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A discrete-time retrial queue with negative customers and unreliable server

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ABSTRACT

This paper treats a discrete-time single-server retrial queue with geometrical arrivals of both positive and negative customers in which the server is subject to breakdowns and repairs. Positive customers who find sever busy or down are obliged to leave the service area and join the retrial orbit. They request service again after some random time. If the server is found idle or busy, the arrival of a negative customer will break the server down and simultaneously kill the positive customer under service if any. But the negative customer has no effect on the system if the server is down. The failed server is sent to repair immediately and after repair it is assumed as good as new. We analyze the Markov chain underlying the queueing system and obtain its ergodicity condition. The generating functions of the number of customers in the orbit and in the system are also obtained along with the marginal distributions of the orbit size when the server is idle, busy or down. Finally, some numerical examples show the influence of the parameters on some crucial performance characteristics of the system.

1. Introduction

Queueing models with negative customers, or G-queues, have found applications in computer communications and manufacturing settings. When a negative customer arrives at the queue, it has the effect of a signal which induces ordinary (positive) customers to leave immediately the node. Artalejo (2000) and Gelenbe (1994, 2000) provided excellent surveys on this topic. For the application on industrial engineering, the reader is referred to the book by Chao, Miyazawa, and Pinedo (1999).

Although many continuous-time queueing models with negative arrivals have been studied extensively in the past years, their discrete-time counterparts received very little attention in the literature. In fact, to the best of our knowledge, the only work about negative customers in discrete-time (without retrials) can be found in Atencia and Moreno(2004a), Atencia and Moreno (2004b), Atencia and Moreno (2005), where the authors considered the single-server discrete-time queue with negative arrivals and various killing disciplines caused by the negative customers. It should be noted that discrete-time queueing models are particularly appropriate to describe the various queueing related phenomena in digital computer and communication systems including mobile and BISDN networks based on asynchronous transfer mode (ATM) technology, due to the packetized nature of these transport protocols, see for instance Bruneel and Kim (1993), Hunter (1983), Takagi (1993), Woodward (1994) and the references therein.

On the other hand, reliability modelling using G-queues has also found applications in communication network. In Harrison, Patel, and Pitel (2000), a breakdown is represented by a negative customer arriving at a queue which removes a random number (batch) of customers, if any are present, chosen according to a given killing strategy. Recently, Wang, Liu, and Li (2008) studied an $M/G/1$ retrial queue subject to disasters and server failures in which the server is subject to breakdowns and repairs because of the arrivals of negative customers or the inherent server's life time.

Nevertheless, discrete-time G-queues taking the retrial phenomenon into account have not been studied up to now. Retrial queues have been well-known in the computer systems literature. With the recent advancements in mobile communications the issue of retrials is becoming more recognized. The main difference between the retrial queues and the standard queues is that a user who is waiting to retry may even miss an idle server since the customer is not always aware of when the server is idle. The reader is referred to Artalejo (1999), Falin (1990), Falin and Templeton (1997), Kulkarni and Liang (1997), Yang and Templeton (1987) for systematic account of the main results on this topic. More recently, a series of work (see Section 14 in Artalejo, 2000) by Artalejo and Gómez-Corral has investigated single-server queues operating under the simultaneous presence of negative arrivals and repeated attempts in continuous-time. However, in some applications, arrivals, service times and other queueing activities, if any, are synchronized, and discrete time models may be more appropriate. In such case, the concurrent movement is the most significant feature.

The paper aims to present a discrete-time single-server retrial queue with two flows of arrivals, both positive and negative, and

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a server subject to breakdowns and repairs. An example of a queueing system with both positive and negative flows of arrivals is computer networks with virus infection. When there are no viruses, computer networks have been modeled and analyzed using conventional queueing networks with or without retrials. The customers represent the execution of algorithmic and logic operations, and the nodes represent CPUs, I/O devices, etc. When a virus enters a node, one or more files may be infected, and the system manager may have to go through a number of backups to recover the infected files. In some cases, they may not be recoverable. A virus may originate from outside the network, e.g., through a floppy disk, or may come from another node in the network, e.g., by an electronic mail. The recover time of the infected files can be regarded as repair time of the server (node) due to an arrival of negative customer (virus).

Another interesting example concerning the presence of disasters in discrete-time queueing systems is studied in a very recent paper published in this journal, see Jolai, Asadzadeh, and Taghizadeh (2008). They modeled an email contact center with disasters as a finite source discrete-time $Geo/Geo/1/\infty/N$ queue with disasters, but without customer retrials. However, it is well known that in a contact (call) center management scenario, the impact of customer retrial phenomenon can not be ignored for the performance of the whole system (see Aguir, Karaesmen, Akşin, & Chauvet, 2004). Jolai, Asadzadeh, and Taghizadeh (2008) also pointed out the flexibility of combining the retrial process in their model at the end of the paper.

Thus, the contribution of this work is not only extending the retrial queueing theory on negative arrivals with two flows of arrivals and reliability modelling using G-queues to the discrete-time situation, but providing a unified way to handle the combinations of the different conditions (such as positive arrival, negative arrival, server breakdowns and customer retrials, etc.). Besides, it should be noted that there are crucial differences in the terminologies, the interpretations, as well as in the technical details.

The rest of this paper is organized as follows. Section 2 gives the mathematical description of the considered queueing model. In Section 3, the Markov chain underlying the model is analyzed, and the ergodicity of the Markov chain is studied. In Section 4, some queueing measures such as the orbit and system size distributions are obtained. Section 5 provides a relation between our discrete-time queueing system to its continuous-time counterpart. In Section 6, some numerical results are presented to illustrate the impact of the negative customers and unreliability factor on the performance of the system, followed by the conclusions in Section 7.

2. Model description

We consider a single-server discrete-time retrial queue where the time axis is divided into equal intervals called slots and all queueing activities (arrivals, departures, retrials, breakdowns and repairs) occur at the slot boundaries. Let the time axis be marked by $0, 1, \ldots, m, \ldots$. Consider the epoch m and suppose that the departures and the completion epochs of the repairs occur in (m^-, m), and the positive arrivals, the negative arrivals, the retrials and the beginning epochs of the repairs in (m, m^+). That is, we consider early arrival system (EAS) policy in the present paper. The model under consideration can be viewed through a self-explanatory figure (see Fig. 1).

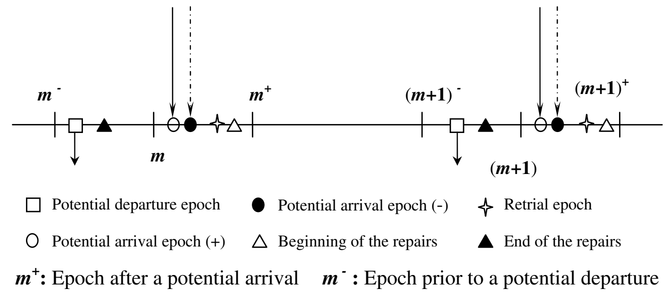

Fig. 1. Various time epochs in an early arrival system (EAS).

Two types of customers, positive and negative, arrive according to geometrical arrival processes with probabilities p and q, respectively. There is no waiting space in front of the server, and therefore, if an arriving positive customer finds the server idle, he commences his service immediately and leaves the system after service completion. Otherwise, if the server is busy or under repair at the arrival epoch, the arriving customer leaves the service area and joins a group of blocked customer (called orbit) in order to try his luck again some time later.

Each customer in the orbit generates a Bernoulli stream with rate $\bar{r} = 1 - r$ independently, where r is the probability that a repeated customer does not make a retrial in a slot. The busy server will break down upon the arrivals of negative customers, and to be sent to repair immediately. The repair times are geometrically distributed with parameter $\bar{\alpha} = 1 - \alpha$, i.e., α is the probability that the repair does not complete in a slot. It is assumed that after repair the server is as good as new.

Negative customers will make the customer being in service lost but has no effect to the orbit. If the sever is already down, the negative customer has no effect on the server. Furthermore, we assume that the negative customer also breaks the server down if the server is found idle upon arrival. If we were to keep the assumption that negative arrivals have no effect on the idle server, this would model systems which break down only when server is busy. Such a system has a similar solution but is not pursued in the present work.

The service times are independent and geometrically distributed with probability $\bar{s} = 1 - s$, where s is the probability that a customer does not conclude its service in a slot.

Finally, various stochastic processes involved in the system are assumed to be independent of each other. To avoid trivial cases, we suppose $0 < p < 1$, $0 < q < 1$, $0 \leqslant r < 1$, $0 < \alpha \leqslant 1$ and $0 < s < 1$.

3. The Markov chain

At time m^+, the system can be described by the process $Y_m = (C_m, N_m)$, where C_m denotes the state of the server (0, 1 or 2 according whether the server is free, busy or down) and N_m is the number of repeated customers in the orbit.

It can be shown that $\{Y_m, m \in \mathbb{N}\}$ is the Markov chain of our queueing system, whose state space is $\{(i,k) : i = 0, 1, 2; k \geqslant 0\}$. Let $\pi_{i,k} = \lim_{m\to\infty} P[C_m = i, N_m = k] (i = 0, 1, 2; k \geqslant 0)$ be the stationary distributions of the Markov chain $\{Y_m, m \in \mathbb{N}\}$. Our object is to find the stationary distributions.

The one-step transition probabilities $p_{yy'} = P\{Y_{m+1} = y' | Y_m = y\}$ for $k \geqslant 0$ are given by:

$$p_{(0,k)(0,k)} = \bar{p}\bar{q}r^k$$
$$p_{(1,k)(0,k)} = \bar{s}\bar{p}\bar{q}r^k$$
$$p_{(2,k)(0,k)} = \bar{\alpha}\bar{p}\bar{q}r^k$$
$$p_{(0,k)(1,k)} = p\bar{q}$$
$$p_{(1,k)(1,k)} = s\bar{p}\bar{q} + \bar{s}p\bar{q}$$
$$p_{(2,k)(1,k)} = \bar{\alpha}p\bar{q}$$
$$p_{(0,k+1)(1,k)} = \bar{p}\bar{q}(1 - r^{k+1})$$
$$p_{(1,k+1)(1,k)} = \bar{s}\bar{p}\bar{q}(1 - r^{k+1})$$
$$p_{(2,k+1)(1,k)} = \bar{\alpha}\bar{p}\bar{q}(1 - r^{k+1})$$

$$p_{(1,k-1)(1,k)} = (1-\delta_{0,k})sp\bar{q}$$
$$p_{(0,k)(2,k)} = q$$
$$p_{(1,k)(2,k)} = \bar{s}q + s\bar{p}q$$
$$p_{(2,k)(2,k)} = \alpha\bar{p} + \bar{\alpha}q$$
$$p_{(2,k-1)(2,k)} = (1-\delta_{0,k})\alpha p$$
$$p_{(1,k-1)(2,k)} = (1-\delta_{0,k})spq,$$

where $\bar{p} \equiv 1-p$, and $\delta_{i,j}$ denotes Kronecker's delta function. The system states and one step transitions are shown in Fig. 2.

By ordering the states as

$$S = \{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),\ldots,(0,N),\\ (1,N),(2,N),\ldots\}$$

we can express the above transition probabilities of the process Y_m in the following matrix-block form, which is called transition probability matrix of a level-dependent Quasi-birth-and-death (QBD) process (see Neuts, 1981):

$$\mathbf{P} = \begin{pmatrix} \mathbf{A_{00}} & \mathbf{A_{01}} & & & \\ \mathbf{A_{10}} & \mathbf{A_{11}} & \mathbf{A_{12}} & & \\ & \mathbf{A_{21}} & \mathbf{A_{22}} & \mathbf{A_{23}} & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $\mathbf{A_{j,j-1}}$, $\mathbf{A_{j,j}}$ and $\mathbf{A_{j,j+1}}$ are the following square matrices of order 3:

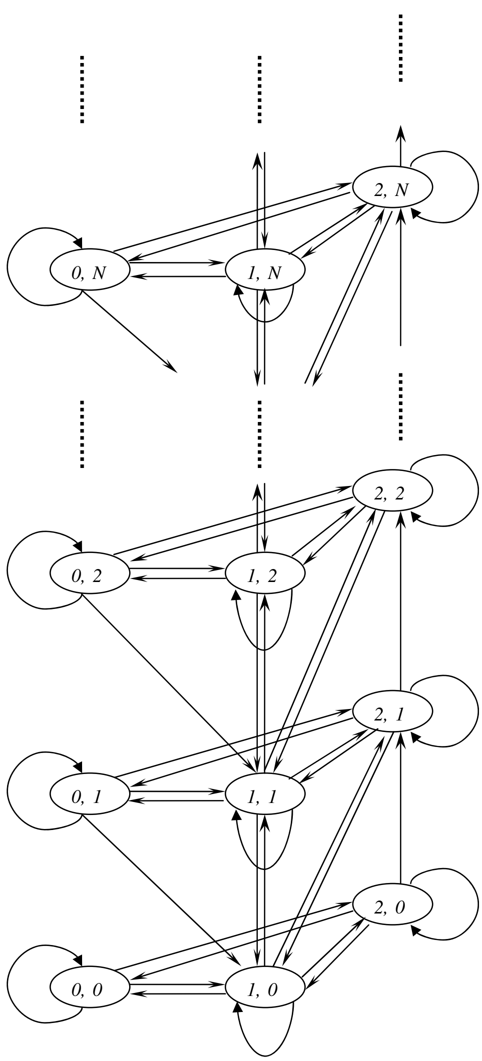

Fig. 2. Transition diagram of the states.

$$\mathbf{A_{j,j-1}} = \begin{pmatrix} 0 & \bar{p}\bar{q}(1-r^j) & 0 \\ 0 & \bar{s}\bar{p}\bar{q}(1-r^j) & 0 \\ 0 & \bar{\alpha}\bar{p}\bar{q}(1-r^j) & 0 \end{pmatrix}, \quad \mathbf{A_{j,j+1}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & sp\bar{q} & spq \\ 0 & 0 & \alpha p \end{pmatrix},$$

$$\mathbf{A_{j,j}} = \begin{pmatrix} \bar{p}\bar{q}r^j & p\bar{q} & q \\ \bar{s}\bar{p}\bar{q}r^j & s\bar{p}\bar{q}+\bar{s}p\bar{q} & \bar{s}q+s\bar{p}q \\ \bar{\alpha}\bar{p}\bar{q}r^j & \bar{\alpha}p\bar{q} & \alpha\bar{p}+\bar{\alpha}q \end{pmatrix}, \quad \text{for } j \geqslant 1.$$

By routine procedure, the Kolmogorov equations for the stationary distribution of our system are given by:

$$\pi_{0,k} = \bar{p}\bar{q}r^k\pi_{0,k} + \bar{s}\bar{p}\bar{q}r^k\pi_{1,k} + \bar{\alpha}\bar{p}\bar{q}r^k\pi_{2,k}, \quad k \geqslant 0, \tag{1}$$

$$\begin{aligned}\pi_{1,k} = {} & p\bar{q}\pi_{0,k} + (s\bar{p}\bar{q}+\bar{s}p\bar{q})\pi_{1,k} + \bar{\alpha}p\bar{q}\pi_{2,k}\bar{p}\bar{q}(1-r^{k+1})\pi_{0,k+1} \\ & + \bar{s}\bar{p}\bar{q}(1-r^{k+1})\pi_{1,k+1} + \bar{\alpha}\bar{p}\bar{q}(1-r^{k+1})\pi_{2,k+1} \\ & + (1-\delta_{0,k})sp\bar{q}\pi_{1,k-1}, \quad k \geqslant 0,\end{aligned} \tag{2}$$

$$\begin{aligned}\pi_{2,k} = {} & q\pi_{0,k} + (\bar{s}q + s\bar{p}q)\pi_{1,k} \\ & + (1-\delta_{0,k})spq\pi_{1,k-1} + (\alpha\bar{p}+\bar{\alpha}q)\pi_{2,k} + (1-\delta_{0,k})\alpha p\pi_{2,k-1}, \quad k \geqslant 0,\end{aligned} \tag{3}$$

where $\bar{p} \equiv 1-p$, and $\delta_{i,j}$ denotes Kronecker's delta function. The normalization condition is given by

$$\sum_{i=0}^{2}\sum_{k=0}^{\infty}\pi_{i,k} = 1.$$

To solve Eqs. (1)–(3), we introduce the following generating functions:

$$\varphi_i(z) = \sum_{k=0}^{\infty}\pi_{i,k}z^k, \quad 0 \leqslant z < 1, \quad i = 0,1,2.$$

Multiplying Eqs. (1)–(3) by z^k and summing over k, these equations become

$$\varphi_0(z) = \bar{p}\bar{q}\varphi_0(rz) + \bar{s}\bar{p}\bar{q}\varphi_1(rz) + \bar{\alpha}\bar{p}\bar{q}\varphi_2(rz), \tag{4}$$

$$\begin{aligned}\varphi_1(z) = {} & p\bar{q}\varphi_0(z) + (s\bar{p}\bar{q}+\bar{s}p\bar{q})\varphi_1(z) + \bar{\alpha}p\bar{q}\varphi_2(z) \\ & + \frac{\bar{p}\bar{q}}{z}[\varphi_0(z) - \varphi_0(rz)] + \frac{\bar{s}\bar{p}\bar{q}}{z}[\varphi_1(z) - \varphi_1(rz)] \\ & + \frac{\bar{\alpha}\bar{p}\bar{q}}{z}[\varphi_2(z) - \varphi_2(rz)] + sp\bar{q}z\varphi_1(z),\end{aligned} \tag{5}$$

$$\begin{aligned}\varphi_2(z) = {} & q\varphi_0(z) + (\bar{s}q + s\bar{p}q)\varphi_1(z) + spqz\varphi_1(z) \\ & + (\alpha\bar{p}+\bar{\alpha}q)\varphi_2(z) + \alpha pz\varphi_2(z).\end{aligned} \tag{6}$$

By substituting (4) into (5) we obtain

$$\begin{aligned}\varphi_1(z) = {} & p\bar{q}\varphi_0(z) + (s\bar{p}\bar{q}+\bar{s}p\bar{q})\varphi_1(z) + \bar{\alpha}p\bar{q}\varphi_2(z) + sp\bar{q}z\varphi_1(z) \\ & + \frac{\bar{p}\bar{q}}{z}\varphi_0(z) + \frac{\bar{s}\bar{p}\bar{q}}{z}\varphi_1(z) + \frac{\bar{\alpha}\bar{p}\bar{q}}{z}\varphi_2(z) - \frac{1}{z}\varphi_0(z).\end{aligned} \tag{7}$$

Solving Eqs. (6) and (7), we find the following generating functions:

$$\varphi_1(z) = \frac{-\alpha p\bar{q}pz + (q+p\bar{q})(\alpha p + \bar{\alpha}\bar{q}) - \bar{\alpha}\bar{p}\bar{q}q}{\mathbf{D}(z)} \cdot \varphi_0(z), \tag{8}$$

$$\varphi_2(z) = \frac{-qs\bar{p}\bar{q}pz + \bar{s}\bar{p}\bar{q}q + (q+p\bar{q})(\bar{s}q + s\bar{p}q)}{\mathbf{D}(z)} \cdot \varphi_0(z), \tag{9}$$

where

$$\begin{aligned}\mathbf{D}(z) \equiv {} & s\bar{p}\bar{q}\alpha pz^2 - [s\bar{p}\bar{q}(\bar{\alpha}\bar{q}+\bar{\alpha}pq) + (1 - s\bar{p}\bar{q} - \bar{s}\bar{p}\bar{q})\alpha p]z \\ & + [\bar{s}\bar{p}\bar{q}(\alpha p + \bar{\alpha}\bar{q}) + \bar{\alpha}\bar{p}\bar{q}(\bar{s}q + s\bar{p}q)].\end{aligned}$$

Before presenting the main results, we give the following lemma whose proof is straightforward and is therefore omitted.

Lemma 1

(1) *The inequality* $\mathbf{D}(z) > 0$ *holds for* $0 \leqslant z < 1$ *if and only if*

$$\frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q}+q)(\bar{s}+s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q}+q}.$$

(2) $\lim_{z\to 1^-}\mathbf{D}(z) = \bar{\alpha}\bar{q}(\bar{p} - s\bar{q} - spq) - \alpha pq$ *if and only if*

$$\frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q} + q}.$$

By Lemma 1, the generating functions $\varphi_1(z), \varphi_2(z)$ are well defined for $0 \leqslant z < 1$, and in $z = 1$ are extended by continuity if and only if

$$\frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q} + q}.$$

From Eqs. (8), (9), we can get

$$\varphi_1(1) = \frac{\bar{\alpha}p\bar{q} + \alpha pq}{\bar{\alpha}\bar{q}(\bar{p} - s\bar{q} - spq) - \alpha pq}\varphi_0(1),$$
$$\varphi_2(1) = \frac{q(1 - s\bar{q} - spq)}{\bar{\alpha}\bar{q}(\bar{p} - s\bar{q} - spq) - \alpha pq}\varphi_0(1).$$

The normalization condition $\sum_{i=0}^{2}\varphi_i(1) = 1$ gives

$$\varphi_0(1) = 1 - \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)} - \frac{q}{\bar{\alpha}\bar{q} + q}, \tag{10}$$
$$\varphi_1(1) = \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}, \tag{11}$$
$$\varphi_2(1) = \frac{q}{\bar{\alpha}\bar{q} + q}. \tag{12}$$

Substituting (8) and (9) into (4), we have

$$\varphi_0(z) = G(rz)\varphi_0(rz), \tag{13}$$

where

$$G(z) \equiv \frac{s\bar{p}\bar{q}\alpha pz^2 + [\alpha p(s\bar{p}\bar{q} - 1) - s\bar{p}\bar{\alpha}\bar{q}]z + [\bar{s}(\alpha p + \bar{\alpha}\bar{q}) + \bar{\alpha}(\bar{s}q + s\bar{p}q)]}{\mathbf{D}(z)}\bar{p}\bar{q}. \tag{14}$$

It follows, by applying (13) recursively, that

$$\varphi_0(z) = \varphi_0(0)\prod_{n=1}^{\infty} G(r^n z). \tag{15}$$

Lemma 2. *The infinite product* $\prod_{n=1}^{\infty} G(r^n z)$ *is convergent if*

$$\frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q} + q}.$$

Proof. Firstly, we give an expression as $G(z) = 1 + F(z)$, where

$$F(z) = \frac{(1 - \bar{p}\bar{q})s\bar{p}\bar{q}\alpha pz^2 + [\alpha pq(1 - s\bar{p}\bar{q}) + \bar{\alpha}p\bar{q}sp]z}{\mathbf{D}(z)}.$$

From Lemma 1 we know that if the condition $\frac{\bar{\alpha}p\bar{q}+\alpha pq}{(\bar{\alpha}\bar{q}+q)(\bar{s}+s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q}+q}$ is satisfied, the function $F(z)$ is nonnegative for $0 \leqslant z \leqslant 1$.

On the other hand, we can represent the infinite product as

$$\prod_{n=1}^{\infty} G(r^n z) = \prod_{n=1}^{\infty}[1 + F(r^n z)].$$

The above infinite product converges if the series $\sum_{n=1}^{\infty} F(r^n z)$ is convergent, which is clear since

$$\lim_{n\to\infty}\frac{F(r^{n+1}z)}{F(r^n z)} = r \leqslant 1.$$

This completes the proof. □

We set $z = 1$ in (15) and it follows that:

$$\varphi_0(0) = \varphi_0(1)[\prod_{n=1}^{\infty} G(r^n)]^{-1},$$

and therefore

$$\varphi_0(z) = \left[1 - \frac{q}{\bar{\alpha}\bar{q} + q} - \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}\right]\frac{\prod_{n=1}^{\infty} G(r^n z)}{\prod_{n=1}^{\infty} G(r^n)}.$$

We summarize the above results in the following theorem.

Theorem 1. *The Markov chain* $Y_m = (C_m, N_m)$ *is ergodic if and only if* $\frac{\bar{\alpha}p\bar{q}+\alpha pq}{(\bar{\alpha}\bar{q}+q)(\bar{s}+s\bar{p}q)} < 1 - \frac{q}{\bar{\alpha}\bar{q}+q}$, *and we have*

$$\varphi_0(z) = \left[1 - \frac{q}{\bar{\alpha}\bar{q} + q} - \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}\right]\cdot\frac{\prod_{n=1}^{\infty} G(r^n z)}{\prod_{n=1}^{\infty} G(r^n)},$$
$$\varphi_1(z) = \frac{-\alpha p\bar{q}pz + (q + p\bar{q})(\alpha p + \bar{\alpha}\bar{q}) - \bar{\alpha}\bar{p}\bar{q}q}{\mathbf{D}(z)}\varphi_0(z),$$
$$\varphi_2(z) = \frac{-qsp\bar{q}\bar{p}z + \bar{s}\bar{p}\bar{q}q + (q + p\bar{q})(\bar{s}q + s\bar{p}q)}{\mathbf{D}(z)}\varphi_0(z),$$

where $G(z)$ *is defined in Eq. (14).*

Corollary 1. *In the steady state, we have*

(1) *The probability generating function of the number of customers in the orbit is given by*

$$\psi(z) = \varphi_0(z) + \varphi_1(z) + \varphi_2(z) = \frac{\varphi_0(z)}{\mathbf{D}(z)}\cdot\{s\bar{p}\bar{q}\alpha pz^2 - [s\bar{p}\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}pq + \bar{p}q) + (1 - s\bar{p}\bar{q} + s\bar{p}\bar{q})\alpha p]z + [(\alpha p + \bar{\alpha}\bar{q} + q)(s\bar{p}\bar{q} + \bar{\alpha}\bar{q} + q) + spq\alpha pq]\}.$$

(2) *The probability generating function of the number of customers in the system, which is denoted by L in the sequel, is given by*

$$\phi(z) = \varphi_0(z) + z\varphi_1(z) + \varphi_2(z) = \frac{\varphi_0(z)}{\mathbf{D}(z)}\cdot\{-\bar{s}\bar{p}\bar{q}\alpha pz^2 - [s\bar{p}\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}pq + \bar{p}q) + (\bar{s}\bar{p}\bar{q} - \bar{s}\bar{p}\bar{q}) - (q + p\bar{q})\bar{\alpha}\bar{q}]z + [(\alpha p + \bar{\alpha}\bar{q} + q)(\bar{s}\bar{p}\bar{q} + \bar{s}pq + \bar{p}q) - \alpha pq(\bar{s}q + s\bar{p}q)]\}.$$

(3) *The probability that the server is idle is*

$$\varphi_0(1) = 1 - \frac{q}{\bar{\alpha}\bar{q} + q} - \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}.$$

(4) *The probability that the server is busy is*

$$\varphi_1(1) = \frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}.$$

(5) *The probability that the server is down is*

$$\varphi_2(1) = \frac{q}{\bar{\alpha}\bar{q} + q}.$$

(6) *The mean number of customers in the orbit is*

$$E(N) = \sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}r^n + \varphi_0(1)\left\{-\frac{\alpha p\bar{q}p + qsp\bar{q}\bar{p}}{\bar{\alpha}\bar{q}(\bar{p} - spq - s\bar{q}) - \alpha pq} + \frac{[(1 - s\bar{q} - p\bar{q})\alpha p + sp\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}p\bar{q})][(\alpha q + \bar{\alpha}\bar{q})p + q(1 - s\bar{q} - spq)]}{[\alpha\bar{q}(\bar{p} - spq - s\bar{q}) - \alpha pq]^2}\right\}.$$

(7) $E(L) = E(N) + \varphi_1(1)$.

Remark 1. To achieve numerical approximations for the performance measures, it is required to estimate the series:

$$\sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}r^n.$$

It is easy to get the following equation:

$$\sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}r^n = \frac{1}{1 - r}\sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}(r^n - r^{n+1}).$$

Hence, the prior series can be approached by an integral in the following way (see Moreno, 2006):

(1) $$\sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}r^n \approx \frac{1}{1-r}\int_0^1 \frac{G'(z)}{G(z)}\mathrm{d}z = \frac{lnG(1)}{1-r}$$

if r is sufficiently near 1.

(2) $$\sum_{n=1}^{\infty}\frac{G'(r^n)}{G(r^n)}r^n \approx \frac{1}{1-r}\sum_{n=1}^{n_0(\varepsilon)}\frac{G'(r^n)}{G(r^n)}(r^n - r^{n+1}) + \int_0^{r^{n_0(\varepsilon)+1}}\frac{G'(z)}{G(z)}\mathrm{d}z = \sum_{n=1}^{n_0(\varepsilon)+1}\frac{G'(r^n)}{G(r^n)}r^n + \frac{lnG(r^{n_0(\varepsilon)+1})}{1-r}$$

where for each $\varepsilon > 0, n_0(\varepsilon)$ is selected such that $r^{n_0(\varepsilon)+1} < \varepsilon$.

Remark 2 (*Special case*). When $r = 0$, $\phi(z)$ reduces to

$$\begin{aligned}\phi(z) = \{&-\bar{s}p\bar{q}\alpha pz^2 - [sp\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}pq + \bar{p}q) + (\bar{s}\bar{p}\bar{q} - \bar{s}p\bar{q})\\ &-(q + p\bar{q})\bar{\alpha}\bar{q}]z + [(\alpha p + \bar{\alpha}\bar{q} + q)(\bar{s}\bar{p}\bar{q} + \bar{s}pq + \bar{p}q)\\ &-\alpha pq(\bar{s}q + s\bar{p}q)]\}\cdot\frac{\varphi_0(1)}{\mathbf{D}(z)},\end{aligned}$$

which is the probability generating function of the number of customers in the $Geo/Geo/1/\infty$ G-queue with an unreliable server, but without retrials. This is not surprising because as $r = 0$, the blocked customers try to get service at every slot boundary. This system is equivalent to the model $Geo/Geo/1/\infty$ G-queue with an unreliable and random service discipline, since the system size distribution does not depend on the service discipline.

4. Stochastic decomposition

The stochastic decomposition law for queueing systems was first given in Fuhrmann and Cooper (1985). Recently, numerous papers, for instance, Atencia and Moreno (2004a), Atencia and Moreno (2004b), Atencia and Moreno(2006), Moreno (2006) investigated the stochastic decomposition property of the discrete-time retrial queueing systems. In this section we analyze the stochastic decomposition law on discrete-time retrial G-queues.

The probability generating function of the number of customers in our system can be expressed as

$$\begin{aligned}\phi(z) = \frac{\varphi_0(1)}{\mathbf{D}(z)}\cdot\{&-\bar{s}p\bar{q}\alpha pz^2 - [sp\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}pq + \bar{p}q) + (\bar{s}\bar{p}\bar{q} - \bar{s}p\bar{q})\\ &-(q + p\bar{q})\bar{\alpha}\bar{q}]z + [(\alpha p + \bar{\alpha}\bar{q} + q)(\bar{s}\bar{p}\bar{q} + \bar{s}pq + \bar{p}q)\\ &-\alpha pq(\bar{s}q + s\bar{p}q)]\}\cdot\frac{\varphi_0(z)}{\varphi_0(1)}.\end{aligned}$$

Let $\Pi(z)$ be the probability generating function of the number of customers in the $Geo/Geo/1/\infty$ G-queue with an unreliable server (without retrials).

$$\begin{aligned}\Pi(z) = \frac{\varphi_0(1)}{\mathbf{D}(z)}\cdot\{&-\bar{s}p\bar{q}\alpha pz^2 - [sp\bar{q}(\bar{\alpha}\bar{q} + \bar{\alpha}pq + \bar{p}q) + (\bar{s}\bar{p}\bar{q} - \bar{s}p\bar{q})\\ &-(q + p\bar{q})\bar{\alpha}\bar{q}]z + [(\alpha p + \bar{\alpha}\bar{q} + q)(\bar{s}\bar{p}\bar{q} + \bar{s}pq + \bar{p}q)\\ &-\alpha pq(\bar{s}q + s\bar{p}q)]\},\end{aligned}$$

and $\Upsilon(z) = \frac{\varphi_0(z)}{\varphi_0(1)}$ the probability generating function of the number of customers in the orbit when the server is idle in our system. Then we have

$$\phi(z) = \Pi(z)\Upsilon(z).$$

This result can be summarized in the following theorem.

Theorem 2. *The total number of customers in our system L can be represented as the sum of two independent random variables. One of them is the total number of the customers in the system $Geo/Geo/1/\infty$ G-queue with unreliable server L_0 and the other is the number of repeated customers given that the server is idle M. That is, $L = L_0 + M$.*

As an application of the theorem, in the following we study the proximity between the steady-state distribution for the standard $Geo/Geo/1/\infty$ G-queue with unreliable server (without retrials) and our queueing system.

Theorem 3. *The following inequalities hold*

$$2[1 - \varphi_1(1) - \varphi_2(1) - \pi_{0,0}] \leqslant \sum_{j=0}^{\infty}|P\{L = j\} - P\{L_0 = j\}| \leqslant 2\frac{1 - \varphi_1(1) - \varphi_2(1) - \pi_{0,0}}{\varphi_0(1)}.$$

The proof of the theorem follow the steps given in Artalejo and Falin (1994) and we omit the details here.

5. Relation to the continuous-time system

In this section we study the relation between our discrete-time retrial G-queue system and its corresponding continuous-time G-queue system. We consider $M/M/1$ retrial G-queue with unreliable server and individual removal killing strategy, which is the special case of the model studied recently by Wu and Yin (2005). The positive (negative, respectively) customers arrive according to a Poisson flow with rate λ (λ^-, respectively). The arrival processes of positive and negative customers are independent. As described in Section 2, negative customers make the server down when they find the server idle or busy, and the failed server is sent to be repaired immediately after the break. The service time, retrial time and repair time are all exponentially distributed with respective parameter μ, δ, θ. Now we prove this continuous-time retrial G-queue system could be approximated by our discrete-time retrial G-queue system.

We suppose that time is divided into intervals of equal length Δ. The continuous-time retrial G-queue system parameters can be approximated by the follows:

$$p = \lambda\Delta, q = \lambda^-\Delta, s = 1 - \mu\Delta, r = 1 - \delta\Delta, \alpha = 1 - \theta\Delta.$$

Firstly, we will prove that $\lim_{\Delta\to 0}\varphi_0(1), \lim_{\Delta\to 0}\varphi_1(1), \lim_{\Delta\to 0}\varphi_2(1)$ are the probabilities that the server is idle, busy and down in continuous-time counterpart, respectively.

According to Corollary 1, we have

$$\begin{aligned}\lim_{\Delta\to 0}\varphi_1(1) &= \lim_{\Delta\to 0}\frac{\bar{\alpha}p\bar{q} + \alpha pq}{(\bar{\alpha}\bar{q} + q)(\bar{s} + s\bar{p}q)}\\ &= \lim_{\Delta\to 0}\frac{\theta\Delta\lambda\Delta(1 - \lambda^-\Delta) + (1 - \theta\Delta)\lambda\Delta\lambda^-\Delta}{[\theta\Delta(1 - \lambda^-\Delta) + \lambda^-\Delta][\mu\Delta + (1 - \mu\Delta)(1 - \lambda\Delta)\lambda^-\Delta]}\\ &= \frac{\theta\lambda + \lambda^-\lambda}{(\theta + \lambda^-)(\mu + \lambda^-)} = \frac{\lambda}{\mu + \lambda^-},\\ \lim_{\Delta\to 0}\varphi_2(1) &= \lim_{\Delta\to 0}\frac{q}{\bar{\alpha}\bar{q} + q} = \lim_{\Delta\to 0}\frac{\lambda^-\Delta}{\theta\Delta(1 - \lambda^-) + \lambda^-\Delta} = \frac{\lambda^-}{\lambda^- + \theta},\\ \lim_{\Delta\to 0}\varphi_0(1) &= 1 - \frac{\lambda}{\mu + \lambda^-} - \frac{\lambda^-}{\lambda^- + \theta},\end{aligned}$$

which agree with the probabilities that the server is idle, busy and down in continuous-time counterpart (see the results in Wu & Yin, 2005).

Using (13) we get

$$\frac{\mathrm{d}}{\mathrm{d}z}\varphi_0(z) = \lim_{\Delta\to 0}\frac{\varphi_0(z) - \varphi_0(rz)}{z - rz} = \lim_{\Delta\to 0}\left\{\frac{F(rz)}{(1 - r)z}\varphi_0(rz)\right\}.$$

Further

$$\lim_{\Delta\to 0}\frac{F(rz)}{(1 - r)z} = \frac{(1 - z)(\lambda + \lambda^-)\lambda\lambda + \lambda[\lambda\theta + \lambda^-(\lambda + \lambda^- + \mu)]}{\lambda\lambda\delta z^2 - \delta[\lambda\theta + \lambda(\lambda + \lambda^- + \mu)]z + \delta[\theta\lambda^- + \mu(\lambda + \theta)]},$$

then we have

$$\frac{d}{dz}\varphi_0(z) = \frac{(1-z)(\lambda+\lambda^-)\lambda\lambda + \lambda[\lambda\theta + \lambda^-(\lambda+\lambda^-+\mu)]}{\lambda\lambda\delta z^2 - \delta[\lambda\theta + \lambda(\lambda+\lambda^-+\mu)]z + \delta[\theta\lambda^- + \mu(\lambda+\theta)]}\varphi_0(z).$$

Solving the above equation yields

$$\lim_{\Delta\to 0}\varphi_0(z) = \varphi_0(1) \times \exp\left\{\int_1^z \frac{1}{\delta}\cdot\frac{\lambda^2(1-z)(\lambda+\lambda^-) + \lambda[\lambda\theta + \lambda^-(\lambda+\lambda^-+\mu)]}{\lambda^2 z^2 - [\lambda\theta + \lambda(\lambda+\lambda^-+\mu)]z + [\theta\lambda^- + \mu(\lambda+\theta)]}dz\right\}.$$

which is the probability generating function of the number of customers in the orbit when the server is idle. This result can be found in Wu and Yin (2005).

6. Numerical examples

In this section, we present some numerical examples to study the effect of the varying parameters on the main performance characteristics of our system. The values of the parameters are chosen so as to satisfy the ergodicity condition.

Fig. 3 describes the behavior of $\phi_0(1)$, $\phi_1(1)$ and $\phi_2(1)$ with varying values of the parameter q. As to be expected, $\phi_0(1)$ and $\phi_1(1)$ are decreasing as functions of the negative customer arrival rate q, whereas the values of $\phi_2(1)$ increase with increasing values of q.

Fig. 4 describes the effect of s on the orbit mean size. We present three curves which correspond to $p = 0.1, 0.2, 0.3$, respectively. As to be expected, they increase with the increasing values of p. On the other hand, when the values of s increases, it is clear that the orbit mean size is increasing.

Fig. 5 depicts the behavior of the mean orbit size against the parameter p. We also present three curves which correspond to the values $s = 0.1, 0.2, 0.25$, respectively. It should be noted that the expectation $E(N)$ is scarcely affected by s when p approaches to 0. Moreover, as p increases, the effect of different values of s on $E(N)$ becomes more diverse.

7. Conclusion

In this paper, we analyzed a discrete-time retrial queue with negative arrivals along with the positive arrivals and the server is subject to breakdowns and repairs due to the negative arrivals. The Markov chain underlying the queueing system was investigated and its ergodicity condition was obtained. By solving the Kolmogorov equations for the system stationary distribution, some queueing measures were derived such as the orbit and system size distribution. A relationship between the discrete-time queueing system and its continuous-time counterpart was provided. Numerical illustrations were presented to show how sensitive the system performance measures are versus changes in parameters of the system.

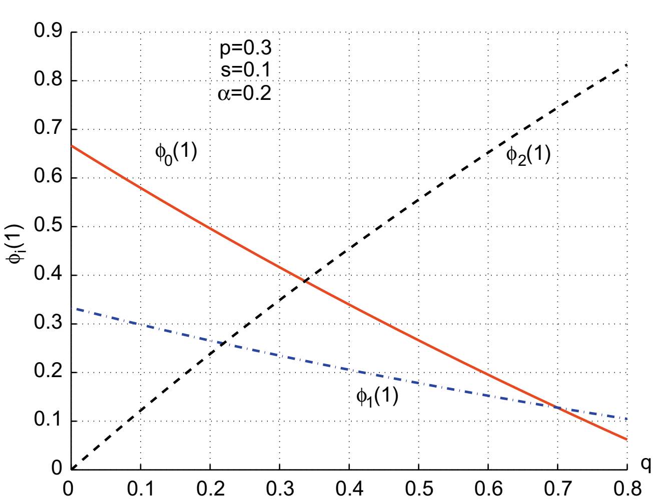

Fig. 3. The effect of q on the sever state.

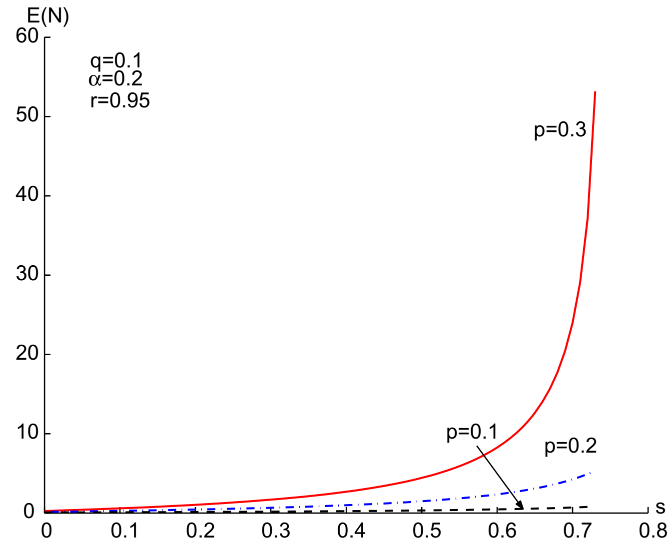

Fig. 4. The effect of s on the sever state.

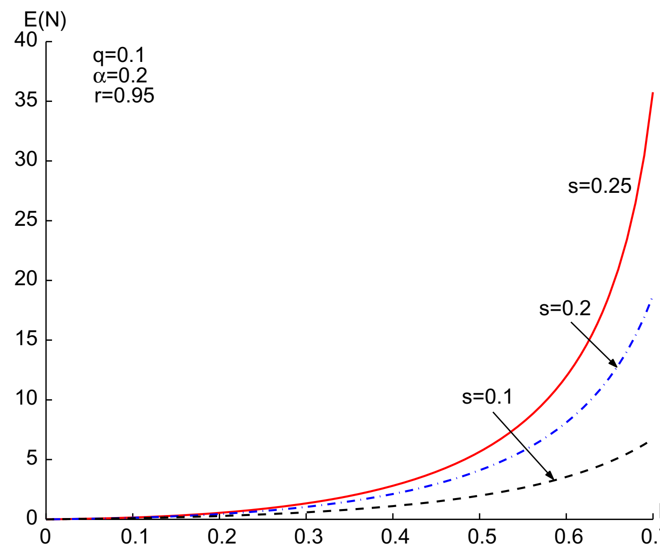

Fig. 5. The effect of p on the sever state.

For future research, one could consider the dependencies between stochastic processes involved in the system, such as dependency between positive and negative arrivals, statistical self-similarity property or long-range dependencies that can govern the analyzed process.

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