

# Algebra Qualifying Exam

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# Contents

<b>1</b>	<b>Group Theory</b>	<b>3</b>
1.1	Sylow Theorems . . . . .	3
1.2	Class Formula, Classification of $p$ -groups . . . . .	11
1.3	Random Problems . . . . .	13
<b>2</b>	<b>Representation Theory</b>	<b>18</b>
2.1	Characters . . . . .	18
2.2	Induced representations . . . . .	20
2.3	Frobenius Reciprocity . . . . .	20

# Chapter 1

## Group Theory

S2013-Q2, S2016-Q1, F2018-Q2, F2001-Q1, F2013-Q  
S2005-Q1, F2009-Q1

### 1.1 Sylow Theorems

We first talk about semidirect products. Let  $G$  be any group, and  $N, H$  be subgroups of  $G$ .

**Definition 1.1.** For  $\varphi : H \rightarrow \text{Aut}(N)$ , define  $N \rtimes H$  by

- (1)  $N \rtimes_\varphi H = N \times H$  as a set.
- (b) Equipped with the group structure

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \varphi(h_1)(n_2), h_1 h_2)$$

The structure  $(N \rtimes_\varphi H, \cdot)$  forms a group.

**Example 1.1.** If  $N$  is a normal subgroup of  $G$ , and  $N \cap H = \{e\}$ , and  $\varphi : H \rightarrow \text{Aut}(N)$  where

$$\varphi : h \mapsto (n \mapsto hnh^{-1})$$

(acting by conjugation), and  $G = NH$ . Then

$$N \rtimes_\varphi H \rightarrow G$$

where

$$(n, h) \mapsto nh$$

is a bijective homomorphism. Hence

$$G \cong N \rtimes_\varphi H$$

Next we present some divisibility results.

**Proposition 1.1 (Lagrange, Orbit-Stabilizer).** We have the following divisibility results:

- Let  $H$  be a subgroup of  $G$ , let  $[G : H]$  denote the number of cosets of  $H$  in  $G$ , then

$$|G| = |H|[G : H]$$

- Let  $G$  be a finite group acting transitively on a finite set  $A$ , then for any  $a \in A$ , we have

$$|\text{Stab}_G(a)| \cdot |O_G(a)| = |G|$$

The class formula is when  $G$  acts on itself by conjugation:

**Proposition 1.2 (class formula).** Let  $G$  act on a finite set  $S$ , and let  $Z$  denote fixed points of this action, then

$$|S| = |Z| + \sum_{a \in A} |O_G(a)|$$

where  $A$  includes exactly one element from each nontrivial orbit.

If  $G$  acts on itself by conjugation, then

$$|G| = |Z(G)| + \sum_g |[g]| = |Z(G)| + \sum_g \frac{|G|}{|C_G(g)|}$$

where  $[g]$  denote the conjugacy class of  $g$ , and the sum includes exactly one from each nontrivial conjugacy class in  $G$ .

**Problem 1.1 (F2019-Q2).** 2. Let  $p, q$  be two prime numbers such that  $p \mid q - 1$ . Prove that

- there exists an integer  $r \not\equiv 1 \pmod{q}$  such that  $r^p \equiv 1 \pmod{q}$ ;
- there exists (up to an isomorphism) only one noncommutative group of order  $pq$ .

*Proof.* (a) We want to show that there exists an element  $r \in (\mathbb{Z}/q\mathbb{Z})^\times$  such that

$$r^p \equiv 1 \pmod{q}$$

We can do this because  $(\mathbb{Z}/q\mathbb{Z})^\times$  has order  $(q - 1)$  and  $p \mid (q - 1)$ . Therefore by Cauchy's theorem, there exists an element of order  $p$  in  $(\mathbb{Z}/q\mathbb{Z})^\times$ .

- Let  $n_p, n_q$  denote the number of  $p, q$ -Sylow subgroups. We see that  $n_q \mid p$  and  $n_q \equiv 1 \pmod{q}$ , since  $p < q$ , we must have  $n_q = 1$ . Now  $n_p = 1$  or  $q$  by the same reasoning. Suppose  $n_q = 1$ , let  $P, Q$  denote the normal subgroups of order  $p, q$ , then

$$G \cong P \times Q$$

by a standard argument (included in the lemma below). Then  $G$  is commutative. Since  $G$  is noncommutative, we have  $n_p = q$ . Choose any  $p$ -Sylow subgroup  $P$ , we know that

$$G \cong Q \rtimes_\theta P$$

where  $Q$  is the normal subgroup of order  $q$  and  $\theta : P \rightarrow \text{Aut}(Q) = (\mathbb{Z}/q\mathbb{Z})^\times$ . We know either  $\theta : 1 \mapsto 1$ , is the trivial map which produces a commutative group; or  $\theta : 1 \mapsto r$ , where  $r \in (\mathbb{Z}/q\mathbb{Z})^\times$  is some element of order  $p$ .

□

**Lemma 1.1.** Let  $p, q$  be two primes such that  $q \nmid (p - 1)$ , and  $N, H$  has order  $p, q$  respectively, suppose that  $N$  is normal in  $G$ , and  $N \cap H = \{e\}$ , then

$$G \cong N \times H$$

*Proof.* We consider the map

$$\psi : N \times H \rightarrow G$$

such that

$$(n, h) \mapsto nh$$

We want to show that  $\psi$  is a homomorphism and  $\psi$  is injective (hence bijective by size argument). It is clearly injective:

$$nh = e \Rightarrow n, h \in N \cap H = \{e\}$$

It suffices to show that  $\psi$  is a homomorphism. We see that this implies

$$n_1 n_2 h_1 h_2 = n_1 h_1 n_2 h_2$$

Therefore it suffices to for any  $n \in N, h \in H$ , one has

$$nh = hn$$

Consider the conjugation action

$$\varphi : H \rightarrow \text{Aut}(N)$$

where

$$h \mapsto (n \mapsto hnh^{-1})$$

Then we claim that  $\varphi$  is trivial. This is because  $\ker(\varphi)$  has size either 1 or  $q$ . If it has size  $q$ , then the map is trivial; if it has size 1, then  $H$  embeds in  $\text{Aut}(N)$ , however,  $|H| = q$ ,  $\text{Aut}(N) = p - 1$ , and  $q \nmid (p - 1)$ , hence impossible. This shows that the map is trivial, i.e., for  $n \in N, h \in H$ ,

$$hn = nh$$

as desired. □

**Problem 1.2 (F2015-Q1).** Prove every group of order 15 is cyclic.

*Proof.* We will show that any group  $G$  of order 15 is isomorphic to

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

For this, using the above lemma, it suffices to show that there is one normal subgroup of order 3 and one normal subgroup of order 5. We repeat the argument above,  $n_5 \mid 3$  and  $n_5 \equiv 1 \pmod{5}$ , hence  $n_5 = 1$ . Moreover,  $n_3 \mid 5$  and  $n_3 \equiv 1 \pmod{3}$ , hence  $n_3 = 1$  as well. By the lemma above, we know that

$$G \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \times \frac{\mathbb{Z}}{5\mathbb{Z}}$$

hence cyclic as desired. □

**Problem 1.3 (S2013-Q2).** Let  $p$  and  $q$  be primes with  $p < q$ . Let  $G$  be a group of order  $pq$ . Prove the following statements:

- (a) If  $p$  does not divide  $q - 1$  (i.e.,  $p \nmid q - 1$ ), then  $G$  is cyclic.
- (b) If  $p$  divides  $q - 1$  (i.e.,  $p \mid q - 1$ ), then  $G$  is either cyclic or isomorphic to a non-abelian group on two generators. Give the presentation of this non-abelian group.

*Proof.* This question is exactly the same as F19-Q2, we will only outline here.

- (a) We have  $n_q = 1$ , and  $n_p \mid q$ , hence  $n_p = 1$  or  $q$ , moreover  $n_p \equiv 1 \pmod{p}$ . If  $n_p = q$ , this implies that  $p \mid (q - 1)$ , hence  $n_p = 1$ . Therefore by the above argument

$$G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$$

- (b) If  $p \mid (q - 1)$ , then  $n_p = 1$  or  $q$ . Hence  $G$  is either of the form above or isomorphic to the non-abelian group

$$G = Q \rtimes_{\theta} P$$

We know from F2019-Q2, the trivial  $\theta$  defines the abelian, hence cyclic group  $G = \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{q\mathbb{Z}}$ . And  $\theta : 1 \mapsto r$ , for some  $r \in (\mathbb{Z}/q\mathbb{Z})^{\times}$  of order  $p$  defines a non-abelian group.

not finished, what are the two generators

□

**Problem 1.4 (F2007-Q1).** Prove that no group of order 148 is simple.

*Proof.* We note the prime factorization of 148 is

$$148 = 2^2 \cdot 37$$

We see that  $n_{37} \mid 4$  and  $n_{37} \equiv 1 \pmod{37}$ , therefore  $n_{37} = 1$ . This shows that there exists a normal subgroup of order 37, i.e., the group is not simple. □

**Problem 1.5 (F2017-Q1).** Show that there is no simple group of order 30.

*Proof.* This is slightly more complicated, and we will use a counting argument. Same reasoning as the above. The prime factorization of 30 is as below:

$$30 = 2 \cdot 3 \cdot 5$$

We see  $n_5 \mid 6$ , and  $n_5 \equiv 1 \pmod{5}$ . Unfortunately,  $n_5$  could either be 1 or 6. Now  $n_3 \mid 10$ , and  $n_3 \equiv 1 \pmod{3}$ , unfortunately again  $n_3$  could be 10. However, we argue that  $n_3 = 10$  and  $n_5 = 6$  cannot happen at the same time. Suppose this is the case, then there are 20 elements of order 2 and 24 elements of order 5, but this is too many! Hence either  $n_3 = 1$  or  $n_5 = 1$ , as desired. □

**Problem 1.6 (F2011-Q1).**

- (a) Let  $G$  be a group of order 5046. Show that  $G$  cannot be a simple group. You may not appeal to the classification of finite simple groups.
- (b) Let  $p$  and  $q$  be prime numbers. Show that any group of order  $p^2q$  is solvable.

*Proof.* The proof is very similar like above.

- (a) The prime factorization of 5049 is as follows:

$$5049 = 2 \cdot 3 \cdot 29^2$$

Hence we see  $n_{29} = 1$ , i.e., there is a normal subgroup of order 29, therefore not simple.

- (b) We will do discussion by cases.

- (1)  $p > q$ . Then  $n_p = 1$  or  $q$  and  $n_p \equiv 1 \pmod{p}$ , therefore  $n_p = 1$ . Let  $P$  be the normal subgroup of  $G$  of order  $p^2$ , we thus have

$$\{e\} \subset P \subset G$$

It is clear that  $|G/P| = q$ , thus abelian, and  $|P| = p^2$  also abelian as well (by the lemma below). This shows that  $G$  is solvable.

- (2)  $p < q$ . Then  $n_p = 1$  or  $q$ , and  $n_q = 1$  or  $p^2$ . Suppose that  $n_q = 1$ , let  $Q$  denote the normal subgroup of order  $q$ , then

$$\{e\} \subset Q \subset G$$

It is clear that  $Q$  and  $G/Q$  are both abelian. Suppose that  $n_q = p^2$  instead, then there are only  $p^2q - p^2(q - 1) = p^2$  elements of order  $\neq q$ . Since any  $p$ -Sylow subgroup has  $p^2$  elements with order  $\neq q$ , we must have  $n_p = 1$ . Hence we are in case (1) again. This shows that  $G$  is solvable in either case  $n_q = 1, p^2$ .

□

**Lemma 1.2 ( $p^2$  abelian).** Fix prime  $p$ , any group of order  $p^2$  is abelian.

*Proof.* For any nontrivial  $p$  group, by the class formula, the center  $Z(G)$  is nontrivial, thus the center has order either  $p$  or  $p^2$ . If it has order  $p^2$ , then the group is abelian. If it has order  $p$ , then

$$|G/Z(G)| = p$$

is also cyclic, therefore  $G$  is abelian (strictly speaking is a contradiction that  $|Z(G)| = p$ ). In either case, we see that  $G$  is abelian. □

**Problem 1.7.** Any  $p$ -group is solvable, for any prime  $p$ .

*Proof.* Suppose  $|G| = p^r$  for some  $r \geq 0$ , we will use induction on  $r$ . If  $r = 0$ , then the trivial group is trivially solvable.

- Base case: if  $r = 1$ ,  $|G| = p$ , then  $G$  is cyclic, hence solvable.

- Induction step: suppose that  $G$  is solvable for all  $|G| = p^k$ , where  $0 \leq k \leq r-1$ . Now we want to show that  $G$  of order  $p^r$  is solvable. We know  $G$  has a nontrivial center, suppose that  $|Z(G)| = p^k$ , where  $1 \leq k \leq r$ , then

$$|G/Z(G)| = p^{r-k}, 0 \leq r-k \leq r-1$$

We know any group  $G$  is solvable if and only if there exists a sequence of subgroups  $G_0, \dots, G_k$

$$\{e\} = G_0 \subset \dots \subset G_k = G$$

such that  $G_{i-1}$  is normal in  $G_i$  and  $G_i/G_{i-1}$  is solvable. Therefore we see when  $|G| = p^r$ ,

$$\{e\} \subset Z(G) \subset G$$

has  $Z(G)$  solvable, and  $G/Z(G)$  also solvable by the induction hypothesis, so we close the induction.  $\square$

**Problem 1.8 (S2016-Q1).** Classify all groups of order 66, up to isomorphism.

*Proof.* By  $66 = 2 \cdot 3 \cdot 11$ , we know  $n_{11} = 1$ . We claim that there is a normal subgroup isomorphic to  $\mathbb{Z}/33\mathbb{Z}$ .

1. First we show that there is a subgroup of order 33. Let  $P_{11}$  denote the normal subgroup of order 11 and let  $P_3$  denote a 3-Sylow subgroup of  $G$ . Then we claim that the following

$$H = \{gh : g \in P_{11}, h \in P_3\}$$

forms a subgroup and is isomorphic to  $\mathbb{Z}/33\mathbb{Z}$ . By the Lemma 1.1, we see that

$$H \cong \frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}} = \frac{\mathbb{Z}}{33\mathbb{Z}}$$

2. Now we show that it is normal. This follows from the following general lemma:

**Lemma 1.3.** Let  $p$  be the smallest prime factor of  $|G|$ , and let  $H$  be a subgroup with index  $p$ , then  $H$  is normal.

*Proof.* We will only prove in the case that  $H$  is a subgroup of index 2, i.e.,  $G = H \sqcup (G \setminus H)$ . We see for all  $g \in G$ ,

$$gH = Hg$$

since if  $g \in H$ , then the equality holds; if  $g \notin H$ , then  $gH = G \setminus H$ , so is  $Hg$ .  $\square$

Now since there is a subgroup of order 2, we can write  $G$  as a semidirect product

$$G = \frac{\mathbb{Z}}{33\mathbb{Z}} \rtimes_{\theta} \frac{\mathbb{Z}}{2\mathbb{Z}}$$

The number of nonisomorphic groups will depend on the choice of  $\theta$ . There are four different choices for  $\theta : H \rightarrow \text{Aut}(\frac{\mathbb{Z}}{11\mathbb{Z}} \times \frac{\mathbb{Z}}{3\mathbb{Z}}) = \frac{\mathbb{Z}}{10\mathbb{Z}} \times \frac{\mathbb{Z}}{2\mathbb{Z}}$

$$\begin{cases} \theta_1 : 1 \mapsto (0, 0) \\ \theta_2 : 1 \mapsto (0, 1) \\ \theta_3 : 1 \mapsto (5, 0) \\ \theta_4 : 1 \mapsto (5, 1) \end{cases}$$

what is happening  $\square$



**Problem 1.9 (S2007-Q2).** Prove that no group of order 224 is simple.

*Proof.* The prime factorization is

$$224 = 2^5 \cdot 7$$

If  $n_2 = 1$  or  $n_7 = 1$ , then we are done; assume that  $n_2 = 7$  instead, then we recall  $G$  has a nontrivial transitive action on the set of 2-Sylow subgroups, i.e., there is a homomorphism  $\varphi : G \rightarrow S_7$ . We know  $\ker(\varphi)$  is a normal subgroup of  $G$ . Since the action is nontrivial transitive, we know  $\ker(\varphi) \neq G$ . If  $\ker(\varphi) = \{e\}$ , then  $\varphi$  produces an embedding of  $G$  into  $S_7$ . However,  $|G| = 224 \nmid |S_7|$ . This shows that  $\ker(\varphi)$  is a nontrivial proper normal subgroup of  $G$ , concluding that  $G$  is not simple.  $\square$

**Problem 1.10 (F2008-Q1).** Show that no group of order 36 is simple.

*Proof.*

$$36 = 2^2 \cdot 3^2$$

We know  $n_2 \mid 9$ ,  $n_2 \equiv 1 \pmod{2}$ , and  $n_3 \mid 4$ ,  $n_3 \equiv 1 \pmod{3}$ . We know  $n_3 = 1$  or 4, suppose that  $n_3 = 4$ , then there is a nontrivial action of  $G$  on the set of 3-Sylow subgroups, i.e.,

$$\varphi : G \rightarrow S_4$$

Suppose that  $G$  is simple, we know  $\ker(\varphi) \neq G$  since the action is nontrivial, by assumption  $\ker(\varphi) = \{e\}$ , which implies that  $\varphi$  is an embedding, but  $|G| = 36 \nmid |S_4|$ , which is a contradiction. This implies that  $G$  is not simple.  $\square$

**Problem 1.11 (S2014-Q2).** All groups of order less than 60 are solvable, i.e., there exists a sequence of subgroups of  $G$ ,  $G_0, \dots, G_k$  such that  $G_i$  is normal in  $G_{i+1}$  and  $G_{i+1}/G_i$  is abelian, and

$$1 = G_0 \subset \dots \subset G_k = G$$

*Proof.* Groups of order  $p, pq, p^2, p^2q$  are solvable.

$$\{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, \\ 31, 33, 34, 35, 37, 38, 39, 41, 43, 44, 45, 46, 47, 49, 50, 51, 52, 53, 55, 57, 58, 59\}$$

And any  $p$ -group is also solvable.

$$\{8, 16, 27, 32\}$$

The remaining groups are

$$\{24, 36, 40, 42, 48, 54, 56\}$$

24: If  $n_2 = 1$  or  $n_3 = 1$ , then we are done. We see  $n_2 = 1$  or 3, consider the action  $\varphi : G \rightarrow S_3$ . We see  $\ker(\varphi)$  is a proper normal subgroup of  $G$ , this implies that

$$\{e\} \subset \ker(\varphi) \subset G$$

where  $|\ker(\varphi)|$  is a known solvable group, hence we are done.

36: Exactly same as above, we assume  $n_3 \neq 1$ , therefore  $n_3 = 4$ , the action  $\varphi : G \rightarrow S_4$  is not injective, hence  $\ker(\varphi)$  is again a proper normal subgroup of  $G$  that is solvable.

40: We see  $n_5 = 1$ , therefore

$$\{e\} \subset \mathbb{Z}/5\mathbb{Z} \subset G$$

42: We see  $n_7 = 1$ .

48: We see  $n_2 = 1$  or 3, the the action  $\varphi : G \rightarrow S_3$  is not injective, hence  $\ker(\varphi)$  is a proper normal subgroup of  $G$  that is solvable.

54: We see  $n_3 = 1$ .

56: We know  $n_7 = 1$  or 8 and  $n_2 = 1$  or 7. The group action argument does not work. We assume  $n_7 = 8$ , then there can be at most  $56 - 8(7 - 1) = 8$  elements of order  $\neq 7$ . This shows that  $n_2 = 1$ . Hence

$$\{e\} \subset P_2 \subset G$$

□

**Problem 1.12 (S2012-Q1).** Let  $G$  be a group of order  $p^3q^2$ , where  $p$  and  $q$  are prime integers. Show that for  $p$  sufficiently large and  $q$  fixed,  $G$  contains a normal subgroup other than  $\{1\}$  and  $G$ .

*Proof.* We want to show that there exists a normal group of size  $p^3$ , i.e.,  $n_p = 1$ . We know  $n_p \mid q^2$ ,  $n_p \equiv 1 \pmod p$ . Let  $p$  be large enough such that  $p > (q^2 - 1)$ , then this forces  $n_p = 1$ , as desired. □

**Problem 1.13 (F2014-Q4).**

- (a) Let  $G$  be a group of order  $p^2q^2$ , where  $p$  and  $q$  are distinct odd primes, with  $p > q$ . Show that  $G$  has a normal subgroup of order  $p^2$ .
- (b) Can a solvable group contain a non-solvable subgroup? Explain.

*Proof.* (a) We know  $n_p = 1$  or  $q$  or  $q^2$ , and  $n_p \equiv 1 \pmod p$ . Since  $p > q$ , we know  $n_p \neq q$ . It suffices to show that  $n_p \neq q^2$ : suppose that  $n_p = q^2$ , then

$$p \mid (q^2 - 1) = (q + 1)(q - 1)$$

Since  $p$  is prime,  $p \mid (q + 1)$  or  $p \mid (q - 1)$ . The latter is impossible since  $q < p$ .  $p \mid (q + 1)$  is also impossible because this implies that  $q = p + 1$ , which implies that  $q$  is even, a contradiction.

- (b) It is not possible. Suppose  $G$  is a solvable group, let  $H$  be a subgroup of  $G$ , then we know there exists sequence

$$\{e\} = G_0 \subset \cdots \subset G_k = G$$

such that  $G_i$  is normal in  $G_{i+1}$  and  $\frac{G_{i+1}}{G_i}$  is abelian. We define  $H_i = G_i \cap H$ , then we see  $H$  is solvable with sequence  $H_0 \subset \cdots \subset H_k$ . □

**Problem 1.14 (F2018-Q2).** Let  $G$  be a group of order 24. Assume that no Sylow subgroup of  $G$  is normal in  $G$ . Show that  $G$  is isomorphic to the symmetric group  $S_4$ .

*Proof.*

□

**Problem 1.15 (F2001-Q1).** Let  $G$  be a finite group and let  $N$  be a normal subgroup of  $G$  such that  $N$  and  $G/N$  have relatively prime orders.

1. Assume that there exists a subgroup  $H$  of  $G$  having the same order as  $G/N$ . Show that  $G = HN$ . (Here  $HN$  denotes the set  $\{xy \mid x \in H, y \in N\}$ .)
2. Show that  $\phi(N) = N$ , for all automorphisms  $\phi$  of  $G$ .

**Problem 1.16 (S2001-Q1).** Let  $G$  be a finite group and  $p$  the smallest prime number dividing the order  $|G|$  of  $G$ . Let  $H$  be a subgroup of  $G$  of index  $p$  in  $G$ . Show that  $H$  is necessarily a normal subgroup of  $G$ .

*Proof.*  $G$  has an action on  $G/H$  by left multiplication:  $\varphi : G \rightarrow \text{Aut}(G/H)$  such that

$$\varphi(g)(\bar{g}H) = g\bar{g}H$$

We will show that  $H = \ker(\varphi)$ . First we see that  $\ker(\varphi) \subset H$ :

$$\ker(\varphi) = \{g \in G : g\bar{g}H = \bar{g}H : \text{for all } \bar{g} \in G\}$$

letting  $\bar{g} \in H$  we see  $g \in \ker(\varphi)$  implies  $g \in H$ , i.e.,  $\ker(\varphi) \subset H$ .

Now we use a size argument to show  $|H| \leq |\ker(\varphi)|$ . We note that  $\text{im}(\varphi)$  is a subgroup of  $\text{Aut}(G/H) = S_p$ , thus

$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p!$$

because  $\frac{|G|}{|\ker(\varphi)|}$  also divides  $|G|$  and  $p$  is the smallest prime that divides  $p$ , we must have

$$\frac{|G|}{|\ker(\varphi)|} \text{ divides } p$$

Note that  $\frac{|G|}{|H|} = p$ , this gives

$$|H| \leq |\ker(\varphi)|$$

which shows  $H \subset \ker(\varphi)$ , hence  $H = \ker(\varphi)$ . □

(End of Page 5)

## 1.2 Class Formula, Classification of $p$ -groups

**Definition 1.2 (nilpotent group).** Let  $G$  be a group. Define inductively an increasing sequence  $\{e\} = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$  of subgroups of  $G$  as follows: for  $i \geq 1$ ,  $Z_i$  is the subgroup of  $G$  corresponding to the center of  $G/Z_{i-1}$ . One can show that  $Z_i$  is normal in  $G$ . A group is *nilpotent* if  $Z_m = G$  for some  $m$ .

**Example 1.2.**

- $p$ -groups are nilpotent.
- Nilpotent groups are solvable.

**Proposition 1.3.** We have the following classification of groups of order  $p, p^2, p^3$ , for prime  $p$ .

- $|G| = p$  implies  $G \cong \mathbb{Z}/p\mathbb{Z}$ .

- $|G| = p^2$  implies

$$G \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}} \quad \text{or} \quad G \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}}$$

- $|G| = p^3$  implies that

$$G \cong \frac{\mathbb{Z}}{p^3\mathbb{Z}} \quad \text{or} \quad G/Z(G) \cong \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p\mathbb{Z}} \quad \text{or} \quad [G, G] = Z(G)$$

**Problem 1.17 (S2010-Q1).** Let  $G$  be a non-abelian group of order  $p^3$ , where  $p$  is prime. Determine the number of distinct conjugacy classes in  $G$ .

*Proof.* We know  $G$  has a nontrivial center, and if  $|Z(G)| = p^2$  or  $p^3$ , then  $G$  is abelian, this shows that  $|Z(G)| = p$ , now let  $g \in G \setminus Z(G)$ , then

$$Z(G) \subsetneq Z_g(G) \subsetneq G$$

where  $Z(G) \subsetneq Z_g(G)$  because  $g \in Z_g(G)$ , and  $Z_g(G) \subsetneq G$  since  $g \notin Z(G)$ . This shows that  $Z_g(G)$  is a subgroup of order  $p^2$ , in other words, the size of the conjugacy class of any  $g \in G \setminus Z(G)$  is

$$|[g]| = \left| \frac{G}{Z_g(G)} \right| = p$$

By the class formula,

$$|G| = |Z(G)| + \sum_{a \in A} |[a]|$$

where  $A$  contains one  $a$  from each nontrivial conjugacy class  $[a]$ . Thus we have

$$p^3 = p + Np \Rightarrow N = p^2 - 1$$

Every element in  $Z(G)$  is its own conjugacy class, thus the total number of conjugacy classes is

$$p^2 + p - 1$$

□

**Problem 1.18 (F2013-Q1).** Let  $p > 2$  be a prime. Classify groups of order  $p^3$  up to isomorphism. The two nonabelian groups of order  $p^3$  (for  $p \neq 2$ ), up to isomorphism, are:

$$\text{Heis}(\mathbb{Z}/(p)) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| a, b, c \in \mathbb{Z}/(p) \right\}$$

and

$$\begin{aligned} G_p &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in \mathbb{Z}/(p^2), a \equiv 1 \pmod{p} \right\} \\ &= \left\{ \begin{pmatrix} 1+pm & b \\ 0 & 1 \end{pmatrix} \middle| m, b \in \mathbb{Z}/(p^2) \right\} \end{aligned}$$

**Problem 1.19** (F2014-Q5).

- (a) Prove that every group of order  $p^2$  (with  $p$  prime) is abelian. Then classify such groups up to isomorphism.
- (b) Give an example of a non-abelian group of order  $p^3$  for  $p = 3$ .  
*Suggestion: Represent the group as a group of matrices.*

*Proof.* (a) See Lemma 1.2. There are two abelian groups:  $\frac{\mathbb{Z}}{p^2\mathbb{Z}}, \frac{\mathbb{Z}}{p\mathbb{Z}} \times \frac{\mathbb{Z}}{p\mathbb{Z}}$ .

(b) See Problem 1.18. □

**Problem 1.20** (F2019-Q4, S2015-Q3). Find all irreducible representations of a finite  $p$ -group over a field of characteristic  $p$ .

*Proof.* Let  $G$  any finite  $p$ -group. Let  $V$  be an irreducible representation over  $\mathbb{F}_p$ , consider the  $[\mathbb{F}_p G]$ -module  $W$  generated by any  $v \in V \setminus \{0\}$ . We see  $W$  is a finite-dimensional vector space over  $\mathbb{F}_p$ , i.e.,

$$|W| = p^d$$

for some  $d \geq 1$ . We consider the action of  $G$  on  $W$ , all the orbits of this action either has size 1 or is a power of  $p$ , since  $G$  is a  $p$ -group, by the class formula, let  $N$  be the number of nontrivial orbits of size 1,

$$|W| \equiv 1 + N \pmod{p} \Rightarrow 1 + N \equiv 0 \pmod{p}$$

Hence there exists at least one nontrivial orbit  $\{v\}$  of size 1. We consider the vector space  $\bar{W}$  generated by  $v$  over  $\mathbb{F}_p$ : it is one-dimensional vector space contained in  $V$ , invariant under  $G$ , since  $V$  is irreducible, we must have  $V = \bar{W}$ . The action of  $G$  on  $\bar{W}$  is the trivial action, thus all irreducible representations of a finite  $p$ -group over  $\mathbb{F}_p$  are trivial. □

## 1.3 Random Problems

**Problem 1.21** (F2010-Q1). Let  $G$  be a group. Let  $H$  be a subset of  $G$  that is closed under group multiplication. Assume that  $g^2 \in H$  for all  $g \in G$ . Show that:

- $H$  is a normal subgroup of  $G$
- $G/H$  is abelian

*Proof.* • We first show that  $H$  is subgroup. It remains to show that if  $h \in H$ , then  $h^{-1} \in H$ , we know  $(h^{-1})^2 \in H$ , thus

$$h(h^{-1})^2 = h^{-1} \in H$$

as desired. Now we show that  $H$  is normal: for any  $h \in H, g \in G$ , we want to show  $ghg^{-1} \in H$ .

$$\begin{aligned} ghg^{-1} &= (gh)^2(gh)^{-1}hg^{-1} \\ &= (gh)^2h^{-1}g^{-1}hg^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2(g^{-1}h)^{-1}g^{-1} \\ &= (gh)^2h^{-1}(g^{-1}h)^2h^{-1} \in H \end{aligned}$$

as desired.

- It suffices to show that for any  $g_1, g_2 \in G$ , we have

$$g_1 g_2 H \subset g_2 g_1 H$$

Take any  $h \in H$ , we want to show  $(g_2 g_1)^{-1} g_1 g_2 h \in H$ ,

$$\begin{aligned} (g_2 g_1)^{-1} g_1 g_2 h &= (g_2 g_1)^{-2} g_2 g_1^2 g_2 h \\ &= (g_2 g_1)^{-2} (g_2 g_1^2)^2 (g_2 g_1^2)^{-1} g_2 h \\ &= (g_2 g_1)^{-2} (g_2 g_1^2)^2 g_1^{-2} h \in H \end{aligned}$$

as desired. □

**Problem 1.22 (S2014-Q1).** Find the number of colorings of the faces of a cube using 3 colors, where two colorings are considered equal if they can be transformed into each other by a rotation of the cube.

[Hint: Use Burnside's formula:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where a group  $G$  acts on a set  $X$ ,  $X/G$  is the set of orbits, and for every  $g \in G$ ,  $X^g$  is the fixed subset of  $g$  in  $X$ .]

*Proof.* Let  $X$  be the set of all possible colorings of the cube (equal cubes allowed), we have  $|X| = 3^6$ . We notice two things:

1. The group of rotations of a cube is  $S_4$ .
2. For  $\sigma_1, \sigma_2 \in S_4$  that are conjugates of each other,  $|X^{\sigma_1}| = |X^{\sigma_2}|$ . Therefore for the Burnside's formula becomes

$$|X/S_4| = \frac{1}{|S_4|} \sum_{[\sigma] \text{ conj classes}} |[ \sigma ]| \cdot |X^\sigma|$$

Now we analyze for each conjugacy class  $[\sigma]$ , what is  $|X^\sigma|$ .

- $(1 + 1 + 1 + 1)$ ,  $|[e]| = 1$  and  $|X^e| = 3^6$ .
- $(1 + 1 + 2)$ ,  $|[\sigma_1]| = 6$  and  $|X^{\sigma_1}| = 3^3$ .
- $(1 + 3)$ ,  $|[\sigma_2]| = 8$ , and  $|X^{\sigma_2}| = 3^2$ .
- $(2 + 2)$ ,  $|[\sigma_3]| = 6$ , and  $|X^{\sigma_3}| = 3^4$ .
- $(4)$ ,  $|[\sigma_4]| = 6$ , and  $|X^{\sigma_4}| = 3^3$ .

Thus combining we get

$$|X/S_4| = \frac{1}{24} (3^6 + 6 \cdot 3^3 + 8 \cdot 3^2 + 6 \cdot 3^4 + 6 \cdot 3^3) = 57$$

□

**Problem 1.23 (S2019-Q4).** Let  $f$  be a polynomial with  $n$  variables and define

$$\text{Sym}(f) = \{\sigma \in S_n \mid f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}) = f(x_1, x_2, \dots, x_n)\}.$$

1. Prove that  $\text{Sym}(f)$  is a subgroup of  $S_n$ .
2. Prove that the dihedral group  $D_4$  (the group of symmetries of the square) is isomorphic to  $\text{Sym}(x_1x_2 + x_3x_4)$ .

*Proof.* 1. The group  $S_n$  acts on the polynomial ring  $k[x_1, \dots, x_n]$ , by permuting the  $x_i$  to  $x_{\sigma(i)}$ , and we see that  $\text{Sym}(f)$  is the centralizer of a fixed element  $f \in k[x_1, \dots, x_n]$ , hence is a subgroup.

2. We have a total of 8 elements in  $\text{Sym}(x_1x_2 + x_3x_4)$ :

$$\{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$$

and we can be drawing a square that this corresponds to the group  $D_4$ . □

**Problem 1.24 (S2011-Q1, F2004-Q1).**

- (a) Let  $H$  be a proper nontrivial subgroup of a finite group  $G$  (i.e.,  $H \neq \{1\}$  and  $H \neq G$ ). Prove that  $G$  is not the union of all conjugates of  $H$  in  $G$ .
- (b) Give an example of an infinite group  $G$  for which the assertion in part (a) fails.

*Proof.* (a) If  $H$  is normal, then all conjugations of  $H$  is equal to  $H$ , but  $H \subsetneq G$ , this  $G$  is not the union of all conjugates of  $H$  in  $G$ . Now suppose  $H$  is not normal, assume the contrary that  $G$  is the union of all conjugates of  $H$ , then the number of distinct conjugates of  $H$  is  $[G : N_G(H)]$ , hence

$$|G| = [G : N_G(H)] \cdot |H| \iff [G : H] = [G : N_G(H)] \iff [N_G(H) : H] = 1$$

this is a contradiction since  $H$  is not normal. Thus  $G$  is not the union of all conjugates of  $H$  in  $G$ .

- (b) Consider

$$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \text{GL}_2(\mathbb{C})$$

It is clear that conjugation of matrices in  $B$  do not give matrices with nonzero left bottom entry. □

**Problem 1.25 (S2009-Q1).** Let  $H$  and  $K$  be two solvable subgroups of a group  $G$  such that  $G = HK$ .

1. Show that if either  $H$  or  $K$  is normal in  $G$ , then  $G$  is solvable.
2. Give an example where  $G$  may not be solvable without the assumption in (a).

*Proof.* 1. WLOG suppose that  $H$  is normal, then the composite map  $\varphi = \pi \circ \iota$ :

$$K \xrightarrow{\iota} G \xrightarrow{\pi} G/H$$

is surjective, therefore

$$\{e\} \subset H \subset G$$

$G/H \cong K/\ker(\varphi)$  is solvable, hence  $G$  is solvable.

2. The smallest nonsolvable group is  $A_5$ , we have

$$A_5 = HK$$

where  $H = \langle (12345) \rangle$ ,  $K = A_4 = \{\sigma \in A_5 : \sigma(5) = 5\}$ . Now  $H, K$  are both solvable, but  $G$  is not.  $\square$

**Problem 1.26 (F2003-Q1).** In a group  $G$ , let 1 denote the identity element and let  $[x, y] = xyx^{-1}y^{-1}$  denote the commutator of elements  $x, y \in G$ .

1. Express  $[z, xy]x$  in terms of  $x$ ,  $[z, x]$ , and  $[z, y]$ .
2. Prove that if the identity  $[[x, y], z] = 1$  holds in  $G$ , then the following identities hold in  $G$ :

$$[x, yz] = [x, y][x, z] \quad \text{and} \quad [xy, z] = [x, z][y, z].$$

*Proof.* 1. We have

$$\begin{aligned} [z, xy]x &= zxyx^{-1}y^{-1}x^{-1}x \\ &= zxx^{-1}x^{-1}xzyx^{-1}y^{-1} \\ &= [z, x]x[z, y] \end{aligned}$$

2. The identity  $[[x, y], z] = 1$  implies

$$[x, y]z = z[x, y]$$

Therefore using the identity in 1, we have

$$\begin{aligned} [x, yz] &= [x, y]y[x, z]y^{-1} \\ &= [x, y]yy^{-1}[x, z] \\ &= [x, y][x, z] \end{aligned}$$

Similarly

$$\begin{aligned} [xy, z] &= xyzy^{-1}x^{-1}z^{-1} \\ &= xyzy^{-1}z^{-1}zx^{-1}z^{-1} \\ &= x[y, z]x^{-1}[x, z] \\ &= [y, z][x, z] \\ &= [x, z][y, z] \end{aligned}$$

$\square$

**Problem 1.27 (S2005-Q1).** Let  $k$  be a field. Let  $G = \text{GL}_n(k)$  be the general linear group, where  $n > 0$ . Let  $D$  be the subgroup of diagonal matrices, and let  $N = N_G(D)$  be the normalizer of  $D$  in  $G$ . Determine the quotient group  $N/D$ .

**Problem 1.28 (F2009-Q1).** Let  $G$  be a finite group, and let  $\text{Aut}(G)$  be its automorphism group. Consider the group action  $\phi: \text{Aut}(G) \times G \rightarrow G$  defined by  $\phi(\sigma, g) = \sigma(g)$ . Assume  $G$  has exactly two orbits under this action.

1. Determine all such groups  $G$  up to isomorphism.
2. For each case from (a), determine when  $\text{Aut}(G)$  is solvable.



**Problem 1.29 (F2016-Q1).** Determine  $\text{Aut}(S_3)$ .

*Proof.* Every element  $\sigma \in \text{Aut}(S_3)$  must send order 2 elements  $\{(12), (23), (13)\}$  to one another, and order 3 elements  $\{(123), (132)\}$  to each other. However,  $\sigma$  is determined by how it permutes

$$\{(12), (23), (13)\}$$

Thus every  $\sigma$  is an inner automorphism of the form  $\sigma_g(h) = ghg^{-1}$  for  $g, h \in S_3$  and  $g$  is some transposition. Hence

$$\text{Aut}(S_3) \cong S_3$$

□

## Chapter 2

# Representation Theory

**Theorem 2.1** (Maschke's theorem).

**Lemma 2.1** (Schur's Lemma).

**Proposition 2.1** (properties of characters).

**Proposition 2.2.** The character tables for  $S_3, S_4, A_5, S_5$  are as follows:

### 2.1 Characters

**Problem 2.1** (S2008-Q4). Let  $V \cong \mathbb{C}^n$  be an  $n$ -dimensional complex vector space with standard basis  $e_1, \dots, e_n$ . Consider the permutation action  $S_n \times V \rightarrow V$  defined by:

$$\sigma \cdot e_i = e_{\sigma(i)} \quad \text{for } \sigma \in S_n$$

Decompose  $V$  into irreducible  $\mathbb{C}[S_n]$ -modules.

**Problem 2.2** (S2014-Q5). Find the table of characters for  $S_4$ .

**Problem 2.3** (F2016-Q6). Find a table of characters for the alternating group  $A_5$ .

**Problem 2.4** (F2015-Q3). Let  $G = S_4$  (the symmetric group on four letters).

- (a) Prove that  $G$  has two non-equivalent irreducible complex representations of dimension 3; call them  $\rho_1$  and  $\rho_2$ .
- (b) Decompose the tensor product representation  $\rho_1 \otimes \rho_2$  into a direct sum of irreducible representations of  $G$ .

**Problem 2.5 (F2011-Q4).** Let  $\rho: S_3 \rightarrow \text{GL}(2, \mathbb{C})$  be a two-dimensional irreducible representation of the symmetric group  $S_3$ .

1. Decompose the tensor square  $\rho^{\otimes 2}$  into irreducible representations of  $S_3$ .
2. Decompose the tensor cube  $\rho^{\otimes 3}$  into irreducible representations of  $S_3$ .

**Problem 2.6 (F2014-Q3).** Let  $G = S_3$  be the symmetric group on three elements.

- (a) Prove that  $G$  has an irreducible complex representation of dimension 2 (call it  $\rho$ ), but none of higher dimension.
- (b) Decompose the triple tensor product  $\rho \otimes \rho \otimes \rho$  into a direct sum of irreducible representations of  $G$ .

**Problem 2.7 (S2006-Q6).** Let  $S_4$  be the symmetric group on four elements.

1. Give an example of a non-trivial 8-dimensional complex representation of  $S_4$ .
2. Show that every 8-dimensional complex representation of  $S_4$  contains a 2-dimensional invariant subspace.

**Problem 2.8 (F2007-Q5).** Prove that every 5-dimensional complex representation of the alternating group  $A_4$  (the alternating group of degree 4) contains a 1-dimensional invariant subspace.

**Problem 2.9 (S2004-Q6).** Consider complex representations of a finite group  $G$ . Let  $\sigma_1, \dots, \sigma_s$  be representatives of the conjugacy classes of  $G$ , and let  $\chi_1, \dots, \chi_s$  be the irreducible characters of  $G$ .

1. Define an inner product on the  $\mathbb{C}$ -vector space of class functions on  $G$  such that  $\{\chi_1, \dots, \chi_s\}$  forms an orthonormal basis.
2. Let  $A = (a_{ij})$  be the character table matrix of  $G$ , where  $a_{ij} = \chi_i(\sigma_j)$  for  $1 \leq i, j \leq s$ . Prove that  $A$  is invertible.

**Problem 2.10 (S2018-Q4, S2007-Q5).** Is  $S_4$  isomorphic to a subgroup of  $\text{GL}_2(\mathbb{C})$ ?

**Problem 2.11 (S2010-Q6).** Let  $G$  be a group of order 24. Using representation theory, prove that  $G \neq [G, G]$ , where  $[G, G]$  denotes the commutator subgroup of  $G$ .

**Problem 2.12 (F2017-Q6).** Let  $G$  be a finite group with center  $Z(G)$ . Show that if  $G$  admits a faithful irreducible representation  $\rho: G \rightarrow \text{GL}_n(k)$  for some positive integer  $n \in \mathbb{Z}^+$  and some field  $k$ , then the center  $Z(G)$  is cyclic.

**Problem 2.13 (S2005-Q6).** Let  $V$  be a finite-dimensional vector space over a field  $k$ , and let  $G$  be a finite group with an irreducible representation  $\varphi: G \rightarrow \text{GL}(V)$ . Suppose  $H$  is a finite abelian subgroup of  $\text{GL}(V)$  contained in the centralizer of  $\varphi(G)$ . Prove that  $H$  must be cyclic.

**Problem 2.14** (F2010-Q6). Let  $G$  be a non-abelian group of order  $p^3$ , where  $p$  is prime.

1. Determine the number of isomorphism classes of irreducible complex representations of  $G$ , and find their dimensions.
2. Which of these irreducible complex representations are faithful? Justify your answer.

**Problem 2.15** (S2011-Q5). Let  $K$  be a field, and let  $\Phi: G \rightarrow \mathrm{GL}_n(K)$  be an  $n$ -dimensional matrix representation of a group  $G$ . Define a  $G$ -action on the matrix ring  $M_n(K)$  by:

$$(g, A) \mapsto \Phi(g) \cdot A \quad (\text{matrix multiplication})$$

for  $g \in G$  and  $A \in M_n(K)$ . This action induces a group homomorphism  $\Psi: G \rightarrow \mathrm{GL}(M_n(K))$ . Express the character  $\chi_\Psi$  of  $\Psi$  in terms of  $\chi_\Phi$  (the character of  $\Phi$ ).

**Problem 2.16** (S2015-Q5). Prove that a tensor product of irreducible representations over an algebraically closed field is irreducible.

**Problem 2.17** (S2001-Q3). Calculate the complete character table for  $\mathbb{Z}/3\mathbb{Z} \times S_3$ , where  $S_3$  is the symmetric group in 3 letters.

## 2.2 Induced representations

**Problem 2.18** (S2009-Q6). Let  $G = S_4$  and consider the subgroup  $H = \langle (1\ 2), (3\ 4) \rangle$ .

- (a) Determine the number of irreducible complex characters of  $H$ .
- (b) Choose a non-trivial irreducible character  $\psi$  of  $H$  over  $\mathbb{C}$  satisfying  $\psi((1\ 2)(3\ 4)) = -1$ . Compute the values of the induced character  $\mathrm{ind}_H^G(\psi)$  on all conjugacy classes of  $G$ , and express it as a sum of irreducible characters of  $G$ .

## 2.3 Frobenius Reciprocity

**Problem 2.19** (S2017-Q6). Let  $G$  be a finite group and  $H$  an abelian subgroup. Show that every irreducible representation of  $G$  over  $\mathbb{C}$  has dimension  $\leq [G : H]$ .

**Problem 2.20** (S2008-Q6). Give an example of non-isomorphic finite groups with same character table. Construct the character table in detail.

**Problem 2.21 (S2012-Q4).** Let  $Q$  be the quaternion group with presentation:

$$Q = \langle t, s_i, s_j, s_k \mid t^2 = 1, s_i^2 = s_j^2 = s_k^2 = s_i s_j s_k = t \rangle.$$

- (a) Find four non-isomorphic 1-dimensional real representations of  $Q$ .
- (b) Prove that the natural embedding  $\rho: Q \rightarrow \mathbb{H}$  given by:

$$\rho(t) = -1, \quad \rho(s_i) = i, \quad \rho(s_j) = j, \quad \rho(s_k) = k$$

defines an irreducible 4-dimensional real representation of  $Q$ , where  $\mathbb{H}$  is the algebra of real quaternions.

- (c) Classify all irreducible complex representations of  $Q$  up to isomorphism.

**Problem 2.22 (F2004-Q6).** Let  $D_8$  be the dihedral group of order 8, with presentation:

$$D_8 = \langle r, s \mid r^4 = 1 = s^2, rs = sr^{-1} \rangle.$$

1. Determine all conjugacy classes of  $D_8$ .
2. Find the commutator subgroup  $D'_8$  of  $D_8$  and determine the number of distinct degree-1 (linear) characters of  $D_8$ .
3. Construct the complete complex character table of  $D_8$ .

**Problem 2.23 (F2000-Q7).** Let  $D_{10}$  be the dihedral group of order 10, with presentation:

$$D_{10} = \langle r, s \mid r^5 = 1 = s^2, rs = sr^{-1} \rangle.$$

1. Determine all conjugacy classes of  $D_{10}$ .
2. Compute the commutator subgroup  $D'_{10}$  of  $D_{10}$ .
3. Prove that  $D_{10}/D'_{10} \cong \mathbb{Z}/2\mathbb{Z}$  and deduce that  $D_{10}$  has exactly two distinct degree-1 characters.
4. Construct the complete complex character table of  $D_{10}$ .