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THE VALUE OF AN ASIAN OPTION

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Abstract

This paper approaches the problem of computing the price of an Asian option in two different ways. Firstly, exploiting a scaling property, we reduce the problem to the problem of solving a parabolic PDE in two variables. Secondly, we provide a lower bound which is so accurate that it is essentially the true price.

BROWNIAN MOTION; FIXED STRIKE; FLOATING STRIKE

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1. Introduction

Let us suppose that the price at time t, S_t, of some risky asset is given by

$$S_t = S_0 \exp(\sigma B_t - \tfrac{1}{2}\sigma^2 t + ct), \tag{1.1}$$

where B is a standard one-dimensional Brownian motion, and c is some constant whose value matters little for now. The problem of computing the value of an Asian (call) option with maturity T and strike price K, written on this risky asset, is mathematically equivalent to calculating

$$\mathbb{E}(Y-K)^+, \tag{1.2}$$

where we define Y by

$$Y \equiv \int_0^T S_u \, \mu(du) \tag{1.3}$$

and assume that $c=r$, the riskless interest rate. In the case of the 'fixed strike' Asian option, the measure μ is given by $\mu(du) = T^{-1} I_{[0,T]}(u)du$, although other candidates for μ are of interest and can be handled without difficulty. For example, if we take $\mu(u) = \delta_T(du)$, then we have the classical European call option, and if we take $\mu(du) = T^{-1} I_{[0,T]}(u)du - \delta_T(du)$ together with $K=0$, then we have the 'floating strike' Asian option, whose price at time zero is therefore

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$$\mathbb{E}\left(T^{-1}\int_0^T S_u\,du - S_T\right)^+. \tag{1.4}$$

Asian options are commonly traded; they were introduced in part to avoid a problem common to European options, that by manipulating the price of an asset near to the maturity date, speculators could drive up the gains from the option. Despite this, there is not yet (and probably never will be) a simple analytical expression for the value, in contrast to the situation for a European call, where the famous Black–Scholes formula is available.

Previous work on this problem has been of three broad types. Firstly, there are numerical studies, such as the work of Kemna and Vorst [6] who use Monte Carlo techniques, and Carverhill and Clewlow [1] who use a Fourier transform method to compute the law of the average. Secondly, there are methods which replace the law of the average (which is hard to specify) by something more tractable; the work of Ruttiens [10], Vorst [12], Levy [7], [8], Levy and Turnbull [9], Turnbull and Wakeman [11] is of this nature, though it has to be admitted that these approaches offer little control on the error produced by the ansatz. Thirdly, there is the determined analysis of Yor [13], and Geman and Yor [3], [4]. This has produced notable expressions for the price as a triple integral, and for the price of an Asian option with 'independent exponential maturity'. Numerical inversion of this Laplace transform seems likely to be slow, and no simple analytic inversion has been found to date.

The present paper is a combination of analysis and numerics. In Section 2, we exploit a scaling property to reduce the calculation of the price of an Asian option to the solution of a parabolic PDE in two variables, rather than the three which at first sight appear necessary. We have learned that a similar scaling property for the floating strike Asian option was observed already by Ingersoll [5], p. 377; one can use this scaling behaviour equally well for fixed strike Asian options, and indeed the formalism we use covers also options whose averaging period starts at a time other than 0, or whose averaging is with respect to a quite general weight function. Numerical solution of this PDE in real time is a practical possibility, and we discuss in Section 4 how this is done. We find that (even without great effort to lubricate the programs) it is possible to compute a value accurate to about 5% in 3 seconds on a SUN SPARC 2 station, provided σ is not too small.

The approach of Section 3 is to try to obtain bounds on the price. We propose only trivial methods for this, based on conditioning firstly on some variable Z. Thus to obtain a lower bound, we have

$$\mathbb{E}(Y^+) = \mathbb{E}(\mathbb{E}(Y^+ \mid Z)) \geqq \mathbb{E}(\mathbb{E}(Y \mid Z)^+). \tag{1.5}$$

We investigated numerically several possible choices for Z, some of them bivariate. However, for the fixed strike Asian option, by far the best choice turned out to be

$$Z = \int_0^T B_u\,du.$$

We obtain a two-dimensional integral as a lower bound, which (at least for a wide range of values of S_0, K, σ, r) is *staggeringly accurate*! (The yardstick here is the set of results of the PDE method of Section 2, which agree with the Monte Carlo results of Kemna and Vorst [7].) The reason why this bound is so good is quite general, and is applicable also to the floating strike Asian option, or an Asian option where the payoff Y is computed as a discrete average of the prices throughout the period. It rests on an analysis of the error committed in making the estimate (1.5), which simplifies very pleasingly.

As we discuss in the final section, Section 4, this bound also computes very quickly, taking less than 1 second, on a SUN SPARC Classic workstation, to get a value accurate to about 1%, comparing very favourably with the PDE method in most cases, except those of large volatility when the PDE method seems to be satisfactory. This estimate for the price has also been used by Curran [2] in the case of fixed strike Asian options.

2. A PDE for the price of an Asian option

We shall assume until further notice that the maturity of the option is T, fixed, and that the probability measure μ has a density ρ_t in $(0, T)$. There is no essential loss of generality in this. If we define

$$\phi(t, x) \equiv \mathbb{E}\left[\left(\int_t^T S_u\,\mu(du) - x\right)^+ \middle| S_t = 1\right], \tag{2.1}$$

where S is given as at (1.1), then we develop the martingale

$$\begin{aligned} M_t &\equiv \mathbb{E}\left[\left(\int_0^T S_u\,\mu(du) - K\right)^+ \middle| \mathscr{F}_t\right] \\ &= \mathbb{E}\left[\left(\int_t^T S_u\,\mu(du) - \left(K - \int_0^t S_u\,\mu(du)\right)\right)^+ \middle| \mathscr{F}_t\right] \\ &= S_t\,\mathbb{E}\left[\left(\int_t^T \frac{S_u}{S_t}\,\mu(du) - \frac{K - \int_0^t S_u\,\mu(du)}{S_t}\right)^+ \middle| \mathscr{F}_t\right] \\ &= S_t\,\phi(t, \xi_t), \end{aligned} \tag{2.2}$$

where

$$\xi_t \equiv \frac{K - \int_0^t S_u\,\mu(du)}{S_t}. \tag{2.3}$$

It is immediate from the definition of ϕ that ϕ is jointly continuous, decreasing in t, and decreasing convex in x. Now by Itô's formula,

$$d\xi_t = -\rho_t\,dt + \xi_t(-\sigma dB_t - r dt + \sigma^2 dt),$$

so assuming that ϕ has enough smoothness to apply Itô's formula to (2.2), we have (with '$\doteq$' signifying that the two sides differ by a local martingale)

$$
\begin{aligned}
dM &= \phi dS + S(\dot{\phi} dt + \phi' d\xi + \tfrac{1}{2}\phi'' d[\xi]) + dS d\phi \\
&\doteq r\phi S dt + S(\dot{\phi} + \phi'(-\rho_t - r\xi + \sigma^2\xi) + \tfrac{1}{2}\sigma^2\xi^2\phi'')dt - \sigma S \cdot \phi'\sigma\xi dt \\
&= S[r\phi + \dot{\phi} - (\rho_t + r\xi)\phi' + \tfrac{1}{2}\sigma^2\xi^2\phi'']dt,
\end{aligned}
$$

which implies that

$$
0 = \dot{\phi} + r\phi + \tfrac{1}{2}\sigma^2\xi^2\phi'' - (\rho_t + r\xi)\phi'. \tag{2.4}
$$

If we now write $f(t, x) \equiv e^{-r(T-t)}\phi(t, x)$, we find that f solves

$$
\dot{f} + \mathscr{G}f = 0, \tag{2.5}
$$

where $\mathscr{G}$ is the operator

$$
\mathscr{G} \equiv \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} - (\rho_t + rx)\frac{\partial}{\partial x}.
$$

The boundary conditions depend on the problem; in the case of the fixed strike Asian option,

$$
f(T, x) = x^-, \tag{2.6}
$$

whereas in the case of the floating strike Asian option, we shall have

$$
f(T, x) = (1 + x)^-. \tag{2.7}
$$

Now the PDE (2.5) is quite simple and can be solved numerically, as we shall discuss in Section 4. Let us denote the solution to the PDE (2.4) with the (fixed strike) boundary condition (2.6) by ϕ, and let the solution to (2.4) with the (floating strike) boundary condition (2.7) be denoted by ψ. Thus in the case where μ is uniform on $[0, T]$, the price of the Asian option with maturity T, fixed strike price K, and initial price S_0 is

$$
\begin{aligned}
e^{-rT}\mathbb{E}\left(\int_0^T (S_u - K)\frac{du}{T}\right)^+ &= S_0 f(0, KS_0^{-1}) \\
&\equiv e^{-rT} S_0 \phi(0, KS_0^{-1}),
\end{aligned}
$$

and the price of the Asian option with maturity T and floating strike is simply

$$
e^{-rT}\mathbb{E}\left(\int_0^T S_u \frac{du}{T} - S_T\right)^+ = e^{-rT} S_0 \psi(0, 0).
$$

Notice that with these parameters, for $x \leqq 0$,

$$
\phi(t, x) = r^{-1}(e^{r(T-t)} - 1) - x,
$$

which makes the solution of the PDE easier in this case. Also, for large negative x, $\psi(t, x)$ is very close to

$$\mathbb{E}\left(\int_t^T S_u \frac{du}{T} - S_T - x \mid S_t = 1\right) = \frac{e^{r(T-t)} - 1}{rT} - e^{r(T-t)} - x,$$

which helps to set boundary values for numerical methods. Other formulae can be derived simply from these; for example, for an Asian option with strike K and maturity T, but whose average is computed over the interval $[T-t, T]$, $0<t<T$, the price is

$$e^{-rT} \int_0^\infty \mathbb{P}(S_{T-t} \in dx) x \phi(T-t, K/x),$$

where ϕ is computed using the measure μ which is uniform on $[T-t, T]$, and this is easily computed once the function $\phi(T-t, \cdot)$ is known. The case where μ puts equal weight on a finite sequence of equally spaced time-points is just as easy.

3. Lower bounds

If we condition the process X on some zero-mean Gaussian variable Z, it remains a Gaussian process, and this is the heart of the lower-bound method used here. To set up some notation, let us write

$$\mathbb{E}(B_t \mid Z) = m_t Z, \qquad \operatorname{cov}(B_s, B_t \mid Z) = v_{st}. \tag{3.1}$$

It is well known that

$$m_t = \mathbb{E}(B_t Z)/\mathbb{E}(Z^2), \qquad v_{st} = s \wedge t - \mathbb{E}(B_s Z)\mathbb{E}(B_t Z)/\mathbb{E}(Z^2). \tag{3.2}$$

In most cases of practical interest, it will prove to be quite easy to compute explicitly what these functions are. For example, if we fix $T=1$ and take $Z = \int_0^1 B_t dt$ then

$$m_t = 3t(2-t)/2, \qquad v_{st} = s \wedge t - 3st(2-s)(2-t)/4, \tag{3.3}$$

and when $Z = \int_0^1 B_t dt - B_1 = \int_0^1 t dB_t$, we obtain likewise

$$m_t = -3t^2/2, \qquad v_{st} = s \wedge t - 3s^2t^2/4. \tag{3.4}$$

The lower bound of (1.5) is not guaranteed to be good, but we can estimate the error made as follows. For any random variable U, we have

$$\begin{aligned} 0 &\leqq \mathbb{E}(U^+) - \mathbb{E}(U)^+ \\ &= \tfrac{1}{2}(\mathbb{E}(|U|) - |\mathbb{E}(U)|) \\ &\leqq \tfrac{1}{2}\mathbb{E}(|U - \mathbb{E}(U)|) \\ &\leqq \tfrac{1}{2}\operatorname{var}(U)^{1/2}. \end{aligned}$$

Accordingly,

$$0 \leqq \mathbb{E}[\mathbb{E}(Y^+ \mid Z) - \mathbb{E}(Y \mid Z)^+] \tag{3.5}$$
$$\leqq \tfrac{1}{2}\mathbb{E}[\mathrm{var}(Y \mid Z)^{1/2}],$$

and it is the variance of Y given Z which we propose to estimate. Firstly, the mean of Y given Z is

$$\mathbb{E}\left[\int_0^1 \exp(\sigma B_t - \tfrac{1}{2}\sigma^2 t + rt)\mu(dt) \mid Z\right] = \int_0^1 \exp(\sigma m_t Z - \tfrac{1}{2}\sigma^2 v m_t^2 + rt)\mu(dt), \tag{3.6}$$

where we have made the abbreviation $v \equiv \mathrm{var}(Z)$ and set $S_0 = 1$, and then similarly we have that

$$\mathrm{var}(Y \mid Z) = \int_0^1 \mu(ds) \int_0^1 \mu(dt) \exp(\sigma Z(m_s + m_t)$$
$$- \tfrac{1}{2}\sigma^2 v(m_s^2 + m_t^2) + r(s+t))(\exp\{\sigma^2 v_{st}\} - 1),$$

after a few rearrangements. Let us now consider what we would have if we used the approximation $e^x \approx 1 + x$ in this integral; we would obtain

$$V \equiv \int_0^1 \mu(ds) \int_0^1 \mu(dt)(1 + \sigma Z(m_s + m_t) - \tfrac{1}{2}\sigma^2 v(m_s^2 + m_t^2) + r(s+t))\sigma^2 v_{st},$$

which *vanishes if* $Z = \int_0^1 B_t \mu(dt)$. To see this, note that the first factor in the integrand may be written in the form

$$1 + f(s, Z) + f(t, Z) \equiv 1 + \{rs + \sigma Z m_s - \tfrac{1}{2}\sigma^2 v m_s^2\} + \{rt + \sigma Z m_t - \tfrac{1}{2}\sigma^2 v m_t^2\},$$

and that

$$\int_0^1 v_{st}\, \mu(ds) = \mathrm{cov}\left(\int_0^1 B_s \mu(ds), B_t \mid Z\right) = \mathrm{cov}(Z, B_t \mid Z) = 0.$$

Accordingly, we may estimate

$$\mathrm{var}(Y \mid Z) = \mathrm{var}(Y \mid Z) - V \leqq I_1 + I_2, \tag{3.7}$$

where

$$I_1 = \int_0^1 \mu(ds) \int_0^1 \mu(dt)(\exp\{f(s, Z) + f(t, Z)\} - 1 - f(s, Z) - f(t, Z))|\exp\{\sigma^2 v_{st}\} - 1|$$

$$I_2 = \int_0^1 \mu(ds) \int_0^1 \mu(dt)(\exp\{\sigma^2 v_{st}\} - 1 - \sigma^2 v_{st})|1 + f(s, Z) + f(t, Z)|.$$

Now the estimate (3.7) depends on Z, and will not be small for all values of Z. However, we have (using (3.5) and (3.7)) the estimates

$$0 \leqq \mathbb{E}[\mathbb{E}(Y^+ \mid Z) - \mathbb{E}(Y \mid Z)^+] \leqq \tfrac{1}{2}(\mathbb{E}(I_1 + I_2))^{1/2}, \tag{3.8}$$

so our task is to estimate $\mathbb{E}I_1$ and $\mathbb{E}I_2$. We shall use the inequalities

$$|e^x - 1| \leqq |x| e^{|x|}, \qquad |e^x - 1 - x| \leqq \tfrac{1}{2} x^2 e^{|x|},$$

valid for all real x. We shall also write $g(s) \equiv rs - \frac{1}{2}\sigma^2 v m_s$ in what follows, and let c, γ_1, γ_2 be constants such that for all $0 \leqq s,\ t \leqq 1$,

$$|v_{st}| \leqq c, \qquad (m_s + m_t)^2 \leqq \gamma_1, \qquad |g_s + g_t| \leqq \gamma_2.$$

Now we estimate

$$\begin{aligned}
\mathbb{E}I_1 &\leqq c\sigma^2 \exp\{c\sigma^2\} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt) \\
&\quad \times \mathbb{E}[\exp(g_s + g_t + Z\sigma(m_s + m_t)) - 1 - g_s - g_t - Z\sigma^2(m_s + m_t)] \\
&= c\sigma^2 \exp\{c\sigma^2\} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt) \\
&\quad \times [\exp(g_s + g_t + \tfrac{1}{2}\sigma^2 v(m_s + m_t)^2) - 1 - g_s - g_t] \\
&= c\sigma^2 \exp\{c\sigma^2\} \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt) \\
&\quad \times [\{\exp\{\tfrac{1}{2}\sigma^2 v(m_s + m_t)^2\} - 1\}\exp\{g_s + g_t\} + \exp\{g_s + g_t\} - 1 - g_s - g_t] \\
&\leqq c\sigma^2 \exp\{c\sigma^2 + \gamma_2\}[\tfrac{1}{2}\sigma^2\gamma_1 v \exp\{\tfrac{1}{2}\sigma^2\gamma_1 v\} + \tfrac{1}{2}\gamma_2^2] \\
&\quad \times \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt) \\
&= c\sigma^2 \exp\{c\sigma^2 + \gamma_2\}[\tfrac{1}{2}\sigma^2\gamma_1 v \exp\{\tfrac{1}{2}\sigma^2\gamma_1 v\} + \tfrac{1}{2}\gamma_2^2]M,
\end{aligned}$$

where we have abbreviated

$$M \equiv \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt).$$

As for the other term, we have

$$\begin{aligned}
\mathbb{E}I_2 &= \int_0^1 \int_0^1 |\mu|(ds)|\mu|(dt)(\exp\{\sigma^2 v_{st}\} - 1 - v_{st}) \cdot |1 + g_s + g_t| \\
&\leqq \tfrac{1}{2}\sigma^4 c^2 \exp\{\sigma^4 c^2\}(1 + \gamma_2)M.
\end{aligned}$$

Assembling this, the estimate on the right of (3.8) becomes

$$\tfrac{1}{2}[c\sigma^2 \exp\{c\sigma^2+\gamma_2\}(\tfrac{1}{2}\sigma^2\gamma_1 v \exp\{\tfrac{1}{2}\sigma^2\gamma_1 v\}+\tfrac{1}{2}\gamma_2^2)+\tfrac{1}{2}\sigma^4 c^2 \exp\{\sigma^4 c^2\}(1+\gamma_2)]^{1/2}M^{1/2}.$$

Let us now see how these estimates shape up in the two examples of main interest to us, the fixed and floating strike Asian options.

Firstly for the fixed strike, we see that $v \equiv \mathrm{var}(Z)=\frac{1}{3}$ and easily that we may take $\gamma_1=9$, $\gamma_2=r+\sigma^2/4$. It is also not hard to establish that we may take $c=\frac{1}{3}$. The integral of $|\mu(ds)|\,|\mu(dt)|$ over the square is 1.

For the floating strike Asian option, the constants are the same except that now we take $c=3^{2/3}/4$ and the integral of the measure over the square comes to 4.

It is now clear that for typical values, say r and σ of the order of 10^{-1}, we shall have that $\mathbb{E}I_1$ is bounded by something that is of the order of 10^{-4} and $\mathbb{E}I_2$ is bounded by something which is of the order of 10^{-4} also, so we can expect that the bound on the error will be something of the order of 10^{-2} at worst; in the next section we discuss the outcome of numerics on the bounds.

4. Computational aspects

Throughout this section, we assume that $T=1$ and $S_0=100$. We tried a variety of methods for solving the PDE (2.5), and report here on what worked well, and on what worked less well. Neither of the authors is an expert in numerical methods, and it was not the goal to obtain the most rapid possible program. However, even the clumsy computing which we carried out showed that it is possible to obtain accuracies of a few percent in times of the order of a few seconds.

It turns out that by treating (2.5) simply as a parabolic PDE and solving it with the NAG routine D03PAF (as the coefficient $\sigma^2x^2/2$ vanishes at $x=0$, we replace it by $\sigma x^2/2+10^{-60}$ to justify the use of D03PAF), quite acceptable combinations of accuracy

TABLE 1
Fixed strike, $\sigma=0.05$

r	Strike	PDE_1	PDE_2	PDE_3	LB_1	LB_2	UB
0.05	95	6.147	6.826	7.157	7.174	7.178	7.183
		(2.07)	(7.95)	(29.4)	(0.03)	(2.95)	(0.61)
	100	3.009	2.577	2.621	2.713	2.716	2.722
		(2.07)	(7.95)	(29.4)	(0.03)	(2.95)	(0.61)
	105	1.289	0.641	0.439	0.334	0.337	0.343
		(2.07)	(7.95)	(29.4)	(0.03)	(2.95)	(0.61)
0.09	95	7.616	8.671	8.823	8.808	8.809	8.821
		(2.16)	(8.26)	(31.3)	(0.03)	(2.96)	(0.56)
	100	4.004	3.931	4.185	4.305	4.308	4.318
		(2.16)	(8.26)	(31.3)	(0.03)	(2.96)	(0.56)
	105	1.848	1.193	1.011	0.955	0.958	0.968
		(2.16)	(8.26)	(31.3)	(0.03)	(2.96)	(0.56)
0.15	95	10.026	11.223	11.090	11.094	11.094	11.114
		(2.47)	(9.28)	(45.0)	(0.02)	(2.96)	(0.59)
	100	5.829	6.447	6.777	6.790	6.794	6.810
		(2.47)	(9.28)	(45.0)	(0.02)	(2.96)	(0.59)
	105	2.990	2.582	2.639	2.741	2.744	2.761
		(2.47)	(9.28)	(45.0)	(0.02)	(2.96)	(0.59)

TABLE 2
Fixed strike, $\sigma = 0.10$

r	Strike	PDE_1	PDE_2	PDE_3	LB_1	LB_2	UB
0.05	90	11.233	11.916	11.942	11.944	11.951	11.973
		(3.05)	(11.1)	(41.1)	(0.04)	(2.67)	(0.56)
	100	3.702	3.58	3.624	3.634	3.641	3.663
		(3.05)	(11.1)	(41.1)	(0.04)	(2.67)	(0.56)
	110	0.800	0.434	0.359	0.324	0.331	0.353
		(3.05)	(11.1)	(41.1)	(0.04)	(2.67)	(0.56)
0.09	90	12.815	13.376	13.382	13.376	13.385	13.410
		(3.05)	(12.5)	(44.6)	(0.04)	(3.02)	(0.57)
	100	4.702	4.804	4.887	4.908	4.915	4.942
		(3.05)	(12.5)	(44.6)	(0.04)	(3.02)	(0.57)
	110	1.147	0.740	0.659	0.623	0.630	0.657
		(3.05)	(12.5)	(44.6)	(0.04)	(3.02)	(0.57)
0.15	90	15.116	15.403	15.398	15.400	15.399	15.445
		(3.44)	(11.8)	(55.4)	(0.03)	(3.14)	(0.56)
	100	6.477	6.899	7.000	7.021	7.028	7.066
		(3.44)	(11.8)	(55.4)	(0.03)	(3.14)	(0.56)
	110	1.876	1.487	1.430	1.406	1.413	1.451
		(3.44)	(11.8)	(55.4)	(0.03)	(3.14)	(0.56)

TABLE 3
Fixed strike, $\sigma = 0.20$

r	Strike	PDE_1	PDE_2	PDE_3	LB_1	LB_2	UB
0.05	90	12.365	12.565	12.589	12.578	12.595	12.687
		(5.89)	(24.6)	(84.3)	(0.04)	(3.03)	(0.61)
	100	5.717	5.751	5.760	5.745	5.762	5.854
		(5.89)	(24.6)	(84.3)	(0.04)	(3.03)	(0.61)
	110	2.158	2.014	1.996	1.971	1.989	2.080
		(5.89)	(24.6)	(84.3)	(0.04)	(3.03)	(0.61)
0.09	90	13.617	13.803	13.825	13.814	13.831	13.927
		(6.14)	(25.9)	(88.1)	(0.05)	(2.75)	(0.59)
	100	6.679	6.758	6.773	6.759	6.777	6.872
		(6.14)	(25.9)	(88.1)	(0.05)	(2.75)	(0.59)
	110	2.684	2.564	2.551	2.528	2.545	2.641
		(6.14)	(25.9)	(88.1)	(0.05)	(2.75)	(0.59)
0.15	90	15.464	15.619	15.636	15.624	15.641	15.748
		(6.45)	(28.1)	(102)	(0.03)	(2.83)	(0.6)
	100	8.245	8.382	8.402	8.391	8.408	8.515
		(6.45)	(28.1)	(102)	(0.03)	(2.83)	(0.6)
	110	3.634	3.563	3.558	3.537	3.554	3.661
		(6.45)	(28.1)	(102)	(0.03)	(2.83)	(0.6)

and speed were obtained, for certain ranges of the parameter values. Tables 1-4 give some specimen results for typical values of the parameters for the fixed strike problem, with maturity time $T = 1$ and initial stock price $S_0 = 100$ (computations carried out on a SUN SPARC Classic workstation). One of the parameters of the NAG routine is the number of points to be used in the spatial grid, which is equivalent to the spacing of the points in the grid. The three columns (PDE_1, PDE_2 and PDE_3) correspond to spacings

TABLE 4
Fixed strike, $\sigma = 0.3$

r	Strike	PDE_1	PDE_2	PDE_3	LB_1	LB_2	UB
0.05	90	13.865	13.940	13.951	13.919	13.952	14.161
		(12.1)	(56.0)	(164)	(0.05)	(5.00)	(0.66)
	100	7.921	7.941	7.944	7.911	7.944	8.153
		(12.1)	(56.0)	(164)	(0.05)	(5.00)	(0.66)
	110	4.124	4.079	4.074	4.037	4.070	4.279
		(12.1)	(56.0)	(164)	(0.05)	(5.00)	(0.66)
0.09	90	14.893	14.970	14.981	14.950	14.983	15.194
		(12.2)	(45.3)	(170)	(0.04)	(3.95)	(0.63)
	100	8.789	8.822	8.827	8.795	8.827	9.039
		(12.2)	(45.3)	(170)	(0.04)	(3.95)	(0.63)
	110	4.735	4.701	4.698	4.662	4.695	4.906
		(12.2)	(45.3)	(170)	(0.04)	(3.95)	(0.63)
0.15	90	16.422	16.500	16.510	16.480	16.512	16.732
		(14.0)	(48.6)	(175)	(0.04)	(5.49)	(0.63)
	100	10.149	10.200	10.208	10.177	10.208	10.429
		(14.0)	(48.6)	(175)	(0.04)	(5.49)	(0.63)
	110	5.746	5.731	5.731	5.696	5.728	5.948
		(14.0)	(48.6)	(175)	(0.04)	(5.49)	(0.63)

TABLE 5
Floating strike

σ	r	PDE_1	PDE_2	PDE_3	LB_1	LB_2	UB
0.1	0.05	1.667	1.312	1.257	1.241	1.245	1.355
		(7.2)	(32.2)	(81.7)	(0.02)	(2.33)	(0.43)
	0.09	1.148	0.791	0.709	0.695	0.699	0.825
		(7.0)	(32.8)	(80.7)	(0.03)	(2.16)	(0.46)
	0.15	0.623	0.333	0.271	0.247	0.252	0.415
		(8.4)	(33.4)	(84.6)	(0.02)	(2.72)	(0.44)
0.2	0.05	3.431	3.405	3.401	3.398	3.404	3.831
		(30.4)	(81.7)	(178)	(0.03)	(2.28)	(0.47)
	0.09	2.683	2.631	2.622	2.616	2.622	3.062
		(29.0)	(82.9)	(192)	(0.02)	(2.35)	(0.45)
	0.15	1.839	1.727	1.723	1.704	1.710	2.187
		(32.1)	(89.4)	(189)	(0.03)	(2.48)	(0.48)
0.3	0.05	5.609	5.625	5.628	5.618	5.625	6.584
		(66.6)	(200)	(219)	(0.04)	(3.49)	(0.51)
	0.09	4.738	4.741	4.736	4.732	4.738	5.706
		(69.5)	(205)	(273)	(0.04)	(3.60)	(0.55)
	0.15	3.630	3.613	3.612	3.602	3.609	4.604
		(73.8)	(222)	(265)	(0.04)	(3.74)	(0.53)

of 0.025, 0.01, and 0.005 respectively. As the spacing increases, the accuracy improves, but the time required increases quite noticeably. The columns LB_1 and LB_2 report the computed values of the lower bound, the second using a smaller mesh in the integration routine. The final column UB computes an upper bound for the price of the option from the estimate (3.5). Throughout, we give in brackets beneath the estimate of the price the amount of time required to compute it.

TABLE 6

Volatility σ	Strike price K	Upper bound	MC result	Lower bound
	95	8.95	8.91	8.91
0.10	100	5.10	4.91	4.92
	105	2.34	2.06	2.07
	90	15.23	14.96	14.98
0.30	100	9.39	8.81	8.83
	110	5.37	4.68	4.70
	90	18.52	18.14	18.18
0.50	100	13.69	12.98	13.02
	110	9.97	9.10	9.18

Certain features are quickly apparent. Except in the cases of options which are out of the money, the lower and upper bounds will give a bracket on the price, accurate to about 2% or better, in under 1s. Even for the options which are out of the money, the accuracy is not too bad (no worse than about 6%), probably comparable with the errors involved in estimating σ. The accuracy decreases slightly as the volatility increases. The PDE results, however, improve as the volatility increases; for volatility 0.05, the PDE answers are unable to get inside the bounds, although they are within 10% for options which are not out of the money, and for volatility 0.1 the PDE results come within 2% if they are allowed a little over 10s and are not required to deal with out-of-the-money cases. But if the volatility is increased to 0.2 then the estimates come within 5% in under 10s of computing time. The results in Table 4 show that the PDE answer is now within about 2% of the bounds, though it requires somewhat over 10s to achieve this; less accuracy needs less time.

Table 5 gives some results for the floating-strike problem. Again, the bounds are evaluated very rapidly, but the gap between upper and lower is larger as a percentage, not least because the actual prices are quite small. Nevertheless, the values obtained agree well with the PDE answers with a small step size.

Table 6 gives some specimen upper and lower bounds for typical values of the parameters with maturity time $T=1$, initial price $S_0=100$ and interest rate $r=0.09$. Comparisons are given with the Monte Carlo results given in Levy and Turnbull [9]. As can be seen, the agreement is excellent.

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