Parametric Curve

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February 5, 2018

Contents

1 Parametric Curve

$$\int_0^1 \sqrt{\left(\frac{d\mathbf{p}}{dt}(t)\right)^2} dt \tag{1}$$

$$\tag{2}$$

2 Bezier Curve

$$\mathbf{p}(t) = (1-t)^3\mathbf{p}_0 + 3(1-t)^2 t\mathbf{p}_1 + 3(1-t)t^2\mathbf{p}_2 + t^3\mathbf{p}_3 \tag{3}$$

Tangent

$$\frac{d\mathbf{p}}{dt}(t) = -3(1-t)^2\mathbf{p}_0 + 3(3t-1)(t-1)\mathbf{p}_1 - 3t(3t-2)\mathbf{p}_2 + 3t^2\mathbf{p}_3 \tag{4}$$

2.1 De Casteljau's algorithm

3 B-Spline Curve

B-Spline curve is constructed from the three ingredients.

- the order of the curve p,
- a set of $n+1$ control points $\{\mathbf{p}_0, \ldots, \mathbf{p}_n\}$,
- a knot vector $\mathbf{u}$ of $m = n + p + 1$ non-descending real number $\{u_0 \leq u_1 \leq u_2 \ldots \leq u_m\}$.

Then, the resulting equation of the curve becomes

$$\mathbf{p}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{p}_i \tag{5}$$

3.1 Knot Vector

The knot vector $\mathbf{u}$ defines segments on the parameter the curve. This allows non-uniformly sized segments and gives the B-Spline higher degree of expression.

3.2 De Boor's algorithm

The basis function is defined hierarchically. In other words, we build up the basis function based on the basis function of one order lower.

We start from 0-th order basis function

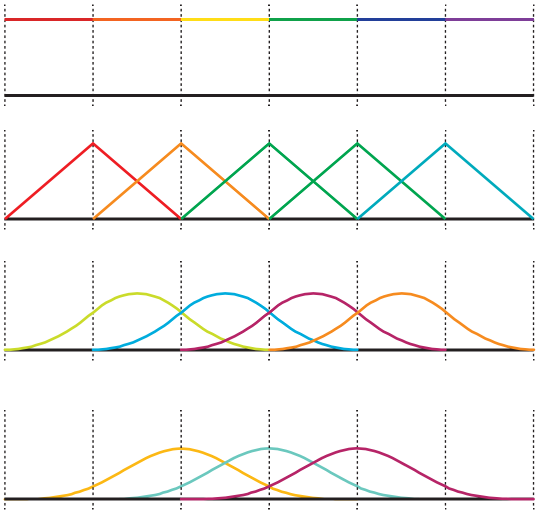
$$N_{i,0}(u) = \begin{cases} 1 & if\ u_i < u < u_{i+1} \\ 0 & otherwize \end{cases} \tag{6}$$

for $p > 0$ we define the basis functions using the basis function of $p-1$-th order as

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u) \tag{7}$$

4 NURBS Curve

The nurbs is the abbreviation of Non-Uniform Rational B-Spline curve. "non-uniform" means that the interval of the knots can be uneven. Rational means there are weight w for the control points.



$t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7$
$N_{0,0}, N_{1,0}, N_{2,0}, N_{3,0}, N_{4,0}, N_{5,0}, N_{6,0}$
$N_{0,1}, N_{1,1}, N_{2,1}, N_{3,1}, N_{4,1}, N_{5,1}$
$N_{0,2}, N_{1,2}, N_{2,2}, N_{3,2}, N_{4,2}$
$N_{0,3}, N_{1,3}, N_{2,3}, N_{3,3}$

4.1 How NURBS can Represent a Perfect Circle

Let's use a homogeneous coordinate to understand how the NURBS can successfully define a perfect circle. Here we think about two dimensional space where. A homogenious coordinate has additional one dimension w in addition to the xy. The point (x, y, w) in homogeneous coordinate corresponds to the two dimensional point $(x/w, y/w)$. Such a homogeneous coordinate is called Affine transformation and is very useful in computer graphics to define perspective transformation.