Minimal viscosity and the transport in cuprate strange metals.

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(Dated: September 24, 2017)

A note explaining some basics (mostly dimensional analysis) associated with the possibility that the famous linear resistivity of the cuprate strange metal is rooted in the counterintuitive properties of densely entangled many body systems.

PACS numbers: 71.27.+a, 74.72.-h, 75.50.Ee

I. WHAT IS AT STAKE?

This is intended to be hopefully a very easy to digest explanation of some expectations as related to the very foundations of the physics of strange metals, especially of the optimally doped cuprate kind. At stake is not so much to explain why Tc is high. This appears as a rather commercial 1980's affair. In 2017 we are in the grip of the realisation inspired by the coming of the quantum computer that we have only seen the tip of the iceberg when it comes to the understanding of reality in general and the nature of matter particular. In the quantum information language the stuff that is dealt with in the condensed matter and high energy text books is called "short ranged entangled product state vacuum". At long distances the superior fashion of quantum physics to process information just disappears and the macroscopic behaviours can be described by the theories of *classical* physics – this includes the Fermi liquid and the BCS superconductor (product states in momentum space).

Let is simplify the microscopic physics by declaring that at the UV cut-off everything is made from quantum bits. In a macroscopic volume one finds $N \simeq 10^{23}$ of such bits and the Hilbert space is spanned up by 2^N configurations of these bits – a classical computer gives up when $N \simeq 50$. We define now many-body entangled matter as being characterized by a ground state ("vacuum") where all these 2^N states are in an *irreducible* coherent superposition: it is impossible to find a representation where it reduces to a short range entangled product state. What do we know about such matter? Especially when it is compressible (no energy gap, unlike the topological systems) the answer is: virtually nothing! This stuff cannot be computed using available computers and by the same token the methods of established quantum field theory (or "many body physics", "condensed matter theory", it is all the same) fall short as well. There is just one circumstance where we have a reasonable "phenomenological" understanding: the strongly interacting "bosonic" quantum critical states [1]. Bosonic means that these map onto an equivalent classical strongly interacting critical state of the Wilson-Fisher variety in Euclidean spacetime. The "critical slowing down" NP-hardness of the classical critical state turns into quantum complexity after Wick rotation.

The new mathematical kid in the block is the AdS/CFT correspondence or "holographic duality". The evidence has been accumulating rapidly over the last few years that it accomplishes the remarkable feat of translating the physics of densely many-body entangled matter onto a gravitational problem living in a spacetime with an extra dimension. To explain this is way beyond the ambition of this short note: to get a sense have a look at Susskind's recent ref. [2] or either the last chapter in the book [3].

By just asking AdS/CFT its opinions about matter, besides reproducing all of the content of conventional theory books it also produced quite a number of surprises [3] in the form of answers that are suggestively similar to the mysteries [4] of the cuprates. At stake is whether AdS/CFT reveals highly *generic* "physical principle" ruling the quantum matter or whether it is tied to the rather strange assumptions regarding the microscopic physics (large N limit of matrix field theory) that have to be made in order to be able to compute. These are in turn deeply rooted in the mysteries of quantum gravity itself ...

From this perspective, the grand outlook of high Tc research changes from "let's make money by inventing a room temperature superconductor" to "let's embarrass the LHC and the Webb space telescope by fooling around in small condensed matter laboratories to shed light on the real big questions of fundamental physics". One would like to nail down by condensed experimentation whether it is indeed the case that especially the strange metals are of the highly entangled kind and whether it is in one or the other way the case that one can learn how these work from black hole quantum physics.

A multi-pronged attack is materialising at the moment – for instance, very recently holography started shedding a very interesting new light on the origin of the highly confusing "intertwined order" [5]. However, here we will focus on the best established part of this agenda: equilibration and transport in the finite temperature strange metals.

Dealing with densely entangled matter one has to unlearn the canonical way of thinking in terms of classical relaxation processes. Take the Fermi liquid; because the vacuum is unentangled the excitations are (quasi-) particles that behave at finite temperatures just as the "cannon balls" of Boltzmannian kinetic gas theory. These fly around, colliding against impurities and against each other and this turns into the transport theory of the con-

ventional text books. In the Galilean continuum the answer is that they will form a hydrodynamical fluid, where the dissipative properties are captured by the viscosity η. As famously checked in ^{3}He, the order of magnitude of the viscosity of a finite temperature Fermi-liquid can be estimated by a simple dimensional analysis,

$$\eta = f \times \tau_P \tag{1}$$

In a Fermi liquid the free energy density is set by $f = nE_F$ (n is the density and E_F the Fermi-energy) while the momentum relaxation time τ_P associated with the momentum diffusion in the continuum is set by the single particle collision time $\tau_P = \tau_c = \hbar E_F/(k_B T)^2$ with the result that $\eta \simeq \hbar n E_F^2/(k_B T)^2$. When the "mean free time" associated with the scattering against impurities $\tau_{mf} = l/v_F$ (l is the mean free path, v_F Fermi velocity) becomes smaller than τ_c the hydrodynamics is destroyed because momentum conservation is lost already on the microscopic (single particle) scale. Bottom line: to find hydrodynamical flows in "product state" matter one has to make it extremely clean: the graphene story.

This is supposed to work in a quite different way in densely many body entangled metals. When everything is entangled with everything there are no particles in the spectrum. Under these circumstances it is just impossible to capture the quantum dynamics in terms of equivalent semi-classical stories like the colliding quasi particles. Instead one has pay tribute to the notions of "eigenstate"-(or "quantum-") thermalisation. This boils down to the very simple idea that the true time evolution takes place in the enormous many body Hilbert space. We have only access to expectation values of local operators (like the "read out" of the quantum computer) only revealing a logarithmically small amount of all the quantum information that is subjected to the unitary time evolution. The result is that we are in no time overwhelmed by the many body quantum info flow which instead appears to us as just randomness, heat, entropy. For a lively illustration of how this hangs together with Boltzmann's cannon balls see ref. [6].

Dealing with densely entangled matter the quantum thermalisation takes over completely and it is no longer possible to find a classical analogy. The case in point is the now well known "Planckian dissipation"[7]. It amounts to the statement that the relaxation time of basically anything that can relax in compressible many body entangled systems is generically of order of

$$\tau_\hbar = \frac{\hbar}{k_B T} \tag{2}$$

First identified in bosonic quantum critical systems [1], evidences has been accumulating that it has the status of general principle: it seems to work always. This now takes the role of the collision time in the particle physics systems. Notice that it is remarkably agnostic, way more simple than the classical stuff: know the temperature and $\hbar$ and you're done.

This in turn underpins also the viscosity, which turns into the "minimal viscosity" first discovered in AdS/CFT. But this can be reproduced easily by dimensional analysis. There are good reasons to argue that *any* compressible highly entangled state of matter can be viewed as a generalisation of the bosonic quantum critical systems. For such a quantum critical system there is no internal energy and the free energy density $f = s\ k_B T$ where s is the entropy density. The relaxation time is just "Planckian" , $\tau_P \simeq \tau_\hbar = \hbar/(k_B T)$. It follows that $\eta = f \times \tau \simeq \hbar \times s$. Thereby $\eta/s = A\hbar$ and according to AdS/CFT the "universal amplitude" A which is pending the particulars of the quantum field theory is equal to $1/(4\pi)$, believed to be a lower bound.

This minimal viscosity is claimed to be at work both in the quark-gluon plasma produced at the heavy ion colliders, as well as the unitary Fermi gas of the cold atom people. At stake is the following basic question: can it be proven by condensed matter experiments that the "minimal viscosity principle" is at work in the cuprates as well, being the key to the anomalous transport properties in the strange metal?

Finally, as a caveat for the advanced reader: notice that this particularly dimensional analysis gets hairy in the finite density system. Given the notion of hyperscaling violation the free energy density should know about $\mu n >> k_B T$ implying that $\eta \sim 1/T$ (instead of the $1/T^2$ of a Fermi liquid). A much more fanciful way to arrive at a similar conclusion is found in ref. [8]. Nevertheless, holography is insisting that η/s is a universal constant also in the finite density systems. The viscosity is dual to the absorption cross section of gravitons by the black hole horizon, and the horizons associated with finite density are in this regard as good as any other horizon. This is presently not at all understood.

II. THE LINEAR RESISTIVITY AND THE PLANCKIAN FLUID.

We suggested a little while ago an explanation of the linear resistivity as observed in the cuprate strange metals which is in a way the simplest possible construction imaginable given elementary holographic wisdoms [9]. A large part of this particular paper is dedicated to the cause of demonstrating that it indeed works using some "big machine" holographic construction. However, the essence is extremely simple and minimal.

It rests on the following set of assumptions: Assumption 1: strange metals are subjected to extremely rapid "hydrodynamization". There are no particles colliding and so forth; this is densely entangled "unparticle matter" giving in to an emergent, ubiquitous quantum critical behaviour which outruns any form of semiclassical thermalisation presumably because we see eigenstate thermalisation at work in an extreme form (see ref. [10]

for a particular extreme case). Even when the system is quite dirty, and one would encounter a "Knudsen flow" type of transport when it would be a Fermi liquid, believing holography such a system could still establish a near-hydrodynamical transport regime. This is actually the most hairy assumption, we really don't know whether this happens in CM systems.

Assumption 2: The "minimal viscosity" principle of holography that became famous due to its apparent success in the quark gluon plasma (heavy ion collisions) and the unitary fermi gas (cold atoms) is operative as well in the cuprate electron fluid.

Assumption 3: The strange metal is "local quantum critical", in the jargon the dynamical critical exponent $z = \infty$. This is required to assure that the disorder potential is not running under renormalisation. This is a no brainer: anything spatial, including disorder, is marginal (not dependent on scale) in such a system. This we need in order to assure that the disorder strength is temperature independent.

The story is now simple. We know from direct measurements that the optical conductivity follows quite precisely a Drude behaviour at least at temperatures below room temperature and at low energy. The DC resistivity is therefore set by

$$\rho_{DC} = 1/(\omega_p^2 \tau_K) \tag{3}$$

where ω_p is the plasma frequency and τ_K is the long wavelength momentum relaxation rate. The latter is directly measured by optical conductivity and there is a general agreement (van der Marel, Tanner, Bazov, ...) that

$$\frac{1}{\tau_K} = B\frac{k_B T}{\hbar} \tag{4}$$

the original "Planckian dissipation" [7], where the experimentalists agree on $B \simeq 2$ at optimal doping, regardless the particulars of the cuprates.

Assuming condition 1 (hydro is established), Stokes himself already figured out that

$$\frac{1}{\tau_K} = \frac{D}{l^2} \tag{5}$$

where D is the momentum diffusivity of the fluid while l is the length where the breaking of Galilean invariance becomes manifest to the fluid. Although there are no particles that have a mean free path, let's call it nevertheless the (elastic) mean free path for reasons of semantic convenience. We now need assumption 3, the local quantum criticality. l encodes for the strength of the disorder potential and in a quantum critical liquid this will in general be a running coupling in the sense of the renormalisation group. The consequence would be that l itself will become temperature dependent. However, in a local quantum critical liquid it follows from elementary scaling considerations that any spatial property will not renormalise; in the jargon the disorder strength is marginal and will therefore be scale-, and thereby temperature independent. All the temperature dependence has to come from the diffusivity.

The diffusivity is in turn proportional to the viscosity. In order to digest the holographic papers, in relativistic (zero-rest mass) hydrodynamics, $D = \eta/(\varepsilon + p)$ where η is the (shear) viscosity and ε and p the energy density and pressure, respectively. In our electron system energy-stress is completely dominated by the electron rest mass $\varepsilon + p = nm_e$, where n and m_e are the electron density and mass, respectively. Summarizing, it directly follows from the "near-hydro" postulate that,

$$\frac{1}{\tau_K} = \frac{\eta}{l^2 n m_e} \tag{6}$$

This is in fact a standard estimate used all the time by engineers to construct fuel pipes, etcetera. But now holography kicks in through the "minimal viscosity" hypothesis, assumption 2:

$$\frac{\eta}{s} = A\,\hbar \tag{7}$$

where $s = nS$ is the *entropy density* and A a constant that is pending the precise nature of the strange metallic quantum critical state. In case of the large N strong 't Hooft coupling Yang Mills theories of holography $A = 1/(4\pi)$; this is believed to be a lower bound and although matters are not that well understood it could well be that the A in cuprate strange metals can be quite a bit larger. Plugging Eq. (7) in Eq. (6) yields *the result*:

$$\frac{1}{\tau_K} = \frac{A}{l^2 m_e}\hbar\, S \tag{8}$$

demonstrating that *the resistivity has to be proportional to the entropy* S. Notice that the density factors n cancel. Given $z \to \infty$ the temperature dependence of the resistivity should come *exclusively* from the entropy. The doping dependence is a bit more tricky since l can vary as well as function of doping.

Now it comes. There is one other interesting linear-in-temperature property measured in the cuprates: the entropy! It was already established in the 1980's by Loram and Tallon that the specific heat is Sommerfeld in the strange metal regime, implying,

$$S = k_B \frac{T}{\mu} \tag{9}$$

where $\mu \simeq 1$ eV according to the data. Surely, the community jumped immediately to the conclusion that this is so because the strange metal is a Fermi-liquid with

a Fermi-energy $E_F = \mu$. Much later we were enlightened by holography where it was demonstrated [3] that in so-called EMD gravities ("Einstein-Maxwell-Dilaton", the dilaton rolls out of string theory) one finds generically "hyperscaling violating" scaling geometries characterized by ""operators that transform co-variantly under scale transformations" (instead of being just invariant). The bottom-line is that it is quite natural to find holographic strange metals having a Sommerfeld specific heat, including the most primitive "Reissner-Nordstrom strange metal" but also the "conformal to AdS_2" fanciful affair considered in [9], both being local quantum critical.

It follows immediately that the resistivity is linear in temperature, since the resistivity is proportional to the entropy which is in turn linear in temperature while there is no other source of temperature dependence!

Give it a minute of thought and you will realise that it actually resolves two of the biggest mysteries associated with the linear resistivity: (a) How can it be that the resistivity stays strictly linear, from low temperatures where the strange metals are very good metals up to high temperatures where it turns into a very bad metal? The good-to-bad metal affair is just defined in the usual way observing that the Ioffe-Regel minimal metallic conductivity is hit at around 400K. According to Eq.'s (8,9) the explanation is very easy. The Sommerfeld entropy is very small at low temperatures, while it becomes large at high temperatures with the effect that the liquid becomes quite viscous causing a bad metal resistivity. (b) More spectacularly, at zero temperature the entropy vanishes and thereby the viscosity becomes zero and the liquid turns into a *perfect fluid*. A perfect fluid is literally unstoppable: regardless whether Galilean invariance is broken or not, the fluid will continue to flow forever. This means that at zero temperature the resistivity has to vanish: *there is no residual resistivity.* Very recently this acquired new impetus by the observation that even in high quality 214 at optimal doping the residual resistivity appears to vanish.

It is a matter of principle: since this zero temperature fluid is *not* a superfluid, it has to be a perfect fluid! The only explanation for this to happen that makes sense is by asserting that the viscosity is proportional to entropy. This is at least my personal reason to take this hypothesis quite seriously.

Let us now zoom in on the numbers. Combining Eq's (8,9),

$$\frac{1}{\tau_K} = \frac{A\hbar}{l^2 m_e}\frac{k_B T}{\mu} \tag{10}$$

observe that this can be written as,

$$\frac{1}{\tau_K} = A\frac{l_\mu^2}{l^2}\frac{1}{\tau_\hbar} \tag{11}$$

We recognise the familiar "Planckian dissipation time",

$$\tau_\hbar = \frac{\hbar}{k_B T} \tag{12}$$

but there is also a characteristic "chemical potential length",

$$l_\mu = \frac{\hbar}{\sqrt{m_e \mu}} \tag{13}$$

balancing the disorder length scale l. This is quite intriguing; notice that this scale originates in the assumptions that the entropy is Sommerfeld while the electrons have a rest mass in combination with the minimal viscosity. It is suggestive to interpret it as the non-relativistic incarnation of the length scale $\sim c/\mu$ (c is velocity of light) characteristic of the holographic local quantum critical metals.

Although we are dealing here with a faithful non-Fermi liquid, lacking quasiparticles and Fermi surfaces, it reproduces a hallmark of Fermi liquid dimensional analysis. According to Eq. (13) $\mu = (\hbar^2/m_e)(1/l_\mu^2)$; identifying this with a Fermi liquid $\mu \to E_F$ and $1/l_\mu \to k_F$. The combination $l/l_\mu \to k_F l$ is of course very familiar but notice that it enters the resistivity like $\rho_{DC} \sim 1/(k_F l)^2 1/\tau_\hbar$, very differently from the Fermi liquid $\sim 1/(k_F l)$. It is not entirely clear what the meaning is of this dimensionless combination; so much seems clear that $l/l_\mu > 1$ since otherwise the disorder potential becomes of order of the chemical potential and one can no longer rely on long wavelength universality ("UV independence").

We are now in the position to quantify this whole affair: the only number we don't know is the mean free path l that we will now estimate. We know the "chemical potential" μ directly from the specific heat measurements: $\mu \simeq 1$ eV. It follows immediately that $l_\mu = \hbar/\sqrt{\mu m_e} = 1.7\ 10^{-9}$ m. According to the optical measurements $1/\tau_K = B/\tau_\hbar$ where $B \simeq 2$ and it follows that

$$l = \sqrt{\frac{A}{B}} l_\mu \tag{14}$$

The main uncertainty is what to take for A: all we seem to know for sure that it has to be "order one" while it is bounded from below by $1/(4\pi)$. Hence, $\sqrt{A/B} > \sqrt{1/(8\pi)} \simeq 0.2$ implying that $l > 0.4$ nanometer; assuming $A \simeq 1$ one finds $l_\mu \simeq 1.2$ nanometer. These estimates are a bit too close to the "UV cut-off" for the peace of mind and perhaps it should be taken with a grain of salt given the parametric factors that are in all likelihood not captured by this dimensional analysis. Whatever, the big deal is that by order of magnitude this estimate for l makes much sense. At the moment that the highly entangled strange metal stuff starts to "fall apart" in more classical forms of matter like the intertwined order in the pseudo gap its presence should become manifest. The

disorder scale associated with the quantum mayonnaise, stripy things and so forth appears to be in this nanometer range.

III. WHAT ARE THE SPATIAL SCALES?

In the unentangled particle physics system it is obvious that the length scale associated with the breakdown of hydrodynamics is obviously the microscopic length l itself. But not so in the strange metal. Let's think again hydrodynamics. When there is hydro there is sound and we have actually direct evidence for the existence of sound in the cuprates. By using EELS one can measure the charge density dynamical susceptibility directly and this was already accomplished by Fink et al. in the late 1980's and very recently reproduced at a much higher energy resolution by Abbamonte et al. This shows directly the existence of a *plasmon*. The plasmon is clearly dispersing with a characteristic velocity v which turns in the velocity of sound in the screened response, in turn directly associated with the conductivity. The magnitude of this velocity is $v \simeq 1$ eV A. This is the natural velocity for a system at this density; the "Fermi-velocity" inferred from ARPES is very similar – I put this in quotation marks because it is rather obscure whether things seen by ARPES should carry any relationship with macroscopic collective properties such as the resistivity. Even the spin wave velocity in the parent Mott-insulators is roughly the same.

Given that the system is hydrodynamical such that the velocity of sound exists, and a momentum relaxation time that is measured to be $\tau_K \sim \tau_\hbar$ it follows immediately that there is a spatial length,

$$\lambda = v\tau_K \tag{15}$$

with the meaning that at length smaller than λ the system behaves hydrodynamically, while hydro breaks down at larger scales. It is very easy to estimate this scale. At 10 K ($\simeq 10^{-3}$ eV) $\lambda = 100$ nanometer, at 1K it becomes a micron and so forth. This is according to the simple intuition following from the resistivity: at temperatures near $T_c \sim 100$K the metal is quite good ($\lambda \simeq 10$ nm) but only when the superconductivity would be suppressed completely one would find extremely small resistivities at the lowest temperatures associated with a "mean free path" of microns.

This puts clearly extreme demands on device fabrication; in order to observe the hydrodynamical transport behaviours one has to build devices with characteristic dimensions in the sub micron regime.

IV. TURBULENCE ON THE NANOSCALE.

Perhaps the most astonishing prediction, very provocative for any fluid mechanics specialist is that our "Planckian fluid" appears to exhibit the potentiality for turbulent flows *even on the nanometer scale*. This is so provocative since normal fluids are very deep in the laminar flow regime already at the micron scale – the famous biophysics stories that e-coli's swim in a very different way than humans .

This is given away by the most famous of all dimensionless parameters, the Reynolds number:

$$Re = \frac{\rho v_{tr} L_{tr}}{\eta} \tag{16}$$

where $\rho = nm_e$ is the mass density (kg/m^3), v_{tr} the characteristic transport velocity (m/s), L_{tr} the characteristic linear dimension (m) and η the (dynamical) viscosity ($kg/(ms)$). The wisdom is that when $Re > 10^3$ the system will behave in a turbulent fashion while laminar flows take over when $Re << 10^3$. In normal fluids η is just so large that $Re << 1000$ already on the micron scales.

In the present context, we only have to insert η, next to identifying m_e as the electron mass. In terms of the quantities defined in Section II,

$$Re = v_{tr}\ L_{tr}\ \frac{1}{A}\ \frac{\tau_\hbar}{l_\mu^2} \tag{17}$$

In order to get a better feeling how this works, let us estimate the sound velocity in a Fermi-liquid guise by asserting $v = v_\mu = \sqrt{2\mu/me}$, this yields the correct order of magnitude for v. Observe that $1/\tau_\mu = v_\mu/l_{mu} = \sqrt{2}\mu/\hbar$ and it follows,

$$Re = \frac{\sqrt{2}}{A}\frac{v_{tr}}{v_\mu}\frac{L_{tr}}{l_\mu}\frac{\mu}{k_B T} \tag{18}$$

This is very elegant: expressing the length (L_{tr}) and velocity (v_{tr}) dimensions of the hydrodynamical flow in the natural units of length $l_\mu \simeq 1$ nm and velocity $v_\mu \simeq 1$ eV A, the Reynolds number is of order $\mu/(k_B T)$!

One way of reading this result is by realising that this is the *smoking gun* to establish the presence of minimal viscosity. The shear magnitude of the resistivity is also sensitive to the strength of the microscopic dirt through the factor $(l_\mu/l)^2$; the Reynolds number is just the direct measure of the viscosity.

How do the numbers work out? In order to observe turbulence one has to be in the hydrodynamical regime and therefore at distances $L \leq \lambda$, the length of the previous section. To find out the best case scenario, let us assume that the spatial scale $L_{tr} \simeq \lambda$. Using that $\lambda/l_\mu \simeq (v_\mu/l_\mu)\ \tau_\hbar$ it follows that $Re \simeq (\mu/k_B T)^2 (v_{tr}/v_\mu)$. This is a quite beneficial scaling. Imagine that one could furnish a strange metal at 1 Kelvin: λ would be of order of a micron while $Re \simeq 10^8 (v_{tr}/v_\mu)$ and v_{tr} could be five orders of magnitude less than the microscopic velocity. On the other hand, take YBCO at $T = 100$ K, just

above T_c: λ is pressing the limits of even silicon CMOS technology (10 nanometers) while $Re = 10^4(v_{tr}/v_\mu)$ and one has to take care that the transport velocity becomes of order of the electronic sound velocity to realise turbulent flows.

The bottom line: the trouble is in the typical magnitude of the superconducting transition temperatures which puts this affair just in a regime where it becomes technologically very hard to construct devices that can detect the turbulent flows. It is close by; as I just argued in the previous paragraph it may well be that one gets into business when a strange metal could be realised at a temperature as "high" as 10 K. I leave it as a challenge to the experimentalists to find ways to make it work. If there is any truth to this story the transport physics of the strange metals is so radically different from anything in the text books that eventually it *has* to manifest itself in do-able laboratory experiments!

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