# Positivity of line bundles in derived categories

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## Motivation

Using the theory of Matsui spectra, the reconstruction theorem of Bondal-Orlov was recently generalized to the following setting.

**Definition 1** *A line bundle $\mathcal{L}$ is said to be* $\otimes$***-ample*** *if* $\langle \mathcal{L}^{\otimes n} \mid n \in \mathbb{Z} \rangle = \mathsf{Perf}\, X$.

On a quasi-projective variety, any (anti-)ample line bundle is $\otimes$-ample.

**Theorem 2 (Ito-Matsui, Ito)** *Let $X$ be a Gorenstein proper variety with $\otimes$-ample canonical bundle $\omega_X$. Then, the following assertions hold:*

- *$X$ can be reconstructed from the triangulated category structure of $\mathsf{Perf}\, X$;*
- *If we have $\mathsf{Perf}\, X \simeq \mathsf{Perf}\, Y$ with a variety $Y$, then $X \cong Y$.*

Here, note we do not require any projectivity, which is more natural from a dg-categorical perspective. Now, a natural question is whether or not this is actually a generalization.

**Question 3** *Are there $\otimes$-ample line bundles that are neither ample nor anti-ample?*

**Main Theorem (Ito-Olander)** Let $X$ be a proper variety and $\mathcal{L}$ a line bundle. TFAE:

1. $\mathcal{L}$ is $\otimes$-ample.
2. $\mathcal{L}|_Z$ is big or anti-big for every closed integral subscheme $Z \subset X$.

The theorem provides a variety of examples and well-behaved theory of $\otimes$-ample line bundles.

## Big line bundles and affine complements

A key technical ingredient is the following characterization of big line bundles, which is interesting in its own right in relation to divisors with affine complements.

**Proposition 4** *Let $X$ be an integral qcqs scheme and $\mathcal{L}$ a line bundle. TFAE:*

1. *There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open $X_s := \{s \neq 0\} = X \setminus V(s)$ is non-empty and quasi-affine.*
2. *There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open $X_s$ is non-empty and affine.*
3. *There exists an integer $n > 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that the open $X_s$ is non-empty and there exists an integral domain $R$ and a morphism*
$$X_s \to \operatorname{Spec}(R)$$
*whose generic fiber $(X_s)_K$ is quasi-affine where $K = \operatorname{Frac}(R)$.*

*If $X$ is a proper variety, then they are further equivalent to:*

4. *$\mathcal{L}$ is big, i.e., there exist constants $m_0, C > 0$ such that $\dim_k \Gamma(X, \mathcal{L}^{\otimes m_0 m}) > C \cdot m^{\dim X}$ for any $m \gg 0$.*

**Definition 5** *Let $X$ be an integral qcqs scheme. A line bundle $\mathcal{L}$ is* ***big*** *if the equivalent conditions 1-3 hold.*

Our main theorem indeed holds in the generality of noetherian schemes with this notion of big line bundles. A proof of the main theorem goes as follows: ($1 \Rightarrow 2$) It is easy to see $\otimes$-amplitude is preserved under quasi-affine pullbacks, so it suffices to show a $\otimes$-ample line bundle on an integral closed subscheme is big or anti-big. Indeed, $\otimes$-amplitude provides a desired section. ($2 \Rightarrow 1$) We can show if there exists $n \neq 0$ and a global section $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that $X_s$ is quasi-affine and $\mathcal{L}|_{V(s^r)}$ is $\otimes$-ample for each $r \geq 0$, then $\mathcal{L}$ is $\otimes$-ample. Now we can conclude by noetherian induction.

## Consequences of the main theorem

Interestingly we have not found a direct proof to the following fundamental facts.

**Lemma 6** *Let $\mathcal{L}$ be a line bundle on a noetherian scheme $X$. Let $0 \neq n \in \mathbf{Z}$. Let $f : Y \to X$ be a finite surjective morphism of schemes.*

1. *$\mathcal{L}$ is $\otimes$-ample if and only if $\mathcal{L}^{\otimes n}$ is $\otimes$-ample.*
2. *$\mathcal{L}$ is $\otimes$-ample if and only if the restriction of $\mathcal{L}$ to the irreducible components of $X$ (with the reduced subscheme structure) are $\otimes$-ample.*
3. *$\mathcal{L}$ is $\otimes$-ample if and only if $f^*\mathcal{L}$ is $\otimes$-ample.*

In practice, the following seems to be a useful way to check if a line bundle is $\otimes$-ample.

**Lemma 7** *Let $X$ be a noetherian scheme and $\mathcal{L}$ a line bundle on $X$. Let $n_1, \ldots, n_k \in \mathbf{Z}$ be integers and $s_i \in \Gamma(X, \mathcal{L}^{\otimes n_i})$ global sections. Assume:*

1. *$X_{s_i}$ have $\otimes$-ample structure sheaf (for example if each $X_{s_i}$ is quasi-affine).*
2. *The restriction of $\mathcal{L}$ to the reduced closed subscheme $V(s_1, \ldots, s_k)_{red} \subset X$ cut out by $s_1, \ldots, s_k$ is $\otimes$-ample.*

*Then $\mathcal{L}$ is $\otimes$-ample.*

## $\otimes$-ample cone in $N^1(X)$

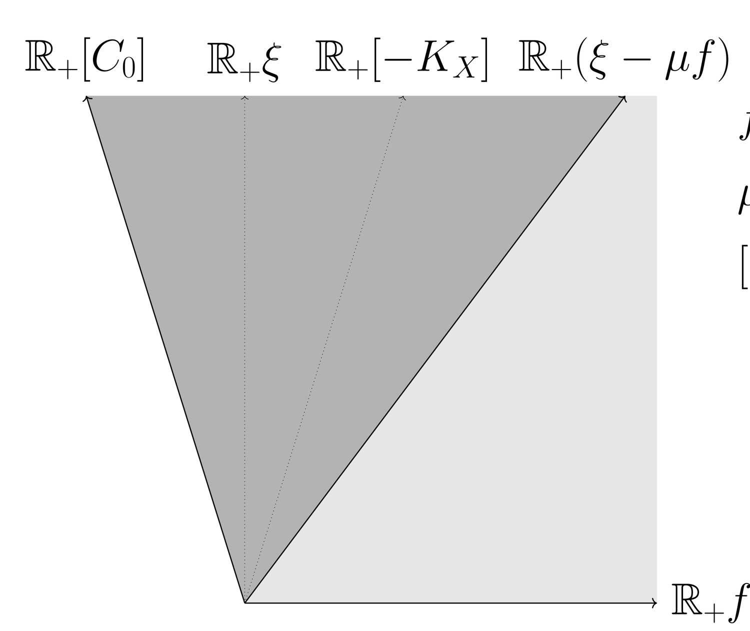
Our main theorem allows numerical studies of $\otimes$-ample line bundles.

**Definition 8** *Let $X$ be a proper variety. We say an $\mathbb{R}$-Cartier divisor on $X$ is* $\otimes$***-ample*** *if its restriction to any closed subvariety is linearly equivalent to a big or anti-big $\mathbb{R}$-Cartier divisor. Let*
$$\otimes\text{-}\mathrm{Amp}(X) \subset N^1(X) = \mathrm{Div}(X)_{\mathbb{R}} / \equiv_{\mathrm{num}}$$
*denote the cone of $\otimes$-ample $\mathbb{R}$-Cartier divisors up to numerical equivalence.*

**Lemma 9** *Let $X$ be a proper variety.*

- *A Cartier divisor is $\otimes$-ample if and only if its numerical class lies in $\otimes$-$\mathrm{Amp}(X)$.*
- *$\otimes$-$\mathrm{Amp}(X) \cap \mathrm{Nef}(X) = \mathrm{Amp}(X)$. In particular, $\otimes$-$\mathrm{Amp}(X) \cap \partial \mathrm{Nef}(X) = \emptyset$.*

Note $\otimes$-$\mathrm{Amp}(X)$ is indeed computable. Let $\pi : X = \mathbb{P}_X(\mathcal{E}) \to C$ be a ruled surface over a projective curve $C$. Suppose $\mathcal{E}$ is unstable with a destabilizing quotient $\mathcal{E} \twoheadrightarrow \mathcal{Q}$, which corresponds to a section $C_0 \subset X$. Then, we get a complete description of cones in $N^1(X)$.




$f = [\text{fiber of } \pi], \xi = [\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)] \in N^1(X)$

$\mu = \deg \mathcal{Q},\ d = \deg \mathcal{E},\ g = g(C)$

$[C_0] = \xi + (\mu - d)f,\ [K_X] = -2\xi + (2g - 2 + d)f$

$\mathrm{Amp}(X)$

$\otimes$-$\mathrm{Amp}_+(X) \setminus \mathrm{Amp}(X)$

$\mathrm{Nef}(X) = \langle \mathbb{R}_+(\xi - \mu f), \mathbb{R}_+ f \rangle$

$\mathrm{PsEff}(X) = \langle \mathbb{R}_+[C_0], \mathbb{R}_+ f \rangle$

Here, $\otimes$-$\mathrm{Amp}_+(X) = \otimes$-$\mathrm{Amp}(X) \cap \mathrm{Big}(X)$.

## Examples of $\otimes$-ample line bundles

The following completely answer Question 3 for smooth projective surfaces.

**Lemma 10** *A line bundle $\mathcal{L}$ on a proper surface is $\otimes$-ample if and only if $\deg \mathcal{L}|_C \neq 0$ for every integral closed curve $C \subset X$ with $C^2 < 0$. Moreover,*
$$\otimes\text{-}\mathrm{Amp}_+(X) = \mathrm{Big}(X) \setminus \bigcup \{C^\perp : C \subset X \text{ an integral curve with } C^2 < 0\}.$$

**Lemma 11** *A smooth projective surface $X$ has a $\otimes$-ample but neither ample nor anti-ample line bundle if and only if there is an integral curve $C \subset X$ with $C^2 < 0$.*

Other types of examples include but not limited to the following.

- Let $X$ be a quasi-projective variety and let $\pi : X \to Y$ be the blow-up at finitely many points with corresponding exceptional divisors $E_i$. Then for any ample line bundle $\mathcal{L}$ and for any $l_i > 0$, $\pi^*\mathcal{L} \otimes \bigotimes \mathcal{O}_X(l_i E_i)$ is is neither ample nor anti-ample, but $\otimes$-ample.
- Let $X = \mathrm{Bl}_{(0,0)} \mathbb{A}^2_k \setminus \{p\}$ where $p$ is a $k$-point of an exceptional divisor. Then the structure sheaf $\mathcal{O}_X$ is $\otimes$-ample but not ample. In particular, being quasi-projective and having a $\otimes$-ample structure sheaf do not imply being quasi-affine.
- The affine space with doubled origin has the $\otimes$-ample structure sheaf. In particular, having a $\otimes$-ample line bundle does not imply neither separated nor having a resolution property.
- If $X$ is a union of two copies of $\mathbf{P}^1$ glued along a node and $\mathcal{L}$ is obtained by gluing $\mathcal{O}(1)$ on one copy with $\mathcal{O}(-1)$ on the other, then $\mathcal{L}$ is $\otimes$-ample.
- Hironaka's example of a non-projective proper variety has a $\otimes$-ample line bundle.

## Examples of $\otimes$-ample canonical bundles

Some of the previous examples provide varieties with $\otimes$-ample (but neither ample nor anti-ample) canonical bundle, to which we can apply Theorem 2.

- For a smooth projective surface $X$, $\omega_X$ is $\otimes$-ample if and only if $\omega_X$ is big or anti-big and $X$ contains no $(-2)$-curve. For example, a smooth projective toric surface with no $(-2)$-curve, a projective bundle of an unstable rank 2 vector bundle over an elliptic curve, and the blow-up of $\mathbb{P}^2$ at r points on a line with $r \neq 3$ have $\otimes$-ample canonical bundles.
- If $X$ is a quasi-projective variety with ample canonical bundle (e.g. a quasi-affine variety), then its blow-up at finitely many closed points has a $\otimes$-ample canonical bundle.
- Take the Fermat hypersurface $X = \{x_0^d + \cdots + x_4^d = 0\} \subset \mathbb{P}^4_k$ with odd $d > 5$. Then, there is a line $l \subset X$ with normal bundle $\mathcal{O}_l(1) \oplus \mathcal{O}_l(2-d)$. Then the blow-up of $X$ along the line $l$ has a $\otimes$-ample canonical bundle.
- A proper toric variety has a $\otimes$-ample canonical bundle if and only if the restriction of the canonical line bundle to the torus boundary divisor is $\otimes$-ample.

## References


- D. Ito and H. Matsui, A new proof of the Bondal-Orlov reconstruction theorem using Matsui spectra (to appear in Bull. Lond. Math. Soc.)
- D. Ito, Polarizations on a triangulated category (available at arXiv:2502.15621)
- D. Ito and N. Olander, On $\otimes$-ample line bundles (coming soon)