Real Analysis 605 MT Review

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Contents

Chapter 1

Definitions

Definition 1.1 (sequence of sets). Let $\{E_k\} \subset \mathbb{R}^n$ be a sequence of sets is said to increase to $\bigcup_k E_k$ if $E_k \subset E_{k+1}$ for all k, and decrease to $\bigcap_k E_k$ if $E_k \supset E_{k+1}$ for all k.

Definition 1.2 (limsup,liminf of sets). Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of sets, we define

$$\limsup E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k=j}^{\infty} E_k\right), \quad \liminf E_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k=j}^{\infty} E_k\right)$$

Definition 1.3 (metric). Let d be a metric on $\mathbb{R}^n$, let $x, y \in \mathbb{R}^n$, then

1. $d(x, y) = d(y, x)$
2. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
3. $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 1.4 (limsup, liminf of sequences). Let $\{a_k\}$ be a sequence of points in $\mathbb{R}$, then

$$\limsup_{k\to\infty} a_k := \lim_{j\to\infty}\{\sup_{k\geq j} a_k\}$$

and

$$\liminf_{k\to\infty} a_k := \lim_{j\to\infty}\{\inf_{k\geq j} a_k\}$$

Definition 1.5 (distance between sets). Let $E_1, E_2 \subset \mathbb{R}^n$, then the distance between E_1 and E_2 is defined as

$$d(E_1, E_2) = \inf\{|x - y| : x \in E_1, y \in E_2\}$$

Definition 1.6 (open set). Let $E \subset \mathbb{R}^n$, then E is called open if for each $x \in E$, there exists δ such that $B_\delta(x) \subset E$.

A subset E_1 of E is said to be relatively open with respect to E if it can be written as $E_1 = E \cap G$ for some open set G.

Definition 1.7 (A_δ, A_σ sets). A set A is said to be of type A_δ if it can be written as a countable intersection of sets and to be of type A_σ if it can be written as a countable union of sets. Then G_δ implies a countable intersection of open sets, and F_σ implies the countable union of closed sets.

Definition 1.8 (perfect set). C is called a perfect set if it is a closed set such that every point in C is a limit point.

Definition 1.9 (compact set). A set E is compact if and only if every open cover of E has a finite subcover.

Definition 1.10 (monotone function). A fuction f defined on $I \subset \mathbb{R}$ is monotone increasing if $f(x) \leq f(y)$ whenever $x < y$. Similarly defined for monotonically decreasing.

Definition 1.11 (continuous). Let f be defined on a neighorhood of x_0, then f is said to be continuous at x_0 if $f(x_0)$ is finite and $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 1.12 (continuous relative to a set). Let f be defined in only a set E containing x_0, f is said to be continuous at x_0 relative to E if $f(x_0)$ is finite and either x_0 is an isolated point of E or x_0 is a limit point of E and for $x \in E$.

$$\lim_{x \to x_0} f(x) = f(x_0)$$

If $E_1 \subset E$, a function is continuous in E_1 relative to E if it is continuous relative to E at every point in E_1.

Definition 1.13 (uniform convergence). A sequence $\{f_k\}$ defined on E is said to uniformly convergence on E to a finite f if given $\varepsilon > 0$, there exists K such that for all $k \geq K$, $x \in E$,

$$|f_k(x) - f(x)| < \epsilon$$

Definition 1.14 (Riemann integral). Let f be bounded on an interval I, partition I into a finite collection Γ of nonoverlapping intervals, denote $|\Gamma| = \max_k diam(I_k)$, select points $\xi_k \in I_k$, let

$$R_\Gamma = \sum_{k=1}^{N} f(\xi_k)|I_k|$$

and

$$U_\Gamma = \sum_{k=1}^{N} (\sup_{x \in I_k} f(x))|I_k|, \quad L_\Gamma = \sum_{k=1}^{N} (\inf_{x \in I_k} f(x))|I_k|$$

The Riemann integral exists if $\lim_{|\Gamma| \to 0} R_\Gamma$ exists and the limit A is the Riemann integral. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\Gamma| < \delta$, we have $|A - R_\Gamma| < \varepsilon$ for any Γ and any chosen $\{\xi_k\}$.

This is equivalent to the statement:

$$\inf_\Gamma U_\Gamma = \sup_\Gamma L_\Gamma = A$$

Definition 1.15 (variation). Let f be defined on $[a, b]$, the variation of f over $[a, b]$ is

$$V(f) = \sup_{\Gamma} \sum_{i=1}^{m} |f(x_i) - f(x_{i-1})|$$

where Γ is any partition $\{x_0, x_1, \ldots, x_m\}$ of $[a, b]$.

Definition 1.16 (Lipschitz). Let f be defined on $[a, b]$, then f is said to be Lipschitz if there exists an absolute constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in [a, b]$.

Definition 1.17 (splitting). For any $x \in \mathbb{R}$, we can write

$$x^+ = \begin{cases} x, x > 0 \\ 0, x \leq 0 \end{cases}$$

$$x^- = \begin{cases} 0, x > 0 \\ -x, x \leq 0 \end{cases}$$

then $|x| = x^+ + x^-, x = x^+ - x^-$.

Definition 1.18 (P_Γ, N_Γ). For any f and any partition Γ, define

$$P_\Gamma = \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})]^+$$

and

$$N_\Gamma = \sum_{i=1}^{m} [f(x_i) - f(x_{i-1})]^-$$

similarly, we define

$$P = \sup_{\Gamma} P_\Gamma, N = \sup_{\Gamma} N_\Gamma$$

Definition 1.19 (rectifiable curve). Let C be a curve, i.e.

$$C : \begin{cases} x = \varphi(t) \\ y = \psi(t) \end{cases}$$

Let Γ be any partition, define

$$L = \sup_{\Gamma} \sum_{i=1}^{m} \left((\phi(t_i) - \phi(t_{i-1}))^2 + (\psi(t_i) - \psi(t_{i-1}))^2 \right)^{1/2}$$

then C is rectifiable if $L < +\infty$.

Definition 1.20 (Riemann-Stieltjes integral). Let f, ϕ be finite on an interval $[a,b]$, let $\Gamma = \{a = x_0 = \cdots < x_m = b\}$ be any partition, define

$$R_\Gamma = \sum_{i=1}^{m} f(\xi_i)\,[\phi(x_i) - \phi(x_{i-1})]$$

If $\lim_{|\Gamma|\to 0} R_\Gamma$ exists, then we call this the Riemann-Stieltjes integral. That is, given any $\varepsilon > 0$, there is $\delta > 0$ such that when $|\Gamma| < \delta$ we have $|I - R_\Gamma| < \varepsilon$. We denote it as

$$I = \int_a^b f(x) d\phi(x) = \int_a^b f d\phi$$

Definition 1.21 (upper, lower R-S sum). Let f be bounded and ϕ be monotonically increasing. Let

$$m_i = \inf_{[x_{i-1},x_i]} f(x), M_i = \sup_{[x_{i-1},x_i]} f(x)$$

then we define the lower and upper Riemann-Stieltjes sums L_Γ, U_Γ as follows:

$$L_\Gamma = \sum_{i=1}^{m} m_i[\phi(x_i) - \phi(x_{i-1})], U_\Gamma = \sum_{i=1}^{m} M_i[\phi(x_i) - \phi(x_{i-1})]$$

Definition 1.22 (Lebesgue outer measure). For let S be a collection of n-dimensional intervals that cover E, then the Lebesgue outer measure of E is given by

$$|E|_e = \inf \sigma(S)$$

where $\sigma(S) = \sum_{I_k \in S} |I_k|$.

Definition 1.23 (Lebesgue measurable). A subset E of $\mathbb{R}^n$ is called Lebesgue measurable if and only if given any $\varepsilon > 0$, there exists an open set G such that

$$E \subset G, |G - E|_e < \varepsilon$$

If E is measurable, then $|E| = |E|_e$.

Definition 1.24 (σ-algebra). A σ-algebra is a collection of sets that is closed under taking complement, countable union, and countable intersection.

The σ-algebra generated by containing all the open sets is called the Borel σ-algebra.

Definition 1.25 (Lebesgue measurable functions). Let E be a measurable set in $\mathbb{R}^n$, f is a mesaurable function if for all finite a, the set

$$\{x \in E : f(x) > a\}$$

is measuarble.

Definition 1.26 (upper,lower semicontinuous). Let f be defined on E, then f is usc at x_0 if for every $M > f(x_0)$, there exists $\delta > 0$ such that when $|x - x_0| < \delta$, we have $f(x) < M$.

f is called usc relative to E if it is usc at every limit point of E.

Definition 1.27 (convergence in measure). Let f, $\{f_k\}$ be defined and a.e. on E, then $f_k \to f$ in measure if for every $\varepsilon > 0$,

$$\lim_{k\to\infty} |\{x \in E : |f(x) - f_k(x)| > \varepsilon\}| = 0$$

Chapter 2

Theorems

Proposition 2.1. $\limsup_{k\to\infty} a_k = L$ if and only if there exists a subsequence $\{a_{k_j}\}$ that converges to L.

Proposition 2.2. For closed and open sets, we have the following:

1. The arbitrary unions of open sets is open, and finite intersections of open sets is open.
2. The arbitrary intersections of closed sets is closed, and finite unions of closed sets is closed.

Proposition 2.3. A set $E_1 \subset E$ is relatively closed with respect to E if and only if

$$E_1 = E \cap \overline{E_1}$$

Proposition 2.4. Every open set in $\mathbb{R}^1$ can be written as a countable union of disjoint open intervals. Moreover, every open set in $\mathbb{R}^n$ can be written as a countable union of nonoverlapping closed cubes.

Theorem 2.1 (Heine-Borel). A set $E \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded. (A set E is compact iff every sequence of points of E has a subsequence that converges to a point of E, i.e., compact implies sequentially compact).

Proposition 2.5. $M = \limsup_{x\to x_0}$ if and only if there exists $\{x_k\}$ in $E - \{x_0\}$ such that $x_k \to x_0$ and $f(x_k) \to M$ and if $M' > M$, there exists $\delta > 0$ such that $f(x) < M'$ for $x \in B(x_0, \delta) \cap E$.

Theorem 2.2. If E is compact and f is continuous in E relative to E, then the following hold:

1. f is bounded on E, $\sup_{x\in E} |f(x)| < \infty$.
2. f attains supremum and infimum on E.
3. F is uniformly continuous on E relative to E.

Theorem 2.3. Let $\{f_k\}$ be a sequence of functions that are continuous in E and converge uniformly to f, then f is continuous on E.

Proposition 2.6. Let $y = Tx$ be a transformation of $\mathbb{R}^n$ that is continuous in E. If E is compact, then the image TE is also compact.

Proposition 2.7. A bounded f is Riemann integral on I if and only if given any $\varepsilon > 0$, there is a partition Γ of I, such that

$$0 \leq U_\Gamma - L_\Gamma < \varepsilon$$

Proposition 2.8. Let f, g be of bounded variation on $[a, b]$, then for any real constant c, we have

$$f + g, fg, cf$$

are of bounded variation. If g is nonvaishing, then f/g is also of bounded variation.

Proposition 2.9. If $[a', b']$ is a subinterval of $[a, b]$, then

$$V[a', b'] \leq V[a, b]$$

Moreover, if $a < c < b$, then

$$V[a, b] = V[a, c] + V[c, b]$$

Proposition 2.10. Let P, N be positive and negative variation defined above, if any of P, N, V is finite, then all three are finite. We have

$$P + N = V, \quad P - N = f(b) - f(a)$$

and

$$P = \frac{1}{2}[V + f(b) - f(a)], \quad N = \frac{1}{2}[V - f(b) + f(a)]$$

Theorem 2.4 (Jordan's theorem). A function f is of bounded variation on $[a, b]$ if and only if it can be written as the difference of two bounded increasing functions on $[a, b]$.

Theorem 2.5. Every function of bounded variation has at most a countable number of discontinuities, and they are all jump or removable discontinuities.

Proposition 2.11. If f is continuous on $[a, b]$, then

$$V = \lim_{|\Gamma| \to 0} S_\Gamma$$

If f has a conitnuous derivative f' on $[a, b]$, then

$$V = \int_a^b |f'|, P = \int_a^b \{f'\}^+, N = \int_a^b \{f'\}^-$$

Proposition 2.12. Let $C := \begin{cases} \varphi(t) \\ \psi(t) \end{cases}$ be a curve, then it is rectifiable if and only if both φ, ψ are of bounded variations.

Proposition 2.13. If $\int_a^b$ exists,

1. For any constant c, we have

$$\int_a^b cfd\phi = \int_a^b fd(c\phi) = c\int_a^b fd\phi$$

2. If $\int_a^b gd\phi$ also exists, then

$$\int_a^b (f+g) = \int fd\phi + \int gd\phi$$

3. If $\int_a^b fd\phi$ exists and $a < c < b$, then two intermediate integrals also exist

$$\int_a^b fd\phi = \int_a^c fd\phi + \int_c^b fd\phi$$

4. $\int_a^b \phi df$ also exists,

$$\int_a^b fd\phi = [f(b)\phi(b) - f(a)\phi(a)] - \int_a^b \phi df$$

Proposition 2.14. Let f be bounded and ϕ be increasing on $[a,b]$,

1. If Γ' is a refinement of Γ, then

$$L_{\Gamma'} \geq L_\Gamma, U_{\Gamma'} \leq U_\Gamma$$

2. If Γ_1, Γ_2 are two partitions, then

$$L_{\Gamma_1} \leq U_{\Gamma_2}$$

Proposition 2.15. If f is continuous on $[a,b]$ and ϕ is of bounded variation on $[a,b]$, then $\int_a^b fd\phi$ exists, and

$$\left|\int_a^b f\phi\right| \leq \sup_{[a,b]f} V[\phi,[a,b]]$$

Theorem 2.6 (Mean-Value Theorem). If f is continuous on $[a,b]$ and ϕ is bounded and increasing on $[a,b]$, there exists $\xi \in [a,b]$ such that

$$\int_a^b d\phi = f(\xi)[\phi(b) - \phi(a)]$$

Proposition 2.16. For an interval I, the exterior measure $|I|_e$ is the volume of I.

Proposition 2.17. If $E_1 \subset E_2$, then $|E_1|_e \leq |E_2|_e$, and if $E = \bigcup_k E_k$ is a countable union of sets, then

$$|E|_e \leq \sum_k |E_k|_e$$

Theorem 2.7. If $E \subset \mathbb{R}^n$, then given $\varepsilon > 0$, there exists an open set G such that $E \subset G$ and $|G|_e \leq |E|_e + \varepsilon$. Hence

$$|E|_e = \inf |G|_e$$

where $\inf$ is taken over all open sets G containing E.

Proposition 2.18. Every open set is measurable, and every set of outer measure zero is measurable. Any interval I is measurable. Let $\{E_k\}$ be measurable sets, then $E = \bigcup_k E_k$ is also measurable, and

$$|E| \leq \sum_k |E_k|$$

Similary, $\bigcap_k E_k$ is also measurable. If E_1, E_2 are measurable, then $E_1 - E_2$ is measurable.

Proposition 2.19. If $\{I_k\}_{k=1}^N$ is a finite collection of nonoverlapping intervals, then $\bigcup_k I_k$ is also measurable, and

$$\left| \bigcup_k I_k \right| = \sum_k |I_k|$$

If $d(E_1, E_2) > 0$, then

$$|E_1 \cup E_2|_e = |E_1|_e + |E_2|_e$$

Proposition 2.20. The collection of measurable sets of $\mathbb{R}^n$ is σ-algebra.

Proposition 2.21. A set $E \subset \mathbb{R}^n$ is measurable if and only if given $\varepsilon > 0$, there exists a closed set $F \subset E$, such that

$$|E - F|_e < \varepsilon$$

Theorem 2.8. If $\{E_k\}$ is a countable collection of disjoint measurable sets, then

$$\left| \bigcup_k E_k \right| = \sum_k |E_k|$$

Proposition 2.22. If E_1, E_2 measurable, and $E_2 \subset E_1$, $|E_2| < \infty$, then

$$|E_1 - E_2| = |E_1| - |E_2|$$

Theorem 2.9. Let $\{E_k\}_{k=1}^\infty$ be a sequence of measurable sets, then

1. If $E_k \nearrow E$, then $\lim_{k\to\infty} |E_k| = |E|$.
2. If $E_k \searrow E$, and $|E_k| < \infty$, then $\lim_{k\to\infty} |E_k| = |E|$.

Theorem 2.10 (Caratheodory). A set E is measurable if and only if for every set A, we have

$$|A|_e = |A \cap E|_e + |A - E|_e$$

Theorem 2.11. If $y = Tx$ is a Lipschitz transformation of $\mathbb{R}^n$, then T maps measurable sets into measurable sets. Recall a Lipschitz transformation is such that there exists a constant c such that

$$|Tx - Ty| \leq c|x - y|$$

where

$$c = \sup_{x \neq y} \frac{|Tx - Ty|}{|x - y|}$$

Theorem 2.12. Let T be a linear transformation of $\mathbb{R}^n$, and let E be a measurable set, then

$$|TE| = \frac{1}{|\det(T)|}|E|$$

Proposition 2.23. Any set in $\mathbb{R}^n$ with positive outer measurable contains a nonmeasurable set.

Proposition 2.24. f is measurable if and only if any of the following statements holds for any finite a:

1. $\{f \geq a\}$ is measurable.
2. $\{f < a\}$ is measurable.
3. $\{f \leq a\}$ is measurable.

Proposition 2.25. If f is measurable, then $\{f > -\infty\}, \{f < \infty\}, \{f = \infty\}, \{a \leq f \leq b\}, \{f = a\}$ is measurable.

Moreover, if $\{f = \infty\}$ or $\{f = -\infty\}$ is measurable, then f is measurable if for every finite a, $\{a < f < \infty\}$ is measurable.

Theorem 2.13. If f is measurable, then for every open G, $f^{-1}(G)$ is measurable. Conversely, if $f^{-1}(G)$ is measurable for every open $G \subset \mathbb{R}^n$ and either $\{f = \infty\}$ or $\{f = -\infty\}$ is measurable.

Theorem 2.14. Let ϕ be continuous on $\mathbb{R}^1$ and let f be finite a.e. in E, in particular, $\phi(f)$ is defined a.e. in E, then $\phi(f)$ is measurable if f is.

Proposition 2.26. If f, g are measurable, then $\{f > g\}$ is measurable. If f is mesaurable, and λ is any real number, then $f + \lambda$ and λf are measurable. If f, g are measurable, then $f + g, fg$ is measurable. If $g \neq 0$ a.e., then f/g also measurable.

If $\{f_k\}$ is a sequence of measurable functions, then $\sup_k f_k(x), \inf_k f_k(x)$ are measurable.

Proposition 2.27. If $\{f_k\}$ is a sequence of measurable functions, then $\limsup_{k\to\infty} f_k, \liminf_{k\to\infty} f_k$ are measurable. In particular, if $f = \lim_{k\to\infty} f_k(x)$, then f is measurable.

Proposition 2.28. We have

1. Every function f can be written as the limit of a sequence $\{f_k\}$ of simple functions.
2. If $f \geq 0$, the sequence can be chosen to increase to f.
3. If f in either 1 or 2 is measurable, then f_k can be chosen to be measurable.

Proposition 2.29. A function f is usc relative to E if and only if $\{x \in E : f(x) \geq a\}$ is relatively closed for all finite a.

A function F is lsc relative to E if and only if $\{x \in E : f(x) \leq a\}$ is relatively closed for all finite a.

Proposition 2.30. A finite function f is continuous relative to E if and only if all sets of the form $\{x \in E : f(x) \geq a\}$ and $\{x \in E : f(x) \leq a\}$ are relatively closed. (or equivalently $\{f > a\}$ and $\{f < a\}$ are relatively open).

Proposition 2.31. Let E be measurable, then f is usc relative to E, then f is measurable.

Theorem 2.15 (Egorov's theorem)**.** Suppose that $\{f_k\}$ is a sequence of measurable functions that converges a.e. to a finite limit f. The given $\varepsilon > 0$, there is a closed subset $F \subset E$ such that $|E - F| < \varepsilon$ and $\{f_k\}$ converges uniformly to f.

Theorem 2.16 (Lusin's Theorem)**.** Let f be defined and finite on a measurable set E, then f is measurable if and only if given $\varepsilon > 0$, there is a closed set F such that $|E - F| < \varepsilon$, and f is continuous on F.

Theorem 2.17. Let $f, \{f_k\}$ be measurable and finite a.e. in E, then if $f_k \to f$ a.e. on E, and $|E| < \infty$, then f_k converges to f in measure.

Theorem 2.18. If f_k converges to f in measure, then there exists a subsequence $\{f_{k_j}\}$ such that $\{f_{k_j}\}$ converges to f a.e. in E.

Theorem 2.19. $\{f_k\}$ converges to f in measure if and only if

$$\lim_{k,l \to \infty} |\{x \in E : |f_k(x) - f_l(x)| > \varepsilon\}| = 0$$

Proposition 2.32. Let f be a nonnegative function defined on a measurable set E, then $\int_E f$ exists if and only if f is measurable.

Proposition 2.33. If f is nonnegative measurable on E, then $\Gamma(f, E)$ has measure zero.

Proposition 2.34. If f is nonnegative, and taking constant values on disjoint sets $E_1, E_2, \ldots$, if $E = \bigcup_j E_j$, then

$$\int_E f = \sum_j a_j |E_j|$$

Proposition 2.35. If f, g are measurable, and $0 \leq g \leq f$ on E, then $\int_E g \leq \int_E f$, in particular, $\int_E \inf f \leq \int_E g$. If f is nonnegative and measurable on E, and $\int_E f$ is finite, then $f < \infty$ a.e. in E. Let E_1, E_2 be measurable and $E_1 \subset E_2$. If f is nonegative and measurable on E_2, then

$$\int_{E_1} f \leq \int_{E_2} f$$

Theorem 2.20 (MCT for nonnegative functions). If $\{f_k\}$ is a sequence of nonnegative functions such that $f_k \nearrow f$ on E, then

$$\lim_{k\to\infty} \int_E f_k = \int_E f$$

Proposition 2.36. Suppose that f is nonnegative and measurable on E such that E is the countable union of disjoint measurable sets, $E = \bigcup_j E_j$, then

$$\int_E f = \sum_j \int_{E_j} f$$

Proposition 2.37. let f be nonnegative on E, if $|E| = 0$, then $\int_E f = 0$.

If f, g are nonnegative and measurable on E, if $g \leq f$ a.e. in E, then

$$\int_E g \leq \int_E f$$

if $f = g$ a.e., then $\int_E f = \int_E g$.

Theorem 2.21 (Chebyshev). Let f be nonnegative, if $\alpha > 0$, then

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha} \int_E f$$

Proposition 2.38. If f is nonnegative, then let c be any nonnegative constant, then

$$\int_E cf = c \int_E f$$

Proposition 2.39. We have the following:

1. If $0 \leq f \leq \phi$, and $\int_E f$ is finite, then

$$\int_E \phi - f = \int_E \phi - \int_E f$$

2. if f_k's are nonnegative, then

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \sum_{k=1}^{\infty} \int_E f_k$$

Theorem 2.22 (Fatou's Lemma)**.** If $\{f_k\}$ is a sequence of nonnegative functions on E, then

$$\int_E (\liminf_{k\to\infty} f_k) \leq \liminf_{k\to\infty} \int_E f_k$$

Proposition 2.40. Let f_k be nonnegative, and let $f_k \to f$ a.e. in E. If $\int_E f_k \leq M$ for all k, then

$$\int_E f \leq M$$

Theorem 2.23 (Lebesgue Dominated Convergence Theorem for nonnegative functions)**.** Let $\{f_k\}$ be nonnegative, and $f_k \to f$ a.e.. If there exists ϕ such that $f_k \leq \phi$ for all k, and if $\int_E \phi$ is finite, then

$$\lim_{k\to\infty} \int_E f_k = \int_E f$$