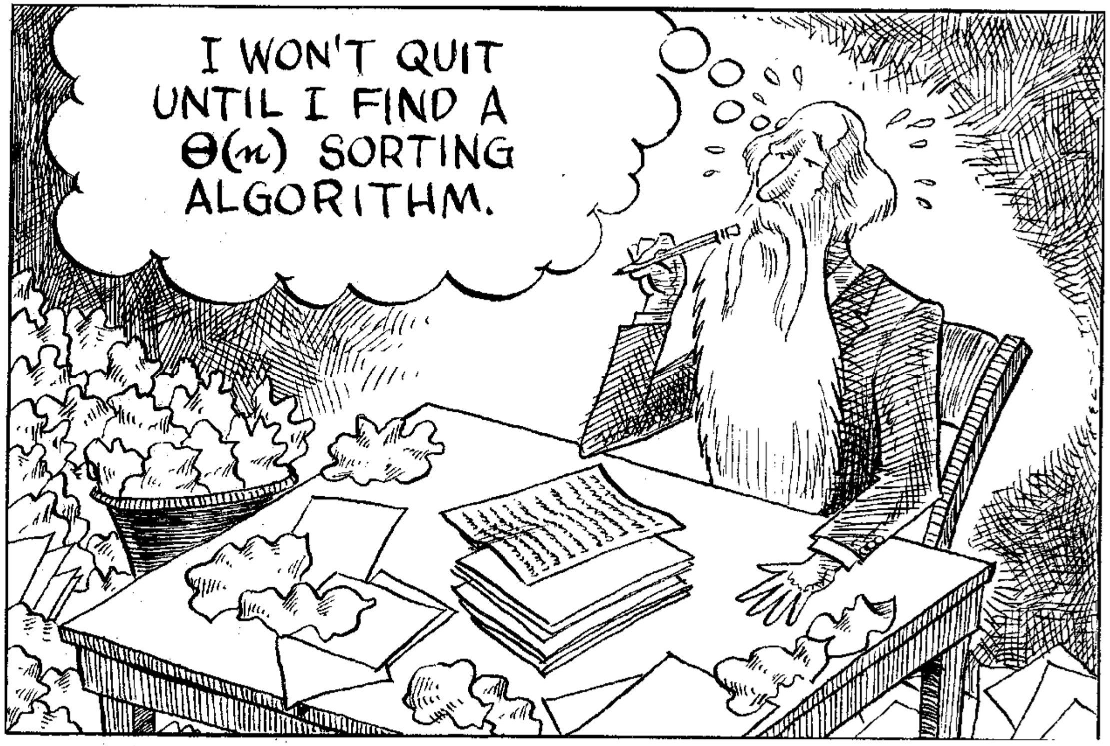
Chapter 7

Introduction to Computational Complexity: The Sorting Problem

Computational Complexity



- Can we find a new sorting algorithm better than *O*(*n*log*n*) ?

Computational Complexity

Computational complexity

- A field of computer science studying a problem itself, not developing efficient algorithms solving the problem.
  - Prove that some problems cannot be solved by computers : halting problem
  - Prove a lower bound of problems
- A computational complexity analysis tries to determine a ***lower bound*** on the efficiency of all algorithms for a given problem.

Computational Complexity

- For example, the lower bound of sorting problem is $\Omega(n\log n)$.
- This implies that it is ***impossible*** to develop an algorithm better than $O(n\log n)$.
- Therefore, merge-sort, quick-sort algorithms are the best algorithms solving the sort problem.

Computational Complexity

❑ Example: Matrix Multiplication Problem

- How fast can we multiply two matrices of size $n \times n$?

- Design an efficient algorithm: basic operation is multiplication of two numbers
  - $O(n^3)$: simple
  - $O(n^{2.81})$: Strassen [1969]
  - $O(n^{2.38})$: Coppersmith, Winograd [1969]

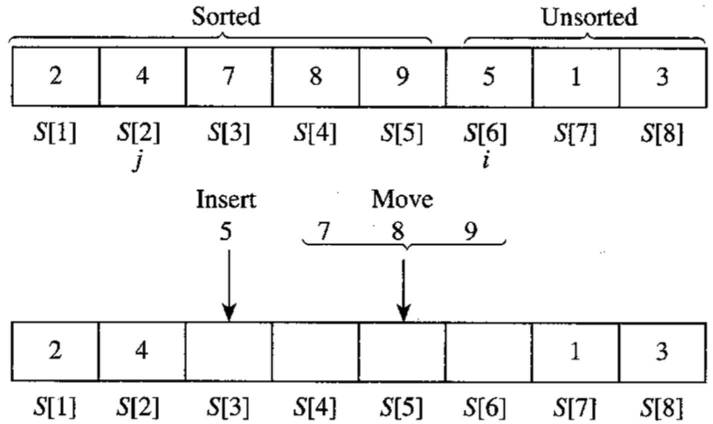
- Develop a lower bound of this problem
  - $\Omega(n^2)$: easy

Computational Complexity

- What do we do next?
  - Fill the gap between the lower bound $\Omega(n^2)$ and the best upper bound $O(n^{2.38})$.
    - Develop a new algorithm better than $O(n^{2.38})$.
    - Prove a new lower bound of the problem better than $\Omega(n^2)$, for example $\Omega(n^{2.3456789})$.

Insertion Sort

- Insertion sorting




```
void insertionsort (int n, keytype S[ ])
{
   index i, j;
   keytype x;

   for (i = 2; i <= n; i++) {
      x = S[i];
      j = i - 1;
      while (j > 0 && S[j] > x) {
         S[j + 1] = S[j];
         j--;
      }
      S[j + 1] = x;
   }
}
```

Insertion Sort

❑ Analysis

- Worst-case Time complexity Analysis of ***Number of Comparisons***
  - Basic operation : the comparison of *S[j]* with *x*.
  - Input size : *n*, the number of keys to be sorted

$$T(n) = \sum_{i=2}^{n} (i-1) = \frac{n(n-1)}{2}$$

Selection Sort

❑ Selection sorting

```
void selectionsort (int n, keytype S[])
{
  index i, j, smallest;

  for (i = 1; i <= n - 1; i++) {
    smallest = i;
    for (j = i + 1; j <= n; j++)
      if (S[j] < S[smallest])
        smallest = j;
    exchange S[i] and S[smallest];
  }
}
```

- Selection sorting is an $O(n\log n)$ algorithm.

Heapsort (Binary Heap)

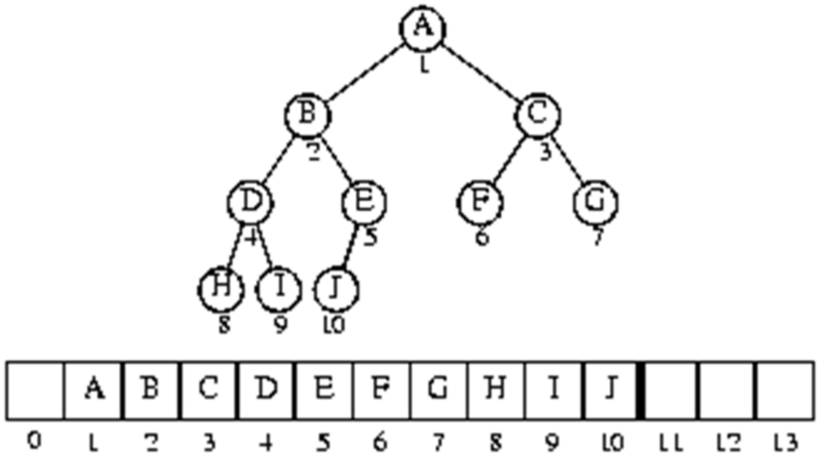
- Categories
  - A Dictionary:
    - Basic Operations
      - Insert
      - Delete
      - Search
  - Data Structures for Dictionary
    - Binary Search Tree,
    - Red-Black Tree,
    - Splay Tree, etc

Heapsort (Binary Heap)

- A Priority Queue
  - Basic Operations
    - Insert
    - Delete Min (or Delete Max)
- Data Structures for Priority Queue
  - The Binary Heap

Complete Binary Tree

- The complete binary tree
  - A tree that is completely filled, with the possible exception of the bottom level, which is filled from left to right.
  - An array of size *N* can represent a complete binary tree with *N* elements.

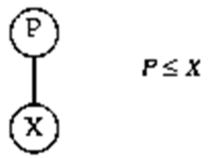


Complete Binary Tree

- **Lemma:**
  - The height (longest path length from the root) of a complete binary tree is $\lfloor \log N \rfloor$.
  - A complete binary tree of height H has between 2^H and $2^{H+1}-1$ nodes.
  - In an array representation of a complete binary tree, for a node of position k,
    - the parent is in position $\lfloor k/2 \rfloor$.
    - the left child is in $2k$
    - the right child is in $2k+1$

Binary Heap

- The binary heap
  - The binary heap has the following properties:
    - It is a complete binary tree
    - (**heap order property**) In a heap, for every node X with parent P, the key in P is smaller than or equal to the key in X.



Heap order property

  - In this case, the heap is called a *min heap*.
  - *Max heaps* have the heap order property in the other way.

Binary Heap

- ❑ Example:

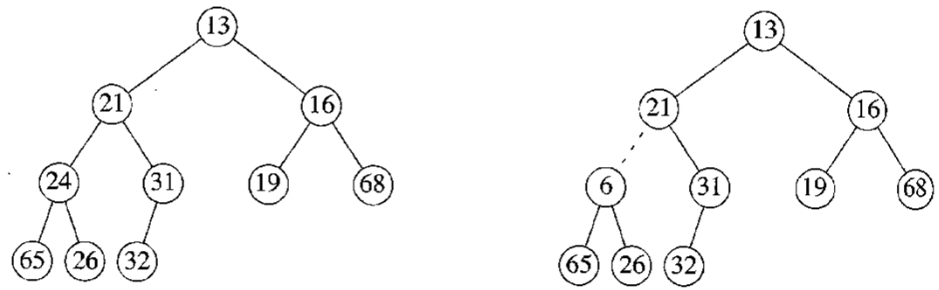


Figure 20.3 Two complete trees (only the left tree is a heap)

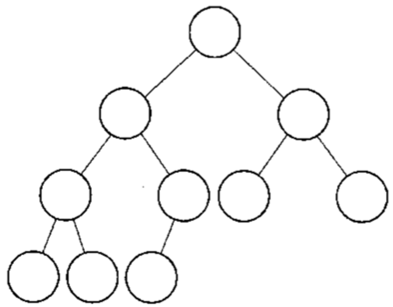


Figure 7.4 An essentially complete binary tree.

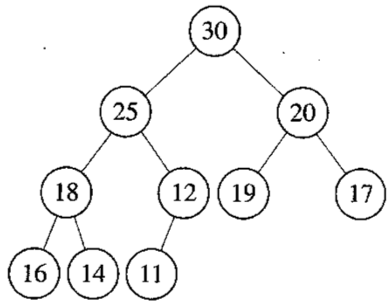
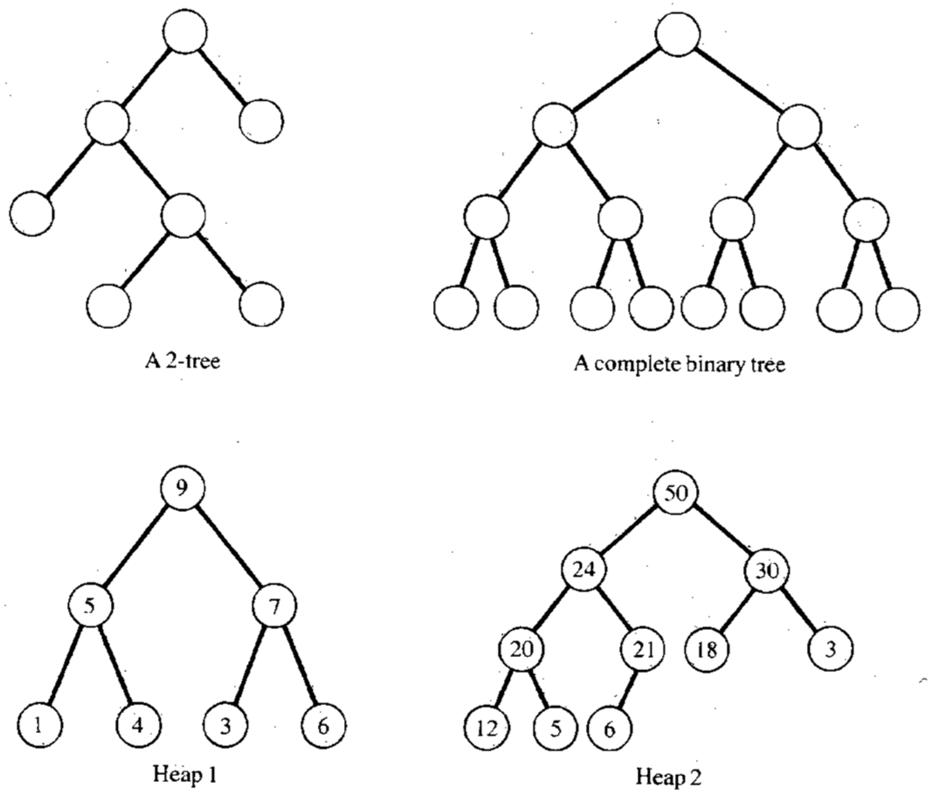


Figure 7.5 A heap.

Binary Heap

- Example:



Binary Heap

- Allowed operations in heap
  - Insert an element in a heap

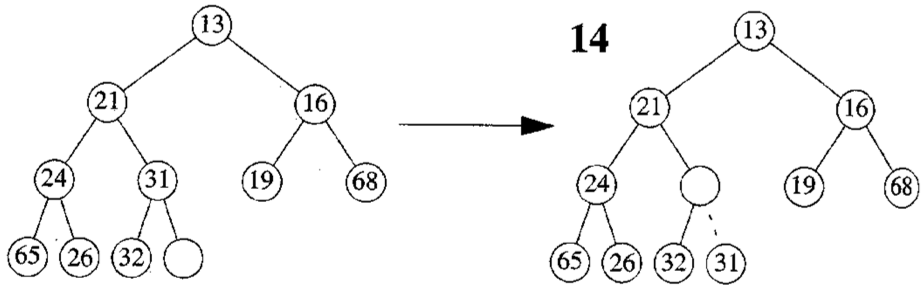


Figure 20.9 Attempt to insert 14, creating the hole and bubbling the hole up

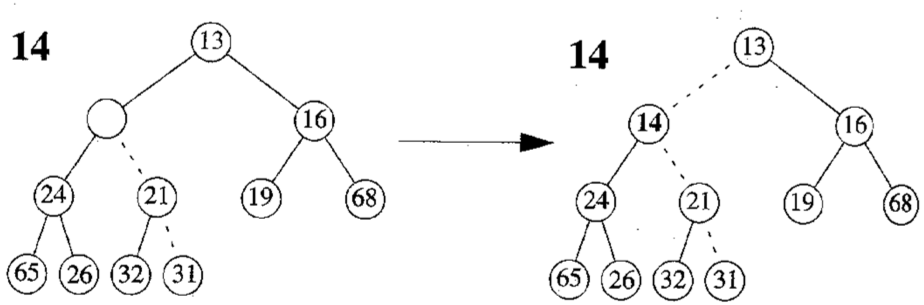


Figure 20.10 The remaining two steps to insert 14 in previous heap

Binary Heap

- Delete a minimum element from a heap

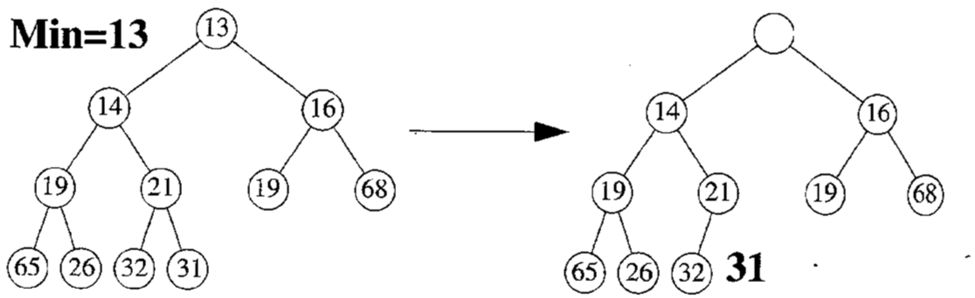


Figure 20.13 Creation of the hole at the root

Binary Heap

- Delete a minimum element from a heap

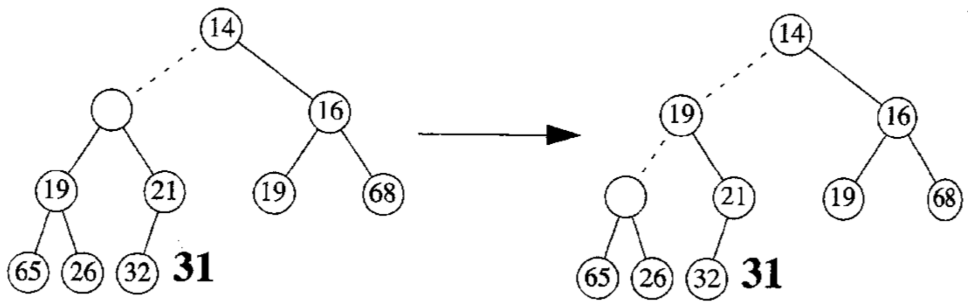


Figure 20.14 Next two steps in `DeleteMin`

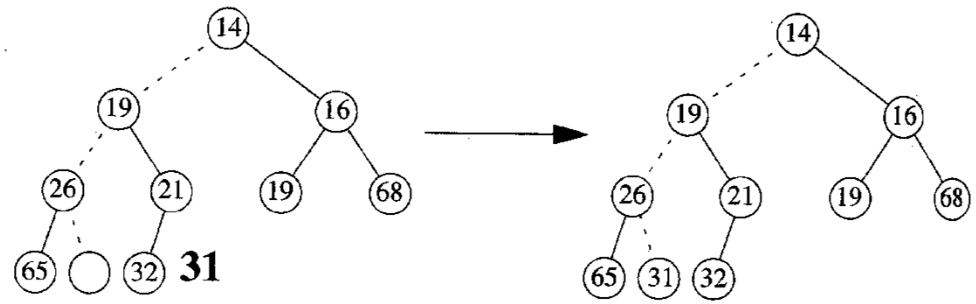


Figure 20.15 Last two steps in `DeleteMin`

Bin

- Delete a maximum element from a heap.

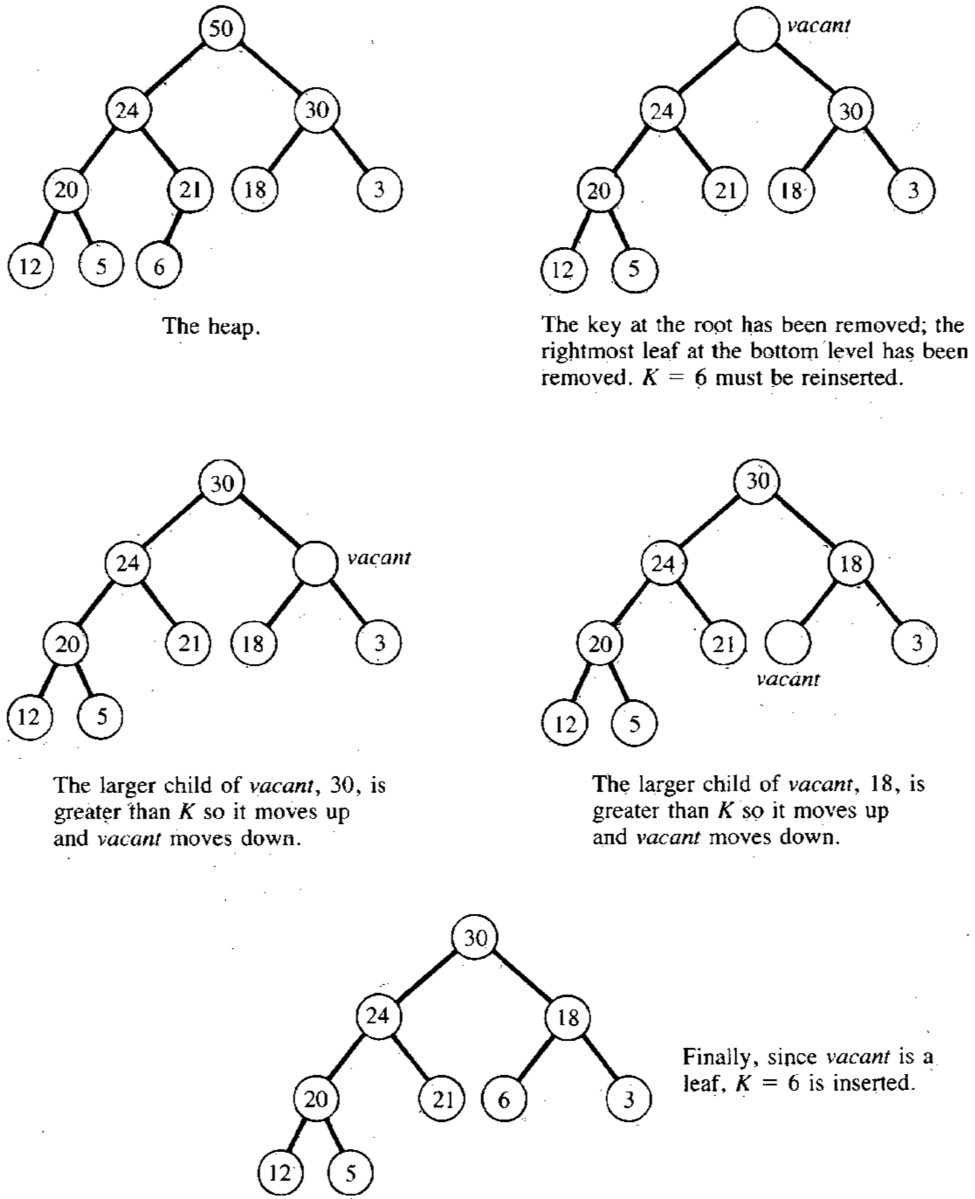


Figure 2.15 Deleting the key at the root and reestablishing the heap property.

Heap Construction

- Heap construction
  - If we are given a complete tree that does not have heap order, we are going to construct a heap.
  - FixHeap Operation:
    - We are given a complete tree that only the root violates the heap order property.

Heap Construction

- Fix heap operation
  - Example for a max heap

Algorithm 2.8 FixHeap

Input: The *root* of a heap and a key *K* to be inserted.

Output: The heap with keys properly rearranged.

```
procedure FixHeap (root: Node; K: Key);
var
    vacant, largerChild : Node;
begin
    vacant := root;
    while vacant is not a leaf do
        largerChild := the child of vacant with the larger key;
        if K < largerChild's key
            then
                copy largerChild's key to vacant;
                vacant := largerChild
            else exitloop
        end { if }
    end { while };
    put K in vacant
end { FixHeap }
```

Heap Construction

- Fix heap operation
  - The FixHeap operation takes $2\lfloor \log N \rfloor$ time, if there are N elements in the heap.

Heap Construction

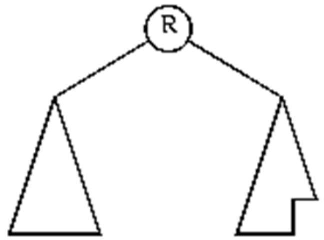
- Shift down operation
  - shiftdown() in the text book is the same as FixHeap() .

```
void siftdown (heap& H)                          // H starts out having the
{                                                // heap property for all
   node parent, largerchild;                     // nodes except the root.
                                                 // H ends up a heap.
   parent = root of H;
   largerchild = parent's child containing larger key;
   while (key at parent is smaller than key at largerchild) {
      exchange key at parent and key at largerchild;
      parent = largerchild;
      largerchild = parent's child containing larger key;
   }
}
```

Heap Construction

❑ Heap construction by divide and conquer

```
procedure ConstructHeap (root: Node);
begin
    if root is not a leaf then
        ConstructHeap (left child of root);
        ConstructHeap (right child of root);
        FixHeap (root, key in root)
    end { if }
end { ConstructHeap }
```



Recursive view of the heap

Heap Construction

- Heap construction by divide and conquer
  - Analysis:

$$T(N) = \begin{cases} 1 & \text{if } N = 1 \\ 2T(N/2) + \log N & \text{otherwise} \end{cases}$$

  - Can you represent the recurrence equation in closed form? (In this case, we cannot apply the master's theorem. Why?)
  - Next time we will show that $T(N)=O(N)$.

Heap Construction

- Iterative version of Heap construction:

Algorithm 2.9 Heap Construction

Input: A heap structure (Property (1)) with keys in arbitrary nodes.

Output: The same structure satisfying the heap-ordering property (Property (2)).

for *level* := *depth*−1 **to** 0 **by** −1 **do**
    for each non-leaf *node* at level *level* **do**
        K := the key at *node*;
        FixHeap(*node*, *K*)
    end { for }
end { for }

Heap Construction

- Iterative version of Heap construction:

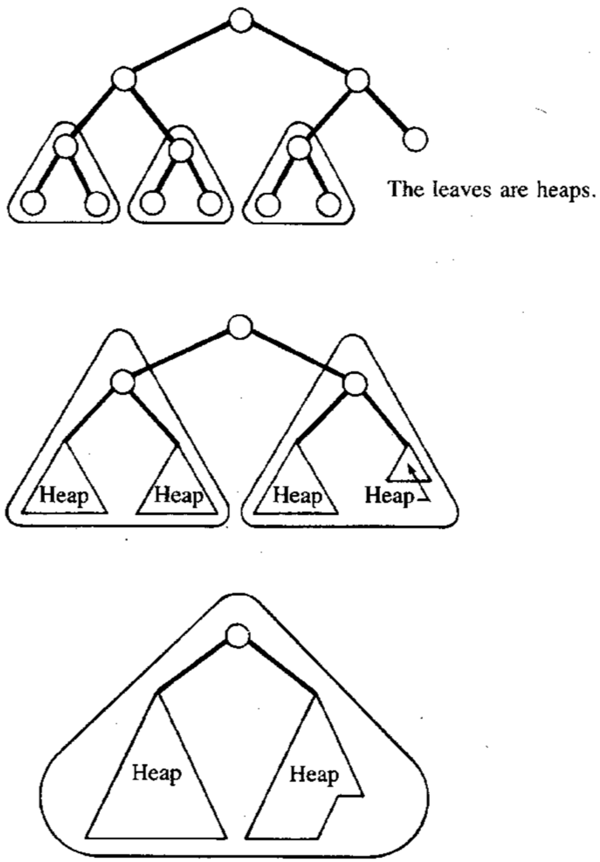


Figure 2.16 Constructing the heap. (*FixHeap* is called for each circled subtree.)

Heap Construction

- ❑ Example

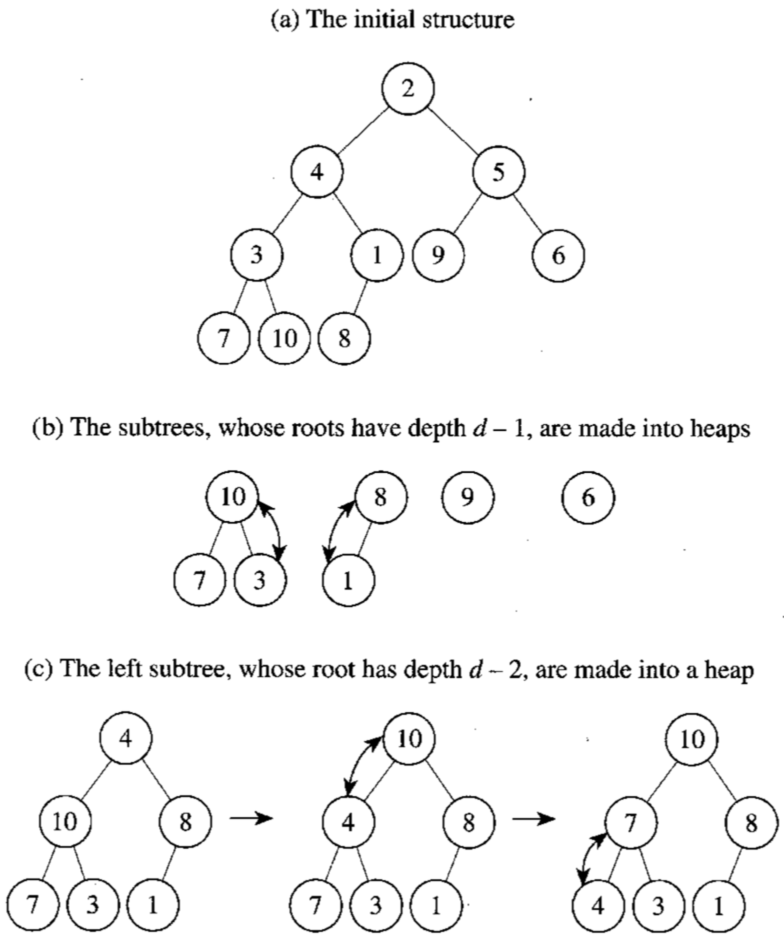


Figure 7.7 Using *siftdown* to make a heap from an essentially complete binary tree. After the steps shown, the right subtree, whose root has depth $d - 2$, must be made into a heap, and finally the entire tree must be made into a heap.

Heap Construction

- ❑ Example

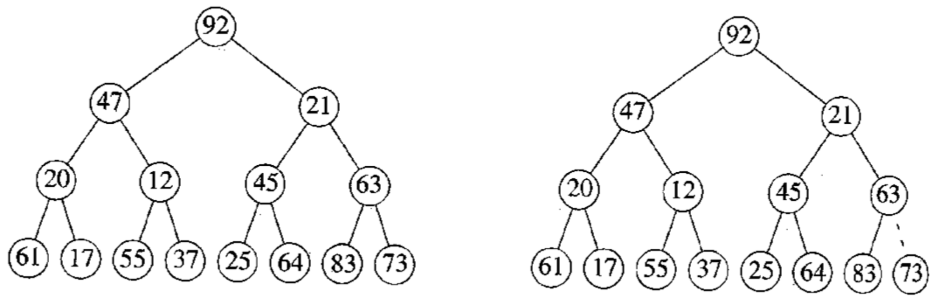


Figure 20.20 Initial heap (left); after `PercolateDown(7)` (right)

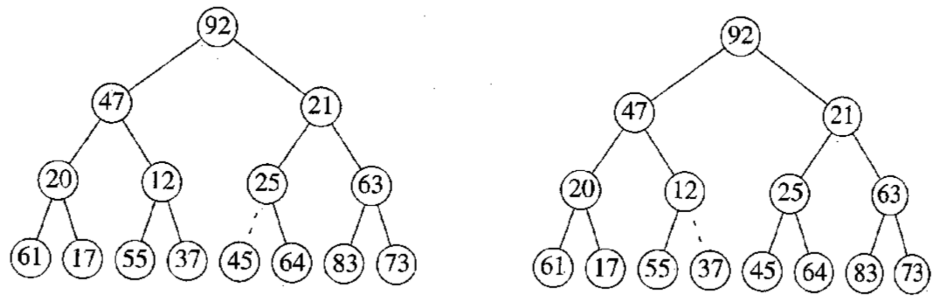


Figure 20.21 After `PercolateDown(6)` (left); after `PercolateDown(5)` (right)

Heap Construction

- Example

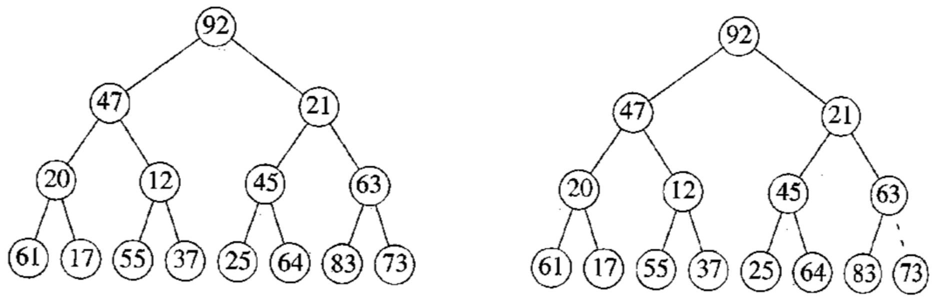


Figure 20.20 Initial heap (left); after `PercolateDown(7)` (right)

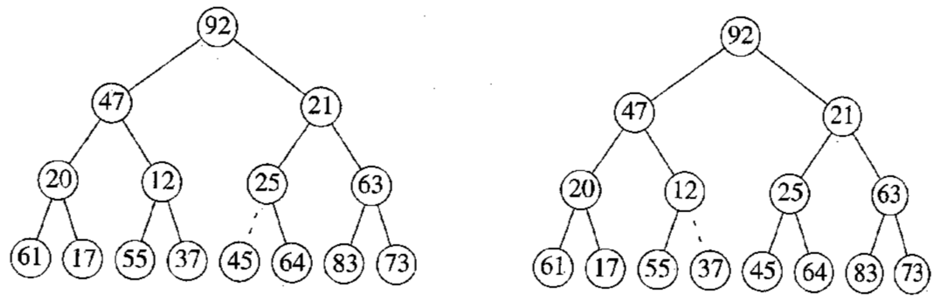


Figure 20.21 After `PercolateDown(6)` (left); after `PercolateDown(5)` (right)

Heap Construction

- Analysis of the heap construction:
  - Let $d = \lfloor \log N \rfloor$
  - Then,

$$
\begin{aligned}
T(N) &= \sum_{k=0}^{d-1} 2(d-k)(\text{the number of nodes at level } k) \\
&= 2\sum_{k=0}^{d-1}(d-k)2^k \\
&= 2^{d+2} - 2d - 4 \\
&= 4N - 2\log N - 4
\end{aligned}
$$

  - Thus the heap is constructed in *T(N) = O(N)*, linear time!

Heap Sorting

Heapsort:

- The priority queue can be used to sort *N* items as follows:
  - Put all the elements in an array of size *N*.
  - Construct a heap
  - Extract every item by calling DeleteMin *N* times. The result is sorted.

Heap Sorting

- Heapsort:

Algorithm 2.10 Heapsort

Input: L, an unsorted array, and $n \geq 1$, the number of keys.

Output: L, with keys in nondecreasing order.

```
procedure Heapsort (var L: Array; n: integer);
var
    i, heapsize : Index;
    max : Key;
begin
    { Heap Construction }
    for i := ⌊n/2⌋ to 1 by −1 do
        FixHeap (i, L[i], n)
    end { for };

    { Repeatedly remove the key at the root and rearrange the heap. }
    for heapsize := n to 2 by −1 do
        max := L[1];
        FixHeap (1, L[heapsize], heapsize−1);
        L[heapsize] := max
    end { for }
end { Heapsort }
```

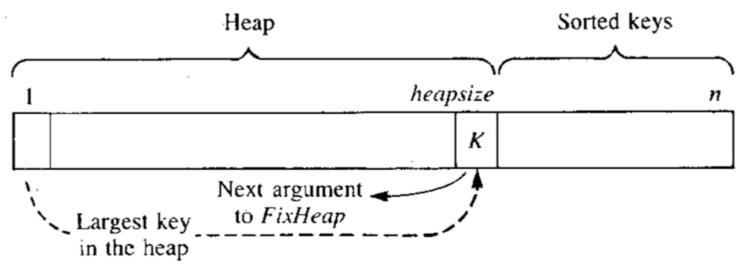


Figure 2.18 The heap and sorted keys in the array.

Heap Sorting

- Save the array space:
  - In heapsort, we construct a max heap, and retract a max value from the heap and put it in the end of the heap. Then we sort elements in increasing order.

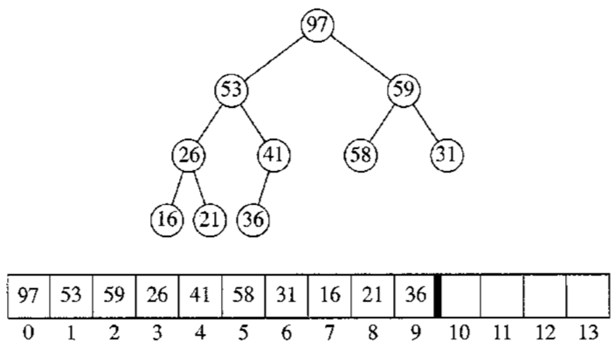


Figure 20.28 (Max) Heap after `FixHeap` phase

Heap Sorting

- Save the array space:

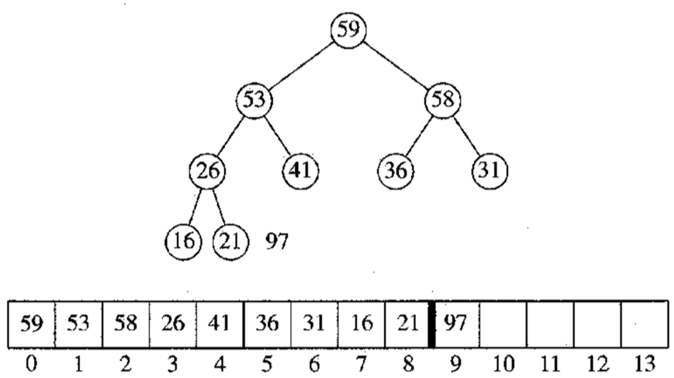
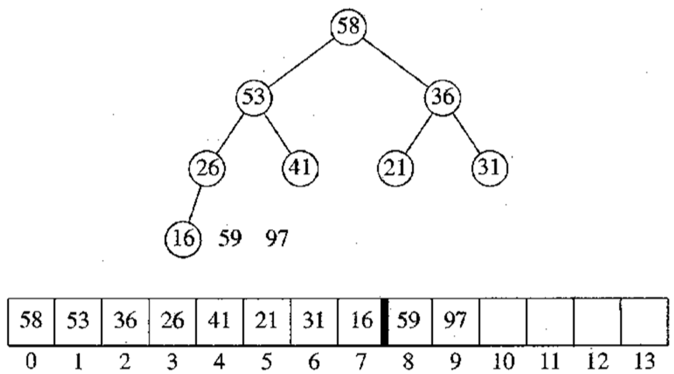
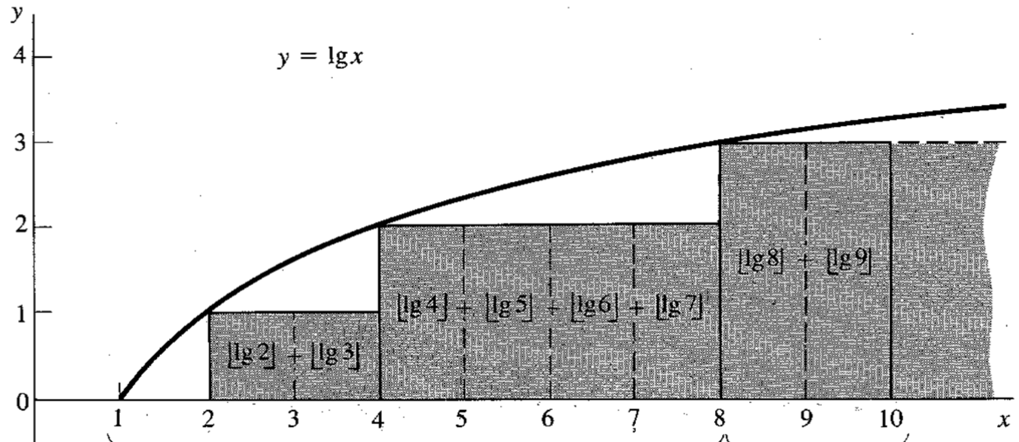


Figure 20.29 Heap after first `DeleteMax`



Heap Sorting

- Analysis of heapsort:
  - Since the number of comparison done by FixHeap on a heap with k elements is at most $2\lfloor \log k \rfloor$, so the total for all deletions is at most $2\sum_{k=1}^{n-1}\lfloor \log k \rfloor$



$$\sum_{i=1}^{n-1}\lfloor \lg i \rfloor = \sum_{j=1}^{\lfloor \lg n \rfloor - 1} j2^j + \lfloor \lg n \rfloor (n-2^{\lfloor \lg n \rfloor})$$

$n = 10$

Heap Sorting

- Analysis of heapsort:
  - Let $d = \lfloor \log N \rfloor$
  - The sum is

$$\sum_{k=1}^{d-1} k2^k + d(N - 2^d)$$
$$= 2(d2^{d+1} - 2^d + 1) + d(N - 2^d)$$
$$= Nd - 2^{d+1} + 2$$
$$= O(N \log N)$$

  - Therefore heapsort takes *O(N log N)* time to sort *N* elements!

Lower Bounds for Sorting

- Lower bounds for sorting only by comparisons of keys
- Decision trees for sorting algorithms
  - An algorithm for sorting three distinct numbers:

```
void sortthree (keytype S[ ])     // S is indexed from 1 to 3.
{
  keytype a, b, c;

  a = S[1]; b = S[2]; c = S[3];
  if (a < b)
    if (b < c)
      S = a, b, c;               // This means S[1] = a; S[2] = b; S[3] = c;
    else if (a < c)
      S = a, c, b;
    else
      S = c, a, b;

  else if (b < c)
    if (a < c)
      S = b, a, c;
    else
      S = b, c, a;
  else
    S = c, b, a;
}
```

Lower Bounds for Sorting

- Lemma 7.1
  - To every algorithm for sorting distinct numbers, there corresponds a decision tree containing exactly $n!$ keys.
- Example:
  - The decision tree corresponding to exchange sort when sorting three numbers.

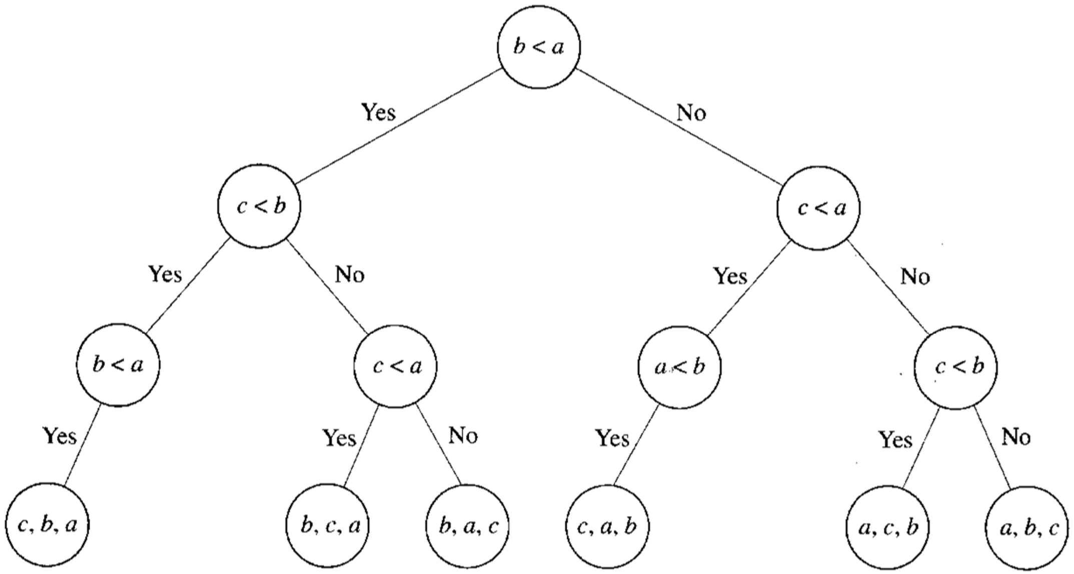


Figure 7.12 The decision tree corresponding to Exchange Sort when sorting three keys.

Lower Bounds for Sorting

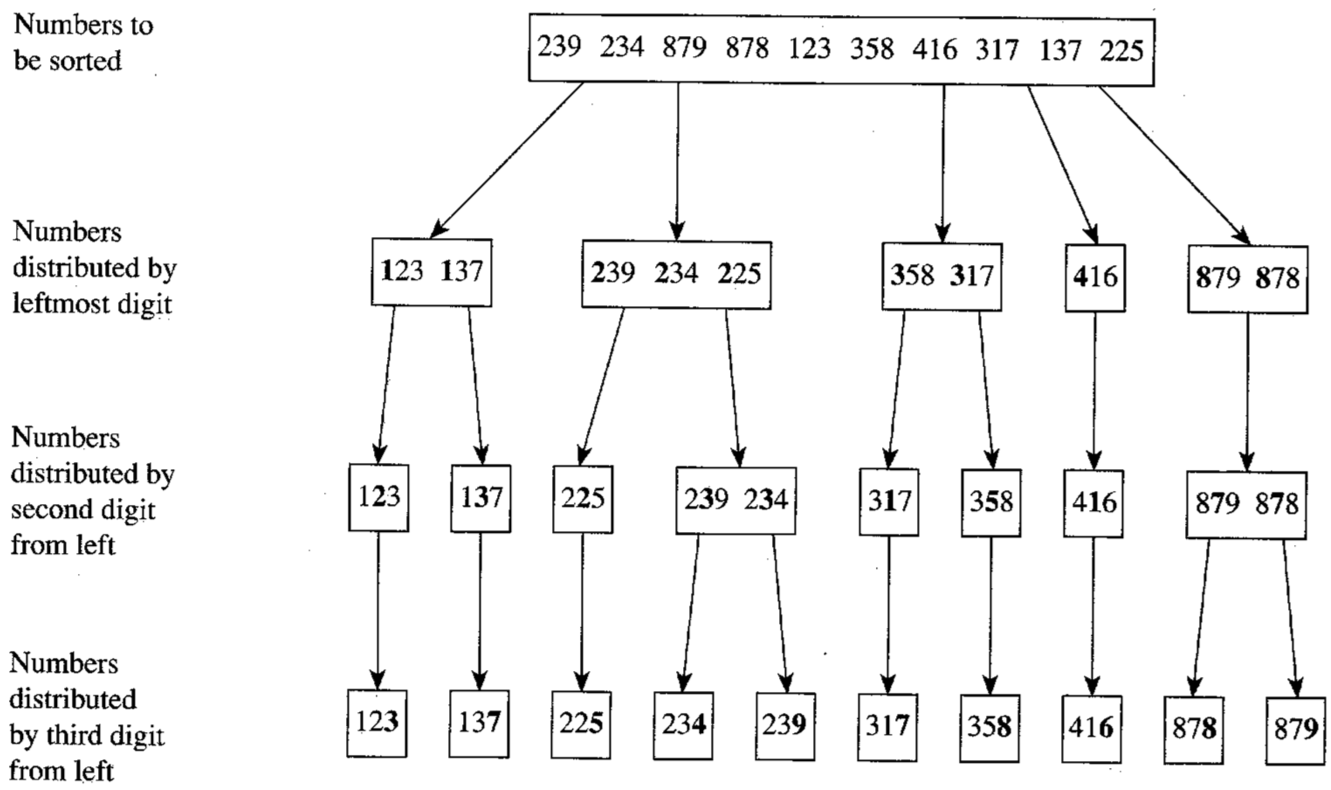
- **Theorem 7.2**
  - Any algorithm that sorts distinct numbers only by comparison of numbers must in the worst case do at least $\lceil \lg(n!) \rceil = O(n\log n)$ comparison of numbers.
- **Proof:**
  - By lemma 7.1 the decision tree has $n!$ leaf nodes
  - Then the depth of the tree is greater than or equal to $\lceil \lg(n!) \rceil$.
  - Note that

$$
\begin{aligned}
\lg(n!) &= \lg[n(n-1)(n-1)\cdots(2)1] \\
&= \sum_{i=1}^{n} \lg i \\
&\geq \int_1^n \lg x dx = \frac{1}{\ln 2}(n \ln n - n + 1) \\
&\geq n \lg n - 1.45n
\end{aligned}
$$

Sorting by Distribution (Radix Sort)

- Method 1

Figure 7.14 Sorting by distribution while inspecting the digits from left to right.



Sorting by Distribution (Radix Sort)

❑ Method 2

Figure 7.15 Sorting by distribution while inspecting the digits from right to left.