The Joint Distribution of Server State and Queue Length of M/M/1/1 Retrial Queue with Abandonment and Feedback

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Abstract— **In this paper, we analyze a continuous time M/M/1/1 queue with retrials,abandonments, and feedback. Calls arrive to a primary queue,and attempt to get served by a single server. If upon arrival, the queue is full and hence the server is busy, the new arriving call either moves into infinite capacity orbit, from which it makes new attempts to reach the primary queue,until it finds the server idle or it abandons and leaves the system. We study the continuous time Markov chain describing this retrial queue system. Consequently, we derive the joint distribution of the server state and retrial queue length using kummer's differential equation. Afterward, we show some numerical results that clarify the relationship between the retrials,arrivals,abandonment rates, and the retrial queue length.**

Keywords : M/M/1/1 Retrial Queue, Server State,Queue Length,Abandonment,Kummer Differential Equations,Joint Distribution Function.

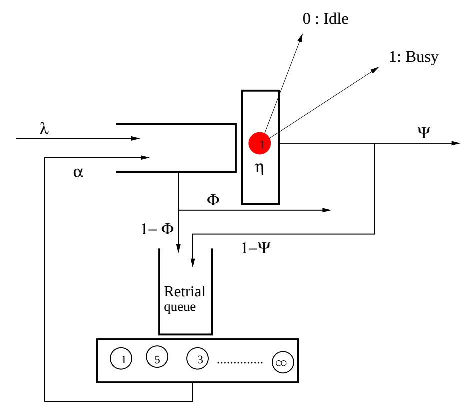

Fig. 1. Retrial Queuing System.

I. Introduction

Any system in which arrivals place demands upon a finite-capacity resource may be termed a queuing system. Certainly, if the arrival times and size of these demands are unpredictable, then conflicts for the use of the resource will arise and queues of waiting customers will form, since arriving customer in the system finds a server busy. In the classical queuing theory, we can mainly describe the solution of this problem in two ways; either a customer leaves the system forever without receiving service which is known by **Erlang loss system**, or the customer waits in line to receive his service which represents **a classical System with queue**.

The length of this queue depends on two aspects of a connection pattern: first, it depends upon the average rate at which connection demands are placed upon resource. Second, it depends upon the statistical fluctuations of this rate. Certainly, when the average rate exceeds the capacity, then the system breaks down and unbounded queues will begin to form, which is a normal phenomenon in many queueing situations such as computer and telecommunication systems, where customers who find all servers busy upon arrival are **obliged to wait in a queue or to leave the service system and to come back to the system after a random amount of time as in a retrial queuing system**.

The retrial queuing system has been introduced as an alternative solution among others to avoid the problems of overloading the classical queuing system, particularly in the switching telephone network. Furthermore, it authorizes a customer to repeat his call after some random time, since it consists of a primary queue where customers are served at their arrival times, and a retrial queue or an orbit where customers can wait to repeat their calls or to choose another type of service (ToS) as it is introduced in [9].

Fig. 1 shows us the mechanism based retrial queuing system. In this, customers arrive at the service station either directly from outside the system or from the retrial queue. If an arriving customer finds the system idle, he receives service immediately and leaves the system after service completion. However, if the customer is blocked from entering the service station because the server is not available, he may abandon and leave the system, or he may join the orbit and attempts service again after some random delay, and continues to do so until the server is found idle.

Retrial queues have been considered as an interesting problem in tele-traffic theory and telephone networks where subscribers redial after receiving a busy signal. In this paper, we are motivated by the need to design and analyze the retrial queuing systems involved in the new developments and technologies, particularly in computer systems and telecommunication networks. For example, peripherals in computer systems may make retrials to receive service from a central processor. Hosts in local area networking (LAN) may make many retrials in order to access the communication medium, which is clearly indicated in the carrier sense multiple access $(CSMA)$ protocol that controls this access.

The remainder of this paper is organized as follows: In section II we discuss the work done around the retrial queues. We also refer to the objective of this paper. In section III, we discuss our model of retrial queuing system and we derive the balanced equations for this system. In section IV, we derive the joint distribution of the queue length and server state.In section V, we discuss some numerical results about the average length of retrial queue, and its relation with other parameters in the system. Finally, section VI will draw a conclusion to this paper, and we will provide some perspective points.

II. Related Work and Objective Statement

In fact, the consideration of service retrials and abandonments complicate the analysis of retrial queuing systems. Hence, some analytical results have been done on determined retrial queuing systems under some assumptions based on the characteristics of retrial times distribution,number of servers,and customers homogeneity. Furthermore, the performance evaluation of the previous work is still limited to numerical algorithms,simulation and approximation methods. further motivations for our paper are inspired by the work of A.Mandelbaum and W.A.Massey in [1], [2]. In these, they develop analytical tools that support the performance analysis of larger telecommunication systems, such as telephone call centers, where abandonment and retrials arise naturally. But, in this paper we use an approach more practical to derive the joint distribution function of server state and queue length rather than the one used in [1], [2].

We analyze an M/M/1/1 retrial queue with both abandonment which may represent the system loss, and feedback where the same customers may return to the system in order to get another type of service as it is explained in [9]. Consequently, this system abstracts and characterizes different practical situations in the telecommunication networks. For example, the mechanism based automatic repeat request (ARQ) protocol in data transmission systems may be modeled by a retrial queue system with feedback, since lost packets are negatively acknowledged by the receivers, then the senders send them once again.

Systems with feedback are introduced and explained in detail by L.Takacs in [3]. This opened a road to researches focused on continuous-time retrial queuing systems with feedback. In [4], [5],Choi and Kulkarni investigate an $M/G/1$ retrial queue with feedback but without loss. In [6], G.I.Falin deeply analyze an $M/M/1$ retrial queue with loss but without feedback.Yang and J.G.C.Templeton in [7] also study an $M/G/1$ retrial queue with loss but without feedback. However, in this paper we derive the joint distribution of the server state and the queue length for a continuous-time $M/M/1/1$ retrial queue with abandonment and feedback by using kummer's differential equation.

III. The model of retrial queuing system

The retrial queuing system that we are going to study throughout this paper is depicted in Fig. 1. In this model new customers or primary calls arrive to the service node following the poisson process with rate λ. If the queue is not full upon primary call arrivals, then the call waits in line, and will served according to the FIFO order, where service times $B(t)$ are assumed to be independent and exponentially distributed with mean $1/\eta$. However, if upon primary call arrivals the queue is full, and hence the server is busy, then these calls move into an infinite capacity orbit with probability $\overline{\phi} = 1 - \phi$ or abandon and leave the system with probability ϕ. Afterward, calls went in the retrial queue make attempts to reach the primary queue, where the attempt times are assumed also to be independent and exponentially distributed with mean $1/\alpha$. Finally, after the customer is served completely, he may decide either to join the retrial group again for another service with probability $\overline{\psi} = 1 - \psi$ or to leave the system forever with probability ψ.

A. System statistical equilibrium equations

Since we would like to derive the relationship between the server state and retrial queue length, so let us suppose that $N(t)$ be the number of repeated calls in the retrial queue at time t, and $C(t)$ represents the server state, where $C(t)$ takes value 1 or 0 at time t when the sever is busy or free respectively. Thus, a process $\{C(t), N(t)\}$ which describes the number of customers in the system is the simplest and simultaneously the most important process associated with the retrial queuing system depicted in Fig. 1. In order to simplify our analysis to this system, we restrict ourselves to the assumption that the service time function $B(t)$ is exponentially distributed. Thus, $\{C(t), N(t)\}$ forms a markov process, where we can consider the markov chain of this process representing this system is embedded at jump customers arrival times rather than a chain embedded at service completion epochs. Hence, the process $\{(C(t), N(t)) : t \succeq 0\}$ forms a Markov chain with a state space $\{0; 1\} \times \{0; 1; ... N\}$, where $\{C, N\} \approx \lim_{t\to\infty}\{C(t), N(t)\}$ in the steady state.

As a result, in the steady state the joint probabilities of server state C and the retrial queue length N, $P_{in} = P\{C = i, N = n\}$, so $P_{0n} = P\{C = 0, N = n\}$ and $P_{1n} = P\{C = 1, N = n\}$ can be characterized through the corresponding partial generating functions for $|z| \leq 1$ by $P_0(z) = \sum_{n=0}^{\infty} P_{on} z^n$ and $P_1(z) = \sum_{n=0}^{\infty} P_{1n} z^n$. Consequently, we can describe the set of statistical equilibrium equations for these probabilities (P_{0n}, P_{1n}) as follows :

$$(\lambda + n\alpha)P_{0n} = \psi\eta P_{1n} + \overline{\psi}\eta P_{1n-1} \tag{1}$$

$$(\lambda\overline{\phi} + \eta + n\phi\alpha)P_{1n} = \overline{\phi}\lambda P_{1n-1} + (n+1)\phi\alpha P_{1n+1} + (n+1)\alpha P_{0n+1} + \lambda P_{0n} \tag{2}$$

Where $P_{1r} = 0$ for r = -1.

IV. Joint Distribution of server state and Queue length

Namely, we investigate the joint distribution of the server state and queue length under the system steady state assumption. Thus, the condition of system stability is assumed to be hold from now on. Further analysis around the stability of retrial queues can be found in [8], where E.Altman and A.A.Borovkov provided the general conditions under which ρ(system's load) < 1 is a sufficient condition for the stability of retrial queuing systems.

Now to continue in deriving the joint distribution, we multiply the equations 1 and 2 by $\sum_{n=0}^{\infty} z^n$, which yields to the following equations :

$$\lambda P_0(z) + \alpha z P_0^{'}(z) = \psi\eta P_1(z) + \overline{\psi}\eta z P_1(z) \tag{3}$$

$$\{\overline{\phi}\lambda(1-z) + \eta\}P_1(z) + \phi\alpha(z-1)P_1^{'}(z) = \alpha P_0^{'}(z) + \lambda P_0(z) \tag{4}$$

Note that the prime indicates the derivative with respect to z. By taking the sum of equation 3 and 4, then divide the sum by $(z-1)$ we obtain

$$(\overline{\phi}\lambda + \overline{\psi}\eta)P_1(z) = \alpha P_0^{'}(z) + \phi\alpha P_1^{'}(z) \tag{5}$$

By substituting equation 5 into 4, we can express $P_0(z)$ in terms of $P_1^{'}(z)$ and $P_1(z)$ as follows:

$$P_0(z) = \frac{\phi\alpha}{\lambda} z P_1^{'}(z) + (\frac{\eta\psi}{\lambda} - \overline{\phi}z)P_1(z) \tag{6}$$

By differentiating equation 6, we get

$$P_0^{'}(z) = \frac{\phi\alpha}{\lambda} z P_1^{''}(z) + (\frac{\phi\alpha + \eta\psi}{\lambda} - \overline{\phi}z)P_1^{'}(z) - \overline{\phi}P_1(z) \tag{7}$$

By Substituting equations 6 and 7 into 3, we obtain a differential equation of $P_1(z)$

$$zP_1^{''}(z) + (\frac{(\lambda+\alpha)\phi + \eta\psi}{\phi\alpha} - \frac{\overline{\phi}\lambda}{\phi\alpha}z)P_1^{'}(z) - \frac{\lambda(\overline{\phi}\lambda + \overline{\phi}\alpha + \overline{\psi}\eta)}{\phi\alpha^2}P_1(z) = 0 \tag{8}$$

Consequently, we transform the equation 8 into Kummer's differential equation, since it has already a solution. Let $Y(x) = P_1(z(x))$ and $z = \frac{\phi\alpha}{\overline{\phi}\lambda}x$ which transforms 8 into

$$xY^{''}(x) + \{\frac{(\lambda+\alpha)\phi + \eta\psi}{\phi\alpha} - x\}Y^{'}(x) - \{\frac{\overline{\phi}\lambda + \overline{\phi}\alpha + \overline{\psi}\eta}{\overline{\psi}\alpha}\}Y(x) = 0 \tag{9}$$

The equation 9 can be rewritten as follows

$$xY^{''}(x) + (d-x)Y^{'}(x) - aY(x) = 0 \tag{10}$$

where $a = \frac{\overline{\phi}(\lambda+\alpha)+\overline{\psi}\eta}{\overline{\phi}\alpha}$ and $d = \frac{(\lambda+\alpha)\phi+\eta\psi}{\phi\alpha}$

Referring to [10], [11], the equation 10 has a regular singular point at $x = 0$, and an irregular singularity at $x = \infty$. Furthermore, the solution of equation 10 is found analytically in a unite circle, $U = \{x : |x| \prec 1\}$ which represents in turn the solution of kummer's function $Y(x)$ and expressed by $Y(x) = m \times F(a; d; x)$, $m \neq 0$, where F(a;d;x) is called a confluent hypergeometric function of the first type and its regular solution is denoted by $F(a; d; x) = \sum_{i=0}^{\infty} \frac{(a)_i}{(d)_i}\frac{x^i}{i!}, \ldots\ldots, d > 0, |x| < \infty$,where $(a)_i$ and $(d)_i$ are called the Pochhammer Symbol [10].

As a result, the equation 8 is solved for $P_1(z)$ as follows

$$P_1(z) = m \times F\{\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}z\}, |z| \preceq 1 \tag{11}$$

Referring to [11], the first derivative of Kummer's function F(a;d;x) is defined as follows : $\frac{dF}{dx} = \frac{a}{d}F(a+1; d+1; x)$, hence $P_1^{'}(z)$ is expressed as follows :

$$P_1^{'}(z) = m\{\frac{\lambda}{\alpha}\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\phi(\lambda+\alpha) + \eta\psi}F\{\frac{\overline{\phi}(\lambda+2\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+2\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}z\}\} \tag{12}$$

To solve equation 6 for $P_0(z)$, we replace $P_1(z)$ and $P_1^{'}(z)$ by their equivalence in equations 11 and 12, and hence $P_0(z)$ is expressed as follows :

$$P_0(z) = \frac{\phi\alpha}{\lambda}zP_1^{'}(z) + \{\frac{\eta\psi}{\lambda} - \overline{\phi}z\}P_1(z) = \frac{\phi\alpha}{\lambda}\{zP_1^{'}(z) + \frac{\lambda\phi + \eta\psi}{\phi\alpha}P_1(z)\} - (\phi + \overline{\phi}z)P_1(z) \tag{13}$$

Where :

$$zP_1^{'}(z) + \frac{\lambda\phi + \eta\psi}{\phi\alpha}P_1(z) = m.\sum_{i=0}^{\infty}(i + \frac{\lambda\phi + \eta\psi}{\phi\lambda})\frac{(\frac{\overline{\phi}(\lambda+\alpha)+\overline{\psi}\eta}{\overline{\phi}\alpha})_i}{(\frac{\phi(\lambda+\alpha)+\eta\psi}{\phi\alpha})_i}\frac{(\frac{\overline{\phi}\lambda}{\phi\lambda}z)^i}{i!} \tag{14}$$

and

$$i + \frac{\lambda\phi + \eta\psi}{\phi\lambda} = \frac{\lambda\phi + \eta\psi}{\phi\lambda}\frac{(\frac{\lambda\phi+\eta\psi}{\phi\lambda} + 1)_i}{(\frac{\lambda\phi+\eta\psi}{\phi\lambda})_i} \tag{15}$$

Consequently, $P_0(z)$ in equation 13 is reexpressed as follows :

$$P_0(z) = m\{\frac{\lambda\phi + \eta\psi}{\lambda}F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\lambda\phi + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}z) - (\phi + \overline{\phi}z)F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}z)\} \tag{16}$$

For all $|z| \leq 1$. Then at the boundary condition where $z = 1$ we can ge the value of m through $P_0(1) + P_1(1) = 1$. Thus $m = \{\frac{\eta\psi+\lambda\phi}{\lambda}F\{\frac{\overline{\phi}(\lambda+\alpha)+\overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\lambda\phi+\eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}\}\}^{-1}$

Theorem 1: For an $M/M/1/1$ retrial queue with impatient customers (i.e., customers abandon) and feedback in the steady state, the generating functions of the joint distribution of server state C and queue length N are given by

$$\begin{aligned} P_0(z) = E(z^N : C = 0) = m\{ & \frac{\lambda\phi + \eta\psi}{\lambda} \\ & F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\lambda\phi + \eta\psi}{\phi\alpha} \\ & ; \frac{\overline{\phi}\lambda}{\phi\alpha} z) - (\phi + \overline{\phi} z) \\ & F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha} z)\} \end{aligned} \tag{17}$$

$$\begin{aligned} P_1(z) = E(z^N : C = 1) = m.F\{ & \frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha} \\ & ; \frac{\phi(\lambda+\alpha) + (1-\overline{\psi})}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha} z\}\} \end{aligned} \tag{18}$$

For all $|z| \leq 1$.

Consequently, the average of the queue length along the idle period of the server is equivalent to $P_0'(1)$, which is expressed by

$$\begin{aligned} E(N : C = 0) = m\{ & \frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\alpha} \\ & F(\frac{\overline{\phi}(\lambda+2\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\lambda}) - \overline{\phi} \\ & F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}) - \\ & \frac{\overline{\phi}\lambda(\lambda+\alpha) + \overline{\psi}\lambda\eta}{\alpha(\phi(\lambda+\alpha) + \eta\psi)} F(\frac{\overline{\phi}(\lambda+2\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha} \\ & ; \frac{\phi(\lambda+2\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha})\} \end{aligned} \tag{19}$$

And the average of the queue length along the busy period of the server is equivalent to $P_1'(1)$, which is expressed by

$$\begin{aligned} E(N : C = 1) = m & \frac{\overline{\phi}\lambda(\lambda+\alpha) + \overline{\psi}\lambda\eta}{\phi\alpha(\lambda+\alpha) + \eta\psi\alpha} \\ & F(\frac{\overline{\phi}(\lambda+2\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+2\alpha) + \eta\psi}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}) \end{aligned} \tag{20}$$

Thus the average of the queue length in the retrial queuing system shown in Fig. 1 is the sum of $P_0'(1)$ and $P_1'(1)$, which is given by

$$\begin{aligned} E(N; C = 0) + E(N; = 1) = m\{ & \frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\alpha} \\ & F(\frac{\overline{\phi}(\lambda+2\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + (1-\overline{\psi})\eta}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha}) - \overline{\phi} \\ & F(\frac{\overline{\phi}(\lambda+\alpha) + \overline{\psi}\eta}{\overline{\phi}\alpha}; \frac{\phi(\lambda+\alpha) + (1-\overline{\psi})\eta}{\phi\alpha}; \frac{\overline{\phi}\lambda}{\phi\alpha})\} \end{aligned} \tag{21}$$

V. Numerical Results

Quality of service to customers in queuing systems with retrials or repeated attempts is characterized by several performance measures. The average waiting time W in the steady state is often considered to be the most important of these performance measures. However, W is an average over all primary calls, including those calls which receive immediate service and really do not wait at all. A better grasp of understanding the waiting time process can be obtained by studying first the relationship between the retrial queue length $E(N)$ and other inputs,outputs and feedback parameters, which we are doing in this paper, second by observing the blocking probability B (the probability that the waiting time is positive) and the conditional mean waiting time $W_B = \frac{W}{B}$ given that the waiting time is positive, which we will do in a future work.

We have conducted some preliminary analysis through some simulations done on the formula described in 21, in order to show the impact of the different parameters indicated in Fig. 1 and its relationship with the retrial queue length $E(N)$. The primary objective behind this was to understand what does happen at telephone switches or in others telecommunication systems where redials or connection retrials arise naturally. This gives us deep insight in the design of optimal telephone call centers including the retrial and main buffer sizes at telephone switches, which may result in solving the problem of extra delay that some connections suffered from. Or even to reduce the percentage of packets loss due to abandonment at the different telephone switching nodes.

These analysis involved three scenarios in order to clarify the relations in different situations between the inputs,output and feedback parameters shown in Fig. 1. In the first scenario, the server service rate η takes these values : $[0, 0.1, 0.2, ..., 1]$ where we evaluate $E(N)$ at different values of service completion probability ψ, while $\alpha = 0.5$, $\phi = 0.7$ and $\lambda = 0.5$. In the second scenario, the service completion probability ψ takes these values : $[0, 0.1, 0.2, ..., 1]$ where we evaluate

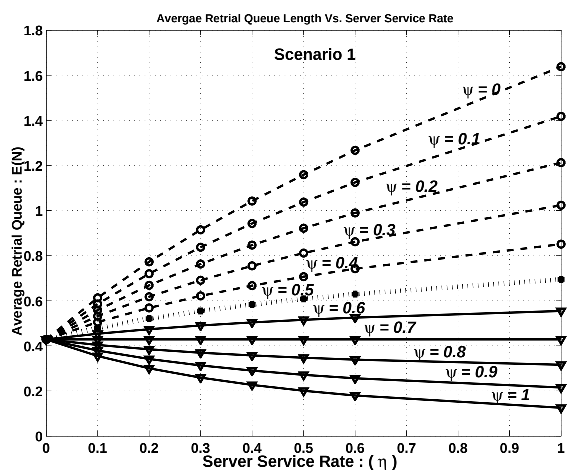

Fig. 2. Average Retrial Queue Length Vs. Service Server rate “η”.

$E(N)$ at different values of abandonment probability ϕ, while $\alpha = 0.5$,$\eta = 0.6$ and $\lambda = 0.5$. In the third scenario, the non-abandonment probability $\overline{\phi} = 1 - \phi$ takes these values : $[0, 0.1, 0.2, ..., 0.9]$ where we evaluate $E(N)$ at different values of retrial probability α, while $\psi = 0.5$, $\eta = 0.8$ and $\lambda = 0.7$. These scenarios were realized through simulations done on a program that computes the confluent hypergeometric function $F\{a;d;x\}$ which we have developed under Matlab program.

As a result, Fig.2 shows that along the increase of η and ψ the retrial queue length decreases. Obviously, this refers to the feedback probability $\overline{\psi}$ which becomes smaller, and the possibility of accepting repeated and primary calls becomes large. This figure shows us also that when ψ becomes greater than $0.5-0.6$ or the feedback probability becomes less than $0.5-0.6$, then $E(N)$ is not affected remarkably or it decreases very slowly.

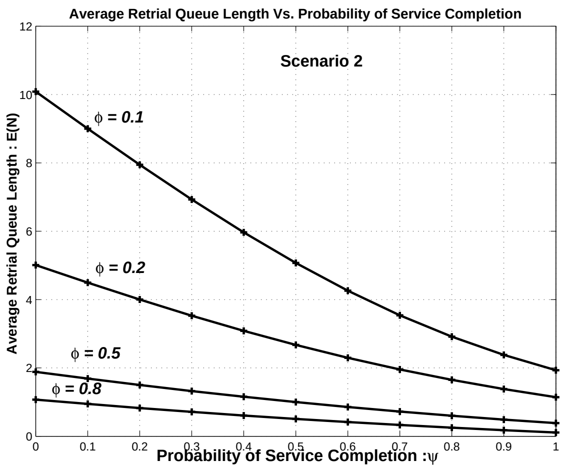

Fig. 3. Average Retrial Queuing System Vs. Probability of Service Completion "ψ".

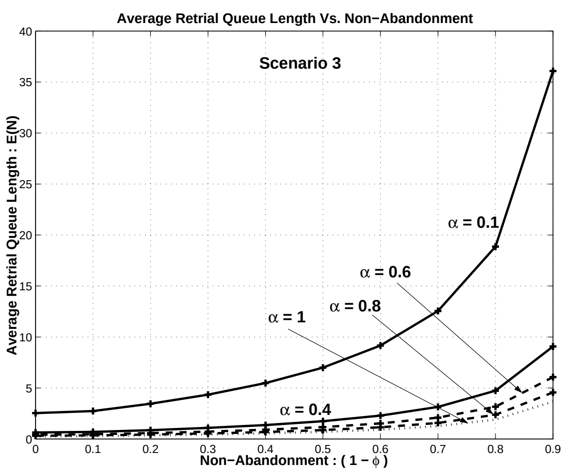

Fig. 4. Average Retrial Queuing System Vs. Non-Abandonment rate "$\overline{\phi}$".

Furthermore, Fig. 3 shows that $E(N)$ for $M/M/1/1$ with abandonment and feedback is not affected by feedback probability $\overline{\psi}$ when the probability $\overline{\phi}$ of non-abandonment or returning to retrial group after customer attempt's failure becomes less than 0.5. However, $E(N)$ increases rapidly as $\overline{\psi}$ and $\overline{\phi}$ become high.

Fig. 4 shows that along the design of retrial queuing system, we have to assign equivalent values for the non-abandonment probability "$\overline{\phi}$" and the retrial probability "α" in order to keep the retrial queue length as short as possible. This can be concluded from the figure since when α takes values greater than 0.5 and "$\overline{\phi}$" gets values less than 0.5 $E(N)$ becomes small.

VI. Conclusion

In this paper, we derived a formula through which we could compute and evaluate in terms of other parameters the retrial queue length of $M/M/1/1$ queuing system shown in Fig. 1. Afterward, we have shown some numerical results through a Matlab program that we have developed for evaluation $F(a;d;x)$, and consequently we could evaluate the average of retrial queue length $E(N)$. These results are promising since they give a correct impression on the process flow that connects all the parameters in formula 21 and $E(N)$. This impression leaves one with sufficient confidence to understand much better the relationship between the retrial queue length,server's state,inputs,outputs and feedback parameters in Fig. 1, which will help us in the design of different network elements such as retrial queue size and number of servers in a retrial queuing system similar to the one shown in Fig. 1.

Consequently, there are two points that require further studies:

- Computation the waiting time in the retrial queuing system shown in Fig. 1.
- Studying and analyzing the systems in Fig. 1 within Jackson networks context, where we see the reference [1] is very important for this point.

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