# 数学方面 (笔记)

姜圣的追随者

2024.7.12

## 摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：B 站账号，姜圣的追随者，

2024.8.18：笔记都是看视频和书记录的。可能会有个别错误。但是我会持续更新，发现错误就会更改。上传频率不太固定。

# 目录

# 1 三角函数

## 1.1 三角恒等式

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \tag{1.1.1}$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B \tag{1.1.2}$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B \tag{1.1.3}$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B \tag{1.1.4}$$

### 1.1.1 和差化积

$$\sin(\alpha)+\sin(\beta)=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.5}$$
$$\sin(\alpha)-\sin(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.6}$$
$$\cos(\alpha)+\cos(\beta)=2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.7}$$
$$\cos(\alpha)-\cos(\beta)=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.8}$$

### 1.1.2 积化和差

$$\cos(A)\sin(B)=\frac{1}{2}\left[\sin(A+B)-\sin(A-B)\right] \tag{1.1.9}$$
$$\sin(A)\cos(B)=\frac{1}{2}\left[\sin(A+B)+\sin(A-B)\right] \tag{1.1.10}$$
$$\sin(A)\sin(B)=\frac{1}{2}\left[\cos(A-B)-\cos(A+B)\right] \tag{1.1.11}$$
$$\cos(A)\cos(B)=\frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \tag{1.1.12}$$

### 1.1.3 降幂

$$\sin^2 x=\frac{1-\cos(2x)}{2} \tag{1.1.13}$$
$$\cos^2 x=\frac{1+\cos(2x)}{2} \tag{1.1.14}$$

### 1.1.4 半角公式

$$\sin\frac{x}{2} = \sqrt{\frac{1-\cos(x)}{2}} \tag{1.1.15}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos(x)}{2}} \tag{1.1.16}$$

$$\tan\frac{x}{2} = \csc x - \cot x \tag{1.1.17}$$

$$\tag{1.1.18}$$

### 1.1.5 倍角公式

$$\sin(2x) = 2\sin x\cos x \tag{1.1.19}$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \tag{1.1.20}$$

$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x} \tag{1.1.21}$$

### 1.1.6 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.22}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.23}$$

### 1.1.7 三角函数恒等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.24}$$

$$1 + \tan^2 x = \sec^2 \tag{1.1.25}$$

$$1 + \cot^2 x = \csc^2 \tag{1.1.26}$$

## 1.2 双曲函数

### 1.2.1 定义

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

### 1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1}) \tag{1.2.1}$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}) \tag{1.2.2}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x}) \tag{1.2.3}$$

### 1.2.3 双曲函数恒等式

$$\sinh(2x) = 2\sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2\sinh^2 \frac{x}{2} \tag{1.2.7}$$

# 2 不等式

$$\frac{x_1+x_2+\cdots+x_n}{n} \geqslant \sqrt[n]{x_1+x_2+\cdots+x_n} \tag{2.0.1}$$

$$|x+y| \leqslant |x|+|y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1+x)^n \leqslant 1+nx \tag{2.0.4}$$

# 3 排列组合

## 3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1) \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

## 3.2 运算

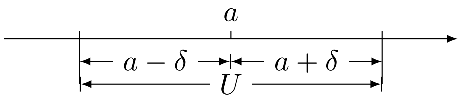
# 4 区间与映射

## 4.1 区间定义

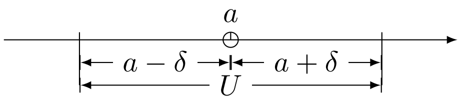
$$
\text{区间定义}\begin{cases}
(a,b)=\{x|a<x<b\}\\
[a,b]=\{x|a\leqslant x\leqslant b\}\\
(a,b]=\{x|a<x\leqslant b\}\\
(a,+\infty)=\{x|a<x\}
\end{cases}
$$

## 4.2 领域定义

点 a 的领域：$U(a,\delta)\begin{cases}\{x|a-\delta<x<a+\delta\}\\ \{x|\ |x-a|<\delta\}\end{cases}$




点 a 的去心领域：$\mathring{U}(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\wedge x\neq 0\}\\ \{x|0<|x-a|<\delta\}\end{cases}$




点 a 的左领域： $(a-\delta,a)$

点 a 的右领域： $(a,a+\delta)$

## 4.3 映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x\in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$\text{记作}\quad f:X\to Y$$

$$
f(x)=y\qquad\begin{cases}
\text{定义域 }(D_f)=X & x\text{-原像}\\
\text{值域 }(R_f)=\{f(x)|x\in X\} & y\text{-像}
\end{cases}
$$

$$
\text{映射类型}\begin{cases}
\text{满射:} & R_f=Y\\
\text{单射:} & x_1\neq x_2\Rightarrow f(x_1)\neq f(x_2)\\
\text{一一映射:} & \text{即使满射又是单射}\Leftrightarrow\text{逆映射:}\begin{cases}f^{(}x) & =y\\ f^{-1}(y) & =x\end{cases}\\
\text{复合映射:} & g\circ f\Leftrightarrow g\left[f\left(x\right)\right]\begin{cases}f:X\to Y_1\\ g:Y_2\to Z\\ g\circ f:X\to Z\qquad (Y_1\subset Y_2)\end{cases}
\end{cases}
$$

# 5 函数

## 5.1 函数相关的定义

### 5.1.1 函数

设数集 $D \in R$ 的映射

$$f : D \to R$$

称 f 为定义在 D 上的函数, 记为

$$y = f(x)\ \{x \in D\}$$

### 5.1.2 驻点

$$Def:\ \ f'(x) = 0$$

### 5.1.3 拐点

$$Def : f''(x) = 0\ (\text{左右两侧凹凸性改变})$$

### 5.1.4 极值点

$$Def : \text{函数} f(x)\ x \in \mathring{U}(x_0), \text{包括可导和不可导的点} \begin{cases} \text{极大值:}\ \ f(x) < f(x_0) \\ \text{极小值:}\ \ f(x) > f(x_0) \end{cases}$$

$$x \in \mathring{U}(x_0) \begin{cases} f(x)\text{可导,}\ \ f'(x_0) = 0 \begin{cases} x_0\text{极大值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) > 0 \\ x \in (x_0, x_0 + \delta), f'(x) < 0 \end{cases} \\ x_0\text{极小值} \begin{cases} x \in (x_0 - \delta, x_0), f'(x) < 0 \\ x \in (x_0, x_0 + \delta), f'(x) > 0 \end{cases} \\ x_0\text{无极值,}\ \ x \in \mathring{U}(x_0) \begin{cases} f'(x) > 0 \\ f'(x) < 0 \end{cases} \end{cases} \\ f(x)\text{二阶可导}, f'(x_0) = 0, f''(x_0) \neq 0 \begin{cases} f''(x) < 0 \Rightarrow x_0\text{极大值} \\ f''(x) > 0 \Rightarrow x_0\text{极小值} \end{cases} \end{cases}$$

#### 5.1.5 最值

$$
\text{最大值或最小值}\begin{cases}\text{驻点}\\\text{端点}\end{cases}
$$

## 5.2 函数的性质

### 5.2.1 函数的有界性

$$
f:D\to R\{D\subset R\}\begin{cases}\text{有界}\begin{cases}\text{有上界}\left\{\exists k_1,\text{使}f(x)\leqslant k_1,\ \forall x\in D\right.\\\text{有下界}\left\{\exists k_1,\text{使}f(x)\geqslant k_1,\ \forall x\in D\right.\end{cases}\\\text{无界}\begin{cases}\text{无上界}\left\{\forall K_1,\ \exists x\in D\text{ 使},\ f(x)\geqslant k_1\right.\\\text{无下界}\left\{\forall K_1,\ \exists x\in D\text{ 使},\ f(x)\leqslant k_1\right.\end{cases}\end{cases}
$$

### 5.2.2 函数的单调性与凹凸性

$$
\text{若}\{x_1,x_2\in D\}\ x_1<x_2\Rightarrow\begin{cases}f(x_1)<f(x_2)\text{称}f(x)\text{在 D 上单调增加}\\f(x_1)>f(x_2)\text{称}f(x)\text{在 D 上单调减少}\\f(x_1)\leqslant f(x_2)\text{称}f(x)\text{在 D 上单调非降}\\f(x_1)\geqslant f(x_2)\text{称}f(x)\text{在 D 上单调非增}\end{cases}
$$

$$
\text{设 }f(x)\text{ 在区间 I 上连续},\forall x_1,x_2\begin{cases}f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2},\text{称 }f(x)\text{ 在 I 上是向上凹}\\f\left(\frac{x_1+x_2}{2}\right)>\frac{f(x_1)+f(x_2)}{2},\text{称 }f(x)\text{ 在 I 上是向上凸}\end{cases}
$$

$$
f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内可导}f'(x)\geqslant 0,\text{有限个点为 0, 单调增}\tag{5.2.1}
$$

$$
f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内可导}f'(x)\leqslant 0,\text{有限个点为 0, 单调减}\tag{5.2.2}
$$

$$
f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内二阶可导}f''(x)\geqslant 0,\text{有限个点为 0, 向上凹}\tag{5.2.3}
$$

$$
f(x)\text{在 }[a,b]\text{ 上连续，在}(a,b)\text{内二阶可导}f''(x)\leqslant 0,\text{有限个点为 0, 向下凸}\tag{5.2.4}
$$

### 5.2.3 函数的奇偶性

$$
\forall x\in D\qquad f(-x)=\begin{cases}f(x)&\text{偶函数}\\-f(x)&\text{奇函数}\end{cases}
$$

$$奇函数 \times 奇函数 = 偶函数 \tag{5.2.5}$$

$$奇函数 \times 偶函数 = 奇函数 \tag{5.2.6}$$

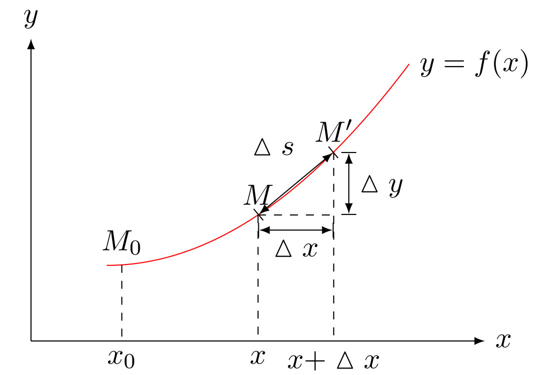
$$偶函数 \times 偶函数 = 偶函数 \tag{5.2.7}$$

### 5.2.4 周期性

$$Def: \quad f(x+L) = f(x)\{L > 0常数, \forall x \in D\} \Rightarrow f(x) \text{ 为 } L \text{ 的周期函数}$$

## 5.3 弧

### 5.3.1 有向曲线弧



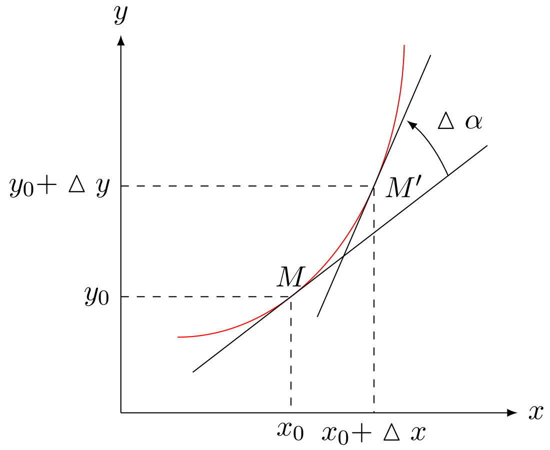

基准点$M_0(x_0, f(x_0))$

以 $x$ 增大的方向为正向, $\overset{\frown}{M_0M} = S$

$S = S(x), S$是关于 x 的单调增加函数

$$\overset{\frown}{M_0M} \begin{cases} \text{绝对值为的长度} \\ \text{与曲线正向一致，取正值} \\ \text{与曲线反向一致，取负值} \end{cases}$$

### 5.3.2 弧微分




$$\begin{aligned} ds &= \sqrt{1+(y')^2}dx \\ &= \sqrt{(dx)^2+(dy)^2} \\ &= \sqrt{(dx)^2+(f'dx)^2} \end{aligned} \tag{5.3.1}$$

$$参数方程 \begin{cases} x = \phi(t) & dx = \phi'(t)dt \\ y = \psi(t) & dy = \psi'(t)dt \end{cases}$$

$$ds = \sqrt{[\phi'(t)]^2 + [\psi'(t)]^2}dt$$

### 5.3.3 曲率

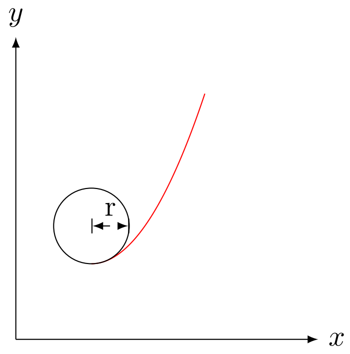
$$M(x_0, y_0), M'(x_0 + \triangle x, y_0 + \triangle y), \triangle s = \overset{\frown}{MM'}$$

$$\text{曲线上弧的}\begin{cases}\text{平均曲率：} & \overline{k}=\left|\frac{\triangle\alpha}{\triangle s}\right| \\ \text{点曲率：} & k=\lim\limits_{\triangle s\to 0}\left|\frac{\triangle\alpha}{\triangle s}\right|=\left|\frac{d\alpha}{ds}\right|\end{cases}$$

$$\left|\frac{d\alpha}{ds}\right|=\frac{|y''|}{[1+(y')^2]^{\frac{3}{2}}} \tag{5.3.2}$$

$$\left|\frac{d\alpha}{ds}\right|\text{的参数方程形式}\begin{cases}x=\phi(t)\\ y=\psi(t)\end{cases}\Rightarrow\left|\frac{d\alpha}{ds}\right|=\frac{\psi''(t)\phi'(t)-\psi'(t)\phi''(t)}{\left\{|\psi'(t)|^2+[\phi'(t)]^2\right\}^{\frac{3}{2}}} \tag{5.3.3}$$

### 5.3.4 曲率圆，曲率半径




$$\text{圆的曲率}k=\left|\frac{\triangle\ \alpha}{\triangle\ s}\right|=\left|\frac{\triangle\ \alpha}{r\ \triangle\ \alpha}\right|=\frac{1}{r}$$

$$\text{曲率半径}r=\frac{1}{k}$$

# 6 并集，交集

## 6.1 定义

$$(\vee\ \text{或},\wedge\ \text{与})$$
$$A \cup B = \{x \in A \vee x \in B\}$$
$$A \cap B = \{x \in A \wedge x \in B\}$$

## 6.2 运算

$$\begin{cases} \text{交换律} \begin{cases} A \cup B &= B \cup A \\ A \cap B &= B \cap A \end{cases} \\ \text{结合律} \begin{cases} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{cases} \\ \text{分配律} \begin{cases} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{cases} \\ \text{对偶律} \begin{cases} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{cases} \end{cases}$$
$$A \cup A = A = A \cap A$$
$$A = B \Leftrightarrow A \subset B \wedge A \supset B$$
$$A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$$

## 6.3 性质

$$A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{6.3.1}$$

$$A \cup B = B \Leftrightarrow A \subset B \tag{6.3.2}$$
$$A \cap B = A \Leftrightarrow A \subset B \tag{6.3.3}$$

$$(n \in N) \qquad A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{6.3.4}$$
$$(n \in N) \qquad A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{6.3.5}$$

## 6.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$
$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

### 6.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{C} = \bigcap_{\alpha}(E_{\alpha}^{C})$$

$$\left(\bigcap_{\alpha} E_{\alpha}\right)^{C} = \bigcup_{\alpha}(E_{\alpha}^{C})$$

# 7 群，环，域

## 7.1 群

### 7.1.1 M1

### 7.1.2 M2

### 7.1.3 M3

### 7.1.4 M4

## 7.2 环

## 7.3 域

内点，外点，边界点，聚点

# 8 极限

## 8.1 数列极限

### 8.1.1 数列的定义

$$Def: \qquad \{x_n\}, x_n = f(n), n \in N^+ \to R$$

### 8.1.2 数列极限的定义

$Def:\ \{x_n\},\ n \in N^+, \exists a,\ \forall \varepsilon > 0, \exists N,\ n > N \Rightarrow |x_n - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_n = a$

极限存在，为收敛，不存在为发散

### 8.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{8.1.1}$$

### 8.1.4 有界数列

$$\text{若}\exists M > 0, \{M \in \text{正数}\}$$
$$\text{使得}\forall n, \quad |x_n| \leqslant M$$
$$\text{则称数列 } \{x_n\} \text{ 为有界数列}$$

### 8.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{8.1.2}$$

$$\text{单调有界数列必收敛} \tag{8.1.3}$$

### 8.1.6 收敛数列的保号性

$$\lim_{n\to\infty} x_n = a \text{ 存在，且 } a > 0, \text{ 则 } \exists N > 0, \{N \in N^+\} \text{ 当 } n > N \text{ 时} \Leftrightarrow x_n > 0 \tag{8.1.4}$$

$$\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} b_n = b, a < b,\ \exists N, n > N, a_n < b_n \tag{8.1.5}$$

### 8.1.7 收敛数列和子数列

$\{x_n\}, \lim\limits_{n\to\infty} x_n = a,\ \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$

证明 $K = N\ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

## 8.2 函数极限

### 8.2.1 极限的定义

$$Def : \forall\varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ \text{当} x < -X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ \text{当}\ |x| > X \quad \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists\delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当} x_0 - \delta < x < x_0, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ \text{当} 0 < |x - x_0| < \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x\to x_0} f(x)\text{存在} \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \tag{8.2.1}$$

图

### 8.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$\lim\limits_{x\to x_0} f(x) = A, \exists M > 0, \delta > 0 \text{使} 0 < |x - x_0| < \delta, |f(x)| \leqslant M$

3 保号性

$\lim\limits_{x\to x_0} f(x) = A,\ A > 0, \exists\delta > 0, \text{当}, 0 < |x - x_0| < \delta \Rightarrow f(x) > 0$

$f(x) > 0, \exists\delta > 0, \text{当}, 0 < |x - x_0| < \delta \Rightarrow \lim\limits_{x\to x_0} f(x) = A,\ A > 0$

4 保序性

$f(x) \geqslant g(x)$, $\lim f(x) = a$, $\lim g(x) = b$, 则$a \geqslant b$

5 函数极限与数列极限的关系

如果 $\lim\limits_{x \to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 $x_0$ 的数列，则满足 $x_n \neq x_0$
则 $\lim\limits_{n \to \infty} f(x_n) = 0 = \lim\limits_{x \to x_0} f(x)$, $x_n \to x_0$

## 8.3 无穷小与无穷大

### 8.3.1 无穷小定义

$Def$ : 如果 $\lim\limits_{x \to x_0} f(x) = 0$则称$f(x)$为$x \to x_0$时的无穷小

$$Def: \forall \varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X \quad \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to +\infty} f(x) = 0 \\ \text{当} x < -X \quad \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to -\infty} f(x) = 0 \\ \text{当}\ |x| > X \quad \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to \infty} f(x) = 0 \end{cases} \\ \exists \delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to x_0^+} f(x) = 0 \\ \text{当} x_0 - \delta < x < x_0, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to x_0^-} f(x) = 0 \\ \text{当} 0 < |x - x_0| < \delta, \text{时}\ |f(x) - 0| < \varepsilon \Leftrightarrow \lim\limits_{x \to x_0} f(x) = 0 \end{cases} \end{cases}$$

### 8.3.2 函数极限与无穷小的关系

$$\text{在自变量的同一变化中。}\alpha\text{ 为无穷小。} \lim f(x) = A \Leftrightarrow f(x) = A + \alpha \tag{8.3.1}$$

### 8.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$\text{如果 } f(x) \text{ 为无穷大，则 } \frac{1}{f(x)} \text{ 为无穷小。} \tag{8.3.2}$$

$$\text{如果 } f(x) \text{ 为无穷小，切 } f(x) \neq 0\text{，则 } \frac{1}{f(x)} \text{ 为无穷小。} \tag{8.3.3}$$

#### 8.3.4 无穷大定义

$$
Def:\forall M>0\begin{cases}
\exists X>0\begin{cases}
\text{当}x>X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=\infty
\end{cases}\\
\text{当}x<-X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=+\infty\\
f(x)<M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=\infty
\end{cases}\\
\text{当}\,|x|>X\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to\infty}f(x)=\infty
\end{cases}
\end{cases}\\
\exists\delta>0\begin{cases}
\text{当}x_0-\delta<x<x_0\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=\infty
\end{cases}\\
\text{当}x_0<x<x_0+\delta\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=\infty
\end{cases}\\
\text{当}0<|x-x_0|<\delta\begin{cases}
f(x)>M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=+\infty\\
f(x)<-M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=-\infty\\
|f(x)|>M\Leftrightarrow\lim\limits_{x\to x_0}f(x)=\infty
\end{cases}
\end{cases}
\end{cases}
$$

$$\lim_{x\to x_0}f(x)=\infty,\ \text{直线}x=x_0\text{是}y=f(x)\text{垂直渐进线}$$

### 8.4 运算

#### 8.4.1 有限个无穷小的和仍为无穷小

$$
\begin{gathered}
\text{设 }\gamma=\alpha+\beta\\
\alpha\text{ 和 }\beta\text{ 同为 }x\to x_0\text{ 时的无穷小}\\
\forall\varepsilon>0,\ \exists\delta_1>0,\ \text{当 }0<|x-x_0|<\delta_1\text{ 时，有 }|\alpha|<\tfrac{\varepsilon}{2}\\
\forall\varepsilon>0,\ \exists\delta_2>0,\ \text{当 }0<|x-x_0|<\delta_2\text{ 时，有 }|\beta|<\tfrac{\varepsilon}{2}\\
\delta=min\{\delta_1,\delta_2\},\text{ 当 }0<|x-x_0|<\delta\text{ 时}\\
0<|x-x_0|<\delta_1,0<|x-x_0|<\delta_2\text{ 同时满足}
\end{gathered}
$$

$$即\ |\alpha| < \frac{\varepsilon}{2}, |\beta| < \frac{\varepsilon}{2}\ 同时成立$$
$$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

### 8.4.2 有界函数与无穷小的乘积仍为无穷小

$$设\ \alpha\ 为\ x \to x_0\ 时的一个无穷小$$
$$g(x)\ 为\ x_0\ 的一个去心邻域\ \mathring{U}(x_0, \delta_1)\ 有界$$
$$f(x) = g(x)\alpha$$
$$证\ f(x)\ 为\ x \to x_0\ 时的无穷小$$
$$因为\ g(x)在\mathring{U}(x_0, \delta_1)\ 有界$$
$$\exists M > 0, 当0 < |x - x_0| < \delta_1\ 时\ |g(x)| < M$$
$$因为\ \alpha\ 是\ x \to x_0\ 的无穷小$$
$$\exists \delta_2 > 0当0 < |x - x_0| < \delta_2\ 时\ |\alpha| < \frac{\varepsilon}{M} < \varepsilon$$
$$取\ \delta = min\{delta, \delta_2\}\ 当\ 0 < |x - x_0| < \delta\ 时$$
$$|g(x)| \geqslant M, |\alpha| < \frac{\varepsilon}{M}\ 同时成立$$
$$|g(x)\alpha| = |g(x)|\ \ |\alpha| < M\frac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

### 8.4.3 极限的四则运算

$\lim f(x) = A,\ \lim g(x) = B$

$$\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) \tag{8.4.1}$$
$$\lim\left(f(x)g(x)\right) = \lim f(x) \lim g(x) \tag{8.4.2}$$
$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)} \tag{8.4.3}$$
$$\lim\left[Cf(x)\right] = C \lim f(x) \tag{8.4.4}$$
$$\lim\left[f(x)\right]^n = \left[\lim f(x)\right]^n \tag{8.4.5}$$

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_m}{a_0 x^n + a_1 x^{n-1} + \cdots + a_n} = \begin{cases} \frac{a}{b} & m = n \\ \infty & m > n \\ 0 & m < n \end{cases} \tag{8.4.6}$$

$$\begin{gathered} \lim_{x \to x_0} g(x) = u_0,\ \lim_{u \to u_0} f(x) = A \\ \exists \delta_0 > 0,\ x \in \mathring{U}\left(x_0, \delta_0\right),\ g(x) \neq u_0 \\ \lim_{x \to x_0} f\left[g(x)\right] = \lim_{u \to u_0} f(u) = A \end{gathered} \tag{8.4.7}$$

### 8.4.4 夹逼定理 (三明治定理)

$$
\begin{aligned}
&x_n \leqslant z_n \leqslant y_n \qquad \forall n > N_0 \\
&\text{若} \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = a \text{则} \lim_{n\to\infty} z_n = a
\end{aligned} \tag{8.4.8}
$$

### 8.4.5 重要极限

$$x \to x_0$$

$$\lim_{x\to x_0} \sin x = \sin x_0 \tag{8.4.9}$$

$$\lim_{x\to x_0} \cos x = \cos x_0 \tag{8.4.10}$$

$$x \to 0$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \tag{8.4.11}$$

$$\lim_{x\to 0} \cos x = 1 \tag{8.4.12}$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1 \tag{8.4.13}$$

$$\lim_{x\to 0} \frac{1-\cos x}{\frac{1}{2}x^2} = 1 \tag{8.4.14}$$

$$\lim_{x\to 0} \frac{\arcsin x}{x} = 1 \tag{8.4.15}$$

$$\lim_{x\to 0} \frac{\arctan x}{x} = 1 \tag{8.4.16}$$

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 \tag{8.4.17}$$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1 \tag{8.4.18}$$

$$\lim_{x\to 0} \frac{(1+x)^n - 1}{nx} = 1 \tag{8.4.19}$$

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \tag{8.4.20}$$

$$x \to \infty$$

$$\{x_n\} \qquad \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \tag{8.4.21}$$

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e \tag{8.4.22}$$

### 8.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

$Def:$ $\alpha, \beta$ 是同一极限过程的无穷小。

(1) 如果 $\lim \frac{\beta}{\alpha} = 0$ 则称 $\beta$ 是 $\alpha$ 的高阶无穷小，记作 $\beta = \circ(\alpha)$

(2) 如果 $\lim \frac{\beta}{\alpha} = \infty$ 则称 $\beta$ 是 $\alpha$ 的底阶无穷小。

(3) 如果 $\lim \frac{\beta}{\alpha} = C$ 则称 $\beta$ 是 $\alpha$ 的同阶无穷小。

(4) 如果 $\lim \frac{\beta}{\alpha^k} = C, k > 0$ 则称 $\beta$ 是 $\alpha$ 的 k 阶无穷小。

(5) 如果 $\lim \frac{\beta}{\alpha} = 1$ 则称 $\beta$ 是 $\alpha$ 的等价阶无穷小。

### 8.4.7 等价无穷小代换，因子代换

$\beta$与$\alpha$是等价无穷小 $\Leftrightarrow \beta = \alpha + \circ(\alpha)$

设$\alpha \sim \alpha'$, $\beta \sim \beta'$, 且 $\lim \frac{\beta'}{\alpha'}$存在, 则 $\lim \frac{\beta}{\alpha} = \lim \frac{\beta'}{\alpha'}$

$\lim \alpha f(x) = \lim \alpha' f(x)$

$\lim \frac{f(x)}{\alpha} = \lim \frac{f(x)}{\alpha'}$

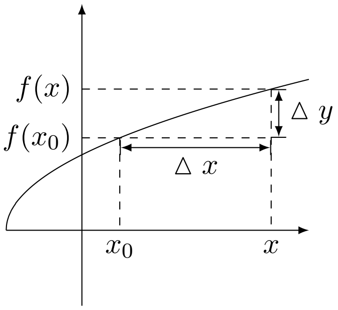
# 9 连续与间断点

## 9.1 定义

### 9.1.1 点连续

$$Def1: 设f(x)在x_0的某邻域内有定义，如果\lim_{x\to x_0} = f(x_0)$$

$$则称f(x)在x_0处连续$$




$$\begin{cases} \triangle\ x = x - x_0 \\ \triangle\ y = \begin{cases} f(x) - f(x_0) \\ f(x_0 + \triangle\ x) - f(x_0) \end{cases} \end{cases}$$

$$Def2: 如果\lim_{\triangle x\to 0} \triangle\ y = \lim_{\triangle x\to 0}[f(x_0+\triangle\ x) - f(x_0)] = 0$$

$$则称f(x)在x_0处连续$$

### 9.1.2 区间连续

$$\forall x_0 \in [a,b] \begin{cases} \lim\limits_{x\to x_0} f(x) = f(x_0) \quad x_0 \in (a,b) \Leftrightarrow f(x_0) = \begin{cases} \lim\limits_{x\to x_0^-} f(x) = f(x_0^-) \\ \lim\limits_{x\to x_0^+} f(x) = f(x_0^+) \end{cases} \\ \lim\limits_{x\to x_0^+} f(x) = f(x_0^+) \quad x_0 = a\ (右连续) \\ \lim\limits_{x\to x_0^-} f(x) = f(x_0^-) \quad x_0 = b\ (左连续) \end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M > 0, x \in [a,b]$ 时，$|f(x)| \geqslant M$

最大值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0 \in [a,b]$ 时，$\forall x \in [a,b], f(x) \geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

### 9.1.3 间断点

1,$f(x)$ 无定义

2,$\lim\limits_{x\to x_0} f(x)$ 不存在

3.$\lim\limits_{x\to x_0} f(x)$ 存在, 但 $\lim\limits_{x\to x_0} f(x) \neq f(x_0)$

第一类间断点：$f(x_0^+)=\lim\limits_{x\to x_0^+}f(x)$ 与 $f(x_0^-)=\lim\limits_{x\to x_0^-}f(x)$
第二类间断点：不是第一类的。

## 9.2 连续函数的运算

函数 $f(x),g(x)$ 在 $x=x_0$ 连续。

$$
\begin{aligned}
\lim_{x\to x_0}[f(x)\pm g(x)]&=\lim_{x\to x_0}f(x)\pm\lim_{x\to x_0}g(x)=f(x_0)\pm g(x_0)\\
\lim_{x\to x_0}[f(x)\cdot g(x)]&=\lim_{x\to x_0}f(x)\cdot\lim_{x\to x_0}g(x)=f(x_0)\cdot g(x_0)\\
\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right]&=\frac{\lim\limits_{x\to x_0}f(x)}{\lim\limits_{x\to x_0}g(x)}=\frac{f(x_0)}{g(x_0)}\qquad(g(x_0)\neq 0)
\end{aligned}
$$

反函数的连续性
若 $y=f(x)$ 在区间 $I_x$ 上单调增加，且连续。
则 $y=f^-1(x)$ 在 $I_y=\{y|y=f(x),x\in I_x\}$ 上也为单调增加，连续

$$
\text{复合函数}\begin{cases}
\text{内外都连续}\begin{cases}
\lim\limits_{x\to x_0}g(x)=g(x_0)=u_0\\
\lim\limits_{u\to u_0}f(x)=f(u_0)\\
\lim\limits_{x\to x_0}f\left[g(x)\right]=f\left[g(x_0)\right]=f(\lim\limits_{x\to x_o}g(x))
\end{cases}\\
\text{外连续}\begin{cases}
x\to x_0\begin{cases}
\lim\limits_{x\to x_0}g(x)=u_0\\
\lim\limits_{u\to u_0}f(x)=f(u_0)\\
\lim\limits_{x\to x_0}f\left[g(x)\right]=f(u_0)=f(\lim\limits_{x\to x_o}g(x))
\end{cases}\\
x\to\infty\begin{cases}
\lim\limits_{x\to\infty}g(x)=u_0\\
\lim\limits_{u\to u_0}f(x)=f(u_0)\\
\lim\limits_{x\to\infty}f\left[g(x)\right]=f(u_0)=f(\lim\limits_{x\to\infty}g(x))
\end{cases}
\end{cases}
\end{cases}
$$

## 9.3 零点定理

2, 设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)\cdot f(b)<0$
则至少存在一点 $\xi\in(a,b)$ 使 $f(\xi)=0$

## 9.4 介质定理

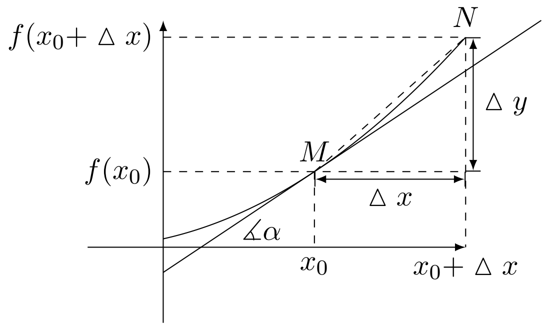
设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)=A,f(b)=B$
$\forall C\in(A,B)$ , 至少有一点 $\xi,f(\xi)=C$

# 10 导数

## 10.1 定义

导数的概念从物理发展出来的。

$$v(t_0) = \lim_{\triangle t \to 0} \frac{s(t_0 + \triangle t) - s(t_0)}{\triangle t}$$




$$NM\text{斜率} = \tan\beta = \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} = \frac{\triangle y}{\triangle x}$$

$$\text{斜率}k = \tan\alpha = \lim_{\triangle x \to 0} \tan\beta = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} = \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x}$$

### 10.1.1 导数定义

$y = f(x)$ 在 $x_0$ 的某邻域内有定义
给自变量的增量 $\triangle x, (x_0 + \triangle x)$ 仍在定义域内
函数得到了相应增量 $\triangle y, \triangle y = f(x_0 + \triangle x)$
如果 $\lim_{\triangle x \to 0} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$ 存在, 称 $y = f(x)$ 在 $x = x_0$ 处可导
(极限值为$y = f(x)$在$x = x_0$处导数)

$$\text{记 } y'|_{x=x_0} = f'(x_0) = \lim_{\triangle x \to 0} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x} = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x}$$

$$\lim_{\triangle x \to 0} \frac{f(x_0 + \triangle x) - f(x_0)}{\triangle x} \Leftrightarrow \lim_{\triangle x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

### 10.1.2 导函数定义

$f(x)$ 在区间 $I$ 内任意一点均可导。
$f'(x) = \lim_{\triangle x \to 0} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$
称 $f'(x)$ 为 $y = f(x)$ 在区间 $I$ 上的导函数

### 10.1.3 闭区间可导定义

$$f(x) \text{ 在 } [a,b] \text{ 可导} \Leftrightarrow \begin{cases} f'(x_0) \quad x_0 \in (a,b) \Leftrightarrow \begin{cases} \text{左导数} f'_-(x_0) = \lim\limits_{\triangle x \to 0^-} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x} \\ \text{右导数} f'_+(x_0) = \lim\limits_{\triangle x \to 0^+} \frac{f(x_0+\triangle x)-f(x_0)}{\triangle x} \end{cases} \\ f'_+(a) \quad x = a \\ f'_-(a) \quad x = b \end{cases}$$

### 10.1.4 导数与连续

$$f'(x)\text{存在} \Rightarrow f(x)\text{在}x = x_0\text{处连续} \tag{10.1.1}$$

## 10.2 幂数，指数，对数

$$(C)' = 0 \tag{10.2.1}$$

$$(x^a)' = ax^{a-1} \tag{10.2.2}$$

$$(a^x)' = a^x \ln a \tag{10.2.3}$$

$$(e^x)' = e^x \tag{10.2.4}$$

$$(\log_a^x)' = \frac{1}{x \ln a} \tag{10.2.5}$$

$$(\ln x)' = \frac{1}{x} \tag{10.2.6}$$

## 10.3 三角函数

$$(\sin x)' = \cos x \tag{10.3.1}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \tag{10.3.2}$$

$$(\csc x)' = -\csc x \cot x \tag{10.3.3}$$

$$(\cos x)' = -\sin x \tag{10.3.4}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \tag{10.3.5}$$

$$(\sec x)' = \sec x \tan x \tag{10.3.6}$$

$$(\operatorname{arcsec} x)' = \frac{1}{|x|\sqrt{x^2-1}} \tag{10.3.7}$$

$$(\tan x)' = \sec^2 x \tag{10.3.8}$$

$$(\arctan x)' = \frac{1}{1+x^2} \tag{10.3.9}$$

$$(\cot x)' = -\csc^2 x \tag{10.3.10}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2} \tag{10.3.11}$$

$$(\sinh x)' = \cosh x \tag{10.3.12}$$

$$(\cosh x)' = \sinh x \tag{10.3.13}$$

$$(\tanh x)' = \frac{1}{cosh^2 x} = 1 - \tanh^2 x \tag{10.3.14}$$

$$(\operatorname{arcsinh} x)' = \frac{1}{\sqrt{x^2+1}} \tag{10.3.15}$$

$$(\operatorname{arccosh} x)' = \frac{1}{\sqrt{x^2-1}} \tag{10.3.16}$$

$$(\operatorname{arctanh} x)' = \frac{1}{1-x^2} \tag{10.3.17}$$

## 10.4 导数运算

$$u = u(x), v = v(x), \text{均在}x\text{点可导}, C\text{为常数}$$

$$(Cu(x))' = Cu'(x) \tag{10.4.1}$$

$$(u(x) \pm v(x))' = u'(x) \pm v'(x) \tag{10.4.2}$$

$$(u(x)\cdot v(x))' = u'(x)v(x) + v'(x)u(x) \tag{10.4.3}$$

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x)v(x) - v'(x)u(x)}{[v(x)]^2} \tag{10.4.4}$$

### 10.5 反函数求导

如果函数 $y=f(x)$ 在区间 $(a,b)$ 内单调可导，且 $f'(y)\neq 0$

$$\begin{cases}\alpha=\min\{f(a)+0,f(b-0)\}\\ \beta=\max\{f(a)+0,f(b-0)\}\end{cases}$$

则它的反函数 $x=f^{-1}(y)$ 在区间 $(\alpha,\beta)$ 内也可导

$$\left[f^{-1}(y)\right]'=\frac{1}{f'(x)}\Leftrightarrow\frac{dy}{\mathrm{d}x}=\frac{1}{\frac{dx}{\mathrm{d}y}}\tag{10.5.1}$$

### 10.6 复合函数求导

$$\text{设函数}\begin{cases}y=f(u)\text{在}U(u_0,\delta_0)\text{处有定义}\\ u=g(x)\text{在}U(x_0,\eta_0)\text{处有定义}\end{cases}$$

$$u_0=g(x_0),\text{且}f'(u)\text{和}g'(x)\text{都存在}$$

$$\text{则复合函数}F(x)=f\left[g(x)\right]\text{在点}x_0\text{可导, 且}$$

$$F'(x_0)=f'\left[g(x_0)\right]g'(x_0)\Leftrightarrow\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}\tag{10.6.1}$$

### 10.7 高阶求导

$$Def:\begin{cases}\text{一阶导数} & y'\Leftrightarrow\frac{dy}{dx}\\ \text{二阶导数} & y''\Leftrightarrow\frac{d^2y}{dx^2}\\ \text{三阶导数} & y'''\Leftrightarrow\frac{d^3y}{dx^3}\\ \text{三阶以上 n 阶导数} & y^{(n)}\Leftrightarrow\frac{d^ny}{dx^n}\end{cases}$$

### 10.8 高阶求导公式

$$(e^x)^{(n)} = e^x \tag{10.8.1}$$
$$(a^x)^{(n)} = a^x\,(lna)^n \tag{10.8.2}$$
$$(x^\mu)^{(n)} = A_\mu^n x^{\mu-n} \tag{10.8.3}$$
$$\left(\frac{1}{x+a}\right)^{(n)} = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{10.8.4}$$
$$[\ln(x+a)]^{(n)} = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{10.8.5}$$
$$(\sin x)^{(n)} = \sin(x+n\frac{\pi}{2}) \tag{10.8.6}$$
$$(\cos x)^{(n)} = \cos(x+n\frac{\pi}{2}) \tag{10.8.7}$$
$$[f(ax+b)]^{(n)} = a^n \cdot f^{(n)}(ax+b) \tag{10.8.8}$$

### 10.9 高阶求导运算法则

$$(u(x)\pm v(x))^{(n)} = u^{(n)}(x) \pm v^{(n)}(x) \tag{10.9.1}$$
$$\text{莱布紫泥公式} \qquad (uv)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(n-k)} \cdot v^(k) \tag{10.9.2}$$

### 10.10 隐函数求导

$F(x,y)=0, y=f(x)$

$F(x,f(x)) \equiv 0$ 可以同时对两面求导

### 10.11 参数方程求导

$x = x(t), y = y(t)$

$$\frac{dy}{dx} = \frac{(\frac{dy}{dt})}{(\frac{dx}{dt})} = \frac{dy}{dt}\cdot\frac{dt}{dx}$$

$$\frac{d^2y}{dx} = \frac{d\frac{dy}{dx}}{dx} = \frac{d}{dt}\left(\frac{(\frac{dy}{dt})}{(\frac{dx}{dt})}\right)\cdot\frac{dt}{dx} = \frac{\frac{d^2y}{dt}\cdot\frac{dx}{dt} - \frac{d^2x}{dt}\cdot\frac{dy}{dt}}{(\frac{dx}{dt})^2}\cdot\frac{dt}{dx} = \frac{\frac{d^2y}{dt}\cdot\frac{dx}{dt} - \frac{d^2x}{dt}\cdot\frac{dy}{dt}}{(\frac{dx}{dt})^3}$$

# 11 微分

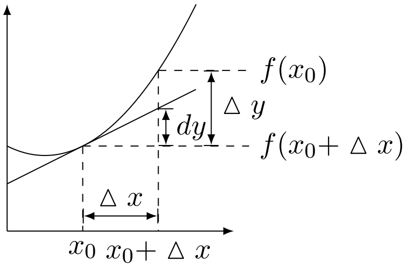
## 11.1 定义

设函数 $f(x)$ 在点 $x_0$ 的一个邻域内有定义。$\triangle y = f(x_0 + \triangle x) - f(x_0)$

如果 $\triangle y$ 可以表示为 $\triangle y = A \triangle x + \circ(\triangle x)$ 其中 $A$ 为与 $\triangle x$ 无关的常数

则称 $f(x)$ 在点 $x_0$ 可微，$A \triangle x$ 称为 $f(x)$ 在点 $x_0$ 处的微分。

记作:$dy = A \triangle x$




$$\text{可微} \Rightarrow \text{可导} \tag{11.1.1}$$

$$\text{可导} \Rightarrow \text{可微} \tag{11.1.2}$$

## 11.2 微分法则

### 11.2.1 核心根本

$$dy = \overset{\text{积分}}{\underset{\text{求导}}{f'(x) \quad d \quad x}}$$

### 11.2.2 四则运算

$$d\left(u \pm v\right) = du \pm dv \tag{11.2.1}$$

$$d(uv) = vdu + udv \tag{11.2.2}$$

$$d\left(\frac{u}{v}\right) = \frac{vdu + udv}{v^2} \tag{11.2.3}$$

### 11.2.3 复合运算

$$\text{可微} \begin{cases} y = f(u) \\ u = g(x) \end{cases} \Rightarrow \begin{cases} dy = f'(u)du \\ du = g'(x)dx \end{cases} \quad \text{则 } y = f(g(x)) \text{ 也可微}$$

且 $dy = f'(u)du = f'(u)g(x)dx$

$u$ 是否为中间变量都成立，微分的不变性。

### 11.2.4 近似计算公式

$$\triangle x \to 0, dy \approx \triangle y \begin{cases} dy = f'(x_0) \triangle x \\ \triangle y = f(x_0 + \triangle x) - f(x_0) \end{cases} \qquad \begin{cases} f(x_0 + \triangle x) \approx f(x_0) + f'(x_0) \triangle x \\ f(x) \approx f(x_0) + f'(x_0)(x - x_0) \\ x_0 = 0 \begin{cases} f(x) \approx f(0) + f'(0)x \\ \sqrt{n} \approx 1 + \frac{1}{n}x \\ \sin x \approx x \\ \tan x \approx x \\ e^x \approx 1 + x \\ \ln(1+n) \approx x \end{cases} \end{cases}$$

### 11.2.5 奇偶函数导数

$$\text{偶函数导数为奇函数 } f(x) = f(-x) \Leftrightarrow f'(x) = -f'(-x)$$
$$\text{奇函数导数为偶函数 } f(x) = -f(-x) \Leftrightarrow f'(x) = f'(-x)$$

### 11.2.6 区间恒为 0

$$\text{若}f'(x)\text{在区间恒为零，则}f(x)\text{在区间}I\text{上为一常数}$$
$$E\text{设}x_1, x_2\text{为区间}I\text{内任意两点}x_1 < x_2$$
$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1) \equiv 0$$
$$f(x_2) \equiv f(x_1) = C$$

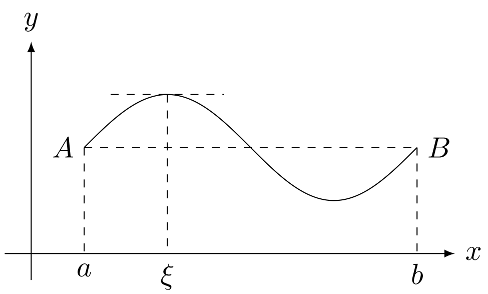
## 11.3 中值定理

### 11.3.1 费马引理

$$f(x), \forall x \in \mathring{U}(x_0) \begin{cases} f(x) \leqslant f(x_0) & f(x)\text{在}x_0\text{处取极大值} \\ f(x) \geqslant f(x_0) & f(x)\text{在}x_0\text{处取极小值} \end{cases}$$

$$\text{如果可导函数}y = f(x)\text{在}x_0\text{取极值，则}f'(x_0) = 0 \tag{11.3.1}$$

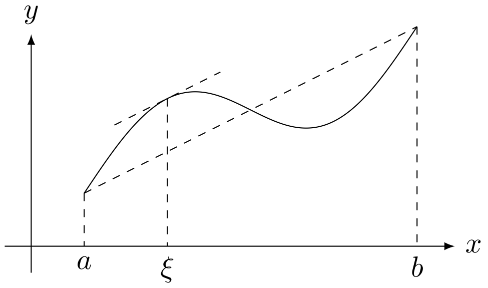
### 11.3.2 罗尔定理




$$f(x) \begin{cases} \text{闭区间 } [a, b] \text{ 上连续} \\ \text{开区间 (a,b) 可导} \\ f(a) = f(b) \end{cases}$$

$$\text{至少有一点}\xi \in (a, b), f'(\xi) = 0 \tag{11.3.2}$$

### 11.3.3 拉格朗日定理 (微分中值定理)




$$f(x)\begin{cases}\text{在闭区间 } [a,b] \text{ 上连续}\\ \text{在开区间 (a,b) 可导}\end{cases}$$

至少有一点$\xi \in (a,b)$

$$f'(\xi)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow f(b)-f(a)=f'(x)(b-a) \tag{11.3.3}$$

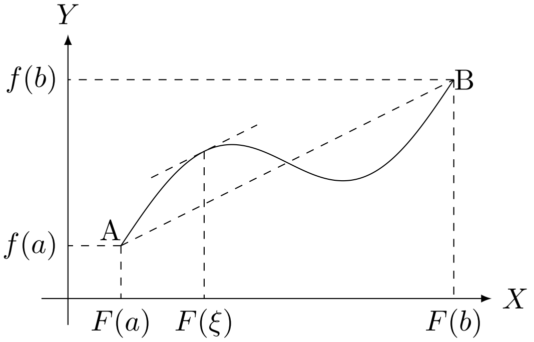
$$[x,x+\triangle x] \text{ 用拉格朗日定理} \quad f(x+\triangle x)-f(x)=f'(\xi)\triangle x$$

$$\xi \in (x,x+\triangle x) \text{ 记作: } \xi = x+\theta\triangle x \qquad 0<\theta<1$$

$$f(x+\triangle x)-f(x)=f(x+\theta\triangle x)\triangle x$$

$$\triangle y = f(x+\theta\triangle x)\triangle x$$

### 11.3.4 柯西定理




$$f(x)\begin{cases}\text{在闭区间 } [a,b] \text{ 上连续}\\ \text{在开区间 (a,b) 可导}\\ F'(x)\neq 0\end{cases}$$

$$\text{参数方程}(a\leqslant x\leqslant b)\begin{cases}X=F(x)\\ Y=f(x)\end{cases}$$

$$\text{至少有一点,}\xi \qquad \frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)} \tag{11.3.4}$$

$$\text{切线斜率}=\frac{dY}{dX}=\frac{df(x)}{dF(x)}=\frac{f'(x)}{F'(x)} \Rightarrow x=\xi\text{时斜率}=\frac{f'(\xi)}{F'(\xi)}$$

$$AB\text{的斜率}=\frac{f(b)-f(a)}{F(b)-F(a)}$$

### 11.3.5 三个定理关系

$$\frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)},(F(x)=x)\Rightarrow f'(\xi)=\frac{f(b)-f(a)}{b-a},(f(b)=f(a))\Rightarrow f'(\xi)=0$$

## 11.4 洛必达法则

$$\text{未定型,}\quad \frac{0}{0},\quad \frac{\infty}{\infty},\quad 0^0,\quad 1^\infty,\quad \infty^0,\quad \infty-\infty$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0}f(x)=0,\lim\limits_{x\to x_0}F(x)=0\\ f(x),F(x)\text{在}x_0\text{的某去心邻域内可导，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\frac{f'(x)}{F'(x)}\text{存在，或无穷小。则}\end{cases}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)}=\lim_{x\to x_0}\frac{f'(x)}{F'(x)} \tag{11.4.1}$$

$$\lim_{x\to\infty}\frac{f(x)}{F(x)}=\lim_{x\to\infty}\frac{f'(x)}{F'(x)}\begin{cases}\lim\limits_{x\to x_0}f(x)=0,\lim\limits_{x\to x_0}F(x)=0\\ \exists N\text{ 当 }|x|>N\text{，时}f'(x),F'(x)\text{存在，且}F'(x)\neq 0\\ \lim\limits_{x\to x_0}\frac{f'(x)}{F'(x)}\text{存在，或为无穷大。则}\end{cases}$$

## 11.5 泰勒公式

$$\begin{aligned}f(x_0+\triangle x)-f(x_0)&=f'(x_0)\triangle x+\circ(\triangle x)\\ x_0+\triangle=x&\qquad \triangle x=x-x_0\\ f(x)-f(x_0)&=f'(x_0)(x-x_0)+\circ(\triangle x)\\ f(x)&=f'(x_0)(x-x_0)+f(x_0)+\circ(\triangle x)\\ f(x)&\approx f'(x_0)(x-x_0)+f(x_0)\end{aligned}\qquad \begin{aligned}P(x_0)&=f(x_0)\\ P'(x_0)&=f'(x_0)\end{aligned}$$

### 11.5.1 泰勒多项式

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots+a_n(x-x_0)^n\text{去近似某个多项式}$$

$$\begin{cases}P_n(x_0)&=f(x_0)=a_0\\ P_n'(x_0)&=f'(x_0)=a_1\\ P_n''(x_0)&=f''(x_0)=a_2\cdot 2!\\ &\vdots\\ P_n^{(n-1)}(x_0)&=f^{(n-1)}(x_0)=a_{n-1}\cdot(n-1)!\\ P_n^{(n)}(x_0)&=f^{(n)}(x_0)=a_n\cdot n!\end{cases}\Rightarrow\begin{cases}a_0&=f_n(x_0)\\ a_1&=f_n'(x_0)\\ a_2&=\frac{f_n''(x_0)}{2!}\\ &\vdots\\ a_{n-1}&=\frac{f_n^{(n-1)}(x_0)}{(n-1)!}\\ a_n&=\frac{f_n^{(n)}(x_0)}{n!}\end{cases}$$

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2\cdots\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}+\frac{f^n(x_0)}{n!}(x-x_0)^n$$

$$f(x)\approx P_n(x)$$

### 11.5.2 泰勒中值定理

如果$f(x)|x_0 \in (a,b)$内有$(n+1)$阶导则

$$f(x) = P_n(x) + R_n(x)$$

$$P_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x-x_0)^n + R_n(x)$$

拉格朗日余项

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \qquad \{\xi \in (x, x_0)\} \tag{11.5.1}$$

皮亚诺于项

$$R_n(x) = \circ(|x-x_0|^n) \tag{11.5.2}$$

$$f(x) \approx P_n(x) \text{ 误差为 } R_n(x)$$

## 11.6 麦克劳林公式

$$x_0 = 0$$

$$P_n(x) = f(0) + f'(0)x + \tfrac{f''(0)}{2!}x^2 + \cdots + \tfrac{f^n(0)}{n!}x^n + \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0<\theta<1 \\ \circ(|x|^n) \end{cases}$$

### 11.6.1 常用的麦克劳林展开

$$e^x = 1 + 1x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n \begin{cases} \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}.\ 0<\theta<1 \\ \circ(|x|^n) \end{cases}$$

$$\sin x = 1x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^{n-1}\frac{1}{(2n-1)!}x^{2n-1} + R_n(x)$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 \cdots + (-1)^n\frac{1}{(2n)!}x^{2n} + R_n(x)$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots + (-1)^{n-1}\frac{1}{n}x^n + R_n(x)$$

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots - \frac{1}{n}x^n + R_n(x)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + R_n(x)$$

# 12 微分方程

## 12.1 基本概念

微分方程，含有自变量，未知函数及导数的方程，称为微分方程。

$$
\text{微分方程}\begin{cases}\text{未知函数为一元函数} & \text{常微分方程}\\ \text{未知函数为多元函数} & \text{偏微分方程 (数理方程)}\end{cases}
$$

### 12.1.1 微分方程的阶

方程中的未知函数的最高阶的导数，阶数称为发挥嗯称的阶。

### 12.1.2 $n$ 阶微分方程解

$$
F(x,y,y',y'',\cdots,y^{(n)})=0,\ y=\varphi(x)
$$

$$
F(x,\varphi(x),\varphi'(x),\quad \varphi''(x),\cdots,\varphi^{(n)})\equiv 0
$$

$$
x\in I,\text{ 称 }\varphi\text{ 为方程在区间 }I\text{ 上的解}\begin{cases}\text{包含有 n 个任意常数称 y=}\varphi(x)\text{ 是方程的通解}\\ \text{不含任意常数称 y=}\varphi(x)\text{ 是方程的特解}\end{cases}
$$

### 12.1.3 齐次方程

如果一阶微分方程可化为 $\frac{dy}{dx}=\varphi\left(\frac{y}{x}\right)$ 则称原方程为齐次方程

## 12.2 一阶线性微分方程

$$
\frac{dy}{dx}+P(x)y=Q(x)\begin{cases}Q(x)\equiv 0, & \text{称一阶线性齐次方程}\\ Q(x)\not\equiv 0, & \text{称一阶线性非齐次方程}\end{cases}
$$

$$
\text{齐次通解}\qquad y=Ce^{-\int P(x)\ dx} \tag{12.2.1}
$$

$$
\text{非齐次通解}\qquad y=e^{-\int P(x)\ dx}\left[\int Q(x)e^{\int P(x)\ dx}\ dx+C\right] \tag{12.2.2}
$$

## 12.3 二阶线性微分方程

$$
y''+P_1(x)y'+P_2(x)y=f(X)\begin{cases}f(x)\equiv 0, & \text{称二阶阶线性齐次方程}\\ f(x)\not\equiv 0, & \text{称二阶阶线性非齐次方程}\end{cases}
$$

### 12.3.1 二阶线性齐次微分方程

$$\begin{aligned}&y_1(x), y_2(x) \text{ 是任意的两个解，} C_1, C_2 \text{ 是任意常数，则} \\ &y = C_1 y_1(x) + C_2 y_2(x) \quad \text{也是的解}\end{aligned} \tag{12.3.1}$$

$$\begin{aligned}&y_1, y_2 \text{ 是两个线性无关解，} C_1, C_2 \text{ 是任意常数，则} \\ &\text{通解为，} \quad y = C_1 y_1 + C_2 y_2\end{aligned} \tag{12.3.2}$$

### 12.3.2 二阶线性非齐次微分方程

$$\text{非齐次通解 } y = Y + y^* \begin{cases} y^* \text{ 非其次特解} \\ Y = C_1 y_1 + C_2 y_2 \text{ 齐次通解} \end{cases}$$

$$y = y_1 - y_2 \text{是对应齐次方程的解} \tag{12.3.3}$$

$$y = \alpha y_1 + (1 - \alpha) y_2 \text{也是解} \tag{12.3.4}$$

$$\begin{aligned}&(1) \quad y'' + P(x)y' + Q(X)y = f_1(x) + f_2(x) \\ &(2) \quad y'' + P(x)y' + Q(X)y = f_1(x) \\ &(3) \quad y'' + P(x)y' + Q(X)y = f_2(x) \\ &(2) \text{ 特解为} y_1^* \quad (3) \text{ 特解为} y_2^* \Rightarrow \quad (1) \text{ 特解为} y_1^* + y_2^*\end{aligned}$$

### 12.3.3 二阶常系数齐次线性微分方程

$$y'' + py' + qy = 0 \qquad (p, q \text{ 属于常数})$$
$$y = e^{rx} \qquad y' = re^{rx} \qquad y'' = r^2 e^{rx}$$

$$\text{特征方程：} r^2 + pr + q = 0 \begin{cases} p^2 - 4q > 0 \quad \text{通解：} C_1 e^{r_1 x} + C_2 e^{r_2 x} \\ p^2 - 4q = 0 \quad \text{通解：} C_1 e^{r_1 x} + C_2 x e^{r_1 x} \\ p^2 - 4q < 0 \quad \text{通解：} e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \end{cases} \tag{12.3.5}$$

### 12.3.4 二阶常系数非齐次线性微分方程

$$y'' + py' + qy = f(x) \qquad (p, q, \text{为常数})$$
$$y'' + py' + qy = 0 \qquad (\text{对应的齐次方程})$$
$$\text{通解} = \text{齐次方程通解} + \text{非齐次特解}$$

$$f(x)=\begin{cases}P_m(x)e^{\lambda x} & (P_m(x)\text{是 } x \text{ 的 } m \text{ 次多项式})\\ \left[P_l(x)\cos\omega x+P_n(x)\sin\omega x\right]e^{\lambda x} & (P_l(x),P_n(x)\text{是 } x \text{ 的 } l.n \text{ 次多项式})\end{cases}$$

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$$f(x)=P_m(x)e^{\lambda x}\text{型}\begin{cases}y &= Q(x)e^{\lambda x}\\ y' &= Q'(x)e^{\lambda x}+\lambda Q(x)e^{\lambda x}\\ &= \left[Q'(x)+\lambda Q(x)\right]e^{\lambda x}\\ y'' &= \left[Q''(x)+\lambda Q'(x)\right]e^{\lambda x}+\left[Q'(x)+\lambda Q(x)\right]\lambda e^{\lambda x}\\ &= \left[Q''(x)+2\lambda Q'(x)+\lambda^2Q(x)\right]e^{\lambda x}\end{cases}$$

$$\begin{aligned}\left[Q''(x)+2\lambda Q'(x)+\lambda^2Q(x)\right]e^{\lambda x}+p\left[Q'(x)+\lambda Q(x)\right]e^{\lambda x}+qQ(x)e^{\lambda x}&=P_m(x)e^{\lambda x}\\ \left[Q''(x)+2\lambda Q'(x)+\lambda^2Q(x)\right]+p\left[Q'(x)+\lambda Q(x)\right]+qQ(x)&=P_m(x)\\ Q''(x)+(2\lambda+p)Q'(x)+(\lambda^2+p\lambda+q)Q(x)&=P_m(x)\end{aligned}$$

$$\lambda^2+p\lambda+q\begin{cases}\neq 0\ (0\text{重根})\Rightarrow\begin{cases}Q(x)=Q_m(x)\\ Q_m(x)=b_0+b_1x+b_2x^2+\cdots+b_mx^m\quad(b_m\neq 0)\\ b_0,b_1,b_2\cdots,b_m\text{待定系数}\\ y^*=Q_m(x)e^{\lambda x}\end{cases}\\ =0\quad 2\lambda+p\begin{cases}\neq 0\ (1\text{重根})\Rightarrow\begin{cases}Q(x)=xQ_m(x)\\ y^*=xQ_m(x)e^{\lambda x}\end{cases}\\ =0\ (2\text{重根})\Rightarrow\begin{cases}Q(x)=x^2Q_m(x)\\ y^*=x^2Q_m(x)e^{\lambda x}\end{cases}\end{cases}\end{cases}$$

---

$$\left[P_l(x)\cos\omega x+P_n(x)\sin\omega x\right]e^{\lambda x}\quad\text{型}\begin{cases}\lambda\text{是常数}\quad P_l(x)\text{是 } x \text{ 的 } l \text{ 次多项式}\\ \omega\text{ 是常数}\quad P_n(x)\text{是 } x \text{ 的 } n \text{ 次多项式}\end{cases}$$

$$y^*=x^ke^{\lambda x}\left[R_n(x)\cos\omega x+R_n(x)\sin\omega x\right]\begin{cases}n=\max\{l,m\}\\ \text{特征方程：}r^2+pr+q=0\\ \lambda+i\omega\begin{cases}\text{不是特征根 } k=0\\ \text{是特征根 } k=1\end{cases}\end{cases}$$

## 12.4 n 阶线性微分方程

$$y^{(n)}+P_1(x)y^{(n-1)}+\cdots+a_ny=f(X)\begin{cases}f(x)\equiv 0,\quad\text{称 n 阶阶线性齐次方程}\\ f(x)\not\equiv 0,\quad\text{称 n 阶阶线性非齐次方程}\end{cases}$$

$y_1, y_2, \cdots, y_n$ 是 $n$ 个线性无关解，$C_1, C_2, \cdots, C_n$ 是任意常数，则
通解为， $y = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ (12.4.1)

$$\text{非齐次通解 } y = Y + y^* \begin{cases} y^* \text{ 非其次特解} \\ Y = C_1y_1 + C_2y_2 + \cdots + C_ny_n \text{ 齐次通解} \end{cases}$$

### 12.4.1 n 阶常系数线性齐次微分方程

$$y^{(n)} + a_1y(n-1) + a_2y^{(n-2)} + \cdots + a_ny = 0$$

$$\text{特征方程：} r^n + a_1rn - 1 + a_2r^{n-2} + \cdots + a_n = 0$$

**不同根对应的通解**

| | |
|---|---|
| 单根 (实) | $Ce^{rx}$ |
| $k$ 个根 (实) | $(C_1 + C_2x + C_3x^2 + \cdots + C_kx^{k-1})e^{rx}$ |
| 单共轭复根 | $e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$ |
| $k$ 个共轭复根 | $e^{\alpha x}\left\{\left[C_1 + C_2x \cdots C_kx^{k-1}\right]\cos\beta x + \left[D_1 + D_2x \cdots D_kx^{k-1}\right]\sin\beta x\right\}$ |

### 12.4.2 n 阶常系数线性非齐次微分方程

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + a_ny = f(X) = P_m(x)e^{\lambda x}$$

$$n\text{阶通解} = n\text{阶齐次通解} + n\text{非齐次特解}$$

$$\text{特解：} y^* = x^kQ_m(x)e^{\lambda x} \quad \text{其中 k 为特征根的重数}$$

## 12.5 全微分方程

$$\text{一阶微分方程对称式} \quad M(x,y)dx + N(x,y)dy = 0$$

$$\text{存在} u(x,y) \text{使} \quad du = M(x,y)dx + N(x,y)dy$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad \begin{cases} M(x,y) = \frac{\partial u}{\partial x} \\ N(x,y) = \frac{\partial u}{\partial y} \end{cases}$$

# 13 不定积分

## 13.1 概念

### 13.1.1 原函数

$$\forall x \in I,\ F'(x) = f(x), \quad F(x)\text{为}f(x)\text{的一个原函数}$$

$$\text{函数}f(x)\text{在区间}I\text{上连续一定有}F(x),\text{使}F'(x) = f(x) \tag{13.1.1}$$

### 13.1.2 不定积分

区间 $I$ 上，$f(x)$ 的带有任意常数的原函数, 称为 $f(x)$ 在区间 I 上的不定积分。
记作:

$$\int f(x)\ dx \quad \begin{cases} \int & \text{积分符号} \\ f(x) & \text{被积函数} \\ f(x)\ dx & \text{被积表达式} \\ x & \text{积分变量} \end{cases}$$

如果$F(x)$是$f(x)$的一个原函数

$$\int f(x)dx = F(x) + C$$

### 13.1.3 不定积分性质

$$\left[\int f(x)\ dx\right]' = f(x)$$

$$d\left[\int f(x)\ dx\right] = f(x)\ dx$$

$$\int\ dF(x) = \int F'(x)\ dx = F(x) + C$$

## 13.2 积分运算

$$\int [f(x)+g(x)]\ dx = \int f(x)\ dx + \int f(x)\ dx \tag{13.2.1}$$

$$\int kf(x)\ dx = k\int f(x)\ dx \quad (k\text{为常数}) \tag{13.2.2}$$

$$\int f[\varphi(x)]\,\varphi'(x)\ dx \xlongequal{u=\varphi(x)} \left[\int f(u)du\right]_{x=\varphi(u)} = F[\varphi(x)]+C \tag{13.2.3}$$

$$\int f(x)\ dx \xlongequal[\varphi'(t)\neq 0]{x=\varphi(t)} \left[\int f[\varphi(t)]\,\varphi'(t)\ dt\right]_{t=\varphi^{-1}(x)} \tag{13.2.4}$$

$$\int f(x)\ dx = \int f(x)\ d(x+C) \tag{13.2.5}$$

### 13.2.1 分部积分法

$$\int u\ dv = uv - \int v\ du \Leftrightarrow \int uv'\ dx = uv - \int u'v\ dx \tag{13.2.6}$$

## 13.3 有理函数积分

### 13.3.1 普通多项式

$$\frac{P(x)}{Q(x)} \qquad P(x),Q(x)\text{是 } x \text{ 多项式, 且没有公因子，称为有理分式}$$

$$\text{有理分式}\begin{cases}\text{真分式} & P(x)\text{次数} < Q(x)\text{次数}\\ \text{假分式} & P(x)\text{次数} \geqslant Q(x)\text{次数}\end{cases}$$

$$\text{如果真分式中}Q(x)=Q_1(x)\cdot Q_2(x),\text{其中}Q_1(x),Q_2(x)\text{都为多项式}$$

$$\frac{P(x)}{Q(x)} = \frac{P_1(x)}{Q_1(x)}\cdot\frac{P_2(x)}{Q_2(x)} \tag{13.3.1}$$

$$\text{假分式} = \text{多项式} + \text{真分式}$$

$$\text{最简分式}\quad \frac{A}{x-a} \quad \frac{A}{(x-a)^2} \quad \frac{Nx+M}{x^2+px+q} \quad \frac{Nx+m}{(x^2+px+q)^k}$$

### 13.3.2 三角函数多项式

$$\text{三角有理分式: } R(\sin x,\cos x)$$

$$\text{万能代换: } \tan\frac{x}{2}=u, x=2\arctan u, dx=\frac{2\ du}{1+u^2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\frac{\tan\frac{x}{2}}{\sec^2\frac{x}{2}} = 2\frac{\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}(1-\tan^2\frac{x}{2}) = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-u^2}{1+u^2}$$

$$\int R(\sin x, \cos x)\ dx = \int R(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2})\frac{2}{1+u^2}\ du = \int Y(u)\ du$$

$Y(u)$ 是 $u$ 的有理函数

## 13.4　积分公式

### 13.4.1　幂数，指数，对数

$$\int k\ dx = kx + C \tag{13.4.1}$$

$$\int x^a\ dx = \frac{x^{a+1}}{a+1} + C \tag{13.4.2}$$

$$\int a^x\ dx = \frac{a^x}{\ln a} + C \tag{13.4.3}$$

$$\int e^x\ dx = e^x + C \tag{13.4.4}$$

$$\int \frac{1}{x}\ dx = \ln|x| + C \tag{13.4.5}$$

$$\int \ln x\ dx = x\ln x - x + C \tag{13.4.6}$$

### 13.4.2 三角函数

$$\int \sin x \ dx = -\cos x + C \tag{13.4.7}$$

$$\int \cos x \ dx = \sin x + C \tag{13.4.8}$$

$$\int \sec x \tan x \ dx = \sec x + C \tag{13.4.9}$$

$$\int \csc x \cot x \ dx = -\csc x + C \tag{13.4.10}$$

$$\int \sec^2 x \ dx = \tan x + C \tag{13.4.11}$$

$$\int \csc^2 x \ dx = -\cot x + C \tag{13.4.12}$$

$$\int \sinh x \ dx = \cosh x + C \tag{13.4.13}$$

$$\int \cosh x \ dx = \sinh x + C \tag{13.4.14}$$

$$\int \tan x \ dx = -\ln|\cos x| + C \tag{13.4.15}$$

$$\int \csc x \ dx = \begin{cases} \ln\left|\tan \frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases} \tag{13.4.16}$$

$$\int \sec x \ dx = \ln|\sec x + \tan x| + C \tag{13.4.17}$$

$$\int \arccos x \ dx = x \arccos x - \sqrt{1-x^2} + C \tag{13.4.18}$$

$$\int \arctan x \ dx = x \arctan x - \frac{1}{2}\ln(1+x^2) + C \tag{13.4.19}$$

### 13.4.3 分式

$$\int \frac{1}{1+x^2}\ dx = \arctan x + C \tag{13.4.20}$$

$$\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\tan\frac{x}{a} + C \tag{13.4.21}$$

$$\int \frac{1}{x^2-a^2}\ dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \tag{13.4.22}$$

$$\int \frac{1}{a^2-x^2}\ dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C \tag{13.4.23}$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \begin{cases} \arcsin x + C \\ -\arccos x + C_1 \end{cases} \tag{13.4.24}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \begin{cases} \arcsin \frac{x}{a} x + C \\ -\arccos \frac{x}{a} + C_1 \end{cases} \tag{13.4.25}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\ dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C \tag{13.4.26}$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\ dx = \ln(x+\sqrt{x^2+a^2}) + C \tag{13.4.27}$$

$$\int \frac{1}{|x|\sqrt{x^2-1}}\ dx = \operatorname{arcsec} x + C \tag{13.4.28}$$

# 14 定积分

## 14.1 定积分的定义

$[a,b]$ 有限区间，，$f(x)$在 $[a,b]$ 上有界

$[a,b]$ 内任找$n-1$个点,分成 $n$ 个区间

$a=x_0<x_1<x_2\cdots x_{n-1}<x_n=b$

$[x_0,x_1],[x_1,x_2]\cdots[x_{n-1},x_n]$

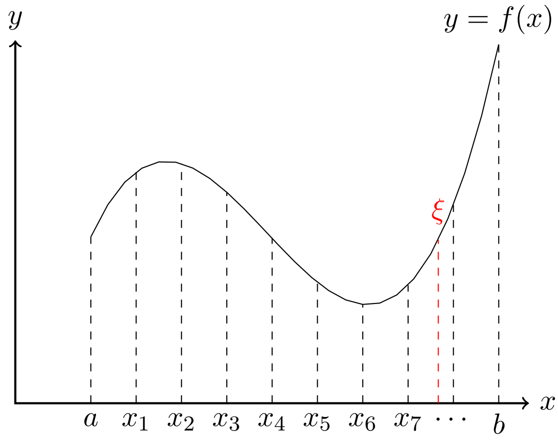
分成 $n$ 个曲边梯形, $[x_{i-1},x_i]$ 为第 $i$ 个

面积 $\triangle S_i$, 对应底 $\triangle x_i=x_i-x_{i-1}$

$\forall \xi_i\in[x_{i-1},x_i], \triangle S_i\approx f(\xi_i)\triangle x_i$

$S=\triangle S_1+\triangle S_2\cdots\triangle S_n=\sum\limits_{i=1}^{n}f(\xi_i)\triangle x_i$

$\lambda=\max\{\triangle x_1,\triangle x_2,\cdots,\triangle x_n\}$, 当$\lambda\to 0$时




$$S=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\triangle x_i$$

$S$是一个定数，则称$f(x)$在 $[a,b]$ 上可积，$S$称为$f(x)$在 $[a,b]$ 上的定积分记作：

$$\int_a^b f(x)\,dx\begin{cases} f(x)\,dx & \text{被积表达式}\left\{ f(x) \quad \text{被积函数}\right. \\ x & \text{积分变量} \\ [a,b] & \text{积分区间}\begin{cases} a & \text{积分下限} \\ b & \text{积分上限}\end{cases}\end{cases}$$

$$\int_a^b f(x)\,dx\triangleq\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\triangle x_i$$

## 14.2 可积的充分条件

如果$f(x)$在 $[a,b]$ 上连续，则$f(x)$在 $[a,b]$ 上可积 (14.2.1)

如果$f(x)$在 $[a,b]$ 上有界，且至多有有限个间断点，则$f(x)$在 $[a,b]$ 上可积 (14.2.2)

## 14.3 定积分的性质

$a<b<c,k$为常数

$$\int_a^a f(x)\ dx = 0 \tag{14.3.1}$$

$$\int_a^b\ dx = b - a \tag{14.3.2}$$

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx \tag{14.3.3}$$

$$\int_a^c f(x)\ dx = \int_a^b f(x)\ dx + \int_b^c f(x)\ dx \tag{14.3.4}$$

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx \tag{14.3.5}$$

$$\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx \tag{14.3.6}$$

$$\int_a^b f(x) \pm g(x)\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx \tag{14.3.7}$$

$$f(x) \geqslant 0 \quad \Rightarrow \int_a^b f(x)\ dx \geqslant 0 \tag{14.3.8}$$

$$f(x) \geqslant g(x) \quad \Rightarrow \int_a^b f(x)\ dx \geqslant \int_a^b g(x)\ dx \tag{14.3.9}$$

$$\left|\int_a^b f(x)\ dx\right| \leqslant \int_a^b |f(x)|\ dx \tag{14.3.10}$$

## 14.4 积分估值公式

$$M\text{为区间 } [a,b] \text{ 最大值}, m\text{为区间 } [a,b] \text{ 最小值}, a < b$$

$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a) \tag{14.4.1}$$

## 14.5 积分中值定理

$$f(x)\text{是 } [a,b] \text{ 上的连续函数，则}, \exists\xi \in [a,b], a < b\text{使}$$

$$\int_a^b f(x)\ dx = f(\xi)(b-a) \tag{14.5.1}$$

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\ dx \qquad \text{称为均值}$$

## 14.6 积分上限函数

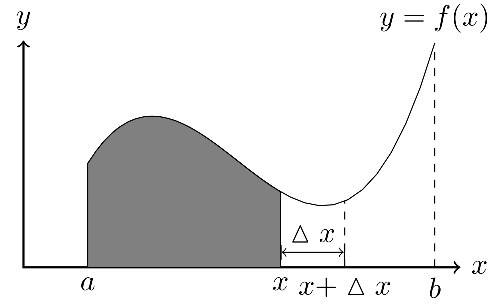
### 14.6.1 定义

$x \in [a,b]$ , $[a,x]$ 对应曲边梯形

$\int_a^x f(x)\ dx = \int_a^x f(u)\ du$

$\phi(x) \triangleq \int_a^x f(u)\ du$

$\phi(x)$是 $[a,b]$ 上函数称为积分上限函数




### 14.6.2 性质

$$\phi'(x) = \frac{d}{dx}\left[\int_a^x f(u)\ du\right] = f(x) \tag{14.6.1}$$

$$\frac{d}{dx}\left[\int_a^{\psi(x)} f(u)\ du\right] = f(\psi(x))\psi'(x) \tag{14.6.2}$$

$$\frac{d}{dx}\left[\int_{v(x)}^{\psi(x)} f(u)\ du\right] = f\left[\psi(x)\right]\psi'(x) - f\left[v(x)\right]v'(x) \tag{14.6.3}$$

若 $f(x)$ 在 $[a,b]$ 上连续，则 $f(x)$ 必存在原函数,$\phi(x) = \int_a^x f(u)\ du$

即为 $f(x)$ 在 $[a,b]$ 上的一个原函数

$$\int f(x)\ dx = \int_a^x f(u)\ du + C$$

## 14.7 微积分基本公式 (牛顿莱布兹尼公式)

$f(x)$ 在 $[a,b]$ 上连续，F(x) 是 f(x) 在 $[a,b]$ 上的的一个原函数

$$\int_a^b f(x)\ dx = F(b) - F(a) \triangleq [F(x)]_a^b = F(x)|_a^b \tag{14.7.1}$$

## 14.8 换元法

$$f(x) \text{ 在 } [a,b] \text{ 上连续，} x = \varphi(t) \begin{cases} \varphi(\alpha) = a \\ \varphi(\beta) = b \end{cases}$$

$\varphi(t)$在 $[\alpha,\beta]$ 上有连续导数，且$R_\varphi = [a,b]$

$$\int_a^b f(x)\ dx = \int_\alpha^\beta f\left[\varphi(t)\right]\varphi'(t)\ dt \tag{14.8.1}$$

### 14.9 分部积分法

$$\int_a^b u\ dv = [uv]_a^b - \int_a^b v\ du \tag{14.9.1}$$

### 14.10 奇偶函数积分

$$(\text{奇函数})\quad \int_{-a}^{a} f(x)\ dx = 2\int_0^a f(x)\ dx \tag{14.10.1}$$

$$(\text{偶函数})\quad \int_{-a}^{a} f(x)\ dx = 0 \tag{14.10.2}$$

### 14.11 周期函数积分

$$\int_a^{a+T} f(x)\ dx = \int_0^T f(x)\ dx \tag{14.11.1}$$

$$\int_a^{a+nT} f(x)\ dx = n\int_0^T f(x)\ dx \tag{14.11.2}$$

### 14.12 积分定理

$$\int_0^{\frac{\pi}{2}} f(\sin x)\ dx = \int_0^{\frac{\pi}{2}} f(\cos x)\ dx \tag{14.12.1}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x\ dx = \int_0^{\frac{\pi}{2}} \cos^n x\ dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} & (\text{n 偶数}) \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}\cdot 1 & (\text{n 奇数}) \end{cases} \tag{14.12.2}$$

### 14.13 积分不等式

$$\left[\int_a^b f(x)g(x)\ dx\right]^2 \leqslant \int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx \tag{14.13.1}$$

$$\left\{\int_a^b [f(x)+g(x)]^2\ \ dx\right\}^{\frac{1}{2}} \leqslant \left[\int_a^b f^2(x)\ dx\right]^{\frac{1}{2}} + \left[\int_a^b g^2(x)\ dx\right]^{\frac{1}{2}} \tag{14.13.2}$$

### 14.14 一些废话 (显而易见的东西)

$$\text{若在 } [a,b] \text{ 上} f(x) \geqslant 0, \text{且} \int_a^b f(x)\ dx = 0, \text{则} f(x) \equiv 0 \tag{14.14.1}$$

$$\text{若在 } [a,b] \text{ 上} f(x) \geqslant 0, \text{且} f(x) \not\equiv 0, \text{则} \int_a^b f(x)\ dx > 0 \tag{14.14.2}$$

$$\text{若在}\,[a,b]\,\text{上}\,f(x) \leqslant g(x),\text{且}\int_a^b f(x)\ dx = \int_a^b g(x)\ dx\text{则}f(x) = g(x), x \in [a,b] \tag{14.14.3}$$

# 15 反常积分 (瑕积分)

## 15.1 有界反常积分

$$f(x)\text{在}[a,+\infty)\text{上连续}, \lim_{t\to+\infty}\int_a^t f(x)\ dx \begin{cases} \text{存在，称：} \int_a^{+\infty} f(x)\ dx\text{收敛} \\ \text{不存在 (或无穷)，称：为发散} \end{cases}$$

$$\lim_{t\to+\infty}\int_a^t f(x)\ dx = \int_a^{+\infty} f(x)\ dx$$

## 15.2 有界反常积分

$f(x)$在$(a,b]$上连续, 且 $\lim\limits_{x\to a^+} f(x) = \infty f(x)$在$(a,b]$上无上界，点 $a$ 称为 $f(x)$ 的一个瑕点

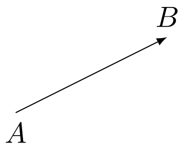
$$\int_a^b f(x)\ dx = \lim_{\varepsilon\to 0^+}\int_{a+\varepsilon}^b f(x)\ dx$$

# 16　向量

## 16.1　向量的概念

向量：既有大小又有方向的量

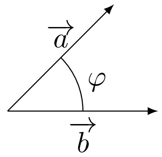
向量表示：$\overrightarrow{AB}$ 或 $\vec{a}$




自由向量：与起点无关的向量

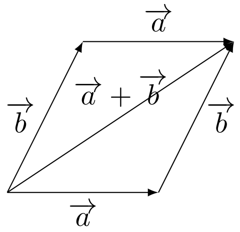
向量的模：$|\overrightarrow{AB}|$　$|\vec{a}|$

单位向量：$|\vec{a}| = 1$




$\vec{a}$单位向量：$\begin{cases} \vec{e}_{\vec{a}} = \dfrac{\vec{a}}{|\vec{a}|} = \dfrac{1}{|\vec{a}|}\vec{a} \\ \vec{a} = |\vec{a}|\vec{e}_{\vec{a}} \end{cases}$

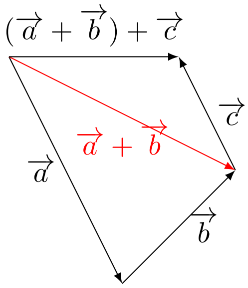
$\vec{0}$向量：$\begin{cases} |\vec{a}| = 0\text{记作：} \vec{a} = \vec{0}\text{方向任意} \\ \vec{0}\text{与任何向量平行} \\ \vec{0}\text{与任何向量垂直} \end{cases}$




向量夹角：$0 \leqslant \varphi \leqslant \pi$　$(\vec{a} \hat{} \vec{b}) = \varphi$

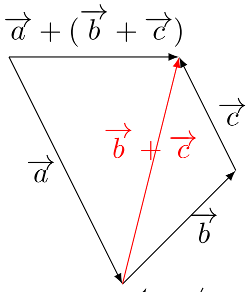
向量平行：$\vec{a} // \vec{b} \Leftrightarrow \begin{cases} (\vec{a} \hat{} \vec{b}) = 0\text{或}(\vec{a} \hat{} \vec{b}) = \pi \\ \vec{a} = \lambda \vec{b} \quad (\text{存在唯一}\lambda) \end{cases}$

向量平行：$\vec{a} \perp \vec{b}$或$(\vec{a} \hat{} \vec{b}) = \dfrac{\pi}{2}$

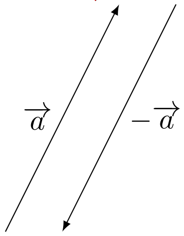



## 16.2　向量的线性运算

三角不等式：$\begin{cases} |\vec{a} + \vec{b}| \leqslant |\vec{a}| + |\vec{b}| \\ |\vec{a} - \vec{b}| \leqslant |\vec{a}| + |\vec{b}| \end{cases}$




向量的加法：$\begin{cases} \vec{a} + \vec{b} \\ \text{交换律：} \vec{a} + \vec{b} = \vec{b} + \vec{a} \\ \text{结合律：} (\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{v}) \end{cases}$




向量的减法：$\begin{cases} \text{负向量：与}\vec{a}\text{大小相等方向相反的向量} - \vec{a} \\ \vec{b} - \vec{a} = \vec{b} + (-\vec{a}) \\ \vec{a} - \vec{a} = \vec{0} \\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \end{cases}$

$$
\text{向量的数乘：}\begin{cases}
\text{大小：} |\lambda\vec{a}| = |\lambda||\vec{a}| \\
\text{方向：}\begin{cases} \lambda > 0 \quad \text{时}\lambda\vec{a}\text{与}\vec{a}\text{方向一致} \\ \lambda < 0 \quad \text{时}\lambda\vec{a}\text{与}\vec{a}\text{方向相反不} \end{cases} \\
\text{特殊数相乘：}\begin{cases} 0\text{：}\ 0\vec{a} = \vec{0} \\ 1\text{：}\ 1\vec{a} = \vec{a} \\ -1\text{：}\ (-1)\vec{a} = -\vec{a} \end{cases} \\
\text{交换律：} \lambda(\mu\vec{a}) = (\lambda\mu)\vec{a} = \mu(\lambda\vec{a}) \\
\text{结合律：}\begin{cases} (\lambda+\mu)\vec{a} = \lambda\vec{a} + \mu\vec{a} \\ \lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b} \end{cases}
\end{cases}
$$

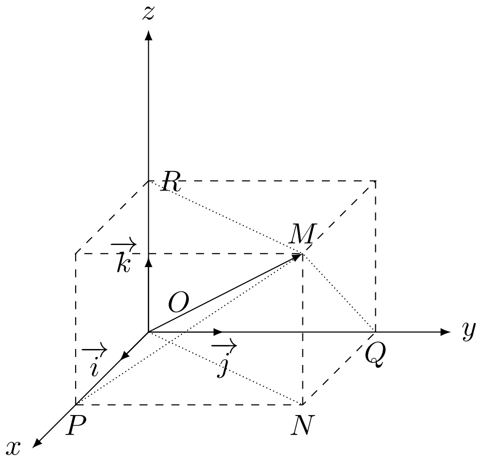
## 16.3 空间直角坐标系

坐标原点：$O$

$$
\text{坐标轴：}\begin{cases} x\text{轴} \quad \text{单位向量}\vec{i} \\ y\text{轴} \quad \text{单位向量}\vec{j} \\ z\text{轴} \quad \text{单位向量}\vec{k} \end{cases}
$$

坐标面：$x \circ y$面 $y \circ z$面 $z \circ x$面

$$
\text{坐标向量运算：}\begin{cases}
\vec{a} = (a_x, a_y, a_z) \quad \vec{b} = (b_x, b_y, b_z) \\
\vec{a} + \vec{b} = (a_x + b_x, a_y + b_y, a_z + b_z) \\
\vec{a} - \vec{b} = (a_x - b_x, a_y - b_y, a_z - b_z) \\
\lambda\vec{a} = (\lambda a_x, \lambda a_y, \lambda a_z)
\end{cases}
$$




$$
\text{方向角：}\begin{cases}
x\text{轴} \quad \cos\alpha = \frac{x}{|\overrightarrow{OM}|} = \frac{x}{\sqrt{x^2+y^2+z^2}} \\
y\text{轴} \quad \cos\beta = \frac{y}{|\overrightarrow{OM}|} = \frac{y}{\sqrt{x^2+y^2+z^2}} \\
z\text{轴} \quad \cos\gamma = \frac{z}{|\overrightarrow{OM}|} = \frac{z}{\sqrt{x^2+y^2+z^2}}
\end{cases}
\qquad
\text{投影：}\begin{cases}
x\text{轴 记：} Prj_x\overrightarrow{OM} = |\overrightarrow{OM}|\cos\alpha \\
y\text{轴 记：} Prj_y\overrightarrow{OM} = |\overrightarrow{OM}|\cos\beta \\
z\text{轴 记：} Prj_z\overrightarrow{OM} = |\overrightarrow{OM}|\cos\gamma
\end{cases}
$$

$$
M\text{点坐标}(x, y, z) \Leftrightarrow \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} \tag{16.3.1}
$$

$$
|\overrightarrow{OM}| = \sqrt{x^2 + y^2 + z^2} \tag{16.3.2}
$$

$$
A(x_1, y_1, z_1) \quad B(x_2, y_2, z_2) \quad |AB| = |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2, (y_2 - y_1)^2, (z_2 - z_1)^2} \tag{16.3.3}
$$

$$(\cos\alpha,\cos\beta,\cos\gamma)=\frac{\overrightarrow{OM}}{|\overrightarrow{OM}|}=\overrightarrow{e}_{\overrightarrow{OM}} \tag{16.3.4}$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1 \tag{16.3.5}$$

$$\text{投影定义}Prj_{\overrightarrow{u}}^{\overrightarrow{a}}=|\overrightarrow{a}|\cos(\overrightarrow{a}\hat{\ }\overrightarrow{u})$$

$$Prj_{\overrightarrow{u}}^{(\overrightarrow{a}+\overrightarrow{b})}=Prj_{\overrightarrow{u}}^{\overrightarrow{a}}+Prj_{\overrightarrow{u}}^{\overrightarrow{b}} \tag{16.3.6}$$

$$Prj_{\overrightarrow{u}}^{\lambda\overrightarrow{a}}=\lambda Prj_{\overrightarrow{u}}^{\overrightarrow{a}} \tag{16.3.7}$$

## 16.4 数量积，向量积，混合积

$$\text{数量积 (内积)}:\ \overrightarrow{a}\bullet\overrightarrow{b}\triangleq|\overrightarrow{a}||\overrightarrow{b}|\cos(\overrightarrow{a}\hat{\ }\overrightarrow{b})\begin{cases}\overrightarrow{a}\bullet\overrightarrow{b}=|\overrightarrow{a}|Prj_{\overrightarrow{a}}^{\overrightarrow{b}}=|\overrightarrow{b}|Prj_{\overrightarrow{b}}^{\overrightarrow{a}}\\ \overrightarrow{a}\bullet\overrightarrow{a}=|\overrightarrow{a}|^2\\ \overrightarrow{a}\perp\overrightarrow{b}\Leftrightarrow\overrightarrow{a}\bullet\overrightarrow{b}=0\\ \overrightarrow{a}\bullet\overrightarrow{b}=\overrightarrow{b}\bullet\overrightarrow{a}\quad(\text{交换律})\\ (\lambda\overrightarrow{a})\bullet\overrightarrow{b}=\overrightarrow{a}\bullet(\lambda\overrightarrow{b})\\ (\overrightarrow{a}+\overrightarrow{b})\bullet\overrightarrow{c}=\overrightarrow{a}\bullet\overrightarrow{c}+\overrightarrow{b}\bullet\overrightarrow{c}\quad\text{分配律}\end{cases}$$

$$\text{坐标表达式：}\overrightarrow{a}\bullet\overrightarrow{b}=a_xb_x+a_yb_y+a_zb_z \tag{16.4.1}$$

$$\text{向量积}:\ |\overrightarrow{a}\times\overrightarrow{b}|\triangleq|\overrightarrow{a}||\overrightarrow{b}|\sin(\overrightarrow{a}\hat{\ }\overrightarrow{b})\begin{cases}\overrightarrow{a}\times\overrightarrow{a}=0\\ \overrightarrow{a}//\overrightarrow{b}\Leftrightarrow\overrightarrow{a}\times\overrightarrow{b}=0\\ \overrightarrow{a}\times\overrightarrow{b}=-\overrightarrow{b}\times\overrightarrow{a}\\ (\lambda\overrightarrow{a})\times\overrightarrow{b}=\overrightarrow{a}\times(\lambda\overrightarrow{b})\\ (\overrightarrow{a}+\overrightarrow{b})\times\overrightarrow{c}=\overrightarrow{a}\times\overrightarrow{c}+\overrightarrow{b}\times\overrightarrow{c}\\ \overrightarrow{a}\times(\overrightarrow{b}+\overrightarrow{c})=\overrightarrow{a}\times\overrightarrow{b}+\overrightarrow{a}\times\overrightarrow{c}\end{cases}$$

$\overrightarrow{a}\times\overrightarrow{b}$是垂直与两个向量的向量，右手 a 到 b，大拇指指的方向

$$\overrightarrow{a}\times\overrightarrow{b}=\begin{vmatrix}a_y&a_z\\b_y&b_z\end{vmatrix}\overrightarrow{i}-\begin{vmatrix}a_x&a_z\\b_x&b_z\end{vmatrix}\overrightarrow{j}+\begin{vmatrix}a_x&a_y\\b_x&b_y\end{vmatrix}\overrightarrow{k} \tag{16.4.2}$$

$$\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$$

$$\text{混合积：}(\overrightarrow{a}\times\overrightarrow{b})\bullet\overrightarrow{c}=\begin{vmatrix}a_y&a_z\\b_y&b_z\end{vmatrix}c_x-\begin{vmatrix}a_x&a_z\\b_x&b_z\end{vmatrix}c_y+\begin{vmatrix}a_x&a_y\\b_x&b_y\end{vmatrix}c_z$$

$$\begin{vmatrix}c_x&c_y&c_z\\a_x&a_y&a_z\\b_x&b_y&b_z\end{vmatrix}$$

## 16.5 空间曲面及其方程

### 16.5.1 概念

$$F(x,y,z)=0$$

与平面垂直的非零向量称为法向量

### 16.5.2 平面表达式

$$\text{法向量}\overrightarrow{n}=(A,B,C)\ \text{平面}\ \pi\ \text{上一点}P_0(x_0,y_0,z_0),M(x,y,z)\text{为平面}\ \pi\ \text{上任意点}$$

$$\begin{aligned}
&\text{平面点法式：}A(x-x_0)+B(y-y_0)+C(z-z_0)=0\\
&\text{平面截距式：}\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1\\
&\text{平面一般方程：}Ax+By+Cz+D=0\begin{cases}D=0\Leftrightarrow\pi\text{过原点}\\A=0\Leftrightarrow\overrightarrow{n}\perp x\text{轴}\Leftrightarrow\pi//x\text{轴}\\B=0\Leftrightarrow\pi//y\text{轴}\\C=0\Leftrightarrow\pi//z\text{轴}\\A=0,B=0\Leftrightarrow\pi//x\circ y\text{面}\end{cases}
\end{aligned}$$

### 16.5.3 平面夹角

$$\begin{gathered}
A_1x+B_1y+C_1z+D_1=0\qquad\overrightarrow{n_1}=(A_1,B_1,C_1)\\
A_2x+B_2y+C_2z+D_2=0\qquad\overrightarrow{n_2}=(A_2,B_2,C_2)\\
\cos\theta=\frac{|\overrightarrow{n_1}\bullet\overrightarrow{n_2}|}{|\overrightarrow{n_1}|\bullet|\overrightarrow{n_2}|}=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{A_1^2+B_1^2+C_1^2}\sqrt{A_2^2+B_2^2+C_2^2}}
\end{gathered}$$

### 16.5.4 点到平面距离

$$\text{平面}\pi:\quad Ax+By+Cz+D=0\ \text{平面外一点}P_0(x_0,y_0,z_0)$$

$$|P_0N|=\left|Prj_{\overrightarrow{n}}\overrightarrow{P_1P_0}\right|=\frac{\left|\overrightarrow{n}\bullet\overrightarrow{P_1P_0}\right|}{|\overrightarrow{n}|}=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

## 16.6 空间线及其方程

### 16.6.1 曲线的一般方程

$$\text{一般方程}\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

### 16.6.2 直线的一般方程

$$\text{一般方程：}\begin{cases} A_1x+B_1y+C_1z+D_1=0 \\ A_2x+B_2y+C_2z+D_2=0 \end{cases}$$

$$\text{对称式：}\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}=t\begin{cases} \overrightarrow{M_0M}=(x-x_0,y-y_0,z-z_0) \\ \text{方向量}\overrightarrow{S}=(m,n,p) \end{cases}$$

$$\text{参数方程：}\begin{cases} x=x_0+mt \\ y=y_0+nt \\ z=z_0+pt \end{cases}$$

### 16.6.3 两直线夹角

两直线的方向向量的夹角称为两直线的夹角 (要求为锐角)
$L_1$： 方向向量$\overrightarrow{S_1}=(m_1,n_1,p_1)$
$L_2$： 方向向量$\overrightarrow{S_2}=(m_2,n_2,p_2)$
$L_1$与$L_2$的夹角$(\overrightarrow{S_1}\hat{\ }\overrightarrow{S_2})$ $(\overrightarrow{S_1}\hat{\ }-\overrightarrow{S_2})$

$$\cos\varphi=\frac{|\overrightarrow{S_1}\bullet\overrightarrow{S_2}|}{|\overrightarrow{S_1}||\overrightarrow{S_2}|}=\frac{|m_1m_2+n_1n_2+p_1p_2|}{\sqrt{m_1^2+n_1^2+p_1^2}\sqrt{m_2^2+n_2^2+p_2^2}}$$

### 16.6.4 线与面夹角

$L$： 方向向量$\overrightarrow{S}=(m,n,p)$
$\pi$： 法向量$\overrightarrow{n}=(A,B,C)$

$$\cos\theta=\frac{|\overrightarrow{S}\bullet\overrightarrow{n}|}{|\overrightarrow{S}||\overrightarrow{n}|}=\frac{|Am+Bn+Cp|}{\sqrt{A^2+B^2+C^2}\sqrt{m^2+n^2+p^2}}$$

$$\sin\varphi=\cos(\frac{\pi-\theta}{2})$$

# 17 多元函数

## 17.1 概念

### 17.1.1 二元函数定义

平面点集 $R^2 = R \cdot R = \{(x,y)|x \in R, y \in R\}$
设$\{D \neq \emptyset\} \subset R^2, \forall P(x,y)$,按照法则$f$都有唯一的实数值与之对应，称$f$是定义在$D$上的二元函数记作

$$z = f(x,y)\begin{cases} x,y\text{自变量} \\ z\text{因变量} \\ D\text{定义域} \\ f(D)\text{值域} \end{cases}$$

### 17.1.2 二元函数极限

$P(x,y) \to P_0(x_0,y_0)$时,$f(x,y) \to A$
$\forall \epsilon > 0, \exists \delta > 0, 0 < |PP_0| < \delta$, 且$P \in D, |f(p) - A| < \delta$成立
称当$P(x,y) \to P_0(x_0,y_0)$时,$f(x,y)$的极限为$A$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = A \quad \text{或} \quad \lim_{P\to P_0} f(P) = A \quad \text{或} \quad \lim_{\substack{x\to x_0 \\ y\to y_0}} f(x,y) = A$$

### 17.1.3 二元函数连续

$f(x,y)$的定义域为$D$ $\quad P_0(x_0,y_0)$为$D$ $\qquad D$

$$\text{如果} \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \text{ 则称} f(x,y)\text{在点}(x_0,y_0)\text{连续}$$

连续函数的和差积商仍为连续函数。连续函数的复合函数仍为连续函数。
多元函数的初等函数：由常数，一元初等函数 (可以不同变量)，有限次四则运算，复合，用一个式子表达的函数。
多元函数在其定义域内都是连续的
断点：$f(x,y)$的定义域为$D, P_0(x_0,y_0)$是$D$的聚点
如果$f(x,y)$在$P_0(x_0,y_0)$点不连续，称$P_0(x_0,y_0)$为$f(x,y)$的一个间断点

### 17.1.4 二元函数偏导

$f(x,y)$ 在 $(x_0,y_0)$ 的某处有定义，固定 $y_0, x$ 在 $x_0$ 处增量 $\Delta x$, 增量 $f(x_0+\Delta x,y_0) - f(x_0,y_0)$ 如果

$$\lim_{\Delta x\to 0} \frac{f(x_0+\Delta x,y_0) - f(x_0,y_0)}{\Delta x}$$

存在则称此极限为 $z=f(x,y)$ 在 $(x_0,y_0)$ 点关于 $x$ 的偏导数, 记作

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} \quad 或 \quad \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} \quad 或 \quad f'_x(x_0,y_0) \quad 或 \quad f'_1(x_0,y_0)$$

$$f'_x(x_0,y_0) \triangleq \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$$

$$f'_y(x_0,y_0) \triangleq \lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta u)-f(x_0,y_0)}{\Delta y}$$

## 17.2 高阶偏导

$$\frac{\partial^2 z}{\partial x^2} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} \triangleq \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} \triangleq \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y\partial x} \triangleq \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial^2 z}{\partial y\partial x} \tag{17.2.1}$$

## 17.3 全微分

$$z=f(x,y)\begin{cases} f(x+\Delta x,y)-f(x,y) & x\text{的偏增量}\\ f(x,y+\Delta y)-f(x,y) & y\text{的偏增量}\end{cases}$$

$\Delta z \triangleq\ f(x+\Delta x,y+\Delta y)-f(x,y)$称为$z=f(x,y)$在点$(x,y)$处的全增量

### 17.3.1 定义

$z=f(x,y)$在点$(x,y)$的某领域有定义, $\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$
如果 $\Delta z=A\Delta x+B\Delta y+\circ(\rho)$, 其中 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$
称 $z=f(x,y)$ 在点 $(x,y)$ 处可微分,$A\Delta x+B\Delta y$ 称为 $z=f(x,y)$ 在 $(x,y)$ 的全微分, 记

$$dz=A\Delta x+B\Delta y$$

一阶全微分

$$dz=\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y \tag{17.3.1}$$

二阶全微分

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2 \tag{17.3.2}$$

## 17.4　多元复合

$$z=f(u(t),v(t))\left\{\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}\right.$$

$$z=f(u(x,y),v(x,y))\begin{cases}\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\\ \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\end{cases}\Rightarrow dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

## 17.5　多元隐函数

### 17.5.1　二元

$F(x,y)$ 在 $(x_0,y_0)$ 的邻域内有连续的偏导数，且 $F(x_0,y_0)=0,F'_y(x_0,y_0)\neq 0$ 则方程 $F(x,y)=0$ 在点 $(x_0,y_0)$ 的邻域内存在唯一的隐函数 $y=y(x)$
$F(x,y)=0,y=y(x)\Rightarrow F(x,y(x))\equiv 0$

$$\frac{dy}{dx}=-\frac{F'_x}{F'_y}$$

### 17.5.2　三元

$F(x,y,z)$ 在 $(x_0,y_0,z_0)$ 的邻域内有连续的偏导数，且 $F(x_0,y_0,z_0)=0,F'_z(x_0,y_0,z_0)\neq 0$ 则方程 $F(x,y,z)=0$ 在点 $(x_0,y_0,z_0)$ 的邻域内存在唯一的隐函数 $z=z(x,y)$
$F(x,y,z)=0,z=z(x,y)\Rightarrow F(x,y,z(x,y))\equiv 0$

$$\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z}\qquad\frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}$$

### 17.5.3　方程组

$$\begin{cases}F(x,y,z)=0\\ G(x,y,z)=0\end{cases}\qquad\frac{dy}{dx}=-\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}}\qquad\frac{dz}{dx}=-\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}\tag{17.5.1}$$

$$\begin{cases}F(x,y,u,v)=0\\ G(x,y,u,v)=0\end{cases}\qquad\frac{dy}{dx}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}\qquad\frac{dz}{dx}=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}\tag{17.5.2}$$

# 18 向量导数

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) \quad t \in [\alpha, \beta] \begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t) \end{cases}$$

$$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k} = (f_1(t), f_2(t), f_3(t))$$

$$\vec{r} = \vec{f}(t) \quad t \in [\alpha, \beta]$$

## 18.1 向量值函数

$$D \in R \quad f : D \to R^n, \text{称} f \text{为一元向量值函数}, \vec{r} = \vec{f}(t) \quad t \in D$$

当 $n = 3$ 的情形 $\quad t \in D \to M(x, y, z)$
$\vec{F} = \overrightarrow{OM}$ t 的变化，M 点轨迹 $\Gamma$(为空间曲线)，称为 $\vec{r} = \vec{f}(t)$ 的图形，$\vec{r} = \vec{f}(t)$ 称为 $\Gamma$ 的方程

## 18.2 极限

$\vec{F} = \vec{f}(t)$ 在某个区新领域有定义 $\vec{r}_0$ 是一个常向量

$$\forall \varepsilon > 0. \exists \delta > 0, 0 < |t - t_0| < \delta, |\vec{f}(t) - \vec{r}_0| < \varepsilon$$

称 $\vec{r}_0$ 为 $\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处的极限 (向量)，记作

$$\lim_{t \to t_0} \vec{f}(t) = \vec{r}_0 \Leftrightarrow \lim_{t \to t_0} \vec{f}(t) = \left( \lim_{t \to t_0} \vec{f}_1(t), \lim_{t \to t_0} \vec{f}_2(t), \lim_{t \to t_0} \vec{f}_3(t) \right)$$

## 18.3 连续

$\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处连续

$$\lim_{t \to t_0} \vec{f}(t) = \lim_{t \to t_0} \vec{f}(t_0)$$

$\vec{r} = \vec{f}(t)$ 在 $D_1 \subset D$, 上每一点连续，称 $\vec{r} = \vec{f}(t)$ 在 $D_1$ 上连续

## 18.4 导数

$\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 某邻域内有定义，如果

$$\lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{f}(t + \Delta t) - \vec{f}(t)}{\Delta t}$$

称此极限为 $\vec{r} = \vec{f}(t)$ 在 $t = t_0$ 处的导向量，记作

$$\vec{f}'(t_0) \quad \text{或} \quad \left. \frac{d\vec{r}}{dt} \right|_{t = t_0}$$

$$\vec{f}'(t) = \left( \vec{f}'_1(t), \vec{f}'_2(t), \vec{f}'_3(t) \right)$$

## 18.5 向量函数求导法则

$$\frac{d\overrightarrow{c}}{dt}=\overrightarrow{0}\quad(\overrightarrow{c}\text{为常数}) \tag{18.5.1}$$

$$\frac{d\left[c\overrightarrow{u}(t)\right]}{dt}=c\frac{\overrightarrow{u}(t)}{dt} \tag{18.5.2}$$

$$\frac{d\left[\overrightarrow{u}(t)\pm\overrightarrow{v}(t)\right]}{dt}=\frac{\overrightarrow{u}(t)}{dt}\pm\frac{\overrightarrow{v}(t)}{dt} \tag{18.5.3}$$

$$\frac{d\left[\varphi(t)\overrightarrow{u}(t)\right]}{dt}=\phi'(t)\overrightarrow{u}(t)+\varphi(t)\overrightarrow{u}'(t) \tag{18.5.4}$$

$$\frac{d\left[\overrightarrow{u}(t)\bullet\overrightarrow{v}(t)\right]}{dt}=\overrightarrow{u}'(t)\bullet\overrightarrow{v}(t)+\overrightarrow{u}(t)\bullet\overrightarrow{v}'(t) \tag{18.5.5}$$

$$\frac{d\left[\overrightarrow{u}(t)\times\overrightarrow{v}(t)\right]}{dt}=\overrightarrow{u}'(t)\times\overrightarrow{v}(t)+\overrightarrow{u}(t)\times\overrightarrow{v}'(t) \tag{18.5.6}$$

$$\frac{d\overrightarrow{u}\left[\varphi(t)\right]}{dt}=\overrightarrow{u}'(\varphi(t))\varphi'(t) \tag{18.5.7}$$

## 18.6 曲线的切线与法平面

$$\Gamma:\begin{cases}x=x(t)\\ y=y(t)\\ z=z(t)\end{cases}\quad t\in[\alpha,\beta],t=t_0\text{对应点}M(x_0,y_0,z_0)$$

$$\text{切向量:}\ \overrightarrow{\Gamma}=(x'(t_0),y'(t_0),z'(t_0))$$

$$\text{切线方程:}\ \frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)}$$

$$\text{法平面方程:}\ x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)$$

## 18.7 曲面的切线与法平面

空间曲面 $\Sigma,F(x,y,z)=0,M(x_0,y_0,z_0)$ 是 $\Sigma$ 上的一点

$$\Gamma\text{参数方程}:\begin{cases}x=x(t)\\ y=y(t)\\ z=z(t)\end{cases}\quad t\in[\alpha,\beta]$$

$$\Gamma\text{在}\Sigma\text{上}\ F(x(t),y(t),z(t))\equiv0,t\in[\alpha,\beta]$$

$$F'_xx'(t)+F'_yy'(t)+F'_zz'(t)=0$$

$t=t_0$对应点M

$$F'_x(x_0,y_0,z_0)x'(t_0)+F'_y(x_0,y_0,z_0)y'(t_0)+F'_z(x_0,y_0,z_0)z'(t_0)=0$$

$\overrightarrow{T}=(x'(t_0),y'(t_0),z'(t_0))$ 是 Γ 过点 M 的一个切向量

$\overrightarrow{n}=(F'_x(x_0,y_0,z_0),F'_y(x_0,y_0,z_0),F'_z(x_0,y_0,z_0))$

$$\overrightarrow{n}\bullet\overrightarrow{T}=0,\ \text{即}\overrightarrow{n}\perp\overrightarrow{T}\quad \overrightarrow{n}\text{与所有切线垂直}\overrightarrow{n}\text{为切平面的法向量}$$

## 18.8 方向导数

$P_0(x_0,y_0)$ 以 $P_0(x_0,y_0)$ 为始的的射线 L，单位向量 $\overrightarrow{e}_l$

$\overrightarrow{e}_l(\cos\alpha,\cos\beta)=\cos\alpha\overrightarrow{i}+\cos\beta\overrightarrow{j}$

$P_0(x,y)$ 是 $L$ 上一点 $\begin{cases} x=x_0+t\cos\alpha \\ y=y_0+t\cos\beta \end{cases}\qquad t>0$

$P_0(x,y)$ 沿 $L$ 方向趋于 $P_0(x_0,y_0)\quad(t\to0^+)\quad z=f(x,y)$

$$\left.\frac{\partial f}{\partial L}\right|_{(x_0,y_0)}\triangleq\lim_{t\to0^+}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}$$

$z=f(x,y)$ 在点 $(x_0,y_0)$ 处沿 L 的方向导数

$$\left.\frac{\partial f}{\partial L}\right|_{(x_0,y_0)}=f'_x(x_0,y_0)\cos\alpha+f'_y(x_0,y_0)\cos\beta \tag{18.8.1}$$

## 18.9 梯度

$$\overrightarrow{\bigtriangledown}f(x_0.y_0)=\overrightarrow{grad}f(x_0.y_0)\triangleq\left(f'_x(x_0,y_0),f'_y(x_0,y_0)\right)$$

$$\overrightarrow{\bigtriangledown}f(x_0.y_0,z_0)=\overrightarrow{grad}f(x_0.y_0,z_0)\triangleq\left(f'_x(x_0,y_0,z_0),f'_y(x_0,y_0,z_0),f'_z(x_0,y_0,z_0)\right)$$

$$\frac{\partial f}{\partial L}_{(x_0,y_0)}=\overrightarrow{\bigtriangledown}f(x_0.y_0)\bullet\overrightarrow{e}_L\qquad\overrightarrow{e}_L=(\cos\alpha,\cos\beta)$$

## 18.10 极值

$$\text{驻点}\begin{cases} f'_x(x,y)=0 \\ f'_y(x,y)=0 \end{cases}$$

$$\text{记}\begin{cases} A=f''_{xx}(x_0,y_0) \\ B=f''_{xy}(x_0,y_0) \\ C=f''_{yy}(y_0,y_0) \end{cases}\qquad\Delta=AC-B^2\begin{cases} \Delta>0,z=f(x,y)\text{在}(x_0,y_0)\text{点}\begin{cases} A>0 & \text{极小值} \\ A<0 & \text{极大值} \end{cases} \\ \Delta=0,\text{未定} \\ \Delta=0,z=f(x,y)\text{在}(x_0,y_0)\text{点无极值} \end{cases}$$

# 19 重积分

## 19.1 二重积分

### 19.1.1 定义

区域 $D$,$D$ 上函数 $z=f(x,y)$,$D$ 分割成 $\Delta\sigma_1,\Delta\sigma_2,\cdots,\Delta\sigma_n$,$(\xi_i,\eta_i)\sigma_i$
如果 $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i$ 存在，称极限为 $f(x,y)$ 在 $D$ 上的二重积分，记作

$$\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i,\eta_i)\Delta\sigma_i\triangleq\iint\limits_D f(x,y)d\sigma\begin{cases}f(x,y) & \text{被积函数}\\ D & \text{积分区域}\\ x,y & \text{积分变量}\\ f(x,y)d\sigma & \text{积分表达式}\\ d\sigma & \text{面积元素}=\begin{cases}dxdy & \text{直角坐标坐标系}\end{cases}\end{cases}$$

### 19.1.2 性质

$$\iint\limits_D \alpha f(x,y)d\sigma=\alpha\iint\limits_D f(x,y)d\sigma\quad \alpha\text{为常数}$$

$$\iint\limits_D[f(x,y)\pm g(x,y)]d\sigma=\iint\limits_D f(x,y)d\sigma\pm\iint\limits_D g(x,y)d\sigma$$

$$\iint\limits_D f(x,y)d\sigma=\iint\limits_{D_1}f(x,y)d\sigma+\iint\limits_{D_2}f(x,y)d\sigma\quad D=D_1+D_2$$

$$\iint\limits_D d\sigma=\iint\limits_D 1d\sigma=\sigma$$

$$f(x,y)\leqslant g(x,y)\Rightarrow\iint\limits_D f(x,y)d\sigma\leqslant\iint\limits_D g(x,y)d\sigma$$

$$\left|\iint\limits_D f(x,y)d\sigma\right|\leqslant\iint\limits_D|f(x,y)|d\sigma$$

$$m,M\text{分别为}f(x,y)\text{在}D\text{上的最小值与最大值，则}\quad m\sigma\leqslant\iint\limits_D f(x,y)d\sigma\leqslant M\sigma$$

$$f(x,y)\text{在}D\text{上连续，则至少存在一点}(\xi,\eta)\text{使}\iint\limits_D f(x,y)d\sigma=f(\xi,\eta)\sigma$$

### 19.1.3 换源法

$f(x,y)$ 在 $D$ 上连续， 变换 $T\begin{cases} x=x(u,v) \\ y=y(u,v)\end{cases} \qquad D'\to D$

满足 $T:D'\to D$ 是一一对应的， $x(u,v),y(uv)$ 在 $D'$ 上有一阶连续偏导

在 $D$‘ 上 $J(u,v)=\frac{\partial(x,y)}{\partial(u,v)}\neq 0$

$$\iint\limits_{D} f(x,y)dxdy=\iint\limits_{D'} f(x(u,v),y(u,v))|J|dudv \tag{19.1.1}$$

### 19.1.4 奇偶性

$f(x,y)$ 关于 x 为奇函数，且 D 关于 y 轴对称，则

$$\iint\limits_{D} f(x,y)dxdy=0$$

$f(x,y)$ 关于 x 为偶函数，且 D 关于 y 轴对称，则

$$D=D_1+D_2 \qquad D_1,D_2\text{关于 y 轴对称} \qquad \iint\limits_{D} f(x,y)dxdy=2\iint\limits_{D_1} f(x,y)dxdy$$

## 19.2 三重积分

空间区域 $\Omega$, 在一个有界函数 $f(x,y,z)$ 分割 $\Omega,\Delta v_1,\Delta v_2,\cdots\Delta V_n$ 任取 $(\xi_i,\eta_i,\zeta_i)\in v_i$

$\lim\limits_{\lambda\to 0}\sum\limits_{n}^{i=1} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$ （$\lambda$ 分割的直径) 若极限存在

则称其为 $f(x,y,z)$ 在 $\Omega$ 上的三重积分, 记作

$$\iiint\limits_{\Omega} f(x,y,z)dv\triangleq\lim_{\lambda\to 0}\sum_{n}^{i=1} f(\xi_i,\eta_i,\zeta_i)\Delta v_i$$

# 20 曲线及曲面积分

## 20.1 曲线积分定义

### 20.1.1 弧长曲线积分

$x \circ y$ 面上曲线 L(分段光滑)，$f(x,y)$ 在 L 上有界，对 L 进行分割，取积，求和，取极限，如果 $\lim\limits_{\lambda\to 0}\sum\limits_{i=0}^{n} f(\xi_i,\eta_i)\Delta S_i$ 存在，称此极限为 $f(x,y)$ 在 L 上对弧长的曲线积分，记作

$$\int_L f(x,y)ds \triangleq \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta S_i$$

### 20.1.2 封闭曲线积分

如果 L 是封闭曲线，记作

$$\oint_L f(x,y)ds$$

## 20.2 性质

$$\int_L [\alpha f(x,y)+\beta g(x,y)]ds = \alpha\int_L f(x,y)ds + \beta\int_L g(x,y)ds$$

$$L = L_1 + L_2 \qquad \int_L f(x,y)ds = \int_{L1} f(x,y)ds + \int_{L2} f(x,y)ds$$

$$f(x,y) \leqslant g(x,y) \qquad \int_L f(x,y)ds \leqslant \int_L g(x,y)ds$$

$$\left|\int_L f(x,y)ds\right| \leqslant \int_L |f(x,y)|ds$$

$$f(x,y)\text{在}L\text{上连续, 则至少窜在一点}(\xi,\eta)\in L\text{使}\int_L f(x,y)ds = f(\xi,\eta)l$$

## 20.3 弧微分计算

$f(x,y)$ 在 L 上连续，L 的参数方程 $\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad (\alpha \leqslant t \leqslant \beta)$

若 $\frac{dx}{dt}, \frac{dy}{dt}$ 在 $[\alpha,\beta]$ 上连续，且 $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 \neq 0$ 则

$$\int_L f(x,y)ds = \int_\alpha^\beta f(x(t),y(t))\sqrt{[x'(t)]^2+[y'(t)]^2}dt$$

$$L\text{极坐标参数方程}\begin{cases} x = \rho(\theta)\cos\theta \\ y = \rho(\theta)\sin\theta \end{cases} \quad ds = \sqrt{(\rho d\theta)^2+(d\rho)^2} = \sqrt{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2}d\theta$$

## 20.4 对坐标曲线积分

### 20.4.1 定义

L 是 $x\circ y$ 面上的有向光滑曲线，$P(x,y),Q(x,y)$ 在 $L$ 有界，若 $\lim\limits_{\lambda\to 0}\sum\limits_{i=1}^{n}P(\xi_i,\eta_i)\Delta x_i$ 存在则称此处极限为 $P(x,y)$ 在 L 上对坐标 x 的曲线积分，记作

$$\int_L P(x,y)dx \triangleq \lim_{\lambda\to 0}\sum_{i=1}^{n}P(\xi_i,\eta_i)\Delta x_i$$

同理定义 $Q(x,y)$ 在 L 上对坐标 y 的曲线积分，记作

$$\int_L Q(x,y)dy \triangleq \lim_{\lambda\to 0}\sum_{i=1}^{n}Q(\xi_i,\eta_i)\Delta y_i$$

$$w=\int_L P(x,y)dx+Q(x,y)dy$$

对坐标的曲线积分称为第二类曲线积分

$$\overrightarrow{r}=\overrightarrow{OM}\qquad d\overrightarrow{r}=(dx,dy)=dx\overrightarrow{i}+dy\overrightarrow{i}$$

$$\int_L Pdx+Qdy=\int_L\overrightarrow{F}\bullet d\overrightarrow{r}$$

### 20.4.2 推广

$$\begin{aligned}
\overrightarrow{F}=&P(x,y,z)\overrightarrow{i}+Q(x,y,z)\overrightarrow{j}+R(x,y,z)\overrightarrow{k}\\
w=&\lim_{\lambda\to 0}\sum_{i=1}^{n}\overrightarrow{F}\bullet d\overrightarrow{r}\\
=&\lim_{\lambda\to 0}\sum_{i=1}^{n}\overrightarrow{F}(\xi_i,\eta_i,\zeta_i)\bullet d\overrightarrow{M_{i-1}M_i}\\
=&\lim_{\lambda\to 0}\sum_{i=1}^{n}P(\xi_i,\eta_i,\zeta_i)\Delta x_i+Q(\xi_i,\eta_i,\zeta_i)\Delta y_i+R(\xi_i,\eta_i,\zeta_i)\Delta z_i\\
=&\int_\tau Pdx+Qdy+Rdz
\end{aligned}$$

### 20.4.3 性质

$$\int_L\left[\alpha\overrightarrow{F_1}(x,y)+\beta\overrightarrow{F_2}(x,y)\right]\bullet d\overrightarrow{r}=\alpha\int_L\overrightarrow{F_1}\bullet d\overrightarrow{r}+\beta\int_L\overrightarrow{F_2}\bullet d\overrightarrow{r}$$

$$L+L_1+L_2\qquad\int_L Pdx+Qdy=\int_{L_1}Pdx+Qdy+\int_{L_2}Pdx+Qdy$$

$$L^-\text{为有向曲线}L\text{的反向曲线}\qquad\int_{L^-}\overrightarrow{F}d\overrightarrow{r}=-\int_L\overrightarrow{F}d\overrightarrow{r}$$

### 20.4.4 计算

$P(x,y),Q(x,y)$ 在有向光滑曲线 $L$ 上连续，$L$ 的参数方程为 $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$

$t:\ \alpha\mapsto\beta$ 起点对应 $\alpha$, 终点对应 $\beta$, $x'(t),y'(t)$ 在 $\alpha$ 到 $\beta$ 区间连续，且 $[x'(t)]^2+[y'(t)]^2\neq 0$ 则

$$\int_L P(x,y)dx+Q(x,y)dy=\int\limits_{\alpha}^{\beta} P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)dt$$

## 20.5 两类曲线积分的关系

$L$曲线的参数方程$\begin{cases} x=x(t) \\ y=y(t) \end{cases}$ $\quad t:\ A\mapsto B$

$[x'(t)]^2+[y'(t)]^2\neq 0$ $\quad\vec{\tau}=(x'(t),y'(t))$为$L$在$(x(t),y(t))$处的一个切向量，方向与增加方向一致

$$\vec{T}\triangleq\frac{\vec{\tau}}{|\vec{\tau}|}=\left(\frac{x'(t)}{\sqrt{[x'(t)]^2+[y'(t)]^2}},\frac{y'(t)}{\sqrt{[x'(t)]^2+[y'(t)]^2}}\right)=(\cos\alpha,\cos\beta)$$

$\int_L P(x,y)dx+Q(x,y)dy$
$=\int\limits_A^B[P(x(t),y(t))x'(t)+Q(x(t),y(t))y'(t)]dt$
$=\int\limits_A^B\left[P(x(t),y(t))\frac{x'(t)}{\sqrt{[x'(t)]^2+[y'(t)]^2}}+Q(x(t),y(t))\frac{y'(t)}{\sqrt{[x'(t)]^2+[y'(t)]^2}}\right]\sqrt{[x'(t)]^2+[y'(t)]^2}dt$
$=\int_L[P\cos\alpha+Q\cos\beta]ds$

## 20.6 格林公式

区域的正向 $\begin{cases} \text{单连通区域} & \text{逆时针方向} \\ \text{复连通区域} & \begin{cases} \text{外边界} & \text{逆时针} \\ \text{内边界} & \text{顺时针} \end{cases} \end{cases}$

$D$ 是由分段光滑曲线 $L$ 围成，P(x,y),Q(x,y) 在 D 上有一阶连续偏导，则

$$\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_L Pdx+Qdy \tag{20.6.1}$$

## 20.7 格林公式求面积

$$A=\frac{1}{2}\oint_L xdy-ydx \tag{20.7.1}$$

## 20.8 曲面积分

$z = z(x, y)$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy \tag{20.8.1}$$

## 20.9 坐标曲面积分

规定了曲面法向量方向的曲面称为：有向曲面，双侧曲面 $\begin{cases} \text{外侧，内侧} \\ \text{上侧，内侧} \\ \text{左侧，右侧} \\ \text{前侧，后侧} \end{cases}$

## 20.10 高斯积分

$$\oiint_{\Sigma} Pdydz + Qdzdx + Rdxdy = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

## 20.11 总结

线 $\alpha \leqslant t \leqslant \beta \begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$$\int_L f(x, y)ds = \int_{\alpha}^{\beta} f(x(t), y(t))\sqrt{(x_t')^2 + \ y_t')^2}dt$$

$a \leqslant t \leqslant b \begin{cases} x = x \\ y = y(t) \end{cases}$

$$\int_L f(x, y)ds = \int_a^b f(x, y(t))\sqrt{1 + \ y_t')^2}dt$$

$\alpha \leqslant t \leqslant \beta \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

$$\int_L f(x, y, z)ds = \int_{\alpha}^{\beta} f(x(t), y(t))\sqrt{(x_t')^2 + \ y_t')^2 + (z_t')^2}dt$$

坐标曲线积分

$$\int_{L^-} P(x.y)dx + Q(x, y)dy = -\int_L P(x.y)dx + Q(x, y)dy$$

$\alpha \leqslant t \leqslant \beta \begin{cases} x = x(t) \\ y = y(t) \end{cases}$

$$\int_L P(x.y)dx + Q(x,y)dy = \int_\alpha^\beta \left[P(x(t).y(t))x'(t) + Q(x(t),y(t))y'(t)\right] dt$$

面 $z = z(x,y)$ $z - z(x,y) = 0$

$$\iint\limits_\Sigma f(x,y,z)ds = \iint_{Dxy} f(x,y,z(x,y))\sqrt{1 + (z_x')^2 + (z_y)'^2}dxdy$$

坐标 $\iint\limits_{\Sigma^-} P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy =$

$-\iint\limits_\Sigma P(x,y,z)dydz + Q(x,y,z)dzdx + R(x,y,z)dxdy$

$\overrightarrow{v} = P\overrightarrow{i} + Q\overrightarrow{j} + R\overrightarrow{k}$

$$z = z(xy) \quad \iint_\Sigma R(x,y,z)dxdy = \pm \int_{Dxy} R(x,y,z(xy))dxd$$

关系

$$\int_L P(x,y)dx + Q(x,y)dy = \int_L \left[P(x,y)\cos\alpha + Q(x,y)\cos\beta\right] ds$$

$\overrightarrow{T} = (\cos\alpha, \cos\beta)$

$$\int_L P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_L \left[P(x,y,z)\cos\alpha + Q(x,y,z)\cos\beta + R(x,y,z)\gamma\right] ds$$

$\overrightarrow{T} = (\cos\alpha, \cos\beta, \cos\gamma)$

$$\iint_\sigma Pdydz + Qdzdx + Rdxdy = \iint\limits_\sigma \left(P\cos\alpha + Q\cos\beta + R\cos\gamma\right) ds$$

$\overrightarrow{n} = (\cos\alpha, \cos\beta, \cos\gamma)$

曲线与二重积分之间的关系格林公式

$$\oint\limits_L Pdx + Qdy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

1)L 是 D 的正向边界 2）PQ 在 D 上有连续一阶偏导曲面积分与三重积分之间的关系高斯公式

$$\oint\limits_\sigma Pdx + QdyRdz = \iiint\limits_\omega \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz$$

1)$\sigma$ 是 $\omega$ 的正向边界曲面

2）PQR 在 $\omega$ 上有连续的一阶偏导

下面 4 个命题等价 (单连通情况下)

1）$\int_L Pdx + Qdy$ 在 D 上路径无关

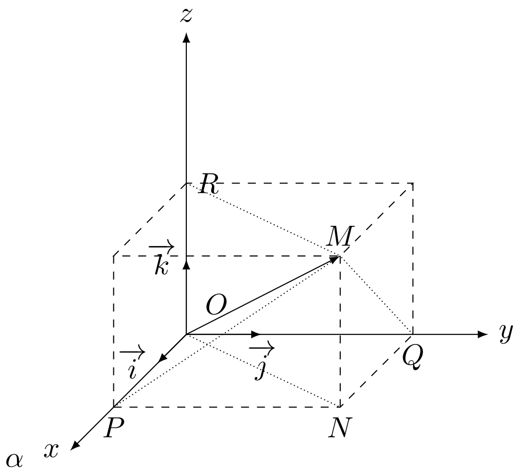
2)$\oint_L Pdx + Qdy = 0$

3)$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}$ $(x,y) \in D$

4) 存在 $u(x,y)$ 使 $du(x,y) = pdx + Qdy$

# 21 极数

## 21.1 常数项级数的概念与性质



# 22 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{22.0.1}$$

---

$$A_N = \sum_{k=0}^{n} q^k \qquad q \cdot A_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - q \cdot A_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1-q^{n+1}}{1-q}$$

---

$$\log_{10} x = \lg_x \tag{22.0.2}$$

$$\log_e x = \ln_x \tag{22.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{22.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{22.0.5}$$

$$\log_b x^n = n \log_b x \tag{22.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{22.0.7}$$

---

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

---

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

---

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

---

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

---

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

---

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a-b) \sum_{m=0}^{n-1} \left(a^{n-m} b^m\right) = (a-b)\left(a^{n-1} + a^{n-2} b + \cdots + a b^{n-2} + b^{n-1}\right)$$

# 23 证明

## 23.1 第 1章

### 1.2.4

$$
\begin{aligned}
\sinh x \cosh x &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= \left(\frac{1}{2}\right)\left(\frac{e^{2x} - e^{-2x}}{2}\right) \\
&= \frac{1}{2}\sinh(2x) \\
\sinh(2x) &= 2\sinh x \cosh x
\end{aligned}
$$

### 1.2.5

$$
\begin{aligned}
\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\
&= e^x \times e^{-x} \\
&= 1
\end{aligned}
$$

### 1.2.6

$$
\begin{aligned}
\cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2 \\
&= \frac{2e^{2x} + 2e^{-2x}}{4} \\
&= \frac{e^{2x} + e^{-2x}}{2} \\
&= \cosh(2x)
\end{aligned}
$$

### 1.2.7

$$
\begin{aligned}
\cosh(2x) &= \cosh^2 x + \sinh^2 x \\
&= \sinh^2 x + 1 + \sinh^2 x \\
&= 2\sinh^2 x + 1 \\
\cosh x &= 2\sinh^2 \frac{x}{2} + 1
\end{aligned}
$$

**1.1.17**

$$\begin{aligned}
\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} &= \sqrt{\frac{1-\cos x}{1+\cos x}} \\
&= \sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1+\cos x}{1+\cos x}} \\
&= \sqrt{\frac{1-\cos^2 x}{(1+\cos x)^2}} \\
&= \frac{\sin x}{1+\cos x} \\
&= \frac{1-cosx}{\sin x} \\
&= \csc x-\cot x
\end{aligned}$$

## 23.2 第 5章

**5.2.1**

$$\begin{aligned}
&\text{设}\forall\ x_1,x_2\in[a,b],x_1<x_2 \\
&f(x_2)-f(x_1)=f'(\xi)(x_2-x_1) \qquad \xi\in(x_1,x_2)\subset[a,b] \\
&f'(\xi)>0,\ (x_2-x_1)>0 \\
&f(x_2)-f(x_1)>0 \\
&f(x_2)>f(x_1)
\end{aligned}$$

**5.2.2**

$$\begin{aligned}
&\text{设}\forall\ x_1,x_2\in[a,b],x_1<x_2 \\
&f(x_2)-f(x_1)=f'(\xi)(x_2-x_1) \qquad \xi\in(x_1,x_2)\subset[a,b] \\
&f'(\xi)<0,\ (x_2-x_1)>0 \\
&f(x_2)-f(x_1)<0 \\
&f(x_2)<f(x_1)
\end{aligned}$$

**5.2.3**

$$\begin{gathered}
\text{设}\forall\ x_1,x_2\in[a,b],x_1<x_2,x_0=\tfrac{x_1+x_2}{2},x_0-x_1=x_2-x_0=h \\
\varphi=\ f(x_0)-f(x_1)=f'(\xi_1)(x_0-x_1) \qquad \xi_1\in(x_1,x_0) \\
\psi=f(x_2)-f(x_0)=f'(\xi_2)(x_2-x_0) \qquad \xi_2\in(x_0,x_2)
\end{gathered}$$

$$\psi - \varphi = f(x_2) + f(x_1) - 2f(x_0) = \left[f'(\xi_2) - f'(\xi_1)\right] h$$
$$= f''(\xi)(\xi_2 - \xi_1)h$$
$$\text{因为 } f''(x) > 0, f''(\xi) > 0, h = x_0 - x_1 > 0$$
$$f(x_2) + f(x_1) - 2f(x_0) > 0$$
$$f(x_2) + f(x_1) > 2f(x_0)$$
$$f(x_0) < \frac{f(x_2) + f(x_1)}{2}$$
$$f(\frac{x_1 + x_2}{2}) < \frac{f(x_2) + f(x_1)}{2}$$

**5.3.1**

$$\triangle\ s = \widehat{M_0M'} - \widehat{M_0M} = \widehat{MM'}, \quad \left|MM'\right|^2 = (\triangle\ x)^2 + (\triangle\ y)^2, \quad \lim_{M' \to M} \frac{\left|\widehat{MM'}\right|}{|MM'|} = 1$$

$$\left(\frac{\triangle\ s}{\triangle\ x}\right)^2 = \left|\frac{\widehat{MM'}}{\triangle\ x}\right|^2 = \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left(\frac{|MM'|}{\triangle\ x}\right)^2$$
$$= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \frac{(\triangle\ x)^2 + (\triangle\ y)^2}{(\triangle\ x)^2}$$
$$= \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$\lim_{\triangle x \to 0} \left(\frac{\triangle\ s}{\triangle\ x}\right)^2 = \lim_{\triangle x \to 0} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$(\triangle\ x \to 0, \triangle\ M' \to M) \qquad = \lim_{M' \to M} \left(\frac{\widehat{MM'}}{|MM'|}\right)^2 \cdot \lim_{\triangle x \to 0} \left[1 + \left(\frac{\triangle\ y}{\triangle\ x}\right)^2\right]$$
$$\left(\frac{ds}{dx}\right)^2 = 1 \cdot (1 + (y')^2)$$
$$\frac{ds}{dx} = \sqrt{1 + (y')^2} = \sqrt{1 + [f'(x)]^2}$$
$$ds = \sqrt{1 + [f'(x)]^2} dx = \sqrt{(dx)^2 + (dy)^2}$$

**5.3.2**

$$\left|\frac{d\alpha}{ds}\right| = \left|\frac{d\alpha}{dx} \cdot \frac{dx}{ds}\right|$$
$$= \left|\frac{d \arctan y'}{dx} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right|$$
$$= \left|\frac{y''}{1 + (y')^2} \cdot \frac{1}{\sqrt{1 + (y')^2}}\right|$$
$$= \frac{|y''|}{(1 + (y')^2)^{\frac{3}{2}}}$$

**5.3.3**

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\psi'(t)}{\phi'(t)} \\
\frac{d^2y}{dx^2} &= \frac{d\frac{\psi'(t)}{\phi'(t)}}{dt} \cdot \frac{dt}{dx} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^2} \cdot \frac{1}{\phi'(t)} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3}
\end{aligned}$$

$$\begin{aligned}
\left|\frac{d\alpha}{ds}\right| &= \frac{|y''|}{(1+(y')^2)^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{[\phi'(t)]^3} \cdot \frac{1}{\left\{1+\left[\frac{\psi'(t)}{\phi'(t)}\right]^2\right\}^{\frac{3}{2}}} \\
&= \frac{\psi''(t)\phi'(t) - \psi'(t)\phi''(t)}{\left\{|\psi'(t)|^2 + [\phi'(t)]^2\right\}^{\frac{3}{2}}}
\end{aligned}$$

## 23.3 第 8章

**8.1.1**

$$\text{反设} \lim_{n\to\infty} x_n = a,\ \lim_{n\to\infty} x_n = b, \text{且} a < b$$

$$\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1,\ n > N_1,\ |x_n - a| < \frac{b-a}{3} \\ \exists N_2,\ n > N_2,\ |x_n - b| < \frac{b-a}{3} \end{cases}$$

$$N = \max\{N_1, N_2\},\ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases}$$

$$\begin{aligned}
b - a &= |(x_n - a) - (x_n - b)| \\
&\leqslant |x_n - a| + |x_n - b| \\
&< \frac{b-a}{3} + \frac{b-a}{3} \\
&< \frac{2(b-a)}{3}
\end{aligned}$$

**8.1.2**

$$\varepsilon = 1,\ \exists N > 0,\ \text{当 n>N 时}\ |X_n - a| < 1$$

$$\begin{aligned}
|X_n| &= |(X_n - a) + a| \\
&\leqslant |x_n - a| + |a| \\
&\leqslant 1 + |a|
\end{aligned}$$

$$M = \max\{|X_n|, |X_2|, \ldots, |X_n|, 1 + |a|\}$$
$$\forall n,\ |X_n| \leqslant M$$

**8.1.4**

1

由于 $\lim\limits_{n\to\infty} x_n = a$, 且$a > 0$
$\varepsilon = \frac{a}{2}$, $\exists N > 0, n > N$
$|x_n - a| < \varepsilon$
$|x_n - a| < \frac{a}{2}$
$-\frac{a}{2} < x_n - a < \frac{a}{2}$
$\frac{a}{2} < x_n < 1$

2

用反证法，反设 $a < 0$. 从某项起 $x_n < 0$ 矛盾

**8.1.5**

$$x_n = b_n - a_n$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} x_n = b - a > 0$$
$$\lim_{n\to\infty} x_n > 0$$
$$b_n - a_n = x_n > 0$$
$$b_n > a_n$$

**8.2.1**

$$\lim_{x\to x_0} f(x)\text{存在} \Rightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$$
$$\text{设} \lim_{x\to x_0} = A$$
$$0 < |x - x_0| < \delta,\ |f(x) - A| < \varepsilon$$
$$0 < |x - x_0| < \delta \Leftrightarrow x \in \mathring{U}(x_0, \delta)$$
$$\begin{cases} \text{当}x_0 < x < x_0 + \delta\text{时}0 < |x - x_0| < \delta,\ |f(x) - A| < \varepsilon,\ \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当}x_0 - \delta < x < x_0\text{时}0 < |x - x_0| < \delta,\ |f(x) - A| < \varepsilon,\ \lim\limits_{x\to x_0^-} f(x) = A \end{cases}$$
$$\lim_{x\to x_0^+} f(x) = A = \lim_{x\to x_0^-} f(x)$$

$$\lim_{x\to x_0} f(x)\text{存在} \Leftarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x)$$

$$A=\begin{cases}\lim\limits_{x\to x_o^+},\forall\varepsilon>0,\exists\delta_1>0,x_0<x<x_0+\delta_1,|f(x)-A|<\varepsilon\\ \lim\limits_{x\to x_o^-},\forall\varepsilon>0,\exists\delta_2>0,x_0-\delta_2<x<x_0,|f(x)-A|<\varepsilon\end{cases}$$

$$\delta=\min\{\delta_1,\delta_2\}$$

$$0<|x-x_0|<\delta\begin{cases}x>x_0,x_0<x<x_0+\delta\leqslant x_0+\delta_1,\ |f(x)-A|<\varepsilon\\ x<x_0,x_0-\delta_2\leqslant x_0+\delta<x<x_0,\ |f(x)-A|<\varepsilon\end{cases}$$

$$\lim_{x\to x_0}f(x)=A$$

**8.3.1**

$$\lim_{x\to x_o}f(x)=A\Rightarrow\begin{cases}\alpha\text{为}x\to x_0\text{时的无穷小}\\ f(x)=\alpha+A\end{cases}$$

$$\text{设}\lim_{x\to x_o}f(x)=A,\ \text{记}f(x)-A=\alpha$$

只需证 $\alpha$ 为无穷小。

$$\forall\varepsilon>0,\exists\delta>0,\text{当}0<|x-x_0|<\delta,\text{时}\,|f(x)-A|<\varepsilon$$

$$\text{即 }|\alpha-0|<\varepsilon$$

$$\alpha\text{为}x\to x_0\text{时的无穷小}$$

$$\lim_{x\to x_o}f(x)=A\Leftarrow\begin{cases}\alpha\text{为}x\to x_0\text{时的无穷小}\\ f(x)=\alpha+A\end{cases}$$

$$\forall\varepsilon>0,\ \exists\delta>0,\ \text{当}0<|x-x+0|<\delta,\ |\alpha|<\varepsilon$$

$$\text{即}\,|f(x)-A|<\varepsilon\ \lim_{x\to x_0}f(x)=A$$

**8.3.2**

$$\text{设}\lim_{x\to x_0}f(x)=\infty$$

$$\text{对 }f(x)\text{ 为 }x\to\text{ 时无穷大}$$

$$\text{对于 }M=\tfrac{1}{\varepsilon}.\text{ 存在 }\delta>0$$

$$\text{当 }0<|x-x_0|<\delta\text{ 时}$$

$$|f(x)|>M=\tfrac{1}{\varepsilon}$$

$$\left|\tfrac{1}{f(x)}\right|<\varepsilon$$

$$\tfrac{1}{f(x)}\text{ 为 }x\to x_0\text{ 时的无穷小}$$

**8.4.2**

$$\begin{aligned}f(x)g(x)&=[A+\alpha]\,[B+\beta]\\&=AB+A\beta+B\alpha+\beta\alpha\\&=AB+\gamma\qquad\qquad(\gamma\text{为无穷小})\\ \lim\,[f(x)g(x)]&=AB+\gamma=\lim f(x)\lim g(x)\end{aligned}$$

**8.4.8**

$$\forall \varepsilon > 0$$
$$|x_n - a| < \varepsilon \qquad \forall n > N_1$$
$$|y_n - a| < \varepsilon \qquad \forall n > N_2$$
$$\text{令} N = \max\{N_1, N_2, N_0\}\text{，则当} n > N \text{时有}$$
$$a - \varepsilon < x_n \le z_n \le y_n < a + \varepsilon$$
$$|z_n - a| < \varepsilon$$
$$\lim_{n\to\infty} z_n = a$$

**8.4.9**

$$\begin{aligned}
|f(x) - \sin x_0| &= |\sin x - \sin x_0| \\
&= \left|2\cos(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\frac{|x-x_0|}{2} = |x-x_0|
\end{aligned}$$

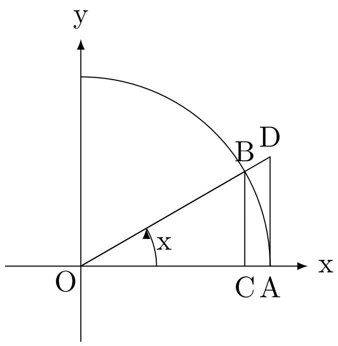
$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\sin x - \sin x_0| \leqslant |x - x_0| < \varepsilon$$

**8.4.10**

$$\begin{aligned}
|f(x) - \cos x_0| &= |\cos x - \cos x_0| \\
&= \left|-2\sin(\frac{x+x_0}{2})\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\left|\sin(\frac{x-x_0}{2})\right| \\
&\leqslant 2\frac{|x-x_0|}{2} = |x-x_0|
\end{aligned}$$

$$\forall \varepsilon, \exists \delta = \varepsilon, \text{当} 0 < |x - x_0| < \delta \text{时}$$
$$|\cos x - \cos x_0| \leqslant |x - x_0| < \varepsilon$$

**8.4.11**




$$
\begin{aligned}
OB = OA = 1 \\
\triangle\ AOB \leqslant \text{扇形面积} \leqslant \triangle\ AOD \\
\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x \\
\sin x \leqslant x \leqslant \tan \\
1 \geqslant \frac{\sin x}{x} \geqslant \cos x \\
\lim_{x\to 0} 1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant \lim_{x\to 0}\cos x \\
1 \geqslant \lim_{x\to 0}\frac{\sin x}{x} \geqslant 1 \\
\lim_{x\to 0}\frac{\sin x}{x} = 1
\end{aligned}
$$

**8.4.12**

$$
\begin{aligned}
|1-\cos x| = 1-\cos x = 2\sin^2\frac{x}{2} \leqslant 2\left(\frac{x}{2}\right)^2 \\
0 \leqslant 1-\cos x \leqslant \frac{x^2}{2} \\
\lim_{x\to 0} 0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant \lim_{x\to 0}\frac{x^2}{2} \\
0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant 0 \\
\lim_{x\to 0}(1-\cos x) = 0 \\
\lim_{x\to 0}\cos x = 1
\end{aligned}
$$

**8.4.13**

$$
\begin{aligned}
\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x} \\
&= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} \\
&= 1
\end{aligned}
$$

**8.4.14**

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2} &= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}\\ &= \lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\\ &= 1\end{aligned}$$

**8.4.15**

$$x=\sin t,\ t=\arcsin x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=\lim_{x\to 0}\frac{t}{\sin t}=1$$

**8.4.16**

$$x=\tan t,\ t=\arctan x$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{\arctan x}{x}=\lim_{t\to 0}\frac{t}{\tan t}=1$$

**8.4.17**

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln e=1$$

**8.4.18**

$$e^x-1=t,\ x=\ln(t+1)$$

$$x\to 0,\ t\to 0$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{t\to 0}\frac{t}{\ln(t+1)}=1$$

**8.4.19**

$$\lim_{x\to 0}\frac{(1+x)^n-1}{nx}=\lim_{x\to 0}\left(\frac{e^{n\ln(1+x)}-1}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right)=1$$

**8.4.21**

$$
\begin{aligned}
x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m}\left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0\left(\frac{1}{n}\right)^0 + C_n^1\left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{n!}{m!\,(n-m)!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
x_{n+1} &= 1+1+\sum_{m=2}^{n+1}\frac{1}{m!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \qquad \text{单调增加} \\
x_n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{n^2} = 1+\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} \\
&< 1+\frac{1}{1-\frac{1}{2}} \\
&< 3 \qquad \text{有界}
\end{aligned}
$$

## 23.4 第 10章

**10.1.1**

$$
\begin{aligned}
f'(x_0) &= \lim_{\triangle x\to 0}\frac{\triangle\, y}{\triangle\, x} \qquad \text{因为极限存在与无穷小的关系} \\
\frac{\triangle\, y}{\triangle\, x} &= f'(x_0)+\alpha \qquad \alpha\text{为}\ \triangle\, x\to 0\text{时的无穷小} \\
\triangle\, y &= f'(x_0)\triangle\, x+\alpha\triangle\, x \\
\lim_{\triangle x\to 0}\triangle\, y &= \lim_{\triangle x\to 0}\left[f'(x_0)\triangle\, x+\alpha\triangle\, x\right] = 0 \\
\lim_{x\to x_0} f(x) = f(x_0) &\Leftrightarrow \lim_{\triangle x\to 0}[f(x_0+\triangle\, x)-f(x_0)=0] \Leftrightarrow \lim_{\triangle x\to 0}\triangle\, y = 0
\end{aligned}
$$

**10.2.1**

$$
\begin{aligned}
(C)' &= \lim_{\triangle x \to 0} \frac{f(x_0 + \triangle\, x) - f(x_0)}{\triangle\, x} \\
&= \lim_{\triangle x \to 0} \frac{C - C}{\triangle\, x} \\
&= 0
\end{aligned}
$$

**10.2.2**

$$
\begin{aligned}
(x^a)' &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \\
&= \frac{x^a - x_0^a}{x - x_0} \\
&= \frac{(x - x_0)\left(x^{a-1} + x^{a-2}x_0 + \cdots + x x_0^{a-2} + x_0^{a-1}\right)}{x - x_0} \\
&= a x_0^{a-1}
\end{aligned}
$$

**10.2.3**

$$
\begin{aligned}
(a^x)' &= \lim_{\triangle x \to 0} \frac{a^{x+\triangle x} - a^x}{\triangle\, x} \\
&= a^x \lim_{\triangle x \to 0} \frac{a^{\triangle x} - 1}{\triangle\, x} \\
&= a^x \lim_{\triangle x \to 0} \frac{e^{\triangle x \ln a} - 1}{\triangle\, x} \\
&= a^x \ln a
\end{aligned}
$$

**10.2.4**

$$
(e^x)' = e^x \ln e = e^x
$$

**10.2.5**

$$
\begin{aligned}
(\log_a^x)' &= \lim_{\triangle x \to 0} \frac{\log_a^{x+\triangle x} - \log_a^x}{\triangle\, x} \\
&= \lim_{\triangle x \to 0} \frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle\, x} \\
&= \lim_{\triangle x \to 0} \frac{\ln 1 + \frac{\triangle x}{x}}{\ln a \triangle\, x} \\
&= \frac{1}{\ln a} \lim_{\triangle x \to 0} \frac{\frac{\triangle x}{x}}{\triangle\, x} \\
&= \frac{1}{x \ln a}
\end{aligned}
$$

**10.2.6**

$$
\begin{aligned}
(\ln^x)' &= \frac{1}{x \ln e} \\
&= \frac{1}{x}
\end{aligned}
$$

**10.3.1**

$$\begin{aligned}(\sin x)' &= \lim_{\triangle x\to 0}\frac{\sin(x_0 + \triangle\ x) - \sin x_0}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\frac{2\cos(x_0+\frac{\triangle x}{2})\sin\frac{\triangle x}{2}}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\cos(x_0+\frac{\triangle\ x}{2})\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\\ &= cosx_0\end{aligned}$$

**10.3.2**

$$\begin{aligned}(\arcsin x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\sin y}{\mathrm{d}y}}\\ &= \frac{1}{\cos y}\\ &= \frac{1}{\sqrt{1-\sin^2 y}}\\ &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

**10.3.3**

$$\begin{aligned}(\csc x)' = (\frac{1}{\sin x})` &= \frac{(1)'\cdot sinx - (\sin x)'\cdot 1}{\sin^2 x}\\ &= \frac{-\cos x}{\sin^2 x}\\ &= -\csc x\cdot\cot x\end{aligned}$$

**10.3.4**

$$\begin{aligned}(\cos x)' &= \lim_{\triangle x\to 0}\frac{\cos(x_0 + \triangle\ x) - cos(x_0)}{\triangle\ x}\\ &= \lim_{\triangle x\to 0}\frac{-2\sin\left(x_0+\frac{\triangle x}{2}\right)\sin\frac{\triangle x}{2}}{\triangle\ x}\\ &= \lim_{\triangle x\to 0} -\sin\left(x_0+\frac{\triangle\ x}{2}\right)\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}}\\ &= -\sin x\end{aligned}$$

**10.3.5**

$$\begin{aligned}(\arccos x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cos y}{\mathrm{d}y}}\\ &= \frac{1}{-\sin y}\\ &= -\frac{1}{\sqrt{1-\cos^2 y}}\\ &= -\frac{1}{\sqrt{1-x^2}}\end{aligned}$$

**10.3.6**

$$
\begin{aligned}
(\sec x)' = \left(\frac{1}{\cos x}\right)' &= \frac{(1)' \cdot \cos x - (\cos x)' \cdot 1}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \sec x \cdot \tan x
\end{aligned}
$$

**10.3.8**

$$
\begin{aligned}
(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' &= \frac{(\sin x)' \cos x - (\cos x)' \sin x}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}
$$

**10.3.9**

$$
\begin{aligned}
(\arctan x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\tan y}{\mathrm{d}y}} \\
&= \frac{1}{\sec y} \\
&= \frac{1}{1+\tan^2 y} \\
&= \frac{1}{1+x^2}
\end{aligned}
$$

**10.3.10**

$$
\begin{aligned}
(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' &= \frac{(\cos x)' \sin x - (\sin x)' \cos x}{\sin^2 x} \\
&= -\frac{\sin^2 x + \cos^2 x}{\cos^2 x} \\
&= -\csc^2 x
\end{aligned}
$$

**10.3.11**

$$
\begin{aligned}
(\operatorname{arccot} x)' = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cot y}{\mathrm{d}y}} \\
&= \frac{1}{-\csc^2 y} \\
&= -\frac{1}{1+\cot^2 y} \\
&= -\frac{1}{1+x^2}
\end{aligned}
$$

**10.3.12**

$$
\begin{aligned}
(\sinh x)' &= \left(\frac{e^x - e^{-x}}{2}\right)' \\
&= \frac{e^x + e^{-x}}{2} \\
&= \cosh x
\end{aligned}
$$

**10.3.13**

$$
\begin{aligned}
(\cosh x)' &= \left(\frac{e^x + e^{-x}}{2}\right)' \\
&= \frac{e^x - e^{-x}}{2} \\
&= \sinh x
\end{aligned}
$$

**10.3.14**

$$
\begin{aligned}
(\tanh x)' &= \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)' \\
&= \frac{\frac{d(e^x - e^{-x})}{\mathrm{d}x}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{\mathrm{d}x}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\
&= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
&= \frac{2^2}{(e^x + e^{-x})^2} \\
&= \frac{1}{\cosh^2 x}
\end{aligned}
$$

**10.3.15**

$$
\begin{aligned}
(\operatorname{arcsinh} x)' &= \left[\ln(x + \sqrt{x^2 + 1})\right]' \\
&= \frac{d \ln(x + \sqrt{x^2 + 1})}{\mathrm{d}\left(x + \sqrt{x^2 + 1}\right)} \cdot \frac{d\left(x + \sqrt{x^2 + 1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2 + 1}\right)}{d\left(x^2 + 1\right)} \cdot \frac{d(x^2 + 1)}{dx}\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x\right) \\
&= \frac{1}{x + \sqrt{x^2 + 1}} \cdot \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} \\
&= \frac{1}{\sqrt{x^2 + 1}}
\end{aligned}
$$

**10.3.16**

$$
\begin{aligned}
(\operatorname{arccosh} x)' &= \left[\ln(x+\sqrt{x^2-1})\right]' \\
&= \frac{d\ln(x+\sqrt{x^2-1})}{\mathrm{d}\left(x+\sqrt{x^2-1}\right)} \cdot \frac{d\left(x+\sqrt{x^2-1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2-1}\right)}{d\left(x^2-1\right)} \cdot \frac{d(x^2-1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

**10.3.17**

$$
\begin{aligned}
(\operatorname{arctanh} x)' &= \left[\frac{1}{2}\ln(\frac{1+x}{1-x})\right]' \\
&= \frac{1}{2} \cdot \frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)} \cdot \frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2} \cdot \frac{1}{\left(\frac{1+x}{1-x}\right)} \cdot \frac{\frac{d(1+x)}{\mathrm{d}x}(1-x) - \frac{d(1-x)}{\mathrm{d}x}(1+x)}{(1-x)^2} \\
&= \frac{1}{2} \cdot \frac{1-x}{1+x} \cdot \frac{(1-x)+(1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

**10.4.1**

$$
\begin{aligned}
[Cu(x)]' &= \lim_{\triangle x \to 0} \frac{Cu(x+\triangle x) - Cu(x)}{\triangle x} \\
&= C \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x)}{\triangle x} \\
&= Cu'(x)
\end{aligned}
$$

**10.4.2**

$$
\begin{aligned}
(u(x) \pm v(x))' &= \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x) \pm v(x+\triangle x) - v(x)}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{u(x+\triangle x) - u(x)}{\triangle x} \pm \lim_{\triangle x \to 0} \frac{v(x+\triangle x) - v(x)}{\triangle x} \\
&= u'(x) \pm v'(x)
\end{aligned}
$$

**10.4.3**

$$
\begin{aligned}
[u(x)\cdot v(x)]' &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)v(x+\triangle x)-u(x)v(x)-u(x)v(x+\triangle x)+u(x)v(x+\triangle x)}{\triangle x}\\
&= \lim_{\triangle x\to 0}\frac{\left[u(x+\triangle x)-u(x)\right]v(x+\triangle x)+u(x)\left[v(x+\triangle x)-v(x)\right]}{\triangle x}\\
&= u'(x)\lim_{\triangle x\to 0}v(x+\triangle x)+u(x)v'(x)\\
&= u'(x)v(x)+v'(x)u(x)
\end{aligned}
$$

**10.5.1**

$$
\begin{aligned}
\left[f^{-1}(y)\right]'|_{y=y_0} &= \lim_{y\to y_o}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\\
&= \lim_{y\to y_o}\frac{x-x_0}{y-y_0}\\
&= \lim_{x\to x_o}\frac{x-x_0}{f(x)-f(x_0)}\\
&= \lim_{x\to x_o}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\\
&= \frac{1}{f'(x)}
\end{aligned}
$$

**10.6.1**

$$
\text{定义函数}\quad A(u)=\begin{cases}\frac{f(u)-f(u_0)}{u-u_0}. & u\neq u_0\\ f'(u_0), & u=u_0\end{cases}
$$

$$
A(u)\text{在}u_o\text{处连续, 既有, }\lim_{u\to u_o}A(u)=A(u_0)=f'(u_0)
$$

$$
\begin{aligned}
&\text{由恒等式}f(u)-f(u_0)=A(u)(u-u_0)\text{我们有}\\
&\frac{F(x)-F(x_0)}{x-x_0}=\frac{f\left[g(x)\right]-f\left[g(x_0)\right]}{x-x_0}\\
&\qquad\qquad\quad =A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\
&\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=\lim_{x\to x_0}A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\\
&F'(x_0)=f'(g(x_0))g'(x_0)
\end{aligned}
$$

## 23.5 第 11章

### 11.1.1

$$\begin{aligned}
\triangle\ y &= A\ \triangle\ x + \circ(\triangle\ x)\\
\frac{\triangle\ y}{\triangle\ x} &= A + \frac{\circ(\triangle\ x)}{\triangle\ x}\\
\lim_{\triangle x\to 0}\frac{\triangle\ y}{\triangle\ x} &= \lim_{\triangle x\to 0}\left[A + \frac{\circ(\triangle\ x)}{\triangle\ x}\right]\\
f'(x_0) &= A + 0\\
f'(x_0) &= A
\end{aligned}$$

### 11.1.2

$$\begin{gathered}
\text{设}f(x)\text{在}x_0\text{点可导}, f'(x_0) = \lim_{\triangle x\to 0}\tfrac{\triangle y}{\triangle x}\text{存在}\\
(\text{极限与无穷小的关系：}\ \lim f(x) = A \Leftrightarrow f(x) = A + \alpha)\\
\tfrac{\triangle y}{\triangle x} = f'(x_0) + \alpha\\
\triangle\ y = f'(x_0)\ \triangle\ x + \alpha\ \triangle\ x\\
\text{其中}\alpha\text{为}\ \triangle\ x \to 0\text{时的无穷小。}\\
\lim_{\triangle x\to 0}\tfrac{\alpha\triangle x}{\triangle x} = \lim_{\vartheta x\to 0}\alpha = 0\\
\alpha\ \triangle\ x = \circ(\triangle\ x)\\
\triangle\ y = f'(x_0)\ \triangle\ x + \circ(\triangle\ x)
\end{gathered}$$

### 11.2.1

$$\begin{aligned}
d(u\pm v) &= (u\pm v)'dx\\
&= (u)'dx \pm (v')dx\\
&= du \pm dv
\end{aligned}$$

### 11.2.2

$$\begin{aligned}
d(u\cdot v) &= (u\cdot v)'dx\\
&= (u)'vdx - (v')udx\\
&= vdu - udv
\end{aligned}$$

### 11.2.3

$$\begin{aligned}
d\left(\frac{u}{v}\right) &= \left(\frac{u}{v}\right)'dx\\
&= \frac{(u)'v-(v')u}{v^2}dx\\
&= \frac{vdu-udv}{v^2}
\end{aligned}$$

### 11.3.1

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}=f'(x_0)$$
$$f(x_0+\Delta x)-f(x_0)\leqslant 0$$
$$\begin{cases}\Delta x>0\left\{\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\leqslant 0\Rightarrow f'(x_0^+)=\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\leqslant 0\right.\\ \Delta x<0\left\{\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\geqslant 0\Rightarrow f'(x_0^-)=\lim\limits_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\geqslant 0\right.\end{cases}$$
$$f'(x_0)=f'(x_0^+)=f'(x_0^-)\Rightarrow f'(x_0)=0$$

### 11.3.2

$$M=\max\{f(x)|x\in[a,b]\},m=\min\{f(x)|x\in[a,b]\}$$
$$\begin{cases}M=m\Rightarrow M=m=f(a)=f(b),\text{此时}f(x)\text{为常数},\forall\xi\in(a,b),f'(\xi)=0\\ M>m\begin{cases}f(a)>m\Rightarrow\exists\xi\in(a,b),f(\xi)=m,\text{根据费马引理},f'(\xi)=0\\ f(a)<M\Rightarrow\exists\xi\in(a,b),f(\xi)=M,\text{根据费马引理},f'(\xi)=0\end{cases}\end{cases}$$

### 11.3.3

$$\varphi(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$$
$$\varphi(a)=f(a)-\frac{f(b)-f(a)}{b-a}a=\frac{bf(a)-af(b)}{b-a}$$
$$\varphi(b)=f(b)-\frac{f(b)-f(a)}{b-a}b=\frac{bf(a)-af(b)}{b-a}$$
$$\varphi(a)=\varphi(b),\exists\xi\in(a,b),\varphi'(\xi)=0$$
$$f'(\xi)=\frac{f(b)-f(a)}{b-a}$$
$$f'(\xi)(b-a)=f(b)-f(a)$$

### 11.3.4

$$\varphi(x)=f(x)-\frac{f(b)-f(a)}{F(b)-F(a)}[F(x)-F(a)]$$
$$\varphi(a)=\varphi(b)=f(a)$$
$$\varphi'(\xi)=0$$
$$f'(\xi)=\frac{f(b)-f(a)}{F(b)-F(a)}F'(\xi)$$
$$\frac{f'(\xi)}{F'(\xi)}=\frac{f(b)-f(a)}{F(b)-F(a)}$$

### 11.4.1

$f(x),F(x)$的去心邻域可导，$\lim\limits_{x\to x_0}\frac{f(x)}{F(x)}$与$f(x_0),F(x_0)$无关。规定$f(x_0)=0,F(x_0)=0$

此时 $\lim\limits_{x\to x_0}f(x)=0=f(x_0)$ $\lim\limits_{x\to x_0}F(x)=0=F(x_0)$ 此时在 $x_0$ 点处也连续

$$\frac{f(x)}{F(x)} = \frac{f(x)-f(x_0)}{F(x)-F(x_0)} = \frac{f'(\xi)}{F'(\xi)}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)} = \lim_{x\to x_0}\frac{f'(\xi)}{F'(\xi)}$$

$$x\to x_0,\text{时}\xi\to x_0 \qquad \text{符号 } \xi \text{ 换成 x}$$

$$\lim_{x\to x_0}\frac{f(x)}{F(x)} = \lim_{\xi\to x_0}\frac{f'(\xi)}{F'(\xi)} = \lim_{x\to x_0}\frac{f'(x)}{F'(x)}$$

**11.5.1**

$$\frac{R_n(x)}{(x-x_0)^{n+1}} = \frac{R_n(x)-R_n(x_0)}{(x-x_0)^{n+1}-(x_0-x_0)^{n+1}} = \frac{R_n'(\xi_1)}{(n+1)(\xi_1-x_0)^n}$$

$$\frac{1}{n+1}\cdot\frac{R_n'(\xi_1)}{(\xi_2-x_0)^n} = \frac{1}{n+1}\cdot\frac{R_n'(\xi_1)-R_n'(x_0)}{(\xi_1-x_0)^n-(x_0-x_0)^n} = \frac{1}{n+1}\cdot\frac{R_n''(\xi_2)}{(n)(\xi_2-x_0)^{n-1}}$$

$$\vdots$$

$$\frac{R_n^{(n)}(\xi_n)}{(n+1)!(\xi_n-x_0)} = \frac{R_n^{(n)}(\xi_n)-R_n^{(n)}(x_0)}{(n+1)!(\xi_n-x_0)-0} = \frac{R_n^{(n+1)}(\xi)}{(n+1)!}$$

$$\frac{R_n^{(n+1)}(\xi)}{(n+1)!} = \frac{f^{(n+1)}(\xi)-P_n^{(n+1)}(\xi)}{(n+1)!} = \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

$$\xi_1\in(x,x_0),\xi_2\in(\xi_1,x_0),\xi_n\in(\xi_{n-1},x_0),\xi\in(\xi_n,x_0)$$

**11.5.2**

$$\begin{aligned}\lim_{x\to x_0}\frac{R_n(x)}{(x-x_0)^n} &= \lim_{x\to x_0}\frac{R_n'(x)}{n(x-x_0)^{n-1}}\\ &= \lim_{x\to x_0}\frac{R_n''(x)}{n(n-1)(x-x_0)^{n-2}}\\ &= \lim_{x\to x_0}\frac{R_n^{(n)}(x)}{n!}\\ &= \frac{1}{n}\cdot 0\\ &= 0\end{aligned}$$

## 23.6 第 12章

### 12.2.1

$$
\begin{aligned}
\frac{dy}{dx} + P(x)y = 0 \\
\frac{dy}{y} &= -P(x)\ dx \\
\int \frac{dy}{y} &= \int -P(x)\ dx \\
\ln|y| &= -\int P(x)\ dx + C_2 \\
|y| &= C_1 e^{-\int P(x)\ dx} \\
y &= Ce^{-\int P(x)\ dx}
\end{aligned}
$$

### 12.2.2

**常数变异法: 假设一个解，包含关于 $x$ 的未知函数 $u(x)$**

$$
\begin{aligned}
y &= Ce^{-\int P(x)\ dx} \\
y &= u(x)e^{-\int P(x)\ dx} \\
\frac{dy}{dx} &= u'(x)e^{-\int P(x)\ dx} + u(x)e^{-\int P(x)\ dx} \cdot [-P(x)] \\
\frac{dy}{dx} &= u'(x)e^{-\int P(x)\ dx} - P(x)u(x)e^{-\int P(x)\ dx}
\end{aligned}
$$

**$u(x)$ 的表达式带入原方程，求出 $u(x)$ 与 $Q(x)$ 的关系**

$$
\begin{aligned}
\frac{dy}{dx} + P(x)y &= Q(x) \\
u'(x)e^{-\int P(x)\ dx} - P(x)u(x)e^{-\int P(x)\ dx} + P(x)u(x)e^{-\int P(x)\ dx} &= Q(x) \\
u'(x)e^{-\int P(x)\ dx} &= Q(x) \\
u'(x) &= Q(x)e^{\int P(x)\ dx} \\
u(x) &= \int Q(x)e^{\int P(x)\ dx}\ dx + C
\end{aligned}
$$

**求出 $u(x)$ 与 $Q(x)$ 的关系，在带回假设的解**

$$
\begin{aligned}
y &= u(x)e^{-\int P(x)\ dx} \\
y &= e^{-\int P(x)\ dx}\left[\int Q(x)e^{\int P(x)\ dx}\ dx + C\right]
\end{aligned}
$$

**12.3.1**

$$y_1''(x)+P_1(x)y_1'+P_2(x)y_1\equiv 0$$
$$y_2''(x)+P_1(x)y_2'+P_2(x)y_2\equiv 0$$

$$y=C_1y_1+C_2y_2$$
$$y'=C_1y_1'+C_2y_2'$$
$$y''=C_1y_1''+C_2y_2''$$

$$\begin{aligned}&y''+P_1(x)y'+P_2(x)y\\=&C_1y_1''+C_2y_2''+P_1(x)\left(C_1y_1'+C_2y_2'\right)+P_2(x)\left(C_1y_1+C_2y_2\right)\\=&C_1\left[y_1''(x)+P_1(x)y_1'+P_2(x)y_1\right]+C_2\left[y_2''(x)+P_1(x)y_2'+P_2(x)y_2\right]\\\equiv&0\end{aligned}$$

**12.3.5**

| | |
|---|---|
| **特征方程** | $y''+py'+qy=0$<br>$r^2e^{rx}+pre^{rx}+qe^{rx}=0$<br>$e^{rx}(r^2+pr+q)=0$<br>$r^2+pr+q=0$ |
| $p^2-4q>0$ | 两个不同实根解：$r_1,r_2$<br>$\dfrac{e^{r_1x}}{e^{r_2x}}=e^{(r_1-r_2)x}\not\equiv$ 常数　　线性无关<br>$y=C_1y_1+C_2y_2=C_1e^{r_1x}+C_2e^{r_2x}$ |
| $p^2-4q=0$ | 两个相同解：$r_1=r_2=-\dfrac{p}{2}$ 设$y_2=u(x)e^{r_1x}$是另一个解<br>$y_2'=u'e^{r_1x}+r_1ue^{r_1x}=(u'+r_1u)e^{r_1x}$<br>$y_2''=(u''+r_1)e^(r_1x)+(u'+r_1u)e^{r_1x}r_1=(u''+2r_1u'+r_1^2u)e^{r_1x}$<br>$y''+py'+qy=0$<br>$(u''+2r_1u'+r_1^2u)e^{r_1x}+p(u'+r_1u)e^{r_1x}+que^{r_1x}=0$<br>$e^{r_1x}\left[u''+(2r_1+p)+(r_1^2+pr_1+q)u\right]=0$<br>$u''=0$<br>$u=C_1x+C_2$　取：$u(x)=x$　$\dfrac{e^{r_1x}}{xe^{r_1x}}=xe^{(r_1-r_2)x}\not\equiv$ 常数<br>$y=C_1y_1+C_2y_2=C_1e^{r_1x}+C_2xe^{r_1x}$ |

| | |
|---|---|
| $p^2-4q<0$ | 两个共轭复根解：$r_{1,2}=\alpha\pm i\beta$<br>$y_1=e^{(\alpha+i\beta)x}=e^{\alpha x}(\cos\beta x+i\sin\beta x)$<br>$y_2=e^{(\alpha=i\beta)x}=e^{\alpha x}(\cos\beta x-i\sin\beta x)$<br>$\overline{y_1}=\frac{1}{2}(y_1+y_2)=e^{\alpha x}\cos\beta x$<br>$\overline{y_2}=\frac{1}{2i}(y_1-y_2)=e^{\alpha x}\sin\beta x$<br>$\frac{\overline{y_1}}{\overline{y_2}}=\cot\beta x\not\equiv$ 常数<br>$y=C_1\overline{y_1}+C_2\overline{y_2}=e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$ |

## 23.7 第 13章

**13.2.1**

$$\left[\int f(x)\ dx\pm\int g(x)\ dx\right]'=\left[\int f(x)\ dx\right]'\pm\left[\int g(x)\ dx\right]'$$
$$=f(x)\pm g(x)$$

**13.2.2**

$$\left[k\int f(x)\ dx\right]'=k\left[\int f(x)\ dx\right]'$$
$$=kf(x)$$

**13.2.3**

$$\{F\left[\varphi(x)\right]\}'=F'\left[\varphi(x)\right]\varphi'(x)$$
$$=f\left[\varphi(x)\right]\varphi'(x)$$

**13.2.4**

$$x=\varphi(t)$$
$$dx=d\varphi(t)$$
$$dx=\varphi'(t)dt$$
$$\int f(x)\ dx=\int f\left[\varphi(t)\right]\varphi'(t)\ dt$$

**13.2.5**

$$\int f(x)\ dx=\int f(x)\cdot(x+C)'\ dx$$
$$=\int f(x)\ d(x+C)$$

**13.4.1**

$$\int k\ dx = \int (kx)'\ dx = kx + C$$

**13.4.2**

$$\int x^a\ dx = \int \left(\frac{1}{a+1}x^{a+1}\right)'\ dx = \frac{x^{a+1}}{a+1} + C$$

**13.4.3**

$$\int a^x\ dx = \int \left(\frac{1}{\ln a}a^x\right)'\ dx = \frac{a^x}{\ln a} + C$$

**13.4.4**

$$\int e^x\ dx = \int (e^x)'\ dx = e^x + C$$

**13.4.5**

$$\int \frac{1}{x}\ dx = \begin{cases} (x > 0) & \int (\ln x)'\ dx = \ln x + C = \ln|x| + C \\ (x < 0) & \int [\ln(-x)]'\ dx = \ln(-x) + C = \ln|x| + C \end{cases}$$

**13.4.6**

$$\begin{aligned} \int \ln x\ dx &= \ln x \cdot x - \int x\ d\ln x \\ &= x\ln x - \int x \cdot \frac{1}{x}\ dx \\ &= x\ln x - x + C \end{aligned}$$

**13.4.7**

$$\int \sin x\ dx = \int (-\cos x)'\ dx = -\cos x + C$$

**13.4.8**

$$\int \cos x\ dx = \int (\sin x)'\ dx = \sin x + C$$

**13.4.9**

$$\int \sec x \tan x\ dx = \int (\sec x)'\ dx = \sec x$$

**13.4.10**

$$\int \csc x \cot x\ dx = -\int (\csc x)'\ dx = -\csc x$$

**13.4.11**

$$\int \sec^2 x \ dx = \int (\tan x)' \ dx = \tan x$$

**13.4.12**

$$\int \csc^2 x \ dx = -\int (\cot x)' \ dx = -\cot x$$

**13.4.15**

$$\begin{aligned}\int \tan x \ dx &= \int \frac{\sin x}{\cos x} \ dx \\ &= -\int \frac{(\cos x)'}{\cos x} \ dx \\ &= -\int \frac{1}{\cos x} \ d(\cos x) \\ &= -\ln|\cos x| + C\end{aligned}$$

**13.4.16**

$$\begin{aligned}\int \csc x \ dx &= \int \frac{1}{\sin x} \ dx \\ &= \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \ dx \\ &= \int \frac{1}{\tan\frac{x}{2}} \cdot \frac{1}{\cos^2\frac{x}{2}} \ d\frac{x}{2} \\ &= \int \frac{1}{\tan\frac{x}{2}} \cdot \sec^2\frac{x}{2} \ d\frac{x}{2} \\ &= \int \frac{1}{\tan\frac{x}{2}} \ d\tan\frac{x}{2} \\ &= \begin{cases} \ln\left|\tan\frac{x}{2}\right| + C \\ \ln|\csc x - \cot x| + C \end{cases}\end{aligned}$$

**13.4.17**

$$\begin{aligned}\int \sec x \ dx &= \int \frac{1}{\cos x} \ dx \\ &= \int \frac{\cos x}{\cos^2 x} \ dx \\ &= \int \frac{1}{\cos^2 x} \ d\sin x \\ &= \int \frac{1}{1-\sin^2 x} \ d\sin x \\ &= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \\ &= \frac{1}{2}\left|\frac{1+\sin x}{\cos x}\right|^2 + C \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

**13.4.18**

$$\begin{aligned}\int \arccos x\ dx &= x\arccos x - \int x\cdot -\frac{1}{\sqrt{1-x^2}}\ dx\\ &= x\arccos x - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\ d(1-x^2)\\ &= x\arccos x - \frac{1}{2}\cdot\frac{1}{-\frac{1}{2}+1}(1-x^2)^{-\frac{1}{2}+1}+C\\ &= x\arccos x - \sqrt{1-x^2}+C\end{aligned}$$

**13.4.19**

$$\begin{aligned}\int \arctan x\ dx &= x\arctan x - \int x\cdot\frac{1}{1+x^2}\ dx\\ &= x\arctan x - \frac{1}{2}\int\frac{1}{1+x^2}\ d(1+x^2)\\ &= x\arctan x - \frac{1}{2}\ln(1+x^2)+C\end{aligned}$$

分式积分公式证明暂时不标号。

$$\int\frac{1}{x^2+1}\ dx = \int(\arctan x)'\ dx = \arctan x + C$$

$$\int\frac{1}{x^2+a^2}\ dx = \frac{1}{a}\int\frac{1}{1+(\frac{x}{a})^2}\ d(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a}+C$$

$$\begin{aligned}\int\frac{1}{x^2-a^2}\ dx &= \int\frac{1}{2a}\cdot\frac{(x+a)-(x-a)}{(x-a)(x+a)}\ dx\\ &= \frac{1}{2a}\cdot\int\frac{1}{x-a}-\frac{1}{x+a}\ dx\\ &= \frac{1}{2a}\cdot\left(\int\frac{1}{x-a}\ dx - \int\frac{1}{x+a}\ dx\right)\\ &= \frac{1}{2a}\left(\ln|x-a|-\ln|x+a|\right)+C\\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\end{aligned}$$

$$\begin{aligned}
\int \frac{1}{a^2-x^2}\ dx &= \int -\frac{1}{2a}\frac{(x-a)-(x+a)}{(a+x)(a-x)}\ dx \\
&= \int \frac{1}{2a}\cdot\frac{(x-a)-(x+a)}{(a+x)(x-a)}\ dx \\
&= \frac{1}{2a}\cdot\left(\int \frac{1}{a+x}\ dx - \int \frac{1}{x-a}\ dx\right) \\
&= \frac{1}{2a}\cdot\left(\int \frac{1}{a+x}\ dx - \int \frac{1}{a-x}\ d(-x)\right) \\
&= \frac{1}{2a}\left(\ln|a+x| - \ln|a-x|\right) + C \\
&= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C
\end{aligned}$$

$$\int \frac{1}{\sqrt{1-x^2}}\ dx = \begin{cases} \int(\arcsin x)'\ dx = \arcsin x + C \\ -\int(\arccos x)'\ dx = -\arccos x + C \end{cases}$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\ dx = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\ d(\frac{x}{a}) = \begin{cases} \arcsin(\frac{x}{a}) + C \\ -\arccos(\frac{x}{a}) + C \end{cases}$$

$$x = a\tan t, -\tfrac{\pi}{2} < t < \tfrac{\pi}{2}, \sec t = \tfrac{\sqrt{x^2+a^2}}{a}, \tan t = \tfrac{x}{a}, dx = a\sec^2 t\ dt$$

$$\begin{aligned}
\int \frac{1}{\sqrt{x^2+a^2}} &= \int \frac{1}{a}\frac{1}{\sqrt{\tan^2 t+1}} a\sec^2 t\ dt \\
&= \int \frac{1}{\sec t}\sec^2 t\ dt \\
&= \int \sec t\ dt \\
&= \ln|\sec t + tant| + C \\
&= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C \\
&= \ln(x+\sqrt{x^2+a^2}) + C_1 \qquad C_1 = C - \ln a
\end{aligned}$$

$$x = a\sec t, a > 0, \sec t = \tfrac{x}{a}, \tan t = \tfrac{\sqrt{x^2-a^2}}{a}, dx = a\sec t\tan t\ dt$$

**x>a**

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\ dx &= \int \frac{1}{a}\frac{1}{\sqrt{\sec^2 t-1}} a\sec t\tan t\ dt \\
&= \int \frac{1}{\tan t}\sec t\tan t\ dt \\
&= \int \sec t\ dt \\
&= \ln|\sec t + tant| + C \\
&= \ln\left|\frac{\sqrt{x^2-a^2}}{a}+\frac{x}{a}\right| + C \\
&= \ln(x+\sqrt{x^2-a^2}) + C_1 \qquad C_1 = C - \ln a
\end{aligned}
$$

**x<-a,x=-t,dx=-dt**

$$
\begin{aligned}
\int \frac{1}{\sqrt{x^2-a^2}}\ dx &= -\int \frac{1}{\sqrt{t^2-a^2}}\ dt \\
&= -\ln\left|t+\sqrt{t^2-a^2}\right| + C \\
&= -\ln\left|-x+\sqrt{x^2-a^2}\right| + C \\
&= -\ln\left|\frac{(-x+\sqrt{x^2-a^2})(x+\sqrt{x^2-a^2})}{x+\sqrt{x^2-a^2}}\right| + C \\
&= -\ln\left|\frac{-a^2}{x+\sqrt{x^2-a^2}}\right| + C \\
&= -\ln\left|-a^2\right| + \ln\left|x+\sqrt{x^2-a^2}\right| + C \\
&= \ln\left|x+\sqrt{x^2-a^2}\right| + C_1 \qquad C_1 = C - \ln\left|-a^2\right|
\end{aligned}
$$

## 23.8 第 15章

**14.3.1**

$$
\begin{aligned}
\int_a^a f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \bigtriangleup x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i)\cdot 0 \\
&= 0
\end{aligned}
$$

**14.3.2**

$$
\begin{aligned}
\int_a^b \ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n} \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n}(x_i - x_{i-1}) \\
&= \lim_{\lambda\to 0}(b-a) \\
&= b-a
\end{aligned}
$$

**14.3.3**

$$
\begin{cases}
\sum_{i=1}^{n} \vartriangle x_i = \sum_{i=1}^{n}(x_i - x_{i-1}) = b-a \\
\sum_{i=1}^{n}(x_{i-1} - x_i) = a-b
\end{cases}
$$

$$
\begin{aligned}
\int_a^b f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to o}\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}) \\
&= -\lim_{\lambda\to o}\sum_{i=1}^{n} f(\xi_i)(x_{i-1} - x_i) \\
&= -\int_b^a f(x)\ dx
\end{aligned}
$$

**14.3.4**

$$
\begin{aligned}
\int_a^c f(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\
&= \lim_{\lambda\to 0}\sum_{i=1}^{b} f(\xi_i) \vartriangle x_i + \lim_{\lambda\to 0}\sum_{i=b+1}^{n} f(\xi_i) \vartriangle x_i \\
&= \int_a^b f(x)\ dx + \int_b^c f(x)\ dx
\end{aligned}
$$

**14.3.5**

$$
\begin{aligned}
\int_a^c f(x)\ dx &= \int_a^b f(x)\ dx + \int_b^c f(x)\ dx \\
\int_a^b f(x)\ dx &= \int_a^c f(x)\ dx - \int_b^c f(x)\ dx \\
&= \int_a^c f(x)\ dx + \int_c^b f(x)\ dx
\end{aligned}
$$

**14.3.6**

$$\begin{aligned}\int_a^b kf(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} kf(\xi_i) \vartriangle x_i \\ &= k\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \\ &= k\int_a^b f(x)\ dx\end{aligned}$$

**14.3.7**

$$\begin{aligned}\int_a^b f(x)\pm g(x)\ dx &= \lim_{\lambda\to 0}\sum_{i=1}^{n} [f(\xi_i)\pm g(\xi_i)] \vartriangle x_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \pm \lim_{\lambda\to 0}\sum_{i=1}^{n} g(\xi_i) \vartriangle x_i \\ &= \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx\end{aligned}$$

**14.3.8**

$$f(x)\geqslant 0, \vartriangle x_i = (x_i - x_{i-1}) > 0 \Rightarrow f(x_i) \vartriangle x_i \geqslant 0$$

$$\int_a^b f(x)\ dx = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i) \vartriangle x_i \geqslant 0$$

**14.3.9**

$$\begin{aligned}f(x)\geqslant g(x) \Rightarrow f(x)-g(x) &\geqslant 0 \\ \int_a^b f(x)-g(x)\ dx &\geqslant 0 \\ \int_a^b f(x)\ dx - \int_a^b g(x)\ dx &\geqslant 0 \\ \int_a^b f(x)\ dx &\geqslant \int_a^b g(x)\ dx\end{aligned}$$

**14.3.10**

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|$$

$$\int_a^b -|f(x)|\ dx \leqslant \int_a^b f(x)\ dx \leqslant \int_a^b |f(x)|\ dx$$

$$\left|\int_a^b f(x)\ dx\right| \leqslant \int_a^b |f(x)|\ dx$$

**14.4.1**

$$m \leqslant f(x) \leqslant M, x\in [a,b]$$

$$\int_a^b m\ dx \leqslant \int_a^b f(x)\ dx \leqslant \int_a^b M\ dx$$
$$m\int_a^b\ dx \leqslant \int_a^b f(x)\ dx \leqslant M\int_a^b\ dx$$
$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a)$$

**14.5.1**

$$M\text{为区间 }[a,b]\text{ 最大值},m\text{为区间 }[a,b]\text{ 最小值},a<b$$
$$m \leqslant f(x) \leqslant M, x \in [a,b]$$
$$m(b-a) \leqslant \int_a^b f(x)\ dx \leqslant M(b-a)$$
$$m \leqslant \frac{1}{(b-a)}\int_a^b f(x)\ dx \leqslant M$$
$$\exists\xi \in [a,b]\,, f(\xi) = \frac{1}{(b-a)}\int_a^b f(x)\ dx$$
$$f(\xi)(b-a) = \int_a^b f(x)\ dx$$

**14.6.1**

$$\begin{aligned}\phi'(x) &= \lim_{\triangle x\to 0}\frac{\phi(x+\triangle x)-\phi(x)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{f(\xi)\triangle x}{\triangle x}\\ &= \lim_{\triangle x\to 0} f(\xi)\quad (\triangle x\to 0,\text{时},\xi\to x)\\ &= f(x)\end{aligned}$$

$$\begin{aligned}&\phi(x+\triangle x)-\phi(x)\\ &=\int_a^{x+\triangle x} f(u)\ du - \int_a^x f(u)\ du\\ &=\int_a^x f(u)\ du + \int_x^{x+\triangle x} f(u)\ du - \int_a^x f(u)\ du\\ &=\int_x^{x+\triangle x} f(u)\ du\\ &=f(\xi)(x+\triangle x - x)\qquad (\xi\in[x,x+\triangle x])\\ &=f(\xi)\triangle x\end{aligned}$$

**14.6.2**

$$\begin{aligned}[\phi(\psi(x))]' &= \frac{d}{dx}\left[\int_a^{\psi(x)} f(u)\ du\right]\\ &= \frac{d}{d\psi(x)}\left[\int_a^{\psi(x)} f(u)\ du\right]\cdot\frac{d\psi(x)}{dx}\\ &= f(\psi(x))\psi'(x)\end{aligned}$$

**14.6.3**

$$\begin{aligned}\frac{d}{dx}\left[\int_{v(x)}^{\psi(x)} f(u)\ du\right] &= \frac{d}{dx}\left[\int_{0}^{\psi(x)} f(u)\ du + \int_{v(x)}^{0} f(u)\ du\right] \\ &= \frac{d}{dx}\left[\int_{0}^{\psi(x)} f(u)\ du\right] - \frac{d}{dx}\left[\int_{0}^{v(x)} f(u)\ du\right] \\ &= f\left[\psi(x)\right]\psi'(x) - f\left[v(x)\right]v'(x)\end{aligned}$$

**14.7.1**

$$\begin{cases} F(x)\text{是}f(x)\text{的原函数} \\ \phi(x) = \int_a^x f(u)\ du\text{也是}f(x)\text{的原函数}\end{cases} \Rightarrow F(x) - \phi(x) \equiv C$$

$$\begin{aligned} x = a \qquad F(a) - \phi(a) &= C \\ F(a) &= C \end{aligned}$$

$$\begin{aligned} F(x) - \phi(x) &= C \\ \phi(x) &= F(x) - C \\ x = b \qquad \phi(b) &= F(b) - F(a) \\ \int_a^b f(x)\ dx &= F(b) - F(a) \end{aligned}$$

**14.8.1**

$$\begin{aligned}\int_a^b f(x)\ dx &= F(b) - F(a) \\ &= F\left[\phi(\beta)\right] - F\left[\phi(\alpha)\right] \\ &= F\left[\phi(t)\right]\Big|_{\alpha}^{\beta} \\ &= \int_{\alpha}^{\beta} \frac{dF\left[\phi(t)\right]}{dt}\ dt \\ &= \int_{\alpha}^{\beta} f\left[\varphi(t)\right]\phi'(t)\ dt\end{aligned}$$

**14.14.1**

$$\text{记}\phi(x) = \int_a^x f(t)\ dt, \quad a \leqslant x \leqslant b$$
$$\phi(a) = 0, \quad \phi(b) = \int_a^b f(t)\ dt = \int_a^b f(x)\ dx$$
$$\phi'(x) = f(x) \geqslant 0, \quad \text{在 } [a, b] \text{ 单调非降 } \phi(a) = 0, \phi(b) = 0$$
$$\phi(x) \equiv 0, x \in [a, b], \phi'(x) = f(x) \equiv 0$$

**14.14.2**

为 14.14.1的逆否命题

**14.14.3**

$$h(x)=g(x)-f(x)\geqslant 0,\quad x\in(a,b)$$
$$\int_a^b h(x)\ dx=\int_a^b g(x)\ dx-\int_a^b f(x)\ dx=0$$
$$h(x)\equiv 0\Rightarrow f(x)=g(x)$$

**14.10.1**

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx&=\int_0^a f(-x)+f(x)\ dx\\&=\int_0^a 2f(x)\ dx\\&=2\int_0^a f(x)\ dx\end{aligned}$$

**14.10.2**

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx&=\int_0^a f(-x)+f(x)\ dx\\&=\int_0^a -f(x)+f(x)\ dx\\&=0\end{aligned}$$

$$\begin{aligned}\int_{-a}^{0} f(x)\ dx&\xlongequal{x=-t}-\int_a^0 f(-t)\ dt\\&=\int_0^a f(-t)\ dt\\&=\int_0^a f(-x)\ dx\end{aligned}$$

$$\begin{aligned}\int_{-a}^{a} f(x)\ dx&=\int_{-a}^{0} f(x)\ dx+\int_0^a f(x)\ dx\\&=\int_0^a f(-x)\ dx+\int_0^a f(x)\ dx\\&=\int_0^a f(-x)+f(x)\ dx\end{aligned}$$

**14.11.1**

$$G(x)=\int_x^{x+T} f(x)\ dx$$
$$G'(x)=f(x+T)-f(x)=0\Rightarrow G(x)\equiv C$$
$$G(a)=G(0)=\int_0^T f(x)\ dx$$

**14.11.2**

$$\begin{aligned}\int_x^{x+nT} f(x)\ dx&=\int_a^T f(x)\ dx+\int_{a+T}^{a+2T} f(x)\ dx+\cdots+\int_{a+(n-1)T}^{a+nT} f(x)\ dx\\&=\int_0^T f(x)\ dx+\int_0^T f(x)\ dx+\cdots+\int_0^T f(x)\ dx\\&=n\int_0^T f(x)\ dx\end{aligned}$$

**14.12.1**

$$\begin{aligned}\int_0^{\frac{\pi}{2}} f(\sin x)\ dx&\xlongequal{\frac{\pi}{2}-x=t}-\int_{\frac{\pi}{2}}^0 f\left[\sin\left(\frac{\pi}{2}-t\right)\right]\ dt\\&=\int_0^{\frac{\pi}{2}} f(\cos t)\ dt\\&=\int_0^{\frac{\pi}{2}} f(\cos x)\ dx\end{aligned}$$

**14.12.2**

$$
\begin{aligned}
I_n = \int_0^{\frac{\pi}{2}} \sin^n x\ dx &= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x\ dx \\
&= -\int_0^{\frac{\pi}{2}} \sin^{n-1} x\ d\cos x \\
&= -\left[sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos x \cdot \sin^{n-2} x \cdot \cos x\ dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \cdot \sin^{n-2} x\ dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x) \cdot \sin^{n-2} x\ dx \\
&= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\ dx \\
n\int_0^{\frac{\pi}{2}} \sin^n x\ dx &= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx \\
\int_0^{\frac{\pi}{2}} \sin^n x\ dx &= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\ dx \\
I_n &= \frac{n-1}{n} I_{n-2} \\
\int_0^{\frac{\pi}{2}} 1\ dx &= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2} \\
\int_0^{\frac{\pi}{2}} \sin x\ dx &= -[\cos x]_0^{\frac{\pi}{2}} = 1 \\
(\text{n 是偶数}) I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \\
(\text{n 是奇数}) I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1
\end{aligned}
$$

**14.13.1**

$$
\begin{aligned}
[f(x) + tg(x)]^2 &\geqslant 0 \\
\int_a^b [f(x) + tg(x)]^2\ dx &\geqslant 0 \\
\int_a^b f^2(x) + 2tf(x)g(x) + t^2 g^2(x)\ dx &\geqslant 0 \\
\int_a^b f^2(x)\ dx + 2t\int_a^b f(x)g(x)\ dx + t^2\int_a^b g^2(x)\ dx &\geqslant 0 \qquad (b^2 - 4ac \leqslant 0) \\
\triangle = \left[\int_a^b f(x)g(x)\ dx\right]^2 - \int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx &\leqslant 0 \\
\left[\int_a^b f(x)g(x)\ dx\right]^2 &\leqslant \int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx
\end{aligned}
$$

**14.13.2**

$$\left[\int_a^b f(x)g(x)\ dx\right]^2 \leqslant \int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx$$

$$\int_a^b f(x)g(x)\ dx \leqslant \left[\int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx\right]^{\frac{1}{2}}$$

$$\int_a^b 2f(x)g(x)\ dx \leqslant 2\left[\int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx\right]^{\frac{1}{2}}$$

$$\int_a^b [f(x)+g(x)]^2\ \ dx \leqslant \int_a^b f^2(x)\ dx + 2\left[\int_a^b f^2(x)\ dx \cdot \int_a^b g^2(x)\ dx\right]^{\frac{1}{2}} + \int_a^b g^2(x)\ dx$$

$$\left\{\int_a^b [f(x)+g(x)]^2\ \ dx\right\}^{\frac{1}{2}} \leqslant \left[\int_a^b f^2(x)\ dx\right]^{\frac{1}{2}} + \left[\int_a^b g^2(x)\ dx\right]^{\frac{1}{2}}$$

## 23.9 第 16章

**16.3.1**

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP} + \overrightarrow{PN} + \overrightarrow{NM}\\ &= \overrightarrow{OP} + \overrightarrow{OQ} + \overrightarrow{OR}\\ &= x\overrightarrow{i} + y\overrightarrow{j} + z\overrightarrow{k}\end{aligned}$$

**16.3.2**

$$\begin{aligned}|OM|^2 =& |ON|^2 + |NM|^2\\ =& |OP|^2 + |PN|^2 + |OR|^2\\ =& |OP|^2 + |OQ|^2 + |OR|^2\\ =& x^2 + y^2 + z^2\\ |OM| =& \sqrt{x^2+y^2+z^2}\end{aligned}$$

**16.3.4**

$$\begin{aligned}(\cos\alpha, \cos\beta, \cos\gamma) &= (\frac{x}{|\overrightarrow{OM}|}, \frac{y}{|\overrightarrow{OM}|}, \frac{z}{|\overrightarrow{OM}|})\\ &= \frac{1}{|\overrightarrow{OM}|}(x, y, z)\\ &= \frac{\overrightarrow{OM}}{|\overrightarrow{OM}|}\end{aligned}$$

**16.4.1**

$$\vec{i}\bullet\vec{j}=0 \qquad \vec{j}\bullet\vec{k}=0 \qquad \vec{k}\bullet\vec{i}=0$$

$$\begin{aligned}\vec{a}\bullet\vec{b}&=(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\bullet(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})\\&=a_xb_x\vec{i}+a_yb_y\vec{j}+a_zb_z\vec{k}+0\\&=a_xb_x+a_yb_y+a_zb_z\end{aligned}$$

**16.4.2**

$$\begin{aligned}\vec{a}\times\vec{b}=&(a_x\vec{i}+a_y\vec{j}+a_z\vec{k})\times(b_x\vec{i}+b_y\vec{j}+b_z\vec{k})\\=&a_xb_x\vec{i}\vec{i}+a_xb_y\vec{i}\vec{j}+a_xb_z\vec{i}\vec{z}\\&+a_yb_x\vec{j}\vec{i}+a_yb_y\vec{j}\vec{j}+a_yb_z\vec{j}\vec{z}\\&+a_zb_x\vec{z}\vec{i}+a_zb_y\vec{z}\vec{j}+a_zb_z\vec{z}\vec{z}\\=&a_xb_y\vec{k}+a_xb_z(-\vec{j})\\&+a_yb_x(-\vec{k})+a_yb_z\vec{i}\\&+a_zb_x\vec{j}+a_zb_y(-\vec{i})\\=&(a_yb_z-a_zb_y)\vec{i}+(a_zb_x-a_xb_z)\vec{j}+(a_xb_y-a_yb_x)\vec{k}\\=&\begin{vmatrix}a_y & a_z\\ b_y & b_z\end{vmatrix}\vec{i}-\begin{vmatrix}a_x & a_z\\ b_x & b_z\end{vmatrix}\vec{j}+\begin{vmatrix}a_x & a_y\\ b_x & b_y\end{vmatrix}\vec{k}\end{aligned}$$

$$\vec{a}=(a_x,a_y,a_z)=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$$

$$\vec{b}=(b_x,b_y,b_z)=b_x\vec{i}+b_y\vec{j}+b_z\vec{k}$$

**17.3.2**

$$\begin{aligned}\Delta z=&f(x+\Delta x,y+\Delta y)-f(x,y)\\\Delta z=&A\Delta x+B\Delta y+\circ(\rho)\\=&\frac{\partial z}{\partial x}\Delta x+\frac{\partial z}{\partial y}\Delta y\end{aligned}$$

取$\Delta y=0$

$$\begin{aligned}f(x+\Delta x,y)-f(x,y)=&A\Delta x+\circ(|\Delta x|)\\\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=&A+\frac{\circ(|\Delta x|)}{\Delta x}\\A=&\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}\\A=&\frac{\partial z}{\partial x}\end{aligned}$$

取$\Delta x=0$

$$\begin{aligned}f(x,y+\Delta y)-f(x,y)=&B\Delta y+\circ(|\Delta y|)\\\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=&B+\frac{\circ(|\Delta y|)}{\Delta y}\\B=&\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}\\B=&\frac{\partial z}{\partial y}\end{aligned}$$

**17.5.1**

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases} \quad \begin{cases} y=y(x) \\ z=z(x) \end{cases} \Rightarrow \begin{cases} F(x,y(x),z(x))\equiv 0 \\ G(x,y(x),z(x))\equiv 0 \end{cases}$$

$$\begin{cases} F'_x+F'_y\frac{dy}{dx}+F'_z\frac{dz}{dx}=0 \\ G'_x+G'_y\frac{dy}{dx}+G'_z\frac{dz}{dx}=0 \end{cases} \Rightarrow \begin{cases} F'_y\frac{dy}{dx}+F'_z\frac{dz}{dx}=-F'_x \\ G'_y\frac{dy}{dx}+G'_z\frac{dz}{dx}=-G'_x \end{cases}$$

$$J=\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}=\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix}\triangleq\frac{\partial(F,G)}{\partial(y,z)}\quad(\text{不等于 0 时有唯一解})$$

$$\frac{dy}{dx}=\frac{\begin{vmatrix} -F'_x & F'_z \\ -G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\frac{\begin{vmatrix} F'_x & F'_z \\ G'_x & G'_z \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(x,z)}}{\frac{\partial(F,G)}{\partial(y,z)}}$$

$$\frac{dz}{dx}=\frac{\begin{vmatrix} F'_y & -F'_x \\ G'_y & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\frac{\begin{vmatrix} F'_y & F'_x \\ G'_y & G'_x \end{vmatrix}}{\begin{vmatrix} F'_y & F'_z \\ G'_y & G'_z \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(y,x)}}{\frac{\partial(F,G)}{\partial(y,z)}}$$

**17.5.2**

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \quad \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases} \Rightarrow \begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

$$\begin{cases} F'_x+F'_u\frac{\partial u}{dx}+F'_v\frac{\partial v}{dx}=0 \\ G'_x+G'_v\frac{\partial u}{dx}+G'_v\frac{\partial v}{dx}=0 \end{cases} \Rightarrow \begin{cases} F'_u\frac{\partial u}{dx}+F'_v\frac{\partial v}{dx}=-F'_x \\ G'_v\frac{\partial u}{dx}+G'_v\frac{\partial v}{dx}=-G'_x \end{cases}$$

$$J=\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}=\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}\triangleq\frac{\partial(F,G)}{\partial(u,v)}\quad(\text{不等于 0 时有唯一解})$$

$$\frac{\partial u}{\partial x}=\frac{\begin{vmatrix} -F'_x & F'_v \\ -G'_x & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\frac{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\frac{\partial v}{\partial x}=\frac{\begin{vmatrix} F'_u & -F'_x \\ G'_u & -G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\frac{\begin{vmatrix} F'_u & F'_x \\ G'_u & G'_x \end{vmatrix}}{\begin{vmatrix} F'_u & F'_v \\ G'_u & G'_v \end{vmatrix}}=-\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

**18.8.1**

$$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}=t \quad (t>0)\begin{cases}\Delta x=t\cos\alpha\\ \Delta y=t\cos\beta\end{cases}$$

$$\begin{aligned}
f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=&f'_x(x_0,y_0)\Delta x+f'_y(x_0,y_0)\Delta y+\circ(\rho)\\
f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)=&f'_x(x_0,y_0)t\cos\alpha+f'_y(x_0,y_0)t\cos\beta+\circ(\rho)\\
\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)}{t}=&\frac{f'_x(x_0,y_0)t\cos\alpha+f'_y(x_0,y_0)t\cos\beta+\circ(\rho)}{t}\\
\lim_{t\to0^+}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)}{t}=&\lim_{t\to0^+}\frac{f'_x(x_0,y_0)t\cos\alpha+f'_y(x_0,y_0)t\cos\beta+\circ(\rho)}{t}\\
\lim_{t\to0^+}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)}{t}=&\lim_{t\to0^+}\left[f'_x(x_0,y_0)\cos\alpha+f'_y(x_0,y_0)\cos\beta+\frac{\circ(\rho)}{t}\right]\\
\lim_{t\to0^+}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)}{t}=&f'_x(x_0,y_0)\cos\alpha+f'_y(x_0,y_0)\cos\beta
\end{aligned}$$

**20.6.1**

$$\begin{aligned}
\oint_L Qdy&=\int_{L_1}Qdy+\int_{L_2}Qdy\\
&=\int_d^c Q(\phi_1(y),y)dy+\int_c^d Q(\phi_2(y),y)dy\\
&=\int_c^d Q(\phi_2(y),y)dy-\int_c^d Q(\phi_1(y),y)dy\\
&=\int_c^d\left[Q(\phi_2(y),y)-\int_c^d Q(\phi_1(y),y)\right]dy\\
&=\int_c^d dy\int_{\phi_2}^{\phi_1}\frac{\partial Q}{\partial x}dx\\
&=\iint\limits_D\frac{\partial Q}{\partial x}dxdy
\end{aligned}$$

同理 $\displaystyle\oint_L Pdx=-=\iint\limits_D\frac{\partial P}{\partial y}dxdy \qquad \iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_L Pdx+Qdy$

**20.7.1**

$$P=-y,Q=x$$

$$\iint\limits_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_L Pdx+Qdy$$

$$\iint\limits_D[1-(-1)]\,dxdy=\oint_L xdy-ydx$$

$$2A=\oint_L xdy-ydx$$

$$A=\frac{1}{2}\oint_L xdy-ydx$$