Advanced Statistical Methods Assignment 4

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5.6

6. If $x \sim \mathrm{Mult}_L(n, \boldsymbol{\pi})$, use the Poisson trick (5.44) to approximate the mean and variance of x_1/x_2. (Here we are assuming that $n\pi_2$ is large enough to ignore the possibility $x_2 = 0$.) Hint: In notation (5.41),

$$\frac{S_1}{S_2} \doteq \frac{\mu_1}{\mu_2}\left(1 + \frac{S_1 - \mu_1}{\mu_1} - \frac{S_2 - \mu_2}{\mu_2}\right).$$

sol)

Let $x = (x_1, \cdots, x_L)^T$, $\pi = (\pi_1, \cdots, \pi_L)^T$, and $N > 0$.

Suppose that $n \sim \mathrm{Poi}(N)$.

Then by Poisson trick (5.44),

$$\mathrm{Mult}_L(n, \pi) \sim \mathrm{Poi}(N\pi)$$

, so $x \mathrel{\dot\sim} \mathrm{Poi}(N\pi)$.

And the approximation $x \mathrel{\dot\sim} \mathrm{Poi}(N\pi)$ removes the correlations, so x_1 and x_2 are independent approximately

Note that $x_1 \mathrel{\dot\sim} \mathrm{Poi}(N\pi_1)$,

and $x_2 \mathrel{\dot\sim} \mathrm{Poi}(N\pi_1)$.

Thus, $E(x_1) \doteq \mathrm{Var}(x_1) \doteq N\pi_1$,

and $E(x_2) \doteq \mathrm{Var}(x_2) \doteq N\pi_2$.

Since x_1 and x_2 are approximately independent Poisson, by Hint,

$$E\left(\frac{x_1}{x_2}\right) \doteq E\left[\frac{\cancel{N}\pi_1}{\cancel{N}\pi_2}\left(1 + \frac{x_1 - N\pi_1}{N\pi_1} - \frac{x_2 - N\pi_2}{N\pi_2}\right)\right] \quad (\text{by Hint})$$

$$\doteq \frac{\pi_1}{\pi_2}\left\{1 + \frac{\cancel{E(x_1)} - \cancel{N\pi_1}}{N\pi_1} - \frac{\cancel{E(x_2)} - \cancel{N\pi_2}}{N\pi_2}\right\}$$

$$\doteq \frac{\pi_1}{\pi_2}$$

and

$$\mathrm{Var}\left(\frac{x_1}{x_2}\right) \doteq \mathrm{var}\left[\frac{\cancel{N}\pi_1}{\cancel{N}\pi_2}\left(1+\frac{x_1-N\pi_1}{N\pi_1}-\frac{x_2-N\pi_2}{N\pi_2}\right)\right]$$

$$\doteq \left(\frac{\pi_1}{\pi_2}\right)^2\left\{\mathrm{var}\left(\frac{x_1-N\pi_1}{N\pi_1}\right)+\mathrm{var}\left(\frac{x_2-N\pi_2}{N\pi_2}\right)\right\}$$

$$\doteq \left(\frac{\pi_1}{\pi_2}\right)^2\left\{\frac{\mathrm{var}(x_1)}{(N\pi_1)^2}+\frac{\mathrm{var}(x_2)}{(N\pi_2)^2}\right\}$$

$$\doteq \left(\frac{\pi_1}{\pi_2}\right)^2\left\{\frac{\cancel{N\pi_1}}{(N\pi_1)^{\cancel{2}}}+\frac{\cancel{N\pi_2}}{(N\pi_2)^{\cancel{2}}}\right\}$$

$$\doteq \left(\frac{\pi_1}{\pi_2}\right)^2\left(\frac{1}{N\pi_1}+\frac{1}{N\pi_2}\right)$$

Therefore, the mean of $\frac{x_1}{x_2}$ is approximately $\frac{\pi_1}{\pi_2}$, and the variance of $\frac{x_1}{x_2}$ is approximately $\left(\frac{\pi_1}{\pi_2}\right)^2\left(\frac{1}{N\pi_1}+\frac{1}{N\pi_2}\right)$.

5.7

7. Show explicitly how the binomial density bi$(12, 0.3)$ is an exponential tilt of bi$(12, 0.6)$.

sol)

Let $p = 0.3$ and $p_0 = 0.6$.

Note that the binomial density bi$(12, p)$ and bi$(12, p_0)$ are

$$f_p(x) = \binom{12}{x} (0.3)^x (0.7)^{12-x}$$

$$f_{p_0}(x) = \binom{12}{x} (0.6)^x (0.4)^{12-x}$$

Then

$$\frac{f_p(x)}{f_{p_0}(x)} = \frac{\cancel{\binom{12}{x}} (0.3)^x (0.7)^{12-x}}{\cancel{\binom{12}{x}} (0.6)^x (0.4)^{12-x}}$$

$$= \left(\frac{1}{2}\right)^x \left(\frac{7}{4}\right)^{12-x}$$

Thus,

$$f_p(x) = \left(\frac{1}{2}\right)^x \left(\frac{7}{4}\right)^{12-x} \cdot f_{p_0}(x)$$

$$= \left(\frac{1}{\cancel{2}} \cdot \frac{\overset{2}{\cancel{4}}}{7}\right)^x \cdot \left(\frac{7}{4}\right)^{12} \cdot f_{p_0}(x)$$

$$= \left(\frac{2}{7}\right)^x \left(\frac{7}{4}\right)^{12} \cdot f_{p_0}(x)$$

$$= e^{\alpha x - \psi(\alpha)} \cdot f_{p_0}(x)$$

where $\alpha = \log\left(\frac{2}{7}\right)$ and $\psi(\alpha) = 12 \log\left(\frac{4}{7}\right)$

$$= 12(\alpha + \log 2)$$

Therefore, the binomial density bi$(12, 0.3)$ is an exponential tilt of bi$(12, 0.6)$.