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# Letters to the Editor

## THE MAXIMUM CAPACITY ROUTE PROBLEM

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THE maximum capacity route problem raised by POLLACK<sup>[1]</sup> can be formulated as follows. An  $n$ -node network consists of nodes  $N_i (i=1, \dots, n)$  and arcs  $A_{ij}$  leading from  $N_i$  to  $N_j$ . Each arc has associated with it a nonnegative number  $c_{ij}$ , called the arc capacity, which denotes the maximum amount of flow that can pass through the arc from  $N_i$  to  $N_j$ . The problem is to find a route from  $N_i$  to  $N_j$  such that

$$\min (c_{ia}, c_{ab}, \dots, c_{dj}) \text{ is maximum,}$$

where  $c_{ia}, c_{ab}, \dots$  denotes arc capacities of the arcs which form a route from  $N_i$  to  $N_j$ .

Now we consider the problem of finding maximum capacity routes between all pairs of nodes in a network. This can be divided into two cases, (1) in which  $c_{ij} = c_{ji}$ , (2)  $c_{ij} \neq c_{ji}$ .

When  $c_{ij} = c_{ji}$ , let us define a maximum spanning tree of the network as a spanning tree with the sum of  $c_{ij}$  associated with the arcs in the tree a maximum. This problem was solved by KRUSKAL<sup>[2]</sup> and PRIM<sup>[3]</sup> and very simple algorithms for constructing the tree can be found in references 2 and 3 or Appendix II of reference 4. For any arc  $A_{ij}$  not in the maximum spanning tree, we have

$$c_{ij} \leq \min (c_{ia}, c_{ab}, \dots, c_{dj}),$$

where  $A_{ia}, A_{ab}, \dots$  are arcs in the tree which form a unique route from  $N_i$  to  $N_j$ . So the whole problem can be summarized as follows. *For a network with  $c_{ij} = c_{ji}$ , maximum capacity routes between all pairs of nodes can be found in a subset of arcs in the network. This subset consists of  $n-1$  arcs which form a maximum spanning tree.* †

When  $c_{ij} \neq c_{ji}$ , we consider the  $n$ -node network consisting of all  $n(n-1)$  arcs. If a route from  $N_a$  to  $N_q$  with  $c_{ij}$  associated with its arcs has the following property

$$c_{ip} \leq \min (c_{ij}, c_{jk}, \dots, c_{qp}),$$

where  $A_{ij}, A_{jk}, \dots$  are arcs in the route and  $A_{ip}$  is any arc that is not in the route, we say that the  $A_{ip}$  is dominated. Clearly, a dominated arc can be omitted with-

† If  $c_{ij} = \min (c_{ia}, \dots, c_{dj})$  we choose the maximum capacity route that uses arcs in the subset. This will not occur if all capacities are different.

out reducing capacities of any maximum capacity route. Now we establish a similar result when  $c_{i,j} \neq c_{j,i}$ .

*For a  $n$ -node directed network, when  $n(n-1)$  maximum capacity routes are to be found, all routes can be found in a subset of the original  $n(n-1)$  arcs. This subset contains at least  $n$  arcs and contains at most  $(n-1)(\frac{1}{2}n+1)$  arcs.*

*Proof.* Assume that the subset of arcs contains  $(n-1)$  arcs or less. Since it takes at least  $n$  arcs to form a strongly connected graph<sup>[5]</sup> containing  $n$  nodes, there must be a route from  $N_i$  to  $N_j$  that uses an arc not in the subset. On the other hand, if  $n$  arcs with the largest  $c_{i,j}$  form a circuit, all other arcs are dominated, and all maximum capacity routes use only the  $n$  arcs.

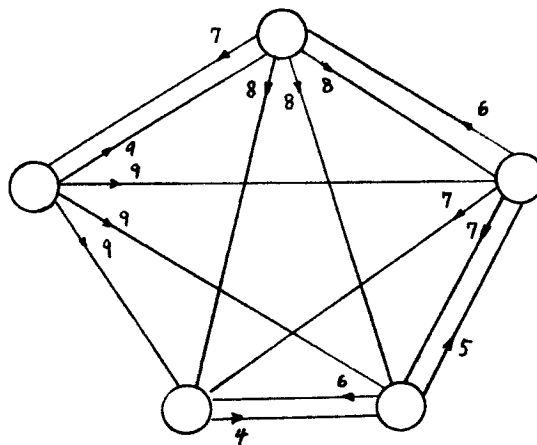
To prove that there are at most  $(n-1)(\frac{1}{2}n+1)$  arcs in the subset  $D$ , let us select the subset of arcs  $A_{i,j}$  by their magnitude, i.e., first select the arc with the largest  $c_{i,j}$ , then the arc with the next largest  $c_{i,j}$ , as each arc is selected as a member of  $D$ , all arcs which become dominated are then eliminated from the network before the next arc is selected. This is continued until all  $A_{i,j}$  are either selected or eliminated. In other words, when the selection process is completed, for any  $A_{k,l}$  not in the subset, there is a route formed by the arcs in the subset with minimum capacity  $\geq c_{k,l}$ . The selected  $A_{i,j}$  will form a graph called the dominating net  $D$ .

Two arcs  $A_{i,j}$  and  $A_{j,i}$  in the dominating net  $D$  are said to form a pair. If the  $A_{i,j}$  are chosen as described, there are at most  $(n-1)$  pairs in  $D$ . Assume that there are  $n$  or more pairs. Since there are at most  $(n-1)$  pairs in a tree,  $n$  or more pairs of them must form a loop  $L$ . Choose  $\min c_{pq}$ , where  $N_p \in L$ ,  $N_q \in L$ . Then this  $A_{pq}$  is dominated, as there is a route  $A_{pa}, A_{ab}, \dots, A_{dq}$  with minimum capacity  $\geq c_{pq}$ . So  $A_{pq}$  does not belong to  $D$ , a contradiction.

Consider all  $n(n-1)$  arcs between  $n$  nodes. They will form  $\frac{1}{2}n(n-1)$  pairs if all are selected. If the dominating subset has no pair at all, it contains at most  $\frac{1}{2}n(n-1)$  arcs, i.e., only one of the  $A_{i,j}$ ,  $A_{j,i}$  is selected. A dominating subset with  $(n-1)$  pairs then contains at most

$$\frac{1}{2}n(n-1) + n - 1 = (n-1)(\frac{1}{2}n + 1) \text{ arcs}$$

The following is a dominating net  $D$  which consists of  $(5-1)(\frac{5}{2}+1) = 14$  arcs



Although the algorithms given yield the maximum capacity route between all node pairs, it is also true that the algorithms can be halted at any time. The routes found thus far are the correct routes for those nodes that are connected. Therefore, even if one does not want the routes between all the node pairs, the algorithms given can still be used and the procedure halted when the node pairs of interest are connected.

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## NOTE ON A PAPER BY HANSSMANN

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(Received May 26, 1961)

IN THE interesting paper "Determination of Optimal Capacities of Service for Facilities with a Linear Measure of Inefficiency,"\* FRIEDRICH HANSSMANN solved the following problem:

A service facility  $F$ , always employs an 'effort'  $x$ , to perform a service  $x$ , is called its capacity. If a customer desires an effort  $x < x_i$ , there will be a wasted effort  $x_i - x$ . If a specified number,  $m$ , of facilities with different capacities  $x_1, \dots, x_m$  is to be installed, what are the capacities  $x_1, \dots, x_m$  that will minimize the wasted effort?

The purpose of this note is to present a corrected version of Table I, the iteration of the example considered, and another observation. Mr. Hanssmann was very fortunate in having the errors of computation in the first two iterations lead him directly to the correct solution. The interpolated solutions do converge quite rapidly, however, an additional iteration is involved.

We have also observed that the equation following Fig. 2 is in error, although the entry in Table II has been calculated by the use of the correct formula.

It follows from Hanssmann's equation (2) that

$$E = [1 - F(x_{n-1})]x_n + [F(x_{n-1}) - F(x_{n-2})]x_{n-1} + \dots + [F(x_2) - F(x_1)]x_2 + [F(x_1)]x_1 - \bar{x}$$

\* *Opns Res* 5, 713-717 (1957)