

数学方面 (笔记)

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历 (-.-) !)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：🔗 <https://github.com/daidongchuixue/jiangping.git>

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，贴吧，姜萍吧，姜圣的追随者，

2024.8.18：笔记都是看视频和书记录的。可能会有个别错误。但是我会持续更新，发现错误就会更改。上传频率不太固定。

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1 三角函数

1.1 三角恒等式

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \quad (1.1.1)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \quad (1.1.2)$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \quad (1.1.3)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \quad (1.1.4)$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.5)$$

$$\sin(\alpha) - \sin(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.6)$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.7)$$

$$\cos(\alpha) - \cos(\beta) = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \quad (1.1.8)$$

1.1.2 积化和差

$$\cos(A) \sin(B) = \frac{1}{2} [\sin(A+B) - \sin(A-B)] \quad (1.1.9)$$

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A+B) + \sin(A-B)] \quad (1.1.10)$$

$$\sin(A) \sin(B) = \frac{1}{2} [\cos(A-B) - \cos(A+B)] \quad (1.1.11)$$

$$\cos(A) \cos(B) = \frac{1}{2} [\cos(A+B) + \cos(A-B)] \quad (1.1.12)$$

1.1.3 降幂

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad (1.1.13)$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad (1.1.14)$$

1.1.4 半角公式

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos(x)}{2}} \quad (1.1.15)$$

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos(x)}{2}} \quad (1.1.16)$$

$$\tan \frac{x}{2} = \csc x - \cot x \quad (1.1.17)$$

$$(1.1.18)$$

1.1.5 倍角公式

$$\sin(2x) = 2 \sin x \cos x \quad (1.1.19)$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x \quad (1.1.20)$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x} \quad (1.1.21)$$

1.1.6 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \quad (1.1.22)$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \quad (1.1.23)$$

1.1.7 三角函数恒等式

$$\sin^2 x + \cos^2 x = 1 \quad (1.1.24)$$

$$1 + \tan^2 x = \sec^2 x \quad (1.1.25)$$

$$1 + \cot^2 x = \csc^2 x \quad (1.1.26)$$

1.2 双曲函数

1.2.1 定义

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}} \end{aligned}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}) \quad (1.2.1)$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \quad (1.2.2)$$

$$\operatorname{artanh} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad (1.2.3)$$

1.2.3 双曲函数恒等式

$$\sinh(2x) = 2 \sinh x \cosh x \quad (1.2.4)$$

$$\cosh^2 x - \sinh^2 x = 1 \quad (1.2.5)$$

$$\cosh^2 x + \sinh^2 x = \cosh(2x) \quad (1.2.6)$$

$$\cosh x = 1 + 2 \sinh^2 \frac{x}{2} \quad (1.2.7)$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 + x_2 + \cdots + x_n} \quad (2.0.1)$$

$$|x + y| \leqslant |x| + |y| \quad (2.0.2)$$

$$\sin x \leqslant x \leqslant \tan x \quad (2.0.3)$$

伯努利不等式

$$(1 + x)^n \leqslant 1 + nx \quad (2.0.4)$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2) \cdots (n-k+1) \quad (3.1.1)$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} \quad (3.1.2)$$

3.2 运算

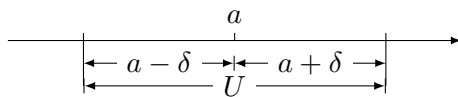
4 区间与映射

4.1 区间定义

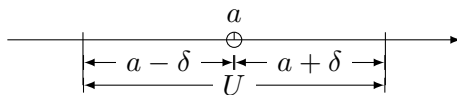
$$\text{区间定义} \begin{cases} (a, b) = \{x | a < x < b\} \\ [a, b] = \{x | a \leq x \leq b\} \\ (a, b] = \{x | a < x \leq b\} \\ (a, +\infty) = \{x | a < x\} \end{cases}$$

4.2 领域定义

$$\text{点 } a \text{ 的领域: } U(a, \delta) \begin{cases} \{x | a - \delta < x < a + \delta\} \\ \{x | |x - a| < \delta\} \end{cases}$$



$$\text{点 } a \text{ 的去心领域: } \dot{U}(a, \delta) = \begin{cases} \{x | a - \delta < x < a + \delta \wedge x \neq 0\} \\ \{x | 0 < |x - a| < \delta\} \end{cases}$$



$$\text{点 } a \text{ 的左领域: } (a - \delta, a)$$

$$\text{点 } a \text{ 的右领域: } (a, a + \delta)$$

4.3 映射定义

定义: X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x \in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$\text{记作 } f: X \rightarrow Y$$

$$f(x) = y \quad \begin{cases} \text{定义域 } (D_f) = X & x\text{-原像} \\ \text{值域 } (R_f) = \{f(x) | x \in X\} & y\text{-像} \end{cases}$$

$$\text{映射类型} \begin{cases} \text{满射:} & R_f = Y \\ \text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\ \text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射: } \begin{cases} f(x) = y \\ f^{-1}(y) = x \end{cases} \\ \text{复合映射:} & g \circ f \Leftrightarrow g[f(x)] \begin{cases} f: X \rightarrow Y_1 \\ g: Y_2 \rightarrow Z \\ g \circ f: X \rightarrow Z \quad (Y_1 \subset Y_2) \end{cases} \end{cases}$$

5 函数

5.1 函数相关的定义

5.1.1 函数

设数集 $D \in R$ 的映射

$$f : D \rightarrow R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x) \{x \in D\}$$

5.1.2 驻点

$$Def : f'(x) = 0$$

5.1.3 拐点

$$Def : f''(x) = 0 \text{ (左右两侧凹凸性改变)}$$

5.1.4 极值点

$$Def : \text{函数 } f(x) \text{ } x \in \mathring{U}(x_0), \text{ 包括可导和不可导的点 } \begin{cases} \text{极大值: } f(x) < f(x_0) \\ \text{极小值: } f(x) > f(x_0) \end{cases}$$

$$x \in \mathring{U}(x_0) \left\{ \begin{array}{l} f(x) \text{ 可导, } f'(x_0) = 0 \left\{ \begin{array}{l} x_0 \text{ 极大值 } \begin{cases} x \in (x_0 - \delta, x_0), f'(x) > 0 \\ x \in (x_0, x_0 + \delta), f'(x) < 0 \end{cases} \\ x_0 \text{ 极小值 } \begin{cases} x \in (x_0 - \delta, x_0), f'(x) < 0 \\ x \in (x_0, x_0 + \delta), f'(x) > 0 \end{cases} \\ x_0 \text{ 无极值, } x \in \mathring{U}(x_0) \begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases} \end{array} \right. \\ f(x) \text{ 二阶可导, } f'(x_0) = 0, f''(x_0) \neq 0 \begin{cases} f''(x) < 0 \Rightarrow x_0 \text{ 极大值} \\ f''(x) > 0 \Rightarrow x_0 \text{ 极小值} \end{cases} \end{array} \right.$$

5.1.5 最值

$$\text{最大值或最小值} \begin{cases} \text{驻点} \\ \text{端点} \end{cases}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f: D \rightarrow R \{D \subset R\} \begin{cases} \text{有界} \begin{cases} \text{有上界} \begin{cases} \exists k_1, \text{使} f(x) \leq k_1, \forall x \in D \\ \text{有下界} \begin{cases} \exists k_1, \text{使} f(x) \geq k_1, \forall x \in D \end{cases} \end{cases} \\ \text{无界} \begin{cases} \text{无上界} \begin{cases} \forall K_1, \exists x \in D \text{ 使, } f(x) \geq k_1 \\ \text{无下界} \begin{cases} \forall K_1, \exists x \in D \text{ 使, } f(x) \leq k_1 \end{cases} \end{cases} \end{cases} \end{cases}$$

5.2.2 函数的单调性与凹凸性

$$\text{若} \{x_1, x_2 \in D\} \ x_1 < x_2 \Rightarrow \begin{cases} f(x_1) < f(x_2) \text{称} f(x) \text{在} D \text{上单调增加} \\ f(x_1) > f(x_2) \text{称} f(x) \text{在} D \text{上单调减少} \\ f(x_1) \leq f(x_2) \text{称} f(x) \text{在} D \text{上单调非降} \\ f(x_1) \geq f(x_2) \text{称} f(x) \text{在} D \text{上单调非增} \end{cases}$$

$$\text{设 } f(x) \text{ 在区间 } I \text{ 上连续, } \forall x_1, x_2 \begin{cases} f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}, \text{称 } f(x) \text{ 在 } I \text{ 上是向上凹} \\ f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}, \text{称 } f(x) \text{ 在 } I \text{ 上是向上凸} \end{cases}$$

$$f(x) \text{在} [a, b] \text{上连续, 在} (a, b) \text{内可导} f'(x) \geq 0, \text{有限个点为 } 0, \text{单调增} \quad (5.2.1)$$

$$f(x) \text{在} [a, b] \text{上连续, 在} (a, b) \text{内可导} f'(x) \leq 0, \text{有限个点为 } 0, \text{单调减} \quad (5.2.2)$$

$$f(x) \text{在} [a, b] \text{上连续, 在} (a, b) \text{内二阶可导} f''(x) \geq 0, \text{有限个点为 } 0, \text{向上凹} \quad (5.2.3)$$

$$f(x) \text{在} [a, b] \text{上连续, 在} (a, b) \text{内二阶可导} f''(x) \leq 0, \text{有限个点为 } 0, \text{向下凸} \quad (5.2.4)$$

5.2.3 函数的奇偶性

$$\forall x \in D \quad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

$$\text{奇函数} \times \text{奇函数} = \text{偶函数} \quad (5.2.5)$$

$$\text{奇函数} \times \text{偶函数} = \text{奇函数} \quad (5.2.6)$$

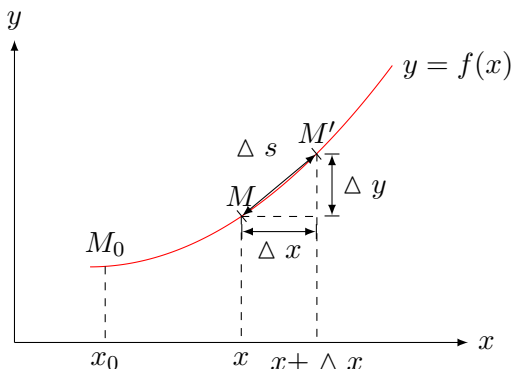
$$\text{偶函数} \times \text{偶函数} = \text{偶函数} \quad (5.2.7)$$

5.2.4 周期性

Def: $f(x+L) = f(x) \{L > 0 \text{ 常数}, \forall x \in D\} \Rightarrow f(x) \text{ 为 } L \text{ 的周期函数}$

5.3 弧

5.3.1 有向曲线弧



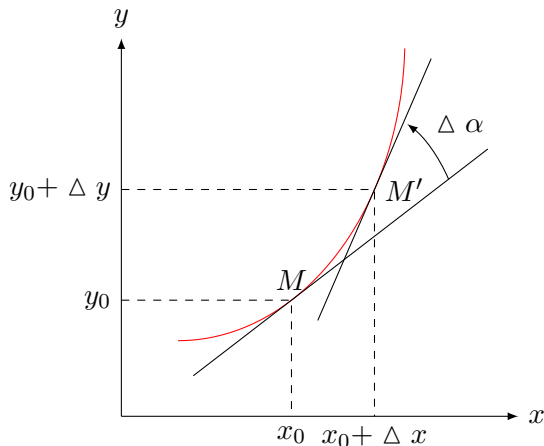
基准点 $M_0(x_0, f(x_0))$

以 x 增大的方向为正向, $\widehat{M_0 M} = S$

$S = S(x)$, S 是关于 x 的单调增加函数

$\widehat{M_0 M} \begin{cases} \text{绝对值为的长度} \\ \text{与曲线正向一致, 取正值} \\ \text{与曲线反向一致, 取负值} \end{cases}$

5.3.2 弧微分



$$\begin{aligned} ds &= \sqrt{1 + (y')^2} dx \\ &= \sqrt{(dx)^2 + (dy)^2} \\ &= \sqrt{(dx)^2 + (f' dx)^2} \end{aligned} \quad (5.3.1)$$

$$\text{参数方程} \begin{cases} x = \phi(t) & dx = \phi'(t) dt \\ y = \psi(t) & dy = \psi'(t) dt \end{cases}$$

$$ds = \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2} dt$$

5.3.3 曲率

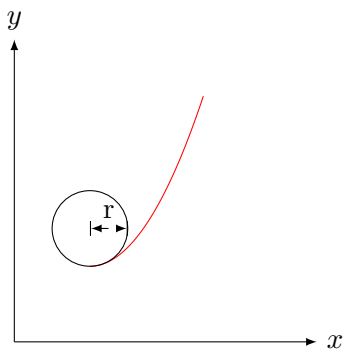
$$M(x_0, y_0), M'(x_0 + \Delta x, y_0 + \Delta y), \Delta s = \widehat{MM'}$$

$$\text{曲线上弧的} \begin{cases} \text{平均曲率:} & \bar{k} = \left| \frac{\Delta \alpha}{\Delta s} \right| \\ \text{点曲率:} & k = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta \alpha}{\Delta s} \right| = \left| \frac{d\alpha}{ds} \right| \end{cases}$$

$$\left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} \quad (5.3.2)$$

$$\left| \frac{d\alpha}{ds} \right| \text{ 的参数方程形式 } \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \left| \frac{d\alpha}{ds} \right| = \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\{|\psi'(t)|^2 + [\phi'(t)]^2\}^{\frac{3}{2}}} \quad (5.3.3)$$

5.3.4 曲率圆，曲率半径



$$\text{圆的曲率} k = \left| \frac{\Delta \alpha}{\Delta s} \right| = \left| \frac{\Delta \alpha}{r \Delta \alpha} \right| = \frac{1}{r}$$

$$\text{曲率半径} r = \frac{1}{k}$$

6 并集, 交集

6.1 定义

(\vee 或, \wedge 与)

$$A \cup B = \{x \in A \vee x \in B\}$$

$$A \cap B = \{x \in A \wedge x \in B\}$$

6.2 运算

$$\left\{ \begin{array}{l} \text{交换律} \left\{ \begin{array}{l} A \cup B = B \cup A \\ A \cap B = B \cap A \end{array} \right. \\ \text{结合律} \left\{ \begin{array}{l} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{array} \right. \\ \text{分配律} \left\{ \begin{array}{l} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{array} \right. \\ \text{对偶律} \left\{ \begin{array}{l} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{array} \right. \end{array} \right.$$

$$A \cup A = A = A \cap A$$

$$A = B \Leftrightarrow A \subset B \wedge A \supset B$$

$$A \cup \emptyset = A \quad A \cap \emptyset = \emptyset$$

6.3 性质

$$A \subset (A \cup B) \quad A \supset (A \cap B) \quad (6.3.1)$$

$$A \cup B = B \Leftrightarrow A \subset B \quad (6.3.2)$$

$$A \cap B = A \Leftrightarrow A \subset B \quad (6.3.3)$$

$$(n \in N) \quad A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \quad (6.3.4)$$

$$(n \in N) \quad A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \quad (6.3.5)$$

6.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

6.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^C = \bigcap_{\alpha} (E_{\alpha}^C)$$

$$\left(\bigcap_{\alpha} E_{\alpha}\right)^C = \bigcup_{\alpha} (E_{\alpha}^C)$$

7 群, 环, 域

7.1 群

7.1.1 M1

7.1.2 M2

7.1.3 M3

7.1.4 M4

7.2 环

7.3 域

内点, 外点, 边界点, 据点

8 极限

8.1 数列极限

8.1.1 数列的定义

$$Def: \quad \{x_n\}, x_n = f(n), n \in N^+ \rightarrow R$$

8.1.2 数列极限的定义

$$Def: \quad \{x_n\}, n \in N^+, \exists a, \forall \varepsilon > 0, \exists N, n > N \Rightarrow |x_n - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} x_n = a$$

极限存在, 为收敛, 不存在为发散

8.1.3 极限的唯一性

数列收敛, 极限的唯一性 (8.1.1)

8.1.4 有界数列

若 $\exists M > 0, \{M \in \text{正数}\}$

使得 $\forall n, |x_n| \leq M$

则称数列 $\{x_n\}$ 为有界数列

8.1.5 收敛数列与有界性

收敛数列必有界 (8.1.2)

单调有界数列必收敛 (8.1.3)

8.1.6 收敛数列的保号性

$$\lim_{n \rightarrow \infty} x_n = a \text{ 存在, 且 } a > 0, \text{ 则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时 } \Leftrightarrow x_n > 0 \quad (8.1.4)$$

$$\lim_{n \rightarrow \infty} x_n = a, \lim_{n \rightarrow \infty} b_n = b, a < b, \exists N, n > N, a_n < b_n \quad (8.1.5)$$

8.1.7 收敛数列和子数列

$$\{x_n\}, \lim_{n \rightarrow \infty} x_n = a, \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim_{n \rightarrow \infty} x_{n_k} = a$$

证明 $K = N \quad k > K$

$$n_k > n_K \geq N$$

$$|x_{n_k} - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} x_{n_k} = a$$

8.2 函数极限

8.2.1 极限的定义

$$Def: \forall \varepsilon > 0 \left\{ \begin{array}{l} \exists X > 0 \left\{ \begin{array}{l} \text{当 } x > X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = A \\ \text{当 } x < -X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = A \\ \text{当 } |x| > X \text{ 时都有 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = A \end{array} \right. \\ \exists \delta > 0 \left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = A \\ \text{当 } x_0 - \delta < x < x_0, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = A \\ \text{当 } 0 < |x - x_0| < \delta, \text{ 时 } |f(x) - A| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = A \end{array} \right. \end{array} \right.$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \rightarrow x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim_{x \rightarrow x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x \rightarrow x_0} f(x) \text{ 存在} \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x) \quad (8.2.1)$$

图

8.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$$\lim_{x \rightarrow x_0} f(x) = A, \exists M > 0, \delta > 0 \text{ 使 } 0 < |x - x_0| < \delta, |f(x)| \leq M$$

3 保号性

$$\lim_{x \rightarrow x_0} f(x) = A, A > 0, \exists \delta > 0, \text{ 当 } 0 < |x - x_0| < \delta \Rightarrow f(x) > 0$$

$$f(x) > 0, \exists \delta > 0, \text{ 当 } 0 < |x - x_0| < \delta \Rightarrow \lim_{x \rightarrow x_0} f(x) = A, A > 0$$

4 保序性

$f(x) \geq g(x)$, $\lim f(x) = a$, $\lim g(x) = b$, 则 $a \geq b$

5 函数极限与数列极限的关系

如果 $\lim_{x \rightarrow x_0} f(x)$ 存在, $\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列, 则满足 $x_n \neq x_0$

则 $\lim_{n \rightarrow \infty} f(x_n) = 0 = \lim_{x \rightarrow x_0} f(x)$, $x_n \rightarrow x_0$

8.3 无穷小与无穷大

8.3.1 无穷小定义

Def: 如果 $\lim_{x \rightarrow x_0} f(x) = 0$ 则称 $f(x)$ 为 $x \rightarrow x_0$ 时的无穷小

$$Def: \forall \varepsilon > 0 \left\{ \begin{array}{l} \exists X > 0 \left\{ \begin{array}{l} \text{当 } x > X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = 0 \\ \text{当 } x < -X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = 0 \\ \text{当 } |x| > X \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = 0 \end{array} \right. \\ \exists \delta > 0 \left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta, \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = 0 \\ \text{当 } x_0 - \delta < x < x_0, \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = 0 \\ \text{当 } 0 < |x - x_0| < \delta, \text{ 时 } |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = 0 \end{array} \right. \end{array} \right.$$

8.3.2 函数极限与无穷小的关系

在自变量的同一变化中。 α 为无穷小。 $\lim f(x) = A \Leftrightarrow f(x) = A + \alpha$ (8.3.1)

8.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

如果 $f(x)$ 为无穷大, 则 $\frac{1}{f(x)}$ 为无穷小。 (8.3.2)

如果 $f(x)$ 为无穷小, 切 $f(x) \neq 0$, 则 $\frac{1}{f(x)}$ 为无穷小。 (8.3.3)

8.3.4 无穷大定义

$$\begin{aligned}
& \left. \begin{aligned} & \exists X > 0 \left\{ \begin{aligned} & \text{当 } x > X \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow +\infty} f(x) = \infty \end{aligned} \right. \\ & \text{当 } x < -X \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow -\infty} f(x) = \infty \end{aligned} \right. \\ & \text{当 } |x| > X \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow \infty} f(x) = \infty \end{aligned} \right. \end{aligned} \right. \\ & \exists \delta > 0 \left\{ \begin{aligned} & \text{当 } x_0 - \delta < x < x_0 \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0^-} f(x) = \infty \end{aligned} \right. \\ & \text{当 } x_0 < x < x_0 + \delta \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \infty \end{aligned} \right. \\ & \text{当 } 0 < |x - x_0| < \delta \left\{ \begin{aligned} & f(x) > M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = +\infty \\ & f(x) < -M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = -\infty \\ & |f(x)| > M \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = \infty \end{aligned} \right. \end{aligned} \right. \end{aligned} \right\} \\
& \lim_{x \rightarrow x_0} f(x) = \infty, \text{ 直线 } x = x_0 \text{ 是 } y = f(x) \text{ 垂直渐近线}
\end{aligned}$$

8.4 运算

8.4.1 有限个无穷小的和仍为无穷小

设 $\gamma = \alpha + \beta$

α 和 β 同为 $x \rightarrow x_0$ 时的无穷小

$\forall \varepsilon > 0, \exists \delta_1 > 0, \text{ 当 } 0 < |x - x_0| < \delta_1 \text{ 时, 有 } |\alpha| < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0, \exists \delta_2 > 0, \text{ 当 } 0 < |x - x_0| < \delta_2 \text{ 时, 有 } |\beta| < \frac{\varepsilon}{2}$

$\delta = \min\{\delta_1, \delta_2\}, \text{ 当 } 0 < |x - x_0| < \delta \text{ 时}$

$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2 \text{ 同时满足}$

$$\begin{aligned} & \text{即 } |\alpha| < \frac{\varepsilon}{2}, |\beta| < \frac{\varepsilon}{2} \text{ 同时成立} \\ & |\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

8.4.2 有界函数与无穷小的乘积仍为无穷小

设 α 为 $x \rightarrow x_0$ 时的一个无穷小
 $g(x)$ 为 x_0 的一个去心邻域 $\dot{U}(x_0, \delta_1)$ 有界

$$f(x) = g(x)\alpha$$

证 $f(x)$ 为 $x \rightarrow x_0$ 时的无穷小

因为 $g(x)$ 在 $\dot{U}(x_0, \delta_1)$ 有界

$$\exists M > 0, \text{ 当 } 0 < |x - x_0| < \delta_1 \text{ 时 } |g(x)| < M$$

因为 α 是 $x \rightarrow x_0$ 的无穷小

$$\exists \delta_2 > 0 \text{ 当 } 0 < |x - x_0| < \delta_2 \text{ 时 } |\alpha| < \frac{\varepsilon}{M} < \varepsilon$$

取 $\delta = \min\{\delta_1, \delta_2\}$ 当 $0 < |x - x_0| < \delta$ 时

$$|g(x)| \geq M, |\alpha| < \frac{\varepsilon}{M} \text{ 同时成立}$$

$$|g(x)\alpha| = |g(x)| |\alpha| < M \frac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

8.4.3 极限的四则运算

$$\lim f(x) = A, \lim g(x) = B$$

$$\lim (f(x) \pm g(x)) = \lim f(x) \pm \lim g(x) \quad (8.4.1)$$

$$\lim (f(x)g(x)) = \lim f(x) \lim g(x) \quad (8.4.2)$$

$$\lim \left(\frac{f(x)}{g(x)} \right) = \frac{\lim f(x)}{\lim g(x)} \quad (8.4.3)$$

$$\lim [Cf(x)] = C \lim f(x) \quad (8.4.4)$$

$$\lim [f(x)]^n = [\lim f(x)]^n \quad (8.4.5)$$

$$\lim_{x \rightarrow \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{a_0 x^n + a_1 x^{n-1} + \cdots + a_n} = \begin{cases} \frac{a}{b} & m = n \\ \infty & m > n \\ 0 & m < n \end{cases} \quad (8.4.6)$$

$$\lim_{x \rightarrow x_0} g(x) = u_0, \lim_{u \rightarrow u_0} f(u) = A$$

$$\exists \delta_0 > 0, x \in \dot{U}(x_0, \delta_0), g(x) \neq u_0 \quad (8.4.7)$$

$$\lim_{x \rightarrow x_0} f[g(x)] = \lim_{u \rightarrow u_0} f(u) = A$$

8.4.4 夹逼定理 (三明治定理)

$$x_n \leq z_n \leq y_n \quad \forall n > N_0$$

$$\text{若 } \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = a \text{ 则 } \lim_{n \rightarrow \infty} z_n = a \quad (8.4.8)$$

8.4.5 重要极限

$$x \rightarrow x_0$$

$$\lim_{x \rightarrow x_0} \sin x = \sin x_0 \quad (8.4.9)$$

$$\lim_{x \rightarrow x_0} \cos x = \cos x_0 \quad (8.4.10)$$

$$x \rightarrow 0$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (8.4.11)$$

$$\lim_{x \rightarrow 0} \cos x = 1 \quad (8.4.12)$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \quad (8.4.13)$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = 1 \quad (8.4.14)$$

$$\lim_{x \rightarrow 0} \frac{\arcsin x}{x} = 1 \quad (8.4.15)$$

$$\lim_{x \rightarrow 0} \frac{\arctan x}{x} = 1 \quad (8.4.16)$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1 \quad (8.4.17)$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \quad (8.4.18)$$

$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{nx} = 1 \quad (8.4.19)$$

$$\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \quad (8.4.20)$$

$$x \rightarrow \infty$$

$$\{x_n\} \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \quad (8.4.21)$$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e \quad (8.4.22)$$

8.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

Def: α, β 是同一极限过程的无穷小。

(1) 如果 $\lim \frac{\beta}{\alpha} = 0$ 则称 β 是 α 的高阶无穷小, 记作 $\beta = o(\alpha)$

(2) 如果 $\lim \frac{\beta}{\alpha} = \infty$ 则称 β 是 α 的底阶无穷小。

- (3) 如果 $\lim \frac{\beta}{\alpha} = C$ 则称 β 是 α 的同阶无穷小。
- (4) 如果 $\lim \frac{\beta}{\alpha^k} = C, k > 0$ 则称 β 是 α 的 k 阶无穷小。
- (5) 如果 $\lim \frac{\beta}{\alpha} = 1$ 则称 β 是 α 的等价阶无穷小。

8.4.7 等价无穷小代换，因子代换

β 与 α 是等价无穷小 $\Leftrightarrow \beta = \alpha + o(\alpha)$

设 $\alpha \sim \alpha', \beta \sim \beta'$, 且 $\lim \frac{\beta'}{\alpha'}$ 存在, 则 $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$

$\lim \alpha f(x) = \lim \alpha' f(x)$

$\lim \frac{f(x)}{\alpha} = \lim \frac{f(x)}{\alpha'}$

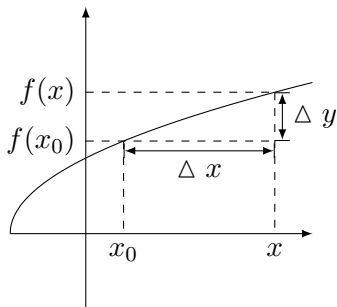
9 连续与间断点

9.1 定义

9.1.1 点连续

Def1: 设 $f(x)$ 在 x_0 的某邻域内有定义, 如果 $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

则称 $f(x)$ 在 x_0 处连续



$$\begin{cases} \Delta x = x - x_0 \\ \Delta y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \Delta x) - f(x_0) \end{cases} \end{cases}$$

Def2: 如果 $\lim_{\Delta x \rightarrow 0} \Delta y = \lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0)] = 0$

则称 $f(x)$ 在 x_0 处连续

9.1.2 区间连续

$$\forall x_0 \in [a, b] \begin{cases} \lim_{x \rightarrow x_0} f(x) = f(x_0) & x_0 \in (a, b) \Leftrightarrow f(x_0) = \begin{cases} \lim_{x \rightarrow x_0^-} f(x) = f(x_0^-) \\ \lim_{x \rightarrow x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim_{x \rightarrow x_0^+} f(x) = f(x_0^+) & x_0 = a \text{ (右连续)} \\ \lim_{x \rightarrow x_0^-} f(x) = f(x_0^-) & x_0 = b \text{ (左连续)} \end{cases}$$

称在 $[a, b]$ 内连续

有界: $\exists M > 0, x \in [a, b]$ 时, $|f(x)| \geq M$

最大值: $\exists x_0 \in [a, b]$ 时, $\forall x \in [a, b], f(x) \leq f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a, b]$ 上的最大值

最小值: $\exists x_0 \in [a, b]$ 时, $\forall x \in [a, b], f(x) \geq f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a, b]$ 上的最小值

1, 闭区间 $[a, b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

9.1.3 间断点

1, $f(x)$ 无定义

2, $\lim_{x \rightarrow x_0} f(x)$ 不存在

3, $\lim_{x \rightarrow x_0} f(x)$ 存在, 但 $\lim_{x \rightarrow x_0} f(x) \neq f(x_0)$

第一类间断点: $f(x_0^+) = \lim_{x \rightarrow x_0^+} f(x)$ 与 $f(x_0^-) = \lim_{x \rightarrow x_0^-} f(x)$

第二类间断点: 不是第一类的。

9.2 连续函数的运算

函数 $f(x), g(x)$ 在 $x = x_0$ 连续。

$$\lim_{x \rightarrow x_0} [f(x) \pm g(x)] = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = f(x_0) \pm g(x_0)$$

$$\lim_{x \rightarrow x_0} [f(x) \cdot g(x)] = \lim_{x \rightarrow x_0} f(x) \cdot \lim_{x \rightarrow x_0} g(x) = f(x_0) \cdot g(x_0)$$

$$\lim_{x \rightarrow x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{f(x_0)}{g(x_0)} \quad (g(x_0) \neq 0)$$

反函数的连续性

若 $y = f(x)$ 在区间 I_x 上单调增加, 且连续。

则 $y = f^{-1}(x)$ 在 $I_y = \{y | y = f(x), x \in I_x\}$ 上也为单调增加, 连续

$$\left. \begin{array}{l} \text{复合函数} \\ \left\{ \begin{array}{l} \text{内外都连续} \\ \text{外连续} \end{array} \right. \end{array} \right\} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} g(x) = g(x_0) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow x_0} f[g(x)] = f[g(x_0)] = f(\lim_{x \rightarrow x_0} g(x)) \end{array} \right. \\ \left\{ \begin{array}{l} x \rightarrow x_0 \left\{ \begin{array}{l} \lim_{x \rightarrow x_0} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow x_0} f[g(x)] = f(u_0) = f(\lim_{x \rightarrow x_0} g(x)) \end{array} \right. \\ \\ \left\{ \begin{array}{l} x \rightarrow \infty \left\{ \begin{array}{l} \lim_{x \rightarrow \infty} g(x) = u_0 \\ \lim_{u \rightarrow u_0} f(x) = f(u_0) \\ \lim_{x \rightarrow \infty} f[g(x)] = f(u_0) = f(\lim_{x \rightarrow \infty} g(x)) \end{array} \right. \end{array} \right. \end{array} \right.$$

9.3 零点定理

2, 设 $f(x)$ 在 $[a, b]$ 上连续, 且 $f(a) \cdot f(b) < 0$

则至少存在一点 $\xi \in (a, b)$ 使 $f(\xi) = 0$

9.4 介值定理

设 $f(x)$ 在 $[a, b]$ 上连续, 且 $f(a) = A, f(b) = B$

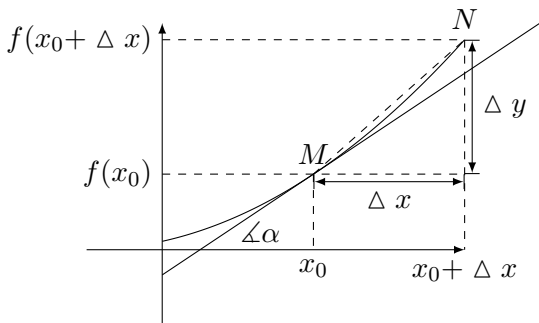
$\forall C \in (A, B)$, 至少有一点 $\xi, f(\xi) = C$

10 导数

10.1 定义

导数的概念从物理发展出来的。

$$v(t_0) = \lim_{\Delta t \rightarrow 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$



$$NM \text{斜率} = \tan \beta = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\Delta y}{\Delta x}$$

$$\text{斜率} k = \tan \alpha = \lim_{\Delta x \rightarrow 0} \tan \beta = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

10.1.1 导数定义

$y = f(x)$ 在 x_0 的某邻域内有定义

给自变量的增量 Δx , $(x_0 + \Delta x)$ 仍在定义域内

函数得到了相应增量 Δy , $\Delta y = f(x_0 + \Delta x)$

如果 $\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ 存在, 称 $y = f(x)$ 在 $x = x_0$ 处可导

(极限值为 $y = f(x)$ 在 $x = x_0$ 处导数)

$$\text{记 } y'|_{x=x_0} = f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \Leftrightarrow \lim_{\Delta x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

10.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

称 $f'(x)$ 为 $y = f(x)$ 在区间 I 上的导函数

10.1.3 闭区间可导定义

$$f(x) \text{ 在 } [a, b] \text{ 可导} \Leftrightarrow \begin{cases} f'(x_0) & x_0 \in (a, b) \\ f'_+(a) & x = a \\ f'_-(b) & x = b \end{cases} \Leftrightarrow \begin{cases} \text{左导数 } f'_-(x_0) = \lim_{\Delta x \rightarrow 0^-} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\ \text{右导数 } f'_+(x_0) = \lim_{\Delta x \rightarrow 0^+} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \end{cases}$$

10.1.4 导数与连续

$$f'(x) \text{ 存在} \Rightarrow f(x) \text{ 在 } x = x_0 \text{ 处连续} \quad (10.1.1)$$

10.2 幂数，指数，对数

$$(C)' = 0 \quad (10.2.1)$$

$$(x^a)' = ax^{a-1} \quad (10.2.2)$$

$$(a^x)' = a^x \ln a \quad (10.2.3)$$

$$(e^x)' = e^x \quad (10.2.4)$$

$$(\log_a^x)' = \frac{1}{x \ln a} \quad (10.2.5)$$

$$(\ln x)' = \frac{1}{x} \quad (10.2.6)$$

10.3 三角函数

$$(\sin x)' = \cos x \quad (10.3.1)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \quad (10.3.2)$$

$$(\csc x)' = -\csc x \cot x \quad (10.3.3)$$

$$(\cos x)' = -\sin x \quad (10.3.4)$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \quad (10.3.5)$$

$$(\sec x)' = \sec x \tan x \quad (10.3.6)$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \quad (10.3.7)$$

$$(\tan x)' = \sec^2 x \quad (10.3.8)$$

$$(\arctan x)' = \frac{1}{1+x^2} \quad (10.3.9)$$

$$(\cot x)' = -\csc^2 x \quad (10.3.10)$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2} \quad (10.3.11)$$

$$(\sinh x)' = \cosh x \quad (10.3.12)$$

$$(\cosh x)' = \sinh x \quad (10.3.13)$$

$$(\tanh x)' = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x \quad (10.3.14)$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}} \quad (10.3.15)$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}} \quad (10.3.16)$$

$$(\operatorname{artanh} x)' = \frac{1}{1-x^2} \quad (10.3.17)$$

10.4 导数运算

$u = u(x), v = v(x)$, 均在 x 点可导, C 为常数

$$(Cu(x))' = Cu'(x) \quad (10.4.1)$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \quad (10.4.2)$$

$$(u(x) \cdot v(x))' = u'(x)v(x) + v'(x)u(x) \quad (10.4.3)$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2} \quad (10.4.4)$$

10.5 反函数求导

如果函数 $y = f(x)$ 在区间 (a, b) 内单调可导, 且 $f'(y) \neq 0$

$$\begin{cases} \alpha = \min\{f(a) + 0, f(b - 0)\} \\ \beta = \max\{f(a) + 0, f(b - 0)\} \end{cases}$$

则它的反函数 $x = f^{-1}(y)$ 在区间 (α, β) 内也可导

$$[f^{-1}(y)]' = \frac{1}{f'(x)} \Leftrightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad (10.5.1)$$

10.6 复合函数求导

设函数 $\begin{cases} y = f(u) \text{ 在 } U(u_0, \delta_0) \text{ 处有定义} \\ u = g(x) \text{ 在 } U(x_0, \eta_0) \text{ 处有定义} \end{cases}$
 $u_0 = g(x_0)$, 且 $f'(u)$ 和 $g'(x)$ 都存在
 则复合函数 $F(x) = f[g(x)]$ 在点 x_0 可导, 且

$$F'(x_0) = f'[g(x_0)] g'(x_0) \Leftrightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad (10.6.1)$$

10.7 高阶求导

$$Def: \begin{cases} \text{一阶导数} & y' \Leftrightarrow \frac{dy}{dx} \\ \text{二阶导数} & y'' \Leftrightarrow \frac{d^2 y}{dx^2} \\ \text{三阶导数} & y''' \Leftrightarrow \frac{d^3 y}{dx^3} \\ \text{三阶以上 } n \text{ 阶导数} & y^{(n)} \Leftrightarrow \frac{d^n y}{dx^n} \end{cases}$$

10.8 高阶求导公式

$$(e^x)^{(n)} = e^x \quad (10.8.1)$$

$$(a^x)^{(n)} = a^x (\ln a)^n \quad (10.8.2)$$

$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \quad (10.8.3)$$

$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \quad (10.8.4)$$

$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1} (n-1)!}{(x+a)^n} \quad (10.8.5)$$

$$(\sin x)^{(n)} = \sin\left(x + n\frac{\pi}{2}\right) \quad (10.8.6)$$

$$(\cos x)^{(n)} = \cos\left(x + n\frac{\pi}{2}\right) \quad (10.8.7)$$

$$[f(ax+b)]^{(n)} = a^n \cdot f^{(n)}(ax+b) \quad (10.8.8)$$

10.9 高阶求导运算法则

$$(u(x) \pm v(x))^{(n)} = u^{(n)}(x) \pm v^{(n)}(x) \quad (10.9.1)$$

$$\text{莱布紫泥公式} \quad (uv)^{(n)} = \sum_{k=0}^n C_n^k u^{(n-k)} \cdot v^{(k)} \quad (10.9.2)$$

10.10 隐函数求导

$$F(x, y) = 0, y = f(x)$$

$F(x, f(x)) \equiv 0$ 可以同时对面求导

10.11 参数方程求导

$$x = x(t), y = y(t)$$

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt} \left(\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \right) \cdot \frac{dt}{dx} = \frac{\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^2} \cdot \frac{dt}{dx} = \frac{\frac{d^2y}{dt^2} \cdot \frac{dx}{dt} - \frac{d^2x}{dt^2} \cdot \frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$$

11 微分

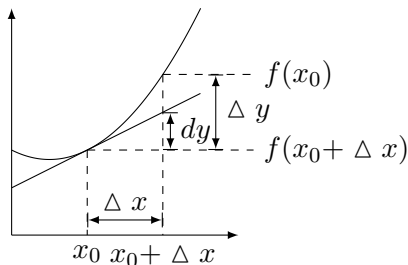
11.1 定义

设函数 $f(x)$ 在点 x_0 的一个邻域内有定义。 $\Delta y = f(x_0 + \Delta x) - f(x_0)$

如果 Δy 可以表示为 $\Delta y = A \Delta x + o(\Delta x)$ 其中 A 为与 Δx 无关的常数

则称 $f(x)$ 在点 x_0 可微, $A \Delta x$ 称为 $f(x)$ 在点 x_0 处的微分。

记作: $dy = A \Delta x$



$$\text{可微} \Rightarrow \text{可导} \quad (11.1.1)$$

$$\text{可导} \Rightarrow \text{可微} \quad (11.1.2)$$

11.2 微分法则

11.2.1 核心根本

$$dy = f'(x) \overset{\text{积分}}{\underset{\text{求导}}{d}} x$$

11.2.2 四则运算

$$d(u \pm v) = du \pm dv \quad (11.2.1)$$

$$d(uv) = vdu + u dv \quad (11.2.2)$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + u dv}{v^2} \quad (11.2.3)$$

11.2.3 复合运算

$$\text{可微} \begin{cases} y = f(u) \\ u = g(x) \end{cases} \Rightarrow \begin{cases} dy = f'(u)du \\ du = g'(x)dx \end{cases} \quad \text{则 } y = f(g(x)) \text{ 也可微}$$

$$\text{且 } dy = f'(u)du = f'(u)g'(x)dx$$

u 是否为中间变量都成立, 微分的不变性。

11.2.4 近似计算公式

$$\Delta x \rightarrow 0, dy \approx \Delta y \begin{cases} dy = f'(x_0) \Delta x \\ \Delta y = f(x_0 + \Delta x) - f(x_0) \end{cases} \left\{ \begin{array}{l} f(x_0 + \Delta x) \approx f(x_0) + f'(x_0) \Delta x \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ x_0 = 0 \left\{ \begin{array}{l} f(x) \approx f(0) + f'(0)x \\ \sqrt{n} \approx 1 + \frac{1}{n}x \\ \sin x \approx x \\ \tan x \approx x \\ e^x \approx 1 + x \\ \ln(1 + n) \approx x \end{array} \right. \end{array} \right.$$

11.2.5 奇偶函数导数

偶函数导数为奇函数 $f(x) = f(-x) \Leftrightarrow f'(x) = -f'(-x)$

奇函数导数为偶函数 $f(x) = -f(-x) \Leftrightarrow f'(x) = f'(-x)$

11.2.6 区间恒为 0

若 $f'(x)$ 在区间恒为零, 则 $f(x)$ 在区间 I 上为一常数

设 x_1, x_2 为区间 I 内任意两点 $x_1 < x_2$

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \equiv 0$$

$$f(x_2) \equiv f(x_1) = C$$

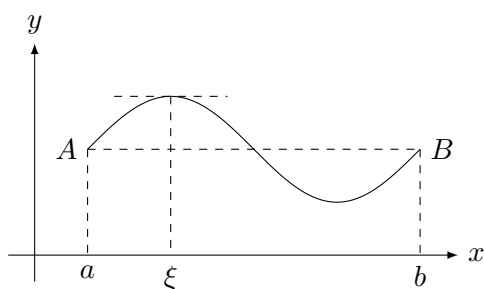
11.3 中值定理

11.3.1 费马引理

$$f(x) \quad \forall x \in \dot{U}(x_0) \begin{cases} f(x) \leq f(x_0) & f(x) \text{ 在 } x_0 \text{ 处取极大值} \\ f(x) \geq f(x_0) & f(x) \text{ 在 } x_0 \text{ 处取极小值} \end{cases}$$

如果可导函数 $y = f(x)$ 在 x_0 取极值, 则 $f'(x_0) = 0$ (11.3.1)

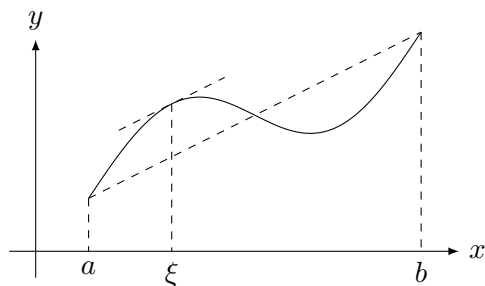
11.3.2 罗尔定理



$$f(x) \begin{cases} \text{闭区间 } [a, b] \text{ 上连续} \\ \text{开区间 } (a, b) \text{ 可导} \\ f(a) = f(b) \end{cases}$$

至少有一点 $\xi \in (a, b)$, $f'(\xi) = 0$ (11.3.2)

11.3.3 拉格朗日定理 (微分中值定理)



$$f(x) \begin{cases} \text{在闭区间 } [a, b] \text{ 上连续} \\ \text{在开区间 } (a, b) \text{ 可导} \end{cases}$$

至少有一点 $\xi \in (a, b)$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow f(b) - f(a) = f'(\xi)(b - a) \quad (11.3.3)$$

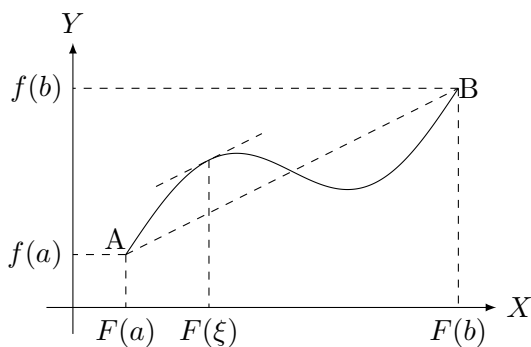
$$[x, x + \Delta x] \text{ 用拉格朗日定理 } f(x + \Delta x) - f(x) = f'(\xi) \Delta x$$

$$\xi \in (x, x + \Delta x) \text{ 记作: } \xi = x + \theta \Delta x \quad 0 < \theta < 1$$

$$f(x + \Delta x) - f(x) = f(x + \theta \Delta x) \Delta x$$

$$\Delta y = f(x + \theta \Delta x) \Delta x$$

11.3.4 柯西定理



$$f(x) \begin{cases} \text{在闭区间 } [a, b] \text{ 上连续} \\ \text{在开区间 } (a, b) \text{ 可导} \\ F'(x) \neq 0 \end{cases}$$

$$\text{参数方程 } (a \leq x \leq b) \begin{cases} X = F(x) \\ Y = f(x) \end{cases}$$

$$\text{至少有一点 } \xi \quad \frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)} \quad (11.3.4)$$

$$\text{切线斜率} = \frac{dY}{dX} = \frac{df(x)}{dF(x)} = \frac{f'(x)}{F'(x)} \Rightarrow x = \xi \text{ 时斜率} = \frac{f'(\xi)}{F'(\xi)}$$

$$AB \text{ 的斜率} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

11.3.5 三个定理关系

$$\frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)}, (F(x) = x) \Rightarrow f'(\xi) = \frac{f(b) - f(a)}{b - a}, (f(b) = f(a)) \Rightarrow f'(\xi) = 0$$

11.4 洛必达法则

$$\text{未定型, } \frac{0}{0}, \frac{\infty}{\infty}, 0^0, 1^\infty, \infty^0, \infty - \infty$$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{F(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{F'(x)} \begin{cases} \lim_{x \rightarrow x_0} f(x) = 0, \lim_{x \rightarrow x_0} F(x) = 0 \\ f(x), F(x) \text{ 在 } x_0 \text{ 的某去心邻域内可导, 且 } F'(x) \neq 0 \\ \lim_{x \rightarrow x_0} \frac{f'(x)}{F'(x)} \text{ 存在, 或无穷小。则} \end{cases}$$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{F(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{F'(x)} \quad (11.4.1)$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{F(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{F'(x)} \begin{cases} \lim_{x \rightarrow x_0} f(x) = 0, \lim_{x \rightarrow x_0} F(x) = 0 \\ \exists N \text{ 当 } |x| > N, \text{ 时 } f'(x), F'(x) \text{ 存在, 且 } F'(x) \neq 0 \\ \lim_{x \rightarrow x_0} \frac{f'(x)}{F'(x)} \text{ 存在, 或为无穷大。则} \end{cases}$$

11.5 泰勒公式

$$f(x_0 + \Delta x) - f(x_0) = f'(x_0) \Delta x + o(\Delta x)$$

$$x_0 + \Delta x = x \quad \Delta x = x - x_0$$

$$f(x) - f(x_0) = f'(x_0)(x - x_0) + o(\Delta x)$$

$$f(x) = f'(x_0)(x - x_0) + f(x_0) + o(\Delta x)$$

$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$P(x_0) = f(x_0)$$

$$P'(x_0) = f'(x_0)$$

11.5.1 泰勒多项式

$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$ 去近似某个多项式

$$\begin{cases} P_n(x_0) &= f(x_0) = a_0 \\ P'_n(x_0) &= f'(x_0) = a_1 \\ P''_n(x_0) &= f''(x_0) = a_2 \cdot 2! \\ \vdots & \\ P_n^{(n-1)}(x_0) &= f^{(n-1)}(x_0) = a_{n-1} \cdot (n-1)! \\ P_n^{(n)}(x_0) &= f^{(n)}(x_0) = a_n \cdot n! \end{cases} \Rightarrow \begin{cases} a_0 &= f_n(x_0) \\ a_1 &= f'_n(x_0) \\ a_2 &= \frac{f''_n(x_0)}{2!} \\ \vdots & \\ a_{n-1} &= \frac{f_n^{(n-1)}(x_0)}{(n-1)!} \\ a_n &= \frac{f_n^{(n)}(x_0)}{n!} \end{cases}$$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \cdots \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^n(x_0)}{n!}(x - x_0)^n$$

$$f(x) \approx P_n(x)$$

11.5.2 泰勒中值定理

如果 $f(x)|x_0 \in (a, b)$ 内有 $(n+1)$ 阶导则

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

拉格朗日余项

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \quad \{\xi \in (x, x_0)\} \quad (11.5.1)$$

皮亚诺余项

$$R_n(x) = o(|x - x_0|^n) \quad (11.5.2)$$

$$f(x) \approx P_n(x) \text{ 误差为 } R_n(x)$$

11.6 麦克劳林公式

$$x_0 = 0$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, & 0 < \theta < 1 \\ o(|x|^n) \end{cases}$$

11.6.1 常用的麦克劳林展开

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, & 0 < \theta < 1 \\ o(|x|^n) \end{cases}$$

$$\sin x = 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^{n-1} \frac{1}{(2n-1)!}x^{2n-1} + R_n(x)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + R_n(x)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1} \frac{1}{n}x^n + R_n(x)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots - \frac{1}{n}x^n + R_n(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \cdots \frac{A_\alpha^n}{n!}x^n + R_n(x)$$

12 微分方程

12.1 基本概念

微分方程，含有自变量，未知函数及导数的方程，称为微分方程。

$$\text{微分方程} \begin{cases} \text{未知函数为一元函数} & \text{常微分方程} \\ \text{未知函数为多元函数} & \text{偏微分方程 (数理方程)} \end{cases}$$

12.1.1 微分方程的阶

方程中的未知函数的最高阶的导数，阶数称为该方程的阶。

12.1.2 n 阶微分方程解

$$F(x, y, y', y'', \dots, y^{(n)}) = 0, \quad y = \varphi(x)$$

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots, \varphi^{(n)}(x)) \equiv 0$$

$$x \in I, \text{ 称 } \varphi \text{ 为方程在区间 } I \text{ 上的解} \begin{cases} \text{包含有 } n \text{ 个任意常数称 } y=\varphi(x) \text{ 是方程的通解} \\ \text{不含任意常数称 } y=\varphi(x) \text{ 是方程的特解} \end{cases}$$

12.1.3 齐次方程

如果一阶微分方程可化为 $\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right)$ 则称原方程为齐次方程

12.2 一阶线性微分方程

$$\frac{dy}{dx} + P(x)y = Q(x) \begin{cases} Q(x) \equiv 0, & \text{称一阶线性齐次方程} \\ Q(x) \not\equiv 0, & \text{称一阶线性非齐次方程} \end{cases}$$

$$\text{齐次通解} \quad y = Ce^{-\int P(x) dx} \quad (12.2.1)$$

$$\text{非齐次通解} \quad y = e^{-\int P(x) dx} \left[\int Q(x)e^{\int P(x) dx} dx + C \right] \quad (12.2.2)$$

12.3 二阶线性微分方程

$$y'' + P_1(x)y' + P_2(x)y = f(x) \begin{cases} f(x) \equiv 0, & \text{称二阶线性齐次方程} \\ f(x) \not\equiv 0, & \text{称二阶线性非齐次方程} \end{cases}$$

12.3.1 二阶线性齐次微分方程

$$\begin{aligned} y_1(x), y_2(x) \text{ 是任意的两个解, } C_1, C_2 \text{ 是任意常数, 则} \\ y = C_1 y_1(x) + C_2 y_2(x) \text{ 也是的解} \end{aligned} \quad (12.3.1)$$

$$\begin{aligned} y_1, y_2 \text{ 是两个线性无关解, } C_1, C_2 \text{ 是任意常数, 则} \\ \text{通解为, } y = C_1 y_1 + C_2 y_2 \end{aligned} \quad (12.3.2)$$

12.3.2 二阶线性非齐次微分方程

$$\text{非齐次通解 } y = Y + y^* \begin{cases} y^* \text{ 非其次特解} \\ Y = C_1 y_1 + C_2 y_2 \text{ 齐次通解} \end{cases}$$

$$y = y_1 - y_2 \text{ 是对应齐次方程的解} \quad (12.3.3)$$

$$y = \alpha y_1 + (1 - \alpha) y_2 \text{ 也是解} \quad (12.3.4)$$

$$(1) \quad y'' + P(x)y' + Q(X)y = f_1(x) + f_2(x)$$

$$(2) \quad y'' + P(x)y' + Q(X)y = f_1(x)$$

$$(3) \quad y'' + P(x)y' + Q(X)y = f_2(x)$$

$$(2) \text{ 特解为 } y_1^* \quad (3) \text{ 特解为 } y_2^* \Rightarrow (1) \text{ 特解为 } y_1^* + y_2^*$$

12.3.3 二阶常系数齐次线性微分方程

$$y'' + py' + qy = 0 \quad (p, q \text{ 属于常数})$$

$$y = e^{rx} \quad y' = r e^{rx} \quad y'' = r^2 e^{rx}$$

$$\text{特征方程: } r^2 + pr + q = 0 \begin{cases} p^2 - 4q > 0 & \text{通解: } C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ p^2 - 4q = 0 & \text{通解: } C_1 e^{r_1 x} + C_2 x e^{r_1 x} \\ p^2 - 4q < 0 & \text{通解: } e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) \end{cases} \quad (12.3.5)$$

12.3.4 二阶常系数非齐次线性微分方程

$$y'' + py' + qy = f(x) \quad (p, q, \text{ 为常数})$$

$$y'' + py' + qy = 0 \quad (\text{对应的齐次方程})$$

$$\text{通解} = \text{齐次方程通解} + \text{非齐次特解}$$

$$f(x) = \begin{cases} P_m(x)e^{\lambda x} & (P_m(x) \text{ 是 } x \text{ 的 } m \text{ 次多项式}) \\ [P_l(x) \cos \omega x + P_n(x) \sin \omega x] e^{\lambda x} & (P_l(x), P_n(x) \text{ 是 } x \text{ 的 } l, n \text{ 次多项式}) \end{cases}$$

$$f(x) = P_m(x)e^{\lambda x} \text{ 型} \begin{cases} y &= Q(x)e^{\lambda x} \\ y' &= Q'(x)e^{\lambda x} + \lambda Q(x)e^{\lambda x} \\ &= [Q'(x) + \lambda Q(x)] e^{\lambda x} \\ y'' &= [Q''(x) + \lambda Q'(x)] e^{\lambda x} + [Q'(x) + \lambda Q(x)] \lambda e^{\lambda x} \\ &= [Q''(x) + 2\lambda Q'(x) + \lambda^2 Q(x)] e^{\lambda x} \end{cases}$$

$$[Q''(x) + 2\lambda Q'(x) + \lambda^2 Q(x)] e^{\lambda x} + p [Q'(x) + \lambda Q(x)] e^{\lambda x} + q Q(x) e^{\lambda x} = P_m(x) e^{\lambda x}$$

$$[Q''(x) + 2\lambda Q'(x) + \lambda^2 Q(x)] + p [Q'(x) + \lambda Q(x)] + q Q(x) = P_m(x)$$

$$Q''(x) + (2\lambda + p)Q'(x) + (\lambda^2 + p\lambda + q)Q(x) = P_m(x)$$

$$\lambda^2 + p\lambda + q \begin{cases} \neq 0 \text{ (0重根)} \Rightarrow \begin{cases} Q(x) = Q_m(x) \\ Q_m(x) = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m \quad (b_m \neq 0) \\ b_0, b_1, b_2, \cdots, b_m \text{ 待定系数} \\ y^* = Q_m(x)e^{\lambda x} \end{cases} \\ = 0 \quad 2\lambda + p \begin{cases} \neq 0 \text{ (1重根)} \Rightarrow \begin{cases} Q(x) = xQ_m(x) \\ y^* = xQ_m(x)e^{\lambda x} \end{cases} \\ = 0 \text{ (2重根)} \Rightarrow \begin{cases} Q(x) = x^2Q_m(x) \\ y^* = x^2Q_m(x)e^{\lambda x} \end{cases} \end{cases} \end{cases}$$

$$[P_l(x) \cos \omega x + P_n(x) \sin \omega x] e^{\lambda x} \text{ 型} \begin{cases} \lambda \text{ 是常数} & P_l(x) \text{ 是 } x \text{ 的 } l \text{ 次多项式} \\ \omega \text{ 是常数} & P_n(x) \text{ 是 } x \text{ 的 } n \text{ 次多项式} \end{cases}$$

$$y^* = x^k e^{\lambda x} [R_n(x) \cos \omega x + R_n(x) \sin \omega x] \begin{cases} n = \max\{l, m\} \\ \text{特征方程: } r^2 + pr + q = 0 \\ \lambda + i\omega \begin{cases} \text{不是特征根 } k = 0 \\ \text{是特征根 } k = 1 \end{cases} \end{cases}$$

12.4 n 阶线性微分方程

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + a_n y = f(x) \begin{cases} f(x) \equiv 0, & \text{称 } n \text{ 阶线性齐次方程} \\ f(x) \not\equiv 0, & \text{称 } n \text{ 阶线性非齐次方程} \end{cases}$$

y_1, y_2, \cdots, y_n 是 n 个线性无关解, C_1, C_2, \cdots, C_n 是任意常数, 则
通解为, $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ (12.4.1)

非齐次通解 $y = Y + y^*$ $\begin{cases} y^* \text{ 非其次特解} \\ Y = C_1y_1 + C_2y_2 + \cdots + C_ny_n \text{ 齐次通解} \end{cases}$

12.4.1 n 阶常系数线性齐次微分方程

$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_ny = 0$

特征方程: $r^n + a_1rn - 1 + a_2r^{n-2} + \cdots + a_n = 0$

不同根对应的通解

单根 (实)	Ce^{rx}
k 个根 (实)	$(C_1 + C_2x + C_3x^2 + \cdots + C_kx^{k-1})e^{rx}$
单共轭复根	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$
k 个共轭复根	$e^{\alpha x} \left\{ [C_1 + C_2x \cdots C_kx^{k-1}] \cos \beta x + [D_1 + D_2x \cdots D_kx^{k-1}] \sin \beta x \right\}$

12.4.2 n 阶常系数线性非齐次微分方程

$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + a_ny = f(X) = P_m(x)e^{\lambda x}$

n 阶通解 = n 阶齐次通解 + n 非齐次特解

特解: $y^* = x^kQ_m(x)e^{\lambda x}$ 其中 k 为特征根的重数

12.5 全微分方程

一阶微分方程对称式 $M(x, y)dx + N(x, y)dy = 0$

存在 $u(x, y)$ 使 $du = M(x, y)dx + N(x, y)dy$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \begin{cases} M(x, y) = \frac{\partial u}{\partial x} \\ N(x, y) = \frac{\partial u}{\partial y} \end{cases}$

13 不定积分

13.1 概念

13.1.1 原函数

$\forall x \in I, F'(x) = f(x), \quad F(x)$ 为 $f(x)$ 的一个原函数

函数 $f(x)$ 在区间 I 上连续一定有 $F(x)$, 使 $F'(x) = f(x)$ (13.1.1)

13.1.2 不定积分

区间 I 上, $f(x)$ 的带有任意常数的原函数, 称为 $f(x)$ 在区间 I 上的不定积分。
记作:

$$\int f(x) dx \quad \left\{ \begin{array}{ll} \int & \text{积分符号} \\ f(x) & \text{被积函数} \\ f(x) dx & \text{被积表达式} \\ x & \text{积分变量} \end{array} \right.$$

如果 $F(x)$ 是 $f(x)$ 的一个原函数

$$\int f(x) dx = F(x) + C$$

13.1.3 不定积分性质

$$\begin{aligned} \left[\int f(x) dx \right]' &= f(x) \\ d \left[\int f(x) dx \right] &= f(x) dx \\ \int dF(x) &= \int F'(x) dx = F(x) + C \end{aligned}$$

13.2 积分运算

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx \quad (13.2.1)$$

$$\int k f(x) dx = k \int f(x) dx \quad (k \text{ 为常数}) \quad (13.2.2)$$

$$\int f[\varphi(x)] \varphi'(x) dx \stackrel{u=\varphi(x)}{=} \left[\int f(u) du \right]_{x=\varphi(u)} = F[\varphi(x)] + C \quad (13.2.3)$$

$$\int f(x) dx \stackrel{\substack{x=\varphi(t) \\ \varphi'(t) \neq 0}}{=} \left[\int f[\varphi(t)] \varphi'(t) dt \right]_{t=\varphi^{-1}(x)} \quad (13.2.4)$$

$$\int f(x) dx = \int f(x) d(x + C) \quad (13.2.5)$$

13.2.1 分部积分法

$$\int u dv = uv - \int v du \Leftrightarrow \int uv' dx = uv - \int u'v dx \quad (13.2.6)$$

13.3 有理函数积分

13.3.1 普通多项式

$\frac{P(x)}{Q(x)}$ $P(x), Q(x)$ 是 x 多项式, 且没有公因子, 称为有理分式

$$\text{有理分式} \begin{cases} \text{真分式} & P(x) \text{ 次数} < Q(x) \text{ 次数} \\ \text{假分式} & P(x) \text{ 次数} \geq Q(x) \text{ 次数} \end{cases}$$

如果真分式中 $Q(x) = Q_1(x) \cdot Q_2(x)$, 其中 $Q_1(x), Q_2(x)$ 都为多项式

$$\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)} + \frac{P_2(x)}{Q_2(x)} \quad (13.3.1)$$

假分式 = 多项式 + 真分式

$$\text{最简分式} \quad \frac{A}{x-a} \quad \frac{A}{(x-a)^2} \quad \frac{Nx+M}{x^2+px+q} \quad \frac{Nx+m}{(x^2+px+q)^k}$$

13.3.2 三角函数多项式

三角有理分式: $R(\sin x, \cos x)$

$$\text{万能代换: } \tan \frac{x}{2} = u, x = 2 \arctan u, dx = \frac{2 du}{1+u^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = 2 \frac{\tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = 2 \frac{\tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{2u}{1 + u^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2} (1 - \tan^2 \frac{x}{2}) = \frac{1 - \tan^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1 - u^2}{1 + u^2}$$

$$\int R(\sin x, \cos x) dx = \int R\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right) \frac{2}{1+u^2} du = \int Y(u) du$$

$Y(u)$ 是 u 的有理函数

13.4 积分公式

13.4.1 幂数, 指数, 对数

$$\int k dx = kx + C \quad (13.4.1)$$

$$\int x^a dx = \frac{x^{a+1}}{a+1} + C \quad (13.4.2)$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \quad (13.4.3)$$

$$\int e^x dx = e^x + C \quad (13.4.4)$$

$$\int \frac{1}{x} dx = \ln |x| + C \quad (13.4.5)$$

$$\int \ln x dx = x \ln x - x + C \quad (13.4.6)$$

13.4.2 三角函数

$$\int \sin x \, dx = -\cos x + C \quad (13.4.7)$$

$$\int \cos x \, dx = \sin x + C \quad (13.4.8)$$

$$\int \sec x \tan x \, dx = \sec x + C \quad (13.4.9)$$

$$\int \csc x \cot x \, dx = -\csc x + C \quad (13.4.10)$$

$$\int \sec^2 x \, dx = \tan x + C \quad (13.4.11)$$

$$\int \csc^2 x \, dx = -\cot x + C \quad (13.4.12)$$

$$\int \sinh x \, dx = \cosh x + C \quad (13.4.13)$$

$$\int \cosh x \, dx = \sinh x + C \quad (13.4.14)$$

$$\int \tan x \, dx = -\ln |\cos x| + C \quad (13.4.15)$$

$$\int \csc x \, dx = \begin{cases} \ln \left| \tan \frac{x}{2} \right| + C \\ \ln |\csc x - \cot x| + C \end{cases} \quad (13.4.16)$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C \quad (13.4.17)$$

$$\int \arccos x \, dx = x \arccos x - \sqrt{1-x^2} + C \quad (13.4.18)$$

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C \quad (13.4.19)$$

13.4.3 分式

$$\int \frac{1}{1+x^2} dx = \arctan x + C \quad (13.4.20)$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan \frac{x}{a} + C \quad (13.4.21)$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \quad (13.4.22)$$

$$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C \quad (13.4.23)$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} \arcsin x + C \\ -\arccos x + C_1 \end{cases} \quad (13.4.24)$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \begin{cases} \arcsin \frac{x}{a} + C \\ -\arccos \frac{x}{a} + C_1 \end{cases} \quad (13.4.25)$$

$$\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln \left| x + \sqrt{x^2-a^2} \right| + C \quad (13.4.26)$$

$$\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln(x + \sqrt{x^2+a^2}) + C \quad (13.4.27)$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} dx = \operatorname{arcsec} x + C \quad (13.4.28)$$

14 定积分

14.1 定积分的定义

$[a, b]$ 有限区间, $f(x)$ 在 $[a, b]$ 上有界

$[a, b]$ 内任找 $n-1$ 个点, 分成 n 个区间

$$a = x_0 < x_1 < x_2 \cdots x_{n-1} < x_n = b$$

$$[x_0, x_1], [x_1, x_2] \cdots [x_{n-1}, x_n]$$

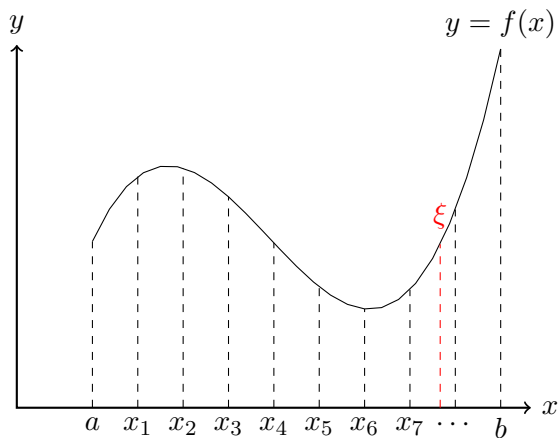
分成 n 个曲边梯形, $[x_{i-1}, x_i]$ 为第 i 个

面积 ΔS_i , 对应底 $\Delta x_i = x_i - x_{i-1}$

$$\forall \xi_i \in [x_{i-1}, x_i], \Delta S_i \approx f(\xi_i) \Delta x_i$$

$$S = \Delta S_1 + \Delta S_2 \cdots \Delta S_n = \sum_{i=1}^n f(\xi_i) \Delta x_i$$

$\lambda = \max \{ \Delta x_1, \Delta x_2, \cdots, \Delta x_n \}$, 当 $\lambda \rightarrow 0$ 时



$$S = \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

S 是一个定数, 则称 $f(x)$ 在 $[a, b]$ 上可积, S 称为 $f(x)$ 在 $[a, b]$ 上的定积分记作:

$$\int_a^b f(x) dx \left\{ \begin{array}{l} f(x) dx \quad \text{被积表达式} \\ x \quad \text{积分变量} \\ [a, b] \quad \text{积分区间} \end{array} \right. \left\{ \begin{array}{l} f(x) \quad \text{被积函数} \\ a \quad \text{积分下限} \\ b \quad \text{积分上限} \end{array} \right.$$

$$\int_a^b f(x) dx \triangleq \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i$$

14.2 可积的充分条件

如果 $f(x)$ 在 $[a, b]$ 上连续, 则 $f(x)$ 在 $[a, b]$ 上可积 (14.2.1)

如果 $f(x)$ 在 $[a, b]$ 上有界, 且至多有有限个间断点, 则 $f(x)$ 在 $[a, b]$ 上可积 (14.2.2)

14.3 定积分的性质

$$a < b < c, k \text{ 为常数}$$

$$\int_a^a f(x) dx = 0 \quad (14.3.1)$$

$$\int_a^b dx = b - a \quad (14.3.2)$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx \quad (14.3.3)$$

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx \quad (14.3.4)$$

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx \quad (14.3.5)$$

$$\int_a^b kf(x) dx = k \int_a^b f(x) dx \quad (14.3.6)$$

$$\int_a^b f(x) \pm g(x) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx \quad (14.3.7)$$

$$f(x) \geq 0 \Rightarrow \int_a^b f(x) dx \geq 0 \quad (14.3.8)$$

$$f(x) \geq g(x) \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx \quad (14.3.9)$$

$$\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx \quad (14.3.10)$$

14.4 积分估值公式

M 为区间 $[a, b]$ 最大值, m 为区间 $[a, b]$ 最小值, $a < b$

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a) \quad (14.4.1)$$

14.5 积分中值定理

$f(x)$ 是 $[a, b]$ 上的连续函数, 则, $\exists \xi \in [a, b], a < b$ 使

$$\int_a^b f(x) dx = f(\xi)(b-a) \quad (14.5.1)$$

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x) dx \quad \text{称为均值}$$

14.6 积分上限函数

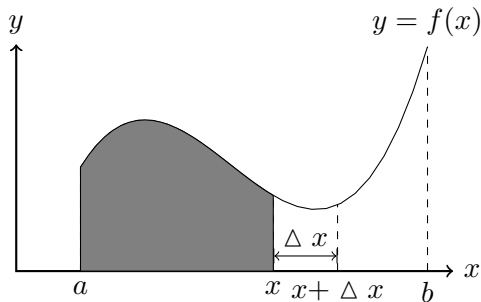
14.6.1 定义

$x \in [a, b], [a, x]$ 对应曲边梯形

$$\int_a^x f(x) dx = \int_a^x f(u) du$$

$$\phi(x) \triangleq \int_a^x f(u) du$$

$\phi(x)$ 是 $[a, b]$ 上函数称为积分上限函数



14.6.2 性质

$$\phi'(x) = \frac{d}{dx} \left[\int_a^x f(u) du \right] = f(x) \quad (14.6.1)$$

$$\frac{d}{dx} \left[\int_a^{\psi(x)} f(u) du \right] = f(\psi(x)) \psi'(x) \quad (14.6.2)$$

$$\frac{d}{dx} \left[\int_{v(x)}^{\psi(x)} f(u) du \right] = f[\psi(x)] \psi'(x) - f[v(x)] v'(x) \quad (14.6.3)$$

若 $f(x)$ 在 $[a, b]$ 上连续, 则 $f(x)$ 必存在原函数, $\phi(x) = \int_a^x f(u) du$

即为 $f(x)$ 在 $[a, b]$ 上的一个原函数

$$\int f(x) dx = \int_a^x f(u) du + C$$

14.7 微积分基本公式 (牛顿莱布兹尼公式)

$f(x)$ 在 $[a, b]$ 上连续, $F(x)$ 是 $f(x)$ 在 $[a, b]$ 上的的一个原函数

$$\int_a^b f(x) dx = F(b) - F(a) \triangleq [F(x)]_a^b = F(x)|_a^b \quad (14.7.1)$$

14.8 换元法

$$f(x) \text{ 在 } [a, b] \text{ 上连续, } x = \varphi(t) \begin{cases} \varphi(\alpha) = a \\ \varphi(\beta) = b \end{cases}$$

$\varphi(t)$ 在 $[\alpha, \beta]$ 上有连续导数, 且 $R_\varphi = [a, b]$

$$\int_a^b f(x) dx = \int_\alpha^\beta f[\varphi(t)] \varphi'(t) dt \quad (14.8.1)$$

14.9 分部积分法

$$\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du \quad (14.9.1)$$

14.10 奇偶函数积分

$$(\text{奇函数}) \quad \int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx \quad (14.10.1)$$

$$(\text{偶函数}) \quad \int_{-a}^a f(x) \, dx = 0 \quad (14.10.2)$$

14.11 周期函数积分

$$\int_a^{a+T} f(x) \, dx = \int_0^T f(x) \, dx \quad (14.11.1)$$

$$\int_a^{a+nT} f(x) \, dx = n \int_0^T f(x) \, dx \quad (14.11.2)$$

14.12 积分定理

$$\int_0^{\frac{\pi}{2}} f(\sin x) \, dx = \int_0^{\frac{\pi}{2}} f(\cos x) \, dx \quad (14.12.1)$$

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} & (\text{n 偶数}) \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1 & (\text{n 奇数}) \end{cases} \quad (14.12.2)$$

14.13 积分不等式

$$\left[\int_a^b f(x)g(x) \, dx \right]^2 \leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \quad (14.13.1)$$

$$\left\{ \int_a^b [f(x) + g(x)]^2 \, dx \right\}^{\frac{1}{2}} \leq \left[\int_a^b f^2(x) \, dx \right]^{\frac{1}{2}} + \left[\int_a^b g^2(x) \, dx \right]^{\frac{1}{2}} \quad (14.13.2)$$

14.14 一些废话 (显而易见的东西)

$$\text{若在 } [a, b] \text{ 上 } f(x) \geq 0, \text{ 且 } \int_a^b f(x) \, dx = 0, \text{ 则 } f(x) \equiv 0 \quad (14.14.1)$$

$$\text{若在 } [a, b] \text{ 上 } f(x) \geq 0, \text{ 且 } f(x) \not\equiv 0, \text{ 则 } \int_a^b f(x) \, dx > 0 \quad (14.14.2)$$

若在 $[a, b]$ 上 $f(x) \leq g(x)$, 且 $\int_a^b f(x) \, dx = \int_a^b g(x) \, dx$ 则 $f(x) = g(x), x \in [a, b]$ (14.14.3)

15 反常积分 (瞎积分)

15.1 有界反常积分

$f(x)$ 在 $[a, +\infty)$ 上连续, $\lim_{t \rightarrow +\infty} \int_a^t f(x) dx \begin{cases} \text{存在, 称: } \int_a^{+\infty} f(x) dx \text{ 收敛} \\ \text{不存在 (或无穷), 称: 为发散} \end{cases}$

$$\lim_{t \rightarrow +\infty} \int_a^t f(x) dx = \int_a^{+\infty} f(x) dx$$

15.2 有界反常积分

$f(x)$ 在 $(a, b]$ 上连续, 且 $\lim_{x \rightarrow a^+} f(x) = \infty$ $f(x)$ 在 $(a, b]$ 上无上界, 点 a 称为 $f(x)$ 的一个瞎点

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{a+\varepsilon}^b f(x) dx$$

16 向量

16.1 向量的概念

向量：既有大小又有方向的量

向量表示： \overrightarrow{AB} 或 \vec{a}

自由向量：与起点无关的向量

向量的模： $|\overrightarrow{AB}|$ $|\vec{a}|$

单位向量： $|\vec{a}| = 1$

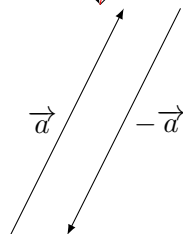
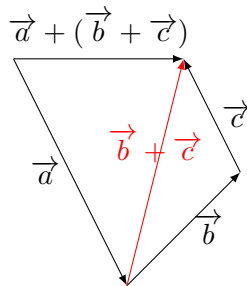
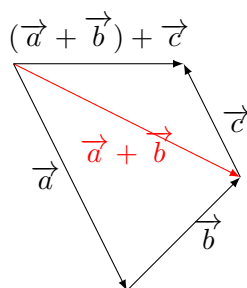
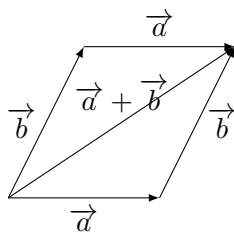
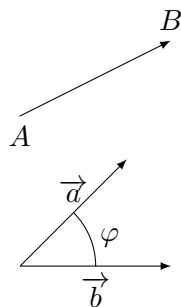
$$\vec{a} \text{ 单位向量: } \begin{cases} \vec{e}_{\vec{a}} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{|\vec{a}|} \vec{a} \\ \vec{a} = |\vec{a}| \vec{e}_{\vec{a}} \end{cases}$$

$$\vec{0} \text{ 向量: } \begin{cases} |\vec{a}| = 0 \text{ 记作: } \vec{a} = \vec{0} \text{ 方向任意} \\ \vec{0} \text{ 与任何向量平行} \\ \vec{0} \text{ 与任何向量垂直} \end{cases}$$

向量夹角： $0 \leq \varphi \leq \pi$ $(\vec{a} \wedge \vec{b}) = \varphi$

向量平行： $\vec{a} // \vec{b} \Leftrightarrow \begin{cases} (\vec{a} \wedge \vec{b}) = 0 \text{ 或 } (\vec{a} \wedge \vec{b}) = \pi \\ \vec{a} = \lambda \vec{b} \quad (\text{存在唯一 } \lambda) \end{cases}$

向量垂直： $\vec{a} \perp \vec{b}$ 或 $(\vec{a} \wedge \vec{b}) = \frac{\pi}{2}$



16.2 向量的线性运算

$$\text{三角不等式: } \begin{cases} |\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}| \\ |\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}| \end{cases}$$

$$\text{向量的加法: } \begin{cases} \vec{a} + \vec{b} \\ \text{交换律: } \vec{a} + \vec{b} = \vec{b} + \vec{a} \\ \text{结合律: } (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \end{cases}$$

$$\text{向量的减法: } \begin{cases} \text{负向量: 与 } \vec{a} \text{ 大小相等方向相反的向量 } -\vec{a} \\ \vec{b} - \vec{a} = \vec{b} + (-\vec{a}) \\ \vec{a} - \vec{a} = \vec{0} \\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \end{cases}$$

$$\text{向量的数乘: } \left\{ \begin{array}{l} \text{大小: } |\lambda \vec{a}| = |\lambda| |\vec{a}| \\ \text{方向: } \begin{cases} \lambda > 0 & \text{时} \lambda \vec{a} \text{与} \vec{a} \text{方向一致} \\ \lambda < 0 & \text{时} \lambda \vec{a} \text{与} \vec{a} \text{方向相反} \end{cases} \\ \text{特殊数相乘: } \begin{cases} 0: 0\vec{a} = \vec{0} \\ 1: 1\vec{a} = \vec{a} \\ -1: (-1)\vec{a} = -\vec{a} \end{cases} \\ \text{交换律: } \lambda(\mu \vec{a}) = (\lambda\mu)\vec{a} = \mu(\lambda \vec{a}) \\ \text{结合律: } \begin{cases} (\lambda + \mu)\vec{a} = \lambda\vec{a} + \mu\vec{a} \\ \lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} \end{cases} \end{array} \right.$$

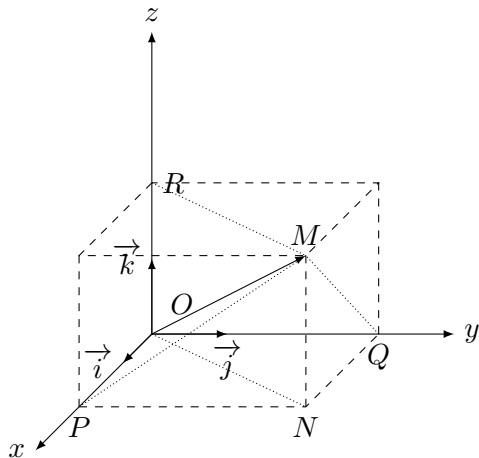
16.3 空间直角坐标系

坐标原点: O

$$\text{坐标轴: } \begin{cases} x\text{轴} & \text{单位向量} \vec{i} \\ y\text{轴} & \text{单位向量} \vec{j} \\ z\text{轴} & \text{单位向量} \vec{k} \end{cases}$$

坐标面: $x \circ y$ 面 $y \circ z$ 面 $z \circ x$ 面

$$\text{坐标向量运算: } \begin{cases} \vec{a} = (a_x, a_y, a_z) & \vec{b} = (b_x, b_y, b_z) \\ \vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z) \\ \vec{a} - \vec{b} = (a_x - b_x, a_y - b_y, a_z - b_z) \\ \lambda \vec{a} = (\lambda a_x, \lambda a_y, \lambda a_z) \end{cases}$$



$$\text{方向角: } \begin{cases} x\text{轴} & \cos \alpha = \frac{x}{|\vec{OM}|} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \\ y\text{轴} & \cos \beta = \frac{y}{|\vec{OM}|} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \\ z\text{轴} & \cos \gamma = \frac{z}{|\vec{OM}|} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \end{cases} \quad \text{投影: } \begin{cases} x\text{轴 记: } Prj_x \vec{OM} = |\vec{OM}| \cos \alpha \\ y\text{轴 记: } Prj_y \vec{OM} = |\vec{OM}| \cos \beta \\ z\text{轴 记: } Prj_z \vec{OM} = |\vec{OM}| \cos \gamma \end{cases}$$

$$M\text{点坐标}(x, y, z) \Leftrightarrow \vec{OM} = x\vec{i} + y\vec{j} + z\vec{k} \quad (16.3.1)$$

$$|\vec{OM}| = \sqrt{x^2 + y^2 + z^2} \quad (16.3.2)$$

$$A(x_1, y_1, z_1) \quad B(x_2, y_2, z_2) \quad |AB| = |\vec{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \quad (16.3.3)$$

$$(\cos \alpha, \cos \beta, \cos \gamma) = \frac{\overrightarrow{OM}}{|\overrightarrow{OM}|} = \vec{e}_{\overrightarrow{OM}} \quad (16.3.4)$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (16.3.5)$$

$$\text{投影定义 } Prj_{\vec{u}} \vec{a} = |\vec{a}| \cos(\vec{a} \wedge \vec{u})$$

$$Prj_{\vec{u}}(\vec{a} + \vec{b}) = Prj_{\vec{u}} \vec{a} + Prj_{\vec{u}} \vec{b} \quad (16.3.6)$$

$$Prj_{\vec{u}} \lambda \vec{a} = \lambda Prj_{\vec{u}} \vec{a} \quad (16.3.7)$$

16.4 数量积, 向量积, 混合积

$$\text{数量积 (内积): } \vec{a} \bullet \vec{b} \triangleq |\vec{a}| |\vec{b}| \cos(\vec{a} \wedge \vec{b}) \left\{ \begin{array}{l} \vec{a} \bullet \vec{b} = |\vec{a}| Prj_{\vec{a}} \vec{b} = |\vec{b}| Prj_{\vec{b}} \vec{a} \\ \vec{a} \bullet \vec{a} = |\vec{a}|^2 \\ \vec{a} \perp \vec{b} \Leftrightarrow \vec{a} \bullet \vec{b} = 0 \\ \vec{a} \bullet \vec{b} = \vec{b} \bullet \vec{a} \quad (\text{交换律}) \\ (\lambda \vec{a}) \bullet \vec{b} = \vec{a} \bullet (\lambda \vec{b}) \\ (\vec{a} + \vec{b}) \bullet \vec{c} = \vec{a} \bullet \vec{c} + \vec{b} \bullet \vec{c} \quad \text{分配律} \end{array} \right.$$

$$\text{坐标表达式: } \vec{a} \bullet \vec{b} = a_x b_x + a_y b_y + a_z b_z \quad (16.4.1)$$

$$\text{向量积: } |\vec{a} \times \vec{b}| \triangleq |\vec{a}| |\vec{b}| \sin(\vec{a} \wedge \vec{b}) \left\{ \begin{array}{l} \vec{a} \times \vec{a} = 0 \\ \vec{a} // \vec{b} \Leftrightarrow \vec{a} \times \vec{b} = 0 \\ \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \\ (\lambda \vec{a}) \times \vec{b} = \vec{a} \times (\lambda \vec{b}) \\ (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \\ \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \end{array} \right.$$

$\vec{a} \times \vec{b}$ 是垂直与两个向量的向量, 右手 a 到 b, 大拇指指的方向

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \quad (16.4.2)$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$\text{混合积: } (\vec{a} \times \vec{b}) \bullet \vec{c} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} c_x - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} c_y + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} c_z$$

$$\begin{vmatrix} c_x & c_y & c_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

16.5 空间曲面及其方程

16.5.1 概念

$$F(x, y, z) = 0$$

与平面垂直的非零向量称为法向量

16.5.2 平面表达式

法向量 $\vec{n} = (A, B, C)$ 平面 π 上一点 $P_0(x_0, y_0, z_0)$, $M(x, y, z)$ 为平面 π 上任意点

平面点法式: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

平面截距式: $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\text{平面一般方程: } Ax + By + Cz + D = 0 \left\{ \begin{array}{l} D = 0 \Leftrightarrow \pi \text{ 过原点} \\ A = 0 \Leftrightarrow \vec{n} \perp x \text{ 轴} \Leftrightarrow \pi // x \text{ 轴} \\ B = 0 \Leftrightarrow \pi // y \text{ 轴} \\ C = 0 \Leftrightarrow \pi // z \text{ 轴} \\ A = 0, B = 0 \Leftrightarrow \pi // x \circ y \text{ 面} \end{array} \right.$$

16.5.3 平面夹角

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 & \vec{n}_1 &= (A_1, B_1, C_1) \\ A_2x + B_2y + C_2z + D_2 &= 0 & \vec{n}_2 &= (A_2, B_2, C_2) \\ \cos \theta &= \frac{|\vec{n}_1 \bullet \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} = \frac{|A_1A_2 + B_1B_2 + C_1C_2|}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \end{aligned}$$

16.5.4 点到平面距离

平面 π : $Ax + By + Cz + D = 0$ 平面外一点 $P_0(x_0, y_0, z_0)$

$$|P_0N| = \left| \text{Proj}_{\vec{n}} \overrightarrow{P_1P_0} \right| = \frac{|\vec{n} \bullet \overrightarrow{P_1P_0}|}{|\vec{n}|} = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

16.6 空间线及其方程

16.6.1 曲线的一般方程

$$\text{一般方程} \begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}$$

16.6.2 直线的一般方程

$$\text{一般方程:} \begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

$$\text{对称式: } \frac{x - x_0}{m} = \frac{y - y_0}{n} = \frac{z - z_0}{p} = t \begin{cases} \overrightarrow{M_0M} = (x - x_0, y - y_0, z - z_0) \\ \text{方向量 } \vec{S} = (m, n, p) \end{cases}$$

$$\text{参数方程:} \begin{cases} x = x_0 + mt \\ y = y_0 + nt \\ z = z_0 + pt \end{cases}$$

16.6.3 两直线夹角

两直线的方向向量的夹角称为两直线的夹角 (要求为锐角)

L_1 : 方向向量 $\vec{S}_1 = (m_1, n_1, p_1)$

L_2 : 方向向量 $\vec{S}_2 = (m_2, n_2, p_2)$

L_1 与 L_2 的夹角 $(\vec{S}_1 \wedge \vec{S}_2) \quad (\vec{S}_1 \wedge -\vec{S}_2)$

$$\cos \varphi = \frac{|\vec{S}_1 \bullet \vec{S}_2|}{|\vec{S}_1||\vec{S}_2|} = \frac{|m_1m_2 + n_1n_2 + p_1p_2|}{\sqrt{m_1^2 + n_1^2 + p_1^2} \sqrt{m_2^2 + n_2^2 + p_2^2}}$$

16.6.4 线与面夹角

L : 方向向量 $\vec{S} = (m, n, p)$

π : 法向量 $\vec{n} = (A, B, C)$

$$\cos \theta = \frac{|\vec{S} \bullet \vec{n}|}{|\vec{S}||\vec{n}|} = \frac{|Am + Bn + Cp|}{\sqrt{A^2 + B^2 + C^2} \sqrt{m^2 + n^2 + p^2}}$$

$$\sin \varphi = \cos\left(\frac{\pi - \theta}{2}\right)$$

17 多元函数

17.1 概念

17.1.1 二元函数定义

平面点集 $R^2 = R \cdot R = \{(x, y) | x \in R, y \in R\}$

设 $\{D \neq \emptyset\} \subset R^2, \forall P(x, y)$, 按照法则 f 都有唯一的实数值与之对应, 称 f 是定义在 D 上的二元函数记作

$$z = f(x, y) \begin{cases} x, y \text{ 自变量} \\ z \text{ 因变量} \\ D \text{ 定义域} \\ f(D) \text{ 值域} \end{cases}$$

17.1.2 二元函数极限

$P(x, y) \rightarrow P_0(x_0, y_0)$ 时, $f(x, y) \rightarrow A$

$\forall \epsilon > 0, \exists \delta > 0, 0 < |PP_0| < \delta$, 且 $P \in D, |f(P) - A| < \delta$ 成立

称当 $P(x, y) \rightarrow P_0(x_0, y_0)$ 时, $f(x, y)$ 的极限为 A

$$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = A \quad \text{或} \quad \lim_{P \rightarrow P_0} f(P) = A \quad \text{或} \quad \lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = A$$

17.1.3 二元函数连续

$f(x, y)$ 的定义域为 D $P_0(x_0, y_0)$ 为 D 内点

如果 $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x, y) = f(x_0, y_0)$ 则称 $f(x, y)$ 在点 (x_0, y_0) 连续

连续函数的和差积商仍为连续函数。连续函数的复合函数仍为连续函数。

多元函数的初等函数: 由常数, 一元初等函数 (可以不同变量), 有限次四则运算, 复合, 用一个式子表达的函数。

多元函数在其定义域内都是连续的

断点: $f(x, y)$ 的定义域为 $D, P_0(x_0, y_0)$ 是 D 的聚点

如果 $f(x, y)$ 在 $P_0(x_0, y_0)$ 点不连续, 称 $P_0(x_0, y_0)$ 为 $f(x, y)$ 的一个间断点

17.1.4 二元函数偏导

$f(x, y)$ 在 (x_0, y_0) 的某处有定义, 固定 y_0, x 在 x_0 处增量 Δx , 增量 $f(x_0 + \Delta x, y_0) - f(x_0, y_0)$ 如果

$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

存在则称此极限为 $z = f(x, y)$ 在 (x_0, y_0) 点关于 x 的偏导数, 记作

$$\left. \frac{\partial z}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}} \quad \text{或} \quad \left. \frac{\partial f}{\partial x} \right|_{\substack{x=x_0 \\ y=y_0}} \quad \text{或} \quad f'_x(x_0, y_0) \quad \text{或} \quad f'_1(x_0, y_0)$$

$$f'_x(x_0, y_0) \triangleq \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$f'_y(x_0, y_0) \triangleq \lim_{\Delta y \rightarrow 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

17.2 高阶偏导

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &\triangleq \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) & \frac{\partial^2 z}{\partial y^2} &\triangleq \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial y} \right) \\ \frac{\partial^2 z}{\partial x \partial y} &\triangleq \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) & \frac{\partial^2 z}{\partial y \partial x} &\triangleq \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) \\ \frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial^2 z}{\partial y \partial x} \end{aligned} \quad (17.2.1)$$

17.3 全微分

$$z = f(x, y) \begin{cases} f(x + \Delta x, y) - f(x, y) & x \text{ 的偏增量} \\ f(x, y + \Delta y) - f(x, y) & y \text{ 的偏增量} \end{cases}$$

$\Delta z \triangleq f(x + \Delta x, y + \Delta y) - f(x, y)$ 称为 $z = f(x, y)$ 在点 (x, y) 处的全增量

17.3.1 定义

$z = f(x, y)$ 在点 (x, y) 的某领域有定义, $\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$

如果 $\Delta z = A\Delta x + B\Delta y + o(\rho)$, 其中 $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

称 $z = f(x, y)$ 在点 (x, y) 处可微分, $A\Delta x + B\Delta y$ 称为 $z = f(x, y)$ 在 (x, y) 的全微分, 记

$$dz = A\Delta x + B\Delta y$$

一阶全微分

$$dz = \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y \quad (17.3.1)$$

二阶全微分

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2 \frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2 \quad (17.3.2)$$

17.4 多元复合

$$z = f(u(t), v(t)) \left\{ \begin{array}{l} \frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \frac{du}{dt} + \frac{\partial z}{\partial v} \frac{dv}{dt} \end{array} \right.$$

$$z = f(u(x, y), v(x, y)) \left\{ \begin{array}{l} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{du}{dx} + \frac{\partial z}{\partial v} \frac{dv}{dx} \\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{du}{dy} + \frac{\partial z}{\partial v} \frac{dv}{dy} \end{array} \right. \Rightarrow dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv$$

17.5 多元隐函数

17.5.1 二元

$F(x, y)$ 在 (x_0, y_0) 的邻域内有连续的偏导数, 且 $F(x_0, y_0) = 0, F'_y(x_0, y_0) \neq 0$ 则方程 $F(x, y) = 0$ 在点 (x_0, y_0) 的邻域内存在唯一的隐函数 $y = y(x)$

$$F(x, y) = 0, y = y(x) \Rightarrow F(x, y(x)) \equiv 0$$

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}$$

17.5.2 三元

$F(x, y, z)$ 在 (x_0, y_0, z_0) 的邻域内有连续的偏导数, 且 $F(x_0, y_0, z_0) = 0, F'_z(x_0, y_0, z_0) \neq 0$ 则方程 $F(x, y, z) = 0$ 在点 (x_0, y_0, z_0) 的邻域内存在唯一的隐函数 $z = z(x, y)$

$$F(x, y, z) = 0, z = z(x, y) \Rightarrow F(x, y, z(x, y)) \equiv 0$$

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \quad \frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}$$

17.5.3 方程组

$$\left\{ \begin{array}{l} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{array} \right. \quad \frac{dy}{dx} = -\frac{\frac{\partial(F, G)}{\partial(x, z)}}{\frac{\partial(F, G)}{\partial(y, z)}} \quad \frac{dz}{dx} = -\frac{\frac{\partial(F, G)}{\partial(y, x)}}{\frac{\partial(F, G)}{\partial(y, z)}} \quad (17.5.1)$$

$$\left\{ \begin{array}{l} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{array} \right. \quad \frac{dy}{dx} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}} \quad \frac{dz}{dx} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, v)}} \quad (17.5.2)$$

18 向量导数

$$\begin{aligned}\vec{r} &= x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \quad t \in [\alpha, \beta] \quad \begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t) \end{cases} \\ \vec{f}(t) &= f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k} = (f_1(t), f_2(t), f_3(t)) \\ \vec{r} &= \vec{f}(t) \quad t \in [\alpha, \beta]\end{aligned}$$

18.1 向量值函数

$D \in R \quad f: D \rightarrow R^n$, 称 f 为一元向量值函数, $\vec{r} = \vec{f}(t) \quad t \in D$

当 $n = 3$ 的情形 $t \in D \rightarrow M(x, y, z)$

$\vec{F} = \overrightarrow{OM}$ t 的变化, M 点轨迹 Γ (为空间曲线), 称为 $\vec{r} = \vec{f}(t)$ 的图形, $\vec{r} = \vec{f}(t)$ 称为 Γ 的方程

18.2 极限

$\vec{F} = \vec{f}(t)$ 在某个区间领域有定义 \vec{r}_0 是一个常向量

$$\forall \varepsilon > 0. \exists \delta > 0, 0 < |t - t_0| < \delta, |\vec{f}(t) - \vec{r}_0| < \varepsilon$$

称 \vec{r}_0 为 $\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处的极限 (向量), 记作

$$\lim_{t \rightarrow t_0} \vec{f}(t) = \vec{r}_0 \Leftrightarrow \lim_{t \rightarrow t_0} \vec{f}(t) = \left(\lim_{t \rightarrow t_0} \vec{f}_1(t), \lim_{t \rightarrow t_0} \vec{f}_2(t), \lim_{t \rightarrow t_0} \vec{f}_3(t) \right)$$

18.3 连续

$\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处连续

$$\lim_{t \rightarrow t_0} \vec{f}(t) = \lim_{t \rightarrow t_0} \vec{f}(t_0)$$

$\vec{r} = \vec{f}(t)$ 在 $D_1 \subset D$, 上每一点连续, 称 $\vec{r} = \vec{f}(t)$ 在 D_1 上连续

18.4 导数

$\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 某邻域内有定义, 如果

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\vec{f}(t + \Delta t) - \vec{f}(t)}{\Delta t}$$

称此极限为 $\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处的导向量, 记作

$$\vec{f}'(t_0) \quad \text{或} \quad \left. \frac{d\vec{r}}{dt} \right|_{t=t_0}$$

$$\vec{f}'(t) = (\vec{f}'_1(t), \vec{f}'_2(t), \vec{f}'_3(t))$$

18.5 向量函数求导法则

$$\frac{d\vec{c}}{dt} = \vec{0} \quad (\vec{c} \text{ 为常数}) \quad (18.5.1)$$

$$\frac{d[c\vec{u}(t)]}{dt} = c \frac{d\vec{u}(t)}{dt} \quad (18.5.2)$$

$$\frac{d[\vec{u}(t) \pm \vec{v}(t)]}{dt} = \frac{d\vec{u}(t)}{dt} \pm \frac{d\vec{v}(t)}{dt} \quad (18.5.3)$$

$$\frac{d[\varphi(t)\vec{u}(t)]}{dt} = \varphi'(t)\vec{u}(t) + \varphi(t)\vec{u}'(t) \quad (18.5.4)$$

$$\frac{d[\vec{u}(t) \bullet \vec{v}(t)]}{dt} = \vec{u}'(t) \bullet \vec{v}(t) + \vec{u}(t) \bullet \vec{v}'(t) \quad (18.5.5)$$

$$\frac{d[\vec{u}(t) \times \vec{v}(t)]}{dt} = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t) \quad (18.5.6)$$

$$\frac{d\vec{u}[\varphi(t)]}{dt} = \vec{u}'(t)\varphi'(t) \quad (18.5.7)$$

18.6 曲线的切线与法平面

$$\Gamma: \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [\alpha, \beta], t = t_0 \text{ 对应点 } M(x_0, y_0, z_0)$$

$$\text{切向量: } \vec{\Gamma} = (x'(t_0), y'(t_0), z'(t_0))$$

$$\text{切线方程: } \frac{x - x_0}{x'(t_0)} = \frac{y - y_0}{y'(t_0)} = \frac{z - z_0}{z'(t_0)}$$

$$\text{法平面方程: } x'(t_0)(x - x_0) + y'(t_0)(y - y_0) + z'(t_0)(z - z_0)$$

18.7 曲面的切线与法平面

空间曲面 $\Sigma, F(x, y, z) = 0, M(x_0, y_0, z_0)$ 是 Σ 上的一点

$$\Gamma \text{ 参数方程: } \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \quad t \in [\alpha, \beta]$$

$$\Gamma \text{ 在 } \Sigma \text{ 上 } F(x(t), y(t), z(t)) \equiv 0, t \in [\alpha, \beta]$$

$$F'_x x'(t) + F'_y y'(t) + F'_z z'(t) = 0$$

$t = t_0$ 对应点 M

$$F'_x(x_0, y_0, z_0)x'(t_0) + F'_y(x_0, y_0, z_0)y'(t_0) + F'_z(x_0, y_0, z_0)z'(t_0) = 0$$

$\vec{T} = (x'(t_0), y'(t_0), z'(t_0))$ 是 Γ 过点 M 的一个切向量

$$\vec{n} = (F'_x(x_0, y_0, z_0), F'_y(x_0, y_0, z_0), F'_z(x_0, y_0, z_0))$$

$\vec{n} \bullet \vec{T} = 0$, 即 $\vec{n} \perp \vec{T}$ \vec{n} 与所有切线垂直 \vec{n} 为切平面的法向量

18.8 方向导数

$P_0(x_0, y_0)$ 以 $P_0(x_0, y_0)$ 为始的射线 L , 单位向量 \vec{e}_l

$$\vec{e}_l(\cos \alpha, \cos \beta) = \cos \alpha \vec{i} + \cos \beta \vec{j}$$

$$P_0(x, y) \text{ 是 } L \text{ 上一点 } \begin{cases} x = x_0 + t \cos \alpha \\ y = y_0 + t \cos \beta \end{cases} \quad t > 0$$

$P_0(x, y)$ 沿 L 方向趋于 $P_0(x_0, y_0)$ ($t \rightarrow 0^+$) $z = f(x, y)$

$$\left. \frac{\partial f}{\partial L} \right|_{(x_0, y_0)} \triangleq \lim_{t \rightarrow 0^+} \frac{f(x_0 + t \cos \alpha, y_0 + t \cos \beta) - f(x_0, y_0)}{t}$$

$z = f(x, y)$ 在点 (x_0, y_0) 处沿 L 的方向导数

$$\left. \frac{\partial f}{\partial L} \right|_{(x_0, y_0)} = f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \cos \beta \quad (18.8.1)$$

18.9 梯度

$$\vec{\nabla} f(x_0, y_0) = \overrightarrow{\text{grad}} f(x_0, y_0) \triangleq (f'_x(x_0, y_0), f'_y(x_0, y_0))$$

$$\vec{\nabla} f(x_0, y_0, z_0) = \overrightarrow{\text{grad}} f(x_0, y_0, z_0) \triangleq (f'_x(x_0, y_0, z_0), f'_y(x_0, y_0, z_0), f'_z(x_0, y_0, z_0))$$

$$\left. \frac{\partial f}{\partial L} \right|_{(x_0, y_0)} = \vec{\nabla} f(x_0, y_0) \bullet \vec{e}_L \quad \vec{e}_L = (\cos \alpha, \cos \beta)$$

18.10 极值

$$\text{驻点} \begin{cases} f'_x(x, y) = 0 \\ f'_y(x, y) = 0 \end{cases}$$

$$\text{记} \begin{cases} A = f''_{xx}(x_0, y_0) \\ B = f''_{xy}(x_0, y_0) \\ C = f''_{yy}(x_0, y_0) \end{cases} \quad \Delta = AC - B^2 \begin{cases} \Delta > 0, z = f(x, y) \text{ 在 } (x_0, y_0) \text{ 点 } \begin{cases} A > 0 & \text{极小值} \\ A < 0 & \text{极大值} \end{cases} \\ \Delta = 0, \text{未定} \\ \Delta = 0, z = f(x, y) \text{ 在 } (x_0, y_0) \text{ 点无极值} \end{cases}$$

19 重积分

19.1 二重积分

19.1.1 定义

区域 D , D 上函数 $z = f(x, y)$, D 分割成 $\Delta\sigma_1, \Delta\sigma_2, \dots, \Delta\sigma_n, (\xi_i, \eta_i)\sigma_i$

如果 $\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta\sigma_i$ 存在, 称极限为 $f(x, y)$ 在 D 上的二重积分, 记作

$$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta\sigma_i \triangleq \iint_D f(x, y) d\sigma \quad \left\{ \begin{array}{ll} f(x, y) & \text{被积函数} \\ D & \text{积分区域} \\ x, y & \text{积分变量} \\ f(x, y) d\sigma & \text{积分表达式} \\ d\sigma & \text{面积元素} = \begin{cases} dxdy & \text{直角坐标坐标系} \end{cases} \end{array} \right.$$

19.1.2 性质

$$\iint_D \alpha f(x, y) d\sigma = \alpha \iint_D f(x, y) d\sigma \quad \alpha \text{ 为常数}$$

$$\iint_D [f(x, y) \pm g(x, y)] d\sigma = \iint_D f(x, y) d\sigma \pm \iint_D g(x, y) d\sigma$$

$$\iint_D f(x, y) d\sigma = \iint_{D_1} f(x, y) d\sigma + \iint_{D_2} f(x, y) d\sigma \quad D = D_1 + D_2$$

$$\iint_D d\sigma = \iint_D 1 d\sigma = \sigma$$

$$f(x, y) \leq g(x, y) \Rightarrow \iint_D f(x, y) d\sigma \leq \iint_D g(x, y) d\sigma$$

$$\left| \iint_D f(x, y) d\sigma \right| \leq \iint_D |f(x, y)| d\sigma$$

$$m, M \text{ 分别为 } f(x, y) \text{ 在 } D \text{ 上的最小值与最大值, 则 } m\sigma \leq \iint_D f(x, y) d\sigma \leq M\sigma$$

$$f(x, y) \text{ 在 } D \text{ 上连续, 则至少存在一点 } (\xi, \eta) \text{ 使 } \iint_D f(x, y) d\sigma = f(\xi, \eta)\sigma$$

19.1.3 换源法

$f(x, y)$ 在 D 上连续, 变换 $T \begin{cases} x = x(u, v) \\ y = y(u, v) \end{cases} \quad D' \rightarrow D$

满足 $T: D' \rightarrow D$ 是一一对应的, $x(u, v), y(u, v)$ 在 D' 上有一阶连续偏导

在 D' 上 $J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \neq 0$

$$\iint_D f(x, y) dx dy = \iint_{D'} f(x(u, v), y(u, v)) |J| du dv \quad (19.1.1)$$

19.1.4 奇偶性

$f(x, y)$ 关于 x 为奇函数, 且 D 关于 y 轴对称, 则

$$\iint_D f(x, y) dx dy = 0$$

$f(x, y)$ 关于 x 为偶函数, 且 D 关于 y 轴对称, 则

$$D = D_1 + D_2 \quad D_1, D_2 \text{ 关于 } y \text{ 轴对称} \quad \iint_D f(x, y) dx dy = 2 \iint_{D_1} f(x, y) dx dy$$

19.2 三重积分

空间区域 Ω , 在一个有界函数 $f(x, y, z)$ 分割 $\Omega, \Delta v_1, \Delta v_2, \dots, \Delta v_n$ 任取 $(\xi_i, \eta_i, \zeta_i) \in v_i$

$\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta v_i$ (λ 分割的直径) 若极限存在

则称其为 $f(x, y, z)$ 在 Ω 上的三重积分, 记作

$$\iiint_{\Omega} f(x, y, z) dv \triangleq \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i, \zeta_i) \Delta v_i$$

20 曲线积分

20.1 曲线积分定义

20.1.1 弧长曲线积分

$x \circ y$ 面上曲线 L (分段光滑), $f(x, y)$ 在 L 上有界, 对 L 进行分割, 取积, 求和, 取极限, 如果 $\lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta S_i$ 存在, 称此极限为 $f(x, y)$ 在 L 上对弧长的曲线积分, 记作

$$\int_L f(x, y) ds \triangleq \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i, \eta_i) \Delta S_i$$

20.1.2 封闭曲线积分

如果 L 是封闭曲线, 记作

$$\oint_L f(x, y) ds$$

20.2 性质

$$\begin{aligned} \int_L [\alpha f(x, y) + \beta g(x, y)] ds &= \alpha \int_L f(x, y) ds + \beta \int_L g(x, y) ds \\ L = L_1 + L_2 \quad \int_L f(x, y) ds &= \int_{L_1} f(x, y) ds + \int_{L_2} f(x, y) ds \end{aligned}$$

$$f(x, y) \leq g(x, y) \quad \int_L f(x, y) ds \leq \int_L g(x, y) ds$$

$$\left| \int_L f(x, y) ds \right| \leq \int_L |f(x, y)| ds$$

$$f(x, y) \text{ 在 } L \text{ 上连续, 则至少存在一点 } (\xi, \eta) \in L \text{ 使 } \int_L f(x, y) ds = f(\xi, \eta) l$$

20.3 弧微分计算

$$f(x, y) \text{ 在 } L \text{ 上连续, } L \text{ 的参数方程 } \begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (\alpha \leq t \leq \beta)$$

若 $\frac{dx}{dt}, \frac{dy}{dt}$ 在 $[\alpha, \beta]$ 上连续, 且 $\left(\frac{dx}{dt}\right)^2 + \left(f \frac{dy}{dt}\right)^2 \neq 0$ 则

$$\int_L f(x, y) ds = \int_{\alpha}^{\beta} f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

21 零散的一些

$$\sum_{k=0}^n q^k = \frac{1 - q^{n+1}}{1 - q} \quad (21.0.1)$$

$$\begin{aligned} A_N &= \sum_{k=0}^n q^k & q \cdot A_N &= \sum_{k=1}^{n+1} q^k \\ A_N - q \cdot A_N &= \sum_{k=0}^n q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1} \\ A_N &= \frac{1 - q^{n+1}}{1 - q} \end{aligned}$$

$$\log_{10} x = \lg_x \quad (21.0.2)$$

$$\log_e x = \ln_x \quad (21.0.3)$$

$$\log_b xy = \log_b x + \log_b y \quad (21.0.4)$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \quad (21.0.5)$$

$$\log_b x^n = n \log_b x \quad (21.0.6)$$

$$\log_b x = \frac{\log_c x}{\log_c b} \quad (21.0.7)$$

$$b^n = x \quad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \quad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = \log_{c(\log_c b)} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^n C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$a^n - b^n = (a-b) \sum_{m=0}^{n-1} (a^{n-m} b^m) = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$$

22 证明

22.1 第 1 章

1.2.4

$$\begin{aligned}
 \sinh x \cosh x &= \left(\frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} \right) \\
 &= \left(\frac{1}{2} \right) \left(\frac{e^{2x} - e^{-2x}}{2} \right) \\
 &= \frac{1}{2} \sinh(2x) \\
 \sinh(2x) &= 2 \sinh x \cosh x
 \end{aligned}$$

1.2.5

$$\begin{aligned}
 \cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \right) \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \right) \\
 &= e^x \times e^{-x} \\
 &= 1
 \end{aligned}$$

1.2.6

$$\begin{aligned}
 \cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} \right)^2 + \left(\frac{e^x - e^{-x}}{2} \right)^2 \\
 &= \frac{2e^{2x} + 2e^{-2x}}{4} \\
 &= \frac{e^{2x} + e^{-2x}}{2} \\
 &= \cosh(2x)
 \end{aligned}$$

1.2.7

$$\begin{aligned}
 \cosh(2x) &= \cosh^2 x + \sinh^2 x \\
 &= \sinh^2 x + 1 + \sinh^2 x \\
 &= 2 \sinh^2 x + 1 \\
 \cosh x &= 2 \sinh^2 \frac{x}{2} + 1
 \end{aligned}$$

1.1.17

$$\begin{aligned}
\tan \frac{x}{2} &= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \sqrt{\frac{1 - \cos x}{1 + \cos x}} \\
&= \sqrt{\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}} \\
&= \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}} \\
&= \frac{\sin x}{1 + \cos x} \\
&= \frac{1 - \cos x}{\sin x} \\
&= \csc x - \cot x
\end{aligned}$$

22.2 第 5 章

5.2.1

$$\begin{aligned}
&\text{设 } \forall x_1, x_2 \in [a, b], x_1 < x_2 \\
&f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \quad \xi \in (x_1, x_2) \subset [a, b] \\
&f'(\xi) > 0, (x_2 - x_1) > 0 \\
&f(x_2) - f(x_1) > 0 \\
&f(x_2) > f(x_1)
\end{aligned}$$

5.2.2

$$\begin{aligned}
&\text{设 } \forall x_1, x_2 \in [a, b], x_1 < x_2 \\
&f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \quad \xi \in (x_1, x_2) \subset [a, b] \\
&f'(\xi) < 0, (x_2 - x_1) > 0 \\
&f(x_2) - f(x_1) < 0 \\
&f(x_2) < f(x_1)
\end{aligned}$$

5.2.3

$$\begin{aligned}
&\text{设 } \forall x_1, x_2 \in [a, b], x_1 < x_2, x_0 = \frac{x_1 + x_2}{2}, x_0 - x_1 = x_2 - x_0 = h \\
&\varphi = f(x_0) - f(x_1) = f'(\xi_1)(x_0 - x_1) \quad \xi_1 \in (x_1, x_0) \\
&\psi = f(x_2) - f(x_0) = f'(\xi_2)(x_2 - x_0) \quad \xi_2 \in (x_0, x_2)
\end{aligned}$$

$$\begin{aligned}\psi - \varphi &= f(x_2) + f(x_1) - 2f(x_0) = [f'(\xi_2) - f'(\xi_1)] h \\ &= f''(\xi)(\xi_2 - \xi_1)h\end{aligned}$$

因为 $f''(x) > 0, f''(\xi) > 0, h = x_0 - x_1 > 0$

$$f(x_2) + f(x_1) - 2f(x_0) > 0$$

$$f(x_2) + f(x_1) > 2f(x_0)$$

$$f(x_0) < \frac{f(x_2) + f(x_1)}{2}$$

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_2) + f(x_1)}{2}$$

5.3.1

$$\Delta s = \widehat{M_0 M'} - \widehat{M_0 M} = \widehat{M M'}, \quad |MM'|^2 = (\Delta x)^2 + (\Delta y)^2, \quad \lim_{M' \rightarrow M} \frac{|\widehat{MM'}|}{|MM'|} = 1$$

$$\begin{aligned}\left(\frac{\Delta s}{\Delta x}\right)^2 &= \left|\frac{\widehat{MM'}}{\Delta x}\right|^2 = \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left(\frac{|MM'|}{\Delta x}\right)^2 \\ &= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta x)^2} \\ &= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]\end{aligned}$$

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \left(\frac{\Delta s}{\Delta x}\right)^2 &= \lim_{\Delta x \rightarrow 0} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\Delta x \rightarrow 0} \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right] \\ (\Delta x \rightarrow 0, \Delta M' \rightarrow M) \quad &= \lim_{M' \rightarrow M} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\Delta x \rightarrow 0} \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]\end{aligned}$$

$$\left(\frac{ds}{dx}\right)^2 = 1 \cdot (1 + (y')^2)$$

$$\frac{ds}{dx} = \sqrt{1 + (y')^2} = \sqrt{1 + [f'(x)]^2}$$

$$ds = \sqrt{1 + [f'(x)]^2} dx = \sqrt{(dx)^2 + (dy)^2}$$

5.3.2

$$\begin{aligned}\left|\frac{d\alpha}{ds}\right| &= \left|\frac{d\alpha}{dx} \cdot \frac{dx}{ds}\right| \\ &= \left|\frac{d \arctan y'}{dx} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right| \\ &= \left|\frac{y''}{1 + (y')^2} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right| \\ &= \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}\end{aligned}$$

5.3.3

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{\psi'(t)}{\phi'(t)} \\
 \frac{d^2y}{dx^2} &= \frac{d\frac{\psi'(t)}{\phi'(t)}}{dt} \cdot \frac{dt}{dx} \\
 &= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^2} \cdot \frac{1}{\phi'(t)} \\
 &= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3}
 \end{aligned}$$

$$\begin{aligned}
 \left| \frac{d\alpha}{ds} \right| &= \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}} \\
 &= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3} \cdot \frac{1}{\left\{ 1 + \left[\frac{\psi'(t)}{\phi'(t)} \right]^2 \right\}^{\frac{3}{2}}} \\
 &= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{ |\psi'(t)|^2 + [\phi'(t)]^2 \right\}^{\frac{3}{2}}}
 \end{aligned}$$

22.3 第 8 章

8.1.1

反设 $\lim_{n \rightarrow \infty} x_n = a$, $\lim_{n \rightarrow \infty} x_n = b$, 且 $a < b$

$$\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1, n > N_1, |x_n - a| < \frac{b-a}{3} \\ \exists N_2, n > N_2, |x_n - b| < \frac{b-a}{3} \end{cases}$$

$$N = \max\{N_1, N_2\}, n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}$$

$$\begin{aligned}
 b - a &= |(x_n - a) - (x_n - b)| \\
 &\leq |x_n - a| + |x_n - b| \\
 &< \frac{b-a}{3} + \frac{b-a}{3} \\
 &< \frac{2(b-a)}{3}
 \end{aligned}$$

8.1.2

$\varepsilon = 1$, $\exists N > 0$, 当 $n > N$ 时 $|X_n - a| < 1$

$$\begin{aligned}
 |X_n| &= |(X_n - a) + a| \\
 &\leq |x_n - a| + |a| \\
 &\leq 1 + |a|
 \end{aligned}$$

$$M = \max\{|X_n|, |X_2|, \dots, |X_n|, 1 + |a|\}$$

$$\forall n, |X_n| \leq M$$

8.1.4

1

由于 $\lim_{n \rightarrow \infty} x_n = a$, 且 $a > 0$

$$\varepsilon = \frac{a}{2}, \exists N > 0, n > N$$

$$|x_n - a| < \varepsilon$$

$$|x_n - a| < \frac{a}{2}$$

$$-\frac{a}{2} < x_n - a < \frac{a}{2}$$

$$\frac{a}{2} < x_n < 1$$

2

用反证法, 反设 $a < 0$. 从某项起 $x_n < 0$ 矛盾

8.1.5

$$x_n = b_n - a_n$$

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} b_n - \lim_{n \rightarrow \infty} a_n$$

$$\lim_{n \rightarrow \infty} x_n = b - a > 0$$

$$\lim_{n \rightarrow \infty} x_n > 0$$

$$b_n - a_n = x_n > 0$$

$$b_n > a_n$$

8.2.1

$$\lim_{x \rightarrow x_0} f(x) \text{ 存在} \Rightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

$$\text{设 } \lim_{x \rightarrow x_0} = A$$

$$0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon$$

$$0 < |x - x_0| < \delta \Leftrightarrow x \in \mathring{U}(x_0, \delta)$$

$$\left\{ \begin{array}{l} \text{当 } x_0 < x < x_0 + \delta \text{ 时 } 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon, \lim_{x \rightarrow x_0^+} f(x) = A \\ \text{当 } x_0 - \delta < x < x_0 \text{ 时 } 0 < |x - x_0| < \delta, |f(x) - A| < \varepsilon, \lim_{x \rightarrow x_0^-} f(x) = A \end{array} \right.$$

$$\lim_{x \rightarrow x_0^+} f(x) = A = \lim_{x \rightarrow x_0^-} f(x)$$

$$\lim_{x \rightarrow x_0} f(x) \text{ 存在} \Leftrightarrow \lim_{x \rightarrow x_0^+} f(x) = \lim_{x \rightarrow x_0^-} f(x)$$

$$A = \begin{cases} \lim_{x \rightarrow x_0^+}, \forall \varepsilon > 0, \exists \delta_1 > 0, x_0 < x < x_0 + \delta_1, |f(x) - A| < \varepsilon \\ \lim_{x \rightarrow x_0^-}, \forall \varepsilon > 0, \exists \delta_2 > 0, x_0 - \delta_2 < x < x_0, |f(x) - A| < \varepsilon \end{cases}$$

$$\delta = \min\{\delta_1, \delta_2\}$$

$$0 < |x - x_0| < \delta \begin{cases} x > x_0, x_0 < x < x_0 + \delta \leq x_0 + \delta_1, |f(x) - A| < \varepsilon \\ x < x_0, x_0 - \delta_2 \leq x_0 + \delta < x < x_0, |f(x) - A| < \varepsilon \end{cases}$$

$$\lim_{x \rightarrow x_0} f(x) = A$$

8.3.1

$$\lim_{x \rightarrow x_0} f(x) = A \Rightarrow \begin{cases} \alpha \text{ 为 } x \rightarrow x_0 \text{ 时的无穷小} \\ f(x) = \alpha + A \end{cases}$$

$$\text{设 } \lim_{x \rightarrow x_0} f(x) = A, \text{ 记 } f(x) - A = \alpha$$

只需证 α 为无穷小。

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ 当 } 0 < |x - x_0| < \delta, \text{ 时 } |f(x) - A| < \varepsilon$$

$$\text{即 } |\alpha - 0| < \varepsilon$$

α 为 $x \rightarrow x_0$ 时的无穷小

$$\lim_{x \rightarrow x_0} f(x) = A \Leftarrow \begin{cases} \alpha \text{ 为 } x \rightarrow x_0 \text{ 时的无穷小} \\ f(x) = \alpha + A \end{cases}$$

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ 当 } 0 < |x - x_0| < \delta, |\alpha| < \varepsilon$$

$$\text{即 } |f(x) - A| < \varepsilon \quad \lim_{x \rightarrow x_0} f(x) = A$$

8.3.2

$$\text{设 } \lim_{x \rightarrow x_0} f(x) = \infty$$

对 $f(x)$ 为 $x \rightarrow$ 时无穷大

对于 $M = \frac{1}{\varepsilon}$. 存在 $\delta > 0$

当 $0 < |x - x_0| < \delta$ 时

$$|f(x)| > M = \frac{1}{\varepsilon}$$

$$\left| \frac{1}{f(x)} \right| < \varepsilon$$

$\frac{1}{f(x)}$ 为 $x \rightarrow x_0$ 时的无穷小

8.4.2

$$f(x)g(x) = [A + \alpha][B + \beta]$$

$$= AB + A\beta + B\alpha + \beta\alpha$$

$$= AB + \gamma \quad (\gamma \text{ 为无穷小})$$

$$\lim [f(x)g(x)] = AB + \gamma = \lim f(x) \lim g(x)$$

8.4.8

$$\forall \varepsilon > 0$$

$$|x_n - a| < \varepsilon \quad \forall n > N_1$$

$$|y_n - a| < \varepsilon \quad \forall n > N_2$$

令 $N = \max \{N_1, N_2, N_0\}$, 则当 $n > N$ 时有

$$a - \varepsilon < x_n \leq z_n \leq y_n < a + \varepsilon$$

$$|z_n - a| < \varepsilon$$

$$\lim_{n \rightarrow \infty} z_n = a$$

8.4.9

$$\begin{aligned} |f(x) - \sin x_0| &= |\sin x - \sin x_0| \\ &= \left| 2 \cos\left(\frac{x+x_0}{2}\right) \sin\left(\frac{x-x_0}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \\ &\leq 2 \frac{|x-x_0|}{2} = |x-x_0| \end{aligned}$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{ 当 } 0 < |x - x_0| < \delta \text{ 时}$$

$$|\sin x - \sin x_0| \leq |x - x_0| < \varepsilon$$

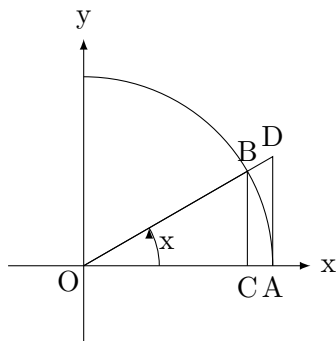
8.4.10

$$\begin{aligned} |f(x) - \cos x_0| &= |\cos x - \cos x_0| \\ &= \left| -2 \sin\left(\frac{x+x_0}{2}\right) \sin\left(\frac{x-x_0}{2}\right) \right| \\ &\leq 2 \left| \sin\left(\frac{x-x_0}{2}\right) \right| \\ &\leq 2 \frac{|x-x_0|}{2} = |x-x_0| \end{aligned}$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{ 当 } 0 < |x - x_0| < \delta \text{ 时}$$

$$|\cos x - \cos x_0| \leq |x - x_0| < \varepsilon$$

8.4.11



$$OB = OA = 1$$

$$\triangle AOB \leq \text{扇形面积} \leq \triangle AOD$$

$$\frac{1}{2} \sin x \leq \frac{1}{2} x \leq \frac{1}{2} \tan x$$

$$\sin x \leq x \leq \tan x$$

$$1 \geq \frac{\sin x}{x} \geq \cos x$$

$$\lim_{x \rightarrow 0} 1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq \lim_{x \rightarrow 0} \cos x$$

$$1 \geq \lim_{x \rightarrow 0} \frac{\sin x}{x} \geq 1$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

8.4.12

$$|1 - \cos x| = 1 - \cos x = 2 \sin^2 \frac{x}{2} \leq 2 \left(\frac{x}{2} \right)^2$$

$$0 \leq 1 - \cos x \leq \frac{x^2}{2}$$

$$\lim_{x \rightarrow 0} 0 \leq \lim_{x \rightarrow 0} (1 - \cos x) \leq \lim_{x \rightarrow 0} \frac{x^2}{2}$$

$$0 \leq \lim_{x \rightarrow 0} (1 - \cos x) \leq 0$$

$$\lim_{x \rightarrow 0} (1 - \cos x) = 0$$

$$\lim_{x \rightarrow 0} \cos x = 1$$

8.4.13

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x}{x} &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \frac{1}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{1}{\cos x} \\ &= 1 \end{aligned}$$

8.4.14

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{1 - \cos x}{\frac{1}{2}x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\frac{1}{2}x^2} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 \\
 &= 1
 \end{aligned}$$

8.4.15

$$\begin{aligned}
 x &= \sin t, \quad t = \arcsin x \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{\arcsin x}{x} &= \lim_{x \rightarrow 0} \frac{t}{\sin t} = 1
 \end{aligned}$$

8.4.16

$$\begin{aligned}
 x &= \tan t, \quad t = \arctan x \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{\arctan x}{x} &= \lim_{t \rightarrow 0} \frac{t}{\tan t} = 1
 \end{aligned}$$

8.4.17

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = \lim_{x \rightarrow 0} \ln(1+x)^{\frac{1}{x}} = \ln e = 1$$

8.4.18

$$\begin{aligned}
 e^x - 1 &= t, \quad x = \ln(t+1) \\
 x &\rightarrow 0, \quad t \rightarrow 0 \\
 \lim_{x \rightarrow 0} \frac{e^x - 1}{x} &= \lim_{t \rightarrow 0} \frac{t}{\ln(t+1)} = 1
 \end{aligned}$$

8.4.19

$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{nx} = \lim_{x \rightarrow 0} \left(\frac{e^{n \ln(1+x)} - 1}{n \ln(1+x)} \cdot \frac{\ln(1+x)}{x} \right) = 1$$

8.4.21

$$\begin{aligned}
x_n &= \left(1 + \frac{1}{n}\right)^n = \sum_{m=0}^n C_n^m 1^{n-m} \left(\frac{1}{n}\right)^m = \sum_{m=0}^n C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0 \left(\frac{1}{n}\right)^0 + C_n^1 \left(\frac{1}{n}\right)^1 + \sum_{m=2}^n C_n^m \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{n!}{m!(n-m)!} \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{\overbrace{(n)(n-1)\cdots(n-m+1)}^m}{m!} \left(\frac{1}{n}\right)^m \\
&= 1 + 1 + \sum_{m=2}^n \frac{1}{m!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-m+1}{n}\right) \\
&= 1 + 1 + \sum_{m=2}^n \frac{1}{m!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right) \\
x_{n+1} &= 1 + 1 + \sum_{m=2}^{n+1} \frac{1}{m!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{m-1}{n+1}\right)
\end{aligned}$$

$$x_n < x_{n+1} \quad \text{单调增加}$$

$$\begin{aligned}
x_n &< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\
&< 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{n^2} = 1 + \frac{1 - \left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&< 3 \quad \text{有界}
\end{aligned}$$

22.4 第 10 章

10.1.1

$$f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \text{因为极限存在与无穷小的关系}$$

$$\frac{\Delta y}{\Delta x} = f'(x_0) + \alpha \quad \alpha \text{ 为 } \Delta x \rightarrow 0 \text{ 时的无穷小}$$

$$\Delta y = f'(x_0) \Delta x + \alpha \Delta x$$

$$\lim_{\Delta x \rightarrow 0} \Delta y = \lim_{\Delta x \rightarrow 0} [f'(x_0) \Delta x + \alpha \Delta x] = 0$$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) \Leftrightarrow \lim_{\Delta x \rightarrow 0} [f(x_0 + \Delta x) - f(x_0) = 0] \Leftrightarrow \lim_{\Delta x \rightarrow 0} \Delta y = 0$$

10.2.1

$$\begin{aligned}
 (C)' &= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{C - C}{\Delta x} \\
 &= 0
 \end{aligned}$$

10.2.2

$$\begin{aligned}
 (x^a)' &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
 &= \frac{x^a - x_0^a}{x - x_0} \\
 &= \frac{(x - x_0)(x^{a-1} + x^{a-2}x_0 + \cdots + xx_0^{a-2} + x_0^{a-1})}{x - x_0} \\
 &= ax_0^{a-1}
 \end{aligned}$$

10.2.3

$$\begin{aligned}
 (a^x)' &= \lim_{\Delta x \rightarrow 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} \\
 &= a^x \lim_{\Delta x \rightarrow 0} \frac{a^{\Delta x} - 1}{\Delta x} \\
 &= a^x \lim_{\Delta x \rightarrow 0} \frac{e^{\Delta x \ln a} - 1}{\Delta x} \\
 &= a^x \ln a
 \end{aligned}$$

10.2.4

$$(e^x)' = e^x \ln e = e^x$$

10.2.5

$$\begin{aligned}
 (\log_a^x)' &= \lim_{\Delta x \rightarrow 0} \frac{\log_a^{x+\Delta x} - \log_a^x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\log_a^{1+\frac{\Delta x}{x}}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\ln 1 + \frac{\Delta x}{x}}{\ln a \Delta x} \\
 &= \frac{1}{\ln a} \lim_{\Delta x \rightarrow 0} \frac{\frac{\Delta x}{x}}{\Delta x} \\
 &= \frac{1}{x \ln a}
 \end{aligned}$$

10.2.6

$$\begin{aligned}
 (\ln^x)' &= \frac{1}{x \ln e} \\
 &= \frac{1}{x}
 \end{aligned}$$

10.3.1

$$\begin{aligned}
 (\sin x)' &= \lim_{\Delta x \rightarrow 0} \frac{\sin(x_0 + \Delta x) - \sin x_0}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{2 \cos(x_0 + \frac{\Delta x}{2}) \sin \frac{\Delta x}{2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \cos(x_0 + \frac{\Delta x}{2}) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \\
 &= \cos x_0
 \end{aligned}$$

10.3.2

$$\begin{aligned}
 (\arcsin x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \sin y}{dy}} \\
 &= \frac{1}{\cos y} \\
 &= \frac{1}{\sqrt{1 - \sin^2 y}} \\
 &= \frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

10.3.3

$$\begin{aligned}
 (\csc x)' &= \left(\frac{1}{\sin x} \right)' = \frac{(1)' \cdot \sin x - (\sin x)' \cdot 1}{\sin^2 x} \\
 &= \frac{-\cos x}{\sin^2 x} \\
 &= -\csc x \cdot \cot x
 \end{aligned}$$

10.3.4

$$\begin{aligned}
 (\cos x)' &= \lim_{\Delta x \rightarrow 0} \frac{\cos(x_0 + \Delta x) - \cos(x_0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2 \sin(x_0 + \frac{\Delta x}{2}) \sin \frac{\Delta x}{2}}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} -\sin(x_0 + \frac{\Delta x}{2}) \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} \\
 &= -\sin x
 \end{aligned}$$

10.3.5

$$\begin{aligned}
 (\arccos x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \cos y}{dy}} \\
 &= \frac{1}{-\sin y} \\
 &= -\frac{1}{\sqrt{1 - \cos^2 y}} \\
 &= -\frac{1}{\sqrt{1 - x^2}}
 \end{aligned}$$

10.3.6

$$\begin{aligned}
 (\sec x)' &= \left(\frac{1}{\cos x} \right)' = \frac{(1)' \cdot \cos x - (\cos x)' \cdot 1}{\cos^2 x} \\
 &= \frac{\sin x}{\cos^2 x} \\
 &= \sec x \cdot \tan x
 \end{aligned}$$

10.3.8

$$\begin{aligned}
 (\tan x)' &= \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} \\
 &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
 &= \sec^2 x
 \end{aligned}$$

10.3.9

$$\begin{aligned}
 (\arctan x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \tan y}{dy}} \\
 &= \frac{1}{\sec y} \\
 &= \frac{1}{1 + \tan^2 y} \\
 &= \frac{1}{1 + x^2}
 \end{aligned}$$

10.3.10

$$\begin{aligned}
 (\cot x)' &= \left(\frac{\cos x}{\sin x} \right)' = \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x} \\
 &= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\
 &= -\csc^2 x
 \end{aligned}$$

10.3.11

$$\begin{aligned}
 (\operatorname{arccot} x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \cot y}{dy}} \\
 &= \frac{1}{-\csc^2 y} \\
 &= -\frac{1}{1 + \cot^2 y} \\
 &= -\frac{1}{1 + x^2}
 \end{aligned}$$

10.3.12

$$\begin{aligned}
 (\sinh x)' &= \left(\frac{e^x - e^{-x}}{2} \right)' \\
 &= \frac{e^x + e^{-x}}{2} \\
 &= \cosh x
 \end{aligned}$$

10.3.13

$$\begin{aligned}
 (\cosh x)' &= \left(\frac{e^x + e^{-x}}{2} \right)' \\
 &= \frac{e^x - e^{-x}}{2} \\
 &= \sinh x
 \end{aligned}$$

10.3.14

$$\begin{aligned}
 (\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)' \\
 &= \frac{\frac{d(e^x - e^{-x})}{dx}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{dx}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= \frac{2^2}{(e^x + e^{-x})^2} \\
 &= \frac{1}{\cosh^2 x}
 \end{aligned}$$

10.3.15

$$\begin{aligned}
 (\operatorname{arcsinh} x)' &= \left[\ln(x + \sqrt{x^2 + 1}) \right]' \\
 &= \frac{d \ln(x + \sqrt{x^2 + 1})}{d(x + \sqrt{x^2 + 1})} \cdot \frac{d(x + \sqrt{x^2 + 1})}{dx} \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{dx}{dx} + \frac{d(\sqrt{x^2 + 1})}{d(x^2 + 1)} \cdot \frac{d(x^2 + 1)}{dx} \right) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \right) \\
 &= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \\
 &= \frac{1}{\sqrt{x^2 + 1}}
 \end{aligned}$$

10.3.16

$$\begin{aligned}
(\operatorname{arccosh} x)' &= \left[\ln(x + \sqrt{x^2 - 1}) \right]' \\
&= \frac{d \ln(x + \sqrt{x^2 - 1})}{d(x + \sqrt{x^2 - 1})} \cdot \frac{d(x + \sqrt{x^2 - 1})}{dx} \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(\frac{dx}{dx} + \frac{d(\sqrt{x^2 - 1})}{d(x^2 - 1)} \cdot \frac{d(x^2 - 1)}{dx} \right) \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x \right) \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} \\
&= \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}$$

10.3.17

$$\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \right]' \\
&= \frac{1}{2} \cdot \frac{d \left[\ln\left(\frac{1+x}{1-x}\right) \right]}{d\left(\frac{1+x}{1-x}\right)} \cdot \frac{d\left(\frac{1+x}{1-x}\right)}{dx} \\
&= \frac{1}{2} \cdot \frac{1}{\left(\frac{1+x}{1-x}\right)} \cdot \frac{\frac{d(1+x)}{dx}(1-x) - \frac{d(1-x)}{dx}(1+x)}{(1-x)^2} \\
&= \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{(1-x) + (1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}$$

10.4.1

$$\begin{aligned}
[Cu(x)]' &= \lim_{\Delta x \rightarrow 0} \frac{Cu(x + \Delta x) - Cu(x)}{\Delta x} \\
&= C \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \\
&= Cu'(x)
\end{aligned}$$

10.4.2

$$\begin{aligned}
(u(x) \pm v(x))' &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x) \pm v(x + \Delta x) - v(x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} \pm \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x) - v(x)}{\Delta x} \\
&= u'(x) \pm v'(x)
\end{aligned}$$

10.4.3

$$\begin{aligned}
[u(x) \cdot v(x)]' &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x)v(x + \Delta x) - u(x)v(x) - u(x)v(x + \Delta x) + u(x)v(x + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{[u(x + \Delta x) - u(x)]v(x + \Delta x) + u(x)[v(x + \Delta x) - v(x)]}{\Delta x} \\
&= u'(x) \lim_{\Delta x \rightarrow 0} v(x + \Delta x) + u(x)v'(x) \\
&= u'(x)v(x) + v'(x)u(x)
\end{aligned}$$

10.5.1

$$\begin{aligned}
[f^{-1}(y)]' |_{y=y_0} &= \lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} \\
&= \lim_{y \rightarrow y_0} \frac{x - x_0}{y - y_0} \\
&= \lim_{x \rightarrow x_0} \frac{x - x_0}{f(x) - f(x_0)} \\
&= \lim_{x \rightarrow x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \\
&= \frac{1}{f'(x)}
\end{aligned}$$

10.6.1

$$\text{定义函数 } A(u) = \begin{cases} \frac{f(u) - f(u_0)}{u - u_0}, & u \neq u_0 \\ f'(u_0), & u = u_0 \end{cases}$$

$A(u)$ 在 u_0 处连续, 既有, $\lim_{u \rightarrow u_0} A(u) = A(u_0) = f'(u_0)$

由恒等式 $f(u) - f(u_0) = A(u)(u - u_0)$ 我们有

$$\begin{aligned}
\frac{F(x) - F(x_0)}{x - x_0} &= \frac{f[g(x)] - f[g(x_0)]}{x - x_0} \\
&= A[g(x)] \frac{g(x) - g(x_0)}{x - x_0} \\
\lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} A[g(x)] \frac{g(x) - g(x_0)}{x - x_0} \\
F'(x_0) &= f'(g(x_0))g'(x_0)
\end{aligned}$$

22.5 第 11 章

11.1.1

$$\begin{aligned}\Delta y &= A \Delta x + o(\Delta x) \\ \frac{\Delta y}{\Delta x} &= A + \frac{o(\Delta x)}{\Delta x} \\ \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \left[A + \frac{o(\Delta x)}{\Delta x} \right] \\ f'(x_0) &= A + 0 \\ f'(x_0) &= A\end{aligned}$$

11.1.2

设 $f(x)$ 在 x_0 点可导, $f'(x_0) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$ 存在
(极限与无穷小的关系: $\lim_{\Delta x \rightarrow 0} f(x) = A \Leftrightarrow f(x) = A + \alpha$)

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= f'(x_0) + \alpha \\ \Delta y &= f'(x_0) \Delta x + \alpha \Delta x\end{aligned}$$

其中 α 为 $\Delta x \rightarrow 0$ 时的无穷小。

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{\alpha \Delta x}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \alpha = 0 \\ \alpha \Delta x &= o(\Delta x)\end{aligned}$$

$$\Delta y = f'(x_0) \Delta x + o(\Delta x)$$

11.2.1

$$\begin{aligned}d(u \pm v) &= (u \pm v)' dx \\ &= (u)' dx \pm (v)' dx \\ &= du \pm dv\end{aligned}$$

11.2.2

$$\begin{aligned}d(u \cdot v) &= (u \cdot v)' dx \\ &= (u)' v dx + (v)' u dx \\ &= v du + u dv\end{aligned}$$

11.2.3

$$\begin{aligned}d\left(\frac{u}{v}\right) &= \left(\frac{u}{v}\right)' dx \\ &= \frac{(u)'v - (v)'u}{v^2} dx \\ &= \frac{v du - u dv}{v^2}\end{aligned}$$

11.3.1

$$\begin{aligned} \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} &= f'(x_0) \\ f(x_0 + \Delta x) - f(x_0) &\leq 0 \\ \begin{cases} \Delta x > 0 & \left\{ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0 \Rightarrow f'(x_0^+) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \leq 0 \right. \\ \Delta x < 0 & \left\{ \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0 \Rightarrow f'(x_0^-) = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \geq 0 \right. \end{cases} \\ f'(x_0) = f'(x_0^+) = f'(x_0^-) &\Rightarrow f'(x_0) = 0 \end{aligned}$$

11.3.2

$$\begin{aligned} M &= \max\{f(x) | x \in [a, b]\}, m = \min\{f(x) | x \in [a, b]\} \\ \begin{cases} M = m \Rightarrow M = m = f(a) = f(b), \text{此时 } f(x) \text{ 为常数, } \forall \xi \in (a, b), f'(\xi) = 0 \\ M > m \begin{cases} f(a) > m \Rightarrow \exists \xi \in (a, b), f(\xi) = m, \text{根据费马引理, } f'(\xi) = 0 \\ f(a) < M \Rightarrow \exists \xi \in (a, b), f(\xi) = M, \text{根据费马引理, } f'(\xi) = 0 \end{cases} \end{cases} \end{aligned}$$

11.3.3

$$\begin{aligned} \varphi(x) &= f(x) - \frac{f(b) - f(a)}{b - a}x \\ \varphi(a) &= f(a) - \frac{f(b) - f(a)}{b - a}a = \frac{bf(a) - af(b)}{b - a} \\ \varphi(b) &= f(b) - \frac{f(b) - f(a)}{b - a}b = \frac{bf(a) - af(b)}{b - a} \\ \varphi(a) &= \varphi(b), \exists \xi \in (a, b), \varphi'(\xi) = 0 \\ f'(\xi) &= \frac{f(b) - f(a)}{b - a} \\ f'(\xi)(b - a) &= f(b) - f(a) \end{aligned}$$

11.3.4

$$\begin{aligned} \varphi(x) &= f(x) - \frac{f(b) - f(a)}{F(b) - F(a)} [F(x) - F(a)] \\ \varphi(a) &= \varphi(b) = f(a) \\ \varphi'(\xi) &= 0 \\ f'(\xi) &= \frac{f(b) - f(a)}{F(b) - F(a)} F'(\xi) \\ \frac{f'(\xi)}{F'(\xi)} &= \frac{f(b) - f(a)}{F(b) - F(a)} \end{aligned}$$

11.4.1

$f(x), F(x)$ 的去心邻域可导, $\lim_{x \rightarrow x_0} \frac{f(x)}{F(x)}$ 与 $f(x_0), F(x_0)$ 无关。规定 $f(x_0) = 0, F(x_0) = 0$
 此时 $\lim_{x \rightarrow x_0} f(x) = 0 = f(x_0)$ $\lim_{x \rightarrow x_0} F(x) = 0 = F(x_0)$ 此时在 x_0 点处也连续

$$\frac{f(x)}{F(x)} = \frac{f(x) - f(x_0)}{F(x) - F(x_0)} = \frac{f'(\xi)}{F'(\xi)}$$

$$\lim_{x \rightarrow x_0} \frac{f(x)}{F(x)} = \lim_{x \rightarrow x_0} \frac{f'(\xi)}{F'(\xi)}$$

$x \rightarrow x_0$, 时 $\xi \rightarrow x_0$ 符号 ξ 换成 x

$$\lim_{x \rightarrow x_0} \frac{f(x)}{F(x)} = \lim_{\xi \rightarrow x_0} \frac{f'(\xi)}{F'(\xi)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{F'(x)}$$

11.5.1

$$\begin{aligned} \frac{R_n(x)}{(x-x_0)^{n+1}} &= \frac{R_n(x) - R_n(x_0)}{(x-x_0)^{n+1} - (x_0-x_0)^{n+1}} = \frac{R'_n(\xi_1)}{(n+1)(\xi_1-x_0)^n} \\ \frac{1}{n+1} \cdot \frac{R'_n(\xi_1)}{(\xi_2-x_0)^n} &= \frac{1}{n+1} \cdot \frac{R'_n(\xi_1) - R'_n(x_0)}{(\xi_1-x_0)^n - (x_0-x_0)^n} = \frac{1}{n+1} \cdot \frac{R''_n(\xi_2)}{(n)(\xi_2-x_0)^{n-1}} \\ &\vdots \end{aligned}$$

$$\begin{aligned} \frac{R_n^{(n)}(\xi_n)}{(n+1)!(\xi_n-x_0)} &= \frac{R_n^{(n)}(\xi_n) - R_n^{(n)}(x_0)}{(n+1)!(\xi_n-x_0) - 0} = \frac{R_n^{(n+1)}(\xi)}{(n+1)!} \\ \frac{R_n^{(n+1)}(\xi)}{(n+1)!} &= \frac{f^{(n+1)}(\xi) - P_n^{(n+1)}(\xi)}{(n+1)!} = \frac{f^{(n+1)}(\xi)}{(n+1)!} \\ R_n(x) &= \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1} \end{aligned}$$

$$\xi_1 \in (x, x_0), \xi_2 \in (\xi_1, x_0), \xi_n \in (\xi_{n-1}, x_0), \xi \in (\xi_n, x_0)$$

11.5.2

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{R_n(x)}{(x-x_0)^n} &= \lim_{x \rightarrow x_0} \frac{R'_n(x)}{n(x-x_0)^{n-1}} \\ &= \lim_{x \rightarrow x_0} \frac{R''_n(x)}{n(n-1)(x-x_0)^{n-2}} \\ &= \lim_{x \rightarrow x_0} \frac{R_n^{(n)}(x)}{n!} \\ &= \frac{1}{n} \cdot 0 \\ &= 0 \end{aligned}$$

22.6 第 12 章

12.2.1

$$\begin{aligned}\frac{dy}{dx} + P(x)y &= 0 \\ \frac{\frac{dy}{dx}}{y} &= -P(x) \, dx \\ \int \frac{dy}{y} &= \int -P(x) \, dx \\ \ln |y| &= -\int P(x) \, dx + C_2 \\ |y| &= C_1 e^{-\int P(x) \, dx} \\ y &= C e^{-\int P(x) \, dx}\end{aligned}$$

12.2.2

常数变异法: 假设一个解, 包含关于 x 的未知函数 $u(x)$

$$\begin{aligned}y &= C e^{-\int P(x) \, dx} \\ y &= u(x) e^{-\int P(x) \, dx} \\ \frac{dy}{dx} &= u'(x) e^{-\int P(x) \, dx} + u(x) e^{-\int P(x) \, dx} \cdot [-P(x)] \\ \frac{dy}{dx} &= u'(x) e^{-\int P(x) \, dx} - P(x) u(x) e^{-\int P(x) \, dx}\end{aligned}$$

$u(x)$ 的表达式代入原方程, 求出 $u(x)$ 与 $Q(x)$ 的关系

$$\begin{aligned}\frac{dy}{dx} + P(x)y &= Q(x) \\ u'(x) e^{-\int P(x) \, dx} - P(x) u(x) e^{-\int P(x) \, dx} + P(x) u(x) e^{-\int P(x) \, dx} &= Q(x) \\ u'(x) e^{-\int P(x) \, dx} &= Q(x) \\ u'(x) &= Q(x) e^{\int P(x) \, dx} \\ u(x) &= \int Q(x) e^{\int P(x) \, dx} \, dx + C\end{aligned}$$

求出 $u(x)$ 与 $Q(x)$ 的关系, 在带回假设的解

$$\begin{aligned}y &= u(x) e^{-\int P(x) \, dx} \\ y &= e^{-\int P(x) \, dx} \left[\int Q(x) e^{\int P(x) \, dx} \, dx + C \right]\end{aligned}$$

12.3.1

$$y_1''(x) + P_1(x)y_1' + P_2(x)y_1 \equiv 0$$

$$y_2''(x) + P_1(x)y_2' + P_2(x)y_2 \equiv 0$$

$$y = C_1y_1 + C_2y_2$$

$$y' = C_1y_1' + C_2y_2'$$

$$y'' = C_1y_1'' + C_2y_2''$$

$$y'' + P_1(x)y' + P_2(x)y$$

$$= C_1y_1'' + C_2y_2'' + P_1(x)(C_1y_1' + C_2y_2') + P_2(x)(C_1y_1 + C_2y_2)$$

$$= C_1[y_1''(x) + P_1(x)y_1' + P_2(x)y_1] + C_2[y_2''(x) + P_1(x)y_2' + P_2(x)y_2]$$

$$\equiv 0$$

12.3.5

特征方程	$y'' + py' + qy = 0$ $r^2e^{rx} + pre^{rx} + qe^{rx} = 0$ $e^{rx}(r^2 + pr + q) = 0$ $r^2 + pr + q = 0$
$p^2 - 4q > 0$	<p>两个不同实根解: r_1, r_2</p> $\frac{e^{r_1x}}{e^{r_2x}} = e^{(r_1-r_2)x} \neq \text{常数} \quad \text{线性无关}$ $y = C_1y_1 + C_2y_2 = C_1e^{r_1x} + C_2e^{r_2x}$
$p^2 - 4q = 0$	<p>两个相同解: $r_1 = r_2 = -\frac{p}{2}$ 设 $y_2 = u(x)e^{r_1x}$ 是另一个解</p> $y_2' = u'e^{r_1x} + r_1ue^{r_1x} = (u' + r_1u)e^{r_1x}$ $y_2'' = (u'' + r_1u')e^{r_1x} + (u' + r_1u)e^{r_1x}r_1 = (u'' + 2r_1u' + r_1^2u)e^{r_1x}$ $y'' + py' + qy = 0$ $(u'' + 2r_1u' + r_1^2u)e^{r_1x} + p(u' + r_1u)e^{r_1x} + qe^{r_1x}u = 0$ $e^{r_1x}[u'' + (2r_1 + p)u' + (r_1^2 + pr_1 + q)u] = 0$ $u'' = 0$ $u = C_1x + C_2 \quad \text{取: } u(x) = x \quad \frac{e^{r_1x}}{xe^{r_1x}} = xe^{(r_1-r_2)x} \neq \text{常数}$ $y = C_1y_1 + C_2y_2 = C_1e^{r_1x} + C_2xe^{r_1x}$

$$p^2 - 4q < 0$$

两个共轭复根解: $r_{1,2} = \alpha \pm i\beta$

$$y_1 = e^{(\alpha+i\beta)x} = e^{\alpha x}(\cos \beta x + i \sin \beta x)$$

$$y_2 = e^{(\alpha-i\beta)x} = e^{\alpha x}(\cos \beta x - i \sin \beta x)$$

$$\overline{y_1} = \frac{1}{2}(y_1 + y_2) = e^{\alpha x} \cos \beta x$$

$$\overline{y_2} = \frac{1}{2i}(y_1 - y_2) = e^{\alpha x} \sin \beta x$$

$$\frac{\overline{y_1}}{\overline{y_2}} = \cot \beta x \neq \text{常数}$$

$$y = C_1 \overline{y_1} + C_2 \overline{y_2} = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$$

22.7 第 13 章

13.2.1

$$\begin{aligned} \left[\int f(x) dx \pm \int g(x) dx \right]' &= \left[\int f(x) dx \right]' \pm \left[\int g(x) dx \right]' \\ &= f(x) \pm g(x) \end{aligned}$$

13.2.2

$$\begin{aligned} \left[k \int f(x) dx \right]' &= k \left[\int f(x) dx \right]' \\ &= k f(x) \end{aligned}$$

13.2.3

$$\begin{aligned} \{F[\varphi(x)]\}' &= F'[\varphi(x)] \varphi'(x) \\ &= f[\varphi(x)] \varphi'(x) \end{aligned}$$

13.2.4

$$x = \varphi(t)$$

$$dx = d\varphi(t)$$

$$dx = \varphi'(t)dt$$

$$\int f(x) dx = \int f[\varphi(t)] \varphi'(t) dt$$

13.2.5

$$\begin{aligned} \int f(x) dx &= \int f(x) \cdot (x + C)' dx \\ &= \int f(x) d(x + C) \end{aligned}$$

13.4.1

$$\int k \, dx = \int (kx)' \, dx = kx + C$$

13.4.2

$$\int x^a \, dx = \int \left(\frac{1}{a+1} x^{a+1} \right)' \, dx = \frac{x^{a+1}}{a+1} + C$$

13.4.3

$$\int a^x \, dx = \int \left(\frac{1}{\ln a} a^x \right)' \, dx = \frac{a^x}{\ln a} + C$$

13.4.4

$$\int e^x \, dx = \int (e^x)' \, dx = e^x + C$$

13.4.5

$$\int \frac{1}{x} \, dx = \begin{cases} (x > 0) & \int (\ln x)' \, dx = \ln x + C = \ln |x| + C \\ (x < 0) & \int [\ln(-x)]' \, dx = \ln(-x) + C = \ln |x| + C \end{cases}$$

13.4.6

$$\begin{aligned} \int \ln x \, dx &= \ln x \cdot x - \int x \, d \ln x \\ &= x \ln x - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - x + C \end{aligned}$$

13.4.7

$$\int \sin x \, dx = \int (-\cos x)' \, dx = -\cos x + C$$

13.4.8

$$\int \cos x \, dx = \int (\sin x)' \, dx = \sin x + C$$

13.4.9

$$\int \sec x \tan x \, dx = \int (\sec x)' \, dx = \sec x$$

13.4.10

$$\int \csc x \cot x \, dx = - \int (\csc x)' \, dx = -\csc x$$

13.4.11

$$\int \sec^2 x \, dx = \int (\tan x)' \, dx = \tan x$$

13.4.12

$$\int \csc^2 x \, dx = - \int (\cot x)' \, dx = - \cot x$$

13.4.15

$$\begin{aligned} \int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ &= - \int \frac{(\cos x)'}{\cos x} \, dx \\ &= - \int \frac{1}{\cos x} \, d(\cos x) \\ &= - \ln |\cos x| + C \end{aligned}$$

13.4.16

$$\begin{aligned} \int \csc x \, dx &= \int \frac{1}{\sin x} \, dx \\ &= \int \frac{1}{2 \sin \frac{x}{2} \cos \frac{x}{2}} \, dx \\ &= \int \frac{1}{\tan \frac{x}{2}} \cdot \frac{1}{\cos^2 \frac{x}{2}} \, d\frac{x}{2} \\ &= \int \frac{1}{\tan \frac{x}{2}} \cdot \sec^2 \frac{x}{2} \, d\frac{x}{2} \\ &= \int \frac{1}{\tan \frac{x}{2}} \, d \tan \frac{x}{2} \\ &= \begin{cases} \ln |\tan \frac{x}{2}| + C \\ \ln |\csc x - \cot x| + C \end{cases} \end{aligned}$$

13.4.17

$$\begin{aligned} \int \sec x \, dx &= \int \frac{1}{\cos x} \, dx \\ &= \int \frac{\cos x}{\cos^2 x} \, dx \\ &= \int \frac{1}{\cos^2 x} \, d \sin x \\ &= \int \frac{1}{1 - \sin^2 x} \, d \sin x \\ &= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C \\ &= \frac{1}{2} \left| \frac{1 + \sin x}{\cos x} \right|^2 + C \\ &= \ln |\sec x + \tan x| + C \end{aligned}$$

13.4.18

$$\begin{aligned}
\int \arccos x \, dx &= x \arccos x - \int x \cdot -\frac{1}{\sqrt{1-x^2}} \, dx \\
&= x \arccos x - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} \, d(1-x^2) \\
&= x \arccos x - \frac{1}{2} \cdot \frac{1}{-\frac{1}{2}+1} (1-x^2)^{-\frac{1}{2}+1} + C \\
&= x \arccos x - \sqrt{1-x^2} + C
\end{aligned}$$

13.4.19

$$\begin{aligned}
\int \arctan x \, dx &= x \arctan x - \int x \cdot \frac{1}{1+x^2} \, dx \\
&= x \arctan x - \frac{1}{2} \int \frac{1}{1+x^2} \, d(1+x^2) \\
&= x \arctan x - \frac{1}{2} \ln(1+x^2) + C
\end{aligned}$$

分式积分公式证明暂时不标号。

$$\int \frac{1}{x^2+1} \, dx = \int (\arctan x)' \, dx = \arctan x + C$$

$$\int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \int \frac{1}{1+(\frac{x}{a})^2} \, d(\frac{x}{a}) = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\begin{aligned}
\int \frac{1}{x^2-a^2} \, dx &= \int \frac{1}{2a} \cdot \frac{(x+a)-(x-a)}{(x-a)(x+a)} \, dx \\
&= \frac{1}{2a} \cdot \int \frac{1}{x-a} - \frac{1}{x+a} \, dx \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{x-a} \, dx - \int \frac{1}{x+a} \, dx \right) \\
&= \frac{1}{2a} (\ln|x-a| - \ln|x+a|) + C \\
&= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{a^2 - x^2} dx &= \int -\frac{1}{2a} \frac{(x-a) - (x+a)}{(a+x)(a-x)} dx \\
&= \int \frac{1}{2a} \cdot \frac{(x-a) - (x+a)}{(a+x)(x-a)} dx \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x} dx - \int \frac{1}{x-a} dx \right) \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x} dx - \int \frac{1}{a-x} d(-x) \right) \\
&= \frac{1}{2a} (\ln|a+x| - \ln|a-x|) + C \\
&= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C
\end{aligned}$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} \int (\arcsin x)' dx = \arcsin x + C \\ -\int (\arccos x)' dx = -\arccos x + C \end{cases}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \int \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} d(\frac{x}{a}) = \begin{cases} \arcsin(\frac{x}{a}) + C \\ -\arccos(\frac{x}{a}) + C \end{cases}$$

$$x = a \tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{\sqrt{x^2+a^2}}{a}, \tan t = \frac{x}{a}, dx = a \sec^2 t dt$$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}} &= \int \frac{1}{a \sqrt{\tan^2 t + 1}} a \sec^2 t dt \\
&= \int \frac{1}{\sec t} \sec^2 t dt \\
&= \int \sec t dt \\
&= \ln |\sec t + \tan t| + C \\
&= \ln \left| \frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a} \right| + C \\
&= \ln(x + \sqrt{x^2+a^2}) + C_1 \quad C_1 = C - \ln a
\end{aligned}$$

$$x = a \sec t, a > 0, \sec t = \frac{x}{a}, \tan t = \frac{\sqrt{x^2-a^2}}{a}, dx = a \sec t \tan t dt$$

$x > a$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - a^2}} dx &= \int \frac{1}{a \sqrt{\sec^2 t - 1}} a \sec t \tan t dt \\
&= \int \frac{1}{\tan t} \sec t \tan t dt \\
&= \int \sec t dt \\
&= \ln |\sec t + \tan t| + C \\
&= \ln \left| \frac{\sqrt{x^2 - a^2}}{a} + \frac{x}{a} \right| + C \\
&= \ln(x + \sqrt{x^2 - a^2}) + C_1 \quad C_1 = C - \ln a
\end{aligned}$$

 $x < -a, x = -t, dx = -dt$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2 - a^2}} dx &= - \int \frac{1}{\sqrt{t^2 - a^2}} dt \\
&= - \ln \left| t + \sqrt{t^2 - a^2} \right| + C \\
&= - \ln \left| -x + \sqrt{x^2 - a^2} \right| + C \\
&= - \ln \left| \frac{(-x + \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \right| + C \\
&= - \ln \left| \frac{-a^2}{x + \sqrt{x^2 - a^2}} \right| + C \\
&= - \ln |-a^2| + \ln \left| x + \sqrt{x^2 - a^2} \right| + C \\
&= \ln \left| x + \sqrt{x^2 - a^2} \right| + C_1 \quad C_1 = C - \ln |-a^2|
\end{aligned}$$

22.8 第 15 章

14.3.1

$$\begin{aligned}
\int_a^a f(x) dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \cdot 0 \\
&= 0
\end{aligned}$$

14.3.2

$$\begin{aligned}
\int_a^b dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n (x_i - x_{i-1}) \\
&= \lim_{\lambda \rightarrow 0} (b - a) \\
&= b - a
\end{aligned}$$

14.3.3

$$\begin{cases} \sum_{i=1}^n \Delta x_i = \sum_{i=1}^n (x_i - x_{i-1}) = b - a \\ \sum_{i=1}^n (x_{i-1} - x_i) = a - b \end{cases}$$

$$\begin{aligned}
\int_a^b f(x) dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \\
&= - \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i)(x_{i-1} - x_i) \\
&= - \int_b^a f(x) dx
\end{aligned}$$

14.3.4

$$\begin{aligned}
\int_a^c f(x) dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^b f(\xi_i) \Delta x_i + \lim_{\lambda \rightarrow 0} \sum_{i=b+1}^n f(\xi_i) \Delta x_i \\
&= \int_a^b f(x) dx + \int_b^c f(x) dx
\end{aligned}$$

14.3.5

$$\begin{aligned}
\int_a^c f(x) dx &= \int_a^b f(x) dx + \int_b^c f(x) dx \\
\int_a^b f(x) dx &= \int_a^c f(x) dx - \int_b^c f(x) dx \\
&= \int_a^c f(x) dx + \int_c^b f(x) dx
\end{aligned}$$

14.3.6

$$\begin{aligned}
\int_a^b k f(x) \, dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n k f(\xi_i) \Delta x_i \\
&= k \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \\
&= k \int_a^b f(x) \, dx
\end{aligned}$$

14.3.7

$$\begin{aligned}
\int_a^b f(x) \pm g(x) \, dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n [f(\xi_i) \pm g(\xi_i)] \Delta x_i \\
&= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \pm \lim_{\lambda \rightarrow 0} \sum_{i=1}^n g(\xi_i) \Delta x_i \\
&= \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx
\end{aligned}$$

14.3.8

$$\begin{aligned}
f(x) \geq 0, \Delta x_i = (x_i - x_{i-1}) > 0 &\Rightarrow f(x_i) \Delta x_i \geq 0 \\
\int_a^b f(x) \, dx &= \lim_{\lambda \rightarrow 0} \sum_{i=1}^n f(\xi_i) \Delta x_i \geq 0
\end{aligned}$$

14.3.9

$$\begin{aligned}
f(x) \geq g(x) &\Rightarrow f(x) - g(x) \geq 0 \\
\int_a^b f(x) - g(x) \, dx &\geq 0 \\
\int_a^b f(x) \, dx - \int_a^b g(x) \, dx &\geq 0 \\
\int_a^b f(x) \, dx &\geq \int_a^b g(x) \, dx
\end{aligned}$$

14.3.10

$$\begin{aligned}
-|f(x)| &\leq f(x) \leq |f(x)| \\
\int_a^b -|f(x)| \, dx &\leq \int_a^b f(x) \, dx \leq \int_a^b |f(x)| \, dx \\
\left| \int_a^b f(x) \, dx \right| &\leq \int_a^b |f(x)| \, dx
\end{aligned}$$

14.4.1

$$m \leq f(x) \leq M, x \in [a, b]$$

$$\begin{aligned}
\int_a^b m \, dx &\leq \int_a^b f(x) \, dx \leq \int_a^b M \, dx \\
m \int_a^b dx &\leq \int_a^b f(x) \, dx \leq M \int_a^b dx \\
m(b-a) &\leq \int_a^b f(x) \, dx \leq M(b-a)
\end{aligned}$$

14.5.1

M 为区间 $[a, b]$ 最大值, m 为区间 $[a, b]$ 最小值, $a < b$

$$\begin{aligned}
m &\leq f(x) \leq M, x \in [a, b] \\
m(b-a) &\leq \int_a^b f(x) \, dx \leq M(b-a) \\
m &\leq \frac{1}{(b-a)} \int_a^b f(x) \, dx \leq M \\
\exists \xi \in [a, b], f(\xi) &= \frac{1}{(b-a)} \int_a^b f(x) \, dx \\
f(\xi)(b-a) &= \int_a^b f(x) \, dx
\end{aligned}$$

14.6.1

$ \begin{aligned} \phi'(x) &= \lim_{\Delta x \rightarrow 0} \frac{\phi(x + \Delta x) - \phi(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(\xi) \Delta x}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} f(\xi) \quad (\Delta x \rightarrow 0 \text{ 时, } \xi \rightarrow x) \\ &= f(x) \end{aligned} $	$ \begin{aligned} &\phi(x + \Delta x) - \phi(x) \\ &= \int_a^{x+\Delta x} f(u) \, du - \int_a^x f(u) \, du \\ &= \int_a^x f(u) \, du + \int_x^{x+\Delta x} f(u) \, du - \int_a^x f(u) \, du \\ &= \int_x^{x+\Delta x} f(u) \, du \\ &= f(\xi)(x + \Delta x - x) \quad (\xi \in [x, x + \Delta x]) \\ &= f(\xi) \Delta x \end{aligned} $
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14.6.2

$$\begin{aligned}
[\phi(\psi(x))]' &= \frac{d}{dx} \left[\int_a^{\psi(x)} f(u) \, du \right] \\
&= \frac{d}{d\psi(x)} \left[\int_a^{\psi(x)} f(u) \, du \right] \cdot \frac{d\psi(x)}{dx} \\
&= f(\psi(x)) \psi'(x)
\end{aligned}$$

14.6.3

$$\begin{aligned}
\frac{d}{dx} \left[\int_{v(x)}^{\psi(x)} f(u) \, du \right] &= \frac{d}{dx} \left[\int_0^{\psi(x)} f(u) \, du + \int_{v(x)}^0 f(u) \, du \right] \\
&= \frac{d}{dx} \left[\int_0^{\psi(x)} f(u) \, du \right] - \frac{d}{dx} \left[\int_0^{v(x)} f(u) \, du \right] \\
&= f[\psi(x)] \psi'(x) - f[v(x)] v'(x)
\end{aligned}$$

14.7.1

$$\begin{cases} F(x) \text{ 是 } f(x) \text{ 的原函数} \\ \phi(x) = \int_a^x f(u) \, du \text{ 也是 } f(x) \text{ 的原函数} \end{cases} \Rightarrow F(x) - \phi(x) \equiv C$$

$$x = a \quad F(a) - \phi(a) = C$$

$$F(a) = C$$

$$F(x) - \phi(x) = C$$

$$\phi(x) = F(x) - C$$

$$x = b \quad \phi(b) = F(b) - F(a)$$

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

14.8.1

$$\begin{aligned}
\int_a^b f(x) \, dx &= F(b) - F(a) \\
&= F[\phi(\beta)] - F[\phi(\alpha)] \\
&= F[\phi(t)]|_{\alpha}^{\beta} \\
&= \int_{\alpha}^{\beta} \frac{dF[\phi(t)]}{dt} \, dt \\
&= \int_{\alpha}^{\beta} f[\phi(t)] \phi'(t) \, dt
\end{aligned}$$

14.14.1

$$\begin{aligned}
&\text{记 } \phi(x) = \int_a^x f(t) \, dt, \quad a \leq x \leq b \\
&\phi(a) = 0, \quad \phi(b) = \int_a^b f(t) \, dt = \int_a^b f(x) \, dx \\
&\phi'(x) = f(x) \geq 0, \quad \text{在 } [a, b] \text{ 单调非降 } \phi(a) = 0, \phi(b) = 0 \\
&\phi(x) \equiv 0, x \in [a, b], \phi'(x) = f(x) \equiv 0
\end{aligned}$$

14.14.2

为 14.14.1 的逆否命题

14.14.3

$$\begin{aligned}
 h(x) &= g(x) - f(x) \geq 0, \quad x \in (a, b) \\
 \int_a^b h(x) \, dx &= \int_a^b g(x) \, dx - \int_a^b f(x) \, dx = 0 \\
 h(x) \equiv 0 &\Rightarrow f(x) = g(x)
 \end{aligned}$$

14.10.1

$$\begin{aligned}
 \int_{-a}^a f(x) \, dx &= \int_0^a f(-x) + f(x) \, dx \\
 &= \int_0^a 2f(x) \, dx \\
 &= 2 \int_0^a f(x) \, dx
 \end{aligned}$$

$$\begin{aligned}
 \int_{-a}^0 f(x) \, dx &\stackrel{x=-t}{=} - \int_a^0 f(-t) \, dt \\
 &= \int_0^a f(-t) \, dt \\
 &= \int_0^a f(-x) \, dx
 \end{aligned}$$

14.10.2

$$\begin{aligned}
 \int_{-a}^a f(x) \, dx &= \int_0^a f(-x) + f(x) \, dx \\
 &= \int_0^a -f(x) + f(x) \, dx \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \int_{-a}^a f(x) \, dx &= \int_{-a}^0 f(x) \, dx + \int_0^a f(x) \, dx \\
 &= \int_0^a f(-x) \, dx + \int_0^a f(x) \, dx \\
 &= \int_0^a f(-x) + f(x) \, dx
 \end{aligned}$$

14.11.1

$$\begin{aligned}
 G(x) &= \int_x^{x+T} f(x) \, dx \\
 G'(x) &= f(x+T) - f(x) = 0 \Rightarrow G(x) \equiv C \\
 G(a) &= G(0) = \int_0^T f(x) \, dx
 \end{aligned}$$

14.11.2

$$\begin{aligned}
 \int_x^{x+nT} f(x) \, dx &= \int_a^T f(x) \, dx + \int_{a+T}^{a+2T} f(x) \, dx + \cdots + \int_{a+(n-1)T}^{a+nT} f(x) \, dx \\
 &= \int_0^T f(x) \, dx + \int_0^T f(x) \, dx + \cdots + \int_0^T f(x) \, dx \\
 &= n \int_0^T f(x) \, dx
 \end{aligned}$$

14.12.1

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} f(\sin x) \, dx &\stackrel{\frac{\pi}{2}-x=t}{=} - \int_{\frac{\pi}{2}}^0 f\left[\sin\left(\frac{\pi}{2}-t\right)\right] \, dt \\
 &= \int_0^{\frac{\pi}{2}} f(\cos t) \, dt \\
 &= \int_0^{\frac{\pi}{2}} f(\cos x) \, dx
 \end{aligned}$$

14.12.2

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \, dx \\
&= - \int_0^{\frac{\pi}{2}} \sin^{n-1} x \, d \cos x \\
&= - \left[\sin^{n-1} x \cos x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos x \cdot \sin^{n-2} x \cdot \cos x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin^{n-2} x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \cdot \sin^{n-2} x \, dx \\
&= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx - (n-1) \int_0^{\frac{\pi}{2}} \sin^n x \, dx \\
n \int_0^{\frac{\pi}{2}} \sin^n x \, dx &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\
\int_0^{\frac{\pi}{2}} \sin^n x \, dx &= \frac{n-1}{n} \int_0^{\frac{\pi}{2}} \sin^{n-2} x \, dx \\
I_n &= \frac{n-1}{n} I_{n-2} \\
\int_0^{\frac{\pi}{2}} 1 \, dx &= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \sin x \, dx &= -[\cos x]_0^{\frac{\pi}{2}} = 1 \\
(n \text{ 是偶数}) I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
(n \text{ 是奇数}) I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1
\end{aligned}$$

14.13.1

$$\begin{aligned}
&[f(x) + tg(x)]^2 \geq 0 \\
&\int_a^b [f(x) + tg(x)]^2 \, dx \geq 0 \\
&\int_a^b f^2(x) + 2tf(x)g(x) + t^2g^2(x) \, dx \geq 0 \\
&\int_a^b f^2(x) \, dx + 2t \int_a^b f(x)g(x) \, dx + t^2 \int_a^b g^2(x) \, dx \geq 0 \quad (b^2 - 4ac \leq 0) \\
\Delta &= \left[\int_a^b f(x)g(x) \, dx \right]^2 - \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \leq 0 \\
&\left[\int_a^b f(x)g(x) \, dx \right]^2 \leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx
\end{aligned}$$

14.13.2

$$\begin{aligned}
\left[\int_a^b f(x)g(x) \, dx \right]^2 &\leq \int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \\
\int_a^b f(x)g(x) \, dx &\leq \left[\int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \right]^{\frac{1}{2}} \\
\int_a^b 2f(x)g(x) \, dx &\leq 2 \left[\int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \right]^{\frac{1}{2}} \\
\int_a^b [f(x) + g(x)]^2 \, dx &\leq \int_a^b f^2(x) \, dx + 2 \left[\int_a^b f^2(x) \, dx \cdot \int_a^b g^2(x) \, dx \right]^{\frac{1}{2}} + \int_a^b g^2(x) \, dx \\
\left\{ \int_a^b [f(x) + g(x)]^2 \, dx \right\}^{\frac{1}{2}} &\leq \left[\int_a^b f^2(x) \, dx \right]^{\frac{1}{2}} + \left[\int_a^b g^2(x) \, dx \right]^{\frac{1}{2}}
\end{aligned}$$

22.9 第 16 章

16.3.1

$$\begin{aligned}
\overrightarrow{OM} &= \overrightarrow{OP} + \overrightarrow{PN} + \overrightarrow{NM} \\
&= \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR} \\
&= x \overrightarrow{i} + y \overrightarrow{j} + z \overrightarrow{k}
\end{aligned}$$

16.3.2

$$\begin{aligned}
|OM|^2 &= |ON|^2 + |NM|^2 \\
&= |OP|^2 + |PN|^2 + |OR|^2 \\
&= |OP|^2 + |OQ|^2 + |OR|^2 \\
&= x^2 + y^2 + z^2 \\
|OM| &= \sqrt{x^2 + y^2 + z^2}
\end{aligned}$$

16.3.4

$$\begin{aligned}
(\cos \alpha, \cos \beta, \cos \gamma) &= \left(\frac{x}{|\overrightarrow{OM}|}, \frac{y}{|\overrightarrow{OM}|}, \frac{z}{|\overrightarrow{OM}|} \right) \\
&= \frac{1}{|\overrightarrow{OM}|} (x, y, z) \\
&= \frac{\overrightarrow{OM}}{|\overrightarrow{OM}|}
\end{aligned}$$

16.4.1

$$\vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\begin{aligned} \vec{a} \bullet \vec{b} &= (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \bullet (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) \\ &= a_x b_x \vec{i} \bullet \vec{i} + a_y b_y \vec{j} \bullet \vec{j} + a_z b_z \vec{k} \bullet \vec{k} + 0 \\ &= a_x b_x + a_y b_y + a_z b_z \end{aligned}$$

16.4.2

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \times (b_x \vec{i} + b_y \vec{j} + b_z \vec{k}) \\ &= a_x b_x \vec{i} \times \vec{i} + a_x b_y \vec{i} \times \vec{j} + a_x b_z \vec{i} \times \vec{k} \\ &\quad + a_y b_x \vec{j} \times \vec{i} + a_y b_y \vec{j} \times \vec{j} + a_y b_z \vec{j} \times \vec{k} \\ &\quad + a_z b_x \vec{k} \times \vec{i} + a_z b_y \vec{k} \times \vec{j} + a_z b_z \vec{k} \times \vec{k} \\ &= a_x b_y \vec{k} + a_x b_z (-\vec{j}) \\ &\quad + a_y b_x (-\vec{k}) + a_y b_z \vec{i} \\ &\quad + a_z b_x \vec{j} + a_z b_y (-\vec{i}) \\ &= (a_y b_z - a_z b_y) \vec{i} + (a_z b_x - a_x b_z) \vec{j} + (a_x b_y - a_y b_x) \vec{k} \\ &= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \vec{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \vec{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \vec{k} \end{aligned}$$

17.3.2

取 $\Delta y = 0$

$$f(x + \Delta x, y) - f(x, y) = A \Delta x + o(|\Delta x|)$$

$$\frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = A + \frac{o(|\Delta x|)}{\Delta x}$$

$$A = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$A = \frac{\partial z}{\partial x}$$

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$\Delta z = A \Delta x + B \Delta y + o(\rho)$$

$$= \frac{\partial z}{\partial x} \Delta x + \frac{\partial z}{\partial y} \Delta y$$

取 $\Delta x = 0$

$$f(x, y + \Delta y) - f(x, y) = B \Delta y + o(|\Delta y|)$$

$$\frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = B + \frac{o(|\Delta y|)}{\Delta y}$$

$$B = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$B = \frac{\partial z}{\partial y}$$

17.5.1

$$\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \quad \begin{cases} y = y(x) \\ z = z(x) \end{cases} \Rightarrow \begin{cases} F(x, y(x), z(x)) \equiv 0 \\ G(x, y(x), z(x)) \equiv 0 \end{cases}$$

$$\begin{cases} F'_x + F'_y \frac{dy}{dx} + F'_z \frac{dz}{dx} = 0 \\ G'_x + G'_y \frac{dy}{dx} + G'_z \frac{dz}{dx} = 0 \end{cases} \Rightarrow \begin{cases} F'_y \frac{dy}{dx} + F'_z \frac{dz}{dx} = -F'_x \\ G'_y \frac{dy}{dx} + G'_z \frac{dz}{dx} = -G'_x \end{cases}$$

$$J = \begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} \triangleq \frac{\partial(F, G)}{\partial(y, z)} \quad (\text{不等于 } 0 \text{ 时有唯一解})$$

$$\frac{dy}{dx} = \frac{\begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(x, z)}}{\frac{\partial(F, G)}{\partial(y, z)}}$$

$$\frac{dz}{dx} = \frac{\begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(y, x)}}{\frac{\partial(F, G)}{\partial(y, z)}}$$

17.5.2

$$\begin{cases} F(x, y, u, v) = 0 \\ G(x, y, u, v) = 0 \end{cases} \quad \begin{cases} u = u(x, y) \\ v = v(x, y) \end{cases} \Rightarrow \begin{cases} F(x, y, u(x, y), v(x, y)) \equiv 0 \\ G(x, y, u(x, y), v(x, y)) \equiv 0 \end{cases}$$

$$\begin{cases} F'_x + F'_u \frac{\partial u}{\partial x} + F'_v \frac{\partial v}{\partial x} = 0 \\ G'_x + G'_u \frac{\partial u}{\partial x} + G'_v \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} F'_u \frac{\partial u}{\partial x} + F'_v \frac{\partial v}{\partial x} = -F'_x \\ G'_u \frac{\partial u}{\partial x} + G'_v \frac{\partial v}{\partial x} = -G'_x \end{cases}$$

$$J = \begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \triangleq \frac{\partial(F, G)}{\partial(u, v)} \quad (\text{不等于 } 0 \text{ 时有唯一解})$$

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -F'_x & F'_v \\ -G'_x & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} = -\frac{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(x, v)}}{\frac{\partial(F, G)}{\partial(u, v)}}$$

$$\frac{\partial v}{\partial x} = \frac{\begin{vmatrix} F'_u & -F'_x \\ G'_u & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} = -\frac{\begin{vmatrix} F'_u & F'_x \\ G'_u & G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}} = -\frac{\frac{\partial(F, G)}{\partial(u, x)}}{\frac{\partial(F, G)}{\partial(u, v)}}$$

18.8.1

$$\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2} = t \quad (t > 0) \begin{cases} \Delta x = t \cos \alpha \\ \Delta y = t \cos \beta \end{cases}$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f'_x(x_0, y_0)\Delta x + f'_y(x_0, y_0)\Delta y + o(\rho)$$

$$f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0) = f'_x(x_0, y_0)t \cos \alpha + f'_y(x_0, y_0)t \cos \beta + o(\rho)$$

$$\frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{t} = \frac{f'_x(x_0, y_0)t \cos \alpha + f'_y(x_0, y_0)t \cos \beta + o(\rho)}{t}$$

$$\lim_{t \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{t} = \lim_{t \rightarrow 0^+} \frac{f'_x(x_0, y_0)t \cos \alpha + f'_y(x_0, y_0)t \cos \beta + o(\rho)}{t}$$

$$\lim_{t \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{t} = \lim_{t \rightarrow 0^+} \left[f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \cos \beta + \frac{o(\rho)}{t} \right]$$

$$\lim_{t \rightarrow 0^+} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)}{t} = f'_x(x_0, y_0) \cos \alpha + f'_y(x_0, y_0) \cos \beta$$