

# Algebraic Topology

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# Chapter 1

## Category Theory

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### 1.1 Lecture 1 8/26

**Definition 1.1 (Category).** A category  $\mathcal{C}$  consists of the following data:

1. A collection of objects denoted as  $\text{Ob}(\mathcal{C})$
2. Given two objects  $X, Y \in \text{Ob}(\mathcal{C})$ , a collection of morphisms between  $X, Y$ ,  $f : X \rightarrow Y$ , denoted as  $\text{mor}_{\mathcal{C}}(X, Y)$ .
3. (Composition) We have  $\text{mor}_{\mathcal{C}}(X, Y) \times \text{mor}_{\mathcal{C}}(Y, Z) \rightarrow \text{mor}_{\mathcal{C}}(X, Z)$  that satisfies associativity

$$f \circ (g \circ h) = (f \circ g) \circ h$$

4. (Identity) There is a distinguished morphism for each  $X$ ,  $\text{id}_{\mathcal{C}}(X, X)$  such that given any  $f \in \text{mor}_{\mathcal{C}}(X, Y)$ , we have  $f \circ \text{id}_X = \text{id}_Y \circ f = f$ .

In this course, we will make the assumption that in all the categories that we work with,  $\text{Ob}(\mathcal{C})$  need not be a set, but given any  $X, Y \in \text{Ob}(\mathcal{C})$ ,  $\text{mor}(X, Y)$  will always be a set. Now we talk about some examples of categories.

**Example 1.1 (Sets).** Let  $\text{Ob}(\text{Sets})$  be all the sets in the universe. Given  $X, Y$  sets,  $\text{mor}(X, Y)$  be all the set maps from  $X$  to  $Y$ , and  $\text{id}_X$  is the identity map.

**Example 1.2 (Top).** Let  $\text{Ob}(\text{Top})$  be all the topological spaces, and  $\text{mor}(X, Y)$  be all the continuous maps from  $X$  to  $Y$ .

**Example 1.3 ( $\text{Vect}_{\mathbb{F}}$ ).** Let  $\mathbb{F}$  be a field, and let  $\text{Ob}$  be all the  $\mathbb{F}$ -vector spaces. Then  $\text{mor}(V, W)$  is all the  $\mathbb{F}$ -linear homomorphisms from  $V$  to  $W$ , where  $\text{id}_V$  is the identity homomorphism.

**Example 1.4 (Posets).** Fix a poset  $P$ , let  $\text{Ob}(P)$  be the collection of elements in  $P$ , and given  $p, q$  we define

$$\text{mor}(p, q) = \begin{cases} *, & \text{if } q \leq p \\ \emptyset, & \text{otherwise} \end{cases}$$

**Problem 1.1. HW(Q1): check this is a category**

**Example 1.5 (Opposite category).** Given a category  $\mathcal{C}$ , there is another category called the opposite category, denoted as  $\mathcal{C}^{op}$ , where

1. The objects are the same as  $\mathcal{C}$
2. Given  $X, Y \in \text{Ob}(\mathcal{C}^{op})$ , we have  $\text{mor}_{op}(X, Y) := \text{mor}_{\mathcal{C}}(Y, X)$ .
3. Moreover, given  $f \in \text{mor}_{op}(X, Y), g \in \text{mor}_{op}(Y, Z)$ , then  $g \circ f$  in  $\mathcal{C}^{op}$  is  $f \circ g : Z \rightarrow X$ .

Naturally, we define isomorphisms now.

**Definition 1.2 (isomorphism).** Given a category  $\mathcal{C}$ , and a morphism  $f \in \text{mor}_{\mathcal{C}}(X, Y)$ , we say  $f$  is an isomorphism if there exists  $g \in \text{mor}_{\mathcal{C}}(Y, X)$  such that

$$f \circ g = \text{Id}_Y, g \circ f = \text{Id}_X$$

Now we introduce maps between categories.

**Definition 1.3 (functor).** Given categories  $\mathcal{C}, \mathcal{D}$ , a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is the following;

1. Given an object  $X$  in  $\mathcal{C}$ ,  $F(X)$  is an object in  $\mathcal{D}$ .
2. Given a morphism  $f : X \rightarrow Y$ ,  $F(f)$  is a functor  $F(f) : F(X) \rightarrow F(Y)$ . Moreover, it satisfies the following:
  - (a)  $F(\text{id}_X) = \text{id}_{F(X)}$
  - (b)  $F(f \circ g) = F(f) \circ F(g)$ . Alternatively, we can rewrite this condition as the following:

$$\begin{array}{ccc} \text{mor}(X, Y) \times \text{mor}(Y, Z) & \longrightarrow & \text{mor}(X, Z) \\ \downarrow \text{mor}(F) \times \text{mor}(F) & & \downarrow \text{mor}(F) \\ \text{mor}(F(X), F(Y)) \times \text{mor}(F(Y), F(Z)) & \longrightarrow & \text{mor}(F(X), F(Z)) \end{array}$$

such that this diagram commutes.

**Problem 1.2. HW(Q2): functors take isomorphisms to isomorphisms.**

Now we talk about some examples of functors.

**Example 1.6.**  $F : \text{Top} \rightarrow \text{Set}$ , where  $X \mapsto X$ , where the latter is a set, and  $f \mapsto f$  as set maps.

**Example 1.7.** Let  $\mathbb{F}$  be a field, and  $F : \text{Sets} \rightarrow \text{Vect}_{\mathbb{F}}$ , where  $X \mapsto \mathbb{F}\langle X \rangle$ , where  $\mathbb{F}\langle X \rangle$  is the free vector space over  $\mathbb{F}$  on the set  $X$ .

**Problem 1.3. HW(Q3): extend this to a functor by defining  $\text{mor}(f)$  and show this is a functor.**

**Example 1.8.** Let  $\mathbb{F}$  be a field, then the following is a functor,  $F : \text{Sets}^{op} \rightarrow \text{Vect}_{\mathbb{F}}$ , where

$$hF : X \mapsto \text{Maps}(X, \mathbb{F})$$

**Problem 1.4. HW(Q4):** show this extends to a functor by defining  $F(f)$ , and show it is a functor.

## 1.2 Lecture 2 8/28

**Definition 1.4 (contravariant functor).** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a contravariant functor from  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ , (equivalently,  $\mathcal{C} \rightarrow \mathcal{D}^{op}$ ).

**Problem 1.5. HW(Q5):** Show that the following functor  $F$  from  $\text{Vect}_{\mathbb{F}}$  to  $\text{Vect}_{\mathbb{F}}$  extends to a contravariant functor, where

$$Ob_F : V \mapsto V^* = \text{Hom}(V, \mathbb{F})$$

i.e., define the morphism function and show it is a contravariant functor.

We remark that we can define a category of categories: let  $Cat$  be the category of categories, with morphisms as functors, and note that objects or morphisms in this case are both not sets!

**Definition 1.5 (natural transformation).** Given functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ , a natural transformation  $T$  from  $F$  to  $G$  is the following:  $T : F \Rightarrow G$ :

1. given object  $X \in Ob(\mathcal{C})$ ,  $T(X) \in mor(F(X), G(X))$
2. Given  $f \in mor(X, Y)$ , the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ T(X) \downarrow & & \downarrow T(Y) \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

where  $mor_F, mor_G$  is the identification function on morphisms by functors  $F, G$

If for all  $X$ ,  $T(X)$  is an isomorphism, then this natural transformation is called a natural isomorphism.

In other words, this natural transformation is how one takes a functor  $F$  and turn it to another functor  $G$ . We will (in a homework) show there exists natural transformation between the following two functors.

**Example 1.9.** Consider  $F, G : \text{Vect}_{\mathbb{F}} \rightarrow \text{Vect}_{\mathbb{F}}$ , define

$$F(V) = V \otimes_{\mathbb{F}} V / \langle a \otimes b - b \otimes a \rangle = V \otimes_{\mathbb{F}} V / \Sigma_2, G(V) = (V \otimes_{\mathbb{F}} V)^{\Sigma_2} = \{ \alpha \in V \otimes_{\mathbb{F}} V : \sigma(\alpha) = \alpha \}$$

Both are vector spaces are fixed under “swaps.” Then a natural transformation can be defined as follows  $T(V) :$

$$T(V) : a \otimes b \mapsto a \otimes b + b \otimes a$$

**Problem 1.6. HW(Q6):** For the above  $F, G$

1. Show that  $T$  defines a natural transformation from  $F$  to  $G$ .
2. Find conditions on  $\mathbb{F}$  for  $T$  being a natural isomorphism.

Next we define limits and colimits. Let  $\mathcal{C}, \mathcal{D}$  be categories,  $d$  be an object in  $\mathcal{D}$ , then we can define a functor  $F_d : \mathcal{C} \rightarrow \mathcal{D}$  such that for any object  $c$  in  $\mathcal{C}$ ,

$$F_d(c) = d, F_d(f) = Id_d$$

In other words, this is the “constant functor” on  $\mathcal{D}$ , i.e., every object is sent to  $d$ , and every morphism is sent to  $id_d$ .

**Definition 1.6 (colimit).** Given any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the colimit of  $F$ , denoted as  $\text{colim}(F)$  is an object in  $\mathcal{D}$  endowed with a natural transformation:

$$\varphi_F : F \Rightarrow F_{\text{colim}(F)}$$

such that given any other object  $d$  in  $\mathcal{D}$  and a natural transformation

$$\varphi : F \Rightarrow F_d$$

there exists a unique morphism in  $\mathcal{D}$ ,  $f : \text{colim}(F) \rightarrow d$  making the following diagram commute: for any  $X, Y, g$ :

$$\begin{array}{ccc} F(X) & \xrightarrow{F(g)} & F(Y) \\ \searrow \varphi_F & & \swarrow \varphi_F \\ & \text{colim}(F) & \\ \searrow \varphi & \downarrow f & \swarrow \varphi \\ & d & \end{array}$$

Next we prove some facts about colimits and give an example, where  $\text{colim}(F)$  exists.

**Proposition 1.1.** If  $\text{colim} F$  exists, then  $\text{colim} F$  is unique up to isomorphisms.

*Proof.* Let  $\text{colim}(F), \text{colim}(F)'$  be two colimits that satisfy the criteria. They are both objects in  $\mathcal{D}$ , then we get a morphism  $f : \text{colim}(F) \rightarrow \text{colim}(F)'$ , and likewise  $g : \text{colim}(F) \rightarrow \text{colim}(F)'$ , then

$$f \circ g : \text{colim}(F)' \rightarrow \text{colim}(F)'$$

is the only morphism, and is the identity morphism. Similarly for  $g \circ f$ . □

Next we demonstrate a fact via an example.

**Theorem 1.1.** Let  $\mathcal{C}$  be a category where  $Ob(\mathcal{C}), mor(X, Y)$  are all sets. Let  $F : \mathcal{C} \rightarrow \text{Top}$  be any functor, then  $\text{colim}(F)$  exists.

*Proof.* Define  $\text{colim}(F) := \bigsqcup_c F(c) / \sim$ , where  $\sim$  is induced by the equivalence relation given by

$$y \sim F(f)y$$

where  $y \in F(C_1), f : C_1 \rightarrow C_2, F(f)x \in F(C_2)$ . The natural transformation we endow on  $F$  as  $\varphi_F : F \Rightarrow F_{\text{colim}(F)}$ :

$$\varphi_F : F(C) \mapsto \bigsqcup_{C \in Ob(\mathcal{C})} F(C) / \sim$$

□

**Problem 1.7. HW(Q7):** Show that  $\text{colim}(F)$ ,  $\varphi_F$  is indeed a colimit.

We note that colimits also exist (the same argument goes through) if we replace Top with groups, sets, but with slightly different constructions, replacing disjoint unions with products, etc.

**Definition 1.7 (limit).** Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the limit of  $F$ , denoted as  $\lim(F)$  is an object of  $\mathcal{D}$ , endowed with a natural transformation:

$$\varphi_F : F_{\lim(F)} \Rightarrow F$$

such that given any other object  $d \in \text{Ob}(\mathcal{D})$  and a natural transformation

$$\varphi : F_d \rightarrow F$$

there exists a unique  $f : \lim F \rightarrow d$  such that the following diagram commutes:

$$\begin{array}{ccc}
 & \lim F & \\
 \varphi_F \swarrow & \downarrow f & \searrow \varphi_F \\
 & d & \\
 \swarrow \varphi & & \searrow \varphi \\
 F(X) & \xrightarrow{F(g)} & F(Y)
 \end{array}$$

Just like colimits, limits are unique up to isomorphisms.

**Problem 1.8. HW(Q8):** Given  $F : \mathcal{C} \rightarrow \mathcal{D}$ , consider  $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ , then

$$\lim F = \text{colim} F^{op}$$

The above problem is interpretation of diagrams and essentially we just reverse all the maps.

## Chapter 2

# Homologies, Cohomologies

### 2.1 Lecture 3 9/4

Today we define (co)chain complexes: let  $R$  be a commutative ring, let  $Mod_R$  denote the category of  $R$ -modules and  $R$ -module maps.

**Definition 2.1 (chain complex).** A chain complex of  $R$ -modules is a collection of  $R$ -modules and  $R$ -modules maps

$$\cdots \rightarrow M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \xrightarrow{\partial_{i-1}} \cdots$$

such that  $\partial_i \circ \partial_{i+1} = 0$  for all  $i$ . In other words, the image of previous map is contained in the kernel of the subsequent map. In short, we have

$$\partial^2 = 0$$

We will denote a chain complex by  $\{M.; \partial.^M\}$ .

Next we introduce morphisms between chain complexes.

**Definition 2.2 (morphism between complexes).** Let  $\{M.; \partial.^M\}, \{N.; \partial.^N\}$ , a morphism  $\{f.\}$  between chain complexes is a “ladder” such that the following commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{i+1} & \xrightarrow{\partial_{i+1}^M} & M_i & \xrightarrow{\partial_i^M} & M_{i-1} \xrightarrow{\partial_{i-1}^M} \cdots \\ & & \downarrow f_{i+1} & & \downarrow f_i & & \downarrow f_{i-1} \\ \cdots & \longrightarrow & N_{i+1} & \xrightarrow{\partial_{i+1}^N} & N_i & \xrightarrow{\partial_i^N} & N_{i-1} \xrightarrow{\partial_{i-1}^N} \cdots \end{array}$$

Moreover, we define composition of morphisms:

$$\{f.\} \circ \{g.\} := \{(f \circ g).\}$$

where  $\{g.\} : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$ , and  $\{f.\} : \{N.; \partial.^N\} \rightarrow \{L.; \partial.^L\}$ , which is simply vertical stacking.

**Problem 2.1. HW(Q9):** Prove that chain complexes of  $R$ -modules form a category  $ch_R$ .

There are interesting functors  $F : ch_R \rightarrow Mod_R$ , and we begin with the following one:



**Definition 2.3** ( $H_n$ ,  $n$ th-homology). Given  $n \in \mathbb{Z}$ , there is a functor

$$H_n : \mathbf{ch}_R \rightarrow \mathbf{Mod}_R$$

defined as follows:

$$H_n(\{M.; \partial.^M\}) := \ker \partial_n^M / \text{Im} \partial_{n+1}^M$$

and for  $f : \{M.; \partial.^M\} \rightarrow \{N.; \partial.^N\}$ , we define:  $H_n(f) : H_n(\{M.; \partial.^M\}) \rightarrow H_n(\{N.; \partial.^N\})$ ,

$$H_n(f)[x] := [f_n(x)]$$

where  $[x] \in H_n(\{M.; \partial.^M\})$ .

*Proof.* We need to show  $H_n$  is well-defined on objects and morphisms. We need to check that  $\text{Im} \partial_{n+1} \subset \ker \partial_n$ . This is a consequence of  $\partial^2 = 0$ .

On morphisms: for  $x \in \ker \partial_n^M$ , we have  $f_n(x) \in \ker \partial_n^N$ . This is we have

$$\partial_n^N(f_n(x)) = f_{n+1}(\partial_n^M(x)) = 0$$

Moreover, we need to check that this doesn't depend on the choice of representatives, i.e., we can check that

$$\text{Im} \partial_{n+1}^M \mapsto 0$$

Let  $x = \partial_{n+1}^M(y)$ , we have

$$f_n(x) = f_n(\partial_{n+1}^M(y)) = \partial_{n+1}^N(f_{n+1}(y)) = 0$$

$$\begin{array}{ccc} M_{n+1} & \xrightarrow{\partial_{n+1}^M} & M_n \\ f_{n+1} \downarrow & & \downarrow f_n \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n \end{array}$$

□

Next we talk about homotopy between morphisms between chain complexes.

**Definition 2.4** (homotopy). Given two morphisms,  $f, g : M. \rightarrow N.$ , a chain homotopy  $h$ . between them is a collection of  $R$ -modules maps, for all  $n \in \mathbb{Z}$ ,

$$h_n : M_n \rightarrow N_{n+1}$$

such that

$$\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n = f_n - g_n$$

denoted as  $\partial h + h \partial = f - g$ .

$$\begin{array}{ccccc} M_{n+1} & \longrightarrow & M_n & \xrightarrow{\partial_n^M} & M_{n-1} \\ f_{n+1} \downarrow & \swarrow h_n & \downarrow f_n/g_n & \swarrow h_{n-1} & \downarrow f_{n-1} \\ N_{n+1} & \xrightarrow{\partial_{n+1}^N} & N_n & \longrightarrow & N_{n-1} \end{array}$$

**Problem 2.2. HW(Q10):** Show that homotopy is an equivalence relation between morphisms. Hint: replace  $h_n$  with  $-h_n : M_n \rightarrow N_{n+1}$ .

*Proof.* Reflexive is shown by defining  $h_n$  to be the zero map. For symmetry, we choose  $-h_n$ . Transitive is a ladder. □

**Proposition 2.1.** Let  $h.$  be a chain homotopy between  $f.$  and  $g.$ , then we have an equality

$$H_n(f.) = H_n(g.)$$

where  $H_n(f.), H_n(g.) : H_n(M.) \rightarrow H_n(N.)$ .

*Proof.* Given  $[x] \in H_n(M.)$ , we have

$$\begin{aligned} H_n(f)[x] &= [f_n(x)] \\ &= [g_n(x) + \partial h.(x) + h.\partial(x)] \\ &= [g_n(x) + \partial h.(x)] \\ &= [g_n(x)] \\ &= H_n(g)[x] \end{aligned}$$

□

Next we define a new category.

**Definition 2.5** ( $Hch_R$ ). Define the category  $Hch_R$  as follows:

1.  $Ob(Hch_R) = Ob(ch_R)$
2.  $mor_{Hch_R}(M., N.) = mor_{ch_R}(M., N.) / \sim$ , where  $\sim$  is the homotopy equivalence.

**Problem 2.3. HW(Q11):** Show that  $Hch_R$  is a category, admitting a functor

$$F : ch_R \rightarrow Hch_R$$

such that the following diagram commutes:

$$\begin{array}{ccc} ch_R & \xrightarrow{F} & Hch_R \\ & \searrow H_n & \swarrow H_n \\ & mod_R & \end{array}$$

Next we introduce long and short exact sequences.

**Definition 2.6 (exactness).** Firstly, given a pair of  $R$ -module maps,

$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

we say that the above is exact at  $X_2$  if  $\ker(g) = \text{im}(f)$ . Hence given a sequence of  $R$ -module maps,

$$\cdots \rightarrow X_{i+1} \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots$$

this is called a long exact sequence if it is exact at all  $X_i$ . Finally, given a pair of  $R$ -module maps,

$$0 \rightarrow X_i \xrightarrow{f} X_2 \xrightarrow{g} X_3 \rightarrow 0$$

This is a short exact sequence, and  $f$  is injective,  $g$  is surjective.

**Problem 2.4. HW(Q12):** Prove the following:

1. Given LES,

$$\cdots \rightarrow X_{i+1} \xrightarrow{f_{i+1}} X_i \xrightarrow{f_i} X_{i-1}$$

show the following is a short exact sequence:

$$0 \rightarrow \ker(f_i) \xrightarrow{i} X_i \xrightarrow{f_i} \ker(f_{i-1}) \rightarrow 0$$

2. Prove the 5-lemma. Given the below sequence, exact at positions  $X_i, Y_i$ , where  $i = 2, 3, 4$ , and assume the diagram commutes and if  $t_1, t_2, t_4, t_5$  are isomorphisms, show that  $t_3$  is also an isomorphism.

$$\begin{array}{ccccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 & \xrightarrow{f_3} & X_4 & \xrightarrow{f_4} & X_5 \\ t_1 \downarrow & & t_2 \downarrow & & t_3 \downarrow & & t_4 \downarrow & & t_5 \downarrow \\ Y_1 & \xrightarrow{g_1} & Y_2 & \xrightarrow{g_2} & Y_3 & \xrightarrow{g_3} & Y_4 & \xrightarrow{g_4} & Y_5 \end{array}$$

Next we state the most important theorem in chain complexes:

**Theorem 2.1 (The snake lemma).** Let  $A \xrightarrow{f} B \xrightarrow{g} C$  be a SES of chain complexes, i.e.,

$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$

is a short exact sequence of all  $n$ . Then there exists a LES of homology groups.

$$\begin{array}{ccccc} & & & & H_{n+1}(C) \\ & & & \swarrow \delta_{n-1} & \\ H_n(A) & \xrightarrow{H_n(f)} & H_n(B) & \xrightarrow{H_n(g)} & H_n(C) \\ & & \searrow \delta_n & & \\ H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) & \xrightarrow{H_{n-1}(g)} & H_{n-1}(C) \\ & & \swarrow \delta_{n-1} & & \\ & & & & H_{n-2}(A) \end{array}$$

## 2.2 Lecture 4 9/9

Today we prove the snake lemma. We will refer to this following diagram throughout the proof.

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

*Proof.* First we define the map  $\delta_n : H_n(C) \rightarrow H_{n-1}(A)$ . Let  $[x] \in H_n(C)$ , then  $x \in \delta^C$ , where  $\delta^C : C_n \rightarrow C_{n-1}$ . We define

$$\delta[x] = [y], y \in A_{n-1}$$

as follows: for  $x \in C_n$ ,  $g_n : B_n \rightarrow C_n$  is surjective, hence there exists  $b \in B_n$  such that  $g_n(b) = x$ . Then consider  $d = \delta^B(b)$ , since the diagram commutes, we have

$$d \in \ker g_{n-1} \Rightarrow d \in \text{im } f_{n-1}$$

Let  $y \in A_{n-1}$  be this unique  $y$  such that  $f_{n-1}(y) = d$ , where uniqueness is by  $f_{n-1}$  is injective. This is indicated in the below diagram:

$$\begin{array}{ccccc} A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_n & \xrightarrow{f_n} & b \in B_n & \xrightarrow{g_n} & x \in C_n \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ y \in A_{n-1} & \xrightarrow{f_{n-1}} & d \in B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} \\ \delta^A \downarrow & & \delta^B \downarrow & & \delta^C \downarrow \\ A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & \xrightarrow{g_{n-2}} & C_{n-2} \end{array}$$

We first need to check that  $[y]$  does not depend on the choice of  $b$ . Let  $g_n(b_1) = g_n(b_2) = x$ , then

$$g(b_1 - b_2) = 0 \Rightarrow b_1 - b_2 = f_n(a), a \in A_n$$

let  $y_1, y_2$  be those determined by  $b_1, b_2$ , then

$$f_{n-1}(y_1 - y_2) = \delta^B(b_1 - b_2) = \delta^B(f_n(a)), a \in A_n$$

Because the following diagram commutes,

$$\begin{array}{ccc} a \in A_n & \xrightarrow{f_n} & B_n \\ \delta^A \downarrow & & \downarrow \delta^B \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} \end{array}$$

we then have

$$y_1 - y_2 = \delta^A(a)$$

i.e.,  $[y_1] = [y_2]$ , as they only differ by  $\text{im } \delta$ .

**Problem 2.5. HW(Q13):** Check that if  $x \in \text{im } \delta^C$ , then  $\delta_n[x] = 0$ .

the proof is not finished, too lazy to tex it up □

Next we review the tensor products of  $R$ -modules. We first review  $R$ -bilinear maps

**Definition 2.7 (bilinear maps).** Let  $M, N, P$  be  $R$ -modules, an  $R$ -bilinear map  $f : M \times N \rightarrow P$  is a map such that

1.  $f$  is linear in both coordinates, we have  $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$ , and similarly,  $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$ .
2. For all  $r \in R$ , we have  $f(rm, n) = f(m, rn) = rf(m, n)$ .

Next we define tensor products.

**Definition 2.8 (tensor product).** A tensor product of  $M \times N$  is an  $R$ -module denoted by  $M \otimes_R N$  such that

1.  $M \otimes_R N$  comes endowed with an  $R$ -bilinear map

$$M \times N \xrightarrow{\varphi} M \otimes_R N$$

2. given any other  $R$ -bilinear map  $f : M \times N \rightarrow P$ , there exists a unique  $R$ -module map  $\psi$  such that the following diagram commutes:

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & M \otimes_R N \\ f \downarrow & \swarrow \psi & \\ P & & \end{array}$$

It is not clear that  $M \otimes_R N$  exists or not. In fact, they exist!

**Theorem 2.2 ( $M \otimes_R N$  exists).** Define  $M \otimes_R N = R\langle M \times N \rangle / K$ , where  $R\langle M \times N \rangle$  is the free  $R$ -module on the set  $M \times N$ . We define  $K$  as the submodule generated by the following four relations:

1.  $\langle (m_1 + m_2, n) \rangle - \langle (m_1, n) \rangle - \langle (m_2, n) \rangle$
2.  $\langle (m, n_1 + n_2) \rangle - \langle (m, n_1) \rangle - \langle (m, n_2) \rangle$
3.  $r\langle (m, n) \rangle - \langle (rm, n) \rangle$
4.  $r\langle (m, n) \rangle - \langle (m, rn) \rangle$

Moreover, the map  $\varphi : M \times N \rightarrow M \otimes_R N$  given by

$$(m, n) \mapsto \langle (m, n) \rangle := m \otimes_R n$$

**Problem 2.6. HW(Q14):** show that  $M \otimes_R N$  is a tensor product.

## 2.3 Lecture 5 9/11

We continue with the tensors of  $R$ -modules. Let  $f : A \rightarrow B$  an  $R$ -module map, let  $N$  be some fixed  $R$ -module, then  $f$  induces maps:  $f \otimes id : A \otimes_R N \rightarrow B \otimes_R N$ ,

$$f \otimes id : a \otimes n \mapsto f(a) \otimes n$$

and  $id \otimes f : N \otimes_R A \rightarrow N \otimes_R B$  :

$$id \otimes f : n \otimes a \mapsto n \otimes f(a)$$

**Problem 2.7. HW(Q15(a)):** Show that the following maps induce functors:

1.  $- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$ , where

$$A \mapsto A \otimes_R N, f \mapsto f \otimes id$$

2.  $N \otimes_R - : \text{Mod}_R \rightarrow \text{Mod}_R$ , where

$$A \mapsto N \otimes_R A, f \mapsto id \otimes f$$

**Problem 2.8. HW(Q15(b)):** Show that one has the following natural isomorphisms:

1.  $0 \otimes_R M \cong 0$ , and  $0 \otimes_R - \cong F_0$  (recall the definition of  $F_0$  as a functor).
2.  $R \otimes_R M \cong M$ , and  $R \otimes_R - \cong id$ .
3.  $M \otimes_R N \cong N \otimes_R M$ , and  $M \otimes_R - \cong - \otimes_R M$ .
4.  $M \otimes_R (N \otimes_R K) \cong (M \otimes_R N) \otimes_R K$ .
5.  $(M \oplus N) \otimes_R K \cong (M \otimes_R K) \oplus (N \otimes_R K)$ .

For convenience, we introduce the following definition:

**Definition 2.9 (positively graded chain complex).** A positively graded chain complex  $\{M.; \partial.^M\}$  is a chain complex so that  $M_i = 0$  for all  $i < 0$ . The category of positively graded chain complexes is denoted as  $ch_R^+$ .

We have our first important theorem for  $ch_R^+$ .

**Theorem 2.3.** There exists a functor  $\otimes_R$  and a natural transformation  $X$  such that the following diagram of functors commutes up to some  $X$ :

$$\begin{array}{ccc} ch_R^+ \times ch + R^+ & \xrightarrow{\otimes_R} & ch_R^+ \\ H_i \times H_j \downarrow & \nearrow X & \downarrow H_{i+j} \\ Mod_R \times Mod_R & \xrightarrow{\otimes_R} & Mod_R \end{array}$$

where  $X : \otimes_R \circ (H_i \times H_j) \Rightarrow H_{i+j} \circ \otimes_R$  is a natural transformation.

We note that the existence of  $X$  means this diagram doesn't commute "on the nose," but these two composition functors are the same up to some natural transformation. Before we given the proof, we recall that  $Ob(C \times D) = Ob(C) \times Ob(D)$ ,  $mor((X, Y), (X', Y')) = mor(X, Y) \times mor(X', Y')$ .

*Proof.* We define  $\otimes_R$  of positively graded chain complexes as follows: let  $\{M.; \partial.^M\}, \{N.; \partial.^N\}$  be two PGCC. Define  $\{M \otimes_R N.; \partial.^{M \otimes_R N}\}$ :

$$(M \otimes_R N) = \bigoplus_{i+j=n} (M_i \otimes_R N_j)$$

note that the RHS is always a finite sum. Moreover,  $\partial^\otimes$  is defined as follows:

$$\partial^\otimes : (M \otimes_R N)_n \rightarrow (M \otimes_R N)_{n-1} \text{ is defined on the component } M_i \otimes_R N_j \text{ (from the RHS)}$$

and

$$\partial^\otimes(m_i \otimes n_j) := \partial^M(m_i) \otimes n_j + (-1)^i m_i \otimes \partial^N(n_j)$$

It is easy to check that  $\partial^\otimes \circ \partial^\otimes = 0$ .

Now we've show  $ch_R^+ \otimes_R ch_R^+$  is well-defined, it remains to define  $X$ , the natural transformation. We define

$$X : H_i(M_\bullet) \otimes_R H_j(N_\bullet) \rightarrow H_{i+j}(M_\bullet \otimes_R N_\bullet)$$

again, it suffices to define  $X$  on elementary tensors.

$$X : [\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$$

we need to check that

1.  $\partial^\otimes(\alpha \otimes \beta) = 0$  if  $\partial^M(\alpha) = 0$  and  $\partial^N(\beta) = 0$ .
2. If  $\alpha = \partial(r)i$ , then notice that  $\partial^\otimes(r \otimes \beta) = \alpha \otimes \beta$ , similarly for  $\beta$ . This would show that  $X$  is well-defined.

It is straightforward to check that  $X$  commutes with morphisms in  $ch_R^+ \times ch_R^+$ . □

Next we define cochain complexes and cohomologies.

**Definition 2.10 (cochain).** A cochain of  $R$ -modules  $(M^\bullet, \partial_M^\bullet)$  is a sequence of  $R$ -module maps:

$$\dots \longrightarrow M^i \xrightarrow{\partial^i} M^{i+1} \xrightarrow{\partial^{i+1}} M^{i+2} \longrightarrow \dots$$

such that  $\partial \circ \partial = 0$ .

Cochain complexes form a category, with morphisms  $\{f^\bullet\}$  form a ladder:

$$\begin{array}{ccccccc} \dots & \longrightarrow & M^i & \xrightarrow{\partial^i} & M^{i+1} & \xrightarrow{\partial^{i+1}} & M^{i+2} \longrightarrow \dots \\ & & \downarrow f^i & & \downarrow f^{i+1} & & \downarrow f^{i+2} \\ \dots & \longrightarrow & N^i & \xrightarrow{\partial^i} & N^{i+1} & \xrightarrow{\partial^{i+1}} & N^{i+2} \longrightarrow \dots \end{array}$$

The  $n$ -th cohomology of a cochain complex  $\{M^\bullet; \partial_M^\bullet\}$  is defined as:

$$H^n(M^\bullet; \partial_M^\bullet) := \frac{\ker \partial^i : M^i \rightarrow M^{i+1}}{\operatorname{im} \partial^{i-1} : M^{i-1} \rightarrow M^i}$$

We remark that there is nothing unexpected here from what we learned about chain complexes. Namely, if we reindex  $\{M^\bullet; \partial_M^\bullet\}$ , this defines a chain complex with  $M'_i = M^{-i}$ . This implies that the snake lemme holds! (with  $\partial^i : H^i(C) \rightarrow H^{i+1}(A)$ ).

**Theorem 2.4.** There is a functor  $D$  and a natural transformation  $\beta$  such that the following diagram of functors commute up to the natural transformation  $\beta$ :

$$\begin{array}{ccc} ch_R^{op} & \xrightarrow{D} & coch_R \\ H_n^{op} \downarrow & \nearrow \beta & \downarrow H^n \\ Mod_R^{op} & \xrightarrow{\bar{D}} & Mod_R \end{array}$$

where  $\bar{D}(M) = Hom_R(M, R)$ , and

$$D(\{M_\bullet; \partial_\bullet^M\}) \text{ is defined as } \{DM^\bullet; \partial^\bullet\}$$

where

$$DM^n := Hom_R(M_n, R), \partial^n : DM^n \rightarrow DM^{n+1} \text{ is the map induced by } \partial_{n+1} : M_{n+1} \rightarrow M_n$$

We observe that  $\partial^{n+1}\partial^n = 0$  since  $\partial_{n+2}\partial_{n+1} = 0$ .

**Problem 2.9. HW(Q16):** Show that  $D$  is a functor.

Next we define the natural transformation  $\beta$ . We have  $\beta : H^n(DM) \rightarrow \text{Hom}_R(H_n(M_\bullet), R)$ , such that

$$\beta : [\varphi] \mapsto \beta[\varphi]$$

let  $[x] \in H_n(M_\bullet)$ , where  $\beta[\varphi]([x]) = \varphi(x)$  (where  $\varphi \in \text{Hom}_R(M_n, \mathbb{R})$ ,  $x \in M_n$ ).

*Proof.* We first need to show that  $\beta$  is well-defined. If  $x = \partial_{n+1}(y)$ , then consider

$$\beta[\varphi][x] = \varphi(x) = \varphi(\partial_{n+1}(y)) = \partial^n(\varphi)(y) = 0, x \in \ker \varphi$$

Conversely, if  $\varphi = \delta^{n-1}(\psi)$ , we have

$$\beta[\varphi][x] = \varphi(x) = \delta^{n-1}\psi(x) = \psi(\partial_n(x)) = 0$$

It remains to check that  $\beta$  commutes with morphisms in  $ch_R^{op}$  (which we will do next time).  $\square$

## 2.4 Lecture 6 9/16

Today we continue our discussion of homological algebra. Let  $M$  be an  $R$ -module.

**Definition 2.11 (resolution).** A resolution of  $M$  is a positively graded chain complex  $\{P_\bullet, \partial_\bullet\}$  such that

1.  $H_n(P_\bullet) = 0$  for all  $n > 0$
2.  $H_0(P_\bullet) = \frac{P_0}{\text{im } \partial_1} \cong M$ , where  $\partial_1 : P_1 \rightarrow P_0$ .

We say  $\{P_\bullet, \partial_\bullet\}$  is a free resolution if  $P_i$  is a free  $R$ -module for each  $i$ .

For resolutions, we prove the following two things: first, free resolutions always exist; second, every  $R$ -module map can be extended to a map between their resolutions (with extra assumptions) and these extensions are unique up to homotopies.

**Proposition 2.2.** For any  $M$ , a free resolution for  $M$  exists.

*Proof.* We proceed this inductively. Define  $P_0$  to be  $R\langle M \rangle$ , where it is the free  $R$ -module defined on the set  $M$ . Note that

$$R\langle M \rangle \twoheadrightarrow M \text{ is surjective : } \langle m \rangle \mapsto m$$

Let  $K$  be the kernel of this map, we have an isomorphism:

$$\epsilon : P_0/K \cong M$$

Define  $P_1$  as  $R\langle K \rangle$ , note that  $P_1 \twoheadrightarrow K$ , then we define

$$\partial_1 : P_1 \rightarrow P_0$$

to be the composite

$$P_1 \twoheadrightarrow K \subset P_0$$

Now we consider  $P_2$ : let  $K_1 \subset P_1$  be the kernel of  $\partial_1$ , define  $P_2 = R\langle K_1 \rangle$ , then define  $\partial_2 : P_2 \rightarrow P_1$  to be the composite"

$$P_2 \twoheadrightarrow K_1 \subset P_1$$

note that  $\ker \partial_1 / \text{im } \partial_2 = K_1/K_1 = 0$ . Then we define  $K_2 = \ker \partial_2$ , define  $P_3 = R\langle K_2 \rangle, \dots$   $\square$

Just like the above proposition, the next theorem uses induction.



**Theorem 2.5 (extension theorem).** Let  $\{P_\bullet^M, \partial_\bullet^M, \epsilon_M\}$  be a free resolution on  $M$ , and let  $\{P_\bullet^N, \partial_\bullet^N, \epsilon_N\}$  be an arbitrary resolution of  $N$ . Then given a map of  $R$ -modules  $f : M \rightarrow N$ , we may extend it to a map of chain complexes:

$$f_\bullet : \{P_\bullet^M, \partial_\bullet^M\} \rightarrow \{P_\bullet^N, \partial_\bullet^N\}$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_0(P_\bullet^M) & \xrightarrow{H_0(f_\bullet)} & H_0(P_\bullet^N) \\ \downarrow \epsilon_M & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

Moreover, given any two extensions  $f_\bullet^1, f_\bullet^2$  of  $f$ , we have a chain homotopy  $h_\bullet$  between  $f_\bullet^1, f_\bullet^2$ .

Remark: if  $f_\bullet$  makes the diagram commute, and  $g_\bullet$  is homotopic to  $f_\bullet$ , then  $g_\bullet$  also makes the diagram commute: homotopy classes work the same on homologies (they are the same on the nose).

*Proof.* We will construct  $f_\bullet$  as follows. We construct  $f_i$  inductively on  $i$ . Consider the diagram:

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

Since  $P_0^M$  is free, and  $\epsilon_N$  is surjective, we may lift  $f$  on generators of  $P_0^M$  by lifting the generators of  $P_0^M$  to elements in  $P_0^N$ . (Note: this lift may not be unique), but this lift extends uniquely to define  $f_0$ . We notice that the bottom square

$$\begin{array}{ccc} P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

commutes on homologies ( $H_0(P_0^M), H_0(P_0^N)$ ). Now we construct  $f_1$ :

$$\begin{array}{ccc} \cdots & & \cdots \\ \downarrow & & \downarrow \\ P_1^M & \xrightarrow{f_1} & P_1^N \\ \downarrow & & \downarrow \\ P_0^M & \xrightarrow{f_0} & P_0^N \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

We will follow the purple path above. Recall that  $\epsilon_M : H_0(P_0) = P_0 / \text{im}(\partial_1^M) \rightarrow M$  is an isomorphism. We

consider the composite:  $f_0 \circ \partial_1^M = g$ , we have

$$\begin{aligned}\epsilon_N \circ g &= \epsilon_N \circ f \circ \partial_1^M \\ &= f \circ \epsilon_M \circ \partial_1^M \\ &= 0\end{aligned}$$

This implies that  $\text{im}(g) \subset \ker(\partial_N) = \text{im}(\partial_1^N)$ . We can lift the generators of  $P_1^M$  to elements of  $P_1^N$ . (Once chosen a lift, one can extend this uniquely to define  $f_1$ ). Then we construct  $f_2, f_3, \dots$  the same way by considering  $f_n \circ \partial_{n+1}$  and show that it is in the kernel of  $\partial_n^N$  and lift it to define  $\partial_{n+1}$ .

Now we homotopy time. Assume  $f_\bullet^1, f_\bullet^2$  are two lifts of  $f$ , we construct  $h : P_\bullet^M \rightarrow P_{\bullet+1}^N$ . We define  $h_\bullet$  inductively, starting with  $h_0$  below:

$$\begin{array}{ccc} P_1^M & \xrightarrow{f_1} & P_1^N \\ \partial_1^M \downarrow & \nearrow h_0 & \downarrow \partial_1^N \\ P_0^M & \xrightarrow{f_0^1, f_0^2} & P_0^N \\ \epsilon_M \downarrow & & \downarrow \epsilon_N \\ M & \xrightarrow{f} & N \end{array}$$

We have  $\epsilon_N(f_0^1 - f_0^2) = 0$ , then

$$f_0^1 - f_0^2 \in \ker \epsilon_N = \text{im } \delta_1^N$$

we may lift  $f_0^1 - f_0^2$  on generators of  $P_0^M$ , where  $h_0 : P_0^M \rightarrow P_1^N$ . Hence

$$(h_{-1} \circ \delta_{-1}^N) + \delta_1^N \circ h_0 = f_0^1 - f_0^2$$

Inductively, we assume  $h_m$  exists for  $m \leq n$ , then

$$\begin{array}{ccc} P_{n+2}^M & \xrightarrow{f_1} & P_{n+2}^N \\ \partial_{n+2}^M \downarrow & \nearrow h_{n-1} & \downarrow \partial_{n+2}^N \\ P_{n+1}^M & \xrightarrow{f_{n+1}^1, f_{n+1}^2} & P_{n+1}^N \\ \partial_{n+1}^M \downarrow & \nearrow h_n & \downarrow \partial_{n+1}^N \\ P_n^M & \xrightarrow{f} & P_n^N \end{array}$$

consider the expressions

$$g_{n+1} := f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

we can check (by diagram chasing),  $\partial_{n+1}^N \circ g = 0$ . This implies that

$$\text{im}(g_{n+1}) \subset \text{im}(\partial_{n+2}^N)$$

we can construct  $h_{n+1}$  to get the map

$$\delta_{n+2}^N \circ h_{n+1} = g_{n+1} = f_{n+1}^1 - f_{n+1}^2 - h_n \circ \partial_{n+1}^M$$

i.e.

$$\partial_{n+2}^N \circ h_{n+1} + h_n \circ \partial_{n+1}^M = f_{n+1}^1 - f_{n+1}^2$$

hence we are done! □

**Corollary 2.1.** Any two free resolutions of an  $R$ -module  $M$  are homotopy equivalent: given two free resolutions  $P_\bullet^M, Q_\bullet^M$ , there exists extension of  $\text{id} : M \rightarrow M$  and such that

$$f_\bullet : P_\bullet^M \rightarrow Q_\bullet^M, g_\bullet : Q_\bullet^M \rightarrow P_\bullet^M$$

such that

$$g_\bullet \circ f_\bullet = \text{id}, f_\bullet \circ g_\bullet = \text{id}$$

**Problem 2.10 (HW(2.1)).** Prove this corollary.

Next we define Tor functors (pretty hard things).

**Definition 2.12 (tor functors).** Let  $N$  be an  $R$ -module, recall the functor

$$- \otimes_R N : \text{Mod}_R \rightarrow \text{Mod}_R$$

we define a collection of functors

$$\text{Tor}_R^i(-, N) : \text{Mod}_R \rightarrow \text{Mod}_R, i \in \mathbb{N}$$

given an object  $M$  in  $\text{Mod}_R$ , let  $\{P_\bullet^M, \partial_\bullet^M, \epsilon_M\}$  be a free resolution of  $M$ , define  $\text{Tor}^i(M, N)$  to be

$$\text{Tor}^i(M, N) = H_i(P_\bullet^M \otimes_R N, \partial_\bullet^M \otimes \text{id}_N)$$

where

$$\cdots \rightarrow P_i^M \otimes N \xrightarrow{\partial_i \otimes \text{id}} P_{i-1}^M \otimes N \rightarrow \cdots$$

We make the remark that there is a choice involved in picking the free resolution, but this is unique since homotopies are the same on homologies.

**Problem 2.11 (HW(2.2)).** For all  $i$ , show that  $\text{Tor}_R^i(M, N)$  is a well-defined functor, and any other choice of free resolution of all objects yields an isomorphic functor. Hint: use the above corollary.

**Problem 2.12 (HW(2.3)).** Show that

1.  $\text{Tor}_R^i(R, N) = 0$  for all  $i > 0$
2.  $\text{Tor}_R^i(M \oplus M', N) \cong \text{Tor}_R^i(M, N) \oplus \text{Tor}_R^i(M', N)$ , given by the natural isomorphism.

We claim that  $\epsilon_M : P_0^M \rightarrow M$  induces the following isomorphism

$$\text{Tor}_R^0(M, N) \cong M \otimes_R N$$

and  $\text{Tor}_R^i(M, N)$ 's are called the higher derived functors of  $- \otimes_R N$ .

## 2.5 Lecture 7 9/18

We continue with our discussion of tor functors. We claim that

**Proposition 2.3.** The natural isomorphism gives the following

$$\text{Tor}_R^0(-, N) \cong - \otimes_R N$$

i.e., for any  $M$ ,

$$\text{Tor}_R^0(M, N) \cong M \otimes_R N$$

*Proof.* By definition,  $\text{Tor}_R^0(M, N)$  is the 0-th homology of

$$\cdots \rightarrow P_1^M \otimes_R N \xrightarrow{\partial_1 \otimes \text{id}_N} P_0^M \otimes_R N \rightarrow 0 \rightarrow 0 \rightarrow \cdots$$

this implies that

$$\text{Tor}_R^0(M, N) = \frac{P_0^M \otimes_R N}{\text{im}(\partial_1 \otimes \text{id}_R)}$$

We complete the proof using the following lemma.

**Lemma 2.1.** We claim that the functor  $-\otimes_R N$  is right exact, meaning that given a sequence of  $R$ -modules,

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \rightarrow 0$$

that is exact at  $A_0$  and  $M$ , the following sequence:

$$A_1 \otimes_R N \xrightarrow{f \otimes \text{id}} A_0 \otimes_R N \xrightarrow{g \otimes \text{id}} M \otimes_R N \rightarrow 0$$

is also exact at  $A_0 \otimes_R N$  and  $M \otimes_R N$ .

If we assume the lemma for now, then applying it to

$$P_1 \xrightarrow{\partial_1^M} P_0 \xrightarrow{\epsilon} M \rightarrow 0$$

then we are done!

We prove the lemma now: exactness of  $M \otimes_R N$  implies that  $g \otimes \text{id}$  is surjective. Given that  $g: A_0 \rightarrow M$  is surjective, every generator of  $M \otimes n$  in  $M \otimes_R N$  is of the form  $g \otimes \text{id}(a \otimes n)$  for some  $a \in A_0$ . This implies that  $g \otimes \text{id}$  is surjective.

Next, we need to show that  $\ker(g \otimes \text{id}) = \text{im}(f \otimes \text{id})$ . It is clear that  $\supset$  holds, hence it suffices to show  $\subset$ . Let  $K = \ker g \otimes \text{id}$ , we need to show that

$$\frac{A_0 \otimes_R N}{K} \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

is surjective. It is enough to construct a map:

$$M \otimes_R N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

by the first isomorphism theorem in algebra and the fact that  $g \otimes \text{id}$  is surjective. To get such a map, we need to construct a bilinear map

$$M \times N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})}$$

defined as

$$(m, n) \mapsto (a, n)$$

and let  $a = g^{-1}(m)$  be a choice of the preimage. We remark that there could be many choices of  $a$ , but the difference  $a_1 - a_2$  comes from  $f$ , since  $A_0$  is exact. This implies that this map is well-defined. This implies that the above map is surjective. There for

$$M \times N \rightarrow \frac{A_0 \otimes_R N}{\text{im}(f \otimes \text{id})} \xrightarrow{g \otimes \text{id}} M \otimes_R N$$

this composition is surjective. (Two surjective maps and maps to identity=isomorphism).  $\square$



**Warning 2.6.** We saw that tensor product preserves surjectivity, but it does not necessarily preserve injectivity. Namely, if we replace the statement of the claim with SES

$$0 \rightarrow A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \rightarrow 0$$

and consider

$$0 \rightarrow A_1 \otimes N \rightarrow \dots$$

$f$  need not to be injective.

Next we see the sufficient condition for  $\text{Tor}_R^i$  to vanish for all  $i \geq 2$ .

**Corollary 2.2.** If  $R$  is a PID, then  $\text{Tor}_R^i(M, N) = 0$  for all  $i \geq 2$ .

*Proof.* Consider the free resolution of  $M$ :

$$0 \rightarrow K \rightarrow R\langle M \rangle \rightarrow 0 \rightarrow \dots$$

such that  $R\langle M \rangle/K \cong M$ . Recall that all submodules of a free module are free, so we can just take  $K = P_1$ , then we have

$$0 \rightarrow K \otimes_R N \rightarrow R\langle M \rangle \otimes_R N \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

so the only homologies are  $\text{Tor}_R^0, \text{Tor}_R^1$ . □

**Problem 2.13 (HW(2.4)).** Calculate  $\text{Tor}_{\mathbb{Z}}^1$  and  $\text{Tor}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$  for  $m, n > 0$ . (Note:  $m, n$  could be equal or not).

**Definition 2.13 (ext functor).** Fix an  $R$ -module  $N$ , consider the functor

$$\text{Hom}_R(-; N) : \text{Mod}_R^{\text{op}} \rightarrow \text{Mod}_R$$

Define the functors  $\text{Ext}_R^i(-, N) : \text{Mod}_R^{\text{op}} \rightarrow \text{Mod}_R$  as follows:

$$\text{Ext}_R^i(M, N) = H^i(\text{Hom}_R(P_{\bullet}^M, N))$$

where  $P_{\bullet}^M$  is a free resolution of  $M$ .

We note that if  $R$  is a PID, then  $\text{Ext}_R^i(M, N) = 0$  for all  $i \geq 2$ .

**Proposition 2.4.** We have

$$\text{Ext}_R^0(M, N) \cong \text{Hom}(M, N)$$

*Proof.* This requires the following lemma:

**Lemma 2.2.** If

$$A_1 \xrightarrow{f} A_0 \xrightarrow{g} M \rightarrow 0$$

is right exact, then

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(A_0, N) \rightarrow 0$$

is exact at  $\text{Hom}_R(M, N)$  and  $\text{Hom}_R(A_0, N)$ . □

**Problem 2.14 (HW(2.5)).** Prove the above lemma.

**Problem 2.15 (HW(2.6)).** Prove the following statements about the Ext functor.

1.

$$\mathrm{Ext}_R^i\left(\bigoplus_{\alpha} M_{\alpha}, N\right) \cong \prod_{\alpha} \mathrm{Ext}_R^i(M_{\alpha}, N)$$

2.

$$\mathrm{Ext}_R^i(M, \prod_{\alpha} N_{\alpha}) \cong \prod_{\alpha} \mathrm{Ext}_R^i(M, N_{\alpha})$$

3. Calculate

$$\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

Next we state and prove Algebraic Kunneth theorem.

**Theorem 2.7 (AKT).** Let  $R$  be a PID, and let  $\{M_{\bullet}, \partial_{\bullet}^M\}, \{N_{\bullet}, \partial_{\bullet}^N\}$  be PGCC of  $R$ -modules such that  $M_i$  is free for all  $i$ . Then there exists a SES:

$$0 \rightarrow \bigoplus_{i+j=n} H_i(M) \otimes_R H_j(N) \xrightarrow{X} H_n((M \otimes_R N)_{\bullet}) \rightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}_R^1(H_i(M_{\bullet}), H_j(N_{\bullet})) \rightarrow 0$$

where  $X$  denotes the algebraic crossproduct.

*Proof.* too long, will type up later

□

## 2.6 Lecture 8 9/23

**Corollary 2.3.** Let  $R$  be a field, then the algebraic crossproduct induces an isomorphism:

$$0 \rightarrow \bigoplus_{i+j=n} H_i(M) \otimes_{\mathbb{F}} H_j(N) \cong H_n(M \otimes_{\mathbb{F}} N)$$

where the isomorphism is given by the algebraic crossproduct  $X$ .

**Corollary 2.4 (Universal Coefficient Theorem).** Let  $\{M_{\bullet}, \partial_{\bullet}^M\}$  be a chain complex of free  $\mathbb{Z}$ -modules, and let  $R$  be any commutative ring, then there is a SES:

$$0 \rightarrow H_n(M_{\bullet}) \otimes_{\mathbb{Z}} R \xrightarrow{f} H_n(M \otimes_{\mathbb{Z}} R) \rightarrow \mathrm{Tor}_{\mathbb{Z}}^1(H_{n-1}(M), R) \rightarrow 0$$

where  $f$  is injective but not necessarily surjective (the failure to be surjective is measured by  $\mathrm{Tor}_R^1$ ).

*Proof.* Use AKT with

$$N_i = \begin{cases} R, & i = 0 \\ 0, & i > 0 \end{cases}, \quad H_i(N) = \begin{cases} R, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

Hence  $(M \otimes_{\mathbb{Z}} N)_{\bullet} = M_{\bullet} \otimes_{\mathbb{Z}} R$ .

□

**Problem 2.16 (HW(2.7)).** Prove the UCT in cohomology: let  $\{M_\bullet, \partial_\bullet^M\}$  be a chain complex of free  $\mathbb{Z}$ -modules, let  $R$  be any commutative ring, then there exists SES

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(M), R) \rightarrow H^n(\text{Hom}_{\mathbb{Z}}(M_\bullet, R)) \xrightarrow{\beta} \text{Hom}_{\mathbb{Z}}(H_n(M), R) \rightarrow 0$$

Hint: use the same proof for AKT, instead of  $\otimes$  with  $N$ , you take the Hom into  $R$ .

## Chapter 3

# Singular Cohomology

We begin with some basic definitions.

**Definition 3.1 ( $n$ -simplex).** The standard  $n$ -simplex  $\Delta_n \subset \mathbb{R}^{n+1}$  is defined as

$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1} : x = \sum_{i=0}^n t_i e_i, t_i \geq 0, \sum_i t_i = 1 \right\}$$

where  $e_i, 0 \leq i \leq n$  are the standard basis vectors of  $\mathbb{R}^{n+1}$ .

**Definition 3.2 (face).** Let  $0 \leq i \leq n$ , then the  $i$ th face  $F_i$  of  $\Delta_n$  is the  $(n-1)$ -simplex

$$F_i = \{x \in \Delta_n : t_i = 0\}$$

**Definition 3.3 (singular chain complex).** Given a topological space  $X$ , the singular chain complex of  $X$ , with  $\mathbb{Z}$  coefficients, denoted as  $S_\bullet(X, \mathbb{Z})$  is defined as

$$S_i(X, \mathbb{Z}) = \begin{cases} 0, & i < 0 \\ \mathbb{Z}\langle \Delta_i(X) \rangle, & i \geq 0 \end{cases}$$

where  $\Delta_i(X)$  is the set of continuous maps from  $\Delta_i \rightarrow X$ . We define  $\partial_n : S_n(X, \mathbb{Z}) \rightarrow S_{n-1}(X, \mathbb{Z})$  as follows:

$$\partial_n \langle f \rangle = \sum_{i=0}^n (-1)^i \langle f \circ F_i \rangle$$

where  $\langle f \rangle$  is a generator of  $\Delta_i(X)$ , and  $f : \Delta_X \rightarrow X$ , where

$$f \circ F_i = \Delta_{n-1} \rightarrow \Delta_n \xrightarrow{f} X$$

Note to complete this definition, one needs to check that  $\partial^2 = 0$ , which we did in class. **might include this later**

### 3.1 Lecture 9 9/25

Recall that last time, we defined the singular chain complexes  $S_\bullet(X, \mathbb{Z})$  with  $\mathbb{Z}$ -coefficients:

$$S_i(X, \mathbb{Z}) = \begin{cases} 0, & i < 0 \\ \mathbb{Z}\langle \Delta_i(X) \rangle, & i \geq 0 \end{cases}$$



where  $\Delta_i(X)$  is the set of continuous maps from  $\Delta_i$  to  $X$ . Now we discuss some variations of this concept.

**Definition 3.4** (relative singular chain complex with  $\mathbb{Z}$ -coefficients). Let  $A \subset X$  be a subspace, define  $S_\bullet(X, A, \mathbb{Z})$  by

$$S_i(X, A, \mathbb{Z}) = \begin{cases} 0, & i < 0 \\ \frac{\mathbb{Z}\langle \Delta_i(X) \rangle}{\mathbb{Z}\langle \Delta_i(A) \rangle}, & i \geq 0 \end{cases}$$

note that the quotient is still free.

We note that  $S_\bullet(X, A, \mathbb{Z})$  is a chain complex with the following  $\partial$  maps such that the following diagram commutes:

$$\begin{array}{ccc} S_i(A, \mathbb{Z}) & \xrightarrow{\partial_i} & S_{i-1}(A, \mathbb{Z}) \\ \downarrow & & \downarrow \\ S_i(X, \mathbb{Z}) & \xrightarrow{\partial_i} & S_{i-1}(X, \mathbb{Z}) \\ \downarrow & & \downarrow \\ S_i(X, A, \mathbb{Z}) & \xrightarrow{\partial_i} & S_{i-1}(X, A, \mathbb{Z}) \end{array}$$

**Definition 3.5** ( $S_\bullet(X, A, R)$ ). We define  $S_\bullet(X, A, R)$ , where  $R$  is any commutative ring, and

$$S_\bullet(X, A, R) = S_\bullet(X, A, \mathbb{Z}) \otimes_{\mathbb{Z}} R$$

it is a chain complex of  $R$ -modules with  $\partial_i$  induced from  $S_\bullet(X, A, \mathbb{Z})$ .

The last variation is as follows:

**Definition 3.6** (singular cochain complex). Define the singular cochain complex of  $R$ -modules  $S^\bullet(X, A, R)$  as follows:

$$S^i(X, A, R) := \text{Hom}_{\mathbb{Z}}(S_i(X, A, \mathbb{Z}), R) = \text{Hom}_R(S_i(X, A, R), R)$$

where  $\partial^i$  is induced from  $\partial_i$  in  $S_\bullet(X, A, R)$ .

We make the following remark: if  $A = \emptyset$ , then  $S_\bullet(X, A, \mathbb{Z}) = S_\bullet(X, \mathbb{Z})$ . Previously, we did UCT for chain complexes of free  $\mathbb{Z}$ -modules, here we state the universal coefficient theorem for singular chain complexes:

**Theorem 3.1 (Universal Coefficient Theorems).** We have some SES's:

1. There exists a short exact sequence

$$0 \rightarrow \bigoplus_{i+j=n} H_i(X, A, \mathbb{Z}) \otimes H_j(Y, B, \mathbb{Z}) \rightarrow H_n(S_\bullet(X, A, \mathbb{Z}) \otimes S_\bullet(Y, B, \mathbb{Z})) \rightarrow \bigoplus_{i+j=n-1} \text{Tor}_{\mathbb{Z}}^1(H_i(X, A, \mathbb{Z}), H_j(Y, B, \mathbb{Z})) \rightarrow 0$$

where  $H_i(X, A, \mathbb{Z}) = H_i(S_\bullet(X, A, \mathbb{Z}))$

2. There exists a SES:

$$0 \rightarrow H_i(X, A, \mathbb{Z}) \otimes_{\mathbb{Z}} R \xrightarrow{f} H_i(X, A, R) \rightarrow \text{Tor}_{\mathbb{Z}}^1(H_{i-1}(X, A, \mathbb{Z}), R) \rightarrow 0$$

again  $f$  is injective, and the failure to be surjective is measured by  $\text{Tor}_{\mathbb{Z}}^1$ .

3. There exists a SES:

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_{n-1}(X, A, \mathbb{Z}), R) \rightarrow H^n(X, A, R) \xrightarrow{\beta} \text{Hom}_{\mathbb{Z}}(H_n(X, A, \mathbb{Z}), R) \rightarrow 0$$

note all the above assumes  $R$  is a PID.

We next introduce the category PTop.

**Definition 3.7 (PTop).** The category PTop has objects pairs  $(X, A)$  where  $A \subset X$  is a subspace of a topological space  $X$ . where

$$\text{mor}_{\text{PTop}}((X, A), (Y, B)) = \text{set of continuous maps from } X \rightarrow Y \text{ that sends } A \text{ to } B$$

i.e., the image of  $f$  in  $A$  is contained in  $B$ .

**Theorem 3.2.**  $S_\bullet(X, A, R)$  is a functor from  $\text{PTop} \rightarrow \text{ch}_R^+$ , and  $S^\bullet(X, A, R)$  is the contravariant functor from PTop to  $\text{coch}_R^+$ .

*Proof.* To show that it is a functor, we know it's defined on objects, we now define it on morphisms. Given  $f : (X, A) \rightarrow (Y, B)$ , we define

$$f_* : S_i(X, A, R) \rightarrow S_i(Y, B, R)$$

as follows:

$$(f_*)_i [\langle g : \Delta_i \rightarrow X \rangle] := [\langle f \circ g : \Delta_i \rightarrow Y \rangle]$$

□

We need to check that it commutes in a ladder as follows:

$$\begin{array}{ccccc} F_j & & & & \\ \downarrow & & & & \\ \Delta_i & \xrightarrow{g} & X & \xrightarrow{f} & Y \end{array}$$

we have

$$\begin{aligned}
 \partial_i \circ (f_*)_i[\langle g : \Delta_i \rightarrow X \rangle] &= \partial_i[\langle f \circ g : \Delta_i \rightarrow Y \rangle] \\
 &= \sum_{j=0}^i (-1)^j (f_*)_{i-1}[\langle g : F_j \rightarrow X \rangle] \\
 &= (f_*)_{i-1} \sum_{j=0}^i (-1)^j [\langle g : F_i \rightarrow X \rangle] \\
 &= (f_*)_{i-1} \circ \partial_i
 \end{aligned}$$

Moreover, it is easy to see that

$$(f \circ g)_* = f_* \circ g_*$$

**Definition 3.8 (singular homology).** The  $n$ th singular homology with coefficients in  $R$  is the composite functor:

$$\mathbf{PTop} \xrightarrow{S_\bullet(X,A,R)} ch_R^+ \xrightarrow{H_n} Mod_R$$

and similarly for cohomologies.

**Example 3.1.** We consider the following simple example  $X = \text{pt}$ , and  $A = \emptyset$ ,  $S_\bullet(\text{pt}, R)$  since the set  $\Delta_i(\text{pt})$  is a singleton  $i \geq 0$ . So  $S_\bullet(\text{pt}, R)$  looks like

$$\cdots \rightarrow R \rightarrow R \xrightarrow{\partial_2} R \xrightarrow{\partial_1} R \rightarrow 0 \rightarrow \cdots$$

where

$$H_i(\text{pt}, R) = \begin{cases} 0, & i \neq 0 \\ R, & i = 0 \end{cases}$$

**Definition 3.9 (path-connected).** A space  $X$  is path-connected if given any  $a, b \in X$ , there exists a continuous path  $\gamma : [0, 1] \rightarrow X$  such that

$$\gamma(0) = a, \quad \gamma(1) = b$$

**Proposition 3.1.** If  $X$  is path-connected, then

$$H_0(X, R) \cong R$$

(implying that  $H_0$  a homology group, could be tiny!)

*Proof.* Recall that by definition, we have

$$H_0(X, R) = \frac{R\langle \Delta_0(X) \rangle}{\text{im}(\partial_1)}$$

where  $\partial_1 : R\langle \Delta_1(X) \rangle \rightarrow R\langle \Delta_0(X) \rangle$ . We consider the homomorphism:

$$\varepsilon : R\langle \Delta_0(X) \rangle \rightarrow R$$

such that  $\varepsilon\langle x \rangle = 1$ , for generator  $x \in X = \Delta_0(X)$ . Notice that

$$\partial_1\langle \gamma \rangle = \langle \gamma(1) \rangle - \langle \gamma(0) \rangle$$

Hence

$$\varepsilon \partial_1 \langle \gamma \rangle = \varepsilon \langle \gamma(1) \rangle - \varepsilon \langle \gamma(0) \rangle = 1 - 1 = 0$$

Hence  $\varepsilon$  extends to a surjective map.

**Problem 3.1** (HW(2.8)). Show that  $\varepsilon$  is also injective.

□

Next we stated Eilenberg-Steenrod Axioms. **will fill in later**

## 3.2 Lecture 10 9/30

**we restated ES axioms** at the beginning of class.

**Definition 3.10 (contractible).** A space  $X$  is said to be contractible if the identity map  $i : X \rightarrow X$  is homotopic to the map that sends all  $X$  to some  $x_0 \in X$ , where  $x_0$  is any point. This means, there exists  $h : X \times [0, 1] \rightarrow X$  such that

$$h(x, 0) = x, h(x, 1) = x_0$$

**Problem 3.2.** Use ES axioms to show that if  $X$  is contractible, then

$$H_n(X) = \begin{cases} 0, & n \neq 0 \\ R, & n = 0 \end{cases}$$

**then we proved this fact without using ES axiom, will fill in later**

**Corollary 3.1.** Let  $B$  be an open ball in  $\mathbb{R}^n$ , then

$$H_n(B) = \begin{cases} R, & n = 0 \\ 0, & n > 0 \end{cases}$$

since the ball is contractible.

Then we consider some spheres and hemispheres.

**Definition 3.11 ( $n$ -sphere, hemisphere).** Let  $S^n \subset \mathbb{R}^{n+1}$  be the  $n$ -sphere

$$S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i^2 = 1\}$$

define the hemispheres:

$$D_+^n = \{(x_0, \dots, x_n) \in S^n : x_0 \geq 0\}, \quad D_-^n = \{(x_0, \dots, x_n) \in S^n : x_0 \leq 0\}$$

notice that

$$D_+^n \cap D_-^n = S^{n-1}$$

We make the following observations.

**Proposition 3.2.** We have the following isomorphism.

1.

$$H_i(S^n, D_-^n) \cong H_i(S^n, \{s\}) := \tilde{H}_i(S^n)$$

where we can choose  $\{s\} \in D_-^n$  to be the south pole.

2.

$$H_i(S^n, D_-^n) \cong H_i(D_+^n, S^{n-1})$$

**Problem 3.3 (HW(2.9)).** Give a proof of 1 in the above proposition using the 5-lemma.

we proved these using ES axioms A2A3, insert later

**Theorem 3.3.** There are the following isomorphisms:

1.

$$H_i(D_+^{n+1}, S^n) = \begin{cases} R, & i = n + 1 \\ 0, & \text{else} \end{cases}$$

for  $n \geq 0$ .

2.

$$H_i(S^n) = \begin{cases} R, & i = 0, n \\ 0, & \text{else} \end{cases}$$

where  $n > 0$ .

*Proof.* We will conduct a simultaneous induction. We will do the base case now: for  $n = 0$ , we prove 1. Consider the LES:

$$\cdots \rightarrow H_{i+1}(D^1, S^0) \rightarrow H_i(S^0) \rightarrow H_i(D^1) \rightarrow H_i(D^1, S^0) \rightarrow \cdots$$

For  $i > 0$ , we have

$$H_i(S^0) = H_i(D^1) = 0$$

this implies that for all  $i > 1$ ,

$$H_i(D^1, S^0) = 0$$

For  $i = 0, 1$ , consider

$$0 \rightarrow H_1(D^1, S^0) \rightarrow H_0(S^0) \rightarrow H_0(S^1) \rightarrow H_0(D^1, S^0) \rightarrow 0$$

this implies that

$$H_0(D^1, S^0) = 0, \quad H_1(D^1, S^0) = R$$

Now we begin the induction step, assume 1,2 holds until  $n$ , we consider  $n + 1$ . Consider the LES:

$$\cdots \rightarrow H_i(D_-^{n+1}) \rightarrow H_i(S^{n+1}) \rightarrow H_i(S^{n+1}, D_-^{n+1}) \rightarrow H_{i-1}(D_-^{n+1}) \rightarrow \cdots$$

where

$$H_i(S^{n+1}, D_-^{n+1}) \cong H_i(D_+^{n+1}, S^n)$$

since  $H_i(D_-^{n+1}) = 0$  for all  $i > 0$ , we have

$$H_i(S^{n+1}) \cong H_i(D_+^{n+1}, S^n) \cong \begin{cases} R, & i = n + 1 \\ 0, & 1 \leq i < n + 1 \end{cases}$$

We only need to understand  $H_1(S^{n+1})$  to fully prove (b). Notice that

$$H_1(D^{n+1}) \rightarrow H_1(S^{n+1}) \rightarrow H_1(S^{n+1}, D^{n+1}) \rightarrow H_0(D^{n+1}) \cong H_0(S^{n+1})$$

and it is clear that  $H_1(S^{n+1}) \cong H_1(D^{n+1}, S^n) = 0$ .

**Problem 3.4 (HW(2.10)).** Prove (a) for  $n + 1$  using (b) and the following LES:

$$\cdots \rightarrow H_{i+1}(D_+^{n+2}, S^{n+1}) \rightarrow H_i(S^{n+1}) \rightarrow H_i(D_+^{n+2}) \rightarrow \cdots$$

□

### 3.3 Lecture 11 10/02

We begin by showing  $S^n$  is not a retraction of  $D^{n+1}$  for  $n > 0$ .

**Corollary 3.2.** There is no map  $\gamma : D^{n+1} \rightarrow S^n$  such that

$$\gamma \circ i = \text{id}$$

where  $i : S^n \rightarrow D_{n+1}$  is the inclusion map.

fill in theorem later

**Theorem 3.4 (Brouwer Fixed point theorem).** Let  $f$  be any continuous map  $f : D^{n+1} \rightarrow D^{n+1}$ , for  $n > 0$ . Then there exists  $x \in D^{n+1}$  such that

$$f(x) = x$$

*Proof.* Assume  $f$  existed without a fixed point, then we use this  $f$  to construct a retraction  $g : D^{n+1} \rightarrow S^n$ . We define  $g$  to be the point on  $S^n$  the line segment  $(x, f(x))$  intersects it at. **incomplete** □

**Problem 3.5.** Show the following:

1. Show that  $\mathbb{R}^n$  and  $\mathbb{R}^m$  are not homeomorphic, if  $m \neq n$ .
2. Show that  $S^n$  is not a retraction of  $S^m$  if  $n < m$ .

**Definition 3.12 (augmented CC).** A augmented chain complex of  $R$ -modules is a postively graded chain complex of  $R$ -modules endowed with an  $R$ -module map:

$$\varepsilon : C_0 \rightarrow R$$

such that

$$\varepsilon \circ \partial_1 = 0$$

i.e.,  $\varepsilon$  factors through  $H_0(C_\bullet)$ .

$$\begin{array}{ccccc} C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\varepsilon} & R \\ & & \downarrow & \nearrow & \\ & & H_0(C_\bullet) & & \end{array}$$

An augmented chain complex is called

1. acyclic. If  $H_i(C_\bullet) = 0$  for all  $i > 0$ , and

$$H_0(C_\bullet) \cong R$$

via the map  $\varepsilon$ .

2. free. If  $C_i$  is a free-module for all  $i$ .

A morphism of two augmented chain complex  $\{C_\bullet, \partial_\bullet^C, \varepsilon^C\}, \{D_\bullet, \partial_\bullet^D, \varepsilon^D\}$  is a morphism  $\{f_\bullet\}$  of chain complexes between  $\{C_\bullet, \partial_\bullet^C\}$  and  $\{D_\bullet, \partial_\bullet^D\}$  such that the following diagram commutes:

$$\begin{array}{ccc} C_0 & \xrightarrow{f_0} & D_0 \\ & \searrow \varepsilon^C & \downarrow \varepsilon^D \\ & & R \end{array}$$

**Definition 3.13 ( $\text{Ach}_R$ ).** Let  $\text{Ach}_R$  be the category of augmented chain complexes of  $R$ -modules.

**Problem 3.6 (HW(2.12)).** Show that if  $\{C_\bullet, \partial_\bullet^C, \varepsilon^C\}$  and  $\{D_\bullet, \partial_\bullet^D, \varepsilon^D\}$  are objects in  $\text{Ach}_R$ , then

$$\{(C \otimes D)_\bullet, \partial_\bullet^{C \otimes D}, \varepsilon^C \otimes \varepsilon^D\}$$

is also an object in  $\text{Ach}_R$ .

*Proof.* It suffices to check that

$$\varepsilon^C \otimes \varepsilon^D : C_0 \otimes D_0 \rightarrow R$$

such that

$$\varepsilon^C \otimes \varepsilon^D \circ \partial_1^{C \otimes D} = 0$$

□

**Example 3.2.** Given a topological space  $X$ ,  $S_\bullet(X, R)$  is an object in  $\text{Ach}_R$ , with

$$\varepsilon : S_0(X, R) \rightarrow R$$

where  $\langle x \rangle \mapsto 1, x \in X$ .

**Definition 3.14 (acyclic functor).** Let  $g : C \rightarrow \text{Ach}_R$  be a functor. Let  $M$  be a collection of objects in  $C$ . We say that  $g$  is acyclic with respect to  $M$  if  $g(\alpha)$  is a cyclic for all  $\alpha \in M$ .

We note that  $S_\bullet(-, R) : \text{Top} \rightarrow \text{Ach}_R$  is acyclic with respect to all contractible spaces. Next we introduce free functors.

**Definition 3.15.** Let  $g : C \rightarrow \text{Ach}_R$  be a functor, and let  $M = \bigsqcup_{k \in \mathbb{N}} M_k$  with

$$M_k = \{(\alpha, k), i_{\alpha, k}\}$$

where  $M_k$  is a collection of objects  $(\alpha, k)$  in  $C$  endowed with a collection of elements

$$\{i_{(\alpha, k)}\} \in g((\alpha, k))_k$$

where  $g((\alpha, k))_k$  means the degree  $k$  of a chain complex, which is an  $R$ -module. We say that  $g$  is free with respect to  $M$  if given any object  $\beta$  in  $C$ , we have

$$g(\beta)_k = \bigoplus_{f \in \text{mor}((\alpha, k), \beta)} R\langle f_*(i_{\alpha, k}) \rangle$$

where  $f_* = g(f)_k$ .

Next we did an example. **fill in later**

Next we prove the Acyclic Model Theorem.

**Theorem 3.5 (AMT).** Let  $g, g'$  be functors from  $C$  to  $\text{Ach}_R$ , let  $M$  be a collection of module in  $C$  such that  $g$  is free and  $g'$  is acyclic. Then the following statements are true:

1. There exists a natural transformation  $T : g \Rightarrow g'$ .
2. If  $T, T'$  are two such transformations, then there exists a natural homotopy  $D$  between them, i.e., given any object  $X$  of  $C$ , there exists a chain homotopy  $D_X : g(X)_k \rightarrow g'(X)_k$  such that

$$\partial D_X + D_X \partial = TX - T'X$$

and the following diagram commutes: for all  $f : X \rightarrow Y$ :

$$\begin{array}{ccc} g(X)_k & \xrightarrow{D_X} & g'(X)_{k+1} \\ g(f) \downarrow & & \downarrow g'(f) \\ g(Y)_k & \xrightarrow{D_Y} & g'(Y)_{k+1} \end{array}$$

## 3.4 Lecture 10/07

We first prove the AMT stated last lecture. **insert the proof here.**

Next we talk about some applications of AMT.

**Theorem 3.6.** Given topological spaces  $X, Y$ , the chain complexes

$$S_\bullet(X, R) \otimes_R S_\bullet(Y, R) \text{ and } S_\bullet(X \times Y, R)$$

are naturally chain homotopic, i.e., the functors from  $\text{Top} \times \text{Top}$  to  $\text{Ach}_R$  are naturally chain homotopically equivalent.



*Proof.* This states that there exists a natural transformation

$$EZ : S_{\bullet}(X, R) \otimes_R S_{\bullet}(Y, R) \rightarrow S_{\bullet}(X \times Y, R)$$

and

$$EZ^{-1} : S_{\bullet}(X \times Y, R) \rightarrow S_{\bullet}(X, R) \otimes_R S_{\bullet}(Y, R)$$

such that

$$EZ^{-1} \circ EZ \cong \text{id}_{S_{\bullet}(X, Y)}$$

via a natural homotopy  $D$ , and similarly for the reverse.

Consider the functors

$$g, g' : \text{Top} \times \text{Top} \rightarrow \text{Ach}_R$$

where

$$g = S_{\bullet}(X) \otimes S_{\bullet}(Y), g' = S_{\bullet}(X \times Y)$$

Consider the models,

$$M = \bigsqcup_{k \in \mathbb{N}} M_k$$

such that

$$M_k = \{(\Delta_p, \Delta_q) : p + q = k, i_{(p,q)} \in g(\Delta_p, \Delta_q) = \langle \text{id}_{\Delta_p} \rangle \otimes \langle \text{id}_{\Delta_q} \rangle\}$$

**Problem 3.7 (HW(3.1)).** Show that  $g$  is free with respect to  $M$ .

We note that  $\Delta_p \times \Delta_q$  is contractible, hence  $S_{\bullet}(\Delta_p \times \Delta_q)$  is acyclic. So  $g'$  is acyclic with respect to  $M$ . This implies that there exists a natural transformation  $EZ : g \Rightarrow g'$  by AMT.

Now we consider  $g = S_{\bullet}(X \times Y)$ ,  $g' = S_{\bullet}(X) \otimes S_{\bullet}(Y)$ , and  $M = \bigsqcup_k M_k$ , where

$$M_k = \{(\Delta_k, \delta_k) : i_{(k,k)} \in S_k(\Delta_k \times \Delta_k) = \langle \Delta \rangle : \Delta_k \rightarrow \Delta_k \times \Delta_k\}$$

i.e.,  $\langle \Delta \rangle$  is the diagonal map. Now we show that  $g$  is free wrt  $M$ .

$$\begin{aligned} g(X, Y)_k &= S_k(X \times Y) \\ &= \bigoplus_{f \in \text{maps}(\Delta_k, X \times Y)} R\langle f \rangle \\ &= \tilde{f} \in \text{maps}(\Delta_k, X) \times \text{maps}(\Delta_k, Y) \end{aligned}$$

with  $f$  given by the composite

$$\Delta_k \xrightarrow{\Delta} \Delta_k \times \Delta_k \xrightarrow{(f_1, f_2) = \tilde{f}} X \times Y$$

By Algebraic Kunnetth Theorem and UCT

$$S_{\bullet}(\Delta_k) \otimes S_{\bullet}(\Delta_k) \text{ is acyclic}$$

we get  $EZ^{-1} : g' \Rightarrow g$  by AMT.

**Problem 3.8 (HW(3.2)).** Finish the proof by showing  $S_{\bullet}(X) \otimes S_{\bullet}(Y)$  is free and acyclic with respect to  $M$ , and similarly  $S_{\bullet}(X \times Y)$  is also free and acyclic. In other words, check free is also acyclic and acyclic is also free. Then conclude by AMT that

$$S_{\bullet}(X, R) \otimes_R S_{\bullet}(Y, R) \text{ and } S_{\bullet}(X \times Y, R)$$

are naturally chain homotopic

□

### 3.5 Lecture 10/09

Last time we showed that

$$S_\bullet(X, R) \otimes_R S_\bullet(Y, R) \text{ and } S_\bullet(X \times Y, R)$$

are chain homotopic. We now make a few remarks:

1. The above theorem extends to a relative natural equivalence: using naturality of the non-relative theorem:  $EZ : S_\bullet(X, A, R) \otimes S_\bullet(Y, B, R) \rightarrow S_\bullet(X \times Y, X \times B \cup A \times Y, R)$
2. On the level of cochain complexes, we have corresponding cochain homotopy equivalences:

$$EZ^* : S^\bullet(X \times Y, R) \rightarrow S^\bullet(X, R) \otimes S^\bullet(Y, R)$$

and more generally for relative equivalences.

3. If  $X = Y$ , then there is a nice classification of the following composite:

$$S_\bullet(X, R) \xrightarrow{\Delta_*} S_\bullet(X \times X, R) \xrightarrow{EZ^{-1}} S_\bullet(X, R) \otimes S_\bullet(X, R)$$

given by the Alexander-Whitney diagonal formula, where  $\Delta_*$  is the map induced by the diagonal map:

$$\Delta : X \rightarrow X \times X$$

where  $x \mapsto (x, x)$ .

Next we talk about the Alexander-Whitney construction.

**Definition 3.16.** The Alexander-Whitney diagonal is a natural map:

$$AW : S_\bullet(X, R) \rightarrow S_\bullet(X, R) \otimes_R S_\bullet(X, R)$$

such that

$$AW\langle f \rangle = \bigoplus_{p+q=n} \langle f \circ Fr^p \rangle \otimes \langle f \circ Bk^q \rangle$$

where  $Fr, Bk$  means front and back, and  $f : \Delta_n \rightarrow X$  is a continuous map and

$$f \circ Fr^p$$

denotes the restriction of  $f$  to the front  $p$ -face given by

$$Fr^p = \{x = (t_0, \dots, t_n) : t_i = 0, \forall i > p\}$$

and the back  $q$ -face is

$$Bk^q = \{x = (t_0, \dots, t_n) : t_i = 0, \forall i < n - q\}$$

Note that

$$\langle f \circ Fr^p \rangle \in S_p(X, R)$$

and

$$\langle f \circ Bk^q \rangle \in S_q(X, R)$$

**Problem 3.9 (HW(3.3)).** Show that  $AW\langle f \rangle$  is a chain map, i.e., it commutes with the differential on both sides.

**Theorem 3.7.** The Alexander-Whitney diagonal is naturally homotopic to the following composite:

$$S_{\bullet}(X, R) \xrightarrow{\Delta_*} S_{\bullet}(X \times X, R) \xrightarrow{EZ^{-1}} S_{\bullet}(X, R) \otimes S_{\bullet}(X, R)$$

*Proof.* Consider the functors:

$$g, g' : Top \rightarrow \text{Ach}_R$$

given by

$$g(X) = S_{\bullet}(X, R), g'(X) = S_{\bullet}(X, R) \otimes S_{\bullet}(X, R)$$

consider the collection

$$M = \bigsqcup_k M_k, M_k = \{\Delta_k\}$$

and let  $i_k \in S_k(\Delta_k, R)$  be  $\langle id_{\Delta_k} \rangle$ . Then notice that  $g$  is free with respect to  $M$  and  $g'$  is acyclic with respect to  $M$  by the algebraic Kunnet theorem. Hence the second part of the AMT says that the two natural transformations  $AW$  and  $EZ^{-1} \circ \Delta_*$  are naturally chain homotopic.  $\square$

Next we prove the ES axiom 2.

**Proposition 3.3.** Singular homology satisfies the homotopy axiom.

*Proof.* fill in later  $\square$

Next we will prove the ES A3, the Excision axiom. We introduce a few definitions before doing so. We will talk about barycentric subdivision.

**Definition 3.17 (barycenter).** Let  $\Delta_n$  denote the  $n$ -simplex, let

$$\{e_{i_0}, \dots, e_{i_k}\}$$

be some collection of vertices of  $\Delta_n$ , where

$$I = \{i_0, \dots, i_k\} \subset \{0, \dots, n\}$$

The barycenter of  $I$  is

$$b(I) = \frac{1}{k+1} \sum_{i \in I} e_i \in \Delta_n$$

Namely,  $b(I)$  is the center of the face spanned by vertices  $e_i$  where  $i \in I$ .

Let  $\sigma \in S_{n+1}$  be a permutation of  $\{0, \dots, n\}$ . Define an  $n$ -simplex  $B(\sigma)$  to be the subspace of  $\Delta_n$  with vertices:

$$\{b(\sigma\{0, \dots, n\}), b(\sigma\{1, \dots, n\}), \dots, b(\sigma\{n\})\}$$

note that  $b$  does not change under permutations, i.e.,

$$B(\sigma) = \left\{ \sum_{j=0}^n t_j b(\sigma\{j, \dots, n\}) : t_j \geq 0, \sum_j t_j = 1 \right\}$$

**Definition 3.18.** The barycentric subdivision is a natural transformation

$$L : S_\bullet(X) \rightarrow S_\bullet(X)$$

such that given  $\langle f \rangle \in S_n(X)$ , we define

$$L(\langle f \rangle) = \sum_{\sigma \in S_{n+1}} (-1)^{\text{sgn}(\sigma)} \langle f \circ B(\sigma) \rangle$$

**Problem 3.10 (HW(3.4)).** Show that  $L$  is a chain map, i.e.,

$$\partial L = L \partial$$

Hint: in  $\partial L$  the internal faces cancel off.

**Corollary 3.3.** By AMT, we have  $L^1, L^2, \dots, L^k$  are all chain homotopic to the identity transformation.

We make the remark that by the naturality of  $L$ , we see that  $L$  extends to a natural transformation

$$L : S_\bullet(X, A, R) \rightarrow S_\bullet(X, A, R)$$

such that  $L^k$  is chain homotopic to the identity for all  $k \in \mathbb{N}$ .

Next time, we will talk about the Excision Axiom A3.

## 3.6 Lecture 10/14

We start the proof of Excision Axiom.

**Definition 3.19 (good cover).** A good cover  $\{U_i\}$  of a topological space  $X$  is a cover such that  $\{\text{int}(U_i)\}$  is an open cover of  $X$ .

We define chain complex

$$\{S_\bullet^U(X, R), \partial_\bullet\}$$

by

$$S_n^U(X, R) = \bigoplus_{f \in \Delta_n^U(X)} R\langle f \rangle$$

where  $\Delta_n^U(X)$  is the set of all continuous maps from  $\Delta_n$  to  $X$  that land in some  $U_i$ , and the differential is given by the usual formula.

The relative version is as follows: if  $A \subset X$  then  $\{A \cap U_i\}$  is a good cover of  $A$ , which we also denote by  $U$ . Define

$$S_\bullet^U(X, A, R) = \{S_\bullet^U(X, R) / S_\bullet^U(A, R); \partial_\bullet\}$$

is the relative version of the definition.

**Proposition 3.4.** The canonical inclusion

$$i : S_\bullet^U(X, A, R) \rightarrow S_\bullet(X, A, R)$$

induces an isomorphism in homology:

$$i_* : H_*(S_\bullet^U(X, A, R)) \cong H_*(X, A, R)$$

*Proof.* First we reduce to the case where  $A = \emptyset$  using the 5-lemma. Consider the ladder of LES:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(S_{\bullet}^U(A, R)) & \longrightarrow & H_n(S_{\bullet}^U(X, R)) & \longrightarrow & H_n(S^U(X, A, R)) \longrightarrow \dots \\ & & \downarrow i_A & & \downarrow i_X & & \downarrow i_{(X, A)} \\ \dots & \longrightarrow & H_n(A, R) & \longrightarrow & H_n(X, R) & \longrightarrow & H_n(X, A, R) \longrightarrow \dots \end{array}$$

So if we show that  $i_A, i_X$  are isomorphisms, then  $i_{(X, A)}$  is also an isomorphism by the 5-lemma. So we consider

$$i_* : H_*(S_{\bullet}^U(X, R)) \rightarrow H_*(X, R)$$

First we show that  $i_*$  is injective, consider

$$\begin{array}{ccc} S_{n+1}^U(X) & \xrightarrow{i} & \beta \in S_{n+1}(X) \\ \downarrow \partial & & \downarrow \partial \\ S_n^U(X) & \xrightarrow{i} & \partial\beta \in S_n(X) \end{array}$$

Let  $[\alpha] \in H_n(S_{\bullet}^U(X))$  such that  $i_*[\alpha] = 0$ . Then

$$\alpha = \partial\beta$$

for some  $\beta \in S_{n+1}(X)$ . Now apply  $L^k$  for some  $k \gg 0$ . Then

$$L^k(\beta) \in S_{n+1}^U(X)$$

We know that

$$L^k(\beta) - \beta = (D\partial + \partial D)(\beta)$$

for some natural homotopy  $D$ , i.e.,

$$\begin{aligned} L^k(\beta) - \beta &= D\partial(\beta) + \partial D(\beta) \\ &= D(\alpha) + \partial D(\beta) \end{aligned}$$

Hence  $\beta = L^k(\beta) - \partial D(\beta) - D(\alpha)$ , hence

$$\partial\beta = \partial(L^k(\beta) - D(\alpha)) = \alpha$$

but  $L^k(\beta) - D(\alpha) \in S_{n+1}^U(X, R)$ , by subdivision and naturality, we see that

$$[\alpha] = 0$$

in  $H_n(S_{\bullet}^U(X, R))$ .

Next we show that  $i_*$  is surjective. Given  $[\alpha] \in H_n(X, R)$ , replace  $[\alpha]$  with  $[L^k(\alpha)]$  for  $k \gg 0$  using the fact that  $L^k$  is chain homotopic to the identity, but

$$L^k(\alpha) \in S_n^U(X, R)$$

so  $i_*$  is surjective. □

**Corollary 3.4.** Singular homology satisfies Excision Axiom A3.

*Proof.* fill in later □

**Theorem 3.8 (Mayer-Vietoris sequence).** Let  $V_1, V_2$  be such that  $\text{int}(V_1) \cup \text{int}(V_2) = X$ , then there is a LES:

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_i(V_1 \cap V_2) & \xrightarrow{s} & H_i(V_1) \oplus H_i(V_2) & \xrightarrow{t} & H_i(X) \\ & & & & \swarrow & & \searrow \\ & & H_{i-1}(V_1 \cap V_2) & \longleftarrow & H_{i-1}(V_1) \oplus H_{i-1}(V_2) & \longleftarrow & H_{i-1}(X) \\ & & & & \swarrow & & \searrow \\ & & \dots & \longleftarrow & & & \end{array}$$

where  $s$  is induced by the map  $(i_1)_* \oplus -(i_2)_*$ , where

$$i_1 : V_1 \cap V_2 \rightarrow V_1$$

and

$$i_2 : V_1 \cap V_2 \rightarrow V_2$$

and  $t$  is induced by the map

$$(j_1)_* + (j_2)_*$$

where

$$j_1 : V_1 \rightarrow X, j_2 : V_2 \rightarrow X$$

**Problem 3.11 (HW(3.5)).** Calculate the homology of the Riemannian surface  $\Sigma_g$ ,  $g$  stands for  $g$ -holes. For example,  $\Sigma_0 = \mathbb{S}^2$ ,  $\Sigma_1 = S^1 \times S^1$ .

Next we talk about products in cohomology. We now introduce and study 4 types of natural pairings.

1. Topological crossproduct in homology:

$$\bigoplus_{i+j=n} H_i(X, A) \otimes_R H_j(Y, B) \xrightarrow{X_\bullet} H_n(X \times Y, A \times Y \cup X \times B)$$

2. Topological cross product in cohomology:

$$\bigoplus_{i+j=n} H^i(X, A) \otimes_R H^j(Y, B) \xrightarrow{X_\bullet} H^n(X \times Y, A \times Y \cup X \times B)$$

3. Cup product in cohomology

$$\bigoplus_{i+j=n} H^i(X, A) \otimes_R H^j(X, B) \xrightarrow{\cup} H^n(X, A \cup B)$$

4. Cap product between (co)homologies:

$$\bigoplus_{i+j=n} H^i(X, A) \otimes_R H_n(X, A \cup B) \xrightarrow{\cap} H_{n-i}(X, B)$$

We start with the first type: topological product in homology. We define the natural pairing

$$\bigoplus_{i+j=n} H_i(X, A) \otimes_R H_j(Y, B) \xrightarrow{X_\bullet} H_n(X \times Y, A \times U \cup X \times B)$$

as the composite:

$$\begin{array}{ccc} H_i(S_\bullet(X, A)) \otimes H_j(S_\bullet(Y, B)) & \xrightarrow{X_\bullet} & H_n(S_\bullet(X \times Y, X \times B \cup A \times Y)) \\ & \searrow X_{atg} & \uparrow EZ(\cong) \\ & & H_n(S_\bullet(X, A) \otimes_R S_\bullet(Y, B)) \end{array}$$

where  $X_{alg}$  denotes the algebraic cross product:

$$X_{alg}([\alpha] \otimes [\beta]) := [\alpha \otimes \beta]$$

We note that  $X$  can be seen as a natural transformation between the functors  $H_i \otimes H_j$  and  $H_{i+j}$  from

$$PTop \times PTop \rightarrow R-Mod$$

(II) Next we talk about topological crossproduct in cohomology. To define topological cross product, we first observe that for any  $R$  modules  $V, W$ , there is a natural map

$$Hom(V, R) \otimes_R Hom(W, R) \xrightarrow{\lambda} Hom(V \otimes W, R)$$

and

$$\lambda(\varphi \otimes \psi)(a \times b) = \varphi(a)\psi(b)$$

We then define

$$\bigoplus_{i+j=n} H^i(X, A) \otimes_R H^j(Y, B) \xrightarrow{X_\bullet} H^n(X \times Y, A \times B \cup X \times Y)$$

as the composite

$$\begin{array}{ccc} H^i(S^\bullet(X, A)) \otimes H^j(S^\bullet(Y, B)) & \xrightarrow{X_\bullet} & H^n(S^\bullet(X \times Y, X \times B \cup A \times Y)) \\ \downarrow X_{alg} & & \uparrow EZ^{-1}(\cong) \\ H^n(S^\bullet(X, A) \otimes S^\bullet(Y, B)) & \xrightarrow{\lambda} & H^n(Hom(S_\bullet(X, A) \otimes_R S_\bullet(Y, B), R)) \end{array}$$

Note: as before,  $X$  can be seen as a natural transformation between the two obvious functors.

Recall the map

$$\beta : H^i(X, A) \rightarrow Hom(H_i(X, A), R)$$

**Problem 3.12 (HW(3.6)).** Show the following diagram commutes:

$$\begin{array}{ccc} H^i(X, A) \otimes H^j(Y, B) & \xrightarrow{X_\bullet} & H^{i+j}(X \times Y, X \times B \cup A \times Y) \\ \beta \otimes \beta \downarrow & & \downarrow \beta \\ Hom(H_i(X, A), R) \otimes Hom(H_j(Y, B), R) & & Hom(H_{i+j}(X \times Y, X \times B \cup A \times Y), R) \\ & \searrow \lambda & \downarrow X_\bullet^* \\ & & Hom(H_i(X, A) \otimes_R H_j(Y, B), R) \end{array}$$

where  $X_\bullet^*$  denotes the dual map induced by the cross product in the homology.

### 3.7 Lecture 10/16

### 3.8 Lecture 10/21

### 3.9 Lecture 10/23