Roughly one page essay: A summary.

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This week I just want to write a summary of the lectures so far. I will focus on the Antiferromagnet, just because it is the most illustrative case we have used in the lectures. With that summary included, I feel ready to face next week's lecture. In addition, it should serve to refresh my memory in the future.

In this roughly one page essay, we start by looking back at the properties of symmetry in quantum mechanics, the emergence of Goldstone modes and their implications for the dispersion relations and finally the order parameters that tell us how to measure order, if we made the right choices.

I. THE HEISENBERG MODEL

First, recall the Heisenberg Model:

$$\mathscr{H} = J \sum_{\langle ij\rangle} \sum_{\alpha\in\{x,y,z\}} S_i^\alpha S_j^\alpha \tag{1}$$

We can define spin-flip operators $S_k^\pm = S_k^x \pm \imath S_k^y$, that flip spins in the z direction. Note that this choice is a manifestation of gauge freedom, in the sense that we can choose any direction (basis). We could have easily chosen the x direction and defined flip operators with the y and z operators.

We can rewrite the Hamiltonian:

$$\mathscr{H} = J \sum_{\langle ij\rangle} \left[S_i^z S_j^z + \frac{1}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right)\right] \tag{2}$$

II. SYMMETRIES

What are the global symmetries of the Heisenberg model? Take a small detour to find the spin-flip and total directional spin commutators:

$$\begin{aligned}
[S^z, S_i^\pm] &= \sum_j \left[S_j^z, S_i^\pm\right] \pm \imath \left[S_j^z, S_i^\pm\right] \\
&= \sum_j \delta_i^j \left[\imath\epsilon_{zx\gamma} S_i^\gamma \pm \imath\epsilon_{zy\gamma} S_i^\gamma\right] \\
&= \pm S_i^\pm
\end{aligned}$$

It is now easy to see that the spin-flip terms of equation 2 are both zero. For example:

$$\begin{aligned}
\left[S_i^+ S_j^-, S^z\right] &= S_i^+ \left[S_j^-, S^z\right] + \left[S_i^+, S^z\right] S_j^- \\
&= -S_i^+ S_j^- + S_i^+ S_j^- \\
&= 0
\end{aligned}$$

Therefore, we find that $[\mathscr{H}, S^z]$ is zero. But remember, z wasn't special at all before we rewrote the Hamiltonian. That was a result of us choosing the spin-flip operators, so we find that $\left[\mathscr{H}, S^{\{x,y,z\}}\right] = 0$.

If the symmetry operator U generated by a Hermitian operator G commutes with the Hamiltonian, we know from the Heisenberg picture that the operators, and hence their expectation values, are conserved in time. Noether's theorem in reverse then implies a global symmetry.

III. THE FERROMAGNET AND THE NÉEL STATE

When $J > 0$, the energy is lowered when all spins are anti-parallel to their neighbours. If you divide the lattice into two sublattices, then you can have all spins on the first sublattice A point up and all on the second sublattice B point down. Such a perfect antiferromagnetic arrangement of spins is known as the Néel state. However, it is not the ground state - finding the ground state for a 3D lattice is an open problem in condensed matter physics.

If all the spins are anti-parallel, then it is easy to consider an interpolating field $\Phi^y = \imath\left(S_a^y - S_b^y\right)$ and see that:

$$\begin{aligned}
\psi &= \langle \text{Néel}\big|\left[S^x, \Phi^y\right]\big|\text{Néel}\rangle \qquad (3)\\
&= \langle \text{Néel}\big|S_b^z - S_a^z\big|\text{Néel}\rangle \qquad (4)\\
&\neq 0 \qquad (5)
\end{aligned}$$

This implies a lot of things. First, Φ^y is an interpolating field that shows the symmetry generated by S^x is broken, while Φ^x shows the same for S^y.

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Second, by Goldstone's Theorem a spontaneously broken symmetry gives rise to excitations whose energy tends to zero in the infinite wave length limit. These are the Nambu-Goldstone modes.

Third, because Φ^x and Φ^y are not themselves part of the symmetry group of $\mathscr{H}$, a type B Goldstone mode will be found. A type B Goldstone mode implies $\omega \propto \kappa^2$, as opposed to the $\omega \propto \kappa$ dispersion of a type A Goldstone mode.

Finally, the Néel state breaks the global $SU(2)$ of the Hamiltonian down to a $U(1)$ global symmetry. In some cases, you can find an additional order parameter. For the partially polarised ferromagnet, we could show that we had the same generator for the two broken symmetries and find a single type B mode. It could be shown, however, that it also had a nematic order parameter that gives rise to a type A mode. This is a massive partner (type A) to the massless mode (type B) that we had already found. A type B mode can be seen as two type A modes with interaction.

For future reference, I want to mention that we also treated the Mermin-Wagner theorem. By looking at the fluctuations of a harmonic crystal, we could show that no ordered state can exist at any non-zero temperature in fewer than three dimensions. Quantum fluctuations will prevent the formation of order in these cases. Here, an ordered state is understood to spontaneously break a continuous symmetry.

IV. CONCLUSION

The Heisenberg model is a proper toy-model for these lectures, as it is possible to cram most keywords into a discussion of the antiferromagnet Heisenberg model. While some things are still unclear, such as the proper meaning of 'spontaneous', the purpose of Goldstone modes, how these dispersions emerge for type A (B) Goldstone modes, and so forth, a picture is starting to become clear.

Especially now that we've found order parameters, we can clearly define what we mean by symmetry breaking and phases of matter.

I would like to finish with a quote from our esteemed lecturer, Dr. Jasper van Wezel:

> Pies are goldstone modes.

REFERENCES