Notation

(X, H) polarized algebraic K3 (resp. (X, ω) Kähler K3)

$\deg_H E := (c_1(E).H)$ (resp. $\deg_\omega(E) := (c_1(E).\omega)$.

<u>Rem.</u> $(X, H) \rightsquigarrow (X, \omega_{FS}|_X)$ w/ $c_1(H) = \omega_{FS}|_X$ $(\because c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = \omega_{FS}) \rightsquigarrow \deg_H = \deg_{\omega_{FS}|_X}$

From now on, we don't specify H or ω, but deg depends on them.

<u>Def.</u> Suppose $\mathrm{rk}(E) \neq 0$.

Define the **slope** of E to be $\mu(E) := \dfrac{\deg E}{\mathrm{rk}(E)}$

A torsion free sheaf E is called **μ-stable** (or **slope stable**) (resp. μ-semistable)
if for any subsheaf $F < E$ w/ $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$,
we have $\mu(F) < \mu(E)$
(resp. $\leq$).

<u>Lem.</u> X smth proj. var.

(i) Any line bundle is μ-stable.

(ii) For a SES $0 \to F \to E \to G \to 0$ w/ $\mathrm{rk}(F), \mathrm{rk}(G) \neq 0$,
$\mu(F) < \mu(E) \iff \mu(E) < \mu(G)$.

<u>Cor.</u> If E is loc. free, E is μ-stable
X smth surface
$\iff$ for any loc. free $F \subset E$, $\mu(F) < \mu(E)$
w/ $\mathrm{rk} F < \mathrm{rk} E$
and tors. free quot.

(iii) μ-stable $\Rightarrow$ simple

(iv) E is μ-stable $\iff E^*$ is μ-stable $\iff E^{**}$ is μ-stable
loc. free and
$\mu(E) = \mu(E^{**})$.

Proof.) (i) vacant

(ii) Note $\deg E = \deg F + \deg G$ and $\mathrm{rk}\, E = \mathrm{rk}\, F + \mathrm{rk}\, G$.

Proof of Cor.)

By (ii), F is μ-stable if $\forall F \subset E$ w/ $0 < \mathrm{rk}\, F < \mathrm{rk}\, E$, $\mu(E) < \mu(E/F)$

$0 \leftarrow (E/F)/T(E/F) \twoheadleftarrow E \leftarrow \ker \longleftarrow 0$

Since $\forall H$, $\mu(H/T(H)) < \mu(H)$. we may assume E/F is torsion free $\rightsquigarrow$ F is loc. free by the following lem.

<u>Lem.</u> $0 \to F \to E \to G \to 0$ on a smooth surface X. If E is loc. free and G is torsion free, then F is loc. free

proof). Take $x \in X$.

<u>Def.</u> $\underset{\mathcal{O}_x\text{-mod}}{\mathrm{depth}\, M} = \min\{ i \mid \mathrm{Ext}^i(k(x), M) \neq 0 \}$ $=$ max lgth. of reg. seq.

$$0 \to F_x \to \underset{\substack{\cong \\ \mathcal{O}_x^{\oplus n}}}{E_x} \to \underset{\text{tors. free.}}{G_x} \to 0$$

Recall $\mathrm{pd}\, F_x + \mathrm{depth}\, F_x = \mathrm{depth}\, \mathcal{O}_x \overset{X \text{ regular } (\Rightarrow \text{Cohen-Mac})}{=} \dim \mathcal{O}_x \overset{X \text{ surface}}{=} 2$,

so it suffices to see $\mathrm{depth}\, F_x = 2$ ($\rightsquigarrow \mathrm{pd}\, F_x = 0$).

Indeed, since G_x is tors. free $\rightsquigarrow$ $\forall a \in \mathcal{O}_x$ is G_x-regular.

$\to$ $\mathrm{depth}\, G_x \geq 1$.

$\to$ $\mathrm{depth}\, F_x \geq 2$. $\square$

(iii) Suppose E is not simple.

Then, $\exists \psi \in \mathrm{Hom}(E,E) \neq k$ w/ non-triv. kernel ($\rightsquigarrow \underset{\overset{\wedge}{0}}{\mathrm{rk}\, \ker} + \underset{\overset{\wedge}{0}}{\mathrm{rk}\, \mathrm{Im}} = \mathrm{rk}\, E$). tors. free

($\because$ pick $k\cdot\mathrm{id} \neq \phi \in \mathrm{Hom}(E,E)$, $x \in X$, and an eigenvalue λ of $\phi_x : T_x \otimes k(x) \to T_x \otimes k(x)$.
Then, $\psi := \phi - \lambda \cdot \mathrm{id}$ is a desired one).

Hence, $0 < \mathrm{rk}\, \mathrm{Im}\psi < \mathrm{rk}\, E$. On the other hand, since E is μ-stable, $\mathrm{Im}\psi \subset E$ and $E \twoheadrightarrow \mathrm{Im}\psi$ imply $\mu(\mathrm{Im}\psi) < \mu(E) < \mu(\mathrm{Im}\psi)$!!

(iv) Corollary of (iii)

E.g. We can produce stable bundles from certain l. bundles on K3.
(i) $\mathcal{L}$ globally generated, ample
Then, $\ker(H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L})$ is μ-stable
(ii) $\mathcal{L}$ globally generated and generates $\mathrm{Pic}(X)$.
Then, $\ker(H^0(X, \mathcal{L}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{L})$ is μ-stable.

Now, we'll show T_X on a complex K3 is μ-stable.

Note T_X is μ-stable $\Leftrightarrow$ $\forall$ l.b. $\mathcal{L} \subset T_X$ w/ tors. free quot., $\mu(\mathcal{L}) = \deg \mathcal{L} < 0 = \mu(T_X)$.

E.g. If $\mathrm{Pic}(X) = 0$, then the only possible testing bundle for T_X is $\mathcal{O}_X$. However, $H^0(X, T_X) = 0$ by Hodge theory, so it's also impossible.
So, T_X is vacantly μ-stable.

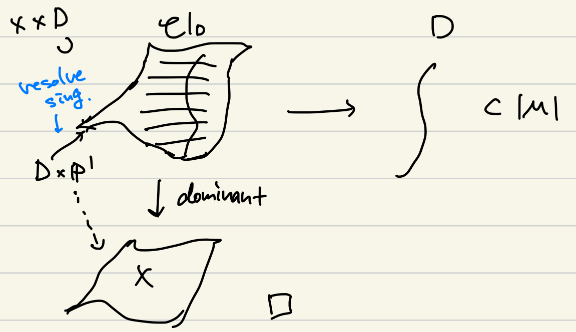
* Algebraic approach.

Thm. (X, H) polarized K3 over an alg. closed field k of char. zero.
If a l.b. $L \subseteq T_X$ has the tors. free quot. w/ $\deg_H \mathcal{L} > 0$,
then for a generic $x \in X$, $\exists$ rat. curve $x \in C \subset X$ s.t. $T_C(x) \subset \mathcal{L}(x) \subset T_X(x)$

Cor. (X, H) as above. T_X does not contain any l.b. of positive degree w/ tor. free quot.

proof.) "K3 surface cannot contain too many rational curves."
Assume $\exists$ $\mathcal{L} \subset T_X$ w/ $\deg \mathcal{L} > 0$ and tor. free quot.
WMA k is uncountable by base change. Then, $\forall\, \mathcal{U} \overset{\text{open}}{\subset} X$, $\mathcal{U}(k)$ cannot be covered by countably many curves. On the other hand, the theorem says $\exists$ $\mathcal{U} \overset{\text{open}}{\subset} X$ that is covered by a rational curves.
Claim. X is uniruled, i.e., $\exists$ rat. dom. map $D \times \mathbb{P}^1 \dashrightarrow X$ for some D.
Idea of proof). Since $\mathrm{Pic}(X) \simeq \mathrm{NS}(X)$ is countable, this implies $\exists$ linear system $|M|$ containing uncountably many rational curves.

We may view $|M|$ as a moduli space w/ univ. family $\mathcal{C} \subseteq X \times |M|$.
Moreover, since being rational is closed condition in $|M|$, there exists a curve $D \subseteq |M|$ whose fibers in X are rational curves. Consider the restriction of $\mathcal{C}$ to D:



$\square$

(cf. Def. 4.1. and Prop. 4.12 for general argument)

Claim. K3 surface X over an alg. closed field of char. 0 is not uniruled.

proof.) Assume X is uniruled.

By resolving indeterminancy of $D \times \mathbb{P}^1 \dashrightarrow X$,

we get $Y \to X$, which is generically étale (by generic smoothness).
dominant

$H^0(U_X, \omega|_U) \hookrightarrow H^0(U_Y, \omega_Y|_U)$ $\quad f^*\omega_X|_{U_X} \cong \omega_Y|_{U_Y}$ $\quad$ or EGA IV_2 Cor. 2.2.8.

Then, $H^0(X, \omega_X) \to H^0(Y, \omega_Y)$ is injective. WMA f surj. $\Rightarrow$ f faith. flat

On there hand, $H^0(X, \omega_X) \neq 0$ while $h^0(Y, \omega_Y) = h^0(\mathbb{P}^1 \times D, \omega_{\mathbb{P}^1 \times D})$
$= h^0(\mathbb{P}^1, \omega) \times h^0(D, \omega_D)$
$= 0$ $\square$

Cor. (X, H) as above. Then, T_X is μ-stable.

proof). Suffices to test w/ l.b. $\mathcal{L} \subset T_X$ w/ tors. free quot.

By Cor. above, $\deg_H \mathcal{L} \leq 0$. Assume $\deg_H \mathcal{L} = 0$. If $\mathcal{L} \not\cong \mathcal{O}$,
then $\exists H'$ s.t. $\deg_{H'} \mathcal{L} > 0$, which is absurd. So, $\mathcal{L} = \mathcal{O}$.
However, since $H^0(X, T_X) = 0$ by Hodge theory, it is also absurd.
Thus, $\deg_H \mathcal{L} < 0$. $\square$