Parametric Curve

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Contents

1 Parametric Curve

Arc-length:

$$\int_0^1 \sqrt{\left(\frac{d\mathbf{p}}{dt}(t)\right)^2} dt \tag{1}$$

$$\tag{2}$$

Tangent

$$\frac{d\mathbf{p}}{dt} \tag{3}$$

2 Bezier Curve

$$\mathbf{p}(t) = (1-t)^3\mathbf{p}_0 + 3(1-t)^2t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3 \tag{4}$$

Tangent

$$\frac{d\mathbf{p}}{dt}(t) = -3(1-t)^2\mathbf{p}_0 + 3(3t-1)(t-1)\mathbf{p}_1 - 3t(3t-2)\mathbf{p}_2 + 3t^2\mathbf{p}_3 \tag{5}$$

2.1 De Casteljau's algorithm

3 B-Spline Curve

B-Spline curve is constructed from the three ingredients.

- the degree of the polynominal basis p.
- a set of n control points $\{\mathbf{p}_0, \ldots, \mathbf{p}_{n-1}\}$,
- a knot vector $\mathbf{u}$ of $m+1$ non-descending real number $\{u_0 \leq u_1 \leq u_2 \ldots \leq u_m\}$ where $m = n + p$.

It is a bit confusing but, if the basis is p degree polynominal, the B-Spline curve is called $p+1$ order. The order of the B-Spline means that up to $p+1$ control points affects a position of a point on the curve.

From the knot vector and p, we can define a basis function $N_{i,p}(u)$ which takes value in the range $u \in [u_0, u_m]$. Then, the resulting equation of the curve becomes

$$\mathbf{p}(u) = \sum_{i=0}^{n-1} N_{i,p}(u)\mathbf{p}_i \tag{6}$$

3.1 Knot Vector

The knot vector $\mathbf{u}$ defines segments on the parameter the curve. This allows non-uniformly sized segments and gives the B-Spline higher degree of expression. Quite frequently, the interval between knots become zero 0 if the knot takes the same value. The number of the time the knot duplicate is called *knots multiplicity*.

3.2 Cox - De Boor's algorithm

The basis function is defined hierarchically. In other words, we build up the basis function based on the basis function of one order lower.

We start from 0 degree basis function which is piecewise constant over the interval

$$N_{i,0}(u) = \begin{cases} 1 & if\ u_i < u < u_{i+1} \\ 0 & otherwize \end{cases} \tag{7}$$

Then, for $p > 0$, we define the basis functions using the basis function of $p - 1$ degree as

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{8}$$

Figure 1 show the basis functions of the B-Spline with different degrees. Please notice that, the degree p basis functions have a domain of $p + 1$ intervals. So if you go up in the higher degree, the influence of a control point gets wider, meaning that the function become smoother. Further more, given a parameter $t \in [t_0, t_m]$, it will cover by up to $p + 1$ basis function. This means that $p + 1$ control points influence a point on the B-Spline curve.

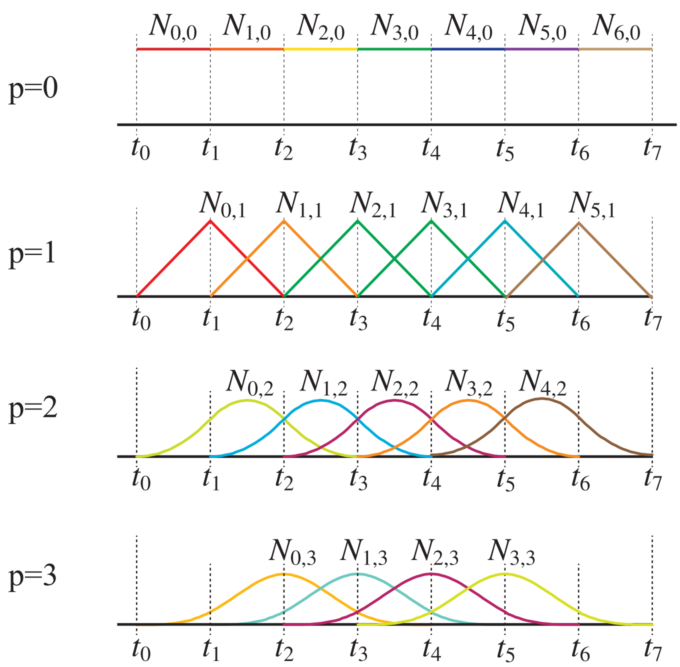


Figure 1: Basis functions of a B-Spline curve with uniform knots. The degree of the basis function increases from top row ($p = 0$) to be bottom row ($p = 3$).

Figure 2 shows that example of the B-Spline basis function construction from the basis functions of $p = 1$ to one of $p = 2$ using the (8). Starting from two neighboring basis functions we call left and right functions, we multiply a linear functions for each. For the left basis function we multiply a linear function that goes from zero to one over the domain of the function. Domain of the function means where the function takes value other than zero. In this case $p = 1$ functions takes two segments as the domain. Similarly, we multiply the right basis function to a linear function that linearly goes down from one to zero over its domain. If we multiply a linear function to p degree function, we get a $p + 1$ degree function. Adding up these two functions, we get a new basis. Because the left and right basis function have domains different by one segment, the new basis function gets a one segment wider.

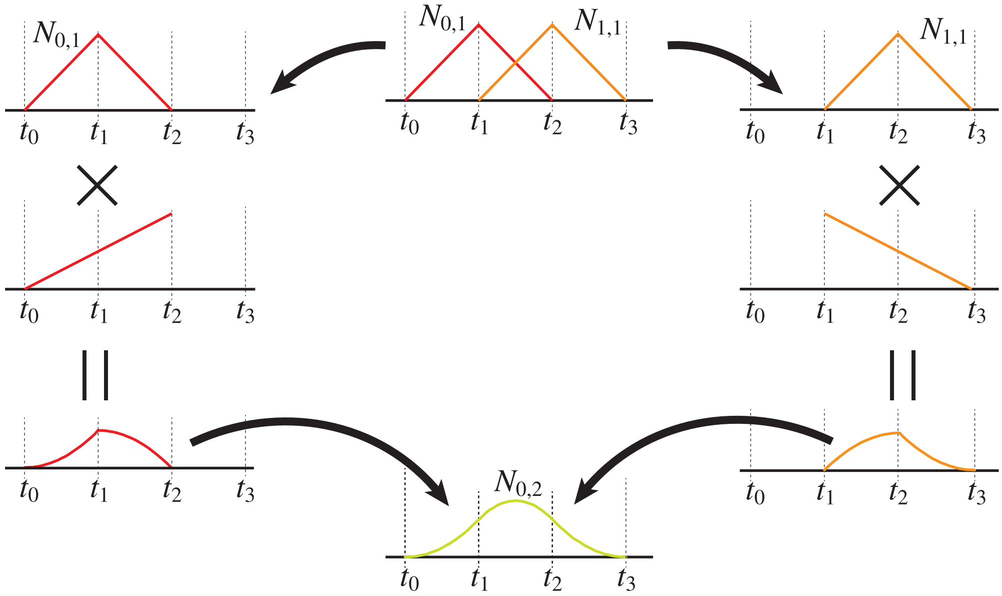


Figure 2: Cox - De Boors algorithm applied to two neighboring piece-wise linear basis functions to produce a piece-wize quadratic basis function.

3.3 Knot Insertion

We can insert a knot in the knot vector without changing the curve shape.

Let's say we insert a knot u in the knot vector. Because the knot values need to be monotonically increase, the u need to be insert in the interval $[u_k, u_{k+1}]$ where $u_k \leq u \leq u_{k+1}$. If we add one knot value, one control point need to be added (remember the relation $n = m + p$). There are $p + 1$ number of basis functions whose domain include u are $N_{k-p,p}, \ldots, N_{k,p}$. Hence by the knot insertion only the corresponding control points $\mathbf{p}_{k-p}, \ldots, \mathbf{p}_k$ need to be considered.

De Boor's algorithm says the new control point locations are on the polyline connecting the old control points. $p - 1$ number of old control points $\mathbf{p}_{k-p+1}, \ldots, \mathbf{p}_{k-1}$ are replaced by p number of new control points $\mathbf{q}_{k-p+1}, \ldots, \mathbf{q}_k$. For example, the new control point $\mathbf{q}_i$ lies on the line segment connecting $\mathbf{p}_{i-1}$ and $\mathbf{p}_i$.

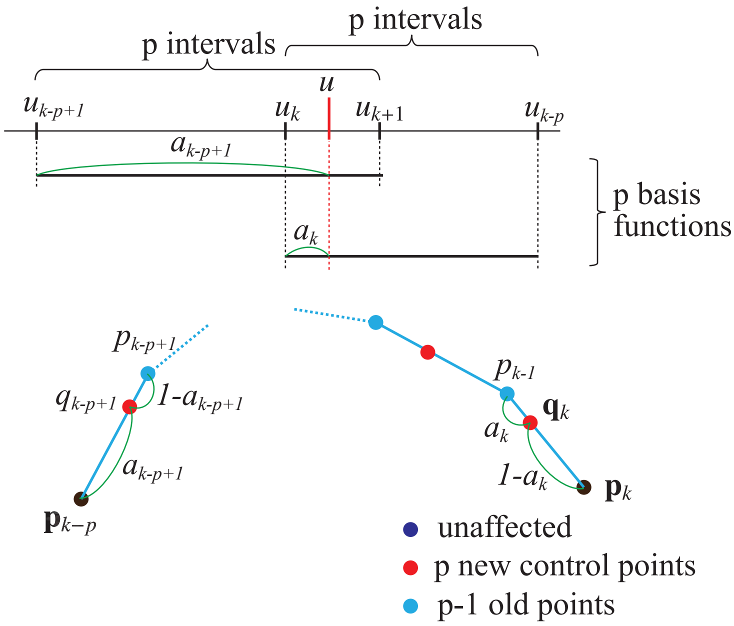


Figure 3: knot insertion algorithm

$$\mathbf{q}_i = (1 - a_i)\mathbf{p}_{i-1} + a_i\mathbf{p}_i \qquad (k - p + 1 \leq i \leq k) \tag{9}$$

$$where \quad a_i = \frac{u - u_i}{u_{i+p} - u_i} \tag{10}$$

3.4 Evaluation of a Point

4 NURBS Curve

The NURBS is the abbreviation of Non-Uniform Rational B-Spline curve. "non-uniform" means that the interval of the knots can be uneven. Rational means there are weight w for the control points.

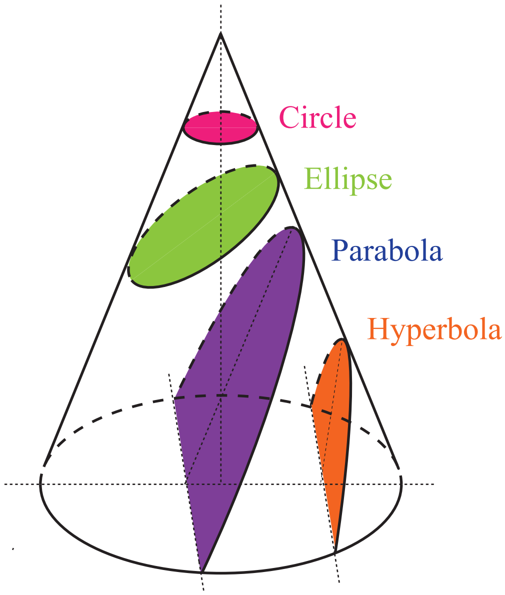
$$\mathbf{p}(u) = \frac{\sum_{i=0}^{n-1} N_{i,p}(u) w_i \mathbf{p}_i}{\sum_{i=0}^{n-1} N_{i,p} w_i} \tag{11}$$

This weight gives further control ability to the curve.

Each control points have a weight w_i. If all the weight is 1 it becomes a B-Spline curve with non-uniform knot interval.

4.1 How NURBS can Represent a Perfect Circle

Let's use a homogeneous coordinate to understand how the NURBS can successfully define a perfect circle. Here we think about two dimensional space where. A homogenious coordinate has additional one dimension w in addition to the xy. The point (x, y, w) in homogeneous coordinate corresponds to the two dimensional point $(x/w, y/w)$. Such a homogeneous coordinate is called Affine transformation and is very useful in computer graphics to define perspective transformation.

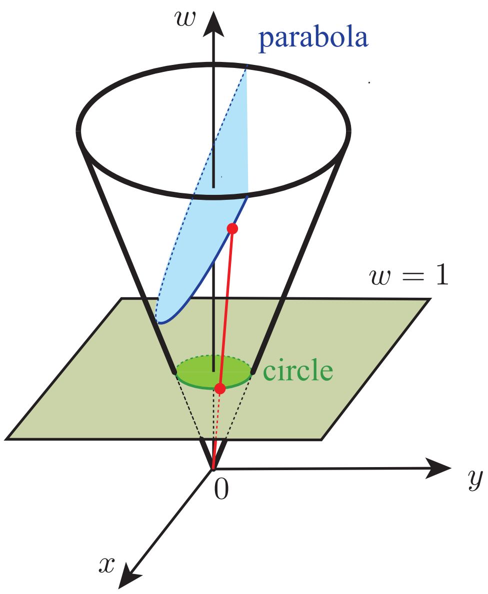


$$u_0, u_1, u_2, u_3, u_4, u_5, u_6, u_7 \tag{12}$$

$$0, x, y, w, w = 1 \tag{13}$$

$$\mathbf{p}_{k-p-1}, \mathbf{p}_{k-p}, \mathbf{p}_k, \mathbf{p}_{k+1} \tag{14}$$

$$\mathbf{q}_{k-p-1}, \mathbf{q}_{k-p}, \mathbf{q}_k, \mathbf{q}_{k+1} \tag{15}$$



w
parabola
$w = 1$
circle
y
0
x