ATTP: Spontaneous symmetry breaking in the context of Condensed Matter Physics

Selected essays

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ABSTRACT: This anthology contains some essays from the first module of the ATTP course on spontanious symmetry braking. The essays were part of the course requirements. This selection is not meant as a 'top 3'; sometimes there was some overlap between the essays and if none of your essays are included it does not mean they were all bad. We just thought it would be nice for you to see what some of the other students wrote. Also, the essays contain some minor errors, so be sure to stay critical when reading them.

We wish you all the best for the next modules.

- Jasper van Wezel, Joris Kattemölle

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Elitzur's theorem

Marc Barroso Mancha - Utrecht University - ID: 6311776

During the first lecture, we have seen a very important result in the field of quantum field theory: Elitzur's theorem. This theorem states that continuous local symmetries cannot be spontaneously broken. More technically, it states that *in a gauge invariant theory a local quantity with vanishing mean value on its orbit under the action of the gauge group has zero ground state expectation value*[1] . Elitzur, in his original work, examined a theory defined on a space-time lattice (such that the ultraviolet cutoff is built-in) with the following action (ϕ_i are the fields defined in the lattice position i as angle variable that are coupled to the gauge field $A_{i,\underline{n}}$, also an angle variable defined as the link between neighboring sites i and $i+\underline{n}$):

$$S = K\sum_{i=1}^{N}\sum_{\underline{m}}^{d}\cos\left(\phi_i - \phi_{i+\underline{m}} - A_{i,\underline{m}}\right) + \frac{1}{g^2}\sum_{i=1}^{N}\sum_{\underline{n},\underline{m},\underline{n}\neq\underline{m}}^{d}\cos\left(A_{i,\underline{n}} + A_{i+\underline{n},\underline{m}} - A_{i+\underline{m},\underline{n}} - A_{i,\underline{m}}\right)$$

where N is the number of sites and d is the dimension of the system. K and g are two constants of our theory. The idea at the time was that some early works on the dynamics of the system in different regimes ($g\to 0$ and $g\to\infty$) were promising and it led people to believe that a phase transition might occur at some critical values. This action is invariant under the following local gauge transformations:

$$\phi_i \to \phi_i + C_i$$
$$A_{i,\underline{m}} \to A_{i,\underline{m}} + C_i - C_{i+\underline{m}}$$

where C_i are arbitrary c-number (complex) functions. Elitzur's point is that *any vacuum expectation value of a gauge variant local operator whose average over the gauge orbit is zero.* This can be seen by writing down the expectation value (we are following the same example as Elitzur did in his original paper, but the result is general) in terms of the path integrals with the auxiliary sources:

$$\langle\cos A_{i,\underline{n}}\rangle = \lim_{J\to 0}\lim_{N\to\infty}\int \mathrm{d}[\phi_i]\mathrm{d}[A_{i,\underline{m}}]e^{-S(\phi,A)}\exp\left\{J\sum_{j,\underline{m}}\cos A_{j,\underline{m}}\right\}\cos A_{i,\underline{n}} Z(J,N)^{-1}$$

$$Z(J,N) = \int \mathrm{d}[\phi_i]\mathrm{d}[A_{i,\underline{m}}]e^{-S(\phi,A)}\exp\left\{J\sum_{j,\underline{m}}\cos A_{j,\underline{m}}\right\}$$

[1] *Impossibility of spontaneously breaking local symmetries*, S. Elitzur, Phys. Rev. D 12 (1975) 1624

Now we can perform one shift in the variables, $l_{i,\underline{m}} \equiv \phi_i - \phi_{i+\underline{m}} - A_{i,\underline{m}}$, such that now the action only depends on l (this is actually a consequence of gauge invariance, since the transformation laws are equivalent to redefining A). Then we can expand the exponential that contains the auxiliary field in powers of J, and find a bound for the value of $\langle \cos A \rangle$.

$$\langle A_{i,\underline{n}} \rangle = \lim_{J\to 0} \lim_{N\to\infty} \int \mathrm{d}[l] e^{-S(l)} \int \mathrm{d}[\phi] \exp\left\{ J \sum_{i,\underline{m}} \cos\left(l_{i,\underline{m}} - \phi_i + \phi_{i+\underline{m}}\right) \right\} \cos A_{i,\underline{n}} Z(J,N)^{-1}$$

$$g_N(l,J) := \int \mathrm{d}[\phi_j]\mathrm{d}[\phi_{j+\underline{n}}] \exp\left\{ J \sum_{i,\underline{m}} \cos\left(l_{i,\underline{m}} - \phi_i + \phi_{i+\underline{m}}\right) \right\} \cos\left(l_{i,\underline{n}} - \phi_i + \phi_{i+\underline{n}}\right) =$$
$$= \sum_{\underline{m}} J 2\pi^2 \cos\left(l_{j,\underline{n}} - l_{j,\underline{m}}\right) + \mathcal{O}(J^2)$$

So what we realize is that actually the integral is independent of the value of N, so that limit becomes trivial (and so, the limits would trivially commute), and taking the second limit will take the expected value to zero (since the result is linear in J), regardless of the values of g and K. A non-zero expectation value of this quantity, that is gauge variant, would have meant that our gauge symmetry has been spontaneously broken. The fact that Elitzur was able to prove that this value is always zero means not such thing can happen (actually this is true for both abelian and compact non-abelian gauge symmetries). Some later developments, such as the paper from Splittorff [2], although, show that gauge invariance is actually not the only condition for this, but we also need $\exp\{-S\}$ to be positive (there are cases in QCD where this quantity can be complex).

It's interesting to compare this situation to the breaking of a global symmetry, as Elitzur explains. If our action is given by

$$S = K \sum_{j,\underline{m}} V(\phi_j - \phi_{j+\underline{m}})$$

then our system shows a global symmetry given by: $\phi_i \to \phi_i + C$. We can write down the expected value for ϕ_i, $\langle \cos \phi_i \rangle$, which again is a gauge variant value, in terms of the path integrals. The key part is that now the shift in variables can be done as: $\phi'_j = \phi_j - \phi_0$, for $j = 0$, such that the action will only depend on ϕ', but the additional part will depend on the number of fields, N.

[2] *Impossibility of spontaneously breaking local symmetries and the sign problem*, K. Splittorff (2003).

Explicitly,

$$\langle \cos\phi_i \rangle = \lim_{J\to 0}\lim_{N\to\infty} Z^{-1}\int \mathrm{d}[\phi] e^{S(\phi)} \exp\left\{ J\sum_j^N \cos\phi_j \right\} \cos\phi_i$$

$$Z = \int \mathrm{d}[\phi] e^{S(\phi)} \exp\left\{ J\sum_j^N \cos\phi_j \right\}$$

$$\to \langle \cos\phi_i \rangle = \lim_{J\to 0}\lim_{N\to\infty} Z^{-1}\int \mathrm{d}[\phi'] e^{S(\phi')} \int \mathrm{d}\phi_0 \exp\left\{ J\sum_j \cos\left(\phi'_j + \phi_0\right) \right\} \cos\left(\phi'_i + \phi_0\right)$$

$$f_N(\phi_j, J) := \int \mathrm{d}\phi_0 \exp\left\{ J\sum_j \cos\left(\phi'_j + \phi_0\right) \right\} \cos\left(\phi'_i + \phi_0\right)$$

So what Elitzur argues is that this $f_N(\phi_j, J)$ depends on N, and does not admit a trivial limit when $N \to \infty$ as in the previous case. This means that the limits will not commute (at least, trivially) and thus we will see a spontaneous global symmetry breaking at some point, characterized by some critical values.

Order and Rigidity

Maria Inês Cravo

The usefulness of studying spontaneous symmetry breaking lies in the fact that a lot of **properties of materials** exist only because symmetries can be broken, regardless of whether or not they have indeed been broken. Two results are important in establishing this: Noether's and Goldstone's theorems. These theorems connect the internal structure of a quantum system with its breaking of a global, continuous symmetry.

Goldstone's theorem states that, for a quantum system that spontaneously breaks a continuous symmetry and which involves no long-range forces between its constituents particles, there is a bosonic excitation whose dispersion $h\omega(k)$ goes to zero for small k. These excitations are the Nambu-Goldstone modes.

Goldstone's theorem leads to the interesting conclusion that solids are rigid. **Rigidity** is defined by the material's response to a perturbation. If the push is soft, the only modes that can be excited are the infinite wavelength phonons close to $k = 0$. That is, in the regime of linear response (small enough perturbation), the only option for a solid piece of material is to be displaced in its entirety. Thus, all phases of matter characterised by a broken symmetry are rigid - for instance, a magnet is rigid in the sense that its magnetisation collectively rotates in the presence of a (small enough) magnetic field.

To clarify why Nambu-Goldstone modes are a **consequence** of the ability of a system to spontaneously break a symmetry, we look at the relation between symmetry and conservation laws, that is, Noether's theorem.

Noether's theorem states that every continuous symmetry corresponds to a conserved charge. We now generalise this to assume that the object breaking the symmetry retains some **translational invariance** in the symmetry-broken state - for example, a crystal, which has unit cells that repeat along a lattice. This allows us to write Noether's theorem as a **continuity equation**, $\partial_\mu j^\mu = 0$, where $j^\mu(x,t) = (Q(x,t), \vec{J}(x,t))$ is a four-vector, $Q(x,t)$ is the local Noether charge density and $\vec{J}(x,t)$ is the expectation value of the local current density.

The common definition of a symmetric state - that its Hamiltonian commutes with a symmetry operator - is not very useful in practice because it does not work for infinitely large systems, the ones studied by field theories. The **new definition** is as follows: if a state $|\Psi\rangle$ breaks the symmetry generated by $\hat{Q}$, it's possible to find an operator Φ such that

$$\langle \hat{O} \rangle = \lim_{\Sigma\to\infty} \sum_{x\in\Sigma} \langle \Psi | [j^0(x,t), \Phi] | \Psi \rangle \neq 0.$$

The operators Φ that cause $\langle \hat{O} \rangle$ to be nonzero are called interpolating fields and $\hat{O}$ is the order parameter associated with the ordered, symmetry-broken state. Physically, the **interpolating field** excites the Nambu-Goldstone modes when acting on the symmetry-broken state! The **order parameter** is the one used in the Landau expansion!

In conclusion, Goldstone's theorem is **constructive**: it establishes the existence of Nambu-Goldstone states and tells us *how* to find them - by applying, to the symmetry-broken state, the operator for the local conserved charge density, j^0, or the interpolating field that defines the order parameter, Φ.

Group Theory and the Order Parameter

Joseph Salaris

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Not so long ago, but before this course, I was having a conversation about symmetry with a math student. He had heard that symmetry is of mayor importance in theoretical physics and he asked me why this is the case. He mentioned that he did not see an obvious connection. My response was some talk about Lagrangians, but as he was not familiar with the term, it probably was not a very satisfactory answer. This week's lecture "Excitations and Fluctuations" , would have given him a good idea how symmetry is utilized, as a very clear connection between the mathematical field of symmetry and physics was discussed.

This field that, as mathematicians like to say, encompasses all the symmetry in the universe is called **group theory**. Though the lecture was geared toward noticing and predicting two types of goldstone bosons, I found a small sidestep concerning group theory particularly interesting.

First let us quickly go over the notion of a group. A group G is a set with an operation, usually called multiplication (though it can be any operation, e.g. addition or matrix multiplication) on G that satisfies four conditions:

1. The group is closed under multiplication: $g_i g_j \in G$, for any $g_i, g_j \in G$
2. The multiplication is associative: $(g_i g_j)g_k = g_i(g_j g_k)$, with $g_i, g_j, g_k \in G$
3. There is an identity element e, such that: $eg_i = g_i e = g_i$ for any $g_i, g_j \in G$
4. Each element of the group has an inverse in the group: for any $g_i \in G$, $\exists\ g_i^{-1}$ such that $g_i g_i^{-1} = e$

Using this mathematical structure, all symmetries of a particular object or state (anything really, that possesses symmetry) can be elegantly described as a group containing all corresponding symmetry transformations. Straightforward examples are geometrical objects with discrete symmetry, such as a tetrahedron where one has twelve rotations (including the identity, i.e. rotation through 2π) that form a group. Ofcourse one also has continuous symmetryies, such as rotations in two dimensions which can be described by the group of rotation matrices known as $SO(2)$. These continuous groups are known as Lie groups.

Now what I found powerful is that one can find all possible values of the order parameter using group theory. Suppose a symmetry of a group G is broken by a state. Then the symmetry-broken state still respects remaining symmetry transformations of G, whose set H forms a subset $H \subset G$ and is itself a group. Note that if no symmetries remain, there is always the identity so that in that case $H = \{e\}$. Then the quotient set $K = G/H$, which is created by partitioning G according to the following equivalence relation:

$$g_i, g_j \in G \text{ are equivalent: } g_i \sim g_j \text{ if } g_i = g_j h_k \text{ for some } h_k \in G$$

describes all possible symmetry-broken states. For example as discussed in the lecture notes, the antiferromagnetic Néel state breaks the spin rotational $SU(2)$ (or equivalently $SO(3)$) symmetry down to the symmetry of rotations in two dimensions $U(1)$ (equivalently $SO(2)$) and the set of symmetry-broken states is $SU(2)/U(1) = S^2$, the two sphere in $\mathbb{R}^3$. Note that if no symmetry remains one imediately knows that the set of symmetry-broken states is obtained from the full symmetry group $G/\{e\} = G$.

Though the quotient set K can be sometimes hard to find for non-mathematicians (or at least for me, because I saw for instance proofs of $SU(2)/U(1) = SO(3)/SO(2) = S^2$ that included mappings like fiber bundles and embeddings), it gives a well-defined prescription of obtaining the full space of the order parameter. It certainly shows a nice connection between the mathematical notion of symmetry and physics.

Gauge freedom in the quantum field theory of Cooper pairs

A.J. Pool
4130987
Universiteit Utrecht

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Elementary particles, and their interactions, are typically described by quantum field theory. By using fields to describe the probability of finding, creating or annihilating a certain type of particle at point x in a 4-dimensional spacetime, one can describe quantum phenomena in a way which is compatible with special relativity. It also provides mathematical tools to study interactions between particles, and I will use it in this essay to describe some interesting phenomena in a superconductor.

Superconductivity arises from the formation of bosonic pairs of electrons called *Cooper pairs*. The formation of these pairs can be described by an attractive interaction between electrons of opposing spin. Electrons also couple to the electromagnetic field because they are charged; this is implemented by *minimal coupling*. The action of this system is then given in imaginary time τ by

$$S[\bar{\psi}, \psi] = \int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ \sum_\sigma \bar{\psi}_\sigma \left(\frac{\partial}{\partial \tau} + \frac{1}{2m} \left(-i\nabla - e\vec{A} \right)^2 - \mu \right) \quad {}_\sigma - g\bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right\}, \tag{1}$$

where $\sigma \in \{\uparrow, \downarrow\}$, e the electron charge, m the electron mass, μ the chemical potential, $\vec{A}$ the electromagnetic vector potential, and g the strength of the interaction creating Cooper pairs. It should be noted that in this action the two spin states are treated as independent particles, instead of using the Dirac Lagrangian which couples the two states. In a gas of electrons however, disregarding short-range interactions is a valid approximation.

The interaction can be rewritten in terms of an *auxiliary field* $\Delta = \langle \bar{\psi}_\downarrow \bar{\psi}_\uparrow \rangle$ by using a *Hubbard–Stratonovich* transformation. The transformation makes use of the following identity:

$$1 = \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-(\bar{\Delta}|A|\Delta)} = \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-(\bar{\Delta} + B^* \bar{\psi}_\uparrow \bar{\psi}_\downarrow | A | \Delta + B\psi_\downarrow \psi_\uparrow)} \tag{2a}$$

$$= e^{-\int_0^{\hbar\beta} d\tau \int d\vec{r} A|B|^2 \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow} \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-\int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ A\bar{\Delta}\Delta + AB^* \Delta \bar{\psi}_\uparrow \bar{\psi}_\downarrow + AB\bar{\Delta}\psi_\downarrow \psi_\uparrow \right\}}, \tag{2b}$$

where A and B are constants, which can be chosen $A = 1/g$ and $B = g$ to let this identity cancel the interaction term in the action.

We can insert this identity in the partition function to rewrite it into the more convenient form

$$\mathcal{Z} = \int \mathcal{D}[\bar{\psi}, \quad] e^{-S[\bar{\psi}, \psi]} = \int \mathcal{D}[\bar{\psi}, \psi, \bar{\Delta}, \Delta] e^{-S[\bar{\psi}, \psi, \bar{\Delta}, \Delta]} \tag{3a}$$

$$= \int \mathcal{D}[\bar{\psi}, \psi] \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-\int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ \frac{1}{g}|\Delta|^2 - \bar{\psi}_\uparrow G_0^{-1} \quad {}_\uparrow - \bar{\psi}_\downarrow G_0^{-1} \quad {}_\downarrow - \bar{\psi}_\uparrow \Delta \bar{\psi}_\downarrow - \psi_\downarrow \bar{\Delta} \psi_\uparrow \right\}}, \tag{3b}$$

with $-G_0^{-1} = \frac{\partial}{\partial \tau} + \frac{1}{2m}(-i\nabla - e\vec{A})^2 - \mu$ the *propagator* of the electron. The action can be written in matrix form as

$$S[\bar{\Psi}, \Psi, \bar{\Delta}, \Delta] = \int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ \frac{1}{g}|\Delta|^2 - \bar{\Psi}\mathcal{G}^{-1} \quad \right\}, \tag{4}$$

with $\Psi = \begin{pmatrix} \psi_\uparrow \\ \bar{\psi}_\downarrow \end{pmatrix}$ and $\mathcal{G}^{-1} = \begin{pmatrix} G_0^{-1} & \Delta \\ \bar{\Delta} & -(G_0^{-1})^\dagger \end{pmatrix}$ a functional of r and τ. The Ψ part can be integrated out to yield

$$S[\bar{\Delta}, \Delta] = \int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ \frac{1}{g} |\Delta|^2 - \ln(\det[-\mathcal{G}^{-1}(r, \tau)]) \right\}. \tag{5}$$

It can be shown that $\ln(\det[-\mathcal{G}^{-1}(r, \tau)]) \approx \lambda |\Delta|^4$; the resulting action describes a system with a *Mexican hat potential* given by

$$V \approx \lambda |\Delta|^4 - \frac{1}{g} |\Delta|^2. \tag{6}$$

This is a situation akin to the pencil balancing on its tip mentioned in the first lecture: there is a ground state with a $U(1)$ symmetry, given by $\Delta = |\Delta_0| e^{i\theta}$ with $\Delta_0 = 1/(2g\lambda)$ for any angle θ. However in this case, the choice of the complex phase of Δ is not a physical symmetry but a gauge freedom because a rotation of a field in the complex plane does not have any physical consequences. We do not observe spontaneously broken symmetry here; a choice of θ merely fixes the gauge.

Just like a system with spontaneously broken symmetry exhibits Goldstone bosons, the θ field also has excitations that can be thought of as a particle. The potential around a ground state of S is approximately quadratic, which means that to find the excitations of the field one can take the second order expansion of the action given by

$$S[\theta, A] = \int_0^{\hbar\beta} d\tau \int d\vec{r} \left\{ \rho \left(\frac{\partial \theta}{\partial \tau} \right)^2 + \frac{n_S}{2m} \left| \nabla \theta - \vec{A} \right|^2 \right\}, \tag{7}$$

with $\vec{A}$ the electromagnetic vector potential, ρ the electronic density of states near the Fermi level, and n_S the number of Cooper pairs. Minimising S, one finds $\frac{\partial \theta}{\partial \tau} = 0$ and $\vec{A} = \nabla \theta$. $\vec{A}$ is thus curl-free by its coupling to the θ field: the superconductor expels the magnetic field. This is known as the Meißner effect.

The same coupling between θ and the electromagnetic field also causes the photons – excitations of the A field – to be massive. This is one particular example of the Higgs mechanism which explains the mass of *gauge bosons*, which are excitations of gauge fields.