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A single server feedback retrial queue with collisions

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ABSTRACT

A Markovian single server feedback retrial queue with linear retrial rate and collisions of customers is studied. Using generating function technique, the joint distribution of the server state and the orbit length under steady-state is investigated. Some interesting and important performance measures of the system are obtained. Finally, numerical illustrations are provided.

1. Introduction

In recent years there have been significant contributions to the retrial queues. Retrial queueing systems are described by the feature that the arriving customers (or calls, packets) who find the server busy join the retrial group/orbit to try again for their requests in a random order and at random time intervals. Retrial queues are widely and successfully used as mathematical models of several computer systems and telecommunication networks. For example, peripherals in computer systems may make retrials to receive service from a central processor. Hosts in Local Area Network (LAN) may make many retrials in order to access the communication medium, which is clearly indicated in the Carrier Sense Multiple Access (CSMA) protocol that controls this access. For excellent and recent bibliographies on retrial queues, the readers are referred to [1–5].

In the retrial queues, the inter-retrial times are modelled according to different disciplines depending on each particular application. In telephone systems, the calls which will be re-attempted can be modelled by an infinite server queue where the time until re-attempt follows exponential distribution with mean $1/\nu$. This is the so-called classical retrial policy whose rate is $j\nu$, when the orbit size is $j \geq 0$ (see [1]). In contrast, there are other types of retrial queueing systems in which the intervals separating successive repeated attempts are independent of the number of customers in the orbit. This discipline is called the constant/control retrial policy, i.e., the retrial rate is $(1-\delta_{j0})\eta$ where δ_{ij} denotes the Kronecker's delta function. This kind of retrial control policy is well known for the stability of the ALOHA protocol in communication systems [6]. Recently, Artalejo and Gomez-Corral [7] unified both policies by defining the linear retrial policy with rate $(1-\delta_{j0})\eta + j\nu$. This type of retrial policy is used widely in the recent literature to investigate the performance measures of the computer and telecommunication networks [5].

In many situations involving data transmission from diverse sources there can be conflict for a limited number of channels or other facilities. Uncoordinated attempts by several sources to use a single server facility can result in "Collision" leading to the loss of the transmission and hence the need for retransmission. An important problem concerns the development of workable procedures for alleviating the conflict and corresponding message delay. For instance, in the unslotted 1- and p_i-persistent Carrier Sense Multiple Access with Collision Detection (CSMA-CD) protocols for a fiber optic bus network with a finite number of stations, each of which has an infinite storage buffer, the collisions occur during the transmission of arbitrary length packets because no slot synchronization is needed. Further, under the unslotted 1-persistent CSMA-CD protocol, retransmission of deferred packets promptly begin at the instant when the channel is sensed idle. Also, because of the unidirectional transmission property of an optical fiber, a solution always has the preference of accessing the channel to its downstream stations. Thus we treat this network as a retrial queueing system with collisions.

Another example for the retrial queue with collision is the operation of 802.11 wireless LANs. In the Medium Access Control (MAC) protocol for IEEE 802.11 wireless LANs, a sender sends a Request-To-Send (RTS) to the receiver, who in turn sends back a Clean-To-Send (CTS) to the sender if it receives the RTS free of errors; the sender can transmit a data packet only after a successful RTS/CTS exchange. This protocol yields good result if the RTS traffic is low. However, if the RTS transmission rate increases, the constant RTS collisions can

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cause the channel to collapse, bring the flow data packets to a halt when no new transmission queues can be started.

Jonin [8] and Falin and Sukharev [9] have analyzed the retrial queueing system with collision, called the queue with double connections, in which, if an arriving customer finds the service facility busy, then the arriving customer interrupts (collides with) a customer in service, both the arriving customer and the served customer join retrial group and the server becomes free immediately. Choi et al. [10] have discussed a retrial queueing system with constant retrial rate and collision in the specific communication protocol CSMA-CD. For this system, the generating function of the limiting distribution of the number of customers in the retrial group is obtained at arbitrary time points.

Most of the papers on retrial queues have analyzed the system without customer feedback. A more practical retrial queue with feedback of the customers occurs in many real world situations: for instance, in communication networks where data transmissions need to be guaranteed error free to within some specified probability, feedback schemes are used to request retransmission of packets that are lost or received in a corrupted form. Specifically, the feedback retrial queue is used to model the Automative Repeat Request (ARQ) protocol in a high frequency communication network. Choi and Kulkarni [11] have studied $M/G/1$ retrial queue with feedback. Falin [12] has discussed $M/M/1$ retrial queue with feedback and geometric loss. Choi et al. [13] have investigated $M/M/c$ retrial queue with geometric loss and feedback. Krishna Kumar et al. [14] have studied $M/G/1$ retrial queue with feedback and starting failures. More recently, Krishna Kumar and Raja [15] have analyzed the multiserver retrial queue with feedback and balking of customers.

Several results have been reported separately on retrial queueing systems with linear retrial rate, retrial queues with feedback and retrial queues with collisions. However, the study of retrial queueing systems taking together the above mentioned features is interesting and not much work in this direction is found in the literature. Based on this observation, we have investigated the feedback retrial queueing system with linear retrial rate and collisions. The novelty of this investigation is the introduction of the linear retrial policy in the Bernoulli feedback retrial queueing system together with service option on arrival and collisions of customers. This makes the system more complex though realistic.

The rest of the paper is organized as follows. In the next section, the mathematical model is described. Section 3 presents the steady-state probabilities of the system size and state of the server. In Section 4, some important and interesting performance measures for the system under investigation are obtained. We present some numerical examples to illustrate the effect of the parameters on the system performance measures in Section 5. In Section 6, cost and sensitivity analyzes are presented. Finally, Section 7 concludes the paper.

2. Model description

Consider a single server retrial queue with Bernoulli feedback and collisions at which customers arrive from outside the system according to a Poisson process with rate λ. These customers are identified as primary customers. It is assumed that there is no waiting space and on arrival, a customer proceeds to the server with probability p $(0 \le p \le 1)$ or enters into the orbit with probability $1-p$ (see [16]). Every customer in the orbit conducts a stream of repeated requests/retrials after an exponentially distributed amount of time and is independent of the number of customers applying for service.

If the server is free, the arriving customer/the customer from the orbit gets served completely and leaves the system with probability θ $(0 < \theta \le 1)$ or joins the retrial group again for another service with probability $(1-\theta)$. The service times of successful customers are assumed to be independent exponentially distributed random variables with mean $1/\mu$. If the server is busy, the arriving customer/retrial customer collides with the customer in service resulting in both being shifted to the orbit.

Here, it is assumed that the retrial rate is $(1-\delta_{j0})\eta+j\nu$ where j the number of customers in the orbit. This linear retrial policy provides a model which incorporates simultaneously the classical and constant retrial policy studied extensively in the literature as we mentioned earlier. From this description, it is clear that the interarrival times, retrial times and service times are assumed to be mutually independent.

Let $X(t)$ be the number of customers (packets or calls) occupying the orbit and $C(t)$ represent the state of the server (service facility) where $C(t)$ takes the value 1 or 0 at time t according as whether the server is busy or free. The state of the system can be described by a bivariate process $N(t)=\{(C(t),X(t));\, t \ge 0\}$. Utilizing the lack of memory property of the exponential distribution, it is easily seen that the process $N(t)$ forms an irreducible, aperiodic and time-homogeneous Markov process with state space $S=\{0,1\}\times Z_+$ (Z_+ is the set of non-negative integers).

3. Analysis of the steady-state distribution

For the bivariate process $N(t)=\{(C(t),X(t));\, t\ge 0\}$, define the joint probabilities of the server state and retrial queue length as

$$Q_j(t)=P\{C(t)=1,X(t)=j\}$$

and

$$P_j(t)=P\{C(t)=0,X(t)=j\}$$

for $t \ge 0$ and $j \ge 0$.

Following routine procedures, the forward Kolmogorov equations for the process can be written in terms of $Q_j(t)$ and $P_j(t)$, for $j \ge 0$, as

$$\frac{dP_0(t)}{dt}=-\lambda P_0(t)+\theta\mu Q_0(t), \tag{3.1}$$

$$\frac{dP_1(t)}{dt}=-(\lambda+\nu+\eta)P_1(t)+\lambda(1-p)P_0(t)+\theta\mu Q_1(t) + (1-\theta)\mu Q_0(t), \tag{3.2}$$

$$\begin{aligned}\frac{dP_j(t)}{dt}=&-(\lambda+j\nu+\eta)P_j(t)+\lambda pQ_{j-2}(t)+((j-1)\nu+\eta)Q_{j-1}(t)\\ &+\lambda(1-p)P_{j-1}(t)+\theta\mu Q_j(t)+(1-\theta)\mu Q_{j-1}(t)\\ &j=2,3,4,\ldots,\end{aligned} \tag{3.3}$$

$$\frac{dQ_0(t)}{dt}=-(\lambda+\mu)Q_0(t)+\lambda pP_0(t)+(\nu+\eta)P_1(t), \tag{3.4}$$

$$\begin{aligned}\frac{dQ_j(t)}{dt}=&-(\lambda+\mu+j\nu+\eta)Q_j(t)+\lambda pP_j(t)+\lambda(1-p)Q_{j-1}(t)\\ &+((j+1)\nu+\eta)P_{j+1}(t), \quad j=1,2,3,\ldots.\end{aligned} \tag{3.5}$$

If the bivariate process $N(t)$ is regular and ergodic, then the limiting joint probabilities, for $j \ge 0$,

$$P_j=\lim_{t\to\infty}P\{C(t)=0,X(t)=j\}$$

and

$$Q_j=\lim_{t\to\infty}P\{C(t)=1,X(t)=j\}$$

exist and the steady-state probability distribution is uniquely determined.

Applying Tweedie's criterion ([17], Theorem 2.3), it can be shown that a sufficient condition for the process $N(t)$ to be regular and ergodic is

$$2\lambda < \theta\mu. \tag{3.6}$$

In fact the same condition is also necessary for ergodicity. To prove this, let t_n be the time when the server is sensed idle for the nth time due to either the service completion of the nth customer or collision of customers. Let $X_n = X(t_n+0)$ $(n \geq 0)$ and $t_0 = 0$. Since all the customers in the system are in the retrial group just after t_n, X_n represents the number of customers in the retrial group just after t_n. It is not difficult to see that the embedded Markov chain $\{X_n; n \in N\}$ is aperiodic and irreducible. Suppose that there exists an integer $j_0 \geq 0$ such that the mean drift $\chi_i = E(X_{n+1} - X_n / X_n = i) = 2\lambda - \theta\mu \geq 0$ for all $i > j_0$. It is noted in our retrial queueing system, there is an integer k such that $P_{ij} = 0$ for all i and j such that $0 < j < i-k$, $i > 0$, where $P = (P_{ij})$ $(i,j = 0,1,2,\ldots)$ is the one-step transition probability matrix associated with the Markov chain $\{X_n; n \in N\}$. By Kaplan's condition, the Markov chain $\{X_n; n \in N\}$ does not have a stationary distribution (see [18]).

To compare the stability conditions of the system with and without collisions, we observe that for a system with collisions, the rate of the customers entry into the orbit is 2λ and the rate of completion of the service being $\theta\mu$, the stability of the system requires the condition $2\lambda < \theta\mu$. However, for the classical feedback retrial queue without collisions, the stability condition is $\lambda < \theta\mu$ (see [19]).

In what follows, it is assumed that the condition (3.6) is fulfilled, so that the limiting joint probabilities P_j and Q_j of the server state and orbit length exist for $j = 0,1,2,\ldots$

Letting $t \to \infty$ in (3.1)–(3.5), the set of statistical equilibrium equations for the probabilities P_j and Q_j are as follows:

$$\lambda P_0 = \theta\mu Q_0, \tag{3.7}$$

$$(\lambda+v+\eta)P_1 = \lambda(1-p)P_0 + \theta\mu Q_1 + (1-\theta)\mu Q_0, \tag{3.8}$$

$$\begin{aligned}(\lambda+jv+\eta)P_j &= \lambda p Q_{j-2} + ((j-1)v+\eta)Q_{j-1} + \lambda(1-p)P_{j-1}\\ &\quad + \theta\mu Q_j + (1-\theta)\mu Q_{j-1}, \quad j = 2,3,4,\ldots,\end{aligned} \tag{3.9}$$

$$(\lambda+\mu)Q_0 = \lambda p P_0 + (v+\eta)P_1, \tag{3.10}$$

$$\begin{aligned}&(\lambda+\mu+jv+\eta)Q_j = \lambda p P_j + \lambda(1-p)Q_{j-1} + ((j+1)v+\eta)P_{j+1},\\ &\quad j = 1,2,3,\ldots.\end{aligned} \tag{3.11}$$

In order to solve Eqs. (3.7)–(3.11), define the partial generating functions for any $z \in [0,1]$ as

$$P(z) = \sum_{j=0}^{\infty} P_j z^j \quad \text{and} \quad Q(z) = \sum_{j=0}^{\infty} Q_j z^j. \tag{3.12}$$

Multiplying throughout by z^n and performing the summations, Eqs. (3.7)–(3.11) are transformed to the first order linear differential equations

$$\begin{aligned}&vzP'(z) + [\lambda+\eta-\lambda(1-p)z]P(z) - \eta P_0\\ &\quad = [\theta\mu + (1-\theta)\mu z + \lambda p z^2 + \eta z]Q(z) + vz^2Q'(z) - \eta z Q_0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&vP'(z) + \left(\lambda p + \frac{\eta}{z}\right)P(z) - \frac{\eta}{z}P_0\\ &\quad = [(\lambda+\mu+\eta) - \lambda(1-p)z]Q(z) - \eta Q_0 + vzQ'(z).\end{aligned} \tag{3.14}$$

Note that the prime indicates the derivative with respect to z. Now multiplying (3.14) throughout by z and subtracting from (3.13), we obtain

$$\lambda P(z) = (\theta\mu - \lambda z)Q(z). \tag{3.15}$$

If we put $z = 1$, this expression results in

$$\lambda[P(1) + Q(1)] = \theta\mu Q(1).$$

As we have assumed that a proper limiting probability distribution exists, i.e., $P(1) + Q(1) = \sum_{j=0}^{\infty}(P_j + Q_j) = 1$, it follows that

$$Q(1) = \sum_{j=0}^{\infty} Q_j = \frac{\lambda}{\theta\mu} \tag{3.16}$$

and that

$$P(1) = \sum_{j=0}^{\infty} P_j = 1 - \frac{\lambda}{\theta\mu}. \tag{3.17}$$

By differentiating (3.15) with respect to z and substituting into (3.13), the differential equation of $Q(z)$ as

$$\begin{aligned}&Q'(z) + \frac{[\lambda^2(1-2p)z^2 + \lambda(p\theta\mu - 2\eta - (\lambda+\mu+v))z + \theta\mu\eta]}{vz(\theta\mu - 2\lambda z)}Q(z)\\ &\quad = \frac{\eta\lambda}{v\theta\mu}\frac{(\theta\mu - \lambda z)}{z(\theta\mu - 2\lambda z)}P_0.\end{aligned} \tag{3.18}$$

The differential equation (3.18) has a singular point at $z = 0$, so the domain of the solution is (0,1]. Solving (3.18) for $Q(z)$, after some mathematical manipulation, for $z \in (0,1]$, we have

$$\begin{aligned}Q(z) &= \frac{\lambda}{\theta\mu} z^{-\eta/v}(\theta\mu - 2\lambda z)^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}\\ &\quad \times \left\{ \frac{e^{-(\lambda(1-2p)/2v)(1-z)}}{[\theta\mu - 2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}} \right.\\ &\quad \left. - P_0 \frac{\eta}{v}\int_z^1 \frac{u^{(\eta/v)-1}[\theta\mu - \lambda u]e^{(\lambda(1-2p)/2v)(z-u)}}{[\theta\mu - 2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du \right\}.\end{aligned} \tag{3.19}$$

As $z \to 0+$, $Q(0) = Q_0 < \infty$ and $z^{-\eta/v}$ diverges. Thus, from (3.19), we must have

$$\begin{aligned}P_0 &= \frac{ve^{-\lambda(1-2p)/2v}}{\eta[\theta\mu - 2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}}\\ &\quad \times \left\{ \lim_{z\to 0+}\int_z^1 \frac{(\theta\mu - \lambda u)u^{(\eta/v)-1}e^{-(\lambda(1-2p)/2v)u}}{[\theta\mu - 2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du \right\}^{-1}.\end{aligned} \tag{3.20}$$

By substituting (3.19) into (3.15), we get $P(z)$ as

$$\begin{aligned}P(z) &= \frac{(\theta\mu - \lambda z)}{\theta\mu} z^{-\eta/v}(\theta\mu - 2\lambda z)^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}\\ &\quad \times \left\{ \frac{e^{-(\lambda(1-2p)/2v)(1-z)}}{[\theta\mu - 2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}} \right.\\ &\quad \left. - P_0 \frac{\eta}{v}\int_z^1 \frac{u^{(\eta/v)-1}[\theta\mu - \lambda u]e^{(\lambda(1-2p)/2v)(z-u)}}{[\theta\mu - 2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du \right\},\end{aligned} \tag{3.21}$$

where P_0 is given in (3.20).

Thus, Eqs. (3.19)–(3.21) provide the steady-state partial generating functions of the joint distribution of server state and orbit queue length for the system under investigation.

Moreover, the probability generating function for the number of customers in the orbit denoted by $K(z)$ is defined as $K(z) = P(z) + Q(z)$.

Substituting expressions for $P(z)$ and $Q(z)$, we get

$$K(z)=z^{-\eta/v}\frac{1}{\theta\mu}[\theta\mu+\lambda(1-z)][\theta\mu-2\lambda z]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}$$
$$\times\left\{\frac{e^{-(\lambda(1-2p)/2v)(1-z)}}{[\theta\mu-2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}}\right.$$
$$\left.-\,P_0\frac{\eta}{v}\int_z^1\frac{u^{(\eta/v)-1}[\theta\mu-\lambda u]e^{(\lambda(1-2p)/2v)(z-u)}}{[\theta\mu-2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du\right\}. \tag{3.22}$$

Define $H(z)=P(z)+zQ(z)$. Consequently, $H(z)$ represents the probability generating function for the number of customers in the system. Using (3.19)–(3.21) and simplifying,

$$H(z)=z^{-\eta/v}[\theta\mu-2\lambda z]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}$$
$$\times\left\{\frac{e^{-(\lambda(1-2p)/2v)(1-z)}}{[\theta\mu-2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}}\right.$$
$$\left.-\,P_0\frac{\eta}{v}\int_z^1\frac{u^{(\eta/v)-1}[\theta\mu-\lambda u]e^{(\lambda(1-2p)/2v)(z-u)}}{[\theta\mu-2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du\right\}, \tag{3.23}$$

where P_0 is given in (3.20).

Observe that $P(z)$ is the probability generating function of the orbit size when the server is free, $Q(z)$ is the probability generating function of the orbit size when server is busy and P_0 is the probability that the server is idle while orbit is empty, i.e., the probability of empty system. These expressions are used in the next section for obtaining measures of the system performance.

4. Performance measures

Quality of service to customers in retrial queueing systems with feedback and collisions can be characterized by certain performance measures.

The performance measures related to the orbit characteristics are of considerable interest in retrial queues. Artalejo and Falin [20] have discussed the orbit busy and idle periods for $M/G/1$ retrial queues with classical retrial rate policy. For the model under consideration, the mean number of customers L_q in the orbit under steady-state condition obtained by differentiating (3.22) with respect to z and evaluating at $z=1$, is

$$L_q=\frac{1}{v}\left\{\frac{\eta(\theta\mu-\lambda)}{(\theta\mu-2\lambda)}P_0+\frac{\lambda(1-2p)}{2}-\eta-\frac{\lambda}{2[\theta\mu-2\lambda]}\right.$$
$$\left.\times[\theta\mu-2(\lambda+\mu+v)]\right\}-\frac{\lambda}{\theta\mu}. \tag{4.1}$$

The probability R of orbit being empty is obtained as

$$R=P_0+Q_0.$$

Using (3.20) and (3.7) in the above and simplifying, we get

$$R=\frac{v(\lambda+\theta\mu)e^{-\lambda(1-2p)/2v}}{\eta\theta\mu[\theta\mu-2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}}$$
$$\times\left\{\lim_{z\to 0+}\int_z^1\frac{(\theta\mu-\lambda u)u^{(\eta/v)-1}e^{(-\lambda(1-2p)/2v)u}}{[\theta\mu-2\lambda u]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]+1}}\,du\right\}^{-1}. \tag{4.2}$$

In a similar manner, the mean number of customers L_s in the system under steady-state condition is established from (3.23) as

$$L_s=\frac{1}{v}\left\{\frac{\eta(\theta\mu-\lambda)}{(\theta\mu-2\lambda)}P_0+\frac{\lambda(1-2p)}{2}-\eta-\frac{\lambda}{2[\theta\mu-2\lambda]}\right.$$
$$\left.\times\,[\theta\mu-2(\lambda+\mu+v)]\right\}, \tag{4.3}$$

where P_0 is given in (3.20). It is interesting to note that

$$L_s=L_q+Q(1),$$

where $Q(1)=\lambda/\theta\mu$ is the probability that the server is busy. The above relation could also be extracted by conditioning the mean number of customers in the orbit, L_q, on the state of the server, that is,

$$L_s=L_qP(1)+(L_q+1)Q(1)=L_q(P(1)+Q(1))+Q(1)=L_q+Q(1),$$

where $Q(1)$ is the probability that the server is busy and $P(1)$ is the probability that the server is idle as given in (3.16) and (3.17).

Furthermore, let M_0' and M_1' denote the partial moments defined by $M_0'=\sum_{j=0}^{\infty}jP_j$ and $M_1'=\sum_{j=0}^{\infty}jQ_j$. For $2\lambda<\theta\mu$, routine differentiation of the partial probability generating functions $Q(z)$ and $P(z)$ yield

$$M_1'=Q'(1)=\frac{\lambda}{\theta\mu}\frac{1}{v}$$
$$\times\left\{\frac{\eta(\theta\mu-\lambda)}{(\theta\mu-2\lambda)}P_0+\frac{\lambda(1-2p)}{2}-\eta-\frac{\lambda}{2[\theta\mu-2\lambda]}\right.$$
$$\left.\times[\theta\mu-2(\lambda+\mu+v)]\right\} \tag{4.4}$$

and

$$M_0'=P'(1)=\frac{(\theta\mu-\lambda)}{\lambda}M_1'-\frac{\lambda}{\theta\mu},$$

so that

$$M_0'=\frac{1}{v}\left(1-\frac{\lambda}{\theta\mu}\right)\left\{\frac{\eta(\theta\mu-\lambda)P_0}{(\theta\mu-2\lambda)}+\frac{\lambda(1-2p)}{2}-\eta\right.$$
$$\left.-\,\frac{\lambda}{2[\theta\mu-2\lambda]}[\theta\mu-2(\lambda+\mu+v)]\right\}-\frac{\lambda}{\theta\mu}. \tag{4.5}$$

It is further observed that the mean number of customers L_s in the system under steady-state can be expressed as

$$L_s=M_0'+M_1'+Q(1),$$

and the mean number of customers L_q in the orbit can be expressed as

$$L_q=M_0'+M_1'.$$

The most important performance measure of a retrial queueing system is the waiting time of a customer in the orbit. To compute the expected waiting time W_q of a customer, we adopt a methodology developed by [21] or [22]. Let t be a long time interval in which a particular system is observed and t_j be the total length of sub-intervals of t in which j customers wait. Since j customers wait for this length of time then the total waiting time expended over all sub-intervals in which j customers wait is jt_j, the average waiting time, is given as

$$W_q=\frac{\sum_{j=0}^{\infty}jt_j}{\lambda t}=\frac{L_q}{\lambda}\quad\text{as } t\to\infty. \tag{4.6}$$

The denominator of the middle term in (4.6) is the number of arrivals in the time interval of length t, which is λt. Hence, from (4.1) and (4.6), we get

$$W_q=\frac{1}{v}\left\{\frac{\eta(\theta\mu-\lambda)}{\lambda(\theta\mu-2\lambda)}P_0+\frac{(1-2p)}{2}-\frac{\eta}{\lambda}-\frac{[\theta\mu-2(\lambda+\mu+v)]}{2(\theta\mu-2\lambda)}\right\}$$
$$-\,\frac{1}{\theta\mu}, \tag{4.7}$$

where P_0 is given in (3.20).

In a similar manner, the average waiting time in the system is obtained as

$$W_s = \frac{1}{v}\left\{\frac{\eta(\theta\mu-\lambda)}{\lambda(\theta\mu-2\lambda)}P_0+\frac{(1-2p)}{2}-\frac{\eta}{\lambda}-\frac{[\theta\mu-2(\lambda+\mu+v)]}{2(\theta\mu-2\lambda)}\right\}. \tag{4.8}$$

Another interesting system performance measure is the average number of look ins per customer. A method similar to that used for the average waiting time is applied here. During the time interval t_j, an average of $[(1-\delta j_0)\eta + jv]t_j$ $(j \geq 1)$ "look ins" occur from the orbit and $\lambda p t_j$ $(j \geq 0)$ new customers apply for service. Assuming the orbit to be empty initially, the total number of customers in orbit or arriving in the time interval t is λt. Hence, the average number of look ins per customer, for sufficiently large t, is given as

$$\Gamma = \frac{\sum_{j=0}^{\infty}[p\lambda + ((1-\delta j_0)\eta + jv)]t_j}{\lambda t},$$

so that

$$\Gamma = p + \frac{\eta}{\lambda} - \frac{\eta}{\lambda}\frac{t_0}{t} + vW_q.$$

It is noted that t_0/t is the probability of the event that the orbit is having no customers (being empty), that is $t_0/t = R$, where R is given in (4.2), so that

$$\Gamma = p + vW_q + \frac{\eta}{\lambda}[1-R]. \tag{4.9}$$

The steady-state interrupted frequency I of the service due to collisions is given by

$$I = \lambda p Q(1) + \sum_{j=0}^{\infty} Q_j(jv+\eta).$$

The first term on the right hand side of the above equation represents the interrupted frequency when a primary arriving customer goes directly to the server when the server is busy while the second term represents the interrupted frequency due to retrial of the customer from the orbit when the server is busy.

On simplification, we get

$$I = \frac{\lambda p^2}{\theta\mu} + vQ'(1) + \eta\left[1 - \frac{\lambda}{\theta\mu}P_0\right].$$

This together with (3.20) and (4.4) yields

$$I = \frac{\lambda p^2}{\theta\mu} + \frac{\lambda^2(1-2p)}{2\theta\mu} - \frac{\lambda^2}{2(\theta\mu-2\lambda)} + \frac{\lambda^2(\lambda+\mu+v)}{\theta\mu(\theta\mu-2\lambda)} + \frac{\eta}{\theta\mu}\left[(\theta\mu-\lambda) + \frac{\lambda^2 p}{(\theta\mu-2\lambda)}\right], \tag{4.10}$$

where P_0 is given in (3.20).

Now consider a busy period of the system for the model under discussion. The mean busy period of the system is an interesting and important performance measure in the retrial context. The system busy period L is defined as the period that starts at an epoch when an arriving customer finds an empty system and ends at the next departure epoch at which the system is empty. The mean length of the system busy period of this model is obtained in a direct way by the theory of regenerative processes which leads to the limiting probability

$$P_0 = \lim_{t\to\infty} P\{C(t)=0, X(t)=0\},$$

as follows:

$$P_0 = \frac{E(T_{00})}{\frac{1}{\lambda} + E(L)},$$

where T_{00} is the amount of time in a regenerative cycle during which the system is in the state (0,0). It is clear that

$$E(T_{00}) = \frac{1}{\lambda}$$

so that

$$E(L) = \frac{1}{\lambda}(P_0^{-1} - 1).$$

Using (3.20), $E(L)$ is obtained as

$$E(L) = \frac{1}{\lambda}\left\{\frac{\eta}{v}\frac{[\theta\mu-2\lambda]^{[(\theta\mu/4v)-((\lambda+\mu+v)/2v)]}}{e^{-\lambda(1-2p)/2v}} \times \left[\lim_{z\to 0+}\int_z^1 \frac{(\theta\mu-\lambda u)u^{\eta/v-1}e^{-(\lambda(1-2p)/2v)u}}{(\theta\mu-2\lambda u)^{[\theta\mu/4v-(\lambda+\mu+v)/2v]+1}}\,du\right] - 1\right\}. \tag{4.11}$$

Remark 1. If $p=1$, $\theta=1$ and $\eta=0$, then the system reduces to classical retrial queue with collisions [8,1]. Thus, under steady-state condition $2\lambda < \mu$, the probability generating function of the number of customers in the system $H(z)$ becomes

$$H(z) = e^{(\lambda/2v)(1-z)}\left[\frac{\mu-2\lambda}{\mu-2\lambda z}\right]^{[(\lambda+v+(\mu/2))/2v]}.$$

The corresponding mean number of customers in the system L_s is obtained as

$$L_s = \frac{\lambda}{\mu-2\lambda}\left[1+\frac{2\lambda}{v}\right],$$

which agrees with Falin and Sukharev [9].

Remark 2. If $p=1$, $\theta=1$ and $v=0$, then the system becomes a retrial queue with constant/control retrial policy and collisions. In this case, under steady-state condition $2\lambda^2/(\mu-2\lambda) < \eta$, the probability generating function of the number of customers in the system $H(z)$ is obtained as

$$H(z) = \frac{(\mu-\lambda z)}{(\mu-\lambda)}\frac{[\eta(\mu-2\lambda)-2\lambda^2]}{[\eta(\mu-2\lambda z)-\lambda^2 z(1+z)]},$$

and the mean number of customers in the system L_s is given as

$$L_s = \frac{\lambda[\eta\mu + \lambda(3\mu-\lambda)]}{(\mu-\lambda)[\eta(\mu-2\lambda)-2\lambda^2]}.$$

5. Numerical results

In this section, we use the performance measures and probability descriptors derived previously to obtain numerical results and to investigate the way the steady-state joint probability P_0 that the server is idle and no customers in the orbit, the mean number of customers L_q in the orbit, the mean waiting time W_q in the orbit and the mean busy period $E(L)$ of the system, are affected when we vary the values of the system parameters.

In Figs. 1–3, we study the behavior of the steady-state joint probability P_0 of the server being idle and no customer in the orbit for $\lambda = 0.1, \mu = 10, \eta = 1.5$ with departure probability $\theta = 0.3, 0.5, 0.7$ and varying values of retrial rate v. Figs. 1–3 indicate that for decreasing departure probabilities, P_0 decreases whereas it increases for increasing values of the retrial rate v as expected. It is also observed in Figs. 1–3 that P_0 increases for increasing values of the probability p.

We show in Figs. 4–6, the influence of the parameters θ and v on the mean number of customers L_q in retrial group. It is noted that L_q decreases for increasing values of retrial rate v and decreases more apparently as the departure probability θ increases. Moreover,

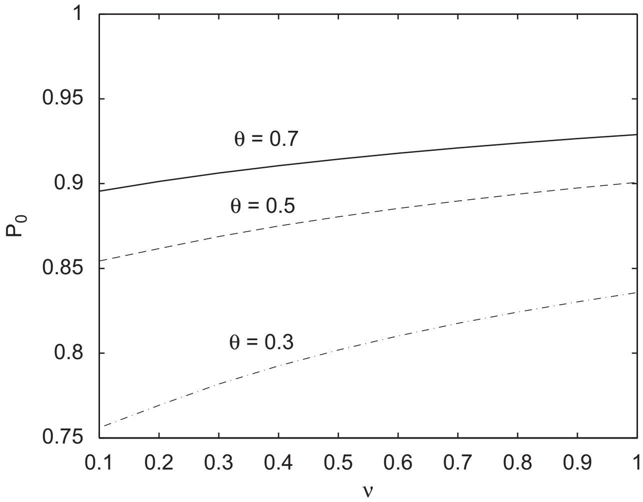

Fig. 1. P_0 versus retrial rate ν for $p=0$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

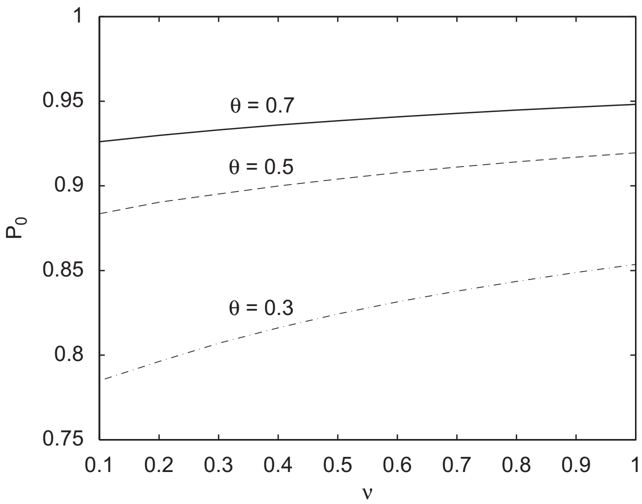

Fig. 2. P_0 versus retrial rate ν for $p=0.5$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

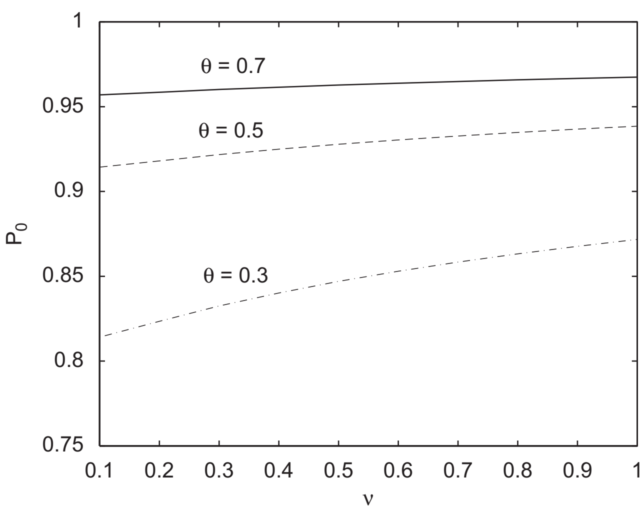

Fig. 3. P_0 versus retrial rate ν for $p=1$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

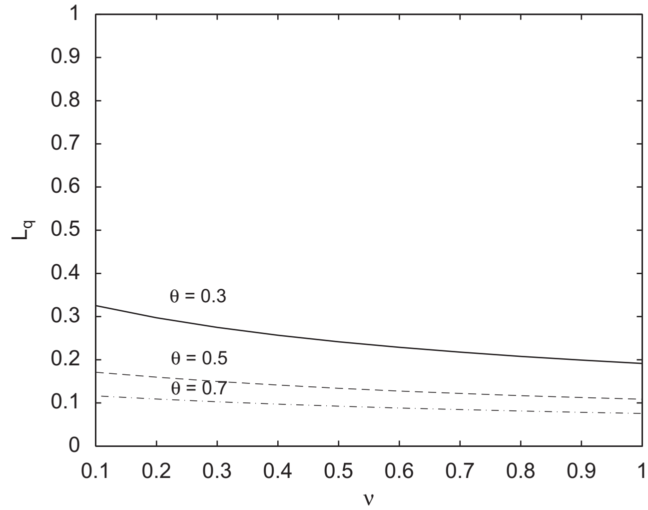

Fig. 4. L_q versus retrial rate ν for $p=0$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

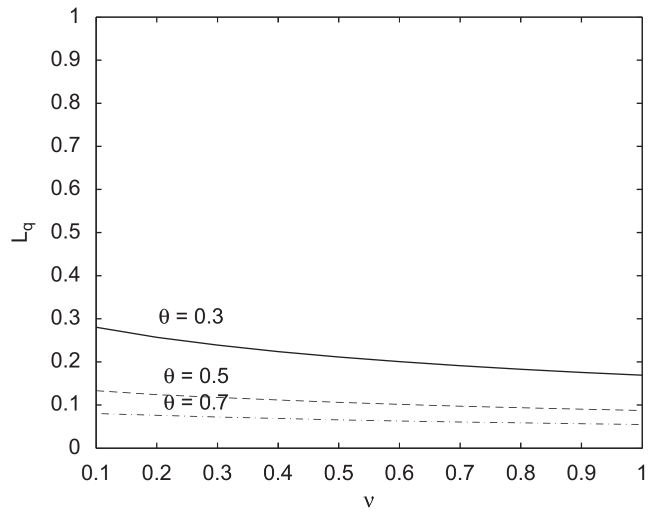

Fig. 5. L_q versus retrial rate ν for $p=0.5$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

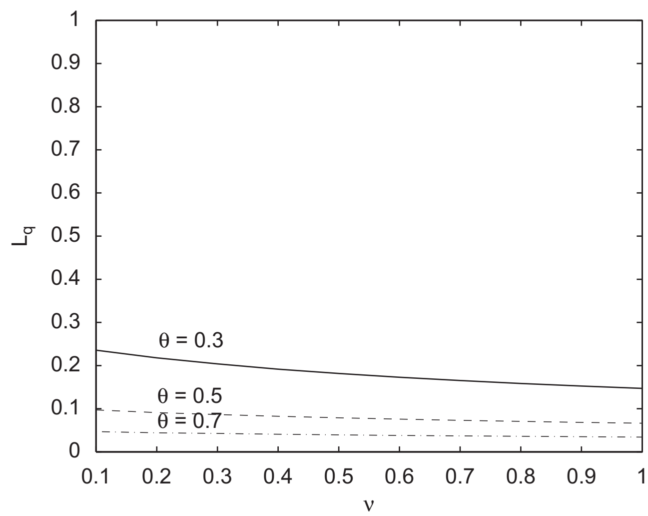

Fig. 6. L_q versus retrial rate ν for $p=1$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

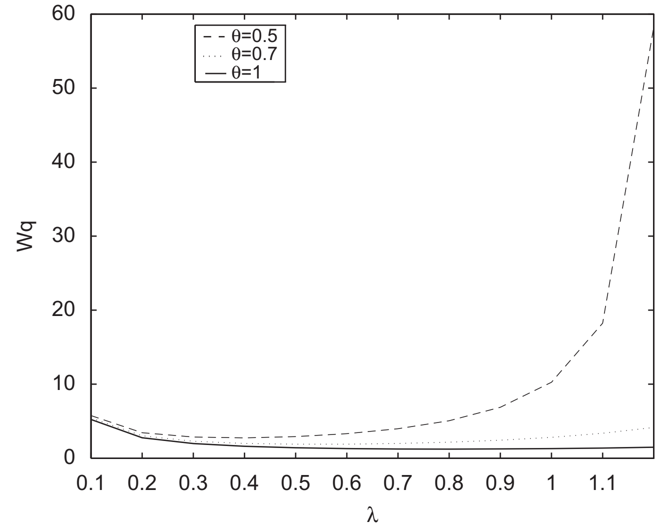

Fig. 7. W_q versus arrival rate λ for $p=0.5$, $\eta=1.5$, $\nu=1$, $\mu=5$.

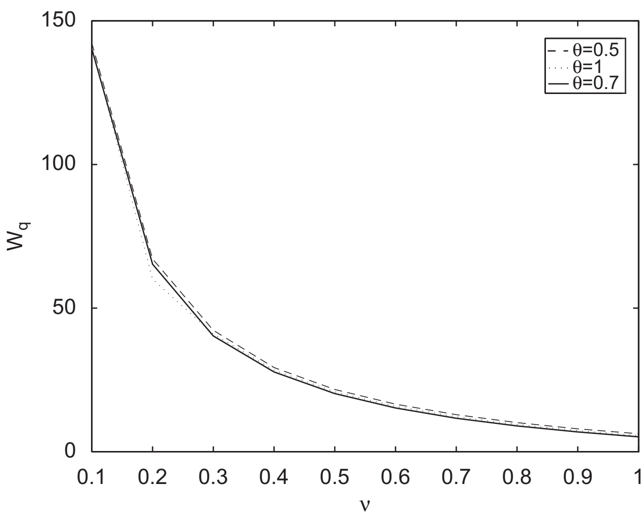

Fig. 8. W_q versus retrial rate ν for $p=0.5$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

as the probability p increases, the mean number of customers L_q in the orbit decreases in contrast to the probability P_0.

In Fig. 7, the mean waiting time W_q in the orbit is plotted versus arrival rate λ. We have presented three curves which correspond to $\theta=0.5, 0.7$ and 1. As expected, W_q drastically increases as λ increases for $\theta=0.5$ and 0.7 (with feedback) while W_q lightly increases as λ increases for $\theta=1$ (without feedback). From Fig. 8, we find that W_q decreases as the retrial rate ν increases for any θ as it must be. In Figs. 9–11, for fixed values of $\lambda=0.1, \mu=10$ and $\eta=1.5$, we plot the mean busy period $E(L)$ of the system for various values of the retrial rate ν. Here also $E(L)$ decreases for increasing values of ν, θ and p as expected.

Now, we present some numerical results that show the effect of the arrival rate λ on the system performance measures. To that end, we show in Tables 1–3, the influence of the parameters λ, θ, p and (ν, η) on P_0, L_q and $E(L)$.

Table 1 contains values of P_0 for varying arrival rates λ with different values of retrial rates (ν, η) and probability p. The computations are carried out for $\theta=0.5$ and $\mu=5$. It is observed that P_0

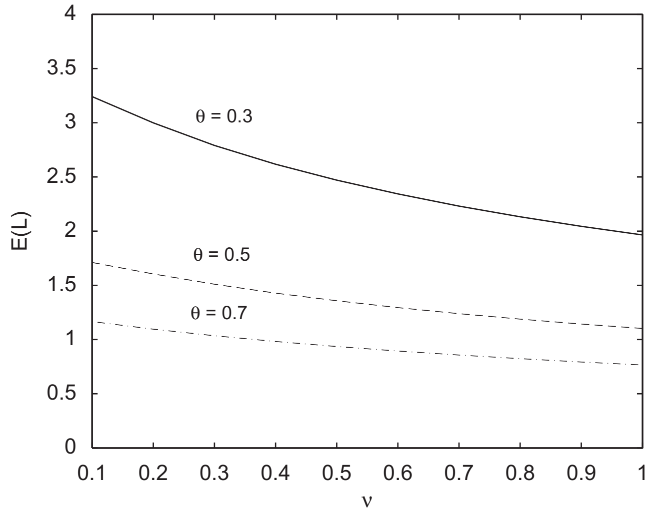

Fig. 9. $E(L)$ versus retrial rate ν for $p=0$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

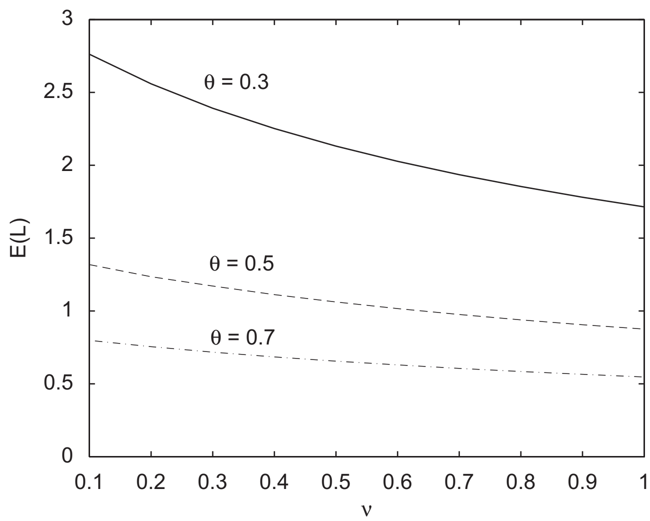

Fig. 10. $E(L)$ versus retrial rate ν for $p=0.5$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

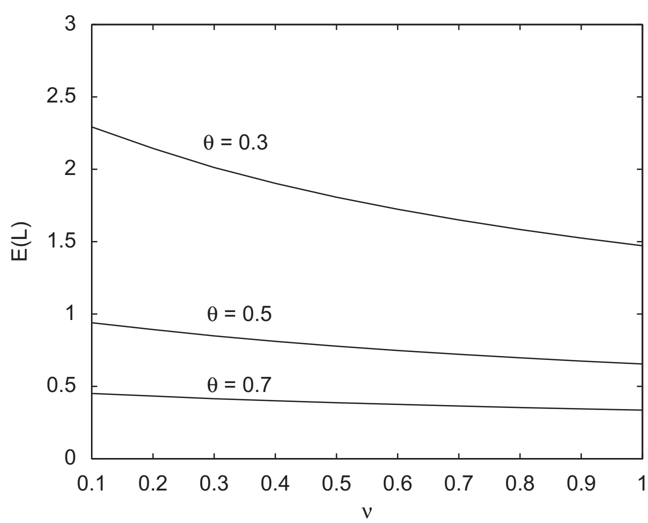

Fig. 11. $E(L)$ versus retrial rate ν for $p=1$, $\eta=1.5$, $\lambda=0.1$, $\mu=10$.

Table 1
P_0 versus arrival rate λ.

λ	$\theta=0.5,\ \mu=5,\ \eta=1.5,\ v=1$			$\theta=0.5,\ \mu=5,\ \eta=2,\ v=1.5$			$\theta=0.5,\ \mu=5,\ \eta=2.5,\ v=2$			$\theta=0.5,\ \mu=5,\ \eta=3,\ v=2.5$		
	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$
0.1	0.87904	0.89736	0.91595	0.90155	0.91479	0.92816	0.91414	0.92450	0.93494	0.92219	0.93070	0.93927
0.2	0.75693	0.78998	0.82391	0.80075	0.82500	0.84969	0.82556	0.84471	0.86421	0.84151	0.85733	0.87335
0.3	0.63533	0.67906	0.72463	0.69833	0.73113	0.76495	0.73453	0.76078	0.78763	0.75809	0.77994	0.80218
0.4	0.51618	0.56621	0.61932	0.59518	0.63385	0.67410	0.64162	0.67300	0.70538	0.69851	0.69851	0.72559
0.5	0.40217	0.45387	0.50992	0.49262	0.53422	0.57814	0.54743	0.58189	0.61795	0.61338	0.61338	0.64369
0.6	0.29646	0.34517	0.39940	0.39234	0.43385	0.47830	0.45295	0.48824	0.52538	0.52489	0.52489	0.55668
0.7	0.20262	0.24424	0.29194	0.29659	0.33490	0.37667	0.35952	0.39335	0.42938	0.43376	0.43376	0.46514
0.8	0.12439	0.15577	0.19304	0.20815	0.24045	0.27636	0.26903	0.29908	0.33151	0.34109	0.34109	0.37009
0.9	0.064913	0.084785	0.10938	0.13048	0.15453	0.18188	0.18408	0.20817	0.23454	0.24875	0.24875	0.27326
1.0	0.025819	0.035302	0.047608	0.067614	0.082284	0.099409	0.10826	0.12473	0.14307	0.14297	0.15964	0.17773

Table 2
L_q versus arrival rate λ.

λ	$\theta=0.5,\ \mu=5,\ \eta=1.5,\ v=1$			$\theta=0.5,\ \mu=5,\ \eta=2,\ v=1.5$			$\theta=0.5,\ \mu=5,\ \eta=2.5,\ v=2$			$\theta=0.5,\ \mu=5,\ \eta=3,\ v=2.5$		
	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$
0.1	0.13676	0.11543	0.094530	0.10940	0.094490	0.079759	0.094530	0.083043	0.071661	0.085177	0.075833	0.066564
0.2	0.31496	0.26925	0.22499	0.24871	0.21746	0.18685	0.21356	0.18978	0.16648	0.19170	0.17249	0.15355
0.3	0.55084	0.47679	0.40594	0.42900	0.37964	0.33185	0.36577	0.32876	0.29262	0.32703	0.29739	0.26829
0.4	0.86622	0.76092	0.65933	0.66657	0.59693	0.52989	0.56427	0.51272	0.46272	0.50226	0.46132	0.42146
0.5	1.3043	1.1577	1.0198	0.98688	0.89417	0.80558	0.82905	0.76148	0.69625	0.73450	0.68141	0.62990
0.6	1.9192	1.7259	1.5448	1.4312	1.3121	1.1987	1.1929	1.1074	1.0252	1.0516	0.98519	0.92095
0.7	2.8155	2.5677	2.3348	2.0713	2.9216	1.7794	1.7127	1.6069	1.5056	1.5021	1.4208	1.3425
0.8	4.1858	3.8747	3.5803	3.0427	2.8574	2.6812	2.4963	2.3673	2.2438	2.1778	2.0798	1.9855
0.9	6.4368	6.0550	5.6893	4.6357	4.4090	4.1924	3.7759	3.6178	3.4701	3.2767	3.1594	3.0467
1.0	10.616	10.159	9.7142	7.6038	7.3291	7.0643	6.1560	5.9677	5.7865	5.3147	5.1747	5.0398

Table 3
$E(L)$ versus arrival rate λ.

λ	$\theta=0.5,\ \mu=5,\ \eta=1.5,\ v=1$			$\theta=0.5,\ \mu=5,\ \eta=2,\ v=1.5$			$\theta=0.5,\ \mu=5,\ \eta=2.5,\ v=2$			$\theta=0.5,\ \mu=5,\ \eta=3,\ v=2.5$		
	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$	$p=0$	$p=0.5$	$p=1$
0.1	1.3760	1.1438	0.91263	1.0920	0.93147	0.77400	0.93924	0.81666	0.69587	0.84375	0.74460	0.64657
0.2	1.6056	1.3293	1.0686	1.2441	1.0606	0.88450	1.0565	0.91919	0.78563	0.94170	0.83206	0.72508
0.3	1.9133	1.5754	1.2667	1.4400	1.2258	1.0242	1.2047	1.0481	0.89877	1.0637	0.94050	0.82201
0.4	2.3433	1.9153	1.5367	1.7004	1.4442	1.2086	1.3964	1.2147	1.0442	1.2194	1.0790	0.94547
0.5	2.9730	2.4065	1.9222	2.0599	1.7438	1.4594	1.6534	1.4371	1.2376	1.4243	1.2606	1.1071
0.6	3.9552	3.1619	2.5063	2.5813	2.1749	1.8179	2.0129	1.7470	1.5056	1.7048	1.5086	1.3273
0.7	5.6219	4.4205	3.4648	3.3881	2.8371	2.3641	2.5450	2.2032	1.8985	2.1088	1.8649	1.6427
0.8	8.7990	6.7747	5.2253	4.7553	3.9486	3.2731	3.3963	2.9295	2.5206	2.7341	2.4147	2.1276
0.9	16.006	11.994	9.0472	7.4045	6.0792	4.9979	4.9249	4.2264	3.6263	3.8079	3.3557	2.9550
1.0	37.731	27.327	20.005	13.790	11.153	9.0595	8.2370	7.0173	5.9896	5.9945	5.2641	4.6265

decreases as λ increases while it increases with increasing values of p and retrial rates (v, η) as expected.

Table 2 shows the way L_q changes for increasing values of arrival rate λ, the other values being fixed as in Table 1. It is noted that L_q increases with increasing λ but decreases for increasing p and (v, η).

Finally, Table 3 summarizes the values of $E(L)$ for various arrival rates λ. The behavior of $E(L)$ is the same as that of L_q for different values of p, and (v, η).

6. Expected total cost

In this section, we construct an expected total cost function per unit time of the system under discussion. First, let us fix the following costs for a classical cost structure:

C_s: the setup cost per opening up the server per unit time.
C_h: the waiting cost per customer per unit time.

The long-run expected total cost function of two decision variables p and θ for the system is defined to be

$$C(\theta, p) = \frac{C_s}{E(T)} + C_h W_s, \tag{6.1}$$

where $E(T)$ is the length of busy cycle defined as $E(T)=E(L)+(1/\lambda)$; W_s and $E(L)$ are given in (4.8) and (4.11), respectively. Since the expected total cost function per unit time is obtained only implicitly, the analytical properties such as convexity of the cost function cannot be studied. However, we present a numerical example to demonstrate the computability of the results derived in our work.

In Table 4, we have given the expected total cost rate in the steady-state. It is assumed that $\lambda=0.1, \mu=2.75, v=0.5, C_s=5$ and $C_h=10$. Table 4 indicates that the total expected cost rate decreases for both increasing values of p and θ. Moreover, the expected total cost rate appears to be more sensitive to θ than to p. It is also observed from Table 4 that the expected total cost rate will be minimized, if

Table 4
Expected total cost for $C_s=5$, $C_h=10$, $\nu=0.5$, $\eta=0.5$, $\lambda=0.1$, $\mu=2.75$.

p	$\theta=0.1$	$\theta=0.2$	$\theta=0.3$	$\theta=0.4$	$\theta=0.5$	$\theta=0.6$	$\theta=0.7$	$\theta=0.8$	$\theta=0.9$	$\theta=1$
0	775.64707	134.60317	69.883893	46.796110	35.120258	28.109601	23.444963	20.121700	17.635727	15.706858
0.1	773.77619	133.19134	68.628665	45.613645	33.979341	26.995459	22.349485	19.039967	16.564533	14.644001
0.2	771.90697	131.78505	62.379683	44.437644	32.844981	25.887923	21.260642	17.964886	15.500003	13.587814
0.3	770.03943	130.38433	66.136965	43.268121	31.717190	24.787001	20.178440	16.896463	14.442141	12.538304
0.4	768.17360	128.98920	64.900530	42.105090	30.595979	23.692704	19.102887	15.834703	13.390953	11.495474
0.5	766.30948	127.59969	63.670395	40.948562	29.481356	22.605037	18.033987	14.779612	12.346442	10.459328
0.6	764.44710	126.21584	62.446575	39.798552	28.373331	21.524010	16.971748	13.731194	11.308613	9.4298687
0.7	762.58647	124..83766	61.229090	38.655070	27.271914	20.449626	15.916176	12.689455	10.277470	8.4070995
0.8	760.72762	123.46517	60.017955	37.518128	26.177111	19.381895	14.867274	11.654397	9.2530148	7.3910226
0.9	758.87056	122.09840	58.813187	36.387735	25.088932	18.320820	13.825046	10.62023	8.2352496	6.3816395
1	757.01531	120.73739	57.614799	35.263904	24.007383	17.266406	12.789496	9.6043357	7.2241764	5.3789512

the arriving primary customers go directly to the server instead of joining immediately the retrial group.

7. Conclusion

The joint steady-state probability generating functions of the server state and the orbit length are derived for Bernoulli feedback retrial queueing system under linear retrial policy together with service option on arrival and collisions of customers. These joint probability generating functions are used to calculate several performance measures relating to the operation of the system. Under the stability condition, numerical results are provided in which several system characteristics are calculated based on assumed numerical values for the system parameters. A cost analysis is discussed to determine the optimal values of the probability of service option on arrival that minimizes the total expected cost per unit time.

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