

Calc III Sections

Fall 2025

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Calc III-Week 5 (9/22-9/26)

Topics: (1) properties of derivatives, (2) directional derivatives, (3) gradient.

Definition 1 (path). A **path** c is a map $c : [a, b] \rightarrow \mathbb{R}^n$. We can write $c(t) = (c_1(t), \dots, c_n(t))$. If c is differentiable, then we can define the **velocity** of c at any $t_0 \in [a, b]$ as

$$c'(t_0) = (c'_1(t_0), \dots, c'_n(t_0))$$

The velocity vector of c at t_0 is also a **tangent** vector to c at t_0 . The **speed** of the path c at t_0 is the length of the velocity vector $\|c'(t_0)\|$.

Definition 2 (tangent line to a path). Let $c : [a, b] \rightarrow \mathbb{R}^n$ be a path, if $c'(t_0) \neq 0$, then the **tangent line** at x_0 is given by

$$l(t) = c(t_0) + c'(t_0)(t - t_0)$$

Proposition 1. Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at x_0 , then the derivative of f at x_0 is an $m \times n$ matrix $Df(x_0) = \left(\frac{\partial f_i}{\partial x_j} \right)_{ij}$. The derivative follows the same properties as derivative for single variable functions:

1. Let $c \in \mathbb{R}$, then

$$D(cf)(x_0) = cDf(x_0) \quad (\text{multiplication of a matrix by constant } c)$$

2. Let $g : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ also be differentiable at x_0 , then

$$D(f + g)(x_0) = Df(x_0) + Dg(x_0) \quad (\text{sum of two matrices})$$

3. Let $h_1 : U \subset \mathbb{R}^n \rightarrow \mathbb{R}, h_2 : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$, then

$$D(h_1 h_2)(x_0) = Dh_1(x_0)h_2(x_0) + h_1(x_0)Dh_2(x_0) \quad (\text{product rule})$$

and if $h_2 \neq 0$ on U .

$$D(h_1/h_2)(x_0) = \frac{Dh_1(x_0)h_2(x_0) - h_1(x_0)Dh_2(x_0)}{h_2^2(x_0)} \quad (\text{quotient rule})$$

4. Let $g : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^m, f : V \subset \mathbb{R}^m \rightarrow \mathbb{R}^p$ such that $g(U) \subset V$, then

$$D(f \circ g)(x_0) = Df(g(x_0))Dg(x_0) \quad (\text{chain rule})$$

Definition 3 (directional derivative). Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$, be differentiable, then the directional derivative at $x_0 \in \mathbb{R}^3$ in the direction of a **unit vector** v is given by

$$\nabla f(x_0) \cdot v = \left[\frac{\partial f}{\partial x_1}(x_0) \right] v_1 + \left[\frac{\partial f}{\partial x_2}(x_0) \right] v_2 + \left[\frac{\partial f}{\partial x_3}(x_0) \right] v_3$$

where $v = (v_1, v_2, v_3)$.

Proposition 2. Suppose that $\nabla f(x_0) \neq 0$, then the direction for which f increases the fastest at x_0 is along $\nabla f(x_0)$.

Proposition 3. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be differentiable, let S be a level surface of f , i.e., S is a surface described by

$$f(x, y, z) = k$$

where k is some constant. Let $(x_0, y_0, z_0) \in S$, then

$$\nabla f(x_0, y_0, z_0) \text{ is normal to the level surface at } (x_0, y_0, z_0)$$

This means if $c(t)$ is a path in S , and $v(0) = (x_0, y_0, z_0)$, and if v is a tangent vector to $c(t)$ at $t = 0$, then

$$\nabla f(x_0, y_0, z_0) \cdot v = 0$$

Moreover, if $\nabla f(x_0, y_0, z_0) \neq 0$, the **tangent plane** of S at (x_0, y_0, z_0) is given by

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

Problem 1. Consider the curve in \mathbb{R} : $c(t) = (2t, t^2, -t)$. Find the speed of the c at $t = 2$ and the tangent line at $t = 1$.

Proof. The velocity vector of c at $t = 2$ is

$$c'(t) = (2, 2t, -1)$$

evaluated at $t = 2$ is $c'(2) = (2, 4, -1)$. Thus the speed is the length of the velocity vector

$$\|c'(2)\| = (2^2 + 4^2 + (-1)^2)^{\frac{1}{2}} = \sqrt{21}$$

For the tangent line: the tangent vector is

$$c'(1) = (2, 2, -1)$$

and $c(1) = (2, 1, -1)$. Thus the tangent line l at $t = 1$ is given by

$$l(t) = (2, 1, -1) + t(2, 2, -1) = (2 + 2t, 1 + 2t, -1 - t)$$

□

Problem 2 (2.5, Q7). Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and

$$g(x, y) = (e^{x-y}, x - y).$$

Calculate $f \circ g$ and

$$D(f \circ g)(1, 1).$$

Proof. We have

$$f \circ g(x, y) = (\tan(e^{x-y} - 1) - e^{x-y}, e^{2(x-y)} - (x - y)^2)$$

and $g(1, 1) = (1, 0)$, thus using chain rule, we have

$$D(f \circ g)(1, 1) = Df(1, 0)Dg(1, 1)$$

where

$$Df(u, v) = \begin{bmatrix} \sec^2(u - 1) & -e^v \\ 2u & -2v \end{bmatrix}, \quad Dg(x, y) = \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}$$

Hence

$$\begin{aligned} D(f \circ g)(1, 1) &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \end{aligned}$$

□

Problem 3 (2.5, Q8). Let $f(u, v, w) = (e^{u-w}, \cos(v+u) + \sin(u+v+w))$ and $g(x, y) = (e^x, \cos(y-x), e^{-y})$. Calculate $f \circ g$ and $D(f \circ g)(0, 0)$.

Proof. We have

$$f \circ g = (e^{e^x - \cos(y-x)}, \cos(e^x + \cos(y-x)), \sin(e^x + e^{-y} + \cos(y-x)))$$

and $g(0, 0) = (1, 1, 1)$. Thus

$$D(f \circ g)(0, 0) = Df(1, 1, 1)Dg(0, 0)$$

where

$$Df(u, v, w) = \begin{bmatrix} e^{u-w} & 0 & -e^{u-w} \\ -\sin(v+u) + \cos(u+v+w) & -\sin(v+u) + \cos(u+v+w) & \cos(u+v+w) \end{bmatrix}$$

and

$$Dg(x, y) = \begin{bmatrix} e^x & 0 \\ \sin(y-x) & -\sin(y-x) \\ 0 & -e^{-y} \end{bmatrix}$$

Thus

$$\begin{aligned} D(f \circ g)(0, 0) &= \begin{bmatrix} 1 & 0 & -1 \\ -\sin 2 + \cos 3 & -\sin 2 + \cos 3 & \cos 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -\sin 2 + \cos 3 & -\cos 3 \end{bmatrix} \end{aligned}$$

□

Problem 4 (2.5, Q11). Let $f(x, y, z) = (3y + 2, x^2 + y^2, x + z^2)$. Let

$$c(t) = (\cos(t), \sin(t), t).$$

(a) Find the path $p = f \circ c$ and the velocity vector

$$p'(\pi).$$

(b) Find $c(\pi)$, $c'(\pi)$ and $Df(-1, 0, \pi)$.

(c) Thinking of $Df(-1, 0, \pi)$ as a linear map, find

$$Df(-1, 0, \pi)(c'(\pi)).$$

Proof. (a) We have

$$p(t) = (3 \sin t + 2, 1, \cos t + t^2)$$

and

$$p'(t) = (3 \cos t + 0, -\sin t + 2t)$$

thus

$$p'(\pi) = (-3, 0, 2\pi)$$

(b) We have $c(\pi) = (-1, 0, \pi)$, and $c'(t) = (-\sin t, \cos t, 1)$, and $c'(\pi) = (0, -1, 1)$. And

$$Df(x, y, z) = \begin{bmatrix} 0 & 3 & 0 \\ 2x & 2y & 0 \\ 1 & 0 & 2z \end{bmatrix}$$

Thus

$$Df(-1, 0, \pi) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}$$

(c) We have

$$Df(-1, 0, \pi)(c'(\pi)) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2\pi \end{bmatrix}$$

□

Problem 5 (2.6, Q3). Compute the directional derivatives of the following functions along unit vectors at the indicated points in directions parallel to the given vector:

(a)

$$f(x, y) = x^y, (x_0, y_0) = (e, e), \quad \mathbf{d} = 5\mathbf{i} + 12\mathbf{j}$$

(b)

$$f(x, y, z) = e^x + yz, (x_0, y_0, z_0) = (1, 1, 1), \quad \mathbf{d} = (1, -1, 1)$$

(c)

$$f(x, y, z) = xyz, (x_0, y_0, z_0) = (1, 0, 1), \quad \mathbf{d} = (1, 0, -1)$$

Proof. We find ∇f for all these functions and find the directional derivative

$$\nabla f(x_0) \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|}$$

(a) We have

$$\nabla f(x, y, z) = (yx^{y-1}, x^y \ln x)$$

hence

$$\nabla f(e, e) \cdot \frac{\mathbf{d}}{\|\mathbf{d}\|} = (e^e, e^e) \cdot \left(\frac{5}{13}, \frac{12}{13} \right) = \frac{17}{13} e^e$$

(b) We have

$$\nabla f(x, y, z) = (e^x, z, y)$$

hence

$$\nabla f(1, 1, 1) \cdot \frac{1}{\sqrt{3}}(1, -1, 1) = \frac{e}{\sqrt{3}}$$

(c) We have

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 0, 1) \cdot \frac{1}{\sqrt{2}}(1, 0, -1) = \frac{1}{\sqrt{2}}(0, 1, 0) \cdot (1, 0, -1) = 0$$

□

Problem 6 (2.6, Q6). Find a vector which is normal to the curve

$$x^3 + xy + y^3 = 11 \text{ at } (1, 2).$$

Proof. Consider the function $f(x, y) = x^3 + xy + y^3$, then the level set of $f(x, y) = 11$ coincides with the curve above. Thus it suffices to compute

$$\nabla f(x, y) = (3x^2 + y, x + 3y^2)$$

and

$$\nabla f(1, 2) = (5, 13)$$

is perpendicular to the the level curve. □

Problem 7 (2.6, Q7). Find the rate of change of $f(x, y, z) = xyz$ in the direction normal to the surface

$$yx^2 + xy^2 + yz^2 = 3 \text{ at } (1, 1, 1).$$

Proof. We first find a normal vector to the surface, consider the surface as a level set of the function

$$g(x, y, z) = yx^2 + xy^2 + yz^2$$

Thus

$$\nabla g(x, y, z) = (2xy + y^2, x^2 + 2xy + z^2, 2yz)$$

hence

$$u = \nabla g(1, 1, 1) = (3, 4, 2)$$

is a normal vector to the surface, and we normalize it to get a unit normal vector $n = \frac{u}{\|u\|} = \frac{1}{\sqrt{29}}(3, 4, 2)$. Now we find the directional derivative of $f(x, y, z) = xyz$ along $(3, 4, 2)$:

$$\nabla f(x, y, z) = (yz, xz, xy)$$

hence

$$\nabla f(1, 1, 1) = (1, 1, 1)$$

and the directional derivative is

$$\nabla f(1, 1, 1) \cdot n = \frac{9}{\sqrt{29}}$$

□

Summary

- If asked to find directional derivative/rate of change along a unit vector v at point x_0 : find $\nabla f(x_0) \cdot v$.
- If asked to find along with direction f increases the fastest: find $\nabla f(x_0)$.
- If asked to find a normal vector to a surface at a point x_0 : construct a function such that the surface is the level set of this function, then find $\nabla f(x_0)$.