Lecture 3

Excitations and Fluctuations

We have seen that a spontaneously broken symmetry is accompanied by conserved quantities, and hence, in the absence of long ranged forces, massless excitations called Nambu-Goldstein modes. We have also seen that these modes come in two types, with either linear or quadratic dispersions in the long wave length limit. We have even seen examples of both types. In the Heisenberg ferromagnet we found magnons with k^2 dispersion, while in the Heisenberg antiferromagnet, we found linear spin waves dispersing as k for low momenta.

But this leaves us with an open question: both types of Heisenberg models have full spin rotational symmetry, and are invariant under rotations generated by S^x_{tot}, S^y_{tot}, and S^z_{tot}. Both symmetry broken states pick out one preferred direction, which we can call z. The rotations generated by S^x_{tot} and S^y_{tot} are then spontaneously broken, while S^z_{tot} still corresponds to an unbroken symmetry. In spite of these similarities, we find two type-A Nambu-Goldstone modes in the antiferromagnetic case, and only a single, type-B, mode in the ferromagnet. So how can we predict how many and which kind of modes arise? What should be expect for example for a Heisenberg ferrimagnet, which again breaks full spin rotational symmetry down to just rotations around the z axis, but which has both a ferromagnetic and an antiferromagnetic moment? Or in the case of a canted antiferromagnet, which breaks the full spin rotational symmetry, leaving nothing? Is there a general way of finding the number and types of Nambu-Goldstone modes that should be expected in any given situation?

Of course there is such a prescription, but in order to define it,

we will need a few concepts from the field of mathematics called group theory. Since one could easily write a textbook just about group theory, and since this is course does not aim to be mathematically complete, we will just define the few things we need from group theory in the next section. The treatment will be summary and very incomplete, but should give you enough of an idea about the mathematical structure to see what's going on in the following section, when we apply group theory to the description of spontaneously broken symmetry. If you are already familiar with some group theory, feel free to skip the next section.

Exercise 3.1
Before reading on, make a guess as to how many Nambu-Goldstone modes will be present in ferrimagnets and canted antiferromagnets. Come up with arguments supporting your guess.

Some elements of group theory

In mathematics, a group G is any set of elements $\{g_1, g_2 \dots g_N\}$ which has three structural properties. First, there should be a rule for multiplication such that the product of any two elements in G is itself also an element of the same group (that is, the group is closed under multiplication). For every pair i and j, there then is some k such that $g_i g_j = g_k$.

Notice that the elements of the group need not be numbers. The can be matrices, vectors, functions, or indeed anything else. Likewise, "multiplication" can be any construction that combines two objects into a third. For example, for matrices it could be given by a commutation relation, while for vectors it could be vector addition. The one restriction, is that the multiplication rule should always be associative, so that $g_i g_j g_k = g_i(g_j g_k) = (g_i g_j)g_k$. In addition, we call a group Abelian if the multiplication rule implies $g_i g_j = g_j g_i$ for all pairs i and j.

The second ingredient of a group is that one of the elements in the group should be the identity element, which is usually denoted by e. The identity behaves under multiplication as $eg_i = g_i e = g_i$ for any i. Finally, a group should also have some notion of an inverse g^{-1}, such that $g_i^{-1} g_i = g_i g_i^{-1} = e$ for any i. If g is in G, then its inverse g^{-1} is

also necessarily an element of the group.

As a simple example, consider the group called $\mathbb{Z}_2$. This group has only two elements, which we can identify for example with the numbers zero and one: $g_1 = 0$ and $g_2 = 1$. As the rule for multiplication we can then take the multiplication of numbers, modulo two, so that we have the following multiplication table:

$\mathbb{Z}_2$	0	1
0	0	1
1	1	0

Clearly, the group is closed under this multiplication rule, and the multiplication is associative. From the first line in the table it can be seen that the number zero acts as the identity element for this rule of multiplication, so that $e = g_1$. The inverse of each element can be defined to be the element itself $g_i^{-1} = g_i$, since the multiplication shows that the product of an element with itself always equals the identity element.

A second useful example is the group of quaternions $\mathbb{Q}$. Its elements are the four directions in quaternion space, as well as there negatives, and the multiplication rule is just the standard quaternion multiplication:

$\mathbb{Q}$	1	-1	i	-i	j	-j	k	-k
1	1	-1	i	-i	j	-j	k	-k
-1	-1	1	-i	i	-j	j	-k	-k
i	i	-i	-1	1	k	-k	-j	j
-i	-i	i	1	-1	-k	k	j	-j
j	j	-j	-k	k	-1	1	i	-i
- j	-j	j	k	-k	1	-1	-i	i
k	k	-k	j	-j	-i	i	-1	1
-k	-k	k	-j	j	i	-i	1	-1

Again, it is easily checked that the group is closed under multiplication, and that the multiplication is associative. In this case, it non-Abelian, since for example $ij = k = -ji$. The number 1 acts as the identity element, and the inverse is given by the quaternion conjugate. Notice that the same multiplication table can also be obtained in a different way. Consider for example the two by two identity matrix, σ_0, along with i times the three Pauli matrices, $i\sigma_1$, $i\sigma_2$, and $i\sigma_3$. We can then replace every 1 in the table above by σ_0, every i by $i\sigma_1$, and so on.

If we consider the multiplication rule in this case to be matrix multiplication, the multiplication table will be exactly the same as that of the quaternions. These two sets of elements thus form the *same* group structure. In such a case, we say that the group $\mathbb{Q}$ has two representations. One is in terms of quaternions and quaternion multiplication, the other is in terms of Pauli matrices and matrix multiplication. But the result of manipulating group elements in each case is the same.

Exercise 3.2

a. Check that the identity matrix, together with the three Pauli matrices multiplied by i, indeed give the same multiplication table as the one above.

b. Can you think of a matrix representation for the group $\mathbb{Z}_2$? What is the dimension of the matrices in your representation?

c. Can you think of representations of $\mathbb{Z}_2$ in terms of transformations of physical objects? How about $\mathbb{Q}$?

Now that we know what a group is, we can go on and define related structures. A group G may contain a subgroup H, denoted as $H \subset G$. The subgroup H consists of some subset of elements from the larger group G, which by themselves also form a group. The subgroup should have the *same* rule for multiplication as its parent group. For example, within the group of quaternions, the subset $\{1, -1, i, -i\}$ by itself forms a subgroup that is closed under multiplication and obeys all other properties of a group. These four elements define the group of complex numbers, denoted as $\mathbb{C}$, and we find that it is a subgroup of the quaternions, $\mathbb{C} \subset \mathbb{Q}$.

If a group G contains a subgroup H, it may be possible to define a third related group K, called the quotient group $K = G/H$. In general the quotient between two groups is just a set, but it in some cases it may possess a group structure itself. The way it can be found, is to take the group G, and *identify* all elements of G related by a multiplication with an element of H. In equations:

$$g_i \sim g_j \text{ if } g_i = g_j h_k \text{ for some } h_k \in H \tag{3.1}$$

Here the symbol $\sim$ means that two elements are defined to be equivalent. Equivalence follows the usual rules. If $g_i \sim g_j$ then also $g_j \sim g_i$. Likewise, if $g_i \sim g_j$ and $g_j \sim g_k$, then also $g_i \sim g_k$. The set of all elements that are equivalent under this definition, is called the coset $Hg_i \equiv \{h_1 g_i, h_2 g_i, \dots\}$. It is easily checked that if two elements g_i and g_j are equivalent, they give rise to the same coset $Hg_i = Hg_j$. The set of all *distinct* cosets, is the quotient set $K = G/H$. If it is a group under the same multiplication rule as that of the original group G, then K is called the quotient group.

As an example, consider the quaternion group $\mathbb{Q}$, and its subgroup $\mathbb{C}$. To find the quotient group, we need to start by identifying elements of $\mathbb{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$ related by multiplication with elements of $\mathbb{C} = \{\pm 1, \pm i\}$. Multiplying all elements of $\mathbb{Q}$ with the first element of $\mathbb{C}$, the number one, yields the equivalence relations $1 \sim 1$, $-1 \sim -1$, $i \sim i$, and so on. So no elements are identified with any others at this point. Multiplying by -1 yields the equivalences $1 \sim -1$, $i \sim -i$, $j \sim -j$, and $k \sim -k$. After that, the multiplication by i yields the additional equivalences $1 \sim i$ and $j \sim k$. The final multiplication by $-i$ then does not yield any new identifications anymore.

After all these identifications, we are left with two distinct cosets: $k_1 = \{\pm 1, \pm i\}$ and $k_2 = \{\pm j, \pm k\}$. These two can now be taken to be the two elements in the quotient set $K = \{k_1, k_2\} = \mathbb{Q}/\mathbb{C}$. To see if the quotient set is also a group, we need to apply the multiplication rule inherited from the parent group G. In other words, to multiply two elements k_i and k_j in the quotient set, take any two representative elements of G contained in the cosets k_i and k_j, multiply them, and then identify the result with an element of K by seeing in which coset it appears. In our case, we find:

$$\begin{aligned} k_1 k_1 &= (+1)(+1) = (+1) = k_1 \\ k_2 k_2 &= (+j)(+j) = (-1) = k_1 \\ k_1 k_2 &= (+1)(+j) = (+j) = k_2 \\ k_2 k_1 &= (+j)(+1) = (+j) = k_2 \end{aligned} \tag{3.2}$$

Looking at this result, you may recognise the multiplication table for the group $\mathbb{Z}_2$, which we say earlier. So indeed the quotient set is a quotient group in this case, and we find the relation $\mathbb{Q}/\mathbb{C} = \mathbb{Z}_2$.

Exercise 3.3

a. Find the quotient set $\mathbb{Q}/\mathbb{Z}_2$.

b. Is this set also a group? If so, which one?

Some properties of continuous groups

We have now the described some of the basic structure of groups, but all of the examples we saw so far concerned discrete groups, containing only a finite number of elements. The symmetries we are interested in describing, however, correspond to continuous and differentiable transformations, generated by a unitary operator. To describe them, we need to consider the more extended machinery of continuous groups, or Lie groups.

The elements of a Lie group G are differentiable functions $g(t_1, t_2, \ldots t_n)$ of a set of n parameters. Here n is called the dimension of the group. Lie groups also always have at least one representation in terms of matrices, with the usual matrix multiplication rules. These matrices are finite dimensional (say with dimension m, which does not have to equal n), which in our case you may think of as the symmetry transformations. We will denote the matrix representation of the element g as $\Gamma_{ij}(g)$

The Lie group written in terms of matrices has the structure of a manifold. To see this, we begin by defining the *distance* d between two group elements g and g' as the sum of the squared distances between all of their corresponding matrix elements: $d^2 = \sum_{i,j}(\Gamma_{ij}(g) - \Gamma_{ij}(g'))^2$. We can then go on and define the *δ-neighbourhood* of an element g to consist of all elements g' which are at a distance less than δ from g.

With these definitions, it can be sown that the neighbourhood of *identity* element in G can be parameterised by n (not m!) real parameters $x_1, x_2, \ldots x_n$. The identity element then corresponds to taking all parameters equal to zero. Additionally, there is guaranteed to be some pair of numbers δ and η such that any element g in the δ-neighbourhood of the identity is represented by a set of parameters $\vec{x}$ with length $|\vec{x}|$ less than η. At the same time the reverse also holds, so that any set of numbers $\vec{x}$ with vector length less than η corresponds to an element g within

the δ-neighbourhood of the identity. These properties are equivalent to the Lie group having a manifold structure.

Using the parameterisation of the matrix representation, and its manifold structure, we can define the differentials or partial derivatives of the Lie group. These are a set of n matrices $\{a_1, a_2, \ldots a_n\}$, all of dimension m, defined as:

$$(a_i)_{jk} \equiv \frac{\partial}{\partial x_i}\Gamma_{jk}(\vec{x})\bigg|_{\vec{x}=0} \tag{3.3}$$

These objects are differentials, because they can be used to express any group elements g (represented as $\Gamma(\vec{x})$) in the neighbourhood of the identity as:

$$\Gamma_{jk}(\vec{x}) = \sum_i x_i (a_i)_{jk} \tag{3.4}$$

The matrices a_i are also known as the generators of the Lie group G, because each element of the group can be generated through the matrix exponential:

$$g = e^{t_1 a_1 + t_2 a_2 + \ldots t_n a_n} \tag{3.5}$$

Notice that the definitions for the generators given in this section, are the one usually encountered in the mathematics literature. In physics, we typically prefer to add an i to the definition of the generators in order to make them Hermitian, and thus observables. This is then compensated by defining the group elements g to be generated by a complex exponent.

Finally, notice that the set of generators $\{a_1, a_2, \ldots a_n\}$ span a matrix-valued vector space[1] with real coefficients $\{t_1, t_2, \ldots t_n\}$. Since matrices in general do not commute, we can use the commutator $[a_i, a_j] = f_{ij}^k a_k$ as the definition of a multiplication rule for generators. A vector space adorned with a multiplication rule defines an algebra, which in this case is called the Lie algebra for the generators a_i of the Lie group.

Examples of continuous groups

As an example of a Lie group, consider rotations in two dimensions. The

[1] A vector space is a set of objects which is closed under addition, contains a zero element, and has a rule for scalar multiplication.

elements $g(\theta)$ represent counterclockwise rotations around the origin in the xy plane, over an angle θ. These transformations can be defined for example by their action on real vectors in two-dimensional space:

$$g(\theta)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\theta)\\ \sin(\theta)\end{pmatrix} \qquad \text{and} \qquad g(\theta)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin(\theta)\\ \cos(\theta)\end{pmatrix} \tag{3.6}$$

These definitions suggest a matrix representation of the two-dimensional rotation group, known as $SO(2)$:

$$g(\theta) = \begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix} \tag{3.7}$$

This is a 2-dimensional representation of a 1-dimensional group, because the matrices have dimension two ($m = 2$), while their elements depend on only one parameter θ (so $n = 1$). The matrix representation is already parameterised, and we can use θ as our t_1 parameter. For $\theta = 0$ the group element $g(\theta)$ becomes the identity matrix, as required. In addition, because all matrix elements are trigonometric functions, it is easily checked that any group element $g(\theta)$ is within the matrix distance $\delta = 2$ of the identity element.

Because the group is one-dimensional, there is only a single generator for rotations in two-dimensional space. It is given by:

$$a = \frac{\partial}{\partial\theta}\begin{pmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{pmatrix}\Bigg|_{\theta=0} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \tag{3.8}$$

Using a Taylor expansion you can straightforwardly convince yourself that this matrix indeed generates rotations in the plane, according to $g(\theta) = e^{\theta a}$. Notice that the generator a can also be written in terms of the Hermitian Pauli matrix as $-i\sigma_y$.

As it turns out, there is also another one-dimensional representation of rotations in the plane, which you are already familiar with. This is the representation $g(\theta) = e^{i\theta}$ acting on numbers in the complex plane. In this case, the generator a can be identified with the complex number i. This representation of rotating complex numbers in the complex plane, is known as $U(1)$. Since the elements of the group $U(1)$ can be mapped one-to-one onto those of $SO(2)$ without changing the group structure, these two groups are called isomorphic, and may be considered equivalent.

As the final example in our whirlwind introduction to group theory, consider the group of two-dimensional unitary matrices with determinant one, known as $SU(2)$. The elements of this group are matrices:

$$g(\alpha, \beta) = \begin{pmatrix} \alpha & -\beta \\ \beta^* & \alpha^* \end{pmatrix} \tag{3.9}$$

Here α and β are both complex numbers, so that at first sight it seems as if group elements are determined by four real numbers. The additional constraint that the determinant of the matrix should equal one, however, implies that we can express one of the four parameters in terms of the other three. There are thus three free parameters left over, and $SU(2)$ is a three-dimensional group, represented in terms of two-dimensional matrices ($n = 3$, and $m = 2$).

A nice choice for the parameterisation of group elements in the neighbourhood of the identity matrix is given by:

$$\begin{aligned} \Im(\beta) &= t_1/2 \\ \Re(\beta) &= t_2/2 \\ \Im(\alpha) &= -t_3/2 \\ \Re(\alpha) &= \sqrt{1 - \Im(\alpha)^2 - \Re(\beta)^2 - \Im(\beta)^2} \end{aligned} \tag{3.10}$$

As expected, taking $t_1 = t_2 = t_3 = 0$ in these equations yields $\alpha = 1$ and $\beta = 0$, so that the group element $g(\alpha, \beta)$ at these parameter values indeed is the identity element. The generators for $SU(2)$ can now be defined in the usual way as derivatives of the matrix representations with respect to the parameters t_i. Doing this explicitly, you can easily check that the three generators a_i precisely equal $-i/2$ times the Pauli matrices. The Pauli matrices and their well-known commutators thus form the $su(2)$ Lie algebra for the Lie group $SU(2)$.

Exercise 3.4
Reproduce the calculations for the $SO(2)$ and $SU(2)$ examples, filling in all gaps in the given derivations.

Spontaneous symmetry breaking in group theoretical terms

Using the formal language of group theory, the broken and unbroken symmetries of any particular system can be precisely described, and using this, it becomes possible to predict both how many and which types of Nambu-Goldstone modes will arise in any given situation. To see how this works, we will begin by reformulating the things we already discussed in terms of group-theoretical concepts.

As we know, a Hamiltonian $\hat{H}$ is said to be symmetric under some unitary transformation $\hat{g}$ if these two operators commute: $[\hat{H}, \hat{g}] = 0$. If a Hamiltonian is symmetric under multiple symmetry transformations, then the set of unitary operators $\{\hat{g}_i\}$ forms a group G. The rule for multiplication here, is simply the quantum mechanical product of operators. If the symmetries under which the Hamiltonian is invariant are continuous symmetries, then the group G must be a Lie group. The elements $\hat{g}_i$ of the Lie group, i.e. the symmetry transformations, may be written in terms of their generators $\hat{a}_i$ as:

$$\hat{g} = e^{\sum_i \theta_i \hat{a}_i} \tag{3.11}$$

Notice that here we adopt the physics convention of defining the generators as Hermitian operators, so that an i appears in the exponentiated expression for the group element. The generators in general do not commute with each other, and their commutator may be used to define a Lie algebra for them. Physically, the symmetry generators correspond to the Noether charges conserved by the symmetry. Assuming some translational invariance of the kind that is a necessary requirement in any field theory, the generators can be written in terms of a spatial integral over the local Noether charge density:

$$\hat{a}_i = \int dx\, \hat{j}_a^0(x) \tag{3.12}$$

Here $\hat{j}^0$ is the zero component of the four-vector whose expectation value contains the local Noether charge density and its associated local current. These are related to each other by the continuity equation $\partial_\mu j^\mu = 0$. Integrating the continuity equation over all of space yields the absolute conservation of the total Noether charge.

As an example, consider the ferromagnetic Heisenberg model. It is invariant under global rotations of all spins simultaneously, which are

generated by the operators S^x_{tot}, S^y_{tot}, and S^z_{tot}. These make up the three-dimensional spin rotation group $SU(2)$. The commutation rules that define the corresponding Lie algebra are the usual commutators of Pauli matrices, and the conserved charges are the x, y, and z components of the total magnetisation. These give rise to a continuity equation for local Noether charge densities defined as the components of local magnetisation averaged over some finite volume, and their corresponding spin currents densities.

Exercise 3.5

a. What are the symmetry group, the symmetry generators, and the conserved charges in the case of an antiferromagnetic Heisenberg model?

b. What about the ferrimagnetic Heisenberg model, defined by $\hat{H} = |J| \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$, with different size spins on the A and B sublattice?

c. And what about the canted (anti)ferrimagnet described by the Hamiltonian $\hat{H} = -|J| \sum_{\langle ij \rangle} \left(\vec{S}_i \cdot \vec{S}_j - |\vec{S}_i \cdot \vec{S}_j|^2 \right)$? In this case, assume that the magnet is built of spin-halves. Notice that the first term in the Hamiltonian favours a ferromagnetic arrangement, while the second term favours spins to be at right angles with their neighbours.

A quantum state $|\psi\rangle$ may be considered symmetric under a symmetry transformation $\hat{g}$, generated by $\hat{a}$, if the expectation value of commutator between the generator and any other operator is zero:

$$|\psi\rangle \text{ symmetric under } \hat{g} \quad \Leftrightarrow \quad \langle\psi|[\hat{a}, \hat{\Phi}]\,|\psi\rangle = 0 \;\; \forall \hat{\Phi} \tag{3.13}$$

Notice that if $|\psi\rangle$ is an eigenstate of the symmetry generator $\hat{a}$, the expectation value will be automatically zero for all $\hat{\Phi}$. This more general definition, however, also includes symmetric states which cannot be simultaneous eigenstates of all of their (non-commuting) symmetry generators.

If there is *any* operator $\hat{\Phi}$ for which the expectation value of the commutator with a particular generator $\hat{a}_i$ is not zero, then the symmetry generated by $\hat{a}_i$ is *broken* in the state $|\psi\rangle$. Notice a state may break some symmetries, while leaving others unbroken. For example, the antiferromagnetic Néel state (with spins alternating between pointing up and down the z-axis) is symmetric under rotations around the z-axis, but breaks the rotations around the x-axis under which the antiferromagnetic Heisenberg Hamiltonian is invariant. The generator of the broken symmetry is the x-component of total spin $S^x_{tot} = S^x_A + S^x_B$. Here S^x_A is the sum of all S^x operators acting on individual spins in the A sublattice. The expectation value of S^x_{tot} is a conserved Noether charge. To see that the Néel state breaks the rotational symmetry generated by S^x_{tot}, consider the interpolating field $\Phi = -i(S^y_A - S^y_B)$. Its commutator with the Noether charge gives the order parameter $O = S^z_A - S^z_B$. This is the staggered magnetisation, which indeed has nonzero expectation value in the Néel state. Notice that the prefactor $-i$ in the definition of the interpolating field was introduced for convenience, to ensure that the order parameter is a Hermitian operator.

Exercise 3.6

a. Check that for a ferromagnet, the symmetry-broken ground state is not an eigenstate of all generators for the $SU(2)$ symmetry of the Heisenberg Hamiltonian.

b. Give arguments supporting the statement that the state $|\psi\rangle = \int d\theta d\eta \, |\theta\eta\rangle$ is also *not* a simultaneous eigenstate of all $su(2)$ generators. Here $|\theta\eta\rangle$ is the fully magnetised state, with magnetisation pointing along the θ and η Euler angles.

c. Without proving it explicitly, argue that the state $|\psi\rangle$ should be symmetric under the interpolating-field definition of symmetric states.

Exercise 3.7

For each symmetry of the ferromagnetic Heisenberg model broken by

the fully magnetised ferromagnetic state, indicate what the corresponding symmetry generator, Noether charge, interpolating field, and order parameter are.

All the symmetry generators that are **not** broken in a particular symmetry-breaking state form a subgroup H of the full symmetry group: $H \subset G$. The quotient set $K = G/H$ can then be identified with all possible values that the order parameter of the symmetry breaking state can take. For example, in the antiferromagnetic Néel state the full $SU(2)$ spin rotational symmetry is broken down to just $U(1)$ rotations around a single axis. The quotient set $SU(2)/U(1) = S^2$ is the set of all points on a two-sphere in $\mathbb{R}^3$. These points can be interpreted for example as indicating all possible directions that the spins on the A sublattice could point (with the B sublattice spins pointing the opposite way), and they can thus be used to label all possible antiferromagnetically ordered Néel states.

An alternative way of viewing the role of the quotient G/H, is to say that all the generators of the broken symmetries (i.e. the ones that are not in H) by themselves form a set. The transformations generated by the operators in that set may be used to act on the symmetry-broken Néel state. Doing so results in a different symmetry-broken state, which in this case is a Néel state with spins parallel to a rotated axis. The set of broken symmetry generators can thus be used to generate a set of broken symmetry transformations, which starting from one particular symmetry-broken state, create others. All possible symmetry-broken states can be reached this way.

Exercise 3.8
Explain why the set S^2 describes all possible symmetry broken states for both the antiferromagnet and the ferromagnet. Does it also apply to ferrimagnets and canted antiferromagnets?

Counting Nambu-Goldstone modes

As we have seen, Nambu-Goldstone modes can be excited by both local interpolating field operators, and local conserved charge density operat-

ors. If a state $|\psi\rangle$ breaks multiple symmetries, it may happen that the expectation value of the commutator between the generators of broken symmetries is non-zero:

$$\langle\psi|[\hat{a}_i, \hat{a}_j]\,|\psi\rangle \neq 0 \tag{3.14}$$

Since the symmetry generators correspond to conserved charges, we can also write this expression in terms of local Noether charge densities.

$$\begin{aligned} &\int_\Omega dx dx'\ \langle\psi|[\hat{j}_i^0(x), \hat{j}_j^0(x')]\,|\psi\rangle \neq 0 \\ \Rightarrow \quad &\int_\Omega dx\,\langle\psi|[\hat{j}_i^0(x), \hat{j}_j^0(x')]\,|\psi\rangle \neq 0 \ \text{ for some } x' \end{aligned} \tag{3.15}$$

The final relation can be interpreted as saying that $\hat{j}_j^0(x')$ acts as an interpolating field for the conserved charge $\hat{a}_i$. Of course we could have just as well written it the other way around, and thus $\hat{j}_i^0(x)$ also acts as an interpolating field for $\hat{a}_j$. But if a Nambu-Goldstone mode may be excited both by applying a conserved charge operators, and its corresponding interpolating field operator, then that implies that in this case $\hat{j}_i^0$ and $\hat{j}_j^0$ excite the *same* Nambu-Goldstone mode. Even though the state $|\psi\rangle$ breaks two symmetries, there is only a single Nambu-Goldstone mode.

If the commutator of two broken symmetry generators has a nonzero expectation value in the symmetry-broken state, we say that they are not independent. The number of distinct Nambu-Goldstone modes in a symmetry broken state corresponds to the number of independent generators for broken symmetries. For example, in the ferromagnet, the rotation symmetries around the x and y axes are both broken. The commutator of the corresponding generators is $[S^x_{tot}, S^y_{tot}] \propto S^z_{tot}$. But the total magnetisations has a nonzero expectation value in the symmetry broken state – it is the order parameter of the ferromagnet. As a result, the modes excited by S^x_{tot} and S^y_{tot} are not independent, and only a single Nambu-Goldstone mode exists in ferromagnets. In contrast, an antiferromagnet breaks the same two symmetries generated by S^x_{tot} and S^y_{tot}. But the total magnetisation S^z_{tot} has zero expectation value in the Néel state, and the modes excited by the generator in this case are independent from one another. We thus find that an antiferromagnet harbours two Nambu-Goldstone modes.

Physically, you can understand this result by rewriting the generators as in terms of S^+_{tot} and S^-_{tot}. It is clear that the local versions of these operators induce spin flips, and may therefore in principle be expected to excite Nambu-Goldstone modes. In the case of the ferromagnet, with all spins pointing up in the symmetry broken state, only the operator S^-_{tot} really does create a spin flip. The operator S^+_{tot} has no effect, since all spins already have maximum magnetisation. For the antiferromagnet on the other hand, there are two sublattices, and we should decomposed the generators into $S^+_A + S^+_B$ and $S^-_A + S^-_B$. The spin raising operator now has an effect on the spins in the B sublattice, while the spin lowering operator affects the A sublattice, and these correspond to two independent spin wave excitations.

Nambu-Goldstone modes that can be excited by independent generators, are called type A Nambu-Goldstone modes, while those excited by a pair of conjugate generators are designated as type B. Generically, type A modes disperse as $\omega_k \propto k$, and type B modes typically obey $\omega \propto k^2$. To prove this observation, group theoretical arguments alone are not enough. Instead, you would have to construct the most general Lagrangian possible, given a particular set of symmetries and a particular symmetry-breaking pattern[2]. If you then take the low-energy limit, you find an effective Lagrangian density of the form:

$$\mathcal{L}_{eff} = \frac{1}{4}\rho_{ij}\left(\dot{\pi}_i\pi_j - \pi_i\dot{\pi}_j\right) + \frac{1}{2}g_{ij}\dot{\pi}_i\dot{\pi}_j - \frac{1}{2}\bar{g}_{ij}\partial_x\pi_i\partial_x\pi_j + \ldots \tag{3.16}$$

Here $\pi_i(x,t)$ is a field corresponding to the generator $\hat{a}_i$ of a *broken* symmetry. The matrix ρ_{ij} turns out to be given precisely by:

$$\rho_{ij}(x) = -i\int dx' \langle [j^0_i(x'), j^0_j(x)]\rangle \tag{3.17}$$

Thus, if there are two generators which excite the same, type B, Nambu-Goldstone mode, the effective Lagrangian describing the spectrum of these modes is to lowest order linear in time derivatives. If the two modes are independent, type A, modes on the other hand, the linear term vanishes and the Lagrangian is quadratic in time derivatives. Either way, the remnant translational symmetry at large distances typically guarantees the Lagrangian to be quadratic in spatial derivatives

[2]This is done by Watanabe and Murayama in *Phys. Rev. X* **4**, 031057 (2014).

to lowest order[3]. The equation of motion then sets $\omega_k \propto k^2$ for type B Nambu-Goldstone modes, while it results in $\omega_k^2 \propto k^2$ for type A excitations. As usual, because the Lagrangian is an effective, low-energy theory, these dispersions are valid in the long wave length limit only.

Exercise 3.9
For the ferrimagnet and canted antiferromagnet, how many type A and type B Nambu-Goldstein modes do you expect, and what are their dispersions in the long wave length limit?

Order parameters and partner modes

Given a Hamiltonian which is invariant under a group of symmetries, and a symmetry-broken state which is invariant under only a subgroup of those symmetries, the choice of order parameters to describe the broken symmetries is often not unique. In other words, a single symmetry-broken state can often be characterised by multiple order parameters, any of which may be used as the expansion parameter in a Ginzburg-Landau theory.

For example, consider a partly polarised ferromagnet, described by the Hamiltonian

$$\hat{H} = -|J| \sum_{\langle ij \rangle} \left(\vec{S}_i \cdot \vec{S}_j - \alpha |\vec{S}_i \cdot \vec{S}_j|^2 \right) \tag{3.18}$$

If the spins in this system are all spin-3/2, it is possible to fine-tune the value of α such that the state $\prod_i |s_i = 3/2, m_i = 1/2\rangle$ is an exact ground state. The Hamiltonian has full $SU(2)$ spin-rotational symmetry, but the ground state only has $U(1)$ symmetry, breaking the rotations around the x and y axes. The symmetry breaking in this case is very much akin to that in ferromagnets, except that the ground state is not fully polarised. The order parameter in this case can be chosen to be the magnetisation S^z, but we could also use the nematicity N^{zz}, with the

[3]For all symmetries that we consider in this course, the Lagrangians are typical. If some very special spacetime symmetries are concerned, however, it is in principle possible that $\bar{g}_{ij}$ is zero, and in that case type A modes have a quartic dispersion, and type B modes quadratic.

nematicity tensor defined as $N^{ij} \equiv (S^i S^j + S^j S^i)/2 - \delta_{ij} s(s+1)/3$. This can be seen for example from the commutators:

$$\begin{aligned} \langle [S^x, S^y] \rangle &\propto \langle S^z \rangle & \langle [S^y, S^x] \rangle &\propto \langle S^z \rangle \\ \langle [S^x, N^{yz}] \rangle &\propto \langle N^{zz} \rangle & \langle [S^y, N^{xz}] \rangle &\propto \langle N^{zz} \rangle \end{aligned} \tag{3.19}$$

We have already seen that if one of the broken symmetry generators can be used as an interpolating field to create an order parameter, the Nambu-Goldstone modes excited by the two broken symmetry generators are in fact the same, type B, mode. In the partly polarised ferromagnet this is the case for the generators S^x and S^y. You might wonder how it can be that two physically distinct observables result in only a single excitation. In fact, this intuition is correct. There is a second mode that is excited by S^x and S^y, but as it turns out, this mode has nonzero energy in the $k \to 0$ limit. Such additional modes are known as massive partner modes to the type B modes. In the case of the partly polarised magnet, the extra mode has a straightforward physical interpretation. We already know that the operator S^- excites spin waves or magnons in the ferromagnet. The operator that is still unused is then S^+, and indeed this one can be seen to excite the massive partner mode in partly polarised ferromagnets. If the ferromagnet becomes fully polarised, the spin raising operator annihilates the symmetry-broken state, and the massive partner mode disappears.

The disappearance of the second mode in fully polarised magnets does not mean that the mode counting is then wrong. In fact, spins are not really bosons, and the fact that spins cannot be raised passed their maximum violation is an indication that precisely the identification of spin excitations with bosonic modes goes awry. If you were to insist on bosonising even S^+ in the ferromagnet, then you need to include an infinity potential to remove the unphysical modes. With some due hand waiving, you can thus interpret the absence of the partner mode in that case by saying that it is pushed to infinite energy. A more mathematically robust observation would be to notice that acting on a fully polarised magnet, the interpolating field N^{yz} excites precisely the same state as the operator S^y, and therefore both create only the Nambu-Goldstone mode[4].

[4]For an altogether more mathematically robust treatment of these issues, see A. Beekman, *Annals of Physics* **361**, 461 (2015).

At this point, you should realise that the existence of the massive partner mode actually poses a bit of a paradox. On the one hand, we have seen that in the partly polarised magnet, S^x and S^y excite both the type B Nambu-Goldstone mode, and the massive partner mode. But looking back at the derivation of Goldstones theorem, the existence of a massive mode, excited by S^x and its interpolating field S^y, would result in a time-dependent order parameter, apparently contradicting the local version of Noethers theorem (the conservation of S^x_{tot} from which the derivation of Goldstones theorem started)!

The resolution to this conundrum lies in the fact that in the symmetry-broken state, in fact $\langle S^x \rangle$ is not conserved by itself. This seeming violation of Noether's theorem is possible, because the actual system is not symmetric. It includes a vanishing symmetry-breaking field, or equivalently, a mean-field order parameter. This order parameter (the magnetisation along the z-axis) rotates the spins around the z-axis, and thus dynamically turns a finite local $\langle S^x(x) \rangle$ expectation value into a finite $\langle S^y(x) \rangle$ expectation value. In other words, in the symmetry broken state, only the combination of the Noether charges S^x and S^y is conserved, because the order parameter itself couples excitations in these sectors. The fact that this possible, is signalled by the fact that S^x and S^y have a non-vanishing expectation value of their commutator, which is precisely the condition for a single type B mode to arise, rather than two type A modes.

The physical picture for all of this, is to say that Goldstone's theorem shows two Nambu-Goldstone modes to exist, which should both have vanishing energy under a completely symmetric Hamiltonian. However, since the symmetry breaking field, or indeed the order parameter field itself, couples the modes, we get level repulsion. As a result, there is one zero-energy mode (it cannot go down any further), and one massive mode. Every type B mode should thus be accompanied by a massive partner mode, unless there is a phase space restriction (which could be said to effectively push it to infinite energy), such as in the fully polarised ferromagnet.

Exercise 3.10
Explain why a canted antiferromagnet should have one type A Nambu-Goldstone mode, as well as one type B mode with one massive partner

mode.

Finally, it is instructive to see how the canted magnet emerges from its two limiting cases, the ferromagnet with polarization along the z-axis, and the antiferromagnet with polarization along the x-axis. In the first case, we start out with a ferromagnet with its type B Nambu-Goldstone mode excited by both S^x and S^y. The expectation value for S^z is maximal, so there is no gapped partner mode. Introducing a canting angle θ will lower this expectation value $\langle S^z \rangle$. As a result, excitations that increase the magnetic quantum number, generated by S^+, become possible, and the gapped partner mode emerges. Furthermore, rotations around the z-axis are now broken as well. This additional breaking of $U(1)$ symmetry leads to an additional type A Nambu-Goldstone mode. We thus finally find one type A mode associated with broken rotations around the magnetization axis, and one type B mode along with its gapped partner mode, associated with the other two broken rotations.

Starting out from the antiferromagnet the reasoning seems different. Taking the staggered magnetisation to run along the x-axis, it breaks rotation generated by S^y and S^z. There are thus two type A Nambu-Goldstone modes. Inducing a canting angle leads to a nonzero magnetisation in a perpendicular direction, say along the z-axis. This causes a ferromagnetic-type symmetry breaking of rotations generated by S^x and S^y. These two generators should conspire to excite one type B Nambu-Goldstone mode and one gapped partner mode. But since S^y already excited a type A mode in the perfect antiferromagnet, we should imagine that in fact this type A mode evolves into the type B mode, rather than the latter being a wholly new type of excitation. Meanwhile, the type A mode excited by S^z persists. We thus finally find the same spectrum of one type A, one type B, and one gapped partner mode in the final state, regardless of whether we take off from a ferromagnetic or antiferromagnetic starting point. From this point of view, there really is no difference between a canted ferromagnet and a canted antiferromagnet.

Mermin-Wagner theorem

According to the Goldstone theorem, low-energy modes are present in all symmetry-breaking systems. It is therefore natural to ask if the necessary presence of such excitations at low energy could have generic

consequences. One of the most basic things we could consider in this regard, is how many Nambu-Goldstone modes are typically occupied at low temperatures.

For sake of clarity, we will focus on the example of a harmonic crystal, breaking translational symmetry. In thus case the Nambu-Goldstone modes are the well-known quantised sound waves, called phonons. Rephrasing our question in terms of crystals, we could ask ourselves: "how large is the typical displacement of atoms in a crystal due to thermal fluctuations?" In fact, rather than considering the exact displacement at any point in time, a more useful thing to consider is the average size of the fluctuations, given by $\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$. If the crystal is in thermal equilibrium, the average displacement $\langle \hat{x}_j \rangle$ in this expression will be zero.

We can in principle calculate the fluctuations of any given atom directly in terms of its corresponding bosonic operators. For our standard example of a one-dimensional chain of atoms, this yields:

$$\begin{aligned}(\Delta X_j)^2 &= \sum_{k,k'} e^{-i(k+k')ja} \langle \hat{X}_k \hat{X}_{k'} \rangle \\ &= \sum_{k,k'} e^{-i(k+k')ja} \frac{\hbar}{2m\sqrt{\omega(k)\omega(k')}} \langle (\hat{a}_k + \hat{a}^\dagger_{-k})(\hat{a}_{k'} + \hat{a}^\dagger_{-k'}) \rangle .\end{aligned} \tag{3.20}$$

The thermal expectation value includes a sum over all possible energy eigenstates of the system. Because the Hamiltonian is diagonal in terms of boson operators in momentum space, this implies that off-diagonal expectation values such as $\langle \hat{a}_k \, \hat{a}_k \rangle$, but also $\langle \hat{a}^\dagger_k \, \hat{a}_{k'} \rangle$ for $k' \neq k$, will always be zero. The only remaining contributions to the termal fluctuations of the atomic position are then:

$$\begin{aligned}(\Delta X_j)^2 &= \sum_{k,k'} e^{-i(k+k')ja} \frac{\hbar}{2m\sqrt{\omega(k)\omega(k')}} \langle \hat{a}_k \, \hat{a}^\dagger_{-k'} + \hat{a}^\dagger_{-k} \, \hat{a}_{k'} \rangle \\ &= \sum_k \frac{\hbar}{2m\omega(k)} \left(1 + 2\langle \hat{a}^\dagger_k \, \hat{a}_k \rangle \right) .\end{aligned} \tag{3.21}$$

The remaining thermal expectation value in the final line is just the Bose-Einstein distribution function. Notice the implication of this equation: the fluctuations in the displacement of a single atom at position

j is determined by the thermal occupation of *all* possible bosons in the material.

Since we are interested in the fluctuations at low temperatures, we can use the so-called Debye approximation for the phonon energies. This means that we approximate the phonon dispersion to be purely linear and isotropic throughout k-space, but we introduce a maximum momentum for phonons called k_D to ensure that the total number of modes does exceed the total number of unit cells (times the number of spatial dimensions considered). Using this to convert the sum over momenta into an integral up to the Debye wave vector, the expression for the fluctuations becomes:

$$(\Delta X_j)^2 = \left(\frac{L}{2\pi}\right) \frac{\hbar}{2mv} \int dk\ \frac{1}{k} \left(1 + \frac{2}{e^{\hbar v k/k_B T} - 1}\right). \tag{3.22}$$

We are interested in the crystal properties at low temperatures. One should be careful to notice however, that because the dispersion of the phonon modes is linear, there are always modes for which $\hbar v k \ll k_B T$, for any non-zero temperature. These modes in fact dominate the integral, and thus the correct expansion of the Bose-Einstein function to use in this case, is in fact the high-temperature expansion $1/(e^{\hbar v k/k_B T} - 1) \approx k_B T/(\hbar v k)$. This yields for the fluctuations:

$$(\Delta X_j)^2 \approx \left(\frac{L}{2\pi}\right) \frac{\hbar}{2mv} \left[\left(\int dk\ \frac{1}{k}\right) + \frac{2k_B T}{\hbar v} \left(\int dk\ \frac{1}{k^2}\right)\right]. \tag{3.23}$$

The second integral arises from the occupation of the boson modes, and can be interpreted as the thermal vibrations of the atoms. The first integral comes directly from the commutation relation of the boson operators, and is present even at zero temperature. It represents the uncertainty in the position of the atom due to the quantum nature of the boson modes, and this part of ΔX is often referred to as the contribution of quantum fluctuations to the atomic position.

The equation (3.23) describes the average mean displacement of any one atom within a chain of atoms separated by the interatomic distance a. However, the integrals in this equation do not converge! The fluctuations of the atom position are thus infinitely large, and certainly much larger than the equilibrium separation between atoms. In other words: the atoms in this material are not localised at all, and the monatomic chain cannot exist in the first place. If we try to construct a material consisting of a one-dimensional chain of atoms, either the thermal

fluctuations or the quantum fluctuations suffice to completely melt the crystal at any temperature.

In fact, we can straightforwardly generalise this argument to higher dimensions. Using the Debye approximation, the fluctuations in d dimensions will be:

$$(\Delta X_j)^2 \propto \left(\frac{L}{2\pi}\right)^d \frac{\hbar}{2mv}\left[\left(\int k^{d-1}dk\ \frac{1}{k}\right) + \frac{2k_BT}{\hbar v}\left(\int k^{d-1}dk\ \frac{1}{k^2}\right)\right].$$

Again, the second integral represents thermal fluctuations, while the first integral arises from the quantum fluctuations. The thermal fluctuations diverge in dimensions $d \leq 2$, while the quantum fluctuations diverge in dimension $d = 1$. The implication is that a crystal consisting of a regular arrangement of atoms can apparently not exist in one dimension even at $T = 0$, while it cannot exist in one or two dimensions for any non-zero temperature, no matter how low.

In fact the arguments leading to this far-reaching conclusion, are very general. All we assumed is the presence of Goldstone modes, and hence a broken continuous symmetry. We can thus formulate an even more general conclusion, which is known as the Mermin-Wagner theorem, and which states that no ordered state can exist at any non-zero temperature in fewer than three dimensions. Quantum fluctuations prevent the formation of order even at zero temperature in fewer than two dimensions. An ordered state is here understood to be anything that spontaneously breaks a continuous symmetry, such as a crystal, a magnet, a superconductor, and so on. The Mermin-Wagner theorem applies to any system with a spontaneously broken continuous symmetry, in the absence of long-ranged interactions.

Exercise 3.11

a. In fact, the Mermin-Wagner theorem leads to slightly different conclusions in the presence of type B, rather than linearly dispersing type A, Nambu-Goldstone modes. Repeat the analysis above for the case of ferromagnets, and argue that these can exist at $t = 0$ even in one spatial dimension.

b. Also show that at finite temperatures, ferromagnets, like crystals, cannot exist in less than three spatial dimensions.

c. Argue what a ferrimagnet should have the same behaviour in terms of critical dimensions as the ferromagnet. What about the canted ferromagnet?
