争取未来开挖掘机

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，姜萍吧，姜圣的追随者，

目录

1 三角函数

1.1 三角恒等式

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{1.1.1}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{1.1.2}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{1.1.3}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{1.1.4}$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.5}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.6}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.7}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.8}$$

1.1.2 积化和差

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \tag{1.1.9}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \tag{1.1.10}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{1.1.11}$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \tag{1.1.12}$$

1.1.3 倍角公式

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x \\ \cos(2x) &= \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \\ \tan(2x) &= \frac{2\tan x}{1-\tan^2 x}\end{aligned}$$

1.1.4 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.13}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.14}$$

1.1.5 三角函数其他等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.15}$$

$$1 + \tan^2 x = \sec^2 \tag{1.1.16}$$

$$1 + \cot^2 x = \csc^2 \tag{1.1.17}$$

1.2 双曲函数

1.2.1 定义

$$\begin{aligned}\sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}\end{aligned}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1}) \tag{1.2.1}$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}) \tag{1.2.2}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x}) \tag{1.2.3}$$

1.2.3 恒等式

$$\sinh(2x) = 2\sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2\sinh^2 \frac{x}{2} \tag{1.2.7}$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 + x_2 + \cdots + x_n} \tag{2.0.1}$$

$$|x + y| \leqslant |x| + |y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1 + x)^n \leqslant 1 + nx \tag{2.0.4}$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1) \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

3.2 运算

4 区间与映射

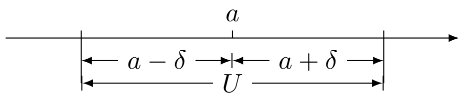
4.1 区间定义

$$
\text{区间定义}\begin{cases}(a,b)=\{x|a<x<b\}\\ [a,b]=\{x|a\leqslant x\leqslant b\}\\ (a,b]=\{x|a<x\leqslant b\}\\ (a,+\infty)=\{x|a<x\}\end{cases}
$$

4.2 领域定义

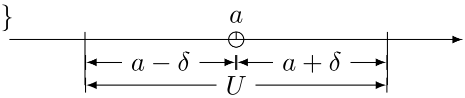
点 a 的领域

$$
U(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\}\\ \{x|\ |x-a|<\delta\}\end{cases}
$$



点 a 的去心领域

$$
\mathring{U}(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\wedge x\neq 0\}\\ \{x|0<|x-a|<\delta\}\end{cases}
$$



点 a 的左领域 $\quad (a-\delta,a)$

点 a 的右领域 $\quad (a,a+\delta)$

4.3 映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x\in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$
\text{记作}\qquad f:X\to Y
$$

$$
f(x)=y\qquad\begin{cases}\text{定义域 }(D_f)=X & x\text{-原像}\\ \text{值域 }(R_f)=\{f(x)|x\in X\} & y\text{-像}\end{cases}
$$

$$
\text{映射类型}\quad
\begin{cases}
\text{满射:} & R_f = Y \\
\text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\
\text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射:} \begin{cases} f^(x) & = y \\ f^{-1}(y) & = x \end{cases} \\
\text{复合映射:} & g \circ f \Leftrightarrow g\left[f\left(x\right)\right] \begin{cases} f : X \rightarrow Y_1 \\ g : Y_2 \rightarrow Z \\ g \circ f : X \rightarrow Z \qquad (Y_1 \subset Y_2) \end{cases}
\end{cases}
$$

5 函数

5.1 函数相关的定义

5.1.1 函数

设数集 $D \in R$ 的映射

$$f : D \to R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x)\ \{x \in D\}$$

5.1.2 驻点

$$Def:\ f'(x) = 0$$

5.1.3 拐点

$$Def : f''(x) = 0\ (\text{左右两侧凹凸性改变})$$

5.1.4 极值点

$$Def : \text{函数} f(x)\ x \in \mathring{U}(x_0), \text{包括可导和不可导的点} \begin{cases} \text{极大值：} f(x) < f(x_0) \\ \text{极小值：} f(x) > f(x_0) \end{cases}$$

$$x \in \mathring{U}(x_0) \begin{cases} f(x)\text{可导，} f'(x_0) = 0 \begin{cases} x_0\text{极大值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) > 0 \\ x \in (x_0, x_0 + \delta), f'(x) < 0 \end{cases} \\ x_0\text{极小值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) < 0 \\ x \in (x_0, x_0 + \delta), f'(x) > 0 \end{cases} \\ x_0\text{无极值，} x \in \mathring{U}(x_0) \begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases} \end{cases} \\ f(x)\text{二阶可导}, f'(x_0) = 0, f''(x_0) \neq 0 \begin{cases} f''(x) < 0 \Rightarrow x_0\text{极大值} \\ f''(x) > 0 \Rightarrow x_0\text{极小值} \end{cases} \end{cases}$$

5.1.5 最值

$$\text{最大值或最小值}\begin{cases}\text{驻点}\\ \text{端点}\end{cases}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f: D \to R\{D \subset R\}\begin{cases}\text{有界}\begin{cases}\text{有上界}\left\{\exists k_1,\ \text{使}f(x) \leqslant k_1,\ \forall x \in D\right.\\ \text{有下界}\left\{\exists k_1,\ \text{使}f(x) \geqslant k_1,\ \forall x \in D\right.\end{cases}\\ \text{无界}\begin{cases}\text{无上界}\left\{\forall K_1,\ \exists x \in D\ \text{使},\ f(x) \geqslant k_1\right.\\ \text{无下界}\left\{\forall K_1,\ \exists x \in D\ \text{使},\ f(x) \leqslant k_1\right.\end{cases}\end{cases}$$

5.2.2 函数的单调性与凹凸性

$$\text{若}\{x_1, x_2 \in D\}\ x_1 < x_2 \Rightarrow \begin{cases} f(x_1) < f(x_2)\text{称}f(x)\text{在 D 上单调增加}\\ f(x_1) > f(x_2)\text{称}f(x)\text{在 D 上单调减少}\\ f(x_1) \leqslant f(x_2)\text{称}f(x)\text{在 D 上单调非降}\\ f(x_1) \geqslant f(x_2)\text{称}f(x)\text{在 D 上单调非增}\end{cases}$$

$$\text{设 } f(x) \text{ 在区间 I 上连续,}\forall x_1, x_2 \begin{cases} f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}, \text{称 } f(x) \text{ 在 I 上是向上凹}\\ f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}, \text{称 } f(x) \text{ 在 I 上是向上凸}\end{cases}$$

$$f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内可导}f'(x) \geqslant 0, \text{有限个点为 0, 单调增} \tag{5.2.1}$$

$$f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内可导}f'(x) \leqslant 0, \text{有限个点为 0, 单调减} \tag{5.2.2}$$

$$f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内二阶可导}f''(x) \geqslant 0, \text{有限个点为 0, 向上凹} \tag{5.2.3}$$

$$f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内二阶可导}f''(x) \leqslant 0, \text{有限个点为 0, 向下凸} \tag{5.2.4}$$

5.2.3 函数的奇偶性

$$\forall x \in D \qquad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

$$\text{奇函数} \times \text{奇函数} = \text{偶函数} \tag{5.2.5}$$

$$\text{奇函数} \times \text{偶函数} = \text{奇函数} \tag{5.2.6}$$

$$\text{偶函数} \times \text{偶函数} = \text{偶函数} \tag{5.2.7}$$

5.2.4 周期性

$$Def: \quad f(x+L) = f(x)\{L > 0\text{常数}, \forall x \in D\} \Rightarrow f(x) \text{ 为 } L \text{ 的周期函数}$$

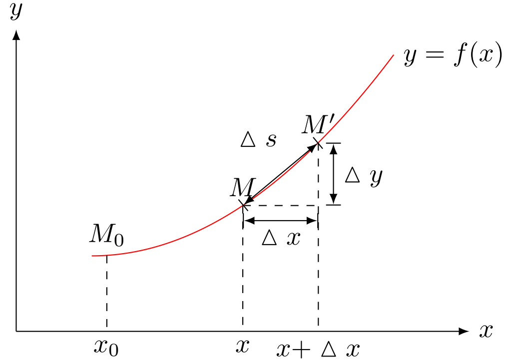
5.3 弧

5.3.1 有向曲线弧

$$\text{基准点}M_0(x_0, f(x_0)), \text{以 } x \text{ 增大的方向为正向}, \overset{\frown}{M_0M} = S$$

$$S = S(x), S\text{是关于 x 的单调增加函数}$$

$$\overset{\frown}{M_0M} \begin{cases} \text{绝对值为的长度} \\ \text{与曲线正向一致，取正值} \\ \text{与曲线反向一致，取负值} \end{cases}$$



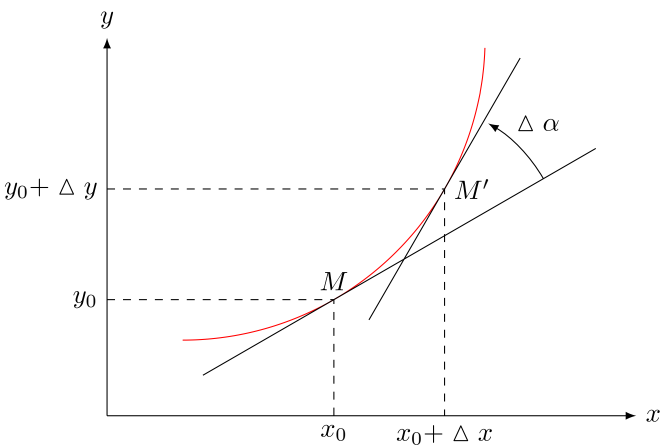
5.3.2 弧微分

$$ds = \sqrt{1+(y')^2}dx \Leftrightarrow ds = \sqrt{(dx)^2+(dy)^2} \Leftrightarrow ds = \sqrt{(dx)^2+(f'dx)^2} \tag{5.3.1}$$

参数方程形式

$$\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \begin{cases} dx = \phi'(t)dt \\ dy = \psi'(t)dt \end{cases} \Rightarrow ds = \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}dt$$

5.3.3 曲率



$M(x_0, y_0), M'(x_0 + \triangle\ x, y_0 + \triangle\ y), \triangle\ s = \widehat{MM'}$

$$\text{曲线上弧的}\begin{cases} \text{平均曲率:} & \overline{k} = \left|\frac{\Delta\alpha}{\Delta s}\right| \\ \text{点曲率:} & k = \lim\limits_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right| = \left|\frac{d\alpha}{ds}\right| \end{cases}$$

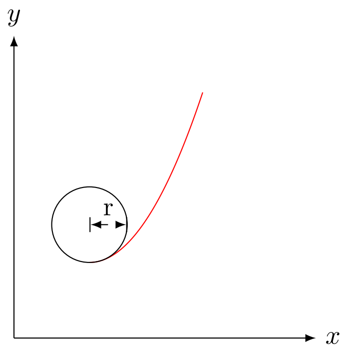
$$\left|\frac{d\alpha}{ds}\right| = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \tag{5.3.2}$$

$$\left|\frac{d\alpha}{ds}\right| \text{的参数方程形式} \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \left|\frac{d\alpha}{ds}\right| = \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{|\psi'(t)|^2 + [\phi'(t)]^2\right\}^{\frac{3}{2}}} \tag{5.3.3}$$

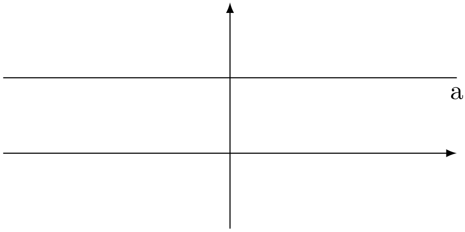
5.3.4 曲率圆，曲率半径

$$圆的曲率k = \left| \frac{\triangle\ \alpha}{\triangle\ s} \right| = \left| \frac{\triangle\ \alpha}{r \triangle\ \alpha} \right| = \frac{1}{r}$$

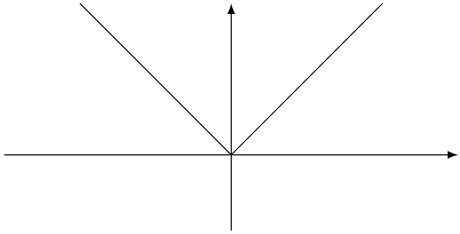
$$曲率半径r = \frac{1}{k}$$



6 图像

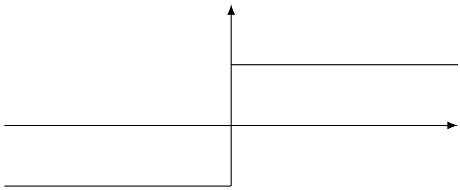


常函数 $f(x)=a\{a\in R\}$

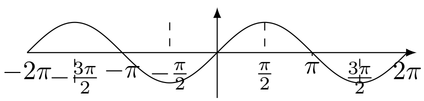


$f(x)=|x|$

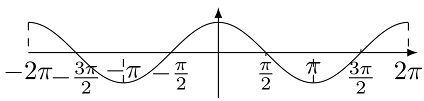
$$f(x)=sgn\ x=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$$



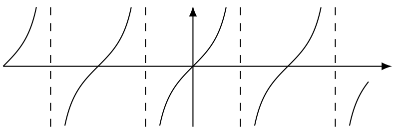
$|x|=x\cdot sgnx$



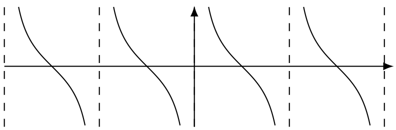
$\sin x$



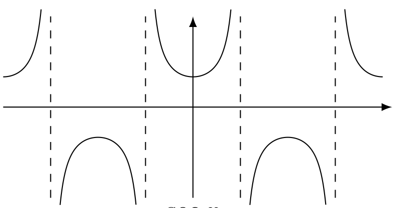
$\cos x$



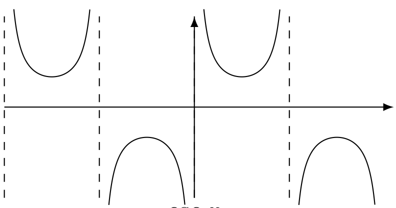
$\tan x$



$\cot x$



$\sec x$



$\csc x$

7 并集，交集

7.1 定义

$$(\vee \text{ 或},\wedge \text{ 与})$$

$$A \cup B = \{x \in A \vee x \in B\}$$

$$A \cap B = \{x \in A \wedge x \in B\}$$

7.2 运算

$$\text{运算满足}\begin{cases}\text{交换律}\begin{cases}A \cup B & = B \cup A \\ A \cap B & = B \cap A\end{cases} \\ \text{结合律}\begin{cases}(A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C)\end{cases} \\ \text{分配律}\begin{cases}(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)\end{cases} \\ \text{对偶律}\begin{cases}(A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C\end{cases}\end{cases}$$

$$A \cup A = A = A \cap A$$

$$A = B \Leftrightarrow A \subset B \wedge A \supset B$$

$$A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$$

7.3 性质

性质 1.

$$A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{7.3.1}$$

性质 2.

$$A \cup B = B \Leftrightarrow A \subset B \tag{7.3.2}$$

性质 3.

$$A \cap B = A \Leftrightarrow A \subset B \tag{7.3.3}$$

性质 4. $(n \in N\ )$

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{7.3.4}$$

性质 5. $(n \in N\ )$

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{7.3.5}$$

7.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

7.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{C} = \bigcap_{\alpha}(E_{\alpha}^{C})$$

$$\left(\bigcap_{\alpha} E_{\alpha}\right)^{C} = \bigcup_{\alpha}(E_{\alpha}^{C})$$

8 群，环，域

8.1 群

8.1.1 M1

8.1.2 M2

8.1.3 M3

8.1.4 M4

8.1.5 sdas

8.2 环

8.3 域

9 极限

9.1 数列极限

9.1.1 数列的定义

$$Def: \qquad \{x_n\}, x_n = f(n), n \in N^+ \to R$$

9.1.2 数列极限的定义

$Def:\ \{x_n\},\ n \in N^+, \exists a,\ \forall \varepsilon > 0, \exists N,\ n > N \Rightarrow |x_n - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_n = a$

极限存在，为收敛，不存在为发散

9.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{9.1.1}$$

9.1.4 有界数列

$$\begin{gathered}\text{若}\exists M > 0, \{M \in \text{正数}\} \\ \text{使得}\forall n, \quad |x_n| \leqslant M \\ \text{则称数列 } \{x_n\} \text{ 为有界数列}\end{gathered}$$

9.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{9.1.2}$$

$$\text{单调有界数列必收敛} \tag{9.1.3}$$

9.1.6 收敛数列的保号性

$$\lim_{n\to\infty} x_n = a \text{ 存在，且 } a > 0, \text{ 则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时 } \Leftrightarrow x_n > 0 \tag{9.1.4}$$

$$\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} b_n = b, a < b,\ \exists N, n > N, a_n < b_n \tag{9.1.5}$$

9.1.7 收敛数列和子数列

$\{x_n\},\ \lim\limits_{n\to\infty} x_n = a,\ \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$

证明 $K = N\ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

9.2 函数极限

9.2.1 极限的定义

$$Def: \forall\varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ \text{当} x < -X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ \text{当}\ |x| > X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists\delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当} x_0 - \delta < x < x_0, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ \text{当} 0 < |x - x_0| < \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x\to x_0} f(x)\text{存在} \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \tag{9.2.1}$$

图

9.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$\lim\limits_{x\to x_0} f(x) = A, \exists M > 0, \delta > 0 \text{使} 0 < |x - x_0| < \delta, |f(x)| \leqslant M$

3 保号性

$\lim\limits_{x\to x_0} f(x) = A,\ A > 0, \exists\delta > 0,$ 当$, 0 < |x - x_0| < \delta \Rightarrow f(x) > 0$

$f(x) > 0, \exists\delta > 0,$ 当$, 0 < |x - x_0| < \delta \Rightarrow \lim\limits_{x\to x_0} f(x) = A,\ A > 0$

4 保序性

$f(x) \geqslant g(x),\ \lim f(x) = a,\ \lim g(x) = b,$ 则$a \geqslant b$

5 函数极限与数列极限的关系

如果 $\lim\limits_{x\to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列，则满足 $x_n \neq x_0$
则 $\lim\limits_{n\to\infty} f(x_n) = 0 = \lim\limits_{x\to x_0} f(x),\ x_n \to x_0$

9.3 无穷小与无穷大

9.3.1 无穷小定义

$Def:$ 如果 $\lim\limits_{x\to x_0} f(x) = 0$则称$f(x)$为$x \to x_0$时的无穷小

$$Def: \forall\varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X & \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = 0 \\ \text{当} x < -X & \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = 0 \\ \text{当}\ |x| > X & \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = 0 \end{cases} \\ \exists\delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = 0 \\ \text{当} x_0 - \delta < x < x_0, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = 0 \\ \text{当} 0 < |x - x_0| < \delta, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = 0 \end{cases} \end{cases}$$

9.3.2 函数极限与无穷小的关系

$$\text{在自变量的同一变化中。}\alpha\ \text{为无穷小。}\lim f(x) = A \Leftrightarrow f(x) = A + \alpha \tag{9.3.1}$$

9.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$\text{如果}\ f(x)\ \text{为无穷大，则}\ \frac{1}{f(x)}\ \text{为无穷小。} \tag{9.3.2}$$

$$\text{如果}\ f(x)\ \text{为无穷小，切}\ f(x) \neq 0\text{，则}\ \frac{1}{f(x)}\ \text{为无穷小。} \tag{9.3.3}$$

9.3.4 无穷大定义

$$
Def:\forall M>0\begin{cases}
\exists X>0\begin{cases}
\text{当}x>X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=\infty
\end{cases}\\
\text{当}x<-X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=+\infty\\
f(x)<M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=\infty
\end{cases}\\
\text{当}\ |x|>X\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=\infty
\end{cases}
\end{cases}\\
\exists\delta>0\begin{cases}
\text{当}x_0-\delta<x<x_0\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=\infty
\end{cases}\\
\text{当}x_0<x<x_0+\delta\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=\infty
\end{cases}\\
\text{当}0<|x-x_0|<\delta\begin{cases}
f(x)>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=+\infty\\
f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=\infty
\end{cases}
\end{cases}
\end{cases}
$$

$$\lim_{x\to x_0}f(x)=\infty\text{，直线}x=x_0\text{是}y=f(x)\text{垂直渐进线}$$

9.4 运算

9.4.1 有限个无穷小的和仍为无穷小

$$\text{设 }\gamma=\alpha+\beta$$
$$\alpha\text{ 和 }\beta\text{ 同为 }x\to x_0\text{ 时的无穷小}$$
$$\forall\varepsilon>0\text{，}\exists\delta_1>0\text{，当 }0<|x-x_0|<\delta_1\text{ 时，有 }|\alpha|<\tfrac{\varepsilon}{2}$$

$$\forall\varepsilon > 0,\ \exists\delta_2 > 0,\ \text{当}\ 0 < |x - x_0| < \delta_2\ \text{时，有}\ |\beta| < \frac{\varepsilon}{2}$$
$$\delta = min\{\delta_1, \delta_2\},\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2\ \text{同时满足}$$
$$\text{即}\ |\alpha| < \frac{\varepsilon}{2}, |\beta| < \frac{\varepsilon}{2}\ \text{同时成立}$$
$$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

9.4.2 有界函数与无穷小的乘积仍为无穷小

$$\text{设}\ \alpha\ \text{为}\ x \to x_0\ \text{时的一个无穷小}$$
$$g(x)\ \text{为}\ x_0\ \text{的一个去心邻域}\ \mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$f(x) = g(x)\alpha$$
$$\text{证}\ f(x)\ \text{为}\ x \to x_0\ \text{时的无穷小}$$
$$\text{因为}\ g(x)\text{在}\mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$\exists M > 0, \text{当} 0 < |x - x_0| < \delta_1\ \text{时}\ |g(x)| < M$$
$$\text{因为}\ \alpha\ \text{是}\ x \to x_0\ \text{的无穷小}$$
$$\exists\delta_2 > 0\text{当}0 < |x - x_0| < \delta_2\ \text{时}\ |\alpha| < \frac{\varepsilon}{M} < \varepsilon$$
$$\text{取}\ \delta = min\{delta_, \delta_2\}\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$|g(x)| \geqslant M, |\alpha| < \frac{\varepsilon}{M}\ \text{同时成立}$$
$$|g(x)\alpha| = |g(x)|\ \ |\alpha| < M\frac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

9.4.3 极限的四则运算

$\lim f(x) = A,\ \lim g(x) = B$

$$\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) \tag{9.4.1}$$
$$\lim\left(f(x)g(x)\right) = \lim f(x) \lim g(x) \tag{9.4.2}$$
$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)} \tag{9.4.3}$$
$$\lim\left[Cf(x)\right] = C\lim f(x) \tag{9.4.4}$$
$$\lim\left[f(x)\right]^n = \left[\lim f(x)\right]^n \tag{9.4.5}$$

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{a_0x^n+a_1x^{n-1}+\cdots+a_n}=\begin{cases}\frac{a}{b} & m=n\\ \infty & m>n\\ 0 & m<n\end{cases} \tag{9.4.6}$$

$$\begin{gathered}\lim_{x\to x_0}g(x)=u_0,\ \lim_{u\to u_0}f(u)=A\\ \exists\delta_0>0,\ x\in\mathring{U}(x_0,\delta_0),\ g(x)\neq u_0\\ \lim_{x\to x_0}f[g(x)]=\lim_{u\to u_0}f(u)=A\end{gathered} \tag{9.4.7}$$

9.4.4 夹逼定理 (三明治定理)

$$\begin{gathered}x_n\leqslant z_n\leqslant y_n \qquad \forall n>N_0\\ \text{若}\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=a\text{则}\lim_{n\to\infty}z_n=a\end{gathered} \tag{9.4.8}$$

9.4.5 重要极限

$$x\to x_0$$

$$\lim_{x\to x_0}\sin x=\sin x_0 \tag{9.4.9}$$

$$\lim_{x\to x_0}\cos x=\cos x_0 \tag{9.4.10}$$

$$x\to 0$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{9.4.11}$$

$$\lim_{x\to 0}\cos x=1 \tag{9.4.12}$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1 \tag{9.4.13}$$

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=1 \tag{9.4.14}$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=1 \tag{9.4.15}$$

$$\lim_{x\to 0}\frac{\arctan x}{x}=1 \tag{9.4.16}$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \tag{9.4.17}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \tag{9.4.18}$$

$$\lim_{x\to 0}\frac{(1+x)^n-1}{nx}=1 \tag{9.4.19}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \tag{9.4.20}$$

$$x \to \infty$$

$$\{x_n\} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{9.4.21}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{9.4.22}$$

9.4.6 无穷小比较

$\frac{0}{0}$ 型未定式
$Def:\ \alpha,\beta$ 是同一极限过程的无穷小。
(1) 如果 $\lim\frac{\beta}{\alpha}=0$ 则称 β 是 α 的高阶无穷小，记作 $\beta=\circ(\alpha)$
(2) 如果 $\lim\frac{\beta}{\alpha}=\infty$ 则称 β 是 α 的底阶无穷小。
(3) 如果 $\lim\frac{\beta}{\alpha}=C$ 则称 β 是 α 的同阶无穷小。
(4) 如果 $\lim\frac{\beta}{\alpha^k}=C, k>0$ 则称 β 是 α 的 k 阶无穷小。
(5) 如果 $\lim\frac{\beta}{\alpha}=1$ 则称 β 是 α 的等价阶无穷小。

9.4.7 等价无穷小代换，因子代换

β与α是等价无穷小 $\Leftrightarrow \beta=\alpha+\circ(\alpha)$
设$\alpha\sim\alpha'$, $\beta\sim\beta'$, 且$\lim\frac{\beta'}{\alpha'}$存在, 则$\lim\frac{\beta}{\alpha}=\lim\frac{\beta'}{\alpha'}$
$\lim\alpha f(x)=\lim\alpha' f(x)$
$\lim\frac{f(x)}{\alpha}=\lim\frac{f(x)}{\alpha'}$

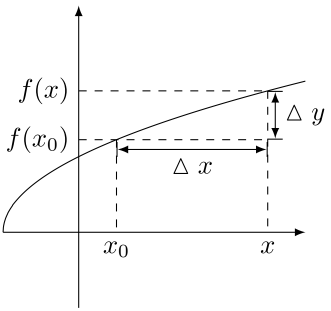
10 连续与间断点

10.1 定义

10.1.1 点连续

$$Def1:\text{设}f(x)\text{在}x_0\text{的某邻域内有定义，如果}\lim_{x\to x_0}=f(x_0)$$

$$\text{则称}f(x)\text{在}x_0\text{处连续}$$



$$\begin{cases}\triangle\ x=x-x_0\\ \triangle\ y=\begin{cases}f(x)-f(x_0)\\ f(x_0+\triangle\ x)-f(x_0)\end{cases}\end{cases}$$

$$Def2:\text{如果}\lim_{\triangle x\to 0}\triangle\ y=\lim_{\triangle x\to 0}[f(x_0+\triangle\ x)-f(x_0)]=0$$

$$\text{则称}f(x)\text{在}x_0\text{处连续}$$

10.1.2 区间连续

$$\forall x_0\in[a,b]\begin{cases}\lim\limits_{x\to x_0}f(x)=f(x_0)\quad x_0\in(a,b)\Leftrightarrow f(x_0)=\begin{cases}\lim\limits_{x\to x_0^-}f(x)=f(x_0^-)\\ \lim\limits_{x\to x_0^+}f(x)=f(x_0^+)\end{cases}\\ \lim\limits_{x\to x_0^+}f(x)=f(x_0^+)\quad x_0=a\ (\text{右连续})\\ \lim\limits_{x\to x_0^-}f(x)=f(x_0^-)\quad x_0=b\ (\text{左连续})\end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M>0,x\in[a,b]$ 时，$|f(x)|\geqslant M$

最大值:$\exists x_0\in[a,b]$ 时，$\forall x\in[a,b],f(x)\leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0\in[a,b]$ 时，$\forall x\in[a,b],f(x)\geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

零点定理

2, 设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)\cdot f(b)<0$
则至少存在一点 $\xi\in(a,b)$ 使 $f(\xi)=0$

介质定理

设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)=A,f(b)=B$
$\forall C\in(A,B)$, 至少有一点 $\xi,f(\xi)=C$

10.1.3 间断点

1,$f(x)$ 无定义
2,$\lim\limits_{x\to x_0} f(x)$ 不存在
3. $\lim\limits_{x\to x_0} f(x)$ 存在, 但 $\lim\limits_{x\to x_0} f(x)\neq f(x_0)$
第一类间断点：$f(x_0^+)=\lim\limits_{x\to x_0^+} f(x)$ 与 $f(x_0^-)=\lim\limits_{x\to x_0^-} f(x)$
第二类间断点：不是第一类的。

10.2 连续函数的运算

函数 $f(x),g(x)$ 在 $x=x_0$ 连续。

$$\lim_{x\to x_0}[f(x)\pm g(x)]=\lim_{x\to x_0}f(x)\pm\lim_{x\to x_0}g(x)=f(x_0)\pm g(x_0)$$
$$\lim_{x\to x_0}[f(x)\cdot g(x)]=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x)=f(x_0)\cdot g(x_0)$$
$$\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}g(x)}=\frac{f(x_0)}{g(x_0)}\qquad(g(x_0)\neq 0)$$

反函数的连续性
若 $y=f(x)$ 在区间 I_x 上单调增加，且连续。
则 $y=f^-1(x)$ 在 $I_y=\{y|y=f(x),x\in I_x\}$ 上也为单调增加，连续

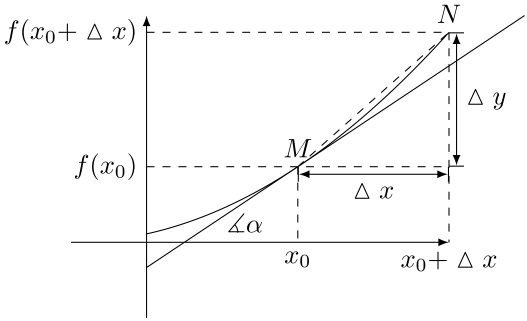
$$
\text{复合函数,}\begin{cases}
\text{内外都连续}\begin{cases}
\lim\limits_{x\to x_0} g(x) = g(x_0) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f\left[g(x_0)\right] = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
\text{外连续}\begin{cases}
x\to x_0 \begin{cases}
\lim\limits_{x\to x_0} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
x\to\infty \begin{cases}
\lim\limits_{x\to\infty} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to\infty} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to\infty} g(x))
\end{cases}
\end{cases}
\end{cases}
$$

11 导数

11.1 定义

导数的概念从物理发展出来的。

$$v\left(t_0\right)=\lim_{\triangle t\to 0}\frac{s\left(t_0+\triangle t\right)-s\left(t_0\right)}{\triangle t}$$



$$NM\text{斜率}=\tan\beta=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}=\frac{\triangle y}{\triangle x}$$

$$\text{斜率}k=\tan\alpha=\lim_{\triangle x\to 0}\tan\beta=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}$$

11.1.1 导数定义

$y=f(x)$ 在 x_0 的某邻域内有定义

给自变量的增量 $\triangle x$, $(x_0+\triangle x)$ 仍在定义域内

函数得到了相应增量 $\triangle y, \triangle y=f\left(x_0+\triangle x\right)$

如果 $\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$ 存在, 称 $y=f(x)$ 在 $x=x_0$ 处可导

(极限值为$y=f(x)$在$x=x_0$处导数)

$$\text{记 } y'|_{x=x_0}=f'(x_0)=\lim_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}=\lim_{\triangle x\to 0}\tfrac{\triangle y}{\triangle x}$$

$$\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\Leftrightarrow\lim_{\triangle x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

11.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。

$f'(x)=\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$

称 $f'(x)$ 为 $y=f(x)$ 在区间 I 上的导函数

11.1.3 闭区间可导定义

$$f(x)\text{ 在 }[a,b]\text{ 可导}\Leftrightarrow\begin{cases} f'(x_0) & x_0\in(a,b)\Leftrightarrow\begin{cases}\text{左导数}f'_-(x_0)=\lim\limits_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\\ \text{右导数}f'_+(x_0)=\lim\limits_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\end{cases}\\ f'_+(a) & x=a\\ f'_-(a) & x=b\end{cases}$$

11.1.4 导数与连续

$$f'(x)\text{存在}\Rightarrow f(x)\text{在}x=x_0\text{处连续}\tag{11.1.1}$$

11.2 幂数，指数，对数

$$(C)'=0\tag{11.2.1}$$

$$(x^a)'=ax^{a-1}\tag{11.2.2}$$

$$(a^x)'=a^x\ln a\tag{11.2.3}$$

$$(e^x)'=e^x\tag{11.2.4}$$

$$(\log_a^x)'=\frac{1}{x\ln a}\tag{11.2.5}$$

$$(\ln x)'=\frac{1}{x}\tag{11.2.6}$$

11.3 三角函数

$$(\sin x)' = \cos x \tag{11.3.1}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \tag{11.3.2}$$

$$(\csc x)' = -\csc x \cot x \tag{11.3.3}$$

$$(\cos x)' = -\sin x \tag{11.3.4}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \tag{11.3.5}$$

$$(\sec x)' = \sec x \tan x \tag{11.3.6}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \tag{11.3.7}$$

$$(\tan x)' = \sec^2 x \tag{11.3.8}$$

$$(\arctan x)' = \frac{1}{1+x^2} \tag{11.3.9}$$

$$(\cot x)' = -\csc^2 x \tag{11.3.10}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2} \tag{11.3.11}$$

$$(\sinh x)' = \cosh x \tag{11.3.12}$$

$$(\cosh x)' = \sinh x \tag{11.3.13}$$

$$(\tanh x)' = \frac{1}{cosh^2 x} = 1 - \tanh^2 x \tag{11.3.14}$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}} \tag{11.3.15}$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}} \tag{11.3.16}$$

$$(\operatorname{arctanh} x)' = \frac{1}{1-x^2} \tag{11.3.17}$$

11.4 导数运算

$$u = u(x), v = v(x), \text{均在}x\text{点可导}, C\text{为常数}$$

$$(Cu(x))' = Cu'(x) \tag{11.4.1}$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \tag{11.4.2}$$

$$(u(x) \cdot v(x))' = u'(x)v(x) + v'(x)u(x) \tag{11.4.3}$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{\left[v(x)\right]^2} \tag{11.4.4}$$

11.5 反函数求导

如果函数 $y=f(x)$ 在区间 (a,b) 内单调可导，且 $f'(y)\neq 0$

$$\begin{cases}\alpha=\min\{f(a)+0,f(b-0)\}\\ \beta=\max\{f(a)+0,f(b-0)\}\end{cases}$$

则它的反函数 $x=f^{-1}(y)$ 在区间 (α,β) 内也可导

$$\left[f^{-1}(y)\right]'=\frac{1}{f'(x)}\Leftrightarrow\frac{dy}{\mathrm{d}x}=\frac{1}{\frac{dx}{\mathrm{d}y}}\tag{11.5.1}$$

11.6 复合函数求导

$$\text{设函数}\begin{cases}y=f(u)\text{在}U(u_0,\delta_0)\text{处有定义}\\ u=g(x)\text{在}U(x_0,\eta_0)\text{处有定义}\end{cases}$$
$$u_0=g(x_0),\text{且}f'(u)\text{和}g'(x)\text{都存在}$$
$$\text{则复合函数}F(x)=f\left[g(x)\right]\text{在点}x_0\text{可导, 且}$$

$$F'(x_0)=f'\left[g(x_0)\right]g'(x_0)\Leftrightarrow\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\tag{11.6.1}$$

11.7 高阶求导

$$Def:\begin{cases}\text{一阶导数} & y'\Leftrightarrow\frac{dy}{dx}\\ \text{二阶导数} & y''\Leftrightarrow\frac{d^2y}{dx^2}\\ \text{三阶导数} & y'''\Leftrightarrow\frac{d^3y}{dx^3}\\ \text{三阶以上 n 阶导数} & y^{(n)}\Leftrightarrow\frac{d^ny}{dx^n}\end{cases}$$

11.8 高阶求导公式

$$(e^x)^{(n)} = e^x \tag{11.8.1}$$

$$(a^x)^{(n)} = a^x\,(lna)^n \tag{11.8.2}$$

$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \tag{11.8.3}$$

$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{11.8.4}$$

$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{11.8.5}$$

$$(\sin x)^{(n)} = \sin(x+n\frac{\pi}{2}) \tag{11.8.6}$$

$$(\cos x)^{(n)} = \cos(x+n\frac{\pi}{2}) \tag{11.8.7}$$

$$[f(ax+b)]^{(n)} = a^n \cdot f^{(n)}(ax+b) \tag{11.8.8}$$

11.9 高阶求导运算法则

$$(u(x) \pm v(x))^{(n)} = u^{(n)}(x) \pm v^{(n)}(x) \tag{11.9.1}$$

$$\text{莱布紫泥公式} \qquad (uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} \cdot v^(k) \tag{11.9.2}$$

11.10 隐函数求导

$F(x,y)=0, y=f(x)$

$F(x,f(x)) \equiv 0$ 可以同时对两面求导

11.11 参数方程求导

$x=x(t), y=y(t)$

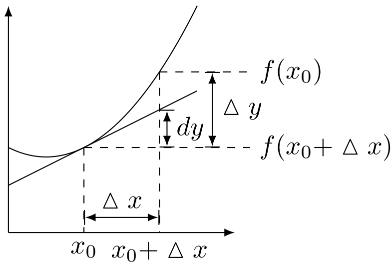
$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{d^2y}{dx} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt}\left(\frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}\right) \cdot \frac{dt}{dx} = \frac{\frac{d^2y}{dt} \cdot \frac{dx}{dt} - \frac{d^2x}{dt} \cdot \frac{dy}{dt}}{(\frac{dx}{dt})^2} \cdot \frac{dt}{dx} = \frac{\frac{d^2y}{dt} \cdot \frac{dx}{dt} - \frac{d^2x}{dt} \cdot \frac{dy}{dt}}{(\frac{dx}{dt})^3}$$

12 微分

12.1 定义

设函数 $f(x)$ 在点 x_0 的一个邻域内有定义。$\triangle\, y = f(x_0 + \triangle\, x) - f(x_0)$
如果 $\triangle\, y$ 可以表示为 $\triangle\, y = A \triangle\, x + \circ(\triangle\, x)$ 其中 A 为与 $\triangle\, x$ 无关的常数
则称 $f(x)$ 在点 x_0 可微，$A \triangle\, x$ 称为 $f(x)$ 在点 x_0 处的微分。
记作:$dy = A \triangle\, x$



$$可微 \Rightarrow 可导 \tag{12.1.1}$$

$$可导 \Rightarrow 可微 \tag{12.1.2}$$

12.2 微分法则

12.2.1 核心根本

积分

$$dy = \quad f'(x) \quad d \quad x$$

求导

12.2.2 四则运算

$$d\left(u \pm v\right) = du \pm dv \tag{12.2.1}$$

$$d(uv) = vdu + udv \tag{12.2.2}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + udv}{v^2} \tag{12.2.3}$$

12.2.3 复合运算

$$\text{可微}\begin{cases} y=f(u) \\ u=g(x) \end{cases} \Rightarrow \begin{cases} dy=f'(u)du \\ du=g'(x)dx \end{cases} \quad \text{则 } y=f(g(x)) \text{ 也可微}$$

$$\text{且 } dy=f'(u)du=f'(u)g(x)dx$$

u 是否为中间变量都成立，微分的不变性。

12.2.4 近似计算公式

$$\triangle x\to 0, dy\approx\triangle y\begin{cases} dy=f'(x_0)\triangle x \\ \triangle y=f(x_0+\triangle x)-f(x_0) \end{cases} \quad \begin{cases} f(x_0+\triangle x)\approx f(x_0)+f'(x_0)\triangle x \\ f(x)\approx f(x_0)+f'(x_0)(x-x_0) \\ x_0=0\begin{cases} f(x)\approx f(0)+f'(0)x \\ \sqrt{n}\approx 1+\frac{1}{n}x \\ \sin x\approx x \\ \tan x\approx x \\ e^x\approx 1+x \\ \ln(1+n)\approx x \end{cases} \end{cases}$$

12.2.5 奇偶函数导数

$$\text{偶函数导数为奇函数 } f(x)=f(-x)\Leftrightarrow f'(x)=-f'(-x)$$
$$\text{奇函数导数为偶函数 } f(x)=-f(-x)\Leftrightarrow f'(x)=f'(-x)$$

12.2.6 区间恒为 0

$$\text{若}f'(x)\text{在区间恒为零，则}f(x)\text{在区间}I\text{上为一常数}$$

设x_1,x_2为区间I内任意两点$x_1<x_2$

$f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)\equiv 0$

$f(x_2)\equiv f(x_1)=C$

12.3 中值定理

12.3.1 费马引理

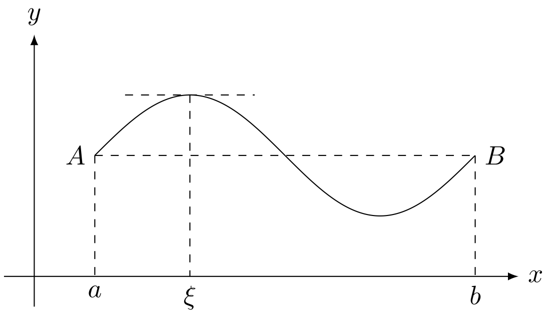
$$f(x) \qquad \forall x\in\mathring{U}(x_0)\begin{cases} f(x)\leqslant f(x_0) & f(x)\text{在}x_0\text{处取极大值} \\ f(x)\geqslant f(x_0) & f(x)\text{在}x_0\text{处取极小值} \end{cases}$$

$$\text{如果可导函数}y=f(x)\text{在}x_0\text{取极值，则}f'(x_0)=0 \tag{12.3.1}$$

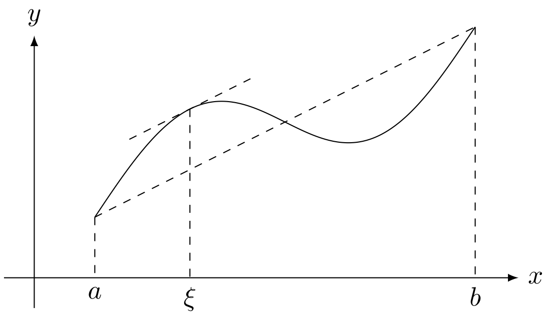
12.3.2 罗尔定理

$$\text{如果函数}f(x)\text{满}\begin{cases}\text{在闭区间 }[a,b]\text{ 上连续}\\ \text{在开区间 (a,b) 可导}\\ f(a)=f(b)\end{cases}$$

$$\text{则至少有一点}\xi\in(a,b),f'(\xi)=0 \tag{12.3.2}$$



12.3.3 拉格朗日定理 (微分中值定理)



$$\text{如果函数}f(x)\text{满}\begin{cases}\text{在闭区间 }[a,b]\text{ 上连续}\\ \text{在开区间 (a,b) 可导}\end{cases}$$
$$\text{则至少有一点}\xi\in(a,b)$$

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow f(b)-f(a)=f'(x)(b-a) \tag{12.3.3}$$

在区间 $[x,x+\triangle x]$ 用拉格朗日定理。

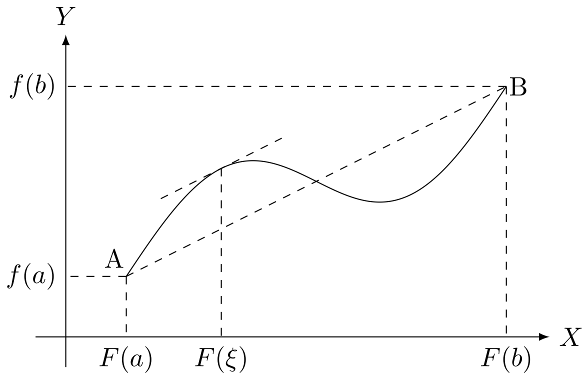
$$f(x+\triangle x)-f(x)=f'(\xi)\triangle x$$

$$\xi\in(x,x+\triangle x)\text{ 记作: }\xi=x+\theta\triangle x\qquad 0<\theta<1$$

$$f(x+\triangle x)-f(x)=f(x+\theta\triangle x)\triangle x$$

$$\triangle y=f(x+\theta\triangle x)\triangle x$$

12.3.4 柯西定理



$$\text{如果函数}f(x)\text{满}\begin{cases}\text{在闭区间 }[a,b]\text{ 上连续}\\ \text{在开区间 (a,b) 可导}\\ F'(x)\neq 0\end{cases}\qquad\text{参数方程}(a\leqslant x\leqslant b)\begin{cases}X=F(x)\\ Y=f(x)\end{cases}$$

$$\text{至少有一点},\xi\qquad \frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)} \tag{12.3.4}$$

$$\text{切线斜率}=\frac{dY}{dX}=\frac{df(x)}{dF(x)}=\frac{f'(x)}{F'(x)}\Rightarrow x=\xi\text{时斜率}=\frac{f'(\xi)}{F'(\xi)}$$

$$AB\text{的斜率}=\frac{f(b)-f(a)}{F(b)-F(a)}$$

12.3.5 三个定理关系

$$\frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)},(F(x)=x)\Rightarrow f'(\xi)=\frac{f(b)-f(a)}{b-a},(f(b)=f(a))\Rightarrow f'(\xi)=0$$

12.4 洛必达法则

$$\text{未定型,}\quad \frac{0}{0},\quad \frac{\infty}{\infty},\quad 0^0,\quad 1^\infty,\quad \infty^0,\quad \infty-\infty$$

$$\lim_{x\to x_0}\tfrac{f(x)}{F(x)}=\lim_{x\to x_0}\tfrac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0}f(x)=0,\lim\limits_{x\to x_0}F(x)=0\\ f(x),F(x)\text{在}x_0\text{的某去心邻域内可导，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\tfrac{f'(x)}{F'(x)}\text{存在，或无穷小。则}\end{cases}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)}\tag{12.4.1}$$

$$\lim_{x\to\infty}\tfrac{f(x)}{F(x)}=\lim_{x\to\infty}\tfrac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0}f(x)=0,\lim\limits_{x\to x_0}F(x)=0\\ \exists N\text{ 当 }|x|>N\text{，时}f'(x),F'(x)\text{存在，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\tfrac{f'(x)}{F'(x)}\text{存在，或为无穷大。则}\end{cases}$$

12.5 泰勒公式

$$f(x_0+\triangle x)-f(x_0)=f'(x_0)\triangle x+\circ(\triangle x)$$

$$x_0+\triangle=x\qquad \triangle x=x-x_0$$

$$f(x)-f(x_0)=f'(x_0)(x-x_0)+\circ(\triangle x)$$

$$f(x)=f'(x_0)(x-x_0)+f(x_0)+\circ(\triangle x)$$

$$f(x)\approx f'(x_0)(x-x_0)+f(x_0)$$

$$P(x_0)=f(x_0)$$

$$P'(x_0)=f'(x_0)$$

12.5.1 泰勒多项式

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n\text{去近似某个多项式}$$

$$\begin{cases}P_n(x_0) & =f(x_0)=a_0\\ P_n'(x_0) & =f'(x_0)=a_1\\ P_n''(x_0) & =f''(x_0)=a_2\cdot 2!\\ & \vdots\\ P_n^{(n-1)}(x_0) & =f^{(n-1)}(x_0)=a_{n-1}\cdot(n-1)!\\ P_n^{(n)}(x_0) & =f^{(n)}(x_0)=a_n\cdot n!\end{cases}\Rightarrow\begin{cases}a_0 & =f_n(x_0)\\ a_1 & =f_n'(x_0)\\ a_2 & =\frac{f_n''(x_0)}{2!}\\ & \vdots\\ a_{n-1} & =\frac{f_n^{(n-1)}(x_0)}{(n-1)!}\\ a_n & =\frac{f_n^{(n)}(x_0)}{n!}\end{cases}$$

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2\cdots\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^n(x_0)}{n!}(x-x_0)^n$$

$$f(x) \approx P_n(x)$$

12.5.2 泰勒中值定理

如果$f(x)|x_0 \in (a,b)$内有$(n+1)$阶导则

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n + R_n(x)$$

拉格朗日余项

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \qquad \{\xi \in (x, x_0)\} \tag{12.5.1}$$

皮亚诺于项

$$R_n(x) = \circ(|x-x_0|^n) \tag{12.5.2}$$

$$f(x) \approx P_n(x) \text{ 误差为 } R_n(x)$$

12.6 麦克劳林公式

$$x_0 = 0$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0 < \theta < 1 \\ \circ(|x|^n) \end{cases}$$

12.6.1 常用的麦克劳林展开

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0 < \theta < 1 \\ \circ(|x|^n) \end{cases}$$

$$\sin x = 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^{n-1}\frac{1}{(2n-1)!}x^{2n-1} + R_n(x)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} + R_n(x)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n + R_n(x)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots - \frac{1}{n}x^n + R_n(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \cdots \frac{A_\alpha^n}{n!}x^n + R_n(x)$$

13 不定积分

13.1 概念

13.1.1 原函数

$$\forall x \in I,\ F'(x) = f(x), \quad F(x)\text{为}f(x)\text{的一个原函数}$$

$$\text{函数}f(x)\text{在区间}I\text{上连续一定有}F(x),\text{使}F'(x) = f(x) \tag{13.1.1}$$

13.1.2 不定积分

区间 I 上，$f(x)$ 带有任意常数的原函数, 称为 $f(x)$ 在区间 I 上的不定积分。
记作:

$$\int f(x)dx \quad \begin{cases} \int & \text{积分符号} \\ f(x) & \text{被积函数} \\ f(x)dx & \text{被积表达式} \\ x & \text{积分变量} \end{cases}$$

如果$F(x)$是$f(x)$的一个原函数

$$\int f(x)dx = F(x) + C$$

13.1.3 不定积分性质

$$\left[\int f(x)dx\right]' = f(x)$$

13.2 幂数，指数，对数

$$\int kdx = kx + C \tag{13.2.1}$$

$$\int x^a \mathrm{d}x = \frac{x^{a+1}}{a+1} + C \tag{13.2.2}$$

$$\int a^x \mathrm{d}x = \frac{a^x}{\ln a} + C \tag{13.2.3}$$

$$\int e^x \mathrm{d}x = e^x + C \tag{13.2.4}$$

$$\int \frac{1}{x} \mathrm{d}x = \ln |x| + C \tag{13.2.5}$$

13.3 三角函数

$$\int \frac{1}{1+x^2} = \arctan x + C \tag{13.3.1}$$

$$\int \sin x \mathrm{d}x = -\cos x + C \tag{13.3.2}$$

$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \begin{cases} \arcsin x + C \\ -\arccos x + C_1 \end{cases} \tag{13.3.3}$$

$$\int \csc x \cot x \mathrm{d}x = -\csc x + C \tag{13.3.4}$$

$$\int \cos x \mathrm{d}x = \sin x + C \tag{13.3.5}$$

$$\int \sec x \tan x \mathrm{d}x = \sec x + C \tag{13.3.6}$$

$$\int \sec^2 x \mathrm{d}x = \tan x + C \tag{13.3.7}$$

$$\int \csc^2 x \mathrm{d}x = -\cot x + C \tag{13.3.8}$$

$$\int \frac{1}{|x|\sqrt{x^2-1}} \mathrm{d}x = \operatorname{arcsec} x + C \tag{13.3.9}$$

$$\int \frac{1}{1+x^2} \mathrm{d}x = \tan x + C \tag{13.3.10}$$

$$\int \sinh x \mathrm{d}x = \cosh x + C \tag{13.3.11}$$

$$\int \cosh x \mathrm{d}x = \sinh x + C \tag{13.3.12}$$

13.4 积分运算

14 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{14.0.1}$$

$$A_N = \sum_{k=0}^{n} q^k \qquad qA_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - qA_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1-q^{n+1}}{1-q}$$

$$\log_{10} x = \lg_x \tag{14.0.2}$$

$$\log_e x = \ln_x \tag{14.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{14.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{14.0.5}$$

$$\log_b x^n = n \log_b x \tag{14.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{14.0.7}$$

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a-b) \sum_{m=0}^{n-1} \left(a^{n-m} b^m\right) = (a-b)\left(a^{n-1} + a^{n-2} b + \cdots + ab^{n-2} + b^{n-1}\right)$$

15 证明

15.1 第 1章

1.2.4

$$\begin{aligned}
\sinh x \cosh x &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{e^{2x} - e^{-2x}}{2}\right) \\
&= \frac{1}{2}\sinh(2x) \\
\sinh(2x) &= 2\sinh x \cosh x
\end{aligned}$$

1.2.5

$$\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\
&= e^x \times e^{-x} \\
&= 1
\end{aligned}$$

1.2.6

$$\begin{aligned}
\cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh(2x)
\end{aligned}$$

1.2.7

$$\begin{aligned}
\cosh(2x) &= \cosh^2 x + \sinh^2 x \\
&= \sinh^2 x + 1 + \sinh^2 x \\
&= 2\sinh^2 x + 1 \\
\cosh x &= 2\sinh^2 \frac{x}{2} + 1
\end{aligned}$$

15.2 第 5章

5.2.1

$$\begin{aligned}
&\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2\\
&f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)\qquad \xi\in(x_1,x_2)\subset[a,b]\\
&f'(\xi)>0,\ (x_2-x_1)>0\\
&f(x_2)-f(x_1)>0\\
&f(x_2)>f(x_1)
\end{aligned}$$

5.2.2

$$\begin{aligned}
&\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2\\
&f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)\qquad \xi\in(x_1,x_2)\subset[a,b]\\
&f'(\xi)<0,\ (x_2-x_1)>0\\
&f(x_2)-f(x_1)<0\\
&f(x_2)<f(x_1)
\end{aligned}$$

5.2.3

$$\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2,x_0=\tfrac{x_1+x_2}{2},x_0-x_1=x_2-x_0=h$$
$$\varphi=\ f(x_0)-f(x_1)=f'(\xi_1)(x_0-x_1)\qquad \xi_1\in(x_1,x_0)$$
$$\psi=f(x_2)-f(x_0)=f'(\xi_2)(x_2-x_0)\qquad \xi_2\in(x_0,x_2)$$

$$\begin{aligned}
\psi-\varphi=f(x_2)+f(x_1)-2f(x_0)&=[f'(\xi_2)-f'(\xi_1)]\,h\\
&=f''(\xi)(\xi_2-\xi_1)h\\
\text{因为 } f''(x)>0,f''(\xi)>0,h&=x_0-x_1>0\\
f(x_2)+f(x_1)-2f(x_0)&>0\\
f(x_2)+f(x_1)&>2f(x_0)\\
f(x_0)&<\frac{f(x_2)+f(x_1)}{2}\\
f(\frac{x_1+x_2}{2})&<\frac{f(x_2)+f(x_1)}{2}
\end{aligned}$$

5.3.1

$$\triangle\ s=\widehat{M_0M'}-\widehat{M_0M}=\widehat{MM'},\quad |MM'|^2=(\triangle\ x)^2+(\triangle\ y)^2,\quad \lim_{M'\to M}\frac{\left|\widehat{MM'}\right|}{|MM'|}=1$$

$$
\begin{aligned}
\left(\frac{\triangle\ s}{\triangle\ x}\right)^2 = \left|\frac{\widehat{MM'}}{\triangle\ x}\right|^2 &= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left(\frac{|MM'|}{\triangle\ x}\right)^2 \\
&= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \frac{(\triangle\ x)^2 + (\triangle\ y)^2}{(\triangle\ x)^2} \\
&= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right] \\
\lim_{\triangle x \to 0} \left(\frac{\triangle\ s}{\triangle\ x}\right)^2 &= \lim_{\triangle x \to 0} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right] \\
(\triangle\ x \to 0, \triangle\ M' \to M) \qquad &= \lim_{M' \to M} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right] \\
\left(\frac{ds}{dx}\right)^2 &= 1 \cdot (1 + (y')^2) \\
\frac{ds}{dx} &= \sqrt{1 + (y')^2} = \sqrt{1 + [f'(x)]^2} \\
ds &= \sqrt{1 + [f'(x)]^2} dx = \sqrt{(dx)^2 + (dy)^2}
\end{aligned}
$$

5.3.2

$$
\begin{aligned}
\left|\frac{d\alpha}{ds}\right| &= \left|\frac{d\alpha}{dx} \cdot \frac{dx}{ds}\right| \\
&= \left|\frac{d\arctan y'}{dx} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right| \\
&= \left|\frac{y''}{1 + (y')^2} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right| \\
&= \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}
\end{aligned}
$$

5.3.3

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\psi'(t)}{\phi'(t)} \\
\frac{d^2y}{dx^2} &= \frac{d\frac{\psi'(t)}{\phi'(t)}}{dt} \cdot \frac{dt}{dx} = \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^2} \cdot \frac{1}{\phi'(t)} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3} \\
\left|\frac{d\alpha}{ds}\right| &= \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3} \cdot \frac{1}{\left\{1+\left[\frac{\psi'(t)}{\phi'(t)}\right]^2\right\}^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{|\psi'(t)|^2 + [\phi'(t)]^2\right\}^{\frac{3}{2}}}
\end{aligned}
$$

15.3 第 9章

9.1.1

$$
\text{反设} \lim_{n\to\infty} x_n = a,\ \lim_{n\to\infty} x_n = b, \text{且} a < b
$$

$$
\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1,\ n > N_1,\ |x_n - a| < \frac{b-a}{3} \\ \exists N_2,\ n > N_2,\ |x_n - b| < \frac{b-a}{3} \end{cases}
$$

$$
N = \max\{N_1, N_2\},\ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}
$$

$$
\begin{aligned}
b - a &= |(x_n - a) - (x_n - b)| \\
&\leqslant |x_n - a| + |x_n - b| \\
&< \frac{b-a}{3} + \frac{b-a}{3} \\
&< \frac{2(b-a)}{3}
\end{aligned}
$$

9.1.2

$$
\varepsilon = 1,\ \exists N > 0,\ \text{当 n>N 时} \, |X_n - a| < 1
$$

$$
\begin{aligned}
|X_n| &= |(X_n - a) + a| \\
&\leqslant |x_n - a| + |a|
\end{aligned}
$$

$$\leqslant 1+|a|$$
$$M=\max\{|X_n|,|X_2|,\ldots,|X_n|,1+|a|\}$$
$$\forall n,\ |X_n|\leqslant M$$

9.1.4

1

由于 $\lim\limits_{n\to\infty} x_n=a$, 且$a>0$
$\varepsilon=\frac{a}{2}$, $\exists N>0,n>N$
$|x_n-a|<\varepsilon$
$|x_n-a|<\frac{a}{2}$
$-\frac{a}{2}<x_n-a<\frac{a}{2}$
$\frac{a}{2}<x_n<1$

2

用反证法，反设 $a<0$. 从某项起 $x_n<0$ 矛盾

9.1.5

$$x_n=b_n-a_n$$
$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}b_n-\lim_{n\to\infty}a_n$$
$$\lim_{n\to\infty}x_n=b-a>0$$
$$\lim_{n\to\infty}x_n>0$$
$$b_n-a_n=x_n>0$$
$$b_n>a_n$$

9.2.1 $\lim\limits_{x\to x_0}f(x)$存在 $\Rightarrow \lim\limits_{x\to x_0^+}f(x)=\lim\limits_{x\to x_0^-}f(x)$

$$\text{设}\lim_{x\to x_0}=A$$
$$0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon$$
$$0<|x-x_0|<\delta\Leftrightarrow x\in\mathring{U}(x_0,\delta)$$
$$\begin{cases}\text{当}x_0<x<x_0+\delta\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^+}f(x)=A\\ \text{当}x_0-\delta<x<x_0\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^-}f(x)=A\end{cases}$$
$$\lim_{x\to x_0^+}f(x)=A=\lim_{x\to x_0^-}f(x)$$

$\lim\limits_{x\to x_0}f(x)$存在 $\Leftarrow \lim\limits_{x\to x_0^+}f(x)=\lim\limits_{x\to x_0^-}f(x)$

$$A=\begin{cases}\lim\limits_{x\to x_o^+},\forall\varepsilon>0,\exists\delta_1>0,x_0<x<x_0+\delta_1,|f(x)-A|<\varepsilon\\ \lim\limits_{x\to x_o^-},\forall\varepsilon>0,\exists\delta_2>0,x_0-\delta_2<x<x_0,|f(x)-A|<\varepsilon\end{cases}$$

$$\delta=\min\{\delta_1,\delta_2\}$$

$$0<|x-x_0|<\delta\begin{cases}x>x_0,x_0<x<x_0+\delta\leqslant x_0+\delta_1,\ |f(x)-A|<\varepsilon\\ x<x_0,x_0-\delta_2\leqslant x_0+\delta<x<x_0,\ |f(x)-A|<\varepsilon\end{cases}$$

$$\lim_{x\to x_0}f(x)=A$$

9.3.1

$$\lim_{x\to x_o}f(x)=A\Rightarrow\begin{cases}\alpha\text{为}x\to x_0\text{时的无穷小}\\ f(x)=\alpha+A\end{cases}$$

$$\text{设}\lim_{x\to x_o}f(x)=A,\ \text{记}f(x)-A=\alpha$$

$$\text{只需证 }\alpha\text{ 为无穷小。}$$

$$\forall\varepsilon>0,\exists\delta>0,\text{当}0<|x-x_0|<\delta,\text{时}\,|f(x)-A|<\varepsilon$$

$$\text{即 }|\alpha-0|<\varepsilon$$

$$\alpha\text{为}x\to x_0\text{时的无穷小}$$

$$\lim_{x\to x_o}f(x)=A\Leftarrow\begin{cases}\alpha\text{为}x\to x_0\text{时的无穷小}\\ f(x)=\alpha+A\end{cases}$$

$$\forall\varepsilon>0,\ \exists\delta>0,\ \text{当}0<|x-x+0|<\delta,\ |\alpha|<\varepsilon$$

$$\text{即}\,|f(x)-A|<\varepsilon\ \lim_{x\to x_0}f(x)=A$$

9.3.2

$$\text{设}\lim_{x\to x_0}f(x)=\infty$$

$$\text{对 }f(x)\text{ 为 }x\to\text{ 时无穷大}$$

$$\text{对于 }M=\tfrac{1}{\varepsilon}.\text{ 存在 }\delta>0$$

$$\text{当 }0<|x-x_0|<\delta\text{ 时}$$

$$|f(x)|>M=\tfrac{1}{\varepsilon}$$

$$\left|\tfrac{1}{f(x)}\right|<\varepsilon$$

$$\tfrac{1}{f(x)}\text{ 为 }x\to x_0\text{ 时的无穷小}$$

9.4.2

$$\begin{aligned}f(x)g(x)&=[A+\alpha]\,[B+\beta]\\&=AB+A\beta+B\alpha+\beta\alpha\\&=AB+\gamma\qquad\qquad(\gamma\text{为无穷小})\end{aligned}$$

$$\lim\,[f(x)g(x)]=AB+\gamma=\lim f(x)\lim g(x)$$

9.4.8

$$\forall \varepsilon > 0$$
$$|x_n - a| < \varepsilon \qquad \forall n > N_1$$
$$|y_n - a| < \varepsilon \qquad \forall n > N_2$$
$$\text{令} N = \max\{N_1, N_2, N_0\}\text{, 则当} n > N \text{时有}$$
$$a - \varepsilon < x_n \leq z_n \leq y_n < a + \varepsilon$$
$$|z_n - a| < \varepsilon$$
$$\lim_{n\to\infty} z_n = a$$

9.4.9

$$\begin{aligned} |f(x) - \sin x_0| &= |\sin x - \sin x_0| \\ &= \left|2\cos(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\ &\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\ &\leqslant 2\frac{|x-x_0|}{2} = |x-x_0| \end{aligned}$$

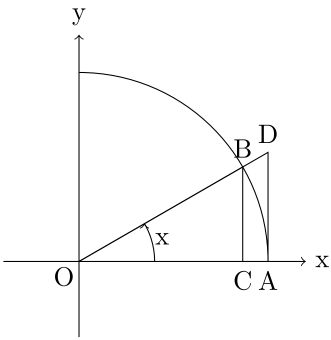
$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\sin x - \sin x_0| \leqslant |x - x_0| < \varepsilon$$

9.4.10

$$\begin{aligned} |f(x) - \cos x_0| &= |\cos x - \cos x_0| \\ &= \left|-2\sin(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\ &\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\ &\leqslant 2\frac{|x-x_0|}{2} = |x-x_0| \end{aligned}$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\cos x - \cos x_0| \leqslant |x - x_0| < \varepsilon$$

9.4.11



$$OB = OA = 1$$

$$\triangle\ AOB \leqslant \text{扇形面积} \leqslant \triangle\ AOD$$

$$\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x$$

$$\sin x \leqslant x \leqslant \tan$$

$$1 \geqslant \frac{\sin x}{x} \geqslant \cos x$$

$$\lim_{x\to 0} 1 \geqslant \lim_{x\to 0} \frac{\sin x}{x} \geqslant \lim_{x\to 0} \cos x$$

$$1 \geqslant \lim_{x\to 0} \frac{\sin x}{x} \geqslant 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

9.4.12

$$\begin{aligned} |1-\cos x| = 1-\cos x &= 2\sin^2\frac{x}{2} \\ &\leqslant 2\left(\frac{x}{2}\right)^2 \end{aligned}$$

$$0 \leqslant 1-\cos x \leqslant \frac{x^2}{2}$$

$$\lim_{x\to 0} 0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant \lim_{x\to 0}\frac{x^2}{2}$$

$$0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant 0$$

$$\lim_{x\to 0}(1-\cos x) = 0$$

$$\lim_{x\to 0}\cos x = 1$$

9.4.13

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x}\\ &= 1\end{aligned}$$

9.4.14

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2} &= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}\\ &= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\\ &= 1\end{aligned}$$

9.4.15

$$x = \sin t,\ t = \arcsin x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arcsin x}{x} = \lim_{x\to 0}\frac{t}{\sin t} = 1$$

9.4.16

$$x = \tan t,\ t = \arctan x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arctan x}{x} = \lim_{t\to 0}\frac{t}{\tan t} = 1$$

9.4.17

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \ln e = 1$$

9.4.18

$$e^x - 1 = t,\ x = \ln(t+1)$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{t\to 0}\frac{t}{\ln(t+1)} = 1$$

9.4.19

$$\lim_{x\to 0}\frac{(1+x)^n-1}{nx} = \lim_{x\to 0}\left(\frac{e^{n\ln(1+x)}-1}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right) = 1$$

9.4.21

$$
\begin{aligned}
x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m}\left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0\left(\frac{1}{n}\right)^0 + C_n^1\left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{n!}{m!\,(n-m)!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
x_{n+1} &= 1+1+\sum_{m=2}^{n+1}\frac{1}{m!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \qquad \text{单调增加} \\
x_n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{n^2} = 1+\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} \\
&< 1+\frac{1}{1-\frac{1}{2}} \\
&< 3 \qquad \text{有界}
\end{aligned}
$$

15.4 第 11章

11.1.1

$$
\begin{aligned}
f'(x_0) &= \lim_{\triangle x\to 0}\frac{\triangle\, y}{\triangle\, x} \qquad \text{因为极限存在与无穷小的关系} \\
\frac{\triangle\, y}{\triangle\, x} &= f'(x_0)+\alpha \qquad \alpha\text{为}\ \triangle\, x\to 0\text{时的无穷小} \\
\triangle\, y &= f'(x_0)\triangle\, x+\alpha\triangle\, x \\
\lim_{\triangle x\to 0}\triangle\, y &= \lim_{\triangle x\to 0}[f'(x_0)\triangle\, x+\alpha\triangle\, x] = 0 \\
\lim_{x\to x_0} f(x) = f(x_0) &\Leftrightarrow \lim_{\triangle x\to 0}[f(x_0+\triangle\, x)-f(x_0)=0] \Leftrightarrow \lim_{\triangle x\to 0}\triangle\, y = 0
\end{aligned}
$$

11.2.1

$$\begin{aligned}(C)' &= \lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{C-C}{\triangle x}\\ &= 0\end{aligned}$$

11.2.2

$$\begin{aligned}(x^a)' &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\ &= \frac{x^a-x_0^a}{x-x_0}\\ &= \frac{(x-x_0)\left(x^{a-1}+x^{a-2}x_0+\cdots+xx_0^{a-2}+x_0^{a-1}\right)}{x-x_0}\\ &= ax_0^{a-1}\end{aligned}$$

11.2.3

$$\begin{aligned}(a^x)' &= \lim_{\triangle x\to 0}\frac{a^{x+\triangle x}-a^x}{\triangle x}\\ &= a^x\lim_{\triangle x\to 0}\frac{a^{\triangle x}-1}{\triangle x}\\ &= a^x\lim_{\triangle x\to 0}\frac{e^{\triangle x\ln a}-1}{\triangle x}\\ &= a^x\ln a\end{aligned}$$

11.2.4

$$(e^x)' = e^x\ln e = e^x$$

11.2.5

$$\begin{aligned}(\log_a^x)' &= \lim_{\triangle x\to 0}\frac{\log_a^{x+\triangle x}-\log_a^x}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{\ln 1+\frac{\triangle x}{x}}{\ln a\triangle x}\\ &= \frac{1}{\ln a}\lim_{\triangle x\to 0}\frac{\frac{\triangle x}{x}}{\triangle x}\\ &= \frac{1}{x\ln a}\end{aligned}$$

11.2.6

$$
\begin{aligned}
(\ln^x)' &= \frac{1}{x \ln e} \\
&= \frac{1}{x}
\end{aligned}
$$

11.3.1

$$
\begin{aligned}
(\sin x)' &= \lim_{\triangle x \to 0} \frac{\sin(x_0 + \triangle\ x) - \sin x_0}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \frac{2\cos(x_0 + \frac{\triangle x}{2}) \sin \frac{\triangle x}{2}}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \cos(x_0 + \frac{\triangle\ x}{2}) \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= cos x_0
\end{aligned}
$$

11.3.2

$$
\begin{aligned}
(\arcsin x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d \sin y}{\mathrm{d}y}} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

11.3.3

$$
\begin{aligned}
(\csc x)' = (\frac{1}{\sin x})` &= \frac{(1)' \cdot sinx - (\sin x)' \cdot 1}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= -\csc x \cdot \cot x
\end{aligned}
$$

11.3.4

$$
\begin{aligned}
(\cos x)' &= \lim_{\triangle x \to 0} \frac{\cos(x_0 + \triangle\ x) - cos(x_0)}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \frac{-2\sin\left(x_0 + \frac{\triangle x}{2}\right) \sin \frac{\triangle x}{2}}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} -\sin\left(x_0 + \frac{\triangle\ x}{2}\right) \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= -\sin x
\end{aligned}
$$

11.3.5

$$
\begin{aligned}
(\arccos x)' &= \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\cos y}{\mathrm{d}y}} \\
&= \frac{1}{-\sin y} \\
&= -\frac{1}{\sqrt{1-\cos^2 y}} \\
&= -\frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

11.3.6

$$
\begin{aligned}
(\sec x)' = \left(\frac{1}{\cos x}\right)' &= \frac{(1)'\cdot\cos x-(\cos x)'\cdot 1}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \sec x\cdot\tan x
\end{aligned}
$$

11.3.8

$$
\begin{aligned}
(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' &= \frac{(\sin x)'\cos x-(\cos x)'\sin x}{\cos^2 x} \\
&= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

11.3.9

$$
\begin{aligned}
(\arctan x)' &= \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\tan y}{\mathrm{d}y}} \\
&= \frac{1}{\sec y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}
\end{aligned}
$$

11.3.10

$$
\begin{aligned}
(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' &= \frac{(\cos x)'\sin x-(\sin x)'\cos x}{\sin^2 x} \\
&= -\frac{\sin^2 x+\cos^2 x}{\cos^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

11.3.11

$$\begin{aligned}(\operatorname{arccot} x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cot y}{\mathrm{d}y}}\\ &= \frac{1}{-\csc^2 y}\\ &= -\frac{1}{1+\cot^2 y}\\ &= -\frac{1}{1+x^2}\end{aligned}$$

11.3.12

$$\begin{aligned}(\sinh x)' &= \left(\frac{e^x - e^{-x}}{2}\right)'\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x\end{aligned}$$

11.3.13

$$\begin{aligned}(\cosh x)' &= \left(\frac{e^x + e^{-x}}{2}\right)'\\ &= \frac{e^x - e^{-x}}{2}\\ &= \sinh x\end{aligned}$$

11.3.14

$$\begin{aligned}(\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)'\\ &= \frac{\frac{d(e^x - e^{-x})}{\mathrm{d}x}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{\mathrm{d}x}(e^x - e^{-x})}{(e^x + e^{-x})^2}\\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}\\ &= \frac{2^2}{(e^x + e^{-x})^2}\\ &= \frac{1}{\cosh^2 x}\end{aligned}$$

11.3.15

$$
\begin{aligned}
(\operatorname{arcsinh} x)' &= \left[\ln(x+\sqrt{x^2+1})\right]' \\
&= \frac{d\ln(x+\sqrt{x^2+1})}{\mathrm{d}\left(x+\sqrt{x^2+1}\right)} \cdot \frac{d\left(x+\sqrt{x^2+1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)} \cdot \frac{d(x^2+1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}
$$

11.3.16

$$
\begin{aligned}
(\operatorname{arccosh} x)' &= \left[\ln(x+\sqrt{x^2-1})\right]' \\
&= \frac{d\ln(x+\sqrt{x^2-1})}{\mathrm{d}\left(x+\sqrt{x^2-1}\right)} \cdot \frac{d\left(x+\sqrt{x^2-1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2-1}\right)}{d\left(x^2-1\right)} \cdot \frac{d(x^2-1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

11.3.17

$$
\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2}\ln(\frac{1+x}{1-x})\right]' \\
&= \frac{1}{2}\cdot\frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)}\cdot\frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2}\cdot\frac{1}{\left(\frac{1+x}{1-x}\right)}\cdot\frac{\frac{d(1+x)}{\mathrm{d}x}(1-x)-\frac{d(1-x)}{\mathrm{d}x}(1+x)}{(1-x)^2} \\
&= \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)+(1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

11.4.1

$$
\begin{aligned}
[Cu(x)]' &= \lim_{\triangle x\to 0}\frac{Cu(x+\triangle x)-Cu(x)}{\triangle x} \\
&= C\lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x} \\
&= Cu'(x)
\end{aligned}
$$

11.4.2

$$
\begin{aligned}
(u(x)\pm v(x))' &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)\pm v(x+\triangle x)-v(x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x}\pm\lim_{\triangle x\to 0}\frac{v(x+\triangle x)-v(x)}{\triangle x} \\
&= u'(x)\pm v'(x)
\end{aligned}
$$

11.4.3

$$
\begin{aligned}
[u(x)\cdot v(x)]' &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)v(x+\triangle x)-u(x)v(x)-u(x)v(x+\triangle x)+u(x)v(x+\triangle x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{\left[u(x+\triangle x)-u(x)\right]v(x+\triangle x)+u(x)\left[v(x+\triangle x)-v(x)\right]}{\triangle x} \\
&= u'(x)\lim_{\triangle x\to 0}v(x+\triangle x)+u(x)v'(x) \\
&= u'(x)v(x)+v'(x)u(x)
\end{aligned}
$$

11.5.1

$$\begin{aligned}\left[f^{-1}(y)\right]'|_{y=y_0} &= \lim_{y\to y_o}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\\ &= \lim_{y\to y_o}\frac{x-x_0}{y-y_0}\\ &= \lim_{x\to x_o}\frac{x-x_0}{f(x)-f(x_0)}\\ &= \lim_{x\to x_o}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\\ &= \frac{1}{f'(x)}\end{aligned}$$

11.6.1

$$\text{定义函数}\quad A(u)=\begin{cases}\frac{f(u)-f(u_0)}{u-u_0}. & u\neq u_0\\ f'(u_0), & u=u_0\end{cases}$$

$$A(u)\text{在}u_o\text{处连续, 既有}$$

$$\lim_{u\to u_o}A(u)=A(u_0)=f'(u_0)$$

$$\text{由恒等式}f(u)-f(u_0)=A(u)(u-u_0)\text{我们有}$$

$$\begin{aligned}\frac{F(x)-F(x_0)}{x-x_0} &= \frac{f\left[g(x)\right]-f\left[g(x_0)\right]}{x-x_0}\\ &= A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\end{aligned}$$

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=\lim_{x\to x_0}A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}$$

$$F'(x_0)=f'(g(x_0))g'(x_0)$$

15.5 第 12章

12.1.1

$$\triangle y = A\triangle x + \circ(\triangle x)$$

$$\frac{\triangle y}{\triangle x}=A+\frac{\circ(\triangle x)}{\triangle x}$$

$$\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\lim_{\triangle x\to 0}\left[A+\frac{\circ(\triangle x)}{\triangle x}\right]$$

$$f'(x_0)=A+0$$

$$f'(x_0)=A$$

12.1.2

设$f(x)$在x_0点可导,$f'(x_0)=\lim\limits_{\triangle x\to 0}\frac{\triangle y}{\triangle x}$存在

(极限与无穷小的关系: $\lim f(x)=A\Leftrightarrow f(x)=A+\alpha$)

$\frac{\triangle y}{\triangle x}=f'(x_0)+\alpha$

$\triangle\ y=f'(x_0)\triangle\ x+\alpha\triangle\ x$

其中α为$\triangle\ x\to 0$时的无穷小。

$\lim\limits_{\triangle x\to 0}\frac{\alpha\triangle x}{\triangle x}=\lim\limits_{\vartheta x\to 0}\alpha=0$

$\alpha\triangle\ x=\circ(\triangle\ x)$

$\triangle\ y=f'(x_0)\triangle\ x+\circ(\triangle\ x)$

12.2.1

$$
\begin{aligned}
d(u\pm v)&=(u\pm v)'dx\\
&=(u)'dx\pm(v')dx\\
&=du\pm dv
\end{aligned}
$$

12.2.2

$$
\begin{aligned}
d(u\cdot v)&=(u\cdot v)'dx\\
&=(u)'vdx-(v')udx\\
&=vdu-udv
\end{aligned}
$$

12.2.3

$$
\begin{aligned}
d\left(\frac{u}{v}\right)&=\left(\frac{u}{v}\right)'dx\\
&=\frac{(u)'v-(v')u}{v^2}dx\\
&=\frac{vdu-udv}{v^2}
\end{aligned}
$$

12.3.1

$$\lim_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)}{\triangle x}=f'(x_0)$$

$$f(x_0+\triangle\ x)-f(x_0)\leqslant 0$$

$$\begin{cases}\triangle\ x>0\left\{\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\leqslant 0\Rightarrow f'(x_0^+)=\lim\limits_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\leqslant 0\right.\\ \triangle\ x<0\left\{\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\geqslant 0\Rightarrow f'(x_0^-)=\lim\limits_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\geqslant 0\right.\end{cases}$$

$$f'(x_0)=f'(x_0^+)=f'(x_0^-)\Rightarrow f'(x_0)=0$$

12.3.2

$$M=\max\{f(x)|x\in[a,b]\},m=\min\{f(x)|x\in[a,b]\}$$

$$\begin{cases} M=m\Rightarrow M=m=f(a)=f(b),\text{此时}f(x)\text{为常数},\forall\xi\in(a,b),f'(\xi)=0 \\ M>m\begin{cases} f(a)>m\Rightarrow\exists\xi\in(a,b),f(\xi)=m,\text{根据费马引理},f'(\xi)=0 \\ f(a)<M5\Rightarrow\exists\xi\in(a,b),f(\xi)=M,\text{根据费马引理},f'(\xi)=0 \end{cases} \end{cases}$$

12.3.3

$$\begin{aligned} \varphi(x)&=f(x)-\frac{f(b)-f(a)}{b-a}x \\ \varphi(a)&=f(a)-\frac{f(b)-f(a)}{b-a}a=\frac{bf(a)-af(b)}{b-a} \\ \varphi(b)&=f(b)-\frac{f(b)-f(a)}{b-a}b=\frac{bf(a)-af(b)}{b-a} \\ &\varphi(a)=\varphi(b),\exists\xi\in(a,b),\varphi'(\xi)=0 \\ f'(\xi)&=\frac{f(b)-f(a)}{b-a} \\ f'(\xi)(b-a)7&=f(b)-f(a) \end{aligned}$$

12.3.4

$$\begin{aligned} &\varphi(x)=f(x)-\frac{f(b)-f(a)}{F(b)-F(a)}[F(x)-F(a)] \\ &\varphi(a)=\varphi(b)=f(a) \\ &\varphi'(\xi)=0 \\ &f'(\xi)=\frac{f(b)-f(a)}{F(b)-F(a)}F'(\xi) \\ &\frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)} \end{aligned}$$

12.4.1

$f(x),F(x)$的去心邻域可导，$\lim\limits_{x\to x_0}\frac{f(x)}{F(x)}$与$f(x_0),F(x_0)$无关。规定$f(x_0)=0,F(x_0)=0$

此时$\lim\limits_{x\to x_0}f(x)=0=f(x_0)$ $\lim\limits_{x\to x_0}F(x)=0=F(x_0)$ 此时在 x_0 点处也连续

$$\frac{f(x)}{F(x)}=\frac{f(x)-f(x_0)}{F(x)-F(x_0)}=\frac{f'(\xi)}{F'(\xi)}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(\xi)}{F'(\xi)}$$

$x\to x_0$,时$\xi\to x_0$ 符号 ξ 换成 x

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{\xi\to x_0}\frac{f'(\xi)}{F'(\xi)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)}$$

12.5.1

$$\frac{R_n(x)}{(x-x_0)^{n+1}} = \frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-(x_0-x_0)^{n+1}} = \frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}$$

$$\frac{1}{n+1}\cdot\frac{R_n'(\xi_1)}{(\xi_2-x_0)^n} = \frac{1}{n+1}\cdot\frac{R_n'(\xi_1)-R_n'(x_0)}{(\xi_1-x_0)^n-(x_0-x_0)^n} = \frac{1}{n+1}\cdot\frac{R_n''(\xi_2)}{(n)(\xi_2-x_0)^{n-1}}$$

$$\vdots$$

$$\frac{R_n^{(n)}(\xi_n)}{(n+1)!(\xi_n-x_0)} = \frac{R_n^{(n)}(\xi_n)-R_n^{(n)}(x_0)}{(n+1)!(\xi_n-x_0)-0} = \frac{R_n^{(n+1)}(\xi)}{(n+1)!}$$

$$\frac{R_n^{(n+1)}(\xi)}{(n+1)!} = \frac{f^{(n+1)}(\xi)-P_n^{(n+1)}(\xi)}{(n+1)!} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$$\xi_1 \in (x, x_0), \xi_2 \in (\xi_1, x_0), \xi_n \in (\xi_{n-1}, x_0), \xi \in (\xi_n, x_0)$$

12.5.2

$$\begin{aligned}
\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n} &= \lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}} \\
&= \lim_{x\to x_0}\frac{R_n''(x)}{n(n-1)(x-x_0)^{n-2}} \\
&= \lim_{x\to x_0}\frac{R_n^{(n)}(x)}{n!} \\
&= \frac{1}{n}\cdot 0 \\
&= 0
\end{aligned}$$