数学方面 (笔记)

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，贴吧，姜萍吧，姜圣的追随者，

2024.8.18：笔记都是看视频和书记录的。可能会有个别错误。但是我会持续更新，发现错误就会更改。上传频率不太固定。

目录

1   三角函数

1.1   三角恒等式

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{1.1.1}$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B \tag{1.1.2}$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{1.1.3}$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B \tag{1.1.4}$$

1.1.1   和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{1.1.5}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{1.1.6}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right) \tag{1.1.7}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right) \tag{1.1.8}$$

1.1.2   积化和差

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right] \tag{1.1.9}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right] \tag{1.1.10}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right] \tag{1.1.11}$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A + B) + \cos(A - B)\right] \tag{1.1.12}$$

1.1.3 降幂

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \tag{1.1.13}$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \tag{1.1.14}$$

1.1.4 半角公式

$$\sin \frac{x}{2} = \sqrt{\frac{1 - \cos(x)}{2}} \tag{1.1.15}$$

$$\cos \frac{x}{2} = \sqrt{\frac{1 + \cos(x)}{2}} \tag{1.1.16}$$

$$\tan \frac{x}{2} = \csc x - \cot x \tag{1.1.17}$$

$$\tag{1.1.18}$$

1.1.5 倍角公式

$$\sin(2x) = 2 \sin x \cos x \tag{1.1.19}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2 \sin^2 x \tag{1.1.20}$$

$$\tan(2x) = \frac{2 \tan x}{1 - \tan^2 x} \tag{1.1.21}$$

1.1.6 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.22}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.23}$$

1.1.7 三角函数恒等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.24}$$

$$1 + \tan^2 x = \sec^2 \tag{1.1.25}$$

$$1 + \cot^2 x = \csc^2 \tag{1.1.26}$$

1.2 双曲函数

1.2.1 定义

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}) \tag{1.2.1}$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \tag{1.2.2}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln(\frac{1 + x}{1 - x}) \tag{1.2.3}$$

1.2.3 双曲函数恒等式

$$\sinh(2x) = 2 \sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2 \sinh^2 \frac{x}{2} \tag{1.2.7}$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 + x_2 + \cdots + x_n} \tag{2.0.1}$$

$$|x + y| \leqslant |x| + |y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1 + x)^n \leqslant 1 + nx \tag{2.0.4}$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1) \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

3.2 运算

4   区间与映射
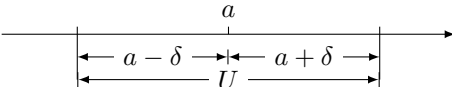
<center>label</center>

4.1   区间定义
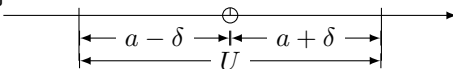
$$区间定义\begin{cases} (a,b) = \{x|a < x < b\} \\ [a,b] = \{x|a \leqslant x \leqslant b\} \\ (a,b] = \{x|a < x \leqslant b\} \\ (a,+\infty) = \{x|a < x\} \end{cases}$$

4.2   领域定义

点 a 的领域

$$U(a,\delta) = \begin{cases} \{x|a - \delta < x < a + \delta\} \\ \{x|\ |x - a| < \delta\} \end{cases}$$

点 a 的去心领域

$$\mathring{U}(a,\delta) = \begin{cases} \{x|a - \delta < x < a + \delta \wedge x \neq 0\} \\ \{x|0 < |x - a| < \delta\} \end{cases}$$

点 a 的左领域 $(a - \delta, a)$
点 a 的右领域 $(a, a + \delta)$

4.3   映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x \in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$记作 \qquad f : X \to Y$$

$$f(x) = y \qquad \begin{cases} 定义域\ (D_f) = X & x\text{-原像} \\ 值域\ (R_f) = \{f(x)|x \in X\} & y\text{-像} \end{cases}$$

<center>13</center>

$$映射类型 \begin{cases} 满射: & R_f = Y \\ 单射: & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\ 一一映射: & 即使满射又是单射 \Leftrightarrow 逆映射: \begin{cases} f^{(}x) & = y \\ f^{-1}(y) & = x \end{cases} \\ 复合映射: & g \circ f \Leftrightarrow g\left[f(x)\right] \begin{cases} f : X \to Y_1 \\ g : Y_2 \to Z \\ g \circ f : X \to Z \quad (Y_1 \subset Y_2) \end{cases} \end{cases}$$

5 函数

5.1 函数相关的定义

5.1.1 函数

设数集 $D \in R$ 的映射

$$f : D \to R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x) \ \{x \in D\}$$

5.1.2 驻点

$$Def : \ f'(x) = 0$$

5.1.3 拐点

$$Def : f''(x) = 0 \ (左右两侧凹凸性改变)$$

5.1.4 极值点

$$Def : 函数f(x) \ x \in \mathring{U}(x_0), 包括可导和不可导的点 \begin{cases} 极大值:\ f(x) < f(x_0) \\ 极小值:\ f(x) > f(x_0) \end{cases}$$

$$x \in \mathring{U}(x_0) \begin{cases} f(x)可导,\ f'(x_0) = 0 \begin{cases} x_0极大值 \begin{cases} x \in (x_0 - \delta, x_0), f'(x) > 0 \\ x \in (x_0, x_0 + \delta), f'(x) < 0 \end{cases} \\ x_0极小值 \begin{cases} x \in (x_0 - \delta, x_0), f'(x) < 0 \\ x \in (x_0, x_0 + \delta), f'(x) > 0 \end{cases} \\ x_0无极值,\ x \in \mathring{U}(x_0) \begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases} \end{cases} \\ f(x)二阶可导, f'(x_0) = 0, f''(x_0) \neq 0 \begin{cases} f''(x) < 0 \Rightarrow x_0极大值 \\ f''(x) > 0 \Rightarrow x_0极小值 \end{cases} \end{cases}$$

5.1.5 最值

$$最大值或最小值\begin{cases} 驻点 \\ 端点 \end{cases}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f : D \to R\{D \subset R\}\begin{cases} 有界\begin{cases} 有上界\left\{\exists k_1,\ 使f(x) \leqslant k_1,\ \forall x \in D\right. \\ 有下界\left\{\exists k_1,\ 使f(x) \geqslant k_1,\ \forall x \in D\right. \end{cases} \\ 无界\begin{cases} 无上界\left\{\forall K_1,\ \exists x \in D\ 使,\ f(x) \geqslant k_1\right. \\ 无下界\left\{\forall K_1,\ \exists x \in D\ 使,\ f(x) \leqslant k_1\right. \end{cases} \end{cases}$$

5.2.2 函数的单调性与凹凸性

$$若\{x_1, x_2 \in D\}\ x_1 < x_2 \Rightarrow \begin{cases} f(x_1) < f(x_2)称f(x)在\ D\ 上单调增加 \\ f(x_1) > f(x_2)称f(x)在\ D\ 上单调减少 \\ f(x_1) \leqslant f(x_2)称f(x)在\ D\ 上单调非降 \\ f(x_1) \geqslant f(x_2)称f(x)在\ D\ 上单调非增 \end{cases}$$

$$设\ f(x)\ 在区间\ I\ 上连续, \forall x_1, x_2 \begin{cases} f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2},称\ f(x)\ 在\ I\ 上是向上凹 \\ f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2},称\ f(x)\ 在\ I\ 上是向上凸 \end{cases}$$

$$f(x)在\,[a,b]\,上连续,\ 在(a,b)内可导f'(x) \geqslant 0, 有限个点为\,0,\ 单调增 \tag{5.2.1}$$

$$f(x)在\,[a,b]\,上连续,\ 在(a,b)内可导f'(x) \leqslant 0, 有限个点为\,0,\ 单调减 \tag{5.2.2}$$

$$f(x)在\,[a,b]\,上连续,\ 在(a,b)内二阶可导f''(x) \geqslant 0, 有限个点为\,0,\ 向上凹 \tag{5.2.3}$$

$$f(x)在\,[a,b]\,上连续,\ 在(a,b)内二阶可导f''(x) \leqslant 0, 有限个点为\,0,\ 向下凸 \tag{5.2.4}$$

5.2.3 函数的奇偶性

$$\forall x \in D \qquad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

$$奇函数 \times 奇函数 = 偶函数 \tag{5.2.5}$$

$$奇函数 \times 偶函数 = 奇函数 \tag{5.2.6}$$

$$偶函数 \times 偶函数 = 偶函数 \tag{5.2.7}$$

5.2.4 周期性

$$Def: \quad f(x+L) = f(x)\{L > 0常数, \forall x \in D\} \Rightarrow f(x) \text{ 为 } L \text{ 的周期函数}$$

5.3 弧

5.3.1 有向曲线弧

$$基准点 M_0(x_0, f(x_0)),\text{以 } x \text{ 增大的方向为正向}, \overset{\frown}{M_0 M} = S$$

$$S = S(x), S\text{是关于 x 的单调增加函数}$$

$$\overset{\frown}{M_0 M} \begin{cases} 绝对值为的长度 \\ 与曲线正向一致，取正值 \\ 与曲线反向一致，取负值 \end{cases}$$
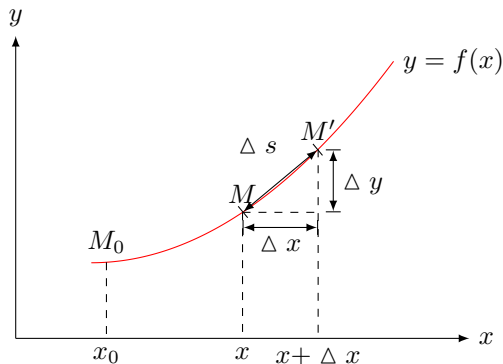
5.3.2　弧微分

$$ds = \sqrt{1 + (y')^2}dx \Leftrightarrow ds = \sqrt{(dx)^2 + (dy)^2} \Leftrightarrow ds = \sqrt{(dx)^2 + (f'dx)^2} \tag{5.3.1}$$

<div align="center">参数方程形式</div>

$$\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \begin{cases} dx = \phi'(t)dt \\ dy = \psi'(t)dt \end{cases} \Rightarrow ds = \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}dt$$

5.3.3　曲率
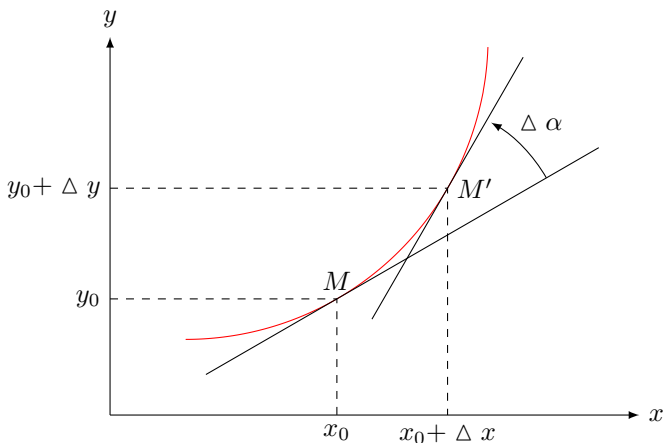
$M(x_0, y_0), M'(x_0 + \triangle x, y_0 + \triangle y), \triangle s = \widehat{MM'}$

$$\text{曲线上弧的} \begin{cases} \text{平均曲率：} \quad \overline{k} = \left| \frac{\triangle \alpha}{\triangle s} \right| \\ \text{点曲率：} \quad k = \lim\limits_{\triangle s \to 0} \left| \frac{\triangle \alpha}{\triangle s} \right| = \left| \frac{d\alpha}{ds} \right| \end{cases}$$

$$\left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{[1 + (y')^2]^{\frac{3}{2}}} \tag{5.3.2}$$

$$\left| \frac{d\alpha}{ds} \right| \text{的参数方程形式} \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \left| \frac{d\alpha}{ds} \right| = \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{ |\psi'(t)|^2 + [\phi'(t)]^2 \right\}^{\frac{3}{2}}} \tag{5.3.3}$$

5.3.4 曲率圆，曲率半径

$$圆的曲率k = \left| \frac{\triangle \alpha}{\triangle s} \right| = \left| \frac{\triangle \alpha}{r \triangle \alpha} \right| = \frac{1}{r}$$

$$曲率半径r = \frac{1}{k}$$
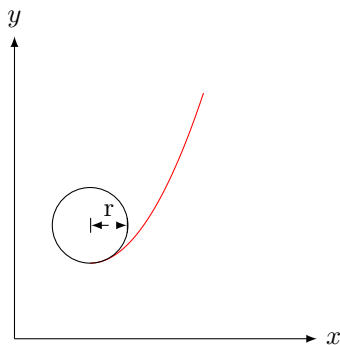
6   图像
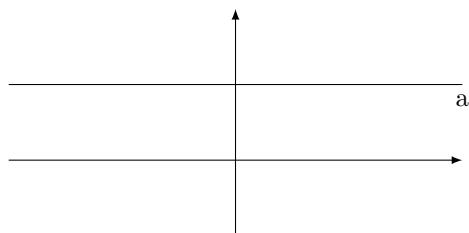
常函数 $f(x) = a\{a \in R\}$
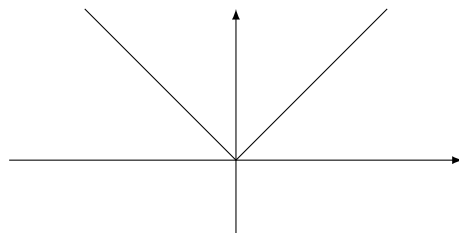
$f(x) = |x|$

$$f(x) = sgn\ x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$
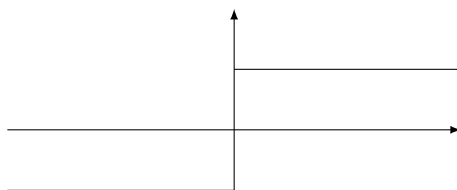
$$|x| = x \cdot sgnx$$
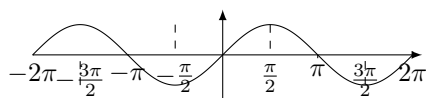
$\sin x$
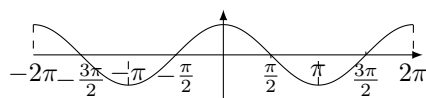
$\cos x$
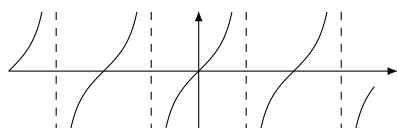
$\tan x$
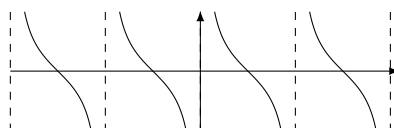
$\cot x$
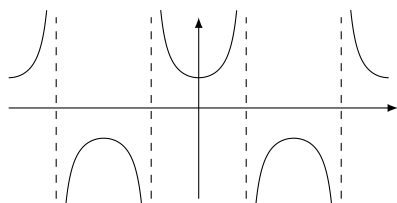
$\sec x$
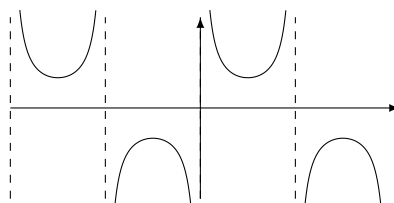
$\csc x$

7 并集，交集

7.1 定义

$$(\vee \text{ 或}, \wedge \text{ 与})$$
$$A \cup B = \{x \in A \vee x \in B\}$$
$$A \cap B = \{x \in A \wedge x \in B\}$$

7.2 运算

$$
\begin{cases}
\text{交换律} \begin{cases} A \cup B \ = B \cup A \\ A \cap B \ = B \cap A \end{cases} \\[2ex]
\text{结合律} \begin{cases} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{cases} \\[2ex]
\text{分配律} \begin{cases} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{cases} \\[2ex]
\text{对偶律} \begin{cases} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{cases}
\end{cases}
$$

$$A \cup A = A = A \cap A$$
$$A = B \Leftrightarrow A \subset B \wedge A \supset B$$
$$A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$$

7.3 性质

性质 1.
$$A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{7.3.1}$$

性质 2.
$$A \cup B = B \Leftrightarrow A \subset B \tag{7.3.2}$$

性质 3.
$$A \cap B = A \Leftrightarrow A \subset B \tag{7.3.3}$$

性质 4. $(n \in N)$

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{7.3.4}$$

性质 5. ($n \in N$)

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{7.3.5}$$

7.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

7.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_\alpha \right)^C = \bigcap_{\alpha} (E_\alpha^C)$$

$$\left(\bigcap_{\alpha} E_\alpha \right)^C = \bigcup_{\alpha} (E_\alpha^C)$$

8 群，环，域

8.1 群

8.1.1 M1

8.1.2 M2

8.1.3 M3

8.1.4 M4

8.1.5 sdas

8.2 环

8.3 域

9 极限

9.1 数列极限

9.1.1 数列的定义

$$Def: \qquad \{x_n\}, x_n = f(n), n \in N^+ \to R$$

9.1.2 数列极限的定义

$Def: \{x_n\},\ n \in N^+, \exists a,\ \forall \varepsilon > 0, \exists N,\ n > N \Rightarrow |x_n - a| < \varepsilon$

$\lim\limits_{n \to \infty} x_n = a$
极限存在，为收敛，不存在为发散

9.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{9.1.1}$$

9.1.4 有界数列

$$若 \exists M > 0, \{M \in 正数\}$$
$$使得 \forall n, \quad |x_n| \leqslant M$$
$$则称数列 \{x_n\} 为有界数列$$

9.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{9.1.2}$$
$$\text{单调有界数列必收敛} \tag{9.1.3}$$

9.1.6 收敛数列的保号性

$$\lim_{n \to \infty} x_n = a \text{ 存在，且 } a > 0, \text{则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时} \Leftrightarrow x_n > 0 \tag{9.1.4}$$

$$\lim_{n \to \infty} x_n = a, \lim_{n \to \infty} b_n = b, a < b,\ \exists N, n > N, a_n < b_n \tag{9.1.5}$$

9.1.7 收敛数列和子数列

$\{x_n\}, \lim\limits_{n\to\infty} x_n = a, \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$

证明 $K = N \ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

9.2 函数极限

9.2.1 极限的定义

$$Def : \forall \varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X \quad \text{时都有} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ \text{当} x < -X \quad \text{时都有} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ \text{当} |x| > X \quad \text{时都有} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists \delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当} x_0 - \delta < x < x_0, \text{时} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ \text{当} 0 < |x - x_0| < \delta, \text{时} \quad |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

<div align="center">注意 1</div>

<div align="center">定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$</div>

<div align="center">注意 2</div>

<div align="center">$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。</div>

$$\lim\limits_{x\to x_0} f(x)\text{存在} \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) \tag{9.2.1}$$

图

9.2.2 极限的性质

<div align="center">1 函数的极限的唯一性</div>

如果 $\lim f(x)$ 存在必唯一。

<div align="center">2 局部有界性</div>

$\lim\limits_{x\to x_0} f(x) = A, \exists M > 0, \delta > 0$使$0 < |x - x_0| < \delta, |f(x)| \leqslant M$

<div align="center">3 保号性</div>

$$\lim_{x \to x_0} f(x) = A, \ A > 0, \exists \delta > 0, 当, 0 < |x - x_0| < \delta \Rightarrow f(x) > 0$$

$$f(x) > 0, \exists \delta > 0, 当, 0 < |x - x_0| < \delta \Rightarrow \lim_{x \to x_0} f(x) = A, \ A > 0$$

<div align="center">4 保序性</div>

$$f(x) \geqslant g(x), \ \lim f(x) = a, \ \lim g(x) = b, \ 则 a \geqslant b$$

<div align="center">5 函数极限与数列极限的关系</div>

如果 $\lim_{x \to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列，则满足 $x_n \neq x_0$
则 $\lim_{n \to \infty} f(x_n) = 0 = \lim_{x \to x_0} f(x), \ x_n \to x_0$

9.3 无穷小与无穷大

9.3.1 无穷小定义

Def : 如果 $\lim_{x \to x_0} f(x) = 0$ 则称 $f(x)$ 为 $x \to x_0$ 时的无穷小

$$Def : \forall \varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} 当 x > X & 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to +\infty} f(x) = 0 \\ 当 x < -X & 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to -\infty} f(x) = 0 \\ 当 |x| > X & 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to \infty} f(x) = 0 \end{cases} \\ \exists \delta > 0 \begin{cases} 当 x_0 < x < x_0 + \delta, 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to x_0^+} f(x) = 0 \\ 当 x_0 - \delta < x < x_0, 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to x_0^-} f(x) = 0 \\ 当 0 < |x - x_0| < \delta, 时 \ |f(x) - 0| < \varepsilon \Leftrightarrow \lim_{x \to x_0} f(x) = 0 \end{cases} \end{cases}$$

9.3.2 函数极限与无穷小的关系

$$在自变量的同一变化中。\alpha \ 为无穷小。\lim f(x) = A \Leftrightarrow f(x) = A + \alpha \tag{9.3.1}$$

9.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$如果 \ f(x) \ 为无穷大，则 \ \frac{1}{f(x)} \ 为无穷小。 \tag{9.3.2}$$

$$如果 \ f(x) \ 为无穷小，切 \ f(x) \neq 0，则 \ \frac{1}{f(x)} \ 为无穷小。 \tag{9.3.3}$$

9.3.4 无穷大定义

$$Def: \forall M > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当}x > X \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to +\infty} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim\limits_{x \to +\infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to +\infty} f(x) = \infty \end{cases} \\ \text{当}x < -X \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to -\infty} f(x) = +\infty \\ f(x) < M \Leftrightarrow \lim\limits_{x \to -\infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to -\infty} f(x) = \infty \end{cases} \\ \text{当}|x| > X \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to \infty} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim\limits_{x \to \infty} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to \infty} f(x) = \infty \end{cases} \end{cases} \\ \exists \delta > 0 \begin{cases} \text{当}x_0 - \delta < x < x_0 \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = \infty \end{cases} \\ \text{当}x_0 < x < x_0 + \delta \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to x_0^+} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim\limits_{x \to x_0^+} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to x_0^+} f(x) = \infty \end{cases} \\ \text{当}0 < |x - x_0| < \delta \begin{cases} f(x) > M \Leftrightarrow \lim\limits_{x \to x_0} f(x) = +\infty \\ f(x) < -M \Leftrightarrow \lim\limits_{x \to x_0} f(x) = -\infty \\ |f(x)| > M \Leftrightarrow \lim\limits_{x \to x_0} f(x) = \infty \end{cases} \end{cases} \end{cases}$$

$$\lim_{x \to x_0} f(x) = \infty, \text{ 直线} x = x_0 \text{是} y = f(x) \text{垂直渐进线}$$

9.4 运算

9.4.1 有限个无穷小的和仍为无穷小

$$设 \ \gamma = \alpha + \beta$$
$$\alpha \ 和 \ \beta \ 同为 \ x \to x_0 \ 时的无穷小$$
$$\forall \varepsilon > 0, \ \exists \delta_1 > 0, \ 当 \ 0 < |x - x_0| < \delta_1 \ 时, \ 有 \ |\alpha| < \frac{\varepsilon}{2}$$

$$\forall \varepsilon > 0,\ \exists \delta_2 > 0,\ \text{当}\ 0 < |x - x_0| < \delta_2\ \text{时,有}\ |\beta| < \tfrac{\varepsilon}{2}$$
$$\delta = min\{\delta_1, \delta_2\},\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2\ \text{同时满足}$$
$$\text{即}\ |\alpha| < \tfrac{\varepsilon}{2}, |\beta| < \tfrac{\varepsilon}{2}\ \text{同时成立}$$
$$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

9.4.2 有界函数与无穷小的乘积仍为无穷小

$$\text{设}\ \alpha\ \text{为}\ x \to x_0\ \text{时的一个无穷小}$$
$$g(x)\ \text{为}\ x_0\ \text{的一个去心邻域}\ \mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$f(x) = g(x)\alpha$$
$$\text{证}\ f(x)\ \text{为}\ x \to x_0\ \text{时的无穷小}$$
$$\text{因为}\ g(x)\text{在}\mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$\exists M > 0, \text{当}0 < |x - x_0| < \delta_1\ \text{时}\ |g(x)| < M$$
$$\text{因为}\ \alpha\ \text{是}\ x \to x_0\ \text{的无穷小}$$
$$\exists \delta_2 > 0\text{当}0 < |x - x_0| < \delta_2\ \text{时}\ |\alpha| < \tfrac{\varepsilon}{M} < \varepsilon$$
$$\text{取}\ \delta = min\{delta, \delta_2\}\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$|g(x)| \geqslant M, |\alpha| < \tfrac{\varepsilon}{M}\ \text{同时成立}$$
$$|g(x)\alpha| = |g(x)|\ |\alpha| < M\tfrac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

9.4.3 极限的四则运算

$\lim f(x) = A,\ \lim g(x) = B$

$$\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) \tag{9.4.1}$$

$$\lim\left(f(x)g(x)\right) = \lim f(x) \lim g(x) \tag{9.4.2}$$

$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)} \tag{9.4.3}$$

$$\lim\left[Cf(x)\right] = C \lim f(x) \tag{9.4.4}$$

$$\lim\left[f(x)\right]^n = \left[\lim f(x)\right]^n \tag{9.4.5}$$

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{a_0 x^n + a_1 x^{n-1} + \cdots + a_n} = \begin{cases} \frac{a}{b} & m = n \\ \infty & m > n \\ 0 & m < n \end{cases} \tag{9.4.6}$$

$$\lim_{x \to x_0} g(x) = u_0, \ \lim_{u \to u_0} f(x) = A$$
$$\exists \delta_0 > 0, \ x \in \mathring{U}\left(x_0, \delta_0\right), \ g(x) \neq u_0 \tag{9.4.7}$$
$$\lim_{x \to x_0} f\left[g(x)\right] = \lim_{u \to u_0} f(u) = A$$

9.4.4 夹逼定理 (三明治定理)

$$x_n \leqslant z_n \leqslant y_n \qquad \forall n > N_0$$
$$\text{若 } \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a \text{则} \lim_{n \to \infty} z_n = a \tag{9.4.8}$$

9.4.5 重要极限

$$x \to x_0$$

$$\lim_{x \to x_0} \sin x = \sin x_0 \tag{9.4.9}$$

$$\lim_{x \to x_0} \cos x = \cos x_0 \tag{9.4.10}$$

$$x \to 0$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{9.4.11}$$

$$\lim_{x \to 0} \cos x = 1 \tag{9.4.12}$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 \tag{9.4.13}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = 1 \tag{9.4.14}$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = 1 \tag{9.4.15}$$

$$\lim_{x \to 0} \frac{\arctan x}{x} = 1 \tag{9.4.16}$$

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1 \tag{9.4.17}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \tag{9.4.18}$$

$$\lim_{x \to 0} \frac{(1 + x)^n - 1}{nx} = 1 \tag{9.4.19}$$

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e \tag{9.4.20}$$

$$x \to \infty$$

$$\{x_n\} \qquad \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \tag{9.4.21}$$

$$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e \tag{9.4.22}$$

9.4.6   无穷小比较

$\frac{0}{0}$ 型未定式

Def : α, β 是同一极限过程的无穷小。

(1) 如果 $\lim \frac{\beta}{\alpha} = 0$ 则称 β 是 α 的高阶无穷小，记作 $\beta = \circ(\alpha)$

(2) 如果 $\lim \frac{\beta}{\alpha} = \infty$ 则称 β 是 α 的底阶无穷小。

(3) 如果 $\lim \frac{\beta}{\alpha} = C$ 则称 β 是 α 的同阶无穷小。

(4) 如果 $\lim \frac{\beta}{\alpha^k} = C, k > 0$ 则称 β 是 α 的 k 阶无穷小。

(5) 如果 $\lim \frac{\beta}{\alpha} = 1$ 则称 β 是 α 的等价阶无穷小。

9.4.7   等价无穷小代换，因子代换

β与α是等价无穷小 $\Leftrightarrow \beta = \alpha + \circ(\alpha)$

设$\alpha \sim \alpha'$, $\beta \sim \beta'$, 且 $\lim \frac{\beta'}{\alpha'}$存在, 则 $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$

$\lim \alpha f(x) = \lim \alpha' f(x)$

$\lim \frac{f(x)}{\alpha} = \lim \frac{f(x)}{\alpha'}$

10 连续与间断点

10.1 定义

10.1.1 点连续

$$Def1: 设f(x)在x_0的某邻域内有定义，如果 \lim_{x \to x_0} = f(x_0)$$
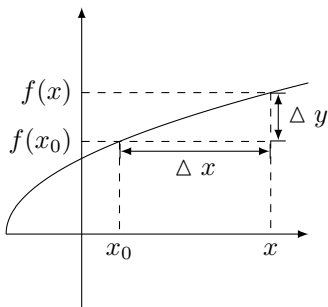
则称$f(x)$在x_0处连续

$$\begin{cases} \triangle x = x - x_0 \\ \triangle y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \triangle x) - f(x_0) \end{cases} \end{cases}$$

$$Def2: 如果 \lim_{\triangle x \to 0} \triangle y = \lim_{\triangle x \to 0} [f(x_0 + \triangle x) - f(x_0)] = 0$$

则称$f(x)$在x_0处连续

10.1.2 区间连续

$$\forall x_0 \in [a,b] \begin{cases} \lim_{x \to x_0} f(x) = f(x_0) \quad x_0 \in (a,b) \Leftrightarrow f(x_0) = \begin{cases} \lim_{x \to x_0^-} f(x) = f(x_0^-) \\ \lim_{x \to x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim_{x \to x_0^+} f(x) = f(x_0^+) \quad x_0 = a \ (右连续) \\ \lim_{x \to x_0^-} f(x) = f(x_0^-) \quad x_0 = b \ (左连续) \end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M > 0, x \in [a,b]$ 时，$|f(x)| \geqslant M$

最大值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

零点定理

2, 设 $f(x)$ 在 $[a, b]$ 上连续，且 $f(a) \cdot f(b) < 0$
则至少存在一点 $\xi \in (a, b)$ 使 $f(\xi) = 0$

<div align="center">介质定理</div>

设 $f(x)$ 在 $[a, b]$ 上连续，且 $f(a) = A, f(b) = B$
$\forall C \in (A, B)$, 至少有一点 $\xi, f(\xi) = C$

10.1.3   间断点

1, $f(x)$ 无定义
2, $\lim\limits_{x \to x_0} f(x)$ 不存在
3. $\lim\limits_{x \to x_0} f(x)$ 存在, 但 $\lim\limits_{x \to x_0} f(x) \neq f(x_0)$
第一类间断点： $f(x_0^+) = \lim\limits_{x \to x_0^+} f(x)$ 与 $f(x_0^-) = \lim\limits_{x \to x_0^-} f(x)$
第二类间断点：不是第一类的。

10.2   连续函数的运算

函数 $f(x), g(x)$ 在 $x = x_0$ 连续。

$$\lim_{x \to x_0} [f(x) \pm g(x)] = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x) = f(x_0) \pm g(x_0)$$

$$\lim_{x \to x_0} [f(x) \cdot g(x)] = \lim_{x \to x_0} f(x) \cdot \lim_{x \to x_0} g(x) = f(x_0) \cdot g(x_0)$$

$$\lim_{x \to x_0} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)} = \frac{f(x_0)}{g(x_0)} \qquad (g(x_0) \neq 0)$$

反函数的连续性
若 $y = f(x)$ 在区间 I_x 上单调增加，且连续。
则 $y = f^-1(x)$ 在 $I_y = \{y | y = f(x), x \in I_x\}$ 上也为单调增加，连续

$$\text{复合函数,}\begin{cases} \text{内外都连续}\begin{cases} \lim\limits_{x \to x_0} g(x) = g(x_0) = u_0 \\ \lim\limits_{u \to u_0} f(x) = f(u_0) \\ \lim\limits_{x \to x_0} f\left[g(x)\right] = f\left[g(x_0)\right] = f(\lim\limits_{x \to x_o} g(x)) \end{cases} \\ \text{外连续}\begin{cases} x \to x_0 \begin{cases} \lim\limits_{x \to x_0} g(x) = u_0 \\ \lim\limits_{u \to u_0} f(x) = f(u_0) \\ \lim\limits_{x \to x_0} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x \to x_o} g(x)) \end{cases} \\ x \to \infty \begin{cases} \lim\limits_{x \to \infty} g(x) = u_0 \\ \lim\limits_{u \to u_0} f(x) = f(u_0) \\ \lim\limits_{x \to \infty} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x \to \infty} g(x)) \end{cases} \end{cases} \end{cases}$$

11 导数

11.1 定义

导数的概念从物理发展出来的。

$$v\left(t_0\right) = \lim_{\triangle t \to 0} \frac{s\left(t_0 + \triangle t\right) - s\left(t_0\right)}{\triangle t}$$
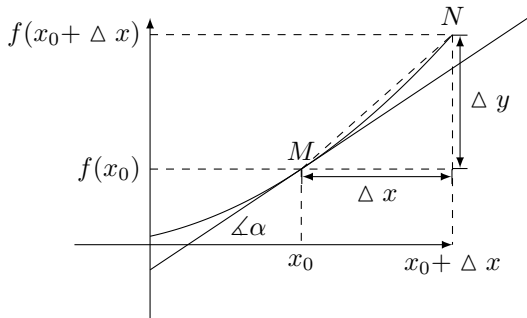
$$NM斜率 = \tan\beta = \frac{f\left(x_0 + \triangle x\right) - f(x_0)}{\triangle x} = \frac{\triangle y}{\triangle x}$$

$$斜率k = \tan\alpha = \lim_{\triangle x \to 0}\tan\beta = \lim_{\triangle x \to 0}\frac{\triangle y}{\triangle x} = \frac{f\left(x_0 + \triangle x\right) - f(x_0)}{\triangle x}$$

11.1.1 导数定义

$y = f(x)$ 在 x_0 的某邻域内有定义

给自变量的增量 $\triangle x, (x_0 + \triangle x)$ 仍在定义域内

函数得到了相应增量 $\triangle y, \triangle y = f\left(x_0 + \triangle x\right)$

如果 $\lim\limits_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x}$ 存在, 称 $y = f(x)$ 在 $x = x_0$ 处可导

(极限值为$y = f(x)$在$x = x_0$处导数)

$$记\ y'|_{x=x_0} = f'(x_0) = \lim_{\triangle x \to 0}\frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} = \lim_{\triangle x \to 0}\frac{\triangle y}{\triangle x}$$

$$\lim_{\triangle x \to 0}\frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \Leftrightarrow \lim_{\triangle x \to x_0}\frac{f(x) - f(x_0)}{x - x_0}$$

11.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。

$f'(x) = \lim\limits_{\triangle x \to 0}\frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x}$

称 $f'(x)$ 为 $y = f(x)$ 在区间 I 上的导函数

11.1.3 闭区间可导定义

$$f(x) \text{ 在 } [a,b] \text{ 可导} \Leftrightarrow \begin{cases} f'(x_0) & x_0 \in (a,b) \Leftrightarrow \begin{cases} \text{左导数} f'_-(x_0) = \lim\limits_{\triangle x \to 0^-} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x} \\ \text{右导数} f'_+(x_0) = \lim\limits_{\triangle x \to 0^+} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x} \end{cases} \\ f'_+(a) & x = a \\ f'_-(a) & x = b \end{cases}$$

11.1.4 导数与连续

$$f'(x)\text{存在} \Rightarrow f(x)\text{在}x = x_0\text{处连续} \tag{11.1.1}$$

11.2 幂数，指数，对数

$$(C)' = 0 \tag{11.2.1}$$

$$(x^a)' = ax^{a-1} \tag{11.2.2}$$

$$(a^x)' = a^x \ln a \tag{11.2.3}$$

$$(e^x)' = e^x \tag{11.2.4}$$

$$(\log_a^x)' = \frac{1}{x \ln a} \tag{11.2.5}$$

$$(\ln x)' = \frac{1}{x} \tag{11.2.6}$$

11.3 三角函数

$$(\sin x)' = \cos x \tag{11.3.1}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1 - x^2}} \tag{11.3.2}$$

$$(\csc x)' = -\csc x \cot x \tag{11.3.3}$$

$$(\cos x)' = -\sin x \tag{11.3.4}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}} \tag{11.3.5}$$

$$(\sec x)' = \sec x \tan x \tag{11.3.6}$$

$$(\text{arcsec}\, x)' = \frac{1}{|x|\sqrt{x^2 - 1}} \tag{11.3.7}$$

$$(\tan x)' = \sec^2 x \tag{11.3.8}$$

$$(\arctan x)' = \frac{1}{1 + x^2} \tag{11.3.9}$$

$$(\cot x)' = -\csc^2 x \tag{11.3.10}$$

$$(\text{arccot}\, x)' = -\frac{1}{1 + x^2} \tag{11.3.11}$$

$$(\sinh x)' = \cosh x \tag{11.3.12}$$

$$(\cosh x)' = \sinh x \tag{11.3.13}$$

$$(\tanh x)' = \frac{1}{cosh^2 x} = 1 - \tanh^2 x \tag{11.3.14}$$

$$(\text{arcsinh}\, x)' = \frac{1}{\sqrt{x^2 + 1}} \tag{11.3.15}$$

$$(\text{arccosh}\, x)' = \frac{1}{\sqrt{x^2 - 1}} \tag{11.3.16}$$

$$(\text{arctanh}\, x)' = \frac{1}{1 - x^2} \tag{11.3.17}$$

11.4 导数运算

$$u = u(x), v = v(x), 均在x点可导, C为常数$$

$$(Cu(x))' = Cu'(x) \tag{11.4.1}$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \tag{11.4.2}$$

$$(u(x) \cdot v(x))' = u'(x)v(x) + v'(x)u(x) \tag{11.4.3}$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2} \tag{11.4.4}$$

11.5 反函数求导

如果函数 $y = f(x)$ 在区间 (a,b) 内单调可导，且 $f'(y) \neq 0$

$$\begin{cases} \alpha = \min\{f(a) + 0, f(b - 0)\} \\ \beta = \max\{f(a) + 0, f(b - 0)\} \end{cases}$$

则它的反函数 $x = f^{-1}(y)$ 在区间 (α, β) 内也可导

$$\left[f^{-1}(y)\right]' = \frac{1}{f'(x)} \Leftrightarrow \frac{dy}{\mathrm{d}x} = \frac{1}{\frac{dx}{\mathrm{d}y}} \tag{11.5.1}$$

11.6 复合函数求导

$$设函数 \begin{cases} y = f(u)在U(u_0, \delta_0)处有定义 \\ u = g(x)在U(x_0, \eta_0)处有定义 \end{cases}$$
$$u_0 = g(x_0), 且f'(u)和g'(x)都存在$$
$$则复合函数F(x) = f\left[g(x)\right] 在点x_0可导, 且$$

$$F'(x_0) = f'\left[g(x_0)\right] g'(x_0) \Leftrightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{11.6.1}$$

11.7 高阶求导

$$Def : \begin{cases} 一阶导数 & y' \Leftrightarrow \frac{dy}{dx} \\ 二阶导数 & y'' \Leftrightarrow \frac{d^2y}{dx^2} \\ 三阶导数 & y''' \Leftrightarrow \frac{d^3y}{dx^3} \\ 三阶以上 \text{ n 阶导数} & y^{(n)} \Leftrightarrow \frac{d^ny}{dx^n} \end{cases}$$

11.8 高阶求导公式

$$(e^x)^{(n)} = e^x \tag{11.8.1}$$

$$(a^x)^{(n)} = a^x \left(lna\right)^n \tag{11.8.2}$$

$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \tag{11.8.3}$$

$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{11.8.4}$$

$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{11.8.5}$$

$$(\sin x)^{(n)} = \sin(x + n\frac{\pi}{2}) \tag{11.8.6}$$

$$(\cos x)^{(n)} = \cos(x + n\frac{\pi}{2}) \tag{11.8.7}$$

$$[f(ax+b)]^{(n)} = a^n \cdot f^{(n)}(ax+b) \tag{11.8.8}$$

11.9 高阶求导运算法则

$$(u(x) \pm v(x))^{(n)} = u^{(n)}(x) \pm v^{(n)}(x) \tag{11.9.1}$$

$$\text{莱布紫泥公式} \qquad (uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} \cdot v^{(}k) \tag{11.9.2}$$

11.10 隐函数求导

$F(x,y) = 0, y = f(x)$

$F(x, f(x)) \equiv 0 \qquad$ 可以同时对两面求导

11.11 参数方程求导

$x = x(t), y = y(t)$

$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2 y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt}\left(\frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}\right) \cdot \frac{dt}{dx} = \frac{\frac{d^2 y}{dt} \cdot \frac{dx}{dt} - \frac{d^2 x}{dt} \cdot \frac{dy}{dt}}{(\frac{dx}{dt})^2} \cdot \frac{dt}{dx} = \frac{\frac{d^2 y}{dt} \cdot \frac{dx}{dt} - \frac{d^2 x}{dt} \cdot \frac{dy}{dt}}{(\frac{dx}{dt})^3}$$

12 微分

12.1 定义

设函数 $f(x)$ 在点 x_0 的一个邻域内有定义。$\triangle y = f(x_0 + \triangle x) - f(x_0)$

如果 $\triangle y$ 可以表示为 $\triangle y = A \triangle x + \circ(\triangle x)$ 其中 A 为与 $\triangle x$ 无关的常数

则称 $f(x)$ 在点 x_0 可微，$A \triangle x$ 称为 $f(x)$ 在点 x_0 处的微分。

记作:$dy = A \triangle x$
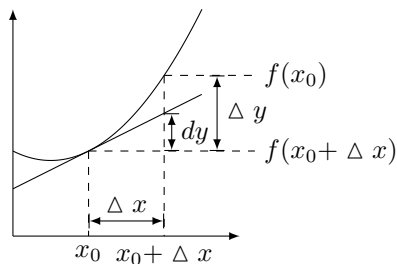
$$可微 \Rightarrow 可导 \tag{12.1.1}$$

$$可导 \Rightarrow 可微 \tag{12.1.2}$$

12.2 微分法则

12.2.1 核心根本

12.2.2 四则运算

$$d\left(u \pm v\right) = du \pm dv \tag{12.2.1}$$

$$d(uv) = vdu + udv \tag{12.2.2}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + udv}{v^2} \tag{12.2.3}$$

12.2.3 复合运算

$$可微 \begin{cases} y = f(u) \\ u = g(x) \end{cases} \Rightarrow \begin{cases} dy = f'(u)du \\ du = g'(x)dx \end{cases} \quad 则\ y = f(g(x))\ 也可微$$

$$且\ dy = f'(u)du = f'(u)g(x)dx$$

$$u\ 是否为中间变量都成立，微分的不变性。$$

12.2.4 近似计算公式

$$\triangle x \to 0, dy \approx \triangle y \begin{cases} dy = f'(x_0) \triangle x \\ \triangle y = f(x_0 + \triangle x) - f(x_0) \end{cases} \qquad \begin{cases} f(x_0 + \triangle x) \approx f(x_0) + f'(x_0) \triangle x \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ x_0 = 0 \begin{cases} f(x) \approx f(0) + f'(0)x \\ \sqrt{n} \approx 1 + \frac{1}{n}x \\ \sin x \approx x \\ \tan x \approx x \\ e^x \approx 1 + x \\ \ln(1 + n) \approx x \end{cases} \end{cases}$$

12.2.5 奇偶函数导数

$$偶函数导数为奇函数\ f(x) = f(-x) \Leftrightarrow f'(x) = -f'(-x)$$

$$奇函数导数为偶函数\ f(x) = -f(-x) \Leftrightarrow f'(x) = f'(-x)$$

12.2.6 区间恒为 0

$$若 f'(x) 在区间恒为零，则 f(x) 在区间 I 上为一常数$$

$$E 设 x_1, x_2 为区间 I 内任意两点 x_1 < x_2$$

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \equiv 0$$

$$f(x_2) \equiv f(x_1) = C$$

12.3 中值定理

12.3.1 费马引理

$$f(x) \qquad \forall x \in \mathring{U}(x_0) \begin{cases} f(x) \leqslant f(x_0) & f(x) 在 x_0 处取极大值 \\ f(x) \geqslant f(x_0) & f(x) 在 x_0 处取极小值 \end{cases}$$

$$如果可导函数 y = f(x) 在 x_0 取极值，则 f'(x_0) = 0 \tag{12.3.1}$$

12.3.2 罗尔定理

$$如果函数 f(x) 满 \begin{cases} 在闭区间 [a,b] 上连续 \\ 在开区间 (a,b) 可导 \\ f(a) = f(b) \end{cases}$$

$$则至少有一点 \xi \in (a,b), f'(\xi) = 0 \tag{12.3.2}$$
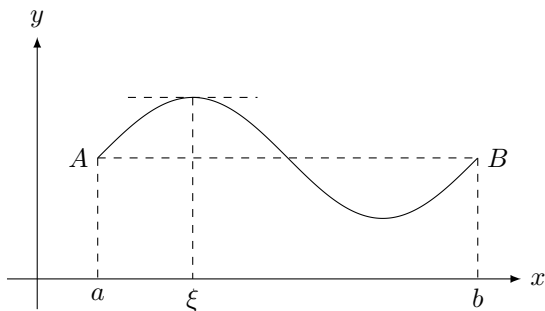
12.3.3 拉格朗日定理 (微分中值定理)
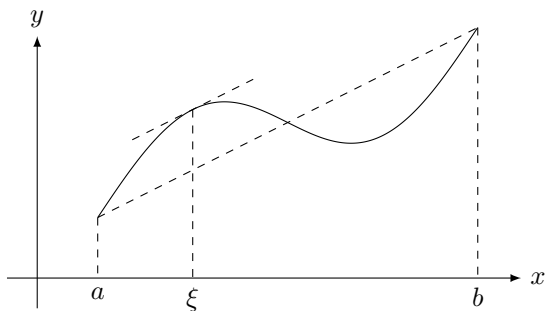
$$如果函数 f(x) 满 \begin{cases} 在闭区间 [a,b] 上连续 \\ 在开区间 (a,b) 可导 \end{cases}$$
$$则至少有一点 \xi \in (a,b)$$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a} \Leftrightarrow f(b) - f(a) = f'(x)(b - a) \tag{12.3.3}$$

在区间 $[x, x+ \triangle x]$ 用拉格朗日定理。

$$f(x+ \triangle x) - f(x) = f'(\xi) \triangle x$$

$$\xi \in (x, x+ \triangle x) \text{ 记作}: \xi = x + \theta \triangle x \qquad 0 < \theta < 1$$

$$f(x+ \triangle x) - f(x) = f(x + \theta \triangle x) \triangle x$$

$$\triangle y = f(x + \theta \triangle x) \triangle x$$

12.3.4 柯西定理
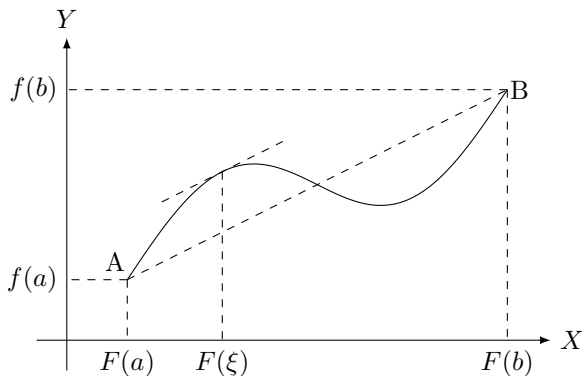
如果函数 $f(x)$ 满 $\begin{cases} \text{在闭区间 } [a,b] \text{ 上连续} \\ \text{在开区间 (a,b) 可导} \\ F'(x) \neq 0 \end{cases}$ $\qquad$ 参数方程 $(a \leqslant x \leqslant b) \begin{cases} X = F(x) \\ Y = f(x) \end{cases}$

$$\text{至少有一点}, \xi \qquad \frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)} \tag{12.3.4}$$

$$\text{切线斜率} = \frac{dY}{dX} = \frac{df(x)}{dF(x)} = \frac{f'(x)}{F'(x)} \Rightarrow x = \xi \text{时斜率} = \frac{f'(\xi)}{F'(\xi)}$$

$$AB\text{的斜率} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

12.3.5 三个定理关系

$$\frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)}, (F(x) = x) \Rightarrow f'(\xi) = \frac{f(b) - f(a)}{b - a}, (f(b) = f(a)) \Rightarrow f'(\xi) = 0$$

12.4 洛必达法则

$$\text{未定型}, \quad \frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0^0, \quad 1^\infty, \quad \infty^0, \quad \infty - \infty$$

$$\lim_{x \to x_0} \frac{f(x)}{F(x)} = \lim_{x \to x_0} \frac{f'(x)}{F'(x)} \begin{cases} \lim_{x \to x_0} f(x) = 0, \lim_{x \to x_0} F(x) = 0 \\ f(x), F(x) \text{在} x_0 \text{的某去心邻域内可导，且} F'(x) \neq 0 \\ \lim_{x \to x_0} \frac{f'(x)}{F'(x)} \text{存在，或无穷小。则} \end{cases}$$

$$\lim_{x \to x_0} \frac{f(x)}{F(x)} = \lim_{x \to x_0} \frac{f'(x)}{F'(x)} \tag{12.4.1}$$

$$\lim_{x \to \infty} \frac{f(x)}{F(x)} = \lim_{x \to \infty} \frac{f'(x)}{F'(x)} \begin{cases} \lim_{x \to x_0} f(x) = 0, \lim_{x \to x_0} F(x) = 0 \\ \exists N \text{ 当} |x| > N, \text{ 时} f'(x), F'(x) \text{存在，且} F'(x) \neq 0 \\ \lim_{x \to x_0} \frac{f'(x)}{F'(x)} \text{存在，或为无穷大。则} \end{cases}$$

12.5 泰勒公式

$$f(x_0 + \triangle x) - f(x_0) = f'(x_0) \triangle x + \circ(\triangle x)$$
$$x_0 + \triangle = x \qquad \triangle x = x - x_0$$
$$f(x) - f(x_0) = f'(x_0)(x - x_0) + \circ(\triangle x)$$
$$f(x) = f'(x_0)(x - x_0) + f(x_0) + \circ(\triangle x)$$
$$f(x) \approx f'(x_0)(x - x_0) + f(x_0)$$

$$P(x_0) = f(x_0)$$
$$P'(x_0) = f'(x_0)$$

12.5.1 泰勒多项式

$P(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n$去近似某个多项式

$$
\begin{cases}
P_n(x_0) & = f(x_0) = a_0 \\
P_n'(x_0) & = f'(x_0) = a_1 \\
P_n''(x_0) & = f''(x_0) = a_2 \cdot 2! \\
& \vdots \\
P_n^{(n-1)}(x_0) & = f^{(n-1)}(x_0) = a_{n-1} \cdot (n-1)! \\
P_n^{(n)}(x_0) & = f^{(n)}(x_0) = a_n \cdot n!
\end{cases}
\Rightarrow
\begin{cases}
a_0 & = f_n(x_0) \\
a_1 & = f_n'(x_0) \\
a_2 & = \frac{f_n''(x_0)}{2!} \\
& \vdots \\
a_{n-1} & = \frac{f_n^{(n-1)}(x_0)}{(n-1)!} \\
a_n & = \frac{f_n^{(n)}(x_0)}{n!}
\end{cases}
$$

$$
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \cdots \frac{f^{(n-1)}(x_0)}{(n-1)!}(x - x_0)^{n-1} + \frac{f^n(x_0)}{n!}(x - x_0)^n
$$

$$
f(x) \approx P_n(x)
$$

12.5.2 泰勒中值定理

如果$f(x)|x_0 \in (a,b)$内有$(n+1)$阶导则

$$
f(x) = P_n(x) + R_n(x)
$$

$$
P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + R_n(x)
$$

拉格朗日余项

$$
R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1} \qquad \{\xi \in (x, x_0)\} \tag{12.5.1}
$$

皮亚诺于项

$$
R_n(x) = \circ(|x - x_0|^n) \tag{12.5.2}
$$

$$
f(x) \approx P_n(x) \text{ 误差为 } R_n(x)
$$

12.6 麦克劳林公式

$$
x_0 = 0
$$

$$
P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n +
\begin{cases}
\frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1} \cdot\ 0 < \theta < 1 \\
\circ(|x|^n)
\end{cases}
$$

12.6.1 常用的麦克劳林展开

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0 < \theta < 1 \\ \circ(|x|^n) \end{cases}$$

$$\sin x = 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^{n-1}\frac{1}{(2n-1)!}x^{2n-1} + R_n(x)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots + (-1)^n\frac{1}{(2n)!}x^{2n} + R_n(x)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n + R_n(x)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots - \frac{1}{n}x^n + R_n(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \cdots \frac{A_\alpha^n}{n!}x^n + R_n(x)$$

13 不定积分

13.1 概念

13.1.1 原函数

$$\forall x \in I, \ F'(x) = f(x), \quad F(x)为f(x)的一个原函数$$

$$函数f(x)在区间I上连续一定有F(x), 使F'(x) = f(x) \tag{13.1.1}$$

13.1.2 不定积分

区间 I 上, $f(x)$ 的带有任意常数的原函数, 称为 $f(x)$ 在区间 I 上的不定积分。
记作:

$$\int f(x) \ dx \quad \begin{cases} \int & 积分符号 \\ f(x) & 被积函数 \\ f(x) \ dx & 被积表达式 \\ x & 积分变量 \end{cases}$$

如果$F(x)$是$f(x)$的一个原函数

$$\int f(x)dx = F(x) + C$$

13.1.3 不定积分性质

$$\left[\int f(x) \ dx \right]' = f(x)$$
$$d\left[\int f(x) \ dx \right] = f(x) \ dx$$
$$\int dF(x) = \int F'(x) \ dx = F(x) + C$$

13.2 积分运算

$$\int [f(x) + g(x)] \ dx = \int f(x) \ dx + \int f(x) \ dx \tag{13.2.1}$$

$$\int kf(x) \ dx = k \int f(x) \ dx \quad (k为常数) \tag{13.2.2}$$

$$\int f[\varphi(x)] \varphi'(x) \ dx \xrightarrow{\underline{u=\varphi(x)}} \left[\int f(u)du\right]_{x=\varphi(u)} = F[\varphi(x)] + C \tag{13.2.3}$$

$$\int f(x) \ dx \xrightarrow{\frac{x=\varphi(t)}{\varphi'(t)\neq 0}} \left[\int f[\varphi(t)]\varphi'(t) \ dt\right]_{t=\varphi^{-1}(x)} \tag{13.2.4}$$

$$\int f(x) \ dx = \int f(x) \ d(x + C) \tag{13.2.5}$$

13.2.1 分部积分法

$$\int u \ dv = uv - \int v \ du \Leftrightarrow \int uv' \ dx = uv - \int u'v \ dx \tag{13.2.6}$$

13.3 有理函数积分

13.3.1 普通多项式

$\dfrac{P(x)}{Q(x)}$ 　　$P(x), Q(x)$是 x 多项式, 且没有公因子，称为有理分式

有理分式 $\begin{cases} 真分式 & P(x)次数 < Q(x)次数 \\ 假分式 & P(x)次数 \geqslant Q(x)次数 \end{cases}$

如果真分式中$Q(x) = Q_1(x) \cdot Q_2(x)$,其中$Q_1(x), Q_2(x)$都为多项式

$$\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)} \cdot \frac{P_2(x)}{Q_2(x)} \tag{13.3.1}$$

假分式 = 多项式 + 真分式

最简分式　$\dfrac{A}{x-a}$　$\dfrac{A}{(x-a)^2}$　$\dfrac{Nx+M}{x^2+px+q}$　$\dfrac{Nx+m}{(x^2+px+q)^k}$

13.3.2 三角函数多项式

$$三角有理分式: R(\sin x, \cos x)$$

$$万能代换: \tan \frac{x}{2} = u, x = 2\arctan u, dx = \frac{2\ du}{1+u^2}$$

$$\sin x = 2\sin \frac{x}{2}\cos \frac{x}{2} = 2\frac{\tan \frac{x}{2}}{\sec^2 \frac{x}{2}} = 2\frac{\tan \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} = \cos^2 \frac{x}{2}(1-\tan^2 \frac{x}{2}) = \frac{1-\tan^2 \frac{x}{2}}{\sec^2 \frac{x}{2}} = \frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}} = \frac{1-u^2}{1+u^2}$$

$$\int R(\sin x, \cos x)\ dx = \int R(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2})\frac{2}{1+u^2}\ du = \int Y(u)\ du$$

$$Y(u)\ 是\ u\ 的有理函数$$

13.4 积分公式

13.4.1 幂数，指数，对数

$$\int k\ dx = kx + C \tag{13.4.1}$$

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C \tag{13.4.2}$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C \tag{13.4.3}$$

$$\int e^x\ dx = e^x + C \tag{13.4.4}$$

$$\int \frac{1}{x}\ dx = \ln|x| + C \tag{13.4.5}$$

$$\int \ln x\ dx = x\ln x - x + C \tag{13.4.6}$$

13.4.2 三角函数

$$\int \sin x \ dx = -\cos x + C \tag{13.4.7}$$

$$\int \cos x \ dx = \sin x + C \tag{13.4.8}$$

$$\int \sec x \tan x \ dx = \sec x + C \tag{13.4.9}$$

$$\int \csc x \cot x \ dx = -\csc x + C \tag{13.4.10}$$

$$\int \sec^2 x \ dx = \tan x + C \tag{13.4.11}$$

$$\int \csc^2 x \ dx = -\cot x + C \tag{13.4.12}$$

$$\int \sinh x \ dx = \cosh x + C \tag{13.4.13}$$

$$\int \cosh x \ dx = \sinh x + C \tag{13.4.14}$$

$$\int \tan x \ dx = -\ln|\cos x| + C \tag{13.4.15}$$

$$\int \csc x \ dx = \begin{cases} \ln\left|\tan \frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases} \tag{13.4.16}$$

$$\int \sec x \ dx = \ln|\sec x + \tan x| + C \tag{13.4.17}$$

$$\int \arccos x \ dx = x \arccos x - \sqrt{1 - x^2} + C \tag{13.4.18}$$

$$\int \arctan x \ dx = x \arctan x - \frac{1}{2}\ln(1 + x^2) + C \tag{13.4.19}$$

13.4.3 分式

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C \tag{13.4.20}$$

$$\int \frac{1}{x^2 + a^2} \, dx = \frac{1}{a} \tan \frac{x}{a} + C \tag{13.4.21}$$

$$\int \frac{1}{x^2 - a^2} \, dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \tag{13.4.22}$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \begin{cases} \arcsin x + C \\ -\arccos x + C_1 \end{cases} \tag{13.4.23}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \begin{cases} \arcsin \frac{x}{a} x + C \\ -\arccos \frac{x}{a} + C_1 \end{cases} \tag{13.4.24}$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left| x + \sqrt{x^2 - a^2} \right| + C \tag{13.4.25}$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln(x + \sqrt{x^2 + a^2}) + C \tag{13.4.26}$$

$$\int \frac{1}{|x| \sqrt{x^2 - 1}} \, dx = \operatorname{arcsec} x + C \tag{13.4.27}$$

14 定积分

15 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1 - q^{n+1}}{1 - q} \tag{15.0.1}$$

$$A_N = \sum_{k=0}^{n} q^k \qquad qA_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - qA_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1 - q^{n+1}}{1 - q}$$

$$\log_{10} x = \lg_x \tag{15.0.2}$$

$$\log_e x = \ln_x \tag{15.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{15.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{15.0.5}$$

$$\log_b x^n = n \log_b x \tag{15.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{15.0.7}$$

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a + b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a - b)(1 + b)$$

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a - b) \sum_{m=0}^{n-1} \left(a^{n-m} b^m\right) = (a - b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)$$

16 证明

16.1 第 1章

1.2.4

$$\sinh x \cosh x = \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$
$$= \left(\frac{1}{2}\right)\left(\frac{e^{2x} - e^{-2x}}{2}\right)$$
$$= \frac{1}{2}\sinh(2x)$$
$$\sinh(2x) = 2\sinh x \cosh x$$

1.2.5

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)$$
$$= e^x \times e^{-x}$$
$$= 1$$

1.2.6

$$\cosh^2 x + \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{2e^{2x} + 2e^{-2x}}{4}$$
$$= \frac{e^{2x} + e^{-2x}}{2}$$
$$= \cosh(2x)$$

1.2.7

$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$= \sinh^2 x + 1 + \sinh^2 x$$
$$= 2\sinh^2 x + 1$$
$$\cosh x = 2\sinh^2 \frac{x}{2} + 1$$

1.1.17

$$
\begin{aligned}
\tan \frac{x}{2} = \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} &= \sqrt{\frac{1 - \cos x}{1 + \cos x}} \\
&= \sqrt{\frac{1 - \cos x}{1 + \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}} \\
&= \sqrt{\frac{1 - \cos^2 x}{(1 + \cos x)^2}} \\
&= \frac{\sin x}{1 + \cos x} \\
&= \frac{1 - cosx}{\sin x} \\
&= \csc x - \cot x
\end{aligned}
$$

16.2　第 5章

5.2.1

$$
\begin{aligned}
&设 \forall \, x_1, x_2 \in [a, b], x_1 < x_2 \\
&f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \qquad \xi \in (x_1, x_2) \subset [a, b] \\
&\quad f'(\xi) > 0, \, (x_2 - x_1) > 0 \\
&f(x_2) - f(x_1) > 0 \\
&\qquad f(x_2) > f(x_1)
\end{aligned}
$$

5.2.2

$$
\begin{aligned}
&设 \forall \, x_1, x_2 \in [a, b], x_1 < x_2 \\
&f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \qquad \xi \in (x_1, x_2) \subset [a, b] \\
&\quad f'(\xi) < 0, \, (x_2 - x_1) > 0 \\
&f(x_2) - f(x_1) < 0 \\
&\qquad f(x_2) < f(x_1)
\end{aligned}
$$

5.2.3

$$
\begin{aligned}
&设 \forall \, x_1, x_2 \in [a, b], x_1 < x_2, x_0 = \frac{x_1 + x_2}{2}, x_0 - x_1 = x_2 - x_0 = h \\
&\varphi = \, f(x_0) - f(x_1) = f'(\xi_1)(x_0 - x_1) \qquad \xi_1 \in (x_1, x_0) \\
&\psi = f(x_2) - f(x_0) = f'(\xi_2)(x_2 - x_0) \qquad \xi_2 \in (x_0, x_2)
\end{aligned}
$$

$$\psi - \varphi = f(x_2) + f(x_1) - 2f(x_0) = [f'(\xi_2) - f'(\xi_1)] \, h$$
$$= f''(\xi)(\xi_2 - \xi_1)h$$

$$\text{因为 } f''(x) > 0, f''(\xi) > 0, h = x_0 - x_1 > 0$$

$$f(x_2) + f(x_1) - 2f(x_0) > 0$$

$$f(x_2) + f(x_1) > 2f(x_0)$$

$$f(x_0) < \frac{f(x_2) + f(x_1)}{2}$$

$$f(\frac{x_1 + x_2}{2}) < \frac{f(x_2) + f(x_1)}{2}$$

5.3.1

$$\triangle s = \widehat{M_0 M'} - \widehat{M_0 M} = \widehat{MM'}, \quad |MM'|^2 = (\triangle x)^2 + (\triangle y)^2, \quad \lim_{M' \to M} \frac{\left|\widehat{MM'}\right|}{|MM'|} = 1$$

$$\left(\frac{\triangle s}{\triangle x}\right)^2 = \left|\frac{\widehat{MM'}}{\triangle x}\right|^2 = \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left(\frac{|MM'|}{\triangle x}\right)^2$$

$$= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \frac{(\triangle x)^2 + (\triangle y)^2}{(\triangle x)^2}$$

$$= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left[1 + \left(\frac{\triangle y}{\triangle x}\right)^2\right]$$

$$\lim_{\triangle x \to 0} \left(\frac{\triangle s}{\triangle x}\right)^2 = \lim_{\triangle x \to 0} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle y}{\triangle x}\right)^2\right]$$

$$(\triangle x \to 0, \triangle M' \to M) \qquad = \lim_{M' \to M} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle y}{\triangle x}\right)^2\right]$$

$$\left(\frac{ds}{dx}\right)^2 = 1 \cdot (1 + (y')^2)$$

$$\frac{ds}{dx} = \sqrt{1 + (y')^2} = \sqrt{1 + [f'(x)]^2}$$

$$ds = \sqrt{1 + [f'(x)]^2} dx = \sqrt{(dx)^2 + (dy)^2}$$

5.3.2

$$\left| \frac{d\alpha}{ds} \right| = \left| \frac{d\alpha}{dx} \cdot \frac{dx}{ds} \right|$$

$$= \left| \frac{d \arctan y'}{dx} \cdot \frac{1}{\sqrt{1 + (y')^2}} \right|$$

$$= \left| \frac{y''}{1 + (y')^2} \cdot \frac{1}{\sqrt{1 + (y')^2}} \right|$$

$$= \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

5.3.3

$$\frac{dy}{dx} = \frac{\psi'(t)}{\phi'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{\psi'(t)}{\phi'(t)}}{dt} \cdot \frac{dt}{dx}$$

$$= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^2} \cdot \frac{1}{\phi'(t)}$$

$$= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3}$$

$$\left| \frac{d\alpha}{ds} \right| = \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

$$= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3} \cdot \frac{1}{\left\{ 1 + \left[\frac{\psi'(t)}{\phi'(t)} \right]^2 \right\}^{\frac{3}{2}}}$$

$$= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{ |\psi'(t)|^2 + [\phi'(t)]^2 \right\}^{\frac{3}{2}}}$$

16.3　第 9章

9.1.1

$$\text{反设} \lim_{n\to\infty} x_n = a, \ \lim_{n\to\infty} x_n = b, \text{且} a < b$$

$$\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1, \ n > N_1, \ |x_n - a| < \frac{b-a}{3} \\ \exists N_2, \ n > N_2, \ |x_n - b| < \frac{b-a}{3} \end{cases}$$

$$N = \max\{N_1, N_2\}, \ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}$$

$$b - a = |(x_n - a) - (x_n - b)|$$
$$\leqslant |x_n - a| + |x_n - b|$$
$$< \frac{b-a}{3} + \frac{b-a}{3}$$
$$< \frac{2(b-a)}{3}$$

9.1.2

$$\varepsilon = 1, \ \exists N > 0, \ \text{当 n>N 时} |X_n - a| < 1$$
$$|X_n| = |(X_n - a) + a|$$
$$\leqslant |x_n - a| + |a|$$
$$\leqslant 1 + |a|$$
$$M = \max\{|X_n|, |X_2|, \ldots, |X_n|, 1 + |a|\}$$
$$\forall n, \ |X_n| \leqslant M$$

9.1.4

1

由于 $\lim_{n\to\infty} x_n = a, \text{且} a > 0$
$\varepsilon = \frac{a}{2}, \ \exists N > 0, n > N$
$|x_n - a| < \varepsilon$
$|x_n - a| < \frac{a}{2}$
$-\frac{a}{2} < x_n - a < \frac{a}{2}$
$\frac{a}{2} < x_n < 1$

2

用反证法, 反设 $a < 0$. 从某项起 $x_n < 0$ 矛盾

9.1.5

$$x_n = b_n - a_n$$

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} b_n - \lim_{n \to \infty} a_n$$

$$\lim_{n \to \infty} x_n = b - a > 0$$

$$\lim_{n \to \infty} x_n > 0$$

$$b_n - a_n = x_n > 0$$

$$b_n > a_n$$

9.2.1

$$\lim_{x \to x_0} f(x)存在 \Rightarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$$

$$设 \lim_{x \to x_0} = A$$

$$0 < |x - x_0| < \delta, \ |f(x) - A| < \varepsilon$$

$$0 < |x - x_0| < \delta \Leftrightarrow x \in \mathring{U}(x_0, \delta)$$

$$\begin{cases} 当x_0 < x < x_0 + \delta时 0 < |x - x_0| < \delta, \ |f(x) - A| < \varepsilon, \ \lim_{x \to x_0^+} f(x) = A \\ 当x_0 - \delta < x < x_0时 0 < |x - x_0| < \delta, \ |f(x) - A| < \varepsilon, \ \lim_{x \to x_0^-} f(x) = A \end{cases}$$

$$\lim_{x \to x_0^+} f(x) = A = \lim_{x \to x_0^-} f(x)$$

$$\lim_{x \to x_0} f(x)存在 \Leftarrow \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x)$$

$$A = \begin{cases} \lim_{x \to x_o^+}, \forall \varepsilon > 0, \exists \delta_1 > 0, x_0 < x < x_0 + \delta_1, |f(x) - A| < \varepsilon \\ \lim_{x \to x_o^-}, \forall \varepsilon > 0, \exists \delta_2 > 0, x_0 - \delta_2 < x < x_0, |f(x) - A| < \varepsilon \end{cases}$$

$$\delta = \min\{\delta_1, \delta_2\}$$

$$0 < |x - x_0| < \delta \begin{cases} x > x_0, x_0 < x < x_0 + \delta \leqslant x_0 + \delta_1, \ |f(x) - A| < \varepsilon \\ x < x_0, x_0 - \delta_2 \leqslant x_0 + \delta < x < x_0, \ |f(x) - A| < \varepsilon \end{cases}$$

$$\lim_{x \to x_0} f(x) = A$$

9.3.1

$$\lim_{x \to x_o} f(x) = A \Rightarrow \begin{cases} \alpha为x \to x_0时的无穷小 \\ f(x) = \alpha + A \end{cases}$$

$$\text{设} \lim_{x \to x_o} f(x) = A, \text{ 记} f(x) - A = \alpha$$
$$\text{只需证 } \alpha \text{ 为无穷小。}$$
$$\forall \varepsilon > 0, \exists \delta > 0, \text{当} 0 < |x - x_0| < \delta, \text{时} |f(x) - A| < \varepsilon$$
$$\text{即 } |\alpha - 0| < \varepsilon$$
$$\alpha \text{为} x \to x_0 \text{时的无穷小}$$
$$\lim_{x \to x_o} f(x) = A \Leftarrow \begin{cases} \alpha \text{为} x \to x_0 \text{时的无穷小} \\ f(x) = \alpha + A \end{cases}$$
$$\forall \varepsilon > 0, \ \exists \delta > 0, \ \text{当} 0 < |x - x + 0| < \delta, \ |\alpha| < \varepsilon$$
$$\text{即 } |f(x) - A| < \varepsilon \lim_{x \to x_0} f(x) = A$$

9.3.2

$$\text{设} \lim_{x \to x_0} f(x) = \infty$$
$$\text{对 } f(x) \text{ 为 } x \to \text{ 时无穷大}$$
$$\text{对于 } M = \frac{1}{\varepsilon}. \text{ 存在 } \delta > 0$$
$$\text{当 } 0 < |x - x_0| < \delta \text{ 时}$$
$$|f(x)| > M = \frac{1}{\varepsilon}$$
$$\left| \frac{1}{f(x)} \right| < \varepsilon$$
$$\frac{1}{f(x)} \text{ 为 } x \to x_0 \text{ 时的无穷小}$$

9.4.2

$$f(x)g(x) = [A + \alpha][B + \beta]$$
$$= AB + A\beta + B\alpha + \beta\alpha$$
$$= AB + \gamma \qquad (\gamma \text{为无穷小})$$
$$\lim[f(x)g(x)] = AB + \gamma = \lim f(x) \lim g(x)$$

9.4.8

$$\forall \varepsilon > 0$$
$$|x_n - a| < \varepsilon \qquad \forall n > N_1$$
$$|y_n - a| < \varepsilon \qquad \forall n > N_2$$
$$\text{令} N = \max\{N_1, N_2, N_0\}, \text{ 则当} n > N \text{时有}$$
$$a - \varepsilon < x_n \leq z_n \leq y_n < a + \varepsilon$$
$$|z_n - a| < \varepsilon$$
$$\lim_{n \to \infty} z_n = a$$

9.4.9

$$
\begin{aligned}
|f(x) - \sin x_0| &= |\sin x - \sin x_0| \\
&= \left| 2\cos(\frac{x + x_0}{2})\sin(\frac{x - x_0}{2}) \right| \\
&\leqslant 2\left| \sin(\frac{x - x_0}{2}) \right| \\
&\leqslant 2\frac{|x - x_0|}{2} = |x - x_0|
\end{aligned}
$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\sin x - \sin x_0| \leqslant |x - x_0| < \varepsilon$$

9.4.10

$$
\begin{aligned}
|f(x) - \cos x_0| &= |\cos x - \cos x_0| \\
&= \left| -2\sin(\frac{x + x_0}{2})\sin(\frac{x - x_0}{2}) \right| \\
&\leqslant 2\left| \sin(\frac{x - x_0}{2}) \right| \\
&\leqslant 2\frac{|x - x_0|}{2} = |x - x_0|
\end{aligned}
$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\cos x - \cos x_0| \leqslant |x - x_0| < \varepsilon$$

9.4.11

$$OB = OA = 1$$

$$\triangle \ AOB \leqslant 扇形面积 \leqslant \triangle \ AOD$$

$$\frac{1}{2} \sin x \leqslant \frac{1}{2} x \leqslant \frac{1}{2} \tan x$$

$$\sin x \leqslant x \leqslant \tan$$

$$1 \geqslant \frac{\sin x}{x} \geqslant \cos x$$

$$\lim_{x \to 0} 1 \geqslant \lim_{x \to 0} \frac{\sin x}{x} \geqslant \lim_{x \to 0} \cos x$$

$$1 \geqslant \lim_{x \to 0} \frac{\sin x}{x} \geqslant 1$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

9.4.12

$$|1 - \cos x| = 1 - \cos x = 2 \sin^2 \frac{x}{2} \leqslant 2 \left(\frac{x}{2} \right)^2$$

$$0 \leqslant 1 - \cos x \leqslant \frac{x^2}{2}$$

$$\lim_{x \to 0} 0 \leqslant \lim_{x \to 0} (1 - \cos x) \leqslant \lim_{x \to 0} \frac{x^2}{2}$$

$$0 \leqslant \lim_{x \to 0} (1 - \cos x) \leqslant 0$$

$$\lim_{x \to 0} (1 - \cos x) = 0$$

$$\lim_{x \to 0} \cos x = 1$$

9.4.13

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{\cos x}$$
$$= \lim_{x \to 0} \frac{\sin x}{x} \lim_{x \to 0} \frac{1}{\cos x}$$
$$= 1$$

9.4.14

$$\lim_{x \to 0} \frac{1 - \cos x}{\frac{1}{2}x^2} = \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{\frac{1}{2}x^2}$$
$$= \lim_{x \to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$
$$= 1$$

9.4.15

$$x = \sin t, \ t = \arcsin x$$

$$x \to 0, \ t \to 0$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = \lim_{x \to 0} \frac{t}{\sin t} = 1$$

9.4.16

$$x = \tan t, \ t = \arctan x$$

$$x \to 0, \ t \to 0$$

$$\lim_{x \to 0} \frac{\arctan x}{x} = \lim_{t \to 0} \frac{t}{\tan t} = 1$$

9.4.17

$$\lim_{x \to 0} \frac{\ln (1 + x)}{x} = \lim_{x \to 0} \ln (1 + x)^{\frac{1}{x}} = \ln e = 1$$

9.4.18

$$e^x - 1 = t, \ x = \ln (t + 1)$$

$$x \to 0, \ t \to 0$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{t \to 0} \frac{t}{\ln (t + 1)} = 1$$

9.4.19

$$\lim_{x \to 0} \frac{(1 + x)^n - 1}{nx} = \lim_{x \to 0} \left(\frac{e^{n \ln(1+x)} - 1}{n \ln (1 + x)} \cdot \frac{\ln (1 + x)}{x} \right) = 1$$

9.4.21

$$x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m} \left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m$$

$$= C_n^0 \left(\frac{1}{n}\right)^0 + C_n^1 \left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m \left(\frac{1}{n}\right)^m$$

$$= 1 + 1 + \sum_{m=2}^{n} \frac{n!}{m!\,(n-m)!} \left(\frac{1}{n}\right)^m$$

$$= 1 + 1 + \sum_{m=2}^{n} \frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!} \left(\frac{1}{n}\right)^m$$

$$= 1 + 1 + \sum_{m=2}^{n} \frac{1}{m!} \left(\frac{n}{n}\right) \left(\frac{n-1}{n}\right) \cdots \left(\frac{n-m+1}{n}\right)$$

$$= 1 + 1 + \sum_{m=2}^{n} \frac{1}{m!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right)$$

$$x_{n+1} = 1 + 1 + \sum_{m=2}^{n+1} \frac{1}{m!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \cdots \left(1 - \frac{m-1}{n+1}\right)$$

$$x_n < x_{n+1} \qquad \text{单调增加}$$

$$x_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$< 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{n^2} = 1 + \frac{1 - \left(\frac{1}{2}\right)^2}{1 - \frac{1}{2}}$$

$$< 1 + \frac{1}{1 - \frac{1}{2}}$$

$$< 3 \qquad \text{有界}$$

16.4 第 11章

11.1.1

$$f'(x_0) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} \qquad \text{因为极限存在与无穷小的关系}$$

$$\frac{\triangle y}{\triangle x} = f'(x_0) + \alpha \qquad \alpha 为 \triangle x \to 0 时的无穷小$$

$$\triangle y = f'(x_0) \triangle x + \alpha \triangle x$$

$$\lim_{\triangle x \to 0} \triangle y = \lim_{\triangle x \to 0} [f'(x_0) \triangle x + \alpha \triangle x] = 0$$

$$\lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow \lim_{\triangle x \to 0} [f(x_0 + \triangle x) - f(x_0) = 0] \Leftrightarrow \lim_{\triangle x \to 0} \triangle y = 0$$

11.2.1

$$\begin{aligned}(C)' &= \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \\ &= \lim_{\triangle x \to 0} \frac{C - C}{\triangle x} \\ &= 0 \end{aligned}$$

11.2.2

$$\begin{aligned}(x^a)' &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \frac{x^a - x_0^a}{x - x_0} \\ &= \frac{(x - x_0)\left(x^{a-1} + x^{a-2}x_0 + \cdots + xx_0^{a-2} + x_0^{a-1}\right)}{x - x_0} \\ &= ax_0^{a-1} \end{aligned}$$

11.2.3

$$\begin{aligned}(a^x)' &= \lim_{\triangle x \to 0} \frac{a^{x+\triangle x} - a^x}{\triangle x} \\ &= a^x \lim_{\triangle x \to 0} \frac{a^{\triangle x} - 1}{\triangle x} \\ &= a^x \lim_{\triangle x \to 0} \frac{e^{\triangle x \ln a} - 1}{\triangle x} \\ &= a^x \ln a \end{aligned}$$

11.2.4

$$(e^x)' = e^x \ln e = e^x$$

11.2.5

$$\begin{aligned}
(\log_a^x)' &= \lim_{\triangle x \to 0} \frac{\log_a^{x+\triangle x} - \log_a^x}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{\ln 1 + \frac{\triangle x}{x}}{\ln a \triangle x} \\
&= \frac{1}{\ln a} \lim_{\triangle x \to 0} \frac{\frac{\triangle x}{x}}{\triangle x} \\
&= \frac{1}{x \ln a}
\end{aligned}$$

11.2.6

$$\begin{aligned}
(\ln^x)' &= \frac{1}{x \ln e} \\
&= \frac{1}{x}
\end{aligned}$$

11.3.1

$$\begin{aligned}
(\sin x)' &= \lim_{\triangle x \to 0} \frac{\sin(x_0 + \triangle x) - \sin x_0}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{2 \cos(x_0 + \frac{\triangle x}{2}) \sin \frac{\triangle x}{2}}{\triangle x} \\
&= \lim_{\triangle x \to 0} \cos(x_0 + \frac{\triangle x}{2}) \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= cos x_0
\end{aligned}$$

11.3.2

$$\begin{aligned}
(\arcsin x)' &= \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d \sin y}{dy}} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}$$

11.3.3

$$\begin{aligned}
(\csc x)' = (\frac{1}{\sin x})' &= \frac{(1)' \cdot sin x - (\sin x)' \cdot 1}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= -\csc x \cdot \cot x
\end{aligned}$$

11.3.4

$$(\cos x)' = \lim_{\triangle x \to 0} \frac{\cos(x_0 + \triangle x) - cos(x_0)}{\triangle x}$$
$$= \lim_{\triangle x \to 0} \frac{-2 \sin\left(x_0 + \frac{\triangle x}{2}\right) \sin \frac{\triangle x}{2}}{\triangle x}$$
$$= \lim_{\triangle x \to 0} - \sin\left(x_0 + \frac{\triangle x}{2}\right) \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}}$$
$$= -\sin x$$

11.3.5

$$(\arccos x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\cos y}{\mathrm{d}y}}$$
$$= \frac{1}{-\sin y}$$
$$= -\frac{1}{\sqrt{1 - \cos^2 y}}$$
$$= -\frac{1}{\sqrt{1 - x^2}}$$

11.3.6

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{(1)' \cdot \cos x - (\cos x)' \cdot 1}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x}$$
$$= \sec x \cdot \tan x$$

11.3.8

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \sec^2 x$$

11.3.9

$$(\arctan x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\tan y}{\mathrm{d}y}}$$
$$= \frac{1}{\sec y}$$
$$= \frac{1}{1 + \tan^2 y}$$
$$= \frac{1}{1 + x^2}$$

11.3.10

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x}$$
$$= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x}$$
$$= -\csc^2 x$$

11.3.11

$$(\operatorname{arccot} x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\cot y}{\mathrm{d}y}}$$
$$= \frac{1}{-\csc^2 y}$$
$$= -\frac{1}{1 + \cot^2 y}$$
$$= -\frac{1}{1 + x^2}$$

11.3.12

$$(\sinh x)' = \left(\frac{e^x - e^{-x}}{2}\right)'$$
$$= \frac{e^x + e^{-x}}{2}$$
$$= \cosh x$$

11.3.13

$$(\cosh x)' = \left(\frac{e^x + e^{-x}}{2}\right)'$$
$$= \frac{e^x - e^{-x}}{2}$$
$$= \sinh x$$

11.3.14

$$
\begin{aligned}
(\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' \\
&= \frac{\frac{d(e^x - e^{-x})}{\mathrm{d}x}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{\mathrm{d}x}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= \frac{2^2}{(e^x + e^{-x})^2} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}
$$

11.3.15

$$
\begin{aligned}
(\operatorname{arcsinh} x)' &= \left[\ln(x + \sqrt{x^2 + 1})\right]' \\
&= \frac{d\ln(x + \sqrt{x^2 + 1})}{\mathrm{d}(x + \sqrt{x^2 + 1})} \cdot \frac{d\left(x + \sqrt{x^2 + 1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2 + 1}\right)}{d\left(x^2 + 1\right)} \cdot \frac{d(x^2 + 1)}{dx}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \\
&= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned}
$$

11.3.16

$$
\begin{aligned}
(\operatorname{arccosh} x)' &= \left[\ln(x + \sqrt{x^2 - 1})\right]' \\
&= \frac{d\ln(x + \sqrt{x^2 - 1})}{\mathrm{d}\left(x + \sqrt{x^2 - 1}\right)} \cdot \frac{d\left(x + \sqrt{x^2 - 1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2 - 1}\right)}{d\left(x^2 - 1\right)} \cdot \frac{d(x^2 - 1)}{dx}\right) \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x\right) \\
&= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}} \\
&= \frac{1}{\sqrt{x^2 - 1}}
\end{aligned}
$$

11.3.17

$$
\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2}\ln(\frac{1 + x}{1 - x})\right]' \\
&= \frac{1}{2} \cdot \frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)} \cdot \frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2} \cdot \frac{1}{\left(\frac{1+x}{1-x}\right)} \cdot \frac{\frac{d(1+x)}{\mathrm{d}x}(1 - x) - \frac{d(1-x)}{\mathrm{d}x}(1 + x)}{(1 - x)^2} \\
&= \frac{1}{2} \cdot \frac{1 - x}{1 + x} \cdot \frac{(1 - x) + (1 + x)}{(1 - x)^2} \\
&= \frac{1}{(1 + x)(1 - x)} \\
&= \frac{1}{1 - x^2}
\end{aligned}
$$

11.4.1

$$
\begin{aligned}
[Cu(x)]' &= \lim_{\triangle x \to 0} \frac{Cu(x + \triangle x) - Cu(x)}{\triangle x} \\
&= C \lim_{\triangle x \to 0} \frac{u(x + \triangle x) - u(x)}{\triangle x} \\
&= Cu'(x)
\end{aligned}
$$

11.4.2

$$(u(x) \pm v(x))' = \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x) \pm v(x+\triangle x) - v(x)}{\triangle x}$$
$$= \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x)}{\triangle x} \pm \lim_{\triangle x \to 0} \frac{v(x+\triangle x) - v(x)}{\triangle x}$$
$$= u'(x) \pm v'(x)$$

11.4.3

$$[u(x) \cdot v(x)]' = \lim_{\triangle x \to 0} \frac{u(x+\triangle x)v(x+\triangle x) - u(x)v(x) - u(x)v(x+\triangle x) + u(x)v(x+\triangle x)}{\triangle x}$$
$$= \lim_{\triangle x \to 0} \frac{[u(x+\triangle x) - u(x)] \, v(x+\triangle x) + u(x) \, [v(x+\triangle x) - v(x)]}{\triangle x}$$
$$= u'(x) \lim_{\triangle x \to 0} v(x+\triangle x) + u(x)v'(x)$$
$$= u'(x)v(x) + v'(x)u(x)$$

11.5.1

$$\left[f^{-1}(y)\right]'|_{y=y_0} = \lim_{y \to y_o} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$$
$$= \lim_{y \to y_o} \frac{x - x_0}{y - y_0}$$
$$= \lim_{x \to x_o} \frac{x - x_0}{f(x) - f(x_0)}$$
$$= \lim_{x \to x_o} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$$
$$= \frac{1}{f'(x)}$$

11.6.1

$$定义函数 \quad A(u) = \begin{cases} \frac{f(u)-f(u_0)}{u-u_0}. & u \neq u_0 \\ f'(u_0), & u = u_0 \end{cases}$$

$$A(u)在u_o处连续, 既有, \lim_{u \to u_o} A(u) = A(u_0) = f'(u_0)$$

由恒等式$f(u) - f(u_0) = A(u)(u - u_0)$我们有

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{f[g(x)] - f[g(x_0)]}{x - x_0}$$

$$= A[g(x)]\frac{g(x) - g(x_0)}{x - x_0}$$

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} A[g(x)]\frac{g(x) - g(x_0)}{x - x_0}$$

$$F'(x_0) = f'(g(x_0))g'(x_0)$$

16.5 第 12章

12.1.1

$$\triangle y = A \triangle x + \circ(\triangle x)$$

$$\frac{\triangle y}{\triangle x} = A + \frac{\circ(\triangle x)}{\triangle x}$$

$$\lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \lim_{\triangle x \to 0}\left[A + \frac{\circ(\triangle x)}{\triangle x}\right]$$

$$f'(x_0) = A + 0$$

$$f'(x_0) = A$$

12.1.2

$$设f(x)在x_0点可导, f'(x_0) = \lim_{\triangle x \to 0}\frac{\triangle y}{\triangle x}存在$$

$$(极限与无穷小的关系: \lim f(x) = A \Leftrightarrow f(x) = A + \alpha)$$

$$\frac{\triangle y}{\triangle x} = f'(x_0) + \alpha$$

$$\triangle y = f'(x_0) \triangle x + \alpha \triangle x$$

$$其中\alpha为 \triangle x \to 0时的无穷小。$$

$$\lim_{\triangle x \to 0} \frac{\alpha \triangle x}{\triangle x} = \lim_{\vartheta x \to 0} \alpha = 0$$

$$\alpha \triangle x = \circ(\triangle x)$$

$$\triangle y = f'(x_0) \triangle x + \circ(\triangle x)$$

12.2.1

$$d(u \pm v) = (u \pm v)'dx$$

$$= (u)'dx \pm (v')dx$$

$$= du \pm dv$$

12.2.2

$$d(u \cdot v) = (u \cdot v)'dx$$
$$= (u)'vdx - (v')udx$$
$$= vdu - udv$$

12.2.3

$$d\left(\frac{u}{v}\right) = \left(\frac{u}{v}\right)' dx$$
$$= \frac{(u)'v - (v')u}{v^2}dx$$
$$= \frac{vdu - udv}{v^2}$$

12.3.1

$$\lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x)}{\triangle x} = f'(x_0)$$
$$f(x_0 + \triangle x) - f(x_0) \leqslant 0$$
$$\begin{cases} \triangle x > 0 \begin{cases} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \leqslant 0 \Rightarrow f'(x_0^+) = \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \leqslant 0 \\ \triangle x < 0 \begin{cases} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \geqslant 0 \Rightarrow f'(x_0^-) = \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \geqslant 0 \end{cases} \end{cases} \end{cases}$$
$$f'(x_0) = f'(x_0^+) = f'(x_0^-) \Rightarrow f'(x_0) = 0$$

12.3.2

$$M = \max\{f(x)|x \in [a,b]\}, m = \min\{f(x)|x \in [a,b]\}$$
$$\begin{cases} M = m \Rightarrow M = m = f(a) = f(b), 此时f(x)为常数, \forall \xi \in (a,b), f'(\xi) = 0 \\ M > m \begin{cases} f(a) > m \Rightarrow \exists \xi \in (a,b), f(\xi) = m, 根据费马引理, f'(\xi) = 0 \\ f(a) < M5 \Rightarrow \exists \xi \in (a,b), f(\xi) = M, 根据费马引理, f'(\xi) = 0 \end{cases} \end{cases}$$

12.3.3

$$\varphi(x) = f(x) - \frac{f(b) - f(a)}{b - a}x$$
$$\varphi(a) = f(a) - \frac{f(b) - f(a)}{b - a}a = \frac{bf(a) - af(b)}{b - a}$$
$$\varphi(b) = f(b) - \frac{f(b) - f(a)}{b - a}b = \frac{bf(a) - af(b)}{b - a}$$
$$\varphi(a) = \varphi(b), \exists \xi \in (a,b), \varphi'(\xi) = 0$$
$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$
$$f'(\xi)(b - a)7 = f(b) - f(a)$$

12.3.4

$$\varphi(x) = f(x) - \frac{f(b) - f(a)}{F(b) - F(a)} \left[F(x) - F(a) \right]$$

$$\varphi(a) = \varphi(b) = f(a)$$

$$\varphi'(\xi) = 0$$

$$f'(\xi) = \frac{f(b) - f(a)}{F(b) - F(a)} F'(\xi)$$

$$\frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

12.4.1

$f(x), F(x)$的去心邻域可导，$\lim\limits_{x \to x_0} \frac{f(x)}{F(x)}$ 与$f(x_0), F(x_0)$无关。规定$f(x_0) = 0, F(x_0) = 0$

此时 $\lim\limits_{x \to x_0} f(x) = 0 = f(x_0)$ $\lim\limits_{x \to x_0} F(x) = 0 = F(x_0)$ 此时在 x_0 点处也连续

$$\frac{f(x)}{F(x)} = \frac{f(x) - f(x_0)}{F(x) - F(x_0)} = \frac{f'(\xi)}{F'(\xi)}$$

$$\lim_{x \to x_0} \frac{f(x)}{F(x)} = \lim_{x \to x_0} \frac{f'(\xi)}{F'(\xi)}$$

$$x \to x_0, \text{时} \xi \to x_0 \qquad \text{符号 } \xi \text{ 换成 x}$$

$$\lim_{x \to x_0} \frac{f(x)}{F(x)} = \lim_{\xi \to x_0} \frac{f'(\xi)}{F'(\xi)} = \lim_{x \to x_0} \frac{f'(x)}{F'(x)}$$

12.5.1

$$\frac{R_n(x)}{(x - x_0)^{n+1}} = \frac{R_n(x) - R_n(x_0)}{(x - x_0)^{n+1} - (x_0 - x_0)^{n+1}} = \frac{R_n'(\xi_1)}{(n + 1)(\xi_1 - x_0)^n}$$

$$\frac{1}{n + 1} \cdot \frac{R_n'(\xi_1)}{(\xi_2 - x_0)^n} = \frac{1}{n + 1} \cdot \frac{R_n'(\xi_1) - R_n'(x_0)}{(\xi_1 - x_0)^n - (x_0 - x_0)^n} = \frac{1}{n + 1} \cdot \frac{R_n''(\xi_2)}{(n)(\xi_2 - x_0)^{n-1}}$$

$$\vdots$$

$$\frac{R_n^{(n)}(\xi_n)}{(n + 1)!(\xi_n - x_0)} = \frac{R_n^{(n)}(\xi_n) - R_n^{(n)}(x_0)}{(n + 1)!(\xi_n - x_0) - 0} = \frac{R_n^{(n+1)}(\xi)}{(n + 1)!}$$

$$\frac{R_n^{(n+1)}(\xi)}{(n + 1)!} = \frac{f^{(n+1)}(\xi) - P_n^{(n+1)}(\xi)}{(n + 1)!} = \frac{f^{(n+1)}(\xi)}{(n + 1)!}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}$$

$$\xi_1 \in (x, x_0), \xi_2 \in (\xi_1, x_0), \xi_n \in (\xi_{n-1}, x_0), \xi \in (\xi_n, x_0)$$

12.5.2

$$
\begin{aligned}
\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} &= \lim_{x \to x_0} \frac{R'_n(x)}{n(x - x_0)^{n-1}} \\
&= \lim_{x \to x_0} \frac{R''_n(x)}{n(n-1)(x - x_0)^{n-2}} \\
&= \lim_{x \to x_0} \frac{R_n^{(n)}(x)}{n!} \\
&= \frac{1}{n} \cdot 0 \\
&= 0
\end{aligned}
$$

16.6　第 13章

13.2.1

$$
\begin{aligned}
\left[\int f(x) \, dx \pm \int g(x) \, dx\right]' &= \left[\int f(x) \, dx\right]' \pm \left[\int g(x) \, dx\right]' \\
&= f(x) \pm g(x)
\end{aligned}
$$

13.2.2

$$
\begin{aligned}
\left[k \int f(x) \, dx\right]' &= k \left[\int f(x) \, dx\right]' \\
&= k f(x)
\end{aligned}
$$

13.2.3

$$
\begin{aligned}
\{F\left[\varphi(x)\right]\}' &= F'\left[\varphi(x)\right] \varphi'(x) \\
&= f\left[\varphi(x)\right] \varphi'(x)
\end{aligned}
$$

13.2.4

$$
\begin{aligned}
x &= \varphi(t) \\
dx &= d\varphi(t) \\
dx &= \varphi'(t) dt \\
\int f(x) \, dx &= \int f\left[\varphi(t)\right] \varphi'(t) \, dt
\end{aligned}
$$

13.2.5

$$
\begin{aligned}
\int f(x) \, dx &= \int f(x) \cdot (x + C)' \, dx \\
&= \int f(x) \, d(x + C)
\end{aligned}
$$

13.4.1

$$\int k \ dx = \int (kx)' \ dx = kx + C$$

13.4.2

$$\int x^a \ dx = \int \left(\frac{1}{a+1} x^{a+1} \right)' \ dx = \frac{x^{a+1}}{a+1} + C$$

13.4.3

$$\int a^x \ dx = \int \left(\frac{1}{\ln a} a^x \right)' \ dx = \frac{a^x}{\ln a} + C$$

13.4.4

$$\int e^x \ dx = \int (e^x)' \ dx = e^x + C$$

13.4.5

$$\int \frac{1}{x} \ dx = \begin{cases} (x > 0) & \int (\ln x)' \ dx = \ln x + C = \ln |x| + C \\ (x < 0) & \int [\ln(-x)]' \ dx = \ln(-x) + C = \ln |x| + C \end{cases}$$

13.4.6

$$\int \ln x \ dx = \ln x \cdot x - \int x \ d\ln x$$
$$= x \ln x - \int x \cdot \frac{1}{x} \ dx$$
$$= x \ln x - x + C$$

13.4.7

$$\int \sin x \ dx = \int (-\cos x)' \ dx = -\cos x + C$$

13.4.8

$$\int \cos x \ dx = \int (\sin x)' \ dx = \sin x + C$$

13.4.9

$$\int \sec x \tan x \ dx = \int (\sec x)' \ dx = \sec x$$

13.4.10

$$\int \csc x \cot x \ dx = -\int (\csc x)' \ dx = -\csc x$$

13.4.11

$$\int \sec^2 x \ dx = \int (\tan x)' \ dx = \tan x$$

13.4.12

$$\int \csc^2 x \ dx = -\int (\cot x)' \ dx = -\cot x$$

13.4.15

$$\begin{aligned}
\int \tan x \ dx &= \int \frac{\sin x}{\cos x} \ dx \\
&= -\int \frac{(\cos x)'}{\cos x} \ dx \\
&= -\int \frac{1}{\cos x} \ d(\cos x) \\
&= -\ln|\cos x| + C
\end{aligned}$$

13.4.16

$$\begin{aligned}
\int \csc x \ dx &= \int \frac{1}{\sin x} \ dx \\
&= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \ dx \\
&= \int \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \ d\frac{x}{2} \\
&= \int \frac{1}{\tan\frac{x}{2}} \cdot \sec^2\frac{x}{2} \ d\frac{x}{2} \\
&= \int \frac{1}{\tan\frac{x}{2}} \ d\tan\frac{x}{2} \\
&= \begin{cases} \ln\left|\tan\frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases}
\end{aligned}$$

13.4.17

$$\int \sec x \ dx = \int \frac{1}{\cos x} \ dx$$
$$= \int \frac{\cos x}{\cos^2 x} \ dx$$
$$= \int \frac{1}{\cos^2 x} \ d\sin x$$
$$= \int \frac{1}{1 - \sin^2 x} \ d\sin x$$
$$= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + C$$
$$= \frac{1}{2} \left| \frac{1 + \sin x}{\cos x} \right|^2 + C$$
$$= \ln |\sec x + \tan x| + C$$

13.4.18

$$\int \arccos x \ dx = x \arccos x - \int x \cdot -\frac{1}{\sqrt{1 - x^2}} \ dx$$
$$= x \arccos x - \frac{1}{2} \int \frac{1}{\sqrt{1 - x^2}} \ d(1 - x^2)$$
$$= x \arccos x - \frac{1}{2} \cdot \frac{1}{-\frac{1}{2} + 1} (1 - x^2)^{-\frac{1}{2} + 1} + C$$
$$= x \arccos x - \sqrt{1 - x^2} + C$$

13.4.19

$$\int \arctan x \ dx = x \arctan x - \int x \cdot \frac{1}{1 + x^2} \ dx$$
$$= x \arctan x - \frac{1}{2} \int \frac{1}{1 + x^2} \ d(1 + x^2)$$
$$= x \arctan x - \frac{1}{2} \ln(1 + x^2) + C$$

分式积分公式证明暂时不标号。

$$\int \frac{1}{x^2 + 1} \ dx = \int (\arctan x)' \ dx = \arctan x + C$$

$$\int \frac{1}{x^2 + a^2} \ dx = \frac{1}{a} \int \frac{1}{1 + (\frac{x}{a})^2} \ d(\frac{x}{a}) = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{1}{x^2 - a^2} \, dx = \int \frac{1}{2a} \cdot \frac{(x+a) - (x-a)}{(x-a)(x+a)} \, dx$$
$$= \frac{1}{2a} \cdot \int \frac{1}{x-a} - \frac{1}{x+a} \, dx$$
$$= \frac{1}{2a} \cdot \left(\int \frac{1}{x-a} \, dx - \int \frac{1}{x+a} \, dx \right)$$
$$= \frac{1}{2a} \left(\ln|x-a| - \ln|x+a| \right) + C$$
$$= \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{1}{a^2 - x^2} \, dx = \int -\frac{1}{2a} \frac{(x-a) - (x+a)}{(a+x)(a-x)} \, dx$$
$$= \int \frac{1}{2a} \cdot \frac{(x-a) - (x+a)}{(a+x)(x-a)} \, dx$$
$$= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x} \, dx - \int \frac{1}{x-a} \, dx \right)$$
$$= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x} \, dx - \int \frac{1}{a-x} \, d(-x) \right)$$
$$= \frac{1}{2a} \left(\ln|a+x| - \ln|a-x| \right) + C$$
$$= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \begin{cases} \int (\arcsin x)' \, dx = \arcsin x + C \\ -\int (\arccos x)' \, dx = -\arccos x + C \end{cases}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \int \frac{1}{\sqrt{1 - (\frac{x}{a})^2}} \, d(\frac{x}{a}) = \begin{cases} \arcsin(\frac{x}{a}) + C \\ -\arccos(\frac{x}{a}) + C \end{cases}$$

$$x = a\tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{\sqrt{x^2+a^2}}{a}, \tan t = \frac{x}{a}, dx = a\sec^2 t \, dt$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} = \int \frac{1}{a} \frac{1}{\sqrt{\tan^2 t + 1}} a \sec^2 t \ dt$$
$$= \int \frac{1}{\sec t} \sec^2 t \ dt$$
$$= \int \sec t \ dt$$
$$= \ln |\sec t + tant| + C$$
$$= \ln \left| \frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a} \right| + C$$
$$= \ln(x + \sqrt{x^2 + a^2}) + C_1 \qquad C_1 = C - \ln a$$

$x = a \sec t, a > 0, \sec t = \frac{x}{a}, \tan t = \frac{\sqrt{x^2 - a^2}}{a}, dx = a \sec t \tan t \ dt$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \ dx = \int \frac{1}{a} \frac{1}{\sqrt{\sec^2 t - 1}} a \sec t \tan t \ dt$$
$$= \int \frac{1}{\tan t} \sec t \tan t \ dt$$
$$= \int \sec t \ dt$$
$$= \ln |\sec t + tant| + C$$
$$= \ln \left| \frac{\sqrt{x^2 - a^2}}{a} + \frac{x}{a} \right| + C$$
$$= \ln(x + \sqrt{x^2 - a^2}) + C_1 \qquad C_1 = C - \ln a$$

$$x < -a, x = -t, dx = -dt$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \ dx = - \int \frac{1}{\sqrt{t^2 - a^2}} \ dt$$
$$= - \ln \left| t + \sqrt{t^2 - a^2} \right| + C$$
$$= - \ln \left| -x + \sqrt{x^2 - a^2} \right| + C$$
$$= - \ln \left| \frac{(-x + \sqrt{x^2 - a^2})(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} \right| + C$$
$$= - \ln \left| \frac{-a^2}{x + \sqrt{x^2 - a^2}} \right| + C$$
$$= - \ln \left| -a^2 \right| + \ln \left| x + \sqrt{x^2 - a^2} \right| + C$$
$$= \ln \left| x + \sqrt{x^2 - a^2} \right| + C_1 \qquad C_1 = C - \ln \left| -a^2 \right|$$