# Capacity Scaling in Ad Hoc Networks With Heterogeneous Mobile Nodes: The Super-Critical Regime

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***Abstract*—We analyze the capacity scaling laws of mobile ad hoc networks comprising heterogeneous nodes and spatial inhomogeneities. Most of previous work relies on the assumption that nodes are identical and uniformly visit the entire network space. Experimental data, however, show that the mobility pattern of individual nodes is usually restricted over the area, while the overall node density is often largely inhomogeneous due to the presence of node concentration points. In this paper we introduce a general class of mobile networks which incorporates both restricted mobility and inhomogeneous node density, and describe a methodology to compute the asymptotic throughput achievable in these networks by the *store-carry-forward* communication paradigm. We show how the analysis can be mapped, under mild assumptions, into a *Maximum Concurrent Flow* (MCF) problem over an associated *Generalized Random Geometric Graph* (GRGG). Moreover, we propose an asymptotically optimal scheduling and routing scheme that achieves the maximum network capacity.**

***Index Terms*—Ad hoc networks, capacity scaling, mobility, delay tolerant networks.**

## I. Introduction

THE development of large-scale ad hoc wireless networks has been so far discouraged by a number of scalability issues. Among several scalability concerns, the most basic one is related to the data transport capability of ad hoc networks comprising a large number of nodes establishing random connections among them. In their seminal work, Gupta and Kumar [3] obtained the disheartening result that, for a wireless network with $n$ static nodes, the per-node throughput decays as $\Theta(1/\sqrt{n})$, even allowing optimal scheduling and node placement.

Later, Grossglauser and Tse [4] showed that a constant per node throughput can be achieved in the case of mobile nodes, by exploiting a novel *store-carry-forward* communication paradigm according to which data are physically carried on the nodes as they move around the network area. Although this communication scheme incurs very large delays, on the time scale of nodes movement across the network space, it has laid the foundation of an entire new area of research, usually referred to as delay-tolerant or disruption-tolerant networking (DTN) [5], which has recently attracted a lot of attention. A typical DTN scenario consists of a mixture of static and mobile nodes, and it is characterized by intermittent connectivity and frequent network partitioning, such that node mobility is indeed essential to guarantee network cohesion and end-to-end communication.

Several interesting applications of DTN have been already envisioned and experimented upon, such as "pocket switched networks" based on human mobility [6], vehicular networks based on public buses [7] or taxi cabs [8], sensor networks for wildlife tracking [9], disaster-relief networks [10], Internet access to remote or rural villages [11]. While the theoretical result of Grossglauser and Tse suggests that delay tolerant networks can indeed scale up to very large sizes, it is unclear whether this result indeed applies to real-life scenarios such as the ones mentioned above. One reason is that the analysis in [4] strongly relies on the assumptions that: i) the mobility pattern is the same for all nodes; ii) each node uniformly visits the entire network area according to an ergodic mobility process; iii) the trajectories of different nodes are independent and identically distributed (i.i.d.). Notice that the same assumptions have been maintained in the several papers that, following [4], have analyzed asymptotic delay-capacity trade-offs, like [12]–[14].

In many practical settings, the above assumptions on the nodes behavior do not hold, and in particular the one that the mobility process of each node uniformly covers the entire space over time, making all nodes basically indistinguishable from each other. Both everyday life experience and campus or city-wide traces containing spatial information (i.e., based on GPS coordinates or radio beacons from base stations and access points) [8], [15]–[17], suggest that a node spends most of the time just in a small portion of the network area. The node rarely goes outside this region that comprises a few frequently visited "significant places" [18]. This in turn demands for more realistic, non-homogeneous mobility models reflecting the characteristics recognized in the traces [19], [20]. Although any two nodes are likely, in the long run, to eventually come in contact with each other, the impact of rare contacts on the overall network capacity has to be carefully investigated.

Moreover, not only nodes are characterized by rather restricted mobility processes, but also the overall node density over the area is largely inhomogeneous, and typically exhibits "concentration points" [8] or hot spots [21] where nodes are more likely to gather. Such clustering behavior has been observed in many different traces related to both human and vehicular movements, and appears to be a quite ubiquitous

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feature of real mobility processes. Examples of concentrations points include workplaces, restaurants, movie theaters, conference rooms (in the case of people), watering holes, oases (in the case of animals), intersections, parking lots, gas stations (in the case of road traffic).

In this paper, we introduce and analyze a new class of mobile networks which incorporates both restricted mobility and inhomogeneous node density. In particular we consider the situation in which the mobility pattern of each node results into a rotationally invariant spatial distribution centered at a given home-point (i.e., a distribution exhibiting simmetry around the central point). The home-points of the nodes are in turn thickened in proximity of a certain number of concentration points uniformly distributed in the area, so as to obtain a desired level of clustering. Our model is, to the best of our knowledge, the first model to account for such non-uniform density.

We show how the problem of establishing the asymptotic capacity of the consider class of mobile ad hoc networks can be mapped, under mild assumptions, into a *Maximum Concurrent Flow* (MCF) problem [22] over an associated *Generalized Random Geometric Graph* (GRGG). Our goal is to provide a general framework for the analysis of the capacity scaling properties in mobile ad hoc networks with *heterogeneous nodes* and *spatial inhomogeneities*, thus extending and generalizing the results in [4].

Our analysis identifies a regime, that we call **super-critical**, in which, despite the fact that individual nodes are characterized by restricted mobility, the node density is asymptotically uniform in the area. This regime, which occurs under a particular scaling of the various network parameters (mobility pattern, area size, number of clusters), is characterized by a smooth degradation of asymptotic capacity which depends only on how the physical network extension scales with respect to the average distance reached by a node from its home-point (not on the details of the node spatial distribution). In the **sub-critical regime**, instead, the network capacity becomes sensitive to the particular node mobility pattern, and a sharp transition between the two regimes occurs when the node spatial distribution has finite support.

Our work is one of the first to study the capacity scaling of heterogeneous, mobile ad hoc networks. Some progress has been already done in the direction of considering special cases of restricted mobility. In [23] the authors have considered a mobility pattern according to which each node independently moves along a randomly chosen great circle on the sphere. Quite surprisingly, even under this one-dimensional mobility pattern a constant throughput per source-destination pair can be sustained. In [24] the network of unit area is partitioned into square cells, and nodes are restricted to move within one randomly chosen cell, whose area is assumed to either scale as $(\log n)/n$ or remain constant.

In the special case of static nodes, several papers have already appeared which generalize the results of Gupta-Kumar to networks of heterogeneous nodes. The simple approach in [25] allows to consider nodes located over straight lines or highly dense neighborhoods. The impact of bottlenecks due to asymmetric traffic conditions or node clusters has been studied in [26]. In [27] the authors analyze the impact of directional antennas on the asymptotic network capacity using a technique similar to ours. Several works have also considered the capacity of hybrid networks comprising a mixture of wired and wireless nodes [28]–[30].

Before presenting the main results of the paper, we need to premise the system assumptions (Section II) and the network scaling considered in our asymptotic analysis (Section III).

## II. System Assumptions and Notation

### A. Mobility Model

We consider a network composed of $n$ nodes moving over a square region of unit area, with wrap-around conditions (to avoid border effects). More formally, the network area is assumed to be a bidimensional Torus surface $\mathcal{O}$ of unit area.[1] We emphasize that the normalization of domain $\mathcal{O}$ is just a technical assumption commonly adopted in previous work [3], [4], which has no impact on the obtained results.

Let $X_i(t)$ denote the position of node $i$ at time $t$ and $X(t) = (X_1(t), X_2(t) \cdots X_n(t))$ be the vector of nodes' positions; we define by $d_{ij}(t)$ the distance between mobile $i$ and mobile $j$ at time $t$, i.e.,[2]: $d_{ij}(t) = \|X_i(t) - X_j(t)\|$.

Heterogeneity of nodes is taken into account at two different levels, based on common features that have been widely recognized in realistic mobility traces.

First, we consider that the stationary spatial distribution of a node is generally non-uniform over the space; rather, a node typically spends most of the time in a small region of the network area, and rarely (or never) visits zones far away from it. We model this behavior assuming that each node $i$ has its own **home-point**, located at $X_i^h$, representing the point most visited by the node. Nodes move "around" their home-points according to independent stationary and ergodic processes, i.e., given any $m$-uple $(B_1, B_2, B_3 \cdots B_m)$ of rectangular subsets of $\mathcal{O}$, it results (almost surely):

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \left[\prod_i \mathbb{I}_{X_i(\tau)\in B_i}|H_n\right] \mathrm{d}\tau = \prod_i E[\mathbb{I}_{X_i(t)\in B_i}|H_n]$$

where $\mathbb{I}$ is the set indicator function [3] and $H_n = \{X_i^h\}_{i=1}^n$.

We describe the density of presence of node $i$ around $X_i^h$ by a function $\phi_i(X) = \phi(X - X_i^h)$, which is assumed to be invariant in all directions. More specifically, we assume that the shape of $\phi_i(X)$ is given by a non-increasing, summable continuous function $s(d)$ of the distance $d \in \mathbb{R}^+$ from the home-point $X_i^h$, with finite first three moments (i.e., $\int_0^\infty x^3 s(x)\mathrm{d}x < \infty$). Without lack of generality, in the following we further suppose that $s(d)$ is normalized in such a way that $\int_{\mathbb{R}^2} s(\|X - X_i^h\|)\mathrm{d}X = 1$; the reasons for this normalization will appear clearer in the fol-

[1]The bidimensional (topological) Torus of unit area is the surface generated by the Cartesian product of two circles of unitary length. The Torus can be equivalently described as a quotient of the Cartesian plane under the identifications $(x, y) \simeq (x+1, y) \simeq (x, y+1)$.

[2]Given any two points $X_1 = (x_1, y_1) \in \mathcal{O}$ and $X_2 = (x_2, y_2) \in \mathcal{O}$ we define their distance according to: $d(X_1, X_2) = \min_{u,v\in\{-1,0,1\}} \sqrt{(x_1+u-x_2)^2 + (y_1+v-y_2)^2}$

[3]$\mathbb{I}_A(\omega)$ is a function that returns 1 for any $\omega \in A$ and 0 for any $\omega \notin A$; with abuse of notation, given a random variable $X$ taking values in $\mathbb{R}$ and any (measurable) sub-set $B$ of real numbers we also write $\mathbb{I}_{X\in B}$ to denote $\mathbb{I}_{X^{-1}(B)}(\omega)$

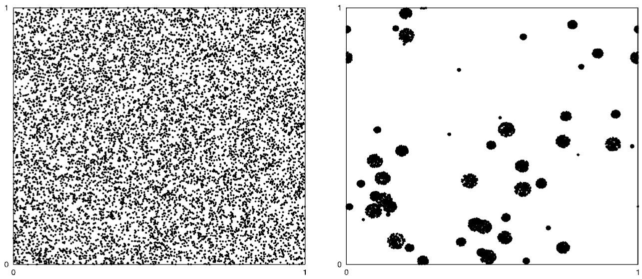

Fig. 1. Examples of home-point distributions according to Uniform model (left plot) and Clustered model (right plot), in the case of $n = 10{,}000$ nodes.

lowing. To obtain a proper probability density function over the network area $\mathcal{O}$, we need to further normalize $\phi_i(X)$ as follows:

$$\begin{aligned}\phi_i(X) &= \phi(X - X_i^h) \\ &= \frac{s(\|X - X_i^h\|)}{\int_{\mathcal{O}} s(\|X - X_i^h\|)\mathrm{d}X}.\end{aligned} \tag{1}$$

Note that, as limit case, we can obtain networks with static nodes; in this case $\phi_i(X) = \delta(X - X_i^h)$, where $\delta(X)$ is the Dirac impulse function. For simplicity, we assume that the mobility of all nodes is characterized by the same function $\phi(X)$. However, our analysis can be easily extended to the case in which nodes with different mobility patterns coexist (e.g., a mixture of fixed nodes and fully mobile nodes, or several classes of nodes with different degrees of mobility around their home-points), as briefly discussed in the conclusions (Section VII).

Second, we account for spatial inhomogeneities of the overall density of nodes over the area. To do so, we distribute the home points of the nodes according to two different models:

- **Uniform**: home-points of nodes are uniformly and independently chosen inside area $\mathcal{O}$.
- **Clustered**: each node, independently of others, chooses one of $m$ clusters, all clusters being equally likely. Each cluster has a middle point which is uniformly located within $\mathcal{O}$. The home-points of nodes belonging to the same cluster are then uniformly and independently placed within a disk of radius $r$ centered at the cluster middle point.

The Uniform model, which is simpler to analyze, has been widely used in the literature to study random networks of static nodes. However, it does not take into account the clustering behavior that has been observed in real traces [8], [21]. The Clustered model, instead, better captures the fact that in realistic scenarios the distribution of users on the territory may be highly inhomogeneous: parts of the network area are more densely populated than others, creating concentration points that are usually well distinct from each other and fairly stable over time. Examples of home-point distributions according to Uniform and Clustered models are shown in Fig. 1.

### B. Interference Model

We assume that interference among simultaneous transmissions is described by the well known *protocol model* [3], which roughly represents the behavior of wireless MAC protocols in the case of omni-directional antennas without power capture.[4] Nodes employ a common range $R_T$ for all their transmissions (equivalently, they employ a common power level, i.e., no power adaptation mechanism is used). Node $i$ is allowed to transmit to node $j$ at time $t$, only if:

i) the distance between $i$ and $j$ is no more than $R_T$, i.e.,

$$d_{ij}(t) < R_T;$$

ii) for every other node $k$ simultaneously transmitting

$$d_{kj}(t) > (1 + \Delta)\, R_T$$

where $\Delta$ is a guard factor. We assume that transmissions occur at fixed rate which is normalized to 1.

### C. Traffic Model

We describe the traffic pattern through its associated $n \times n$ traffic matrix whose entries represent the average data rate exchanged by node pairs. Conventionally we express the traffic matrix in the form $\lambda\mathbf{\Lambda}$, the elements of matrix $\mathbf{\Lambda} = [\lambda_{sd}]$ being re-scaled in such a way that $\sum_s \sum_d \lambda_{sd} = n$. We assume that traffic is generated at each source node $s$ according to a stationary and ergodic process with average rate $\lambda\lambda_{sd}$.

Let $Z(t)$ be the network backlog, that is, the number of data units already generated by sources which have not yet been delivered to destinations at time $t$. We say that traffic $\lambda\mathbf{\Lambda}$ is *sustainable* if there exists a scheduling-routing policy such that $\limsup_{t\to\infty} Z(t)/t = 0$, w.p.1.

Similarly to previous work we focus on *random permutation traffic patterns*, i.e., traffic patterns whose associated matrix $\mathbf{\Lambda}$ corresponds to a randomly selected $n \times n$ permutation matrix.[5]

## III. Asymptotic Analysis

To analyze asymptotic properties as the network grows large, we progressively increase the number of nodes $n$, generating a sequence of systems indexed by $n$. We are essentially interested in establishing how the network capacity scales with $n$ under the assumptions we have introduced on node mobility, interference and traffic. We say that the *per-node capacity* (or maximum per-node throughput) of the system is[6] $\Theta(h(n))$ if, given a sequence of random permutation traffic patterns with rate $\lambda^{(n)} = h(n)$, there exist two constants $c, c'$ such that $c < c'$ and both the following properties hold:

$$\begin{cases}\lim_{n\to\infty} \Pr\{c\lambda^{(n)} \text{ is sustainable}\} = 1 \\ \lim_{n\to\infty} \Pr\{c'\lambda^{(n)} \text{ is sustainable}\} = 0.\end{cases}$$

Equivalently, we say in this case that the *network capacity* (or maximum network throughput) is $\Theta(nh(n))$.

[4]The protocol model has been proven to be pessimistic with respect to the physical model employing power control (see Theorem 4.1, pg. 174 in [31]). Thus the results obtained in this paper can be regarded as lower bounds of the network capacity achievable under the physical model employing power control.

[5]$\mathbf{\Lambda} = [\lambda_{sd}]$ is a permutation matrix if $\forall s, d,\ \lambda_{sd} \in \{0, 1\}$; $\forall d,\ \sum_s \lambda_{sd} = 1$; $\forall s,\ \sum_d \lambda_{sd} = 1$.

[6]Given two functions $f(n) \geq 0$ and $g(n) \geq 0$: $f(n) = o(g(n))$ means $\lim_{n\to\infty} f(n)/g(n) = 0$; $f(n) = O(g(n))$ means $\limsup_{n\to\infty} f(n)/g(n) = c < \infty$; $f(n) = \omega(g(n))$ is equivalent to $g(n) = o(f(n))$; $f(n) = \Omega(g(n))$ is equivalent to $g(n) = O(f(n))$; $f(n) = \Theta(g(n))$ means $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

### A. Scaling the Network Size

While the mobility pattern of nodes can well be considered to be independent of $n$ (i.e., the mobility process of users is exogenous to the system), the physical extension of the network (in our case, the edge length of the considered square region) depends on $n$, and typically increases as $n$ increases. This is actually the key point that determines how the network capacity scales with $n$, as we will see. Let $f(n)$ be a non decreasing function which characterizes how the physical extension of the network scales with $n$. In this paper, we will mainly consider the case in which $f(n)$ has the form $f(n) = n^{\alpha}$ with $\alpha \in [0, 1/2]$.[7]

When $\alpha = 0$ we obtain the special case in which the network physical extension remains constant, while the node density increases linearly with $n$. In this scenario a node gets in contact with $\Theta(n)$ other nodes (a constant fraction of the entire population of users). We will see that, in this case, the network asymptotic capacity scales as if the nodes mobility process was uniform over the area (Grossglauser-Tse case).

When $\alpha = 1/2$, or, more generally, $f(n) = \Theta(\sqrt{n})$, we obtain the other extreme case in which the area size increases linearly with $n$, while the node density is kept independent on $n$. In this scenario a node gets in contact with $\Theta(1)$ other nodes (a finite number of nodes). We will see that, in this case, the asymptotic network capacity scales as if nodes were static (Gupta-Kumar case).

In this paper we are concerned with all intermediate cases as well. Since we normalize the network area to 1, we need to properly scale down the spatial stationary distribution of the users in (1), as the number of nodes increases. In particular, we need to progressively reduce the area spanned by each node, in accordance to how the physical extension of the network scales up with $n$. We obtain the sequence of spatial distributions

$$\begin{aligned} \phi_i^n(X) &= \phi^n\left(X - X_i^h\right) \\ &= \frac{s\left(f(n)\left\|X - X_i^h\right\|\right)}{\int_{\mathcal{O}} s\left(f(n)\left\|X - X_i^h\right\|\right) \mathrm{d}X} \\ &= \frac{f^2(n) s\left(f(n)\left\|X - X_i^h\right\|\right)}{G_n} \end{aligned} \tag{2}$$

with $G_n = f^2(n) \int_{\mathcal{O}} s(f(n)\|X - X_i^h\|)\mathrm{d}X = f^2(n) \int_{\mathcal{O}} s(f(n)\|X\|)\mathrm{d}X$ where the last equality holds in light of the fact that $\mathcal{O}$ is a Torus.

When $f(n) = \omega(1)$, for $n \to \infty$ we obtain that $G_n \to 1$, as it can be easily verified through a change of variable inside the integral. In general, defined:

$$G = \begin{cases} 1 & \text{if } f(n) = \omega(1) \\ \hat{f}^2 \int_{\mathcal{O}} s(\hat{f}\|X\|)\mathrm{d}X & \text{if } f(n) \to \hat{f} < \infty \end{cases}$$

we have

$$\phi_i^n(X) \sim \frac{f^2(n)}{G} s\left(f(n)\left\|X - X_i^h\right\|\right)$$

where the symbol $\sim$ stands for asymptotically equivalent.[8]

[7]We do not consider cases in which $\alpha > 1/2$, which, however, can be proved to be equivalent to the case $\alpha = 1/2$ in terms of capacity scaling properties.

[8]$f(n) \sim g(n)$ means $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$

### B. Scaling the Clusters

In the case of the Clustered model, we also need to specify how the number of clusters $m$ and the cluster radius $r$ scale with $n$. In this paper we will focus on Clustered models in which $m(n) = \Theta(n^{\nu})$ with $\nu \in [0, 1)$ and $r(n) = \Theta(n^{-\beta})$, with $\beta \in [0, \infty)$, under the constraint that $\nu - 2\beta < 0$. In this case the different clusters are, with high probability, non-overlapping. Note that the Uniform model can be regarded as a limit case of the Clustered model where the number of clusters is $n$ (i.e., $\nu = 1$), and, in addition, every cluster contains deterministically a single node.

### C. Scaling the Transmission Range

A final consideration regards how the transmission range $R_T$ scales with $n$. Previous investigations [1], [3], [4], [32] have shown that transmission ranges should be reduced as much as possible so as to increase the overall network capacity (and, at the same time, reduce the power consumption of nodes), while maintaining network connectivity.

In this paper we will generalize previous findings showing that, in a wide range of contexts, choosing $R_T = \Theta(\frac{1}{\sqrt{n}})$ indeed maximizes the overall network capacity.

## IV. Summary of Results

Consider a sequence of networks with increasing number of nodes $n$ in which nodes' home-points are placed either according to the Uniform model or to a $(m, r)$ Clustered model. We define $\gamma(n) = \frac{\log(m(n))}{m(n)}$ ($m(n) = n$ in the Uniform model). Quantity $\sqrt{\gamma(n)}$ represents the minimal transmission range that would guarantee network connectivity in the case nodes remained still at their home-points, for $r(n) = 0$. Two different asymptotic regimes appear depending on how $f(n)$ scales with respect to $\sqrt{\gamma(n)}$, as $n \to \infty$.

### A. Super-Critical Regime

This regime occurs when $f(n)\sqrt{\gamma(n)} = o(1)$. Here, mobility plays a fundamental role in favoring the exchange of information between nodes. Indeed node pairs whose home-points are separated by a distance $O(\frac{1}{f(n)})$, because of mobility can get in contact with each other and thus have the opportunity of directly exchanging data.

The main properties of this regime are as follows.

- The per-node capacity is $\Theta(\frac{1}{f(n)})$ with high probability[9] (w.h.p.), *independently* of both the shape $s(d)$ of the nodes' spatial distribution and of the parameters $m(n), r(n)$ of the particular Clustered model.
- The maximum network capacity is achieved by an implementable scheduling/routing scheme (specified later) in which the transmission range is $\Theta(\frac{1}{\sqrt{n}})$.
- The implementation of the proposed optimal scheduling/routing scheme does not require the nodes to know the shape of $s(d)$, neither the parameters $m(n)$ and $r(n)$ associated to the distribution of home-points. They only need to know how $f(n)$ scales with $n$.

[9]I.e., with a probability tending to 1 as $n to \infty$

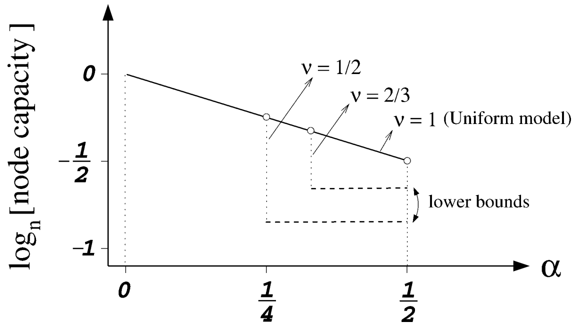


Fig. 2. Per-node asymptotic capacity as a function of $\alpha$ for Uniform and Clustered models. Lower bounds are tight when function $s(d)$ has finite support.

### B. Sub-Critical Regime

This regime occurs when $f(n)\sqrt{\gamma(n)} = \omega(1)$. Here, the impact of mobility on system performance is much less important, and a transmission range $R_T(n) = \Theta(\sqrt{\gamma(n)})$ may be required to guarantee network connectivity. In general, the network capacity becomes sensitive to both the shape of $s(d)$ and to the parameters $m(n)$, $r(n)$ of the specific Clustered model. In this paper, being mainly interested in studying the asymptotic properties of the super-critical phase, we limit our investigation on the sub-critical regime by providing the following lower bound to the per-node capacity:

$$\Theta\left(\frac{1}{n\sqrt{\gamma(n)}}\right) = \Theta\left(\sqrt{\frac{m(n)}{n^2 \log(m(n))}}\right)$$

and showing that such lower bound is tight at least for one significant class of spatial distributions $\phi(X)$, i.e., the ones associated to functions $s(d)$ having finite support. An in-depth analysis of the sub-critical capacity resulting from arbitrary spatial distributions $\phi(X)$ can be found in [33].

### C. Graphical Representation of Capacity Results

A graphical representation of our findings on asymptotic network capacity is reported in Fig. 2, where we show on log-log scale the per-node capacity as a function of $\alpha$, for different values of the parameter $\nu$ of the Clustered model (recall that the number of clusters is $m = \Theta(n^{\nu})$). The super-critical regime occurs when $\alpha < \nu/2$ (neglecting logarithmic factors). The per-node capacity in super-critical regime exhibits a log-linear decay while increasing $\alpha$, independently of the specific clustered model and of the shape of $s(d)$. In sub-critical regime ($\alpha > \nu/2$) the behavior is more complex: the per-node capacity becomes sensitive to both the parameters of the clustered model and the shape of $s(d)$. In the worst case (i.e., when $s(d)$ has finite support) the per-node capacity in the sub-critical region does not depend on $\alpha$, and a sharp transition occurs between the two regimes (represented in the figure by dotted vertical lines).

## V. Analysis in Super-Critical Regime

As first step of our analysis we evaluate how nodes are distributed within the domain $\mathcal{O}$. Given any point $X_0 \in \mathcal{O}$, we formally define the local density of nodes in $X_0$ as

$$\rho^n(X_0) = \sum_{i=1}^{n} E[\mathbb{1}_{X_i^{(n)}(t)\in B(X_0,1/\sqrt{n})}|H_n] \tag{3}$$

where $B(X_0, 1/\sqrt{n})$ is the closed disk centered in $X_0$, of radius $1/\sqrt{n}$. Note that $\rho^n(X_0)$ represents the average number of nodes falling at steady state within the disk $B(X_0, 1/\sqrt{n})$. We emphasize that the value of $\rho^n(X_0)$ in general depends on the home-points location $H_n$, i.e., $\rho^n(X_0)$ is a random variable over the probabilistic space defined by $H_n = \{X_i^h\}_{i=1}^n$. However, averaging over all possible instances of home point locations, we obtain $E_{H_n}[\rho^n(X_0)] = \pi$ independently of $X_0$, thanks to the spatial homogeneity (on average) of both the Uniform and Clustered model.

In addition, for a given instance of home point locations $H_n$, the local density may significantly vary from point to point, potentially becoming infinite at some points and null at other points. This happens, for example, when nodes are assumed to be still at their home-points. In this case, indeed, the existing inhomogeneities in the home-points distribution (especially under the clustered model) directly reflects into local density inhomogeneities. The impact of home-points inhomogeneities on the local density becomes weaker and weaker as we increase the area spanned by each node around its home-point for effect of mobility (i.e., increasing $1/f(n)$). As limit case, in the Grossglauser-Tse scenario the local density becomes constant and independent of $H_n$. We will see in the following that the local density plays a key role in establishing the capacity scaling properties of the network. Theorem 1 below provides the conditions on $f(n)$ under which the local density at any point $X_0$ for $n \to \infty$ is finite and different from 0:

*Theorem 1:* If $f(n)\sqrt{\gamma(n)} = o(1)$, then for any $X_0 \in \mathcal{O}$ it is possible to find two positive constants $q, Q$ such that $q \le \pi \le Q$ and

$$q < \rho^n(X_0) < Q \quad \text{w.h.p.} \tag{4}$$

To prove the theorem we need to premise a useful result whose proof is reported in Appendix A.

*Lemma 1:* Suppose that $\{X_i^h\}_{i=1}^n$ are displaced on $\mathcal{O}$ either according to the Uniform model or according to a $(m(n), r(n))$ Clustered model such that $\lim_{n\to\infty} m(n)r^2(n) = 0$. Let $\mathcal{A}^n$ be a sequence of regular tessellations of $\mathcal{O}$ (or any its sub-region), with the property that the area of each element of the tessellation $|A^n| \ge (16+\delta)\gamma(n)$, for some small $\delta > 0$. Let $N(A_k^n)$ be the number of home-points inside $A_k^n$. Then uniformly over the tessellations w.h.p. $N(A_k^n)$ is comprised between $\frac{n|A^n|}{2}$ and $2n|A^n|$, i.e., $\frac{n|A^n|}{2} < \inf_k N(A_k^n) \le \sup_k N(A_k^n) < 2n|A^n|$.

*Proof:* (Theorem 1). The main steps of this proof are: i) the local density is first expressed as a sum of contributions, each contribution given by the nodes whose home-points fall within the same squarelet; ii) applying Lemma 1, every contribution can be bounded w.h.p. (both from below and from above); iii) the upper (lower) bound is shown to converge w.h.p. to some constant value as $n \to \infty$.

More in detail, consider a generic point $X_0 \in \mathcal{O}$. By combining (2) and (3):

$$\rho^n(X_0) = \frac{f^2(n)}{G_n} \sum_{i=1}^{n} \int_{B(X_0,1/\sqrt{n})} s\left(f(n)\left\|X - X_i^h\right\|\right) \mathrm{d}X.$$

Since $s(d)$ is decreasing and continuous:

$$\frac{\pi}{n} s\left(f(n)\left(\|X_0 - X_i^h\| + \frac{1}{\sqrt{n}}\right)\right) \leq \int_{B(X_0, 1/\sqrt{n})} s\left(f(n)\|X - X_i^h\|\right) \mathrm{d}X \leq \frac{\pi}{n} s\left(f(n)\left[\max\left(0, \|X_0 - X_i^h\| - \frac{1}{\sqrt{n}}\right)\right]\right).$$

Thus, having defined $\underline{s^n}(d) = s(d + \frac{f(n)}{\sqrt{n}})$ and $\overline{s^n}(d) = s(\max(0, d - \frac{f(n)}{\sqrt{n}}))$ we have

$$\frac{\pi f^2(n)}{nG_n} \sum_{i=1}^{n} \underline{s^n}\left(f(n)\|X_0 - X_i^h\|\right) \leq \rho^n(X_0) \leq \frac{\pi f^2(n)}{nG_n} \sum_{i=1}^{n} \overline{s^n}\left(f(n)\|X_0 - X_i^h\|\right).$$

Now, let $\mathcal{A}^n$ denote a regular square tessellation of $\mathcal{O}$, such that each squarelet $A_k^n$ of $\mathcal{A}^n$ has area $|A^n| = (16 + \delta)\gamma(n)$. Let $\underline{d}_{0k}$ and $\bar{d}_{0k}$ be, respectively, the inferior and the superior of the distances between points $X \in A_k$ and $X_0$; i.e., $\underline{d}_{0k} = \inf_{X \in A_k^n} \|X - X_0\|$; and $\bar{d}_{0k} = \sup_{X \in A_k^n} \|X - X_0\|$; at last, let $\underline{N}(A_k^n)$ and $\bar{N}(A_k^n)$ be, respectively, a lower bound and an upper bound of the number of nodes whose home-point falls within $A_k^n$. It results:

$$\frac{\pi f^2(n)}{nG_n} \sum_k \underline{s^n}(f(n)\bar{d}_{0k})\underline{N}(A_k^n) < \rho^n(X_0) < \frac{\pi f^2(n)}{nG_n} \sum_k \overline{s^n}(f(n)\underline{d}_{0k})\bar{N}(A_k^n).$$

Since w.h.p., uniformly over $k$, $\underline{N}(A_k^n) \geq n|A^n|/2$ and $\bar{N}(A_k) \leq 2n|A^n|$ (Lemma 1):

$$\frac{\pi f^2(n)}{2G_n} \sum_k \underline{s^n}(f(n)\bar{d}_{0k})|A^n| < \rho^n(X_0) < \frac{2\pi f^2(n)}{G_n} \sum_k \overline{s^n}(f(n)\underline{d}_{0k})|A^n|.$$

Observe that: i) $f^2(n)\sum_k \underline{s^n}(f(n)\bar{d}_{0k})|A^n|$ and $f^2(n)\sum_k \overline{s^n}(f(n)\underline{d}_{0k})|A^n|$ can be interpreted, respectively, as lower Riemann sum of $\int_{\mathcal{O}} \underline{s^n}(f(n)\|X - X_0\|)\mathrm{d}X$ and upper Riemann sum of $\int_{\mathcal{O}} \overline{s^n}(f(n)\|X - X_0\|)\mathrm{d}X$; ii) $\int_{\mathcal{O}} \underline{s^n}(f(n)\|X - X_0\|)\mathrm{d}X \sim \int_{\mathcal{O}} s(f(n)\|X - X_0\|)\mathrm{d}X \sim \int_{\mathcal{O}} \overline{s^n}(f(n)\|X - X_0\|)\mathrm{d}X$; iii) since $f(n)\sqrt{\gamma(n)} = o(1)$, the mesh size of the partitions associated to Riemann sums vanishes to 0 as $n \to \infty$. As a result:

$$\pi f^2(n) \sum_k \underline{s^n}(f(n)\bar{d}_{0k})|A^n| \sim \pi f^2(n) \sum_k \overline{s^n}(f(n)\underline{d}_{0k})|A^n| \sim \pi f^2(n) \int_{\mathcal{O}} s(f(n)\|X - X_0\|)\mathrm{d}X = \pi G.$$

Thus for any $0 < \epsilon < \pi/2$ choosing $q = \pi/2 - \epsilon$ and $Q = 2\pi + \epsilon$, we obtain the assertion. ∎

### A. Scheduling Policy

The scheduling policy $S$ is in charge of selecting at every time an implementable transmission configuration, i.e., to select a set $\pi^S(t)$ of non-interfering node pairs, which are allowed to simultaneously communicate. In this paper we restrict our investigation to stationary and ergodic scheduling policies, i.e., policies for which the capacity $\mu_{ij}^S$ between any two nodes $i$ and $j$ satisfies (w.p.1):

$$\mu_{ij}^S = E[\mathbb{1}_{(i,j) \in \pi^S(t)} | H_n] = \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathbb{1}_{(i,j) \in \pi^S(\tau)} \mathrm{d}\tau.$$

In general, the selection of $\pi^S(t)$ may be influenced by several parameters, including instantaneous queues lengths, age of stored information at nodes, etc. In [1] it has been proven that the class of position-based scheduling policies, i.e., those policies in which the selection of the transmission configuration $\pi^S(t)$ is driven only by the vector of instantaneous nodes' positions $X(t)$, achieves the maximum network capacity.

In particular, in this work we consider the following position-based scheduling policy, which complies with the constraints imposed by the protocol interference model:

*Definition 1:* Given a network comprising $n$ nodes, policy $S^*$ enables transmission between node $i$ and node $j$ when the following conditions are satisfied:

$$d_{ij}(t) < R_T(n) = \frac{c_T}{\sqrt{n}}$$
$$\min(d_{kj}(t), d_{ki}(t)) > (1 + \Delta)R_T(n)$$

for every other node $k$ in the network (regardless of node $k$ activity), where $c_T > 0$ is any constant. Moreover, the transmission bandwidth between $i$ and $j$ is equally shared in the two directions.

In simple words, nodes $i$ and $j$ communicate when i) one node comes within the transmission range $R_T(n)$ of the other; ii) all of the other nodes are at a distance greater than $(1 + \Delta)R_T(n)$ from both $i$ and $j$.

Long term capacities achieved by $S^*$ in a network comprising $n$ nodes will be denoted by $\mu_{ij}^{*n}$. They can be expressed as function of the stationary spatial distribution of all $n$ nodes as follows:

$$\mu_{ij}^{*n} = \Pr\{d_{ij}(t) < R_T(n)\} \prod_{k \neq i,j} \Pr\{\min(d_{ki}(t), d_{kj}(t)) > (1 + \Delta)R_T(n)\} = \frac{1}{2} \int_{X_i \in \mathcal{O}} \int_{X_j \in B(X_i, c_T/\sqrt{n})} \times \left[\prod_{k \neq i,j} \int_{X_k \in A_\Delta(X_i, X_j)} \phi_k(X_k)\mathrm{d}X_k\right] \phi_j(X_j)\mathrm{d}X_j \phi_i(X_i)\mathrm{d}X_i \quad (5)$$

where $A_\Delta(X_i, X_j)$ is the region defined by the set of points $\{X : \min(\|X - X_j\|, \|X - X_i\|) > (1 + \Delta)\frac{c_T}{\sqrt{n}}\}$.

However, the following Theorem, whose proof is reported in Appendix B, permits to simplify the analysis:

*Theorem 2:* Under policy $S^*$, if $f(n)\sqrt{\gamma(n)} = o(1)$, for any pair of nodes $(i,j)$ and any finite $c_T > 0$:

$$\mu_{ij}^{*n} = \Theta\left(\Pr\left\{d_{ij} \le \frac{c_T}{\sqrt{n}} | H_{ij}\right\}\right)$$

where $H_{ij} = \{X_i^h, X_j^h\}$

Theorem 2 allows us to say that, in the super-critical regime, if policy $S^*$ is adopted, the long-term capacity between two nodes is of the same order as the fraction of time during which one node falls within the transmission range of the other. This because $\prod_{k \ne i,j} \Pr\{\min(d_{ki}(t), d_{kj}(t)) > (1+\Delta)R_T(n)\}$ turns out to be $\Theta(1)$. As a consequence: i) no other scheduling policy employing a transmission range $R_T(n) = \Theta(\frac{1}{\sqrt{n}})$ can achieve an asymptotically higher throughput (in order sense) than $S^*$; ii) to evaluate $\mu_{ij}^{*n}$ (in order sense), we can reduce ourselves to computing the 'contact probabilities' $\Pr\{d_{ij} \le R_T(n)\}$. It results:

$$\begin{aligned}
&\Pr\{d_{ij} \le R_T(n)\} \\
&= \frac{f^4(n)}{G_n^2} \int_{X_i \in \mathcal{O}} \int_{X_j \in B(X_i, R_T(n))} s\left(f(n)\left\|X_j - X_j^h\right\|\right) \\
&\quad s\left(f(n)\left\|X_i - X_i^h\right\|\right) \mathrm{d}X_i \mathrm{d}X_j \\
&\sim \frac{f^4(n)}{G^2} \int_{X \in \mathcal{O}} \int_{Y \in B(X, R_T(n))} s\left(f(n)\left\|X - \left(X_j^h - X_i^h\right)\right\|\right) \\
&\quad s(f(n)\|Y\|) \mathrm{d}X \mathrm{d}Y \\
&\sim \frac{f^4(n)}{G^2} \int_{X \in \mathbb{R}^2} s\left(f(n)\left\|X - \left(X_j^h - X_i^h\right)\right\|\right) \\
&\quad \int_{Y \in B(X, R_T(n))} s(f(n)\|Y\|) \mathrm{d}Y \mathrm{d}X \\
&= \frac{1}{G^2} \int_{X \in \mathbb{R}^2} s\left(\left\|X - f(n)\left(X_j^h - X_i^h\right)\right\|\right) \\
&\quad \int_{Y \in B(X, f(n) R_T(n))} s(\|Y\|) \mathrm{d}Y \mathrm{d}X
\end{aligned} \tag{6}$$

where the asymptotic equivalence $(\sim)$ holds because $s(d)$ is summable.

Now in light of the fact that $s(d)$ is decreasing and continuous, it follows immediately that

$$\begin{aligned}
&s(\|X\| + R_T(n)) |B(X, f(n) R_T(n))| \\
&\quad < \int_{Y \in B(X, f(n) R_T(n))} s(\|Y\|) \mathrm{d}Y \\
&\quad < s(\max[0, \|X\| - R_T(n)]) \\
&\qquad \times |B(X, f(n) R_T(n))|
\end{aligned} \tag{7}$$

being $R_T(n) = \frac{c_T}{\sqrt{n}}$ and $f(n) = o(\sqrt{n})$. Thanks to Theorem 2, it results:

$$\mu_{ij}^{*n} = \Theta\left(g(n)\eta\left(f(n)\left\|X_j^h - X_i^h\right\|\right)\right) \tag{8}$$

having defined:

$$\eta(\|Z\|) = \int_{X \in \mathbb{R}^2} s(\|X - Z\|) s(\|X\|) \mathrm{d}X$$

and $g(n) = \pi c_T^2 \frac{f^2(n)}{n}$. Note that according to (8) the long term capacity between nodes $i$ and $j$ is a function of the relative distance between their home-points $d_{ij}^h$; therefore in the following we will use interchangebly both $\mu_{ij}^{*n}$ and $\mu^{*n}(d_{ij}^h)$ to denote such capacity.

In addition, with abuse of notation, we define also $\eta(d)$, with $d \in \mathbb{R}^+$, as $\eta(\|Z\|)$ for any $Z$ such that $\|Z\| = d$. Note that $\eta$ is a non-increasing, summable function $\mathbb{R}^+ \to \mathbb{R}^+$ which is everywhere continuous and derivable (with continuous derivative) at every point.

At last we remark that, as consequence of the fact that the third moment of $s(d)$ is supposed to be finite, it can be shown that the first three moments of $\eta(d)$ are finite as well.

### B. Mapping Over Generalized Random Geometric Graph

As result of previous findings, the asymptotic capacity achievable by a mobile wireless network under the constraints: i) $\sqrt{\gamma(n)}f(n) = o(1)$; ii) the scheduling policy selects transmission ranges $O(1/\sqrt{n})$, can be studied by representing the network as a Generalized Random Geometric Graph $G(n,m,r,\mu)$ defined as follows[10]: the $n$ vertices of the graph stand for the home-points of the nodes, which are randomly located according to an $(m,r)$ Clustered model (conventionally if $m = n$ we assume the home-points to be located according to the Uniform model); the pair of vertices $(i,j)$ are connected by an edge (link) of capacity $\mu_{ij}^n$ which depends only on the distance between the corresponding home-points. In particular, we define:

$$\begin{aligned}
\mu_{ij}^n = \mu^n\left(d_{ij}^h\right) &\triangleq \mu_{ij}^{*n} \\
&= \Theta\left(g(n)\eta\left(f(n) d_{ij}^h\right)\right)
\end{aligned}$$

with $d_{ij}^h = \|X_i^h - X_j^h\|$ and $g(n) = \pi c_T^2 \frac{f^2(n)}{n}$, when policy $S^*$ is applied.

The network capacity under traffic patterns represented by $\lambda\mathbf{\Lambda} = \lambda[\lambda_{sd}]$, (we remind that $\mathbf{\Lambda}$ is normalized in such a way that $\sum_s \sum_d \lambda_{sd} = n$) can be found by solving a Maximum Concurrent Flow (MCF) problem [22] over the associated GRGG, i.e., by solving the following multi-commodity flow problem:

$$\begin{cases} \max \lambda \\ \lambda \sum_s \sum_d \lambda_{sd} f_{ij}^{sd} \le \mu_{ij}^n \end{cases}$$

with $f_{ij}^{sd} \in [0,1]$ denoting the average fraction of traffic from node $s$ to node $d$, which is routed through link $(i,j)$, i.e., $j$ follows $i$ as relay node along the path. We have the following routing continuity constraints:

$$\sum_i f_{ij}^{sd} - \sum_k f_{jk}^{sd} = \begin{cases} 1, & \text{for } j = d \\ 0, & \text{for } j \ne d \text{ and } j \ne s \\ -1, & \text{for } j = s. \end{cases} \tag{9}$$

Note that $n\lambda$ represents the maximum total traffic the network/graph can transport; the set $\{f_{ij}^{sd}\}$ univocally defines the corresponding routing strategy in the network/graph.

[10]The class of GRGG generalizes the class of Random Geometric Graphs [32]

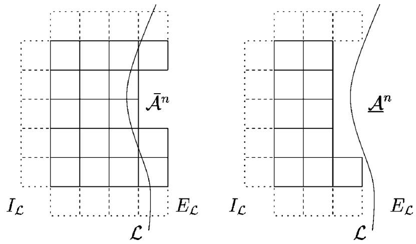


Fig. 3. Example of outer (left) and inner (right) tessellations of $I_{\mathcal{L}}$. Similar tessellations can be drawn for $E_{\mathcal{L}}$.

In general, MCF problems are hard to solve; an upper bound to $\lambda$ can be obtained in terms of graph cuts:

*Proposition 1:* Traffic $\lambda\mathbf{\Lambda}$ is sustainable only if, for any partition $(\mathcal{S},\mathcal{D})$ of the nodes, it results:

$$\lambda \sum_{s\in\mathcal{S}} \sum_{d\in\mathcal{D}} \lambda_{sd} \le \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{D}} \mu_{ij}^n. \tag{10}$$

It was proven in [22] that, in undirected graphs, traffic is guaranteed to be sustainable if the ratio between the minimum value of the r.h.s. and the maximum value of the l.h.s. in (10) is $\Omega(\log k)$, where $k$ is the number of flows.

In the case of GRGG, node partitions can be obtained by physically partitioning the area $\mathcal{O}$ into two disjoint regions $\mathcal{S}'$ and $\mathcal{D}'$:

$$\lambda \le \frac{\sum_{i:X_i^h\in\mathcal{S}'} \sum_{j:X_j^h\in\mathcal{D}'} \mu_{ij}^n}{\sum_{s:X_s^h\in\mathcal{S}'} \sum_{d:X_d^h\in\mathcal{D}'} \lambda_{sd}}. \tag{11}$$

Since $\mathbf{\Lambda}$ is a permutation traffic matrix, the denominator of (11) counts the number of traffic relations with source $s\in\mathcal{S}'$ and destination $d\in\mathcal{D}'$.

A different upper-bound on the maximum achievable throughput can be obtained observing that:

*Proposition 2:* Defined $d_{ij}^h = \|X_i^h - X_j^h\|$, if traffic $\lambda\mathbf{\Lambda}$ is sustainable, then:

$$\lambda \sum_{sd} \lambda_{sd} d_{sd}^h \le \sum_{ij} \mu_{ij}^n d_{ij}^h.$$

*Proof:* The proof is a consequence of continuity constraints on routing. Indeed for any source-destination pair $(s,d)$, traffic $\lambda_{sd}$ must be routed in graph $G(n,m,r,\mu)$ through paths connecting $s$ and $d$. The length of these paths, by triangular inequality, cannot be shorter than $d_{sd}^h$, thus the assertion follows.■

Let us introduce the cumulative quantity

$$T^n(d) = -\sum_i \sum_j \mu_{ij}^n \mathbb{1}_{d_{ij}^h > d}$$

which represents the sum of all link capacities $i$ and $j$ such that $d_{ij}^h > d$, changed in sign. From its definition $T(d+\Delta)-T(d) = \sum_i \sum_j \mu_{ij}^n \mathbb{1}_{d<d_{ij}^h<d+\Delta}$, and thus

$$\sum_i \sum_j \mu_{ij}^n d_{ij}^h = \int x \mathrm{d}T^n(x).$$

This implies

$$\lambda \sum_s \sum_d \lambda_{sd} d_{sd}^h \le \int x \mathrm{d}T^n(x).$$

In our case, for sufficiently large $n$, we have $d_{sd}^h = \Theta(1)$ w.h.p., since we are dealing with a random permutation traffic pattern. Therefore w.h.p., it results:

$$\lambda = O\left(\frac{1}{n}\int x \mathrm{d}T^n(x)\right).$$

Furthermore, decomposing $\frac{1}{n}\int x\mathrm{d}T^n(x)$ as sum of two contributions $\frac{1}{n}(\int_{x\le D} x\mathrm{d}T^n(x) + \int_{x>D} x\mathrm{d}T^n(x))$, we have that, by construction, the latter contribution:

$$\frac{1}{n}\int_{x>D} x\mathrm{d}T^n(x) \tag{12}$$

represents an upper bound to the network throughput that can be sustained by transmissions between pairs of nodes $(i,j)$ with $d_{ij}^h > D$.

In Section V.C we will show how (11) can be used to easily compute an upper bound to the overall transport capacity of the network. Bound (12) will instead be exploited in Section V.D to prove that transmission ranges $R_T = \Theta(\frac{1}{\sqrt{n}})$ allow to achieve the maximum asymptotic network capacity.

### C. Computation of the Capacity of GRGG

Given a random graph $G(n,m,r,\mu)$ and any simple, closed, convex curve $\mathcal{L}$, dividing $\mathcal{O}$ in two regions $I_{\mathcal{L}}$ and $E_{\mathcal{L}}$, we define the capacity crossing $\mathcal{L}$ as

$$\mu_{\mathcal{L}}^n = \sum_{i\in I_{\mathcal{L}}} \sum_{j\in E_{\mathcal{L}}} \mu^n\left(d_{ij}^h\right).$$

Note that $\mu_{\mathcal{L}}^n$ depends on the locations of nodes' home-points $H_n = \{X_i^h\}_{i=1}^n$.

Averaging over all possible $H_n$, by spatial uniformity (on average) of Uniform and Clustered models, it immediately follows that:

$$E[\mu_{\mathcal{L}}^n] = \frac{n^2}{|I_{\mathcal{L}}||E_{\mathcal{L}}|}\int_{I_{\mathcal{L}}}\int_{E_{\mathcal{L}}} \mu^n(\|X-Y\|)\mathrm{d}X\mathrm{d}Y$$

where $|I_{\mathcal{L}}|$ and $|E_{\mathcal{L}}|$ stand respectively for the area of $I_{\mathcal{L}}$ and $E_{\mathcal{L}}$. One fundamental point that we need to establish is:

*Proposition 3:* In super-critical regime,

$$\mu_{\mathcal{L}}^n = \Theta(E[\mu_{\mathcal{L}}^n]) \quad \text{w.h.p.}$$

*Proof:* By defining: $\bar{\mathcal{A}}^n$ and $\bar{\mathcal{B}}^n$ as square outer tessellations of $I_{\mathcal{L}}$ and $E_{\mathcal{L}}$, respectively (see Fig. 3); $\bar{N}(A_k)$ and $\bar{N}(B_h)$ as upper bounds on the number of vertices within $A_k\in\bar{\mathcal{A}}^n$ and $B_h\in\bar{\mathcal{B}}^n$; $\underline{d}_{A_k,B_h}$ as the minimum distance between points of $A_k$ and $B_h$. Then, $\mu_{\mathcal{L}}^n$ can be upper bounded by:

$$\sum_{A_k\in\bar{\mathcal{A}}^n} \sum_{B_h\in\bar{\mathcal{B}}^n} \mu^n(\underline{d}_{A_k,B_h})\bar{N}(A_k)\bar{N}(B_h). \tag{13}$$

Analogously, by defining: $\underline{\mathcal{A}}^n$ and $\underline{\mathcal{B}}^n$ as square inner tessellations of $I_{\mathcal{L}}$ and $E_{\mathcal{L}}$, respectively (see again Fig. 3); $\underline{N}(A_k)$ and $\underline{N}(B_h)$ as lower bounds on the number of vertices within $A_k\in\underline{\mathcal{A}}^n$ and $B_h\in\underline{\mathcal{B}}^n$; $\bar{d}_{A_k,B_h}$ as the maximum distance between points of $A_k$ and $B_h$. Then, $\mu_{\mathcal{L}}^n$ can be lower bounded by

$$\sum_{A_k\in\underline{\mathcal{A}}^n} \sum_{B_h\in\underline{\mathcal{B}}^n} \mu^n(\bar{d}_{A_k,B_h})\underline{N}(A_k)\underline{N}(B_h). \tag{14}$$

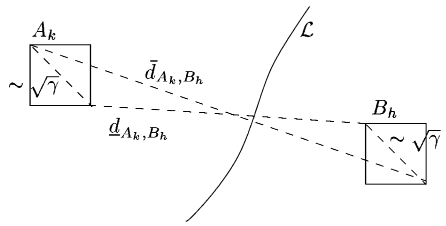


Fig. 4. Definitions of the distances between $A_k$ and $B_h$.

If we select the tessellations $\underline{\mathcal{A}}^n$, $\underline{\mathcal{B}}^n$, $\bar{\mathcal{A}}^n$, $\bar{\mathcal{B}}^n$, in such a way that their elements have area $(16+\delta)\gamma(n)$, combining bounds (13) and (14), according to Lemma 1, it results:

$$\frac{(16+\delta)^2 n^2}{4} \sum_{A_k \in \underline{\mathcal{A}}^n} \sum_{B_h \in \underline{\mathcal{B}}^n} \mu^n(\bar{d}_{A_k,B_h})\gamma^2(n) \le \mu^n_{\mathcal{L}}$$
$$\le (16+\delta)^2 4n^2 \sum_{A_k \in \bar{\mathcal{A}}^n} \sum_{B_h \in \bar{\mathcal{B}}^n} \mu^n(\underline{d}_{A_k,B_h})\gamma^2(n).$$

Note that $\bar{d}_{A_k^n,B_h^n} - \underline{d}_{A_k^n,B_h^n} \le 2\sqrt{2(16+\delta)\gamma(n)}$. Then under the condition $f(n)\sqrt{\gamma(n)} = o(1)$, it results:

$$\sum_{A_k \in \bar{\mathcal{A}}^n} \sum_{B_h \in \bar{\mathcal{B}}^n} \eta(f(n)\underline{d}_{A_k,B_h})\gamma^2(n) \sim \quad (15)$$
$$\sim \sum_{A_k \in \underline{\mathcal{A}}^n} \sum_{B_h \in \underline{\mathcal{B}}^n} \eta(f(n)\bar{d}_{A_k,B_h})\gamma^2(n) \sim \quad (16)$$
$$\sim \int_{X\in I_{\mathcal{L}}} \int_{Y \in E_{\mathcal{L}}} \eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y \quad (17)$$

being $\eta(d)$ non increasing and summable. Indeed, observe that, i) (15) and (16) can be interpreted, respectively, as lower Riemann sum and upper Riemann sum of $\int_{X\in I_{\mathcal{L}}} \int_{Y\in E_{\mathcal{L}}} \eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y$; ii) since $f(n)\sqrt{\gamma(n)} = o(1)$, the mesh size of the partitions associated to Riemann sums vanishes to 0 as $n \to \infty$. ■

According to Proposition 1, the evaluation of $\mu^n_{\mathcal{L}}$ permits to obtain an upper bound to the transport capacity of the GRGG. Considering, indeed, any simple, regular, closed curve $\mathcal{L}$, under a random permutation traffic, the number of flows crossing $\mathcal{L}$ (i.e., connections whose source lies in $I_{\mathcal{L}}$ and destination in $E_{\mathcal{L}}$) w.h.p. is $\Theta(n)$, thus by (11) it results:

$$\lambda \le \Theta\left(ng(n)\int_{X\in I_{\mathcal{L}}} \int_{Y\in E_{\mathcal{L}}} \eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y\right).$$

To upper bound the order of magnitude of $\lambda$, the following result (proven in Appendix C) comes in handy:

*Proposition 4:* For any convex, simple, closed curve $\mathcal{L}$ regular at every point:

$$f^2(n)\int_{X\in I_{\mathcal{L}}} \int_{Y\in E_{\mathcal{L}}} \eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y = \Theta\left(\frac{1}{f(n)}\right)$$

from which we obtain $\lambda = O(\frac{1}{f(n)})$.

### D. Effect of Transmission Range

Looking at (6), it can be noticed that, by increasing the transmission range, one can increase the contact probability between any two nodes. However the potential advantage deriving from augmenting the transmission range is totally offset by the increased interference produced over possible concurrent transmissions. In this section we restrict our analysis to the interesting case $R_T = o(\frac{1}{f(n)})$ (note that previous analysis has been carried out under such assumption).

The complementary case $R_T = \omega(\frac{1}{f(n)})$ can be easily verified to be largely sub-optimal due to interference. Moreover, note that in the case $R_T = \omega(\frac{1}{f(n)})$ node mobility is not anymore efficiently exploited to reduce the distance traversed by data on the wireless channel (because the typical distance reached by nodes from their home-points, for effect of mobility, is $\frac{1}{f(n)} = o(R_T)$), and the system behaves as if nodes were fixed at their home-points.

*Theorem 3:* In super-critical regime, the network capacity cannot be increased by increasing the transmission range beyond $\Theta(\frac{1}{\sqrt{n}})$, if $\eta$ has finite support.

*Proof:* Since there exists a finite $y_0$ such that $\eta(y) = 0$ for $y \ge y_0$, $\Pr\{d_{ij} \le R_T(n)\} > 0$ only if $\|X_j^h - X_i^h\| \le \frac{y_0}{f(n)}$. The radio interface of a node can, at most, be fully utilized all the time. Thus we have $\mu_i^n = \sum_j \mu_{ij}^n \le 1$. Consider a line $\mathcal{L}$ dividing $\mathcal{O}$ in two regions $A_0$ and $B_0$. The capacity flow through the corresponding cut is $\sum_{i\in A_0}\sum_{j\in B_0}\mu_{ij}^n$. However, by construction, no edges exist between node $i \in A_0$ and any $j \in B_0$ if the distance between $X_i^h$ and line $\mathcal{L}$ is greater than $\frac{y_0}{f(n)}$. Thus defining with $C_0$ the region of points belonging to $A_0$ whose distance from $\mathcal{L}$ is less than $\frac{y_0}{f(n)}$.

$$\sum_{i\in A_0}\sum_{j\in B_0}\mu_{ij}^n = \sum_{i\in C_0}\sum_{j\in B_0}\mu_{ij}^n \le \sum_{i\in C_0}\mu_i^n$$
$$\le N(C_0) \le 2n|C_0| = \Theta\left(\frac{n}{f(n)}\right)$$

because of Lemma 1 and $|C_0| = \Theta(\frac{1}{f(n)}) = \omega(\sqrt{\gamma(n)})$ with $\sqrt{\gamma(n)} = \omega(\gamma(n))$. We conclude that $\lambda = O(1/f(n))$. Comparing this capacity with that provided by the proposed scheduling/routing scheme (described in the next section) the assertion follows. ■

*Theorem 4:* In super-critical regime, the network capacity cannot be increased by increasing the transmission range beyond $\Theta(\frac{1}{\sqrt{n}})$, if the scheduling policy allows only transmissions between node pairs $(i,j)$ with $d_{ij}^h = O(\frac{1}{f(n)})$.

*Proof:* If the scheduling policy allows transmissions only between node pairs $(i,j)$ with $d_{ij}^h = O(\frac{1}{f(n)})$; then an $M_0$ can be found, such that, for large $n$, no transmissions are allowed between node pairs $(i,j)$, when $d_{ij}^h = \frac{M_0}{f(n)}$.

Repeating exactly the same scheme as in the proof of Theorem 3, where $y_0$ is substituted by $M_0$, the assertion follows. ■

Previous results are complemented by the following theorem exploiting (12); the proof is reported in Appendix D.

*Theorem 5:* Consider any possible scheduling/routing scheme employing a transmission range $R_T = o(\frac{1}{f(n)})$. At most a network capacity $O(\frac{1}{f(n)})$ can be sustained by transmissions between node pairs $(i,j)$ with $d_{ij}^h > \frac{10}{f(n)}$.

As immediate consequence of Theorem 4 and Theorem 5 we conclude that no capacity gain can be achieved in the super-critical regime by increasing the transmission range $R_T$ beyond $\Theta(\frac{1}{\sqrt{n}})$.

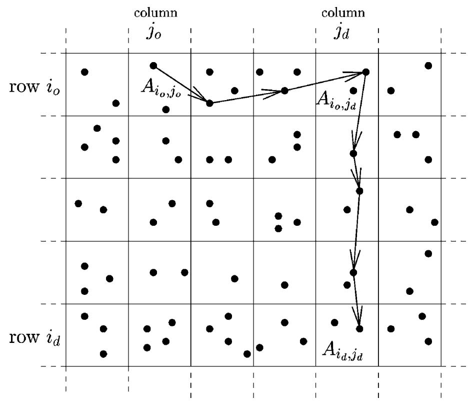


Fig. 5. Possible logical route from a source located in squarelet $A_{i_o,j_o}$ to a destination located in squarelet $B_{i_d,j_d}$. Dots denote home-points and arrows represent logical hops of the path: the actual forwarding occurs when nodes corresponding to the selected home-points get in contact with each other.

### *E. Optimal Routing Scheme*

Here we show that the upper bound on network capacity given in Proposition 4 is tight, presenting a simple routing scheme that achieves per-node throughput $\Theta(\frac{1}{f(n)})$.

We partition the network area $\mathcal{O}$ into a regular square tessellation $\mathcal{A}^n$ whose elements have area equal to $c^2/f^2(n)$, being $c$ an appropriate constant. Squarelets $A^n_{i,j}$ are indexed by tuple $(i,j)$, where $i$ is the row index, and $j$ is the column index. We adopt the following routing scheme, illustrated in Fig. 5: traffic originated by a node in $A^n_{i_o,j_o}$ and destined to a node in $A^n_{i_d,j_d}$ is first routed along a (logical) horizontal path through a sequence of randomly selected relay home-points residing in contiguous squarelets up to reach a node whose home-point is in $A^n_{i_o,j_d}$, and then routed through a similar (logical) vertical path up to its final destination.

*Theorem 6:* Employing the routing scheme described above, a per-node throughput $\lambda = \Theta(\frac{1}{f(n)})$ can be achieved.

*Proof:* An upper bound to the total traffic flowing between two adjacent squarelets $A$ and $B$ can be easily obtained as follows. The traffic between two adjacent horizontal squarelets cannot exceed the total traffic produced by all sources lying in the same row. Similarly, the traffic between two adjacent vertical squarelets cannot exceed the total traffic sent to destinations lying in the same column. Since $\gamma(n) = o(1/f^2(n))$, thanks to Lemma 1 we can upper bound the traffic flowing between any two adjacent squarelets by $\lambda\frac{2cn}{f(n)}$.

On the other hand, considering the GRGG associated to the network, the number of edges connecting vertices (home-points) in $A$ to vertices (home-points) in $B$ is equal to $N(A)N(B)$. It follows that an upper bound to the traffic traversing each individual edge connecting squarelets $A$ and $B$ is given by

$$\frac{\lambda\frac{2cn}{f(n)}}{\underline{N}(A)\underline{N}(B)}$$

where w.h.p., uniformly over the tessellation, $\underline{N}(A) = \underline{N}(B) = \frac{c^2 n}{2f^2(n)}$, since $\gamma(n) = o(1/f^2(n))$. Traffic can be sustained if no edge is overloaded; this is true w.h.p. if

$$\frac{\lambda\frac{2cn}{f(n)}}{\underline{N}(A)\underline{N}(B)} = \lambda\frac{8f^3(n)}{c^3 n} \leq \mu^n(\bar{d}_{A,B}) = g(n)\eta(\sqrt{5}c)$$

where $\bar{d}_{A,B} = \frac{\sqrt{5}c}{f(n)}$. Note that $c$ must be chosen is such a way that $\eta(\sqrt{5}c) > 0$. As a result, the maximum sustainable per-node traffic with the proposed routing scheme is

$$\lambda = \Theta\left(\frac{g(n)\eta(\sqrt{5}c)}{\frac{8f^3(n)}{c^3 n}}\right) = \Theta\left(\frac{1}{f(n)}\right).$$

■

## VI. Analysis in Sub-Critical Regime

In this section we briefly discuss what happens when the condition $f(n)\sqrt{\gamma(n)} = o(1)$ is violated. More specifically, we limit our discussion to the strictly sub-critical regime (i.e., when $f(n)\sqrt{\gamma(n)} = \omega(1)$) leaving for future investigations the critical case $f(n)\sqrt{\gamma(n)} = \Theta(1)$.

Unfortunately, the insensitiveness of network capacity to the shape of $s(d)$ is lost in sub-critical regime. Here we focus on the worst case among all possible shapes $s(d)$, which provides a lower bound to network capacity in sub-critical regime. First we introduce the following result.

*Proposition 5:* If $f(n)\sqrt{\gamma(n)} = \omega(1)$, values of $R_T(n) < \sqrt{\gamma(n)}$ may fail even to guarantee network connectivity.

*Proof:* We restrict our investigation to the case in which $s(d)$ has finite support, i.e., $s(d) = 0$ for $d > d_0 \in \mathbb{R}^+$; in this case, considering two nodes $i$ and $j$ and applying the triangular inequality to the distance between them we obtain

$$\|X_i(t) - X_j(t)\| \geq \|X_i^h - X_j^h\| - 2\frac{d_0}{f(n)} \tag{18}$$

which implies that, whenever $\|X_i^h - X_j^h\| = \omega(\frac{1}{f(n)})$, then $\|X_i(t) - X_j(t)\| \sim \|X_i^h - X_j^h\| = \omega(\frac{1}{f(n)})$, i.e., for sufficiently large $n$, the transmission capacity $\mu^n_{ij}$ between node pairs $(i,j)$ such that $\|X_i^h - X_j^h\| = \omega(\frac{1}{f(n)})$ is null unless $R_T(n) = \Theta(\|X_i^h - X_j^h\|)$.

On the other hand, it was shown in [3] for the Uniform model that the probability that every node $i$ in the network finds at least another node $j$ whose home-point $X_j^h$ has distance from $X_i^h$ less than $k(n)$, for any $k(n) \leq \sqrt{\frac{\log(n)}{n}}$, asymptotically satisfies

$$\Pr\left\{\forall i, \exists j \neq i : \left\|X_i^h - X_j^h\right\| < k(n)\right\} < \beta < 1.$$

As a consequence, the network may result asymptotically disconnected with strictly positive probability as long as $R_T(n) \leq \sqrt{\frac{\log(n)}{n}}$.

The previous result can be easily extended to the clustered model, since the $m(n)$ cluster centers are uniformly and independently located within the area. It turns out that the probability that every node $i$ in the network finds another node $j$ belonging to a different cluster, such that the distance between the corresponding home points is less than $k(n)$, is asymptotically strictly less than 1 when $k(n) < \sqrt{\frac{\log(m(n))}{m(n)}}$:

$$\Pr\left\{\forall \text{ cluster } c, \forall \text{ node } i \in c, \exists \text{ node } j \notin c : \left\|X_i^h - X_j^h\right\| < k(n)\right\} < \beta < 1$$

for any $k(n) < \sqrt{\frac{\log(m(n))}{m(n)}}$. Thus also in this case the network is disconnected with strictly positive probability as long as $R_T(n) = o(\sqrt{\frac{\log(n)}{n}})$. ■

As a consequence, node mobility plays no significant role, and the system behaves as if nodes were fixed. Therefore, the Gupta–Kumar result (whose extension to the clustered model is rather straightforward) can be applied in sub-critical regime to estimate the network capacity when $s(d_0) = 0$, for some $d_0$. The resulting per node capacity is

$$\Theta\left(\frac{1}{nR_T(n)}\right) = \Theta\left(\sqrt{\frac{m(n)}{n^2\log(m(n))}}\right).$$

In the case $s(d) > 0$ for any $d$, then all pairs of nodes occasionally meet, thus network connectivity is guaranteed also by using transmission ranges $R_T(n) = o(\sqrt{\gamma(n)})$. As a consequence, the system capacity can be in general increased with respect to the case in which $s(d)$ has finite support. In [33], the interested reader can find an exhaustive analysis of the behavior in sub-critical regime when $s(d)\sim d^{-\zeta}$.

## VII. Conclusion

In this paper, we have extended the analysis of the capacity scaling properties of mobile ad hoc networks to the important case of systems including heterogeneous nodes and spatial inhomogeneities, two common features widely recognized in realistic mobility traces. We have identified two asymptotic regimes which appear for different scaling of the various system parameters. In the super-critical regime, the network area expands slowly enough for all local phenomena to be evened out at the level of the optimal communication range. As a consequence, the network capacity in the super-critical regime turns out to be insensitive to the details of the node mobility pattern and to the degree of node clustering; in the sub-critical regime, instead, the capacity becomes sensitive to the specific node mobility pattern and to the level of clustering in the network. Focusing on the super-critical regime, we have computed the exact scaling law of network capacity, and we have proposed an implementable, asymptotically optimal, scheduling and routing scheme.

As final remark, our analysis can be extended to the case in which nodes with different mobility patterns coexist (e.g., a mixture of fixed nodes and fully mobile nodes, or several classes of nodes with different degrees of mobility around their home-points). The basic idea of such extension is to consider multiple classes of nodes, each characterized by its own mobility function $\phi(X)$, and to separately compute inter-capacities or intra-capacities for all pairs of classes. An example of this technique in the case of two classes has been presented in [1], from which it is possible to derive easily the extension to any (finite) number of classes.

## Appendix A
## Proof of Lemma 1

The proof is a standard application of the Chernoff bound [34]. We first consider the Uniform model and then generalize to the Clustered model.

In the case of the Uniform model, given an element $A_k^n$ of tessellation $\mathcal{A}^n$, by definition:

$$N\left(A_k^n\right) = \sum_{i=1}^{n} \mathbb{1}_{X_i^h \in A_k^n}$$

with $\mathbb{1}_{X_i^h \in A_k^n}$ i.i.d. Bernoullian random variables with mean $p_1 = E[\mathbb{1}_{X_i^h \in A_k^n}] = |A^n|$. By applying Chernoff bounds [34] we get

$$\Pr\left\{N(A_k^n) < \frac{1}{2}E\left[N\left(A_k^n\right)\right] = \frac{1}{2}np_1\right\} < e^{-np_1/8}$$
$$\Pr\left\{N\left(A_k^n\right) > 2E\left[N\left(A_k^n\right)\right]\right\} < (e/4)^{np_1} < e^{-np_1/8}.$$

Thus, setting $|A^n| = 16\frac{\log(n)}{n}$, we obtain

$$\Pr\left\{\frac{1}{2}E\left[N\left(A_k^n\right)\right] < N\left(A_k^n\right) < 2E\left[N\left(A_k^n\right)\right]\right\} \geq 1 - 2e^{-2\log n} = 1 - 2n^{-2}.$$

At last, by sub-additivity of probability measures:

$$\begin{aligned}\Pr&\left\{\cap_k\left\{\frac{1}{2}E\left[N\left(A_k^n\right)\right] < N\left(A_k^n\right) < 2E\left[N\left(A_k^n\right)\right]\right\}\right\}\\ &\geq 1 - \sum_k \Pr\left\{N(A_k^n) < \frac{1}{2}E\left[N\left(A_k^n\right)\right] \text{ or } N\left(A_k^n\right) > 2E\left[N\left(A_k^n\right)\right]\right\}\\ &\geq 1 - n2n^{-2} = 1 - \frac{2}{n}.\end{aligned}$$

In the case of Clustered model with $m(n) = n^\nu$, $\nu \in [0,1)$, and $\lim_{n\to\infty} m(n)r^2(n) = 0$, consider a sequence of regular tessellations $\mathcal{A}_n$, such that $|A^n| > 16\gamma(n)$. Since centers of clusters are distributed according to a uniform distribution, the number of clusters $N_c(A_k^n)$ whose center fall within $A_k^n$, satisfies, uniformly over the tessellation, w.h.p.:

$$E\left[N_c\left(A_k^n\right)\right]/2 < N_c\left(A_k^n\right) < 2E\left[N_c\left(A_k^n\right)\right].$$

Let $n_c$ be the number of nodes belonging to cluster $c$, $1 < c < m(n)$; for any $\epsilon > 0$ with high probability, $\frac{(1-\epsilon)}{m(n)} < n_c < \frac{(1+\epsilon)}{m(n)}$, uniformly over $c$. Indeed, considering a particular cluster $c$, by definition it results: $n_c = \sum_1^n \mathbb{1}_{i\in c}$, being $\mathbb{1}_{i\in c}$ i.i.d. Bernoullian random variables with average $p_1 = 1/m(n)$; thus applying again the Chernoff bound

$$\Pr\left\{\frac{(1-\epsilon)n}{m(n)} < n_c < \frac{(1+\epsilon)n}{m(n)}\right\} \geq 1 - 2\exp\left(-\frac{\left(\epsilon\frac{n}{m(n)}\right)^2}{2\frac{n}{m(n)}}\right) = 1 - 2\exp\left(-\frac{\epsilon n^{1-\nu}}{2}\right).$$

As a consequence:

$$\begin{aligned}\Pr&\left\{\cap_c\left\{\frac{(1-\epsilon)n}{m(n)} < n_c < \frac{(1+\epsilon)n}{m(n)}\right\}\right\} \geq 1\\ &- \sum_c \Pr\left\{n_c \notin \left[\frac{(1-\epsilon)n}{m(n)}, \frac{(1+\epsilon)n}{m(n)}\right]\right\}\\ &\geq 1 - 2n\exp\left(-\frac{\epsilon n^{1-\nu}}{2}\right)\end{aligned}$$

which tends to 0 when $n \to \infty$.

Moreover, for the generic element $A_k^n$ of the tessellation, we denote by $\underline{A}_k^n$ the subset of its points whose distance from its frontier is greater than $r(n)$, and by $\bar{A}_k^n$ the superset of $A_k^n$ comprising points in $A_k^n$ plus those points whose distance from points of $A_k^n$ is not greater than $r(n)$. Under the condition $|A^n| \geq (16+\delta)\gamma(n)$, for any small $\delta > 0$, it is of immediate verification that

$$\begin{aligned}\sum_c \sum_i &\mathbb{1}_{i\in c}\mathbb{1}_{c\in \underline{A}_k^n}\\ &\leq N(A_k^n)\\ &\leq \sum_c \sum_i \mathbb{1}_{i\in c}\mathbb{1}_{c\in \bar{A}_k^n}.\end{aligned}$$

Since $|\underline{A}_k^n| \sim |\bar{A}_k^n| \sim |A_k^n|$, it follows that $|\underline{A}_k^n| > 16\frac{\log(m(n))}{m(n)}$, for sufficiently large $n$; thus previously obtained results can be applied to bound $\sum_c \mathbb{1}_{c\in \underline{A}_k^n}$, $\sum_i \mathbb{1}_{i\in c}$, and $\sum_c \mathbb{1}_{c\in \bar{A}_k^n}$, and the assertion follows, letting $\epsilon \to 0$. ■

## APPENDIX B
## PROOF OF THEOREM 2

The goal of this proof is essentially to show that:

$$\prod_{k\neq i,j} \Pr\{\min(d_{ki}(t), d_{kj}(t)) > (1+\Delta)R_T\} = \Theta(1) \text{ w.h.p.}$$

i.e., for some $\epsilon > 0$, w.h.p.:

$$\begin{aligned}\lim_{n\to\infty} \prod_{k\neq i,j} &\int_{X_k \in A_\Delta(X_i,X_j)} \phi^n\left(X_k - X_k^h\right) \mathrm{d}X_k\\ &= \prod_{k\neq i,j} E\left[\mathbb{1}_{X_k^{(n)}(t)\in A_\Delta(X_i,X_j)}|H_n\right] > \epsilon\end{aligned}$$

for every $X_i \in \mathcal{O}$ and $X_j \in B(X_i, c_T/\sqrt{n})$. Note that, by triangular inequality,

$$\begin{aligned}\lim_{n\to\infty} &\int_{X_k \notin A_\Delta(X_i,X_j)} \phi^n\left(X_k - X_k^h\right) \mathrm{d}X_k\\ &\leq \int_{X_k\in B\left(X_i, 2\frac{(1+\Delta)c_T}{\sqrt{n}}\right)} \phi^n(X_k - X_k^h)\mathrm{d}X_k \to 0\end{aligned}$$

since, by hypothesis, $\frac{f(n)}{\sqrt{n}} = o(1)$. Thus for any $X_i$ and $X_j \in B(X_i, c_T/\sqrt{n})$, defined for short $p_k^n = E[\mathbb{1}_{X_k^{(n)}\notin A_\Delta(X_i,X_j)}|H_n]$, it results:

$$\lim_{n\to\infty} p_k^n = 0 \quad \forall\, k. \tag{19}$$

In addition, thanks to (4), with high probability, uniformly over $X_i$ and $X_j \in B(X_i, (1+\Delta)c_T/\sqrt{n})$:

$$\lim_{n\to\infty} \sum_{k\neq i,j}^{n} p_k^n < 4(1+\Delta)^2 c_T^2 Q. \tag{20}$$

To conclude the proof, observe that

$$\lim_{n\to\infty} \prod_{k\neq i,j} (1-p_k^n) > \epsilon > 0 \tag{21}$$

by the continuity of log function is equivalent to:

$$\lim_{n\to\infty} \sum_{k\neq i,j} \log\left(1-p_k^n\right) > \log\epsilon > -\infty \tag{22}$$

and that $\log(1-x) > -2x$ for any $0 \leq x \leq x_0$, with $x_0 \approx 0.8$ solution of equation $1 - x_0 = \exp(-2x_0)$. At last, for $n$ sufficiently large, thanks to (19) and (20), we can assume all $p_k^n < x_0$ and get

$$\lim_{n\to\infty} \sum_{k\neq i,j} \log(1-p_k^n) \geq -\sum_{k\neq i,j} 2p_k^n > -8(1+\Delta)^2 c_T^2 Q.$$

■

## APPENDIX C
## PROOF OF PROPOSITION 4

In this proof we provide upper and lower bounds to the quantity $\int_{I_\mathcal{L}}\int_{E_\mathcal{L}} \eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y$. We first consider the case $f(n) = \Theta(1)$ and then generalize the approach to the case $f(n) = \omega(1)$.

In the case $f(n) = \Theta(1)$, without lack of generality we can assume $f(n) = 1$. In the particular case in which $\eta(d) = 1$ for $d \leq \sqrt{2}$ (which is essentially the Grossglauser-Tse case), the result is immediate, since, for any $\mathcal{L}$:

$$\int_{I_\mathcal{L}}\int_{E_\mathcal{L}} \eta(\|X-Y\|)\mathrm{d}X\mathrm{d}Y = \int_{I_\mathcal{L}} \mathrm{d}X \int_{E_\mathcal{L}} \mathrm{d}Y = \Theta(1).$$

More generally, let $D = \sup\{d|\eta(d) > \frac{\eta(0)}{2}\}$. For any point $X \in I_\mathcal{L}$, let $X_\mathcal{L}$ be its projection on curve $\mathcal{L}$. At last, let $I_\mathcal{L}^{D/2}$ be the set of points in $I_\mathcal{L}$ whose distance from $\mathcal{L}$ is less or equal than $D/2$. Note that $\eta(d) \geq \eta(0)/2\mathbb{1}_{d<D}$ and remind that by triangular inequality $\|X - X_\mathcal{L}\| + \|Y - X_\mathcal{L}\| \geq \|X-Y\|$, thus:

$$\begin{aligned}\int_{X\in I_\mathcal{L}}&\int_{Y\in E_\mathcal{L}} \eta(\|X-Y\|)\mathrm{d}X\mathrm{d}Y\\ &\geq \int_{X\in I_\mathcal{L}}\int_{Y\in E_\mathcal{L}} \eta(\|X-Y\|)\\ &\quad\times \mathbb{1}_{\|X-Y\|\leq D}\mathrm{d}X\mathrm{d}Y\\ &\geq \int_{X\in I_\mathcal{L}}\int_{Y\in E_\mathcal{L}} \frac{\eta(0)}{2}\\ &\quad\times \mathbb{1}_{\|X-X_\mathcal{L}\|\leq D/2}\mathbb{1}_{\|Y-X_\mathcal{L}\|\leq D/2}\mathrm{d}X\mathrm{d}Y\\ &= \int_{X\in I_\mathcal{L}^{D/2}} \frac{\eta(0)}{2}\left[\int_{E_\mathcal{L}} \mathbb{1}_{\|Y-X_\mathcal{L}\|\leq D/2}\mathrm{d}Y\right]\\ &\quad\times \mathrm{d}X \geq \frac{\eta(0)}{2}\left|I_\mathcal{L}^{D/2}\right|\pi\frac{D^2}{4}\end{aligned}$$

where the last inequality holds since the set of points in $E_\mathcal{L}$ whose distance from $X_\mathcal{L}$ is not greater than $D/2$ covers a semicircle of radius $D/2$, being $\mathcal{L}$ convex.

On the other side it is immediate to verify

$$\int_{I_\mathcal{L}}\int_{E_\mathcal{L}} \eta(\|X-Y\|)\mathrm{d}X\mathrm{d}Y \leq \eta(0)|I_\mathcal{L}||E_\mathcal{L}|$$

and since both $D$ and and $|\mathcal{L}|$ are finite (remind that $\mathcal{L}$ is regular at every point), it results $|I_\mathcal{L}^{D/2}| = \Theta(1)$, thus we get the assertion.

In case $f(n) = \omega(1)$, to obtain a lower bound it is sufficient to redefine $D = \sup\{d|\eta(f(n)d) > \frac{\eta(0)}{2}\}$ (note that $D$ scales as $1/f(n)$) and repeat the previous arguments.

To obtain an upper bound, first suppose the support of $\eta(d)$ is finite, i.e., there exists $d_0$ such that $\eta(d)=0$ for $d>d_0$. In such a case $\eta(d)\le\eta(0)\mathbb{1}_{d<d_0}$. Define $I_{\mathcal{L}}^{\frac{d_0}{f(n)}}$ the set of points in $I_{\mathcal{L}}$ whose distance from $\mathcal{L}$ is less or equal than $d_0/f(n)$. It results:

$$\begin{aligned}\int_{X\in I_{\mathcal{L}}}\int_{Y\in E_{\mathcal{L}}}&\eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y\\ &\le\int_{X\in I_{\mathcal{L}}}\int_{Y\in E_{\mathcal{L}}}\eta(0)\mathbb{1}_{\|X-Y\|\le\frac{d_0}{f(n)}}\mathrm{d}X\mathrm{d}Y\\ &\le\int_{X\in I_{\mathcal{L}}}\int_{Y\in E_{\mathcal{L}}}\eta(0)\mathbb{1}_{\|X-X_{\mathcal{L}}\|\le\frac{d_0}{f(n)}}\mathbb{1}_{\|Y-X_{\mathcal{L}}\|\le\frac{d_0}{f(n)}}\mathrm{d}X\mathrm{d}Y\\ &=\int_{X\in I_{\mathcal{L}}^{\frac{d_0}{f(n)}}}\eta(0)\left[\int_{E_{\mathcal{L}}}\mathbb{1}_{\|Y-X_{\mathcal{L}}\|\le\frac{d_0}{f(n)}}\mathrm{d}Y\right]\mathrm{d}X\\ &\le\eta(0)\left|I_{\mathcal{L}}^{\frac{d_0}{f(n)}}\right|2\pi\frac{d_0^2}{4f^2(n)}\end{aligned}$$

given that $|I_{\mathcal{L}}^{\frac{d_0}{f(n)}}|\approx|\mathcal{L}|\frac{d_0}{f(n)}$.

Consider now the case in which the support of $\eta(d)$ is unlimited; by assumption $\sum_i(i+1)^3\eta(i)<\infty$. To get a new upper bound, note that

$$\eta(d)\le\sum_i\eta(i)\mathbb{1}_{i\le d\le(i+1)}\le\sum_i\eta(i)\mathbb{1}_{d\le(i+1)}$$

now

$$\begin{aligned}\int_{I_{\mathcal{L}}}\int_{E_{\mathcal{L}}}&\eta(f(n)\|X-Y\|)\mathrm{d}X\mathrm{d}Y\\ &\le\int_{I_{\mathcal{L}}}\int_{E_{\mathcal{L}}}\sum_i\eta(i)\mathbb{1}_{f(n)\|X-Y\|\le i+1}\mathrm{d}X\mathrm{d}Y\\ &\le\sum_i\eta(i)\int_{I_{\mathcal{L}}}\int_{E_{\mathcal{L}}}\mathbb{1}_{f(n)\|X-Y\|<i+1}\mathrm{d}X\mathrm{d}Y\end{aligned}$$

where the last equality is made possible by the monotone convergence theorem. Bounding each term as before we get the assertion. ∎

## APPENDIX D
## PROOF OF THEOREM 5

Essentially we apply (12) to show that the network throughput can not be increased (in order sense) by exploiting transmissions between node pairs $(i,j)$ with $d_{ij}^h=\omega(\frac{1}{\sqrt{f(n)}})$.

To directly apply (12), an exact expression for $T^n(d)$ is needed, however we can simplify our task by lower bounding $T^n(d)$ with a function $S^n(d)$; i.e., by finding an increasing function $S^n(d)$ such that: i) $\sup S(d)=0$ ii) $T^n(d)\ge S^n(d)$ $\forall d$. By integration per parts it can be easily shown that:

$$\int_{x>D}x\mathrm{d}T^n(x)\le\int_{x>D}x\mathrm{d}S^n(x)\quad\forall\, D\ge 0$$

To obtain an expression for $S(d)$, consider two nodes $i$ and $j$ whose home-points are $X_i^h$ and $X_j^h$ with $d_{ij}^h=\|X_i^h-X_j^h\|$; by construction, the event

$$\{d_{ij}(t)<R_T\}\subseteq\left\{\left\|X_i(t)-X_i^h\right\|>\frac{d_{ij}^h-R_T}{2}\right\}$$
$$\cup\left\{\left\|X_j(t)-X_j^h\right\|>\frac{d_{ij}^h-R_T}{2}\right\}$$

i.e.,

$$\begin{aligned}\Pr\{d_{ij}(t)<R_T\}&\le\Pr\left\{\left\|X_i(t)-X_i^h\right\|>\frac{d_{ij}^h-R_T}{2}\right\}\\ &\quad+\Pr\left\{\left\|X_j(t)-X_j^h\right\|>\frac{d_{ij}^h-R_T}{2}\right\}\\ &=2\Pr\left\{\left\|X_i(t)-X_i^h\right\|>\frac{d_{ij}^h-R_T}{2}\right\}\\ &=2\int_{\mathcal{O}}\phi^n(x)\mathbb{1}_{x>\frac{d_{ij}^h-R_T}{2}}\mathrm{d}x.\end{aligned}$$

Indeed, the aggregate number of transmissions $T(d)$ that can potentially occur between nodes whose home-points are separated by a distance greater than or equal to $d$, satisfies:

$$T(d)\ge-\sum_i E\left[\mathbb{1}_{\|X_i(t)-X_i^h\|>\frac{d-R_T}{2}}\right]$$

since for any such transmission at least one of the two nodes must be at a distance greater than or equal to $\frac{d-R_T}{2}$ from its home-point. Therefore,

$$T(d)\ge-2n\int_{\mathcal{O}}\phi(x)\mathbb{1}_{x>\frac{d-R_T}{2}}\mathrm{d}x=S(d)$$

(observe that by construction $\int_{\mathcal{O}}\phi^n(x)\mathbb{1}_{x>\frac{d-R_T}{2}}\mathrm{d}x\rightarrow 0$ as $d\rightarrow\frac{1}{\sqrt{2}}$).

Now, to conclude the proof, it is enough to show that for some $0<M<\infty$:

$$\frac{1}{n}\int_{\frac{M}{f(n)}}y\mathrm{d}S^n(y)=O\left(\frac{n}{f(n)}\right)$$

i.e., the throughput sustained by transmissions between node pairs $(i,j)$, with $d_{ij}^h>\frac{M}{f(n)}$, is $O(\frac{n}{f(n)})$. This because, as shown in Section V.E, a scheduling/routing scheme achieving a per node throughput $\Theta(\frac{n}{f(n)})$ can be found according to which only transmissions between nodes $(i,j)$ with $d_{ij}^h=\Theta(\frac{1}{f(n)})$ occur.

It results

$$\lambda\le-\frac{1}{n}\int_{y>\frac{M}{f(n)}}2ny\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_{X\in\mathcal{O}}\phi^n(\|X\|)\mathbb{1}_{\|X\|>\frac{y-R_T}{2}}\mathrm{d}X\right)\mathrm{d}y$$

Now selecting for example $M=10$, since $R_T=o(\frac{1}{f(n)})$, for sufficiently large $n$ it results $\frac{y-R_T}{2}>\frac{y}{3}$, for any $y\ge\frac{10}{f(n)}$. Thus:

$$\begin{aligned}\lambda&\le-\frac{1}{n}\int_{y>\frac{10}{f(n)}}2ny\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_{X\in\mathcal{O}}\phi^n(\|X\|)\mathbb{1}_{\|X\|>\frac{y-R_T}{2}}\mathrm{d}X\right)\mathrm{d}y\\ &\le-\int_{y>\frac{10}{f(n)}}2y\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_{X\in\mathcal{O}}\phi^n(\|X\|)\mathbb{1}_{\|X\|>\frac{y}{3}}\mathrm{d}X\right)\mathrm{d}y\\ &\approx 2\int_{y>\frac{10}{f(n)}}y\left[2\pi y\phi^n\left(\frac{y}{3}\right)\right]\mathrm{d}y\\ &<4\pi\int_y f^2(n)y^2\phi\left(f(n)\frac{y}{3}\right)\mathrm{d}y=\Theta\left(\frac{1}{f(n)}\right).\end{aligned}$$

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