Calc III Section Notes with Answers

Fall 2025

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Contents

Chapter 1

The Geometry of Euclidean Spaces

Week 1 (8/25-29)

Logistics

- TA: Hui.
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- Office Hours: Tuesday 4-6 PM, Krieger 211; Friday 1-2 PM Zoom.
- Biweekly Quizzes: 15 min, 10%.
- Attendance: 5%. (If you can't make it, email me).

Icebreaking Activity

- In a group of three or four:
  1. Learn each other names, year, pronouns.
  2. Find something in common and different among you and share with the entire class.
  3. Play Buzz if you have time, with prime 7: say the number if it doens't contain or is not divisible by 7, say buzz otherwise.

Problem 1. Draw the following vectors in $\mathbb{R}^2$:

$$u = (1, 2), \quad v = (3, -2)$$

Compute $u + v, u - v$, and draw them in the plane.

Proof.

$$u + v = (4, 0), \quad u - v = (-2, 4)$$

□

Problem 2. Consider the following vectors in $\mathbb{R}^3$:

$$u = (1, 2, 3), \quad , v = (-2, 1, 4)$$

1. Compute their norms.
2. Two vectors $a, b \in \mathbb{R}^3$ are called **orthognal** if $a \cdot b = 0$. Are u, v orthogonal? If not, find a nonzero vector orthogonal to u.

Proof. 1.

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{14}, \quad \|v\| = \sqrt{21}$$

2. We check

$$u \cdot v = -2 + 2 + 12 = 12 \neq 0$$

thus not orthogonal. A vector that is orthognal to u: $(-3, 0, 1)$. Note that this vector is **not** unique! For example, $(-1, -1, 1)$ is another such vector.

$\square$

Problem 3. Let $u, v \in \mathbb{R}^3$, suppose that u, v are orthongal, show that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Bonus: is the converse true? (meaning assuming $\|u + v\|^2 = \|u\|^2 + \|v\|^2$, is it true that $u \cdot v = 0$?)

Proof. We have

$$\begin{aligned}\|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + u \cdot v + v \cdot u + v \cdot v \\ &= \|u\|^2 + \|v\|^2\end{aligned}$$

because $u \cdot v = v \cdot u = 0$. The converse is also true: we know by definition that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2 + 2u \cdot v$$

given the assumption, we also have

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Thus equating them we get

$$\|u\|^2 + \|v\|^2 + 2u \cdot v = \|u\|^2 + \|v\|^2 \Rightarrow u \cdot v = 0$$

$\square$

Reminders

1. First HW due this Friday.
2. First Quiz next Tuesday.

Week 2 (9/1-5)

Topics: (1) cross product, (2) plane in $\mathbb{R}^3$.

Definition 1 (cross product)**.** Let $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ be vectors in $\mathbb{R}^3$, the cross product of a, b is the vector $a \times b$,

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where i, j, k are the standard vectors in $\mathbb{R}^3$.

Definition 2 (Plane in three dimensions)**.** A perpendicular vector and a normal vector uniquely define a plane in $\mathbb{R}^3$: given the plane $\mathcal{P}$ passing containing the point (x_0, y_0, z_0) that has a normal vector (A, B, C) is given by the equation:

$$\mathcal{P} : A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Proposition 1. Here are some properties of the cross product:

1. $a \times b$ is perpendicular to vectors a, b.
2. The length of the cross product is the area of the parallelogram:

   $$\|a \times b\| = \|a\|\|b\| \sin \theta$$

   where θ is the angle between them. (Compare this with the dot product).
3. $a \times b = -b \times a$, and $a \times (b + c) = a \times b + a \times c$. Moreover, $a \times b = 0$ iff a, b are parallel or either a or b are 0.
4. (HW) The cross product is **not** associative! For example, compute

   $$(i \times i) \times j, \quad i \times (i \times j)$$

Problem 4. Let $\vec{u} = (1, 2, 3), \vec{v} = (0, 1, 1)$ be vectors in $\mathbb{R}^3$, compute the area of the parallelogram spanned by these two vectors.

Proof.

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = -i - j + k = (-1, -1, 1)$$

Thus the area of the parallelogram is

$$\|u \times v\| = \sqrt{3}$$

□

Problem 5. Compute the plane containg all three points:

$$(1, 0, 2), \quad (2, -1, 0), \quad (-1, 2, 3)$$

Proof. Let $A = (1, 0, 2), B = (2, -1, 0), C = (-1, 2, 3)$, then consider two vectors in this plane

$$AB = (1, -1, -2), AC = (-2, 2, 1)$$

Then taking their cross product we find a normal vector to this plane:

$$AB \times AC = \begin{bmatrix} i & j & k \\ 1 & -1 & -2 \\ -2 & 2 & 1 \end{bmatrix} = 3i + 3j + 0k = (3, 3, 0)$$

Thus using the definition above, and point A, we know the formula is given by

$$3(x - 1) + 3(y) = 0$$

One can simplify this to

$$x + y - 1 = 0$$

$\square$

Reminders HW is due Sunday 11:59PM.

Chapter 2

Differentiation

Week 3 (9/8-9/12)

Topics: (1) graphing multivariable functions, (2) limits and continuity.

Definition 3 (graph)**.** The **image** of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subset of $\mathbb{R}^m$,

$$\text{Image}(f) = \{f(x) \in \mathbb{R}^m : x \in \mathbb{R}^n\}$$

and the **graph** of f is a subset of $\mathbb{R}^{n+m}$,

$$\text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{R}^n\}$$

Definition 4 (limit)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is open, let x_0 be in A or be a boundary point of A and N be a neighborhood of a point $b \in \mathbb{R}^m$. Now let x approach x_0, f is said to be **eventually in** N if there exists a neighborhood U of x_0 such that

$$\text{if } x \in U, \text{ then } f(x) \in N$$

If f is eventually in N for *any* neighborhood N around b, then the **limit** of f as $x \to x_0$ exists, denoted as

$$\lim_{x \to x_0} f(x) = b$$

Definition 5 (limit')**.** If the limit exists, then $\lim_{x \to x'} f(x) = b$ is when $x = (x_1, x_2, \ldots, x_n) \to x' = (x'_1, x'_2, \ldots, x'_n)$ from **all directions**, and $f(x)$ approaches $b = (b_1, \ldots, b_m)$.

Definition 6 (continuity)**.** Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is said to be **continuous** at $x_0 \in A$ if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

And f is called continuous if f is continuous at every $x_0 \in A$.

Example 1. The limit doesn't need to exist! For example, let

$$H(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases}$$

Note the limit doesn't exist at $x = 0$.

Problem 6. For the following functions, find their (1) image, (2) graph, (3) draw their graphs.

1. Let $f : \mathbb{R} \to \mathbb{R}$, and $f(x) = x^2 + 1$.
2. Let $g : \mathbb{R}^2 \to \mathbb{R}$, and $g(x, y) = x^2 + y^2$.

Proof.

1. Image$(f) = \{x^2 + 1 : x \in \mathbb{R}\}$, and Graph$(f) = \{(x, x^2 + 1) : x \in \mathbb{R}\}$.
2. Image$(g) = \{x^2 + y^2, (x, y) \in \mathbb{R}^2\}$, and Graph$(g) = \{(x, y, x^2 + y^2) : (x, y) \in \mathbb{R}^2\}$.

$\square$

Problem 7. Compute the following limits:

1. $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y}$$
   (Hint: try writing $\frac{\sin xy}{y} = \frac{\sin xy}{xy} \cdot x$, and recall $\lim_{t\to 0} \frac{\sin t}{t} = 1$).
2. $$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{y}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{(x - y)^2}{x^2 + y^2}$$

Proof.

1. Following the hint, we see
   $$\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = \lim_{(x,y)\to(0,0)} \frac{\sin xy}{xy} x = \lim_{x\to 0} x = 0$$
2. This one uses the exact same trick:
   $$\lim_{(x,y)\to(0,0)} \frac{e^{xy} - 1}{xy} \cdot y = 0$$
3. First letting $x \to 0$ along $y = 0$, we see the limit is 1; letting $x = y \to 0$, we see the limit is 0, thus the limit doesn't exist!

$\square$

Problem 8. Compute the limit of the following functions:

1. $$\lim_{(x,y)\to(0,0)} \frac{x}{x+y}$$

2. $$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y}$$

   (Hint: try considering $y = x^2 - x$ and $y = x$)

3. $$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$$

Proof. 1. First fix $x = 0$, let $y \to 0$, then the limit is 0; now fix $y = 0$, let $x \to 0$, the limit is 1. The limit doesn't exist!

2. Consider $y = x^2 - x$, (as $x \to 0, y \to 0$), then

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x+y} = \lim_{x\to 0} \frac{x^3 - x^2}{x^2} = \lim_{x\to 0} x - 1 = -1$$

and consider $y = x$, we see the limit is 0, thus the limit doesn't exist!

Warning 1. 2 does not follow from 1! A student suggests a proof: $\lim_{(x,y)\to(0,0)} = \frac{x}{x+y} \cdot y$, and by 1, the limit $\frac{x}{x+y}$ doesn't exist, this implies the limit of $\frac{xy}{x+y}$ also doesn't exist. This argument is not correct! Consider the following counterexample: $\lim_{y\to 0} \frac{1}{y}$ doesn't exist, but the limit

$$\lim_{y\to 0} \frac{1}{y} \cdot y = 1$$

exists! More concretely, if you multiply by any function that doesn't tend to 0, the argument follows, but it doesn't work when the function tends to 0! (Sorry I wasn't able to give a concrete counterexample in class other than saying this gives "bad and untrue vibes"). Thank you (the student) who brought it up, your attempt still remains very very good.

3. We see that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy} \frac{xy}{x+y}$$

Note that the limit of $\sin(xy)/(xy) = 1$, but the second one doesn't exist, thus the limit doesn't exist! $\square$

How to find a a limit $\lim_{x\to x_0} f(x)$:

- Step 1: Guess what the limit should be.
- Step 2: Try from approaching x_0 from different directions.
- Step 3: Try to replace terms with expressions you are familiar with.

Week 4 (9/15-9/19)

Topics: (1) Partial derivatives. (2) Definition of total derivatives.

Problem 9. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ for the following functions:

1.
$$x^3y^4 - xy^2$$

2.
$$x^2\sin(2y) + 3$$

3.
$$\ln\left(\frac{y}{x}\right) + \ln\left(\frac{1}{x+y}\right) - \ln\left(\frac{x}{2}\right)$$

You may use the following identities to simply the equation first:

$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b, \quad \ln\left(\frac{1}{a}\right) = -\ln a$$

Proof. We have

1.
$$\partial x : 3x^2y^4 - y^2, \quad \partial y : 4x^3y^3 - 2xy$$

2.
$$\partial x : 2x\sin(2y), \quad \partial y : 2x^2\cos(2y)$$

3.
$$\partial x : -\frac{2}{x} - \frac{1}{x+y}, \quad \partial y : \frac{1}{y} - \frac{1}{x+y}$$

□

Definition 7 (tangent plane)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at (x_0, y_0), then the **tangent plane** to the graph f in $\mathbb{R}^3$ is the given by

$$z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)$$

Problem 10. Compute the plane tangent to the graph of $f(x, y) = x^2y + 2xy - y^2$ at $(1, 2)$.

Proof. We have

$$\frac{\partial f}{\partial x}(1,2) = 2xy + 2y|_{(1,2)} = 8, \quad \frac{\partial f}{\partial y}(1,2) = x^2 + 2x - 2y|_{(1,2)} = -1$$

and $f(1, 2) = 2$, thus the plane is given by

$$z = 2 + 8(x - 1) - (y - 2)$$

i.e., $z = 8x - y - 4$. □

Definition 8 (derivative for two variables)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$, then f is said to be **differentiable** at (x_0, y_0) if $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (x_0, y_0) and if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - \mathcal{P}(x,y)}{\|(x,y) - (x_0,y_0)\|} = 0$$

where $\mathcal{P}(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{(x_0,y_0)}(y - y_0)$ is the tangent plane to f at (x_0, y_0).

Definition 9 (derivative for n variables)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$, then the **gradient** of f, denoted as ∇f is given by

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right]$$

is a $1 \times n$ matrix. And f is said to be **differentiable** at $x_0 \in \mathbb{R}^n$ if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - \nabla f(x_0)(x - x_0)\|}{\|x - x_0\|} = 0$$

and the derivative of f is exactly the gradient ∇f at x_0.

Definition 10 (derivative for m outputs)**.** Let $f : \mathbb{R} \to \mathbb{R}^m$, where $f(x) = (f_1(x), \dots, f_m(x))$, then let T denote the $n \times 1$ matrix

$$T = \begin{bmatrix} \frac{df_1}{dx}(x_0) \\ \frac{df_2}{dx}(x_0) \\ \dots \\ \frac{df_m}{dx}(x_0) \end{bmatrix}$$

Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{|f(x) - f(x_0) - T(x - x_0)|}{|x - x_0|} = 0$$

and the matrix T is the derivative at x_0.

Example 2. Let $f(x) = (x^2, 2x, -x)$, then

$$T = Df(1) = \begin{bmatrix} 2x \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Definition 11 (derivative for general functions)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$, let T be the $m \times n$ matrix with entries $\partial f_i / \partial x_j$ evaluated at $x_0 \in \mathbb{R}^n$. Then f is said to be **differentiable** at x_0 if

$$\lim_{x\to x_0} \frac{\|f(x) - f(x_0) - T(x - x_0)\|}{\|x - x_0\|} = 0$$

then f is differentiable at x_0, and the matrix T is the derivative at x_0. Note that T loosk like

$$T = Df(x_0), \quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \dots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Example 3. Let $f(x, y, z) = (ze^x, -ye^z)$, then

$$Df(x, y, z) = \begin{bmatrix} ze^x & 0 & e^x \\ 0 & -e^z & -ye^z \end{bmatrix}$$

Problem 11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$f(x, y, z) = x^2y + y\sin(z) + ze^x.$$

Compute the gradient of f at $(1, 2, 0)$.

Proof. You can compute the partial derivatives

$$\frac{\partial f}{\partial x} = 2xy + ze^x,$$
$$\frac{\partial f}{\partial y} = x^2 + \sin(z),$$
$$\frac{\partial f}{\partial z} = y\cos(z) + e^x.$$

thus the gradient is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xy + ze^x, x^2 + \sin z, y\cos z + e^x)$$

Hence

$$\nabla f(1, 2, 0) = (4, 1, 2 + e)$$

□

Week 5 (9/22-9/26)

Topics: (1) properties of derivatives, (2) directional derivatives, (3) gradient.

Definition 12 (path). A **path** c is a map $c : [a, b] \to \mathbb{R}^n$. We can write $c(t) = (c_1(t), \ldots, c_n(t))$. If c is differentiable, then we can define the **velocity** of c at any $t_0 \in [a, b]$ as

$$c'(t_0) = (c_1'(t_0), \ldots, c_n'(t_0))$$

The velocity vector of c at t_0 is also a **tangent** vector to c at t_0. The **speed** of the path c at t_0 is the length of the velocity vector $\|c'(t_0)\|$.

Definition 13 (tangent line to a path). Let $c : [a, b] \to \mathbb{R}^n$ be a path, if $c'(t_0) \neq 0$, then the **tangent line** at x_0 is given by

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

Proposition 2. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, then the derivative of f at x_0 is an $m \times n$ matrix $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$. The derivative follows the same properties as derivative for single variable functions:

1. Let $c \in \mathbb{R}$, then

$$D(cf)(x_0) = cDf(x_0) \qquad \text{(multiplication of a matrix by constant } c\text{)}$$

2. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ also be differentiable at x_0, then

$$D(f + g)(x_0) = Df(x_0) + Dg(x_0) \qquad \text{(sum of two matrices)}$$

3. Let $h_1 : U \subset \mathbb{R}^n \to \mathbb{R}, h_2 : U \subset \mathbb{R}^n \to \mathbb{R}$,then

$$D(h_1h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \qquad \text{(product rule)}$$

   and if $h_2 \neq 0$ on U.

$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \qquad \text{(quotient rule)}$$

4. Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m, f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ such that $g(U) \subset V$, then

$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \qquad \text{(chain rule)}$$

Definition 14 (directional derivative). Let $f : \mathbb{R}^3 \to \mathbb{R}$, be differentiable, then the directional directive at $x_0 \in \mathbb{R}^3$ in the direction of a **unit vector** v is given by

$$\nabla f(x_0) \cdot v = \left[\frac{\partial f}{\partial x_1}(x_0)\right] v_1 + \left[\frac{\partial f}{\partial x_2}(x_0)\right] v_2 + \left[\frac{\partial f}{\partial x_3}(x_0)\right] v_3$$

where $v = (v_1, v_2, v_3)$.

Proposition 3. Suppose that $\nabla f(x_0) \neq 0$, then the direction for which f increases the fastest at x_0 is along $\nabla f(x_0)$.

Proposition 4. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable, let S be a level surface of f, i.e., S is a surface described by

$$f(x, y, z) = k$$

were k is some constant. Let $(x_0, y_0, z_0) \in S$, then

$$\nabla f(x_0, y_0, z_0) \text{ is normal to the level surface at } (x_0, y_0, z_0)$$

This means if $c(t)$ is a path in S, and $v(0) = (x_0, y_0, z_0)$, and if v is a tangent vector to $c(t)$ at $t = 0$, then

$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$

Moreover, if $\nabla f(x_0, y_0, z_0) \neq 0$, the **tangent plane** of S at (x_0, y_0, z_0) is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Problem 12. Consider the curve in $\mathbb{R}$: $c(t) = (2t, t^2, -t)$. Find the speed of the c at $t = 2$ and the tangent line at $t = 1$.

Proof. The velocity vector of c at $t = 2$ is

$$c'(t) = (2, 2t, -1)$$

evaluated at $t = 2$ is $c'(2) = (2, 4, -1)$. Thus the speed is the length of the velocity vector

$$\|c'(2)\| = \left(2^2 + 4^2 + (-1)^2\right)^{\frac{1}{2}} = \sqrt{21}$$

For the tangent line: the tangent vector is

$$c'(1) = (2, 2, -1)$$

and $c(1) = (2, 1, -1)$. Thus the tangent line l at $t = 1$ is given by

$$l(t) = (2, 1, -1) + t(2, 2, -1) = (2 + 2t, 1 + 2t, -1 - t)$$

□

Problem 13 (2.5, Q7). Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and

$$g(x, y) = (e^{x-y}, x - y).$$

Calculate $f \circ g$ and

$$D(f \circ g)(1, 1).$$

Proof. We have

$$f \circ g(x, y) = (\tan(e^{x-y} - 1) - e^{x-y}, e^{2(x-y)} - (x - y)^2)$$

and $g(1, 1) = (1, 0)$, thus using chain rule, we have

$$D(f \circ g)(1, 1) = Df(1, 0)Dg(1, 1)$$

where

$$Df(u, v) = \begin{bmatrix} \sec^2(u - 1) & -e^v \\ 2u & -2v \end{bmatrix}, \quad Dg(x, y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

Hence

$$D(f\circ g)(1,1) = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$$

□

Problem 14 (2.5, Q8). Let $f(u,v,w) = (e^{u-w}, \cos(v+u)+\sin(u+v+w))$ and $g(x,y) = (e^x, \cos(y-x), e^{-y})$. Calculate $f\circ g$ and $D(f\circ g)(0,0)$.

Proof. We have

$$f\circ g = (e^{e^x - \cos(y-x)}, \cos(e^x + \cos(y-x)), \sin(e^x + e^{-y} + \cos(y-x)))$$

and $g(0,0) = (1,1,1)$. Thus

$$D(f\circ g)(0,0) = Df(1,1,1)Dg(0,0)$$

where

$$Df(u,v,w) = \begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u)+\cos(u+v+w) & -\sin(v+u)+\cos(u+v+w) & \cos(u+v+w) \end{bmatrix}$$

and

$$Dg(x,y) = \begin{bmatrix} e^x & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix}$$

Thus

$$D(f\circ g)(0,0) = \begin{bmatrix} 1 & 0 & -1 \\ -\sin 2+\cos 3 & -\sin 2 + \cos 3 & \cos 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\sin 2 + \cos 3 & -\cos 3 \end{bmatrix}$$

□

Problem 15 (2.5, Q11). Let $f(x,y,z) = (3y+2, x^2+y^2, x+z^2)$. Let

$$c(t) = (\cos(t), \sin(t), t).$$

(a) Find the path $p = f\circ c$ and the velocity vector

$$p'(\pi).$$

(b) Find $c(\pi), c'(\pi)$ and $Df(-1,0,\pi)$.
(c) Thinking of $Df(-1,0,\pi)$ as a linear map, find

$$Df(-1,0,\pi)\ (c'(\pi)).$$

Proof. (a) We have

$$p(t) = (3\sin t + 2, 1, \cos t + t^2)$$

and

$$p'(t) = (3\cos t + 0, -\sin t + 2t)$$

thus

$$p'(\pi) = (-3, 0, 2\pi)$$

(b) We have $c(\pi) = (-1, 0, \pi)$, and $c'(t) = (-\sin t, \cos t, 1)$, and $c'(\pi) = (0, -1, 1)$. And

$$Df(x, y, z) = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix}$$

Thus

$$Df(-1, 0, \pi) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}$$

(c) We have

$$Df(-1, 0, \pi)(c'(\pi)) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}$$

□

Problem 16 (2.6, Q3)**.** Compute the directional derivatives of the following functions along unit vectors at the indicated points in directions parallel to the given vector:

(a)

$$f(x, y) = x^y, (x_0, y_0) = (e, e), \quad \mathbf{d} = 5\mathbf{i} + 12\mathbf{j}$$

(b)

$$f(x, y, z) = e^x + yz, \ (x_0, y_0, z_0) = (1, 1, 1), \quad \mathbf{d} = (1, -1, 1)$$

(c)

$$f(x, y, z) = xyz, \ (x_0, y_0, z_0) = (1, 0, 1), \quad \mathbf{d} = (1, 0, -1)$$

Proof. We find ∇f for all these functions and find the directional directive

$$\nabla f(x_0) \cdot \frac{d}{\|d\|}$$

(a) We have

$$\nabla f(x, y, z) = (yx^{y-1}, x^y \ln x)$$

hence

$$\nabla f(e, e) \cdot \frac{d}{\|d\|} = (e^e, e^e) \cdot \left(\frac{5}{13}, \frac{12}{13}\right) = \frac{17}{13} e^e$$

(b) We have

$$\nabla f(x, y, z) = (e^x, z, y)$$

hence

$$\nabla f(1, 1, 1) \cdot \frac{1}{\sqrt{3}}(1, -1, 1) = \frac{e}{\sqrt{3}}$$

(c) We have

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 0, 1) \cdot \frac{1}{\sqrt{2}}(1, 0, -1) = \frac{1}{\sqrt{2}}(0, 1, 0) \cdot (1, 0, -1) = 0$$

□

Problem 17 (2.6, Q6). Find a vector which is normal to the curve

$$x^3 + xy + y^3 = 11 \text{ at } (1, 2).$$

Proof. Consider the function $f(x, y) = x^3 + xy + y^3$, then the level set of $f(x, y) = 11$ coincides with the curve above. Thus it suffices to compute

$$\nabla f(x, y) = (3x^2 + y, x + 3y^2)$$

and

$$\nabla f(1, 2) = (5, 13)$$

is perpendicular to the the level curve. □

Problem 18 (2.6, Q7). Find the rate of change of $f(x, y, z) = xyz$ in the direction normal to the surface

$$yx^2 + xy^2 + yz^2 = 3 \text{ at } (1, 1, 1).$$

Proof. We first find a normal vector to the surface, consider the surface as a level set of the function

$$g(x, y, z) = yx^2 + xy^2 + yz^2$$

Thus

$$\nabla g(x, y, z) = (2xy + y^2, x^2 + 2xy + z^2, 2yz)$$

hence

$$u = \nabla g(1, 1, 1) = (3, 4, 2)$$

is a normal vector to the surface, and we normalize it to get a unit normal vector $n = \frac{u}{\|u\|} = \frac{1}{\sqrt{29}}(3, 4, 2)$. Now we find the directional derivative of $f(x, y, z) = xyz$ along $(3, 4, 2)$:

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 1, 1) = (1, 1, 1)$$

and the directional derivative is

$$\nabla f(1, 1, 1) \cdot n = \frac{9}{\sqrt{29}}$$

□

Summary

- If asked to find directional derivative/rate of change along a unit vector v at point x_0: find $\nabla f(x_0) \cdot v$.
- If asked to find along with direction f increases the fastest: find $\nabla f(x_0)$.
- If asked to find a normal vector to a surface at a point x_0: construct a function such that the surface is the level set of this function, then find $\nabla f(x_0)$.

Chapter 3

Higher-Order Derivatives: Maxima and Minima

Week 6 (9/29-10/3)

Topics: (1) Higher-order derivatives, (2) Taylor expansion.

Proposition 5. Let $f(x, y)$ be twice continuously differentiable, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Definition 15 (First order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + R_1(a, x)$$

where

$$\frac{R_1(a, x)}{\|x - a\|} \to 0 \text{ as } x \to a$$

Definition 16 (Alternative Definition (First-order))**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $a \in U$. Then

$$f(a + h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + R_1(a, h)$$

where $R_1(a, h)/\|h\| \to 0$ as $h \to 0$.

Definition 17 (Second order Taylor expansion)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable at $a \in U$, then

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + R_2(a, x)$$

where

$$\frac{R_2(a, x)}{\|x - a\|} \to 0 \text{ as } x \to a$$

Definition 18 (Alternative Definition (Second-order)). Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order. Then we can write

$$f(a+h) = f(a) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(a) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a) + R_2(a,h)$$

where $R_2(a,h)/\|h\|^2 \to 0$ as $h \to 0$.

Problem 19. Find all the second partial derivatives of $f(x,y) = xy + \ln(x-y)$.
(This includes $\partial^2 f/\partial x^2, \partial^2 f/\partial x \partial y, \partial^2 f/\partial y \partial x, \partial^2 f/\partial y^2$).

Proof. For first-order derivatives:

$$\frac{\partial f}{\partial x} = y + \frac{1}{x-y}, \quad \frac{\partial f}{\partial y} = x - \frac{1}{x-y}$$

Then

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= -\frac{1}{(x-y)^2} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = 1 + \frac{1}{(x-y)^2} \\ \frac{\partial^2 f}{\partial y^2} &= -\frac{1}{(x-y)^2} \end{aligned}$$

□

Problem 20. Write the second-order Taylor expansion for the following function,

$$f(x,y) = e^{x+y}$$

centered at $(x,y) = (0,0)$.

Proof. We first compute all the partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} = e^{x+y}$$

then

$$f(x) = 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + R_2(x)$$

where $R_2(x)/\|x\| \to 0$ as $x \to 0$. □

Week 7 (10/6-10/10)

Topics: Extremum.

Definition 19 (quadratic function)**.** A function $g : \mathbb{R}^n \to \mathbb{R}$ is called a **quadratic function** if it is given by

$$g(h_1, \ldots, h_n) = \sum_{i,j=1}^{n} a_{ij} h_i h_j$$

where (a_{ij}) is an $n \times n$ matrix. We can also write g as follows:

$$g(h_1, \ldots, h_n) = [h_1, \ldots, h_n] \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n_1} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$$

Definition 20 (Hessian matrix)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$, and suppose all the second-order partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ exist, then the Hessian matrix of f is the $n \times n$ matrix given by

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \ldots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n x_1} & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}$$

The Hessian as a quadratic function is defined by

$$Hf(x)(h) = \frac{1}{2} \begin{bmatrix} h_1 & \ldots & h_n \end{bmatrix} Hf(x) \begin{bmatrix} h_1 \\ \ldots \\ h_n \end{bmatrix}$$

where $h = (h_1, \ldots, h_n)$.

Definition 21 (degenerate/nondegenerate points)**.** Let $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, let (x_0, y_0) be a critical point. We define the **discriminant**, $\boldsymbol{\mathcal{D}}$,of the Hessian by

$$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $\mathcal{D} \neq 0$, the critical point (x_0, y_0) is called **nondegenerate**; if $\mathcal{D} = 0$, the point (x_0, y_0) is called **degenerate**.

Definition 22 (positive, negative-definite)**.** A quadratic function $g : \mathbb{R}^n \to \mathbb{R}$ is called **positive-definite** if $g(h) \geq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. Similarly, g is **negative-definite** if $g(h) \leq 0$ for all $h \in \mathbb{R}^n$ and $g(h) = 0$ implies $h = 0$. (The matrix is positive-definite iff it is symmetric $A^T = A$ and the eigenvalues are nonnegative).

Definition 23 (bounded set)**.** A set $A \subset \mathbb{R}^n$ is said to be **bounded** if there is a number $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Proposition 6 (extremums are critical points)**.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable, where U is open. If x_0 is a local extremum, then $Df(x_0) = 0$.

Proposition 7 (extremum). Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be in C^3, and x_0 is a critical point of f. If the Hessian $Hf(x_0)$ is positive-definite, then x_0 is a local minimum of f; if $Hf(x_0)$ is negative-definite, then x_0 is a local maximum.

Proposition 8 (local minimum). Let $f(x, y)$ be of C^2, and U is open in $\mathbb{R}^2$. A point (x_0, y_0) is a strict local **minimum** of f if the following conditions hold:

1. $$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$$

2. $$\mathcal{D}(x_0, y_0) > 0$$

   where $\mathcal{D}$ is the **discriminant** of the Hessian, defined by

   $$\mathcal{D} = \det(Hf) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

   where Hf is the 2×2 Hessian matrix.

3. $$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) > 0$$

If $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) < 0$ in 3, then it becomes a local maximum.

Proposition 9 (saddle points). Let $f(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}$ be of C^2, if $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0$, and $\mathcal{D}(x_0, y_0) < 0$, where $\mathcal{D}$ is the discriminant, then the critical point (x_0, y_0) is a saddle point, i.e., neither a maximum or a minimum.

Proposition 10 (continuous functions attain extremum on closed bounded sets). Let $f : D \to \mathbb{R}$ be continuous, where D is closed and bounded in $\mathbb{R}^n$. Then f assumes its absolute maximum and absolute minimum values at some point $x_0, x_1 \in D$.

Problem 21. Is the following matrix positive-definite?

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Proof. It is not! Consider the vector $(0, 1)$, we have

$$[0, 1] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = [0, 1] \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1$$

□

Problem 22. Find the critical point of $f(x, y) = y + x \sin y$ and classify whether it is a local max/min or a saddle point.

Proof. We compute

$$f_x = \sin y, \quad f_y = 1 + x \cos y$$

Setting them both to 0 gives

$$(1, n\pi), \text{ when } n \text{ is odd}, \quad (-1, n\pi) \text{ when } n \text{ is even}$$

Now we compute the discriminant:

$$f_{xx} = 0, \quad f_{xy} = f_{yx} = \cos y, \quad f_{yy} = -x \sin y$$

Thus

$$\mathcal{D} = \det \begin{bmatrix} 0 & \cos y \\ \cos y & -x \sin y \end{bmatrix} = -\cos^2 y$$

Thus $\mathcal{D} < 0$ for all the critical points, hence they are saddle points! $\square$

Week 7 Additional content

Proposition 11. If f has a maximum or minimum at x_0 when constrained to a surface S, then $\nabla f(x_0)$ is perpendicular to S at x_0.

Consequence of Proposition 11:

Proposition 12. Let $f, g : U \subset \mathbb{R}^n \to \mathbb{R}$ and $\vec{x_0} \in U$ such that $g(x_0) = c$. Let $\mathcal{L}_c$ be the level set for g with value c and assume

$$\nabla g(\vec{x_0}) \neq \vec{0}.$$

If f restricted to $\mathcal{L}_c$ has a local minimum or maximum on $\mathcal{L}_c$ at x_0, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda \nabla g(x_0).$$

Problem 23 (Marsden-Tromba, III. 2)**.** Let $f(x, y, z) = x - y + z$, find the extremum of f subject to the constraint $x^2 + y^2 + z^2 = 2$.

Proof. We compute the gradient of f and $g(x, y, z) = x^2 + y^2 + z^2$:

$$\nabla f(x, y, z) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \quad \nabla g(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$

By the proposition above, we need to have

$$\begin{aligned} \lambda &= 2x \\ -\lambda &= 2y \\ \lambda &= 2z \end{aligned}$$

Equating all λ gives

$$x = z = -y$$

Plugging in the constraint we get

$$3x^2 = 2$$

Hence two critical points

$$\left(\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}} \right), \quad \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}} \right)$$

Plugging them in we find the extremums are $\sqrt{6}$ and $-\sqrt{6}$. □

Problem 24 (Marsden-Tromba, III. 16)**.** Use Lagrange multipliers to find the distance from the point $(2, 0, -1)$ to the plane $3x - 2y + 8z + 1 = 0$.

Proof. Let (x, y, z) be a point on the plane, then we would like the minimize the function

$$f(x, y, z) = (x - 2)^2 + y^2 + (z + 1)^2$$

with the constraint

$$3x - 2y + 8z + 1 = 0$$

Writing $g(x, y, z) = 3x - 2y + 8z + 1$, we do the exact same thing as we did above:

$$\nabla f = \begin{bmatrix} 2x - 4 & 2y & 2z + 2 \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 3 & -2 & 8 \end{bmatrix}$$

Then setting

$$\begin{aligned} 2x - 4 &= 3\lambda \\ 2y &= -2\lambda \\ 2z + 2 &= 8\lambda \end{aligned}$$

One can then replace x, y, z with expressions in λ in the constraint:

$$3x - 2y + 8z + 1 = 0$$

which gives

$$\lambda = 2/77$$

Then plugging in λ to solve for x, y, z, the $\sqrt{f(x, y, z)}$ is the final answer. (I am too lazy to do the computation). $\square$

Chapter 4

Vector-Valued Functions

Week 8 (Fall Break)

Topics: (1) Acceleration and Arc Length, (2) Vector Fields.

Proposition 13 (Newton's second law)**.** Let $c(t)$ be a path of a particle with mass m and $a(t) = c''(t)$ be the acceleration, then

$$F(c(t)) = ma(t)$$

where F is the force applying on the particle.

Definition 24 (arc length)**.** Let $c(t) = (x(t), y(t), z(t))$ be a path, then the length of the path in $\mathbb{R}^3$ from $t_0 \leq t \leq t_1$ is

$$\begin{aligned} L_{t_0 \to t_1}(c) &= \int_{t_0}^{t_1} \left(x'(t)^2 + y'(t)^2 + z'(t)^2\right)^{\frac{1}{2}} dt \\ &= \int_{t_0}^{t_1} \|c'(t)\| dt \end{aligned}$$

More generally, if $c(t) = (x_1(t), \ldots, x_n(t))$ is a path in $\mathbb{R}^n$, then

$$L_{t_0 \to t_1}(c) = \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} x_i'(t)^2\right)^{\frac{1}{2}} dt$$

Definition 25 (vector field)**.** A vector field is a function $F : A \subset \mathbb{R}^n \to \mathbb{R}^n$ that assigns $x \in \mathbb{R}^n$ to another vector $F(x) \in \mathbb{R}^n$.

Problem 25. Find the velocity, speed, and acceleration of the following path at $t = 0$:

$$c(t) = (\cos t, 2t, -\sin t)$$

Proof. The velocity is

$$c'(t) = (-\sin t, 2, -\cos t), \quad c'(0) = (0, 2, -1)$$

And the speed is

$$\|c'(t)\| = \sqrt{5}$$

□

Problem 26. Find the length of the curve above from $t = 0$ to $t = 2$.

Proof.

$$\begin{aligned} L_{0\to 2}\|c'(t)\|dt &= \int_0^2 \left(\sin^2 t + 4 + \cos^2 t\right)^{\frac{1}{2}} dt \\ &= \int_0^2 \sqrt{5}dt \\ &= 2\sqrt{5} \end{aligned}$$

□

Chapter 5

Double and Triple Integrals

Week 10 (10/27-31)

Topics: (1) Divergence and Curl, (2) Double integrals.

Definition 26 (flow line)**.** Let F be a vector field, a flow line of F is a path $c(t)$ satisfying

$$c'(t) = F(c(t))$$

(Tangent vector of the path coincides with the given vector field F).

Definition 27 (divergence)**.** Let F be a vector field in $\mathbb{R}^3$ $F = (F_1, F_2, F_3)$, the divergence of F is the **scalar field** (assigns one number to an given point (x, y, z)),

$$\operatorname{div} F := \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

More generally, if $F = (F_1, \ldots, F_n)$ is a vector field on $\mathbb{R}^n$, its divergence is

$$\operatorname{div} F = \sum_{i=1}^{n} \frac{\partial F_i}{\partial x_i} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$$

Remark 1. *We write the divergence as $\nabla \cdot F$ because*

$$\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$$

and if $F = (F_1, \ldots, F_n)$,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n} = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) \cdot (F_1, \ldots, F_n) = \nabla \cdot F$$

Definition 28 (curl)**.** Let F be a vector field in $\mathbb{R}^3$, writing $F = (F_1, F_2, F_3)$, the **curl** of F is the vector field

$$\operatorname{curl} F := \nabla \times F = \det \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

If $\operatorname{curl} F = 0$, then we say the vector field is **irrotational**.

Proposition 14 (gradient is irrotational)**.** Let $f \in C^2$, viewing ∇f as a vector field, then

$$\nabla \times (\nabla f) = 0$$

Proposition 15 (divergence of a curl vanishes)**.** For any C^2 vector field F,

$$\nabla \cdot (\nabla \times F) = 0$$

Proposition 16 (Fubini's Theorem for rectangles)**.** Let f be a continuous function on a rectangular domain $R = [a, b] \times [c, d]$, then

$$\int_a^b \int_c^d f(x, y)dydx = \int_c^d \int_a^b f(x, y)dxdy$$

Proposition 17 (Fubini's Theorem for general regions)**.** Suppose D is a set of points (x, y) such that $y \in [c, d]$ and $\psi_1(y) \leq x \leq \psi_2(y)$, and similarly for $x \in [a, b]$, $\varphi_1(x) \leq y \leq \varphi_2(x)$. If f is continuous on D, then

$$\iint_D f(x, y)dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)dydx = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)dxdy$$

Problem 27. Change the order of integration to $dydx$ for the following function:

$$\int_0^1 \int_{e^y}^e \frac{x}{\ln x}dxdy$$

Proof. It should be

$$\int_1^e \int_0^{\ln x} \frac{x}{\ln x}dydx$$

□

Problem 28. Change the order of integrations for the following functions:

1. $$\int_0^1 \int_x^1 f(x,y)dydx$$

2. $$\int_0^1 \int_{-y}^{y^3} f(x,y)dxdy$$

3. $$\int_0^3 \int_{2x}^{6} f(x,y)dydx$$

4. $$\int_0^1 \int_{-\sqrt{y}}^{y^2} f(x,y)dxdy$$

5. $$\int_0^8 \int_{\sqrt[3]{y}}^{2} f(x,y)dxdy$$

6. $$\int_0^1 \int_{\ln y}^{1} f(x,y)dxdy$$

Proof. 1. It should be

$$\int_0^1 \int_0^y f(x,y)dxdy$$

2. It should be

$$\int_{-1}^0 \int_{-x}^1 f(x,y)dydx + \int_0^1 \int_{\sqrt{x}}^1 f(x,y)dydx$$

3. It should be

$$\int_0^6 \int_0^{\frac{y}{2}} f(x,y)dxdy$$

4. It should be

$$\int_0^1 \int_{\sqrt{x}}^1 f(x,y)dydx + \int_{-1}^0 \int_{x^2}^1 f(x,y)dydx$$

5. It should be

$$\int_0^2 \int_0^{x^3} f(x,y)dydx$$

6. It should be

$$\int_{-\infty}^0 \int_0^{e^x} f(x,y)dydx + \int_0^1 \int_0^1 f(x,y)dydx$$

□

Problem 29. Show that the vector field $V(x,y,z) = (x^2, -y, z)$ is not the curl of any vector field F. In other words, there is no vector field F such that

$$V = \operatorname{curl} F$$

Proof. By the above proposition, if there exists F such that

$$V = \operatorname{curl} F$$

then

$$\nabla \cdot V = 0$$

However,

$$\nabla \cdot V(x,y,z) = 2x \neq 0 \text{ if } x \neq 0$$

this is a contradiction, hence no such F exists. □

Problem 30 (Marsden-Tromba, IV.4, 21)**.** Let $F(x,y,z) = (x^2, x^2y, z+zx)$. Can there exist a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $F = \nabla f$?

Proof. No. Suppose there exists such f, then the gradient must be irrotational. In other words, we must have

$$\nabla \times F = 0$$

Thus we compute the curl of F:

$$\nabla \times F = (-2yz, z, 0)$$

and it is not identically 0. □

Chapter 6

Change of Variables Formula

Week 11 (11/3-7)

Topic: theorem of change of variables.

Definition 29 (injective, surjective). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a map, then T is **injective, or one-to-one,** if for $x, y \in \mathbb{R}^2$

$$Tx = Ty$$

then

$$x = y.$$

T is called **surjective or onto** if for all $y \in \mathbb{R}^2$, there exists $x \in \mathbb{R}^2$ such that

$$Tx = y$$

If T is both injective and bijective, then we say T is **bijective**.

Definition 30 (linear map). Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$, then T is linear if and only if for all $x, y \in \mathbb{R}^2, \lambda \in \mathbb{R}$.

$$\begin{cases} T(x+y) = Tx + Ty \\ T(\lambda x) = \lambda Tx \end{cases}$$

Definition 31 (Jacobian Determinant). Let $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be of C^1 defined by

$$T : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$$

The **Jacobian determinant** of T, denoted as $\frac{\partial(x,y)}{\partial(u,v)}$ is the determinant of the matrix $DT(u,v)$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Proposition 18. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, then there exists a 2×2 matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ such that

$$Tx = Ax$$

Proposition 19. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, then T is injective if and only if it is also surjective if and only if $\det(A) \neq 0$, where A is the matrix associated to T.

Theorem 2 (change of variables formula)**.** Let D, D^* be elementary regions in $\mathbb{R}^2$, suppose $T : D^* \to D$ is bijective. Then for any integral function $f : D \to \mathbb{R}$, the **change of variable formula** states

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(x(u,v), y(u,v)) \left|\det(J)\right| dudv$$

where

$$\det(J) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$

is the determinant of the Jacobian matrix.

Corollary 1. An immediate corollary of the above theorem is the change of variables into polar cooridnates:

$$\iint_D f(x,y)dxdy = \iint_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Corollary 2. Let W, W^* be elementary regions in $\mathbb{R}^3$, and suppose $T : W^* \to W$ is bijective. Then the change of varialbes formula for triple integrals states:

$$\iiint_W f(x,y,z)dxdydz = \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \mid \det(J) \mid dudvdw$$

where

$$\det(J) = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

Problem 31. Compute the following integral:

$$\iint_D (x^2+y^2)^{1/2} dxdy$$

where D is the disk $D = \{(x,y) : x^2+y^2 \leq 9\}$.

Proof. Writing this in polar coordinates and using the change of variables formula, we have

$$\begin{aligned} \iint_D (x^2+y^2)^{1/2} dxdy &= \int_0^{2\pi}\int_0^3 (r^2)^{1/2} rdrd\theta \\ &= \int_0^{2\pi}\int_0^3 r^2 drd\theta \\ &= 18\pi \end{aligned}$$

□

Problem 32 (Marsden-Tromba, 6.2, Exercise 19)**.** Compute $\iint_R (x+y)^2 e^{x-y} dxdy$, where R is the region bounded by $x+y=1, x+y=4, x-y=-1, x-y=1$.

Proof. Writing $u = x + y, v = x - y$, by the change of variables formula, we have

$$\begin{aligned}\iint_R (x+y)^2 e^{x-y} dxdy &= \int_{-1}^{1}\int_{1}^{4} u^2 e^v |\det(J)| dudv \\ &= \frac{1}{2}\left(\int_1^4 u^2 du\right)\left(\int_{-1}^{1} e^v dv\right) \\ &= \frac{21}{2}\left(e - \frac{1}{e}\right)\end{aligned}$$

□

Problem 33. Compute $\iint_R xy^2 dA$ where R is the region bounded by $xy = 1, xy = 3, y = 2, y = 6$, using the transformation $x = \frac{v}{6u}, y = 2u$.

Proof.

$$\begin{aligned}\iint_R xy^2 dA &= \int_1^3 \int_3^9 \frac{v}{6u} 4u^2 |\det(J)| dvdu \\ &= \int_1^3 \int_3^9 \frac{2v}{9} dvdu \\ &= 2\int_3^9 \frac{2v}{9} dv \\ &= 16\end{aligned}$$

where

$$\det(J) = \det \begin{bmatrix} -\frac{v}{6u^2} & \frac{1}{6u} \\ 2 & 0 \end{bmatrix} = -\frac{1}{3u}$$

□

Chapter 7

Integrals over Paths and Surfaces

Week 12 (11/10-14)

Definition 32 (path integral)**.** Let $c : [a,b] \to \mathbb{R}^3$ be a path of C^1 and $f : \mathbb{R}^3 \to \mathbb{R}$ is such that $f \circ c$ is continuous on $[a,b]$, The **path integral** of $f(x,y,z)$ along the path c is given by

$$\begin{aligned}\int_c f ds &= \int_a^b f(c(t))\|c'(t)\|dt \\ &= \int_a^b f(x(t), y(t), z(t))\|c'(t)\|dt\end{aligned}$$

Definition 33 (line integral)**.** Let F be a vector field on $\mathbb{R}^3$ that is continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$, where $c(t) = (x(t), y(t), z(t))$. We define $\int_c F \cdot ds$, the **line integral** of F along c by the following

$$\begin{aligned}\int_c F \cdot ds &= \int_a^b F(c(t)) \cdot c'(t)dt \\ &= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt \\ &:= \int_c F_1 dx + F_2 dy + F_3 dz\end{aligned}$$

the expression $F_1 dx + F_2 dy + F_3 dz$ is called the **differential form**.

Example 4 (work done along a path)**.** The force field F on a particle moving along a path $c : [a,b] \to \mathbb{R}^3$ is given by

$$\text{work done by } F = \int_a^b F(c(t)) \cdot c'(t)dt$$

Definition 34 (reparametrization)**.** Let $h : I \to I_1$ be a C^1 real-valued bijective function. Let $c : I_1 \to \mathbb{R}^3$ be a piecewise C^1 path. Then we call the composition

$$p = c \circ h : I \to \mathbb{R}^3$$

a **reparametrization** of c.

Example 5. Let $c : [0,1] \to \mathbb{R}^3$ be a C^1 path, then consider $h : [0,1] \to [0,1]$, where $h(t) = 1 - t$. Then the path

$$c_{\text{op}} = c \circ h(t) = c(1-t)$$

is the same path in the opposite direction.

Proposition 20 (reparametrization for path integrals)**.** Let c be a C^1 path and c' be any reparametrization of c, and let f be a continuous function on the image of c, then

$$\int_c f(x,y,z)ds = \int_{c'} f(x,y,z)ds$$

Proposition 21 (reparametrization for line integrals)**.** Let F be a vector feld continuous on the C^1 path $c : [a,b] \to \mathbb{R}^3$, and let $c' : [a', b'] \to \mathbb{R}^3$ be a reparametrization of c. If c' is orientation-preserving, then

$$\int_{c'} F \cdot ds = \int_c F \cdot ds$$

If c' is orientation-reversing, then

$$\int_{c'} F \cdot ds = -\int_c F \cdot ds$$

Proposition 22 (fundamental theorem of line integrals)**.** Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is of C^1 and that $c : [a,b] \to \mathbb{R}^3$ is piecewise C^1. Then

$$\int_c \nabla f \cdot ds = f(c(b)) - f(c(a))$$

Problem 34. Let $c : [a,b] \to \mathbb{R}^3$ be a C^1 path, find a reparametrization $\tilde{c} = c \circ h$, such that $\tilde{c} : [0,1] \to \mathbb{R}^3$.

Proof. Define $h(t) = a + (b-a)t$, where $0 \le t \le 1$. □

Problem 35. Suppose that $\nabla f(x,y,z) = (2x^2, y, -z)$, and that $f(0,1,-1) = 2$. What is the value of $f(2,1,1)$?

Proof. By the fundamental theorem of line integrals, we know

$$\begin{aligned}
f(2,1,1) &= \int_\ell \nabla f \cdot ds + f(0,1,-1) \\
&= \int_0^1 \nabla f(2t, 1, 2t-1) \cdot (2,0,2)dt + f(0,1,-1) \\
&= \int_0^1 \left(16t^2 + 2 - 4t\right) dt + 2 \\
&= \frac{22}{3}
\end{aligned}$$

□

Problem 36. Compute the path integral of $f(x, y, z) = 2x + y - z$ over the curve c, where c is the intersection of the two surfaces below:

$$y = x, \quad y^2 + z^2 = 4$$

Hint: first find a parametrization of c, then use the formula of path integral.

Proof. Using polar coordinates, we can write $y = 2\cos\theta$, $z = 2\sin\theta$, and since $y = x$, we know $x = 2\cos\theta$, thus the parametrization is given by

$$c(\theta) = (2\cos\theta, 2\cos\theta, 2\sin\theta), \quad 0 \le \theta \le 2\pi$$

Now

$$\begin{aligned} \int_c f ds &= \int_0^{2\pi} f(c(\theta))\|c'(\theta)\|d\theta \\ &= \int_0^{2\pi} (4\cos\theta + 2\cos\theta - 2\sin\theta)\,\|c'(\theta)\|d\theta \\ &= \text{complicated} \end{aligned}$$

□

Problem 37. Suppose path c has length l, and F is a vector field such that $\|F\| \le M$. Prove that

$$\left|\int_c F \cdot ds\right| \le Ml$$

(Hint: Cauchy-Schwarz.)

Proof.

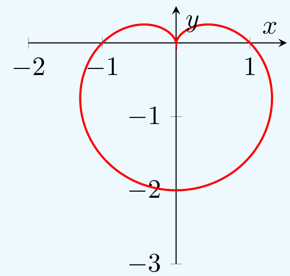
$$\begin{aligned} \left|\int_c F \cdot ds\right| &= \left|\int_a^b F(c(t)) \cdot c'(t)dt\right| \\ &\le \int_a^b |F(c(t)) \cdot c'(t)dt|\, dt \\ &\le \int_a^b \|F(c(t))\| \cdot \|c'(t)\|dt \\ &\le Ml \end{aligned}$$

□

Problem 38 (Group work). Given vector field $F(x, y) = (1, 1)$, and the following heart is given by the parametrization:

$$c(t) = ((1 - \sin t)\cos t, (1 - \sin t)\sin t) \quad \text{where } 0 \leq t \leq 2\pi$$

Group 1: what is the work done by going up the top of the heart? ($0 \leq t \leq \pi$).
Group 2: what is the work done by going up the bottom of the heart? ($\pi \leq t \leq 2\pi$).



Proof. No need to actually compute $c'(t)$! For Group 1, we see that

$$\begin{aligned} \int_0^\pi F(c(t)) \cdot c'(t)dt &= \int_0^\pi x'(t) + y'(t)dt & (c(t) = (x(t), y(t))) \\ &= x(\pi) + y(\pi) - x(0) - y(0) \\ &= -2 \end{aligned}$$

Similarly,

$$\int_\pi^{2\pi} F(c(t)) \cdot c'(t)dt = 2$$

Thus combined

$$\int_0^{2\pi} F(c(t)) \cdot c'(t)dt = 0$$

□

Week 13 (11/17-21)

Definition 35 (parametrization of surface). Let S be a surface in $\mathbb{R}^3$, a **surface parametrization** is a map $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, where

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

Definition 36 (regular surface, tangent plane). Let $\Phi(u, v)$ be a parametrization of a surface $S \subset \mathbb{R}^3$. We say S is **regular** at $\Phi(u_0, v_0)$ if

$$T_u \times T_v \neq 0 \text{ at } (u_0, v_0)$$

If S is regular at $\Phi(u_0, v_0)$, then we can find the tangent plane by first finding a normal vector to the surface at this point: $n = T_u \times T_v$, then the tangent plane at $(x_0, y_0, z_0) = \Phi(u_0, v_0)$ is given by

$$(x - x_0, y - y_0, z - z_0) \cdot n = 0$$

Definition 37 (surface area). Let $S \subset \mathbb{R}^3$ be a parametrized surface, then the **surface area** $A(S)$ of S is given by

$$\begin{aligned} A(S) &= \iint_D \|T_u \times T_v\| dudv \\ &= \iint_D \left(\left[\frac{\partial(x, y)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(y, z)}{\partial(u, v)}\right]^2 + \left[\frac{\partial(x, z)}{\partial(u, v)}\right]^2 \right)^{1/2} dudv \end{aligned}$$

where $\|T_u \times T_v\|$ is the norm of $T_u \times T_v$, and

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}, \quad \ldots$$

Definition 38 (integral over a surface). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be continuous, i.e., f is a scalar-valued continuous function defined on a parametrized surface S by $\Phi : D \to S \subset \mathbb{R}^3$, we define the integral of f over S as

$$\iint_S f dS = \iint_D f(\Phi(u, v)) \|T_u \times T_v\| dudv$$

A special case is when we take S as the graph of some function $g(x, y)$. Then we have

$$\iint f dS = \iint_D \frac{f(x, y, g(x, y))}{\cos\theta} dxdy$$

where θ is the angle between the unit vector $(x, y, g(x, y))$ and the normal vector the surface. (Recall that the normal vector of a graph is given by $n = -\frac{\partial g}{\partial x} i - \frac{\partial g}{\partial y} + k$).

Problem 39. Find the tangent plane at $(1, 1, 2)$ for the following surface parametrized by:

$$\Phi(x, y) = (x, y, x^2 + y)$$

Proof. We compute

$$T_x = (1, 0, 2x), \quad T_y = (0, 1, 1)$$

Then

$$T_x \times T_y = \det \begin{pmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 1 \end{pmatrix} = (-2x, -1, 1)$$

At $(1, 1, 2)$ the tangent plane is given by

$$(x-1, y-1, z-2) \cdot (-2, -1, 1) = 0$$

Simplifying we get

$$2x + y - z = 1$$

□

Problem 40 (Marsden-Tromba, 7.5, Exercise 4)**.** Evalute the following integral:

$$\iint_S (x+z)dS$$

where S is the part of the cylinder $y^2 + z^2 = 4$ with $x \in [0, 5]$.

Proof. Let's first parametrize this surface:

$$\Phi(t, \theta) = (t, 2\cos\theta, 2\sin\theta), \quad 0 \leq t \leq 5, 0 \leq \theta \leq 2\pi$$

and we compute

$$T_t = (1, 0, 0), T_\theta = (0, -2\sin\theta, 2\cos\theta)$$

and

$$\|T_t \times T_\theta\| = \|(0, -2\cos\theta, -2\sin\theta)\| = 2$$

Thus we have

$$\begin{aligned} \iint_S x + z dS &= \iint_D t + 2\sin\theta \|T_t \times T_\theta\| dt d\theta \\ &= 2\int_0^{2\pi} \int_0^5 t + 2\sin\theta dt d\theta \\ &= 4\pi \int_0^5 t dt \\ &= 50\pi \end{aligned}$$

□