Functional Analysis

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Chapter 1

Preliminary

1.1 9/3 lecture

Definition 1.1 (orthonormal basis). Let S be an orthonormal set in the Hilbert space such that no other orthonormal set contains S as a proper subset. Then S is called an orthonormal basis.

Proposition 1.1. Every Hilbert space admits an orthonormal basis.

Proof. Zorn's lemma. □

Remark: if H is separable, i.e., H has a countable dense subset, then the proof does not require Zorn's lemma. For example, L^2 is separable.

Proposition 1.2 (II.6, Parsevel's formula). Let $\mathcal{H}$ be a Hilbert space, and $S = \{x_n\}$ be an orthonormal basis, then for each $y \in \mathcal{H}$,

$$y = \sum_{\alpha \in \mathcal{A}} (x_\alpha, y) x_\alpha, \quad \|y\|^2 = \sum |(x_n, y)|^2$$

where $\mathcal{A}$ is an index set.

Proof. Bessel's inequality states that for any $\mathcal{A}' \subset \mathcal{A}$ finite, we have

$$\sum_{\alpha \in \mathcal{A}'} |(x_\alpha, y)|^2 \leq \|y\|^2 < \infty$$

It follows that $|(x_\alpha, y)| > \frac{1}{n}$ for at most finitely many α's, and $|(x_\alpha, y)| \neq 0$ for at most countably many α's. Let $\{\alpha_i\}_{i=1}^\infty$ be an enumeration of such α's. Then

$$\sum_{i=1}^N |(x_{\alpha_i}, y)|^2 \leq \|y\|^2 < \infty$$

which implies

$$\sum_{i=1}^\infty |(x_{\alpha_i}, y)|^2 < \infty$$

Let

$$y_n = \sum_{i=1}^n (x_{\alpha_i}, y) x_{\alpha_i},$$

we would like to show that the sequence $\{y_n\}$ is cauchy,

$$\|y_n - y_m\|^2 = \left\| \sum_{i=m+1}^{n} (x_{\alpha_i}, y) x_{\alpha_i} \right\|^2 \to 0 \text{ as } m \to \infty$$

Thus $\{y_n\}$ is Cauchy. In other words,

$$y_n \to y = \sum_{i=1}^{\infty} (x_{\alpha_i}, y) x_{\alpha_i}$$

□

Definition 1.2. A metric space is separable if it has a countable dense subset.

Proposition 1.3 (II.7). Let $\mathcal{H}$ be a Hilbert space, then it is separable iff it has a countable orthonormal basis.

Proof. Suppose $\mathcal{H}$ is separable, let $\{x_n\}$ be a countable dense set, then we throw out terms in $\{x_n\}$ until we get a linearly indepedent dense subset $\{u_n\} \subset \{x_n\}$. Applying Gram-Schmidt, we can assume $\{u_n\}$ to be countable and orthonormal. Conversely, if $\{u_n\}$ is a countable orthonormal basis, then the set of linear combinations of $\{u_n\}$ with rational coefficients forms a countable dense subset of $\mathcal{H}$. □

Definition 1.3 (Fourier Coefficient). The nth Fourier coefficient of a 2π-periodic function f is

$$c_n = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{-inx} f(x) dx$$

The Fourier series of f is

$$\tilde{f}(x) = \lim_{M \to \infty} \sum_{M=-N}^{N} \frac{1}{\sqrt{2\pi}} c_n e^{inx}$$

Proposition 1.4. The Fourier series $\sum_k c_k$ converges if $f \in L^2$. Moreover, the series converges uniformly to a continuous function if $\sum |c_k| < \infty$

I am too lazzy to type it up, but it uses the fun lemma below:

Lemma 1.1. Suppose f is 2π-periodic, and $(f, e^{inx}) = 0$ for all n, then $f \equiv 0$. (In other words, if all the Fourier coefficients are 0, then the function must be identically zero).

1.2 9/8 Lecture

Definition 1.4 (Banach space). A complete normed linear space is called a Banach space.

Example 1.1.
1. $L^\infty(\mathbb{R}) = \{f : f(x) \leq M \text{ a.e. }\}$, where $\|f\|_\infty$ is the smallest such M, is a Banach space.
2. Let $C(\mathbb{R})$ be the bounded continuous functions on $\mathbb{R}$. Let $C(\mathbb{R}) \subset L^\infty(\mathbb{R})$ and equip it with the same norm. Moreover, $C(\mathbb{R})$ is also a Banach space (due to the uniform convergence of continuous functions is still continuous).
3. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support, and this is not a Banach space under $\|\cdot\|_\infty$.
4. L^p is complete for all $1 \leq p < \infty$.
5. Let $a = \{a_n\}$ be a sequence of complex numbers, ad
$$\|a\| = \sup_n |a_n| < \infty$$
let $c_0 = \{\lim_{n\to\infty} a_n = 0\}$, $s = \{\lim_{n\to\infty} n^N a_n = 0 \forall N\}$, and $l_p = \{\|a\|_p^p = \sum_{n=1}^\infty |a_n|^p < \infty\}$. Note that the space
$$f = \{a_n = 0 \text{ for al but finitely many } n\}$$
is not complete! However, it is a dense subset in l^p. Morever, the set of elements in f with rational coefficients, and the closure of f in s, l^p, c_0 are exactly the whole spaces, i.e., s, l^p, c_0 are separable.
6. Let $L(X, Y)$ be bounded linear operators from X, Y, with the operator norm, and $L(X, Y)$ is a Banach psace.

Proposition 1.5. Let $L^p(\mathbb{R})$, where $1 \leq p < \infty$ be the space of functions with the norm
$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^p$$
then

1. (Minkowski's inequality) $\|f\|_p \leq \|f\|_p + \|g\|_p$.
2. (Riesz-Fischer) L^p is complete.
3. (Holder) Given $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have
$$\|fg\|_r \leq \|f\|_p \|g\|_q$$
if $f \in L^p, g \in L^q$.

Proposition 1.6. If Y is complete, then $L(X, Y)$ is a Banach space.

Proof. Suppose $\{A_n\}$ is Cauchy, now we construct the limit: for each x, $A_n x = y_n$ is a Cauchy sequence:
$$\|y_n - y_m\| \leq \|A_n - Am\| \cdot \|x\|$$
Now since Y is complete, we know that $A_n x \to y$. Let $Ax = y$. (This is our limit)! Now $\|A_n\| \leq C$ for all n, which implies $\|A\| \leq C$. Thus $L(X, Y)$ is complete! □

1.2.1 Duals

Definition 1.5 (dual space)**.** The space of bounded linear functionals $L(X, \mathbb{C})$, where X is Banach, is called the dual space to X, denoted by X^*. Let $f \in X^*$, then define the norm

$$\|f\| = \sup_{x \in X, \|x\| \leq 1} |f(x)|$$

Example 1.2.

1. Suppose that $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and let $g \in L^q$, then

$$G(f) = \int_{-\infty}^{\infty} \bar{g}(x) f(x) dx$$

   Then G is in $(L^p)^*$. Moreover, any such linear functional on L^p can be written in this way for some $g \in L^q$. And

$$|G(f)| \leq \|f\|_p \|g\|_q$$

   by Holder. Moreover,

$$L^q(\mathbb{R})^* = L^p, (L^q(\mathbb{R})^*)^* = L^q$$

   because L^q is reflexive! In particular, L^2 is its own dual space.

2. Suppose $\{\lambda_k\} \subset l^q$, then

$$\Lambda(\{a_k\}) = \sum_k \lambda_k a_k$$

   is a bounded linear functional on l^p. Thus

$$l_q \subset (l^p)^*$$

for $1 \leq p \leq \infty$. It turns out every linear functional on l^p can be written in this form.

Example 1.3. Let $p = 1$, we have

$$L^1(\mathbb{R})^* = L^\infty, \text{ but } L^\infty(\mathbb{R})^* \neq L^1(\mathbb{R})$$

in fact $L^\infty(\mathbb{R})^*$ is bigger.