The minimal model program and resolution of cDV singularity

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Abstract

Our goal is to compute resolution of various types of singularities with a view towards the minimal model program. This thesis contains a general overview of the minimal model program together with practical techniques to deal with specific varieties and singularities. Thoroughly explained examples include the affine quadric cone, Del Pezzo surfaces, the cone over the Veronese surface, and Du Val singularities. At the end, we will observe a specific example of a resolution of a cDV singularity. Throughout the thesis, I also put emphasis on clearly stating techniques and results that are often taken for granted by people with working knowledge.

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Contents

Introduction

Our goal is to compute resolution of various types of singularities with a view towards the minimal model program. This thesis contains a general overview of the minimal model program together with many working techniques through various examples to practically deal with specific varieties and singularities.

In Chapter 1, I explain concepts related to normal varieties and introduce my favorite example "the affine quadric cone". Normal varieties provide a huge class of singular varieties, but they are still fairly similar to non-singular varieties in the sense that we can generalize many notions and results for non-singular varieties to normal varieties. (Indeed, they are equivalent for curves!) Hence, in this thesis, we will focus on study of singularities in normal varieties. To get familiar with normal varieties, I first explain geometric intuitions of them via Serre's normality criterion (Corollary 1.1.13) and then introduce a notion of a local complete intersection to provide a tool to show that a given variety is normal in practice (Example 1.2.7,1.2.16). Then, I move on to generalization of concepts defined in the non-singular case to the normal case. Here, I refer to Appendix A a lot, where I summarize standard facts about divisors. In particular, we see why a normal variety is a suitable setting to talk about divisors while discussing what kind of modifications we need due to an unfortunate fact that Weil divisors are not necessarily Cartier on normal varieties (Lemma A.1.11). Finally, we apply previous definitions and arguments to the affine quadric cone (§1.4), which is one of the simplest singular varieties, but clearly illustrates our departure from the non-singular category. Here, I also introduce a lot of useful techniques that will be used again and again throughout the thesis.

In Chapter 2, I give an overview of the minimal model program. The whole story begins with Castelnuovo's contraction theorem (Theorem 2.0.3), which claims that we can smoothly blow-down any (-1)-curve in a non-singular surface, i.e. any (-1)-curve in a non-singular surface can be regarded as the exceptional divisor of the blow-up of a smooth surface at a point. One of the nicest things about this result is that the resultant surface is still non-singular despite the fact that intuitively contraction of a curve into a point introduces a singularity. Now, by contracting all the (-1)-curves to points, we obtain a surface with a simpler global structure, which is called a relatively minimal model. Although this naive notion needs to be modified a little bit to define "genuine" minimal models, the modifications give us satisfactory results in the surface minimal model program (Construction 2.1.19). Most importantly for us, everything has been done in the smooth category. However, if we try to generalize the surface minimal model program to 3-folds, we face the problem that some contractions of curves inevitably introduce singularities in the resultant variety (Theorem 2.2.1). Here, we need to start thinking about singularity. In the rest of Chapter 2, I introduce several notions to describe how singular a singularity is, motivate some examples (Example 2.2.13) in the following chapters, and finally give a strategy of the higher dimensional minimal model program (Construction 2.2.16).

In chapter 3 and 4, we work on resolutions of various types of singularities. After introducing notions to describe singularities locally, we consider some standard examples of surface singularities such as the origin of the affine cone of a non-singular projective hypersurface (Lemma 3.2.5) and the vertex of quotient varieties (Example 3.3.6) by recalling techniques in our first example "the affine quadric cones". Then, we deal with a little bit more complicated surface singularities, so called Du Val singularities. I thoroughly worked through the step-by-step process of resolutions (Lemma 3.4.3). Finally, in Chapter 4, we will deal with 3-fold singularities, so called compound Du Val singularities with an introduction to weighted blow-ups.

Throughout the thesis, I also put emphasis on clearly stating results and techniques (e.g. Lemma 1.1.10, §1.2, §1.4, §4.1, Lemma 4.1.10), which are often taken for granted by people with working knowledge.

In this thesis, we use the following notations and definitions.

Notation 0.0.1.

(i) The end of a remark, an example, and a construction is marked with $\blacksquare$.

(ii) I will use indented versions of Theorem, Lemma, Example, etc. with ends marked with $\diamondsuit$. Also, the end of a proof for a indented claim is indicated by $\heartsuit$.

(iii) $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the fields of complex numbers, real numbers, rational numbers, and the ring of integers, respectively. Furthermore, $\mathbb{Z}_{>0}$ denotes the set of positive integers and $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. Also, k denotes an algebraically closed field and R denotes a ring unless otherwise specified.

(iv) For $f \in k[x_1, \ldots, x_n]$, $\{f = 0\}$ (or just $f = 0$) denotes the scheme $\operatorname{Spec} k[x_1, \ldots, x_n]/\langle f \rangle$.

Definition 0.0.2.

(i) A **variety** is a separated scheme of finite type over a field. In this thesis, we suppose that a variety is integral (i.e. reduced and irreducible) and over $\mathbb{C}$ unless otherwise specified.

(ii) A **curve**, **surface**, and **3-fold** are varieties of dimension 1, 2, and 3, respectively. In particular, we assume these are integral unless otherwise specified (by equations).

(iii) A closed subset all of whose irreducible components are of codimension 1 in an ambient scheme X is said to be a **hypersurface** in X.

(iv) A variety X is **non-singular (regular)** if all of its local rings are regular. Note that a smooth variety over a field k is always non-singular and the converse is true if k is a perfect field (e.g. fields of characteristic 0, finite fields, algebraically closed fields, etc.) [Vak17, 12.2.10.]. Hence, I will *not* distinguish smoothness from non-singularity in this thesis when working over $\mathbb{C}$.

(v) A point $x \in X$ of a variety X is said to be a **singularity** if the local ring is not regular. A variety is said to be a **singular** variety if it admits a singularity. A singularity $x \in X$ is said to be **isolated** if there exists an open neighborhood U of x such that x is the only singularity.

(vi) A (-1)**-curve** is a non-singular rational curve with self-intersection $Y^2 = -1$.

(vii) A **rational map** $f : X \dashrightarrow Y$ between schemes is an equivalence class of pairs (U, ϕ) of an open dense subset U of X and a morphism $\phi : U \to Y$ where $(U, \phi) \sim (V, \psi)$ if there exists an open dense subset $W \subset U \cap V$ such that $\phi|_W = \psi|_W$. The following is motivating and useful:

Lemma 0.0.3 (Reduced-to-Separated Theorem)**.** [Vak17, 10.2.2.] Let $f, g : X \to Y$ be morphisms of schemes and suppose X is reduced and Y is separated. If $f|_U = g|_U$ for an open dense subset $U \subset X$, then $f = g$. In particular, for a rational map $h : X \dashrightarrow Y$, there exists the unique maximal element in $\{V \subset X \mid (V, \psi) \text{ represents } h\}$ with respect to the inclusion relation. The maximal element is called the **domain of definition** for h and the complement is called the **locus of indeterminacy**. ♢

(viii) Let $f : X \to Y$ be a (dominant) morphism of irreducible schemes. Then, f is said to be a **birational morphism** if f is a **birational map**, i.e. there exists a dominant rational map $g : Y \dashrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ as rational maps. Note in particular that if X and Y are integral and separated, then a birational morphism whose inverse is a morphism is an isomorphism.

Remark 0.0.4. In some literature (e.g. [Liu02]), a birational morphism $f : X \to Y$ of integral schemes over a scheme S is defined to be an S-morphism such that $f_\eta^\# : K(Y) \to K(X)$ is an isomorphism of function fields for the generic point η of X. These definitions are equivalent if X and Y are of finite type over S. Thus, a birational morphism works well for integral and separated schemes of finite type (e.g. varieties). ♢

It is often useful to consider a proper birational morphism for example by the following result:

Lemma 0.0.5. [Liu02, Corollary 4.4.3.] Let Y be a normal locally noetherian scheme, X an integral scheme, and $f : X \to Y$ a proper birational morphism.

(a) The canonical homomorphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.

(b) There exists an open subset V of Y such that $f^{-1}(V) \to V$ is an isomorphism, and X_y has no isolated point. Furthermore, the complement of V has codimension ≥ 2.

♢

(ix) Let $f : X \to Y$ be a birational morphism.

(a) The **exceptional set** $\mathrm{Ex}(f) \subset X$ is the set of points $x \in X$ where $f(x)$ is in the locus of indeterminacy for $f^{-1} : Y \dashrightarrow X$. We usually view $\mathrm{Ex}(f)$ as a closed subscheme of X with the induced reduced structure.

(b) An ***f*-exceptional divisor** is a Weil divisor D on X such that $f(\operatorname{Supp} D)$ is of codimension ≥ 2 in Y.

Chapter 1

Preliminaries

We will review some important results mainly regrading normal varieties and introduce an enlightening example in §1.4.

1.1 Serre's normality criterion

First, we are going to show a criterion for a locally noetherian scheme to be normal (Corollary 1.1.13), which is helpful for intuition.

Definition 1.1.1 (Serre's condition)**.** Let R be a noetherian ring. Then, R is said to satisfy:

(i) $\boldsymbol{R_n}$ if $R_\mathfrak{p}$ is regular for every prime ideal $\mathfrak{p}$ in R with $\operatorname{ht}\mathfrak{p} = \dim R_\mathfrak{p} \leq n$.

(ii) $\boldsymbol{S_n}$ if $\operatorname{depth} R_\mathfrak{p} \geq \min(\dim R_\mathfrak{p}, n)$ for all prime ideals $\mathfrak{p}$.

> **Definition 1.1.2.** Let R be a ring, and $I \subset R$ an ideal. Let M be a finitely generated R-module. The $\boldsymbol{I}$**-depth** $\operatorname{depth}_I(M)$ is defined as follows:
>
> (a) If $IM \neq M$, then $\operatorname{depth}_I(M)$ is the supremum of the lengths of M**-regular sequences** in I, i.e. sequences $r_1, \ldots, r_d \in I$ such that r_i is not a zero-divisor on $M/\langle r_1, \ldots, r_{i-1}\rangle M$ for all $i = 1, \ldots, d$ and $M/\langle r_1, \ldots, r_d\rangle M \neq 0$.
>
> (b) If $IM = M$, then $\operatorname{depth}_I(M) = \infty$.
>
> If $(R, \mathfrak{m})$ is a local ring, then the **depth** of M is $\operatorname{depth} M := \operatorname{depth}_\mathfrak{m}(M)$. ♢

Remark 1.1.3. For a general ring R and an R-module M, a reordering of an M-regular sequence is not necessarily a regular sequence (e.g. [Vak17, 8.4.5.]). However, if $(R, \mathfrak{m})$ is a local regular ring and M is a finitely generated R-module, then any M-regular sequence (in $\mathfrak{m}$) remains a regular sequence upon any reordering (cf. [Vak17, 8.4.6.]). ■

Example 1.1.4. [Mat87, p.183] Let R be a noetherian ring.

(i) R_n (resp. S_n) implies R_m (resp. S_m) for all $n \geq m$.

(ii) S_0 holds for any R.

(iii) S_1 holds if and only if all the associated primes of R are minimal, i.e. R does not have embedded associated primes.

(iv) If R is an integral domain, then S_2 holds if and only if every prime divisor of a non-zero principal ideal has height 1.

(v) If $(R, \mathfrak{m})$ is a local ring, then S_n holds for all $n \geq 0$ if and only if R is **Cohen-Macaulay**, i.e. $\operatorname{depth} R = \dim R$.

(vi) S_1 and R_0 hold if and only if R is reduced.

■

Theorem 1.1.5 (Serre's normality criterion)**.** [Mat87, Theorem 23.8.] *Let R be a noetherian ring. Then, R is a normal ring if and only if S_2 and R_1 hold.*

Now, let us generalize this result to schemes.

Definition 1.1.6.

(i) A scheme X is $\boldsymbol{R_n}$ if every local ring $\mathcal{O}_{X,x}$ of dimension n is regular.

(ii) A scheme X is $\boldsymbol{S_n}$ if $\operatorname{depth} \mathcal{O}_{X,x} \geq \min(\dim \mathcal{O}_{X,x}, n)$ for all $x \in X$.

Example 1.1.7. A scheme X is $\boldsymbol{R_1}$ **(regular in codimension one)** if every local ring $\mathcal{O}_{X,x}$ of dimension 1 is regular, i.e. a DVR. ■

Example 1.1.8. A locally noetherian scheme is said to be **Cohen-Macaulay** if it is S_n for all $n \geq 0$. ■

Example 1.1.9. Since normality is local, a locally noetherian scheme is normal if and only if it is R_1 and S_2 by Serre's normality condition. ■

Although the geometric meaning of R_1 is clear, we want more geometric intuition for S_2, which is given in the following.

Lemma 1.1.10 (Algebraic Hartogs's Lemma)**.** *Let X be a scheme. If X is S_2, then for any closed subset $Y \subset X$ of codimension ≥ 2, the natural map $\mathcal{O}_X \to \iota_* \mathcal{O}_{X \setminus Y}$ is an isomorphism where $\iota : X \setminus Y \hookrightarrow X$. In particular, we have $\mathcal{O}_X(X) \cong \mathcal{O}_X(X \setminus Y)$, i.e. we can extend any regular function on the complement of a codimension ≥ 2 closed subset to the whole, which is analogous to Hartogs's lemma in complex geometry (cf.* [Vak17, 11.3.11.]*). The converse holds if X is Cohen-Macaulay in codimension 1 (e.g. R_1).*

Proof. The idea of the proof is due to Sándor Kovács's answer to a mathoverflow question [Kova]. First, note the following two claims from [Har67].

> **Lemma 1.1.11.** [Har67, Proposition 1.11.] Let Y be a closed subset of a topological space X, let $\mathcal{F}$ be a sheaf of abelian groups on X, and let n be an integer. Then, the following are equivalent:
>
> (i) For all $i \leq n$, $\underline{\mathrm{H}}^i_Y(\mathcal{F}) = 0$, where $\underline{\mathrm{H}}^i_Y(\mathcal{F})$ is **the cohomology sheaf of X with coefficients in $\mathcal{F}$ and support in Z** (cf. [Har67, pp.1-2]).

(ii) For all open subsets U of X, the natural map

$$\alpha_i : \mathrm{H}^i(U, \mathcal{F}) \to \mathrm{H}^i(U \cap (X \setminus Y), \mathcal{F})$$

is injective for $i = 0$ and an isomorphism for all $i < n$.

$\diamondsuit$

Lemma 1.1.12. [Har67, Theorem 3.8.] Let X be a locally noetherian scheme, let Y be a closed subset, $\mathcal{F}$ be a coherent sheaf on X, and let n be an integer. Then, the following are equivalent:

(i) For all $i < n$, $\underline{\mathrm{H}}^i_Y(\mathcal{F}) = 0$.

(ii) $\operatorname{depth}_Y \mathcal{F} := \inf_{x \in Y} \operatorname{depth} \mathcal{F}_x \geq n$.

$\diamondsuit$

First, by the preceding lemmas note that for a closed subset $Z \subset X$, we have $\operatorname{depth}_Z \mathcal{O}_X \geq 2$ if and only if the natural map $\mathcal{O}_X \to \iota_* \mathcal{O}_{X \setminus Z}$ is an isomorphism. Hence, if X is S_2, then $\operatorname{depth}_Y \mathcal{O}_X \geq 2$ for any closed subset Y of codimension 2 as desired. Conversely, suppose that X is Cohen-Macaulay in codimension 1 and that $\operatorname{depth}_Y \mathcal{O}_X \geq 2$ for any closed subset Y of codimension 2. Choose a point $x \in X$. If $\dim \mathcal{O}_{X,x} \leq 1$, then $\operatorname{depth} \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} = \min(2, \dim \mathcal{O}_{X,x})$ by the first supposition. If $\dim \mathcal{O}_x \geq 2$, then $\operatorname{depth} \mathcal{O}_{X,x} \geq 2 = \min(2, \dim \mathcal{O}_{X,x})$ by the second supposition, which suffices for the proof. $\square$

The preceding arguments can be summarized as follows:

Corollary 1.1.13. *A locally noetherian scheme X is normal if and only if the following hold:*

(R_1): *The singular locus of X is of codimension ≥ 2.*

(S_2'): *For any closed subset $Y \subset X$ of codimension ≥ 2, the natural map $\mathcal{O}_X \to \iota_* \mathcal{O}_{X \setminus Y}$ is an isomorphism.*

1.2 Local complete intersection

Although Serre's normality criterion offers some intuition for normal varieties, it would be nice to have a more handy condition to actually check whether a variety is normal. Since the Jacobian criterion (e.g. [Liu02, Theorem 4.2.19]) enables us to compute the singular locus, what we care is the S_2 condition. By definition, to see a variety is S_2 it suffices to show that it is Cohen-Macaulay. For this purpose, we introduce the notion of a local complete intersection, which implies Cohen-Macaulay and hence produces many examples of S_2 varieties (e.g. Example 1.2.5, 1.2.7, and 1.2.16). This section also serves to clarify the relation between the algebraic notion (Definition 1.2.1) and the geometric notion (Definition 1.2.12) of a local complete intersection.

Definition 1.2.1.

(i) A noetherian local ring $(R, \mathfrak{m})$ is said to be a **complete intersection** if its completion $\hat{R}$ can be written as the quotient of a regular local ring by an ideal generated by a regular sequence, i.e. if there exists a surjective morphism $A \to \hat{R}$ with A a regular local ring such that the kernel is generated by a regular sequence of $\hat{R}$.

(ii) A locally noetherian scheme (or a noetherian ring) is said to be a **local complete intersection** if all of its local rings are complete intersections.

Remark 1.2.2. Here are some reasons why we take the completion in the definition. First, note that for any noetherian complete local ring $(R, \mathfrak{m})$, there exists a surjection $A \to R$ with A a regular local ring by the Cohen structure theorem ([Sta21, Tag 032A]). Also, note that the fact that the kernel is generated by a regular sequence does not depend on a choice of surjections ([Sta21, Tag 09PZ]). ■

First of all, we have the following as desired.

Lemma 1.2.3. [Mat87, Theorem 18.1, Theorem 21.3.] *For a noetherian local ring, we have the following implications:*

$$\textit{regular} \Rightarrow \textit{complete intersection}(\Rightarrow \textit{Gorenstein}) \Rightarrow \textit{Cohen-Macaulay} \Rightarrow S_n.$$

Therefore, to check that a locally noetherian scheme X is S_2, it suffices to show that every local ring $\mathcal{O}_{X,x}$ is a complete intersection. Now, let us see examples of a complete intersection. First, the following sanity check produces a lot of examples.

Lemma 1.2.4. [Sta21, Tag 09Q0] *Let R be a regular ring and let $\mathfrak{p} \subset R$ be a prime ideal. Suppose $f_1, \dots, f_l \in \mathfrak{p}$ is a regular sequence. Then, the localization*

$$A = (R/\langle f_1, \dots, f_l \rangle_R)_{\mathfrak{p}} = R_{\mathfrak{p}}/\langle f_1, \dots, f_l \rangle_{R_{\mathfrak{p}}}$$

is a complete intersection.

The lemma follows because the completion of A is $\hat{A} = \hat{R}_{\mathfrak{p}}/\langle f_1, \dots, f_l \rangle_{\hat{R}_{\mathfrak{p}}}$ and we can show that $\hat{R}_{\mathfrak{p}}$ is a regular local ring and the image of the sequence $f_1, \dots, f_l$ is a regular sequence.

Example 1.2.5. For any regular ring R and a non-zero divisor $f \in R$, the quotient ring $R/\langle f \rangle$ is a local complete intersection. In particular, any subscheme of $\mathbb{A}^n_k$ (or $\mathbb{P}^n$) cut out by a single (homogeneous) polynomial is a local complete intersection. ■

Example 1.2.6. Let $R = k[X_1, \dots, X_n]$ and $\mathfrak{p} = \langle X_1, \dots, X_n \rangle$. Then, R is in particular a regular ring and a R-regular sequence $X_1, \dots, X_l$ in $\mathfrak{p}$ $(l < n)$ defines a complete intersection at $\mathfrak{p}$. Geometrically, it says that the intersection of hyperplanes $X_i = 0$ at the origin is a complete intersection. ■

More generally, we have the following.

Example 1.2.7. Let $R = k[X_1, \dots, X_n]$ (or just a regular ring) and $Y = \operatorname{Spec} R$. Then, if $f_1, \dots, f_l \in R$ is an R-regular sequence, then the local ring $\mathcal{O}_{X,x}$ of a subvariety

$$X = \operatorname{Spec} R/\langle f_1, \dots, f_l \rangle \subset Y$$

at any $x \in X$ is a complete intersection, noting a point in X corresponds to a prime ideal of Y containing $f_1, \dots, f_l$. Thus, X is a local complete intersection scheme. ■

The preceding lemma and examples suggest that the following relative notion gives a tautological criterion for a local complete intersection.

Definition 1.2.8. Let $f : X \hookrightarrow Y$ be a locally closed immersion of schemes with Y a locally noetherian scheme. Then, f is said to be a **regular immersion (of codimension l) at $p \in X$** if in the local ring $\mathcal{O}_{Y,p}$, the ideal of X is generated by a regular sequence (of length l). If f is a regular immersion (of codimension l) at every $x \in X$, then f is said to be a **regular immersion (of codimension l)**.

Remark 1.2.9. [Vak17, 8.4.G.] The locally noetherian hypothesis ensures that if f is a regular immersion at $x \in X$, then f is a regular immersion in an open neighborhood x. ■

Example 1.2.10. A closed immersion $f : X \hookrightarrow Y$ of locally noetherian schemes is a regular immersion of codimension 1 iff X is an effective Cartier divisor of Y (cf. Remark A.1.16). Hence, we can think of a regular closed immersion locally as a finite sequence of iterate operations of taking an effective Cartier divisor in an effective Cartier divisor. ■

By Lemma 1.2.4 and definition, we tautologically have the following criterion for a local intersection.

Corollary 1.2.11. *Let $f : X \hookrightarrow Y$ be a locally closed immersion of locally noetherian schemes with Y non-singular. If f is a regular immersion at $x \in X$, then the local ring $\mathcal{O}_{X,x}$ is a complete intersection.*

For schemes locally of finite type, we have another characterization of local complete intersections.

Definition 1.2.12. Let S be a finitely generated k-algebra for a field k.

(i) S is said to be a **global complete intersection over** k if there exists an isomorphism $S \cong k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_l \rangle$ such that $\dim S = n - l$.

(ii) S is said to be a **local complete intersection over** k if there exists a covering $\operatorname{Spec} S = \bigcup_i \operatorname{Spec} S_{g_i}$ such that each S_{g_i} is a global intersection over k.

Indeed, this definition is compatible with the previous one (although it is not as obvious as it may seem).

Lemma 1.2.13. [Sta21, Tag 09Q6] *Let S be a finitely generated k-algebra for a field k.*

(i) *For a prime ideal $\mathfrak{p} \subset S$, the local ring $S_{\mathfrak{p}}$ is a global complete intersection in the sense of Definition 1.2.12 if and only if $S_{\mathfrak{p}}$ is a complete intersection in the sense of Definition 1.2.1.*

(ii) *S is a local complete intersection in the sense of Definition 1.2.12 if and only if S is a local complete intersection in the sense of Definition 1.2.1.*

In particular, a local complete intersection scheme locally of finite type is locally isomorphic to $\operatorname{Spec} S$ *with S a global complete intersection.*

Remark 1.2.14. In [Har77, II.8.], a closed subscheme Y of a non-singular variety X over k is said to be a local complete intersection in X if the ideal sheaf $\mathcal{I}_Y$ of Y in X can be locally generated by $r = \operatorname{codim}_X(Y)$ elements at every point. Hence, the preceding lemma shows that our definitions are also compatible with this definition. Note that this also follows directly from Corollary 1.2.11 together with the following result:

Lemma 1.2.15. [Sta21, Tag 02JN] Let $(R, \mathfrak{m})$ be a noetherian local Cohen-Macaulay ring and $x_1, \ldots, x_l \in \mathfrak{m}$. Then, $x_1, \ldots, x_l$ is a regular sequence if and only if $\dim R/\langle x_1, \ldots, x_l \rangle = \dim R - l$. ♢

■

Now, we have the following examples of local complete intersection varieties.

Example 1.2.16. A subvariety X of $\mathbb{A}_k^n$ (resp. $\mathbb{P}_k^n$) is often said to be a **(global) complete intersection** (by abuse of notation for us) if the ideal of X is generated by precisely $(n - \dim X)$ polynomials (resp. homogeneous polynomials). A (global) complete intersection in this sense is a local complete intersection by the preceding lemma. ■

A subvariety of $\mathbb{P}_k^n$ that is a local complete intersection is not necessarily a (global) complete intersection.

Example 1.2.17. [Har77, Exercise I.2.17.] Consider the **twisted cubic** C in $\mathbb{P}_k^3$, which is given by the embedding associated to the very ample line bundle $\mathcal{O}_{\mathbb{P}^1}(3)$ (cf. Lemma A.3.2), i.e. by the embedding

$$\mathbb{P}_k^1 \hookrightarrow \mathbb{P}_k^3, \quad [s:t] \mapsto [s^3 : s^2 t : s t^2 : t^3].$$

Note if we write $\mathbb{P}_k^3 = \operatorname{Proj} k[x_0, x_1, x_2, x_3]$, then we clearly have

$$C = \operatorname{Proj} k[x_0, x_1, x_2, x_3]/\langle x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3 \rangle.$$

Therefore, we can see that in each standard chart of $\mathbb{P}_k^3$, C is cut out by two polynomials, i.e. C is a local complete intersection. Thus, it suffices to show that the homogeneous ideal I of C cannot be generated by two elements. Let $\mathfrak{m} = \langle x_0, x_1, x_2, x_3 \rangle$. Then, by Nakayama's lemma, homogeneous polynomials $f_1, \ldots, f_l \in \mathfrak{m}$ minimally generates I if and only if $f_1, \ldots, f_l$ descend to a base of $I/\mathfrak{m}I$. Since $x_0x_3 - x_1x_2, x_1^2 - x_0x_2, x_2^2 - x_1x_3$ descend to linearly independent elements (indeed a base), we are done. ■

1.3 Divisors on a normal variety

In this section, we quickly introduce some notions regarding divisors on a normal variety. A normal variety is one of the most natural settings for Weil divisors; R_1 allows us to define principal Weil divisors (cf. Example 1.1.7) and S_2 ensures that rational functions only depend on subsets of codimension 1 by Hartogs's lemma (Lemma 1.1.10). However, since a normal variety is in general not locally factorial, the natural injective map $\operatorname{CaCl}(X) \hookrightarrow \operatorname{Cl}(X)$ (cf. Lemma A.1.11 (iv)) is in general not surjective. In particular, it is possible that the canonical divisor is not Cartier, which is very inconvenient for intersection theory and hence for studies of positivity. This problem is sometimes (partially) solved by extending coefficients from $\mathbb{Z}$ to $\mathbb{Q}$.

Definition 1.3.1. Let X be a normal variety.

(i) A **$\mathbb{Q}$-divisor** is a formal linear combination of prime divisors of X over $\mathbb{Q}$.

(ii) A $\mathbb{Q}$-divisor D of X is said to be **$\mathbb{Q}$-Cartier** if there exists $n \in \mathbb{Z}_{>0}$ such that nD is Cartier.

(iii) Let $f : Z \to Y$ be a morphism of schemes and D a $\mathbb{Q}$-Cartier divisor with mD Cartier. Then, define the pull-back of D via f to be $f^*D := \frac{1}{m} f^*(mD)$.

(iv) A normal variety X is said to be **$\mathbb{Q}$-factorial** (or **with only $\mathbb{Q}$-factorial singularities**) if all the Weil divisors (i.e. $\mathbb{Q}$-divisors) are $\mathbb{Q}$-Cartier.

(v) Let $Z \subset X$ be a closed scheme of dimension k and let $D_1, \ldots, D_k$ be $\mathbb{Q}$-Cartier divisors on X with $n_i D_i$ Cartier. Then, we define the intersection number of $D_1, \ldots, D_k$ with Z to be

$$(D_1 \cdots D_k \cdot Z) := \frac{(n_1 D_1 \cdots n_k D_k \cdot Z)}{n_1 \cdots n_k},$$

where $(n_1 D_1 \cdots n_k D_k \cdot Z)$ denotes the usual intersection number (e.g. [KM98, 1.34.]).

Also, we can define the canonical divisor on a normal variety as follows:

Definition 1.3.2. Let X be a normal variety and let U denote the smooth locus of X, whose complement is of codimension ≥ 2 by normality (cf. Corollary 1.1.13). The **canonical divisor** K_X of X is a Weil divisor defined as the image of K_U under the isomorphism $\mathrm{Cl}(U) \xrightarrow{\sim} \mathrm{Cl}(X)$ by Lemma A.1.3.

Hence, we can compute the canonical class of a normal variety just as in the non-singular case except that we need to take the closure at the end. In particular, we can apply the adjunction formula (Theorem A.2.3). We can also define the relative canonical divisors (Definition A.2.5).

Definition 1.3.3. Let Y be a normal variety such that mK_X is Cartier for some $m > 0$ and $f : X \to Y$ be a birational morphism from a normal variety X. Then, the relative canonical divisor is defined to be $K_{X/Y} := K_X - f^* K_Y$.

The example in the next section sheds light on many aspects of the preceding definitions.

1.4 Example: the affine quadric cone

Let

$$Q = \operatorname{Spec} k[x, y, z]/\langle z^2 - x^2 - y^2 \rangle$$

be the **affine quadric cone** in $\mathbb{A}^3_k$. First, Q is normal by Serre's normality criterion (Theorem 1.1.5), noting that Q is a complete intersection (cf. Example 1.2.5), i.e. S_2 (cf. Lemma 1.2.3) and that the origin is the only singularity, i.e. Q is R_1. For a more direct proof, see [Sha88, p.125]. The following lemma shows that $\mathcal{O}_Q(Q)$ is not locally factorial by Lemma A.1.11. (We can also show it directly, noting the factorization $y^2 = (z + x)(z - x)$.)

Lemma 1.4.1. *Consider a line $L = \operatorname{Spec} k[x, y, z]/\langle z - x, y \rangle \subset Q$. The Weil divisor L is not Cartier. However, the Weil divisor $2L$ is a principal Cartier divisor. In particular, L is not Cartier but $\mathbb{Q}$-Cartier.*

Remark 1.4.2. Note that we have the following chain of ideals in $k[x, y, z]$:

$$\langle z^2 - x^2 - y^2 \rangle \subsetneq \langle z - x, z^2 - x^2 - y^2 \rangle = \langle z - x, y^2 \rangle \subsetneq \langle z - x, y \rangle.$$

Set-theoretically the last two ideals define the same subset of Q, but scheme-theoretically they define different subschemes of Q. ■

Proof of Lemma. There are several ways to prove the first part, where the proof of the second part is contained in the first case:

(Definition): To see L is not Cartier, it suffices to show that L is not locally principal at the origin, i.e. for any open neighborhood U of the origin and any rational function f on U,

$$[L \cap U] \neq \sum_{[Z \cap U]:\text{prime divisor}} v_{Z \cap U}(f)[Z \cap U].$$

Indeed, since a uniformizer of $\mathcal{O}_{Q,L\cap U}$ is given by the equivalence class of y, we have $v_{L\cap U}(y) = 1$ and $v_{L\cap U}(z-x) = v_{L\cap U}(y^2/(z+x)) = 2$. Hence, if $v_{L\cap U}(f) > 0$, then we have $v_{L\cap U}(f) > 1$ or $v_{L'\cap U}(f) > 0$ for $L' = \langle z+x, y\rangle$. On the other hand, since $2L = v_L(z-x)L$ as we saw, $2L$ is a principal Cartier divisor associated to $z - x$.

(Tangent space): To see L is not principal, it suffices to show that L is not cut out by a single polynomial around the origin. Indeed, since $\dim Q = 2$ and Q is singular at the origin, $\dim T_{\langle x,y,z\rangle} Q = 3$ and clearly $\dim T_{\langle x,y,z\rangle} L = 1$. On the other hand, for any $f \in \langle x, y, z\rangle$, we have

$$\dim T_{\langle x,y,z\rangle}(k[x, y, z]/\langle f\rangle) \geq 3 - 1 = 2.$$

Therefore L cannot be cut out by a single equation around the origin. Morally speaking, any smooth Weil divisor that contains a singular point of the ambient scheme Q is not locally principal around the point.

(Intersection theory): (cf. [Har77, Example A.1.1.2.]) Assume L is a Cartier divisor for the sake of contradiction. Note L is linearly equivalent to the line $L' = \{z + x = 0, y = 0\}$ by a rational function $f(x, y, z) = y/(z + x)$ on Q. Thus, $(L \cdot L) = (L \cdot L')$. Now also note that $2L' \sim L + L'$ is linearly equivalent to a circle $C = \{z - 1 = 0\}$ by a rational function $g(x, y, z) = y/(z - 1)$. Then, since C intersects with L transversely at one point, we have

$$2(L \cdot L') = (L \cdot (2L')) = (L \cdot C) = 1,$$

which is absurd since an intersection number of Cartier divisors is integer. Note this computation is correct in the sense of Definition 1.3.1 (iv) since $(2L \cdot 2L) = ((L + L') \cdot C) = 2$ in the usual sense.

$\square$

Now, by the adjunction formula (Theorem A.2.3) and the fact that $\mathrm{Cl}(\mathbb{A}^n_k) = 0$ (Lemma A.1.6), the canonical class is given as follows.

Lemma 1.4.3. *For the affine quadric cone Q, we have $K_Q = 0$ and in particular K_Q is ($\mathbb{Q}$-)Cartier. In particular, K_Q can be represented by $Q \cap H$ for any hyperplane $H \subset \mathbb{A}^3_k$, where H may contain the origin.*

Indeed, $\mathrm{Cl}(Q)$ consists of the classes of the line $L = \langle z - x, y\rangle$ and the canonical class K_Q.

Lemma 1.4.4. [Har77, Example II.6.5.2.] *For the affine quadric cone Q, we have $\mathrm{Cl}(Q) = \mathbb{Z}/2\mathbb{Z}$ with the line $L = \operatorname{Spec} k[x, y, z]/\langle z - x, y\rangle$ a generator. In particular, Q is $\mathbb{Q}$-factorial.*

Proof. By Lemma A.1.3, we have an exact sequence

$$\mathbb{Z} \xrightarrow{1\mapsto L} \mathrm{Cl}(Q) \longrightarrow \mathrm{Cl}(Q \setminus L) \to 0\ .$$

Then, $\mathrm{Cl}(Q \setminus L) = 0$ by Lemma A.1.6 since $Q \setminus L \cong \mathrm{Spec}(k[x, y, z]/\langle z^2 - x^2 - y^2\rangle)_{(z-x)}$ (cf. Remark 1.4.2) and $(k[x, y, z]/\langle z^2 - x^2 - y^2\rangle)_{(z-x)} \cong (k[a, b, c]/\langle ab - c^2\rangle)_b \cong k[b, b^{-1}, c]$ is a unique factorization domain. Hence, L generates $\mathrm{Cl}(Q)$ and since $2L = 0$ by Lemma 1.4.1, we have $\mathrm{Cl}(Q) \cong \mathbb{Z}/2\mathbb{Z}$. □

Lemma 1.4.5. *Let $\pi_Q : \tilde{Q} \to Q$ be the blow-up of the quadric cone Q at the origin with exceptional divisor E. Then, $\tilde{Q}$ is non-singular, $(E \cdot E) = -2$, and $K_{\tilde{Q}/Q} = 0$.*

Proof. For the purpose of symmetry, we write $Q = \{f(x, y, z) = x^2 + y^2 + z^2 = 0\}$. First of all, note the blow-up $\pi : \mathrm{Bl}_0\, \mathbb{A}^3 \to \mathbb{A}^3$ is given by

$$\mathrm{Bl}_0\, \mathbb{A}^3 = \{((x, y, z), [p_x : p_y : p_z]) \in \mathbb{A}^3 \times \mathbb{P}^2 \mid p_x y = p_y x, p_y z = p_z y, p_z x = p_x z\}.$$

Now, consider the affine chart U_x of $\mathrm{Bl}_0\, \mathbb{A}^3$ with $p_x \neq 0$, which is isomorphic to $\mathrm{Spec}\, k[x, y', z']$ via $y' = \frac{p_y}{p_x}$ and $z' = \frac{p_z}{p_x}$. Then, we have

$$\pi^{-1}(Q) \cap U_x = \{f(x, y'x, z'x) = x^2(1 + {y'}^2 + {z'}^2) = 0\}.$$

Here, the factor x^2 vanishes on the exceptional plane $\pi^{-1}(0) \cap U_x = \{x = 0\}$ and the other component $\{1 + {y'}^2 + {z'}^2 = 0\} \cap U_x$ is the birational transform of Q. Then, the exceptional fiber in U_x is given by

$$E \cap U_x = \pi^{-1}(0) \cap \{1 + {y'}^2 + {z'}^2 = 0\} = \{1 + {y'}^2 + {z'}^2 = 0, x = 0\}.$$

Therefore, $\tilde{Q} \cap U_x = \mathrm{Spec}\, k[x, y', z']/\langle 1 + {y'}^2 + {z'}^2\rangle \cong \mathrm{Spec}\, k[x, {y'}^2]$ is non-singular and hence $\tilde{Q}$ is non-singular by symmetry. Furthermore, by the local expressions, the exceptional divisor is the non-singular projective conic in $E_0 = \mathrm{Proj}\, k[p_x, p_y, p_z]$:

$$E = \mathrm{Proj}\, k[p_x, p_y, p_z]/\langle p_x^2 + p_y^2 + p_z^2\rangle \cong \mathbb{P}^1.$$

To see $(E \cdot E) = -2$ (cf. [Bur, §4.]), consider a principal divisor $(y \circ \pi_Q)$, where

$$y \in K(Q) = \mathrm{Frac}(k[x, y, z]/\langle x^2 + y^2 + z^2\rangle)$$

is a rational function on Q. Then, for example by [Har77, Lemma V.1.3.], note that

$$((y \circ \pi_Q) \cdot E) = \deg_E(\mathcal{O}_E \otimes \mathcal{O}_{\tilde{Q}}(y \circ \pi_Q)) = \deg_E(\mathcal{O}_E) = 0.$$

Now, consider the chart $\tilde{Q} \cap U_x$ as above. Then, $y \circ \pi_Q = y'x$ vanishes (with multiplicity 1) on $\{x = 0\} = E \cap U_x$ and

$$\{y' = 0\} = \{z' = \sqrt{-1}\} \cup \{z' = -\sqrt{-1}\} = (C_+ \cap U_x) \cup (C_- \cap U_x),$$

where

$$C_\pm = \{p_z = \pm\sqrt{-1}p_x, z = \pm\sqrt{-1}x, y = p_y = 0\} \subset \mathrm{Bl}_0\, \mathbb{A}^3.$$

Since we can do the similar arguments in the other two charts, we see that

$$(y \circ \pi_Q) = E + C_+ + C_-.$$

Hence, $0 = ((y \circ \pi_Q) \cdot E) = (E \cdot E) + (C_+ \cdot E) + (C_- \cdot E) = (E \cdot E) + 2$ as desired, where $(C_\pm \cdot E) = 1$ follows by the local expressions given above. See also the following Figure 1.1.

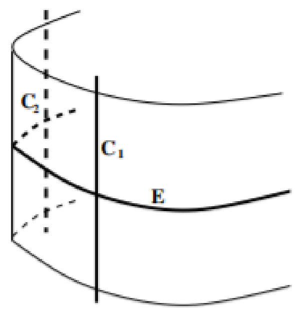


Figure 1.1: The vanishing locus of y [Bur, §4.]

Finally, let us see $K_{\tilde{Q}/Q} = 0$. I present two ways of computations.

(Adjunction formula): ([oM18, p.10]) By the adjunction formula and Lemma A.2.8, we have

$$\begin{aligned} K_Q &= (K_{\mathbb{A}^3} + Q)|_Q, \\ K_{\tilde{Q}} &= (K_{\mathrm{Bl}_0 \mathbb{A}^3} + \tilde{Q})|_{\tilde{Q}}, \\ K_{\mathrm{Bl}_0 \mathbb{A}^3} &= \pi^* K_{\mathbb{A}^3} + 2E_0. \end{aligned}$$

First of all, by the second and the third equations together with $E_0|_{\tilde{Q}} = E$, we have

$$\begin{aligned} K_{\tilde{Q}} &= K_{\mathrm{Bl}_0 \mathbb{A}^3}|_{\tilde{Q}} + \tilde{Q}|_{\tilde{Q}} \\ &= (\pi^* K_{\mathbb{A}^3} + 2E_0)|_{\tilde{Q}} + \tilde{Q}|_{\tilde{Q}} \\ &= (\pi^* K_{\mathbb{A}^3})|_{\tilde{X}} + 2E + \tilde{Q}|_{\tilde{Q}} \\ &= \pi_Q^*(K_{\mathbb{A}^3}|_Q) + 2E + \tilde{Q}|_{\tilde{Q}}, \end{aligned}$$

where the third equality follows by the commutativity of the summation of divisors and the restriction of divisors and the last equation follows from the following commutative diagram:

$$\begin{tikzcd} \tilde{Q} \arrow[r, hook] \arrow[d, "\pi_Q"'] & \mathrm{Bl}_0 \mathbb{A}^n \arrow[d, "\pi"] \\ Q \arrow[r, hook] & \mathbb{A}^3 \end{tikzcd}$$

Now, note we have $K_{\mathbb{A}^3}|_Q = K_Q - Q|_Q$ by the first adjunction above. Hence,

$$\begin{aligned} K_{\tilde{Q}} &= \pi_Q^*(K_Q - Q|_Q) + \tilde{Q}|_{\tilde{Q}} + 2E \\ &= \pi_Q^* K_Q - \pi_Q^*(Q|_Q) + \tilde{Q}|_{\tilde{Q}} + 2E \\ &= \pi_Q^* K_Q - (\pi^* Q)|_Q + \tilde{Q}|_{\tilde{Q}} + 2E \\ &= \pi_Q^*(K_Q) - (\pi^*(Q) - \tilde{Q})|_{\tilde{Q}} + 2E. \end{aligned}$$

Now, since we can write $\pi^*(Q) - \tilde{Q} = nE_0$, where n corresponds to the multiplicity of the origin in Q, which is 2, we have

$$K_{\tilde{Q}} = \pi_Q^*(K_Q) - (2E_0)|_{\tilde{Q}} + 2E = \pi_Q^*(K_Q).$$

(Canonical forms): ([Mil, Proof of Lemma 2.2.]) We can also see this by more explicit computations on rational forms. First, since $f(x, y, z) = x^2 + y^2 + z^2$ vanishes on Q, we have

$$df = \partial_x f dx + \partial_y f dy + \partial_z f dz = 0$$

as a 1-form on Q. Then, $S = \frac{dx \wedge dy \wedge dz}{f} \in \Omega^3_{\mathbb{A}^3}(Q)$ is a basis for 3-forms on $\mathbb{A}^3$ with pole of order 1 along Q. By the usual computation, we observe that each expression on the right of

$$s = \operatorname{Res}_{\mathbb{A}^3|X} S := \frac{dx \wedge dy}{\partial_z f} = \frac{dy \wedge dz}{\partial_x f} = \frac{dz \wedge dx}{\partial_y f}$$

coincides, which defines a rational canonical form on Q and is called the **Poincaré residue** of S. Now, as in complex analysis, since the vanishing of all the partial derivatives implies that the point is singular, s is regular and non-zero at every non-singular point. Now, as above consider an affine chart $\operatorname{Spec} k[x', y', z]$ of $\operatorname{Bl}_0 \mathbb{A}^3$ given by coordinates $x = x'z, y = y'z, z = z$ (i.e. the chart with $p_z \neq 0$ and $x' = p_x/p_z, y' = p_y/p_z$). Then, the strict transform is given by $\{f_z(x', y', z) := \frac{f(x'z, y'z, z)}{z^2} = 0\}$. Consider a 3-form

$$S'_z = \frac{dx' \wedge dy' \wedge dz}{f_z}$$

on $\operatorname{Spec} k[x', y', z]$. Then, noting $x = x'z, y = y'z, z = z$ imply $dx = zdx_1 +$ (a multiple of dz) and similarly for dy, we see that

$$\pi^*(dx \wedge dy \wedge dz)|_{\operatorname{Spec} k[x', y', z]} = z^2 dx' \wedge dy' \wedge dz$$

and hence $\pi^* S|_{\operatorname{Spec} k[x', y', z]} = S'_z$. Therefore, the symmetric arguments in the other two charts shows that we can obtain a 3-form S' on $\tilde{Q}$ from S'_x, S'_y, S'_z in the obvious way and then obtain the corresponding canonical form $s' = \operatorname{Res}_{\operatorname{Bl}_0 \mathbb{A}^3|\tilde{Q}} S'$ on $\tilde{Q}$. Since we have $\pi_Q^* s = s'$ by construction, we have $\pi_Q^* K_Q = K_{\tilde{Q}}$

□

I will omit details of computations that are similar to what we did in the preceding proof (for example, the standard coordinates in an affine chart of the blow-up, computations of the relative canonical divisors by the adjunction formula, and the derivation of Poincaré residue+) in the later computations.

Chapter 2

The minimal model program

Although studies of singularities are ubiquitous in all times and areas of mathematics, let me introduce the minimal model program as one algebro-geometric context where studies of singularities are vital. In short, the minimal model program aims at the birational classification of projective varieties by "minimal models" that have simpler global structure in the sense that they contain fewer rational curves. To obtain a minimal model that corresponds to the birational class of X, we perform a sequence of "reasonable" birational modifications to X. Therefore, our goal is to identify what it means to have a simpler global structure and what kind of birational modifications are reasonable ones.

First, let us see the case of classical surface minimal models (called relatively minimal models) as a warm-up in birational geometry and a motivation towards generalization.

Definition 2.0.1. A non-singular projective surface X is called a **relatively minimal model** if any birational morphism $f : X \to X'$ to another non-singular projective surface X' is necessarily an isomorphism.

Remark 2.0.2. Let $(\Sigma, \geq)$ be a poset on the set of all isomorphism classes of projective varieties where $X \geq Y$ iff there exists a birational morphism $X \to Y$. Then, a relatively minimal model is a minimal element in $(\Sigma, \geq)$. ■

To obtain a relatively minimal model we set the reasonable birational modification to be a blow-down whose existence is guaranteed by the following result.

Theorem 2.0.3 (Castelnuovo's contraction theorem)**.** [Har77, Theorem 5.7] *Suppose Y is a (-1)-curve on a non-singular projective surface X. Then, there exists a morphism $f : X \to X_0$ to a non-singular projective surface X_0 and a point $p \in X_0$ such that f is the blow-up of X_0 at p with exceptional divisor Y.*

Indeed, the (relatively) minimal model program for non-singular projective surfaces is complete in the following sense. We will go through a proof since it gives insight towards generalization and provides some remarks with examples.

Theorem 2.0.4. [Har77, Theorem V.5.8.] *Every non-singular projective surface admits a birational morphism to a relatively minimal model.*

Proof. First of all, I claim that a non-singular projective surface is a relatively minimal model if and only if it contains no (-1)-curve. One direction is obvious by Castelnuovo's contraction theorem.

Conversely, suppose a non-singular projective surface X contains no (-1)-curve and let $f : X \to Y$ be a birational morphism. Then, we first note the following lemma, which characterizes a birational contraction of irreducible curves as a sequence blow-downs and in particular guarantees the existence of a (-1)-curve.

Lemma 2.0.5. [Har77, Corollary V.5.4.] *Let $f : X' \to X$ be a birational morphism of non-singular projective surfaces and let $n(f)$ be the number of irreducible curves $C' \subset X'$ such that $f(C')$ is a point. Then, $n(f)$ is finite and f can be factored into the composition of exactly $n(f)$ blow-downs.* ♢

The lemma in particular says if $n(f) > 0$, then X' contains at least one (-1)-curve since the curve blown-down at the last step is a (-1)-curve. Therefore, we must have $n(f) = 0$. Hence, it suffices to show that if $n(f) = 0$, then f is an isomorphism, which follows by the following:

Theorem 2.0.6 (Zariski's Main Theorem)**.** [Har77, Theorem V.5.2.] *Let $f : X \dashrightarrow Y$ be a birational map of projective varieties with Y normal. If P is in the locus of indeterminacy of f, then the total transform $T(P)$ is connected and of dimension ≥ 1. Here, for any subset $Z \subset X$, the* ***total transform*** *$T(Z)$ is defined to be $p_2(p_1^{-1}(Z))$, where p_1 and p_2 are the projections of the graph $\Gamma \subset X \times Y$ of f, i.e the closure of the graph of the representation (U, ϕ) of f in $X \times Y$, onto X and Y.* ♢

Indeed, Zariski's main theorem shows that if $n(f) = 0$, then the birational map $f^{-1} : Y \dashrightarrow X$ is also a morphism, i.e. f is an isomorphism as desired.

Hence, given a non-singular projective surface X, it suffices to show that there exists a birational morphism $X \to Y$, where Y is a non-singular projective surface with no (-1)-curve. If X contains no (-1)-curve, we are done. Thus, suppose X contains a (-1)-curve. Then, by Castelnuovo's contraction theorem we obtain a sequence of blow-downs

$$X = X_0 \to X_1 \to X_2 \to \cdots .$$

Now, it suffices to show that the sequence eventually stops. There are several ways to see this. One way is to observe that (-1)-curves $E_i \subset X_i$ we contract descend to linearly independent elements of a finite dimensional vector space $\mathrm{H}^1(X, \Omega_X)$ since the cohomology class $e_i = c(E_i)$ of E_i in $\mathrm{H}^1(X, \Omega_X)$ satisfy the intersection relations $\langle e_i, e_i \rangle = 0$ and $\langle e_i, e_j \rangle = 1$ (cf. [Har67, Ex. V.1.8.]).

Example 2.0.7. This does not mean that the number of (-1)-curves is bounded (by $\mathrm{H}^1(X, \Omega_X)$) since it is possible that some contraction contracts infinitely many (-1)-curves. Indeed, we can obtain a surface with infinity many (-1)-curves by blowing up nine points on $\mathbb{P}^2$ with eight of them in a nice configuration (Example 2.1.15) so that the blow-down at the other point gives us a del Pezzo surface of degree 1, which only contains 280 (-1)-curves (cf. Lemma 2.1.13). ♢

Another way is to notice that the Picard number (Construction A.4.5) drops by one at each blow-down, which suffices for the proof since the Picard number of a non-singular projective surface is finite by the Néron-Severi theorem. □

Remark 2.0.8. Now, what we should learn from the proof are the following:

(i) Relatively minimal surfaces can be characterized by the non-existence of (-1)-curves. Thus, we expect that a minimal model would be defined as a variety that contains fewer rational curves and hence has a simpler global structure.

(ii) We were able to birationally contract (-1)-curves to obtain a relatively minimal surface. Hence, we expect that a reasonable birational modification would look like the contraction of a rational curve. In particular, we need to identify what kind of rational curves we should contract and what kind of variety we obtain after contraction. Furthermore, we want the contraction to respect some invariants like the Picard number to ensure the finiteness of the number of contractions to obtain a minimal model.

Here, I also point out some problems in the preceding classical approach.

(i) We may end up non-isomorphic relatively minimal models even if we start with the same non-singular projective surface and perform blow-downs as in the proof.

Example 2.0.9. Let X be a surface given by blowing up two distinct points on $\mathbb{P}^2$. We can obtain two non-isomorphic relatively minimal models by blowing down different (-1)-curves. Indeed, if we blow down the two exceptional divisors E_1, E_2 corresponding to those two points, then we obtain a relatively minimal model $\mathbb{P}^2$ of X. On the other hand, we can also blow down the proper transform L of the line in $\mathbb{P}^2$ joining the two points. Then, we obtain a relatively minimal model $\mathbb{P}^1 \times \mathbb{P}^1 (\ncong \mathbb{P}^2)$. Note L is indeed a (-1)-curve since we have $((H - E_1 - E_2) \cdot (H - E_1 - E_2)) = 1 - 1 - 1 = -1$ for a hyperplane section H of $\mathbb{P}^2$. Note we have an isomorphism

$$\mathbb{P}^2 \times \mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1] \times \operatorname{Proj} k[x_2, x_3] \cong \operatorname{Proj} k[x_0, x_1, x_2, x_3]/\langle x_0x_3 - x_1x_2 \rangle =: Q$$

and a (bi)rational map

$$Q \dashrightarrow \operatorname{Proj} k[x_0, x_1, x_2] = \mathbb{P}^2$$

by projection. Then, consider the graph $\Gamma \subset Q \times \mathbb{P}^2$ of π. By explicit computation, we can see that the first projection $\pi_1 : \Gamma \to Q$ is the blow-up of Q at $[0 : 0 : 0 : 1]$ and the second projection $\pi_2 : \Gamma \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at $[0 : 0 : 1]$ and $[0 : 1 : 0]$. For details of computations, see [Har92, Example 7.22.]

$\diamondsuit$

In the surface minimal model program, we have the uniqueness of a minimal model in the sense that a minimal model obtained from a non-singular surface does not depend on the choice of contraction. In the higher dimensional minimal model program, we do not have the uniqueness.

(ii) Since a (-1)-curve is a notion specific to surfaces, it would be better to have more general languages to describe rational curves to contract.

$\blacksquare$

Remark 2.0.10. Keeping the previous remarks in mind, we are interested in finding rational curves on a variety to begin with. For example, if X is a smooth variety that is Fano, i.e. with the ample anti-canonical divisor $-K_X$, then we can show that through any point of X there is a rational curve D such that

$$0 < -(D \cdot K_X) \leq \dim X + 1$$

([KM98, Theorem 1.10]). In particular, a smooth Fano variety is far from what we expect for a minimal model. For details of the proof, see section 1.1 of [KM98]. $\blacksquare$

2.1 The surface minimal model program

To be more precise with Remark 2.0.8, let us introduce notions of cones and extremal faces.

Definition 2.1.1. Let $K = \mathbb{Q}$ or $K = \mathbb{R}$ and V a K-vector space. A subset $N \subset V$ is called a **cone** if $0 \in N$ and N is closed under multiplication by positive scalars.

A subcone $M \subset N$ is called **extremal** or said to be an **external face** of N if $u, v \in N$ and $u + v \in M$ imply $u, v \in M$. A 1-dimensional extremal subcone is called an **extremal ray**.

Now, we can define contractions.

Definition 2.1.2. Let X be a projective variety and $F \subset \overline{\mathrm{NE}}(X)$ an extremal face, where $\mathrm{NE}(X)$ is the **(Kleiman-Mori) cone of curves**, i.e.

$$\mathrm{NE}(X) := \{\sum a_i[C_i] \mid C_i \subset X, 0 \leq a_i \in \mathbb{R}\} \subset N_1(X),$$
$$\overline{\mathrm{NE}}(X) := \text{the closure of } \mathrm{NE}(X) \text{ in } N_1(X).$$

A morphism $\mathsf{cont}_F : X \to Z$ is called the **contraction** of F if the following hold:

(i) $\mathsf{cont}_F(C)$ is a point for an irreducible curve $C \subset X$ iff $[C] \in F$,

(ii) $(\mathsf{cont}_F)_*\mathcal{O}_X = \mathcal{O}_Z$.

Note that the contraction does not necessarily exist.

The following result regarding extremal rays is quite useful.

Theorem 2.1.3 (The Cone Theorem)**.** [KM98, Theorem 1.24. (i)] *Let X be a non-singular projective variety. Then, there are countably many rational curves $C_i \subset X$ such that $0 < -(C_i \cdot K_X) \leq \dim X + 1$, and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X > 0} + \sum_i \mathbb{R}_{\geq 0}[C_i],$$

where $\overline{\mathrm{NE}}(X)_{K_X>0} := \overline{\mathrm{NE}(X)} \cap K_{X>0}$ with $K_{X>0} := \{x \in N_1(X) \mid (x \cdot K_X) > 0\}$.

Remark 2.1.4. The significance of the result in terms of the minimal model program is that the theorem in particular says if K_X is not nef, then $\overline{\mathrm{NE}}(X)_{K_X \leq 0} \neq 0$ and hence there exists a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$, i.e., $(R \cdot K_X) < 0$. In other words, this tells us which extremal face we should contract. ■

Now, in the case of surfaces, we have the following classification of contractions.

Theorem 2.1.5. [KM98, Theorem 1.28.] *Let X be a smooth projective surface and $R \subset \overline{\mathrm{NE}}(X)$ a K_X-negative extremal ray, i.e. $(R \cdot K_X) < 0$. Then, the contraction $\mathsf{cont}_R : X \to Z$ exists and is one of the following types:*

(i) *Z is a point, $\rho(X) = 1$ and $-K_X$ is ample. (In fact, $X \cong \mathbb{P}^2$)*

(ii) *Z is a smooth surface and X is a minimal ruled surface over Z, i.e. each fiber is an integral rational curve; $\rho(X) = 2$.*

(iii) *Z is a smooth surface and X is obtained from Z by blowing up a closed point; $\rho(Z) = \rho(X) - 1$, where ρ is the Picard number.*

Sketch of a proof. Let $C \subset X$ be an irreducible curve such that $[C] \in R$. The above three cases correspond to the sign of the self-intersection $(C \cdot C)$.

(i) First assume $(C \cdot C) > 0$. Then, by [KM98, Corollary 1.21.], $[C]$ is an interior point of $\overline{\mathrm{NE}}(X)$. Since $[C]$ also generates an extremal ray, we have $N_1(X) \cong \mathbb{R}$, i.e. $\rho(X) = 1$. By supposition we have $(C \cdot K_X) < 0$ and hence K_X is negative on $\overline{\mathrm{NE}}(X) \setminus \{0\}$. Thus, $-K_X$ is ample by Kleiman's ampleness criterion (Theorem A.4.6).

(ii) Next assume $(C \cdot C) = 0$. Then, we can show that $|mC|$ is base point free for $m \gg 1$ (cf. Definition A.3.3). Now, let $\mathsf{cont}_R : X \to Z$ be the Stein factorization ([Har67, Corollary 11.5.]) of the corresponding morphism, i.e. cont_R is a projective morphism with connected fibers and Z is a normal curve and hence is non-singular. Let $\sum a_i C_i$ be a fiber of cont_R. Then, we have $\sum a_i [C_i] = [C] \in R$. Since R is an extremal ray, we have $[C_i] \in R$ for all i. Therefore, we have $(C_i \cdot C_i) = 0$ and $(C_i \cdot K_X) < 0$. By the adjunction formula (Theorem A.2.4), $2g(C) - 2 = (C \cdot (C + K_X))$, where $g(C)$ denotes the genus of C, we have $C_i \cong \mathbb{P}^1$ and $(C_i \cdot K_i) = -2$. Therefore,

$$-2 = (C \cdot K_X) = \left(\sum a_i C_i \cdot K_X\right) = -2 \sum a_i,$$

i.e. $\sum a_i C_i$ is an integral rational curve as desired.

(iii) Finally assume $(C \cdot C) < 0$. By the adjunction formula and $(C \cdot K_X) < 0$, C is a (-1)-curve and hence we apply Castelnuovo's contraction theorem (Theorem 2.0.3).

□

Note that the contraction in (i) and (ii) yields the explicit structure of X while the contraction in (iii) introduces the new surface Z. Now, before discussing the minimal model program, let us appreciate the cone theorem and (the proof of) Theorem 2.1.5 by observing the following example of computations of $\overline{\mathrm{NE}}(X)$.

Example 2.1.6 (Del Pezzo Surfaces)**.** In this example, we compute the cone of curves of del Pezzo surfaces. Since del Pezzo surfaces are interesting in its own right and appear later, I will go through basics of them. See for example [Man86, §§24-26], [Liu17, §§3.6-3.8.], and [Deb16, Example 5.15.].

Definition 2.1.7. A non-singular projective surface V is said to be a **del Pezzo surface** if it is a **Fano variety**, i.e. if the anti-canonical divisor $-K_V$ is ample. A **degree** of a del Pezzo surface V is defined to be the self-intersection $(K_V \cdot K_V)$. ♢

First, del Pezzo surfaces have the following nice descriptions.

Lemma 2.1.8. [Man86, Theorem 24.3, 24.4.], [Har77, Corollary V.4.7.] Let V be a del Pezzo surface of degree d. Then, we have $1 \leq d \leq 9$.

(i) If $d = 9$, then $V \cong \mathbb{P}^2$.

(ii) If $d = 8$, then either $V \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $V \cong \mathrm{Bl}_p \mathbb{P}^2$ for $p \in \mathbb{P}^1$.

(iii) If $1 \leq d \leq 7$, then $V \cong \mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$ for some $p_1, \ldots, p_{9-d} \in \mathbb{P}^2$ **in general position**, i.e. with no triple colinear and no 6-tuple lying on a conic.

Conversely, suppose $3 \leq d \leq 9$ and $p_1, \ldots, p_{9-d} \in \mathbb{P}^1$ are in general position. Then, $V'_d := \mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are del Pezzo surfaces of the corresponding degree; furthermore, the ample anti-canonical divisor $-K_{V'_d}$ gives the anti-canonical embedding $V'_d \hookrightarrow \mathbb{P}^d$ as a surface of degree d, whose canonical sheaf $\omega_{V'_d}$ is isomorphic to $\mathcal{O}_{V'}(-1)$. ♢

Remark 2.1.9. [Man86, Remark 24.4.1, 24.4.2.]

(i) As we see in $d = 8$, it is not necessarily true that del Pezzo surfaces of the same degree are all isomorphic. However, since the projective automorphism of $\mathbb{P}^2$ acts transitively on systems of ≥ 4 points in general position, all del Pezzo surfaces of degree $5 \leq d \leq 7$ are isomorphic.

(ii) We also have sufficient conditions for a configuration of $p_1, \ldots, p_{9-d}$ with $d = 1, 2$ so that $\mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$ is a del Pezzo surface. See [Man86, Remark 26.2.].

♢

Corollary 2.1.10. [Har77, Proposition V.4.8.] Let $V \ncong \mathbb{P}^1 \times \mathbb{P}^1$ be a del Pezzo surface of degree d and write $V \cong \mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$.

(i) $\operatorname{Pic} V \cong \mathbb{Z}^{10-d}$, generated by $l, e_1, \ldots, e_{9-d}$, where l corresponds to the hyperplane section of $\mathbb{P}^2$ and e_i corresponds to the exceptional divisor.

(ii) The intersection pairing on X is given by $l^2 = 1$, $e_i^2 = -1$, $l \cdot e_i = 0$, $e_i \cdot e_j = 0$ for $i \neq j$.

(iii) The canonical class is $K_V = -3l - \sum e_i$. Note since we anti-canonically embed $i : V \hookrightarrow \mathbb{P}^d$, i.e. $i^*\mathcal{O}_{\mathbb{P}^d}(1) = \mathcal{O}_V(-K_V)$, the hyperplane section is $-K_V$.

♢

Proof. Part (i) follows from Corollary A.1.4. Hence, we have part (ii) by Lemma A.2.7. Finally, (iii) follows by noting that $K_{\mathbb{P}^2} = -3H$ (Example A.2.2) and Lemma A.2.6. ♡

Remark 2.1.11. Part (ii) in the corollary has the following geometric interpretations:

(i) $l^2 = 1$ (two lines in $\mathbb{P}^2$ intersects at one point),

(ii) $l.e_i = 0$ (l does not pass through the $(9 - d)$ points we blow up),

(iii) $e_i.e_j = 0$ for distinct i, j (e_i and e_j are disjoint),

(iv) $e_i^2 = -1$ (a line passing though p_i and e_i intersects at one point, i.e., $(l - e_i).e_i = 1$).

♢

Now, by Lemma 2.1.8, we in particular see that the blow-up of $\mathbb{P}^2$ at 6 points in general position is a non-singular cubic surface in $\mathbb{P}^3$ and vice versa. Hence, by Corollary 2.1.10 we obtain the following well-known result.

Lemma 2.1.12. [Har77, Theorem V.4.9.] Any non-singular cubic surface X in $\mathbb{P}^3$ contains exactly 27 (-1)-curves. Writing $X \cong \mathrm{Bl}_{p_1,\ldots,p_6} \mathbb{P}^2$ with corresponding exceptional curves E_i, these (-1)-curves can be listed as follows:

(i) the 6 exceptional curves E_i $(i = 1, \ldots, 6)$;

(ii) the 15 strict transform F_{ij} of the line in $\mathbb{P}^2$ containing p_i and p_j $(1 \leq i < j \leq 6)$;

(iii) the 6 strict transform G_j of of the conic containing the five p_i ($i \neq j$ and $j = 1, \ldots, 6$).

Here, note an irreducible curve $C \subset V$ is a (-1)-curve if and only if C is a **line**, i.e. a curve of degree 1 and genus 0, by the adjunction formula $2g - 2 = -\deg_V(C) + (C \cdot C)$, where we use the fact that the hyperplane section is given by $-K_X$ (Lemma 2.1.10) to see $\deg_V(C) = -(C \cdot K_V)$. ♢

Proof. First, noting that $E_i \sim e_i$, $F_{ij} \sim l - e_i - e_j$, and $G_j \sim 2l - \sum_{i \neq j} e_i$, we see that $E_i^2 = F_{ij}^2 = G_j^2 = -1$ by Lemma 2.1.10. Hence, it remains to show that any curve C on X with $C^2 = -1$ (and hence $\deg C = 1$) is one of the lines listed above. Assume C is not any of the exceptional curves E_i and write

$$C \sim al - \sum_i b_i e_i$$

with $a > 0$ and $b_i \geq 0$. Then, $\deg C = (C \cdot (-K_X)) = 1$ and $C^2 = -1$ give the following equations:

$$3a - \sum_i b_i = 1$$
$$a^2 - \sum_i b_i^2 = -1.$$

Now, by the Cauchy-Schwarz's inequality, we have

$$\left(\sum_i b_i\right)^2 \leq 6 \left(\sum_i b_i^2\right).$$

Therefore, we have

$$3a^2 - 6a - 5 \leq a,$$

which implies $a = 1$ or $a = 2$. Then, when $a = 1$, we get $C = F_{ij}$ and when $a = 2$, we get $C = G_j$ as desired. ♡

Now, by the same strategy, we can show the following:

Lemma 2.1.13. [Man86, Theorem 26.2.] Let V be a del Pezzo surface of degree d given by the blow-up $\mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$ and let h and e_i denote the generators of $\mathrm{Pic}(V)$ as in Lemma 2.1.10. Then, an exceptional curve in V is given as one of the following:

(i) the exceptional divisor corresponding to p_i: e_i;

(ii) the strict transform of a line passing through two of the points p_i: $l - e_i - e_j$;

(iii) the strict transform of a conic passing through five of the points p_i: $2l - \sum_5 e_i$;

(iv) the strict transform of a cubic passing through seven of the points p_i such that one of them is a double point: $3l - 2e_i - \sum_6 e_j$;

(v) the strict transform of a quartic passing though eight of the points p_i such that three of them are double points: $4l - 2\sum_3 e_i - \sum_5 e_j$;

(vi) the strict transform of a quintic passing through eight of the points p_i such that six of them are double points: $5l - 2\sum_6 e_i - \sum_2 e_j$;

(vii) the strict transform of a sextic passing through eight of the points p_i such that seven of the are double and one of them is a triple point: $6l - 3e_i - 2\sum_7 e_j$.

In particular, the number n_d of exceptional curves on $V \cong \mathrm{Bl}_{p_1,\ldots,p_{9-d}} \mathbb{P}^2$ is given as follows:

d	1	2	3	4	5	6	7	8	9
n_d	240	56	27	16	10	6	3	1	0

$\diamondsuit$

We have prepared tools to compute the cone of curves for del Pezzo surfaces. First of all, if $V = \mathbb{P}^2$, then since $N_1(\mathbb{P}^2) = \mathbb{R}[L]$ for a line L, we have $\mathrm{NE}(\mathbb{P}^2) = \overline{\mathrm{NE}}(\mathbb{P}^2) = \mathbb{R}_{>0}[L]$. Similarly, if $V = \mathbb{P}^1 \times \mathbb{P}^1$, then since $N_1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{R}[L_1] + \mathbb{R}[L_2]$ for lines $L_1 = \mathbb{P}^1 \times \{p_2\}$ and $L_2 = \{p_1\} \times \mathbb{P}^1$, we have $\mathrm{NE}(\mathbb{P}^1 \times \mathbb{P}^1) = \overline{\mathrm{NE}}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{R}_{>0}[L_1] + \mathbb{R}_{>0}[L_2]$. The rest are given as follows:

Lemma 2.1.14. Let $V \ncong \mathbb{P}^1 \times \mathbb{P}^1$ be a del Pezzo surface of degree $d \leq 8$. Then, there exists n_d (-1)-curves $C_i \subset V$ such that

$$\mathrm{NE}(V) = \overline{\mathrm{NE}}(V) = \sum_{i=1}^{n_d} \mathbb{R}_{>0}[C_i],$$

where n_d is the integer given in Lemma 2.1.13.

$\diamondsuit$

Proof. First of all, since $-K_V$ is ample by definition, $\overline{\mathrm{NE}}(X) \setminus \{0\} \subset N_1(X)_{-K_X > 0}$ by Kleiman's ampelness criterion (Theorem A.4.6). Then, by the cone theorem, we have countably many $C_i \subset V$ such that

$$\mathrm{NE}(V) = \overline{\mathrm{NE}}(V) = \sum_i \mathbb{R}_{>0}[C_i].$$

Now, since V is neither isomorphic to $\mathbb{P}^2$ nor a ruled surface, the proof of Theorem 2.1.5 shows that C_i is a (-1)-curve. Hence, there are only n_d curves C_i by Lemma 2.1.13. $\heartsuit$

$\blacksquare$

Now, it is natural to ask what happens if we blow up 9 points.

Example 2.1.15. [KM98, Example 1.23 (4)] Let $X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the nine base points of a pencil of cubic curves, i.e. at $\{p_0, \ldots, p_8\} = C_1 \cap C_2$ for cubic curves C_1, C_2 and suppose all members of the pencil are irreducible. Then, we have infinitely many (-1)-curves on X. Let $\pi : X \to \mathbb{P}^1$ be the morphism given by the pencil of cubics. Then, $E_0, \ldots, E_8$ are sections. Now, since generic fibers of π are elliptic curves, they become an abelian group by choosing the intersection with E_0 to be 0. Hence, the translation by (intersections with) $E_1, \ldots, E_8$ generates a subgroup of the automorphim group of generic fibers which is isomorphic to $\mathbb{Z}^8$ and extends to a subgroup of $\mathrm{Aut}(X)$. Hence, $f(E_0)$ for $f \in \mathrm{Aut}(X)$ give infinitely many (-1)-curves that are all in distinct classes in $N_1(X)$ by the construction of the automorphism. $\blacksquare$

Now, as in the case of relatively minimal models, we can consider a sequence of contractions and we can see such a sequence terminates by looking at the Picard number. More precisely:

Theorem 2.1.16. [KM98, Theorem 1.29.] *Let X be a smooth projective surface. Then, there is a sequence of contractions $X \to X_1 \to \cdots \to X_n = X'$ such that X' satisfies exactly one of the following conditions:*

(i) *$K_{X'}$ is nef;*

(ii) *X' is a minimal ruled surface;*

(iii) *$X' \cong \mathbb{P}^2$.*

A proof is done in the following construction.

Definition 2.1.17. If $K_{X'}$ is nef above, then X' is called a **(surface) minimal model** of X. It turns out that in this case the morphism $X \to X'$ is unique (in particular, does not depend on the choice of extremal rays), i.e., X' is determined by X.

Remark 2.1.18.

(i) Suppose X is a non-singular surface containing a (-1)-curve. Then, since we have $(K_X \cdot C) = -1$ by the adjunction formula, X is not nef. Hence, a minimal model of surface is a relatively minimal model.

(ii) The counter-example for the uniqueness of relatively minimal model in Remark 2.0.8 does not work here since neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$ is a minimal model.

■

Construction 2.1.19. [KM98, Summary 1.31.] The following is the complete lists of the 2-fold minimal model program:

Step 0. We have a smooth projective surface X.

Step 1. If K_X is nef, then go to Step 5. Otherwise, Theorem 2.1.3 yields a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$.

Step 2. By Theorem 2.1.5, the contraction $\mathsf{cont}_R : X \to Z$ exists. We have two possibilities.

Step 3. If $\dim Z = \dim X$, then $\rho(Z) = \rho(X) - 1$, so replace X with Z and go back to Step 1.

Step 4. If $\dim Z < \dim X$, then Theorem 2.1.5 (ii),(iii) determine the structure of X.

Step 5. If K_X is nef, then we stop. These surfaces should be investigated by other methods.

■

The minimal model program aims at listing up similar step-by-step approaches for higher dimensional varieties. Of course there are several difficulties. For example, as we will see in the next section, in the 3-fold case we need to deal with singular points while in the 2-fold case we stayed in the smooth category.

2.2 The 3-fold minimal model program and singularities

We are going to follow the surface minimal model program, but along the way, we need a lot of modifications. The most remarkable one for us is that we need to consider singularities. We can observe this by the classification of contractions of extremal rays, which corresponds to Theorem 2.1.5 in the surface minimal program.

Theorem 2.2.1. [KM98, Theorem 1.32.] *Let X be a* ***non-singular*** *projective 3-fold (over $\mathbb{C}$) and $\mathsf{cont}_R : X \to Y$ the contraction of a K_X-negative extremal ray $R \subset \overline{\mathrm{NE}}(X)$. The following is the list of all possibilities for cont_R:*

- E: (Exceptional) $\dim Y = 3$, cont_R *is birational and there are five types of local behaviour near the contracted surface:*
  - E_1: cont_R *is the inverse of the blow-up of a smooth curve in the smooth 3-fold Y.*
  - E_2: cont_R *is the inverse of the blow-up of a smooth point in the smooth 3-fold Y.*
  - E_3: cont_R *is the inverse of the blow-up of an ordinary double point of Y (Definition 3.2.2).*
  - E_4: cont_R *is the inverse of the blow-up of a point, which is formally isomorphic to an isolated cA_1-singularity $(0 \in \{x^2 + y^2 + z^2 + w^3 = 0\})$ (cf. Definition 3.1.1 and 4.2.1).*
  - E_5: cont_R *contracts a smooth $\mathbb{P}^2$ a point of multiplicity 4 on Y, which is formally isomorphic to the germ $(0 \in \mathbb{A}^3/\frac{1}{2}(1,1,1))$ (Example 3.3.6).*
- C: (Conic bundle) *$dimY = 2$ and cont_R is a fibration whose fibers are plane conics.*
- D: (Del Pezzo fibration) *$dimY = 1$ and general fibers of cont_R are Del Pezzo surfaces (Example 2.1.6).*
- F: (Fano variety) $\dim Y = 0$*, X is a* ***Fano variety****, i.e. $-K_X$ is ample.*

Remark 2.2.2. [KM98, p.29] The cases C and D give structures that we understand sufficiently well. Case F is also sufficient since we have a complete list of the occurring Fano 3-folds (e.g. [Isk80]). The cases E_1 and E_2 are analogous to Theorem 2.1.5 (i). In particular, the resulting variety Y remains in the smooth category, i.e. we can apply Theorem 2.2.1 again. However, our strategy does not work when we encounter E_3, E_4, and E_5 since they produce a singular variety Y. Hence, although the surface minimal model program can be done in the smooth category, we need to consider some singularities for the higher dimensional generalization. ■

When we allow singularities, the classification of contractions is given as follows:

Theorem 2.2.3. [KM98, Proposition 2.5.] *Let X be a normal $\mathbb{Q}$-factorial projective variety and let $\mathsf{cont}_R : X \to Y$ be the contraction of an extremal ray $R \subset \overline{\mathrm{NE}}(X)$. Then, we have one of the following:*

(i) (Fiber type contraction) $\dim X > \dim Y$.

(ii) (Divisorial contraction) *f is birational and $\mathrm{Ex}(f)$ is an irreducible divisor.*

(iii) (Small contraction) *f is birational and $\mathrm{Ex}(f)$ has codimension ≥ 2.*

Remark 2.2.4. [KM98, 2.6.] Let me comment on some implications of each type of contractions:

(i) (Fiber type contraction) We can interpret this as reducing the problem of X to the study of the lower dimensional variety Y together with the fibers of f (cf. Construction 2.1.19 Step 4). Moreover, the fibers are nice in the sense that they are analogous to $\mathbb{P}^1$ and the Del Pezzo surfaces in the previous cases.

(ii) (Divisorial contraction) In this case we can see that Y is again $\mathbb{Q}$-factorial and hence $\rho(Y) \leq \rho(X) - 1$ (indeed $\rho(A) = \rho(X) - 1$). Hence, we can apply Theorem 2.2.3 (cf. Contraction 2.1.19 Step 3).

(iii) (Small contraction) This is the worst case since K_Y is not even $\mathbb{Q}$-Cartier. Hence, we cannot apply Theorem 2.2.3. In this case, we apply an operation called a **flip**, which is an algebraic analogue of topological surgery; instead of contracting the codimension ≥ 1 subvariety $E = \mathrm{Ex}(f) \subset X$, we replace E with another codimension ≥ 2 subvariety E^+ to obtain a new variety $(X \setminus E) \cup E^+$. The details of flips are beyond the scope of this thesis. For example, see [KM98, Example 2.7.] and later parts of the same textbook.

■

Now, let us introduce some terminology regarding singularity (See also §3.1).

Definition 2.2.5. Let X be a variety.

(i) A **resolution** of X is a non-singular variety Y together with a birational proper morphism $f : Y \to X$.

(ii) A resolution $f : Y \to X$ is said to be a **log resolution** if $\mathrm{Ex}(f)$ is a **simple normal crossing** divisor, i.e. $\mathrm{Ex}(f)$ is a Weil divisor and we can write $\mathrm{Ex}(f) = \sum_i d_i E_i$ with E_i non-singular and intersecting everywhere transversely.

(iii) A resolution $f : Y \to X$ is said to be a **minimal resolution** if for any resolution $f' : Y' \to X$ of X, there exists a morphism $\sigma : Y' \to Y$ such that $f' = f \circ \sigma$.

We have the following existence results.

Theorem 2.2.6 (Hironaka)**.** [KM98, Theorem 0.2.] *A log resolution exists for any variety over a field of characteristic zero.*

Lemma 2.2.7. [Kol07, Theorem 2.26] *Let Y be a surface. Then, a minimal resolution of Y exists and is unique up to isomorphism. Furthermore, a resolution $f : X \to Y$ is minimal if and only if K_X is f-**nef**, i.e. $(K_X \cdot E) \geq 0$ for every f-exceptional divisor. In particular, we can obtain a minimal resolution by taking one resolution $f : X_0 \to Y$ and contracting all the f-exceptional curves E with $(E \cdot K_X) < 0$ by Castelnuovo's contraction theorem.*

Example 2.2.8. Note by the adjunction formula for curves, if $(C \cdot C) < -1$, then $(C \cdot K_X) \geq 0$. Hence, a resolution $f : Y \to X$ is the minimal resolution if $(E \cdot E) \leq -2$ for every f-exceptional curve E. In particular, the resolution of the affine quadric cone in Lemma 1.4.5 is the minimal resolution. ■

If Y is a normal variety such that mK_Y is Cartier for some $m > 0$, then the (relative) canonical class allows us to measure how singular Y is (a priori) with respect to a resolution $X \to Y$ as follows.

Definition 2.2.9. Let X be a normal variety with mK_X Cartier for $m > 0$, Y a normal variety, and $f : Y \to X$ a (not necessarily proper) birational morphism. Then, for a Weil divisor E on Y, we define the **discrepancy of** E **with respect to** X to be the rational number $a(E, X)$ (with $m \cdot a(E, X) \in \mathbb{Z}$) given as the coefficient of E in $K_{Y/X} = K_Y - f^*K_X$. Note that if f is a resolution (i.e. f is proper and Y is non-singular), then we can write

$$K_{Y/X} = \sum_{E\text{:irreducible } f\text{-exceptional}} a(E, X)E.$$

Now, define the **discrepancy of** X to be

$$\mathrm{discrep}(X) := \min_{E\text{:exceptional}} a(E, X),$$

where E runs through all the irreducible exceptional divisors for all birational morphisms $f : Y \to X$ with Y normal. As we will see, the larger discrep(X) implies that the milder singularities. Keeping this in mind, we say X is:

(i) **terminal** if $\mathrm{discrep}(X) > 0$.

(ii) **canonical** if $\mathrm{discrep}(X) \geq 0$.

(iii) **log terminal** if $\mathrm{discrep}(X) > -1$.

(iv) **log canonical** if $\mathrm{discrep}(X) \geq -1$.

Remark 2.2.10. [KM98, Remark 2.23.] Under the notation of the preceding definition, we can also define the discrepancy $a(E, X)$ of the irreducible exceptional divisor E with respect to Y as follows: Take a general point $e \in E$ and local coordinates $\{y_i\}$ so that $E = \{y_1 = 0\}$. Then, locally near e,

$$f^*(\text{local generator of } \mathcal{O}_X(mK_X)) = y_1^{m\cdot a(E,X)}(\text{unit})(dy_1 \wedge \cdots \wedge dy_n)^{\otimes m},$$

which is a local illustration of the previous definition (e.g. see the last part of the proof of Lemma 1.4.5 or Example 3.3.6). Now, note the local ring $\mathcal{O}_{E,Y} \subset k(Y) = k(X)$ is a discrete valuation ring, where $k(X)$ denotes the field of rational functions on X, which corresponds to a valuation $v(E, Y)$ of $k(X)$. Note if we have another data (f', Y', E') with $v(Y', E') = v(Y, E)$, then the rational map $Y \to X \dashrightarrow Y'$ is an isomorphism at the generic points of E and E'. Hence, by the local definition of the discrepancy, we see that $a(E, X) = a(E', X)$ and hence $a(E, X)$ only depends on the valuation $v(E, Y)$. In this sense, we omit f and Y from the notation of $a(E, X)$. ■

Remark 2.2.11. [KM98, Corollary 2.31.] The reason why we do not consider the case when $\mathrm{discrep}(X) < -1$ is because once you have $a(E, X) = -1 - c$ for some E and $c > 0$, you can construct a resolution $Y_n \to X$ with exceptional divisor E_n and $a(E_n, X) = -cn$ for any n by repeatedly blowing up, i.e. $\mathrm{discrep}(X) = -\infty$. Indeed, we can show that we have either

(i) $\mathrm{discrep}(X) < -\infty$, or

(ii) $-1 \leq \mathrm{discrep}(X) \leq 1$.

■

Now, since it is quite hard to consider all the possible exceptional divisors to compute the discrepancy, we appreciate the following result in this thesis:

Theorem 2.2.12. [KM98, Corollary 2.32.] *Let Y be a normal variety such that mK_X is Cartier for some $m > 0$ and $f : X \to Y$ be a resolution with exceptional divisors $E_i \subset \mathrm{Ex}(f)$.*

(i) *Assume that $0 \leq \min_i\{a(E_i, X)\} \leq 1$. Then,*

$$\mathrm{discrep}(X) = \min_i\{a(E_i, X)\}.$$

(ii) *Assume f is a log resolution and assume $a(E_i, X) \geq -1$ for all i. Then,*

$$\mathrm{discrep}(X) = \min\left\{\min_i\{a(E_i, X)\}, 1\right\}.$$

Hence, it is in general sufficient to find one log resolution (and such a resolution exists by Hironaka).

Example 2.2.13. Here is a list of some examples we have in this thesis.

(i) For any non-singular variety X, we have $\mathrm{discrep}(X) = 1$ by considering the blow-up in codimension 2 (Lemma A.2.8).

(ii) For the quotient variety $\mathbb{A}^3/\frac{1}{2}(1, 1, 1)$ (or the cone over the Veronese surface), we have $\mathrm{discrep}(X) = \frac{1}{2}$, i.e. X has a terminal singularity at the origin (Example 3.3.6).

(iii) For the affine quadric cone X (or more generally Du Val singularities (Theorem 3.4.1)), we have $\mathrm{discrep}(X) = 0$, i.e. X has a canonical singularity at the origin (Lemma 1.4.5).

(iv) For the affine cone of a non-singular hypersurface X of degree d in $\mathbb{P}^n$, we have $\mathrm{discrep}(X) = n - 1 - d$ if $d \leq n$ or $\mathrm{discrep}(X) = -\infty$ otherwise (Example 3.2.5). Hence, X has:

   (a) a terminal singularity if $d \leq n - 2$,

   (b) a canonical singularity if $d = n - 1$,

   (c) a log terminal singularity if $d < n - 1$,

   (d) a log canonical singularity if $d = n$,

   (e) a "bad" singularity if $d > n$.

   Note this is quite intuitive since the higher multiplicity of a singularity implies the less mild singularity.

■

Now, we go back to the minimal model program. In analogy with the surface minimal model program, we define higher dimensional minimal models as follows:

Definition 2.2.14. Let X be a normal and proper variety. Then, X is said to be **minimal** or a **minimal model** if

(i) X has terminal singularities, and

(ii) K_X is nef.

If Y is a smooth proper variety birational to X, then X is also said to be a minimal model of Y.

Remark 2.2.15. Unlike surface minimal models, higher dimensional are not unique. ■

Now, we have introduced all the terminology we need to sketch the strategy of the higher dimensional minimal model program.

Construction 2.2.16 (Minimal Model Program)**.**

Step 0. (Initial datum) We have a projective variety $X = X_0$ with only $\mathbb{Q}$-factorial and terminal singularities. We inductively construct intermediate varieties X_i and then stop with a finial variety X^*. Suppose that we have constructed X_i.

Step 1. (Preparation) If K_{X_i} is nef, then there is nothing to do and go to Step 3.2. Otherwise, we establish two results:

(a) (Cone Theorem) A generalization of the cone theorem for non-singular projective varieties (Theorem 2.1.3) thankfully holds ([KM98, Theorem 3.7.]), which locates a K_{X_i}-negative extremal ray $R_i \subset \overline{\mathrm{NE}}(X_i)$ if K_{X_i} is not nef. (We try to construct a theory that works with any choice of R_i, but it is sometime convenient to choose R_i cleverly.)

(b) (Contraction of an extremal ray) Let $\mathsf{cont}_{R_i} : X_i \to Y_i$ denote the contraction of R_i in Theorem 2.2.3.

Step 2. By Theorem 2.2.3, we have three possible types of contractions $\mathsf{cont}_{R_i} : X_i \to Y_i$, two of which are used to produce a new variety X_{i+1} as follows.

(a) (Divisorial contraction) If $\mathsf{cont}_{R_i} : X_i \to Y_i$ is a divisorial contraction as in Theorem 2.2.3 (ii), then set $X_{i+1} = Y_i$. We can show that Y_i only admits $\mathbb{Q}$-factorial and terminal singularities, i.e. we can go back to Step 0.

(b) (Flipping contraction) If $\mathsf{cont}_{R_i} : X_i \to Y_i$ is a small contraction as in Theorem 2.2.3 (iii), then set $X_{i+1} = X_i^+$, where X_i^+ is obtained by the flip of cont_{R_i}. We can show that X_i^+ exists and only admits $\mathbb{Q}$-factorial and terminal singularities (for 3-folds), i.e. we can go back to Step 0.

Step 3. (Final outcome) We hope that the procedure eventually stops and we get one of the following two possibilities.

(a) (Fano fiber space) $\mathsf{cont}_{R_i} : X_i \to Y_i$ is a fiber type contraction as in Theorem 2.2.3 (i), then set $X^* = X_i$. We hope that studies of the lower dimensional variety Y_i an of the fibers provide new methods to study X^*.

(b) (Minimal model) If K_{X_i} is nef, then we set $X^* = X_i$. We hope that the semi-positivity of the canonical class helps us to understand X^*.

■

Remark 2.2.17. First of all, this strategy is known to work only for surfaces and 3-folds. In higher dimensions, it is not known whether we can always perform the flip operation in Step 2 (b). It is also possible that we encounter an infinite sequence of flips. Finally, we also do not know whether the resulting X^* is indeed of any use since we improve the global structure at the cost of introducing singularities. ■

Although there are my uncertainties in higher dimensions, these go beyond the scope of the thesis. In the rest of the thesis, we focus on studies of singularities in surfaces and 3-folds to make sure that we are comfortable with the terminal (or canonical) singularity category.

Chapter 3

Resolution of surface singularity

3.1 Terminology

First, let me introduce some basic languages to talk about the local behavior of singularities.

Definition 3.1.1.

(i) Consider the category whose objects are the pairs $(x \in X)$ of a variety X and a point $x \in X$ and whose morphism $(x \in X) \to (y \in Y)$ is an equivalence class of morphisms $f : U \to Y$ of varieties with U an open neighborhood of x and $f(x) = y$, where $(f, U) \sim (f', U')$ if $f_x = f'_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$, i.e. $f|_V = f'|_V$ for some open neighborhood $V \subset U \cap U'$ of x. An isomorphism class of objects $(x \in X)$ of the category is called a **germ of a point** and denoted by $(x \in X)$ by abuse of notation. If a point is a singularity, then $(x \in X)$ is said to be a **germ of a singularity**. Note for any open neighborhood U of $x \in X$, we have $(x \in X) \cong (x \in U)$.

(ii) Let $f : (x \in X) \to (y \in Y)$ be a morphism of germs of a point. Then, f is said to be a **formal isomorphism** if f induces an isomorphism $\hat{\mathcal{O}}_{Y,y} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,x}$ of the completions. If there exists a formal isomorphism between germs $(x \in X)$ and $(y \in Y)$, then $(x \in X)$ and $(y \in Y)$ are said to be **formally isomorphic** and denoted by $(x \in X) \cong^{\mathsf{fm}} (y \in Y)$.

Remark 3.1.2. Let me quickly mention some relations between singularities in algebraic varieties and those in analytic spaces. First, we can define germs in an analytic space in the same way as above. Then, we have the following.

> **Theorem 3.1.3** (Hironaka, Rossi, Artin)**.** [Ish14, Theorem 4.2.3.] Let $(x \in X)$ and $(y \in Y)$ be germs of analytic spaces. Then, the following are equivalent:
>
> (i) $(x \in X) = (y \in Y)$.
>
> (ii) There exists an isomorphism $\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y}$ of $\mathbb{C}$-algebras.
>
> (iii) There exists an isomorphism $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,y}$ of $\mathbb{C}$-algebras.
>
> In other words, germs of a point in an analytic space are isomorphic if and only if they are formally isomorphic. ♢

Now, note that an algebraic variety X over $\mathbb{C}$ can be also viewed as an analytic space X^{an}.

Theorem 3.1.4 (Artin's Algebrization Theorem)**.** [Ish14, Theorem 4.2.4.] Let $(x \in X)$ be a germ of an isolated singularity in an analytic space. Then, there exists an algebraic variety Y over $\mathbb{C}$ and a point $y \in Y$ such that $(x \in X) = (y \in Y^{\mathsf{an}})$. $\diamondsuit$

Hence, as long as we consider an isolated singularity of an analytic space, we may view it as an isolated singularity in an algebraic variety. ■

The following corollary of [Eis95, Theorem 7.16.] shows how we can produce formally isomorphic singular points.

Lemma 3.1.5. *Let $h_1, \dots, h_n \in k[[x_1, \dots, x_n]]$ be non-unit formal power series whose linear terms are linearly independent, i.e. generate $\langle x_1, \dots, x_n \rangle$. Then,*

$$\phi : k[[x_1, \dots, x_n]] \to k[[x_1, \dots, x_n]], \quad f \mapsto f(h_1, \dots, h_n)$$

defines a k-homomorphism and furthermore ϕ is an k-isomorphism.

First, we have some mundane examples.

Example 3.1.6. Let $h_1, \dots, h_n \in \mathfrak{m} := \langle x_1, \dots, x_n \rangle_{k[x_1, \dots, x_n]}$ be polynomials whose linear terms are linearly independent. Then, for any $f \in \mathfrak{m}$, we have a k-isomorphism:

$$k[[x_1, \dots, x_n]]/\langle f \rangle \xrightarrow{\sim} k[[x_1, \dots, x_n]]/\langle f(h_1, \dots, h_n) \rangle,$$

i.e. a formal isomorphism

$$(0 \in \operatorname{Spec} k[x_1, \dots, x_n]/\langle f \rangle) \overset{\mathsf{fm}}{\xrightarrow{\sim}} (0 \in \operatorname{Spec} k[x_1, \dots, x_n]/\langle f(h_1, \dots, h_n) \rangle).$$

■

We also have a bit more interesting example where h_i are genuinely formal power series.

Example 3.1.7. [Har77, Example I.5.6.3.] Let $f(x, y) = y^2 - x^2(x + 1)$ and $g(x, y) = xy$. I claim that we have

$$(0 \in \operatorname{Spec} k[x, y]/\langle f \rangle) \cong^{\mathsf{fm}} (0 \in \operatorname{Spec} k[x, y]/\langle g \rangle).$$

First, notice $f(x, y) = (y + x\sqrt{1 + x})(y - x\sqrt{1 + x})$. Then, the formal Taylor expansion of $\sqrt{1 + x}$ and Lemma 3.1.5 suffice for a proof. We can also construct a formal isomorphism directly, which is essentially the same as above but also illustrates an idea behind Lemma 3.1.5. To construct an isomorphism, first notice that we have $f(x, y) = (y + x)(y - x) - x^3$ and let us write

$$\begin{aligned} h_1(x, y) &= y + x + h_{1,2} + h_{1,3} + \cdots \\ h_2(x, y) &= y - x + h_{2,2} + h_{2,3} + \cdots, \end{aligned}$$

where $h_{i,n}$ denotes the homogeneous part of h_i of degree n. Then, we have

$$g(h_1, h_2) = (y^2 - x^2) + ((y - x)h_{1,2} + (y + x)h_{2,2}) + ((y - x)h_{1,3} + h_{1,2}h_{2,2} + (y + x)h_{2,3}) + \cdots,$$

where each parenthesis indicates the homogeneous part. Now, comparing f and $g(h_1, h_2)$, we need to have

$$(y - x)h_{1,2} + (y + x)h_{2,2} = -x^3, \quad (y - x)h_{1,3} + h_{1,2}h_{2,2} + (y + x)h_{2,3} = 0, \quad \dots,$$

which can be inductively constructed because $y-x$ and $y+x$ generate $\langle x_1, \ldots, x_n \rangle$. Note that the reason why the induction works well in the preceding example is that in each homogeneous part there are only two undetermined homogeneous polynomials that furthermore appear as coefficients of the generators $(x+y)$ and $(x-y)$. This illustrates the importance of the supposition in Lemma 3.1.5. ■

In the following sections, we observe resolutions of some basic surface singularities.

3.2 Cones over hypersurfaces

First, let us recall the following definition.

Definition 3.2.1. Let $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$ be homogeneous polynomials and let $X_0 = \operatorname{Proj} k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$ be a projective variety. Then, the **(affine) cone** X over the projective variety X_0 is defined to be $X_0 = \operatorname{Spec} k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$.

Now, we introduce the following simplest cone singularities.

Definition 3.2.2. A germ of singularity $(x \in X)$ is said to be an **ordinary double point** if it is formally isomorphic to the germ $(0 \in \{x_1^2 + \cdots + x_n^2 = 0\} \subset \mathbb{A}^n)$.

The singularity $(0 \in Q)$ of the affine quadric cone Q (Example 1.4) at the origin is a surface ordinary double point. Indeed, an ordinary double point (in any dimension) can be resolved just by blowing up the singular point once as we saw in Lemma 1.4.5. Recalling the way got the affine chart description, we indeed have the following.

Lemma 3.2.3. *Let $f_1, \ldots, f_l \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial and suppose the projective variety $X_0 = \operatorname{Proj} k[x_0, \ldots, x_n]/\langle f_1, \ldots, f_l \rangle$ is non-singular. Then, the affine cone X over X_0 has the isolated singularity at the origin and the blow-up of X at the origin gives a resolution. Furthermore, the exceptional divisor is isomorphic to X_0.*

Example 3.2.4. Consider the cone $X = \operatorname{Spec} k[x, y, z]/\langle x^d + y^d + z^d \rangle$ and let $\pi : \tilde{X} \to X$ be the blow-up of X at the origin. Then, since the only thing that changes in the computation of $K_{\tilde{X}/X}$ is the multiplicity of the origin, we see that $K_{\tilde{X}/X} = 2 - d$. Hence, if $d = 3$, then X is log canonical. However, even if $d = 4$, which looks not too bad, we obtain a bad singularity with $\operatorname{discrep}(X) = -\infty$. ■

Now, by varying the dimension of the ambient space and the multiplicity of the origin in the proof of Lemma 1.4.5 (also cf. Lemma A.2.8), we obtain the following:

Lemma 3.2.5. [oM18, p.11] *Let $f \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d and suppose the hypersurface $X_0 = \operatorname{Proj} k[x_1, \ldots, x_n]/\langle f \rangle$ is non-singular. Let $\widetilde{X} \to X$ be the blow up of the affine cone X over X_0 at the origin with exceptional divisor E. Then, we have*

$$K_{\tilde{X}/X} = ((n-1) - d)E = (n - 1 - d)E$$

Remark 3.2.6. [Mil80, Example 1.5.(1)] Again we have a direct proof of Lemma 3.2.5 as in the proof of Lemma 1.4.5. First, take a canonical form

$$s = \operatorname{Res}_{\mathbb{A}^n|X} \frac{dx_1 \wedge \cdots \wedge dx_n}{f}.$$

Also, in the affine chart $\operatorname{Spec} k[x_1, x_2', \dots, x_n']$ of $\mathrm{Bl}_0 \mathbb{A}^n$ we can take a canonical form

$$s' = \operatorname{Res}_{\mathrm{Bl}_0 \mathbb{A}^n | \tilde{X}} \frac{dx_1 \wedge dx_2' \wedge \cdots \wedge dx_n'}{f_1'},$$

where $f'(x_1, x_2', \dots, x_n') = \frac{f(x_1, x_2' x_1, \dots, x_n' x_1)}{x_1^d}$. Hence, noting

$$\pi^*(dx_1 \wedge \cdots \wedge dx_n) = x_1^{n-1} dx_1 \wedge dx_2' \wedge \cdots \wedge dx_n'$$

we have a relation

$$\pi^* s|_{\operatorname{Spec} k[x_1, x_2', \dots, x_n']} = x_1^{n-1-d} s'$$

as desired. ■

3.3 Quotient singularities

We can also create a singularity from a variety with a group action by taking the quotient. Indeed, we can also view the ordinary double point as a quotient singularity. In this thesis, we will only consider quotients of $\mathbb{C}$-varieties by complex reductive groups, e.g. $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, and finite groups (cf. Appendix B.2).

Definition 3.3.1. Let $(x \in X)$ be a germ of the singularity in a (complex) variety X. Then, X is said to have a **quotient singularity** if there are a smooth germ $(0 \in Y)$ and a finite group G acting on some neighborhood $U \subset Y$ of 0 such that $(x \in X) \cong^{\mathsf{fm}} (0 \in U/G) =: (0 \in Y)/G$.

Example 3.3.2. [Kol08, Exercise 72] Let G be a finite group. Then, giving a finite-dimensional representation $\rho : G \to \mathrm{GL}_n$ is equivalent to defining a linear G-action on $\mathbb{A}^n$, which gives the quotient variety $\mathbb{A}^n/G$. Suppose that the G-action is effective and fixed point free outside of a codimension 2 set. Then, $\mathbb{A}^n/G$ is log terminal. Furthermore, if $G \subset \mathrm{SL}_n$, then the canonical class of $\mathbb{A}^n/G$ is Cartier and in particular $\mathbb{A}^n/G$ is canonical. ■

Construction 3.3.3. [Kol08, Exercise 72] Let $\mu_m = \langle g \rangle$ denote a cyclic group of order m. Then, any cyclic action ρ on $\mathbb{A}^n$ can be diagonalized and written as

$$\rho(g)(x_1, \dots, x_n) = (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_n} x_n),$$

where $\varepsilon = e^{2\pi i/m}$ and $0 \le a_i < m$ for $i = 1, \dots, n$. Define the **age** of g (with respect to ρ) as $\operatorname{age}(g) = \frac{1}{m}(a_1 + \cdots + a_n)$. The quotient by this action is often denoted by

$$\mathbb{A}^n / \frac{1}{m}(a_1, \dots, a_n).$$

A singularity of this type is called a **cyclic quotient singularity**. We have a nice characterization criterion for cyclic quotient singularities.

> **Lemma 3.3.4** (Reid-Tai Criterion)**.** A cyclic quotient singularity $\mathbb{A}^n/\mu_m$ is canonical (resp. terminal) iff the age of every non-identity element $g \in \mu_m$ is ≥ 1 (resp. > 1). ♢

■

Example 3.3.5. The quotient variety $\mathbb{A}^2/\frac{1}{2}(1,1)$ is isomorphic to the affine quadric cone Q since with respect to this μ_2-action,

$$k[x_1,x_2]^{\mu_2} = k[x_1^2, x_1y_1, y_1^2] \cong k[y_1,y_2,y_3]/\langle y_1y_3 - y_2^2\rangle.$$

Hence, $\mathbb{A}^2/\frac{1}{2}(1,1)$ has a canonical singularity at the origin and its age is indeed 1. ■

Example 3.3.6 (The cone over the Veronese surface)**.** (There are many interesting observations related to this example. See [oM18, p.12], [Mil87, Example 1.3.], [Mil87, Example 1.8.(ii)], [Mil80, Example 1.5.(ii)], and mathoverflow answers [Kovb] and [FL]).) Consider the quotient space $X = \mathbb{A}^3/\frac{1}{2}(1,1,1)$. Then, since the age is $3/2$, X should have a terminal singularity. Now, as above

$$\begin{aligned} k[x_1,x_2,x_3]^{\mu_2} &= k[x_1^2,x_2^2,x_3^2,x_1x_2,x_2x_3,x_3x_1] \\ &\cong k[y_0,y_1,y_2,y_3,y_4,y_5]/I, \end{aligned}$$

where $I = \langle y_0y_1 - y_3^2, y_1y_2 - y_4^2, y_2y_0 - y_5^2, y_0y_4 - y_5y_3, y_1y_5 - y_3y_4, y_2y_3 - y_4y_5\rangle$. In particular, X is isomorphic to the affine cone over the Veronese surface (i.e. the image of the embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ by the very ample line bundle $\mathcal{O}_{\mathbb{P}^2}(2)$). Let $p : \mathbb{A}^3 \to X = \operatorname{Spec} k[y_0,y_1,y_2,y_3,y_4,y_5]/I$ be the quotient.

First of all, $2K_X$ is a Cartier divisor, noting that the rational 3-form

$$\omega = \frac{(dy_0 \wedge dy_3 \wedge dy_5)^{\otimes 2}}{y_0^3}$$

on X gives a basis of $(\Omega_X^3)^{\otimes 2}$ since

$$p^*\omega = \frac{(2x_1^3 dx_1 \wedge dx_2 \wedge dx_3)^{\otimes 2}}{x_1^6} = 4(dx_1 \wedge dx_2 \wedge dx_3)^{\otimes 2},$$

which is a basis of $(\Omega^3_{\mathbb{A}^3})^{\otimes 2}$ and p is étale outside of the vertex. Now, consider the blow-up $\pi : \tilde{X} \to X = \operatorname{Spec} k[y_0,y_1,y_2,y_3,y_4,y_5]/I$ at the origin with exceptional divisor E. We can check $\tilde{X}$ is non-singular as usual. Now, we are going to see that the discrepancy $a(E,X)$ is $\frac{1}{2} > 0$ in two ways.

(Canonical forms): Take the standard affine chart U_0 of $\operatorname{Bl}_0 \mathbb{A}^6_{y_0,\dots,y_5}$ with $p_{y_0} \neq 0$, i.e. the chart given by the substitutions $y_0 = y_0, y_1 = y_1'y_0, \dots, y_5 = y_5'y_0$. Then, $\tilde{X} \cap U_0$ is given by

$$\{y_1' - {y_3'}^2 = y_1'y_2' - {y_4'}^2 = y_2' - {y_5'}^2 = y_4' - y_5'y_3' = y_5' - y_3'y_4' = y_2'y_3' - y_4'y_5' = 0\} \subset U_0.$$

Hence, we have an isomorphism $U_0 \cong \operatorname{Spec} k[y_0, y_3', y_5']$ given by

$$(y_0, y_3', y_5') \mapsto (y_0, {y_3'}^2, {y_5'}^2, y_3', y_3'y_5', y_5').$$

Here, note that the exceptional divisor is still given by $\{y_0 = 0\}$ in $\operatorname{Spec} k[y_0, y_3', y_5']$. Thus, since we have

$$\pi^*\omega|_{\operatorname{Spec} k[y_0,y_3',y_5']} = \frac{(y_0^2 dy_0 \wedge dy_3' \wedge dy_5')^{\otimes 2}}{y_0^3} = y_0(dy_0 \wedge dy_3' \wedge dy_5')^{\otimes 2},$$

we see the discrepancy is $\frac{1}{2}$.

(Adjunction formula): First, write $K_{\tilde{X}} = \pi^* K_X + dE$. Then, we have $K_{\tilde{X}} + E = \pi^* K_X + (d+1)E$ and hence by the adjunction formula and the fact that $\pi^* K_X|_E = 0$, we have $K_E = (d+1)E|_E$. Now, since E is isomorphic to the Veronese surface by Lemma 3.2.3, we have $\mathcal{O}_E(K_E) \cong \mathcal{K}_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. Furthermore, since $-E|_E$ is the restriction of the hyperplane section of the exceptional divisor $\mathbb{P}^5$ to E, it is a conic on E, i.e. $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$. Hence, we have $-3 = -2(d+1)$, i.e. $d = \frac{1}{2}$.

■

3.4 Du Val singularities

Theorem 3.4.1. [KM98, Theorem 4.20, 4.22.] *Let $(0 \in X)$ be a germ of a normal surface singularity. We say $(0 \in X)$ is a* ***Du Val singularity*** *if one (and hence all) of the following equivalent conditions:*

(i) *$(0 \in X)$ is a canonical singularity.*

(ii) *For the minimal resolution $f : Y \to X$ (cf. Lemma 2.2.7), we have $(K_Y \cdot E_i) = 0$ for every f-exceptional curve $E_i \subset Y$.*

(iii) *$(0 \in X)$ is formally isomorphic to a singularity defined by one of the following equations:*

   A. The singularity A_n $(n \geq 1)$ has equation $x^2 + y^2 + z^{n+1}$.

   D. The singularity D_n $(n \geq 4)$ has equation $x^2 + y^2 z + z^{n-1} = 0$.

   E. The singularity E_6 (resp. E_7, resp. E_8) has equation $x^2 + y^3 + z^4 = 0$, (resp. $z^2 + y^3 + yz^3 = 0$, resp. $x^2 + y^3 + z^5 = 0$).

Now, the following notion enables us to describe a resolution of a singularity:

Definition 3.4.2. Let $C = \bigcup_i C_i$ be a collection of proper curves on a smooth surface U. The **dual graph** Γ of C is defined as follows:

(i) The vertices of Γ are the curves C_i.

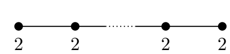
(ii) The vertex C_i is labeled by $b_i = -(C_i \cdot C_i)$.

(iii) The vertices C_i and C_j are connected with $(C_i \cdot C_j)$ edges.

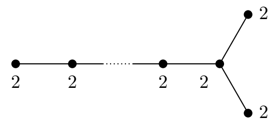
In particular, we can define the dual graph of exceptional curves for a resolution of surface singularity.

Lemma 3.4.3. [Kol08, Exercise 66.] *The minimal resolutions of Du Val singularities have the following dual graphs:*

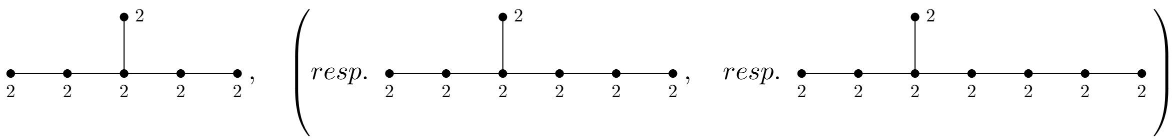
A. The singularity A_n $(n \geq 1)$ has equation $x^2 + y^2 + z^{n+1}$ and dual graph with n vertices:



D. *The singularity D_n $(n \geq 4)$ has equation $x^2 + y^2 z + z^{n-1} = 0$ and dual graph with n vertices:*



E. *The singularity E_6 (resp. E_7, resp. E_8) has equation $x^2+y^3+z^4 = 0$, (resp. $z^2+y^3+yz^3 = 0$, resp. $x^2 + y^3 + z^5 = 0$) and dual graph with 6 (resp. 7, resp. 8) vertices:*



Proof. First, note a resolution represented by one of the dual graphs above is indeed the minimal resolution by Example 2.2.8. Also, taking partial derivatives, it is clear that the affine variety defined by one of the equations above has the only singularity at the origin. In the following, I will only show some cases, but these contain all the necessary ideas to do other resolutions that I will not write down.

A_1: This is the affine quadric cone. Hence, by Lemma 1.4.5, the dual graph is

As I remarked, I will omit some details already computed in the proof of Lemma 1.4.5.

A_2 Let $X = k[x, y, z]/\langle x^2 + y^2 + z^3\rangle$ and let $\pi : \tilde{X} \to X$ be the blow-up at the origin. Consider a standard affine chart $\operatorname{Spec} k[x, y', z']$ given by $x = x, y = y'x, z = z'x$. Then, the birational transform of X is given by

$$\{1 + {y'}^2 + x{z'}^3 = 0\},$$

which is non-singular, and the exceptional divisor is given by two lines

$$\{1 + {y'}^2 + x{z'}^3 = 0\} \cap \{x = 0\} \cong \{y' = \sqrt{-1}, x = 0\} \sqcup \{y' = -\sqrt{-1}, x = 0\}.$$

By symmetry, in the affine chart $k[x', y, z']$, the birational transform is given by

$$\{{x'}^2 + 1 + y{z'}^3 = 0\},$$

which is non-singular, and the exceptional divisor is given by two lines

$$\{{x'}^2 + 1 + y{z'}^3 = 0\} \cap \{y = 0\} \cong \{x' = \sqrt{-1}, y = 0\} \sqcup \{x' = -\sqrt{-1}, y = 0\}.$$

Finally, in the affine chart $k[x', y', z]$, the birational transform is given by

$$\{{x'}^2 + {y'}^2 + z = 0\},$$

which is non-singular, and the exceptional divisor is given by two lines intersecting at the origin:

$$\{x'^2 + y'^2 + z = 0\} \cap \{z = 0\} \cong \{(x' + \sqrt{-1}y')(x' - \sqrt{-1}y') = 0, z = 0\}.$$

Hence, noting that globally in $\mathrm{Bl}_0 \mathbb{A}^3$, the two exceptional lines are given by

$$E_\pm : \{p_x \pm \sqrt{-1}p_y = 0, x = y = z = 0\} \subset \mathrm{Bl}_0 \mathbb{A}^3,$$

the birational transform of X has one A_{n-2} singularity and the two exceptional lines intersect at the singularity transversely. Thus, we obtain the following dual graph for this resolution:

$$\bullet\!\!-\!\!\bullet$$

Now, it remains to show $(E_\pm \cdot E_\pm) = -2$. As in Lemma 1.4.5, consider a principal divisor $(z \circ \pi)$, where z is a rational function on X and hence $((z \circ \pi) \cdot E_\pm) = 0$. Then, in the affine chart $U_x = k[x, y', z']$, the rational function $z \circ \pi = z'x$ vanishes (with multiplicity 1) on $\{x = 0\} = E \cap U_x$ and

$$\{z' = 0\} \cap U_x = \{y' = \sqrt{-1}, z' = 0\} \cup \{y' = -\sqrt{-1}, z' = 0\} = (C_+ \cap U_x) \cup (C_- \cap U_x),$$

where

$$C_\pm = \{p_y = \pm\sqrt{-1}p_x, y = \pm\sqrt{-1}x, p_z = z = 0\} \subset \mathrm{Bl}_0 \mathbb{A}^3.$$

Similarly, in $k[x', y, z']$, we see that $z \circ \pi$ vanishes on E and $C_\pm$ with multiplicity 1. Since the union of these charts only miss a point in $\tilde{X}$, which is of codimension 2, we conclude

$$(z \circ \pi) = E + C_+ + C_-.$$

Hence,

$$\begin{aligned} 0 &= ((z \circ \pi) \cdot E) \\ &= (E \cdot E) + (C_+ \cdot E) + (C_- \cdot E) \\ &= ((E_+ + E_-) \cdot (E_+ + E_-)) + 2 \\ &= (E_+ \cdot E_-) + 2(E_+ \cdot E_-) + (E_- \cdot E_-) + 2 \\ &= (E_+ \cdot E_-) + (E_+ \cdot E_-) + 4. \end{aligned}$$

Since we have $(E_+ \cdot E_+) = (E_- \cdot E_-)$ by symmetry, we are done and obtain the A_2 diagram:

$$\underset{2}{\bullet}\!\!-\!\!\underset{2}{\bullet}$$

Note that $C_+ \cup C_-$ is the birational transform of $\{z = 0\} \cap X$; in general, to compute (z) it suffices to count the multiplicity of the strict transform of $\{z = 0\} \cap X$ and exceptional curves as these are all the possibilities where z vanishes.

A_n: Let $X = k[x, y, z]/\langle x^2 + y^2 + z^{n+1}\rangle$ with $n > 2$ and let $\pi : \tilde{X} \to X$ be the blow-up at the origin. For the affine charts $k[x', y, z']$ and $k[x, y', z']$, we can do the exactly the same computations as in A_2. In the affine chart $k[x', y', z]$, the birational transform is given by

$$\{x'^2 + y'^2 + z^{n-1} = 0\},$$

which has an A_{n-2} singularity at the origin, and the exceptional divisor is given by two lines intersecting at the origin:

$$\{x'^2 + y'^2 + z^{n-1} = 0\} \cap \{z = 0\} \cong \{(x' + \sqrt{-1}y')(x' - \sqrt{-1}y') = 0, z = 0\}.$$

Hence, noting that globally in $\mathrm{Bl}_0 \mathbb{A}^3$, the exceptional lines glue together and are given by

$$E_\pm : \{p_x \pm \sqrt{-1}p_y = 0, x = y = z = 0\} \subset \mathrm{Bl}_0 \mathbb{A}^3,$$

the birational transform of X has one A_{n-2} singularity and the two exceptional lines intersect at the singularity. Now, to complete the induction steps, it remains to observe what happens to the intersection theory of new and old exceptional divisors after the blow-up at the A_{n-2} singularity. We consider the blow-up as the blow-up $\pi' : \tilde{X}' \to X'$ of $X' := \operatorname{Spec} k[x', y', z]/\langle x'^2 + y'^2 + z^{n-1}\rangle$ at the origin. Setting $u = x', v = y', w = z$, the defining equation becomes

$$\{u^2 + v^2 + w^{n-1} = 0\}$$

and the old exceptional divisor E becomes

$$\{u^2 + v^2 = 0, w = 0\} = \{u = \sqrt{-1}v, w = 0\} \cup \{u = -\sqrt{-1}v, w = 0\}.$$

Now, in exactly the same way, the exceptional divisor of the blow up at the origin is given by the two lines

$$E'_\pm : \{p_u = \pm\sqrt{-1}p_v, u = v = w = 0\} \subset \mathrm{Bl}_0 \mathbb{A}^3,$$

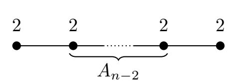
which intersect at $([p_u : p_v : p_w], u, v, w) = ([0 : 0 : 1], (0, 0, 0))$ (which is an A_{n-4} singularity if $n > 4$ and a smooth point otherwise). Also, the birational transform of the old exceptional divisor E is given by the two lines

$$\tilde{E}_\pm : \{p_u = \pm\sqrt{-1}p_v, u = \pm\sqrt{-1}v, p_w = w = 0\} \subset \mathrm{Bl}_0 \mathbb{A}^3,$$

which are disjoint, i.e. $(\tilde{E}_+ \cdot \tilde{E}_-) = 0$. Furthermore, since $\tilde{E}_\pm$ and $E'_\mp$ are disjoint respectively, we have $(\tilde{E}_\pm \cdot E'_\mp) = 0$, and since $\tilde{E}_\pm$ and $E'_\pm$ intersect at $([p_u : p_v : p_w], u, v, w) = ([\pm\sqrt{-1} : 1 : 0], (0, 0, 0))$ transversely, we have $(\tilde{E}_\pm \cdot E'_\pm) = 1$. Hence, it remains to show $(\tilde{E}_\pm \cdot \tilde{E}_\pm) = (E'_\pm \cdot E'_\pm) = -2$. Now, comparing the equations of $\tilde{E}_\pm$ with the equations of $C_\pm$ in the A_2 case, we see that for a rational function z on X, we have

$$(z \circ \pi') = (w) = E'_+ + E'_- + \tilde{E}_+ + \tilde{E}_-.$$

Then, considering the intersection of $(x \circ \pi')$ with each of the exceptional lines as before, we obtain $(\tilde{E}_\pm \cdot \tilde{E}_\pm) = (E'_\pm \cdot E'_\pm) = -2$ as desired. In summary, we have completed the following induction step:



where the outer vertices correspond to $\tilde{E}_\pm$ and the inner vertices correspond to $E'_\pm$.

D_4: Let $X = k[x, y, z]/\langle x^2 + zy^2 + z^3\rangle$ and let $f = x$ be a rational function on X with the vanishing locus $C = \{x = 0\} \cap X = \{z(z + \sqrt{-1}y)(z - \sqrt{-1}y) = 0\}$, which we will keep track of to compute the self-intersections of exceptional divisors.

Step 1 Let $\pi_0 : \tilde{X} \to X$ be the blow-up at the origin with exceptional divisor E_0.

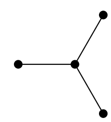
(a). In the standard affine chart $\operatorname{Spec} k[x, y', z']$, the birational transform of X is given by
$$\{1 + xz'({y'}^2 + {z'}^2) = 0\},$$
which is non-singular, and does not intersect the exceptional divisor as
$$\{1 + xz'({y'}^2 + {z'}^2) = 0, x = 0\} = \emptyset.$$
We are not interested in this chart and in the sequel I will omit non-singular charts that do note have useful information about exceptional divisors.

(b). In the affine chart $\operatorname{Spec} k[x', y, z']$, the birational transform is given by
$$\{{x'}^2 + yz'(1 + {z'}^2) = 0\},$$
which has an A_1 singularity at the origin as we can ignore the factor $(1+{z'}^2)$ around the origin (or more rigorously cf. Example 3.1.7), and the exceptional divisor E_0 is given by the z'-axis
$$\{{x'}^2 + yz'(1 + {z'}^2) = 0\} \cap \{y = 0\} = \{x' = y = 0\},$$
which in particular contains the A_1 singularity. Here, the strict transform of C is given by
$$\{z'(z' + \sqrt{-1})(z' - \sqrt{-1}) = 0\}$$
and $f = x'y$.

(c). Finally, in the affine chart $\operatorname{Spec} k[x', y', z]$, the birational transform is given by
$$\{{x'}^2 + z({y'}^2 + 1) = 0\},$$
which has A_1 singularities at $(0, \pm\sqrt{-1}, 0)$, and the exceptional divisor E_0 is given by the y'-axis
$$\{{x'}^2 + z({y'}^2 + 1) = 0\} \cap \{z = 0\} = \{x' = 0, z = 0\},$$
which in particular contain both of the A_1 singularities. Here, the strict transform of C is given by
$$\{x = 0\} \cap X = \{(1 + \sqrt{-1}y')(1 - \sqrt{-1}y') = 0\}$$
and $f = x'z$.

Step 2: Now, note that the three A_1 singularities in Step 1 are all distinct since in $\operatorname{Bl}_0 \mathbb{A}^3$ they are
$$([p_x : p_y : p_z], (x, y, z)) = ([0 : 1 : 0], (0, 0, 0)), ([0 : \pm\sqrt{-1} : 1], (0, 0, 0))$$
and they all lie on the exceptional divisor E_0 as we observed. Also, note that the strict transform C_0 of C intersect with E_0 precisely at these point. Hence, considering the blow-ups π_i at these A_1 singularities with exceptional divisors E_i ($i = 1, 2, 3$) (say π_1 for

$([0:1:0],(0,0,0))$, π_2 for $([0:\sqrt{-1}:1],(0,0,0))$, and π_3 for $([0:-\sqrt{-1}:1],(0,0,0)))$, we obtain the following dual graph:



where the central vertex corresponds to the initial exceptional divisor E_0 and the other three correspond to E_1, E_2, E_3. Noting the symmetry given by $x^2+zy^2+z^3 = x^2+z(z+\sqrt{-1}y)(z-\sqrt{-1}y)$, to see the vanishing orders of f on E_i, it suffices to observe one of the blow-ups, say π_1. Write $k[x',y,z'] = k[x_1,y_1,z_1]$ in Step 1 (b). Then, we have:

$$\begin{aligned} \text{the strict transform } X_1\text{:} \quad & \{x_1^2 + y_1 z_1(1+{z_1}^2) = 0\} \\ \text{the exceptional divisor } E_0\text{:} \quad & \{x_1 = 0, y_1 = 0\} \\ \text{the rational function:} \quad & f = x_1 y_1. \end{aligned}$$

Then, $\pi_1 : \tilde{X}_1 \to X_1$ is the blow-up of X_1 at the origin with exceptional divisor E_1.

(a). In the affine chart $k[x_1, y_1', z_1']$, the strict transform is given by

$$1 + y_1' z_1'(1 + z_1' {x_1}^2) = 0$$

and the exceptional divisor E_1 is given by

$$\{1 + y_1' z_1' = 0, x_1 = 0\}.$$

Note this chart does not intersect with E_0 since $E_0 = \{x_1 = 0, x_1 y_1' = 0\}$ implies $E_0 = E_1$ in this chart, which is absurd. Here, $f = x_1^2 y_1'$.

(b). In the affine chart $k[x_1', y_1, z_1']$, the strict transform is given by

$${x_1'}^2 + z_1'(1 + (z_1' y_1)^2) = 0$$

and the exceptional divisor E_1 is given by

$$\{{x_1'}^2 + z_1' = 0, y_1 = 0\}.$$

This chart does not intersect with E_0 as above. Here, $f = x_1' y_1^2$.

(c). In the affine chart $k[x_1', y_1', z_1]$, the strict transform is given by

$$\{{x_1'}^2 + y_1'(1 + z_1^2) = 0\}$$

and the exceptional divisor E_1 is given by

$$\{{x_1'}^2 + y_1' = 0, z_1 = 0\}.$$

In this chart, the exceptional curve E_0 is given by

$$\{x_1' = 0, y_1' = 0\}$$

and $f = x_1' y_1' z_1^2$.

Step 3: Now, by Step 2 (c), $f = x_1'y_1'z_1^2(\sim {x_1'}^3z_1^2 \in (k[x_1', y_1', z_1]/\langle {x_1'}^2 + y_1'(1+z_1^2)\rangle)_0)$ vanishes with multiplicity 3 on $E_0 = \{x_1' = y_1' = 0\}$ and with multiplicity 2 on $E_1 = \{{x_1'}^2 + y_1' = 0, z_1 = 0\}$. Since f also vanishes with multiplicity 1 on the strict transform $\tilde{C}$ of the vanishing locus $C \subset X$ of f in X, we have

$$(f') = 3\tilde{E}_0 + 2E_1 + 2E_2 + 2E_3 + \tilde{C},$$

where $\tilde{E}_0$ is the strict transform of E_0 under the three blow-ups and we have $(\tilde{E}_0 \cdot \tilde{C}) = 0$ and $(E_i \cdot \tilde{C}) = 1$ for $i = 1, 2, 3$ since the intersections of E_0 and the strict transform C_0 of C under π are precisely blown up by π_i. Therefore, since $((f) \cdot E_i) = 0$ for every i, we are done.

E_6: [Bur, §5.] Let $X = \operatorname{Spec} k[x, y, z]/\langle x^2 + y^3 + z^4\rangle$ and let $f = x$ be a rational function on X.

Step 1: Let $\pi_0 : \tilde{X}_0 \to X$ be the blow-up at the origin with exceptional divisor E_0. In the affine chart $\operatorname{Spec} k[x, y', z']$, the strict transform is given by

$$\{1 + x{y'}^3 + x^2{z'}^4 = 0\},$$

which is non-singular. In the affine chart $\operatorname{Spec} k[x', y, z']$, the strict transform is given by

$$\{{x'}^2 + y + y^2{z'}^4 = 0\},$$

which is non-singular. In the affine chart, $\operatorname{Spec} k[x', y', z]$, the strict transform is given by

$$\{{x'}^2 + z{y'}^3 + z^2 = 0\},$$

which has the only singularity at the origin, and the exceptional divisor is given by the y'-axis

$$\{x' = 0, z = 0\}.$$

In this chart, $f = x'z$.

Step 2: Write $k[x', y', z] = k[x_1, y_1, z_1]$. Then, we have:

$$\begin{aligned} \text{the strict transform } X_1\text{:} &\quad \{x_1^2 + z_1y_1^3 + z_1^2 = 0\} \\ \text{the exceptional divisor } E_0\text{:} &\quad \{x_1 = 0, z_1 = 0\} \\ \text{the rational function:} &\quad f = x_1z_1. \end{aligned}$$

Let $\pi_1 : \tilde{X}_1 \to X_1$ be the blow-up at the origin with exceptional divisor E_1. Then, the only affine chart with a singularity is $\operatorname{Spec} k[x_1', y_1, z_1']$, where the strict transform is given by

$$\{{x_1'}^2 + y_1^2z_1' + {z_1'}^2 = 0\},$$

which has the only singularity at the origin, and the exceptional divisor E_1 is given by

$$\{{x_1'}^2 + {z_1'}^2 = 0, y_1 = 0\} = \{x_1' = \sqrt{-1}z_1', y_1 = 0\} \cup \{x_1' = -\sqrt{-1}z_1', y_1 = 0\}.$$

In this chart, E_0 is given by

$$\{x_1' = 0, z_1' = 0\}$$

and $f = x_1'y_1^2z_1'$.

Step 3: Write $k[x_1', y_1, z_1'] = k[x_2, y_2, z_2]$. Then, we have:

$$\begin{aligned} \text{the strict transform } X_2&: \quad \{x_2^2 + y_2^2 z_2 + z_2^2 = 0\} \\ \text{the exceptional divisor } E_0&: \quad \{x_2 = 0, z_2 = 0\} \\ \text{the exceptional divisor } E_1&: \quad \{x_2^2 + z_2^2 = 0, y_2 = 0\} \\ \text{the rational function}&: \quad f = x_2 y_2^2 z_2. \end{aligned}$$

Let $\pi_2 : \tilde{X}_2 \to X_2$ be the blow-up at the origin with exceptional divisor E_2.

(a). In the affine chart $\operatorname{Spec} k[x_2', y_2, z_2']$, the strict transform is given by

$$\{{x_2'}^2 + y_2 z_2' + {z_2'}^2 = 0\},$$

which has an A_1 singularity at the origin, and the exceptional divisor E_2 is given by

$$\{{x_2'}^2 + {z_2'}^2 = 0, y_2 = 0\} = \{z_2' = \sqrt{-1}x_2', y_2 = 0\} \cup \{z_2' = -\sqrt{-1}x_2', y_2 = 0\} = E_{2,+} \cup E_{2,-}.$$

In this chart, E_0 is given by

$$\{x_2' = 0, z_2' = 0\}$$

and $f = x_2' y^4 z_2'$. Note E_1 does not intersect this affine chart since $y_2 = 0$ implies $E_1 = E_2$ in this chart, which is absurd. To get information of E_1, consider another chart.

(b). In the affine chart $\operatorname{Spec} k[x_2, y_2', z_2']$, the strict transform is given by

$$\{1 + x_2 {y_2'}^2 z_2' + {z_2'}^2 = 0\},$$

which is non-singular, and the exceptional divisor E_2 is given by

$$\{x_2 = 0, 1 + {z_2'}^2 = 0\} = \{x_2 = 0, z_2' = \sqrt{-1}\} \cup \{x_2 = 0, z_2' = -\sqrt{-1}\} = E_{2,+} \cup E_{2,-}.$$

In this affine chart, E_1 is given by

$$\{1 + z_2'^2 = 0, y_2' = 0\} = \{y_2' = 0, z_2' = \sqrt{-1}\} \cup \{y_2' = 0, z_2' = -\sqrt{-1}\} = E_{1,+} \cup E_{1,-}.$$

Hence, all the intersections are transversal. Also, we have $f = x_2^4 {y_2'}^2 z_2'$.

Step 4: Write $k[x_2', y_2, z_2'] = k[x_3, y_3, z_3]$. Then, we have:

$$\begin{aligned} \text{the strict transform } X_3&: \quad \{x_3^2 + y_3 z_3 + z_3^2 = 0\} \\ \text{the exceptional divisor } E_0&: \quad \{x_3 = 0, z_3 = 0\} \\ \text{the exceptional divisor } E_2&: \quad \{{x_3}^2 + {z_3}^2 = 0, y_3 = 0\} \\ \text{the rational function}&: \quad f = x_3^2 y_3^4 z_3. \end{aligned}$$

Let $\pi_3 : \tilde{X}_3 \to X_3$ be the blow-up at the origin.

(a). In the affine chart $\operatorname{Spec} k[x_3, y_3', z_3']$, the strict transform is given by

$$\{1 + y_3' z_3' + {z_3'}^2 = 0\}$$

and the exceptional divisor E_3 is given by

$$\{1 + y_3' z_3' + {z_3'}^2 = 0, x_3 = 0\}.$$

In this chart, E_2 is given by

$$\{1 + {z_3'}^2 = 0, y_3' = 0\} = \{y_3' = 0, z_3' = \sqrt{-1}\} \cup \{y_3' = 0, z_3' = -\sqrt{-1}\} = E_{2,+} \sqcup E_{2,-}.$$

Hence, E_3, $E_{2,\pm}$ intersect transversely. Also, $f = x_3^6 {y_3}^4 z_3$.

(b). In the affine chart $\operatorname{Spec} k[x_3', y_3, z_3']$, the strict transform is given by

$$\{{x_3'}^2 + z_3' + {z_3'}^2 = 0\}$$

and the exceptional divisor E_3 is given by

$$\{{x_3'}^2 + z_3' + {z_3'}^2 = 0, y_3 = 0\}.$$

In this chart, E_0 is given by

$$\{x_3' = 0, z_3' = 0\}.$$

Hence, E_3 and E_0 intersect transversely. Also, $f = x_3' y_3^6 z_3'$.

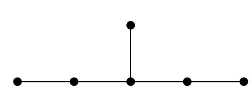
Step 5: Let $\pi : \tilde{X} \to X$ be the resolution of a singularity we constructed above. By abuse of notations, Step 3 (b) and Step 4 give the following charts of $\tilde{X}$:

the strict transform $X_{3,b}$:	$\{1 + xy^2 z + z^2 = 0\}$
the exceptional divisor E_2:	$\{x = 0, z = \sqrt{-1}\} \cup \{x = 0, z = -\sqrt{-1}\} = E_{2,+} \cup E_{2,-}$
the exceptional divisor E_1:	$\{y = 0, z = \sqrt{-1}\} \cup \{y = 0, z = -\sqrt{-1}\} = E_{1,+} \cup E_{1,-}$
the rational function:	$f = x^4 y^2 z$.

the strict transform $X_{4,a}$:	$\{1 + yz + z^2 = 0\}$
the exceptional divisor E_3:	$\{1 + yz + z^2 = 0, x = 0\}$
the exceptional divisor E_2:	$\{y = 0, z = \sqrt{-1}\} \cup \{y = 0, z = -\sqrt{-1}\} = E_{2,+} \cup E_{2,-}$.
the rational function:	$f = x^6 y^4 z$.

the strict transform $X_{4,b}$:	$\{x^2 + z + z^2 = 0\}$
the exceptional divisor E_3:	$\{x^2 + z + z^2 = 0, y = 0\}$
the exceptional divisor E_0:	$\{x = 0, z = 0\}$.
the rational function:	$f = xy^6 z$.

Hence, although we have skipped some proofs of disjointness of exceptional divisors (which can be done just by checking in all the charts as in the A_n case), we obtain the following dual graph:



where the central vertex corresponds to E_3, the upper vertex to E_0, the outer vertices to $E_{1,\pm}$, and the rest to $E_{2,\pm}$. Now, it suffices to compute the self-intersections of these exceptional curves. Hence, we compute the principal divisor $(f\circ\pi)$ on $\tilde{X}$. First, in the chart $X_{3,b}$, $f\circ\pi = x^4y^2z$ and hence $f\circ\pi$ vanishes with multiplicity 4 on $\{x=0\}=E_2$ and with multiplicity 2 on $\{y=0\}=E_1$. Note $z=0$ cannot happen. Next, in the chart $X_{4,a}$, $f\circ\pi = x^6y^4z$ vanishes with multiplicity 6 on $\{x=0\}=E_3$ and with multiplicity 4 on $\{y=0\}=E_2$. Note $z=0$ cannot happen. Finally, in the chart $X_{4,b}$, $f\circ\pi = xy^6z(\sim x^3y^6 \in (k[x,y,z]/x^2+z+z^2)_0)$ and hence $f\circ\pi$ vanishes with multiplicity 3 on $\{x=z=0\}=E_0$, with multiplicity 6 on $\{y=0\}=E_3$, with multiplicity 1 on $\{x=0,z=-1\}=C$, where C is the strict transform of the vanishing locus $\{z=0\}\subset X$ of f in X. In summary, we have

$$(f\circ\pi) = 3E_0 + 2(E_{1,+}+E_{1,-}) + 4(E_{2,+}+E_{2,-}) + 6E_3 + C.$$

Hence, noting $((f\circ\pi)\cdot E_\bullet)=0$ for all $\bullet$, $(C\cdot E_3)=1$, and $(C\cdot E_i)=0$ for all $\bullet\neq 3$, we can compute all the self-intersections and get $(E_\bullet\cdot E_\bullet)=-2$ for all $\bullet$ as desired.

□

Remark 3.4.4. As we might expect from the case of A_1 (Example 3.3.5), Du Val singularities can be also expressed as quotient singularities. Indeed, we have the following classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to conjugation, each of which corresponds to one of Du Val singularities [Bur, §2,§3]:

(i) Cyclic groups μ_n correspond to A_{n-1}.

(ii) Binary dihedral groups $\mathbb{D}_n$ ($|\mathbb{D}_n|=4n$) correspond to D_{n+2}.

(iii) The binary tetrahedral group $\mathbb{T}$ ($|\mathbb{T}|=24$) corresponds to E_6.

(iv) The binary octahedral group $\mathbb{O}$ ($|\mathbb{O}|=48$) corresponds to E_7.

(v) The binary icosahedral subgroup $\mathbb{I}$ ($|I|=120$) corresponds to E_8.

One fascinating fact about this correspondence is that we can recover the dual graph of the minimal resolution as the McKay graph of the corresponding group, which can be computed in terms of representation theory (cf. [Bur, §6]). Also, we have the following classification of surface singularity that extends the preceding correspondence:

Lemma 3.4.5. [Kol95, Theorem 3.6.] Let $(0\in X)$ be a germ of a normal surface singularity over $\mathbb{C}$. Then, the germ $(0\in X)$ is:

$$\begin{aligned}
\text{terminal} &\iff \text{smooth;}\\
\text{canonical} &\iff \mathbb{C}^2/(\text{finite subgroup of } \mathrm{SL}_2(\mathbb{C}));\\
\text{log terminal} &\iff \mathbb{C}^2/(\text{finite subgroup of } \mathrm{GL}_2(\mathbb{C}));\\
\text{log canonical} &\iff \text{simple elliptic, cusp, smooth, or a quotient of these by a finite group.}
\end{aligned}$$

◊

■

Chapter 4

Resolution of cDV singularity

In this chapter, we set $k = \mathbb{C}$ unless otherwise specified.

4.1 Weighted projective spaces and weighted blow-ups

First, I introduce a useful tool for resolving cDV singularities and singularities in general.

Definition 4.1.1. Let $(a_0, \dots, a_n) \in \mathbb{Z}_{>0}^{n+1}$. The **weighted projective space** $\mathbb{P}(a_0, \dots, a_n)$ **with weight** $(a_0, \dots, a_n)$ is the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/k^*$ of $\mathbb{A}^{n+1}$, where $\xi \in k^*$ acts by

$$\xi \cdot (x_0, \dots, x_n) = (\xi^{a_0} x_0, \dots, \xi^{a_n} x_n).$$

We can make this construction scheme-theoretic by writing $\mathbb{A}_k^{n+1} \setminus \{0\} = \bigcup_{i=0}^n \operatorname{Spec} k[x_0, \dots, x_n]_{x_i}$ and gluing the affine quotients $\operatorname{Spec} k[x_0, \dots, x_n]_{x_i}/\mathbb{G}_m$ (cf. Definition B.2.7). We also have

$$\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} k[x_0, \dots, x_n],$$

where we assign $\deg(x_i) = a_i$ for $i = 0, \dots, n$.

Example 4.1.2. If $(a_0, \dots, a_n) = (1, \dots, 1)$, then we clearly have the usual $\mathbb{P}(a_0, \dots, a_n) = \mathbb{P}^n$. ■

Lemma 4.1.3. [Hos20, Lemma 2.2.3.] *Let $(a_0, \dots, a_n) \in \mathbb{Z}_{>0}^{n+1}$ and let $\mu_{a_i} = \langle g \rangle$ denote the cyclic group of order a_i. An action ρ of μ_{a_i} on $\mathbb{A}^n$ is said to be **of type** $\frac{1}{a_i}(a_0, \dots, \hat{a}_i, \dots, a_n)$ if*

$$\rho(g)(x_0, \dots \hat{x}_i, \dots, x_n) = (g^{a_0} x_0, \dots, \widehat{g^{a_i} x_i} \dots, g^{a_n} x_n).$$

Now, if we write $U_i = \{[x_0 : \dots : x_n] \in \mathbb{P}(a_0, \dots, a_n) \mid x_i \neq 0\}$, then we have

$$U_i \cong \mathbb{A}^n/\mu_{a_i},$$

where the action of μ_{a_i} on $\mathbb{A}^n$ is of type $\frac{1}{a_i}(a_0, \dots, \hat{a}_i, \dots, a_n)$.

Definition 4.1.4. Fix $w_1, \dots, w_n \in \mathbb{Z}_{>0}$. Let $R := k[x_1, \dots, x_n]$. For a monomial in R, set $w(\prod x_i^{m_i}) = \sum m_i w_i$. More generally, for $f \in R$ set $w(f) = \min_{a_M \neq 0} w(M)$ where we write $f = \sum_M a_M M$ as the sum of monomials M. (Note that $w(0) = \infty$.) We obtain ideals $m^w(n) =$

$\{f \in R | w(f) \geq n\}$. The **weighted blow-up of $\mathbb{A}_k^n$ with weights** w_i (or the w**-blow-up of** $\mathbb{A}_k^n$) is defined as

$$\mathrm{Bl}_0^w \mathbb{A}_k := \mathrm{Proj}_R \oplus_{n\geq 0} m^w(n).$$

For any $X \subset \mathbb{A}^n$ this defines $\mathrm{Bl}_0^w X$ as the birational transform of X in $\mathrm{Bl}_0^w \mathbb{A}_k^n$, i.e. the closure of $X \setminus \{0\}$ in $\mathrm{Bl}_0^w \mathbb{A}_k^n$.

Remark 4.1.5. Note that this is a generalization of the blow-up of $\mathbb{A}_k^n$ at the origin. Recall that the Proj construction of the blow up at the origin is given as follows. Let $R = k[x_1, \ldots, x_n]$ and $I = \langle x_1, \ldots, x_n \rangle$. Then, the blow-up of $\mathbb{A}_k^n = \mathrm{Spec}\, R$ at $\{0\} = I$ is given by

$$\mathrm{Bl}_0 \mathbb{A}_k^n = \mathrm{Proj}_R \oplus_{n\geq 0} I^n,$$

where we set $I^0 = R$. Indeed, if $w = (1, \ldots, 1)$, then we have $w(f) = \min_{a_M \neq 0} \deg(M)$ and $m^w(n) = I^n$. Now, note we have a natural surjective homomorphism

$$R[T_1, \ldots, T_n] \twoheadrightarrow \oplus_{n\geq 0} I^n, \quad T_i \mapsto x_i$$

of graded rings with kernel given by the homogeneous ideal $I := \langle x_j T_i - x_i T_j \rangle_{i,j}$. Hence, we have

$$\mathrm{Bl}_0 \mathbb{A}^n = \mathrm{Proj}\, R[T_1, \ldots, T_n]/\langle x_j T_i - x_i T_j \rangle_{i,j} \subset \mathbb{P}_R^{n-1} = \mathbb{P}_k^{n-1} \times_k \mathbb{A}_k^n,$$

which recovers a classical definition by using coordinates. ■

Remark 4.1.6. Note that although the usual blow-up of $R \cong k[X_1, \ldots, X_n]$ along an ideal does not depend on the choice of isomorphisms $R \cong k[X_1, \ldots, X_n]$, weighted blow-ups depend on a specific choice of indeterminates with weights. ■

Keeping the preceding remark in mind, I give another characterization of the weighted blow-up $\mathrm{Bl}_0^w \mathbb{A}_k^n$ by using coordinates.

Construction 4.1.7. We use the same notations as in Definition 4.1.4. First, let $R = k[x_1, \ldots, x_n]$ and consider a weighted polynomial ring $R[T_1, \ldots, T_n]^w$ with $\deg T_i = w_i$. Then, we have a surjective homomorphism he surjective homomorphism

$$\phi : R[T_1, \ldots, T_n] \to \oplus_{n\geq 0} m^w(n), \quad T_i \mapsto x_i$$

of graded rings. Then, we clearly have

$$I^w := \langle T_i^{w_j} x_j^{w_i} - T_j^{w_i} x_i^{w_j} \rangle_{i,j} \subset \mathrm{Ker}\, \phi.$$

■

In analogy with the usual blow-up, we have the following:

Lemma 4.1.8. *Suppose k is an algebraically closed field of characteristic* 0. *Then, with the notations in Construction 4.1.7, the natural morphism*

$$F : \mathrm{Bl}_0^w \mathbb{A}_k^n \cong \mathrm{Proj}\, R[T_1, \ldots, T_n]/\,\mathrm{Ker}\, \phi \to \mathrm{Proj}\, R[T_1, \ldots, T_n]/I^w$$

is an isomorphism.

Proof. To begin with, let us get familiar with the target variety:

$$V := \operatorname{Proj} R[T_1, \ldots, T_n]/I^w \subset \operatorname{Proj} R[T_1, \ldots, T_n] = \mathbb{P}_k(w_1, \ldots, w_n) \times_k \operatorname{Spec} k[x_1, \ldots, x_n].$$

Lemma 4.1.9. Under the closed immersion above, V can be locally described as follows: we have an isomorphism

$$V \cap (\{T_i \neq 0\} \times_k \operatorname{Spec} k[x_1, \ldots, x_n]) \cong \operatorname{Spec} k[x'_1, \ldots, x'_n]/\frac{1}{w_1}(0, w_2, \ldots, w_n),$$

which gives the following relations:

$$x_i = {x'_i}^{w_i}, \quad x_j = x'_j {x'_i}^{w_j} \quad \text{for all } j \neq i.$$

$\diamondsuit$

Proof. For brevity, suppose $i = 1$. Then, by Lemma 4.1.3, we have

$$\begin{aligned}
&V \cap (\{T_1 \neq 0\} \times \operatorname{Spec} k[x_1, \ldots, x_n]) \\
&\cong \{(T_2, \ldots, T_n), (x_1, \ldots, x_n)) \in \mathbb{A}^{n-1} \times \mathbb{A}^n \mid \forall i, j : T_i^{w_j} x_j^{w_i} = T_j^{w_i} x_i^{w_j}, T_1 = 1\}/\frac{1}{w_1}(w_2, \ldots, w_n) \\
&= \{(T_2, \ldots, T_n), (x_1, \ldots, x_n)) \in \mathbb{A}^{n-1} \times \mathbb{A}^n \mid \forall i : x_i^{w_1} = T_i^{w_1} x_1^{w_i}\}/\frac{1}{w_1}(w_2, \ldots, w_n) \\
&\cong \{(T_2, \ldots, T_n), (x_1, \ldots, x_n)) \in \mathbb{A}^{n-1} \times \mathbb{A}^n \mid \forall i : x_i = T_i x_1^{w_i/w_j}\}/\frac{1}{w_1}(w_2, \ldots, w_n) \\
&\cong \operatorname{Spec} k[x'_1, \ldots, x'_n]/\frac{1}{w_1}(0, w_2, \ldots, w_n),
\end{aligned}$$

where each μ_{w_1}-action is the obvious one and the last isomorphism is given by

$$x_1 = {x'_1}^{w_1}, \quad \forall i \geq 2 : x'_i = T_i.$$

$\heartsuit$

Now, by Lemma B.2.10, the preceding lemma in particular shows that V is normal. Also, since $\oplus_{n \geq 0} m^w(n)$ is an integral domain and hence $\operatorname{Ker} \phi$ is a prime ideal, $\operatorname{Bl}_0^w \mathbb{A}_k^n$ is integral (in particular irreducible). Hence, it suffices to show that F induces a bijection on closed points by the following claim:

Lemma 4.1.10. Let X and Y be varieties over an algebraically closed field k of characteristic 0. Suppose X is irreducible and Y is normal. If a morphism $f : X \to Y$ induces a bijection on closed points, then it is an isomorphism. $\diamondsuit$

Proof. The idea of a proof is due to a mathoverflow answer ([Ele]) by Georges Elencwajg. Since the claim is local on the target, we may assume Y is integral by normality. Now, I claim f is a quasi-finite morphism, which is equivalent to showing that f has finite fibers by [Sta21, Tag 02NH] since f is of finite type. Indeed, since the fiber $f^{-1}(x)$ of any closed point $x \in Y$ consists of a single (closed) point of X, we see that f has finite fibers by [GW10, Remark 12.16], which essentially says that since the set S of points in Y with finite fibers is a constructible set and contains the set of closed points, we have $S = Y$, where we also use the fact that k is algebraically closed in a crucial way. Now, note the following claim in a mathoverflow answer. (In the proof, we may skip the reduction to a finite morphism since a quasi-finite projective morphism is finite (e.g. [Liu02, Corollary 4.4.8.])):

Lemma 4.1.11. [use] (cf.[Liu02, Exercise 6.2.9.]) Let $f : X \to Y$ be a dominant morphism of integral algebraic varieties over an algebraically closed field. Suppose the field extension $K(X)/K(Y)$ of function fields is finite separable and $[K(X) : K(Y)] = n$. Then, there exists a dense open subset of U of Y such that $f^{-1}(y)$ consists of n points for all $y \in Y$. $\diamondsuit$

Since f is bijective on closed points, f is dominant. Hence, noting that k is of characteristic 0, i.e. $K(X)/K(Y)$ is separable, we see that the degree $[K(X) : K(Y)]$ is 1 and hence f is birational. Then, note the following variant of Zariski's main theorem:

Lemma 4.1.12 (Zariski's Main Theorem)**.** [Liu02, Corollary 4.4.6.] Let X be an irreducible scheme and Y a normal, locally noetherian, integral scheme. Let $f : X \to Y$ be a separated birational morphism of finite type. If f is moreover quasi-finite, then f is an open immersion. $\diamondsuit$

Since we suppose and have shown all the conditions, f is an open immersion. Now, since f is surjective on closed points, f is surjective by Chevalley's theorem ([Vak17, 7.4.2,7.4.E.]), which suffices for a proof. $\heartsuit$

Now, first since $I^w \subset \operatorname{Ker}\phi$, we see that F is a closed immersion $F : \operatorname{Bl}_0^w \mathbb{A}_k^n \hookrightarrow V$ and is in particular injective on closed points. Hence, it suffices to see F is surjective on closed points. Consider the following chart of $\mathbb{P}_k(w_1, \dots, w_n) \times_k \mathbb{A}_k^n$:

$$\begin{aligned} U_1 &:= (\mathbb{P}_k(w_1, \dots, w_n) \times_k \mathbb{A}_k^n) \cap (\{T_1 \neq 0\} \times_k \mathbb{A}_k^n) \\ &\cong \{((T_2, \dots, T_n), (x_1, \dots, x_n)) \in \mathbb{A}^{n-1} \times \mathbb{A}^n\} / \frac{1}{w_1}(w_2, \dots, w_n) \end{aligned}$$

and assume there exists a closed point $P \in V \cap U_1$ that is not lying in $\operatorname{Bl}_0^w \mathbb{A}_k^n$ for the sake of contradiction. Then, since $\operatorname{Bl}_0^w \mathbb{A}_k^n$ is a closed subscheme of V, there exists a polynomial $f \in \operatorname{Ker}\phi \subset R[T_1, \dots, T_n]$ such that f is w-weighted homogeneous and $f|_{U_1}$ cuts out a closed subset of $V \cap U_1$ that contain $\operatorname{Bl}_0^w \mathbb{A}^n \cap U_1$, but does not contain P. Then, by the equations $x_i^{w_1} = T_i^{w_1} x_1^{w_i}$ on $V \cap U_1$ given in the proof of Lemma 4.1.9, we can write the equation $f|_{U_1} = 0$ purely in terms of x_i's on $V \cap U_1 \cap \{x_1 \neq 0\}$. Note in general that by writing a w-homogeneous equation E of T_j's and x_i's on $V \cap U_1$ purely in terms of x_i's, we see that if E becomes trivial in $\oplus_{n \geq 0} m^w(n)$ via $T_k \mapsto x_k$, then E is trivialized by the relations $x_i^{w_1} = T_i^{w_1} x_1^{w_i}$ on $V \cap U_1$.

Example 4.1.13. Note the w-homogeneous equation E

$$T_i^{w_j w_k} x_j^{w_k w_i} x_k^{w_i w_j} + x_i^{w_j w_k} T_j^{w_k w_i} x_k^{w_i w_j} - 2 x_i^{w_j w_k} x_j^{w_k w_i} T_k^{w_i w_j} = 0$$

gets mapped to a trivial equation by $T_i \mapsto x_i$, but E is trivial from the beginning by relations $x_i^{w_1} = T_i^{w_1} x_1^{w_i}$. This must be true in general since if we use $T_i = \frac{x_i}{x_1^{w_i/w_1}}$, then the denominator of each monomial is the same multiple of x_1 (as E is w-homogeneous) and hence we can take them away, which gives us exactly the image of the map $T_i \mapsto x_i$. $\diamondsuit$

Hence, since f cuts out a proper subset of $V \cap U_1$, f gives a non-trivial relation of x_i's in $\oplus_{n \geq 0} m^w(n)$, which is absurd since $f \in \operatorname{Ker}\phi$ and $R[T_1, \dots, T_n]/\operatorname{Ker}\phi \cong \oplus_{n \geq 0} m^w(n)$. $\square$

In the sequel, we may use Lemma 4.1.9 as a local parameter of the weighted blow-up.

4.2 Resolution of cDV singularity

Definition 4.2.1. A germ $(x \in X)$ of 3-fold singularity is called a **compound Du Val (cDV)** singularity if it is formally isomorphic to the germ $(0 \in \{F = 0\})$ for

$$F(x, y, z, t) = f(x, y, z) + tg(x, y, z, t),$$

where f is the equation of a Du Val singularity (Theorem 3.4.1) and g is an arbitrary polynomial, or equivalently a 3-fold singularity in $\mathbb{A}^3$ for which there exists a hyperplane $H \subset \mathbb{A}^3_k$ such that $(x \in H)$ is a Du Val singularity.

As we expect, cDV singularities give a nice class of 3-fold singularities.

Theorem 4.2.2. [Mil83, Theorem 1.1.] *Let $(p \in X)$ be a germ of a point in 3-fold. Then, $(p \in X)$ is an isolated cDV singularity if and only if $p \in X$ is a terminal of index 1, i.e. K_X is principal around p (cf. [Mil83, 0.12.(e)]).*

Now, we consider a resolution of an isolated cE_7 singularity to see it is indeed terminal.

Lemma 4.2.3. [Kol08, Example 49] *Let X be the cE_7-type singularity given by $f(x, y, z, t) = x^2 + y^3 + yg_3(z, t) + h_5(z, t)$ in $\mathbb{A}^4$, where g_3 (resp. h_5) is homogeneous of degree 3 (resp. 5) and they do not have a common factor. Then, X has the only singularity at the origin and the singularity is terminal.*

Proof. First, to see X has the only one singularity at the origin, note that we have

$$\partial_x f = 2x, \quad \partial_y f = 3y^2 + g_3(z, t), \quad \partial_z f = y\partial_z g_3 + \partial_z h_5, \quad \partial_t f = y\partial_t g_3 + \partial_t h_5.$$

Hence, if (x_0, y_0, z_0, t_0) is a singularity, then $x_0 = 0$. Therefore, we have

$$\begin{aligned}
&f(0, y_0, z_0, t_0) = y_0^3 + y_0 g_3(z_0, t_0) + h_5(z_0, t_0) = 0,\\
&3y_0^2 + g_3(z_0, t_0) = 0,\\
&y_0\partial_z g_3(z_0, t_0) + \partial_z h_5(z_0, t_0) = 0,\\
&y_0\partial_t g_3(z_0, t_0) + \partial_t h_5(z_0, t_0) = 0.
\end{aligned}$$

First, by the first and the second equations, we have $h_5(z_0, t_0) = -2y_0^3$. Also, by the third and the fourth equations together with Euler's homogeneous function formula, we have $3y_0 g_3(z_0, t_0) + 5h_5(z_0, t_0) = 0$, which, combined with the second equation, yields $h_5(z_0, t_0) = \frac{9}{5}y_0^3$. Hence, we have $-2y_0^3 = \frac{9}{5}y_0^3$, i.e. $y_0 = 0$. Therefore, we have $g_3(z_0, t_0) = 0$ and $h_5(z_0, t_0) = 0$. Since g_3 and h_5 do not have a common factor, we have $z_0 = t_0 = 0$ as desired.

Now, to see that the singularity is terminal, consider the $(3, 2, 1, 1)$-blow up $Y \to X$ with exceptional divisor E. Write $w = (3, 2, 1, 1)$. Now, we consider the strict transforms of X in the 4 standard charts of $\mathrm{Bl}_0^w \mathbb{A}^4$ by Lemma 4.1.9.

Chart U_t: First, consider the chart $U_t = \operatorname{Spec} k[x', y', z', t]$ given by $x = x't^3, y = y't^2, z = z't, t = t$. Then, since

$$f(x't^3, y't^2, z't, t) = t^5(tx'^2 + ty'^3 + y'g_3(z', 1) + h_5(z', 1)),$$

$Y \cap U_t$ is given by

$$\{f_t(x', y', z', t) := tx'^2 + ty'^3 + y'g_3(z', 1) + h_5(z', 1) = 0\} \subset U_t.$$

Then, the exceptional divisor is given by

$$\{f_t(x', y', z', t) = 0, t = 0\} \subset U_t.$$

Now, we compute the discrepancy $a(X, E)$ with respect to $\pi : Y \to X$. As before, we have the following canonical 3-form on X:

$$s = \mathrm{Res}_{\mathbb{A}^4|X} \frac{dx \wedge dy \wedge dz \wedge dt}{f} = \frac{dy \wedge dz \wedge dt}{\partial_x f} = \frac{dy \wedge dz \wedge dt}{2x}.$$

Then, we have

$$\pi^* s|_{U_t} = \frac{t^3 dy' \wedge dz' \wedge dt}{2x' t^3} = t \frac{dy' \wedge dz' \wedge dt}{\partial_x f_t} = t \cdot \mathrm{Res}_{U_t|Y \cap U_t} \frac{dx \wedge dy \wedge dz \wedge dt}{f_t}$$

and hence we see the discrepancy $a(E, X)$ of E with respect to X is given by

$$a(E, X) = 1.$$

In particular, to see X has a terminal singularity, it suffices to show that Y has a terminal singularity. To begin with, note

$$\partial_{x'} f_t = 2tx', \quad \partial_{y'} f_t = 3{y'}^2 + g_3(z', 1), \quad \partial_{z'} f_t = y' d_{z'} g_3(z', 1) + d_{z'} h_5(z', 1), \quad \partial_t f_t = {x'}^2 + {y'}^3.$$

If $x' = 0$, then $y' = 0$. Hence, $g_3(z', 1) = h_5(z', 1) = 0$, which is absurd since g_3 and h_5 do not have a common factor. In particular, there is no singularity with $x' = y' = z' = 0$. The case when $t = 0$ (i.e. a point lying in E) will be examined in the charts U_y and U_x.

Chart U_z: Next, consider the chart $U_z = \operatorname{Spec} k[x', y', z, t']$. Since the polynomial f and the weight $(3, 2, 1, 1)$ are symmetric with respect to z and t, the only points we concern are points with $z = 0$ (lying in E), which will be examined in the charts U_y and U_x.

Chart U_y: Now, consider the chart $U_y = \operatorname{Spec} k[x', y', z', t']/\frac{1}{2}(1, 0, 1, 1)$ given by the substitutions $x = x'{y'}^3$, $y = {y'}^2$, $z = z'y'$, $t = t'y'$. Then, $Y \cap U_y$ is given by

$$\{f'_y(x', y', z', t') := ({x'}^2 + 1)y' + g_3(z', t') + h_5(z', t') = 0\}/\frac{1}{2}(1, 0, 1, 1) \subset \mathbb{A}^4_{x', y', z', t'}/\frac{1}{2}(1, 0, 1, 1).$$

(cA_2). First, we consider singularities that are not the vertex of the quotient. First, consider $\tilde{U}_y = \{f'_y = 0\} \subset \mathbb{A}^4$. Then, note

$$\partial_{x'} f'_y = 2x'y', \quad \partial_{y'} f'_y = {x'}^2 + 1, \quad \partial_{z'} f_y = \partial_{z'} g_3 + \partial_{z'} h_5, \quad \partial_{t'} f'_y = \partial_{t'} g_3 + \partial_{t'} h_5.$$

Hence, by the same argument using Euler's homogeneous function formula as in the first part, we see that the only singularities are $(x', y', z', t') = (\pm\sqrt{-1}, 0, 0, 0)$ (lying in L), which are clearly cA_2 singularities. Since $\tilde{U}_y$ is a μ_2-subvariety and the quotient is étale outside of the origin, these singularities remain to be cA_2 singularities of the form

$$0 \in \{x'y' + g_3(z', t') + h_5(z', t') = 0\}$$

after the quotient and they indeed become the same point. Since the singularity is isolated by the same argument as above, we know that it is terminal by Theorem 4.2.2.

To actually get a resolution, we first take the $(1,2,1,1)$-blow up at the origin (cf. [Che13, Theorem 3.2.] for the reason of this specific choice). Then, it is easy to see that the charts corresponding to x' and y' are non-singular and the charts corresponding to z' and t' have cA_1 singularities. For example, in the affine chart $U_{t'}$ corresponding to t', i.e. the chart given by substitutions $x' = x''t', y' = y''{t'}^2, z' = z''t', t' = t'$, the strict transform is given by

$$\{x''y'' + g_3(z'',1) + {t'}^2 g_5(z'',1) = 0\} \subset U_{t'}.$$

It is also straightforward to check that the discrepancy with respect to this weighted blow-up is positive as above. Now, since we may assume the singularity is given by

$$0 \in \{x''y'' + {t'}^2 + {z''}^3 = 0\},$$

we immediately see the usual blow-up resolves the singularity.

(Vertex). Now, it suffices to see that the vertex of the quotient is terminal. From now on, I will write $x' = x_0, y' = x_1, z' = x_2, t' = x_3$. Then, note a basis of $(\Omega^3_{\tilde{U}_y})^{\otimes 2}$ can be given by

$$\left(\mathrm{Res}_{\mathbb{A}^4|\tilde{U}_y} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}{{f'_y}^2}\right)^{\otimes 2} = \frac{(dx_0 \wedge dx_2 \wedge dx_3)^{\otimes 2}}{(x_0^2+1)^2}.$$

Hence, noting that the μ_2-action acts on these three coordinates as $\frac{1}{2}(1,1,1)$ and trivially on the other, we can see that the blow-up at the vertex has positive discrepancy by the similar argument as in Example 3.3.6 with $\tilde{U}_y$ in the place of $\mathbb{A}^3$. Since $\tilde{U}_y$ is non-singular at the origin and hence the vertex can be viewed as the vertex of the affine cone over a non-singular projective variety via the Veronese embedding, the blow-up resolves the singularity (Lemma 3.2.3) and hence the singularity is terminal.

Chart U_x: Now, consider the chart $U_x = \mathrm{Spec}\, k[x',y',z',t']/\frac{1}{3}(0,2,1,1)$ given by the substitutions $x = x'{y'}^3$, $y = {y'}^2$, $z = z'y'$, $t = t'y'$. Then, $Y \cap U_x$ is given by

$$\{f'_x(x',y',z',t') = x'(1+{y'}^3) + y'g_3(z',t') + h_5(z',t') = 0\}/\frac{1}{3}(0,2,1,1).$$

By the same arguments as above we have one cA_2 singularity in E and outside of the vertex, which should be resolved and shown to be terminal as above, noting that this is an isolated cDV singularity and hence must be terminal by Theorem 4.2.2 although a resolution should be trickier since we have $y'g_3(z',t')$ instead of $g_3(z',t')$ (in particular, we are expected to consider the $(1,3,1,1)$-blow-up again by [Che13, Theorem 3.2.]). Finally, the singularity at the vertex should be resolved as in the case of $\mathbb{A}^3/\frac{1}{3}(2,1,1)$. Although I did not present a resolution of this specific quotient singularity, it should be resolved as in the case of $\mathbb{A}^3/\frac{1}{2}(1,1,1)$, where we are expected to consider $\left(\Omega^3_{\mathbb{A}^3/\frac{1}{3}(2,1,1)}\right)^{\otimes 3}$ instead of $\left(\Omega^3_{\mathbb{A}^3/\frac{1}{3}(2,1,1)}\right)^{\otimes 2}$. Since we know that the singularity at the vertex of $\mathbb{A}^3/\frac{1}{3}(2,1,1)$ is terminal by Reid-Tai criterion (Lemma 3.3.4), these computations must suffice for a proof.

□

Appendix A

Divisors

A.1 Relations among several notions

Although we will focus on the case of normal varieties in this thesis, I will give more general settings to observe subtlety regrading normality.

Notation A.1.1. In this section, let $(*)$ denote the following condition for a scheme X:

> $(*)$ X is a noetherian integral separated scheme which is regular in codimension one.

In particular, a normal variety satisfies $(*)$.

Definition A.1.2. Suppose that a scheme X satisfies $(*)$.

(i) A **prime divisor** is a closed integral subscheme of codimension 1.

(ii) A **Weil divisor** is a formal linear combination of prime divisors over $\mathbb{Z}$. Let $\mathrm{Div}(X)$ denote the abelian group of Weil divisors.

(iii) An **effective Weil divisor** is a Weil divisor D whose coefficients of each generator is non-negative. We write $D \geq 0$.

(iv) A **principal Weil divisor** is a Weil divisor defined by

$$(f) := \sum_{[Z]:\text{prime divisor}} v_Z(f)[Z],$$

where f is a non-zero rational function on X and v_Z is the discrete valuation of $\mathcal{O}_{X,\eta}$ with η the generic point of Z. Let $\mathrm{Princ}(X) \subset \mathrm{Div}(X)$ denote the subgroup of principal Weil divisors.

(v) A **linear equivalence** $\sim$ on $\mathrm{Div}(X)$ is defined so that $D_1 \sim D_2$ iff $D_1 - D_2 \in \mathrm{Princ}(X)$.

(vi) A **Weil divisor class group** is the quotient $\mathrm{Cl}(X) := \mathrm{Div}(X)/\mathrm{Princ}(X)$.

(vii) The **support** $\mathrm{Supp}\, D$ of $D \in \mathrm{Div}\, X$ is the union of prime divisors that appear in the summation.

The following are useful for computation.

Lemma A.1.3. [Har77, Proposition II.6.5.] *Let X be a scheme that satisfies $(*)$, Z a proper closed subset of X, and $U = X \setminus Z$.*

(i) *There is a surjection $\operatorname{Div} X \twoheadrightarrow \operatorname{Div} U$ defined by $\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$, where we ignore those $(Y_i \cap U)$ which are empty, which also induces a surjection $\operatorname{Cl} X \twoheadrightarrow \operatorname{Cl} U$.*

(ii) *If $\operatorname{codim}_X Z \geq 2$, then the surjections $\operatorname{Div} X \twoheadrightarrow \operatorname{Div} U$ and $\operatorname{Cl} X \twoheadrightarrow \operatorname{Cl} U$ are isomorphisms.*

(iii) *If Z is an irreducible subset of codimension 1, then there are exact sequences:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Div} X \longrightarrow \operatorname{Div} U \longrightarrow 0$$

$$\mathbb{Z} \xrightarrow{1 \mapsto [Z]} \operatorname{Cl} X \longrightarrow \operatorname{Cl} U \longrightarrow 0$$

Proof.

(i) The only nontrivial claim is the well-definedness of $\operatorname{Cl} X \to \operatorname{Cl} U$. Indeed, if f is a non-zero rational function on X with $(f) = \sum n_i Y_i$, then $(f|_U) = \sum n_i (Y_i \cap U)$.

(ii) The groups $\operatorname{Div} X$ and $\operatorname{Cl} X$ only depend on subsets of codimension 1. The inverse is given by taking the closure of each prime divisor.

(iii) The kernel of $\operatorname{Div} X \to \operatorname{Div} U$ consists of divisors whose support is in Z. Since Z irreducible, we obtain the desired exact sequences.

□

Corollary A.1.4. [Har77, II.Ex.8.5(a)] *Let X be a nonsingular variety, Y a non-singular subvariety of codimension $r \geq 2$, $\pi : \tilde{X} \to X$ the blow-up of X along Y, and $Y' = \pi^{-1}(Y)$ (cf.* [Har67, Definition in p.163 and Proposition II.7.13.(a)]*). Then, we have the spilt short exact sequence:*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1 \mapsto [Y']} \operatorname{Pic} \tilde{X} \xrightarrow{\iota^*} \operatorname{Pic} X \longrightarrow 0,$$

where ι^* is given by the composition $\operatorname{Pic} \tilde{X} \to \operatorname{Pic}(\tilde{X} \setminus Y') \cong \operatorname{Pic}(X \setminus Y) \cong \operatorname{Pic} X$ and the pull-back $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$ gives a section. In particular, $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Proof. By Lemma A.1.3 and Corollary A.1.21, it suffices to show that $\mathbb{Z} \to \operatorname{Pic} \tilde{X}$ is injective. Indeed, by [Har77, Theorem II.8.24.], we have $\mathcal{O}_{\tilde{X}}(-nY') \cong \mathcal{O}_{Y'}(-n)$ and hence $n[Y'] = [nY'] = 0$ implies $n = 0$. □

Remark A.1.5. If X is a non-singular surface and Y is a point on X above, then the injectivity of $\mathbb{Z} \to \operatorname{Pic} \tilde{X}$ simply follows by $(nY' \cdot nY') = -n^2$. ■

Lemma A.1.6. [Har77, II.6.2.] *Let R be a noetherian domain. Then, A is a unique factorization domain if and only if $X = \operatorname{Spec} R$ is normal and $\operatorname{Cl}(X) = 0$. In particular, $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.*

We now define Cartier divisors, which are generalization of Weil divisors to arbitrary schemes.

Definition A.1.7. Let X be a scheme.

(i) The **sheaf $\mathcal{K}_X$ of stalks of meromorphic functions on X** (or the **sheaf of total quotient rings on X**) is defined as follows: first, for each open subset U of X, let $\mathcal{R}_X(U) = \{a \in \mathcal{O}_X(U) \mid a_x \in R(\mathcal{O}_{X,x}) \text{ for all } x \in U\}$, where $R(\mathcal{O}_{X,x})$ denotes the multiplicative group of the regular elements (i.e. non-zero divisors) of $\mathcal{O}_{X,x}$, which clearly defines a sheaf $\mathcal{R}$ on X.

Lemma A.1.8. [Liu02, Lemma 7.1.12.] There exists a unique presheaf of algebras $\mathcal{K}'_X$ on X containing $\mathcal{O}_X$ with the following properties:

(a) For any open subset U of X, we have $\mathcal{K}'_X(U) = \mathcal{R}_X(U)^{-1}\mathcal{O}_X(U)$. In particular, $\mathcal{K}'_X(U)$ is the total ring of fraction of $\mathcal{O}_X(U)$, denoted by $\mathrm{Frac}(\mathcal{O}_X(U))$, if U is affine.

(b) For any open subset U of X, the canonical homomorphism $\mathcal{K}'_X(U) \to \prod_{x \in U} \mathcal{K}'_{X,x}$ is injective.

(c) If X is locally noetherian, then for any $x \in X$, $\mathcal{K}'_{X,x} \cong \mathrm{Frac}(\mathcal{O}_{X,x})$.

$\diamondsuit$

Then, we define the sheaf $\mathcal{K}_X$ to be the sheafification of $\mathcal{K}'_X$.

(ii) A **Cartier divisor** is a global section of the quotient sheaf $\mathcal{K}^*/\mathcal{O}^*$, which can be represented by $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open covering of X, f_i is the quotient of two regular elements of $\mathcal{O}(U_i)$, and $f_i|_{U_i \cap U_j} \in f_j|_{U_i \cap U_j} \mathcal{O}(U_i \cap U_j)^*$. Write

$$\mathrm{CaDiv}(X) := \mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*).$$

We consider the multiplicative group structure, but the group operation is denoted additively.

(iii) An **effective Cartier divisor** is an element D of the image of the canonical map

$$\mathrm{H}^0(X, (\mathcal{K}^* \cap \mathcal{O})/\mathcal{O}^*) \to \mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*).$$

In particular, it can be represented by $\{(U_i, f_i)\}$ with $f_i \in \mathcal{K}^*(U_i) \cap \mathcal{O}(U_i)$ for each i. We write $D \geq 0$. See also Remark A.1.16.

(iv) A **principal Cartier divisor** is an element of the image

$$\mathrm{CaPrinc}(X) := \mathrm{Im}(\mathrm{H}^0(X, \mathcal{K}^*) \to \mathrm{H}^0(X, \mathcal{K}^*/\mathcal{O}^*)).$$

In particular, it can be represented by $\{(U_i, f_i)\}$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for any i, j.

(v) A **linear equivalence** $\sim$ on $\mathrm{CaDiv}(X)$ is defined so that $D_1 \sim D_2$ iff $D_1 - D_2 \in \mathrm{CaPrinc}(X)$.

(vi) A **Cartier divisor class group** is the quotient $\mathrm{CaCl}(X) = \mathrm{CaDiv}(X)/\mathrm{CaPrinc}(X)$.

Now, we compare Weil divisors with Cartier divisors. For sufficiently good schemes, they are the same.

Construction A.1.9. [Har77, Proposition 6.11] Suppose X satisfies $(*)$. Then, we have the natural map

$$\Phi : \mathrm{CaDiv}(X) \to \mathrm{Div}(X)$$

defined as follows. Take a Cartier divisor $\{(U_i, f_i)\} \in \mathrm{CaDiv}(X)$. Then, note that for a prime divisor $[Z] \in \mathrm{Div}(X)$ and any U_i, U_j that intersect Z, we have

$$v_Z(f_i) = v_Z(f_j)$$

since f_i/f_j is invertible on $U_i \cap U_j$, i.e., $v_Z(f_i/f_j) = 0$. Hence, by fixing $U_i \cap Z \neq \emptyset$, if exists, for each prime divisor $[Z]$ we have a well-defined Weil divisor

$$\Phi(\{(U_i, f_i)\}) = \sum v_Z(f_i)[Z] \in \mathrm{Div}(X)$$

where the sum is finite since X is noetherian. In particular, Φ is a group homomorphism and sends effective Cartier divisors to effective Weil divisors. ■

Remark A.1.10. We say a Weil divisor D is **locally principal** if X can be covered by open sets U such that $D|_U = D \cap U \subset U$ is a principal Weil divisor for each U. Therefore, by construction, the natural map Φ maps Cartier divisors to locally principal Weil divisors. ■

Lemma A.1.11. [Liu02, Proposition 7.2.14] *Suppose X satisfies $(*)$ and is moreover normal.*

(i) *The natural map Φ in Construction A.1.9 induces an isomorphism*

$$\mathrm{CaPrinc}(X) \xrightarrow{\sim} \mathrm{Princ}(X).$$

(ii) *The natural map Φ is injective and induces the injective homomorphism*

$$\mathrm{CaCl}(X) \hookrightarrow \mathrm{Cl}(X).$$

Moreover, $\{(U_i, f_i)\}$ is effective iff $\Phi(\{(U_i, f_i)\})$ is effective.

(iii) *In particular, the natural map Φ induces an isomorphism*

$$\mathrm{CaDiv}(X) \xrightarrow{\sim} \mathrm{Im}(\Phi) = \{\textit{the group of locally pricincipal Weil divisors}\}.$$

(iv) *Suppose X is moreover **locally factorial**, i.e., all of whose local rings are unique factorization domains (in particular normal domains). Then, the natural map Φ induces isomorphisms*

$$\mathrm{CaDiv}(X) \xrightarrow{\sim} \mathrm{Div}(X), \quad \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Cl}(X).$$

Remark A.1.12. In the proof, we first show that Φ induces a surjective homomorphism $\mathrm{CaPrinc}(X) \twoheadrightarrow \mathrm{Princ}(X)$ and then show Φ is injective, which implies the isomorphism of the principal divisor groups and hence part (iii). ■

Remark A.1.13.

(i) We need the normality of X for the injectivity of Φ. In other words, if X is not normal, then there can be a non-zero principal Cartier divisor that gets mapped to 0 in $\mathrm{Div}(X)$.

Example A.1.14. [Liu02, Example 7.2.15] Let $X = \operatorname{Spec} k[s,t]/\langle s^2 - t^3 \rangle$ be an integral curve and let $p = \langle t, s \rangle \in X$. Then, we have

$$\operatorname{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\langle t \rangle) = \operatorname{length}_{\mathcal{O}_{X,p}}(\mathcal{O}_{X,p}/\langle t - s \rangle) = 2.$$

Hence, for a principal Cartier divisor D defined by $f := (t-s)/t \in K(X)^*$, we have $\Phi(D) = 0$ since $f_x \in \mathcal{O}_{X,x}^*$, i.e., $v_x(f) = 0$ for all points $x \neq p$ and in particular for all closed points (i.e., closed subschemes of codimension 1). On the other hands, $D \neq 0$ since $f_p \notin \mathcal{O}_{X,p}$. ♢

(ii) We need the local factoriality of X for the isomorphicity of Φ. See Lemma 1.4.1.

■

Finally, we will see the relation between Cartier divisors and line bundles.

Construction A.1.15. Let X be a scheme. We construct a map $\operatorname{CaDiv}(X) \to \operatorname{Lin}(X)$ as follows. Take a Cartier divisor $D = \{(U_i, f_i)\} \in \operatorname{CaDiv}(X)$. Then, define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$ by

$$\mathcal{O}_X(D)|_{U_i} := f_i^{-1}\mathcal{O}_X|_{U_i},$$

which is well-defined since f_i/f_j is invertible on $U_i \cap U_j$ and hence $f_i^{-1}\mathcal{O}_X|_{U_i \cap U_j} \cong f_j^{-1}\mathcal{O}_X|_{U_i \cap U_j}$ and clearly does not depend on the choice of the representation $\{(U_i, f_i)\}$. Noting that

$$\mathcal{O}_{U_i} \to \mathcal{O}_X(D)|_{U_i}; \quad 1 \mapsto f_i^{-1}$$

is an isomorphism, we have obtained a map

$$\operatorname{CaDiv}(X) \to \operatorname{Lin}(X); \quad D \mapsto \mathcal{O}_X(D).$$

■

Remark A.1.16. Note that D is an effective Cartier divisor iff $\mathcal{O}_X(-D)$ is a subsheaf of $\mathcal{O}_X$. Since $\mathcal{O}_X(-D)$ is locally generated by the non-zero divisors f_i by construction, the ideal sheaf $\mathcal{O}_X(-D)$ corresponds to the closed subscheme of X locally cut out by the non-zero divisors f_i. Therefore, an effective Cartier divisor corresponds to a closed subscheme of codimension 1 whose ideal sheaf is invertible. Here, we used Krull's principal ideal theorem to compute the codimension:

> **Theorem A.1.17** (Krull's Principal Ideal Theorem)**.** [Vak17, 11.3.3.] Let A be a noetherian ring and $f \in A$. Then, every prime $\mathfrak{p}$ minimal among containing f has codimension at most 1. If f is furthermore not a zero divisor, then every such prime $\mathfrak{p}$ containing f has codimension precisely 1. ♢

■

Remark A.1.18. If X satisfies $(*)$ and is locally factorial, then we obtain a homomorphism

$$\operatorname{Div}(X) \cong \operatorname{CaDiv}(X) \to \operatorname{Lin}(X),$$

which sends $D \in \operatorname{Div}(X)$ to an invertible sheaf $\mathcal{O}(D)$ defined by

$$\mathcal{O}_X(D)(U) = \{f \in K(X) \mid ((f) + D)|_U \geq 0\}.$$

Note we can always obtain a (not necessarily invertible) sheaf $\mathcal{O}_X(D)$ in this way with the following nice properties:

Lemma A.1.19. [Fer01, Proposition 3.3.] Let X be a normal scheme satisfying $(*)$.

(i) For every Weil divisor D, the sheaf is **reflexive**, i.e. coherent and with the canonical map to its second dual $\mathcal{O}_X(D) \to \mathcal{O}_X(D))^{\vee\vee}$ being an isomorphism, and locally free of rank one at every generic point and at every point of codimension 1.

(ii) Conversely, any reflexive sheaf locally free of rank one at every generic point and at every point of codimension 1 is isomorphic to $\mathcal{O}_X(D)$ for some Weil divisor D.

(iii) If D_1 and D_2 are Weil divisors on X, then $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$ as $\mathcal{O}_X$-modules.

(iv) If D, D_1, D_2 are Weil divisors on X, we have $\mathcal{O}_X(-D) \cong \mathcal{O}_X(D)^{\vee}$ and $\mathcal{O}_X(D_1+D_2) = (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee}$.

$\diamondsuit$

$\blacksquare$

We can always regard a Cartier divisor as an invertible sheaf, but the converse is true only for a sufficiently nice scheme.

Lemma A.1.20. [Liu02, Proposition 7.1.18, Corollary 7.1.19] *Let X be a scheme.*

(i) *The natural map* $\mathrm{CaDiv}(X) \to \mathrm{Lin}(X); D \mapsto \mathcal{O}_X(D)$ *induces an injective homomorphism*

$$\Psi : \mathrm{CaCl}(X) \to \mathrm{Pic}(X),$$

where $\mathrm{Pic}(X)$ *denote the* ***Picard group*** *of X, i.e., the group of isomorphism classes of invertible sheaves on X.*

(ii) *In particular, we have a natural isomorphism*

$$\Psi : \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Im}(\Phi) = \{\textit{invertible sheaves contained in } \mathcal{K}_X\}.$$

(iii) *If X is moreover a noetherian scheme without embedded point (e.g., reduced); then the natural map Ψ is an isomorphism:*

$$\Psi : \mathrm{CaCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X).$$

Corollary A.1.21. *If X is a noetherian, integral, separated locally factorial scheme (e.g. a non-singular variety), then there are natural isomorphisms*

$$\mathrm{Cl}(X) \cong \mathrm{CaCl}(X) \cong \mathrm{Pic}(X).$$

A.2 Canonical divisors on a non-singular variety

Definition A.2.1. Let X be a non-singular k-variety of dimension n. The **canonical sheaf** is defined to be the line bundle $\mathcal{K}_X := \det \Omega_{X/k} = \Omega^n_{X/k}$, where $\Omega_{X/k}$ denotes the sheaf of relative differentials (e.g. [Liu02, Definition 6.1.19.]) and is locally free of rank n as X is non-singular (e.g. [Liu02, Proposition 6.2.2.]). The corresponding Weil divisor K_X is called the **canonical divisor** of X (cf. Corollary A.1.21).

Example A.2.2. First of all, note we have $\mathrm{Cl}(\mathbb{P}^n_k) \cong \mathbb{Z}$ ([Har77, Proposition II.6.4.]), where a generator is the hyperplane section, i.e. the divisor class given by hyperplanes, and in particular for any divisor D of degree d as a hypersurface, we have $D \sim dH$. In other words, the invertible sheaf $\mathcal{O}_{\mathbb{P}^n_k}(1)$ generates $\mathrm{Pic}(X)$ and any invertible sheaf is isomorphic to $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \in \mathbb{Z}$. Now, we have $K_{\mathbb{P}^n_k} = -(n+1)H$. For example, see [Har77, Theorem II.8.20.1.] where we use a suitable exact sequence. Otherwise, observe that the n-form

$$w_0 = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

on $\{x_0 \neq 0\} \subset \mathbb{P}^n_{x_0,\dots,x_n}$ extends to a canonical n-form form w on $\mathbb{P}^n_k$ in the obvious way. Since w does not have zeros and has poles of multiplicity 1 along each hypersurface, we are done. ∎

Let us recall some useful techniques and computations.

Theorem A.2.3 (Adjunction Formula)**.** [Vak17, 21.5.B.] *Suppose that X is a smooth variety and Z is a smooth subvariety of X. Then, we have*

$$\mathcal{K}_Z \cong \mathcal{K}_X|_Z \otimes \det \mathcal{N}_{Z/X},$$

where $\mathcal{N}_{Z/X} := (\mathcal{I}/\mathcal{I}^2)^\vee$ is the ***normal sheaf*** *for a closed immersion $Z \hookrightarrow X$ with ideal sheaf $\mathcal{I}$. In particular, if D is a smooth effective Cartier divisor of X, then $\mathcal{N}_{D/X} \cong \mathcal{O}_X(D)|_D$ (cf. Remark A.1.16 or* [Vak17, 21.2.H.]*) and hence*

$$\mathcal{K}_D \cong (\mathcal{K}_X \otimes \mathcal{O}_X(D))|_D$$

or in terms of divisors

$$K_D = (K_X + D)|_D.$$

Corollary A.2.4 (Adjunction Formula for Curves)**.** [Har67, Proposition V.1.5.] *If C is a non-singular curve of genus g on a non-singular surface X. Then, $2g-2 = (C \cdot (C + K_X))$.*

Definition A.2.5. Let $f : X \to Y$ be a birational morphism of non-singular varieties. Then, the **relative canonical divisor** with respect to f is defined to be $K_{X/Y} := K_X - f^*K_Y$.

Example A.2.6. [Har77, Proposition V.3.3.] Let X be a non-singular surface and $\pi : \tilde{X} \to X$ be the blow-up at a point $P \in X$ with the exceptional divisor E. Then, $K_{\tilde{X}/X} = E$. First, note the following straightforward formulas:

> **Lemma A.2.7.** [Har77, Proposition V.3.2.] Let X be a non-singular surface and $\pi : \tilde{X} \to X$ be the blow-up at a point $P \in X$ with the exceptional divisor E. If $C, D \in \mathrm{Pic}\, X$, then $((\pi^*C) \cdot (\pi^*D)) = (C \cdot D)$ and $((\pi^*C) \cdot E) = 0$. ♢

Now, since π induces an isomorphism $\tilde{X} \setminus E \xrightarrow{\sim} X \setminus \{P\}$, we have $K_{\tilde{X}} = \pi^*K_X + D$, where $\mathrm{Supp}\, D \subset \mathrm{Ex}(\pi) = E$. Since E is irreducible, we have $K_{\tilde{X}} = \pi^*K_X + nE$ for some $n \in \mathbb{Z}$ (which also follows directly from Lemma A.1.4 and Lemma A.2.7). Now, by the adjunction formula for curves, we have $-2 = (E \cdot (E + K_{\tilde{X}})) = -1 + (E \cdot K_{\tilde{X}})$. Hence, by Lemma A.2.7, $n = 1$. There are other ways to see this (e.g. [oM18, p.7]). ∎

This is generalized as follows:

Lemma A.2.8. [Har77, II.Ex.8.5(b)] *Let X be a nonsingular variety, Y a non-singular subvariety of codimension $r \geq 2$, $\pi : \tilde{X} \to X$ the blow-up of X along Y, and $Y' = \pi^{-1}(Y)$. Then, we have $\mathcal{K}_{\tilde{X}} \cong f^*\mathcal{K}_X \otimes \mathcal{O}_{\tilde{X}}((r-1)Y')$, i.e. $K_{\tilde{X}/X} = (r-1)Y'$.*

A.3 Positivity of divisors

We first review how we obtain a morphism to a projective space.

Definition A.3.1. Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then, $\mathcal{F}$ is said to be **generated by global sections** $s_i \in \mathcal{F}(X)$ $(i \in I)$ if $\mathcal{F}_x$ is generated by $(s_i)_x$ as an $\mathcal{O}_{X,x}$-module, or equivalently if the corresponding morphism $\bigoplus_{i\in I} \mathcal{O}_X \to F$ is surjective. We say $\mathcal{F}$ is **globally generated** if $\mathcal{F}$ is generated by some global sections.

For a line bundle on an R-scheme for a ring R, being generated by finitely many global sections is equivalent to having sufficiently many global sections to define a morphism into a projective space in the following sense.

Lemma A.3.2. [Har77, Theorem II.7.1.] *Let R be a ring and X an R-scheme.*

(i) *Suppose we have an R-morphism $\phi : X \to \mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$. Then, $\phi^*(\mathcal{O}_{\mathbb{P}^n_R}(1))$ is an invertible sheaf on X, which is globally generated by $s_i = \phi^*(x_i)$.*

(ii) *Conversely, if $\mathcal{L}$ is an invertible sheaf on X and generated by global sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$, then there exists an unique R-morphisms $X \to \mathbb{P}^n_R$ such that $\mathcal{L} \cong \phi^*\mathcal{O}_{\mathbb{P}^n_R}$ and $s_i\phi^*(x_i)$ under the isomorphism.*

Proof. Part (i) follows from the fact that $\mathcal{O}_{\mathbb{P}^n_A}$ is generated by global sections $x_i \in \Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(1))$. To see part (ii), let

$$X_i = \{p \in X \mid (s_i)_p \notin \mathfrak{m}_p\mathcal{L}_p\}.$$

Then, X_i is the complement of the vanishing scheme of s_i and is an open subscheme of X. Since s_i generates $\mathcal{L}$, we have obtained an open cover $\{X_i\}$ of X. Now, we define a morphism from X_i to the standard chart $U_i = \operatorname{Spec} R[y_0, \dots, y_n]$ with $y_j = x_j/x_i$ by the ring homomorphism

$$R[y_0, \dots, y_n] \to \Gamma(X_i, \mathcal{O}_{X_i}); \quad y_j \mapsto s_j/s_i.$$

It is straightforward to check that this gives the desired morphism. $\square$

On a smooth projective variety, we have another characterization of having enough global sections to define a morphism to a projective space.

Definition A.3.3. Let X be a non-singular projective variety over k (so that $\Gamma(X, \mathcal{O}_X) = k$).

(i) A **complete linear system** $|D_0|$ of $D_0 \in \operatorname{Div}(X)$ on X is the (possibly empty) set of effective divisors linearly equivalent to D_0. One can easily see that $|D_0|$ is in one-to-one correspondence with the set $(\Gamma(X, \mathcal{O}(D_0)) \setminus \{0\})/k^*$, which gives $|D_0|$ a structure of the set of closed points of a projective space over k. ([Har77, Proposition II.7.7.])

(ii) A **linear system** $\mathfrak{d}$ on X is a subset of a complete linear system $|D_0|$ that is a linear subspace with respect to the projective space structure of $|D_0|$. In particular, $\mathfrak{d}$ corresponds to a vector subspace $V \subset \Gamma(X, \mathcal{O}(D_0))$, where $V = \{s \in \Gamma(X, \mathcal{O}(D_0)) \mid (s)_0 \in \mathfrak{d}\} \cup \{0\}$, where $(s)_0$ denote the **divisors of zeros** of s defined as a Cartier divisor $\{U, \phi_U(s)\}$ for trivialization $\phi_U : \mathcal{O}(D_0)|_U \xrightarrow{\sim} \mathcal{O}_U$.

(iii) A point $x \in X$ is a **base point** of a linear system $\mathfrak{d}$ if $p \in \operatorname{Supp} D$ for all $D \in \mathfrak{d}$. The set of base points is called the **base locus** of $\mathfrak{d}$. A linear system $\mathfrak{d}$ is said to be **base-point-free** if the base locus is empty.

Lemma A.3.4. [Har77, Lemma II.7.8.] *Let X be a smooth projective variety and $\mathfrak{d}$ a linear system on X corresponding to the subspace $V \subset \Gamma(X, \mathcal{L})$, Then, a point $p \in X$ is a base point if and only if $s_p \in \mathfrak{m}_p \mathcal{L}_p$ for all $s \in V$. In particular, $\mathfrak{d}$ is base-point-free if and only if $\mathcal{L}$ is generated by global sections in V.*

Remark A.3.5. In short, to give a morphism from X to a projective space is equivalent to giving a base-point-free linear system whose corresponding vector space is finite-dimensional. ■

We review definitions/defining properties of positivity of divisors.

Definition A.3.6. Let X be an algebraic variety and $\mathcal{L}$ be a line bundle on X.

(i) $\mathcal{L}$ is **base-point-free** if the corresponding complete linear space is base-point-free, i.e., for any $x \in X$ there exists $E \in \mathrm{H}^0(X, \mathcal{L})$ such that $x \notin \operatorname{Supp} E$.

(ii) $\mathcal{L}$ is **very ample** if $\mathcal{L}$ is base-point-free and the corresponding morphism is an embedding into a projective space.

(iii) $\mathcal{L}$ is **ample** if $\mathcal{L}^{\otimes n}$ is very ample for $n \gg 0$, which is equivalent to the following if X is moreover proper:

  (a) (Cohomological criterion) For any coherent sheaves $\mathcal{F}$ on X, $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for $i > 0$ and $n \gg 0$ [Har77, Proposition III.5.3.].

  (b) (Intersection theory) For every closed integral subscheme $Z \subset X$, $(\mathcal{L}^{\otimes \dim Z} \cdot Z) > 0$.

  In particular, (b) is called the Nakai-Moishezon criterion [KM98, Theorem 1.37.].

(iv) $\mathcal{L}$ is **nef** if $(\mathcal{L} \cdot C) \geq 0$ for every irreducible curve $C \subset X$. More generally, one can define nef $\mathbb{Q}$-Cartier divisors in the same way.

A.4 Cones in $N_1(X)$

Now, let us observe the relations among these concepts by considering corresponding cones (see Definition 2.1.1) in a suitable vector space. Other than linear equivalence, we can also consider several equivalence relations on $\mathrm{Div}(X)$.

Definition A.4.1. Let X be a noetherian, integral, separated locally factorial scheme.

(i) Divisors $D_1, D_0 \in \mathrm{Div}(X)$ are **numerically equivalent** if $(D_1 \cdot C) = (D_0 \cdot C)$ for any curve $C \subset X$.

(ii) Divisors $D_1, D_0 \in \mathrm{Div}(X)$ are **algebraically equivalent** if there is a curve C and a divisor D on $X \times C$ flat over C, such that $[D \cap X \times \{0\}] - [D \cap X \times \{1\}] = [D_1] - [D_0]$ for some two points 0 and 1 on the curve.

Lemma A.4.2. *Linear equivalence implies algebraic equivalence and algebraic equivalence implies numerical equivalence.*

We can also consider the quotient group with respect to algebraic equivalence.

Definition A.4.3. Let X be a noetherian, integral, separated locally factorial scheme. We define the **Néron-Severi group** $\mathrm{NS}(X)$ to be the quotient group $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$, where $\mathrm{Pic}^0(X)$ is the connected component. Since closed points of $\mathrm{Pic}^0(X)$ are algebraically trivial divisors (i.e., algebraically equivalent to 0), $\mathrm{NS}(X)$ geometrically represents the algebraic equivalence classes of divisors on X. To observe cones, we usually consider the real vector space $N^1(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Remark A.4.4. We have the following diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & \mathrm{CaPrinc}(X) \arrow[d] & & \\
 & \mathrm{CaDiv}(X) \arrow[d] & & \\
0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & \mathrm{NS}(X) \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

■

Construction A.4.5. Let $N_1(X)$ denote the quotient of the formal $\mathbb{R}$-linear group of 1-cycles (i.e. irreducible, reduced curves) by the numerical equivalence, where two 1-cycles C and C' are said to be **numerically equivalent** if $(C \cdot D) = (C' \cdot D)$ for any Cartier divisor D. Then, we have a perfect paring

$$N_1(X) \times N^1(X) \to \mathbb{R}, \quad (Y, Z) \mapsto (Y \cdot Z),$$

which is well-defined by the definition of numerical equivalence of cycles and the fact that algebraic equivalence implies numerical equivalence. Hence, by the theorem of the base of Néron-Severi, which says $N^1(X)$ is finite-dimensional, we see that $N_1(X)$ is finite dimensional, whose dimension is called the **Picard number** and denoted by $\rho(X)$. ■

Now, notice that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are nef, effective, ample, then so is any positive linear combination of $\mathcal{L}_1$ and $\mathcal{L}_2$. Hence, they define cones of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Furthermore, since they also respect the numerical equivalence, we see that the corresponding cones of $\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ descend to the cones of $N^1(X)$. Note we can also consider the corresponding dual cones in $N_1(X)$. There are several criteria that can be written with the language of cones in $N^1(X)$ and $N_1(X)$ and the following result is particularly useful in this thesis:

Theorem A.4.6 (Kleiman's Ampleness Criterion)**.** [KM98, Theorem 1.18.] *Let X be a projective variety and D a Cartier divisor. Then, D is ample if and only if*

$$N_1(X)_{D>0} \supset \overline{\mathrm{NE}}(X) \setminus \{0\},$$

where $N_1(X)_{D>0}$ denotes $\{x \in N_1(X) \mid (x \cdot D) > 0\}$ and $\mathrm{NE}(X)$ denotes the cone of curves (Definition 2.1.2).

Appendix B

Algebraic groups and quotients

B.1 Linear algebraic groups

We quickly review some facts regarding linear algebraic groups (e.g. cf. [Dré00, Section 2.3.]).

Definition B.1.1. An **algebraic group** is a group object in the category of algebraic varieties. A closed subscheme of an algebraic group is called an **algebraic subgroup** if it inherits the algebraic group structure.

Construction B.1.2. The general linear group $\mathrm{GL}_n(k)$ over a field k can be equipped with an affine scheme structure by regarding it as a subscheme of $\mathbb{A}^{n^2+1} = \operatorname{Spec} k[\{T_{ij}\}_{i,j}, T]$ cut out by the equation

$$(\det(T_{ij})_{i,j}) \cdot T = 1.$$

In other words, we have $\mathrm{GL}_n(k) = \operatorname{Spec} k[\{T_{ij}\}_{i,j}, \det^{-1}]$, where $\det = \det(T_{ij})_{i,j}$. For example,

$$\mathrm{GL}_1(k) = \operatorname{Spec} k[T, T^{-1}]$$

is called **the multiplicative algebraic group** and denoted by $\mathbb{G}_{m/k}$. Now, $\mathrm{GL}_n(k)$ has the structure of an algebraic group given by the comultiplication

$$\Delta : k[\{T_{ij}\}_{i,j}, \det{}^{-1}] \to k[\{T_{ij}\}_{i,j}, \det{}^{-1}] \otimes_k k[\{T_{ij}\}_{i,j}, \det{}^{-1}] : \quad T_{ij} \mapsto \sum_{1 \le l \le n} T_{il} \otimes T_{lj}.$$

Note the comultiplication Δ induces the usual matrix multiplication on (the closed points of) $\mathrm{GL}_n(k)$. An algebraic subgroup of $\mathrm{GL}_n(k)$ is called a **linear algebraic group**. ■

Example B.1.3. Any finite group G with $|G| = n$ can be viewed as a closed subgroup of $\mathrm{GL}_n(k)$ as permutations of n elements, which gives a reduced k-scheme structure to G. ■

Definition B.1.4. Let G be an algebraic group with multiplication morphism $m : G \times_k G \to G$ and identity element $e \in G$.

(i) A variety X over k together with a morphism $\psi : G \times_k X \to X$ of varieties over k is said to be a **G-variety** if $\psi(g_2, \psi(g_1, x)) = \psi(m(g_2, g_1), x)$ and $\psi(e, x) = x$ for all $x \in X$ and $g_1, g_2 \in G$. The morphism ψ is often omitted from the notation and $\psi(g, x)$ is often denoted by $g \cdot x$.

(ii) Let X and Y be G-varieties over k. Then, a morphism $f : X \to Y$ of varieties over k is said to be a **G-morphism** if $f(g \cdot x) = g \cdot f(x)$ for all $x \in X$ and $g \in G$.

(iii) Let X be a G-variety over k and Y a variety over k. Then, a morphism $f : X \to Y$ of varieties over k is said to be **G-invariant** if $f(g \cdot x) = f(x)$ for all $x \in X$ and $g \in G$.

B.2 Complex reductive groups and quotients

Notions of quotients of G-varieties by the G-action could be troublesome in some cases, but here we will only focus on some simple cases, in particular over $\mathbb{C}$.

Definition B.2.1. Let X be a real algebraic variety. Then, the **complexification** $X_{\mathbb{C}}$ of X is the complex algebraic variety $X \times_{\mathbb{R}} \operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{C}$.

Since the category of algebraic groups is closed under fiber product, the complexification of an algebraic group has a canonical group structure.

Definition B.2.2. Let K be a real algebraic group whose real points $K(\mathbb{R})$ form a real compact Lie group. The complexification $G = K_{\mathbb{C}}$ of K is called a **complex reductive group** and K is a maximal compact subgroup of G.

Remark B.2.3. The preceding definition of reductive groups is obviously not general. Generally, a connected algebraic group over a field k of characteristic zero is said to be **reductive** if the category of finite dimensional representations of G over k is **semi-simple**, i.e. every finite dimensional representation is isomorphic to the direct sum of irreducible subrepresentations ([Mil15, Theorem 22.138.]). These definitions indeed coincide over $\mathbb{C}$ (cf. [Kam, Section 5.2]). ■

Example B.2.4. Note by Cartan's closed subgroup theorem, any closed subgroup of a Lie group is a Lie subgroup. Hence, compact subgroups of GL_n produce a lot of examples. For example, $\mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}_n(\mathbb{C})$ (see below when $n = 1$), $\mathrm{SU}(n)_{\mathbb{C}} = \mathrm{SL}_n(\mathbb{C})$, $\mathrm{Sp}(n)_{\mathbb{C}} = \mathrm{Sp}_{2n}(\mathbb{C})$, etc. Furthermore, finite groups are also complex reductive groups with trivial $\mathbb{C}$-action. ■

Example B.2.5. Let $K = \mathrm{U}(1) = S^1 = \operatorname{Spec} \mathbb{R}[X, Y]/\langle X^2 + Y^2 - 1\rangle_{\mathbb{R}}$. Then,

$$K_{\mathbb{C}} \cong \operatorname{Spec}\big((\mathbb{R}[X, Y]/\langle X^2 + Y^2 - 1\rangle_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}\big) \cong \operatorname{Spec} \mathbb{C}[u, v]/\langle uv - 1\rangle_{\mathbb{C}} \cong \operatorname{Spec} \mathbb{C}[X]_X \cong \mathbb{G}_{m/\mathbb{C}}.$$

Since $\mathbb{G}_{m/\mathbb{R}} \times_{\mathbb{R}} \mathbb{C} \cong \mathbb{G}_{m/\mathbb{C}}$, there are non-isomorphic real algebraic groups whose complexifications are isomorphic. ■

One of the most useful facts, due to Nagata, about complex reductive groups is the following, which is false for a general G.

Lemma B.2.6. [Nag64, Main Theorem] *If G is a complex reductive group and A is a finitely generated $\mathbb{C}$-algebra, then A^G is a finitely generated $\mathbb{C}$-algebra, where A^G denotes the subring of G-invariant elements.*

Hence, we can define the following

Definition B.2.7. Let G be a reductive group and $X = \operatorname{Spec} A$ be an affine G-variety. Then, the inclusion $A^G \hookrightarrow A$ induces a morphism of affine varieties by Lemma B.2.6:

$$\pi : X \to X//G := \operatorname{Spec} A^G,$$

The pair $(X//G, \pi)$ is called the **affine (GIT) quotient** of X by G.

More generally, we can construct a (GIT) quotient when affine quotients glue successfully. Let me quickly explain some important properties of affine quotients.

Definition B.2.8. Let G be an algebraic group and X a G-variety. Then, a pair (Y, p) of a variety Y together with a G-invariant morphisms $p : X \to Y$ is said to be:

(i) a **categorical quotient** if for every G-invariant morphism $f : X \to Z$, there exists a unique morphism $\overline{f} : Y \to Z$ such that $\overline{f} \circ p = f$.

(ii) a **good quotient** if p satisfies the following:

  (a) p is affine and surjective;

  (b) the natural homomorphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(\pi^{-1}(U))^G$ is an isomorphism for every open subset $U \subset Y$;

  (c) If V_1 and V_2 are G-invariant closed subsets of X with $W_1 \cap W_2 = \emptyset$, then $p(V_1) \cap p(V_2) = \emptyset$.

Note that a good quotient is local on the target. Also, a good quotient is a categorical quotient ([New06, Proposition 1.11]).

Lemma B.2.9. [Dré00, Theorem 2.16.] *An affine quotient $(X//G, \pi)$ of an affine G-variety X by a (complex) reductive group G is a good quotient.*

Since a good quotient is local on the target, general (GIT) quotients are good quotients and in particular categorical quotients. Now, good quotients are "good" for example in the following sense:

Lemma B.2.10. [Dré00, Proposition 2.15.] *Let G be an algebraic group and X a G-variety. Suppose a good quotient (Y, p) of X by G exists. Then, if X is normal (resp. reduced, resp. irreducible, resp. connected), so is Y.*

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