争取未来开挖掘机

姜圣的追随者

2024.7.12

摘要

沉迷游戏的我无意间看见关于姜圣的新闻。深感愧疚，幼儿班的我就已经熟练的掌握了的九九乘法表。而现在我却每天沉迷于提瓦特大陆，天天只知道打丘丘人。

从今天开始我也要努力学习数学，希望姜圣以后当上院士的时候能带我一起开发挖掘机。

(本书内容：仅有公式，定理及证明)

(作者文凭：中专学历，混的文凭，简单理解就是初中学历（-。-）!)

(公式及证明出处：公式及证明都是在别的书里参考过来的，极个别公式证明是我自己瞎写的。)

本书的 pdf, 及 latex 源码地址：https://github.com/daidongchuixue/jiangping.git

2024.7.31：本书几乎是跟着 B 站高数视频记录的。记录完，会作为第一版。(预计时间几个月) 然后参考数学分析书籍重新整理，为第二版。

2024.8.5：联系方式，姜萍吧，姜圣的追随者，

目录

1 三角函数

1.1 三角恒等式

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \tag{1.1.1}$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \tag{1.1.2}$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \tag{1.1.3}$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \tag{1.1.4}$$

1.1.1 和差化积

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.5}$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.6}$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.7}$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{1.1.8}$$

1.1.2 积化和差

$$\cos(A)\sin(B) = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right] \tag{1.1.9}$$

$$\sin(A)\cos(B) = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] \tag{1.1.10}$$

$$\sin(A)\sin(B) = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \tag{1.1.11}$$

$$\cos(A)\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \tag{1.1.12}$$

1.1.3 倍角公式

$$
\begin{aligned}
\sin(2x) &= 2\sin x\cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x = 1 - 2\sin^2 x \\
\tan(2x) &= \frac{2\tan x}{1-\tan^2 x}
\end{aligned}
$$

1.1.4 反三角函数

$$\arcsin x + \arccos x = \frac{\pi}{2} \tag{1.1.13}$$

$$\arctan x + \operatorname{arccot} x = \frac{\pi}{2} \tag{1.1.14}$$

1.1.5 三角函数其他等式

$$\sin^2 x + \cos^2 = 1 \tag{1.1.15}$$

$$1 + \tan^2 x = \sec^2 \tag{1.1.16}$$

$$1 + \cot^2 x = \csc^2 \tag{1.1.17}$$

1.2 双曲函数

1.2.1 定义

$$
\begin{aligned}
\sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} \\
\tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}}
\end{aligned}
$$

1.2.2 反双曲函数

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1}) \tag{1.2.1}$$

$$\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}) \tag{1.2.2}$$

$$\operatorname{artanh} x = \frac{1}{2}\ln(\frac{1+x}{1-x}) \tag{1.2.3}$$

1.2.3 恒等式

$$\sinh(2x) = 2\sinh x \cosh x \tag{1.2.4}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{1.2.5}$$

$$\cosh^2 x + \sinh^2 x = cosh(2x) \tag{1.2.6}$$

$$\cosh x = 1 + 2\sinh^2 \frac{x}{2} \tag{1.2.7}$$

2 不等式

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 + x_2 + \cdots + x_n} \tag{2.0.1}$$

$$|x + y| \leqslant |x| + |y| \tag{2.0.2}$$

$$\sin x \leqslant x \leqslant \tan x \tag{2.0.3}$$

伯努利不等式

$$(1 + x)^n \leqslant 1 + nx \tag{2.0.4}$$

3 排列组合

3.1 定义

$$\mathbb{A}_n^k = \frac{n!}{(n-k)!} \tag{3.1.1}$$

$$\mathbb{C}_n^k = \frac{\mathbb{A}_n^k}{k!} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{3.1.2}$$

3.2 运算

4 区间与映射

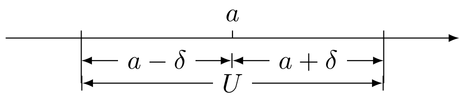
4.1 区间定义

$$
\text{区间定义}\begin{cases}(a,b)=\{x|a<x<b\}\\ [a,b]=\{x|a\leqslant x\leqslant b\}\\ (a,b]=\{x|a<x\leqslant b\}\\ (a,+\infty)=\{x|a<x\}\end{cases}
$$

4.2 领域定义

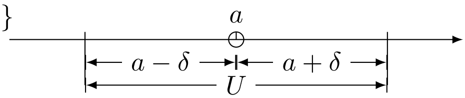
点 a 的领域

$$
U(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\}\\ \{x|\ |x-a|<\delta\}\end{cases}
$$



点 a 的去心领域

$$
\mathring{U}(a,\delta)=\begin{cases}\{x|a-\delta<x<a+\delta\wedge x\neq 0\}\\ \{x|0<|x-a|<\delta\}\end{cases}
$$



点 a 的左领域 $\quad (a-\delta,a)$

点 a 的右领域 $\quad (a,a+\delta)$

4.3 映射定义

定义:X 与 Y 是两个非空集合, 如果存在一个法则对任一 $x\in X$, 都有确定的 y 与之对应。则称 f 为从 X 到 Y 的一个映射。

$$
\text{记作}\qquad f:X\rightarrow Y
$$

$$
f(x)=y\qquad\begin{cases}\text{定义域 }(D_f)=X & x\text{-原像}\\ \text{值域 }(R_f)=\{f(x)|x\in X\} & y\text{-像}\end{cases}
$$

$$
\text{映射类型}\begin{cases}
\text{满射:} & R_f = Y \\
\text{单射:} & x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \\
\text{一一映射:} & \text{即使满射又是单射} \Leftrightarrow \text{逆映射:} \begin{cases} f^{(}x) & = y \\ f^{-1}(y) & = x \end{cases} \\
\text{复合映射:} & g \circ f \Leftrightarrow g\left[f\left(x\right)\right] \begin{cases} f : X \to Y_1 \\ g : Y_2 \to Z \\ g \circ f : X \to Z \qquad (Y_1 \subset Y_2) \end{cases}
\end{cases}
$$

5 并集，交集

5.1 定义

$$
\begin{gathered}
(\vee \text{ 或}, \wedge \text{ 与}) \\
A \cup B = \{x \in A \vee x \in B\} \\
A \cap B = \{x \in A \wedge x \in B\}
\end{gathered}
$$

5.2 运算

$$
\text{运算满足}\begin{cases}
\text{交换律}\begin{cases} A \cup B & = B \cup A \\ A \cap B & = B \cap A \end{cases} \\
\text{结合律}\begin{cases} (A \cup B) \cup C = A \cup (B \cup C) \\ (A \cap B) \cap C = A \cap (B \cap C) \end{cases} \\
\text{分配律}\begin{cases} (A \cup B) \cap C = (A \cap C) \cup (B \cap C) \\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C) \end{cases} \\
\text{对偶律}\begin{cases} (A \cup B)^C = A^C \cap B^C \\ (A \cap B)^C = A^C \cup B^C \end{cases}
\end{cases}
$$

$$
\begin{gathered}
A \cup A = A = A \cap A \\
A = B \Leftrightarrow A \subset B \wedge A \supset B \\
A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing
\end{gathered}
$$

5.3 性质

性质 1.

$$A \subset (A \cup B) \qquad A \supset (A \cap B) \tag{5.3.1}$$

性质 2.

$$A \cup B = B \Leftrightarrow A \subset B \tag{5.3.2}$$

性质 3.

$$A \cap B = A \Leftrightarrow A \subset B \tag{5.3.3}$$

性质 4. $(n \in N\ )$

$$A \cup (B_1 \cap B_2 \cap \cdots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \cdots \cap (A \cup B_n) \tag{5.3.4}$$

性质 5. $(n \in N\ )$

$$A \cap (B_1 \cup B_2 \cup \cdots \cup B_n) = (A \cap B_1) \cup (A \cap B_2) \cup \cdots \cup (A \cap B_n) \tag{5.3.5}$$

5.4 gustus De Morgan 定理

$$\neg(A \vee B) \Leftrightarrow (\neg A) \wedge (\neg B)$$

$$\neg(A \wedge B) \Leftrightarrow (\neg A) \vee (\neg B)$$

5.5 德摩根律 定理

$$\left(\bigcup_{\alpha} E_{\alpha}\right)^{C} = \bigcap_{\alpha}(E_{\alpha}^{C})$$

$$\left(\bigcap_{\alpha} E_{\alpha}\right)^{C} = \bigcup_{\alpha}(E_{\alpha}^{C})$$

6 群，环，域

6.1 群

6.1.1 M1

6.1.2 M2

6.1.3 M3

6.1.4 M4

6.1.5 sdas

6.2 环

6.3 域

7 极限

7.1 数列极限

7.1.1 数列的定义

$Def:\ \ \{x_n\}: N^+ \to R$

$$x_n = f(n)$$

7.1.2 数列极限的定义

$Def:\ \{x_n\},\ n \in N^+, \exists a,\ \forall \varepsilon > 0, \exists N,\ n > N \Rightarrow |x_n - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_n = a$

极限存在，为收敛，不存在为发散

7.1.3 极限的唯一性

$$\text{数列收敛，极限的唯一性} \tag{7.1.1}$$

7.1.4 有界数列

若$\exists M > 0, \{M \in \text{正数}\}$

使得$\forall n,\quad |x_n| \leqslant M$

则称数列 $\{x_n\}$ 为有界数列

7.1.5 收敛数列与有界性

$$\text{收敛数列必有界} \tag{7.1.2}$$

$$\text{单调有界数列必收敛} \tag{7.1.3}$$

7.1.6 收敛数列的保号性

$$\text{如果 } \lim_{n\to\infty} x_n = a \text{ 存在，且 } a > 0, \text{ 则 } \exists N > 0\{N \in N^+\} \text{ 当 } n > N \text{ 时，} \Leftrightarrow x_n > 0 \tag{7.1.4}$$

$$\lim_{n\to\infty} x_n = a, \lim_{n\to\infty} b_n = b, a < b,\ \exists N, n > N, a_n < b_n \tag{7.1.5}$$

7.1.7 收敛数列和子数列

$\{x_n\}, \lim\limits_{n\to\infty} x_n = a,\ \{x_{n_k}\} \subset \{x_n\} \Rightarrow \lim\limits_{n\to\infty} x_{n_k} = a$

证明 $K = N\ k > K$

$n_k > n_K \geqslant N$

$|x_{n_k} - a| < \varepsilon$

$\lim\limits_{n\to\infty} x_{n_k} = a$

7.2 函数极限

7.2.1 极限的定义

$$Def: \forall \varepsilon > 0 \begin{cases} \exists X > 0 \begin{cases} \text{当} x > X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to+\infty} f(x) = A \\ \text{当} x < -X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to-\infty} f(x) = A \\ \text{当 } |x| > X & \text{时都有} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to\infty} f(x) = A \end{cases} \\ \exists \delta > 0 \begin{cases} \text{当} x_0 < x < x_0 + \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^+} f(x) = A \\ \text{当} x_0 - \delta < x < x_0, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0^-} f(x) = A \\ \text{当} 0 < |x - x_0| < \delta, \text{时} & |f(x) - A| < \varepsilon \Leftrightarrow \lim\limits_{x\to x_0} f(x) = A \end{cases} \end{cases}$$

注意 1

定义中 $0 < |x - x_0|$ 表示 $x \neq x_0$ 讨论 $x \to x_0$, 只考虑 $x \neq x_0$

注意 2

$\lim\limits_{x\to x_0} f(x)$ 是否存在与 $f(x_0)$ 是否有定义取什么值无关。

$$\lim_{x\to x_0} f(x)\text{存在} \Leftrightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) \tag{7.2.1}$$

图

7.2.2 极限的性质

1 函数的极限的唯一性

如果 $\lim f(x)$ 存在必唯一。

2 局部有界性

$\lim\limits_{x\to x_0} f(x) = A \Rightarrow \exists M > 0, \delta > 0 \text{使} 0 < |x - x_0| < \delta, |f(x)| \leqslant M$

3 保号性

$\lim\limits_{x\to x_0} = A,\ A>0, \Rightarrow \exists\delta>0,$ 当$,0<|x-x_0|<\delta$时$f(x)>0$

4 保序性

$f(x)\geqslant g(x),\ \lim f(x)=a,\ \lim g(x)=b,$ 则$a\geqslant b$

4 函数极限与数列极限的关系

如果 $\lim\limits_{x\to x_0} f(x)$ 存在，$\{x_n\}$ 为 $f(x)$ 定义域的任一收敛于 x_0 的数列，则满足 $x_n\neq x_0$

则 $\lim\limits_{n\to\infty} f(x_n)=0=\lim\limits_{x\to x_0} f(x),\ x_n\to x_0$

7.3 无穷小与无穷大

7.3.1 无穷小定义

$Def:$ 如果 $\lim\limits_{x\to x_0} f(x)=0$则称$f(x)$为$x\to x_0$时的无穷小

$$Def:\forall\varepsilon>0\begin{cases}\exists X>0\begin{cases}\text{当}x>X & \text{时}\ |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to+\infty}f(x)=0\\ \text{当}x<-X & \text{时}\ |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to-\infty}f(x)=0\\ \text{当}\ |x|>X & \text{时}\ |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to\infty}f(x)=0\end{cases}\\ \exists\delta>0\begin{cases}\text{当}x_0<x<x_0+\delta,\text{时} & |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0^+}f(x)=0\\ \text{当}x_0-\delta<x<x_0,\text{时} & |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0^-}f(x)=0\\ \text{当}0<|x-x_0|<\delta,\text{时} & |f(x)-0|<\varepsilon\Leftrightarrow\lim\limits_{x\to x_0}f(x)=0\end{cases}\end{cases}$$

7.3.2 函数极限与无穷小的关系

$$\text{在自变量的同一变化中。}\alpha\ \text{为无穷小。}\lim f(x)=A\Leftrightarrow f(x)=A+\alpha \tag{7.3.1}$$

7.3.3 无穷大与无穷小的关系

在自变量同一变化过程中

$$\text{如果}\ f(x)\ \text{为无穷大，则}\ \tfrac{1}{f(x)}\ \text{为无穷小。} \tag{7.3.2}$$

$$\text{如果}\ f(x)\ \text{为无穷小，切}\ f(x)\neq 0\text{，则}\ \tfrac{1}{f(x)}\ \text{为无穷小。} \tag{7.3.3}$$

7.3.4 无穷大定义

$$
Def:\forall M>0\begin{cases}
\exists X>0\begin{cases}
\text{当}x>X\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=+\infty\\ f(x)<-M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to+\infty}f(x)=\infty\end{cases}\\
\text{当}x<-X\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=+\infty\\ f(x)<M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to-\infty}f(x)=\infty\end{cases}\\
\text{当}\ |x|>X\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=+\infty\\ f(x)<-M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to\infty}f(x)=\infty\end{cases}
\end{cases}\\
\exists\delta>0\begin{cases}
\text{当}x_0-\delta<x<x_0\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=+\infty\\ f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^-}f(x)=\infty\end{cases}\\
\text{当}x_0<x<x_0+\delta\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=+\infty\\ f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0^+}f(x)=\infty\end{cases}\\
\text{当}0<|x-x_0|<\delta\begin{cases} f(x)>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=+\infty\\ f(x)<-M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=-\infty\\ |f(x)|>M\Leftrightarrow \lim\limits_{x\to x_0}f(x)=\infty\end{cases}
\end{cases}
\end{cases}
$$

$$\lim_{x\to x_0}f(x)=\infty,\ \text{直线}x=x_0\text{是}y=f(x)\text{垂直渐进线}$$

7.4 运算

7.4.1 有限个无穷小的和仍为无穷小

$$\text{设}\ \gamma=\alpha+\beta$$

$$\alpha\ \text{和}\ \beta\ \text{同为}\ x\to x_0\ \text{时的无穷小}$$

$$\forall\varepsilon>0,\ \exists\delta_1>0,\ \text{当}\ 0<|x-x_0|<\delta_1\ \text{时，有}\ |\alpha|<\tfrac{\varepsilon}{2}$$

$$\forall\varepsilon > 0,\ \exists\delta_2 > 0,\ \text{当}\ 0 < |x - x_0| < \delta_2\ \text{时，有}\ |\beta| < \tfrac{\varepsilon}{2}$$
$$\delta = min\{\delta_1, \delta_2\},\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$0 < |x - x_0| < \delta_1, 0 < |x - x_0| < \delta_2\ \text{同时满足}$$
$$\text{即}\ |\alpha| < \tfrac{\varepsilon}{2}, |\beta| < \tfrac{\varepsilon}{2}\ \text{同时成立}$$
$$|\gamma| = |\alpha + \beta| < |\alpha| + |\beta| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

7.4.2 有界函数与无穷小的乘积仍为无穷小

$$\text{设}\ \alpha\ \text{为}\ x \to x_0\ \text{时的一个无穷小}$$
$$g(x)\ \text{为}\ x_0\ \text{的一个去心邻域}\ \mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$f(x) = g(x)\alpha$$
$$\text{证}\ f(x)\ \text{为}\ x \to x_0\ \text{时的无穷小}$$
$$\text{因为}\ g(x)\text{在}\mathring{U}(x_0, \delta_1)\ \text{有界}$$
$$\exists M > 0, \text{当} 0 < |x - x_0| < \delta_1\ \text{时}\ |g(x)| < M$$
$$\text{因为}\ \alpha\ \text{是}\ x \to x_0\ \text{的无穷小}$$
$$\exists\delta_2 > 0\text{当}0 < |x - x_0| < \delta_2\ \text{时}\ |\alpha| < \tfrac{\varepsilon}{M} < \varepsilon$$
$$\text{取}\ \delta = min\{delta_, \delta_2\}\ \text{当}\ 0 < |x - x_0| < \delta\ \text{时}$$
$$|g(x)| \geqslant M, |\alpha| < \tfrac{\varepsilon}{M}\ \text{同时成立}$$
$$|g(x)\alpha| = |g(x)|\ \ |\alpha| < M\tfrac{\varepsilon}{M} = \varepsilon$$

推论 1. 常数与无穷小的乘积为无穷小

推论 2. 有限个无穷小的乘积为无穷小

7.4.3 极限的四则运算

$\lim f(x) = A,\ \lim g(x) = B$

$$\lim\left(f(x) \pm g(x)\right) = \lim f(x) \pm \lim g(x) \tag{7.4.1}$$
$$\lim\left(f(x)g(x)\right) = \lim f(x) \lim g(x) \tag{7.4.2}$$
$$\lim\left(\frac{f(x)}{g(x)}\right) = \frac{\lim f(x)}{\lim g(x)} \tag{7.4.3}$$
$$\lim\left[Cf(x)\right] = C\lim f(x) \tag{7.4.4}$$
$$\lim\left[f(x)\right]^n = \left[\lim f(x)\right]^n \tag{7.4.5}$$

$$\lim_{x\to\infty}\frac{a_0x^m+a_1x^{m-1}+\cdots+a_m}{a_0x^n+a_1x^{n-1}+\cdots+a_n}=\begin{cases}\frac{a}{b} & m=n\\ \infty & m>n\\ 0 & m<n\end{cases} \tag{7.4.6}$$

$$\begin{gathered}\lim_{x\to x_0}g(x)=u_0,\ \lim_{u\to u_0}f(x)=A\\ \exists\delta_0>0,\ x\in\mathring{U}\left(x_0,\delta_0\right),\ g(x)\neq u_0\\ \lim_{x\to x_0}f\left[g(x)\right]=\lim_{u\to u_0}f(u)=A\end{gathered} \tag{7.4.7}$$

7.4.4 夹逼定理 (三明治定理)

$$\begin{aligned}&x_n\leqslant z_n\leqslant y_n \qquad \forall n>N_0\\ &\text{若}\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=a\text{则}\lim_{n\to\infty}z_n=a\end{aligned} \tag{7.4.8}$$

7.4.5 重要极限

趋向 0

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \tag{7.4.9}$$

$$\lim_{x\to 0}\cos x=1 \tag{7.4.10}$$

$$\lim_{x\to 0}\frac{\tan x}{x}=1 \tag{7.4.11}$$

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{1}{2}x^2}=1 \tag{7.4.12}$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=1 \tag{7.4.13}$$

$$\lim_{x\to 0}\frac{\arctan x}{x}=1 \tag{7.4.14}$$

$$\lim_{x\to 0}\frac{\ln\left(1+x\right)}{x}=1 \tag{7.4.15}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1 \tag{7.4.16}$$

$$\lim_{x\to 0}\frac{\left(1+x\right)^n-1}{nx}=1 \tag{7.4.17}$$

$$\lim_{x\to 0}\left(1+x\right)^{\frac{1}{x}}=e \tag{7.4.18}$$

趋向 ∞

$$\{x_n\} \qquad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{7.4.19}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e \tag{7.4.20}$$

7.4.6 无穷小比较

$\frac{0}{0}$ 型未定式

$Def:$ α,β 是同一极限过程的无穷小。

(1) 如果 $\lim\frac{\beta}{\alpha}=0$ 则称 β 是 α 的高阶无穷小，记作 $\beta=\circ(\alpha)$

(2) 如果 $\lim\frac{\beta}{\alpha}=\infty$ 则称 β 是 α 的底阶无穷小。

(3) 如果 $\lim\frac{\beta}{\alpha}=C$ 则称 β 是 α 的同阶无穷小。

(4) 如果 $\lim\frac{\beta}{\alpha^k}=C, k>0$ 则称 β 是 α 的 k 阶无穷小。

(5) 如果 $\lim\frac{\beta}{\alpha}=1$ 则称 β 是 α 的等价阶无穷小。

7.4.7 等价无穷小代换，因子代换

β与α是等价无穷小 $\Leftrightarrow \beta=\alpha+\circ(\alpha)$

设$\alpha\sim\alpha'$, $\beta\sim\beta'$,且$\lim\frac{\beta'}{\alpha'}$存在, 则$\lim\frac{\beta}{\alpha}=\lim\frac{\beta'}{\alpha'}$

$\lim\alpha f(x)=\lim\alpha' f(x)$

$\lim\frac{f(x)}{\alpha}=\lim\frac{f(x)}{\alpha'}$

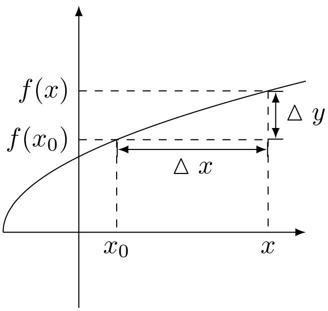
8 连续与间断点

8.1 定义

8.1.1 点连续

$$Def1:设f(x)在x_0的某邻域内有定义，如果\lim_{x\to x_0}=f(x_0)$$

$$则称f(x)在x_0处连续$$



$$\begin{cases}\triangle\ x=x-x_0\\ \triangle\ y=\begin{cases}f(x)-f(x_0)\\ f(x_0+\triangle\ x)-f(x_0)\end{cases}\end{cases}$$

$$Def2:如果\lim_{\triangle x\to 0}\triangle\ y=\lim_{\triangle x\to 0}[f(x_0+\triangle\ x)-f(x_0)]=0$$

$$则称f(x)在x_0处连续$$

8.1.2 区间连续

$$\forall x_0\in[a,b]\begin{cases}\lim\limits_{x\to x_0}f(x)=f(x_0) & x_0\in(a,b)\Leftrightarrow f(x_0)=\begin{cases}\lim\limits_{x\to x_0^-}f(x)=f(x_0^-)\\ \lim\limits_{x\to x_0^+}f(x)=f(x_0^+)\end{cases}\\ \lim\limits_{x\to x_0^+}f(x)=f(x_0^+) & x_0=a\ (右连续)\\ \lim\limits_{x\to x_0^-}f(x)=f(x_0^-) & x_0=b\ (左连续)\end{cases}$$

称在 $[a,b]$ 内连续

有界:$\exists M>0,x\in[a,b]$ 时，$|f(x)|\geqslant M$

最大值:$\exists x_0\in[a,b]$ 时，$\forall x\in[a,b],f(x)\leqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最大值

最小值:$\exists x_0\in[a,b]$ 时，$\forall x\in[a,b],f(x)\geqslant f(x_0)$ 称 $f(x_0)$ 为 $f(x)$ 在 $[a,b]$ 上的最小值

1, 闭区间 $[a,b]$ 上的连续函数 $f(x)$ 有界, 一定取得最大值与最小值。

零点定理

2, 设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)\cdot f(b)<0$
则至少存在一点 $\xi\in(a,b)$ 使 $f(\xi)=0$

介质定理

设 $f(x)$ 在 $[a,b]$ 上连续，且 $f(a)=A, f(b)=B$
$\forall C\in(A,B)$，至少有一点 $\xi, f(\xi)=C$

8.1.3 间断点

1,$f(x)$ 无定义
2,$\lim\limits_{x\to x_0} f(x)$ 不存在
3. $\lim\limits_{x\to x_0} f(x)$ 存在, 但 $\lim\limits_{x\to x_0} f(x)\neq f(x_0)$
第一类间断点：$f(x_0^+)=\lim\limits_{x\to x_0^+} f(x)$ 与 $f(x_0^-)=\lim\limits_{x\to x_0^-} f(x)$
第二类间断点：不是第一类的。

8.2 连续函数的运算

函数 $f(x), g(x)$ 在 $x=x_0$ 连续。

$$\lim_{x\to x_0}[f(x)+g(x)]=\lim_{x\to x_0} f(x)+\lim_{x\to x_0} g(x)=f(x_0)+g(x_0)$$
$$\lim_{x\to x_0}[f(x)-g(x)]=\lim_{x\to x_0} f(x)-\lim_{x\to x_0} g(x)=f(x_0)-g(x_0)$$
$$\lim_{x\to x_0}[f(x)\cdot g(x)]=\lim_{x\to x_0} f(x)\cdot\lim_{x\to x_0} g(x)=f(x_0)\cdot g(x_0)$$
$$\lim_{x\to x_0}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}=\frac{f(x_0)}{g(x_0)}\qquad(g(x_0)\neq 0)$$

反函数的连续性
若 $y=f(x)$ 在区间 I_x 上单调增加，且连续。
则 $y=f^-1(x)$ 在 $I_y=\{y|y=f(x),x\in I_x\}$ 上也为单调增加，连续

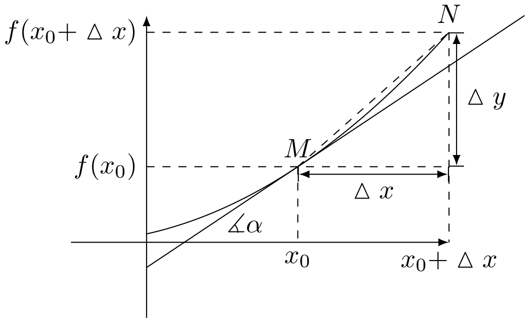
$$
\text{复合函数,}\begin{cases}
\text{内外都连续}\begin{cases}
\lim\limits_{x\to x_0} g(x) = g(x_0) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f\left[g(x_0)\right] = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
\text{外连续}\begin{cases}
x \to x_0 \begin{cases}
\lim\limits_{x\to x_0} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to x_0} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to x_o} g(x))
\end{cases} \\
x \to \infty \begin{cases}
\lim\limits_{x\to \infty} g(x) = u_0 \\
\lim\limits_{u\to u_0} f(x) = f(u_0) \\
\lim\limits_{x\to \infty} f\left[g(x)\right] = f(u_0) = f(\lim\limits_{x\to \infty} g(x))
\end{cases}
\end{cases}
\end{cases}
$$

9 导数

9.1 定义

导数的概念从物理发展出来的。

$$v\left(t_0\right)=\lim_{\triangle t\to 0}\frac{s\left(t_0+\triangle t\right)-s\left(t_0\right)}{\triangle t}$$



$$NM斜率=\tan\beta=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}=\frac{\triangle y}{\triangle x}$$

$$斜率k=\tan\alpha=\lim_{\triangle x\to 0}\tan\beta=\lim_{\triangle x\to 0}\frac{\triangle y}{\triangle x}=\frac{f\left(x_0+\triangle x\right)-f(x_0)}{\triangle x}$$

9.1.1 导数定义

$y=f(x)$ 在 x_0 的某邻域内有定义
给自变量的增量 $\triangle x,(x_0+\triangle x)$ 仍在定义域内
函数得到了相应增量 $\triangle y,\triangle y=f\left(x_0+\triangle x\right)$
如果 $\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$ 存在, 称 $y=f(x)$ 在 $x=x_0$ 处可导
(极限值为$y=f(x)$在$x=x_0$处导数)

$$记\ y'|_{x=x_0}=f'(x_0)=\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$$

$$\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\Leftrightarrow\lim_{\triangle x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

9.1.2 导函数定义

$f(x)$ 在区间 I 内任意一点均可导。
$f'(x)=\lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}$
称 $f'(x)$ 为 $y=f(x)$ 在区间 I 上的导函数

9.1.3 闭区间可导定义

$$f(x)\ 在\ [a,b]\ 可导 \Leftrightarrow \begin{cases} f'(x_0) & x_0 \in (a,b) \Leftrightarrow \begin{cases} 左导数 f'_-(x_0) = \lim\limits_{\Delta x \to 0^-} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \\ 右导数 f'_+(x_0) = \lim\limits_{\Delta x \to 0^+} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \end{cases} \\ f'_+(a) & x = a \\ f'_-(a) & x = b \end{cases}$$

9.1.4 导数与连续

$$f'(x)存在 \Rightarrow f(x)在x = x_0处连续 \tag{9.1.1}$$

9.2 幂数，指数，对数

$$\frac{d}{\mathrm{d}x}C = 0 \tag{9.2.1}$$

$$\frac{d}{\mathrm{d}x}x^a = ax^{a-1} \tag{9.2.2}$$

$$\frac{d}{\mathrm{d}x}a^x = a^x \ln a \tag{9.2.3}$$

$$\frac{d}{\mathrm{d}x}e^x = e^x \tag{9.2.4}$$

$$\frac{d}{\mathrm{d}x}\log_a^x = \frac{1}{x\ln a} \tag{9.2.5}$$

$$\frac{d}{\mathrm{d}x}\ln x = \frac{1}{x} \tag{9.2.6}$$

9.3 三角函数

$$\frac{d}{\mathrm{d}x}\sin x = \cos x \tag{9.3.1}$$

$$\frac{d}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}} \tag{9.3.2}$$

$$\frac{d}{\mathrm{d}x}\csc x = -\csc x \cot x \tag{9.3.3}$$

$$\frac{d}{\mathrm{d}x}\cos x = -\sin x \tag{9.3.4}$$

$$\frac{d}{\mathrm{d}x}\arccos x = -\frac{1}{\sqrt{1-x^2}} \tag{9.3.5}$$

$$\frac{d}{\mathrm{d}x}\sec x = \sec x \tan x \tag{9.3.6}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}} \tag{9.3.7}$$

$$\frac{d}{\mathrm{d}x}\tan x = \sec^2 x \tag{9.3.8}$$

$$\frac{d}{\mathrm{d}x}\arctan = \frac{1}{1+x^2} \tag{9.3.9}$$

$$\frac{d}{\mathrm{d}x}\cot x = -\csc^2 x \tag{9.3.10}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arccot} x = -\frac{1}{1+x^2} \tag{9.3.11}$$

$$\frac{d}{\mathrm{d}x}\sinh x = \cosh x \tag{9.3.12}$$

$$\frac{d}{\mathrm{d}x}\cosh x = \sinh x \tag{9.3.13}$$

$$\frac{d}{\mathrm{d}x}\tanh x = \frac{1}{cosh} \tag{9.3.14}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arcsinh} x = \tag{9.3.15}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arccosh} x = \tag{9.3.16}$$

$$\frac{d}{\mathrm{d}x}\operatorname{arctanh} x = \tag{9.3.17}$$

9.4 导数运算

$$U = u(x), V = v(x), \text{均在}x\text{点可导}, C\text{为常数}$$

$$\frac{d(CU)}{\mathrm{d}x} = C\frac{d(U)}{\mathrm{d}x} \tag{9.4.1}$$

$$\frac{d(U+V)}{\mathrm{d}x} = \frac{dU}{\mathrm{d}x} \pm \frac{dV}{\mathrm{d}x} \tag{9.4.2}$$

$$\frac{d(UV)}{\mathrm{d}x} = \frac{dU}{\mathrm{d}x}V + \frac{dV}{\mathrm{d}x}U \tag{9.4.3}$$

$$\frac{d(\frac{U}{V})}{\mathrm{d}x} = \frac{\frac{dU}{\mathrm{d}x}V - \frac{dV}{\mathrm{d}x}U}{V^2} \tag{9.4.4}$$

9.5 反函数求导

如果函数 $y = f(x)$ 在区间 (a,b) 内单调可导，且 $f'(y) \neq 0$

$$\begin{cases} \alpha = \min\{f(a)+0, f(b-0)\} \\ \beta = \max\{f(a)+0, f(b-0)\} \end{cases}$$

则它的反函数 $x = f^{-1}(y)$ 在区间 (α, β) 内也可导

$$\left[f^{-1}(y)\right]' = \frac{1}{f'(x)} \tag{9.5.1}$$

$$\frac{dy}{\mathrm{d}x} = \frac{1}{\frac{dx}{\mathrm{d}y}}$$

9.6 复合函数求导

$$\text{设函数}\begin{cases} y = f(u)\text{在}U(u_0, \delta_0)\text{处有定义} \\ u = g(x)\text{在}U(x_0, \eta_0)\text{处有定义} \end{cases}$$

$$u_0 = g(x_0), \text{且} f'(u)\text{和}g'(x)\text{都存在}$$

$$\text{则复合函数}F(x) = f\left[g(x)\right]\text{在点}x_0\text{可导, 且}$$

$$F'(x_0) = f'\left[g(x_0)\right]g'(x_0) \tag{9.6.1}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

9.7 高阶求导

$$Def: \begin{cases} \text{一阶导数} & y' \Leftrightarrow \frac{dy}{dx} \\ \text{二阶导数} & y'' \Leftrightarrow \frac{d^2y}{dx^2} \\ \text{三阶导数} & y''' \Leftrightarrow \frac{d^3y}{dx^3} \\ \text{三阶以上 n 阶导数} & y^{(n)} \Leftrightarrow \frac{d^ny}{dx^n} \end{cases}$$

9.8 高阶求导公式

$$\frac{d^n}{dx}e^x = e^x \tag{9.8.1}$$

$$\frac{d^n}{dx}a^x = a^x\left(lna\right)^n \tag{9.8.2}$$

$$\frac{d^n}{dx}x^\mu = A_\mu^n x^{\mu-n} \tag{9.8.3}$$

$$\frac{d^n}{dx}\left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n+1}} \tag{9.8.4}$$

$$\frac{d^n}{dx}\ln(x+a) = \frac{(-1)^{n-1}(n-1)!}{(x+a)^n} \tag{9.8.5}$$

$$\frac{d^n}{dx}\sin x = \sin(x+n\frac{\pi}{2}) \tag{9.8.6}$$

$$\frac{d^n}{dx}\cos x = \cos(x+n\frac{\pi}{2}) \tag{9.8.7}$$

$$\frac{d^n}{dx}\left[f(ax+b)\right] = a^n \cdot \frac{d^n f(ax+b)}{d(ax+n)} \tag{9.8.8}$$

9.9 高阶求导运算法则

$$\frac{d^n}{dx}(u \pm v) = \frac{d^n u}{dx} \pm \frac{d^n v}{dx} \tag{9.9.1}$$

$$\text{莱布紫泥公式} \qquad (uv)^n = \sum_{k=0}^{n} C_n^k u^{(n-k)} \cdot v^k \tag{9.9.2}$$

9.10 隐函数求导

9.11 参数方程求导

10 积分

10.1 幂数，指数，对数

$$\int x^a \mathrm{d}x = \frac{1}{a-1} x^{a-1} + C \tag{10.1.1}$$

$$\int b^x \mathrm{d}x = \frac{b^x}{\ln b} + C \tag{10.1.2}$$

$$\int e^x \mathrm{d}x = e^x + C \tag{10.1.3}$$

$$\int \frac{1}{x} \mathrm{d}x = \ln x + C \tag{10.1.4}$$

10.2 三角函数

$$\int \sin x \mathrm{d}x = -\cos x + C \tag{10.2.1}$$

$$\int \frac{1}{\sqrt{1-x^2}} \mathrm{d}x = \arcsin x + C \tag{10.2.2}$$

$$\int \csc x \cot x \mathrm{d}x = -\csc x + C \tag{10.2.3}$$

$$\int \cos x \mathrm{d}x = \sin x + C \tag{10.2.4}$$

$$\int \sec x \tan x \mathrm{d}x = \sec x + C \tag{10.2.5}$$

$$\int \sec^2 x \mathrm{d}x = \tan x + C \tag{10.2.6}$$

$$\int \csc^2 x \mathrm{d}x = -\cot x + C \tag{10.2.7}$$

$$\int \frac{1}{|x| \sqrt{x^2-1}} \mathrm{d}x = \operatorname{arcsec} x + C \tag{10.2.8}$$

$$\int \frac{1}{1+x^2} \mathrm{d}x = \tan x + C \tag{10.2.9}$$

$$\int \sinh x \mathrm{d}x = \cosh x + C \tag{10.2.10}$$

$$\int \cosh x \mathrm{d}x = \sinh x + C \tag{10.2.11}$$

10.3 积分运算

11 零散的一些

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q} \tag{11.0.1}$$

$$A_N = \sum_{k=0}^{n} q^k \qquad qA_N = \sum_{k=1}^{n+1} q^k$$

$$A_N - qA_N = \sum_{k=0}^{n} q^k - \sum_{k=1}^{n+1} q^k = 1 - q^{n+1}$$

$$A_N = \frac{1-q^{n+1}}{1-q}$$

$$\log_{10} x = \lg_x \tag{11.0.2}$$

$$\log_e x = \ln_x \tag{11.0.3}$$

$$\log_b xy = \log_b x + \log_b y \tag{11.0.4}$$

$$\log_{(b^n)} x = \frac{1}{n} \log_b x \tag{11.0.5}$$

$$\log_b x^n = n \log_b x \tag{11.0.6}$$

$$\log_b x = \frac{\log_c x}{\log_c b} \tag{11.0.7}$$

$$b^n = x \qquad b^m = y$$

$$b^{n+m} = xy$$

$$\log_b xy = n + m = \log_b x + \log_b y$$

$$b^n = x$$

$$\log_b x = n$$

$$\frac{1}{n} \log_b x = 1 = \log_{(b^n)} x$$

$$b^1 = x^n \qquad b^{\frac{1}{n}} = x$$

$$n \log_b x = 1 = \log_b x^n$$

$$\log_b x = log_{c^{(\log_c b)}} c^{(\log_c x)} = \frac{\log_c x}{\log_c b}$$

$$(a+b)^n = \sum_{m=0}^{n} C_n^m a^{n-m} b^m$$

$$a^2 - b^2 = (a-b)(1+b)$$

$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$$

$$a^n - b^n = (a-b)\sum_{m=0}^{n-1}\left(a^{n-m}b^m\right) = (a-b)\left(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}\right)$$

12 证明

12.1 第 1章

1.2.4

$$\begin{aligned}\sinh x\cosh x &= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \left(\frac{1}{2}\right)\left(\frac{e^{2x} - e^{-2x}}{2}\right)\\ &= \frac{1}{2}\sinh(2x)\\ \sinh(2x) &= 2\sinh x\cosh x\end{aligned}$$

1.2.5

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)\\ &= e^x \times e^{-x}\\ &= 1\end{aligned}$$

1.2.6

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 + \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{2e^{2x} + 2e^{-2x}}{4}\\ &= \frac{e^{2x} + e^{-2x}}{2}\\ &= \cosh(2x)\end{aligned}$$

1.2.7

$$\begin{aligned}\cosh(2x) &= \cosh^2 x + \sinh^2 x\\ &= \sinh^2 x + 1 + \sinh^2 x\\ &= 2\sinh^2 x + 1\\ \cosh x &= 2\sinh^2\frac{x}{2} + 1\end{aligned}$$

12.2 第 7章

7.1.1

$$
\begin{aligned}
&\text{反设} \lim_{n\to\infty} x_n = a,\ \lim_{n\to\infty} x_n = b, \text{且} a < b \\
&\varepsilon = \frac{b-a}{3} \begin{cases} \exists N_1,\ n > N_1,\ |x_n - a| < \frac{b-a}{3} \\ \exists N_2,\ n > N_2,\ |x_n - b| < \frac{b-a}{3} \end{cases} \\
&N = \max\{N_1, N_2\},\ n > N \Rightarrow \begin{cases} n > N_1 \\ n > N_2 \end{cases} \\
b - a &= |(x_n - a) - (x_n - b)| \\
&\leqslant |x_n - a| + |x_n - b| \\
&< \frac{b-a}{3} + \frac{b-a}{3} \\
&< \frac{2(b-a)}{3}
\end{aligned}
$$

7.1.2

$$
\begin{gathered}
\varepsilon = 1,\ \exists N > 0,\ \text{当 n>N 时} \, |X_n - a| < 1 \\
|X_n| = |(X_n - a) + a| \\
\leqslant |x_n - a| + |a| \\
\leqslant 1 + |a| \\
M = \max\{|X_n|, |X_2|, \ldots, |X_n|, 1 + |a|\} \\
\forall n,\ |X_n| \leqslant M
\end{gathered}
$$

7.1.4

1

由于 $\lim_{n\to\infty} x_n = a$, 且$a > 0$
$\varepsilon = \frac{a}{2},\ \exists N > 0, n > N$
$|x_n - a| < \varepsilon$
$|x_n - a| < \frac{a}{2}$
$-\frac{a}{2} < x_n - a < \frac{a}{2}$
$\frac{a}{2} < x_n < 1$

2

用反证法，反设 $a<0$. 从某项起 $x_n<0$ 矛盾

7.1.5

$$x_n = b_n - a_n$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n$$
$$\lim_{n\to\infty} x_n = b - a > 0$$
$$\lim_{n\to\infty} x_n > 0$$
$$b_n - a_n = x_n > 0$$
$$b_n > a_n$$

7.2.1 $\lim\limits_{x\to x_0} f(x)$存在 $\Rightarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x)$

$$\text{设} \lim_{x\to x_0} = A$$
$$0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon$$
$$0<|x-x_0|<\delta \Leftrightarrow x\in \mathring{U}(x_0,\delta)$$
$$\begin{cases} \text{当}x_0<x<x_0+\delta\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^+} f(x)=A \\ \text{当}x_0-\delta<x<x_0\text{时}0<|x-x_0|<\delta,\ |f(x)-A|<\varepsilon,\ \lim\limits_{x\to x_0^-} f(x)=A \end{cases}$$
$$\lim_{x\to x_0^+} f(x) = A = \lim_{x\to x_0^-} f(x)$$

$\lim\limits_{x\to x_0} f(x)$存在 $\Leftarrow \lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x)$

$$A=\begin{cases} \lim\limits_{x\to x_o^+}, \forall\varepsilon>0, \exists\delta_1>0, x_0<x<x_0+\delta_1, |f(x)-A|<\varepsilon \\ \lim\limits_{x\to x_o^-}, \forall\varepsilon>0, \exists\delta_2>0, x_0-\delta_2<x<x_0, |f(x)-A|<\varepsilon \end{cases}$$
$$\delta=\min\{\delta_1,\delta_2\}$$
$$0<|x-x_0|<\delta \begin{cases} x>x_0, x_0<x<x_0+\delta\leqslant x_0+\delta_1,\ |f(x)-A|<\varepsilon \\ x<x_0, x_0-\delta_2\leqslant x_0+\delta<x<x_0,\ |f(x)-A|<\varepsilon \end{cases}$$
$$\lim_{x\to x_0} f(x) = A$$

7.3.1

$$\lim_{x\to x_o} f(x) = A \Rightarrow \begin{cases} \alpha\text{为}x\to x_0\text{时的无穷小} \\ f(x)=\alpha+A \end{cases}$$
$$\text{设} \lim_{x\to x_o} f(x)=A,\ \text{记}f(x)-A=\alpha$$

只需证 α 为无穷小。

$$\forall\varepsilon>0,\exists\delta>0,当0<|x-x_0|<\delta,时\,|f(x)-A|<\varepsilon$$
$$即\ |\alpha-0|<\varepsilon$$
$$\alpha为x\to x_0时的无穷小$$
$$\lim_{x\to x_o}f(x)=A\Leftarrow\begin{cases}\alpha为x\to x_0时的无穷小\\ f(x)=\alpha+A\end{cases}$$
$$\forall\varepsilon>0,\ \exists\delta>0,\ 当0<|x-x+0|<\delta,\ |\alpha|<\varepsilon$$
$$即\,|f(x)-A|<\varepsilon\ \lim_{x\to x_0}f(x)=A$$

7.3.2

$$设\ \lim_{x\to x_0}f(x)=\infty$$
$$对\ f(x)\ 为\ x\to\ 时无穷大$$
$$对于\ M=\tfrac{1}{\varepsilon}.\ 存在\ \delta>0$$
$$当\ 0<|x-x_0|<\delta\ 时$$
$$|f(x)|>M=\tfrac{1}{\varepsilon}$$
$$\left|\tfrac{1}{f(x)}\right|<\varepsilon$$
$$\tfrac{1}{f(x)}\ 为\ x\to x_0\ 时的无穷小$$

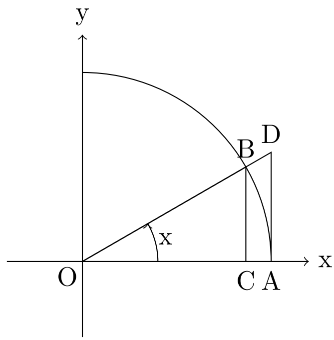
7.4.2

$$\begin{aligned}f(x)g(x)&=[A+\alpha]\,[B+\beta]\\&=AB+A\beta+B\alpha+\beta\alpha\\&=AB+\gamma\qquad\qquad(\gamma为无穷小)\end{aligned}$$
$$\lim\,[f(x)g(x)]=AB+\gamma=\lim f(x)\lim g(x)$$

7.4.8

$$\forall\varepsilon>0$$
$$|x_n-a|<\varepsilon\qquad\forall n>N_1$$
$$|y_n-a|<\varepsilon\qquad\forall n>N_2$$
$$令N=\max\{N_1,N_2,N_0\}\,,\ 则当n>N时有$$
$$a-\varepsilon<x_n\le z_n\le y_n<a+\varepsilon$$
$$|z_n-a|<\varepsilon$$
$$\lim_{n\to\infty}z_n=a$$

7.4.9



$$OB = OA = 1$$

$$\triangle\ AOB \leqslant \text{扇形面积} \leqslant \triangle\ AOD$$

$$\frac{1}{2}\sin x \leqslant \frac{1}{2}x \leqslant \frac{1}{2}\tan x$$

$$\sin x \leqslant x \leqslant \tan$$

$$1 \geqslant \frac{\sin x}{x} \geqslant \cos x$$

$$\lim_{x\to 0} 1 \geqslant \lim_{x\to 0} \frac{\sin x}{x} \geqslant \lim_{x\to 0} \cos x$$

$$1 \geqslant \lim_{x\to 0} \frac{\sin x}{x} \geqslant 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

7.4.10

$$\begin{aligned} |1-\cos x| = 1-\cos x &= 2\sin^2\frac{x}{2} \\ &\leqslant 2\left(\frac{x}{2}\right)^2 \end{aligned}$$

$$0 \leqslant 1-\cos x \leqslant \frac{x^2}{2}$$

$$\lim_{x\to 0} 0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant \lim_{x\to 0}\frac{x^2}{2}$$

$$0 \leqslant \lim_{x\to 0}(1-\cos x) \leqslant 0$$

$$\lim_{x\to 0}(1-\cos x) = 0$$

$$\lim_{x\to 0}\cos x = 1$$

7.4.11

$$\begin{aligned}\lim_{x\to0}\frac{\tan x}{x}&=\lim_{x\to0}\frac{\sin x}{x}\frac{1}{\cos x}\\&=\lim_{x\to0}\frac{\sin x}{x}\lim_{x\to0}\frac{1}{\cos x}\\&=1\end{aligned}$$

7.4.12

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos x}{\frac{1}{2}x^2}&=\lim_{x\to0}\frac{2\sin^2\frac{x}{2}}{\frac{1}{2}x^2}\\&=\lim_{x\to0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\\&=1\end{aligned}$$

7.4.13

$$x=\sin t,\ t=\arcsin x$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{\arcsin x}{x}=\lim_{x\to0}\frac{t}{\sin t}=1$$

7.4.14

$$x=\tan t,\ t=\arctan x$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{\arctan x}{x}=\lim_{t\to0}\frac{t}{\tan t}=1$$

7.4.15

$$\lim_{x\to0}\frac{\ln(1+x)}{x}=\lim_{x\to0}\ln(1+x)^{\frac{1}{x}}=\ln e=1$$

7.4.16

$$e^x-1=t,\ x=\ln(t+1)$$

$$x\to0,\ t\to0$$

$$\lim_{x\to0}\frac{e^x-1}{x}=\lim_{t\to0}\frac{t}{\ln(t+1)}=1$$

7.4.17

$$\lim_{x\to0}\frac{(1+x)^n-1}{nx}=\lim_{x\to0}\left(\frac{e^{n\ln(1+x)}-1}{n\ln(1+x)}\cdot\frac{\ln(1+x)}{x}\right)=1$$

7.4.19

$$
\begin{aligned}
x_n &= \left(1+\frac{1}{n}\right)^n = \sum_{m=0}^{n} C_n^m 1^{n-m}\left(\frac{1}{n}\right)^m = \sum_{m=0}^{n} C_n^m \left(\frac{1}{n}\right)^m \\
&= C_n^0\left(\frac{1}{n}\right)^0 + C_n^1\left(\frac{1}{n}\right)^1 + \sum_{m=2}^{n} C_n^m\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{n!}{m!\,(n-m)!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{\overbrace{(n)\,(n-1)\cdots(n-m+1)}^{m}}{m!}\left(\frac{1}{n}\right)^m \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\cdots\left(\frac{n-m+1}{n}\right) \\
&= 1+1+\sum_{m=2}^{n}\frac{1}{m!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right) \\
x_{n+1} &= 1+1+\sum_{m=2}^{n+1}\frac{1}{m!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{m-1}{n+1}\right) \\
x_n &< x_{n+1} \qquad \text{单调增加} \\
x_n &< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \\
&< 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{n^2} = 1+\frac{1-\left(\frac{1}{2}\right)^2}{1-\frac{1}{2}} \\
&< 1+\frac{1}{1-\frac{1}{2}} \\
&< 3 \qquad \text{有界}
\end{aligned}
$$

12.3 第 9章

9.1.1

$$
\begin{aligned}
f'(x_0) &= \lim_{\triangle x\to 0}\frac{\triangle\, y}{\triangle\, x} \qquad \text{因为极限存在与无穷小的关系} \\
\frac{\triangle\, y}{\triangle\, x} &= f'(x_0)+\alpha \qquad \alpha\text{为}\triangle\, x\to 0\text{时的无穷小} \\
\triangle\, y &= f'(x_0)\triangle\, x+\alpha\triangle\, x \\
\lim_{\triangle x\to 0}\triangle\, y &= \lim_{\triangle x\to 0}[f'(x_0)\triangle\, x+\alpha\triangle\, x] = 0 \\
\lim_{x\to x_0} f(x) = f(x_0) &\Leftrightarrow \lim_{\triangle x\to 0}[f(x_0+\triangle\, x)-f(x_0)=0] \Leftrightarrow \lim_{\triangle x\to 0}\triangle\, y = 0
\end{aligned}
$$

9.2.1

$$\begin{aligned}\frac{d}{\mathrm{d}x}C &= \lim_{\triangle x\to 0}\frac{f(x_0+\triangle x)-f(x_0)}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{C-C}{\triangle x}\\ &= 0\end{aligned}$$

9.2.2

$$\begin{aligned}\frac{d}{\mathrm{d}x}x^a &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\ &= \frac{x^a-x_0^a}{x-x_0}\\ &= \frac{(x-x_0)\left(x^{a-1}+x^{a-2}x_0+\cdots+xx_0^{a-2}+x_0^{a-1}\right)}{x-x_0}\\ &= ax_0^{a-1}\end{aligned}$$

9.2.3

$$\begin{aligned}\frac{d}{\mathrm{d}x}a^x &= \lim_{\triangle x\to 0}\frac{a^{x+\triangle x}-a^x}{\triangle x}\\ &= a^x\lim_{\triangle x\to 0}\frac{a^{\triangle x}-1}{\triangle x}\\ &= a^x\lim_{\triangle x\to 0}\frac{e^{\triangle x\ln a}-1}{\triangle x}\\ &= a^x\ln a\end{aligned}$$

9.2.4

$$\frac{d}{\mathrm{d}x}e^x = e^x\ln e = e^x$$

9.2.5

$$\begin{aligned}\frac{d}{\mathrm{d}x}\log_a^x &= \lim_{\triangle x\to 0}\frac{\log_a^{x+\triangle x}-\log_a^x}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{\log_a^{1+\frac{\triangle x}{x}}}{\triangle x}\\ &= \lim_{\triangle x\to 0}\frac{\ln 1+\frac{\triangle x}{x}}{\ln a\triangle x}\\ &= \frac{1}{\ln a}\lim_{\triangle x\to 0}\frac{\frac{\triangle x}{x}}{\triangle x}\\ &= \frac{1}{x\ln a}\end{aligned}$$

9.2.6

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\ln^{x} &= \frac{1}{x\ln e} \\
&= \frac{1}{x}
\end{aligned}
$$

9.3.1

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\sin x &= \lim_{\triangle x\to 0}\frac{\sin(x_0+\triangle\ x)-\sin x_0}{\triangle\ x} \\
&= \lim_{\triangle x\to 0}\frac{2\cos(x_0+\frac{\triangle x}{2})\sin\frac{\triangle x}{2}}{\triangle\ x} \\
&= \lim_{\triangle x\to 0}\cos(x_0+\frac{\triangle\ x}{2})\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= cosx_0
\end{aligned}
$$

9.3.2

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\arcsin x = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\sin y}{\mathrm{d}y}} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin^2 y}} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}
$$

9.3.3

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\csc x = \frac{d\frac{1}{\sin x}}{\mathrm{d}x} &= \frac{\frac{d1}{\mathrm{d}x}\sin x-\frac{d\sin x}{\mathrm{d}x}\cdot 1}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= -\csc x\cdot\cot x
\end{aligned}
$$

9.3.4

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\cos x &= \lim_{\triangle x\to 0}\frac{\cos(x_0+\triangle\ x)-cos(x_0)}{\triangle\ x} \\
&= \lim_{\triangle x\to 0}\frac{-2\sin\left(x_0+\frac{\triangle x}{2}\right)\sin\frac{\triangle x}{2}}{\triangle\ x} \\
&= \lim_{\triangle x\to 0}-\sin\left(x_0+\frac{\triangle\ x}{2}\right)\frac{\sin\frac{\triangle x}{2}}{\frac{\triangle x}{2}} \\
&= -\sin x
\end{aligned}
$$

9.3.5

$$\frac{d}{\mathrm{d}x}\arccos x=\frac{1}{\frac{dx}{\mathrm{d}y}}=\frac{1}{\frac{d\cos y}{\mathrm{d}y}}$$
$$=\frac{1}{-\sin y}$$
$$=-\frac{1}{\sqrt{1-\cos^2 y}}$$
$$=-\frac{1}{\sqrt{1-x^2}}$$

9.3.6

$$\frac{d}{\mathrm{d}x}\sec x=\frac{d\frac{1}{\cos x}}{\mathrm{d}x}=\frac{\frac{d1}{\mathrm{d}x}\cos x-\frac{d\cos x}{\mathrm{d}x}\cdot 1}{\cos^2 x}$$
$$=\frac{\sin x}{\cos^2 x}$$
$$=\sec x\cdot\tan x$$

9.3.8

$$\frac{d}{\mathrm{d}x}\tan x=\frac{d\frac{\sin x}{\cos x}}{\mathrm{d}x}=\frac{\frac{d\sin x}{\mathrm{d}x}\cos x-\frac{d\cos x}{\mathrm{d}x}\sin x}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}$$
$$=\sec^2 x$$

9.3.9

$$\frac{d}{\mathrm{d}x}\arctan x=\frac{1}{\frac{dx}{\mathrm{d}y}}=\frac{1}{\frac{d\tan y}{\mathrm{d}y}}$$
$$=\frac{1}{\sec y}$$
$$=\frac{1}{1+\tan^2 y}$$
$$=\frac{1}{1+x^2}$$

9.3.10

$$\frac{d}{\mathrm{d}x}\cot x=\frac{d\frac{\cos x}{\sin x}}{\mathrm{d}x}=\frac{\frac{d\cos x}{\mathrm{d}x}\sin x-\frac{d\sin x}{\mathrm{d}x}\cos x}{\sin^2 x}$$
$$=-\frac{\sin^2 x+\cos^2 x}{\cos^2 x}$$
$$=-\csc^2 x$$

9.3.11

$$\begin{aligned}\frac{d}{\mathrm{d}x}\operatorname{arccot} x = \frac{1}{\frac{dx}{\mathrm{d}y}} &= \frac{1}{\frac{d\cot y}{\mathrm{d}y}} \\ &= \frac{1}{-\csc^2 y} \\ &= -\frac{1}{1+\cot^2 y} \\ &= -\frac{1}{1+x^2}\end{aligned}$$

9.3.12

$$\begin{aligned}\frac{d}{\mathrm{d}x}\sinh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{e^x + e^{-x}}{2} \\ &= \cosh x\end{aligned}$$

9.3.13

$$\begin{aligned}\frac{d}{\mathrm{d}x}\cosh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{e^x - e^{-x}}{2} \\ &= \sinh x\end{aligned}$$

9.3.14

$$\begin{aligned}\frac{d}{\mathrm{d}x}\tanh x &= \frac{d}{\mathrm{d}x}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \\ &= \frac{\frac{d(e^x - e^{-x})}{\mathrm{d}x}(e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{\mathrm{d}x}(e^x - e^{-x})}{(e^x + e^{-x})^2} \\ &= \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\ &= \frac{2^2}{(e^x + e^{-x})^2} \\ &= \frac{1}{\cosh x^2}\end{aligned}$$

9.3.15

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arcsinh} x &= \frac{d}{\mathrm{d}x}\left[\ln(x+\sqrt{x^2+1})\right] \\
&= \frac{d\ln(x+\sqrt{x^2+1})}{\mathrm{d}\big(x+\sqrt{x^2+1}\big)} \cdot \frac{d\left(x+\sqrt{x^2+1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2+1}\right)}{d\left(x^2+1\right)} \cdot \frac{d(x^2+1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2+1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2+1}} \cdot \frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}
$$

9.3.16

$$
\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arccosh} x &= \frac{d}{\mathrm{d}x}\left[\ln(x+\sqrt{x^2-1})\right] \\
&= \frac{d\ln(x+\sqrt{x^2-1})}{\mathrm{d}\big(x+\sqrt{x^2-1}\big)} \cdot \frac{d\left(x+\sqrt{x^2-1}\right)}{\mathrm{d}x} \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(\frac{dx}{dx} + \frac{d\left(\sqrt{x^2-1}\right)}{d\left(x^2-1\right)} \cdot \frac{d(x^2-1)}{dx}\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2-1}} \cdot 2x\right) \\
&= \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}} \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}
$$

9.3.17

$$\begin{aligned}
\frac{d}{\mathrm{d}x}\operatorname{arctanh} x &= \frac{d}{\mathrm{d}x}\left[\frac{1}{2}\ln(\frac{1+x}{1-x})\right] \\
&= \frac{1}{2}\cdot\frac{d\left[\ln(\frac{1+x}{1-x})\right]}{\mathrm{d}\left(\frac{1+x}{1-x}\right)}\cdot\frac{d\left(\frac{1+x}{1-x}\right)}{\mathrm{d}x} \\
&= \frac{1}{2}\cdot\frac{1}{\left(\frac{1+x}{1-x}\right)}\cdot\frac{\frac{d(1+x)}{\mathrm{d}x}(1-x)-\frac{d(1-x)}{\mathrm{d}x}(1+x)}{(1-x)^2} \\
&= \frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{(1-x)+(1+x)}{(1-x)^2} \\
&= \frac{1}{(1+x)(1-x)} \\
&= \frac{1}{1-x^2}
\end{aligned}$$

9.4.1

$$\begin{aligned}
\frac{d(CU)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{Cu(x+\triangle x)-Cu(x)}{\triangle x} \\
&= C\lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x} \\
&= C\frac{d(U)}{\mathrm{d}x}
\end{aligned}$$

9.4.2

$$\begin{aligned}
\frac{d(U\pm V)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)\pm v(x+\triangle x)-v(x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)-u(x)}{\triangle x}\pm\frac{v(x+\triangle x)-v(x)}{\triangle x} \\
&= \frac{d(U)}{\mathrm{d}x}\pm\frac{d(V)}{\mathrm{d}x}
\end{aligned}$$

9.4.3

$$\begin{aligned}
\frac{d(UV)}{\mathrm{d}x} &= \lim_{\triangle x\to 0}\frac{u(x+\triangle x)v(x+\triangle x)-u(x)v(x)-u(x)v(x+\triangle x)+u(x)v(x+\triangle x)}{\triangle x} \\
&= \lim_{\triangle x\to 0}\frac{[u(x+\triangle x)+u(x)]\,v(x+\triangle x)+u(x)\,[v(x+\triangle x)-v(x)]}{\triangle x} \\
&= \frac{dU}{\mathrm{d}x}\lim_{\triangle x\to 0}v(x+\triangle x)+u(x)\frac{dV}{\mathrm{d}x} \\
&= \frac{dU}{\mathrm{d}x}v(x)+u(x)\frac{dV}{\mathrm{d}x}
\end{aligned}$$

9.5.1

$$\begin{aligned}\left[f^{-1}(y)\right]'|_{y=y_0} &= \lim_{y\to y_o}\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0}\\ &= \lim_{y\to y_o}\frac{x-x_0}{y-y_0}\\ &= \lim_{x\to x_o}\frac{x-x_0}{f(x)-f(x_0)}\\ &= \lim_{x\to x_o}\frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}\\ &= \frac{1}{f'(x)}\end{aligned}$$

9.6.1

$$\text{定义函数}\quad A(u)=\begin{cases}\frac{f(u)-f(u_0)}{u-u_0}. & u\neq u_0\\ f'(u_0), & u=u_0\end{cases}$$

$$A(u)\text{在}u_o\text{处连续, 既有}$$

$$\lim_{u\to u_o}A(u)=A(u_0)=f'(u_0)$$

$$\text{由恒等式}f(u)-f(u_0)=A(u)(u-u_0)\text{我们有}$$

$$\begin{aligned}\frac{F(x)-F(x_0)}{x-x_0} &= \frac{f\left[g(x)\right]-f\left[g(x_0)\right]}{x-x_0}\\ &= A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}\end{aligned}$$

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0}=\lim_{x\to x_0}A\left[g(x)\right]\frac{g(x)-g(x_0)}{x-x_0}$$

$$F'(x_0)=f'(g(x_0))g'(x_0)$$