Projectile Motion Lesson

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May 17, 2022

Contents

We structure this lesson following Section 12.6 (Motion of a Projectile) from the classic Hibbler text "Engineering Mechanics Dynamnics, 10th edition".

1 Learning Objectives

- Derive kinematic equations for 2D projectile motion from kinematic equations from 1D rectilinear motion
- Identify the assumptions required for the projectile motion equations to hold:
- Air resistance is neglected

  - Gravitational acceleration acts downward and is constant, regardless of altitude

- Solve any given (well-defined) free-flight projectile motion problems by:
  - Able to select an appropriate Cartesian coordinate system to simplify the problem as much as possible
  - Able to identify the known variables
  - Able to identify the unknown variables
  - Able to write projectile motion equations for the given problem
  - Able to solve the projectile motion equations for the unknown quantities

2 Rectilinear Kinematics: Continuous Motion (Recap)

As covered previously, the equations relating velocity (v), position (p) and time (t) for motion in one dimension with constant acceleration (a) are as follows:

$$v = v^i + at \tag{1}$$

$$p = p^i + v^i t + \frac{1}{2}at^2 \tag{2}$$

$$v^2 = (v^i)^2 + 2a(p - p^i) \tag{3}$$

where v^i and p^i are the initial velocity and position, respectively.

Only two of these equations are independent, since the third equation can always be derived from the other two.

3 Motion of a Projectile

The free flight motion of a projectile is often studied in terms of its rectangular components in the $x - y$ plane, since the projectile's acceleration **always** acts in the vertical direction. To illustrate the kinematic analysis, consider a projectile launched at point (p_x^i, p_y^i) with the initial velocity $\mathbf{v}^i$ having components v_x^i and v_y^i, as shown in Figure 1. The position vector $\mathbf{p}$ changes over time. At any instant of time t the position is represented by the components p_x and p_y. When we assume that air resistance is neglected, the only force acting on the projectile is its weight, which causes the projectile to have a **constant downward acceleration** of approximately $a = a_y = -g = -9.81\text{m/s}^2$ or $g = 32.2\text{ft/s}^2$.[1]

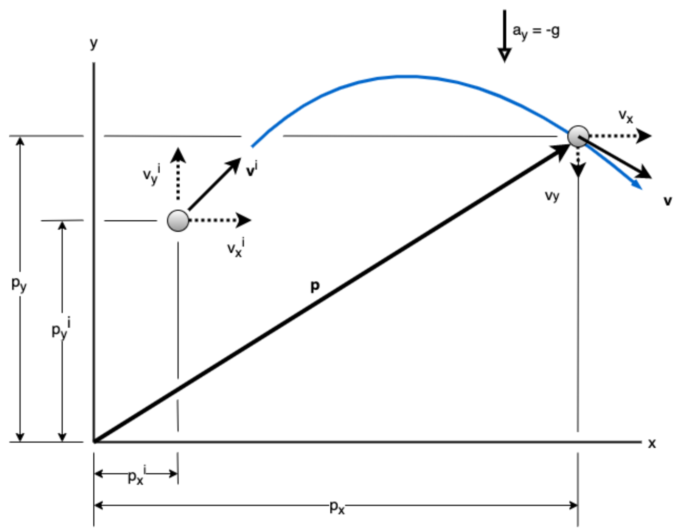

Figure 1: Coordinate System and Definition of Symbols

The equations for rectilinear kinematics given above (Equations 1, 2 and 3) are in one dimension. These equations can be applied for both the vertical motion and the horizontal directions, as follows:

3.1 Horizontal Motion

For projectile motion the acceleration in the horizontal direction is constant and equal to zero ($a_x = 0$). This value can be substituted in the equations for constant acceleration given above to yield the following:

From Equation 1:

$$v_x = v_x^i \tag{4}$$

From Equation 2:

$$p_x = p_x^i + v_x^i \tag{5}$$

From Equation 3:

$$v_x = v_x^i \tag{6}$$

Since the acceleration in the x direction (a_x) is zero, the horizontal component of velocity always remains constant during motion.

[1]: This assumes that the earth's gravitational field does not vary with altitude

3.2 Vertical Motion

Since the positive y axis is directed upward, the acceleration in the vertical direction is $a_y = -g$. This value can be substituted in the equations for constant acceleration given above to yield the following:

From Equation 1:

$$v_y = v_y^i - gt \tag{7}$$

From Equation 2

$$p_y = p_y^i + v_y^i t - \frac{1}{2}gt^2 \tag{8}$$

From Equation 3:

$$v_y^2 = (v_y^i)^2 - 2g(p_y - p_y^i) \tag{9}$$

Recall that the last equation can be formulated on the basis of eliminating the time t between the first two equations and therefore **only two of the above three equations are independent of one another**.

3.3 Summary

In addition to knowing that the horizontal component of velocity is constant, problems involving the motion of a projectile can have at most three unknowns since only three independent equations can be written: that is, **one** equation in the **horizontal direction** and **two** in the **vertical direction**. Once v_x and v_y are obtained, the resultant velocity $\mathbf{v}$, which is **always tangent** to the path, is defined by the **vector sum** as shown in Figure 1.

3.4 Procedure for Analysis

Free-flight projectile motion problems can be solved using the following procedure.

3.4.1 Step 1: Coordinate System

- Establish the fixed x, y coordinate axes and sketch the trajectory of the particle. Between any **two points** on the path specify the given problem data and the **three unknowns**. In all cases the acceleration of gravity acts downward. The particle's initial and final velocities should be represented in terms of their x and y components.

- Remember that positive and negative position, velocity, and acceleration components always act in accordance with their associated coordinate directions.

- The two points that are selected should be significant points where something about the motion of the particle is known. Potential significant points include the initial point of launching the projectile and the final point where it lands. The landing point often has a known y value.

- The variables in the equation may need to be changed to match the notation of the specific problem. For instance, a distinction may need to be made between the x-coordinate of points A and B, via notation like p_x^A and p_x^B.

3.4.2 Step 2: Identify Knowns

Using the notation for the problem in question, write out the known variables and their values. The known variables will be a subset of the following: $p_x^i, p_x, p_y^i, p_y, v_x^i, v_x, v_y^i, v_y$ and t. The knowns should be written in the notation adopted for the particular problem.

3.4.3 Step 3: Identify Unknowns

Each problem will have at most 4 unknowns that need to be determined, selected from the variables listed in the Step 2 that are not known. The number of relevant unknowns will usually be less than 4, since questions will often focus on one or two unknowns. As an example, the equation that horizontal velocity is constant is so trivial that most problems will not look for this as an unknown. The unknowns should be written in the notation adopted for the particular problem.

3.4.4 Step 4: Kinematic Equations

Depending upon the known data and what is to be determined, a choice should be made as to which four of the following five equations should be applied between the two points on the path to obtain the most direct solution to the problem.

1. Step 4.1: Horizontal Motion

   From Equation 4: $v_x = v_x^i$ (The **velocity** in the horizontal or x direction is **constant**)

   From Equation 5: $p_x = p_x^i + v_x^i t$

2. Step 4.2: Vertical Motion

   In the vertical or y direction **only two** of the following three equations (using $a_y = -g$) can be used for solution. (The sign of g will change to positive if the positive y axis is downward.)

From Equation 7: $v_y = v_y^i - gt$

From Equation 8: $p_y = p_y^i + v_y^i t - \frac{1}{2}gt^2$

From Equation 9: $v_y^2 = (v_y^i)^2 - 2g(p_y - p_y^i)$

For example, if the particle's final velocity v_y is not needed, then the first and third of these questions (for y) will not be useful.

3.4.5 Step 5: Solve for Unknowns

Use the equations from Step 4, together with the known values from Step 2 to find the unknown values from Step 3. We can do this systematically by going through each equation and determining how many unknowns are in that equation. Any equations with one unknown can be used to solve for that unknown directly.

4 Example (Sack Slides Off of Ramp)

A sack slides off the ramp, shown in Figure 1. We can ignore the physics of the sack sliding down the ramp and just focus on its exit velocity from the ramp. There is initially no vertical component of velocity and the horizontal velocity is:

```
horiz_velo = 17 #m/s.
```

Task: Determine the time needed for the sack to strike the floor and the range R where sacks begin to pile up.

The acceleration due to gravity g is assumed to have the following value.

```
g = 9.81 #m/s^2
```

4.1 Solution

Step 1: Coordinate System. The origin of the coodinates is established at the beginning of the path, point A (Figure 2). The initial positions and velocities will be taken at Point A and the final positions and velocities will be taken at Point B. Points A and B were selected because we know values at the launch point (Point A) and we wish to find values related to the time of flight and the landing point (Point B).

With respect to notation, we identify point A as the initial point and point B as the final point. Therefore, in equations **??**, 5, **??**, 8 and 9 we have the following new notation:

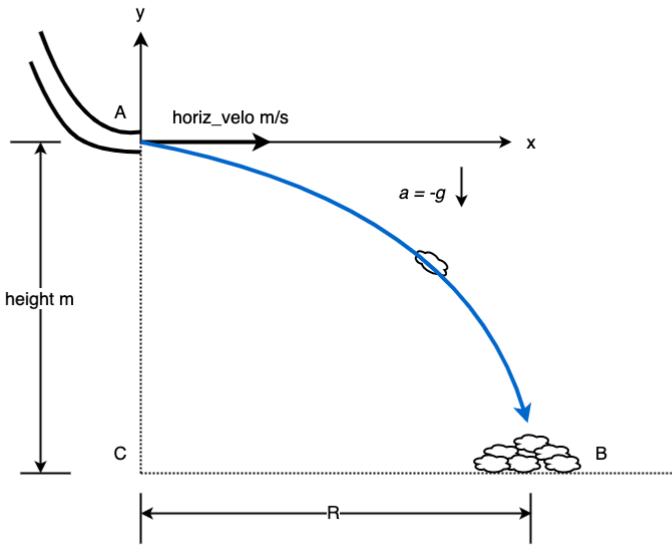

Figure 2: Coordinate System and Definition of Symbols

$p_x^i = p_x^A, p_x = p_x^B, p_y^i = p_y^A, p_y = p_y^B, v_x^i = v_x^A, v_x = v_x^B, v_y^i = v_y^A, v_y = v_y^B$
and t_{AB} refers to the time that passes when the particle moves from Point A to Point B.

Step 2: Identify Knowns. We know values for 5 of the 9 possible variables:

- $p_x^A = 0$
- $p_y^A = 0$

- $v_x^A =$

```
pAx = 0
pAy = 0
vAx = horiz_velo
vAy = 0
pBy = -height
```

Step 3: Identify Unknowns. According to the original question our goal is to find 2 unknowns:

- t_{AB}: the time needed for the sack to strike the floor.
- p_x^B: the range R, which is the x-coordinate of the final position.

(We also have 2 other unknowns that are not specifically asked for: v_x^B and v_y^B. We may need to solve for these as part of the solution for the requested unknown values.)

Step 4: Kinematic Equations.

Step 4.1: Horizonal Motion. From 4 we know:

$$v_x^B = v_x^A \tag{10}$$

From 5 we know:

$$p_x^B = p_x^A + v_x^A t_{AB} \tag{11}$$

Step 4.2: Vertical Motion. From 7 we know:

$$v_y^B = v_y^A - g t_{AB} \tag{12}$$

From 8 we know:

$$p_y^B = p_y^A + v_y^A t_{AB} - \frac{1}{2} g t_{AB}^2 \tag{13}$$

From 9 we know:

$$(v_y^B)^2 = (v_y^A)^2 - 2g(p_y^B - p_y^A) \tag{14}$$

Step 5: Solve for the Unknowns. We can go through each of the above 5 equations to see how many unknowns are in each equation:

- Eq_1 has one unknown: v_x^B
- Eq_2 has two unknowns: p_x^B, t_{AB}
- Eq_3 has two unknowns: v_y^B, t_{AB}
- Eq_4 has one unknown: t_{AB}
- Eq_5 has one unknown: v_y^B

From Step 3, we know that our goal is to find t_{AB} and p_x^B. We can see that 13 allows us solve for t_{AB}. Once we have this value, we can use 11 to solve for p_x^B. (Although we could also solve for v_x^B and v_y^B, using 10 and 12 (or 14), respectively, we are not asked to do so by the question.)

Find t_{AB} from 13:
$p_y^B = p_y^A + v_y^A t - \frac{1}{2} g t_{AB}^2$
Since $v_y^A = 0$,

$p_y^B = p_y^A - \frac{1}{2}gt_{AB}^2$

We can rearrange the above equation to solve for t_{AB}:

$t_{AB} = \sqrt{(p_y^A - p_y^B)/\frac{1}{2}(g)}$

The following code solves for t_{AB}.

```
import math
tAB = math.sqrt((pAy - pBy)/(0.5*(g)))
print("ANSWER tAB = ", tAB, "s")
```