{D} & coch_R \\ H_n^{op} \downarrow & \nearrow \beta & \downarrow H^n \\ Mod_R^{op} & \xrightarrow{\bar{D}} & Mod_R \end{array}$$

where $\bar{D}(M) = Hom_R(M, R)$, and

$$D(\{M_\bullet; \partial_\bullet^M\}) \text{ is defined as } \{DM^\bullet; \partial^\bullet\}$$

where

$$DM^n := Hom_R(M_n, R), \partial^n : DM^n \rightarrow DM^{n+1} \text{ is the map induced by } \partial_{n+1} : M_{n+1} \rightarrow M_n$$

We observe that $\partial^{n+1}\partial^n = 0$ since $\partial_{n+2}\partial_{n+1} = 0$.

Problem 2.9. HW(Q16): Show that D is a functor.

Next we define the natural transformation β . We have $\beta : H^n(DM) \rightarrow \text{Hom}_R(H_n(M_\bullet), R)$, such that

$$\beta : [\varphi] \mapsto \beta[\varphi]$$

let $[x] \in H_n(M_\bullet)$, where $\beta[\varphi]([x]) = \varphi(x)$ (where $\varphi \in \text{Hom}_R(M_n, \mathbb{R}), x \in M_n$).

Proof. We first need to show that β is well-defined. If $x = \partial_{n+1}(y)$, then consider

$$\beta[\varphi][x] = \varphi(x) = \varphi(\partial_{n+1}(y)) = \partial^n(\varphi)(y) = 0, x \in \ker \varphi$$

Conversely, if $\varphi = \delta^{n-1}(\psi)$, we have

$$\beta[\varphi][x] = \varphi(x) = \delta^{n-1}\psi(x) = \psi(\partial_n(x)) = 0$$

It remains to check that β commutes with morphisms in ch_R^{op} (which we will do next time). \square

2.4 Lecture 6 9/16

Today we continue our discussion of homological algebra. Let M be an R -module.

Definition 2.11 (resolution). A resolution of M is a positively graded chain complex $\{P_\bullet, \partial_\bullet\}$ such that

1. $H_n(P_\bullet) = 0$ for all $n > 0$
2. $H_0(P_\bullet) = \frac{P_0}{\text{im } \partial_1} \cong M$, where $\partial_1 : P_1 \rightarrow P_0$.

We say $\{P_\bullet, \partial_\bullet\}$ is a free resolution if P_i is a free R -module for each i .

For resolutions, we prove the following two things: first, free resolutions always exist; second, every R -module map can be extended to a map between their resolutions (with extra assumptions) and these extensions are unique up to homotopies.

Proposition 2.2. For any M , a free resolution for M exists.

Proof. We proceed this inductively. Define P_0 to be $R\langle M \rangle$, where it is the free R -module defined on the set M . Note that

$$R\langle M \rangle \twoheadrightarrow M \text{ is surjective : } \langle m \rangle \mapsto m$$

Let K be the kernel of this map, we have an isomorphism:

$$\epsilon : P_0/K \cong M$$

Define P_1 as $R\langle K \rangle$, note that $P_1 \twoheadrightarrow K$, then we define

$$\partial_1 : P_1 \rightarrow P_0$$

to be the composite

$$P_1 \twoheadrightarrow K \subset P_0$$

Now we consider P_2 : let $K_1 \subset P_1$ be the kernel of δ_1 , define $P_2 = R\langle K_1 \rangle$, then define $\partial_2 : P_2 \rightarrow P_1$ to be the composite"

$$P_2 \twoheadrightarrow K_1 \subset P_1$$

note that $\ker \delta_1 / \text{im } \delta_2 = K_1/K_1 = 0$. Then we define $K_2 = \ker \delta_2$, define $P_3 = R\langle K_2 \rangle, \dots$ \square

Just like the above proposition, the next theorem uses induction.

Theorem 2.5 (extension theorem). Let $\{P_\bullet^M, \delta_\bullet^M, \epsilon_M\}$ be a free resolution on M , and let $\{P_\bullet^N, \delta_\bullet^N, \epsilon_N\}$ be an arbitrary resolution of N . Then given a map of R -modules $f : M \rightarrow N$, we may extend it to a map of chain complexes:

$$f_\bullet : \{P_\bullet^M, \delta_\bullet^M\} \rightarrow \{P_\bullet^N, \delta_\bullet^N\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_0(P_\bullet^M) & \xrightarrow{H_0(f_\bullet)} & H_0(P_\bullet^N) \\ \downarrow \epsilon_M & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

Moreover, given any two extensions f_\bullet^1, f_\bullet^2 of f , we have a chain homotopy h_\bullet between f_\bullet^1, f_\bullet^2 .

Remark: if f_\bullet makes the diagram commute, and g_\bullet is homotopic to f_\bullet , then g_\bullet also makes the diagram commute: homotopy classes work the same on homologies (they are the same on the nose).

Proof. We will construct f_\bullet as follows. We construct f_i inductively on i . Consider the diagram:

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Since P_0^M is free, and ϵ_N is surjective, we may lift f on generators of P_0^M by lifting the generators of P_0^M to elements in P_0^N . (Note: this lift may not be unique), but this lift extends uniquely to define f_0 . We notice that the bottom square

$$\begin{array}{ccc} P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commutes on homologies ($H_0(P_0^M), H_0(P_0^N)$). Now we construct f_1 :

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & \xrightarrow{f_1} & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

We will follow the purple path above. Recall that $\epsilon_M : H_0(P_0) = P_0 / \text{im}(\partial_1^M) \rightarrow M$ is an isomorphism. We

consider the composite: $f_0 \circ \partial_1^M = g$, we have

$$\begin{aligned}\epsilon_N \circ g &= \epsilon_N \circ f \circ \partial_1^M \\ &= f \circ \epsilon_M \circ \partial_1^M \\ &= 0\end{aligned}$$

This implies that $\text{im}(g) \subset \ker(\partial_N) = \text{im}(\partial_1^N)$. We can lift the generators of P_1^M to elements of P_1^N . (Once chosen a lift, one can extend this uniquely to define f_1). Then we construct f_2, f_3, \dots the same way by considering $f_n \circ \partial_{n+1}$ and show that it is in the kernel of ∂_n^N and lift it to define ∂_{n+1} .

Now we homotopy time. Assume f_\bullet^1, f_\bullet^2 are two lifts of f , we construct $h : P_\bullet^M \rightarrow P_{\bullet+1}^N$. We define h_\bullet inductively, starting with h_0 below:

$$\begin{array}{ccc} P_1^M & \xrightarrow{f_1} & P_1^N \\ \partial_1^M \downarrow & \nearrow h_0 & \downarrow \partial_1^N \\ P_0^M & \xrightarrow{f_0^1, f_0^2} & P_0^N \\ \epsilon_M \downarrow & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

We have $\epsilon_N(f_0^1 - f_0^2) = 0$, then

$$f_0^1 - f_0^2 \in \ker \epsilon_N = \text{im } \delta_1^N$$

we may lift $f_0^1 - f_0^2$ on generators of P_0^M , where $h_0 : P_0^M \rightarrow P_1^N$. Hence

$$(h_{-1} \circ \delta_{-1}^N) + \delta_1^N \circ h_0 = f_0^1 - f_0^2$$

Inductively, we assume h_m exists for $m \leq n$, then

$$\begin{array}{ccc} P_{n+2}^M & \xrightarrow{f_1} & P_{n+2}^N \\ \partial_{n+2}^M \downarrow & \nearrow h_{n-1} & \downarrow \partial_{n+2}^N \\ P_{n+1}^M & \xrightarrow{f_{n+1}^1, f_{n+1}^2} & P_{n+1}^N \\ \partial_{n+1}^M \downarrow & \nearrow h_n & \downarrow \partial_{n+1}^N \\ P_n^M & \xrightarrow{f} & P_n^N \end{array}$$

consider the expressions

$$g_{n+1} := f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

we can check (by diagram chasing), $\partial_{n+1}^N \circ g = 0$. This implies that

$$\text{im}(g_{n+1}) \subset \text{im}(\partial_{n+2}^N)$$

we can construct h_{n+1} to get the map

$$\delta_{n+2}^N \circ h_{n+1} = g_{n+1} = f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

i.e.

$$\partial_{n+2}^N \circ h_{n+1} + h_n \circ \partial_{n+1}^M = f_{n+1}^1 - f_{n+1}^2$$

hence we are done! □

Corollary 2.1. Any two free resolutions of an R -module M are homotopy equivalent: given two free resolutions P_\bullet^M, Q_\bullet^M , there exists extension of $\text{id} : M \rightarrow M$ and such that

$$f_\bullet : P_\bullet^M \rightarrow Q_\bullet^M, g_\bullet : Q_\bullet^M \rightarrow P_\bullet^M$$

such that

$$g_\bullet \circ f_\bullet = \text{id}, f_\bullet \circ g_\bullet = \text{id}$$

Problem 2.10 (HW(2.1)). Prove this corollary.

Next we define Tor functors (pretty hard things).

Definition 2.12 (tor functors). Let N be an R -module, recall the functor

$$- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$$

we define a collection of functors

$$\text{Tor}_R^i(-, N) : \text{Mod}_R \rightarrow \text{Mod}_R, i \in \mathbb{N}$$

given an object M in Mod_R , let $\{P_\bullet^M, \partial_\bullet^M, \epsilon_M\}$ be a free resolution of M , define $\text{Tor}^i(M, N)$ to be

$$\text{Tor}^i(M, N) = H_i(P_\bullet^M \otimes_R N, \partial_\bullet^M \otimes \text{id}_N)$$

where

$$\dots \rightarrow P_i^M \otimes N \xrightarrow{\partial_i \otimes \text{id}} P_{i-1}^M \otimes N \rightarrow \dots$$

We make the remark that there is a choice involved in picking the free resolution, but this is unique since homotopies are the same on homologies.

Problem 2.11 (HW(2.2)). Show that $\text{Tor}_R^i(M, N)$ is a well-defined functor for all i , and any other choice of free resolution of all objects yields an isomorphic functor. Hint: use the above corollary.

Problem 2.12 (HW(2.3)). Show that

1. $\text{Tor}_R^i(R, N) = 0$ for all $i > 0$
2. $\text{Tor}_R^i(M \oplus M', N) \cong \text{Tor}_R^i(M, N) \oplus \text{Tor}_R^i(M', N)$, given by the natural isomorphism.

We claim that $\epsilon_M : P_0^M \rightarrow M$ induces the following isomorphism

$$\text{Tor}_R^0(M, N) \cong M \otimes_R N$$

and $\text{Tor}_R^i(M, N)$'s are called the highest derived functors of $(-) \otimes_R N$.