STOCHASTIC PROPERTIES OF WAITING LINES

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The stochastic properties of waiting lines may be analyzed by a two-stage process: first solving the time-dependent equations for the state probabilities and then utilising these transient solutions to obtain the auto-correlation function for queue length and the root-mean-square frequency spectrum of its fluctuations from mean length. The procedure is worked out in detail for the one-channel, exponential service facility with Poisson arrivals, and the basic solutions for the m-channel exponential service case are given. The analysis indicates that the transient behavior of the queue length $n(t)$ may be measured by a 'relaxation time,' the mean time any deviation of $n(t)$ away from its mean value L takes to return $(1/\varepsilon)$ of the way back to L. This relaxation time increases as $(1-\rho)^{-2}$ as the utilization factor ρ approaches unity, whereas the mean length L increases as $(1-\rho)^{-1}$. In other words, as saturation of the facility is approached, the mean length of line increases; but, what is often more detrimental, the length of time for the line to return to average, once it diverges from average, increases even more markedly.

MANY of the measures of effectiveness of an operation involving queuing can be computed from the steady-state solutions developed by Erlang,[1] Pollaczeck,[2] Crommelin[3] and others.[4] But a number of properties of practical importance require the consideration of the time dependence of the operation.[2] The length of queue $n(t)$ is, of course, a stochastic function of time; an expression of its behavior in terms of its auto-correlation function[5] and its related frequency spectrum would be the appropriate means of representing its behavior as a time series. But the usual computation[5] of the auto-correlation function, by averaging over the distribution of arrivals, is not applicable here because of the non-linear behavior of the service function.

Another approach to the problem is to compute the transient behavior of the state probabilities for various initial conditions. This appears to be the easier approach; once the transient solutions are obtained the auto-correlation function can be computed from these transient solutions.

To explain the technique in more detail, we will first consider the case of a single-channel, exponential-time service facility, of mean service rate μ, with a simple, strict-discipline queue having Poisson arrivals at a mean rate λ. The utilization factor ρ is thus (λ/μ). The steady-state solution is, of course, obtained from the state probabilities P_n^0, the probability that n units will be found in the system:

Abstract

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