数学方面 (笔记)

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，贴吧，姜萍吧，姜圣的追随者，

2024.8.18：笔记都是看视频和书记录的。可能会有个别错误。但是我会持续更新，发现错误就会更改。上传频率不太固定。

目录

1 三角函数

1.1 三角恒等式

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{1.1.1}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B \tag{1.1.2}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{1.1.3}$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B \tag{1.1.4}$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.5}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.6}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.7}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.8}$$

1.1.2 积化和差

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \tag{1.1.9}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \tag{1.1.10}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{1.1.11}$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \tag{1.1.12}$$

1.1.3 降幂

$$\sin^2 x = \frac{1-\cos(2x)}{2} \tag{1.1.13}$$
$$\cos^2 x = \frac{1+\cos(2x)}{2} \tag{1.1.14}$$

1.1.4 半角公式

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos(x)}{2}} \tag{1.1.15}$$
$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos(x)}{2}} \tag{1.1.16}$$
$$\tan\frac{x}{2} = \csc x - \cot x \tag{1.1.17}$$
$$\tag{1.1.18}$$

1.1.5 倍角公式

$$\sin(2x) = 2\sin x\cos x \tag{1.1.19}$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \tag{1.1.20}$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x} \tag{1.1.21}$$

1.1.6 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.22}$$
$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.23}$$

1.1.7 三角函数恒等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.24}$$

$$1 + \tan^2 x = \sec^2 \tag{1.1.25}$$

$$1 + \cot^2 x = \csc^2 \tag{1.1.26}$$

1.2 双曲函数

1.2.1 定义

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2 + 1}) \tag{1.2.1}$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2 - 1}) \tag{1.2.2}$$

$$\operatorname{artanh} x = \frac{1}{2} \ln(\frac{1 + x}{1 - x}) \tag{1.2.3}$$

1.2.3 双曲函数恒等式

$$\sinh(2x) = 2 \sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2 \sinh^2 \frac{x}{2} \tag{1.2.7}$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 + x_2 + \cdots + x_n} \tag{2.0.1}$$

$$|x + y| \leqslant |x| + |y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1 + x)^n \leqslant 1 + nx \tag{2.0.4}$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1) \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

3.2 运算

4 区间与映射

label

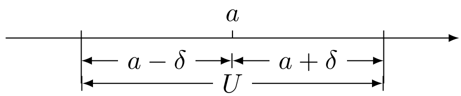
4.1 区间定义

$$\text{区间定义}\begin{cases}(a,b)=\{x|a<x<b\}\\ [a,b]=\{x|a\leqslant x\leqslant b\}\\ (a,b]=\{x|a<x\leqslant b\}\\ (a,+\infty)=\{x|a<x\}\end{cases}$$

4.2 领域定义

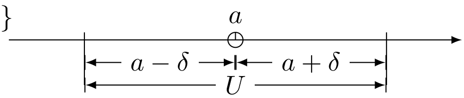
点 a 的领域

$$U(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\}\\ \{x|\ |x-a|<\delta\}\end{cases}$$



点 a 的去心领域

$$\mathring{U}(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\wedge x\neq 0\}\\ \{x|0<|x-a|<\delta\}\end{cases}$$



点 a 的左领域 $(a-\delta,a)$

点 a 的右领域 $(a,a+\delta)$

4.3 映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x\in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$\text{记作}\qquad f:X\to Y$$

$$f(x)=y\qquad\begin{cases}\text{定义域 }(D_f)=X & x\text{-原像}\\ \text{值域 }(R_f)=\{f(x)|x\in X\} & y\text{-像}\end{cases}$$

$$
\text{映射类型}\quad\begin{cases}
\text{满射:} & R_f = Y \\
\text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\
\text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射:} \begin{cases} f^(x) & = y \\ f^{-1}(y) & = x \end{cases} \\
\text{复合映射:} & g \circ f \Leftrightarrow g\left[f\left(x\right)\right] \begin{cases} f : X \to Y_1 \\ g : Y_2 \to Z \\ g \circ f : X \to Z \qquad (Y_1 \subset Y_2) \end{cases}
\end{cases}
$$

5 函数

5.1 函数相关的定义

5.1.1 函数

设数集 $D \in R$ 的映射

$$f : D \to R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x)\ \{x \in D\}$$

5.1.2 驻点

$$Def:\ f'(x) = 0$$

5.1.3 拐点

$$Def : f''(x) = 0\ (\text{左右两侧凹凸性改变})$$

5.1.4 极值点

$$Def : \text{函数} f(x)\ x \in \mathring{U}(x_0), \text{包括可导和不可导的点} \begin{cases} \text{极大值:}\ f(x) < f(x_0) \\ \text{极小值:}\ f(x) > f(x_0) \end{cases}$$

$$x \in \mathring{U}(x_0) \begin{cases} f(x)\text{可导,}\ f'(x_0) = 0 \begin{cases} x_0\text{极大值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) > 0 \\ x \in (x_0, x_0 + \delta), f'(x) < 0 \end{cases} \\ x_0\text{极小值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) < 0 \\ x \in (x_0, x_0 + \delta), f'(x) > 0 \end{cases} \\ x_0\text{无极值,}\ x \in \mathring{U}(x_0) \begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases} \end{cases} \\ f(x)\text{二阶可导}, f'(x_0) = 0, f''(x_0) \neq 0 \begin{cases} f''(x) < 0 \Rightarrow x_0\text{极大值} \\ f''(x) > 0 \Rightarrow x_0\text{极小值} \end{cases} \end{cases}$$

5.1.5 最值

$$\text{最大值或最小值}\begin{cases}\text{驻点}\\ \text{端点}\end{cases}$$

5.2 函数的性质

5.2.1 函数的有界性

$$f:D\to R\{D\subset R\}\begin{cases}\text{有界}\begin{cases}\text{有上界}\left\{\exists k_1,\ \text{使}f(x)\leqslant k_1,\ \forall x\in D\right.\\ \text{有下界}\left\{\exists k_1,\ \text{使}f(x)\geqslant k_1,\ \forall x\in D\right.\end{cases}\\ \text{无界}\begin{cases}\text{无上界}\left\{\forall K_1,\ \exists x\in D\ \text{使},\ f(x)\geqslant k_1\right.\\ \text{无下界}\left\{\forall K_1,\ \exists x\in D\ \text{使},\ f(x)\leqslant k_1\right.\end{cases}\end{cases}$$

5.2.2 函数的单调性与凹凸性

$$\text{若}\{x_1,x_2\in D\}\ x_1<x_2\Rightarrow\begin{cases}f(x_1)<f(x_2)\text{称}f(x)\text{在 D 上单调增加}\\ f(x_1)>f(x_2)\text{称}f(x)\text{在 D 上单调减少}\\ f(x_1)\leqslant f(x_2)\text{称}f(x)\text{在 D 上单调非降}\\ f(x_1)\geqslant f(x_2)\text{称}f(x)\text{在 D 上单调非增}\end{cases}$$

$$\text{设 }f(x)\text{ 在区间 I 上连续},\forall x_1,x_2\begin{cases}f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2},\text{称 }f(x)\text{ 在 I 上是向上凹}\\ f\left(\frac{x_1+x_2}{2}\right)>\frac{f(x_1)+f(x_2)}{2},\text{称 }f(x)\text{ 在 I 上是向上凸}\end{cases}$$

$$f(x)\text{在}[a,b]\text{上连续，在}(a,b)\text{内可导}f'(x)\geqslant 0,\text{有限个点为 0, 单调增} \tag{5.2.1}$$

$$f(x)\text{在}[a,b]\text{上连续，在}(a,b)\text{内可导}f'(x)\leqslant 0,\text{有限个点为 0, 单调减} \tag{5.2.2}$$

$$f(x)\text{在}[a,b]\text{上连续，在}(a,b)\text{内二阶可导}f''(x)\geqslant 0,\text{有限个点为 0, 向上凹} \tag{5.2.3}$$

$$f(x)\text{在}[a,b]\text{上连续，在}(a,b)\text{内二阶可导}f''(x)\leqslant 0,\text{有限个点为 0, 向下凸} \tag{5.2.4}$$

5.2.3 函数的奇偶性

$$\forall x \in D \qquad f(-x) = \begin{cases} f(x) & \text{偶函数} \\ -f(x) & \text{奇函数} \end{cases}$$

$$\text{奇函数} \times \text{奇函数} = \text{偶函数} \tag{5.2.5}$$

$$\text{奇函数} \times \text{偶函数} = \text{奇函数} \tag{5.2.6}$$

$$\text{偶函数} \times \text{偶函数} = \text{偶函数} \tag{5.2.7}$$

5.2.4 周期性

$$Def: \quad f(x+L) = f(x)\{L > 0\text{常数}, \forall x \in D\} \Rightarrow f(x)\ \text{为}\ L\ \text{的周期函数}$$

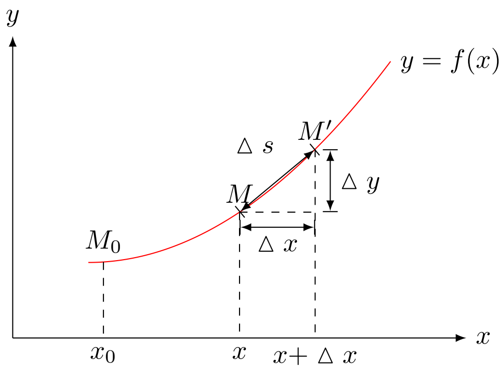
5.3 弧

5.3.1 有向曲线弧

$$\text{基准点} M_0(x_0, f(x_0)), \text{以}\ x\ \text{增大的方向为正向}, \overset{\frown}{M_0M} = S$$

$$S = S(x), S\text{是关于 x 的单调增加函数}$$

$$\overset{\frown}{M_0M} \begin{cases} \text{绝对值为的长度} \\ \text{与曲线正向一致，取正值} \\ \text{与曲线反向一致，取负值} \end{cases}$$



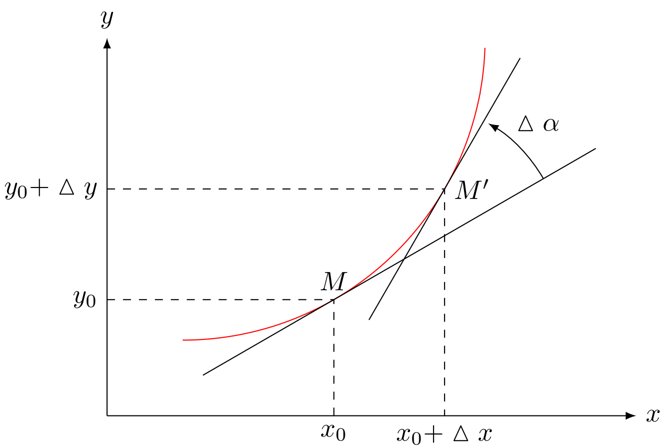
5.3.2 弧微分

$$ds = \sqrt{1+(y')^2}dx \Leftrightarrow ds = \sqrt{(dx)^2+(dy)^2} \Leftrightarrow ds = \sqrt{(dx)^2+(f'dx)^2} \tag{5.3.1}$$

参数方程形式

$$\begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \begin{cases} dx = \phi'(t)dt \\ dy = \psi'(t)dt \end{cases} \Rightarrow ds = \sqrt{[\phi'(t)]^2+[\psi'(t)]^2}dt$$

5.3.3 曲率



$M(x_0, y_0), M'(x_0+\triangle x, y_0+\triangle y), \triangle s = \widehat{MM'}$

曲线上弧的 $\begin{cases} \text{平均曲率:} & \bar{k} = \left|\frac{\Delta\alpha}{\Delta s}\right| \\ \text{点曲率:} & k = \lim\limits_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right| = \left|\frac{d\alpha}{ds}\right| \end{cases}$

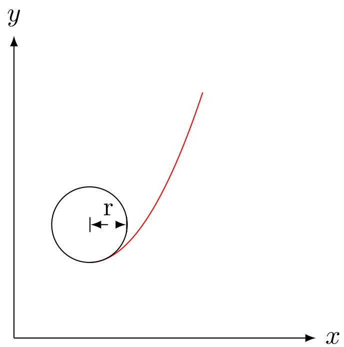
$$\left|\frac{d\alpha}{ds}\right| = \frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \tag{5.3.2}$$

$$\left|\frac{d\alpha}{ds}\right| \text{的参数方程形式} \begin{cases} x = \phi(t) \\ y = \psi(t) \end{cases} \Rightarrow \left|\frac{d\alpha}{ds}\right| = \frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left\{|\psi'(t)|^2+[\phi'(t)]^2\right\}^{\frac{3}{2}}} \tag{5.3.3}$$

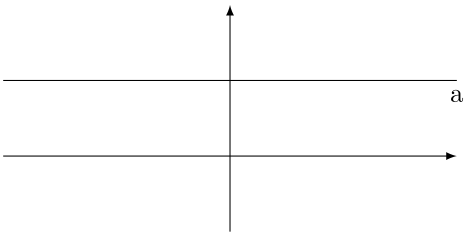
5.3.4 曲率圆，曲率半径

$$\text{圆的曲率}k = \left|\frac{\triangle\ \alpha}{\triangle\ s}\right| = \left|\frac{\triangle\ \alpha}{r\ \triangle\ \alpha}\right| = \frac{1}{r}$$

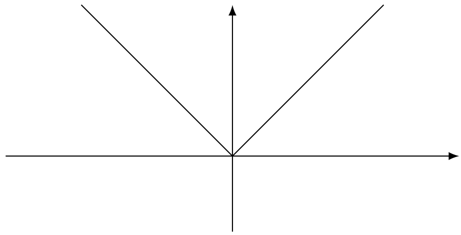
$$\text{曲率半径}r = \frac{1}{k}$$



6 图像

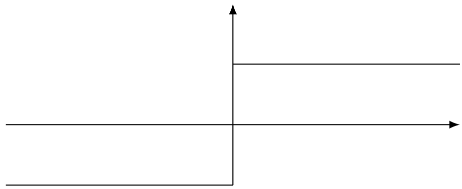


常函数 $f(x) = a\{a \in R\}$

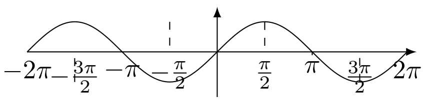


$f(x) = |x|$

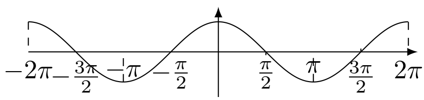
$$f(x) = sgn\ x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



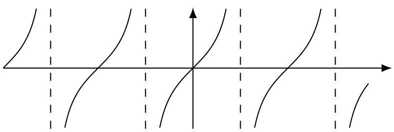
$|x| = x \cdot sgnx$



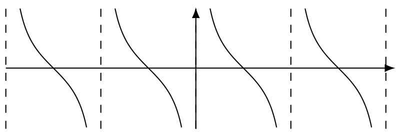
$\sin x$



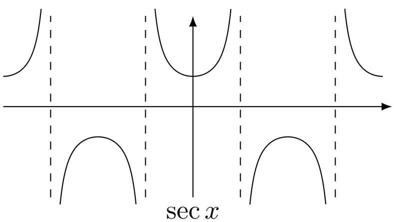
$\cos x$



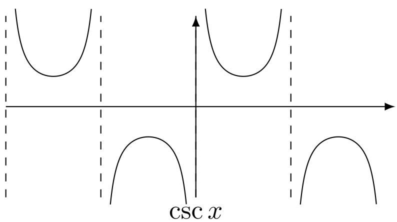
$\tan x$



$\cot x$



$\sec x$



$\csc x$

7 并集，交集

7.1 定义

$$
\begin{gathered}
(\vee \text{ 或},\wedge \text{ 与}) \\
A \cup B = \{x \in A \vee x \in B\} \\
A \cap B = \{x \in A \wedge x \in B\}
\end{gathered}
$$

7.2 运算

$$
\begin{cases}
\text{交换律} \begin{cases} A \cup B & = B \cup A \\ A \cap B & = B \cap A \end{cases} \\
\text{结合律} \begin{cases} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{cases} \\
\text{分配律} \begin{cases} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{cases} \\
\text{对偶律} \begin{cases} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{cases}
\end{cases}
$$

$$
\begin{gathered}
A \cup A = A = A \cap A \\
A = B \Leftrightarrow A \subset B \wedge A \supset B \\
A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing
\end{gathered}
$$

7.3 性质

性质 1.

$$
A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{7.3.1}
$$

性质 2.

$$
A \cup B = B \Leftrightarrow A \subset B \tag{7.3.2}
$$

性质 3.

$$
A \cap B = A \Leftrightarrow A \subset B \tag{7.3.3}
$$

性质 4. $(n \in N\ )$

$$
A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{7.3.4}
$$

性质 5. ($n \in N$)

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{7.3.5}$$

7.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

7.5 德摩根律 定理

$$\left(\bigcup_\alpha E_\alpha\right)^C = \bigcap_\alpha (E_\alpha^C)$$

$$\left(\bigcap_\alpha E_\alpha\right)^C = \bigcup_\alpha (E_\alpha^C)$$

8 群，环，域

8.1 群

8.1.1 M1

8.1.2 M2

8.1.3 M3

8.1.4 M4

8.1.5 sdas

8.2 环

8.3 域

9 极限

9.1 数列极限

9.1.1 数列的定义

$$Def: \qquad \{x_n\}, x_n = f(n), n \in N^+ \to R$$

9.1.2 数列极限的定义

$Def:\ \{x_n\},\ n \in N^+, \exists a,\ \forall \varepsilon > 0, \exists N,\ n > N \Rightarrow |x_n - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_n = a$

极限存在，为收敛，不存在为发散

9.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{9.1.1}$$

9.1.4 有界数列

$$\begin{gathered} \text{若}\exists M > 0, \{M \in \text{正数}\} \\ \text{使得}\forall n, \quad |x_n| \leqslant M \\ \text{则称数列 } \{x_n\} \text{ 为有界数列} \end{gathered}$$

9.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{9.1.2}$$

$$\text{单调有界数列必收敛} \tag{9.1.3}$$

9.1.6 收敛数列的保号性

$$\lim_{n\to\infty} x_n = a \text{ 存在，且 } a > 0, \text{ 则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时 } \Leftrightarrow x_n > 0 \tag{9.1.4}$$

$$\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} b_n = b, a < b,\ \exists N, n > N, a_n < b_n \tag{9.1.5}$$

9.1.7 收敛数列和子数列

$\{x_n\}, \lim\limits_{n\to\infty} x_n = a,\ \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$
证明 $K = N\ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

9.2 函数极限

9.2.1 极限的定义

$$Def: \forall\varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ \text{当} x < -X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ \text{当}\ |x| > X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists\delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当} x_0 - \delta < x < x_0, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ \text{当} 0 < |x - x_0| < \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x\to x_0} f(x)\text{存在} \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \tag{9.2.1}$$

图

9.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$\lim\limits_{x\to x_0} f(x) = A, \exists M > 0, \delta > 0 \text{使} 0 < |x - x_0| < \delta, |f(x)| \leqslant M$

3 保号性

$\lim\limits_{x\to x_0} f(x)=A,\ A>0,\exists\delta>0,$当$,0<|x-x_0|<\delta\Rightarrow f(x)>0$

$f(x)>0,\exists\delta>0,$当$,0<|x-x_0|<\delta\Rightarrow \lim\limits_{x\to x_0} f(x)=A,\ A>0$

4 保序性

$f(x)\geqslant g(x),\ \lim f(x)=a,\ \lim g(x)=b,$ 则$a\geqslant b$

5 函数极限与数列极限的关系

如果 $\lim\limits_{x\to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列，则满足 $x_n\neq x_0$ 则 $\lim\limits_{n\to\infty} f(x_n)=0=\lim\limits_{x\to x_0} f(x),\ x_n\to x_0$

9.3 无穷小与无穷大

9.3.1 无穷小定义

$Def:$如果 $\lim\limits_{x\to x_0} f(x)=0$则称$f(x)$为$x\to x_0$时的无穷小

$$Def:\forall\varepsilon>0\begin{cases}\exists X>0\begin{cases}\text{当}x>X & \text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to+\infty} f(x)=0\\ \text{当}x<-X & \text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to-\infty} f(x)=0\\ \text{当 }|x|>X & \text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to\infty} f(x)=0\end{cases}\\ \exists\delta>0\begin{cases}\text{当}x_0<x<x_0+\delta,\text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0^+} f(x)=0\\ \text{当}x_0-\delta<x<x_0,\text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0^-} f(x)=0\\ \text{当}0<|x-x_0|<\delta,\text{时 } |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0} f(x)=0\end{cases}\end{cases}$$

9.3.2 函数极限与无穷小的关系

$$\text{在自变量的同一变化中。}\alpha\text{ 为无穷小。}\lim f(x)=A\Leftrightarrow f(x)=A+\alpha \tag{9.3.1}$$

9.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$\text{如果 } f(x) \text{ 为无穷大，则 } \tfrac{1}{f(x)} \text{ 为无穷小。} \tag{9.3.2}$$

$$\text{如果 } f(x) \text{ 为无穷小，切 } f(x)\neq 0\text{，则 } \tfrac{1}{f(x)} \text{ 为无穷小。} \tag{9.3.3}$$

9.3.4 无穷大定义

$$
Def:\forall M>0\begin{cases}
\exists X>0\begin{cases}
当x>X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=\infty
\end{cases}\\
当x<-X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=+\infty\\
f(x)<M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=\infty
\end{cases}\\
当\ |x|>X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=\infty
\end{cases}
\end{cases}\\
\exists\delta>0\begin{cases}
当x_0-\delta<x<x_0\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=\infty
\end{cases}\\
当x_0<x<x_0+\delta\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=\infty
\end{cases}\\
当0<|x-x_0|<\delta\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=\infty
\end{cases}
\end{cases}
\end{cases}
$$

$$\lim_{x\to x_0}f(x)=\infty，直线x=x_0是y=f(x)垂直渐进线$$

9.4 运算

9.4.1 有限个无穷小的和仍为无穷小

$$设\ \gamma=\alpha+\beta$$

$$\alpha\ 和\ \beta\ 同为\ x\to x_0\ 时的无穷小$$

$$\forall\varepsilon>0，\exists\delta_1>0，当\ 0<|x-x_0|<\delta_1\ 时，有\ |\alpha|<\tfrac{\varepsilon}{2}$$

$$\forall\varepsilon>0,\ \exists\delta_2>0,\ \text{当 } 0<|x-x_0|<\delta_2 \text{ 时，有 } |\beta|<\frac{\varepsilon}{2}$$
$$\delta=min\{\delta_1,\delta_2\},\ \text{当 } 0<|x-x_0|<\delta \text{ 时}$$
$$0<|x-x_0|<\delta_1, 0<|x-x_0|<\delta_2 \text{ 同时满足}$$
$$\text{即 } |\alpha|<\frac{\varepsilon}{2}, |\beta|<\frac{\varepsilon}{2} \text{ 同时成立}$$
$$|\gamma|=|\alpha+\beta|<|\alpha|+|\beta|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

9.4.2 有界函数与无穷小的乘积仍为无穷小

$$\text{设 } \alpha \text{ 为 } x\to x_0 \text{ 时的一个无穷小}$$
$$g(x) \text{ 为 } x_0 \text{ 的一个去心邻域 } \mathring{U}(x_0,\delta_1) \text{ 有界}$$
$$f(x)=g(x)\alpha$$
$$\text{证 } f(x) \text{ 为 } x\to x_0 \text{ 时的无穷小}$$
$$\text{因为 } g(x)\text{在}\mathring{U}(x_0,\delta_1) \text{ 有界}$$
$$\exists M>0, \text{当}0<|x-x_0|<\delta_1 \text{ 时 } |g(x)|<M$$
$$\text{因为 } \alpha \text{ 是 } x\to x_0 \text{ 的无穷小}$$
$$\exists\delta_2>0\text{当}0<|x-x_0|<\delta_2 \text{ 时 } |\alpha|<\frac{\varepsilon}{M}<\varepsilon$$
$$\text{取 } \delta=min\{delta_,\delta_2\} \text{ 当 } 0<|x-x_0|<\delta \text{ 时}$$
$$|g(x)|\geqslant M, |\alpha|<\frac{\varepsilon}{M} \text{ 同时成立}$$
$$|g(x)\alpha|=|g(x)|\ \ |\alpha|<M\frac{\varepsilon}{M}=\varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

9.4.3 极限的四则运算

$\lim f(x)=A,\ \lim g(x)=B$

$$\lim\left(f(x)\pm g(x)\right)=\lim f(x)\pm\lim g(x) \tag{9.4.1}$$
$$\lim\left(f(x)g(x)\right)=\lim f(x)\lim g(x) \tag{9.4.2}$$
$$\lim\left(\frac{f(x)}{g(x)}\right)=\frac{\lim f(x)}{\lim g(x)} \tag{9.4.3}$$
$$\lim\left[Cf(x)\right]=C\lim f(x) \tag{9.4.4}$$
$$\lim\left[f(x)\right]^n=\left[\lim f(x)\right]^n \tag{9.4.5}$$

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{a_0 x^n + a_1 x^{n-1} + \cdots + a_n} = \begin{cases} \frac{a}{b} & m = n \\ \infty & m > n \\ 0 & m < n \end{cases} \tag{9.4.6}$$

$$\begin{gathered} \lim_{x \to x_0} g(x) = u_0,\ \lim_{u \to u_0} f(x) = A \\ \exists \delta_0 > 0,\ x \in \mathring{U}\left(x_0, \delta_0\right),\ g(x) \neq u_0 \\ \lim_{x \to x_0} f\left[g(x)\right] = \lim_{u \to u_0} f(u) = A \end{gathered} \tag{9.4.7}$$

9.4.4 夹逼定理 (三明治定理)

$$\begin{gathered} x_n \leqslant z_n \leqslant y_n \qquad \forall n > N_0 \\ \text{若} \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a \text{则} \lim_{n \to \infty} z_n = a \end{gathered} \tag{9.4.8}$$

9.4.5 重要极限

$$x \to x_0$$

$$\lim_{x \to x_0} \sin x = \sin x_0 \tag{9.4.9}$$

$$\lim_{x \to x_0} \cos x = \cos x_0 \tag{9.4.10}$$

$$x \to 0$$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{9.4.11}$$

$$\lim_{x \to 0} \cos x = 1 \tag{9.4.12}$$

$$\lim_{x \to 0} \frac{\tan x}{x} = 1 \tag{9.4.13}$$

$$\lim_{x \to 0} \frac{1 - \cos x}{\frac{1}{2} x^2} = 1 \tag{9.4.14}$$

$$\lim_{x \to 0} \frac{\arcsin x}{x} = 1 \tag{9.4.15}$$

$$\lim_{x \to 0} \frac{\arctan x}{x} = 1 \tag{9.4.16}$$

$$\lim_{x \to 0} \frac{\ln\left(1 + x\right)}{x} = 1 \tag{9.4.17}$$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \tag{9.4.18}$$

$$\lim_{x \to 0} \frac{\left(1 + x\right)^n - 1}{nx} = 1 \tag{9.4.19}$$

$$\lim_{x \to 0} \left(1 + x\right)^{\frac{1}{x}} = e \tag{9.4.20}$$

$$x \to \infty$$

$$\{x_n\} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{9.4.21}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{9.4.22}$$

9.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

$Def:$ α, β 是同一极限过程的无穷小。

(1) 如果 $\lim\frac{\beta}{\alpha}=0$ 则称 β 是 α 的高阶无穷小，记作 $\beta=\circ(\alpha)$

(2) 如果 $\lim\frac{\beta}{\alpha}=\infty$ 则称 β 是 α 的底阶无穷小。

(3) 如果 $\lim\frac{\beta}{\alpha}=C$ 则称 β 是 α 的同阶无穷小。

(4) 如果 $\lim\frac{\beta}{\alpha^k}=C, k>0$ 则称 β 是 α 的 k 阶无穷小。

(5) 如果 $\lim\frac{\beta}{\alpha}=1$ 则称 β 是 α 的等价阶无穷小。

9.4.7 等价无穷小代换，因子代换

β与α是等价无穷小 $\Leftrightarrow \beta=\alpha+\circ(\alpha)$

设$\alpha\sim\alpha'$，$\beta\sim\beta'$, 且$\lim\frac{\beta'}{\alpha'}$存在, 则$\lim\frac{\beta}{\alpha}=\lim\frac{\beta'}{\alpha'}$

$\lim\alpha f(x)=\lim\alpha' f(x)$

$\lim\frac{f(x)}{\alpha}=\lim\frac{f(x)}{\alpha'}$

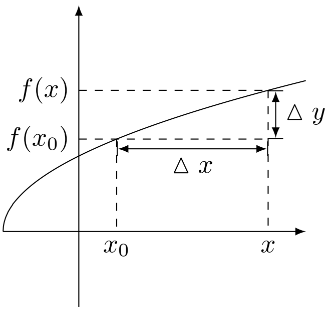
10 连续与间断点

10.1 定义

10.1.1 点连续

$$Def1: 设f(x)在x_0的某邻域内有定义，如果 \lim_{x\to x_0} = f(x_0)$$

$$则称f(x)在x_0处连续$$



$$\begin{cases} \triangle\ x = x - x_0 \\ \triangle\ y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \triangle\ x) - f(x_0) \end{cases} \end{cases}$$

$$Def2: 如果 \lim_{\triangle x\to 0} \triangle\ y = \lim_{\triangle x\to 0} [f(x_0 + \triangle\ x) - f(x_0)] = 0$$

$$则称f(x)在x_0处连续$$

10.1.2 区间连续

$$\forall x_0 \in [a,b] \begin{cases} \lim\limits_{x\to x_0} f(x) = f(x_0) \quad x_0 \in (a,b) \Leftrightarrow f(x_0) = \begin{cases} \lim\limits_{x\to x_0^-} f(x) = f(x_0^-) \\ \lim\limits_{x\to x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim\limits_{x\to x_0^+} f(x) = f(x_0^+) \quad x_0 = a\ (右连续) \\ \lim\limits_{x\to x_0^-} f(x) = f(x_0^-) \quad x_0 = b\ (左连续) \end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M > 0, x \in [a,b]$ 时，$|f(x)| \geqslant M$

最大值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

10.1.3 间断点

1,$f(x)$ 无定义

2, $\lim\limits_{x\to x_0} f(x)$ 不存在

3. $\lim\limits_{x\to x_0} f(x)$ 存在, 但 $\lim\limits_{x\to x_0} f(x) \neq f(x_0)$

第一类间断点：$f(x_0^+) = \lim\limits_{x\to x_0^+} f(x)$ 与 $f(x_0^-) = \lim\limits_{x\to x_0^-} f(x)$

第二类间断点：不是第一类的。

10.2 连续函数的运算

函数 $f(x), g(x)$ 在 $x = x_0$ 连续。

$$\lim_{x\to x_0} [f(x) \pm g(x)] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = f(x_0) \pm g(x_0)$$

$$\lim_{x\to x_0} [f(x) \cdot g(x)] = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = f(x_0) \cdot g(x_0)$$

$$\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)} = \frac{f(x_0)}{g(x_0)} \qquad (g(x_0) \neq 0)$$

反函数的连续性

若 $y = f(x)$ 在区间 I_x 上单调增加，且连续。

则 $y = f^-1(x)$ 在 $I_y = \{y|y = f(x), x \in I_x\}$ 上也为单调增加，连续

$$\text{复合函数,}\begin{cases} \text{内外都连续}\begin{cases} \lim\limits_{x\to x_0} g(x) = g(x_0) = u_0 \\ \lim\limits_{u\to u_0} f(x) = f(u_0) \\ \lim\limits_{x\to x_0} f[g(x)] = f[g(x_0)] = f(\lim\limits_{x\to x_o} g(x)) \end{cases} \\ \text{外连续}\begin{cases} x \to x_0 \begin{cases} \lim\limits_{x\to x_0} g(x) = u_0 \\ \lim\limits_{u\to u_0} f(x) = f(u_0) \\ \lim\limits_{x\to x_0} f[g(x)] = f(u_0) = f(\lim\limits_{x\to x_o} g(x)) \end{cases} \\ x \to \infty \begin{cases} \lim\limits_{x\to \infty} g(x) = u_0 \\ \lim\limits_{u\to u_0} f(x) = f(u_0) \\ \lim\limits_{x\to \infty} f[g(x)] = f(u_0) = f(\lim\limits_{x\to \infty} g(x)) \end{cases} \end{cases} \end{cases}$$

10.3 零点定理

2, 设 $f(x)$ 在 $[a, b]$ 上连续，且 $f(a) \cdot f(b) < 0$

则至少存在一点 $\xi \in (a, b)$ 使 $f(\xi) = 0$

10.4 介质定理

设 $f(x)$ 在 $[a, b]$ 上连续，且 $f(a) = A, f(b) = B$

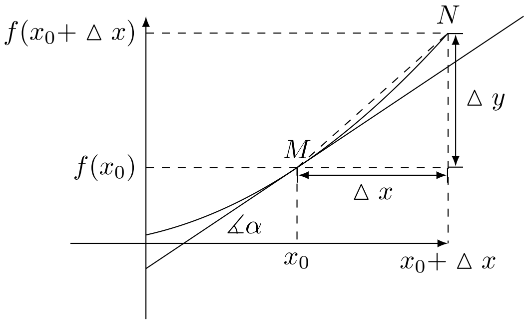
$\forall C \in (A, B)$, 至少有一点 $\xi, f(\xi) = C$

11 导数

11.1 定义

导数的概念从物理发展出来的。

$$v\left(t_{0}\right)=\lim_{\triangle t\to 0}\frac{s\left(t_{0}+\triangle t\right)-s\left(t_{0}\right)}{\triangle t}$$



$$NM\text{斜率}=\tan\beta=\frac{f\left(x_{0}+\triangle x\right)-f(x_{0})}{\triangle x}=\frac{\triangle y}{\triangle x}$$

$$\text{斜率}k=\tan\alpha=\lim_{\triangle x\to 0}\tan\beta=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\frac{f\left(x_{0}+\triangle x\right)-f(x_{0})}{\triangle x}$$

11.1.1 导数定义

$y=f(x)$ 在 x_0 的某邻域内有定义

给自变量的增量 $\triangle x,(x_0+\triangle x)$ 仍在定义域内

函数得到了相应增量 $\triangle y,\triangle y=f\left(x_0+\triangle x\right)$

如果 $\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$ 存在, 称 $y=f(x)$ 在 $x=x_0$ 处可导

(极限值为$y=f(x)$在$x=x_0$处导数)

$$\text{记 } y'|_{x=x_0}=f'(x_0)=\lim_{\triangle x\to 0}\tfrac{f(x_0+\triangle x)-f(x_0)}{\triangle x}=\lim_{\triangle x\to 0}\tfrac{\triangle y}{\triangle x}$$

$$\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\Leftrightarrow\lim_{\triangle x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

11.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。

$f'(x)=\lim\limits_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$

称 $f'(x)$ 为 $y=f(x)$ 在区间 I 上的导函数

11.1.3 闭区间可导定义

$$f(x)\text{ 在 }[a,b]\text{ 可导}\Leftrightarrow\begin{cases} f'(x_0) & x_0\in(a,b)\Leftrightarrow\begin{cases}\text{左导数}f'_-(x_0)=\lim\limits_{\Delta x\to 0^-}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\\ \text{右导数}f'_+(x_0)=\lim\limits_{\Delta x\to 0^+}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\end{cases}\\ f'_+(a) & x=a\\ f'_-(a) & x=b\end{cases}$$

11.1.4 导数与连续

$$f'(x)\text{存在}\Rightarrow f(x)\text{在}x=x_0\text{处连续} \tag{11.1.1}$$

11.2 幂数，指数，对数

$$(C)'=0 \tag{11.2.1}$$

$$(x^a)'=ax^{a-1} \tag{11.2.2}$$

$$(a^x)'=a^x\ln a \tag{11.2.3}$$

$$(e^x)'=e^x \tag{11.2.4}$$

$$(\log_a^x)'=\frac{1}{x\ln a} \tag{11.2.5}$$

$$(\ln x)'=\frac{1}{x} \tag{11.2.6}$$

11.3 三角函数

$$(\sin x)' = \cos x \tag{11.3.1}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \tag{11.3.2}$$

$$(\csc x)' = -\csc x \cot x \tag{11.3.3}$$

$$(\cos x)' = -\sin x \tag{11.3.4}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \tag{11.3.5}$$

$$(\sec x)' = \sec x \tan x \tag{11.3.6}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \tag{11.3.7}$$

$$(\tan x)' = \sec^2 x \tag{11.3.8}$$

$$(\arctan x)' = \frac{1}{1+x^2} \tag{11.3.9}$$

$$(\cot x)' = -\csc^2 x \tag{11.3.10}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2} \tag{11.3.11}$$

$$(\sinh x)' = \cosh x \tag{11.3.12}$$

$$(\cosh x)' = \sinh x \tag{11.3.13}$$

$$(\tanh x)' = \frac{1}{cosh^2 x} = 1 - \tanh^2 x \tag{11.3.14}$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}} \tag{11.3.15}$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}} \tag{11.3.16}$$

$$(\operatorname{arctanh} x)' = \frac{1}{1-x^2} \tag{11.3.17}$$

11.4 导数运算

$$u = u(x), v = v(x), \text{均在}x\text{点可导}, C\text{为常数}$$

$$(Cu(x))' = Cu'(x) \tag{11.4.1}$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \tag{11.4.2}$$

$$(u(x) \cdot v(x))' = u'(x)v(x) + v'(x)u(x) \tag{11.4.3}$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2} \tag{11.4.4}$$

11.5 反函数求导

如果函数 $y=f(x)$ 在区间 (a,b) 内单调可导，且 $f'(y)\neq 0$

$$\begin{cases}\alpha=\min\{f(a)+0,f(b-0)\}\\ \beta=\max\{f(a)+0,f(b-0)\}\end{cases}$$

则它的反函数 $x=f^{-1}(y)$ 在区间 (α,β) 内也可导

$$\left[f^{-1}(y)\right]'=\frac{1}{f'(x)}\Leftrightarrow\frac{dy}{\mathrm{d}x}=\frac{1}{\frac{dx}{\mathrm{d}y}} \tag{11.5.1}$$

11.6 复合函数求导

$$\text{设函数}\begin{cases}y=f(u)\text{在}U(u_0,\delta_0)\text{处有定义}\\ u=g(x)\text{在}U(x_0,\eta_0)\text{处有定义}\end{cases}$$
$$u_0=g(x_0),\text{且}f'(u)\text{和}g'(x)\text{都存在}$$
$$\text{则复合函数}F(x)=f\left[g(x)\right]\text{在点}x_0\text{可导, 且}$$

$$F'(x_0)=f'\left[g(x_0)\right]g'(x_0)\Leftrightarrow\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \tag{11.6.1}$$

11.7 高阶求导

$$Def:\begin{cases}\text{一阶导数} & y'\Leftrightarrow\frac{dy}{dx}\\ \text{二阶导数} & y''\Leftrightarrow\frac{d^2y}{dx^2}\\ \text{三阶导数} & y'''\Leftrightarrow\frac{d^3y}{dx^3}\\ \text{三阶以上 n 阶导数} & y^{(n)}\Leftrightarrow\frac{d^ny}{dx^n}\end{cases}$$

11.8 高阶求导公式

$$(e^x)^{(n)} = e^x \tag{11.8.1}$$

$$(a^x)^{(n)} = a^x\,(lna)^n \tag{11.8.2}$$

$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \tag{11.8.3}$$

$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{11.8.4}$$

$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{11.8.5}$$

$$(\sin x)^{(n)} = \sin(x+n\frac{\pi}{2}) \tag{11.8.6}$$

$$(\cos x)^{(n)} = \cos(x+n\frac{\pi}{2}) \tag{11.8.7}$$

$$[f(ax+b)]^{(n)} = a^n\cdot f^{(n)}(ax+b) \tag{11.8.8}$$

11.9 高阶求导运算法则

$$(u(x)\pm v(x))^{(n)} = u^{(n)}(x)\pm v^{(n)}(x) \tag{11.9.1}$$

$$\text{莱布紫泥公式}\qquad (uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)}\cdot v^(k) \tag{11.9.2}$$

11.10 隐函数求导

$F(x,y)=0, y=f(x)$

$F(x,f(x))\equiv 0$　　可以同时对两面求导

11.11 参数方程求导

$x=x(t), y=y(t)$

$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt}\left(\frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}\right)\cdot\frac{dt}{dx} = \frac{\frac{d^2y}{dt}\cdot\frac{dx}{dt}-\frac{d^2x}{dt}\cdot\frac{dy}{dt}}{(\frac{dx}{dt})^2}\cdot\frac{dt}{dx} = \frac{\frac{d^2y}{dt}\cdot\frac{dx}{dt}-\frac{d^2x}{dt}\cdot\frac{dy}{dt}}{(\frac{dx}{dt})^3}$$

12 微分

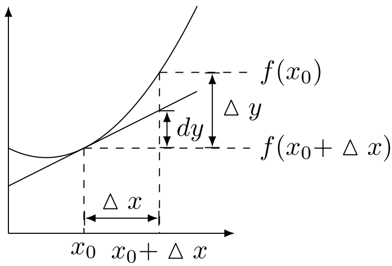
12.1 定义

设函数 $f(x)$ 在点 x_0 的一个邻域内有定义。$\triangle y = f(x_0 + \triangle x) - f(x_0)$

如果 $\triangle y$ 可以表示为 $\triangle y = A \triangle x + \circ(\triangle x)$ 其中 A 为与 $\triangle x$ 无关的常数

则称 $f(x)$ 在点 x_0 可微，$A \triangle x$ 称为 $f(x)$ 在点 x_0 处的微分。

记作:$dy = A \triangle x$



$$\text{可微} \Rightarrow \text{可导} \tag{12.1.1}$$

$$\text{可导} \nRightarrow \text{可微} \tag{12.1.2}$$

12.2 微分法则

12.2.1 核心根本

积分

$$dy = \; f'(x) \quad d \quad x$$

求导

12.2.2 四则运算

$$d\left(u \pm v\right) = du \pm dv \tag{12.2.1}$$

$$d(uv) = vdu + udv \tag{12.2.2}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + udv}{v^2} \tag{12.2.3}$$

12.2.3 复合运算

$$\text{可微}\begin{cases} y = f(u) \\ u = g(x) \end{cases} \Rightarrow \begin{cases} dy = f'(u)du \\ du = g'(x)dx \end{cases} \quad \text{则 } y = f(g(x)) \text{ 也可微}$$

$$\text{且 } dy = f'(u)du = f'(u)g(x)dx$$

$$u \text{ 是否为中间变量都成立，微分的不变性。}$$

12.2.4 近似计算公式

$$\triangle x \to 0, dy \approx \triangle y \begin{cases} dy = f'(x_0) \triangle x \\ \triangle y = f(x_0 + \triangle x) - f(x_0) \end{cases} \quad \begin{cases} f(x_0 + \triangle x) \approx f(x_0) + f'(x_0) \triangle x \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ x_0 = 0 \begin{cases} f(x) \approx f(0) + f'(0)x \\ \sqrt{n} \approx 1 + \frac{1}{n}x \\ \sin x \approx x \\ \tan x \approx x \\ e^x \approx 1 + x \\ \ln(1+n) \approx x \end{cases} \end{cases}$$

12.2.5 奇偶函数导数

$$\text{偶函数导数为奇函数 } f(x) = f(-x) \Leftrightarrow f'(x) = -f'(-x)$$
$$\text{奇函数导数为偶函数 } f(x) = -f(-x) \Leftrightarrow f'(x) = f'(-x)$$

12.2.6 区间恒为 0

$$\text{若} f'(x) \text{在区间恒为零，则} f(x) \text{在区间} I \text{上为一常数}$$
$$E\text{设} x_1, x_2 \text{为区间} I \text{内任意两点} x_1 < x_2$$
$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \equiv 0$$
$$f(x_2) \equiv f(x_1) = C$$

12.3 中值定理

12.3.1 费马引理

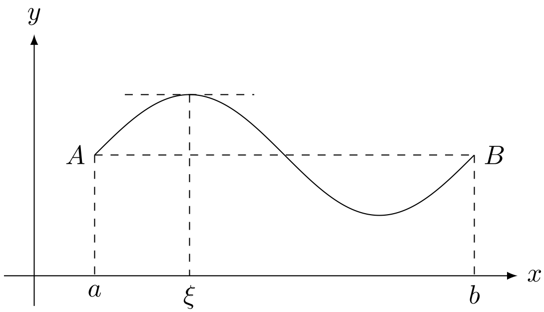
$$f(x) \qquad \forall x \in \mathring{U}(x_0) \begin{cases} f(x) \leqslant f(x_0) & f(x)\text{在} x_0 \text{处取极大值} \\ f(x) \geqslant f(x_0) & f(x)\text{在} x_0 \text{处取极小值} \end{cases}$$

$$\text{如果可导函数} y = f(x) \text{在} x_0 \text{取极值，则} f'(x_0) = 0 \tag{12.3.1}$$

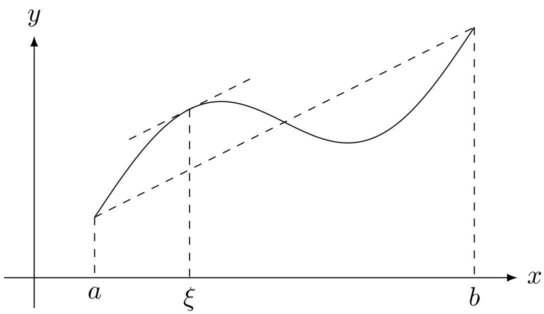
12.3.2 罗尔定理

$$\text{如果函数}f(x)\text{满}\begin{cases}\text{在闭区间 }[a,b]\text{ 上连续}\\ \text{在开区间 (a,b) 可导}\\ f(a)=f(b)\end{cases}$$

$$\text{则至少有一点}\xi\in(a,b),f'(\xi)=0 \tag{12.3.2}$$



12.3.3 拉格朗日定理 (微分中值定理)

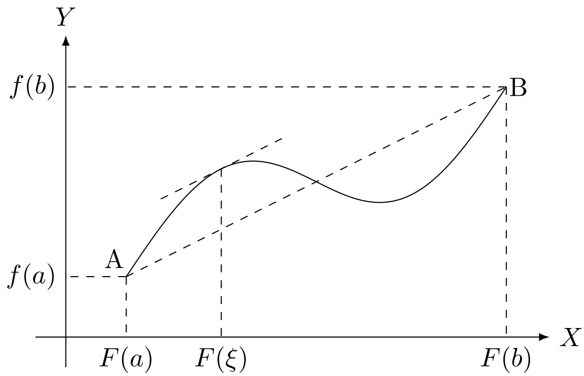


$$\text{如果函数}f(x)\text{满}\begin{cases}\text{在闭区间 }[a,b]\text{ 上连续}\\ \text{在开区间 (a,b) 可导}\end{cases}$$
$$\text{则至少有一点}\xi\in(a,b)$$

$$f'(\xi)=\frac{f(b)-f(a)}{b-a}\Leftrightarrow f(b)-f(a)=f'(x)(b-a) \tag{12.3.3}$$

$$\text{在区间 } [x, x+\triangle x] \text{ 用拉格朗日定理。}$$
$$f(x+\triangle x) - f(x) = f'(\xi)\triangle x$$
$$\xi \in (x, x+\triangle x) \text{ 记作: } \xi = x + \theta \triangle x \qquad 0 < \theta < 1$$
$$f(x+\triangle x) - f(x) = f(x+\theta\triangle x)\triangle x$$
$$\triangle y = f(x+\theta\triangle x)\triangle x$$

12.3.4 柯西定理



$$\text{如果函数} f(x) \text{满} \begin{cases} \text{在闭区间 } [a, b] \text{ 上连续} \\ \text{在开区间 (a,b) 可导} \\ F'(x) \neq 0 \end{cases} \qquad \text{参数方程} (a \leqslant x \leqslant b) \begin{cases} X = F(x) \\ Y = f(x) \end{cases}$$

$$\text{至少有一点}, \xi \qquad \frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)} \tag{12.3.4}$$

$$\text{切线斜率} = \frac{dY}{dX} = \frac{df(x)}{dF(x)} = \frac{f'(x)}{F'(x)} \Rightarrow x = \xi \text{时斜率} = \frac{f'(\xi)}{F'(\xi)}$$
$$AB\text{的斜率} = \frac{f(b) - f(a)}{F(b) - F(a)}$$

12.3.5 三个定理关系

$$\frac{f'(\xi)}{F'(\xi)} = \frac{f(b) - f(a)}{F(b) - F(a)}, (F(x) = x) \Rightarrow f'(\xi) = \frac{f(b) - f(a)}{b - a}, (f(b) = f(a)) \Rightarrow f'(\xi) = 0$$

12.4 洛必达法则

$$\text{未定型,}\quad \frac{0}{0},\quad \frac{\infty}{\infty},\quad 0^0,\quad 1^\infty,\quad \infty^0,\quad \infty-\infty$$

$$\lim_{x\to x_0}\tfrac{f(x)}{F(x)}=\lim_{x\to x_0}\tfrac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0} f(x)=0, \lim\limits_{x\to x_0} F(x)=0\\ f(x),F(x)\text{在}x_0\text{的某去心邻域内可导，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\tfrac{f'(x)}{F'(x)}\text{存在，或无穷小。则}\end{cases}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)} \tag{12.4.1}$$

$$\lim_{x\to \infty}\tfrac{f(x)}{F(x)}=\lim_{x\to \infty}\tfrac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0} f(x)=0, \lim\limits_{x\to x_0} F(x)=0\\ \exists N\text{ 当 }|x|>N\text{，时}f'(x),F'(x)\text{存在，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\tfrac{f'(x)}{F'(x)}\text{存在，或为无穷大。则}\end{cases}$$

12.5 泰勒公式

$$\begin{aligned}f(x_0+\triangle x)-f(x_0)&=f'(x_0)\triangle x+\circ(\triangle x)\\ x_0+\triangle=x&\qquad \triangle x=x-x_0\\ f(x)-f(x_0)&=f'(x_0)(x-x_0)+\circ(\triangle x)\\ f(x)&=f'(x_0)(x-x_0)+f(x_0)+\circ(\triangle x)\\ f(x)&\approx f'(x_0)(x-x_0)+f(x_0)\end{aligned}\qquad \begin{aligned}P(x_0)&=f(x_0)\\ P'(x_0)&=f'(x_0)\end{aligned}$$

12.5.1 泰勒多项式

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n\text{去近似某个多项式}$$

$$\begin{cases} P_n(x_0) & = f(x_0) = a_0 \\ P_n'(x_0) & = f'(x_0) = a_1 \\ P_n''(x_0) & = f''(x_0) = a_2 \cdot 2! \\ & \vdots \\ P_n^{(n-1)}(x_0) & = f^{(n-1)}(x_0) = a_{n-1} \cdot (n-1)! \\ P_n^{(n)}(x_0) & = f^{(n)}(x_0) = a_n \cdot n! \end{cases} \Rightarrow \begin{cases} a_0 & = f_n(x_0) \\ a_1 & = f_n'(x_0) \\ a_2 & = \frac{f_n''(x_0)}{2!} \\ & \vdots \\ a_{n-1} & = \frac{f_n^{(n-1)}(x_0)}{(n-1)!} \\ a_n & = \frac{f_n^{(n)}(x_0)}{n!} \end{cases}$$

$$P_n(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \cdots \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^n(x_0)}{n!}(x-x_0)^n$$

$$f(x) \approx P_n(x)$$

12.5.2 泰勒中值定理

如果$f(x)|x_0 \in (a,b)$内有$(n+1)$阶导则

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n + R_n(x)$$

拉格朗日余项

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \qquad \{\xi \in (x, x_0)\} \tag{12.5.1}$$

皮亚诺于项

$$R_n(x) = \circ(|x-x_0|^n) \tag{12.5.2}$$

$$f(x) \approx P_n(x) \text{ 误差为 } R_n(x)$$

12.6 麦克劳林公式

$$x_0 = 0$$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^n(0)}{n!}x^n + \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}. \ 0 < \theta < 1 \\ \circ(|x|^n) \end{cases}$$

12.6.1 常用的麦克劳林展开

$$
\begin{aligned}
e^x &= 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0 < \theta < 1 \\ \circ(|x|^n) \end{cases} \\
\sin x &= 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^{n-1}\frac{1}{(2n-1)!}x^{2n-1} + R_n(x) \\
\cos x &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots + (-1)^n\frac{1}{(2n)!}x^{2n} + R_n(x) \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n + R_n(x) \\
\ln(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots - \frac{1}{n}x^n + R_n(x) \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 \cdots \frac{A_\alpha^n}{n!}x^n + R_n(x)
\end{aligned}
$$

13 不定积分

13.1 概念

13.1.1 原函数

$$\forall x \in I,\ F'(x) = f(x), \quad F(x)\text{为}f(x)\text{的一个原函数}$$

$$\text{函数}f(x)\text{在区间}I\text{上连续一定有}F(x),\text{使}F'(x) = f(x) \tag{13.1.1}$$

13.1.2 不定积分

区间 I 上，$f(x)$ 的带有任意常数的原函数, 称为 $f(x)$ 在区间 I 上的不定积分。
记作:

$$\int f(x)\ dx \quad \begin{cases} \int & \text{积分符号} \\ f(x) & \text{被积函数} \\ f(x)\ dx & \text{被积表达式} \\ x & \text{积分变量} \end{cases}$$

如果$F(x)$是$f(x)$的一个原函数

$$\int f(x)dx = F(x) + C$$

13.1.3 不定积分性质

$$\left[\int f(x)\ dx\right]' = f(x)$$

$$d\left[\int f(x)\ dx\right] = f(x)\ dx$$

$$\int\ dF(x) = \int F'(x)\ dx = F(x) + C$$

13.2 积分运算

$$\int [f(x)+g(x)]\ dx = \int f(x)\ dx + \int f(x)\ dx \tag{13.2.1}$$

$$\int kf(x)\ dx = k\int f(x)\ dx \quad (k\text{为常数}) \tag{13.2.2}$$

$$\int f\left[\varphi(x)\right]\varphi'(x)\ dx \xlongequal{u=\varphi(x)} \left[\int f(u)du\right]_{x=\varphi(u)} = F\left[\varphi(x)\right]+C \tag{13.2.3}$$

$$\int f(x)\ dx \xlongequal[\varphi'(t)\neq 0]{x=\varphi(t)} \left[\int f\left[\varphi(t)\right]\varphi'(t)\ dt\right]_{t=\varphi^{-1}(x)} \tag{13.2.4}$$

$$\int f(x)\ dx = \int f(x)\ d(x+C) \tag{13.2.5}$$

13.2.1 分部积分法

$$\int u\ dv = uv - \int v\ du \Leftrightarrow \int uv'\ dx = uv - \int u'v\ dx \tag{13.2.6}$$

13.3 有理函数积分

13.3.1 普通多项式

$$\frac{P(x)}{Q(x)} \qquad P(x),Q(x)\text{是 } x \text{ 多项式, 且没有公因子，称为有理分式}$$

$$\text{有理分式}\begin{cases}\text{真分式} & P(x)\text{次数} < Q(x)\text{次数}\\ \text{假分式} & P(x)\text{次数} \geqslant Q(x)\text{次数}\end{cases}$$

$$\text{如果真分式中}Q(x)=Q_1(x)\cdot Q_2(x),\text{其中}Q_1(x),Q_2(x)\text{都为多项式}$$

$$\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)}\cdot\frac{P_2(x)}{Q_2(x)} \tag{13.3.1}$$

$$\text{假分式} = \text{多项式} + \text{真分式}$$

$$\text{最简分式} \quad \frac{A}{x-a} \quad \frac{A}{(x-a)^2} \quad \frac{Nx+M}{x^2+px+q} \quad \frac{Nx+m}{(x^2+px+q)^k}$$

13.3.2 三角函数多项式

$$\text{三角有理分式: } R(\sin x, \cos x)$$

$$\text{万能代换: } \tan\frac{x}{2} = u, x = 2\arctan u, dx = \frac{2\ du}{1+u^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = 2\frac{\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}(1-\tan^2\frac{x}{2}) = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-u^2}{1+u^2}$$

$$\int R(\sin x, \cos x)\ dx = \int R(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2})\frac{2}{1+u^2}\ du = \int Y(u)\ du$$

$$Y(u) \text{ 是 } u \text{ 的有理函数}$$

13.4 积分公式

13.4.1 幂数，指数，对数

$$\int k\ dx = kx + C \tag{13.4.1}$$

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C \tag{13.4.2}$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C \tag{13.4.3}$$

$$\int e^x\ dx = e^x + C \tag{13.4.4}$$

$$\int \frac{1}{x}\ dx = \ln|x| + C \tag{13.4.5}$$

$$\int \ln x\ dx = x\ln x - x + C \tag{13.4.6}$$

13.4.2 三角函数

$$\int \sin x \; dx = -\cos x + C \tag{13.4.7}$$

$$\int \cos x \; dx = \sin x + C \tag{13.4.8}$$

$$\int \sec x \tan x \; dx = \sec x + C \tag{13.4.9}$$

$$\int \csc x \cot x \; dx = -\csc x + C \tag{13.4.10}$$

$$\int \sec^2 x \; dx = \tan x + C \tag{13.4.11}$$

$$\int \csc^2 x \; dx = -\cot x + C \tag{13.4.12}$$

$$\int \sinh x \; dx = \cosh x + C \tag{13.4.13}$$

$$\int \cosh x \; dx = \sinh x + C \tag{13.4.14}$$

$$\int \tan x \; dx = -\ln|\cos x| + C \tag{13.4.15}$$

$$\int \csc x \; dx = \begin{cases} \ln\left|\tan \frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases} \tag{13.4.16}$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C \tag{13.4.17}$$

$$\int \arccos x \; dx = x \arccos x - \sqrt{1 - x^2} + C \tag{13.4.18}$$

$$\int \arctan x \; dx = x \arctan x - \frac{1}{2}\ln(1 + x^2) + C \tag{13.4.19}$$

13.4.3 分式

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C \tag{13.4.20}$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan\frac{x}{a} + C \tag{13.4.21}$$

$$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \tag{13.4.22}$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \tag{13.4.23}$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} \arcsin x + C \\ -\arccos x + C_1 \end{cases} \tag{13.4.24}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \begin{cases} \arcsin \frac{x}{a} x + C \\ -\arccos \frac{x}{a} + C_1 \end{cases} \tag{13.4.25}$$

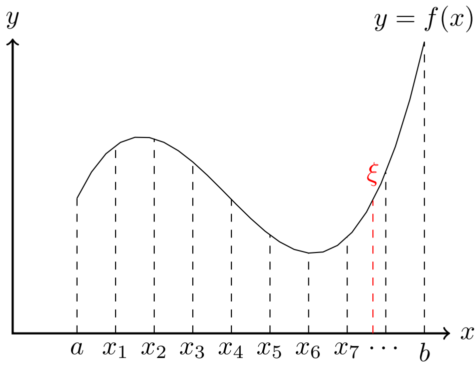
$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{13.4.26}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2}) + C \tag{13.4.27}$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \operatorname{arcsec} x + C \tag{13.4.28}$$

14 定积分

14.1 定积分的定义



$$\begin{cases} [a,b]\text{ 内任找}n-1\text{个点,分成 }n\text{ 个区间} \\ a=x_0<x_1<x_2\cdots x_{n-1}<x_n=b \\ [x_0,x_1],[x_1,x_2]\cdots[x_{n-1},x_n] \\ \text{分成 }n\text{ 个曲边梯形,}[x_{i-1},x_i]\text{ 为第 }i\text{ 个} \\ \text{面积 }\triangle S_i\text{,对应底 }\triangle x_i=x_i-x_{i-1} \\ \forall\xi_i\in[x_{i-1},x_i],\triangle S_i\approx f(\xi_i)\triangle x_i \\ S=\triangle S_1+\triangle S_2\cdots\triangle S_n=\sum_{i=1}^{n}f(\xi_i)\triangle x_i \end{cases}$$

$$\lambda=\max\{\triangle x_1,\triangle x_2,\cdots,\triangle x_n\},\text{当}\lambda\to 0\text{时}$$

$$S=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\triangle x_i$$

S是一个定数，则称$f(x)$在$[a,b]$上可积，S称为$f(x)$在$[a,b]$上的定积分记作：

$$\int_a^b f(x)\,dx\begin{cases} f(x)\,dx & \text{被积表达式}\begin{cases} f(x) & \text{被积函数}\end{cases} \\ x & \text{积分变量} \\ [a,b] & \text{积分区间}\begin{cases} a & \text{积分下限} \\ b & \text{积分上限}\end{cases} \end{cases}$$

$$\int_a^b f(x)\,dx\triangleq\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\triangle x_i$$

14.2 可积的充分条件

$$\text{如果}f(x)\text{在}[a,b]\text{上连续，则}f(x)\text{在}[a,b]\text{上可积} \tag{14.2.1}$$

$$\text{如果}f(x)\text{在}[a,b]\text{上有界，且至多有有限个间断点，则}f(x)\text{在}[a,b]\text{上可积} \tag{14.2.2}$$

14.3 定积分的性质

$$a<b<c, k\text{为常数}$$

$$\int_a^a f(x)\ dx=0 \tag{14.3.1}$$

$$\int_a^b\ dx=b-a \tag{14.3.2}$$

$$\int_a^b f(x)\ dx=-\int_b^a f(x)\ dx \tag{14.3.3}$$

$$\int_a^c f(x)\ dx=\int_a^b f(x)\ dx+\int_b^c f(x)\ dx \tag{14.3.4}$$

$$\int_a^b f(x)\ dx=\int_a^c f(x)\ dx+\int_c^b f(x)\ dx \tag{14.3.5}$$

$$\int_a^b kf(x)\ dx=k\int_a^b f(x)\ dx \tag{14.3.6}$$

$$\int_a^b f(x)\pm g(x)\ dx=\int_a^b f(x)\ dx\pm\int_a^b g(x)\ dx \tag{14.3.7}$$

$$f(x)\geqslant 0\quad\Rightarrow\int_a^b f(x)\ dx\geqslant 0 \tag{14.3.8}$$

$$f(x)\geqslant g(x)\quad\Rightarrow\int_a^b f(x)\ dx\geqslant\int_a^b g(x)\ dx \tag{14.3.9}$$

$$\left|\int_a^b f(x)\ dx\right|\leqslant\int_a^b |f(x)|\ dx \tag{14.3.10}$$

14.4 积分估值公式

$$M\text{为区间 }[a,b]\text{ 最大值}, m\text{为区间 }[a,b]\text{ 最小值}, a<b$$

$$m(b-a)\leqslant\int_a^b f(x)\ dx\leqslant M(b-a) \tag{14.4.1}$$

14.5 积分中值定理

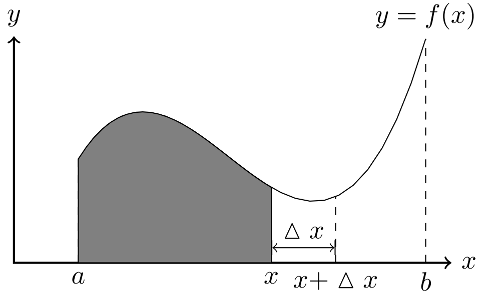
$$f(x)\text{是 }[a,b]\text{ 上的连续函数，则}, \exists\xi\in[a,b], a<b\text{使}$$

$$\int_a^b f(x)\ dx=f(\xi)(b-a) \tag{14.5.1}$$

$$f(\xi)=\frac{1}{b-a}\int_a^b f(x)\ dx\qquad\text{称为均值}$$

14.6 积分上限函数

14.6.1 定义



$x \in [a,b]$, $[a,x]$ 对应曲边梯形
$\int_a^x f(x)\ dx = \int_a^x f(u)\ du$
$\phi(x) \triangleq \int_a^x f(u)\ du$
$\phi(x)$是 $[a,b]$ 上函数称为积分上限函数

14.6.2 性质

$$\phi'(x) = \frac{d}{dx}\left[\int_a^x f(u)\ du\right] = f(x) \tag{14.6.1}$$

$$\frac{d}{dx}\left[\int_a^{\psi(x)} f(u)\ du\right] = f(\psi(x))\psi'(x) \tag{14.6.2}$$

$$\frac{d}{dx}\left[\int_{v(x)}^{\psi(x)} f(u)\ du\right] = f\left[\psi(x)\right]\psi'(x) - f\left[v(x)\right]v'(x) \tag{14.6.3}$$

若 $f(x)$ 在 $[a,b]$ 上连续，则 $f(x)$ 必存在原函数,$\phi(x) = \int_a^x f(u)\ du$
即为 $f(x)$ 在 $[a,b]$ 上的一个原函数

$$\int f(x)\ dx = \int_a^x f(u)\ du + C$$

15 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{15.0.1}$$

$$A_N = \sum_{k=0}^{n} q^k \qquad q \cdot A_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - q \cdot A_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1-q^{n+1}}{1-q}$$

$$\log_{10} x = \lg_x \tag{15.0.2}$$

$$\log_e x = \ln_x \tag{15.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{15.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{15.0.5}$$

$$\log_b x^n = n \log_b x \tag{15.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{15.0.7}$$

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a-b) \sum_{m=0}^{n-1} \left(a^{n-m} b^m\right) = (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)$$

16 证明

16.1 第 1章

1.2.4

$$
\begin{aligned}
\sinh x \cosh x &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{e^{2x} - e^{-2x}}{2}\right) \\
&= \frac{1}{2}\sinh(2x) \\
\sinh(2x) &= 2\sinh x \cosh x
\end{aligned}
$$

1.2.5

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\
&= e^x \times e^{-x} \\
&= 1
\end{aligned}
$$

1.2.6

$$
\begin{aligned}
\cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh(2x)
\end{aligned}
$$

1.2.7

$$
\begin{aligned}
\cosh(2x) &= \cosh^2 x + \sinh^2 x \\
&= \sinh^2 x + 1 + \sinh^2 x \\
&= 2\sinh^2 x + 1 \\
\cosh x &= 2\sinh^2 \frac{x}{2} + 1
\end{aligned}
$$

1.1.17

$$\begin{aligned}\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}&=\sqrt{\frac{1-\cos x}{1+\cos x}}\\&=\sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1+\cos x}{1+\cos x}}\\&=\sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}}\\&=\frac{\sin x}{1+\cos x}\\&=\frac{1-cosx}{\sin x}\\&=\csc x-\cot x\end{aligned}$$

16.2 第 5章

5.2.1

$$\begin{aligned}&\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2\\f(x_2)-f(x_1)&=f'(\xi)(x_2-x_1)\qquad\xi\in(x_1,x_2)\subset[a,b]\\f'(\xi)&>0,\ (x_2-x_1)>0\\f(x_2)-f(x_1)&>0\\f(x_2)&>f(x_1)\end{aligned}$$

5.2.2

$$\begin{aligned}&\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2\\f(x_2)-f(x_1)&=f'(\xi)(x_2-x_1)\qquad\xi\in(x_1,x_2)\subset[a,b]\\f'(\xi)&<0,\ (x_2-x_1)>0\\f(x_2)-f(x_1)&<0\\f(x_2)&<f(x_1)\end{aligned}$$

5.2.3

$$\text{设}\forall\ x_1,x_2\in[a,b]\,,x_1<x_2,x_0=\tfrac{x_1+x_2}{2},x_0-x_1=x_2-x_0=h$$
$$\varphi=\ f(x_0)-f(x_1)=f'(\xi_1)(x_0-x_1)\qquad\xi_1\in(x_1,x_0)$$
$$\psi=f(x_2)-f(x_0)=f'(\xi_2)(x_2-x_0)\qquad\xi_2\in(x_0,x_2)$$

$$\psi-\varphi=f(x_2)+f(x_1)-2f(x_0)=[f'(\xi_2)-f'(\xi_1)]\,h$$
$$=f''(\xi)(\xi_2-\xi_1)h$$

因为 $f''(x)>0, f''(\xi)>0, h=x_0-x_1>0$

$$f(x_2)+f(x_1)-2f(x_0)>0$$
$$f(x_2)+f(x_1)>2f(x_0)$$
$$f(x_0)<\frac{f(x_2)+f(x_1)}{2}$$
$$f(\frac{x_1+x_2}{2})<\frac{f(x_2)+f(x_1)}{2}$$

5.3.1

$$\triangle\ s=\widehat{M_0M'}-\widehat{M_0M}=\widehat{MM'},\quad |MM'|^2=(\triangle\ x)^2+(\triangle\ y)^2,\quad \lim_{M'\to M}\frac{\left|\widehat{MM'}\right|}{|MM'|}=1$$

$$\left(\frac{\triangle\ s}{\triangle\ x}\right)^2=\left|\frac{\widehat{MM'}}{\triangle\ x}\right|^2=\left(\frac{\widehat{MM'}}{|MM'|}\right)^2\cdot\left(\frac{|MM'|}{\triangle\ x}\right)^2$$
$$=\left(\frac{\widehat{MM'}}{|MM'|}\right)^2\cdot\frac{(\triangle\ x)^2+(\triangle\ y)^2}{(\triangle\ x)^2}$$
$$=\left(\frac{\widehat{MM'}}{|MM'|}\right)^2\cdot\left[1+\left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$\lim_{\triangle x\to 0}\left(\frac{\triangle\ s}{\triangle\ x}\right)^2=\lim_{\triangle x\to 0}\left(\frac{\widehat{MM'}}{|MM'|}\right)^2\cdot\lim_{\triangle x\to 0}\left[1+\left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$(\triangle\ x\to 0,\triangle\ M'\to M)\qquad =\lim_{M'\to M}\left(\frac{\widehat{MM'}}{|MM'|}\right)^2\cdot\lim_{\triangle x\to 0}\left[1+\left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$\left(\frac{ds}{dx}\right)^2=1\cdot(1+(y')^2)$$
$$\frac{ds}{dx}=\sqrt{1+(y')^2}=\sqrt{1+[f'(x)]^2}$$
$$ds=\sqrt{1+[f'(x)]^2}dx=\sqrt{(dx)^2+(dy)^2}$$

5.3.2

$$
\begin{aligned}
\left|\frac{d\alpha}{ds}\right| &= \left|\frac{d\alpha}{dx}\cdot\frac{dx}{ds}\right| \\
&= \left|\frac{d\arctan y'}{dx}\cdot\frac{1}{\sqrt{1+(y')^2}}\right| \\
&= \left|\frac{y''}{1+(y')^2}\cdot\frac{1}{\sqrt{1+(y')^2}}\right| \\
&= \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}}
\end{aligned}
$$

5.3.3

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\psi'(t)}{\phi'(t)} \\
\frac{d^2y}{dx^2} &= \frac{d\frac{\psi'(t)}{\phi'(t)}}{dt}\cdot\frac{dt}{dx} \\
&= \frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left[\phi'(t)\right]^2}\cdot\frac{1}{\phi'(t)} \\
&= \frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left[\phi'(t)\right]^3}
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{d\alpha}{ds}\right| &= \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left[\phi'(t)\right]^3}\cdot\frac{1}{\left\{1+\left[\frac{\psi'(t)}{\phi'(t)}\right]^2\right\}^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left\{\left|\psi'(t)\right|^2+\left[\phi'(t)\right]^2\right\}^{\frac{3}{2}}}
\end{aligned}
$$

16.3 第 9章

9.1.1

$$\text{反设} \lim_{n\to\infty} x_n = a, \ \lim_{n\to\infty} x_n = b, \text{且} a < b$$

$$\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1, \ n > N_1, \ |x_n - a| < \frac{b-a}{3} \\ \exists N_2, \ n > N_2, \ |x_n - b| < \frac{b-a}{3} \end{cases}$$

$$N = \max\{N_1, N_2\}, \ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}$$

$$\begin{aligned} b - a &= |(x_n - a) - (x_n - b)| \\ &\leqslant |x_n - a| + |x_n - b| \\ &< \frac{b-a}{3} + \frac{b-a}{3} \\ &< \frac{2(b-a)}{3} \end{aligned}$$

9.1.2

$$\varepsilon = 1, \ \exists N > 0, \ \text{当 n>N 时} \, |X_n - a| < 1$$
$$|X_n| = |(X_n - a) + a|$$
$$\leqslant |x_n - a| + |a|$$
$$\leqslant 1 + |a|$$
$$M = \max\{|X_n|, |X_2|, \ldots, |X_n|, 1 + |a|\}$$
$$\forall n, \ |X_n| \leqslant M$$

9.1.4

1

由于 $\lim_{n\to\infty} x_n = a$, 且$a > 0$
$\varepsilon = \frac{a}{2}, \ \exists N > 0, n > N$
$|x_n - a| < \varepsilon$
$|x_n - a| < \frac{a}{2}$
$-\frac{a}{2} < x_n - a < \frac{a}{2}$
$\frac{a}{2} < x_n < 1$

2

用反证法，反设 $a<0$. 从某项起 $x_n<0$ 矛盾

9.1.5

$$x_n = b_n - a_n$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} x_n = b - a > 0$$
$$\lim_{n\to\infty} x_n > 0$$
$$b_n - a_n = x_n > 0$$
$$b_n > a_n$$

9.2.1

$$\lim_{x\to x_0} f(x)\text{存在} \Rightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$$

$$\text{设} \lim_{x\to x_0} = A$$
$$0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon$$
$$0<|x-x_0|<\delta \Leftrightarrow x\in \mathring{U}(x_0,\delta)$$
$$\begin{cases} \text{当}x_0<x<x_0+\delta\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^+} f(x)=A \\ \text{当}x_0-\delta<x<x_0\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^-} f(x)=A \end{cases}$$
$$\lim_{x\to x_0^+} f(x) = A = \lim_{x\to x_0^-} f(x)$$

$$\lim_{x\to x_0} f(x)\text{存在} \Leftarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$$

$$A=\begin{cases} \lim\limits_{x\to x_o^+}, \forall\varepsilon>0, \exists\delta_1>0, x_0<x<x_0+\delta_1, |f(x)-A|<\varepsilon \\ \lim\limits_{x\to x_o^-}, \forall\varepsilon>0, \exists\delta_2>0, x_0-\delta_2<x<x_0, |f(x)-A|<\varepsilon \end{cases}$$
$$\delta=\min\{\delta_1,\delta_2\}$$
$$0<|x-x_0|<\delta \begin{cases} x>x_0, x_0<x<x_0+\delta\leqslant x_0+\delta_1,\ |f(x)-A|<\varepsilon \\ x<x_0, x_0-\delta_2\leqslant x_0+\delta<x<x_0,\ |f(x)-A|<\varepsilon \end{cases}$$
$$\lim_{x\to x_0} f(x) = A$$

9.3.1

$$\lim_{x\to x_o} f(x) = A \Rightarrow \begin{cases} \alpha\text{为}x\to x_0\text{时的无穷小} \\ f(x)=\alpha+A \end{cases}$$

$$\text{设} \lim_{x\to x_o} f(x) = A,\ \text{记} f(x) - A = \alpha$$

$$\text{只需证 } \alpha \text{ 为无穷小。}$$

$$\forall\varepsilon > 0, \exists\delta > 0, \text{当} 0 < |x - x_0| < \delta, \text{时} |f(x) - A| < \varepsilon$$

$$\text{即 } |\alpha - 0| < \varepsilon$$

$$\alpha \text{为} x \to x_0 \text{时的无穷小}$$

$$\lim_{x\to x_o} f(x) = A \Leftarrow \begin{cases} \alpha \text{为} x \to x_0 \text{时的无穷小} \\ f(x) = \alpha + A \end{cases}$$

$$\forall\varepsilon > 0,\ \exists\delta > 0,\ \text{当} 0 < |x - x + 0| < \delta,\ |\alpha| < \varepsilon$$

$$\text{即} |f(x) - A| < \varepsilon \lim_{x\to x_0} f(x) = A$$

9.3.2

$$\text{设} \lim_{x\to x_0} f(x) = \infty$$

$$\text{对 } f(x) \text{ 为 } x \to \text{ 时无穷大}$$

$$\text{对于 } M = \tfrac{1}{\varepsilon}.\ \text{存在 } \delta > 0$$

$$\text{当 } 0 < |x - x_0| < \delta \text{ 时}$$

$$|f(x)| > M = \tfrac{1}{\varepsilon}$$

$$\left|\tfrac{1}{f(x)}\right| < \varepsilon$$

$$\tfrac{1}{f(x)} \text{ 为 } x \to x_0 \text{ 时的无穷小}$$

9.4.2

$$\begin{aligned} f(x)g(x) &= [A + \alpha]\,[B + \beta] \\ &= AB + A\beta + B\alpha + \beta\alpha \\ &= AB + \gamma \qquad\qquad (\gamma \text{为无穷小}) \end{aligned}$$

$$\lim\,[f(x)g(x)] = AB + \gamma = \lim f(x) \lim g(x)$$

9.4.8

$$\forall\varepsilon > 0$$

$$|x_n - a| < \varepsilon \qquad \forall n > N_1$$

$$|y_n - a| < \varepsilon \qquad \forall n > N_2$$

$$\text{令} N = \max\{N_1, N_2, N_0\},\ \text{则当} n > N \text{时有}$$

$$a - \varepsilon < x_n \leq z_n \leq y_n < a + \varepsilon$$

$$|z_n - a| < \varepsilon$$

$$\lim_{n\to\infty} z_n = a$$

9.4.9

$$
\begin{aligned}
|f(x)-\sin x_0| &= |\sin x-\sin x_0| \\
&= \left|2\cos(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\frac{|x-x_0|}{2}=|x-x_0|
\end{aligned}
$$

$$
\forall\varepsilon,\exists\delta=\varepsilon,\text{当}0<|x-x_0|<\delta\text{时}
$$

$$
|\sin x-\sin x_0|\leqslant|x-x_0|<\varepsilon
$$

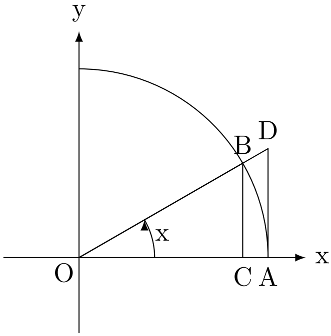
9.4.10

$$
\begin{aligned}
|f(x)-\cos x_0| &= |\cos x-\cos x_0| \\
&= \left|-2\sin(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\frac{|x-x_0|}{2}=|x-x_0|
\end{aligned}
$$

$$
\forall\varepsilon,\exists\delta=\varepsilon,\text{当}0<|x-x_0|<\delta\text{时}
$$

$$
|\cos x-\cos x_0|\leqslant|x-x_0|<\varepsilon
$$

9.4.11



$$OB = OA = 1$$

$$\triangle\ AOB \leqslant \text{扇形面积} \leqslant \triangle\ AOD$$

$$\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x$$

$$\sin x \leqslant x \leqslant \tan$$

$$1 \geqslant \frac{\sin x}{x} \geqslant \cos x$$

$$\lim_{x\to 0} 1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant \lim_{x\to 0}\cos x$$

$$1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant 1$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

9.4.12

$$|1-\cos x| = 1-\cos x = 2\sin^2\frac{x}{2} \leqslant 2\left(\frac{x}{2}\right)^2$$

$$0 \leqslant 1-\cos x \leqslant \frac{x^2}{2}$$

$$\lim_{x\to 0} 0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant \lim_{x\to 0}\frac{x^2}{2}$$

$$0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant 0$$

$$\lim_{x\to 0}(1-\cos x) = 0$$

$$\lim_{x\to 0}\cos x = 1$$

9.4.13

$$\begin{aligned}\lim_{x\to0}\frac{\tan x}{x}&=\lim_{x\to0}\frac{\sin x}{x}\frac{1}{\cos x}\\&=\lim_{x\to0}\frac{\sin x}{x}\lim_{x\to0}\frac{1}{\cos x}\\&=1\end{aligned}$$

9.4.14

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos x}{\frac{1}{2}x^2}&=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}\\&=\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\\&=1\end{aligned}$$

9.4.15

$$x=\sin t,\ t=\arcsin x$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{\arcsin x}{x}=\lim_{x\to0}\frac{t}{\sin t}=1$$

9.4.16

$$x=\tan t,\ t=\arctan x$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{\arctan x}{x}=\lim_{t\to0}\frac{t}{\tan t}=1$$

9.4.17

$$\lim_{x\to0}\frac{\ln(1+x)}{x}=\lim_{x\to0}\ln(1+x)^{\frac{1}{x}}=\ln e=1$$

9.4.18

$$e^x-1=t,\ x=\ln(t+1)$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{e^x-1}{x}=\lim_{t\to0}\frac{t}{\ln(t+1)}=1$$

9.4.19

$$\lim_{x\to0}\frac{(1+x)^n-1}{nx}=\lim_{x\to0}\left(\frac{e^{n\ln(1+x)}-1}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right)=1$$

9.4.21

$$
\begin{aligned}
x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m}\left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0\left(\frac{1}{n}\right)^0 + C_n^1\left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{n!}{m!\,(n-m)!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
x_{n+1} &= 1+1+\sum_{m=2}^{n+1}\frac{1}{m!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \qquad \text{单调增加} \\
x_n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{n^2} = 1+\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} \\
&< 1+\frac{1}{1-\frac{1}{2}} \\
&< 3 \qquad \text{有界}
\end{aligned}
$$

16.4 第 11章

11.1.1

$$
\begin{aligned}
f'(x_0) &= \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} \qquad \text{因为极限存在与无穷小的关系} \\
\frac{\triangle y}{\triangle x} &= f'(x_0) + \alpha \qquad \alpha\text{为} \triangle x \to 0\text{时的无穷小} \\
\triangle y &= f'(x_0) \triangle x + \alpha \triangle x \\
\lim_{\triangle x \to 0} \triangle y &= \lim_{\triangle x \to 0} [f'(x_0) \triangle x + \alpha \triangle x] = 0 \\
\lim_{x \to x_0} f(x) = f(x_0) &\Leftrightarrow \lim_{\triangle x \to 0} [f(x_0 + \triangle x) - f(x_0) = 0] \Leftrightarrow \lim_{\triangle x \to 0} \triangle y = 0
\end{aligned}
$$

11.2.1

$$
\begin{aligned}
(C)' &= \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{C - C}{\triangle x} \\
&= 0
\end{aligned}
$$

11.2.2

$$
\begin{aligned}
(x^a)' &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \frac{x^a - x_0^a}{x - x_0} \\
&= \frac{(x - x_0)\left(x^{a-1} + x^{a-2}x_0 + \cdots + x x_0^{a-2} + x_0^{a-1}\right)}{x - x_0} \\
&= a x_0^{a-1}
\end{aligned}
$$

11.2.3

$$
\begin{aligned}
(a^x)' &= \lim_{\triangle x \to 0} \frac{a^{x+\triangle x} - a^x}{\triangle x} \\
&= a^x \lim_{\triangle x \to 0} \frac{a^{\triangle x} - 1}{\triangle x} \\
&= a^x \lim_{\triangle x \to 0} \frac{e^{\triangle x \ln a} - 1}{\triangle x} \\
&= a^x \ln a
\end{aligned}
$$

11.2.4

$$
(e^x)' = e^x \ln e = e^x
$$

11.2.5

$$
\begin{aligned}
(\log_a^x)' &= \lim_{\triangle x \to 0} \frac{\log_a^{x+\triangle x} - \log_a^x}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \frac{\ln 1 + \frac{\triangle x}{x}}{\ln a \triangle\ x} \\
&= \frac{1}{\ln a} \lim_{\triangle x \to 0} \frac{\frac{\triangle x}{x}}{\triangle\ x} \\
&= \frac{1}{x \ln a}
\end{aligned}
$$

11.2.6

$$
\begin{aligned}
(\ln^x)' &= \frac{1}{x \ln e} \\
&= \frac{1}{x}
\end{aligned}
$$

11.3.1

$$
\begin{aligned}
(\sin x)' &= \lim_{\triangle x \to 0} \frac{\sin(x_0 + \triangle\ x) - \sin x_0}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \frac{2\cos(x_0 + \frac{\triangle x}{2}) \sin \frac{\triangle x}{2}}{\triangle\ x} \\
&= \lim_{\triangle x \to 0} \cos(x_0 + \frac{\triangle\ x}{2}) \frac{\sin \frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= cosx_0
\end{aligned}
$$

11.3.2

$$
\begin{aligned}
(\arcsin x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d \sin y}{\mathrm{d}y}} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1 - \sin^2 y}} \\
&= \frac{1}{\sqrt{1 - x^2}}
\end{aligned}
$$

11.3.3

$$
\begin{aligned}
(\csc x)' = (\frac{1}{\sin x})` &= \frac{(1)' \cdot sinx - (\sin x)' \cdot 1}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= -\csc x \cdot \cot x
\end{aligned}
$$

11.3.4

$$\begin{aligned}(\cos x)' &= \lim_{\triangle x\to 0}\frac{\cos(x_0+\triangle\ x)-cos(x_0)}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\frac{-2\sin\left(x_0+\frac{\triangle x}{2}\right)\sin\frac{\triangle x}{2}}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}-\sin\left(x_0+\frac{\triangle\ x}{2}\right)\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\\ &= -\sin x\end{aligned}$$

11.3.5

$$\begin{aligned}(\arccos x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cos y}{\mathrm{d}y}}\\ &= \frac{1}{-\sin y}\\ &= -\frac{1}{\sqrt{1-\cos^2 y}}\\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

11.3.6

$$\begin{aligned}(\sec x)' = \left(\frac{1}{\cos x}\right)' &= \frac{(1)'\cdot\cos x-(\cos x)'\cdot 1}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x}\\ &= \sec x\cdot\tan x\end{aligned}$$

11.3.8

$$\begin{aligned}(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' &= \frac{(\sin x)'\cos x-(\cos x)'\sin x}{\cos^2 x}\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

11.3.9

$$
\begin{aligned}
(\arctan x)' &= \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\tan y}{\mathrm{d}y}} \\
&= \frac{1}{\sec y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}
\end{aligned}
$$

11.3.10

$$
\begin{aligned}
(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' &= \frac{(\cos x)'\sin x - (\sin x)'\cos x}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

11.3.11

$$
\begin{aligned}
(\operatorname{arccot} x)' &= \frac{1}{\frac{dx}{\mathrm{d}y}} = \frac{1}{\frac{d\cot y}{\mathrm{d}y}} \\
&= \frac{1}{-\csc^2 y} \\
&= -\frac{1}{1+\cot^2 y} \\
&= -\frac{1}{1+x^2}
\end{aligned}
$$

11.3.12

$$
\begin{aligned}
(\sinh x)' &= \left(\frac{e^x - e^{-x}}{2}\right)' \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}
$$

11.3.13

$$
\begin{aligned}
(\cosh x)' &= \left(\frac{e^x + e^{-x}}{2}\right)' \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}
$$

11.3.14

$$\begin{aligned}
(\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' \\
&= \frac{\frac{d(e^x - e^{-x})}{\mathrm{d}x}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{\mathrm{d}x}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= \frac{2^2}{(e^x + e^{-x})^2} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}$$

11.3.15

$$\begin{aligned}
(\operatorname{arcsinh} x)' &= \left[\ln(x + \sqrt{x^2+1})\right]' \\
&= \frac{d\ln(x + \sqrt{x^2+1})}{\mathrm{d}\big(x + \sqrt{x^2+1}\big)} \cdot \frac{d\left(x + \sqrt{x^2+1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x + \sqrt{x^2+1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)} \cdot \frac{d(x^2+1)}{dx}\right) \\
&= \frac{1}{x + \sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right) \\
&= \frac{1}{x + \sqrt{x^2+1}} \cdot \frac{x + \sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

11.3.16

$$
\begin{aligned}
(\operatorname{arccosh} x)' &= \left[\ln(x+\sqrt{x^2-1})\right]' \\
&= \frac{d\ln(x+\sqrt{x^2-1})}{\mathrm{d}\big(x+\sqrt{x^2-1}\big)} \cdot \frac{d\left(x+\sqrt{x^2-1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2-1}\right)}{d\left(x^2-1\right)} \cdot \frac{d(x^2-1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

11.3.17

$$
\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2}\ln(\frac{1+x}{1-x})\right]' \\
&= \frac{1}{2} \cdot \frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)} \cdot \frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2} \cdot \frac{1}{\left(\frac{1+x}{1-x}\right)} \cdot \frac{\frac{d(1+x)}{\mathrm{d}x}(1-x) - \frac{d(1-x)}{\mathrm{d}x}(1+x)}{(1-x)^2} \\
&= \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{(1-x)+(1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

11.4.1

$$
\begin{aligned}
[Cu(x)]' &= \lim_{\triangle x \to 0} \frac{Cu(x+\triangle x) - Cu(x)}{\triangle x} \\
&= C \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x)}{\triangle x} \\
&= Cu'(x)
\end{aligned}
$$

11.4.2

$$\begin{aligned}(u(x)\pm v(x))' &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)\pm v(x+\triangle x)-v(x)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x}\pm\lim_{\triangle x\to 0}\frac{v(x+\triangle x)-v(x)}{\triangle x}\\ &= u'(x)\pm v'(x)\end{aligned}$$

11.4.3

$$\begin{aligned}[u(x)\cdot v(x)]' &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)v(x+\triangle x)-u(x)v(x)-u(x)v(x+\triangle x)+u(x)v(x+\triangle x)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{[u(x+\triangle x)-u(x)]\,v(x+\triangle x)+u(x)\,[v(x+\triangle x)-v(x)]}{\triangle x}\\ &= u'(x)\lim_{\triangle x\to 0}v(x+\triangle x)+u(x)v'(x)\\ &= u'(x)v(x)+v'(x)u(x)\end{aligned}$$

11.5.1

$$\begin{aligned}\left[f^{-1}(y)\right]'|_{y=y_0} &= \lim_{y\to y_o}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\\ &= \lim_{y\to y_o}\frac{x-x_0}{y-y_0}\\ &= \lim_{x\to x_o}\frac{x-x_0}{f(x)-f(x_0)}\\ &= \lim_{x\to x_o}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\\ &= \frac{1}{f'(x)}\end{aligned}$$

11.6.1

$$\text{定义函数}\quad A(u)=\begin{cases}\frac{f(u)-f(u_0)}{u-u_0}. & u\neq u_0\\ f'(u_0), & u=u_0\end{cases}$$

$A(u)$在u_o处连续, 既有, $\lim_{u\to u_o}A(u)=A(u_0)=f'(u_0)$

由恒等式$f(u)-f(u_0)=A(u)(u-u_0)$我们有

$$\begin{aligned}\frac{F(x)-F(x_0)}{x-x_0}&=\frac{f\left[g(x)\right]-f\left[g(x_0)\right]}{x-x_0}\\&=A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}&=\lim_{x\to x_0}A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\F'(x_0)&=f'(g(x_0))g'(x_0)\end{aligned}$$

16.5 第 12章

12.1.1

$$\begin{aligned}\triangle\ y&=A\triangle\ x+\circ(\triangle\ x)\\\frac{\triangle\ y}{\triangle\ x}&=A+\frac{\circ(\triangle\ x)}{\triangle\ x}\\\lim_{\triangle x\to 0}\frac{\triangle\ y}{\triangle\ x}&=\lim_{\triangle x\to 0}\left[A+\frac{\circ(\triangle\ x)}{\triangle\ x}\right]\\f'(x_0)&=A+0\\f'(x_0)&=A\end{aligned}$$

12.1.2

设$f(x)$在x_0点可导, $f'(x_0)=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}$存在

(极限与无穷小的关系: $\lim f(x)=A\Leftrightarrow f(x)=A+\alpha$)

$$\frac{\triangle y}{\triangle x}=f'(x_0)+\alpha$$

$$\triangle\ y=f'(x_0)\triangle\ x+\alpha\triangle\ x$$

其中α为$\triangle\ x\to 0$时的无穷小。

$$\lim_{\triangle x\to 0}\frac{\alpha\triangle x}{\triangle x}=\lim_{\vartheta x\to 0}\alpha=0$$

$$\alpha\triangle\ x=\circ(\triangle\ x)$$

$$\triangle\ y=f'(x_0)\triangle\ x+\circ(\triangle\ x)$$

12.2.1

$$\begin{aligned}d(u\pm v)&=(u\pm v)'dx\\&=(u)'dx\pm(v')dx\\&=du\pm dv\end{aligned}$$

12.2.2

$$\begin{aligned} d(u \cdot v) &= (u \cdot v)' dx \\ &= (u)' v dx - (v') u dx \\ &= v du - u dv \end{aligned}$$

12.2.3

$$\begin{aligned} d\left(\frac{u}{v}\right) &= \left(\frac{u}{v}\right)' dx \\ &= \frac{(u)' v - (v') u}{v^2} dx \\ &= \frac{v du - u dv}{v^2} \end{aligned}$$

12.3.1

$$\lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x)}{\triangle x} = f'(x_0)$$

$$f(x_0 + \triangle x) - f(x_0) \leqslant 0$$

$$\begin{cases} \triangle x > 0 \left\{ \frac{f(x_0+\triangle x) - f(x_0)}{\triangle x} \leqslant 0 \Rightarrow f'(x_0^+) = \lim\limits_{\triangle x \to 0} \frac{f(x_0+\triangle x) - f(x_0)}{\triangle x} \leqslant 0 \right. \\ \triangle x < 0 \left\{ \frac{f(x_0+\triangle x) - f(x_0)}{\triangle x} \geqslant 0 \Rightarrow f'(x_0^-) = \lim\limits_{\triangle x \to 0} \frac{f(x_0+\triangle x) - f(x_0)}{\triangle x} \geqslant 0 \right. \end{cases}$$

$$f'(x_0) = f'(x_0^+) = f'(x_0^-) \Rightarrow f'(x_0) = 0$$

12.3.2

$$M = \max\{f(x) | x \in [a,b]\}, m = \min\{f(x) | x \in [a,b]\}$$

$$\begin{cases} M = m \Rightarrow M = m = f(a) = f(b), \text{此时} f(x) \text{为常数}, \forall \xi \in (a,b), f'(\xi) = 0 \\ M > m \begin{cases} f(a) > m \Rightarrow \exists \xi \in (a,b), f(\xi) = m, \text{根据费马引理}, f'(\xi) = 0 \\ f(a) < M5 \Rightarrow \exists \xi \in (a,b), f(\xi) = M, \text{根据费马引理}, f'(\xi) = 0 \end{cases} \end{cases}$$

12.3.3

$$\varphi(x) = f(x) - \frac{f(b) - f(a)}{b - a} x$$

$$\varphi(a) = f(a) - \frac{f(b) - f(a)}{b - a} a = \frac{bf(a) - af(b)}{b - a}$$

$$\varphi(b) = f(b) - \frac{f(b) - f(a)}{b - a} b = \frac{bf(a) - af(b)}{b - a}$$

$$\varphi(a) = \varphi(b), \exists \xi \in (a,b), \varphi'(\xi) = 0$$

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

$$f'(\xi)(b - a)7 = f(b) - f(a)$$

12.3.4

$$\varphi(x)=f(x)-\frac{f(b)-f(a)}{F(b)-F(a)}\left[F(x)-F(a)\right]$$
$$\varphi(a)=\varphi(b)=f(a)$$
$$\varphi'(\xi)=0$$
$$f'(\xi)=\frac{f(b)-f(a)}{F(b)-F(a)}F'(\xi)$$
$$\frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)}$$

12.4.1

$f(x),F(x)$的去心邻域可导， $\lim\limits_{x\to x_0}\frac{f(x)}{F(x)}$与$f(x_0),F(x_0)$无关。规定$f(x_0)=0,F(x_0)=0$

此时 $\lim\limits_{x\to x_0}f(x)=0=f(x_0)$ $\lim\limits_{x\to x_0}F(x)=0=F(x_0)$ 此时在 x_0 点处也连续

$$\frac{f(x)}{F(x)}=\frac{f(x)-f(x_0)}{F(x)-F(x_0)}=\frac{f'(\xi)}{F'(\xi)}$$
$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(\xi)}{F'(\xi)}$$

$x\to x_0$,时$\xi\to x_0$ 符号 ξ 换成 x

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{\xi\to x_0}\frac{f'(\xi)}{F'(\xi)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)}$$

12.5.1

$$\frac{R_n(x)}{(x-x_0)^{n+1}}=\frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-(x_0-x_0)^{n+1}}=\frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}$$
$$\frac{1}{n+1}\cdot\frac{R_n'(\xi_1)}{(\xi_2-x_0)^n}=\frac{1}{n+1}\cdot\frac{R_n'(\xi_1)-R_n'(x_0)}{(\xi_1-x_0)^n-(x_0-x_0)^n}=\frac{1}{n+1}\cdot\frac{R_n''(\xi_2)}{(n)(\xi_2-x_0)^{n-1}}$$
$$\vdots$$
$$\frac{R_n^{(n)}(\xi_n)}{(n+1)!(\xi_n-x_0)}=\frac{R_n^{(n)}(\xi_n)-R_n^{(n)}(x_0)}{(n+1)!(\xi_n-x_0)-0}=\frac{R_n^{(n+1)}(\xi)}{(n+1)!}$$
$$\frac{R_n^{(n+1)}(\xi)}{(n+1)!}=\frac{f^{(n+1)}(\xi)-P_n^{(n+1)}(\xi)}{(n+1)!}=\frac{f^{(n+1)}(\xi)}{(n+1)!}$$
$$R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$$\xi_1\in(x,x_0),\xi_2\in(\xi_1,x_0),\xi_n\in(\xi_{n-1},x_0),\xi\in(\xi_n,x_0)$$

12.5.2

$$\begin{aligned}\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n} &= \lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}}\\ &= \lim_{x\to x_0}\frac{R_n''(x)}{n(n-1)(x-x_0)^{n-2}}\\ &= \lim_{x\to x_0}\frac{R_n^{(n)}(x)}{n!}\\ &= \frac{1}{n}\cdot 0\\ &= 0\end{aligned}$$

16.6 第 13章

13.2.1

$$\begin{aligned}\left[\int f(x)\ dx \pm \int g(x)\ dx\right]' &= \left[\int f(x)\ dx\right]' \pm \left[\int g(x)\ dx\right]'\\ &= f(x)\pm g(x)\end{aligned}$$

13.2.2

$$\begin{aligned}\left[k\int f(x)\ dx\right]' &= k\left[\int f(x)\ dx\right]'\\ &= kf(x)\end{aligned}$$

13.2.3

$$\begin{aligned}\left\{F\left[\varphi(x)\right]\right\}' &= F'\left[\varphi(x)\right]\varphi'(x)\\ &= f\left[\varphi(x)\right]\varphi'(x)\end{aligned}$$

13.2.4

$$\begin{aligned}x &= \varphi(t)\\ dx &= d\varphi(t)\\ dx &= \varphi'(t)dt\\ \int f(x)\ dx &= \int f\left[\varphi(t)\right]\varphi'(t)\ dt\end{aligned}$$

13.2.5

$$\begin{aligned}\int f(x)\ dx &= \int f(x)\cdot(x+C)'\ dx\\ &= \int f(x)\ d(x+C)\end{aligned}$$

13.4.1

$$\int k\ dx = \int (kx)'\ dx = kx + C$$

13.4.2

$$\int x^a\ dx = \int \left(\frac{1}{a+1}x^{a+1}\right)'\ dx = \frac{x^{a+1}}{a+1} + C$$

13.4.3

$$\int a^x\ dx = \int \left(\frac{1}{\ln a}a^x\right)'\ dx = \frac{a^x}{\ln a} + C$$

13.4.4

$$\int e^x\ dx = \int (e^x)'\ dx = e^x + C$$

13.4.5

$$\int \frac{1}{x}\ dx = \begin{cases} (x > 0) & \int (\ln x)'\ dx = \ln x + C = \ln|x| + C \\ (x < 0) & \int [\ln(-x)]'\ dx = \ln(-x) + C = \ln|x| + C \end{cases}$$

13.4.6

$$\begin{aligned} \int \ln x\ dx &= \ln x \cdot x - \int x\ d\ln x \\ &= x\ln x - \int x \cdot \frac{1}{x}\ dx \\ &= x\ln x - x + C \end{aligned}$$

13.4.7

$$\int \sin x\ dx = \int (-\cos x)'\ dx = -\cos x + C$$

13.4.8

$$\int \cos x\ dx = \int (\sin x)'\ dx = \sin x + C$$

13.4.9

$$\int \sec x \tan x\ dx = \int (\sec x)'\ dx = \sec x$$

13.4.10

$$\int \csc x \cot x \; dx = -\int (\csc x)' \; dx = -\csc x$$

13.4.11

$$\int \sec^2 x \; dx = \int (\tan x)' \; dx = \tan x$$

13.4.12

$$\int \csc^2 x \; dx = -\int (\cot x)' \; dx = -\cot x$$

13.4.15

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x} \; dx \\ &= -\int \frac{(\cos x)'}{\cos x} \; dx \\ &= -\int \frac{1}{\cos x} \; d(\cos x) \\ &= -\ln|\cos x| + C\end{aligned}$$

13.4.16

$$\begin{aligned}\int \csc x \; dx &= \int \frac{1}{\sin x} \; dx \\ &= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \; dx \\ &= \int \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \; d\frac{x}{2} \\ &= \int \frac{1}{\tan\frac{x}{2}} \cdot \sec^2\frac{x}{2} \; d\frac{x}{2} \\ &= \int \frac{1}{\tan\frac{x}{2}} \; d\tan\frac{x}{2} \\ &= \begin{cases} \ln\left|\tan\frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases}\end{aligned}$$

13.4.17

$$\begin{aligned}\int \sec x \ dx &= \int \frac{1}{\cos x} \ dx \\ &= \int \frac{\cos x}{\cos^2 x} \ dx \\ &= \int \frac{1}{\cos^2 x} \ d\sin x \\ &= \int \frac{1}{1-\sin^2 x} \ d\sin x \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ &= \frac{1}{2}\left|\frac{1+\sin x}{\cos x}\right|^2 + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

13.4.18

$$\begin{aligned}\int \arccos x \ dx &= x\arccos x - \int x \cdot -\frac{1}{\sqrt{1-x^2}} \ dx \\ &= x\arccos x - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \ d(1-x^2) \\ &= x\arccos x - \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}(1-x^2)^{-\frac{1}{2}+1} + C \\ &= x\arccos x - \sqrt{1-x^2} + C\end{aligned}$$

13.4.19

$$\begin{aligned}\int \arctan x \ dx &= x\arctan x - \int x \cdot \frac{1}{1+x^2} \ dx \\ &= x\arctan x - \frac{1}{2}\int \frac{1}{1+x^2} \ d(1+x^2) \\ &= x\arctan x - \frac{1}{2}\ln(1+x^2) + C\end{aligned}$$

分式积分公式证明暂时不标号。

$$\int \frac{1}{x^2+1} \ dx = \int (\arctan x)' \ dx = \arctan x + C$$

$$\int \frac{1}{x^2+a^2} \ dx = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2} \ d(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\begin{aligned}
\int \frac{1}{x^2-a^2}\ dx &= \int \frac{1}{2a} \cdot \frac{(x+a)-(x-a)}{(x-a)(x+a)}\ dx \\
&= \frac{1}{2a} \cdot \int \frac{1}{x-a} - \frac{1}{x+a}\ dx \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{x-a}\ dx - \int \frac{1}{x+a}\ dx \right) \\
&= \frac{1}{2a} \left(\ln|x-a| - \ln|x+a| \right) + C \\
&= \frac{1}{2a} \ln\left| \frac{x-a}{x+a} \right| + C
\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{a^2-x^2}\ dx &= \int -\frac{1}{2a} \frac{(x-a)-(x+a)}{(a+x)(a-x)}\ dx \\
&= \int \frac{1}{2a} \cdot \frac{(x-a)-(x+a)}{(a+x)(x-a)}\ dx \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x}\ dx - \int \frac{1}{x-a}\ dx \right) \\
&= \frac{1}{2a} \cdot \left(\int \frac{1}{a+x}\ dx - \int \frac{1}{a-x}\ d(-x) \right) \\
&= \frac{1}{2a} \left(\ln|a+x| - \ln|a-x| \right) + C \\
&= \frac{1}{2a} \ln\left| \frac{a+x}{a-x} \right| + C
\end{aligned}$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \begin{cases} \int (\arcsin x)'\ dx = \arcsin x + C \\ -\int (\arccos x)'\ dx = -\arccos x + C \end{cases}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\ d(\frac{x}{a}) = \begin{cases} \arcsin(\frac{x}{a}) + C \\ -\arccos(\frac{x}{a}) + C \end{cases}$$

$x = a\tan t, -\frac{\pi}{2} < t < \frac{\pi}{2}, \sec t = \frac{\sqrt{x^2+a^2}}{a}, \tan t = \frac{x}{a}, dx = a\sec^2 t\ dt$

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}} &= \int \frac{1}{a}\frac{1}{\sqrt{\tan^2 t+1}} a\sec^2 t\ dt \\
&= \int \frac{1}{\sec t}\sec^2 t\ dt \\
&= \int \sec t\ dt \\
&= \ln|\sec t + tant| + C \\
&= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right| + C \\
&= \ln(x+\sqrt{x^2+a^2}) + C_1 \qquad C_1 = C - \ln a
\end{aligned}
$$

$$x = a\sec t, a>0, \sec t = \tfrac{x}{a}, \tan t = \tfrac{\sqrt{x^2-a^2}}{a}, dx = a\sec t\tan t\ dt$$

x>a

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\ dx &= \int \frac{1}{a}\frac{1}{\sqrt{\sec^2 t-1}} a\sec t\tan t\ dt \\
&= \int \frac{1}{\tan t}\sec t\tan t\ dt \\
&= \int \sec t\ dt \\
&= \ln|\sec t + tant| + C \\
&= \ln\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right| + C \\
&= \ln(x+\sqrt{x^2-a^2}) + C_1 \qquad C_1 = C - \ln a
\end{aligned}
$$

x<-a,x=-t,dx=-dt

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\ dx &= -\int \frac{1}{\sqrt{t^2-a^2}}\ dt \\
&= -\ln\left|t+\sqrt{t^2-a^2}\right| + C \\
&= -\ln\left|-x+\sqrt{x^2-a^2}\right| + C \\
&= -\ln\left|\frac{(-x+\sqrt{x^2-a^2})(x+\sqrt{x^2-a^2})}{x+\sqrt{x^2-a^2}}\right| + C \\
&= -\ln\left|\frac{-a^2}{x+\sqrt{x^2-a^2}}\right| + C \\
&= -\ln\left|-a^2\right| + \ln\left|x+\sqrt{x^2-a^2}\right| + C \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| + C_1 \qquad C_1 = C - \ln\left|-a^2\right|
\end{aligned}
$$

16.7 第 14章

14.3.1

$$
\begin{aligned}
\int_a^a f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\cdot 0 \\
&= 0
\end{aligned}
$$

14.3.2

$$
\begin{aligned}
\int_a^b \ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= \lim_{\lambda\to 0}(b-a) \\
&= b-a
\end{aligned}
$$

14.3.3

$$
\begin{cases}
\sum_{i=1}^{n} \vartriangle x_i = \sum_{i=1}^{n}(x_i - x_{i-1}) = b-a \\
\sum_{i=1}^{n}(x_{i-1} - x_i) = a-b
\end{cases}
$$

$$
\begin{aligned}
\int_a^b f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to o}\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \\
&= -\lim_{\lambda\to o}\sum_{i=1}^{n} f(\xi_i)(x_{i-1} - x_i) \\
&= -\int_b^a f(x)\ dx
\end{aligned}
$$

14.3.4

$$\begin{aligned}\int_a^c f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{b} f(\xi_i) \vartriangle x_i + \lim_{\lambda\to 0}\sum_{i=b+1}^{n} f(\xi_i) \vartriangle x_i \\ &= \int_a^b f(x)\ dx + \int_b^c f(x)\ dx\end{aligned}$$

14.3.5

$$\begin{aligned}\int_a^c f(x)\ dx &= \int_a^b f(x)\ dx + \int_b^c f(x)\ dx \\ \int_a^b f(x)\ dx &= \int_a^c f(x)\ dx - \int_b^c f(x)\ dx \\ &= \int_a^c f(x)\ dx + \int_c^b f(x)\ dx\end{aligned}$$

14.3.6

$$\begin{aligned}\int_a^b kf(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} kf(\xi_i) \vartriangle x_i \\ &= k\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\ &= k\int_a^b f(x)\ dx\end{aligned}$$

14.3.7

$$\begin{aligned}\int_a^b f(x) \pm g(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} [f(\xi_i) \pm g(\xi_i)] \vartriangle x_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \pm \lim_{\lambda\to 0}\sum_{i=1}^{n} g(\xi_i) \vartriangle x_i \\ &= \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx\end{aligned}$$

14.3.8

$$f(x) \geqslant 0, \vartriangle x_i = (x_i - x_{i-1}) > 0 \Rightarrow f(x_i) \vartriangle x_i \geqslant 0$$

$$\int_a^b f(x)\ dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \geqslant 0$$

14.3.9

$$f(x) \geqslant g(x) \Rightarrow f(x) - g(x) \geqslant 0$$
$$\int_a^b f(x) - g(x)\ dx \geqslant 0$$
$$\int_a^b f(x)\ dx - \int_a^b g(x)\ dx \geqslant 0$$
$$\int_a^b f(x)\ dx \geqslant \int_a^b g(x)\ dx$$

14.3.10

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|$$
$$\int_a^b -|f(x)|\ dx \leqslant \int_a^b f(x)\ dx \leqslant \int_a^b |f(x)|\ dx$$
$$\left|\int_a^b f(x)\ dx\right| \leqslant \int_a^b |f(x)|\ dx$$

14.4.1

$$m \leqslant f(x) \leqslant M, x \in [a,b]$$

$$\int_a^b m\ dx \leqslant \int_a^b f(x)\ dx \leqslant \int_a^b M\ dx$$
$$m\int_a^b\ dx \leqslant \int_a^b f(x)\ dx \leqslant M\int_a^b\ dx$$
$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a)$$

14.5.1

$$M\text{为区间 } [a,b] \text{ 最大值}, m\text{为区间 } [a,b] \text{ 最小值}, a < b$$
$$m \leqslant f(x) \leqslant M, x \in [a,b]$$
$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a)$$
$$m \leqslant \frac{1}{(b-a)}\int_a^b f(x)\ dx \leqslant M$$
$$\exists\xi \in [a,b], f(\xi) = \frac{1}{(b-a)}\int_a^b f(x)\ dx$$
$$f(\xi)(b-a) = \int_a^b f(x)\ dx$$

14.6.1

$$\phi'(x) = \lim_{\triangle x \to 0} \frac{\phi(x+\triangle x) - \phi(x) = \begin{cases} \int_a^{x+\Delta x} f(u)\ du - \int_a^x f(u)\ du \\ \int_a^x f(u)\ du + \int_x^{x+\Delta x} f(u)\ du - \int_a^x f(u)\ du \\ \int_x^{x+\Delta x} f(u)\ du \\ f(\xi)(x+\triangle x - x) \qquad (\xi \in [x, x+\triangle x]) \\ f(\xi)\triangle x \end{cases}}{\triangle x}$$

$$= \lim_{\triangle x \to 0} \frac{f(\xi)\triangle x}{\triangle x}$$

$$= \lim_{\triangle x \to 0} f(\xi) \quad (\triangle x \to 0, \text{时}, \xi \to x)$$

$$= f(x)$$

$$\begin{aligned} \phi'(x) &= \lim_{\triangle x \to 0} \frac{\phi(x+\triangle x) - \phi(x)}{\triangle x} \\ &= \lim_{\triangle x \to 0} \frac{f(\xi)\triangle x}{\triangle x} \\ &= \lim_{\triangle x \to 0} f(\xi) \quad (\triangle x \to 0, \text{时}, \xi \to x) \\ &= f(x) \end{aligned}$$

$$\begin{aligned} &\phi(x+\triangle x) - \phi(x) \\ &= \int_a^{x+\Delta x} f(u)\ du - \int_a^x f(u)\ du \\ &= \int_a^x f(u)\ du + \int_x^{x+\Delta x} f(u)\ du - \int_a^x f(u)\ du \\ &= \int_x^{x+\Delta x} f(u)\ du \\ &= f(\xi)(x+\triangle x - x) \qquad (\xi \in [x, x+\triangle x]) \\ &= f(\xi)\triangle x \end{aligned}$$

14.6.2

$$\begin{aligned} [\phi(\psi(x))]' &= \frac{d}{dx}\left[\int_a^{\psi(x)} f(u)\ du\right] \\ &= \frac{d}{d\psi(x)}\left[\int_a^{\psi(x)} f(u)\ du\right] \cdot \frac{d\psi(x)}{dx} \\ &= f(\psi(x))\psi'(x) \end{aligned}$$

14.6.3

$$\begin{aligned}\frac{d}{dx}\left[\int_{v(x)}^{\psi(x)} f(u)\ du\right] &= \frac{d}{dx}\left[\int_{0}^{\psi(x)} f(u)\ du + \int_{v(x)}^{0} f(u)\ du\right] \\ &= \frac{d}{dx}\left[\int_{0}^{\psi(x)} f(u)\ du\right] - \frac{d}{dx}\left[\int_{0}^{v(x)} f(u)\ du\right] \\ &= f\left[\psi(x)\right]\psi'(x) - f\left[v(x)\right]v'(x)\end{aligned}$$