Black-Scholes Formula

Assumption
1. short term interest rates are constant
2. stocks pay no dividends
3. no transaction costs
4. Can borrow fraction of stock
5. short selling allowed

Method Overview

1. Price derivative using replication
2. construct risk free portfolio
3. $C_t = C(S, t)$ smooth function of C. use Ito's Rule to express portfolio drift as C's partial derivatives
4. find $C(S,t)$ that satisfy with this.

Construct portfolio with option + stocks

assume 1 options $-\alpha_t$ shares

this portfolio consisted of 1 share option and α share of stocks

Because this is a risk-free portfolio, α is identical to $\boxed{\text{delta}}$ $\frac{\partial C_t}{\partial S}$

the ratio: when stock price go up \$1 the price impact on option price

$$V_t = C_t - \alpha S_t$$

$$dV_t = dC_t - \alpha\, dS_t \qquad (1)$$

Assume underlying follows GBM geometric brownian motion

$$dS_t = \underbrace{\mu S_t dt}_{\text{drift}} + \underbrace{\sigma S_t dW_t}_{\text{diffusion}}$$

As $C_t = C(S_t, t)$

discard

$$\partial C_t = \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS + \frac{1}{2}\left(\frac{\partial^2 C}{\partial S^2} dS^2 + \frac{\partial^2 C}{\partial t^2} dt^2\right) \qquad (2)$$

plug (2) into (1)

$$dV_t = \frac{\partial C_t}{\partial t} dt + \frac{\partial C_t}{\partial S} dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} dS^2 - \alpha\, dS_t \qquad (3)$$

assume $\alpha = \frac{\partial C_t}{\partial S}$, number of shares have in portfolio is equivalent to delta risk free portfolio

⊛ Using GBM, $dS_t^2 = (\mu S_t dt + \sigma S_t dW)^2$ (4)

$$= \underbrace{\mu^2 S_t^2 dt^2}_{\text{discard}} + \underbrace{2\mu S_t^2 \sigma \, dt dW}_{\text{discard}} + \sigma^2 S_t^2 \underbrace{dW^2}_{=dt}$$

plug (4) into (3)

$$dV_t = \left[\frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\alpha^2 S_t^2\right] dt$$

in (3) $\boxed{\alpha dS_t}$ disappear because $\alpha = \frac{\partial C_t}{\partial S}$

$$\frac{\partial C_t}{\partial S} dS - \alpha dS_t = \frac{\partial C_t}{\partial S} dS - \frac{\partial C_t}{\partial S} dS = 0$$

Arbitrage free portfolio

$$dV_t = rV_t dt \quad \text{where } dB_t = rB_t dt$$

$$= r\underbrace{(C_t - \alpha S_t)}_{\text{portfolio above}} dt \qquad r \text{ is the risk-free interest rate}$$

$$r(C_t - \alpha S_t) dt = \left[\frac{\partial C_t}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\cancel{\alpha^2}\sigma^2 S_t^2\right] dt$$

α is delta $\frac{\partial C}{\partial t}$

rearrange PDE is as follows

$$\frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S_t^2 + rS_t\frac{\partial C}{\partial S} = rC_t \qquad (\text{BS formula})$$

Boundary Condition

expiry payoff of call option is

$$C(S,t) = (S-K)^+ = \max(0, S-K)$$