Calc III Section Notes with Answers

Fall 2025

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Calc III-Week 1 (8/25-29)

Logistics

- TA: Hui.
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- Office Hours: Tuesday 4-6 PM, Krieger 211; Friday 1-2 PM Zoom.
- Biweekly Quizzes: 15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Icebreaking Activity

- In a group of three or four:
  1. Learn each other names, year, pronouns.
  2. Find something in common and different among you and share with the entire class.
  3. Play Buzz if you have time, with prime 7: say the number if it doens't contain or is not divisible by 7, say buzz otherwise.

Problem 1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 3. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned} \|u+v\|^2 &= (u+v)\cdot(u+v) \\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u\cdot v$$

given the assumption, we also have

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u\cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u\cdot v = 0$$

$\square$

Reminders

1. First HW due this Friday.
2. First Quiz next Tuesday.

Calc III-Week 2 (9/1-5)

Topics: (1) cross product, (2) plane in $\mathbb{R}^3$.

Definition 1 (cross product). Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$, the cross product of a, b is the vector $a \times b$,

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 2 (Plane in three dimensions). A perpendicular vector and a normal vector uniquely define a plane in $\mathbb{R}^3$: given the plane $\mathcal{P}$ passing containing the point (x_0, y_0, z_0) that has a normal vector (A, B, C) is given by the equation:

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1. Here are some properties of the cross product:

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:

$$\|a \times b\| = \|a\|\|b\| \sin\theta$$

   where θ is the angle between them. (Compare this with the dot product).
3. $a \times b = -b \times a$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. (HW) The cross product is **not** associative! For example, compute

$$(i \times i) \times j, \quad i \times (i \times j)$$

Problem 4. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 5. Compute the plane containg all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

□

Reminders HW is due Sunday 11:59PM.

Calc III-Week 3 (9/8-9/12)

Topics: (1) graphing multivariable functions, (2) limits and continuity.

Definition 3 (graph). The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 4 (limit). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, ($A$ is open), let N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0 ($x_0 \in \bar{A}$,) f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that whenever $x \in U$, then $f(x) \in N$ as well.

The **limit** of f as $x \to x_0$, if it exists, is $\lim_{x\to x_0} f(x) := b \in \mathbb{R}^m$ such that f is eventually in N, for every neighborhood N of b.

Definition 5 (limit'). If the limit exists, then $\lim_{x\to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x'_1, x'_2, \ldots, x'_n)$ from **all directions**, and $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 6 (continuity). Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x\to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 6. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x, y) = x^2 + y^2$.

Proof.

1. $\text{Image}(f) = \{x^2 + 1 : x \in \mathbb{R}\}$, and $\text{Graph}(f) = \{(x, x^2 + 1) : x \in \mathbb{R}\}$.
2. $\text{Image}(g) = \{x^2 + y^2, (x, y) \in \mathbb{R}^2\}$, and $\text{Graph}(g) = \{(x, y, x^2 + y^2) : (x, y) \in \mathbb{R}^2\}$.

$\square$

Problem 7. Compute the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$$
   (Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$).
2. $$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}$$

Proof. 1. Following the hint, we see

$$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = \lim_{x\to 0} x = 0$$

2. This one uses the exact same trick:

$$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} \cdot y = 0$$

3. First letting $x \to 0$ along $y = 0$, we see the limit is 1; letting $x = y \to 0$, we see the limit is 0, thus the limit doesn't exist!

□

Problem 8. Compute the limit of the following functions:

1. $$\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$$
2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$
   (Hint: try considering $y = x^2 - x$ and $y = x$)
3. $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$$

Proof. 1. First fix $x = 0$, let $y \to 0$, then the limit is 0; now fix $y = 0$, let $x \to 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \to 0, y \to 0$), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3-x^2}{x^2} = \lim_{x\to 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

Warning 0.1. 2 does not follow from 1! A student suggests a proof: $\lim_{(x,y)\to(0,0)} = \frac{x}{x+y} \cdot y$, and by 1, the limit $\frac{x}{x+y}$ doesn't exist, this implies the limit of $\frac{xy}{x+y}$ also doesn't exist. This argument is not correct! Consider the following counterexample: $\lim_{y\to 0} \frac{1}{y}$ doesn't exist, but the limit

$$\lim_{y\to 0} \frac{1}{y} \cdot y = 1$$

exists! More concretely, if you multiply by any function that doesn't tend to 0, the argument follows, but it doesn't work when the function tends to 0! (Sorry I wasn't able to give a concrete counterexample in class other than saying this gives "bad and untrue vibes"). Thank you (the student) who brought it up, your attempt still remains very very good.

3. We see that
$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} \frac{xy}{x+y}$$
Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist!

$\square$

How to find a a limit $\lim_{x\to x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.

Calc III-Week 4 (9/15-9/19)

Topics: (1) Partial derivatives. (2) Definition of total derivatives.

Problem 9. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the following functions:

1. $$x^3y^4 - xy^2$$
2. $$x^2\sin(2y) + 3$$
3. $$\ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{2}\right)$$

   You may use the following identities to simply the equation first:

   $$\ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln\left(\frac{1}{a}\right) = -\ln a$$

Proof. We have

1. $$\partial x : 3x^2y^4 - y^2, \quad \partial y : 4x^3y^3 - 2xy$$
2. $$\partial x : 2x\sin(2y), \quad \partial y : 2x^2\cos(2y)$$
3. $$\partial x : -\frac{2}{x} - \frac{1}{x+y}, \quad \partial y : \frac{1}{y} - \frac{1}{x+y}$$

□

Definition 7 (tangent plane). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0), then the **tangent plane** to the graph f in $\mathbb{R}^3$ is the given by

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)$$

Problem 10. Compute the plane tangent to the graph of $f(x, y) = x^2y + 2xy - y^2$ at $(1, 2)$.

Proof. We have

$$\frac{\partial f}{\partial x}(1,2) = 2xy + 2y|_{(1,2)} = 8, \quad \frac{\partial f}{\partial y}(1,2) = x^2 + 2x - 2y|_{(1,2)} = -1$$

and $f(1, 2) = 2$, thus the plane is given by

$$z = 2 + 8(x - 1) - (y - 2)$$

i.e., $z = 8x - y - 4$. □

Definition 8 (derivative for two variables)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$, then f is said to be **differentiable** at (x_0, y_0) if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{\|(x,y) - (x_0,y_0)\|} = 0$$

where $\mathcal{P}(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$ is the tangent plane to f at (x_0, y_0).

Definition 9 (derivative for n variables)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **gradient** of f, denoted as ∇f is given by

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right]$$

is a $1 \times n$ matrix. And f is said to be **differentiable** at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)\|}{\|x - x_0\|} = 0$$

and the derivative of f is exactly the gradient ∇f at x_0.

Definition 10 (derivative for m outputs)**.** Let $f : \mathbb{R} \to \mathbb{R}^m$, where $f(x) = (f_1(x), \ldots, f_m(x))$, then let T denote the $n \times 1$ matrix

$$T = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \cdots \\ \frac{df_m}{dx}(x_0) \end{bmatrix}$$

Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - T(x - x_0)|}{|x - x_0|} = 0$$

and the matrix T is the derivative at x_0.

Example 2. Let $f(x) = (x^2, 2x, -x)$, then

$$T = Df(1) = \begin{bmatrix} 2x \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Definition 11 (derivative for general functions)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$, let T be the $m \times n$ matrix with entries $\partial f_i / \partial x_j$ evaluated at $x_0 \in \mathbb{R}^n$. Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

then f is differentiable at x_0, and the matrix T is the derivative at x_0. Note that T loosk like

$$T = Df(x_0), \quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 3. Let $f(x,y,z) = (ze^x, -ye^z)$, then

$$Df(x,y,z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$

Problem 11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x,y,z) = x^2y + y\sin(z) + ze^x.$$

Compute the gradient of f at $(1,2,0)$.

Proof. You can compute the partial derivatives

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2xy + ze^x, \\ \frac{\partial f}{\partial y} &= x^2 + \sin(z), \\ \frac{\partial f}{\partial z} &= y\cos(z) + e^x. \end{aligned}$$

thus the gradient is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xy + ze^x, x^2 + \sin z, y\cos z + e^x)$$

Hence

$$\nabla f(1,2,0) = (4, 1, 2 + e)$$

□