Let's say we want to value some financial contract. $C_t = f(S_t, t)$

$$d\,f(S_t) = f(S_{t+dt}) - f(S_t) = f(S_t + dS_t) - f(S_t)$$

$$d\,f(S_t) = \frac{\partial f}{\partial S}(dS_t) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS_t)^2 + \frac{1}{6}\cdot\frac{\partial^3 f}{\partial S^3}(dS_t)^3 + \dots$$

as $dt \to 0$ $\quad dt^k \to 0$ as $k > 1$

$dt\, dW_s$

$(dW_s)^k$, $k > 3 \to 0$ quickly

Keep dt term dW_t terms

$$dt = (dW_t)^2$$

$$df(S_t) = \frac{\partial f}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial S^2} d(S_t)^2 \Rightarrow d(f(S_t)) = \left[\mu_t \frac{\partial f}{\partial S}(S_t) + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial S^2}(S_t)\right] dt$$

$$+ \sigma_t \cdot \frac{\partial f}{\partial S}(S_t)\, dW_t$$

for Generic Itô drift diffusion process,

$$dS_t = \mu_t\, dt + \sigma_t\, dW_t$$

$$(dS_t)^2 = (\mu_t\, dt + \sigma_t\, dW_t)^2 = (\sigma_t\, dW_t)^2 = \sigma_t^2 dt \qquad (3)$$

$$= \underbrace{\mu_t^2 dt^2}_{\text{discard}} + \sigma_t^2 (dW_t)^2 + \underbrace{2\mu_t dt \cdot \sigma_t W_t}_{\text{discard}}$$

$$d(S_t)^k = (\mu S_t\, dt + \sigma S_t\, dW_t)^k$$

as $dt \to 0$

discard	keep
dt^k $(k>1)$	dt term
$dt \cdot dW_s$	dW_t terms
$(dW_s)^k$ $(k>3)$	$dt = (dW_t)^2$

$$df(S_t) = \frac{\partial f}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_t)^2$$

(1) plug in the formula of Geometric Brownian Motion (2)

$$\frac{dS_t}{S_t} = \mu_t \, dt + \sigma_t \, dW_t$$

plug in formula (3)

we have

$$df(S_t) = \frac{\partial f}{\partial S} (\mu_t dt + \sigma_t dW_t) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \cdot \sigma_t^2 dt$$

$$= [\mu_t \frac{\partial f}{\partial S} (S_t) + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial S^2} (S_t)] \, dt + \sigma_t \cdot \frac{\partial f}{\partial S} (S_t) \, dW_t$$

$$f(S_t) = f(S_0) + \int_0^t \frac{\partial f}{\partial S} (S_t) \, dS_t + \frac{1}{2} \int_0^t \frac{\partial f}{\partial S^2} (S_t) \, d\underbrace{[S, S]_t}_{\text{quadratic variation}}$$

Accumulates at rate σ_t^2

apply (2)

we have

$$f(S_t) = f(S_0) + \int_0^t \frac{\partial f}{\partial S} (S_t) \mu_t \, dt + \int_0^t \sigma_t \frac{\partial f}{\partial S} (S_t) \, dW_t + \frac{1}{2} \int_0^t (\sigma_t)^2 \frac{\partial^2 f}{\partial S^2} (S_t) \, dt$$

let $f(S_t) = \ln (S_t)$ $\quad f'(s) = \frac{1}{s}$ $\quad f''(s) = -\frac{1}{s^2}$

$$\ln S_t = \ln S_0 + \int_0^t u_t \, dt + \int_0^t \sigma \, dW_t + \int_0^t (-\frac{1}{2} \sigma^2) \, dt$$

$$\frac{S_t}{S_0} = \exp(\mu_t \cdot t - \frac{1}{2} \sigma^2 t + \sigma \cdot W_t)$$