Questions

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Problem 0.1. To see whether a polynomial is irreducible over $\mathbb{Q}$, is it sufficient to test whether $f \mod p$ is irreducible over any prime p?

For example, $x^5 - 5x^3 + 1$.

Problem 0.2. In the above example, how do we know that the Galois group contains an element of order 5? (It is clear why it contains a transposition because there exists complex roots).

Proof. This is because the Galois group G acts transitively on the set of roots, by the Orbit stabilizer theorem, we know
$$|G| = |\text{Orbit}(\alpha)| \cdot |\text{Stab}(\alpha)| = 5 \cdot |\text{Stab}(\alpha)|$$
i.e., 5 divides $|G|$. By Cauchy's theorem, there exists an element of order 5 in G, i.e., a 5-cycle. □

Problem 0.3. The Galois action on the set of roots implies for any root r of the polynomial (where G is the splitting field of), we must have

$$\text{Orbit}(r) = \{ \text{ set of all roots}\}$$