A strengthening of the Murty-Simon Conjecture for diameter 2 critical graphs

Antoine Dailly — Univ Lyon, Université Lyon 1, LIRIS UMR CNRS 5205, F-69621, Lyon, France.
Florent Foucaud — LIMOS, Université Clermont Auvergne, Aubière, France
Adriana Hansberg — Instituto de Matemáticas, UNAM, Mexico

Abstract

A *diameter 2 edge-critical graph*, noted D2C graph, is a graph of diameter 2 and such that the deletion of any edge increases the diameter. The Murty-Simon Conjecture states that all D2C graphs of order n have at most $\lfloor \frac{n^2}{4} \rfloor$ edges and that this bound is only reached by the balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. The conjecture has been proved for several families (triangle-free [8], high maximum degree [6]...) and when the order is either small [2] ($n \leq 24, n = 26$) or large [3] (n greater than a tower of powers of 2 of size 10^{14}).

In 2015, a smaller bound of $\lfloor (n-1)^2/4 \rfloor + 1$ was proved by Balbuena *et al.* [1] for non-bipartite triangle-free D2C graphs, the extremal family being certain inflations of C_5. This, along with several observations, opens the question of strengthening the Murty-Simon Conjecture. We propose the following strengthening: for every positive integer c, there exists an integer n_0 such that every non-bipartite D2C graph of order $n \geq n_0$ has less than $\lfloor \frac{n^2}{4} \rfloor - c$ edges. We prove this strengthened conjecture for $c = 1$ on D2C graphs with a dominating edge.

1 Introduction

A graph is *diameter d edge-critical*, denoted DdC, if it has diameter d and the deletion of any edge increases the diameter. In 1975, Plesník [9] studied D2C graphs and found that all the known examples had at most $\lfloor \frac{n^2}{4} \rfloor$ edges. D2C graphs include several well-known graphs, as shown in Figure 1. Murty and Simon independently made the following conjecture (according to Füredi [3], Erdős said that this was also formulated by Ore in the 1960s):

Conjecture 1 (Murty-Simon Conjecture)**.** *Let G be a D2C graph of order n with m edges. We have $m \leq \lfloor \frac{n^2}{4} \rfloor$, with equality if and only if $G = K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.*

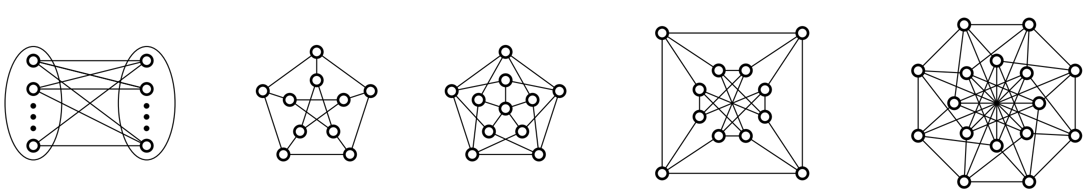

Figure 1: Examples of well-known graphs that are D2C: a complete bipartite graph, the Petersen Graph, the Grötzsch Graph, the Chvátal Graph and the Clebsch Graph.

As an example, the validity of Conjecture 1 is proved for triangle-free graphs by Mantel's Theorem [8]. A history and a summary of progress on the topic of the Murty-Simon Conjecture, as well as its link with total domination can be found in [7]. In particular, in 1992 Füredi proved the conjecture for D2C graphs with order more than a tower of powers of 2 of size 10^{14} [3]. In his article, Füredi states, still in terms of very large order, that the bound of Conjecture 1 can be improved to $\lfloor (n-1)^2/4 \rfloor + 1$ by excluding complete bipartite graphs and that the extremal graphs for the improved bound are those constructed by removing an edge xy from a complete bipartite graph before adding a vertex z and the edges xz and yz. In 2015 [1], the same improved bound was confirmed for non-bipartite triangle-free D2C graphs of any order and it was shown that the extremal graphs for this bound consist of a certain family of inflated C_5's (an inflation of C_5 consists of five nonempty independent vertex sets $X_0, \ldots, X_4$ such that all the possible edges between X_i and $X_{i+1 \bmod 5}$ exist), that is, to a much wider family than the one described by Füredi (proving that his statement was not correct). With this in mind, the authors of [1] conjectured that their result can be extended to the family of non-bipartite D2C graphs with no dominating edge. Furthermore, computer searches show that, up to order 11, there is only one small D2C graph with a dominating edge, called H_5 in [7] (depicted in Figure 2), for which the better bound of $\lfloor (n-1)^2/4 \rfloor + 1$ is wrong. This, together with the conjecture stated in [1], leads us to propose the following strengthening of the Murty-Simon Conjecture:

Conjecture 2. *Let G be a non-bipartite D2C graph of order n with m edges. If $G \neq H_5$, then we have $m \leq \lfloor (n-1)^2/4 \rfloor + 1$.*

As mentioned above, Conjecture 2 holds for triangle-free D2C graphs [1]. However, reaching this stronger bound may be hard to study in the general case, so we propose a weaker conjecture:

Conjecture 3. *For every positive integer c, there exists an integer n_0 such that for every non-bipartite D2C graph of order $n \geq n_0$ with m edges, we have $m < \lfloor \frac{n^2}{4} \rfloor - c$.*

In this paper, we prove Conjecture 3 for $c = 1$, with $n_0 = 7$, for D2C graphs with a dominating edge, a class of graphs for which the problem was studied in [4, 5, 10] (examples of D2C graph with a dominating edge are depicted in Figure 2):

Theorem 4. *Let G be a non-bipartite D2C graph of order n, with a dominating edge and m edges. If $G \neq H_5$, then we have $m < \lfloor \frac{n^2}{4} \rfloor - 1$.*

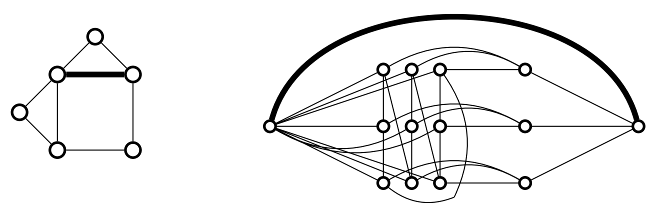

Figure 2: Two D2C graphs with a dominating edge (the dominating edge is bolded). On the left hand side is H_5.

2 Proof of Theorem 4

The validity of the Murty-Simon Conjecture for D2C graphs with a dominating edge was proved in several papers: first, the bound was proved in 2003 [4], then the fact that the only graphs reaching the bound were complete bipartite was proved in [5, 10].

We improve the method used in [4], which consists in extracting several strong properties of D2C graphs with a dominating edge before partitioning them into two parts, and proving that one can associate a unique non-edge between the parts to each edge inside a part. This proves that the graph has at most as many edges as a complete bipartite graph. Our method also allows for a shorter proof of the validity of Conjecture 1 for this family.

Our proof is based on the following definition:

Definition 5. *Let $G(V, E)$ be a D2C graph. An edge $e \in E$ is* critical *for a pair of vertices $\{x, y\} \in V^2$ if and only if either $e = xy$ and $N(x) \cap N(y) = \emptyset$ or $xy \notin E$, $N(x) \cap N(y) = \{z\}$ and $e \in \{xz, yz\}$.*

We notice that every edge of a D2C graph is critical for some pair of vertices. Also, since a D2C graph has diameter 2, an edge xy can only be critical for a pair $\{x, z\}$ with $z \in N[y]$ or $\{y, z\}$ with $z \in N[x]$. Now we give the outline of the proof:

Outline of the proof. Let $G(V, E)$ be a non-bipartite D2C graph with a dominating edge uv. We partition the vertices of $V \setminus \{u, v\}$ into four sets:

1. $P(uv)$ is the set of all vertices x such that uv is critical for either $\{u, x\}$ or $\{v, x\}$;

2. S_{uv} is the set of common neighbours of u and v;

3. S_u and S_v are the sets of the remaining neighbours of u and v, respectively.

We partition the graph into two sets X and Y: $X = \{v\} \cup S_u \cup P(uv) \cup S_{uv}$ and $Y = \{u\} \cup S_v$. Using some properties of those four sets allows us to define a function f that assigns every edge in $E(X) \cup E(Y)$ to a unique non-edge in $E(X \times Y)$: for every edge $xy \in E(X)$, we select a pair of nonadjacent vertices $\{y, z\}$ with $z \in Y$ for which the edge xy is critical and set $f(xy) = \overline{yz}$, where $\overline{yz}$ denotes the non-edge between y and z. Note that xy cannot be critical for a pair of vertices in X since all vertices in X have u as a common neighbour. The function f is defined in the same way for all edges in $E(Y)$. This is depicted in Figure 3.

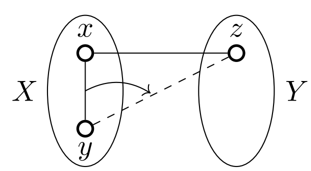

Figure 3: The definition of the function f.

We then prove that the function f is injective. From now on, we will refer to the non-edges in $E(X) \times E(Y)$ not having a preimage by f as the *f-free non-edges* and denote $\text{free}(f)$ be the number of f-free non-edges. We notice the following:

Claim 5.1. *G has exactly $\frac{n^2 - ||X| - |Y||^2}{4} - \text{free}(f)$ edges.*

The next step is to assume by contradiction that G, which is neither bipartite nor H_5, has at least $\frac{n^2}{4} - 1$ edges. This implies, together with Claim 5.1, that $\text{free}(f) \leq 1$.

First, we prove that $P(uv)$ and S_{uv} are empty. Note that by definition of the partition, this implies that the only edges within X and Y are in S_u and S_v.

We then define a partial orientation $\overrightarrow{G}$ on the edges of G. In particular, we orient all the edges within X and Y with respect to their assignment by f: let xy be an edge such that $f(xy) = \overline{yz}$, then we orient xy from x to y. This allows us to prove several very important properties:

Claim 5.2. *$\overrightarrow{G}$ has no directed cycle.*

Claim 5.3. *Let s be a source of $\overrightarrow{G}$. There is at least one f-free non-edge incident with a vertex in $N^+[s]$.*

Claim 5.4. *Let t be a sink of $\overrightarrow{G}$. There is at least one f-free non-edge incident with a vertex in $N^-[t]$.*

Since X and Y are acyclic, each nontrivial component within them has at least one source and one sink. Furthermore, if there are two nontrivial components then there is a contradiction with the fact that $\text{free}(f) \leq 1$. So there is exactly one nontrivial component in both X and Y, and the only f-free non-edge is incident with some vertex in the in-neighbourhood of every sink and with some vertex in the out-neighbourhood of every source. Without loss of generality, let r be the vertex in X such that for every source $s \in X$ and every sink $t \in X$, we have $t \in N^+[r], s \in N^-[r]$. In particular, if r is a source then it is the only source, and if r is a sink then it is the only sink. Furthermore, the following claims hold:

Claim 5.5. *Either r is a sink, or r is the only in-neighbour of all sinks and r only has sinks as out-neighbours.*

Claim 5.6. *There is exactly one source.*

Those last two properties allow us to find a contradiction to the fact that $\text{free}(f) \leq 1$, and thus to prove Theorem 4. $\square$

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