Queries

LLOYD S. NELSON, *Editor**

QUERY 2 The Sum of Values from a Normal and a Truncated Normal Distribution

An item which we make has, among others, two parts which are assembled additively with regard to length. The lengths of both parts are normally distributed but, before assembly, one of the parts is subjected to an inspection which removes all individuals below a specified length. As an example, suppose that X comes from a normal distribution with a mean of 100 and a standard deviation of 6, and Y comes from a normal distribution with a mean of 50 and a standard deviation of 3, but with the restriction that $Y \geq 44$. How can I find the chance that $X + Y$ is equal to or less than a given value?

Editor's Note: Query 2 was originally answered by M. A. Weinstein [1] who derived the required cumulative distribution function and showed how it can be approximated using only a table of the normal distribution. Three correspondents have suggested the following procedures which, by the use of additional tables, obviate the need for such numerical computation.

Myron Lipow writes the required probability density functions as:

$$f(X) = (2\pi\sigma_X^2)^{-\frac{1}{2}} \exp\left[-\frac{(X-\mu_X)^2}{2\sigma_X^2}\right], \qquad -\infty < X < \infty \tag{1}$$

and

$$g(Y) = \begin{cases} \dfrac{(2\pi\sigma_Y^2)^{-\frac{1}{2}}}{\Phi\left(\dfrac{\mu_Y - Y_0}{\sigma_Y}\right)} \exp\left[-\dfrac{(Y-\mu_Y)^2}{2\sigma_Y^2}\right], & Y \geq Y_0 \\ 0, & Y < Y_0 \end{cases} \tag{2}$$

where

$$\Phi(Z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{Z} \exp\left[-\frac{u^2}{2}\right] du. \tag{3}$$

The probability $Q(t)$ that $X + Y \leq t$ is reduced to the form:

$$\Phi(h)Q(t) = \frac{1}{2\pi(1-r^2)^{\frac{1}{2}}} \int_{-h}^{\infty} dX \int_{k}^{\infty} \exp\left[-\frac{1}{2(1-r^2)}(X^2 + 2rXY + Y^2)\right] dY \tag{4}$$

where

$$h \equiv (\mu_Y - Y_0)/\sigma_Y\ , \tag{5}$$

$$k \equiv (\mu_X + \mu_Y - t)/(\sigma_X^2 + \sigma_Y^2)^{\frac{1}{2}}, \tag{6}$$

and

$$r \equiv \sigma_Y/(\sigma_X^2 + \sigma_Y^2)^{\frac{1}{2}}. \tag{7}$$

From [2, pp. vi–vii], one can write

$$\Phi(h)Q(t) = L(-h, k, -r) \equiv -L(h, k, r) + 1 - \Phi(k). \tag{8}$$

$L(h, k, r)$ is tabulated for positive h and k in [2] for $[h, k = 0(0.1)4; r = \pm 0(0.05)\, 0.95\, (0.01)1]$, and $2\, \Phi(k) - 1$ is tabulated in [3].

For the example in the original answer, $h = 2$, $k = 1.7889$, and $r = 0.4472$. Hence, by linear interpolation, [2] gives

$$L(2, 1.7889, 0.4472) = 0.00481$$

and [3] gives

$$\Phi(2) = 0.97725$$

$$\Phi(1.7889) = 0.96318.$$

Thus

$$Q(138) = (-0.00481 + 1 - 0.96318)/0.97725 = 0.03276.$$

This procedure was also suggested by Nathan Mantel.

J. W. Wilkinson derives a form analogous to equation (4) above and indicates that, in addition to solution by the use of references [2] or [4], the right-hand side of the equation can also be evaluated by expressing it in terms of the normal integral $\Phi(Z)$ and $T(h, a)$, where

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp\,[-h^2(1 + x^2)/2]}{1 + x^2}\, dx. \tag{9}$$

$T(h, a)$ is described and tabulated by Owen [5, 6] for $[a = 0.25(0.25)1.00; h = 0(0.01)3.14]$, $[a = 0(0.01)1.00(0.25)2.00, \infty; h = 0(0.25)3.25]$, and $[a = 0.10, 0.20(0.05)0.50(0.10)0.80, 1.00, \infty; h = 3.00(0.05)3.50(0.10)4.00(0.20)4.60, 4.76]$.

For all values of t,

$$\Phi(h)Q(t) = (\tfrac{1}{2})\Phi(-k) - (\tfrac{1}{2})\Phi(-h) + T(k, n) + T(h, q) + \delta_{k,h}\,, \tag{10}$$

where

$$n = [(t - \mu_X - Y_0)\sigma_Y/\sigma_X + (\mu_Y - Y_0)\sigma_X/\sigma_Y]/(\mu_X + \mu_Y - t), \tag{11}$$

$$q = [(\mu_X + Y_0 - t)/(\mu_Y - Y_0)]\sigma_Y/\sigma_X\,, \tag{12}$$

and

$$\delta_{k,h} = \begin{cases} 0, & (k < 0, h \le 0) \quad \text{or} \quad (k > 0, h \ge 0) \\ \tfrac{1}{2}, & \text{otherwise.} \end{cases} \tag{13}$$

In the solution of the example, $t = 138$, $h = 2$, $k = 1.7889$, $n = 0.75$, $q = 0.5$. Linear interpolation in [3] and [5] gives

$$\begin{aligned} \Phi(2)Q(138) &= (\tfrac{1}{2})\Phi(-1.7889) - (\tfrac{1}{2})\Phi(-2) + T(1.7889, 0.75) + T(2, 0.5) \\ &= (\tfrac{1}{2})(0.03682) - (\tfrac{1}{2})(0.02275) + 0.01636 + 0.00862 \\ &\text{and} \quad Q(138) = (0.03201)/0.97725 = 0.03276. \end{aligned}$$

References

[1] Weinstein, M. A., 1964. The Sum of Values from a Normal and a Truncated Normal Distribution, *Technometrics*, *6*, 104–105.

[2] Tables of the Bivariate Normal Distribution Function and Related Functions, National Bureau of Standards, Applied Mathematics Series 50, 1959.

[3] Tables of Normal Probability Functions, National Bureau of Standards, Applied Mathematics Series 23, 1953.

[4] Zelen, M. and Severo, N. C., 1960. Graphs for Bivariate Normal Probabilities, *Ann. Math. Stat.*, *31*, 619–624.

[5] Owen, D. B., 1962. Handbook of Statistical Tables, Addison-Wesley Publishing Company, Inc., Reading, Mass., pp. 184–202.
[6] Owen, D. B., 1956. Tables for Computing Bivariate Normal Probabilities, *Ann. Math. Stat.*, *27*, 1075–1090.

QUERY 7 Savings in Test Time When Comparing Weibull Scale Parameters

A comparison of the scale parameters of two independent Weibull distributions can be made using the fact that, when $\alpha_1 = \alpha_2$, $\hat{\alpha}_1/\hat{\alpha}_2$ is distributed as the distribution F with $2r_1$ and $2r_2$ degrees of freedom, where r_1 and r_2 are the first failures in samples of n_1 and n_2, respectively. Since the sample sizes do not enter into the analysis, statistically nothing is gained by choosing $n_1 > r_1$ or $n_2 > r_2$. For particular values of r_1 and r_2, however, the larger the sample sizes the less the time required to complete the experiment. What is the relationship between sample size and time required to obtain a certain number of failures?

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Your statement is correct for known shape parameters [1]. For known shape parameters, say β_1 and β_2, the transformations $Y_1 = X_1^{\beta_1}$ and $Y_2 = X_2^{\beta_2}$, where X_1 and X_2 are the two independent Weibull variables, yield two independent exponential variables. Tables 2B-2(a), 2B-2(b), 2B-3 and 2B-4 on pages 2.32–2.39 of [2] tabulate the relationship between sample size n, termination number γ, and an expected relative saving in time required to complete the experiment. Using these tables one can compute the savings in Y. Expressions on the relative saving in X are not simple and are as yet unknown. Other comparison procedures between two Weibull processes can be found in [3], where consideration is also given to savings in experimentation time.

References

[1] Kao, John H. K., 1961. The Beta Distribution in Reliability and Quality Control, the Proceedings of Seventh National Symposium on Reliability and Quality Control, 496–511.
[2] Quality Control and Reliability Handbook (Interim) H-108, 1960. Sampling Procedures and Tables for Life and Reliability Testing (Based on Exponential Distribution), Office of the Assistant Secretary of Defense (Supply and Logistics), U. S. Government Printing Office.
[3] Qureishi, A. S., 1964. The Discrimination Between Two Weibull Processes, *Technometrics*, *6*, 57–75.

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