Calc III Section Notes with Answers

Spring 2026

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Contents

Chapter 1

The Geometry of Euclidean Spaces

Week 1 (1/19-23)

Logistics

- TA: Hui.
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- Office Hours: Tuesday 2-3 PM, 4-5 pm, Krieger 211.
- Biweekly Quizzes: 10%.
- Attendance: 5%. (If you can't make it, email me).

Definition 1.1 (standard basis of $\mathbb{R}^3$)**.** The following vectors

$$i = (1, 0, 0), j = (0, 1, 0), k = (0, 0, 1)$$

are called the **standard basis** vectors of $\mathbb{R}^3$, and for any vector $a = (a_1, a_2, a_3) \in \mathbb{R}^3$, we can write

$$a = a_1 i + a_2 j + a_3 k$$

Definition 1.2 (dot product)**.** Let $v = (v_1, v_2, v_3), w = (w_1, w_2, w_3) \in \mathbb{R}^3$, the **dot product** $v \cdot w$ is given by

$$v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Alternatively,

$$v \cdot w = \|v\| \|w\| \cos \theta$$

where

$$\theta = \arccos\left(\frac{v \cdot w}{\|v\| \|w\|}\right)$$

Definition 1.3 (length of vector)**.** Let $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, the **length** or **norm** of v, denoted as $\|v\|$, is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{v \dot{v}}$$

Definition 1.4 (linear combination)**.** Let $v, w \in \mathbb{R}^3$, a **linear combination** of v, w is a sum

$$av + bw$$

for some $a, b \in \mathbb{R}$. One can generalize this definition to n vectors: let $v_1, v_2 \ldots, v_n \in \mathbb{R}^3$, a linear combination of these vectors is a finite sum

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

for some $a_i \in \mathbb{R}, 1 \leq i \leq n$.

Proposition 1.1 (properties of the dot product)**.** Let $a, b, c \in \mathbb{R}^n$, then

(a) Nonnegativity: $a \cdot a \geq 0$, and $a \cdot a = 0$ if and only if $a = 0$.

(b) Scalar multiplication: let $\lambda \in \mathbb{R}$, then

$$\lambda(a \cdot b) = \lambda a \cdot b = a \cdot \lambda b$$

(c) Distributivity:

$$a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c$$

(d) Symmetry: $a \cdot b = b \cdot a$.

Problem 1.1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 1.2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

□

Problem 1.3. Can you express $w = (1, 2)$ as a linear combination of v_1, v_2 for difference choices of v_1, v_2?

1. $v_1 = (1, 1), v_2 = (-2, -2)$.
2. $v_1 = (2, 1), v_2 = (-1, 0)$.

Proof. 1. We first note that $(1, 1), (-2, -2)$ lie on the same line through the origin. Hence, any linear combination of v_1, v_2 will stay in this line, i.e., of the form (a, a), for some $a \in \mathbb{R}$. Therefore, it is impossible to write $w = (1, 2)$ as a linear combination of v_1, v_2.

2. Suppose $w = a_1 v_1 + a_2 v_2$ for some $a_1, a_2 \in \mathbb{R}$, then

$$\begin{cases} 2a_1 - a_2 = 1 \\ a_1 = 2 \end{cases} \Rightarrow \begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

Thus we can write w as a linear combination of v_1, v_2:

$$w = 2v_1 + 3v_2$$

$\square$

Problem 1.4. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u + v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned} \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

given the assumption, we also have

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u \cdot v = 0$$

$\square$

Week 2 (1/26-30)

Definition 1.5 (determinant)**.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix, the **determinant** of A is given by

$$\det(A) = ad - bc$$

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ be a 3×3 matrix, the **determinant** of A is given by

$$\det(A) = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

Definition 1.6 (cross product)**.** Let $a, b \in \mathbb{R}^3$, write $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$, then the **cross product**

$$a \times b = \det \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Proposition 1.2 (properties of the cross product)**.** We have the following properties regarding the cross product: let $a, b \in \mathbb{R}^3$,

1. $a \times a = 0$.
2. $a \times b = -b \times a$.
3. $(a + b) \times c = a \times c + b \times c$, and $a \times (b + c) = a \times b + a \times c$.
4. $(\alpha a) \times b = \alpha(a \times b)$ for any $a \in \mathbb{R}$.
5. $a \times b$ is perpendicular to vectors a, b.
6. The length of the cross product is the area of the parallelogram spanned by a, b:
$$\|a \times b\| = \|a\|\|b\| \sin\theta$$
where $0 \leq \theta \leq \pi$ is the angle between them.
7. $a \times b = 0$ iff a, b are parallel or either a or b are 0.
8. The cross product is **not** associative! For example, compute
$$(i \times i) \times j, \quad i \times (i \times j)$$

Proposition 1.3 (determinant and linear combination)**.** Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ be a 3×3 matrix, let

$$a = (a_1, a_2, a_3), b = (b_1, b_2, b_3), c = (c_1, c_2, c_3)$$

If any of a, b, or c is a linear combination of the other two vectors, then $\det(A) = 0$. (Relevant topic: linear independence).

Problem 1.5. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 1.6. Compute the determinant of the following matrix A:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

Proof. Notice that the third row vector $(3, 1, 1)$ is the sum of the two row vectors above, hence by Proposition 1.3, we know we must have $\det(A) = 0$. □