Waiting Time Distributions for Processor-Sharing Systems

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ABSTRACT. A basic probability model that has arisen in the study of time-shared or multi-programmed systems is the so-called processor-sharing model. In this model it is assumed that the processor is shared simultaneously by each unit in the system (e.g. job or program in computer systems, or message in communication systems). In particular, if there are n units in the system, then any given unit is being processed at a rate which is $(1/n)$-th the rate at which it would be processed if it had the system to itself.

We make the assumptions of a Poisson arrival process and exponential service times and then derive an expression for the Laplace transform of the waiting time distribution of an arriving unit conditioned on the service it requires and the number it finds in the system on arrival. From this result we obtain the first two moments of the waiting times, the Laplace transform of the equilibrium waiting time distribution, and the first two moments of this latter distribution. The paper concludes with a discussion of the results, especially as they compare with similar results for the first-come-first-served discipline.

KEY WORDS AND PHRASES: time-sharing analysis, processor-sharing analysis, pure time-sharing analysis, multiprogramming analysis, operating systems studies, time-division multiplexing, supervisor systems, shared computer systems

CR CATEGORIES: 3.80, 3.81, 4.32

Introduction

A queueing system of considerable interest in both computer and communications engineering is the so-called *processor-sharing* or *pure time-sharing* system [1–3]. In this type of system the server (processor or channel) shares its (fixed) capacity equally among all units present in the system. Thus, as shown in Figure 1, there is no waiting line. However, the service received by individual units at any point in time is inversely proportional to the total number in the system at that point in time. In other words, the total rate at which the server performs work is constant, but if there are n units in the system then each unit is receiving service at $(1/n)$-th the rate at which it would receive service if it had the server to itself. Of course, the rate at which units receive service changes each time a new arrival joins the system and each time a completed unit departs

A direct physical realization of the processor-sharing (PS) model may be difficult to visualize; so to help understand the applications of this discipline we briefly examine the so-called *round-robin* (RR) model. As shown in Figure 2, units arrive to the RR system and join the end of an ordered queue. When a given unit reaches the service point it is allocated a fixed quantum (q) of service time. If the unit completes within this time it simply leaves the system. If after q seconds it still requires more service it is immediately returned to the end of the queue. In due course, after the other units have received a quantum of service, the given unit returns to the service point and receives another quantum of service.

Such a procedure is used for sequencing programs in time-sharing systems in

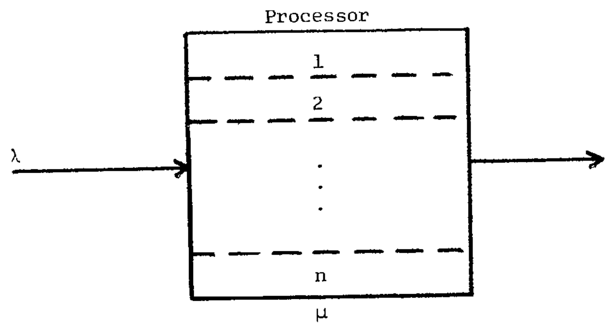

FIG. 1. The processor-sharing model

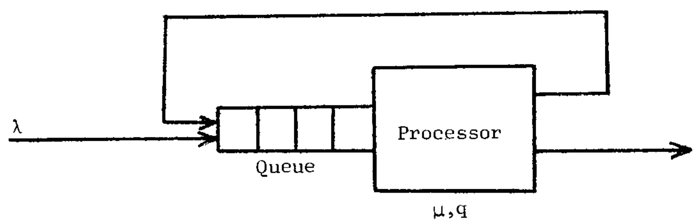

FIG. 2. The round-robin model

order to provide a smaller waiting time in system for those programs having smaller service requirements. This scheme also corresponds to time-multiplexed message transmission in communication systems. In this case, small segments (quanta) of each message are alternately sent over the channel. At the receiver the channel output is decoded so that each "quantum" is identified with the proper message.

If we now examine the behavior of the RR system as q approaches zero, we observe that, in the limit, the RR system becomes the PS system. Thus, the PS model provides a good approximation to an RR system in which the quantum size is very small relative to the service requirements of the units being served.

With this motivation, the object of this paper is to compute the waiting time distribution for the PS model under exponential assumptions for interarrival and service times. In particular, we obtain the Laplace transform of this distribution conditioned on the service requirement and the number in system at arrival. From this result we calculate the first two moments, the equilibrium (i.e. steady-state) waiting time distribution, and the first two moments of this latter distribution. The significance of the results is discussed, especially as they compare with those of the classical first-come-first-served (FCFS) system.

Previous work concerned with the PS model [1–3] has succeeded in finding only the first moment of the equilibrium waiting time distribution. Specifically, this result has been computed as the limit of the corresponding result for the RR system when the quantum size is allowed to approach zero. Sakata, et al [6] have also studied the PS model under the assumption of a general service time distribution. Their results relate to the distribution of the number in system and the overall flow time, but not to conditional waiting time distributions. Interesting deterministic scheduling problems connected with the PS discipline have also been studied quite recently [4, 7].

Waiting Time Distribution for the PS Model

For the PS model of Figure 1 let the interarrival time distribution be given by

$$A(t) = \begin{cases} 1 - \epsilon^{-\lambda t}, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \quad \lambda > 0, \tag{1}$$

with the mean value $1/\lambda$ seconds. Similarly, let the service time distribution be given by

$$B(t) = \begin{cases} 1 - \epsilon^{-\mu t}, & t > 0, \\ 0, & \text{otherwise}, \end{cases} \quad \mu > 0, \tag{2}$$

with the mean value $1/\mu$ seconds. According to our description of the PS model, we see that the capacity (processing rate) of the server is effectively determined by the parameter μ. That is, if $1/\mu$ is the mean processing time of a unit then μ is the maximum mean rate at which units are processed by the server. Moreover, the departure process from the PS system is Poisson and identical to that of the Poisson FCFS queue. Thus, for both the PS and FCFS systems the properties of the process $\omega(t)$, whose value at time t is the number in the system, are precisely the same (and, of course, well-known [5]). Of particular interest below will be the equilibrium or stationary probability distribution $\{p_n\}$ for the number in system:

$$p_n = (1 - \rho)\rho^n, \qquad n = 0, 1, 2, \cdots, \tag{3}$$

where

$$\rho = \lambda/\mu, \tag{4}$$

and where the condition for the existence of this distribution (i.e. the existence of a steady-state) is $\rho < 1$; i.e. the arrival rate λ must be less than the maximum departure rate μ.

Now let $X_n(\tau)$ denote a random variable whose distribution is that of the "delay" experienced by a unit requiring τ seconds of service and arriving when there are n others already in the system. Thus $X_n(\tau) + \tau$ is the total time spent in system by the given arrival. We define $w_n(\tau, s) = E[\epsilon^{-sX_n(\tau)}]$, the Laplace transform [5] of the distribution for $X_n(\tau)$. Formally, if $F_n(\tau, x)$ denotes the cumulative distribution function then

$$w_n(\tau, s) = \int_0^\infty \epsilon^{-sx}\, d_x F_n(\tau, x), \qquad s \geq 0, \quad \tau \geq 0, \quad n = 0, 1, 2, \cdots. \tag{5}$$

We now proceed to obtain an expression for the $w_n(\tau, s)$. We begin with the following lemma.

LEMMA 1. *The transform $w_n \equiv w_n(\tau, s)$ is governed by the differential-difference equation*

$$\partial w_n/\partial \tau = \lambda(n + 1)w_{n+1} - [\lambda(n + 1) + (\mu + s)n]w_n + \mu n w_{n-1}, \qquad n = 0, 1, 2, \cdots, \tag{6}$$

with the boundary condition

$$w_n(0, s) = 1, \qquad s \geq 0, \qquad n = 0, 1, 2, \cdots. \tag{7}$$

(Note that when $n = 0$, the last term on the right of eq. (6) drops out so that w_{-1} does not actually appear.)

PROOF. We use the basic theory of the birth and death process $\omega(t)$ as described by Feller [5]. Let $\Delta\tau$ be a differential element of (service) time. Then an arriving unit finding n in the PS system must wait $(n + 1)\Delta\tau$ seconds before it has received its first $\Delta\tau$ seconds of service. Neglecting terms of the order of $(\Delta\tau)^2$, the probability of another arrival during this time interval is $\lambda(n + 1)\Delta\tau$, the probability of a departure is $\mu(n + 1)(n/(n + 1))\Delta\tau = \mu n\Delta\tau$, and the probability of no change is $1 - [\lambda(n + 1) + \mu n]\Delta\tau$. Also, we observe that the accumulated delay when there is no change is $n\Delta\tau$ seconds and is of the order of $\Delta\tau$ seconds in any case. Hence, again neglecting terms of the order of $(\Delta\tau)^2$, we may write

$$E[\epsilon^{-sX_n(\tau+\Delta\tau)}] = (1 - [\lambda(n + 1) + \mu n]\Delta\tau)\epsilon^{-sn\Delta\tau}E[\epsilon^{-sX_n(\tau)}] + \lambda(n + 1)\Delta\tau E[\epsilon^{-sX_{n+1}(\tau)}] + \mu n\Delta\tau E[\epsilon^{-sX_{n-1}(t)}]$$

or

$$\begin{aligned} w_n(\tau + \Delta\tau, s) = w_n(\tau, s) &- ([\lambda(n + 1) + (\mu + s)n]w_n(\tau, s) \\ &- \lambda(n + 1)w_{n+1}(\tau, s) - \mu n w_{n-1}(\tau, s))\Delta\tau, \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{8}$$

Rearranging and taking the limit $\Delta\tau \to 0$ we obtain eq. (6). Finally, eq. (7) follows from the obvious fact that $X_n(0) = 0$ for all n. Q.E.D.

In order to solve eq. (6) under the boundary conditions eq. (7), it is convenient to introduce the generating function

$$W(\tau, s, z) = \sum_{n=0}^{\infty} w_n(\tau, s)z^n. \tag{9}$$

Applying eq. (9) to eqs. (6) and (7) and comparing coefficients in the usual way yields for $W \equiv W(\tau, s, z)$,

$$\partial W/\partial\tau - (\mu z^2 - (\lambda + \mu + s)z + \lambda)\partial W/\partial z = (\mu z - \lambda)W \tag{10}$$

with the boundary condition

$$W(0, s, z) = 1/(1 - z). \tag{11}$$

The basic result now follows.

THEOREM 1. *Let r be defined as a root of the quadratic $\lambda z^2 - z(\lambda + \mu + s) + \mu$. For example, if r is taken as the smaller root*

$$r = (\lambda + \mu + s - [(\lambda + \mu + s)^2 - 4\mu\lambda]^{1/2})/2\lambda \tag{12}$$

or

$$r = 2\mu(\lambda + \mu + s + [(\lambda + \mu + s)^2 - 4\mu\lambda]^{1/2})^{-1} \tag{13}$$

where we have made use of the fact that the product of the two roots of the polynomial is $1/\rho$.

The solution of eq. (10) satisfying eq. (11) is

$$W = (1 - \rho r^2)\epsilon^{-\lambda\tau(1-r)}/((1 - \rho r)(1 - rz) - (1 - r)(z - \rho r)\epsilon^{-\mu\tau(1-\rho r^2)/r}). \tag{14}$$

Expanding eq. (14) *in powers of z we obtain*

$$w_n(\tau, s) = ((1 - \rho r^2)\epsilon^{-\lambda\tau(1-r)}/((1 - \rho r) + \rho r(1 - r)\epsilon^{-\mu\tau(1-\rho r^2)/r}))\beta^n, \quad n = 0, 1, 2, \cdots, \tag{15}$$

where

$$\beta = (r(1 - \rho r) + (1 - r)\epsilon^{-\mu\tau(1-\rho r^2)/r})/((1 - \rho r) + \rho r(1 - r)\epsilon^{-\mu\tau(1-\rho r^2)/r}). \tag{16}$$

PROOF. In terms of r, eq. (10) may be rewritten

$$\partial W/\partial\tau - \mu(z - r^{-1})(z - \rho r)\partial W/\partial z = \mu(z - \rho)W. \tag{17}$$

Equation (17) is obtained from eq. (10) by substituting $s = (\lambda r - \mu)(r - 1)/r$, which is obtained from $\lambda r^2 - r(\lambda + \mu + s) + \mu = 0$.

Now the general solution of eq. (17) is given by the solution of the following system of ordinary differential equations,

$$d\tau/1 = dz/(-\mu(z - r^{-1})(z - \rho r)) = dW/(\mu(z - \rho)W). \tag{18}$$

Using the first equation, $d\tau = -dz/\mu[(z - r^{-1})(z - \rho r)]$, one obtains

$$((z - r^{-1})/(z - \rho r))\epsilon^{\alpha\tau} = c_1 \tag{19}$$

where

$$\alpha = \mu(r^{-1} - \rho r). \tag{20}$$

For a second solution we use

$$dW/W = \mu(z - \rho)\, d\tau. \tag{21}$$

Eliminating z from eq. (21) by solving for it in eq. (19) we get

$$\log W = \mu\int ((c_1\rho(r - 1) - (r^{-1} - \rho)\epsilon^{\alpha\tau})/(c_1 - \epsilon^{\alpha\tau}))\, d\tau + \text{const.} \tag{22}$$

Integrating and eliminating c_1 from the result we obtain the second solution:

$$((z - r^{-1})/(\rho r - r^{-1}))\epsilon^{\lambda(1-r)\tau}W = c_2. \tag{23}$$

The general solution of eq. (17) can now be represented as

$$((z - r^{-1})/(\rho r - r^{-1}))\epsilon^{\lambda(1-r)\tau}W = f(((z - r^{-1})/(z - \rho r))\epsilon^{\alpha\tau})$$

or

$$W = ((\rho r - r^{-1})/(z - r^{-1}))\epsilon^{-\lambda(1-r)\tau}f(((z - r^{-1})/(z - \rho r))\epsilon^{\alpha\tau}) \tag{24}$$

where f is an arbitrary function which, in our case, will be determined from the boundary condition of eq. (11). From eq. (24) this requires

$$f(u) = (z - r^{-1})/((\rho r - r^{-1})(1 - z)) \tag{25}$$

where

$$u = (z - r^{-1})/(z - \rho r). \tag{26}$$

Solving for z in terms of u in eq. (26) and substituting into eq. (25) gives

$$f(u) = 1/((1 - \rho r) - (1 - r^{-1})u^{-1})$$

from which

$$\begin{aligned} f(((z - r^{-1})/(z - \rho r))\epsilon^{\alpha\tau}) \\ = (z - r^{-1})/((1 - \rho r)(z - r^{-1}) - (1 - r^{-1})(z - \rho r)\epsilon^{-\alpha\tau}). \end{aligned} \tag{27}$$

Finally, substitution into eq. (24) gives, after some simplification, the result of eq. (14). Q.E.D.

As indicated in eqs. (12) and (13), $r = 1/(\rho r')$ where r and r' are the two roots of the polynomial $\lambda z^2 - (\lambda + \mu + s)z + \mu$. It is readily verified from eq. (14) that r may be taken as either the smaller or larger root. That is, substitution of $1/(\rho r')$ for r in eq. (14) will, after rearrangement, yield precisely the same function in terms of r'. However, in the form in which eq. (14) is written, it is more convenient to take r as the smaller root in order to verify that the solution has the correct properties.

Taking r as the smaller root, we note from eqs. (12) and (13) that $\lim_{s\to 0} r = 1$ and $\lim_{s\to\infty} r = 0$. From these observations it is easy to show $\lim_{\tau\to\infty} W(\tau, s, z) = 0$ and

$$\lim_{s\to 0} W(\tau, s, z) = 1/(1 - z). \tag{28}$$

Equation (28) verifies that $w_n(\tau, 0) = \int_0^\infty d_x F_n(\tau, x) = 1$ for all n. Also,

$$\lim_{s\to\infty} W(\tau, s, z) = \epsilon^{-\lambda\tau}. \tag{29}$$

This can be seen to correspond to the probability that an arriving unit finds an empty system and experiences no delay (beyond its τ-second service requirement).

COROLLARY 1. *Under the condition* $\rho < 1$, *the Laplace transform,* $w(\tau, s)$, *of the equilibrium waiting time distribution is given by*

$$w(\tau, s) = (1 - \rho)(1 - \rho r^2)\epsilon^{-\lambda(1-r)\tau}/((1 - \rho r)^2 - \rho(1 - r)^2\epsilon^{-\mu\tau(1-\rho r^2)/r}). \tag{30}$$

Equation (30) can be obtained by averaging eq. (15) over the distribution $p_n = (1 - \rho)\rho^n$. Examining eq. (9) we see that this averaging has effectively been done for us; we simply substitute ρ for z in eq. (14) and multiply by $(1 - \rho)$.

Let $E(X)$ and $\sigma^2(X)$ denote the mean and variance, respectively, of the equilibrium waiting time distribution. We have from eqs. (5) and (30)

$$E(X) = -\lim_{s\to 0} \partial W/\partial s = \rho\tau/(1 - \rho) \tag{31}$$

and

$$\begin{aligned} \sigma^2(X) &= \lim_{s\to 0} \partial^2 W/\partial s^2 - [E(X)]^2 \\ &= (2\rho\tau/(\mu(1 - \rho)^3)) - (2\rho/(\mu(1 - \rho)^4))[1 - \epsilon^{-(1-\rho)\mu\tau}]. \end{aligned} \tag{32}$$

Now let $E_n(X)$ and $\sigma_n^2(X)$ denote the mean and variance, respectively, of the waiting time distribution conditioned on n (eq. (15)). Then using the same methods we obtain

$$E_n(X) = \rho\tau/(1 - \rho) + [n(1 - \rho) - \rho]((1 - \epsilon^{-(1-\rho)\mu\tau})/(\mu(1 - \rho)^2)), \tag{33}$$

$$\begin{aligned} \sigma_n^2(X) = {} & (2\rho\tau/(\mu(1 - \rho)^3))[1 + (1 + \rho)\epsilon^{-a\tau}] - (\rho^2/(\mu^2(1 - \rho)^4))(1 - \epsilon^{-2a\tau}) \\ & - (4\rho/(\mu^2(1 - \rho)^4))(1 - \epsilon^{-a\tau}) + n((1 + \rho)/(\mu^2(1 - \rho)^3)) \\ & \cdot[(1 - \epsilon^{-2a\tau}) - 2(1 - \rho)\mu\tau\epsilon^{-a\tau}], \end{aligned} \tag{34}$$

where $a = \mu(1 - \rho)$.

To verify that eqs. (33) and (34) are analytic for all $\rho \geq 0$ it can be seen from their form that we must dispose of the case $\rho = 1$. Accordingly, it is not difficult to compute the following limits:

$$\lim_{\rho \to 1} E_n(X) = n\tau + \mu\tau^2/2,$$

$$\lim_{\rho \to 1} \sigma^2(X) = (\mu\tau^3/3)(2n + 1).$$

Remark. If we add the service requirement τ to eq. (31), we obtain $\tau/(1 - \rho)$ as the mean time in system for the PS system in equilibrium. This is the result obtained by others [1–3] as a limit of the corresponding round-robin result.

Discussion

We have developed results for the PS model which consist of the Laplace transforms of the equilibrium waiting time distribution and the waiting time distribution conditioned on the number in system at arrival. From these results the first two mo-

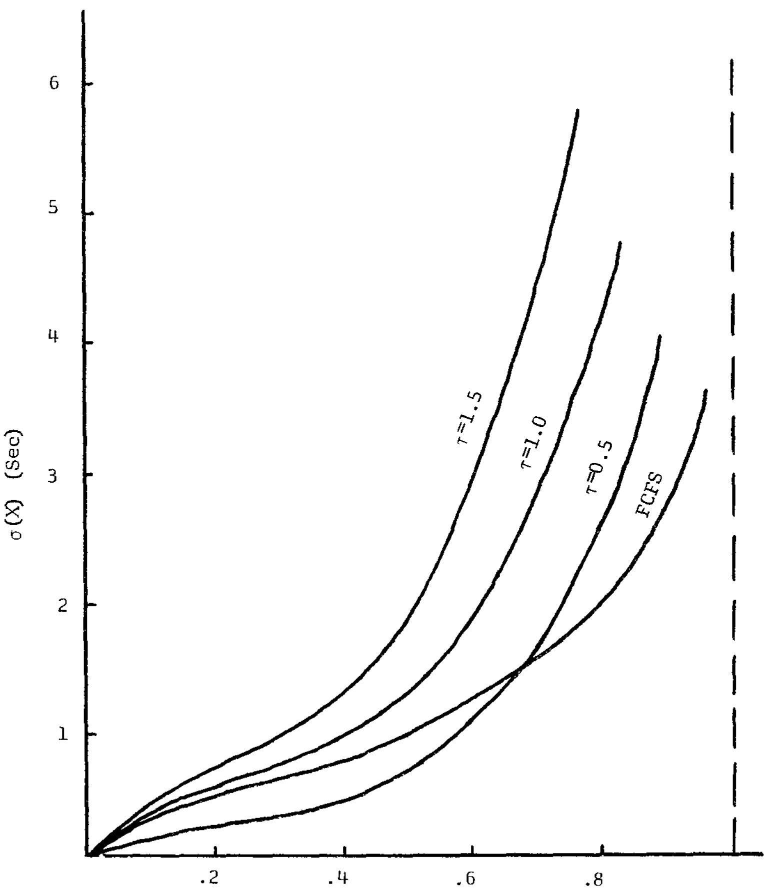

FIG. 3. Standard deviation vs. loading, $\mu = 1.0$/sec

ments were calculated. It remains to discuss the characteristics of the PS discipline, particularly as they compare with those of the Poisson FCFS system.

For the Poisson FCFS system the waiting time (in queue) conditioned on the number (n) in system at arrival is distributed as the sum of n independent, identical exponential service times. Thus, from eqs. (2) and (3), the mean and variance of the equilibrium waiting time distribution are given, respectively, by

$$E_F(X) = \rho(1/\mu)/(1 - \rho), \tag{35}$$

$$\sigma_F^2(X) = \rho(1/\mu^2)/(1 - \rho). \tag{36}$$

Equation (35) is to be compared with eq. (31) for the PS model. Note that the mean wait in the PS system varies linearly with the service requirement τ. Comparison shows that for arrivals having a service requirement less than the average (i.e. $\tau < 1/\mu$), the mean waiting time is less in the PS system than in the FCFS system. However, for larger-than-average service requirements, the mean wait is larger in the PS system. This demonstrates, of course, the well-known property of round-robin-type disciplines, according to which the jobs or customers with lesser service requirements are favored in terms of the queueing time. The mean wait conditioned on n also shows the expected behavior as a function of n. That is, the mean wait will be greater or less than that in eq. (31) according to whether the number in system at arrival is greater or less than its mean value ($\rho/(1 - \rho)$).

Comparison of eqs. (32) and (34) with eq. (36) shows that the variance of waiting times in the PS system is potentially very much larger than that in the FCFS system. Figure 3 has been provided in order to obtain a better feeling for this disparity. We have chosen a mean service requirement of $1/\mu = 1$ second and graphed the standard deviation $\sigma(X)$ versus the loading (ρ) with τ as a parameter.

REFERENCES

1. Coffman, E. G., and Kleinrock, L. Feedback queueing models for time-shared systems. *J. ACM 15*, 4 (Oct. 1968), 549–576.
2. Kleinrock, L. Time-shared systems: a theoretical treatment. *J. ACM 14*, 2 (Apr. 1967), 242–261.
3. Schrage, L. E. Some queueing models for a time-shared facility. Ph.D. Diss., Dep. of Indust. Eng., Cornell U., Ithaca, N. Y., Feb. 1966.
4. Muntz, R. R., and Coffman, E. G. Multiprocessor scheduling of computation graphs. Tech. Rep. No. 63, Computer Sciences Laboratory, Elec. Eng. Dep., Princeton U., Princeton, N. J., June 1968.
5. Feller, W. *An Introduction to Probability Theory and Its Applications, Vol. I*, 1957; *Vol. II*, 1966, Wiley, New York.
6. Sakata, M., Noguchi, S., and Oizumi, J. Analysis of a processor shared queueing model for time sharing systems. Proc. Second Hawaii Int. Conf. on System Sciences, 1969, University of Hawaii, pp. 625–627.
7. Coffman, E. G., and Muntz, R. R. Models of resource allocation using pure time-sharing disciplines. Proc. 24th ACM Nat. Conf., 1969, pp. 217–228.

RECEIVED FEBRUARY, 1969; REVISED JUNE, 1969