

争取未来开挖掘机

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历 (-。 -) !)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：<https://github.com/daidongchuixue/jiangping.git>

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，姜萍吧，姜圣的追随者，

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1 三角函数

1.1 三角恒等式

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \quad (1.1.1)$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \quad (1.1.2)$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \quad (1.1.3)$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \quad (1.1.4)$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.5)$$

$$\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.6)$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.7)$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.8)$$

1.1.2 积化和差

$$\cos(A) \sin(B) = \frac{1}{2} [\sin(A + B) - \sin(A - B)] \quad (1.1.9)$$

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \quad (1.1.10)$$

$$\sin(A) \sin(B) = \frac{1}{2} [\cos(A - B) - \cos(A + B)] \quad (1.1.11)$$

$$\cos(A) \cos(B) = \frac{1}{2} [\cos(A + B) + \cos(A - B)] \quad (1.1.12)$$

1.1.3 倍角公式

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x}$$

1.1.4 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \quad (1.1.13)$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \quad (1.1.14)$$

1.1.5 三角函数其他等式

$$\sin^2 x + \cos^2 x = 1 \quad (1.1.15)$$

$$1 + \tan^2 x = \sec^2 x \quad (1.1.16)$$

$$1 + \cot^2 x = \csc^2 x \quad (1.1.17)$$

1.2 双曲函数

1.2.1 定义

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}) \quad (1.2.1)$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad (1.2.2)$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (1.2.3)$$

1.2.3 恒等式

$$\sinh(2x) = 2 \sinh x \cosh x \quad (1.2.4)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad (1.2.5)$$

$$\cosh^2 x + \sinh^2 x = \cosh(2x) \quad (1.2.6)$$

$$\cosh x = 1 + 2 \sinh^2 \frac{x}{2} \quad (1.2.7)$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 + x_2 + \cdots + x_n} \quad (2.0.1)$$

$$|x + y| \leq |x| + |y| \quad (2.0.2)$$

$$\sin x \leq x \leq \tan x \quad (2.0.3)$$

伯努利不等式

$$(1 + x)^n \leq 1 + nx \quad (2.0.4)$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} \quad (3.1.1)$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \quad (3.1.2)$$

3.2 运算

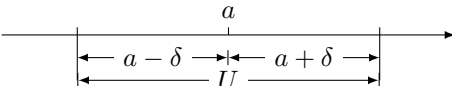
4 区间与映射

4.1 区间定义

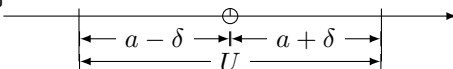
$$\text{区间定义} \begin{cases} (a, b) = \{x | a < x < b\} \\ [a, b] = \{x | a \leq x \leq b\} \\ (a, b] = \{x | a < x \leq b\} \\ (a, +\infty) = \{x | a < x\} \end{cases}$$

4.2 领域定义

点 a 的领域

$$U(a, \delta) = \begin{cases} \{x | a - \delta < x < a + \delta\} \\ \{x | |x - a| < \delta\} \end{cases}$$


点 a 的去心领域

$$\dot{U}(a, \delta) = \begin{cases} \{x | a - \delta < x < a + \delta \wedge x \neq a\} \\ \{x | 0 < |x - a| < \delta\} \end{cases}$$


点 a 的左领域 $(a - \delta, a)$

点 a 的右领域 $(a, a + \delta)$

4.3 映射定义

定义: X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x \in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

记作 $f : X \rightarrow Y$

$$f(x) = y \quad \begin{cases} \text{定义域 } (D_f) = X & x\text{-原像} \\ \text{值域 } (R_f) = \{f(x) | x \in X\} & y\text{-像} \end{cases}$$

$$\text{映射类型} \left\{ \begin{array}{ll}
 \text{满射:} & R_f = Y \\
 \text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\
 \text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射:} \begin{cases} f(x) = y \\ f^{-1}(y) = x \end{cases} \\
 \text{复合映射:} & g \circ f \Leftrightarrow g[f(x)] \begin{cases} f: X \rightarrow Y_1 \\ g: Y_2 \rightarrow Z \\ g \circ f: X \rightarrow Z \end{cases} \quad (Y_1 \subset Y_2)
 \end{array} \right.$$

5 函数与图像

5.1 函数的定义

设数集 $D \in R$ 的映射

$$f : D \rightarrow R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x) \{x \in D\}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f : D \rightarrow R \{D \subset R\} \begin{cases} \text{有界} \begin{cases} \text{有上界} \begin{cases} \exists k_1, \text{ 使 } f(x) \leq k_1, \forall x \in D \\ \text{有下界} \begin{cases} \exists k_1, \text{ 使 } f(x) \geq k_1, \forall x \in D \end{cases} \end{cases} \\ \text{无界} \begin{cases} \text{无上界} \begin{cases} \forall K_1, \exists x \in D \text{ 使, } f(x) \geq k_1 \\ \text{无下界} \begin{cases} \forall K_1, \exists x \in D \text{ 使, } f(x) \leq k_1 \end{cases} \end{cases} \end{cases} \end{cases}$$

5.2.2 函数的单调性

$$\text{单调增加 若 } \{x_1, x_2 \in D\} \ x_1 < x_2 \Rightarrow \begin{cases} f(x_1) < f(x_2) \text{ 称 } f(x) \text{ 在 } D \text{ 上单调增加} \\ f(x_1) > f(x_2) \text{ 称 } f(x) \text{ 在 } D \text{ 上单调减少} \\ f(x_1) \leq f(x_2) \text{ 称 } f(x) \text{ 在 } D \text{ 上单调非降} \\ f(x_1) \geq f(x_2) \text{ 称 } f(x) \text{ 在 } D \text{ 上单调非增} \end{cases}$$

5.2.3 函数的奇偶性

定义域

$$\forall x \in D \quad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

奇偶性运算

$$\text{奇函数} \times \text{奇函数} = \text{偶函数} \quad (5.2.1)$$

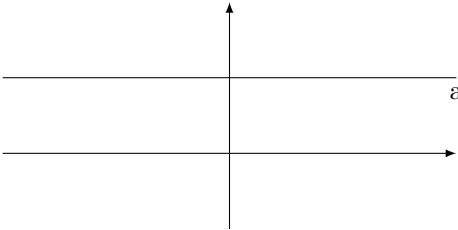
$$\text{奇函数} \times \text{偶函数} = \text{奇函数} \quad (5.2.2)$$

$$\text{偶函数} \times \text{偶函数} = \text{偶函数} \quad (5.2.3)$$

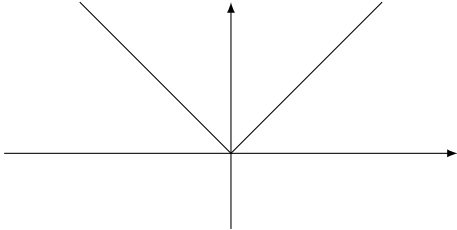
5.2.4 周期性

Def: $f(x + L) = f(x)\{L > 0\text{常数}, \forall x \in D\} \Rightarrow f(x)$ 为 L 的周期函数

5.3 函数图像

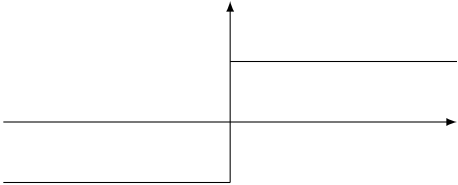


常函数 $f(x) = a\{a \in R\}$

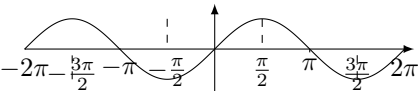


$f(x) = |x|$

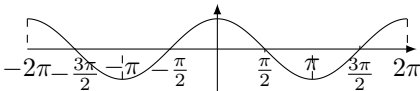
$$f(x) = \operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



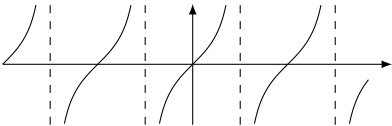
$$|x| = x \cdot \operatorname{sgn} x$$



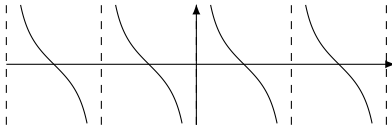
$\sin x$



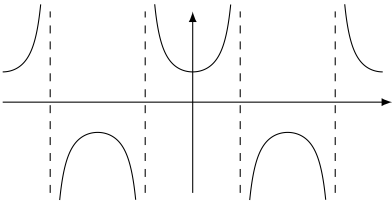
$\cos x$



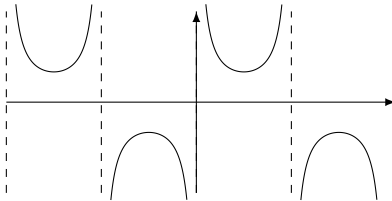
$\tan x$



$\cot x$



$\sec x$



$\csc x$

6 并集, 交集

6.1 定义

(\vee 或, \wedge 与)

$$A \cup B = \{x \in A \vee x \in B\}$$

$$A \cap B = \{x \in A \wedge x \in B\}$$

6.2 运算

$$\text{运算满足} \left\{ \begin{array}{l} \text{交换律} \left\{ \begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap A \end{array} \right. \\ \text{结合律} \left\{ \begin{array}{l} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{array} \right. \\ \text{分配律} \left\{ \begin{array}{l} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{array} \right. \\ \text{对偶律} \left\{ \begin{array}{l} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{array} \right. \end{array} \right.$$

$$A \cup A = A = A \cap A$$

$$A = B \Leftrightarrow A \subset B \wedge A \supset B$$

$$A \cup \emptyset = A \quad A \cap \emptyset = \emptyset$$

6.3 性质

性质 1.

$$A \subset (A \cup B) \quad A \supset (A \cap B) \quad (6.3.1)$$

性质 2.

$$A \cup B = B \Leftrightarrow A \subset B \quad (6.3.2)$$

性质 3.

$$A \cap B = A \Leftrightarrow A \subset B \quad (6.3.3)$$

性质 4. ($n \in \mathbb{N}$)

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \quad (6.3.4)$$

性质 5. ($n \in \mathbb{N}$)

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \quad (6.3.5)$$

6.4 *gustus De Morgan* 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

6.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha} \right)^C = \bigcap_{\alpha} (E_{\alpha}^C)$$

$$\left(\bigcap_{\alpha} E_{\alpha} \right)^C = \bigcup_{\alpha} (E_{\alpha}^C)$$

7 群，环，域

7.1 群

7.1.1 M1

7.1.2 M2

7.1.3 M3

7.1.4 M4

7.1.5 sdas

7.2 环

7.3 域

8 极限

8.1 数列极限

8.1.1 数列的定义

$$Def: \quad \{x_n\}, x_n = f(n), n \in N^+ \rightarrow R$$

8.1.2 数列极限的定义

$$Def: \quad \{x_n\}, n \in N^+, \exists a, \forall \varepsilon > 0, \exists N, n > N \Rightarrow |x_n - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} x_n = a$$

极限存在，为收敛，不存在为发散

8.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \quad (8.1.1)$$

8.1.4 有界数列

若 $\exists M > 0, \{M \in \text{正数}\}$
使得 $\forall n, |x_n| \leq M$
则称数列 $\{x_n\}$ 为有界数列

8.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \quad (8.1.2)$$

$$\text{单调有界数列必收敛} \quad (8.1.3)$$

8.1.6 收敛数列的保号性

$$\lim_{n \rightarrow \infty} x_n = a \text{ 存在, 且 } a > 0, \text{ 则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时 } \Leftrightarrow x_n > 0 \quad (8.1.4)$$

$$\lim_{n \rightarrow \infty} x_n = a, \lim_{n \rightarrow \infty} b_n = b, a < b, \exists N, n > N, a_n < b_n \quad (8.1.5)$$

8.1.7 收敛数列和子数列

$$\{x_n\}, \lim_{n \rightarrow \infty} x_n = a, \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim_{n \rightarrow \infty} x_{n_k} = a$$

证明 $K = N \quad k > K$

$$n_k > n_K \geq N$$

$$|x_{n_k} - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} x_{n_k} = a$$

8.2 函数极限

8.2.1 极限的定义

$$Def: \forall \varepsilon > 0 \left\{ \begin{array}{l} \exists X > 0 \left\{ \begin{array}{l} \text{当 } x > X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = A \\ \text{当 } x < -X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = A \\ \text{当 } |x| > X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = A \end{array} \right. \\ \exists \delta > 0 \left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = A \\ \text{当 } x_0 - \delta < x < x_0, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = A \\ \text{当 } 0 < |x - x_0| < \delta, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = A \end{array} \right. \end{array} \right.$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \rightarrow x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim_{x \rightarrow x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x \rightarrow x_0} f(x) \text{ 存在} \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) \quad (8.2.1)$$

图

8.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$$\lim_{x \rightarrow x_0} f(x) = A, \exists M > 0, \delta > 0 \text{ 使 } 0 < |x - x_0| < \delta, |f(x)| \leq M$$

3 保号性

$$\lim_{x \rightarrow x_0} f(x) = A, A > 0, \exists \delta > 0, \text{当 } 0 < |x - x_0| < \delta \Rightarrow f(x) > 0$$

$$f(x) > 0, \exists \delta > 0, \text{当 } 0 < |x - x_0| < \delta \Rightarrow \lim_{x \rightarrow x_0} f(x) = A, A > 0$$

4 保序性

$$f(x) \geq g(x), \lim f(x) = a, \lim g(x) = b, \text{则 } a \geq b$$

5 函数极限与数列极限的关系

如果 $\lim_{x \rightarrow x_0} f(x)$ 存在, $\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列, 则满足 $x_n \neq x_0$

$$\text{则 } \lim_{n \rightarrow \infty} f(x_n) = 0 = \lim_{x \rightarrow x_0} f(x), x_n \rightarrow x_0$$

8.3 无穷小与无穷大

8.3.1 无穷小定义

Def: 如果 $\lim_{x \rightarrow x_0} f(x) = 0$ 则称 $f(x)$ 为 $x \rightarrow x_0$ 时的无穷小

$$Def: \forall \varepsilon > 0 \left\{ \begin{array}{l} \exists X > 0 \left\{ \begin{array}{l} \text{当 } x > X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = 0 \\ \text{当 } x < -X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = 0 \\ \text{当 } |x| > X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = 0 \end{array} \right. \\ \exists \delta > 0 \left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta, \text{时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = 0 \\ \text{当 } x_0 - \delta < x < x_0, \text{时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = 0 \\ \text{当 } 0 < |x - x_0| < \delta, \text{时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = 0 \end{array} \right. \end{array} \right.$$

8.3.2 函数极限与无穷小的关系

$$\text{在自变量的同一变化中。}\alpha \text{ 为无穷小。}\lim f(x) = A \Leftrightarrow f(x) = A + \alpha \quad (8.3.1)$$

8.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$\text{如果 } f(x) \text{ 为无穷大, 则 } \frac{1}{f(x)} \text{ 为无穷小。} \quad (8.3.2)$$

$$\text{如果 } f(x) \text{ 为无穷小, 切 } f(x) \neq 0, \text{ 则 } \frac{1}{f(x)} \text{ 为无穷小。} \quad (8.3.3)$$

8.3.4 无穷大定义

$$\begin{aligned}
& \left. \begin{array}{l} \text{Def: } \forall M > 0 \end{array} \right\} \left\{ \begin{array}{l} \text{当 } x > X \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = \infty \end{array} \right. \\ \text{当 } x < -X \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = \infty \end{array} \right. \\ \text{当 } |x| > X \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = \infty \end{array} \right. \end{array} \right. \\
& \left. \begin{array}{l} \exists \delta > 0 \end{array} \right\} \left\{ \begin{array}{l} \text{当 } x_0 - \delta < x < x_0 \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = \infty \end{array} \right. \\ \text{当 } x_0 < x < x_0 + \delta \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \infty \end{array} \right. \\ \text{当 } 0 < |x - x_0| < \delta \left\{ \begin{array}{l} f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = \infty \end{array} \right. \end{array} \right. \\
& \lim_{x \rightarrow x_0} f(x) = \infty, \text{ 直线 } x = x_0 \text{ 是 } y = f(x) \text{ 垂直渐近线}
\end{aligned}$$

8.4 运算

8.4.1 有限个无穷小的和仍为无穷小

设 $\gamma = \alpha + \beta$

α 和 β 同为 $x \rightarrow x_0$ 时的无穷小

$\forall \varepsilon > 0, \exists \delta_1 > 0, \text{ 当 } 0 < |x - x_0| < \delta_1 \text{ 时, 有 } |\alpha| < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0, \exists \delta_2 > 0$, 当 $0 < |x - x_0| < \delta_2$ 时, 有 $|\beta| < \frac{\varepsilon}{2}$

$\delta = \min\{\delta_1, \delta_2\}$, 当 $0 < |x - x_0| < \delta$ 时

$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2$ 同时满足

即 $|\alpha| < \frac{\varepsilon}{2}, |\beta| < \frac{\varepsilon}{2}$ 同时成立

$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

8.4.2 有界函数与无穷小的乘积仍为无穷小

设 α 为 $x \rightarrow x_0$ 时的一个无穷小

$g(x)$ 为 x_0 的一个去心邻域 $\dot{U}(x_0, \delta_1)$ 有界

$f(x) = g(x)\alpha$

证 $f(x)$ 为 $x \rightarrow x_0$ 时的无穷小

因为 $g(x)$ 在 $\dot{U}(x_0, \delta_1)$ 有界

$\exists M > 0$, 当 $0 < |x - x_0| < \delta_1$ 时 $|g(x)| < M$

因为 α 是 $x \rightarrow x_0$ 的无穷小

$\exists \delta_2 > 0$ 当 $0 < |x - x_0| < \delta_2$ 时 $|\alpha| < \frac{\varepsilon}{M} < \varepsilon$

取 $\delta = \min\{\delta_1, \delta_2\}$ 当 $0 < |x - x_0| < \delta$ 时

$|g(x)| \geq M, |\alpha| < \frac{\varepsilon}{M}$ 同时成立

$|g(x)\alpha| = |g(x)| |\alpha| < M \frac{\varepsilon}{M} = \varepsilon$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

8.4.3 极限的四则运算

$\lim f(x) = A, \lim g(x) = B$

$$\lim (f(x) \pm g(x)) = \lim f(x) \pm \lim g(x) \quad (8.4.1)$$

$$\lim (f(x)g(x)) = \lim f(x) \lim g(x) \quad (8.4.2)$$

$$\lim \left(\frac{f(x)}{g(x)} \right) = \frac{\lim f(x)}{\lim g(x)} \quad (8.4.3)$$

$$\lim [Cf(x)] = C \lim f(x) \quad (8.4.4)$$

$$\lim [f(x)]^n = [\lim f(x)]^n \quad (8.4.5)$$

$$\lim_{x \rightarrow \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{a_0 x^n + a_1 x^{n-1} + \cdots + a_n} = \begin{cases} \frac{a}{b} & m = n \\ \infty & m > n \\ 0 & m < n \end{cases} \quad (8.4.6)$$

$$\begin{aligned} \lim_{x \rightarrow x_0} g(x) = u_0, \quad \lim_{u \rightarrow u_0} f(x) = A \\ \exists \delta_0 > 0, \quad x \in \overset{\circ}{U}(x_0, \delta_0), \quad g(x) \neq u_0 \\ \lim_{x \rightarrow x_0} f[g(x)] = \lim_{u \rightarrow u_0} f(u) = A \end{aligned} \quad (8.4.7)$$

8.4.4 夹逼定理 (三明治定理)

$$\begin{aligned} x_n \leq z_n \leq y_n \quad \forall n > N_0 \\ \text{若 } \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = a \text{ 则 } \lim_{n \rightarrow \infty} z_n = a \end{aligned} \quad (8.4.8)$$

8.4.5 重要极限

$$\begin{aligned} x \rightarrow x_0 \\ \lim_{x \rightarrow x_0} \sin x = \sin x_0 \end{aligned} \quad (8.4.9)$$

$$\lim_{x \rightarrow x_0} \cos x = \cos x_0 \quad (8.4.10)$$

$$\begin{aligned} x \rightarrow 0 \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \end{aligned} \quad (8.4.11)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (8.4.12)$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (8.4.13)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = 1 \quad (8.4.14)$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1 \quad (8.4.15)$$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{x} = 1 \quad (8.4.16)$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad (8.4.17)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad (8.4.18)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{nx} = 1 \quad (8.4.19)$$

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \quad (8.4.20)$$

$$x \rightarrow \infty$$

$$\{x_n\} \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad (8.4.21)$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad (8.4.22)$$

8.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

Def : α, β 是同一极限过程的无穷小。

- (1) 如果 $\lim \frac{\beta}{\alpha} = 0$ 则称 β 是 α 的高阶无穷小, 记作 $\beta = o(\alpha)$
- (2) 如果 $\lim \frac{\beta}{\alpha} = \infty$ 则称 β 是 α 的底阶无穷小。
- (3) 如果 $\lim \frac{\beta}{\alpha} = C$ 则称 β 是 α 的同阶无穷小。
- (4) 如果 $\lim \frac{\beta}{\alpha^k} = C, k > 0$ 则称 β 是 α 的 k 阶无穷小。
- (5) 如果 $\lim \frac{\beta}{\alpha} = 1$ 则称 β 是 α 的等价阶无穷小。

8.4.7 等价无穷小代换, 因子代换

β 与 α 是等价无穷小 $\Leftrightarrow \beta = \alpha + o(\alpha)$

设 $\alpha \sim \alpha', \beta \sim \beta'$, 且 $\lim \frac{\beta'}{\alpha'}$ 存在, 则 $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$

$\lim \alpha f(x) = \lim \alpha' f(x)$

$\lim \frac{f(x)}{\alpha} = \lim \frac{f(x)}{\alpha'}$

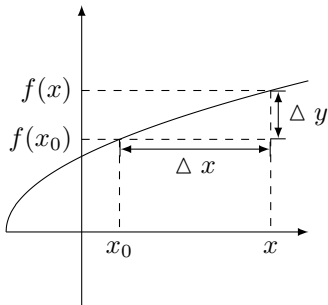
9 连续与间断点

9.1 定义

9.1.1 点连续

Def1: 设 $f(x)$ 在 x_0 的某邻域内有定义, 如果 $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

则称 $f(x)$ 在 x_0 处连续



$$\begin{cases} \Delta x = x - x_0 \\ \Delta y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \Delta x) - f(x_0) \end{cases} \end{cases}$$

Def2: 如果 $\lim_{\Delta x \rightarrow 0} \Delta y = \lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0)] = 0$

则称 $f(x)$ 在 x_0 处连续

9.1.2 区间连续

$$\forall x_0 \in [a, b] \begin{cases} \lim_{x \rightarrow x_0} f(x) = f(x_0) & x_0 \in (a, b) \Leftrightarrow f(x_0) = \begin{cases} \lim_{x \rightarrow x_0^-} f(x) = f(x_0^-) \\ \lim_{x \rightarrow x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim_{x \rightarrow x_0^+} f(x) = f(x_0^+) & x_0 = a \text{ (右连续)} \\ \lim_{x \rightarrow x_0^-} f(x) = f(x_0^-) & x_0 = b \text{ (左连续)} \end{cases}$$

称在 $[a, b]$ 内连续

有界: $\exists M > 0, x \in [a, b]$ 时, $|f(x)| \geq M$

最大值: $\exists x_0 \in [a, b]$ 时, $\forall x \in [a, b], f(x) \leq f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a, b]$ 上的最大值

最小值: $\exists x_0 \in [a, b]$ 时, $\forall x \in [a, b], f(x) \geq f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a, b]$ 上的最小值

1, 闭区间 $[a, b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

零点定理

2, 设 $f(x)$ 在 $[a, b]$ 上连续, 且 $f(a) \cdot f(b) < 0$

则至少存在一点 $\xi \in (a, b)$ 使 $f(\xi) = 0$

介值定理

设 $f(x)$ 在 $[a, b]$ 上连续, 且 $f(a) = A, f(b) = B$

$\forall C \in (A, B)$, 至少有一点 $\xi, f(\xi) = C$

9.1.3 间断点

1, $f(x)$ 无定义

2, $\lim_{x \rightarrow x_0} f(x)$ 不存在

3. $\lim_{x \rightarrow x_0} f(x)$ 存在, 但 $\lim_{x \rightarrow x_0} f(x) \neq f(x_0)$

第一类间断点: $f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x)$ 与 $f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x)$

第二类间断点: 不是第一类的。

9.2 连续函数的运算

函数 $f(x), g(x)$ 在 $x = x_0$ 连续。

$$\lim_{x \rightarrow x_0} [f(x) \pm g(x)] = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = f(x_0) \pm g(x_0)$$

$$\lim_{x \rightarrow x_0} [f(x) \cdot g(x)] = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = f(x_0) \cdot g(x_0)$$

$$\lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{f(x_0)}{g(x_0)} \quad (g(x_0) \neq 0)$$

反函数的连续性

若 $y = f(x)$ 在区间 I_x 上单调增加, 且连续。

则 $y = f^{-1}(x)$ 在 $I_y = \{y | y = f(x), x \in I_x\}$ 上也为单调增加, 连续

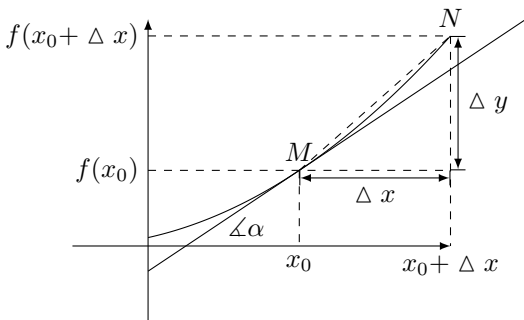
$$\text{复合函数,} \left\{ \begin{array}{l} \text{内外都连续} \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} g(x) = g(x_0) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow x_0} f[g(x)] = f[g(x_0)] = f(\lim_{x \rightarrow x_0} g(x)) \end{array} \right. \\ \\ \text{外连续} \left\{ \begin{array}{l} x \rightarrow x_0 \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow x_0} f[g(x)] = f(u_0) = f(\lim_{x \rightarrow x_0} g(x)) \end{array} \right. \\ \\ x \rightarrow \infty \left\{ \begin{array}{l} \lim_{x \rightarrow \infty} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow \infty} f[g(x)] = f(u_0) = f(\lim_{x \rightarrow \infty} g(x)) \end{array} \right. \end{array} \right.
 \end{array} \right.$$

10 导数

10.1 定义

导数的概念从物理发展出来的。

$$v(t_0) = \lim_{\Delta t \rightarrow 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$



$$NM \text{斜率} = \tan \beta = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

$$\text{斜率 } k = \tan \alpha = \lim_{\Delta x \rightarrow 0} \tan \beta = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

10.1.1 导数定义

$y = f(x)$ 在 x_0 的某邻域内有定义

给自变量的增量 Δx , $(x_0 + \Delta x)$ 仍在定义域内

函数得到了相应增量 Δy , $\Delta y = f(x_0 + \Delta x) - f(x_0)$

如果 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 存在, 称 $y = f(x)$ 在 $x = x_0$ 处可导
(极限值为 $y = f(x)$ 在 $x = x_0$ 处导数)

$$\text{记 } y'|_{x=x_0} = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \Leftrightarrow \lim_{\Delta x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

10.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

称 $f'(x)$ 为 $y = f(x)$ 在区间 I 上的导函数

10.1.3 闭区间可导定义

$$f(x) \text{ 在 } [a, b] \text{ 可导} \Leftrightarrow \begin{cases} f'(x_0) & x_0 \in (a, b) \\ f'_+(a) & x = a \\ f'_-(b) & x = b \end{cases} \Leftrightarrow \begin{cases} \text{左导数 } f'_-(x_0) = \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ \text{右导数 } f'_+(x_0) = \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \end{cases}$$

10.1.4 导数与连续

$$f'(x) \text{ 存在} \Rightarrow f(x) \text{ 在 } x = x_0 \text{ 处连续} \quad (10.1.1)$$

10.2 幂数, 指数, 对数

$$(C)' = 0 \quad (10.2.1)$$

$$(x^a)' = ax^{a-1} \quad (10.2.2)$$

$$(a^x)' = a^x \ln a \quad (10.2.3)$$

$$(e^x)' = e^x \quad (10.2.4)$$

$$(\log_a^x)' = \frac{1}{x \ln a} \quad (10.2.5)$$

$$(\ln x)' = \frac{1}{x} \quad (10.2.6)$$

10.3 三角函数

$$(\sin x)' = \cos x \quad (10.3.1)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad (10.3.2)$$

$$(\csc x)' = -\csc x \cot x \quad (10.3.3)$$

$$(\cos x)' = -\sin x \quad (10.3.4)$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad (10.3.5)$$

$$(\sec x)' = \sec x \tan x \quad (10.3.6)$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \quad (10.3.7)$$

$$(\tan x)' = \sec^2 x \quad (10.3.8)$$

$$(\arctan)' = \frac{1}{1+x^2} \quad (10.3.9)$$

$$(\cot x)' = -\csc^2 x \quad (10.3.10)$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2} \quad (10.3.11)$$

$$(\sinh x)' = \cosh x \quad (10.3.12)$$

$$(\cosh x)' = \sinh x \quad (10.3.13)$$

$$(\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x \quad (10.3.14)$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}} \quad (10.3.15)$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}} \quad (10.3.16)$$

$$(\operatorname{artanh} x)' = \frac{1}{1-x^2} \quad (10.3.17)$$

10.4 导数运算

$u = u(x), v = v(x)$, 均在 x 点可导, C 为常数

$$(Cu(x))' = Cu'(x) \quad (10.4.1)$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \quad (10.4.2)$$

$$(u(x) \cdot v(x))' = u'(x)v(x) + v'(x)u(x) \quad (10.4.3)$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2} \quad (10.4.4)$$

10.5 反函数求导

如果函数 $y = f(x)$ 在区间 (a, b) 内单调可导, 且 $f'(y) \neq 0$

$$\begin{cases} \alpha = \min\{f(a) + 0, f(b - 0)\} \\ \beta = \max\{f(a) + 0, f(b - 0)\} \end{cases}$$

则它的反函数 $x = f^{-1}(y)$ 在区间 (α, β) 内也可导

$$[f^{-1}(y)]' = \frac{1}{f'(x)} \Leftrightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad (10.5.1)$$

10.6 复合函数求导

设函数 $\begin{cases} y = f(u) \text{ 在 } U(u_0, \delta_0) \text{ 处有定义} \\ u = g(x) \text{ 在 } U(x_0, \eta_0) \text{ 处有定义} \end{cases}$
 $u_0 = g(x_0)$, 且 $f'(u)$ 和 $g'(x)$ 都存在
 则复合函数 $F(x) = f[g(x)]$ 在点 x_0 可导, 且

$$F'(x_0) = f'[g(x_0)] g'(x_0) \Leftrightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (10.6.1)$$

10.7 高阶求导

$$Def: \begin{cases} \text{一阶导数} & y' \Leftrightarrow \frac{dy}{dx} \\ \text{二阶导数} & y'' \Leftrightarrow \frac{d^2 y}{dx^2} \\ \text{三阶导数} & y''' \Leftrightarrow \frac{d^3 y}{dx^3} \\ \text{三阶以上 } n \text{ 阶导数} & y^{(n)} \Leftrightarrow \frac{d^n y}{dx^n} \end{cases}$$

10.8 高阶求导公式

$$(e^x)^{(n)} = e^x \quad (10.8.1)$$

$$(a^x)^{(n)} = a^x (\ln a)^n \quad (10.8.2)$$

$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \quad (10.8.3)$$

$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \quad (10.8.4)$$

$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1} (n-1)!}{(x+a)^n} \quad (10.8.5)$$

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right) \quad (10.8.6)$$

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right) \quad (10.8.7)$$

$$[f(ax+b)]^{(n)} = a^n \cdot f^{(n)}(ax+b) \quad (10.8.8)$$

10.9 高阶求导运算法则

$$(u(x) \pm v(x))^{(n)} = u^{(n)}(x) \pm v^{(n)}(x) \quad (10.9.1)$$

$$\text{莱布紫泥公式} \quad (uv)^{(n)} = \sum_{k=0}^n C_n^k u^{(n-k)} \cdot v^{(k)} \quad (10.9.2)$$

10.10 隐函数求导

$$F(x, y) = 0, y = f(x)$$

$$F(x, f(x)) \equiv 0 \quad \text{可以同时对面求导}$$

10.11 参数方程求导

$$x = x(t), y = y(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d \frac{dy}{dx}}{dx} = \frac{d}{dt} \left(\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \right) \cdot \frac{dt}{dx} = \frac{\frac{d^2 y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2 x}{dt^2} \cdot \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{dt}{dx} = \frac{\frac{d^2 y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2 x}{dt^2} \cdot \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

11 微分

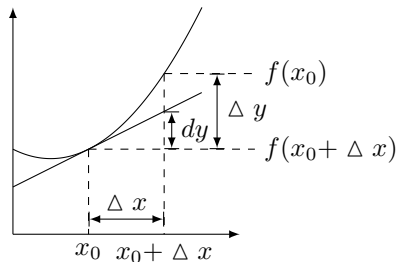
11.1 定义

设函数 $f(x)$ 在点 x_0 的一个邻域内有定义。 $\Delta y = f(x_0 + \Delta x) - f(x_0)$

如果 Δy 可以表示为 $\Delta y = A \Delta x + o(\Delta x)$ 其中 A 为与 Δx 无关的常数

则称 $f(x)$ 在点 x_0 可微, $A \Delta x$ 称为 $f(x)$ 在点 x_0 处的微分。

记作: $dy = A \Delta x$



$$\text{可微} \Rightarrow \text{可导} \quad (11.1.1)$$

$$\text{可导} \Rightarrow \text{可微} \quad (11.1.2)$$

11.2 微分法则

11.2.1 核心根本

$$dy = \overset{\text{积分}}{\overbrace{f'(x) \, d \, x}^{\text{求导}}}$$

11.2.2 四则运算

$$d(u \pm v) = du \pm dv \quad (11.2.1)$$

$$d(uv) = vdu + u dv \quad (11.2.2)$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + u dv}{v^2} \quad (11.2.3)$$

11.2.3 复合运算

$$\text{可微} \begin{cases} y = f(u) \\ u = g(x) \end{cases} \Rightarrow \begin{cases} dy = f'(u)du \\ du = g'(x)dx \end{cases} \quad \text{则 } y = f(g(x)) \text{ 也可微}$$

且 $dy = f'(u)du = f'(u)g'(x)dx$

u 是否为中间变量都成立, 微分的不变性。

11.2.4 近似计算公式

$$\Delta x \rightarrow 0, dy \approx \Delta y \begin{cases} dy = f'(x_0) \Delta x \\ \Delta y = f(x_0 + \Delta x) - f(x_0) \end{cases} \left\{ \begin{array}{l} f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ \left. \begin{array}{l} f(x) \approx f(0) + f'(0)x \\ \sqrt{n} \approx 1 + \frac{1}{n}x \\ \sin x \approx x \\ \tan x \approx x \\ e^x \approx 1 + x \\ \ln(1 + n) \approx x \end{array} \right\} \right. \\ x_0 = 0 \end{array} \right.$$

11.2.5 奇偶函数导数

偶函数导数为奇函数 $f(x) = f(-x) \Leftrightarrow f'(x) = -f'(-x)$

奇函数导数为偶函数 $f(x) = -f(-x) \Leftrightarrow f'(x) = f'(-x)$

11.2.6 区间恒为 0

若 $f'(x)$ 在区间恒为零, 则 $f(x)$ 在区间 I 上为一常数

设 x_1, x_2 为区间 I 内任意两点 $x_1 < x_2$

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \equiv 0$$

$$f(x_2) \equiv f(x_1) = C$$

11.3 中值定理

11.3.1 费马引理

$$f(x) \quad \forall x \in \dot{U}(x_0) \begin{cases} f(x) \leq f(x_0) & f(x) \text{ 在 } x_0 \text{ 处取极大值} \\ f(x) \geq f(x_0) & f(x) \text{ 在 } x_0 \text{ 处取极小值} \end{cases}$$

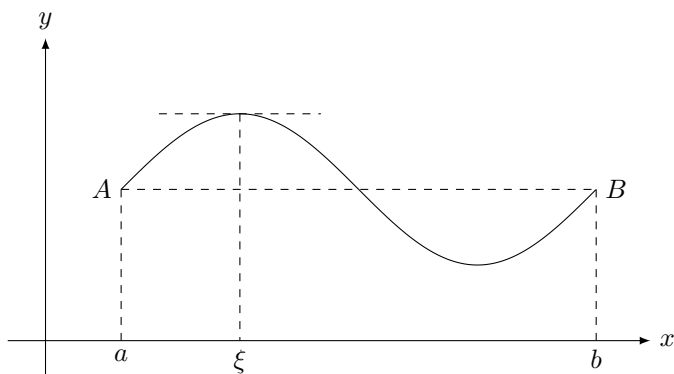
如果可导函数 $y = f(x)$ 在 x_0 取极值, 则 $f'(x_0) = 0$ (11.3.1)

11.3.2 罗尔定理

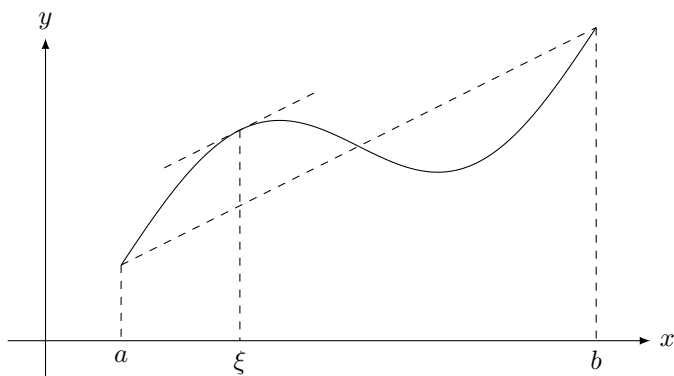
如果函数 $f(x)$ 满

$$\left\{ \begin{array}{l} \text{在闭区间 } [a, b] \text{ 上连续} \\ \text{在开区间 } (a, b) \text{ 可导} \\ f(a) = f(b) \end{array} \right.$$

则至少有一点 $\xi \in (a, b)$, $f'(\xi) = 0$ (11.3.2)



11.3.3 拉格朗日定理 (微分中值定理)



如果函数 $f(x)$ 满

$$\left\{ \begin{array}{l} \text{在闭区间 } [a, b] \text{ 上连续} \\ \text{在开区间 } (a, b) \text{ 可导} \end{array} \right.$$

则至少有一点 $\xi \in (a, b)$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow f(b) - f(a) = f'(\xi)(b - a) \quad (11.3.3)$$

在区间 $[x, x + \Delta x]$ 用拉格朗日定理。

$$f(x + \Delta x) - f(x) = f'(\xi) \Delta x$$

$$\xi \in (x, x + \Delta x) \text{ 记作: } \xi = x + \theta \Delta x \quad 0 < \theta < 1$$

$$f(x + \Delta x) - f(x) = f(x + \theta \Delta x) \Delta x$$

$$\Delta y = f(x + \theta \Delta x) \Delta x$$

11.3.4 柯西定理

12 积分

12.1 幂数, 指数, 对数

$$\int x^a dx = \frac{1}{a-1} x^{a-1} + C \quad (12.1.1)$$

$$\int b^x dx = \frac{b^x}{\ln b} + C \quad (12.1.2)$$

$$\int e^x dx = e^x + C \quad (12.1.3)$$

$$\int \frac{1}{x} dx = \ln x + C \quad (12.1.4)$$

12.2 三角函数

$$\int \sin x dx = -\cos x + C \quad (12.2.1)$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C \quad (12.2.2)$$

$$\int \csc x \cot x dx = -\csc x + C \quad (12.2.3)$$

$$\int \cos x dx = \sin x + C \quad (12.2.4)$$

$$\int \sec x \tan x dx = \sec x + C \quad (12.2.5)$$

$$\int \sec^2 x dx = \tan x + C \quad (12.2.6)$$

$$\int \csc^2 x dx = -\cot x + C \quad (12.2.7)$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C \quad (12.2.8)$$

$$\int \frac{1}{1+x^2} dx = \tan x + C \quad (12.2.9)$$

$$\int \sinh x dx = \cosh x + C \quad (12.2.10)$$

$$\int \cosh x dx = \sinh x + C \quad (12.2.11)$$

12.3 积分运算

13 零散的一些

$$\sum_{k=0}^n q^k = \frac{1 - q^{n+1}}{1 - q} \quad (13.0.1)$$

$$\begin{aligned} A_N &= \sum_{k=0}^n q^k & qA_N &= \sum_{k=1}^{n+1} q^k \\ A_N - qA_N &= \sum_{k=0}^n q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1} \\ A_N &= \frac{1 - q^{n+1}}{1 - q} \end{aligned}$$

$$\log_{10} x = \lg_x \quad (13.0.2)$$

$$\log_e x = \ln_x \quad (13.0.3)$$

$$\log_b xy = \log_b x + \log_b y \quad (13.0.4)$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \quad (13.0.5)$$

$$\log_b x^n = n \log_b x \quad (13.0.6)$$

$$\log_b x = \frac{\log_c x}{\log_c b} \quad (13.0.7)$$

$$b^n = x \quad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \quad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = \log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^n C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^n - b^n = (a-b) \sum_{m=0}^{n-1} (a^{n-m} b^m) = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

14 证明

14.1 第 1 章

1.2.4

$$\begin{aligned}
 \sinh x \cosh x &= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \left(\frac{1}{2} \right) \left(\frac{e^{2x} - e^{-2x}}{2} \right) \\
 &= \frac{1}{2} \sinh(2x) \\
 \sinh(2x) &= 2 \sinh x \cosh x
 \end{aligned}$$

1.2.5

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right) \\
 &= e^x \times e^{-x} \\
 &= 1
 \end{aligned}$$

1.2.6

$$\begin{aligned}
 \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 + \left(\frac{e^x - e^{-x}}{2} \right)^2 \\
 &= \frac{2e^{2x} + 2e^{-2x}}{4} \\
 &= \frac{e^{2x} + e^{-2x}}{2} \\
 &= \cosh(2x)
 \end{aligned}$$

1.2.7

$$\begin{aligned}
 \cosh(2x) &= \cosh^2 x + \sinh^2 x \\
 &= \sinh^2 x + 1 + \sinh^2 x \\
 &= 2 \sinh^2 x + 1 \\
 \cosh x &= 2 \sinh^2 \frac{x}{2} + 1
 \end{aligned}$$

14.2 第 8 章

8.1.1

反设 $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} x_n = b$, 且 $a < b$

$$\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1, n > N_1, |x_n - a| < \frac{b-a}{3} \\ \exists N_2, n > N_2, |x_n - b| < \frac{b-a}{3} \end{cases}$$

$$N = \max\{N_1, N_2\}, n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}$$

$$\begin{aligned} b-a &= |(x_n - a) - (x_n - b)| \\ &\leq |x_n - a| + |x_n - b| \\ &< \frac{b-a}{3} + \frac{b-a}{3} \\ &< \frac{2(b-a)}{3} \end{aligned}$$

8.1.2

$\varepsilon = 1$, $\exists N > 0$, 当 $n > N$ 时 $|X_n - a| < 1$

$$\begin{aligned} |X_n| &= |(X_n - a) + a| \\ &\leq |x_n - a| + |a| \\ &\leq 1 + |a| \end{aligned}$$

$$\begin{aligned} M &= \max\{|X_n|, |X_2|, \dots, |X_n|, 1 + |a|\} \\ &\forall n, |X_n| \leq M \end{aligned}$$

8.1.4

1

由于 $\lim_{n \rightarrow \infty} x_n = a$, 且 $a > 0$

$\varepsilon = \frac{a}{2}$, $\exists N > 0, n > N$

$$|x_n - a| < \varepsilon$$

$$|x_n - a| < \frac{a}{2}$$

$$-\frac{a}{2} < x_n - a < \frac{a}{2}$$

$$\frac{a}{2} < x_n < 1$$

2

用反证法, 反设 $a < 0$. 从某项起 $x_n < 0$ 矛盾

8.1.5

$$\begin{aligned}x_n &= b_n - a_n \\ \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n \\ \lim_{n \rightarrow \infty} x_n &= b - a > 0 \\ \lim_{n \rightarrow \infty} x_n &> 0 \\ b_n - a_n &= x_n > 0 \\ b_n &> a_n\end{aligned}$$

$$8.2.1 \quad \lim_{x \rightarrow x_0} f(x) \text{ 存在} \Rightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

$$\begin{aligned}&\text{设 } \lim_{x \rightarrow x_0} f(x) = A \\&0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon \\&0 < |x - x_0| < \delta \Leftrightarrow x \in \mathring{U}(x_0, \delta) \\&\left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta \text{ 时 } 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon, \lim_{x \rightarrow x_0^+} f(x) = A \\ \text{当 } x_0 - \delta < x < x_0 \text{ 时 } 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon, \lim_{x \rightarrow x_0^-} f(x) = A \end{array} \right. \\&\lim_{x \rightarrow x_0^+} f(x) = A = \lim_{x \rightarrow x_0^-} f(x)\end{aligned}$$

$$\lim_{x \rightarrow x_0} f(x) \text{ 存在} \Leftarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

$$\begin{aligned}A &= \left\{ \begin{array}{l} \lim_{x \rightarrow x_0^+} f(x), \forall \varepsilon > 0, \exists \delta_1 > 0, x_0 < x < x_0 + \delta_1, |f(x) - A| < \varepsilon \\ \lim_{x \rightarrow x_0^-} f(x), \forall \varepsilon > 0, \exists \delta_2 > 0, x_0 - \delta_2 < x < x_0, |f(x) - A| < \varepsilon \end{array} \right. \\&\delta = \min\{\delta_1, \delta_2\} \\&0 < |x - x_0| < \delta \left\{ \begin{array}{l} x > x_0, x_0 < x < x_0 + \delta \leq x_0 + \delta_1, |f(x) - A| < \varepsilon \\ x < x_0, x_0 - \delta_2 \leq x_0 + \delta < x < x_0, |f(x) - A| < \varepsilon \end{array} \right. \\&\lim_{x \rightarrow x_0} f(x) = A\end{aligned}$$

8.3.1

$$\begin{aligned}\lim_{x \rightarrow x_0} f(x) = A &\Rightarrow \left\{ \begin{array}{l} \alpha \text{ 为 } x \rightarrow x_0 \text{ 时的无穷小} \\ f(x) = \alpha + A \end{array} \right. \\&\text{设 } \lim_{x \rightarrow x_0} f(x) = A, \text{ 记 } f(x) - A = \alpha \\&\text{只需证 } \alpha \text{ 为无穷小}.\end{aligned}$$

$$\forall \varepsilon > 0, \exists \delta > 0, \text{当 } 0 < |x - x_0| < \delta, \text{时 } |f(x) - A| < \varepsilon$$

$$\text{即 } |\alpha - 0| < \varepsilon$$

α 为 $x \rightarrow x_0$ 时的无穷小

$$\lim_{x \rightarrow x_0} f(x) = A \Leftarrow \begin{cases} \alpha \text{ 为 } x \rightarrow x_0 \text{ 时的无穷小} \\ f(x) = \alpha + A \end{cases}$$

$$\forall \varepsilon > 0, \exists \delta > 0, \text{当 } 0 < |x - x_0| < \delta, |\alpha| < \varepsilon$$

$$\text{即 } |f(x) - A| < \varepsilon \quad \lim_{x \rightarrow x_0} f(x) = A$$

8.3.2

$$\text{设 } \lim_{x \rightarrow x_0} f(x) = \infty$$

对 $f(x)$ 为 $x \rightarrow$ 时无穷大

对于 $M = \frac{1}{\varepsilon}$. 存在 $\delta > 0$

当 $0 < |x - x_0| < \delta$ 时

$$|f(x)| > M = \frac{1}{\varepsilon}$$

$$\left| \frac{1}{f(x)} \right| < \varepsilon$$

$\frac{1}{f(x)}$ 为 $x \rightarrow x_0$ 时的无穷小

8.4.2

$$f(x)g(x) = [A + \alpha][B + \beta]$$

$$= AB + A\beta + B\alpha + \beta\alpha$$

$$= AB + \gamma \quad (\gamma \text{ 为无穷小})$$

$$\lim [f(x)g(x)] = AB + \gamma = \lim f(x) \lim g(x)$$

8.4.8

$$\forall \varepsilon > 0$$

$$|x_n - a| < \varepsilon \quad \forall n > N_1$$

$$|y_n - a| < \varepsilon \quad \forall n > N_2$$

令 $N = \max \{N_1, N_2, N_0\}$, 则当 $n > N$ 时有

$$a - \varepsilon < x_n \leq z_n \leq y_n < a + \varepsilon$$

$$|z_n - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} z_n = a$$

8.4.9

$$\begin{aligned}
 |f(x) - \sin x_0| &= |\sin x - \sin x_0| \\
 &= \left| 2 \cos\left(\frac{x+x_0}{2}\right) \sin\left(\frac{x-x_0}{2}\right) \right| \\
 &\leq 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \\
 &\leq 2 \frac{|x-x_0|}{2} = |x-x_0|
 \end{aligned}$$

$\forall \varepsilon, \exists \delta = \varepsilon$, 当 $0 < |x - x_0| < \delta$ 时

$$|\sin x - \sin x_0| \leq |x - x_0| < \varepsilon$$

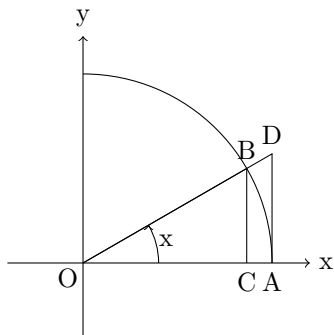
8.4.10

$$\begin{aligned}
 |f(x) - \cos x_0| &= |\cos x - \cos x_0| \\
 &= \left| -2 \sin\left(\frac{x+x_0}{2}\right) \sin\left(\frac{x-x_0}{2}\right) \right| \\
 &\leq 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \\
 &\leq 2 \frac{|x-x_0|}{2} = |x-x_0|
 \end{aligned}$$

$\forall \varepsilon, \exists \delta = \varepsilon$, 当 $0 < |x - x_0| < \delta$ 时

$$|\cos x - \cos x_0| \leq |x - x_0| < \varepsilon$$

8.4.11



$$OB = OA = 1$$

$$\triangle AOB \leq \text{扇形面积} \leq \triangle AOD$$

$$\frac{1}{2} \sin x \leq \frac{1}{2} x \leq \frac{1}{2} \tan x$$

$$\sin x \leq x \leq \tan x$$

$$1 \geq \frac{\sin x}{x} \geq \cos x$$

$$\lim_{x \rightarrow 0} 1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq \lim_{x \rightarrow 0} \cos x$$

$$1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

8.4.12

$$|1 - \cos x| = 1 - \cos x = 2 \sin^2 \frac{x}{2} \\ \leq 2 \left(\frac{x}{2} \right)^2$$

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

$$\lim_{x \rightarrow 0} 0 \leq \lim_{x \rightarrow 0} (1 - \cos x) \leq \lim_{x \rightarrow 0} \frac{x^2}{2}$$

$$0 \leq \lim_{x \rightarrow 0} (1 - \cos x) \leq 0$$

$$\lim_{x \rightarrow 0} (1 - \cos x) = 0$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

8.4.13

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{1}{\cos x} \\ = \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{1}{\cos x} \\ = 1$$

8.4.14

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\frac{1}{2}x^2} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \\
 &= 1
 \end{aligned}$$

8.4.15

$$\begin{aligned}
 x &= \sin t, \quad t = \arcsin x \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{\arcsin x}{x} &= \lim_{x \rightarrow 0} \frac{t}{\sin t} = 1
 \end{aligned}$$

8.4.16

$$\begin{aligned}
 x &= \tan t, \quad t = \arctan x \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{\arctan x}{x} &= \lim_{t \rightarrow 0} \frac{t}{\tan t} = 1
 \end{aligned}$$

8.4.17

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}} = \ln e = 1$$

8.4.18

$$\begin{aligned}
 e^x - 1 &= t, \quad x = \ln(t+1) \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{t \rightarrow 0} \frac{t}{\ln(t+1)} = 1
 \end{aligned}$$

8.4.19

$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{nx} = \lim_{x \rightarrow 0} \left(\frac{e^{n \ln(1+x)} - 1}{n \ln(1+x)} \cdot \frac{\ln(1+x)}{x} \right) = 1$$

8.4.21

$$\begin{aligned}
x_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{m=0}^n C_n^m 1^{n-m} \left(\frac{1}{n}\right)^m = \sum_{m=0}^n C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0 \left(\frac{1}{n}\right)^0 + C_n^1 \left(\frac{1}{n}\right)^1 + \sum_{m=2}^n C_n^m \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{n!}{m! (n-m)!} \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{\overbrace{(n)(n-1)\cdots(n-m+1)}^m}{m!} \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{1}{m!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-m+1}{n}\right) \\
&= 1 + 1 + \sum_{m=2}^n \frac{1}{m!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) \\
x_{n+1} &= 1 + 1 + \sum_{m=2}^{n+1} \frac{1}{m!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \quad \text{单调增加} \\
x_n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{n^2} = 1 + \frac{1 - \left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&< 3 \quad \text{有界}
\end{aligned}$$

14.3 第 10 章

10.1.1

$$\begin{aligned}
f'(x_0) &= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \text{因为极限存在与无穷小的关系} \\
\frac{\Delta y}{\Delta x} &= f'(x_0) + \alpha \quad \alpha \text{ 为 } \Delta x \rightarrow 0 \text{ 时的无穷小} \\
\Delta y &= f'(x_0) \Delta x + \alpha \Delta x \\
\lim_{\Delta x \rightarrow 0} \Delta y &= \lim_{\Delta x \rightarrow 0} [f'(x_0) \Delta x + \alpha \Delta x] = 0 \\
\lim_{x \rightarrow x_0} f(x) &= f(x_0) \Leftrightarrow \lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0) = 0] \Leftrightarrow \lim_{\Delta x \rightarrow 0} \Delta y = 0
\end{aligned}$$

10.2.1

$$\begin{aligned}
 (C)' &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{C - C}{\Delta x} \\
 &= 0
 \end{aligned}$$

10.2.2

$$\begin{aligned}
 (x^a)' &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
 &= \frac{x^a - x_0^a}{x - x_0} \\
 &= \frac{(x - x_0)(x^{a-1} + x^{a-2}x_0 + \cdots + xx_0^{a-2} + x_0^{a-1})}{x - x_0} \\
 &= ax_0^{a-1}
 \end{aligned}$$

10.2.3

$$\begin{aligned}
 (a^x)' &= \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \\
 &= a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} \\
 &= a^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x \ln a} - 1}{\Delta x} \\
 &= a^x \ln a
 \end{aligned}$$

10.2.4

$$(e^x)' = e^x \ln e = e^x$$

10.2.5

$$\begin{aligned}
 (\log_a^x)' &= \lim_{\Delta x \rightarrow 0} \frac{\log_a^{x+\Delta x} - \log_a^x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\log_a^{1+\frac{\Delta x}{x}}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\ln 1 + \frac{\Delta x}{x}}{\ln a \Delta x} \\
 &= \frac{1}{\ln a} \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{x}}{\Delta x} \\
 &= \frac{1}{x \ln a}
 \end{aligned}$$

10.2.6

$$\begin{aligned}
 (\ln x)' &= \frac{1}{x \ln e} \\
 &= \frac{1}{x}
 \end{aligned}$$

10.3.1

$$\begin{aligned}
 (\sin x)' &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2 \cos(x_0 + \frac{\Delta x}{2}) \sin \frac{\Delta x}{2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \cos(x_0 + \frac{\Delta x}{2}) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \\
 &= \cos x_0
 \end{aligned}$$

10.3.2

$$\begin{aligned}
 (\arcsin x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \sin y}{dy}} \\
 &= \frac{1}{\cos y} \\
 &= \frac{1}{\sqrt{1 - \sin^2 y}} \\
 &= \frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

10.3.3

$$\begin{aligned}
 (\csc x)' &= \left(\frac{1}{\sin x} \right)' = \frac{(1)' \cdot \sin x - (\sin x)' \cdot 1}{\sin^2 x} \\
 &= \frac{-\cos x}{\sin^2 x} \\
 &= -\csc x \cdot \cot x
 \end{aligned}$$

10.3.4

$$\begin{aligned}
 (\cos x)' &= \lim_{\Delta x \rightarrow 0} \frac{\cos(x_0 + \Delta x) - \cos(x_0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2 \sin(x_0 + \frac{\Delta x}{2}) \sin \frac{\Delta x}{2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} -\sin(x_0 + \frac{\Delta x}{2}) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \\
 &= -\sin x
 \end{aligned}$$

10.3.5

$$\begin{aligned}
 (\arccos x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \cos y}{dy}} \\
 &= \frac{1}{-\sin y} \\
 &= -\frac{1}{\sqrt{1 - \cos^2 y}} \\
 &= -\frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

10.3.6

$$\begin{aligned}
 (\sec x)' &= \left(\frac{1}{\cos x} \right)' = \frac{(1)' \cdot \cos x - (\cos x)' \cdot 1}{\cos^2 x} \\
 &= \frac{\sin x}{\cos^2 x} \\
 &= \sec x \cdot \tan x
 \end{aligned}$$

10.3.8

$$\begin{aligned}
 (\tan x)' &= \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= \sec^2 x
 \end{aligned}$$

10.3.9

$$\begin{aligned}
 (\arctan x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \tan y}{dy}} \\
 &= \frac{1}{\sec y} \\
 &= \frac{1}{1 + \tan^2 y} \\
 &= \frac{1}{1 + x^2}
 \end{aligned}$$

10.3.10

$$\begin{aligned}
 (\cot x)' &= \left(\frac{\cos x}{\sin x} \right)' = \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x} \\
 &= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\
 &= -\csc^2 x
 \end{aligned}$$

10.3.11

$$\begin{aligned}
 (\operatorname{arccot} x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \cot y}{dy}} \\
 &= \frac{1}{-\csc^2 y} \\
 &= -\frac{1}{1 + \cot^2 y} \\
 &= -\frac{1}{1 + x^2}
 \end{aligned}$$

10.3.12

$$\begin{aligned}
 (\sinh x)' &= \left(\frac{e^x - e^{-x}}{2} \right)' \\
 &= \frac{e^x + e^{-x}}{2} \\
 &= \cosh x
 \end{aligned}$$

10.3.13

$$\begin{aligned}
 (\cosh x)' &= \left(\frac{e^x + e^{-x}}{2} \right)' \\
 &= \frac{e^x - e^{-x}}{2} \\
 &= \sinh x
 \end{aligned}$$

10.3.14

$$\begin{aligned}
 (\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)' \\
 &= \frac{\frac{d(e^x - e^{-x})}{dx}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{dx}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{2^2}{(e^x + e^{-x})^2} \\
 &= \frac{1}{\cosh^2 x}
 \end{aligned}$$

10.3.15

$$\begin{aligned}
 (\operatorname{arcsinh} x)' &= \left[\ln(x + \sqrt{x^2 + 1}) \right]' \\
 &= \frac{d \ln(x + \sqrt{x^2 + 1})}{d(x + \sqrt{x^2 + 1})} \cdot \frac{d(x + \sqrt{x^2 + 1})}{dx} \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{dx}{dx} + \frac{d(\sqrt{x^2 + 1})}{d(x^2 + 1)} \cdot \frac{d(x^2 + 1)}{dx} \right) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \right) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{x^2 + 1}}
 \end{aligned}$$

10.3.16

$$\begin{aligned}
 (\operatorname{arccosh} x)' &= \left[\ln(x + \sqrt{x^2 - 1}) \right]' \\
 &= \frac{d \ln(x + \sqrt{x^2 - 1})}{d(x + \sqrt{x^2 - 1})} \cdot \frac{d(x + \sqrt{x^2 - 1})}{dx} \\
 &= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(\frac{dx}{dx} + \frac{d(\sqrt{x^2 - 1})}{d(x^2 - 1)} \cdot \frac{d(x^2 - 1)}{dx} \right) \\
 &= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x \right) \\
 &= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} \\
 &= \frac{1}{\sqrt{x^2 - 1}}
 \end{aligned}$$

10.3.17

$$\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \right]' \\
&= \frac{1}{2} \cdot \frac{d \left[\ln \left(\frac{1+x}{1-x} \right) \right]}{d \left(\frac{1+x}{1-x} \right)} \cdot \frac{d \left(\frac{1+x}{1-x} \right)}{dx} \\
&= \frac{1}{2} \cdot \frac{1}{\left(\frac{1+x}{1-x} \right)} \cdot \frac{\frac{d(1+x)}{dx}(1-x) - \frac{d(1-x)}{dx}(1+x)}{(1-x)^2} \\
&= \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{(1-x) + (1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}$$

10.4.1

$$\begin{aligned}
[Cu(x)]' &= \lim_{\Delta x \rightarrow 0} \frac{Cu(x + \Delta x) - Cu(x)}{\Delta x} \\
&= C \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \\
&= Cu'(x)
\end{aligned}$$

10.4.2

$$\begin{aligned}
(u(x) \pm v(x))' &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x) \pm v(x + \Delta x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \pm \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} \\
&= u'(x) \pm v'(x)
\end{aligned}$$

10.4.3

$$\begin{aligned}
[u(x) \cdot v(x)]' &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x) - u(x)v(x + \Delta x) + u(x)v(x + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]v(x + \Delta x) + u(x)[v(x + \Delta x) - v(x)]}{\Delta x} \\
&= u'(x) \lim_{\Delta x \rightarrow 0} v(x + \Delta x) + u(x)v'(x) \\
&= u'(x)v(x) + v'(x)u(x)
\end{aligned}$$

10.5.1

$$\begin{aligned}
[f^{-1}(y)]' \big|_{y=y_0} &= \lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\
&= \lim_{y \rightarrow y_0} \frac{x - x_0}{y - y_0} \\
&= \lim_{x \rightarrow x_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{x \rightarrow x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \\
&= \frac{1}{f'(x)}
\end{aligned}$$

10.6.1

$$\text{定义函数 } A(u) = \begin{cases} \frac{f(u) - f(u_0)}{u - u_0}, & u \neq u_0 \\ f'(u_0), & u = u_0 \end{cases}$$

$A(u)$ 在 u_0 处连续, 既有

$$\lim_{u \rightarrow u_0} A(u) = A(u_0) = f'(u_0)$$

由恒等式 $f(u) - f(u_0) = A(u)(u - u_0)$ 我们有

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} &= \frac{f[g(x)] - f[g(x_0)]}{x - x_0} \\
&= A[g(x)] \frac{g(x) - g(x_0)}{x - x_0} \\
\lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} A[g(x)] \frac{g(x) - g(x_0)}{x - x_0} \\
F'(x_0) &= f'(g(x_0))g'(x_0)
\end{aligned}$$

14.4 第 11 章

11.1.1

$$\begin{aligned}
\Delta y &= A \Delta x + o(\Delta x) \\
\frac{\Delta y}{\Delta x} &= A + \frac{o(\Delta x)}{\Delta x} \\
\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \left[A + \frac{o(\Delta x)}{\Delta x} \right] \\
f'(x_0) &= A + 0 \\
f'(x_0) &= A
\end{aligned}$$

11.1.2

设 $f(x)$ 在 x_0 点可导, $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ 存在
 (极限与无穷小的关系: $\lim f(x) = A \Leftrightarrow f(x) = A + \alpha$)

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \alpha$$

$$\Delta y = f'(x_0) \Delta x + \alpha \Delta x$$

其中 α 为 $\Delta x \rightarrow 0$ 时的无穷小。

$$\lim_{\Delta x \rightarrow 0} \frac{\alpha \Delta x}{\Delta x} = \lim_{\vartheta x \rightarrow 0} \alpha = 0$$

$$\alpha \Delta x = o(\Delta x)$$

$$\Delta y = f'(x_0) \Delta x + o(\Delta x)$$

11.2.1

$$\begin{aligned} d(u \pm v) &= (u \pm v)' dx \\ &= (u)' dx \pm (v)' dx \\ &= du \pm dv \end{aligned}$$

11.2.2

$$\begin{aligned} d(u \cdot v) &= (u \cdot v)' dx \\ &= (u)' v dx + (v)' u dx \\ &= v du + u dv \end{aligned}$$

11.2.3

$$\begin{aligned} d\left(\frac{u}{v}\right) &= \left(\frac{u}{v}\right)' dx \\ &= \frac{(u)'v - (v)'u}{v^2} dx \\ &= \frac{v du - u dv}{v^2} \end{aligned}$$

11.3.1

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &= f'(x_0) \\ f(x_0 + \Delta x) - f(x_0) &\leq 0 \\ \left\{ \begin{array}{l} \Delta x > 0 \\ \Delta x < 0 \end{array} \right\} &\left\{ \begin{array}{l} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0 \Rightarrow f'(x_0^+) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0 \\ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0 \Rightarrow f'(x_0^-) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0 \end{array} \right. \\ f'(x_0) &= f'(x_0^+) = f'(x_0^-) \Rightarrow f'(x_0) = 0 \end{aligned}$$

11.3.2

$$\begin{aligned}
 & M = \max\{f(x)|x \in [a, b]\}, m = \min\{f(x)|x \in [a, b]\} \\
 & \left\{ \begin{array}{l} M = m \Rightarrow M = m = f(a) = f(b), \text{此时 } f(x) \text{ 为常数, } \forall \xi \in (a, b), f'(\xi) = 0 \\ M > m \left\{ \begin{array}{l} f(a) > m \Rightarrow \exists \xi \in (a, b), f(\xi) = m, \text{根据费马引理, } f'(\xi) = 0 \\ f(a) < M \Rightarrow \exists \xi \in (a, b), f(\xi) = M, \text{根据费马引理, } f'(\xi) = 0 \end{array} \right. \end{array} \right.
 \end{aligned}$$

11.3.3

$$\varphi(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$

$$\varphi(a) = f(a) - \frac{f(b) - f(a)}{b - a}a = \frac{bf(a) - af(b)}{b - a}$$

$$\varphi(b) = f(b) - \frac{f(b) - f(a)}{b - a}b = \frac{bf(a) - af(b)}{b - a}$$

$$\varphi(a) = \varphi(b), \exists \xi \in (a, b), \varphi'(\xi) = 0$$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

$$f'(\xi)(b - a) = f(b) - f(a)$$