争取未来开挖掘机

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，姜萍吧，姜圣的追随者，

目录

1 三角函数

1.1 三角恒等式

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \tag{1.1.1}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \tag{1.1.2}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{1.1.3}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{1.1.4}$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.5}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.6}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.7}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.8}$$

1.1.2 积化和差

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \tag{1.1.9}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \tag{1.1.10}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{1.1.11}$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \tag{1.1.12}$$

1.1.3 倍角公式

$$\sin(2x) = 2\sin x\cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

1.1.4 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.13}$$
$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.14}$$

1.1.5 三角函数其他等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.15}$$
$$1 + \tan^2 x = \sec^2 \tag{1.1.16}$$
$$1 + \cot^2 x = \csc^2 \tag{1.1.17}$$

1.2 双曲函数

1.2.1 定义

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1}) \tag{1.2.1}$$
$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}) \tag{1.2.2}$$
$$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x}) \tag{1.2.3}$$

1.2.3 恒等式

$$\sinh(2x) = 2\sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2\sinh^2 \frac{x}{2} \tag{1.2.7}$$

2 不等式

$$\frac{x_1+x_2+\cdots+x_n}{n} \geqslant \sqrt[n]{x_1+x_2+\cdots+x_n} \tag{2.0.1}$$

$$|x+y| \leqslant |x|+|y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1+x)^n \leqslant 1+nx \tag{2.0.4}$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

3.2 运算

4 区间与映射

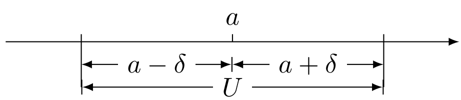
4.1 区间定义

$$\text{区间定义}\begin{cases}(a,b)=\{x|a<x<b\}\\ [a,b]=\{x|a\leqslant x\leqslant b\}\\ (a,b]=\{x|a<x\leqslant b\}\\ (a,+\infty)=\{x|a<x\}\end{cases}$$

4.2 领域定义

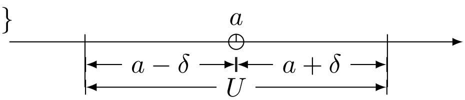
点 a 的领域

$$U(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\}\\ \{x|\ \ |x-a|<\delta\}\end{cases}$$



点 a 的去心领域

$$\mathring{U}(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\wedge x\neq 0\}\\ \{x|0<|x-a|<\delta\}\end{cases}$$



点 a 的左领域 $\quad (a-\delta,a)$

点 a 的右领域 $\quad (a,a+\delta)$

4.3 映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x\in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$\text{记作}\qquad f:X\to Y$$

$$f(x)=y\qquad\begin{cases}\text{定义域 }(D_f)=X & x\text{-原像}\\ \text{值域 }(R_f)=\{f(x)|x\in X\} & y\text{-像}\end{cases}$$

$$
\text{映射类型}\begin{cases}
\text{满射:} & R_f = Y \\
\text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\
\text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射:} \begin{cases} f^(x) & = y \\ f^{-1}(y) & = x \end{cases} \\
\text{复合映射:} & g \circ f \Leftrightarrow g\left[f\left(x\right)\right] \begin{cases} f : X \to Y_1 \\ g : Y_2 \to Z \\ g \circ f : X \to Z \qquad (Y_1 \subset Y_2) \end{cases}
\end{cases}
$$

5 函数与图像

5.1 函数的定义

设数集 $D \in R$ 的映射

$$f : D \to R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x)\ \{x \in D\}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f : D \to R\{D \subset R\}\begin{cases} \text{有界}\begin{cases}\text{有上界}\left\{\exists k_1,\ \text{使}f(x) \leqslant k_1,\ \forall x \in D\right. \\ \text{有下界}\left\{\exists k_1,\ \text{使}f(x) \geqslant k_1,\ \forall x \in D\right.\end{cases} \\ \text{无界}\begin{cases}\text{无上界}\left\{\forall K_1,\ \exists x \in D\ \text{使},\ f(x) \geqslant k_1\right. \\ \text{无下界}\left\{\forall K_1,\ \exists x \in D\ \text{使},\ f(x) \leqslant k_1\right.\end{cases}\end{cases}$$

5.2.2 函数的单调性

$$\text{单调增加 若}\{x_1, x_2 \in D\}\ x_1 < x_2 \Rightarrow \begin{cases} f(x_1) < f(x_2)\text{称}f(x)\text{在 D 上单调增加} \\ f(x_1) > f(x_2)\text{称}f(x)\text{在 D 上单调减少} \\ f(x_1) \leqslant f(x_2)\text{称}f(x)\text{在 D 上单调非降} \\ f(x_1) \geqslant f(x_2)\text{称}f(x)\text{在 D 上单调非增} \end{cases}$$

5.2.3 函数的奇偶性

定义域

$$\forall x \in D \qquad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

奇偶性运算

$$\text{奇函数} \times \text{奇函数} = \text{偶函数} \tag{5.2.1}$$

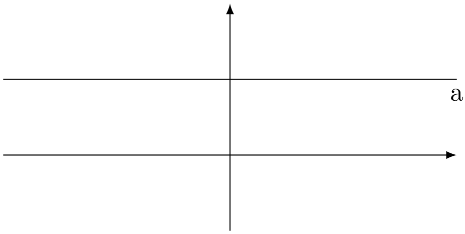
$$\text{奇函数} \times \text{偶函数} = \text{奇函数} \tag{5.2.2}$$

$$\text{偶函数} \times \text{偶函数} = \text{偶函数} \tag{5.2.3}$$

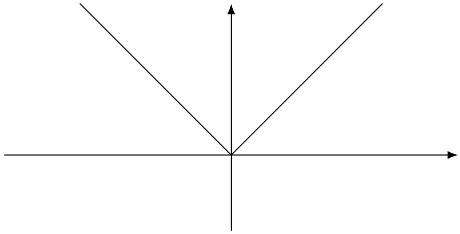
5.2.4 周期性

$Def: \quad f(x+L)=f(x)\{L>0常数, \forall x \in D\} \Rightarrow f(x)$ 为 L 的周期函数

5.3 函数图像

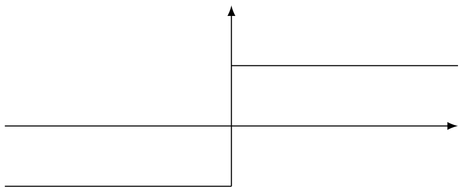


常函数 $f(x)=a\{a \in R\}$

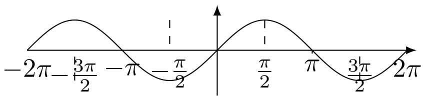


$f(x)=|x|$

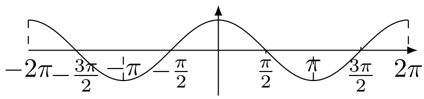
$$f(x)=sgn\ x=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$



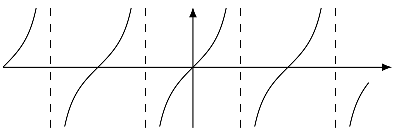
$|x|=x\cdot sgnx$



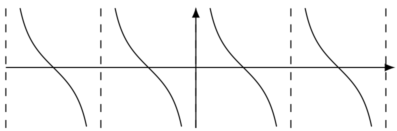
$\sin x$



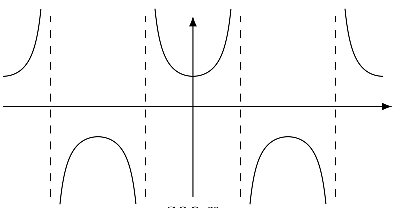
$\cos x$



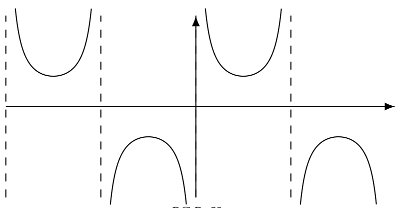
$\tan x$



$\cot x$



$\sec x$



$\csc x$

6 并集，交集

6.1 定义

$$
\begin{gathered}
(\vee \text{ 或}, \wedge \text{ 与}) \\
A \cup B = \{x \in A \vee x \in B\} \\
A \cap B = \{x \in A \wedge x \in B\}
\end{gathered}
$$

6.2 运算

$$
\text{运算满足}\begin{cases}
\text{交换律}\begin{cases} A\cup B & = B\cup A \\ A\cap B & = B\cap A \end{cases} \\
\text{结合律}\begin{cases} (A\cup B)\cup C = A\cup(B\cup C) \\ (A\cap B)\cap C = A\cap(B\cap C) \end{cases} \\
\text{分配律}\begin{cases} (A\cup B)\cap C = (A\cap C)\cup(B\cap C) \\ (A\cap B)\cup C = (A\cup C)\cap(B\cup C) \end{cases} \\
\text{对偶律}\begin{cases} (A\cup B)^C = A^C\cap B^C \\ (A\cap B)^C = A^C\cup B^C \end{cases}
\end{cases}
$$

$$
\begin{gathered}
A \cup A = A = A \cap A \\
A = B \Leftrightarrow A \subset B \wedge A \supset B \\
A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing
\end{gathered}
$$

6.3 性质

性质 1.

$$A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{6.3.1}$$

性质 2.

$$A \cup B = B \Leftrightarrow A \subset B \tag{6.3.2}$$

性质 3.

$$A \cap B = A \Leftrightarrow A \subset B \tag{6.3.3}$$

性质 4. $(n \in N\ )$

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{6.3.4}$$

性质 5. $(n \in N\ )$

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{6.3.5}$$

6.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

6.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{C} = \bigcap_{\alpha} (E_{\alpha}^{C})$$

$$\left(\bigcap_{\alpha} E_{\alpha}\right)^{C} = \bigcup_{\alpha} (E_{\alpha}^{C})$$

7 群，环，域

7.1 群

7.1.1 M1

7.1.2 M2

7.1.3 M3

7.1.4 M4

7.1.5 sdas

7.2 环

7.3 域

8 极限

8.1 数列极限

8.1.1 数列的定义

$Def:\ \ \{x_n\}:N^+\to R$

$$x_n=f(n)$$

8.1.2 数列极限的定义

$Def:\ \{x_n\},\ n\in N^+,\exists a,\ \forall\varepsilon>0,\exists N,\ n>N\Rightarrow|x_n-a|<\varepsilon$
$\lim\limits_{n\to\infty}x_n=a$
极限存在，为收敛，不存在为发散

8.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{8.1.1}$$

8.1.4 有界数列

若$\exists M>0,\{M\in\text{正数}\}$
使得$\forall n,\quad|x_n|\leqslant M$
则称数列 $\{x_n\}$ 为有界数列

8.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{8.1.2}$$

$$\text{单调有界数列必收敛} \tag{8.1.3}$$

8.1.6 收敛数列的保号性

$$\text{如果}\lim_{n\to\infty}x_n=a\text{ 存在，且 }a>0,\text{ 则 }\exists N>0\{N\in N^+\}\text{ 当 }n>N\text{ 时，}\Leftrightarrow x_n>0 \tag{8.1.4}$$

$$\lim_{n\to\infty}x_n=a,\lim_{n\to\infty}b_n=b,a<b,\ \exists N,n>N,a_n<b_n \tag{8.1.5}$$

8.1.7 收敛数列和子数列

$\{x_n\}, \lim\limits_{n\to\infty} x_n = a,\ \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$

证明 $K = N\ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

8.2 函数极限

8.2.1 极限的定义

$$Def: \forall\varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} 当 x > X & 时都有 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ 当 x < -X & 时都有 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ 当 |x| > X & 时都有 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists\delta > 0 \begin{cases} 当 x_0 < x < x_0 + \delta, 时 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ 当 x_0 - \delta < x < x_0, 时 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ 当 0 < |x - x_0| < \delta, 时 & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x\to x_0} f(x) 存在 \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \tag{8.2.1}$$

图

8.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$\lim\limits_{x\to x_0} f(x) = A \Rightarrow \exists M > 0, \delta > 0 使 0 < |x - x_0| < \delta, |f(x)| \leqslant M$

3 保号性

$\lim_{x\to x_0} = A,\ A>0, \Rightarrow \exists\delta>0, 当, 0<|x-x_0|<\delta 时 f(x)>0$

4 保序性

$f(x)\geqslant g(x),\ \lim f(x)=a,\ \lim g(x)=b,\ 则 a\geqslant b$

4 函数极限与数列极限的关系

如果 $\lim_{x\to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列，则满足 $x_n\neq x_0$

则 $\lim_{n\to\infty} f(x_n)=0=\lim_{x\to x_0} f(x),\ x_n\to x_0$

8.3 无穷小与无穷大

8.3.1 无穷小定义

$Def: 如果 \lim_{x\to x_0} f(x)=0 则称 f(x) 为 x\to x_0 时的无穷小$

$$Def:\forall\varepsilon>0\begin{cases}\exists X>0\begin{cases}当 x>X & 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to+\infty}f(x)=0\\ 当 x<-X & 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to-\infty}f(x)=0\\ 当\ |x|>X & 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to\infty}f(x)=0\end{cases}\\ \exists\delta>0\begin{cases}当 x_0<x<x_0+\delta, 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to x_0^+}f(x)=0\\ 当 x_0-\delta<x<x_0, 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to x_0^-}f(x)=0\\ 当 0<|x-x_0|<\delta, 时\ |f(x)-0|<\varepsilon\Leftrightarrow\lim_{x\to x_0}f(x)=0\end{cases}\end{cases}$$

8.3.2 函数极限与无穷小的关系

$$在自变量的同一变化中。\alpha 为无穷小。\lim f(x)=A\Leftrightarrow f(x)=A+\alpha \tag{8.3.1}$$

8.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$如果\ f(x)\ 为无穷大，则\ \frac{1}{f(x)}\ 为无穷小。\tag{8.3.2}$$

$$如果\ f(x)\ 为无穷小，切\ f(x)\neq 0，则\ \frac{1}{f(x)}\ 为无穷小。\tag{8.3.3}$$

8.3.4 无穷大定义

$$
Def:\forall M>0\begin{cases}
\exists X>0\begin{cases}
\text{当}x>X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=\infty
\end{cases}\\
\text{当}x<-X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=+\infty\\
f(x)<M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=\infty
\end{cases}\\
\text{当}\ |x|>X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=\infty
\end{cases}
\end{cases}\\
\exists\delta>0\begin{cases}
\text{当}x_0-\delta<x<x_0\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=\infty
\end{cases}\\
\text{当}x_0<x<x_0+\delta\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=\infty
\end{cases}\\
\text{当}0<|x-x_0|<\delta\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=\infty
\end{cases}
\end{cases}
\end{cases}
$$

$$\lim_{x\to x_0}f(x)=\infty\text{，直线}x=x_0\text{是}y=f(x)\text{垂直渐进线}$$

8.4 运算

8.4.1 有限个无穷小的和仍为无穷小

$$\text{设 }\gamma=\alpha+\beta$$

$$\alpha\text{ 和 }\beta\text{ 同为 }x\to x_0\text{ 时的无穷小}$$

$$\forall\varepsilon>0\text{，}\exists\delta_1>0\text{，当 }0<|x-x_0|<\delta_1\text{ 时，有 }|\alpha|<\tfrac{\varepsilon}{2}$$

$$\forall\varepsilon > 0,\ \exists\delta_2 > 0,\ \text{当}\ 0 < |x - x_0| < \delta_2\ \text{时，有}\ |\beta| < \frac{\varepsilon}{2}$$
$$\delta = min\{\delta_1, \delta_2\},\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2\ \text{同时满足}$$
$$\text{即}\ |\alpha| < \frac{\varepsilon}{2}, |\beta| < \frac{\varepsilon}{2}\ \text{同时成立}$$
$$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

8.4.2 有界函数与无穷小的乘积仍为无穷小

$$\text{设}\ \alpha\ \text{为}\ x \to x_0\ \text{时的一个无穷小}$$
$$g(x)\ \text{为}\ x_0\ \text{的一个去心邻域}\ \mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$f(x) = g(x)\alpha$$
$$\text{证}\ f(x)\ \text{为}\ x \to x_0\ \text{时的无穷小}$$
$$\text{因为}\ g(x)\text{在}\mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$\exists M > 0, \text{当} 0 < |x - x_0| < \delta_1\ \text{时}\ |g(x)| < M$$
$$\text{因为}\ \alpha\ \text{是}\ x \to x_0\ \text{的无穷小}$$
$$\exists\delta_2 > 0 \text{当} 0 < |x - x_0| < \delta_2\ \text{时}\ |\alpha| < \frac{\varepsilon}{M} < \varepsilon$$
$$\text{取}\ \delta = min\{delta_, \delta_2\}\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$|g(x)| \geqslant M, |\alpha| < \frac{\varepsilon}{M}\ \text{同时成立}$$
$$|g(x)\alpha| = |g(x)|\ \ |\alpha| < M\frac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

8.4.3 极限的四则运算

$\lim f(x) = A,\ \lim g(x) = B$

$$\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) \tag{8.4.1}$$
$$\lim\left(f(x)g(x)\right) = \lim f(x) \lim g(x) \tag{8.4.2}$$
$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)} \tag{8.4.3}$$
$$\lim\left[Cf(x)\right] = C\lim f(x) \tag{8.4.4}$$
$$\lim\left[f(x)\right]^n = \left[\lim f(x)\right]^n \tag{8.4.5}$$

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{a_0x^n+a_1x^{n-1}+\cdots+a_n}=\begin{cases}\frac{a}{b} & m=n\\ \infty & m>n\\ 0 & m<n\end{cases} \tag{8.4.6}$$

$$\begin{gathered}\lim_{x\to x_0}g(x)=u_0,\ \lim_{u\to u_0}f(x)=A\\ \exists\delta_0>0,\ x\in\mathring{U}\left(x_0,\delta_0\right),\ g(x)\neq u_0\\ \lim_{x\to x_0}f\left[g(x)\right]=\lim_{u\to u_0}f(u)=A\end{gathered} \tag{8.4.7}$$

8.4.4 夹逼定理 (三明治定理)

$$\begin{aligned}&x_n\leqslant z_n\leqslant y_n\qquad\forall n>N_0\\ &\text{若}\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=a\text{则}\lim_{n\to\infty}z_n=a\end{aligned} \tag{8.4.8}$$

8.4.5 重要极限

趋向 0

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{8.4.9}$$

$$\lim_{x\to 0}\cos x=1 \tag{8.4.10}$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1 \tag{8.4.11}$$

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=1 \tag{8.4.12}$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=1 \tag{8.4.13}$$

$$\lim_{x\to 0}\frac{\arctan x}{x}=1 \tag{8.4.14}$$

$$\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}=1 \tag{8.4.15}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \tag{8.4.16}$$

$$\lim_{x\to 0}\frac{\left(1+x\right)^n-1}{nx}=1 \tag{8.4.17}$$

$$\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}=e \tag{8.4.18}$$

趋向 ∞

$$\{x_n\} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{8.4.19}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{8.4.20}$$

8.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

$Def:\ \alpha, \beta$ 是同一极限过程的无穷小。

(1) 如果 $\lim \frac{\beta}{\alpha} = 0$ 则称 β 是 α 的高阶无穷小，记作 $\beta = \circ(\alpha)$

(2) 如果 $\lim \frac{\beta}{\alpha} = \infty$ 则称 β 是 α 的底阶无穷小。

(3) 如果 $\lim \frac{\beta}{\alpha} = C$ 则称 β 是 α 的同阶无穷小。

(4) 如果 $\lim \frac{\beta}{\alpha^k} = C, k > 0$ 则称 β 是 α 的 k 阶无穷小。

(5) 如果 $\lim \frac{\beta}{\alpha} = 1$ 则称 β 是 α 的等价阶无穷小。

8.4.7 等价无穷小代换，因子代换

β与α是等价无穷小 $\Leftrightarrow \beta = \alpha + \circ(\alpha)$

设$\alpha \sim \alpha'$, $\beta \sim \beta'$, 且$\lim \frac{\beta'}{\alpha'}$存在, 则$\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$

$\lim \alpha f(x) = \lim \alpha' f(x)$

$\lim \frac{f(x)}{\alpha} = \lim \frac{f(x)}{\alpha'}$

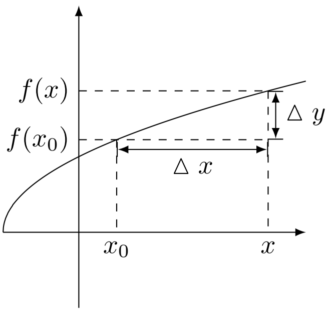
9 连续与间断点

9.1 定义

9.1.1 点连续

$$Def1: 设f(x)在x_0的某邻域内有定义，如果 \lim_{x\to x_0} = f(x_0)$$

$$则称f(x)在x_0处连续$$



$$\begin{cases} \triangle\ x = x - x_0 \\ \triangle\ y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \triangle\ x) - f(x_0) \end{cases} \end{cases}$$

$$Def2: 如果 \lim_{\triangle x\to 0} \triangle\ y = \lim_{\triangle x\to 0} [f(x_0 + \triangle\ x) - f(x_0)] = 0$$

$$则称f(x)在x_0处连续$$

9.1.2 区间连续

$$\forall x_0 \in [a,b] \begin{cases} \lim_{x\to x_0} f(x) = f(x_0) \quad x_0 \in (a,b) \Leftrightarrow f(x_0) = \begin{cases} \lim_{x\to x_0^-} f(x) = f(x_0^-) \\ \lim_{x\to x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim_{x\to x_0^+} f(x) = f(x_0^+) \quad x_0 = a \text{（右连续）} \\ \lim_{x\to x_0^-} f(x) = f(x_0^-) \quad x_0 = b \text{（左连续）} \end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M > 0, x \in [a,b]$ 时，$|f(x)| \geqslant M$

最大值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

零点定理

2, 设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)\cdot f(b)<0$
则至少存在一点 $\xi\in(a,b)$ 使 $f(\xi)=0$

介质定理

设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)=A, f(b)=B$
$\forall C\in(A,B)$, 至少有一点 $\xi, f(\xi)=C$

9.1.3 间断点

1,$f(x)$ 无定义
2,$\lim\limits_{x\to x_0} f(x)$ 不存在
3. $\lim\limits_{x\to x_0} f(x)$ 存在, 但 $\lim\limits_{x\to x_0} f(x)\neq f(x_0)$
第一类间断点：$f(x_0^+)=\lim\limits_{x\to x_0^+} f(x)$ 与 $f(x_0^-)=\lim\limits_{x\to x_0^-} f(x)$
第二类间断点：不是第一类的。

9.2 连续函数的运算

函数 $f(x), g(x)$ 在 $x=x_0$ 连续。

$$\lim_{x\to x_0}[f(x)+g(x)]=\lim_{x\to x_0} f(x)+\lim_{x\to x_0} g(x)=f(x_0)+g(x_0)$$

$$\lim_{x\to x_0}[f(x)-g(x)]=\lim_{x\to x_0} f(x)-\lim_{x\to x_0} g(x)=f(x_0)-g(x_0)$$

$$\lim_{x\to x_0}[f(x)\cdot g(x)]=\lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0} g(x)=f(x_0)\cdot g(x_0)$$

$$\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}=\frac{f(x_0)}{g(x_0)}\qquad(g(x_0)\neq 0)$$

反函数的连续性
若 $y=f(x)$ 在区间 I_x 上单调增加，且连续。
则 $y=f^-1(x)$ 在 $I_y=\{y|y=f(x), x\in I_x\}$ 上也为单调增加，连续

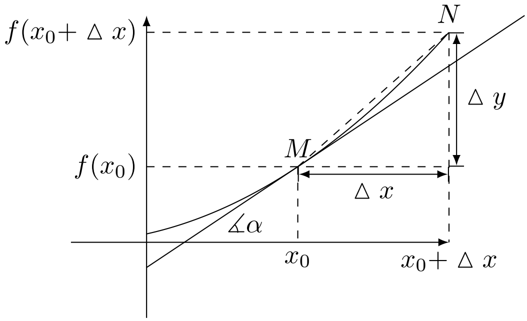
$$
\text{复合函数,}\begin{cases}
\text{内外都连续}\begin{cases}
\lim\limits_{x\to x_0} g(x) = g(x_0) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f\left[g(x_0)\right] = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
\text{外连续}\begin{cases}
x\to x_0 \begin{cases}
\lim\limits_{x\to x_0} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
x\to\infty \begin{cases}
\lim\limits_{x\to\infty} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to\infty} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to\infty} g(x))
\end{cases}
\end{cases}
\end{cases}
$$

10 导数

10.1 定义

导数的概念从物理发展出来的。

$$v\left(t_0\right)=\lim_{\triangle t\to 0}\frac{s\left(t_0+\triangle t\right)-s\left(t_0\right)}{\triangle t}$$



$$NM\text{斜率}=\tan\beta=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}=\frac{\triangle y}{\triangle x}$$

$$\text{斜率}k=\tan\alpha=\lim_{\triangle x\to 0}\tan\beta=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}$$

10.1.1 导数定义

$y=f(x)$ 在 x_0 的某邻域内有定义
给自变量的增量 $\triangle x,(x_0+\triangle x)$ 仍在定义域内
函数得到了相应增量 $\triangle y,\triangle y=f\left(x_0+\triangle x\right)$
如果 $\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$ 存在, 称 $y=f(x)$ 在 $x=x_0$ 处可导
(极限值为$y=f(x)$在$x=x_0$处导数)

$$\text{记 } y'|_{x=x_0}=f'(x_0)=\lim_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$$

$$\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\Leftrightarrow\lim_{\triangle x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

10.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。
$f'(x)=\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$
称 $f'(x)$ 为 $y=f(x)$ 在区间 I 上的导函数

10.1.3 闭区间可导定义

$$f(x) \text{ 在 } [a,b] \text{ 可导} \Leftrightarrow \begin{cases} f'(x_0) & x_0 \in (a,b) \Leftrightarrow \begin{cases} \text{左导数} f'_-(x_0) = \lim\limits_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ \text{右导数} f'_+(x_0) = \lim\limits_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \end{cases} \\ f'_+(a) & x = a \\ f'_-(a) & x = b \end{cases}$$

10.1.4 导数与连续

$$f'(x)\text{存在} \Rightarrow f(x)\text{在}x = x_0\text{处连续} \tag{10.1.1}$$

10.2 幂数，指数，对数

$$\frac{d}{\mathrm{d}x}C = 0 \tag{10.2.1}$$

$$\frac{d}{\mathrm{d}x}x^a = ax^{a-1} \tag{10.2.2}$$

$$\frac{d}{\mathrm{d}x}a^x = a^x \ln a \tag{10.2.3}$$

$$\frac{d}{\mathrm{d}x}e^x = e^x \tag{10.2.4}$$

$$\frac{d}{\mathrm{d}x}\log_a^x = \frac{1}{x \ln a} \tag{10.2.5}$$

$$\frac{d}{\mathrm{d}x}\ln x = \frac{1}{x} \tag{10.2.6}$$

10.3 三角函数

$$\frac{d}{\mathrm{d}x}\sin x = \cos x \tag{10.3.1}$$

$$\frac{d}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}} \tag{10.3.2}$$

$$\frac{d}{\mathrm{d}x}\csc x = -\csc x\cot x \tag{10.3.3}$$

$$\frac{d}{\mathrm{d}x}\cos x = -\sin x \tag{10.3.4}$$

$$\frac{d}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}} \tag{10.3.5}$$

$$\frac{d}{\mathrm{d}x}\sec x = \sec x\tan x \tag{10.3.6}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \tag{10.3.7}$$

$$\frac{d}{\mathrm{d}x}\tan x = \sec^2 x \tag{10.3.8}$$

$$\frac{d}{\mathrm{d}x}\arctan = \frac{1}{1+x^2} \tag{10.3.9}$$

$$\frac{d}{\mathrm{d}x}\cot x = -\csc^2 x \tag{10.3.10}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arccot} x = -\frac{1}{1+x^2} \tag{10.3.11}$$

$$\frac{d}{\mathrm{d}x}\sinh x = \cosh x \tag{10.3.12}$$

$$\frac{d}{\mathrm{d}x}\cosh x = \sinh x \tag{10.3.13}$$

$$\frac{d}{\mathrm{d}x}\tanh x = \frac{1}{cosh^2 x} = 1-\tanh^2 x \tag{10.3.14}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arcsinh} x = \frac{1}{\sqrt{x^2+1}} \tag{10.3.15}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arccosh} x = \frac{1}{\sqrt{x^2-1}} \tag{10.3.16}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arctanh} x = \frac{1}{1-x^2} \tag{10.3.17}$$

10.4 导数运算

$$U = u(x), V = v(x), \text{均在}x\text{点可导}, C\text{为常数}$$

$$\frac{d(CU)}{\mathrm{d}x}=C\frac{d(U)}{\mathrm{d}x} \tag{10.4.1}$$

$$\frac{d(U+V)}{\mathrm{d}x}=\frac{dU}{\mathrm{d}x}\pm\frac{dV}{\mathrm{d}x} \tag{10.4.2}$$

$$\frac{d(UV)}{\mathrm{d}x}=\frac{dU}{\mathrm{d}x}V+\frac{dV}{\mathrm{d}x}U \tag{10.4.3}$$

$$\frac{d(\frac{U}{V})}{\mathrm{d}x}=\frac{\frac{dU}{\mathrm{d}x}V-\frac{dV}{\mathrm{d}x}U}{V^2} \tag{10.4.4}$$

10.5 反函数求导

如果函数 $y=f(x)$ 在区间 (a,b) 内单调可导，且 $f'(y)\neq 0$

$$\begin{cases}\alpha=\min\{f(a)+0,f(b-0)\}\\ \beta=\max\{f(a)+0,f(b-0)\}\end{cases}$$

则它的反函数 $x=f^{-1}(y)$ 在区间 (α,β) 内也可导

$$\left[f^{-1}(y)\right]'=\frac{1}{f'(x)} \tag{10.5.1}$$

$$\frac{dy}{\mathrm{d}x}=\frac{1}{\frac{dx}{\mathrm{d}y}}$$

10.6 复合函数求导

$$\text{设函数}\begin{cases}y=f(u)\text{在}U(u_0,\delta_0)\text{处有定义}\\ u=g(x)\text{在}U(x_0,\eta_0)\text{处有定义}\end{cases}$$

$$u_0=g(x_0),\text{且}f'(u)\text{和}g'(x)\text{都存在}$$

$$\text{则复合函数}F(x)=f\left[g(x)\right]\text{在点}x_0\text{可导, 且}$$

$$F'(x_0)=f'\left[g(x_0)\right]g'(x_0) \tag{10.6.1}$$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

10.7 高阶求导

$$Def:\begin{cases}\text{一阶导数} & y'\Leftrightarrow\frac{dy}{dx}\\ \text{二阶导数} & y''\Leftrightarrow\frac{d^2y}{dx^2}\\ \text{三阶导数} & y'''\Leftrightarrow\frac{d^3y}{dx^3}\\ \text{三阶以上 n 阶导数} & y^{(n)}\Leftrightarrow\frac{d^ny}{dx^n}\end{cases}$$

10.8 高阶求导公式

$$\frac{d^n}{dx}e^x = e^x \tag{10.8.1}$$

$$\frac{d^n}{dx}a^x = a^x\left(lna\right)^n \tag{10.8.2}$$

$$\frac{d^n}{dx}x^\mu = A_\mu^n x^{\mu-n} \tag{10.8.3}$$

$$\frac{d^n}{dx}\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{10.8.4}$$

$$\frac{d^n}{dx}\ln(x+a) = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{10.8.5}$$

$$\frac{d^n}{dx}\sin x = \sin(x+n\frac{\pi}{2}) \tag{10.8.6}$$

$$\frac{d^n}{dx}\cos x = \cos(x+n\frac{\pi}{2}) \tag{10.8.7}$$

$$\frac{d^n}{dx}\left[f(ax+b)\right] = a^n \cdot \frac{d^n f(ax+b)}{d(ax+n)} \tag{10.8.8}$$

10.9 高阶求导运算法则

$$\frac{d^n}{dx}(u\pm v) = \frac{d^n u}{dx} \pm \frac{d^n v}{dx} \tag{10.9.1}$$

$$\text{莱布紫泥公式} \qquad (uv)^n = \sum_{k=0}^{n} C_n^k u^{(n-k)} \cdot v^k \tag{10.9.2}$$

10.10 隐函数求导

10.11 参数方程求导

11 积分

11.1 幂数，指数，对数

$$\int x^a \mathrm{d}x = \frac{1}{a-1} x^{a-1} + C \tag{11.1.1}$$

$$\int b^x \mathrm{d}x = \frac{b^x}{\ln b} + C \tag{11.1.2}$$

$$\int e^x \mathrm{d}x = e^x + C \tag{11.1.3}$$

$$\int \frac{1}{x} \mathrm{d}x = \ln x + C \tag{11.1.4}$$

11.2 三角函数

$$\int \sin x \mathrm{d}x = -\cos x + C \tag{11.2.1}$$

$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \arcsin x + C \tag{11.2.2}$$

$$\int \csc x \cot x \mathrm{d}x = -\csc x + C \tag{11.2.3}$$

$$\int \cos x \mathrm{d}x = \sin x + C \tag{11.2.4}$$

$$\int \sec x \tan x \mathrm{d}x = \sec x + C \tag{11.2.5}$$

$$\int \sec^2 x \mathrm{d}x = \tan x + C \tag{11.2.6}$$

$$\int \csc^2 x \mathrm{d}x = -\cot x + C \tag{11.2.7}$$

$$\int \frac{1}{|x| \sqrt{x^2-1}} \mathrm{d}x = \operatorname{arcsec} x + C \tag{11.2.8}$$

$$\int \frac{1}{1+x^2} \mathrm{d}x = \tan x + C \tag{11.2.9}$$

$$\int \sinh x \mathrm{d}x = \cosh x + C \tag{11.2.10}$$

$$\int \cosh x \mathrm{d}x = \sinh x + C \tag{11.2.11}$$

11.3 积分运算

12 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{12.0.1}$$

$$A_N = \sum_{k=0}^{n} q^k \qquad qA_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - qA_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1-q^{n+1}}{1-q}$$

$$\log_{10} x = \lg_x \tag{12.0.2}$$

$$\log_e x = \ln_x \tag{12.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{12.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{12.0.5}$$

$$\log_b x^n = n \log_b x \tag{12.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{12.0.7}$$

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a-b)\sum_{m=0}^{n-1}\left(a^{n-m}b^m\right) = (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)$$

13 证明

13.1 第 1章

1.2.4

$$\begin{aligned}\sinh x\cosh x &= \left(\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}\right)\\ &= \left(\frac{1}{2}\right)\left(\frac{e^{2x}-e^{-2x}}{2}\right)\\ &= \frac{1}{2}\sinh(2x)\\ \sinh(2x) &= 2\sinh x\cosh x\end{aligned}$$

1.2.5

$$\begin{aligned}\cosh^2 x-\sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}\right)\left(\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}\right)\\ &= e^x\times e^{-x}\\ &= 1\end{aligned}$$

1.2.6

$$\begin{aligned}\cosh^2 x+\sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2+\left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{2e^{2x}+2e^{-2x}}{4}\\ &= \frac{e^{2x}+e^{-2x}}{2}\\ &= \cosh(2x)\end{aligned}$$

1.2.7

$$\begin{aligned}\cosh(2x) &= \cosh^2 x+\sinh^2 x\\ &= \sinh^2 x+1+\sinh^2 x\\ &= 2\sinh^2 x+1\\ \cosh x &= 2\sinh^2\frac{x}{2}+1\end{aligned}$$

13.2 第 8章

8.1.1

$$
\begin{aligned}
&\text{反设} \lim_{n\to\infty} x_n = a,\ \lim_{n\to\infty} x_n = b, \text{且} a < b \\
&\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1,\ n > N_1,\ |x_n - a| < \frac{b-a}{3} \\ \exists N_2,\ n > N_2,\ |x_n - b| < \frac{b-a}{3} \end{cases} \\
&N = \max\{N_1, N_2\},\ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases} \\
b - a &= |(x_n - a) - (x_n - b)| \\
&\leqslant |x_n - a| + |x_n - b| \\
&< \frac{b-a}{3} + \frac{b-a}{3} \\
&< \frac{2(b-a)}{3}
\end{aligned}
$$

8.1.2

$$
\begin{aligned}
\varepsilon = 1,\ \exists N > 0,\ \text{当 n>N 时} \, |X_n - a| < 1 \\
|X_n| = |(X_n - a) + a| \\
\leqslant |x_n - a| + |a| \\
\leqslant 1 + |a| \\
M = \max\{|X_n|, |X_2|, \ldots, |X_n|, 1 + |a|\} \\
\forall n,\ |X_n| \leqslant M
\end{aligned}
$$

8.1.4

1

由于 $\lim\limits_{n\to\infty} x_n = a$, 且$a > 0$

$\varepsilon = \frac{a}{2},\ \exists N > 0, n > N$

$|x_n - a| < \varepsilon$

$|x_n - a| < \frac{a}{2}$

$-\frac{a}{2} < x_n - a < \frac{a}{2}$

$\frac{a}{2} < x_n < 1$

2

用反证法，反设 $a<0$. 从某项起 $x_n<0$ 矛盾

8.1.5

$$x_n = b_n - a_n$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} x_n = b - a > 0$$
$$\lim_{n\to\infty} x_n > 0$$
$$b_n - a_n = x_n > 0$$
$$b_n > a_n$$

8.2.1 $\lim\limits_{x\to x_0} f(x)$存在 $\Rightarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x)$

$$\text{设}\lim_{x\to x_0} = A$$
$$0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon$$
$$0<|x-x_0|<\delta \Leftrightarrow x\in \mathring{U}(x_0,\delta)$$
$$\begin{cases} \text{当}x_0<x<x_0+\delta\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^+} f(x)=A \\ \text{当}x_0-\delta<x<x_0\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^-} f(x)=A \end{cases}$$
$$\lim_{x\to x_0^+} f(x) = A = \lim_{x\to x_0^-} f(x)$$

$\lim\limits_{x\to x_0} f(x)$存在 $\Leftarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x)$

$$A=\begin{cases} \lim\limits_{x\to x_o^+}, \forall\varepsilon>0, \exists\delta_1>0, x_0<x<x_0+\delta_1, |f(x)-A|<\varepsilon \\ \lim\limits_{x\to x_o^-}, \forall\varepsilon>0, \exists\delta_2>0, x_0-\delta_2<x<x_0, |f(x)-A|<\varepsilon \end{cases}$$
$$\delta=\min\{\delta_1,\delta_2\}$$
$$0<|x-x_0|<\delta\begin{cases} x>x_0, x_0<x<x_0+\delta\leqslant x_0+\delta_1,\ |f(x)-A|<\varepsilon \\ x<x_0, x_0-\delta_2\leqslant x_0+\delta<x<x_0,\ |f(x)-A|<\varepsilon \end{cases}$$
$$\lim_{x\to x_0} f(x) = A$$

8.3.1

$$\lim_{x\to x_o} f(x) = A \Rightarrow \begin{cases} \alpha\text{为}x\to x_0\text{时的无穷小} \\ f(x)=\alpha+A \end{cases}$$
$$\text{设}\lim_{x\to x_o} f(x) = A,\ \text{记}f(x)-A=\alpha$$
$$\text{只需证 }\alpha\text{ 为无穷小。}$$

$$\forall\varepsilon>0,\exists\delta>0,\text{当}0<|x-x_0|<\delta,\text{时}\,|f(x)-A|<\varepsilon$$
$$\text{即 }|\alpha-0|<\varepsilon$$
$$\alpha\text{为}x\to x_0\text{时的无穷小}$$
$$\lim_{x\to x_o}f(x)=A\Leftarrow\begin{cases}\alpha\text{为}x\to x_0\text{时的无穷小}\\ f(x)=\alpha+A\end{cases}$$
$$\forall\varepsilon>0,\ \exists\delta>0,\ \text{当}0<|x-x+0|<\delta,\ |\alpha|<\varepsilon$$
$$\text{即}\,|f(x)-A|<\varepsilon\ \lim_{x\to x_0}f(x)=A$$

8.3.2

$$\text{设 }\lim_{x\to x_0}f(x)=\infty$$
$$\text{对 }f(x)\text{ 为 }x\to\text{ 时无穷大}$$
$$\text{对于 }M=\tfrac{1}{\varepsilon}.\text{ 存在 }\delta>0$$
$$\text{当 }0<|x-x_0|<\delta\text{ 时}$$
$$|f(x)|>M=\tfrac{1}{\varepsilon}$$
$$\left|\tfrac{1}{f(x)}\right|<\varepsilon$$
$$\tfrac{1}{f(x)}\text{ 为 }x\to x_0\text{ 时的无穷小}$$

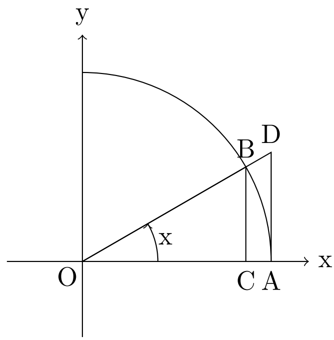
8.4.2

$$\begin{aligned}f(x)g(x)&=[A+\alpha]\,[B+\beta]\\&=AB+A\beta+B\alpha+\beta\alpha\\&=AB+\gamma\qquad\qquad(\gamma\text{为无穷小})\\ \lim\,[f(x)g(x)]&=AB+\gamma=\lim f(x)\lim g(x)\end{aligned}$$

8.4.8

$$\forall\varepsilon>0$$
$$|x_n-a|<\varepsilon\qquad\forall n>N_1$$
$$|y_n-a|<\varepsilon\qquad\forall n>N_2$$
$$\text{令}N=\max\{N_1,N_2,N_0\}\,,\text{则当}n>N\text{时有}$$
$$a-\varepsilon<x_n\le z_n\le y_n<a+\varepsilon$$
$$|z_n-a|<\varepsilon$$
$$\lim_{n\to\infty}z_n=a$$

8.4.9



$$OB = OA = 1$$

$$\triangle\ AOB \leqslant \text{扇形面积} \leqslant \triangle\ AOD$$

$$\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x$$

$$\sin x \leqslant x \leqslant \tan$$

$$1 \geqslant \frac{\sin x}{x} \geqslant \cos x$$

$$\lim_{x\to 0} 1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant \lim_{x\to 0}\cos x$$

$$1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant 1$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

8.4.10

$$|1-\cos x| = 1-\cos x = 2\sin^2\frac{x}{2}$$

$$\leqslant 2\left(\frac{x}{2}\right)^2$$

$$0 \leqslant 1-\cos x \leqslant \frac{x^2}{2}$$

$$\lim_{x\to 0} 0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant \lim_{x\to 0}\frac{x^2}{2}$$

$$0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant 0$$

$$\lim_{x\to 0}(1-\cos x) = 0$$

$$\lim_{x\to 0}\cos x = 1$$

8.4.11

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x}&=\lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x}\\&=\lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x}\\&=1\end{aligned}$$

8.4.12

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}&=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}\\&=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\\&=1\end{aligned}$$

8.4.13

$$x=\sin t,\ t=\arcsin x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{x\to 0}\frac{t}{\sin t}=1$$

8.4.14

$$x=\tan t,\ t=\arctan x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{t\to 0}\frac{t}{\tan t}=1$$

8.4.15

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln e=1$$

8.4.16

$$e^x-1=t,\ x=\ln(t+1)$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{t\to 0}\frac{t}{\ln(t+1)}=1$$

8.4.17

$$\lim_{x\to 0}\frac{(1+x)^n-1}{nx}=\lim_{x\to 0}\left(\frac{e^{n\ln(1+x)}-1}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right)=1$$

8.4.19

$$
\begin{aligned}
x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m}\left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0\left(\frac{1}{n}\right)^0 + C_n^1\left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{n!}{m!\,(n-m)!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
x_{n+1} &= 1+1+\sum_{m=2}^{n+1}\frac{1}{m!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \qquad \text{单调增加} \\
x_n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{n^2} = 1+\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} \\
&< 1+\frac{1}{1-\frac{1}{2}} \\
&< 3 \qquad \text{有界}
\end{aligned}
$$

13.3 第 10章

10.1.1

$$
\begin{aligned}
f'(x_0) &= \lim_{\triangle x\to 0}\frac{\triangle\ y}{\triangle\ x} \qquad \text{因为极限存在与无穷小的关系} \\
\frac{\triangle\ y}{\triangle\ x} &= f'(x_0)+\alpha \qquad \alpha\text{为}\ \triangle\ x\to 0\text{时的无穷小} \\
\triangle\ y &= f'(x_0)\triangle\ x+\alpha\triangle\ x \\
\lim_{\triangle x\to 0}\triangle\ y &= \lim_{\triangle x\to 0}[f'(x_0)\triangle\ x+\alpha\triangle\ x] = 0 \\
\lim_{x\to x_0} f(x) = f(x_0) &\Leftrightarrow \lim_{\triangle x\to 0}[f(x_0+\triangle\ x)-f(x_0)=0] \Leftrightarrow \lim_{\triangle x\to 0}\triangle\ y = 0
\end{aligned}
$$

10.2.1

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}C &= \lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{C-C}{\triangle x}\\
&= 0
\end{aligned}
$$

10.2.2

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}x^a &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\
&= \frac{x^a-x_0^a}{x-x_0}\\
&= \frac{(x-x_0)\left(x^{a-1}+x^{a-2}x_0+\cdots+xx_0^{a-2}+x_0^{a-1}\right)}{x-x_0}\\
&= ax_0^{a-1}
\end{aligned}
$$

10.2.3

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}a^x &= \lim_{\triangle x\to 0}\frac{a^{x+\triangle x}-a^x}{\triangle x}\\
&= a^x\lim_{\triangle x\to 0}\frac{a^{\triangle x}-1}{\triangle x}\\
&= a^x\lim_{\triangle x\to 0}\frac{e^{\triangle x\ln a}-1}{\triangle x}\\
&= a^x\ln a
\end{aligned}
$$

10.2.4

$$
\frac{d}{\mathrm{d}x}e^x = e^x\ln e = e^x
$$

10.2.5

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\log_a^x &= \lim_{\triangle x\to 0}\frac{\log_a^{x+\triangle x}-\log_a^x}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\ln 1+\frac{\triangle x}{x}}{\ln a\triangle x}\\
&= \frac{1}{\ln a}\lim_{\triangle x\to 0}\frac{\frac{\triangle x}{x}}{\triangle x}\\
&= \frac{1}{x\ln a}
\end{aligned}
$$

10.2.6

$$\begin{aligned}\frac{d}{\mathrm{d}x}\ln^{x} &= \frac{1}{x\ln e}\\ &= \frac{1}{x}\end{aligned}$$

10.3.1

$$\begin{aligned}\frac{d}{\mathrm{d}x}\sin x &= \lim_{\triangle x\to 0}\frac{\sin(x_0+\triangle\ x)-\sin x_0}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\frac{2\cos(x_0+\frac{\triangle x}{2})\sin\frac{\triangle x}{2}}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\cos(x_0+\frac{\triangle\ x}{2})\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\\ &= cosx_0\end{aligned}$$

10.3.2

$$\begin{aligned}\frac{d}{\mathrm{d}x}\arcsin x = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\sin y}{\mathrm{d}y}}\\ &= \frac{1}{\cos y}\\ &= \frac{1}{\sqrt{1-\sin^2 y}}\\ &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

10.3.3

$$\begin{aligned}\frac{d}{\mathrm{d}x}\csc x = \frac{d\frac{1}{\sin x}}{\mathrm{d}x} &= \frac{\frac{d1}{\mathrm{d}x}\sin x-\frac{d\sin x}{\mathrm{d}x}\cdot 1}{\sin^2 x}\\ &= \frac{-\cos x}{\sin^2 x}\\ &= -\csc x\cdot\cot x\end{aligned}$$

10.3.4

$$\begin{aligned}\frac{d}{\mathrm{d}x}\cos x &= \lim_{\triangle x\to 0}\frac{\cos(x_0+\triangle\ x)-cos(x_0)}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\frac{-2\sin\left(x_0+\frac{\triangle x}{2}\right)\sin\frac{\triangle x}{2}}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}-\sin\left(x_0+\frac{\triangle\ x}{2}\right)\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\\ &= -\sin x\end{aligned}$$

10.3.5

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\arccos x = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cos y}{\mathrm{d}y}} \\
&= \frac{1}{-\sin y} \\
&= -\frac{1}{\sqrt{1-\cos^2 y}} \\
&= -\frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

10.3.6

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\sec x = \frac{d\frac{1}{\cos x}}{\mathrm{d}x} &= \frac{\frac{d1}{\mathrm{d}x}\cos x - \frac{d\cos x}{\mathrm{d}x}\cdot 1}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \sec x \cdot \tan x
\end{aligned}
$$

10.3.8

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\tan x = \frac{d\frac{\sin x}{\cos x}}{\mathrm{d}x} &= \frac{\frac{d\sin x}{\mathrm{d}x}\cos x - \frac{d\cos x}{\mathrm{d}x}\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

10.3.9

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\arctan x = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\tan y}{\mathrm{d}y}} \\
&= \frac{1}{\sec y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}
\end{aligned}
$$

10.3.10

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\cot x = \frac{d\frac{\cos x}{\sin x}}{\mathrm{d}x} &= \frac{\frac{d\cos x}{\mathrm{d}x}\sin x - \frac{d\sin x}{\mathrm{d}x}\cos x}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

10.3.11

$$\begin{aligned}\frac{d}{\mathrm{d}x}\operatorname{arccot}x &= \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\cot y}{\mathrm{d}y}}\\ &= \frac{1}{-\csc^2 y}\\ &= -\frac{1}{1+\cot^2 y}\\ &= -\frac{1}{1+x^2}\end{aligned}$$

10.3.12

$$\begin{aligned}\frac{d}{\mathrm{d}x}\sinh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x-e^{-x}}{2}\right)\\ &= \frac{e^x+e^{-x}}{2}\\ &= \cosh x\end{aligned}$$

10.3.13

$$\begin{aligned}\frac{d}{\mathrm{d}x}\cosh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x+e^{-x}}{2}\right)\\ &= \frac{e^x-e^{-x}}{2}\\ &= \sinh x\end{aligned}$$

10.3.14

$$\begin{aligned}\frac{d}{\mathrm{d}x}\tanh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x-e^{-x}}{e^x+e^{-x}}\right)\\ &= \frac{\frac{d(e^x-e^{-x})}{\mathrm{d}x}(e^x+e^{-x}) - \frac{d(e^x+e^{-x})}{\mathrm{d}x}(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}\\ &= \frac{2^2}{(e^x+e^{-x})^2}\\ &= \frac{1}{\cosh^2 x}\end{aligned}$$

10.3.15

$$\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arcsinh}x &= \frac{d}{\mathrm{d}x}\left[\ln(x+\sqrt{x^2+1})\right] \\
&= \frac{d\ln(x+\sqrt{x^2+1})}{\mathrm{d}\big(x+\sqrt{x^2+1}\big)}\cdot\frac{d\left(x+\sqrt{x^2+1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2+1}}\cdot\left(\frac{dx}{dx}+\frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)}\cdot\frac{d(x^2+1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

10.3.16

$$\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arccosh}x &= \frac{d}{\mathrm{d}x}\left[\ln(x+\sqrt{x^2-1})\right] \\
&= \frac{d\ln(x+\sqrt{x^2-1})}{\mathrm{d}\big(x+\sqrt{x^2-1}\big)}\cdot\frac{d\left(x+\sqrt{x^2-1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\left(\frac{dx}{dx}+\frac{d\left(\sqrt{x^2-1}\right)}{d\left(x^2-1\right)}\cdot\frac{d(x^2-1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{1}{2\sqrt{x^2-1}}\cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}$$

10.3.17

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arctanh} x &= \frac{d}{\mathrm{d}x}\left[\frac{1}{2}\ln(\frac{1+x}{1-x})\right] \\
&= \frac{1}{2}\cdot\frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)}\cdot\frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2}\cdot\frac{1}{\left(\frac{1+x}{1-x}\right)}\cdot\frac{\frac{d(1+x)}{\mathrm{d}x}(1-x)-\frac{d(1-x)}{\mathrm{d}x}(1+x)}{(1-x)^2} \\
&= \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)+(1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

10.4.1

$$
\begin{aligned}
\frac{d(CU)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{Cu(x+\triangle x)-Cu(x)}{\triangle x} \\
&= C\lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x} \\
&= C\frac{d(U)}{\mathrm{d}x}
\end{aligned}
$$

10.4.2

$$
\begin{aligned}
\frac{d(U\pm V)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)\pm v(x+\triangle x)-v(x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x}\pm\frac{v(x+\triangle x)-v(x)}{\triangle x} \\
&= \frac{d(U)}{\mathrm{d}x}\pm\frac{d(V)}{\mathrm{d}x}
\end{aligned}
$$

10.4.3

$$
\begin{aligned}
\frac{d(UV)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)v(x+\triangle x)-u(x)v(x)-u(x)v(x+\triangle x)+u(x)v(x+\triangle x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{\left[u(x+\triangle x)+u(x)\right]v(x+\triangle x)+u(x)\left[v(x+\triangle x)-v(x)\right]}{\triangle x} \\
&= \frac{dU}{\mathrm{d}x}\lim_{\triangle x\to 0}v(x+\triangle x)+u(x)\frac{dV}{\mathrm{d}x} \\
&= \frac{dU}{\mathrm{d}x}v(x)+u(x)\frac{dV}{\mathrm{d}x}
\end{aligned}
$$

10.5.1

$$\begin{aligned}\left[f^{-1}(y)\right]'|_{y=y_0} &= \lim_{y\to y_o}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\\ &= \lim_{y\to y_o}\frac{x-x_0}{y-y_0}\\ &= \lim_{x\to x_o}\frac{x-x_0}{f(x)-f(x_0)}\\ &= \lim_{x\to x_o}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\\ &= \frac{1}{f'(x)}\end{aligned}$$

10.6.1

$$\text{定义函数}\quad A(u)=\begin{cases}\frac{f(u)-f(u_0)}{u-u_0}. & u\neq u_0\\ f'(u_0), & u=u_0\end{cases}$$

$$A(u)\text{在}u_o\text{处连续, 既有}$$

$$\lim_{u\to u_o}A(u)=A(u_0)=f'(u_0)$$

$$\begin{aligned}\text{由恒等式}f(u)-f(u_0) &= A(u)(u-u_0)\text{我们有}\\ \frac{F(x)-F(x_0)}{x-x_0} &= \frac{f\left[g(x)\right]-f\left[g(x_0)\right]}{x-x_0}\\ &= A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\ \lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} &= \lim_{x\to x_0}A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\ F'(x_0) &= f'(g(x_0))g'(x_0)\end{aligned}$$