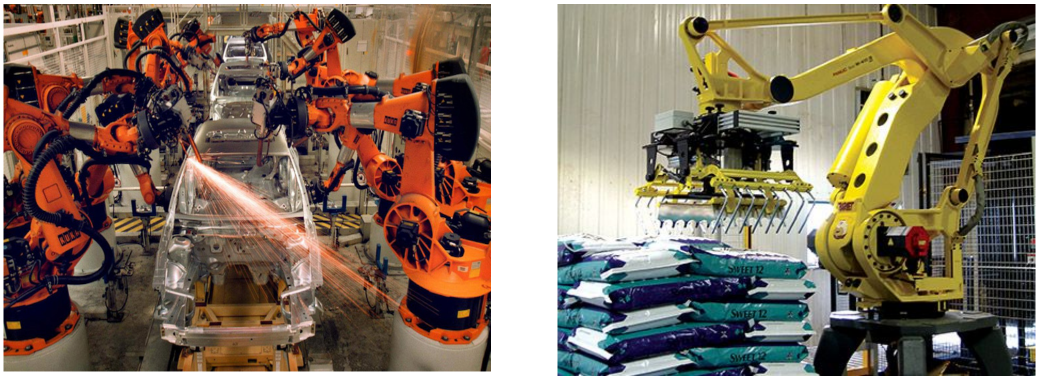
Figure H.1: Examples of Industrial Robots

(a) Spot Welding (b) Material Handling



Sources: Autobot Systems and Automation (https://www.autobotsystems.com) and PaR Systems (https://www.par.com)

Online Appendix (Not for Publication)

H Further Details in Data and Empirical Setting

H.1 Robot Definition and Examples

As defined in Footnote 1, industrial robots are defined as multiple-axes manipulators. More formally, following International Organization for Standardization (ISO), I define robots as "automatically controlled, reprogrammable, multipurpose manipulator, programmable in three or more axes, which can be either fixed in place or mobile for use in industrial automation applications" (ISO 8373:2012). This section gives a detailed discussion about such industrial robots. Figure H.1 shows the pictures of examples of industrial robots that are intensively used in the production process and considered in this paper. The left panel shows spot-welding robots, while the right panel shows the material-handling robots. The spot welding robots are an example of robots in routine-production occupations, while the material-handling robots are that in routine-transportation (material-moving) occupations.

It is also worthwhile to give an example of technologies that are *not* robots according to the definition in this paper. An example of a growth in technology in the material-handling area is autonomous driving. Mehta and Levy (2020) predicts that such automation will grow strong and result in the reduction of total number of jobs in this area in eight to ten years since 2020. However, since autonomous vehicles do not operate multiple-axes, they are not treated in this paper at all. A similar observation applies for computers or artificial intelligence more generally.

H.2 Methods for Adjusting the Robot Prices

In the paper, I use the general equilibrium model to control for the quality component of robot prices. However, there are other methods proposed in the literature of measuring the price of capital goods. In this subsecsion, I briefly describe these methods and their limitations.

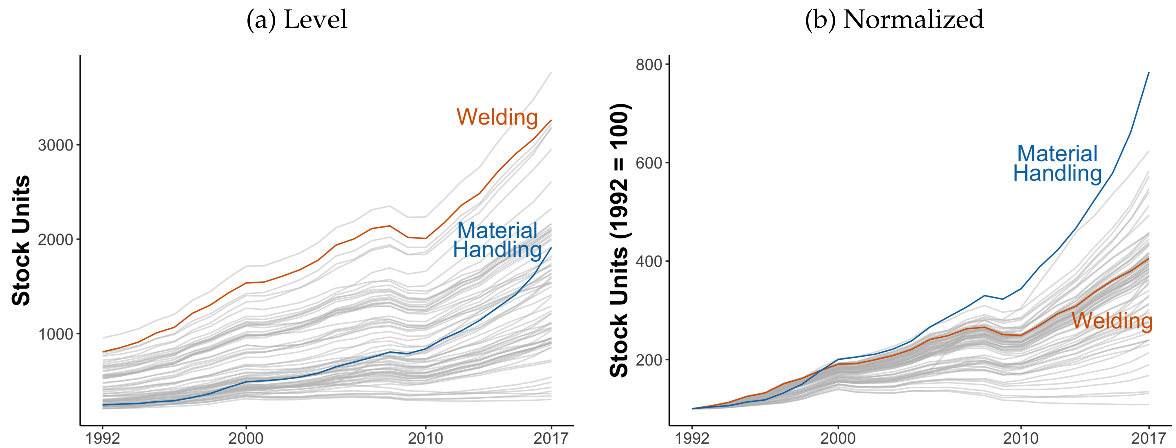
Another approach to solving this issue is to control for the quality change by the hedonic approach as in Timmer, Van Ark, et al. (2007), and in the application to digital capital in Tambe et al. (2019). The hedonic approach requires detailed information about the detailed specification of each robot. Unfortunately, it is difficult to keep track of the detailed specifications of commonly used robots as the robotics industry is rapidly changing.

Another method is a more data-driven one. Specifically, the Bank of Japan (BoJ) provides the quality-controlled price index. However, the method is not clearly declared. In fact, it is claimed to be "cost-evaluation method," in which the BoJ asks producer firms to measure the component of quality upgrading for price changes between periods. Unfortunately, I do not know the survey firms and quality components. Therefore, it is hard for me to determine better measures, and so I stick to use my raw price measure based on the representativeness of my data.

H.3 Trends of Robot Stocks and Prices

In this section, I show that each occupation experienced different trends in robot adoption. Figure H.2 shows the trend of US robot stocks at the occupation level. In the left panel, I show the trend of raw stock, which reveals the following two facts. Firstly, it shows that the overall robot stocks increased rapidly in the period, as found in the previous literature. Second, the panel also depicts that the increase occurred at different speeds

Figure H.2: US Robot Stocks at the Occupation Level



Note: The author's calculation based on JARA and O*NET. The figure shows the trend of stocks of robots in the US for each occupation. The left panel shows the level, whereas the right panel shows the normalized trend at 100 in 1992. In both panels, I highlight two occupations. "Welding" corresponds the occupation code in IPUMS USA, OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds the occupation code OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises.

across occupation. To highlight such a difference, in the right panel, I plot the normalized trend at 100 in the initial year. There is significant heterogeneity in the growth rates, ranging from a factor of one to eight.

To further emphasize the different speed, in these two figures, I color the following two occupations: "Welding, Soldering, and Brazing Workers" (or "Welding") and "Laborers and Freight, Stock, and Material Movers, Hand" (or "Material handling"). On the one hand, the stock of welding robots grew continuously throughout the period, as can be confirmed in the left panel. However, the growth rate is not outstanding but within the range of growth rates of other occupations. On the other hand, material handling was not a majority occupation as of the initial year, but it grew at the most rapid pace in the sample period. These findings indicate the difference between the automation shocks to each occupation. Some occupations were already somewhat automated by robots as of the initial year, and the automation process continued afterward (e.g., welding). There are a few occupations where robotics automation had not occurred initially, but the adoption proceeded rapidly in the sample period (e.g., material handling). This observation bases the model that incorporates the heterogeneity across occupations in the Model section.

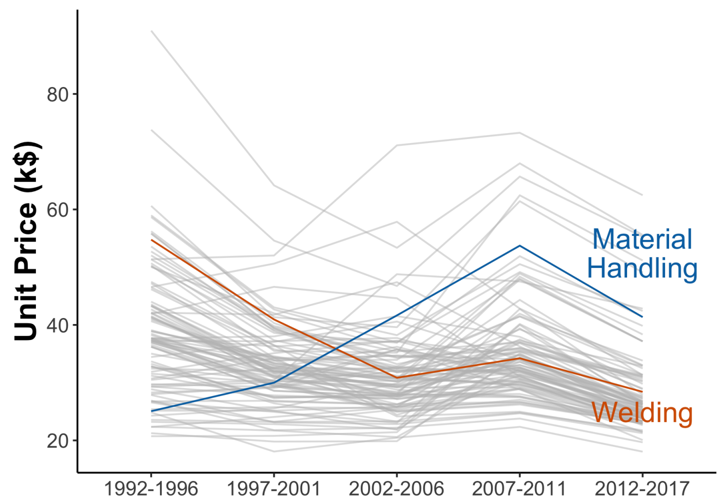
Next, Figure H.3 shows the trend of prices of robots in the US for each occupation. In addition to the overall decreasing trend, there is significant heterogeneity in the pattern of price falls across occupations. For instance, although the welding robots saw a large drop in the price during the 1990s, the material handling robots did not but increased the price over the sample period. These patterns are strongly correlated across countries, as indicated by the correlation coefficient of 0.968 between the US and non-US prices at the occupation-year level. Based on this finding, I use the non-US countries' prices as the Japan robot shock to the US in the Data section.

In Figure H.3, one might wonder if there is an anomaly to the overall decreasing trend in the 2007-2011 period, in which the trends of robot prices halt dropping across the board. This pattern emerges because of the method for generating these series. The average price is measured by total sales divided by total units. During the Great Recession period, the total units decreased more than the total sales did. The relative pattern caused a temporary increase in average robot prices. In addition, after the Great Recession, both the growth of sales and units of robots accelerated. These observations suggest the structural break of the robot industry during the Great Recession, which is out of the scope of the paper.

H.4 Labor Market Outcomes and Various Robot Measures

Figure H.4 plots the correlation between the changes in robot measures and the changes in log labor market outcomes in the US at the occupation level, weighted by the size of occupation measured by initial the employment level. The top two panels take the occupational wage as a labor market outcome on the y-axis, while the bottom two take the occupational employment. The left two panels take log monetary value of robot stock in US as a robot measure on the left panel, and the right two take the log robot prices from Japan. As expected, the correlation between the two labor market outcomes and the robot stock measure is negative, while that with the Japanese robot price is positive.

Figure H.3: Robot Prices at the Occupation Level



Note: The author's calculation based on JARA and O*NET. The figure shows the trend of prices of robots in the US for each occupation. I highlight two occupations. "Welding" corresponds the occupation code in IPUMS USA, OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds the occupation code OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises. The dollars are converted to 2000 real US dollar using CPI.

Next, I consider the role of control variables. Figure H.5 shows the results of a set of robustness checks. Each panel shows the same result as the ones corresponding to the same panel in Figure H.4, but after residualizing all variables with respect to the demographic control variables (initial-year female share, college graduates share, age 35-49 share, age 50-64 share, and foreign-born share in each occupation). The main result is robust to the control of these demographic variables.

Note that, the robot adoption in the US, as used in Figure H.4 and H.5, might suffer from the endogeneity bias due to the demand factors that drive both the labor market outcome and robot adoption. This concern is addressed in Appendix E.6. Figure H.6 shows the result without controling for the demographic variables, which shows a qualitatively similar pattern as the main results.

Next, one may be concerned that robot quality changes over year. Specifically, if the per-unit efficiency of robots increases over year, the average unit price may understate the decrease in the price of robots. To deal with this concern, I consider the following method of quality adjustment, based on the spirit of Khandelwal, Schott, and Wei (2013).

Figure H.4: Correlation between US Occupational Wage and US Robot Measures (Raw)

Note: The author's calculation based on JARA, O*NET, and the US Census/ACS. The figures show the scatterplot, weighted fit line, and the 95 percent confidence interval of the changes in log robot measures and changes in log labor market outcomes. Each bubble represents a 4-digit occupation. The bubble size reflects the employment in the baseline year (1990, which is the closest Census year to the initial year that I observe the robot adoption, 1992). All variables are partialled out by control variables (the occupational female share, college share, age distribution, and foreign born share). In panel (a), y-axis and x-axis are log occupational wages and the change of log US robot stock measured by the monetary value. In panel (b), y-axis and x-axis are log occupational wages and the change of log US robot price. In panel (c), y-axis and x-axis are log occupational wages and the change of log US robot stock measured by the monetary value. In panel (d), y-axis and x-axis are log occupational wages and the change of log US robot prices.

Namely, I fit the following equation with the fixed-effect regression:

$$\ln\left(X^R_{JPN\to i,o,t}\right) = -\varsigma \ln\left(p^R_{JPN\to i,o,t}\right) + a^R_{o,t} + e^R_{i,o,t},$$

from which I obtain the fixed effect $a^R_{o,t}$, which absorbs the occupation-o specific log sales component that is not explained by the prices. I then proxy the quality change by the change in such fixed effects, $\Delta a^R_{o,t} \equiv a^R_{o,t} - a^R_{o,t_0}$. The (log) quality-adjusted price is then ob-

Figure H.5: Correlation between US Occupational Wage and US Robot Measures (Controlled)

(a) Wage and Robot Values

(b) Wage and Robot Prices

(c) Employment and Robot Values

(d) Employment and Robot Prices

Note: The author's calculation based on JARA, O*NET, and the US Census/ACS. The figures show the scatterplot, weighted fit line, and the 95 percent confidence interval of the changes in log robot measures and changes in log labor market outcomes. Each bubble represents a 4-digit occupation. The bubble size reflects the employment in the baseline year (1990, which is the closest Census year to the initial year that I observe the robot adoption, 1992). All variables are partialled out by control variables (the occupational female share, college share, age distribution, and foreign born share). In panel (a), y-axis and x-axis are log occupational wages and the change of log US robot stock measured by the monetary value. In panel (b), y-axis and x-axis are log occupational wages and the change of log US robot price. In panel (c), y-axis and x-axis are log occupational wages and the change of log US robot stock measured by the monetary value. In panel (d), y-axis and x-axis are log occupational wages and the change of log US robot prices.

tained by $\ln\left(p^R_{JPN\to i,o,t}\right) - \Delta a^R_{o,t}$. Figure H.7 shows the result of correlation using quality-adjusted robot prices All the results are robust to these considerations–wage growths are negatively correlated with stock growths, and positively correlated with price growths, both across occupations.

Figure H.6: Correlation between US Occupational Wage and Non-US Robot Measures (Raw)

(a) Wage and Robot Values

(b) Wage and Robot Prices

(c) Employment and Robot Values

(d) Employment and Robot Prices

Note: The author's calculation based on JARA, O*NET, and the US Census/ACS. The figures show the scatterplot, weighted fit line, and the 95 percent confidence interval of the changes in log robot measures and changes in log labor market outcomes. Each bubble represents a 4-digit occupation. The bubble size reflects the employment in the baseline year (1990, which is the closest Census year to the initial year that I observe the robot adoption, 1992). In panel (a), y-axis and x-axis are log occupational wages and the change of log non-US robot stock measured by the monetary value. In panel (b), y-axis and x-axis are log occupational wages and the change of log non-US robot price. In panel (c), y-axis and x-axis are log occupational wages and the change of log non-US robot stock measured by the monetary value. In panel (d), y-axis and x-axis are log occupational wages and the change of log non-US robot prices.

H.5 Robot Price Trends by Occupation Groups

In this section, I examine the facts discussed in Section 2.3 for each occupation group described in Section 4.1. Figure H.8 shows the plot of the trend of the robot price distribution since 1992 for each occupation group, a version of Figure 1a, disaggregated by occupation groups. The top three panels show the trends for routine occupations, namely, from the left, routine-production, routine-transportation, and routine-others. The bottom two panels show the trends for service occupations and abstract occupations, from the left.

Figure H.7: Correlation between US Occupational Wage and Non-US Quality-adjusted Robot Prices

(a) Wage and Robot Price, Raw

(b) Wage and Robot Price, Controlled

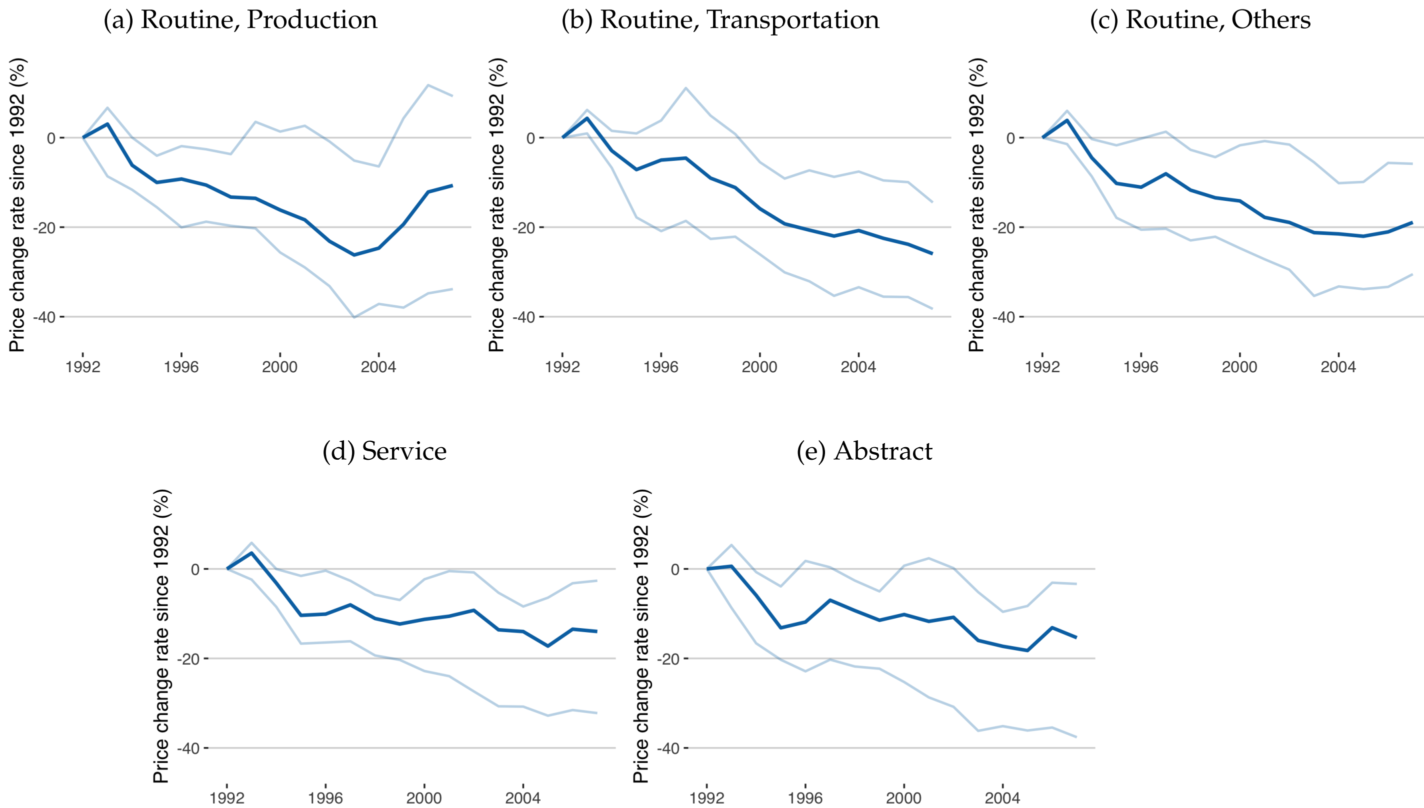
(c) Employment and Robot Price, Raw

(d) Employment and Robot Price, Control

Note: The author's calculation based on JARA, O*NET, and the US Census/ACS. The figures show the scatterplot, weighted fit line, and the 95 percent confidence interval of the changes in log robot measures and changes in log labor market outcomes. Each bubble represents a 4-digit occupation. The bubble size reflects the employment in the baseline year (1990, which is the closest Census year to the initial year that I observe the robot adoption, 1992). In panel (a), y-axis and x-axis are log occupational wages and the change of log non-US quality-adjusted robot price measured in raw values. In panel (b), y-axis and x-axis are log occupational wages and the change of log non-US quality-adjusted robot price, both partialled out by control variables (the occupational female share, college share, age distribution, and foreign born share). In panel (c), y-axis and x-axis are log occupational wages and the change of log non-US quality-adjusted robot price in raw values. In panel (d), y-axis and x-axis are log occupational wages and the change of log non-US quality-adjusted robot prices, both partialled out by control variables.

All these panels show the overall decreasing trend of robot prices, and the dispersion of prices within each occupation group. Having such a dispersion is important because in Section 4 when I estimate heterogeneous EoS between robots and labor, I use the price variation within each occupation group.

Figure H.8: Robot Price Trends by Occupation Groups



Note: The author's calculation based on JARA and O*NET. Panels show the trend of robot prices by occupation groups defined in the main text.

H.6 The Effect of Robots from Japan and Other Countries

A potential concern for my empirical setting is the selection issue regarding the robot origin country of Japan. Specifically, robots from Japan may be different from those from other countries, so the labor market implication may also differ between them. Unfortunately, it is hard to directly compare the effects of these two different groups of robots due to the data limitation, so I will focus on the best comparable measures of robotization between Japan-sourced robots and robots from all countries, which is the quantity of robot stock. Specifically, I take the total stock of robots in the US from the IFR data. This measure does not contain the monetary value at the occupation level for all the sample periods, but it is the number of units. Note also that the IFR variable is the total number that does not specify the source country. I then convert the IFR application codes to the JARA application codes to use the allocation rule for matching the JARA application codes and the occupation codes. As a result, I obtain the robots used in the US and sourced from any countries at the occupation level. I then run the following regression to examine the correlations between labor market outcomes and the robot measures and compare them depending on if the measure is robots from Japan or robots from all over the world:

$$\Delta Y_o = \beta^Q \Delta K_o^{R,Q} + X_o \gamma^Q + \varepsilon_o^Q, \tag{H.1}$$

where ΔY_o is either the changes in wages or employment at the occupation-o level, ΔK_o^Q is the measure of the number of robots taken either from JARA (so robots from Japan) or IFR (so robots from the world), and ε_o^Q is the error term. The coefficient of interest is β^Q, which gives us an insight about the correlation between the changes in labor market outcomes and the changes in robot quantity, depending on if the robots are conditioned to be sourced from Japan or not. Specifically, if robots from Japan may substitute workers stronger than robots from the other countries, coefficient β^Q is expected to be larger when we use the JARA robot measure than IFR.

Table H.1 shows the regression result of equation (H.1). Columns 1-4 considers the changes in occupational wage in the outcome variable, while columns 5-8 take occupational employment. Columns 1, 2, 5, and 6 do not include the demographic control variables (female share, age distribution, college-graduate share, and foreign-born share), while columns 3, 4, 7, and 8 do. Columns 1, 3, 5, and 7 take the robots from Japan from JARA data, while columns 2, 4, 6, and 8 take the robots from the world from IFR data. Table H.1 reveals that both the JARA- and IFR-based robot measures capture the substitution of workers with robots. The result for the IFR data is in line with the previous finding, such as Acemoglu and Restrepo (2020). In contrast, comparing the size of coefficients, one can find that the coefficient is somewhat stronger for JARA robot measures than for IFR. Overall, I find some evidence that Japanese robots substitute workers stronger than other countries' robots, while all sorts of robots do seem to have some substitution effect on

Table H.1: Regression Result of Labor Market Outcome on JARA and IFR Robot Stocks

VARIABLES	(1) $\Delta \ln(w)$	(2) $\Delta \ln(w)$	(3) $\Delta \ln(w)$	(4) $\Delta \ln(w)$	(5) $\Delta \ln(L)$	(6) $\Delta \ln(L)$	(7) $\Delta \ln(L)$	(8) $\Delta \ln(L)$
$\Delta \ln(K^{R,Q}_{JPN\to USA})$	-0.372***		-0.264***		-0.765***		-0.636***	
	(0.0311)		(0.0317)		(0.0903)		(0.0830)	
$\Delta \ln(K^{R,Q}_{USA})$		-0.144***		-0.110***		-0.311***		-0.461***
		(0.0161)		(0.0196)		(0.0447)		(0.0461)
Observations	324	324	324	324	324	324	324	324
R-squared	0.307	0.200	0.334	0.262	0.182	0.131	0.179	0.260
Demographic controls			✓	✓			✓	✓

Note: The author's calculation based on JARA, IFR, O*NET, and US Census/ACS. Observations are 4-digit level occupations, and the sample is all occupations that existed throughout 1970 and 2007. Columns 1-4 considers the changes in occupational wage in the outcome variable, while columns 5-8 take occupational employment. Columns 1, 2, 5, and 6 do not include the demographic control variables (female share, age distribution, college-graduate share, and foreign-born share), while columns 3, 4, 7, and 8 do. Columns 1, 3, 5, and 7 take the robots from Japan from JARA data, while columns 2, 4, 6, and 8 take the robots from the world from IFR data. *** p<0.01, ** p<0.05, * p<0.1.

Table H.2: List of Data Sources

Variable	Description	Source
$\widetilde{y}^G_{ij,t_0}, \widetilde{x}^G_{ij,t_0}, \widetilde{y}^R_{ij,t_0}, \widetilde{x}^R_{ij,t_0}$	Trade shares of goods and robots	BACI, IFR
$\widetilde{x}^O_{i,o,t_0}$	Occupation cost shares	IPUMS
l_{i,o,t_0}	Labor shares within occupation	JARA, IFR, IPUMS
$s^G_{i,t_0}, s^V_{i,t_0}, s^R_{i,t_0}$	Robot expenditure shares	BACI, IFR, WIOT
$\alpha_{i,M}$	Intermediate input share	WIOT

workers.

H.7 Data about Initial Shares

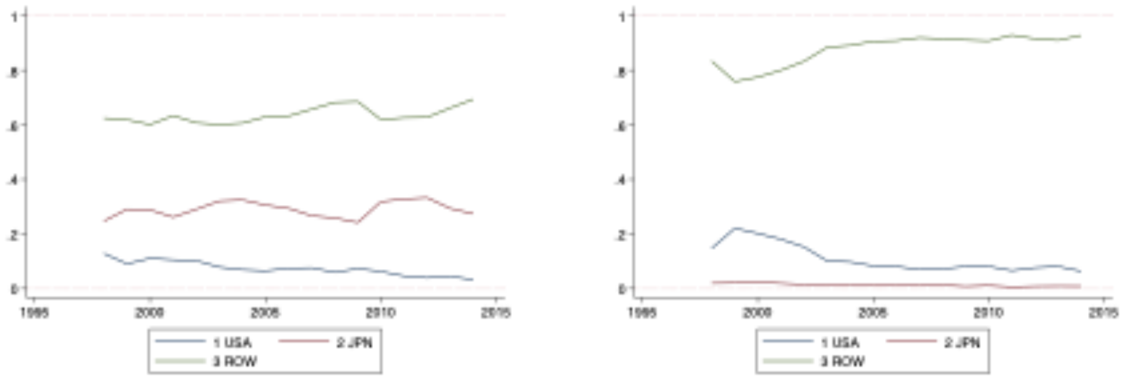
Since the log-linearized sequential equilibrium solution depends on several initial share data generated from the initial equilibrium, I discuss the data sources and methods for measuring these shares. I define $t_0 = 1992$ and the time frequency is annual. I consider the world that consists of three countries $\{USA, JPN, ROW\}$. Table H.2 summarizes overview of the variable notations, descriptions, and data sources.

I take matrices of trade of goods and robots by BACI data. As in Humlum (2019), I measure robots by HS code 847950 ("Industrial Robots For Multiple Uses") and approximate the initial year value by year of 1998, in which the robot HS code is first available. Figure H.9 shows the trend of export and import shares of robots among the world for the US, Japan, and the Rest Of the World. The trends are fairly stable for the three regions of the world, except that the import share of the US has declined relative to the ROW.

To obtain the domestic robot absorption data, I take from IFR data the flow quantity variable and the aggregate price variable for a selected set of countries. I then multiply these to obtain USA and JPN robot adoption value. For robot prices in ROW, I take the simple average of the prices among the set of countries (France, Germany, Italy, South Korea, and the UK, as well as Japan and the US) for which the price is available in 1999, the earliest year in which the price data are available. Graetz and Michaels (2018) discuss prices of robots with the same data source. Figure H.10 shows the comparison of the US price index measure available between JARA and IFR. The JARA measures are

Figure H.9: Robot Trade Share Trends

(a) Exports (b) Imports

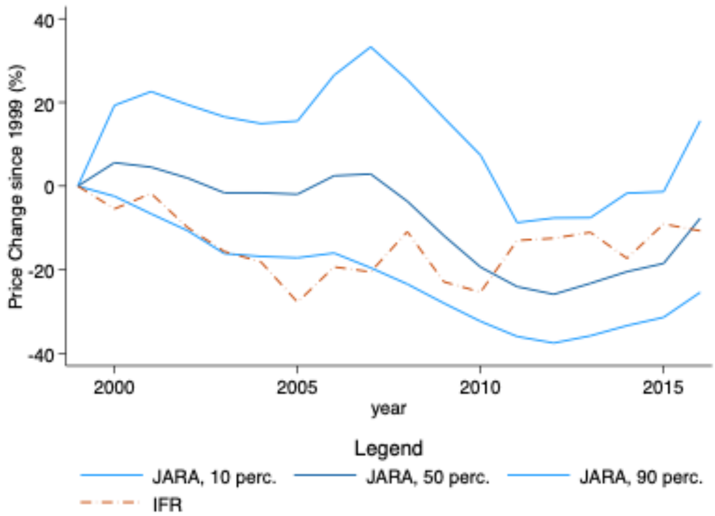


Note: The author's calculation of world trade shares based on the BACI data. Industrial robots are measured by HS code 847950 (Industrial robots for multiple uses).

disaggregated by 4-digit occupations. The figure shows the 10th, 50th (median), and 90th percentiles each year, as in Figure 1a. All measures are normalized at 1999, the year in which the first price measure is available in the IFR data. Overall, the JARA price trend variation tracks the overall price evolution measured by IFR reasonably well: The long-run trends from 1999 to the late 2010s are similar between the JARA median price and the IFR price index. During the 2000s, the IFR price index drops faster than the median price in the JARA data. It compares with the JARA 10th percentile price, which could be due to robotic technological changes in other countries than Japan in the corresponding period.

I construct occupation cost shares $\widetilde{x}^{O}_{i,o,t_0}$ and labor shares within occupation l_{i,o,t_0} as follows. To measure $\widetilde{x}^{O}_{i,o,t_0}$, I aggregate the total wage income of workers that primarily works in each occupation o in year 1990, the Census year closest to t_0. I then take the share of this total compensation measure for each occupation. To measure l_{i,o,t_0}, I take the total compensation as the total labor cost and a measure of the user cost of robots for each occupation. The user cost of robots is calculated with the occupation-level robot price data available in IFR and the set of calibrated parameters in Section 4.1. Table H.3 summarizes these statistics for the aggregated 5 occupation groups in the US. One can see that the cost for production occupations and transportation occupations comprise 18% and 8% of the

Figure H.10: Comparison of US Price Indices between JARA and IFR



Note: The author's calculation of US robot price measures in JARA and IFR. The JARA measures are disaggregated by 4-digit occupations, and the figure shows the 10th, 50th (median), and 90th percentiles each year. All measures are normalized at 1999, the year in which the first price measure is available in the IFR data.

Table H.3: Baseline Shares by 5 Occupation Group

Occupation Group	$\widetilde{x}^{O}_{1,o,t_0}$	l^{O}_{1,o,t_0}	y^{R}_{2,o,t_0}	x^{R}_{1,o,t_0}	x^{R}_{2,o,t_0}	x^{R}_{3,o,t_0}
Routine, Production	17.58%	99.81%	64.59%	67.49%	62.45%	67.06%
Routine, Transportation	7.82%	99.93%	12.23%	11.17%	13.09%	11.04%
Routine, Others	28.78%	99.99%	10.88%	9.52%	11.68%	10.40%
Service	39.50%	99.99%	8.87%	8.58%	9.17%	8.32%
Abstract	6.32%	99.97%	3.43%	3.24%	3.60%	3.18%

Note: The author's calculation of initial-year share variables based on the US Census, IFR, and JARA. As in the main text, country 1 indicates the US, country 2 Japan, and country 3 the rest of the world. See the main text for the construction of each variable.

US economy, respectively, totaling more than one-fourth. Furthermore, the share of robot cost in all occupations is still quite low with the highest share of 0.19% in production occupations, revealing still small-scale adoption of robots from the overall US economy perspective.

To calculate the effect on total income, I also need to compute the sales share of robots by occupations $y^{R}_{i,o,t_0} \equiv Y^{R}_{i,o,t_0} / \sum_o Y^{R}_{i,o,t_0}$ and the absorption share $x^{R}_{i,o,t_0} \equiv X^{R}_{i,o,t_0} / \sum_o X^{R}_{i,o,t_0}$. To obtain y^{R}_{i,o,t_0}, I compute the share of robots by occupations produced in Japan $y^{R}_{2,o,t_0} = Y^{R}_{2,o,t_0} / \sum_o Y^{R}_{2,o,t_0}$ and assume the same distribution for other countries due to the data limitation: $y^{R}_{i,o,t_0} = y^{R}_{2,o,t_0}$ for all i. To have x^{R}_{i,o,t_0}, I compute the occupational robot adoption in each country by $X^R_{i,o,t_0} = P^R_{i,t_0} Q^R_{i,o,t_0}$, where Q^R_{i,o,t_0} is the occupation-level robot quantity obtained by the O*NET concordance generated in Section 2.2 applied to the IFR application classification. As mentioned above, the robot price index P^R_{i,t_0} is available for a selected set of countries. To compute the rest-of-the-world price index P^R_{3,t_0}, I take the average of all available countries weighted by the occupational robot values each year. The summary table for these variables y^R_{i,o,t_0} and x^R_{i,o,t_0} at 5 occupation groups are shown in Table H.3. All values in Table H.3 are obtained by aggregating 4-digit-level occupations, and raw and disaggregated data are available upon request.

I take a more standard measure, the intermediate input share $\alpha_{i,M}$, from World Input-Output Tables (WIOT Timmer, Dietzenbacher, et al. 2015). Finally, I combine the trade matrix generated above and WIOT to construct the good and robot expenditure shares s^G_{i,t_0}, s^V_{i,t_0}, and s^R_{i,t_0}. In particular, with the robot trade matrix, I take the total sales value by summing across importers for each exporter, and total absorption value by summing across exporters for each importers. I also obtain the total good absorption by WIOT. From these total values, I compute expenditure shares. are obtained by aggregating 4-digit occupations, and the disaggregated data are available upon request.

As initial year occupation switching probabilities μ_{i,oo',t_0}, I take 1990 flow Markov transition matrix from the cleaned CPS-ASEC data created in Section E.1. Table H.4 shows this initial-year conditional switching probability. The matrix for the other years are available upon request. As for other countries than the US, although Freeman, Ganguli, and Handel (2020) has begun to develop occupational wage measures consistent across country, world-consistent occupation employment data are hard to obtain. Therefore, I assign the same flow probabilities for other countries in my estimation.

I Further Details in Theory

Table H.4: 1990 Occupation Group Switching Probability

		Routine Production	Routine Transportation	Routine Others	Service	Abstract
Routine	Production	0.961	0.011	0.010	0.006	0.012
Routine	Transportation	0.020	0.926	0.020	0.008	0.025
Routine	Others	0.005	0.006	0.955	0.020	0.014
Service		0.003	0.002	0.020	0.967	0.007
Abstract		0.014	0.014	0.036	0.015	0.922

Note: The author's calculation from the CPS-ASEC 1990 data. The conditional switching probability to column occupation group conditional on being in each row occupation.

I.1 Adjustment Cost of Robot Capital

Another key feature of the model is the convex adjustment cost of robot adoption. To interpret this, consider the cost of adopting new technology and integration. With the convex adjustment cost, the model predicts the staggered adoption of robots over years that I observe in the data (see Figure 3b), and implies a rich prediction about the short- and long-run effects of robotization.

First, when adopting new technology including robots, it is necessary to re-optimize the overall production process since the production process is typically optimized to employ workers. More generally, the literature of organizational dynamics studies the difficulty, not to say the impossibility, of changing strategies of a company due to complementarities (see Brynjolfsson and Milgrom 2013 for a review). Such a re-optimization incurs an additional cost of adoption in addition to the purchase of robot arms. Moreover, even within a production unit, there is a variation of this difficulty of adopting robots across production processes. In this case, the part where the adjustment is easy adopts the robots first, and vice versa. This allocation of robot adoptions over years may aggregate to make the robot stock increase slowly (Baldwin and Lin 2002). Waldman-Brown (2020) also finds that the incremental and sluggish automation is particularly well-observed in small and medium-sized firms, as they add "a machine here or there, rather than installing whole new systems that are more expensive to buy and integrate" (Autor, Mindell, and Reynolds 2020).

The second component of the adjustment cost may come from the cost of integration as discussed in Section 2.1. The marginal integration cost may increase as the massive upgrading of robotics system may require large-scale overhaul of production process, which increases the complexity and so is costly. The adjustment cost may capture the increasing marginal cost component of the integration cost. It explains an additional component of the integration cost implied by constant returns-to-scale robot aggregation in equation (9).

Another potential choice of modeling a staggered growth of robot stocks is to assume a fixed cost of robot adoption and lumpy investment. Humlum (2019) finds that many plants buy robots only once during the sample period. Since JARA data does not observe plant-level adoptions, I do not separately identify lumpy investment from the staggered growth of robot stocks in the data. To the extent that fixed cost of investment may make the policy intervention less effective (e.g., Koby and Wolf 2019), the counterfactual analysis in this paper may overestimate the effect of robot taxes since it does not take into account the fixed cost and lumpiness of investment.

I.2 Derivation of Worker's Optimality Conditions

In this section, I formalize the assumptions behind the derivation and show equations (B.2) and (B.3). One trick new to the below discussion is that I characterize the switching cost by an ad-valorem term, which makes the log-linearization simpler when solving the model.

Fix country i and period t. There is a mass $\overline{L}_{i,t}$ of workers. In the beginning of each period, worker $\omega \in \left[0, \overline{L}_{i,t}\right]$ draws a multiplicative idiosyncratic preference shock $\left\{Z_{i,o,t}\left(\omega\right)\right\}_{o}$ that follows an independent Fréchet distribution with scale parameter $A^{V}_{i,o,t}$ and shape parameter $1/\phi$. Note that one can simply extend that the idiosyncratic preference follows a correlated Fréchet distribution to allow correlated preference across occupations, as in Lind and Ramondo (2018). To keep the expression simple, I focus on the case of independent distribution. A worker ω then works in the current occupation, earns income, consumes and derives logarithmic utility, and then chooses the next period's occupation with discount rate ι. When choosing the next period occupation o', she pays an ad-valorem switching cost $\chi_{i,oo',t}$ in terms of consumption unit that depends on current occupation o. She consumes her income in each period. Thus, worker ω who currently works in occupation o_t maximizes the following objective function over the future stream of utilities by choosing occupations $\{o_s\}_{s=t+1}^{\infty}$:

$$E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[\ln\left(C_{i,o_s,s}\right) + \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right]$$

where $C_{i,o,s}$ is a consumption bundle when working in occupation o in period $s \geq t$, and E_t is the expectation conditional on the value of $Z_{i,o_t,t}\left(\omega\right)$. Each worker owns occupation-specific labor endowment $l_{i,o,t}$. I assume that her income is comprised of labor income $w_{i,o,t}$ and occupation-specific ad-valorem government transfer with rate $T_{i,o,t}$. Given the consumption price $P_{i,t}^{G}$, the budget constraint is

$$P_{i,t}^{G} C_{i,o,t} = w_{i,o,t} l_{i,o,t} \left(1 + T_{i,o,t}\right)$$

for any worker, with $P_{i,t}^{G}$ being the price index of the non-robot good G.

By linearity of expectation,

$$\begin{aligned} & E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[\ln\left(C_{i,o_s,s}\right) + \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right] \\ &= \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right]. \end{aligned}$$

By monotone transformation with exponential function,

$$\begin{aligned} & \exp\left\{\sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right].\right\} \\ &= \prod_{s=t}^{\infty} \exp\left\{\left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right]\right\}. \end{aligned}$$

Write the value function conditional on the realization of shocks at period t as follows:

$$V_{i,o_t,t}(\omega) \equiv \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp\left\{ \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \right\}.$$

I apply Bellman's principle of optimality as follows:

$$\begin{aligned}
&V_{i,o_t,t}(\omega) \\
&= \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp\left\{ \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \right\} \\
&= \max_{o_{t+1}} \exp\left\{ \ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right) + \ln\left(Z_{i,o_{t+1},t}(\omega)\right) \right\} \times \\
&\max_{\{o_s\}_{s=t+2}^{\infty}} \prod_{s=t+1}^{\infty} \exp\left\{ \left(\frac{1}{1+\iota}\right)^{s-(t+1)} \left[E_{t+1} \ln\left(C_{i,o_s,s}\right) + E_{t+1} \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_{t+1} \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \right\} \\
&= \max_{o_{t+1}} \exp\left\{ \ln\left(Z_{i,o_t,t}(\omega)\right) + \ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right) \right\} V_{i,o_{t+1},t+1},
\end{aligned}$$

where $V_{i,o_t,t}$ is the unconditional expected value function $V_{i,o_t,t} \equiv E_{t-1} V_{i,o_t,t}(\omega)$. Changing the notation from (o_t, o_{t+1}) into (o, o'), I have

$$V_{i,o,t}(\omega) = \max_{o'} C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1}.$$

Solving the worker's maximization problem is equivalent to finding:

$$\begin{aligned}
\mu_{i,oo',t} &\equiv \Pr\left(\text{worker } \omega \text{ in } o \text{ chooses occupation } o'\right) \\
&= \Pr\left(\max_{o''} C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) Z_{i,o'',t}(\omega) V_{i,o'',t+1} \leq C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1} \right).
\end{aligned}$$

By the independent Fréchet assumption, I have the maximum value distribution

$$\begin{aligned}
\Pr\left(\max_{o''} C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}\left(\omega\right) V_{i,o',t+1} \leq v\right) &= \prod_{o'} \Pr\left(Z_{i,o',t}\left(\omega\right) \leq \frac{v}{C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}}\right) \\
&= \prod_{o''} \exp\left(\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} v^{-\phi}\right) \\
&= \exp\left(\sum_{o''} \left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} v^{-\phi}\right).
\end{aligned}$$

Therefore, the conditional choice probability satisfies, again by the independent Fréchet assumption,

$$\begin{aligned}
&\mu_{i,oo',t} \\
=&\int_0^{\infty} \Pr\left(\max_{o''\neq o'} C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) Z_{i,o',t}\left(\omega\right) V_{i,o'',t+1} \leq v\right) d\Pr\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}\left(\omega\right) V_{i,o',t+1} \geq v\right) \\
=&\int_0^{\infty} \exp\left(\sum_{o''\neq o'} \left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} v^{-\phi}\right) \times \\
&\qquad \left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} \exp\left(\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} v^{-\phi}\right) \times \left(-\phi v^{-\phi-1}\right) dv \\
=&\frac{\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}} \times \\
&\qquad \int_0^{\infty} \exp\left(\sum_{o'''} \left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi} v^{-\phi}\right) \sum_{o''} \left(C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi} \times \left(-\phi v^{-\phi-1}\right) dv.
\end{aligned}$$

The last integral term is one by integration and the definition of distribution. Therefore, I arrive at

$$\mu_{i,oo',t} = \frac{\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}} = \frac{\left(\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}},$$

$$V_{i,o,t+1} = \widetilde{\Gamma} C_{i,o,t}\left(\sum_{o'}\left(\left(1-\chi_{i,oo',t+1}\right) V_{i,o',t+2}\right)^{\phi}\right)^{\frac{1}{\phi}}.$$

I.3 Equilibrium Characterization

To characterize the producer problem, I show the static optimization conditions and then the dynamic ones. For simplicity, I focus on the case with $\vartheta = 1$, or Cobb-Douglas in the mix of occupation aggregates, intermediates, and non-robot capital. To solve for the static problem of labor, intermediate goods, and non-robot capital, consider the FOCs of equation (7)

$$p^G_{i,t}\alpha_{i,L}\frac{Y^G_{i,t}}{T^O_{i,t}}\left(b_{i,o,t}\frac{T^O_{i,t}}{T^O_{i,o,t}}\right)^{\frac{1}{\beta}}\left((1-a_{o,t})\frac{T^O_{i,o,t}}{L_{i,o,t}}\right)^{\frac{1}{\theta_o}} = w_{i,o,t}, \tag{I.2}$$

where $T^O_{i,t}$ is the aggregated occupations $T^O_{i,t} \equiv \left[\sum_o \left(T^O_{i,o,t}\right)^{(\beta-1)/\beta}\right]^{\beta/(\beta-1)}$,

$$p^G_{i,t}\alpha_{i,M}\frac{Y^G_{i,t}}{M_{i,t}}\left(\frac{M_{i,t}}{M_{li,t}}\right)^{\frac{1}{\varepsilon}} = p^G_{li,t}, \tag{I.3}$$

and

$$p^G_{i,t}\alpha_{i,K}\frac{Y^G_{i,t}}{K_{i,t}} = r_{i,t}, \tag{I.4}$$

where $\alpha_{i,K} \equiv 1 - \alpha_{i,L} - \alpha_{i,M}$. Note also that by the envelope theorem,

$$\frac{\partial \pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}\right)}{\partial K^R_{i,o,t}} = p^G_{i,t}\frac{\partial Y_{i,t}}{\partial K^R_{i,o,t}} = p^G_{i,t}\left(\alpha_L\frac{Y^G_{i,t}}{T^O_{i,t}}\left(b_{i,o,t}\frac{T^O_{i,t}}{T^O_{i,o,t}}\right)^{\frac{1}{\beta}}\left(a_{o,t}\frac{T^O_{i,o,t}}{K^R_{i,o,t}}\right)^{\frac{1}{\theta}}\right). \tag{I.5}$$

Another static problem of producers is robot purchase. Define the "before-integration" robot aggregate $Q^{R,BI}_{i,o,t} \equiv \left[\sum_l \left(Q^R_{li,o,t}\right)^{\frac{\varepsilon^R-1}{\varepsilon^R}}\right]^{\frac{\varepsilon^R}{\varepsilon^R-1}}$ and the corresponding price index $P^{R,BI}_{i,o,t}$. By the first order condition with respect to $Q^R_{li,o,t}$ for equation (9), I have $p^R_{li,o,t}Q^R_{li,o,t} = \left(\frac{p^R_{li,o,t}}{P^{R,BI}_{i,o,t}}\right)^{1-\varepsilon^R} P^{R,BI}_{i,o,t}Q^{R,BI}_{i,o,t}$, and $P^{R,BI}_{i,o,t}Q^{R,BI}_{i,o,t} = \alpha P^R_{i,o,t}Q^R_{i,o,t}$. Thus $p^R_{li,o,t}Q^R_{li,o,t} = \alpha\left(\frac{p^R_{li,o,t}}{P^{R,BI}_{i,o,t}}\right)^{1-\varepsilon^R} P^R_{i,o,t}Q^R_{i,o,t}$. Hence

$$Q^R_{li,o,t} = \alpha\left(p^R_{li,o,t}\right)^{-\varepsilon^R}\left(P^{R,BI}_{i,o,t}\right)^{\varepsilon^R-1}P^R_{i,o,t}Q^R_{i,o,t}.$$

Writing $P^R_{i,o,t} = \left(P^{R,BI}_{i,o,t}\right)^{\alpha^R} (P_{i,t})^{1-\alpha^R}$, I have

$$Q^R_{li,o,t} = \alpha \left(\frac{p^R_{li,o,t}}{P^{R,BI}_{i,o,t}}\right)^{-\varepsilon^R} \left(\frac{P^{R,BI}_{i,o,t}}{P_{i,t}}\right)^{-\left(1-\alpha^R\right)} Q^R_{i,o,t}.$$

Alternatively, one can define the robot price index by $\widetilde{P}^R_{i,o,t} = \alpha^{\frac{1}{\varepsilon^R}} \left(P^{R,BI}_{i,o,t}\right)^{\frac{\varepsilon^R - \left(1-\alpha^R\right)}{\varepsilon^R}} P_{i,t}^{\frac{1-\alpha^R}{\varepsilon^R}}$ and show

$$Q^R_{li,o,t} = \left(\frac{p^R_{li,o,t}}{\widetilde{P}^R_{i,o,t}}\right)^{-\varepsilon^R} Q^R_{i,o,t}, \tag{I.6}$$

which is a standard gravity representation of robot trade.

To solve the dynamic problem, set up the (current-value) Lagrangian function for non-robot goods producers

$$\mathcal{L}_{i,t} = \sum_{t=0}^{\infty} \left\{ \left(\frac{1}{1+\iota}\right)^t \left[\pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}_o\right) - \sum_{l,o} \left(p^R_{li,o,t}\left(1+u_{li,t}\right) Q^R_{li,o,t} + P^G_{i,t} I^{int}_{i,o,t} + \gamma P^R_{i,o,t} Q^R_{i,o,t} \frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\right] \right\}.$$
$$- \lambda^R_{i,o,t} \left\{K^R_{i,o,t+1} - (1-\delta) K^R_{i,o,t} - Q^R_{i,o,t}\right\}$$

Taking the FOC with respect to the hardware from country l, $Q^R_{li,o,t}$, I have

$$p^R_{li,o,t}\left(1+u_{li,t}\right) + 2\gamma P^R_{i,o,t} \left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right) \frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}} = \lambda^R_{i,o,t} \frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}}. \tag{I.7}$$

Taking the FOC with respect to the integration input $I^{int}_{i,o,t}$, I have

$$P^G_{i,t} + 2\gamma P^R_{i,o,t} \left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right) \frac{\partial Q^R_{i,o,t}}{\partial I^{int}_{i,o,t}} = \lambda^R_{i,o,t} \frac{\partial Q^R_{i,o,t}}{\partial I^{int}_{i,o,t}}, \tag{I.8}$$

Taking the FOC with respect to $K^R_{i,o,t+1}$, I have

$$\left(\frac{1}{1+\iota}\right)^{t+1}\left[\frac{\partial \pi_{i,t+1}\left(\left\{K^R_{i,o,t+1}\right\}_o\right)}{\partial K^R_{i,o,t+1}}+\gamma P^R_{i,o,t+1}\left(\frac{Q^R_{i,o,t+1}}{K^R_{i,o,t+1}}\right)^2+(1-\delta)\,\lambda^R_{i,o,t+1}\right]-\left(\frac{1}{1+\iota}\right)^t\lambda^R_{i,o,t}=0, \tag{I.9}$$

and the transversality condition: for any j and o,

$$\lim_{t\to\infty} e^{-\iota t}\lambda^R_{j,o,t}K^R_{j,o,t+1}=0. \tag{I.10}$$

Rearranging equation (I.9), I obtain the following Euler equation.

$$\lambda^R_{i,o,t}=\frac{1}{1+\iota}\left[(1-\delta)\,\lambda^R_{i,o,t+1}+\frac{\partial}{\partial K^R_{i,o,t+1}}\pi_{i,t+1}\left(\left\{K^R_{i,o,t+1}\right\}\right)+\gamma p^R_{i,o,t+1}\left(\frac{Q^R_{i,o,t+1}}{K^R_{i,o,t+1}}\right)^2\right]. \tag{I.11}$$

Turning to the demand for non-robot good, I will characterize bilateral intermediate good trade demand and total expenditure. Write $X^G_{j,t}$ the total purchase quantity (but not value) of good G in country j in period t. By equation (B.1), the bilateral trade demand is given by

$$p^G_{ij,t}Q^G_{ij,t}=\left(\frac{p^G_{ij,t}}{P^G_{j,t}}\right)^{1-\varepsilon}P^G_{j,t}X^G_{j,t}, \tag{I.12}$$

for any i, j, and t. In this equation, $P^G_{j,t}X^G_{j,t}$ is the total expenditures on non-robot goods. The total expenditure is the sum of final consumption $I_{j,t}$, payment to intermediate goods $\alpha_M p^G_{j,t}Y^G_{j,t}$, input to robot productions $\sum_o P^G_{j,t}I^R_{j,o,t}=\sum_{o,k}p^R_{jk,o,t}Q^R_{jk,o,t}$, and payment to robot integration $\sum_o P^G_{j,t}I^{int}_{j,o,t}=\left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}$. Hence

$$P^G_{j,t}X^G_{j,t}=I_{j,t}+\alpha_M p^G_{j,t}Y^G_{j,t}+\sum_{o,k}p^R_{jk,o,t}Q^R_{jk,o,t}+\left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}.$$

For country j and period t, by substituting into income $I_{j,t}$ the period cash flow of non-robot good producer that satisfies

$$\Pi_{j,t} \equiv \pi_{j,t}\left(\left\{K^R_{j,o,t}\right\}_o\right) - \sum_{i,o}\left(p^R_{ij,o,t}\left(1+u_{ij,t}\right)Q^R_{ij,o,t} + \sum_o P^G_{j,t} I^{int}_{j,o,t} + \gamma P^R_{j,o,t} Q^R_{j,o,t}\left(\frac{Q^R_{j,o,t}}{K^R_{j,o,t}}\right)\right)$$

and robot tax revenue $T_{j,t} = \sum_{i,o} u_{ij,t} p^R_{ij,o,t} Q^R_{ij,o,t}$, I have

$$I_{j,t} = (1-\alpha_M)\sum_k p^G_{jk,t} Q^G_{jk,t} - \left(\sum_{i,o} p^R_{ij,o,t} Q^R_{ij,o,t} + \left(1-\alpha^R\right)\sum_o P^R_{j,o,t} Q^R_{j,o,t}\right), \tag{I.13}$$

or in terms of variables in the definition of equilibrium,

$$I_{j,t} = (1-\alpha_M)\sum_k p^G_{jk,t} Q^G_{jk,t} - \frac{1}{\alpha^R}\sum_{i,o} p^R_{ij,o,t} Q^R_{ij,o,t}.$$

Hence, the total expenditure measured in terms of the production side as opposed to income side is

$$P^G_{j,t} X^G_{j,t} = \sum_k p^G_{jk,t} Q^G_{jk,t} - \sum_{i,o} p^R_{ij,o,t} Q^R_{ij,o,t}\left(1 + \gamma\frac{Q^R_{ij,o,t}}{K^R_{j,o,t}}\right). \tag{I.14}$$

Note that this equation embeds the balanced-trade condition. By substituting equation (I.14) into equation (I.12), I have

$$p^G_{ij,t} Q^G_{ij,t} = \left(\frac{p^G_{ij,t}}{P^G_{j,t}}\right)^{1-\varepsilon^G}\left(\sum_k p^G_{jk,t} Q^G_{jk,t} + \sum_{k,o} p^R_{jk,o,t} Q^R_{jk,o,t} - \sum_{i,o} p^R_{ij,o,t} Q^R_{ij,o,t}\right). \tag{I.15}$$

The good and robot-o market-clearing conditions are given by,

$$Y^R_{i,t} = \sum_j Q^G_{ij,t} \tau^G_{ij,t}, \tag{I.16}$$

for all i and t, and

$$p^R_{i,o,t} = \frac{P^G_{i,t}}{A^R_{i,o,t}} \tag{I.17}$$

for all i, o, and t, respectively.

Conditional on state variables $\boldsymbol{S}_t = \left\{\boldsymbol{K}_t^R, \boldsymbol{\lambda}_t^R, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$, equations (B.2), (I.2), (I.7), (I.15), (I.16), and (I.17) characterize the temporary equilibrium $\left\{\boldsymbol{p}_t^G, \boldsymbol{p}_t^R, \boldsymbol{w}_t, \boldsymbol{Q}_t^G, \boldsymbol{Q}_t^R, \boldsymbol{L}_t\right\}$. In addition, conditional on initial conditions $\left\{\boldsymbol{K}_0^R, \boldsymbol{L}_0\right\}$, equations (8), (I.11), and (I.10) characterize the sequential equilibrium.

Finally, the steady state conditions are given by imposing the time-invariance condition to equations (8) and (I.11):

$$Q_{i,o}^R = \delta K_{i,o}^R, \tag{I.18}$$

$$\frac{\partial}{\partial K_{i,o}^R}\pi_i\left(\left\{K_{i,o}^R\right\}\right) = (\iota + \delta)\,\lambda_{i,o}^R - \sum_l \gamma p_{li,o}^R \left(\frac{Q_{li,o}^R}{K_{i,o}^R}\right)^2 \equiv c_{i,o}^R. \tag{I.19}$$

Note that equation (I.19) can be interpreted as the flow marginal profit of capital must be equalized to the marginal cost term. Thus I define the steady state marginal cost of robot capital $c_{i,o}^R$ from the right-hand side of equation (I.19). Note that if there is no adjustment cost $\gamma = 0$, the steady state Euler equation (I.19) implies

$$\frac{\partial}{\partial K_{i,o}^R}\pi_i\left(\left\{K_{i,o}^R\right\}\right) = c_{i,o}^R = (\iota + \delta)\,\lambda_{i,o}^R,$$

which states that the marginal profit of capital is the user cost of robots in the steady state (Hall and Jorgenson 1967).

I.4 Remaining Proofs

Proof of Proposition F.1 To prove Proposition I follow the arguments made in Sections 2 and 3 of Newey and McFadden (1994). The proof consists of four sub results in the following Lemma. Proposition F.1 can be obtained as a combination of the four results. The formal statement requires the following additional assumptions.

Assumption I.1. *(i) A function of $\widetilde{\boldsymbol{\Theta}}$, $\mathbb{E}\left[H_o\left(\boldsymbol{\psi}_{t_1}^J\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right)\right] \neq 0$ for any $\widetilde{\boldsymbol{\Theta}} \neq \boldsymbol{\Theta}$. (ii) $\underline{\theta} \leq \theta_o \leq \overline{\theta}$ for any o, $\underline{\beta} \leq \beta \leq \overline{\beta}$, $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$, and $\underline{\phi} \leq \phi \leq \overline{\phi}$ for some positive values $\underline{\theta}, \underline{\beta}, \underline{\gamma}, \underline{\phi}, \overline{\theta}, \overline{\beta}, \overline{\gamma}, \overline{\phi}$. (iii) $\mathbb{E}\left[\sup_{\boldsymbol{\Theta}} \| H_o\left(\boldsymbol{\psi}_{t_1}^J\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right)\|\right] < \infty$. (iv) $\mathbb{E}\left[\| H_o\left(\boldsymbol{\psi}_{t_1}^J\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right)\|^2\right] < \infty$ (v)*

$\mathbb{E}\left[\sup_{\mathbf{\Theta}} \parallel H_o\left(\boldsymbol{\psi}_{t_1}^J\right) \nabla_{\widetilde{\mathbf{\Theta}}} \nu_o\left(\widetilde{\mathbf{\Theta}}\right) \parallel\right] < \infty.$

Lemma I.1. *Assume Assumptions 1 and I.1(i)-(iii).*

(a) The estimator of the form (F.14) is consistent.

Additionally, assume Assumptions I.1(iv)-(v).

(b) The estimator of the form (F.14) is asymptotically normal.

(c) $\sqrt{O}\left(\mathbf{\Theta}_{\boldsymbol{H}^*} - \mathbf{\Theta}\right) \to_d \mathcal{N}\left(0, \left(\boldsymbol{G}^\top \mathbf{\Omega}^{-1} \boldsymbol{G}\right)^{-1}\right)$*, and the asymptotic variance is the minimum of that of the estimator of the form (F.14) for any function* $\boldsymbol{H}$.

Proof. (a) I follow Theorems 2.6 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(iv) of this theorem are satisfied. Assumption I.1(i) guarantees condition (i). Condition (ii) is implied by Assumption I.1(ii). Condition (iii) follows because all supply and demand functions in the model is continuous. Condition (iv) is implied by Assumption I.1(iii).

(b) I follow Theorem 3.4 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(v) of this theorem are satisfied. Condition (i) is satisfied by Assumption I.1(i). Condition (ii) follows because all supply and demand functions in the model is continuously differentiable. Condition (iii) is implied by Assumption 1 and Assumption I.1(iv). Assumption I.1(v) implies condition (iv). Finally, the gradient vectors of the structural residual is linear independent, guaranteeing the non-singularity of the variance matrix and condition (v).

(c) By Theorem 3.4 of Newey and McFadden (1994), for an arbitrary IV-generating function $\boldsymbol{H}$, the asymptotic variance of the GMM estimator $\mathbf{\Theta}_{\boldsymbol{H}}$ is

$$\left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}_{t_1}^J\right) G_o\right]\right)^{-1} \mathbb{E}\left[H_o\left(\boldsymbol{\psi}_{t_1}^J\right) \nu_o \nu_o^\top \left(H_o\left(\boldsymbol{\psi}_{t_1}^J\right)\right)^\top\right] \left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}_{t_1}^J\right) G_o\right]\right)^{-1},$$

where $G_o \equiv \mathbb{E}\left[\nabla_{\mathbf{\Theta}} \nu_o\left(\mathbf{\Theta}\right) | \boldsymbol{\psi}_{t_1}^J\right]$. Therefore, if $H_o\left(\boldsymbol{\psi}_{t_1}^J\right) = Z_o \equiv \mathbb{E}\left[\nabla_{\mathbf{\Theta}} \nu_o\left(\mathbf{\Theta}\right) | \boldsymbol{\psi}_{t_1}^J\right] \mathbb{E}\left[\nu_o\left(\mathbf{\Theta}\right) \left(\nu_o\left(\mathbf{\Theta}\right)\right)^\top | \boldsymbol{\psi}_{t_1}^J\right]^{-1}$, then this expression is equal to

$\left(\boldsymbol{G}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$, where

$$\boldsymbol{G} \equiv \mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)\right] \text{ and } \boldsymbol{\Omega} \equiv \mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top\right].$$

To show that this variance is minimal, I will check that

$$\boldsymbol{\Delta} \equiv \left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\right)^{-1}\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\nu_o\nu_o^\top\left(H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\right)^\top\right]\left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\right)^{-1}$$
$$-\left(\boldsymbol{G}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$$

is positive semi-definite. In fact, note that

$$\boldsymbol{\Delta} = \left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\right)^{-1}\times$$
$$\left\{\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\nu_o\nu_o^\top\left(H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\right)^\top\right]-\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\left(\boldsymbol{G}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\right\}\times$$
$$\left(\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\right)^{-1}.$$

Define

$$\widetilde{\nu}_o = H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\nu_o - \mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\nu_o\left(\left(G_o\right)^\top\Omega_o^{-1}\nu_o\right)^{-1}\right]\mathbb{E}\left(\left(G_o\right)^\top\Omega_o^{-1}\nu_o\right)^{-1}\left(G_o\right)^\top\Omega_o^{-1}\nu_o,$$

where $\Omega_o \equiv \mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\boldsymbol{\psi}^J_{t_1}\right]$. Applying Theorem 5.3 of Newey and McFadden (1994), I have

$$\mathbb{E}\left[\widetilde{\nu}_o\left(\widetilde{\nu}_o\right)^\top\right] = \mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\nu_o\nu_o^\top\left(H_o\left(\boldsymbol{\psi}^J_{t_1}\right)\right)^\top\right]-\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right]\left(\boldsymbol{G}^\top \boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\boldsymbol{\psi}^J_{t_1}\right)G_o\right].$$

Since $\mathbb{E}\left[\widetilde{\nu}_o\left(\widetilde{\nu}_o\right)^\top\right]$ is positive semi-definite, so is $\boldsymbol{\Delta}$, which completes the proof. □

Proof of Proposition C.1 I apply arguments in Section 6.1 of Newey and McFadden (1994). Namely, I define the joint estimator of the first-step and second-step estimator in Proposition C.1 that falls into the class of general GMM estimation, and discuss the asymptotic property using the general result about GMM estimation. In the proof, I modify the notation of the set of functions that yield optimal IV, $\boldsymbol{H}^*$, to clarify that it depends on parameters $\boldsymbol{\Theta}$ as follows:

$$H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}\right)=\mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)|\boldsymbol{\psi}_{t_1}^J\right]\mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\boldsymbol{\psi}_{t_1}^J\right]^{-1}.$$

Define the joint estimator as follows:

$$\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix}\equiv\arg\min_{\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1}\left[\sum_o e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right]^\top\left[\sum_o e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right],$$

where

$$e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\equiv\begin{pmatrix}H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\\ H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\end{pmatrix}.$$

Since for any $\boldsymbol{\Theta}$, IV-generating function $H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_0\right)$ gives the consistent estimator for $\boldsymbol{\Theta}$, I have $\boldsymbol{\Theta}_1\to\boldsymbol{\Theta}$ and $\boldsymbol{\Theta}_2\to\boldsymbol{\Theta}$. I also have the asymptotic variance

$$\operatorname{Var}\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix}=\left[\left(\widetilde{\boldsymbol{G}}\right)^\top\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{G}}\right]^{-1},$$

where

$$\begin{aligned}\widetilde{\boldsymbol{G}}&\equiv\mathbb{E}\left[\nabla_{\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)^\top}e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right]\\ &=\mathbb{E}\begin{bmatrix}H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_1\right)\nabla\nu_o\left(\boldsymbol{\Theta}_2\right) & \nabla H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\\ \mathbf{0} & H_o^*\left(\boldsymbol{\psi}_{t_1}^J;\boldsymbol{\Theta}_0\right)\nabla\nu_o\left(\boldsymbol{\Theta}_1\right)\end{bmatrix}\end{aligned}$$

and

$$\widetilde{\boldsymbol{\Omega}} \equiv \mathbb{E}\left[e_o\left(\boldsymbol{\Theta}_2, \boldsymbol{\Theta}_1\right)\left[e_o\left(\boldsymbol{\Theta}_2, \boldsymbol{\Theta}_1\right)\right]^{\top}\right]$$
$$= \mathbb{E}\left[\begin{array}{cc} H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_1\right) \nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_1\right) \nu_o\left(\boldsymbol{\Theta}_2\right)\right]^{\top} & H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_1\right) \nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_0\right) \nu_o\left(\boldsymbol{\Theta}_1\right)\right]^{\top} \\ H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_0\right) \nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_1\right) \nu_o\left(\boldsymbol{\Theta}_2\right)\right]^{\top} & H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_0\right) \nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_0\right) \nu_o\left(\boldsymbol{\Theta}_1\right)\right]^{\top} \end{array}\right]$$

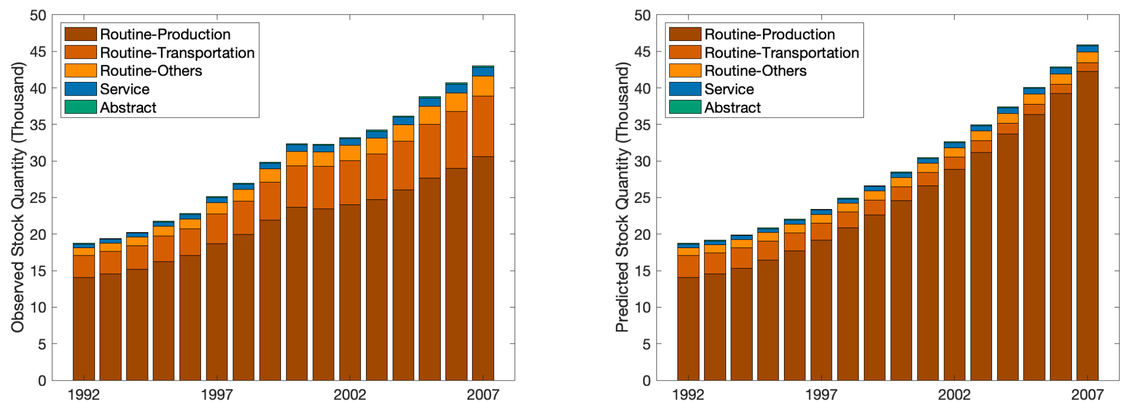
Note that Assumption 1 implies that any function of $\boldsymbol{\psi}_{t_1}^J$ is orthogonal to ν_o, implying $\mathbb{E}\left[\nabla H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}_1\right) \nu_o\left(\boldsymbol{\Theta}_2\right)\right] = 0$. Therefore, $\widetilde{\boldsymbol{G}}$ is a block-diagonal matrix and thus the marginal asymptotic distribution of $\boldsymbol{\Theta}_2$ is normal with variance $\mathrm{Var}\left(\boldsymbol{\Theta}_2\right) = \left(\boldsymbol{G}^{\top}\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$, noting that $\boldsymbol{G} = \mathbb{E}\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}\right) \nabla\nu_o\left(\boldsymbol{\Theta}\right)\right]$ and $\boldsymbol{\Omega} \equiv \mathbb{E}\left[H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}\right) \nu_o\left(\boldsymbol{\Theta}\right)\left(H_o^*\left(\boldsymbol{\psi}_{t_1}^J; \boldsymbol{\Theta}\right) \nu_o\left(\boldsymbol{\Theta}\right)\right)^{\top}\right]$. By Proposition F.1, this asymptotic variance is minimal among the GMM estimator (F.14).

I.5 On the Choice of the Steady-State Matrix in Equation (22)

In equation (22), I use the steady-state matrix $\overline{\boldsymbol{E}}$ instead of the transitional dynamics matrix $\overline{\boldsymbol{F}_t}$ for a computational reason. Since I have annual observation for occupational robot costs, it is potentially possible to leverage this rich variation for the structural estimation, which may permit me to estimate the EoS θ_o at a narrower occupation group level. However, the bottleneck of this approach is the computational burden to compute the dynamic solution matrix $\overline{\boldsymbol{F}_t}$. Specifically, dynamic substitution matrix $\overline{\boldsymbol{F}_{t+1}^{y}}$ in equation (15) is based on the conditions of Blanchard and Kahn (1980). This requires computing the eigenspace, as described in detail in Section K. This is computationally hard since I cannot rely on the sparse structure of the matrix $\overline{\boldsymbol{F}_{t+1}^{y}}$. In contrast, the estimation method in Proposition C.1 does not involve such computation, but only requires computing the steady-state solution matrix $\overline{\boldsymbol{E}}$. Then I only need to invert steady-state substitution matrix $\overline{\boldsymbol{E}^{y}}$, which is feasible given the sparse structure of $\overline{\boldsymbol{E}^{y}}$.

Figure J.11: Trends of Robot Stocks

(a) Data (b) Model



Note: Figures show the trend of the observed (left) and predicted (right) stock of robots for each occupation group measured by quantities. The predicted robot stocks are computed by shocks backed out from the estimated model and applying the first-order solution to the general equilibrium described in equation (16).

J Further Details in Simulation Results

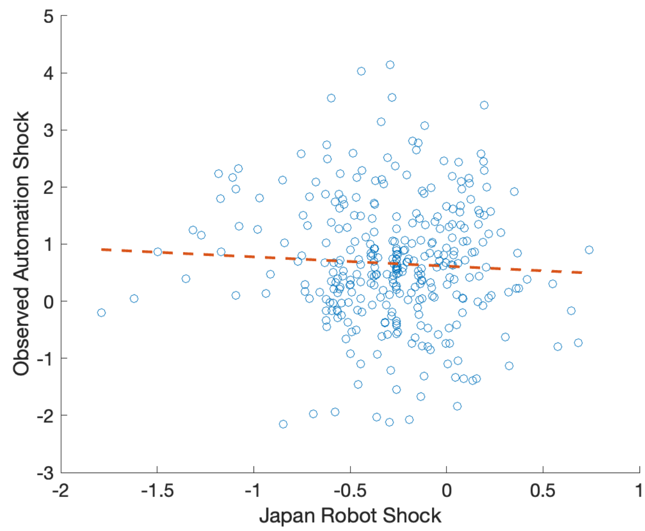
J.1 Actual and Predicted Robot Accumulation Dynamics

Figure J.11 shows the trends of robot stock in the US in the data and the model. Although I do not match the overall robot capital stocks, the estimated model tracks the observed pattern well between 1992 and the late 2010s, consistent with the fact that I target the changes between 1992 and 2007. Although there is over-prediction of the growth of production robots and under-prediction of the growth of transportation (material moving) robots between occupation groups, the overall predicted stock of robots matches well with the actual data.

J.2 The Japan Robot Shock and The Implied Automation Shock

In turn, Figure J.12 shows a further detailed scatter plot between the two shocks, delivering a mild negative relationship. This negative correlation is consistent with the example of robotic innovations in Appendix E.2.

Figure J.12: Correlation between Japan Robot Shock ψ_o^J and Automation Shock $\widehat{a_o^{obs}}$



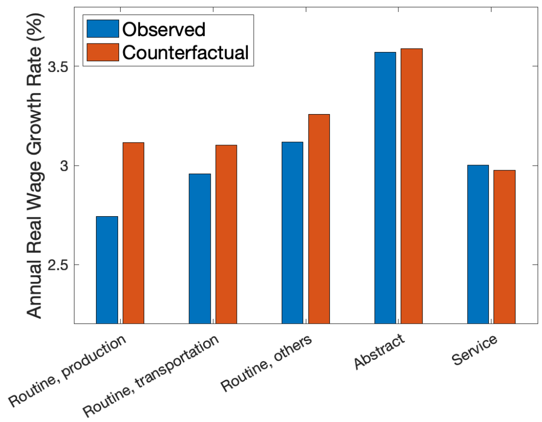
Note: The author's calculation based on JARA, O*NET, and US Census/ACS. The x-axis shows the Japan robot shock, and is taken from the regression of equation (1). The y-axis shows the implied automation shock, and is backed out from equation (21) with the estimated parameters in Table 2. Each circle is 4-digit occupation and dashed line is the fitted line.

J.3 Automation and Wages for each Occupation

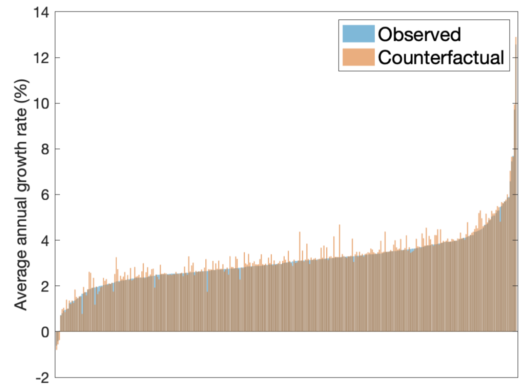
Figure J.13 shows the observed and counterfactual growth rate of real wages for each occupation, where the counterfactual change means the simulated change when there is no the automation shock. Figure J.13a shows the results aggregated at the 5 occupations groups defined in Section 4.1. I compute the counterfactual growth rate from the observed rate of the wage change, subtracted by the change predicted by the first-order steady-state solution $\overline{\boldsymbol{E}}$ and the implied automation shock $\widehat{\boldsymbol{a}^{\text{imp}}}$. The result is based on the observed high growth rates of robots in routine production and transportation (material moving) occupations, and these occupations' high EoS estimates between robots and workers. In particular, at the 5-occupation aggregate level, most of the observed differences in the real wage growth rates in the three routine occupation groups are closed absent the automation shock. Applying the similar exercise for all occupations in my sample, Figure J.13b shows a more granular result, where occupations are sorted by the observed changes of wages from 1990-2007.

Figure J.13: The Steady-state Effect of Robots on Wages

(a) Occupation Groups



(b) All Occupations



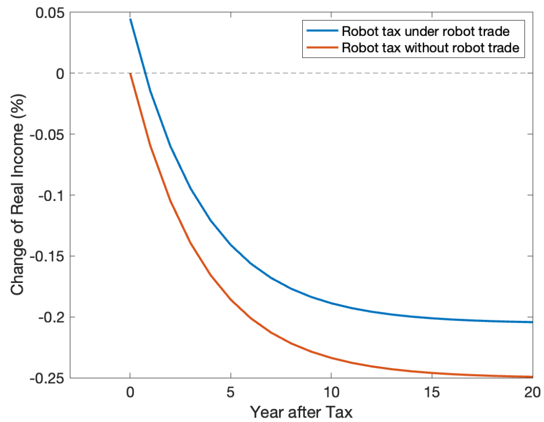
J.4 Trade and the Effect of the Robot Tax

Figure J.14 shows the dynamic effect of the robot tax on the US real income. If the robot trade is not allowed, the robot tax does not increases the real income in any period since the terms-of-trade effect does not show up, but only the long-run capital decumulation effect does. On the other hand, once I allow the robot trade as observed in the data, the robot tax may increase the real income because it decreases the price of imported robots. The effect is concentrated in the short-run before the capital decumulation process matures. In the long run, the negative decumulation effect dominates the positive terms-of-trade effect.

J.5 The Effect of Robot Tax on Occupations

In Figure J.15a, I show two scenarios of the steady-state changes in occupational real wages. On the one hand, I shock the economy only with the automation shocks. On the other hand, I shock the economy with both the automation shocks and the robot tax. The result shows heterogenous effects on occupational real wages of the robot tax. The tax mitigates the negative effect of automation on routine production workers and routine transportation workers, while the tax marginally decreases the small gains that workers

Figure J.14: Effects of the Robot Tax on the US Real Income



in the other occupations would have enjoyed. Overall, the robot tax mitigates the large heterogeneous effects of the automation shocks, that could go negative and positive directions depending on occupation groups, and compresses the effects towards zero. Figure J.15b shows the dynamics of the effects of robot tax, net of the effects of automation shocks. Although the steady-state effects of robot tax were heterogeneous as shown in Figure J.15a, the effect is not immediate but materializes after around 10 years, due to the sluggish adjustment in the accumulation of the robot capital stock. Overall, I find that since the robot tax slows down the adoption of robots, it rolls back the real wage effect of automation–workers in occupations that experienced significant automation shocks (e.g., production and transportation in the routine occupation groups) benefit from the tax, while the others lose.

To study how the occupational effects unfold over time and if the US policy affects third countries, I study occupational value evolution given the US general robot tax. Figure J.16 shows the impact of the US's unilateral, unexpected, and permanent 6% general robot tax on the world's occupational values in the short run and the long run. In the first row, panels show the US occupational values and corroborate the finding in Figure J.15 that production and transportation workers gain from the robot tax but not other workers. As can be seen from the figure, it takes roughly 10 years until the worker values reach steady states. In other countries than the US, the US robot tax effect is negative but

Figure J.15: Effects of the Robot Tax on Occupational Real Wages

(a) Steady-state Comparison

(b) Transitional Effect of Tax

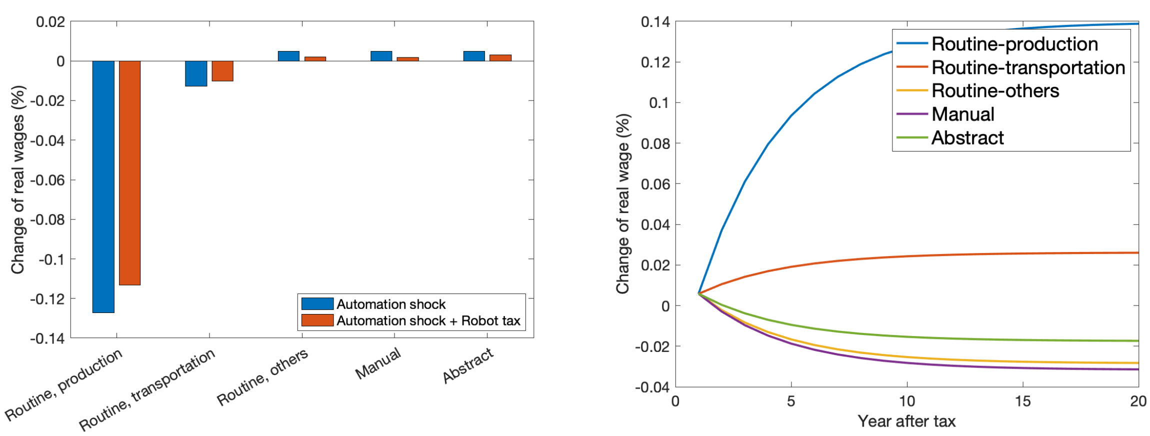
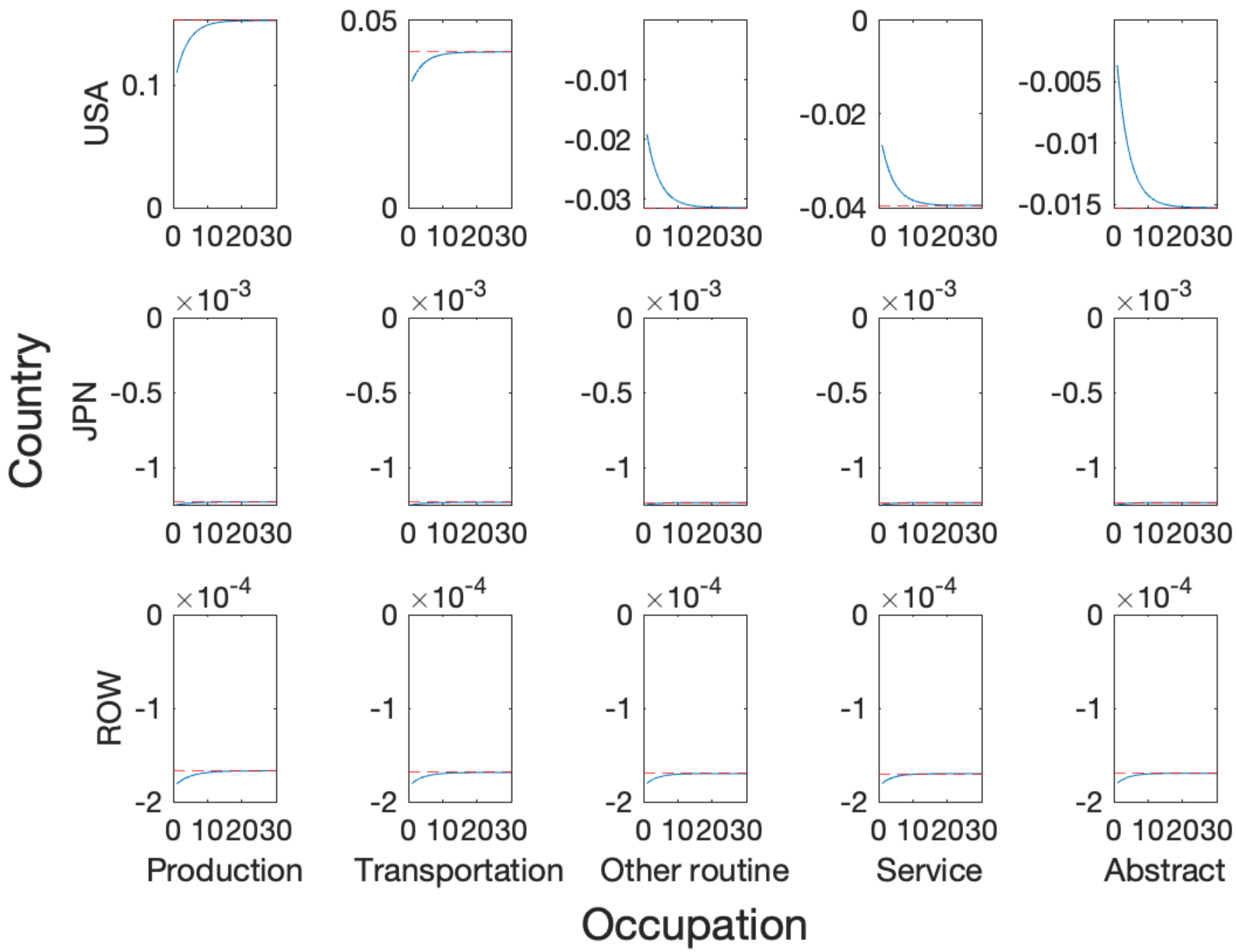
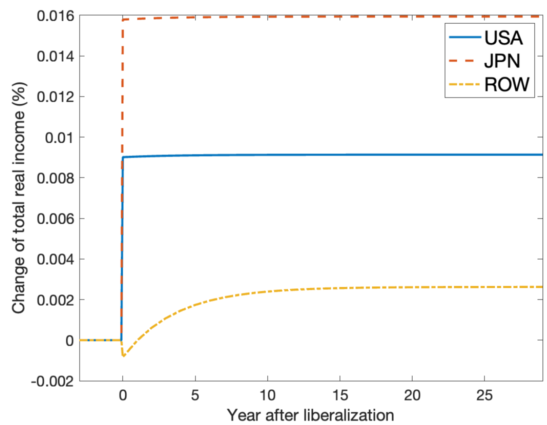


Figure J.16: US General Robot Tax and Global Occupational Value Evolution



Note: Transition dynamics of workers' occupation-specific values given the US's unexpected, unilateral, and permanent 6% general robot tax for all occupational robots at the initial steady-state (period 0) are shown. Blue solid lines are the transitional dynamics, and red dashed lines are the steady-state values.

Figure J.17: The Effect of Robot Trade Cost Reduction



quantitatively limited.

J.6 Trade Liberalization of Robots

What is the effect of liberalizing the trade of robots? To approach this question and gain insights about dynamics gains from trade, I consider unexpected and permanent 20% reduction in the bilateral trade costs, following Ravikumar, Santacreu, and Sposi (2019). Figure J.17 shows the result of such a simulation for a 20-years time horizon. All country groups in the model gain from the trade liberalization. The US gain materialize almost immediately after the trade cost change. A possible explanation is the combination of the following two observation. First, it takes time to accumulate robots after the trade liberalization, which makes the gains from trade liberalization sluggish. Second, by exporting robots to ROW, the US increases the revenue of robot sales immediately after the trade cost drop, improving the short-run real income gain. The real income gain is the largest for Japan, a large net robot exporter. It is noteworthy that ROW loses from the reduction in the robot trade cost, possibly due to the terms-of-trade deterioration in the short-run.

K Expressions of the GE Solution

I discuss the derivation log-linearization in equations (12), (14), and (16), so that I can bring the theory with computation. Throughout the section, relational operator $\circ$ is Hadamard product, $\oslash$ indicates Hadamard division, and $\otimes$ means Kronecker product. In this section, I use θ_o to denote the elasticity of substitution between robots and workers for each occupation

It is useful to show that the CES production structure implies the share-weighted log-change expression for both prices and quantities. Namely, I have a formula for the change in destination price index $\widehat{P_{j,t}^G} = \sum_i \widetilde{x}_{ij,t_0}^G \widehat{p_{ij,t}^G}$ and one for the change in destination expenditure $\widehat{P_{j,t}^G} + \widehat{Q_{j,t}^G} = \sum_i \widetilde{x}_{ij,t_0}^G \left(\widehat{p_{ij,t}^G} + \widehat{Q_{ij,t}^G} \right)$. These imply that

$$\widehat{Q_{j,t}^G} = \sum_i \widetilde{x}_{ij,t_0}^G \widehat{Q_{ij,t}^G},$$

or the changes of quantity aggregate $\widehat{Q_{j,t}^G}$ are also share-weighted average of changes of origin quantity $\widehat{Q_{ij,t}^G}$.

By log-linearizing equation (I.16) for any i ,

$$\begin{aligned}
&-\alpha_M \widehat{p_{i,t}^G} + \alpha_M \sum_l \widetilde{x}_{li,t_0}^G \widehat{p_{l,t}^G} + (1-\alpha_M) \sum_j \widetilde{y}_{ij,t_0}^G \widehat{Q_{ij,t}^G} - \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O l_{i,o,t_0}^O \widehat{L_{i,o,t}} \\
&= \frac{\alpha_L}{\theta_o - 1} \sum_o \frac{\widetilde{x}_{i,o,t_0}^O}{1-a_{o,t_0}} \left(-a_{o,t_0} l_{i,o,t_0}^O + (1-a_{o,t_0}) \left(1 - l_{i,o,t_0}^O\right)\right) \widehat{a_{o,t}} + \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O \frac{1}{\beta - 1} \widehat{b_{i,o,t}} \\
&\quad + \widehat{A_{i,t}^G} + (1-\alpha_L-\alpha_M) \widehat{K_{i,t}} - \alpha_M \sum_l \widetilde{x}_{li,t_0}^G \widehat{\tau_{li,t}^G} - (1-\alpha_M) \sum_j \widetilde{y}_{ij,t_0}^G \widehat{\tau_{ij,t}^G} + \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O \left(1 - l_{i,o,t_0}^O\right) \widehat{K_{i,o,t}^R},
\end{aligned}$$

To write a matrix notation, write

$$\overline{\boldsymbol{M}^{yG}} \equiv \begin{bmatrix} \left[\widetilde{y}_{11,t_0}^G, \ldots, \widetilde{y}_{1N,t_0}^G\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{y}_{N1,t_0}^G, \ldots, \widetilde{y}_{NN,t_0}^G\right] \end{bmatrix}$$

a $N \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xOl}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{N,\cdot,t_0}\right)^\top \end{bmatrix}$$

a $N \times NO$ matrix where

$$\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \equiv \left(\widetilde{x}^O_{1,o,t_0}\right)_o \text{ and } \widetilde{\boldsymbol{l}}_{1,\cdot,t_0} \equiv \left(l^O_{1,o,t_0}\right)_o \tag{K.20}$$

are $O \times 1$ vectors, $\overline{\boldsymbol{M}^{al}}$ as a matrix with its element

$$M^{al}_{i,o} = \frac{-a_{o,t_0} l^O_{i,o,t_0} + (1 - a_{o,t_0}) \left(1 - l^O_{i,o,t_0}\right)}{1 - a_{o,t_0}},$$

and a $N \times O$ matrix,

$$\overline{\boldsymbol{M}^{xO}} \equiv \begin{bmatrix} \left[\widetilde{x}^O_{1,1,t_0}, \ldots, \widetilde{x}^O_{1,O,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{x}^O_{N,1,t_0}, \ldots, \widetilde{x}^O_{N,O,t_0}\right] \end{bmatrix},$$

a $N \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xG}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}^G_{1\cdot,t_0}\right) & \ldots & \operatorname{diag}\left(\widetilde{x}^G_{N\cdot,t_0}\right) \end{bmatrix},$$

a $N \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xOl,2}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \left(\mathbf{1}_O - \widetilde{\boldsymbol{l}}_{1,\cdot,t_0}\right)\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \left(\mathbf{1}_O - \widetilde{\boldsymbol{l}}_{N,\cdot,t_0}\right)\right)^\top \end{bmatrix},$$

a $N \times NO$ matrix where $\widetilde{\boldsymbol{x}}_{1,\cdot,t_0}$ and $\widetilde{\boldsymbol{l}}_{1,\cdot,t_0}$ are defined in equation (K.20). Then I have

$$
\begin{aligned}
&-\alpha_M \left(\overline{\boldsymbol{I}} - \left(\overline{\widetilde{\boldsymbol{x}}_{t_0}^G}\right)^\top\right) \widehat{\boldsymbol{p}_t^G} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\widehat{\boldsymbol{Q}_t^G} - \alpha_L \overline{\boldsymbol{M}^{xOl}}\widehat{\boldsymbol{L}}_t \\
&= \frac{\alpha_L}{\theta_o - 1}\left(\widetilde{\boldsymbol{x}}_{t_0}^O \circ \overline{\boldsymbol{M}^{al}}\right)\widehat{\boldsymbol{a}}_t + \frac{\alpha_L}{\beta - 1}\overline{\boldsymbol{M}^{xO}}\widehat{\boldsymbol{b}}_t + \widehat{\boldsymbol{A}_t^G} + (1-\alpha_L-\alpha_M)\,\widehat{\boldsymbol{K}_t} \\
&\quad - \left[\alpha_M \overline{\boldsymbol{M}^{xG}} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\right]\widehat{\boldsymbol{\tau}_t^G} + \alpha_L \overline{\boldsymbol{M}^{xOl,2}}\widehat{\boldsymbol{K}_t^R},
\end{aligned}
$$

By log-linearizing equation (I.17) for any i and o,

$$
\widehat{p_{i,o,t}^R} = \widehat{P_{i,t}^G} - \widehat{A_{i,o,t}^R}
$$

$$
-\sum_l \widetilde{x}_{li,t_0}^G \widehat{p_{l,t}^G} + \widehat{p_{i,o,t}^R} = -\widehat{A_{i,o,t}^R} + \sum_l \widetilde{x}_{li,t_0}^G \widehat{\tau_{li,t}^G}.
$$

In matrix notation, write

$$
\overline{\boldsymbol{M}^{xG,2}} \equiv \begin{bmatrix} \boldsymbol{1}_O \left[\widetilde{x}_{11,t_0}^G, \ldots, \widetilde{x}_{N1,t_0}^G\right] \\ \vdots \\ \boldsymbol{1}_O \left[\widetilde{x}_{1N,t_0}^G, \ldots, \widetilde{x}_{NN,t_0}^G\right] \end{bmatrix}
$$

a $NO \times N$ matrix, and

$$
\overline{\boldsymbol{M}^{xG,3}} \equiv \begin{bmatrix} \widetilde{x}_{11,t_0}^G & \ldots & \widetilde{x}_{N1,t_0}^G & & & \boldsymbol{0} \\ & & & \ddots & & \\ & \boldsymbol{0} & & \widetilde{x}_{1N,t_0}^G & \ldots & \widetilde{x}_{NN,t_0}^G \end{bmatrix} \otimes \boldsymbol{1}_O
$$

a $NO \times N^2$ matrix. Then I have

$$
-\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}_t^G} + \widehat{\boldsymbol{p}_t^R} = -\widehat{\boldsymbol{A}_t^R} + \overline{\boldsymbol{M}^{xG,3}}\widehat{\boldsymbol{\tau}_t^G}.
$$

By log-linearizing equations (B.2), (B.3), and (B.4) for any i, o, and o', I have

$$\widehat{\mu_{i,oo',t}} = \phi\left(-d\chi_{i,oo',t} + \frac{1}{1+\iota}\widehat{V_{i,o',t+1}}\right) - \sum_{o''}\mu_{i,oo'',t_0}\left(-d\chi_{i,oo'',t} + \frac{1}{1+\iota}\widehat{V_{i,o'',t+1}}\right), \tag{K.21}$$

$$\widehat{V_{i,o,t+1}} = \widehat{w_{i,o,t+1}} + dT_{i,o,t+1} - \widehat{P_{i,t+1}} + \sum_{o'}\mu_{i,oo',t_0}\left(-d\chi_{i,oo',t+1} + \frac{1}{1+\iota}\widehat{V_{i,o',t+2}}\right), \tag{K.22}$$

and

$$\widehat{L_{i,o,t+1}} = \sum_{o'}\frac{L_{i,o',t_0}}{L_{i,o,t_0}}\mu_{i,o'o,t_0}\left(\widehat{\mu_{i,o'o,t}} + \widehat{L_{i,o',t}}\right). \tag{K.23}$$

In matrix notation, by equation (K.21),

$$\widehat{\boldsymbol{\mu}_t^{\text{vec}}} = -\phi\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)d\boldsymbol{\chi}_t^{\text{vec}} + \frac{\phi}{1+\iota}\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)\left(\overline{\boldsymbol{I}_{NO}} \otimes \mathbf{1}_O\right)\widehat{\boldsymbol{V}_{t+1}}.$$

where

$$\overline{\boldsymbol{M}^{\mu}} \equiv \overline{\boldsymbol{M}^{\mu,3}} \otimes \mathbf{1}_O,$$

$$\overline{\boldsymbol{M}^{\mu,3}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{i,1\cdot,t_0}\right)^{\top} & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & \left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right)^{\top} & & & & \\ & & & \ddots & & & \\ & & & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^{\top} & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & \left(\boldsymbol{\mu}_{i,O\cdot 1,t_0}\right)^{\top} \end{bmatrix},$$

$$d\boldsymbol{\chi}_t^{\text{vec}} \equiv \begin{bmatrix} d\boldsymbol{\chi}_{1,1\cdot,t} & \dots & d\boldsymbol{\chi}_{1,O\cdot,t} & \dots & d\boldsymbol{\chi}_{N,1\cdot,t} & \dots & d\boldsymbol{\chi}_{N,O\cdot,t} \end{bmatrix}^{\top},$$

and

$$\boldsymbol{\mu}_{i,o\cdot,t_0} \equiv \left(\mu_{i,oo',t_0}\right)_{o'} \text{ and } d\boldsymbol{\chi}_{1,o\cdot,t} \equiv \left(d\chi_{1,oo',t}\right)_{o'} \tag{K.24}$$

are $O \times 1$ vectors. By equation (K.22),

$$\frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\check{\boldsymbol{V}_{t+2}} = \overline{\boldsymbol{M}^{xG,2}}\boldsymbol{p}_{t+1}^{\check{G}} - \check{\boldsymbol{w}_{t+1}} + \check{\boldsymbol{V}_{t+1}}.$$

where

$$\overline{\boldsymbol{M}^{\mu,2}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right)^\top & & \\ \vdots & & \mathbf{0} \\ \left(\boldsymbol{\mu}_{1,O\cdot,t_0}\right)^\top & & \\ & \ddots & \\ & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^\top \\ \mathbf{0} & & \vdots \\ & & \left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right)^\top \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (K.24) for any i and o. By equation (K.22),

$$\check{\boldsymbol{L}_{t+1}} = \overline{\boldsymbol{M}^{\mu L,2}}\boldsymbol{\mu}_t^{\check{\mathrm{vec}}} + \overline{\boldsymbol{M}^{\mu L}}\check{\boldsymbol{L}}_t$$

where $\overline{\boldsymbol{M}^{\mu L}}$ being the $NO \times NO$ matrix

$$\overline{\boldsymbol{M}^{\mu L}} = \overline{\boldsymbol{M}^{\mu,2}} \circ \left(\begin{bmatrix} \left(\boldsymbol{L}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\boldsymbol{L}_{N,\cdot,t_0}\right)^\top \end{bmatrix} \otimes \mathbf{1}_O \right) \oslash \left(\begin{bmatrix} \boldsymbol{L}_{1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{L}_{N,\cdot,t_0} \end{bmatrix} \otimes \left(\mathbf{1}_O\right)^\top \right)$$

and $\overline{\boldsymbol{M}^{\mu L,2}}$ being the $NO \times NO^2$ matrix

$$\overline{\boldsymbol{M}^{\mu L,2}} = \overline{\boldsymbol{M}^{\mu,4}} \circ \left(\begin{bmatrix} \left(\boldsymbol{L}_{1,\cdot,t_0}\right)^\top & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left(\boldsymbol{L}_{N,\cdot,t_0}\right)^\top \end{bmatrix} \otimes \overline{\boldsymbol{I}_O} \right) \oslash$$
$$\begin{pmatrix} \left(\boldsymbol{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{1,o,t_0}\right) & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left(\boldsymbol{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{N,o,t_0}\right) \end{pmatrix},$$

where

$$\overline{\boldsymbol{M}^{\mu,4}} \equiv \begin{bmatrix} \operatorname{diag}\left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right) & & & \boldsymbol{0} & \\ & & & \ddots & & & \\ & \boldsymbol{0} & & & \operatorname{diag}\left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right) \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (K.24) for any i and o.

By log-linearizing equation (I.15) for each i and j,

$$\widehat{Q^G_{ij,t}} = -\varepsilon^G \widehat{p^G_{ij,t}} - \left(1-\varepsilon^G\right) \widehat{P^G_{j,t}} + \left[s^G_{j,t_0} \sum_k \widehat{p^G_{jk,t} Q^G_{jk,t}} + s^V_{j,t_0} \sum_{i,o} \widehat{p^R_{ij,o,t} Q^R_{ij,o,t}} + s^R_{j,t_0} \sum_{o,k} \widehat{p^R_{jk,o,t} Q^R_{jk,o,t}} \right]$$

where

$$s^G_{j,t_0} \equiv \frac{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}$$

is the baseline share of non-robot good production in income,

$$s^R_{j,t_0} \equiv \frac{\sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the baseline share of robot production, and

$$s^V_{j,t_0} \equiv -\frac{\sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the (negative) baseline absorption share of robots. Thus

$$\begin{aligned}
&\left[\varepsilon^G \widehat{p^G_{i,t}} + \left(1-\varepsilon^G\right)\sum_l \widetilde{x}^G_{lj,t_0}\widehat{p^G_{l,t}} - s^G_{j,t_0}\widehat{p^G_{j,t}}\right] - \left[s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,o,t_0}\widetilde{x}^R_{j,o,t_0}\widehat{p^R_{l,o,t}} + s^R_{t_0}\sum_o \widetilde{y}^R_{j,o,t_0}\widehat{p^R_{j,o,t}}\right] \\
&+\left(\widehat{Q^G_{ij,t}} - s^G_{j,t_0}\sum_k \widetilde{y}^G_{jk,t_0}\widehat{Q^G_{jk,t}}\right) - \left(s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,o,t_0}\widetilde{x}^R_{j,o,t_0}\widehat{Q^R_{lj,o,t}} + s^R_{j,t_0}\sum_{k,o}\widetilde{y}^R_{jk,o,t_0}\widetilde{y}^R_{j,o,t_0}\widehat{Q^R_{jk,o,t}}\right) \\
&= -\left[\varepsilon^G\widehat{\tau^G_{ij,t}} + \left(1-\varepsilon^G\right)\sum_l \widetilde{x}^G_{lj,t_0}\widehat{\tau^G_{lj,t}} - s^G_{j,t_0}\sum_k \widetilde{y}^G_{jk,t_0}\widehat{\tau^G_{jk,t}}\right] + \left[s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,t_0}\widehat{\tau^R_{lj,t}} + s^R_{j,t_0}\sum_{k,o}\widetilde{y}^R_{jk,t_0}\widehat{\tau^R_{jk,t}}\right]
\end{aligned}$$

where

$$\widetilde{x}^R_{ij,o,t_0} \equiv \frac{p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{P^R_{j,o,t_0} Q^R_{j,o,t_0}}, \widetilde{x}^R_{j,o,t_0} \equiv \frac{P^R_{j,o,t_0} Q^R_{j,o,t_0}}{P^R_{j,t_0} Q^R_{j,t_0}}, \widetilde{x}^R_{ij,t_0} \equiv \frac{\sum_o p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{P^R_{j,t_0} Q^R_{j,t_0}},$$

$$\widetilde{y}^R_{ij,o,t_0} \equiv \frac{p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{\sum_k p^R_{ik,o,t_0} Q^R_{ik,o,t_0}}, \widetilde{y}^R_{i,o,t_0} \equiv \frac{\sum_k p^R_{ik,o,t_0} Q^R_{ik,o,t_0}}{\sum_{k,o'} p^R_{ik,o',t_0} Q^R_{ik,o',t_0}}, \widetilde{y}^R_{ij,t_0} \equiv \frac{\sum_o p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{\sum_{k,o} p^R_{ik,o,t_0} Q^R_{ik,o,t_0}}.$$

In matrix notation, define

$$\overline{\boldsymbol{M}^{xR}} \equiv \mathbf{1}_N \otimes \left[\begin{array}{ccc} \widetilde{\boldsymbol{x}}^R_{t_0} \circ \widetilde{\boldsymbol{x}}^R_{\cdot 1,\cdot,t_0} & \ldots & \widetilde{\boldsymbol{x}}^R_{t_0} \circ \widetilde{\boldsymbol{x}}^R_{\cdot N,\cdot,t_0} \end{array}\right],$$

a $N^2 \times NO$ matrix,

$$\overline{\boldsymbol{M}^{yR}} \equiv \mathbf{1}_N \otimes \left[\begin{array}{ccccc} \widetilde{y}^R_{1,1} & \cdots & \widetilde{y}^R_{1,O} & & \mathbf{0} \\ & & \ddots & & \\ \mathbf{0} & & \widetilde{y}^R_{N,1} & \cdots & \widetilde{y}^R_{N,O} \end{array}\right],$$

a $N^2 \times NO$ matrix,

$$\overline{\boldsymbol{M}^{yG,2}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{yG}}.$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,2}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{x}^R_{1,o,t_0}\widetilde{x}^R_{11,o,t_0}\right]_o & & \mathbf{0} & \left[\widetilde{x}^R_{1,o,t_0}\widetilde{x}^R_{N1,o,t_0}\right]_o & & \mathbf{0} \\ & \ddots & & \cdots & \ddots & \\ \mathbf{0} & & \left[\widetilde{x}^R_{N,o,t_0}\widetilde{x}^R_{1N,o,t_0}\right]_o & \mathbf{0} & & \left[\widetilde{x}^R_{N,o,t_0}\widetilde{x}^R_{NN,o,t_0}\right]_o \end{bmatrix}$$

a $N^2 \times N^2O$ matrix ,

$$\overline{\boldsymbol{M}^{yR,2}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{11,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{1N,o,t_0}\right]_o & & \mathbf{0} & \\ & & & \ddots & & \\ & \mathbf{0} & & \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{N1,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{NN,o,t_0}\right]_o \end{bmatrix}$$

a $N^2 \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xG,4}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xG}}$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,3}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}^R_{1\cdot,t_0}\right) & \cdots & \operatorname{diag}\left(\widetilde{x}^R_{N\cdot,t_0}\right) \end{bmatrix}$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{yR,3}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{11,t_0},\ldots,\widetilde{y}^R_{1N,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{y}^R_{N1,t_0},\ldots,\widetilde{y}^R_{NN,t_0}\right] \end{bmatrix}$$

a $N^2 \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xO,2}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xO}},$$

a $N^2 \times NO$ matrix. Then I have

$$\left(\varepsilon^G \left[\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_N\right] + \left(1-\varepsilon^G\right)\left[\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}^G_{t_0}\right)^\top\right] - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\left[\mathbf{1}_N \otimes \overline{\boldsymbol{I}_N}\right]\right)\widehat{\boldsymbol{p}^G_t}$$
$$-\left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR}}\right)\widehat{\boldsymbol{p}^R_t}$$
$$+\left(\overline{\boldsymbol{I}_{N^2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{Q}^G_t} - \left[\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,2}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,2}}\right]\widehat{\boldsymbol{Q}^R_t}$$
$$= -\left(\varepsilon^G + \left(1-\varepsilon^G\right)\overline{\boldsymbol{M}^{xG,4}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{\tau}^G_t}$$
$$+\left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,3}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,3}}\right)\widehat{\boldsymbol{\tau}^R_t}$$

By log-linearizing equation (I.7) for each i, j, and o,,

$$\left(1-\alpha^R\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\sum_l \widetilde{x}^G_{lj,t_0}\widehat{p^G_{l,t}} + \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\widehat{p^R_{i,o,t}}$$
$$+\left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta} - \left(1-\alpha^R\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_l \widetilde{x}^R_{lj,o,t_0}\widehat{p^R_{l,o,t}}$$
$$+\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\frac{1}{\varepsilon^R}\widehat{Q^R_{ij,o,t}} + \left[-\frac{1}{\varepsilon^R}\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} + \frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_l \widetilde{x}^R_{lj,o,t_0}\widehat{Q^R_{lj,o,t}}$$
$$= -\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}du_{ij,t} - \left(1-\alpha^R\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\sum_l \widetilde{x}^G_{lj,t_0}\widehat{\tau^G_{lj,t}} - \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\widehat{\tau^R_{ij,t}}$$
$$-\left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta} - \left(1-\alpha^R\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_l \widetilde{x}^R_{lj,o,t_0}\widehat{\tau^R_{lj,t}} + \widehat{\lambda^R_{j,o,t}} + \frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}\widehat{K^R_{j,o,t}}.$$

In matrix notation, write a preliminary $N \times N$ matrix $\overline{\widetilde{\boldsymbol{u}_{t_0}}}$ as such that the (i,j)-element is

$$\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}.$$

Then $\mathbf{1}_N\left(\mathbf{1}_N\right)^\top - \overline{\widetilde{\boldsymbol{u}_{t_0}}}$ is a matrix that is filled with $2\gamma\delta/\left(1+u_{ij,t_0}+2\gamma\delta\right)$ for its (i,j) element and

$$\overline{\boldsymbol{M}^u} \equiv \operatorname{diag}\left(\left[\widetilde{u_{1\cdot,t_0}}, \ldots, \widetilde{u_{N\cdot,t_0}}\right]^\top\right).$$

Using these, write

$$\overline{\boldsymbol{M}^{xG,5}} \equiv \left(\overline{\boldsymbol{M}^{u}} \otimes \overline{\boldsymbol{I}_O}\right)\left(\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top} \otimes \mathbf{1}_O\right)$$

a $N^2O \times N$ matrix,

$$\overline{\boldsymbol{M}^{u,2}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{u}_{1\cdot,t_0}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{u}_{N\cdot,t_0}} \end{bmatrix} \otimes \overline{\boldsymbol{I}_O},$$

a $N^2O \times NO$ matrix where $\widetilde{\boldsymbol{u}_{i\cdot,t_0}} \equiv \left(\widetilde{u_{i\cdot,t_0}}\right)_j$ is a $N \times 1$ vector,

$$\overline{\boldsymbol{M}^{xR,4}} \equiv \left\{\left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^{u}}\right) - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{u}}\right] \otimes \overline{\boldsymbol{I}_O}\right\} \times$$
$$\left(\mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\left\{\widetilde{x}^{R}_{11,o,t_0}\right\}_o\right) & \ldots & \operatorname{diag}\left(\left\{\widetilde{x}^{R}_{N1,o,t_0}\right\}_o\right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\left\{\widetilde{x}^{R}_{1N,o,t_0}\right\}_o\right) & \ldots & \operatorname{diag}\left(\left\{\widetilde{x}^{R}_{NN,o,t_0}\right\}_o\right) \end{bmatrix}\right)$$

a $N^2O \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xR,5}} \equiv \left\{\left[-\frac{1}{\varepsilon^R}\overline{\boldsymbol{M}^{u}} + \left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^{u}}\right)\right] \otimes \overline{\boldsymbol{I}_O}\right\} \times$$
$$\left\{\mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^{R}_{11,1,t_0} \\ \vdots \\ \widetilde{x}^{R}_{11,O,t_0} \\ \vdots \\ \widetilde{x}^{R}_{1N,O,t_0} \end{bmatrix}\right) & \ldots & \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^{R}_{N1,1,t_0} \\ \vdots \\ \widetilde{x}^{R}_{N1,O,t_0} \\ \vdots \\ \widetilde{x}^{R}_{NN,O,t_0} \end{bmatrix}\right) \end{bmatrix}\right\}$$

a $N^2O \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xG,6}} \equiv \left(\overline{\boldsymbol{M}^{u}} \otimes \overline{\boldsymbol{I}_O}\right)\left\{\mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^{G}_{11,t_0} \\ \vdots \\ \widetilde{x}^{G}_{1N,t_0} \end{bmatrix}\right) & \ldots & \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^{G}_{N1,t_0} \\ \vdots \\ \widetilde{x}^{G}_{NN,t_0} \end{bmatrix}\right) \end{bmatrix} \otimes \mathbf{1}_O\right\}$$

a $N^2O \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,6}} \equiv \left\{ \left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^{u}} \right) - \left(1 - \alpha^R \right) \overline{\boldsymbol{M}^{u}} \right] \otimes \overline{\boldsymbol{I}_O} \right\}$$
$$\times \left\{ \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{x}^R_{11,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} & \cdots & \left[\widetilde{x}^R_{N1,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{1N,o,t_0} \right]_o & & \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{N3,o,t_0} \right]_o \end{bmatrix} \right\}$$

a $N^2O \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{u,3}} \equiv \begin{bmatrix} 1 - \widetilde{u_{11,t_0}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 - \widetilde{u_{1N,t_0}} \\ & \vdots & \\ 1 - \widetilde{u_{N1,t_0}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1 - \widetilde{u_{NN,t_0}} \end{bmatrix} \otimes \overline{\boldsymbol{I}_O}$$

a $N^2O \times NO$ matrix. Finally, I have

$$\left(1 - \alpha^R \right) \overline{\boldsymbol{M}^{xG,5}} \widehat{\boldsymbol{p}^G_t} + \left[\overline{\boldsymbol{M}^{u,2}} + \overline{\boldsymbol{M}^{xR,4}} \right] \widehat{\boldsymbol{p}^R_t} + \left\{ \frac{1}{\varepsilon^R} \left(\overline{\boldsymbol{M}^{u}} \otimes \overline{\boldsymbol{I}_O} \right) + \overline{\boldsymbol{M}^{xR,5}} \right\} \widehat{\boldsymbol{Q}^R_t}$$
$$= - \left(\overline{\boldsymbol{M}^{u}} \otimes \mathbf{1}_O \right) d\boldsymbol{u}_t - \left(1 - \alpha^R \right) \overline{\boldsymbol{M}^{xG,6}} \widehat{\boldsymbol{\tau}^G_t} - \left[\left(\overline{\boldsymbol{M}^{u}} \otimes \mathbf{1}_O \right) + \overline{\boldsymbol{M}^{xR,6}} \right] \widehat{\boldsymbol{\tau}^R_t} + \left(\mathbf{1}_N \otimes \overline{\boldsymbol{I}_{NO}} \right) \widehat{\boldsymbol{\lambda}^R_t} + \overline{\boldsymbol{M}^{u,3}} \widehat{\boldsymbol{K}^R_t}.$$

By log-linearizing equation (I.2) for each i and o,

$$
\begin{aligned}
&\widehat{p^G_{i,t}} + \sum_j \widetilde{y}^G_{ij,t_0} \widehat{Q^G_{ij,t}} - \widehat{w_{i,o,t}} + \left[-\frac{1}{\theta_o} + \left(-\frac{1}{\beta} + \frac{1}{\theta_o} \right) l^O_{i,o,t_0} \right] \widehat{L_{i,o,t}} + \left(-1 + \frac{1}{\beta} \right) \sum_{o'} \widetilde{x}^O_{i,o',t_0} l^O_{i,o',t_0} \widehat{L_{i,o',t}} \\
&= -\frac{1}{\beta} \widehat{b_{i,o,t}} + \frac{1}{\theta_o} \frac{a_{o,t_0}}{1 - a_{o,t_0}} \widehat{a_{o,t}} + \left(-\frac{1}{\beta} + \frac{1}{\theta_o} \right) \frac{1}{\theta_o - 1} \left[-\left(1 - l^O_{i,o,t_0} \right) + l^O_{i,o,t_0} \frac{a_{o,t_0}}{1 - a_{o,t_0}} \right] \widehat{a_{o,t}} \\
&\quad + \left(-1 + \frac{1}{\beta} \right) \frac{1}{\theta_o - 1} \sum_{o'} \widetilde{x}^O_{i,o',t_0} \left[-\left(1 - l^O_{i,o',t_0} \right) + l^O_{i,o',t_0} \frac{a_{o',t_0}}{1 - a_{o',t_0}} \right] \widehat{a_{o',t}} \\
&\quad - \sum_j y^G_{ij,t_0} \widehat{\tau^G_{ij,t}} - \left(-\frac{1}{\beta} + \frac{1}{\theta_o} \right) \left(1 - l^O_{i,o,t_0} \right) \widehat{K^R_{i,o,t}} - \left(-1 + \frac{1}{\beta} \right) \sum_{o'} \widetilde{x}^O_{i,o',t_0} \left(1 - l^O_{i,o',t_0} \right) \widehat{K^R_{i,o',t}},
\end{aligned}
$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{yG,3}} \equiv \overline{\boldsymbol{M}^{yG}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xOl,3}} \equiv \overline{\boldsymbol{M}^{xOl}} \otimes \mathbf{1}_O$$

a $NO \times NO$ matrix,

$$\overline{\boldsymbol{M}^{a}} \equiv \mathbf{1}_N \otimes \operatorname{diag}\left(\frac{a_{o,t_0}}{1 - a_{o,t_0}} \right)$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,2}} \equiv \begin{bmatrix} \operatorname{diag}\left(-\left(1 - l^O_{1,o,t_0} \right) + l^O_{1,o,t_0} \frac{a_{o,t_0}}{1 - a_{o,t_0}} \right) \\ \vdots \\ \operatorname{diag}\left(-\left(1 - l^O_{N,o,t_0} \right) + l^O_{N,o,t_0} \frac{a_{o,t_0}}{1 - a_{o,t_0}} \right) \end{bmatrix}$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,3}} \equiv \left(\widetilde{\boldsymbol{x}}^O_{t_0} \circ \overline{\boldsymbol{M}^{al}} \right) \otimes \mathbf{1}_O$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{xOl,4}} \equiv \overline{\boldsymbol{M}^{xOl,2}} \otimes \mathbf{1}_O,$$

a $NO \times NO$ matrix. I have

$$
\begin{aligned}
&\left(\boldsymbol{I}_N \otimes \mathbf{1}_O\right) \widehat{\boldsymbol{p}_t^G} - \widehat{\boldsymbol{w}_t} + \overline{\boldsymbol{M}^{yG,3}} \widehat{\boldsymbol{Q}_t^G} + \left(-\frac{1}{\theta_o} \overline{\boldsymbol{I}_{NO}} + \left(-\frac{1}{\beta} + \frac{1}{\theta_o}\right) \operatorname{diag}\left(\boldsymbol{l}_{t_0}^O\right) + \left(-1 + \frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,3}}\right) \widehat{\boldsymbol{L}}_t \\
&= -\frac{1}{\beta} \widehat{\boldsymbol{b}}_t + \left[\frac{1}{\theta_o} \overline{\boldsymbol{M}^a} + \left(-\frac{1}{\beta} + \frac{1}{\theta_o}\right) \frac{1}{\theta - 1} \overline{\boldsymbol{M}^{al,2}} + \left(-1 + \frac{1}{\beta}\right) \frac{1}{\theta - 1} \overline{\boldsymbol{M}^{al,3}}\right] \widehat{\boldsymbol{a}}_t - \overline{\boldsymbol{M}^{yG,3}} \widehat{\boldsymbol{\tau}_t^G} \\
&\quad + \left[-\left(-\frac{1}{\beta} + \frac{1}{\theta_o}\right) \operatorname{diag}\left(1 - l_{i,o,t_0}^O\right) - \left(-1 + \frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,4}}\right] \widehat{\boldsymbol{K}_t^R}.
\end{aligned}
$$

Hence the log-linearized temporary equilibrium system is

$$
\overline{\boldsymbol{D}^x} \widehat{\boldsymbol{x}} = \overline{\boldsymbol{D}^A} \widehat{\boldsymbol{A}}
$$

where matrices $\overline{\boldsymbol{D}^x}$ and $\overline{\boldsymbol{D}^A}$ are defined as

$$
\overline{\boldsymbol{D}^x} \equiv \begin{bmatrix}
\overline{\boldsymbol{D}_{11}^x} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{14}^x} & \mathbf{0} & \overline{\boldsymbol{D}_{16}^x} \\
-\overline{\boldsymbol{M}^{xG,2}} & \overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\phi \overline{\boldsymbol{M}^{xG,2}} & \mathbf{0} & -\phi \overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{M}^l} \\
\overline{\boldsymbol{D}_{41}^x} & \overline{\boldsymbol{D}_{42}^x} & \mathbf{0} & \overline{\boldsymbol{D}_{44}^x} & \overline{\boldsymbol{D}_{45}^x} & \mathbf{0} \\
\overline{\boldsymbol{D}_{51}^x} & \overline{\boldsymbol{D}_{52}^x} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{55}^x} & \mathbf{0} \\
\overline{\boldsymbol{D}_{61}^x} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \overline{\boldsymbol{M}^{yG,3}} & \mathbf{0} & \overline{\boldsymbol{D}_{66}^x}
\end{bmatrix},
$$

where

$$
\overline{\boldsymbol{D}_{11}^x} \equiv -\alpha_M \left(\overline{\boldsymbol{I}_N} - \left(\widetilde{\boldsymbol{x}}_{t_0}^G\right)^\top\right), \ \overline{\boldsymbol{D}_{14}^x} \equiv \left(1 - \alpha_M\right) \overline{\boldsymbol{M}^{yG}}, \ \overline{\boldsymbol{D}_{16}^x} \equiv -\alpha_L \overline{\boldsymbol{M}^{xOl}},
$$

$$
\overline{\boldsymbol{D}_{41}^x} \equiv \varepsilon^G \left[\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_N\right] + \left(1 - \varepsilon^G\right) \left[\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}_{t_0}^G\right)^\top\right] - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^G\right) \left[\mathbf{1}_N \otimes \overline{\boldsymbol{I}_N}\right],
$$

$$
\overline{\boldsymbol{D}_{42}^x} \equiv \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^V\right) \overline{\boldsymbol{M}^{xR}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^R\right) \overline{\boldsymbol{M}^{yR}},
$$

$$
\overline{\boldsymbol{D}_{44}^x} \equiv \overline{\boldsymbol{I}_{N^2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^G\right) \overline{\boldsymbol{M}^{yG,2}},
$$

$$
\overline{\boldsymbol{D}_{45}^x} \equiv -\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^V\right) \overline{\boldsymbol{M}^{xR,2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}_{t_0}^R\right) \overline{\boldsymbol{M}^{yR,2}},
$$

$$\overline{\boldsymbol{D}_{51}^{x}} \equiv \left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xG,5}},\ \overline{\boldsymbol{D}_{52}^{x}} \equiv \overline{\boldsymbol{M}^{u,2}}+\overline{\boldsymbol{M}^{xR,4}},\ \overline{\boldsymbol{D}_{55}^{x}} \equiv \frac{1}{\varepsilon^{R}}\left(\overline{\boldsymbol{M}^{u}}\otimes\overline{\boldsymbol{I}_{O}}\right)+\overline{\boldsymbol{M}^{xR,5}},$$

$$\overline{\boldsymbol{D}_{61}^{x}} \equiv \boldsymbol{I}_{N}\otimes\boldsymbol{1}_{N},\ \overline{\boldsymbol{D}_{66}^{x}} \equiv -\frac{1}{\theta_{o}}+\left(-\frac{1}{\beta}+\frac{1}{\theta_{o}}\right)\operatorname{diag}\left(\boldsymbol{l}_{t_0}^{O}\right)+\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}},$$

and

$$\overline{\boldsymbol{D}^{A}} \equiv \begin{bmatrix} \boldsymbol{0} & \overline{\boldsymbol{D}_{12}^{A}} & \overline{\boldsymbol{D}_{13}^{A}} & \overline{\boldsymbol{I}_{N}} & \boldsymbol{0} & \overline{\boldsymbol{D}_{16}^{A}} & \overline{\boldsymbol{D}_{17}^{A}} & \boldsymbol{0} & \alpha_{L}\overline{\boldsymbol{M}^{xOl,2}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} & \overline{\boldsymbol{M}^{xG}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\phi\overline{\boldsymbol{M}^{xG,3}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \overline{\boldsymbol{D}_{47}^{A}} & \overline{\boldsymbol{D}_{48}^{A}} & \boldsymbol{0} & \boldsymbol{0} \\ \overline{\boldsymbol{D}_{51}^{A}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \overline{\boldsymbol{D}_{57}^{A}} & \overline{\boldsymbol{D}_{58}^{A}} & \overline{\boldsymbol{M}^{u,3}} & \overline{\boldsymbol{D}_{5,10}^{A}} \\ \boldsymbol{0} & \overline{\boldsymbol{D}_{62}^{A}} & -\frac{1}{\beta}\overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\overline{\boldsymbol{M}^{yG,3}} & \boldsymbol{0} & \overline{\boldsymbol{D}_{69}^{A}} & \boldsymbol{0} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{12}^{A}} \equiv \frac{\alpha_{L}}{\theta-1}\left(\widetilde{\boldsymbol{x}}_{t_0}^{O}\otimes\overline{\boldsymbol{M}^{al}}\right),\ \overline{\boldsymbol{D}_{13}^{A}} \equiv \frac{\alpha_{L}}{\beta-1}\overline{\boldsymbol{M}^{xO}},$$

$$\overline{\boldsymbol{D}_{16}^{A}} \equiv \left(1-\alpha_{L}-\alpha_{M}\right)\overline{\boldsymbol{I}_{N}},\ \overline{\boldsymbol{D}_{17}^{A}} \equiv -\left[\alpha_{M}\overline{\boldsymbol{M}^{xG}}+\left(1-\alpha_{M}\right)\overline{\boldsymbol{M}^{yG}}\right],$$

$$\overline{\boldsymbol{D}_{47}^{A}} \equiv -\varepsilon^{G}+\left(1-\varepsilon^{G}\right)\overline{\boldsymbol{M}^{xG,4}}+\operatorname{diag}\left(\boldsymbol{1}_{N}\otimes\boldsymbol{s}_{t_0}^{G}\right)\overline{\boldsymbol{M}^{yG,2}},$$

$$\overline{\boldsymbol{D}_{48}^{A}} \equiv \operatorname{diag}\left(\boldsymbol{1}_{N}\otimes\boldsymbol{s}_{t_0}^{V}\right)\overline{\boldsymbol{M}^{xR,3}}+\operatorname{diag}\left(\boldsymbol{1}_{N}\otimes\boldsymbol{s}_{t_0}^{R}\right)\overline{\boldsymbol{M}^{yR,3}},$$

$$\overline{\boldsymbol{D}_{51}^{A}} \equiv -\left(\overline{\boldsymbol{M}^{u}}\otimes\boldsymbol{1}_{O}\right),\ \overline{\boldsymbol{D}_{57}^{A}} \equiv -\left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xG,6}},$$

$$\overline{\boldsymbol{D}_{58}^{A}} \equiv -\left[\left(\overline{\boldsymbol{M}^{u}}\otimes\boldsymbol{1}_{O}\right)+\overline{\boldsymbol{M}^{xR,6}}\right],\ \overline{\boldsymbol{D}_{5,10}^{A}} \equiv \boldsymbol{1}_{N}\otimes\overline{\boldsymbol{I}_{NO}},$$

$$\overline{\boldsymbol{D}_{62}^{A}} \equiv \frac{1}{\theta}\overline{\boldsymbol{M}^{a}}+\left(-\frac{1}{\beta}+\frac{1}{\theta_{o}}\right)\frac{1}{\theta_{o}-1}\overline{\boldsymbol{M}^{al,2}}+\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta_{o}-1}\overline{\boldsymbol{M}^{al,3}},$$

and

$$\overline{\boldsymbol{D}_{69}^{A}} \equiv -\left(-\frac{1}{\beta}+\frac{1}{\theta_{o}}\right)\operatorname{diag}\left(1-l_{i,o,t_0}^{O}\right)-\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}}.$$

To normalize the price, one of the good-demand equation must be replaced with log-linearized numeraire condition $\widehat{P_{1,t}^{G}} = \sum_i x_{i1,t_0}^{G}\left(\widehat{p_{i,t}^{G}} + \widehat{\tau_{i1,t}^{G}}\right) = 0$, or

$$\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{p}_t^{G}} = -\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{\tau}_t^{G}},$$

where $\overline{\boldsymbol{M}^{xG,num}} \equiv \left[x_{11,t_0}^{G}, x_{21,t_0}^{G}, x_{31,t_0}^{G}\right]$.

To analyze the steady state conditions, first note that the steady state accumulation condition (I.18) implies $\widehat{Q_{i,o}^{R}} = \widehat{K_{i,o}^{R}}$. Using robot integration function, integration demand and unit cost formula, I have

$$\widehat{Q_{i,o}^{R}} = \sum_l x_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}} + \left(1-\alpha^{R}\right)\left(\sum_l \tilde{x}_{ij,o,t_0}^{R}\widehat{p_{li,o}^{R}} - \sum_l \tilde{x}_{li,t_0}^{G}\widehat{p_{li,t}^{G}}\right) \tag{K.25}$$

Thus the condition is

$$\begin{aligned}
&\sum_l \tilde{x}_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}} + \left(1-\alpha^{R}\right)\sum_l \tilde{x}_{li,o,t_0}^{R}\widehat{p_{l,o}^{R}} - \left(1-\alpha^{R}\right)\sum_l \tilde{x}_{li,t_0}^{G}\widehat{p_{l}^{G}} - \widehat{K_{i,o}^{R}}\\
&= \left(1-\alpha^{R}\right)\sum_l \tilde{x}_{li,t_0}^{G}\widehat{\tau_{li}^{G}} - \left(1-\alpha^{R}\right)\sum_l \tilde{x}_{li,o,t_0}^{R}\widehat{\tau_{li}^{R}}.
\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{xR,7}} \equiv \begin{bmatrix} \operatorname{diag}\left(\tilde{x}_{1\cdot,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\tilde{x}_{N\cdot,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xR,8}} \equiv \begin{bmatrix} \operatorname{diag}\left(\tilde{x}_{11,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\tilde{x}_{N1,\cdot,t_0}^{R}\right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\tilde{x}_{1N,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\tilde{x}_{NN,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times NO$ matrix, and

$$\overline{\boldsymbol{M}^{xG,7}} \equiv \begin{bmatrix} \widetilde{x}^{G}_{11,t_0} & & & \cdots & \widetilde{x}^{G}_{N1,t_0} & & \mathbf{0} \\ & \ddots & & & & \ddots & \\ \mathbf{0} & & \widetilde{x}^{G}_{1N,t_0} & & & & \widetilde{x}^{G}_{NN,t_0} \end{bmatrix} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix.

$$\overline{\boldsymbol{M}^{xR,9}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{x}}^{R}_{11,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^{R}_{N1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & & \cdots & & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{x}}^{R}_{1N,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^{R}_{NN,\cdot,t_0} \end{bmatrix},$$

a $NO \times N^2$ matrix, where $\widetilde{\boldsymbol{x}}^{R}_{ij,\cdot,t_0} \equiv \left(\widetilde{x}^{R}_{ij,o,t_0}\right)_o$ is an $O \times 1$ vector for any i and j. Then I have

$$-\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} + \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\widehat{\boldsymbol{p}^R} + \overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} - \widehat{\boldsymbol{K}^R} = \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}$$

Next, to study the steady state Euler equation (I.19) , note that by equation (I.5),

$$\begin{aligned}
\frac{\partial \pi_{i,t}\left(\widehat{\left\{K^R_{i,o,t}\right\}}\right)}{\partial K^R_{i,o,t}} &= \sum_j \widetilde{y}^G_{ij,t}\left(\widehat{p^G_{ij,t}} + \widehat{Q^G_{ij,t}}\right) + \left[-\frac{1}{\beta}\sum_{o'} x^O_{i,o',t_0}\widehat{b_{i,o',t}} + \frac{1}{\beta}\widehat{b_{i,o,t}}\right] \\
&+ \left\{\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta_o - 1}\sum_{o'}\frac{\widetilde{x}^O_{i,o',t_0}}{1-a_{o,t_0}}\left[-l^O_{i,o',t_0}a_{o,t_0} + \left(1-l^O_{i,o',t_0}\right)\left(1-a_{o,t_0}\right)\right]\widehat{a_{o',t}}\right. \\
&+ \left.\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\frac{1}{\theta_o - 1}\frac{-l^O_{i,o,t_0}a_{o,t_0} + \left(1-l^O_{i,o,t_0}\right)\left(1-a_{o,t_0}\right)}{1-a_{o,t_0}} + \frac{1}{\theta_o}\right\}\widehat{a_{o,t}}\right\} \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} x^O_{i,o',t_0} l^O_{i,o',t_0}\widehat{L_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right) l^O_{i,o,t_0}\widehat{L_{i,o,t}}\right] \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}^O_{i,o',t_0}\left(1-l^O_{i,o',t_0}\right)\widehat{K^R_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\left(1-l^O_{i,o,t_0}\right)\widehat{K^R_{i,o,t}} + \left(-\frac{1}{\theta_o}\right)\widehat{K^R_{i,o,t}}\right].
\end{aligned} \tag{K.26}$$

Note that by the steady state accumulation condition (I.18), $Q^R_{i,o,t_0}/K^R_{i,o,t_0} = \delta$. Note also that investment function implies that, in the steady state,

$$\frac{\lambda_{j,o}^{R}}{P_{j,o}^{R}} = \left(\sum_{i} \frac{x_{ij,o}^{R}}{\left(1+u_{ij}\right)^{1-\varepsilon^{R}}}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}} + 2\gamma\delta. \tag{K.27}$$

To simplify the notation, set

$$\widetilde{u}_{j,o,t_0}^{SS} \equiv \frac{(\iota+\delta)\left[\left(\sum_i x_{ij,o,t_0}^{R}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^{R}\right)}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}} + 2\gamma\delta\right]}{(\iota+\delta)\left[\left(\sum_i x_{ij,o,t_0}^{R}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^{R}\right)}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}} + 2\gamma\delta\right] - \gamma\delta^{2}},$$

Then, by log-linearizing equation (I.19), after rearranging, I have:

$$\begin{aligned}
&\left[\widehat{p_i^G} + 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{p_{l,t}^{G}}\right] - \left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\widehat{p_{i,o}^{R}} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{ij,o,t_0}^{R}\widehat{p_{l,o}^{R}} \\
&+\sum_j \widetilde{y}_{ij,t_0}^{G}\widehat{Q_{ij}^{G}} - 2\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}} + \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}_{i,o',t_0}^{O} l_{i,o',t_0}^{O}\widehat{L_{i,o'}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right) l_{i,o,t_0}^{O}\widehat{L_{i,o}}\right] \\
&+\Bigg[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}_{i,o',t_0}^{O}\left(1-l_{i,o',t_0}^{O}\right)\widehat{K_{i,o'}^{R}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\left(1-l_{i,o,t_0}^{O}\right)\widehat{K_{i,o}^{R}} + \left(-\frac{1}{\theta_o}\right)\widehat{K_{i,o}^{R}} + 2\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\widehat{K} \\
&-\widetilde{u}_{i,o,t_0}^{SS}\widehat{\lambda_{i,o}^{R}} \\
&= -\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta_o-1}\sum_{o'}\frac{\widetilde{x}_{i,o',t_0}^{O}}{1-a_{o,t_0}}\left[\left(1-l_{i,o',t_0}^{O}\right)\left(1-a_{o',t_0}\right) - l_{i,o',t_0}^{O}a_{o',t_0}\right]\widehat{a_{o'}} \\
&-\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\frac{1}{\theta_o-1}\frac{1}{1-a_{o,t_0}}\left[\left(1-l_{i,o,t_0}^{O}\right)\left(1-a_{o,t_0}\right) - l_{i,o,t_0}^{O}a_{o,t_0}\right] + \frac{1}{\theta_o}\right\}\widehat{a_o} \\
&-\left[-\frac{1}{\beta}\sum_{o'}\widetilde{x}_{i,o',t_0}^{O}\widehat{b_{i,o'}} + \frac{1}{\beta}\widehat{b_{i,o}}\right] + \left[-\sum_j \widetilde{y}_{ij,t_0}^{G}\widehat{\tau_{ij}^{G}} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{\tau_{li,t}^{G}}\right] \\
&+2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{ij,o,t_0}^{R}\widehat{\tau_{li}^{R}}
\end{aligned}$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{xO,3}} \equiv \overline{\boldsymbol{M}^{xO}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix. Then

$$\left[\left(\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_O\right) + 2\left(1-\alpha^R\right) \operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right) \overline{\boldsymbol{M}^{xG,2}}\right] \widehat{\boldsymbol{p}^G} - \operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\left(\overline{\boldsymbol{I}_{NO}} - 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\widehat{\boldsymbol{p}^R}$$
$$+\overline{\boldsymbol{M}^{yG,3}}\widehat{\boldsymbol{Q}^G} - 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} + \left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(l^O_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{L}}$$
$$+\left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(1-l^O_{\cdot,\cdot,t_0}\right) - \frac{1}{\theta_o}\overline{\boldsymbol{I}_{NO}} + 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{K}^R} - \operatorname{diag}\left(\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)$$
$$= -\left[\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta_o-1}\overline{\boldsymbol{M}^{al,3}} - \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\frac{1}{\theta_o-1}\overline{\boldsymbol{M}^{al,2}}\right]\widehat{\boldsymbol{a}} - \frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right)\widehat{\boldsymbol{b}}$$
$$+\left[-\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,7}}\right]\widehat{\boldsymbol{\tau}^G} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}$$

In the steady state, I write equations (K.22) and (K.23) as

$$\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} - \widehat{\boldsymbol{w}} + \left[\overline{\boldsymbol{I}_{NO}} - \frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\right]\widehat{\boldsymbol{V}} = -\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} + d\boldsymbol{T} - \overline{\boldsymbol{M}^{\mu,3}}d\boldsymbol{\chi}^{\text{vec}}$$

and

$$\left[\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{\mu L}}\right]\widehat{\boldsymbol{L}} - \overline{\boldsymbol{M}^{\mu L,2}}\widehat{\boldsymbol{\mu}^{\text{vec}}} = \mathbf{0}.$$

respectively.

Hence the log-linearized steady state system is

$$\overline{\boldsymbol{E}^y}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}^\Delta}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{E}^y} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,SS}}} \end{bmatrix}, \text{ and } \overline{\boldsymbol{E}^\Delta} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \overline{\boldsymbol{D}^{\Delta,SS}} \end{bmatrix},$$

$\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,T}} & \overline{\boldsymbol{D}^{A,\Delta}} \end{bmatrix}$, and matrices $\overline{\boldsymbol{D}^{y,SS}}$ and $\overline{\boldsymbol{D}^{\Delta,SS}}$ are defined as

$$\overline{\boldsymbol{D}^{y,SS}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{y,SS}_{11}} & \overline{\boldsymbol{D}^{y,SS}_{12}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{M}^{xR,7}} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \mathbf{0} \\ \overline{\boldsymbol{D}^{y,SS}_{21}} & \overline{\boldsymbol{D}^{y,SS}_{22}} & \mathbf{0} & \overline{\boldsymbol{M}^{yG,3}} & \overline{\boldsymbol{D}^{y,SS}_{25}} & \overline{\boldsymbol{D}^{y,SS}_{26}} & \overline{\boldsymbol{D}^{y,SS}_{27}} & \overline{\boldsymbol{D}^{y,SS}_{28}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{11}^{y,SS}} \equiv -\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{12}^{y,SS}} \equiv \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}},$$

$$\overline{\boldsymbol{D}_{21}^{y,SS}} \equiv \left(\overline{\boldsymbol{I}_N}\otimes \mathbf{1}_O\right) + 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{22}^{y,SS}} \equiv -\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right),$$

$$\overline{\boldsymbol{D}_{25}^{y,SS}} \equiv -2\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,7}},$$

$$\overline{\boldsymbol{D}_{26}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\text{diag}\left(l^O_{\cdot,\cdot,t_0}\right),$$

$$\overline{\boldsymbol{D}_{27}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\text{diag}\left(1-l^O_{\cdot,\cdot,t_0}\right) - \frac{1}{\theta_o}\overline{\boldsymbol{I}_{NO}} + 2\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right),$$

$$\overline{\boldsymbol{D}_{28}^{y,SS}} \equiv -\text{diag}\left(\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right),$$

and

$$\overline{\boldsymbol{D}^{\Delta,SS}} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{17}^{\Delta,SS}} & \overline{\boldsymbol{D}_{18}^{\Delta,SS}} \\ \mathbf{0} & \overline{\boldsymbol{D}_{22}^{\Delta,SS}} & \overline{\boldsymbol{D}_{23}^{\Delta,SS}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{27}^{\Delta,SS}} & \overline{\boldsymbol{D}_{28}^{\Delta,SS}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{17}^{\Delta,SS}} \equiv \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,7}},$$

$$\overline{\boldsymbol{D}_{18}^{\Delta,SS}} \equiv -\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,9}},$$

$$\overline{\boldsymbol{D}_{22}^{\Delta,SS}} \equiv \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\frac{1}{\theta_o-1}\overline{\boldsymbol{M}^{al,2}} - \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta_o-1}\overline{\boldsymbol{M}^{al,3}},$$

$$\overline{\boldsymbol{D}_{23}^{\Delta,SS}} \equiv -\frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right),$$

$$\overline{\boldsymbol{D}_{27}^{\Delta,SS}} \equiv -\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,7}},$$

and

$$\overline{\boldsymbol{D}_{28}^{\Delta,SS}} \equiv 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,9}}.$$

If $\overline{\boldsymbol{E}^{y}}$ is invertible, I have $\overline{\boldsymbol{E}} \equiv \left(\overline{\boldsymbol{E}^{y}}\right)^{-1} \overline{\boldsymbol{E}^{\Delta}}$ such that $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}} \boldsymbol{\Delta}$. Write dimensions of $\boldsymbol{y}$ and $\boldsymbol{\Delta}$ as $n_y \equiv N + 3NO + N^2 + N^2 O$ and $n_\Delta \equiv 3N^2 + O + 2NO + 2N$, respectively.

Finally, to study the transitional dynamics, the capital accumulation dynamics (8) implies

$$\check{K}^{R}_{i,o,t+1} = -\delta \left(1 - \alpha^R\right) \sum_l \widetilde{x}^{G}_{li,t_0} \check{p}^{G}_{l,t} + \delta \left(1 - \alpha^R\right) \sum_l \widetilde{x}^{R}_{li,o} p^{\check{R}}_{l,o,t} + \delta \sum_l \widetilde{x}^{R}_{li,o} Q^{\check{R}}_{li,o,t} + (1 - \delta) K^{\check{R}}_{i,o,t}.$$

In a matrix form, write

$$\boldsymbol{K}^{\check{R}}_{t+1} = -\delta \left(1 - \alpha^R\right) \overline{\boldsymbol{M}^{xG,2}} \check{\boldsymbol{p}}^{G}_{t} + \delta \left(1 - \alpha^R\right) \overline{\boldsymbol{M}^{xR,8}} \check{\boldsymbol{p}}^{R}_{t} + \delta \overline{\boldsymbol{M}^{xR,7}} \check{\boldsymbol{Q}}^{R}_{t} + (1 - \delta) \overline{\boldsymbol{I}_{NO}} \check{\boldsymbol{K}}^{R}_{t}.$$

Next, to study the Euler equation, define

$$\widetilde{u}^{TD,1}_{i,o} \equiv \frac{-(\iota + \delta) \left[\left(\sum_l x^{R}_{li,o} \left(1 + u_{li}\right)^{-\left(1 - \varepsilon^R\right)} \right)^{\frac{1}{1 - \varepsilon^R} \alpha^R} + 2\gamma\delta \right] + \gamma\delta^2}{(1 - \delta) \left[\left(\sum_l x^{R}_{li,o} \left(1 + u_{li}\right)^{-\left(1 - \varepsilon^R\right)} \right)^{\frac{1}{1 - \varepsilon^R} \alpha^R} + 2\gamma\delta \right]}$$

and

$$\widetilde{u}^{TD,2}_{i,o} \equiv \frac{-\gamma\delta^2}{(1 - \delta) \left[\left(\sum_l x^{R}_{li,o} \left(1 + u_{li}\right)^{-\left(1 - \varepsilon^R\right)} \right)^{\frac{1}{1 - \varepsilon^R} \alpha^R} + 2\gamma\delta \right]}.$$

Then I have

$$\begin{aligned}
&\left[-\widetilde{u}^{TD,1}_{i,o} \check{p}^{G}_{i,t+1} + 2 \left(1 - \alpha^R\right) \widetilde{u}^{TD,2}_{i,o} \sum_l \widetilde{x}^{G}_{li} \check{p}^{G}_{l,t+1} \right] + \left[-\widetilde{u}^{TD,2}_{i,o} \check{p}^{R}_{i,o,t+1} - 2 \left(1 - \alpha^R\right) \widetilde{u}^{TD,2}_{i,o} \sum_l \widetilde{x}^{R}_{li,o} \check{p}^{R}_{l,o,t+1} \right] \\
&- \widetilde{u}^{TD,1}_{i,o} \sum_j \widetilde{y}^{G}_{ij} \check{Q}^{G}_{ij,t+1} - 2\widetilde{u}^{TD,2}_{i,o} \sum_l \widetilde{x}^{R}_{li,o} \check{Q}^{R}_{li,o,t+1} - \widetilde{u}^{TD,1}_{i,o} \left(-1 + \frac{1}{\beta}\right) \sum_{o'} x^{O}_{i,o'} \left(1 - l^{O}_{i,o'}\right) \check{K}^{R}_{i,o',t+1} \\
&- \widetilde{u}^{TD,1}_{i,o} \left[\left(-1 + \frac{1}{\beta}\right) \sum_{o'} x^{O}_{i,o'} l^{O}_{i,o'} \check{L}_{i,o',t+1} + \left(-\frac{1}{\beta} + \frac{1}{\theta_o}\right) l^{O}_{i,o} \check{L}_{i,o,t+1} \right] \\
&- \left[\widetilde{u}^{TD,1}_{i,o} \left\{ \left(-\frac{1}{\beta} + \frac{1}{\theta_o}\right) \left(1 - l^{O}_{i,o}\right) + \left(-\frac{1}{\theta_o}\right) \right\} - 2\widetilde{u}^{TD,2}_{i,o} \right] \check{K}^{R}_{i,o,t+1} + \check{\lambda}^{R}_{i,o,t+1} = \frac{1 + \iota}{1 - \delta} \lambda^{\check{R}}_{i,o,t}
\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{u,4}} = \begin{bmatrix} \widetilde{\boldsymbol{u}}_{1,\cdot}^{TD,1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{u}}_{N,\cdot}^{TD,1} \end{bmatrix},$$

a $NO \times N$ matrix where $\widetilde{\boldsymbol{u}}_{i,\cdot}^{TD,1} \equiv \left(\widetilde{u}_{i,o}^{TD,1}\right)_o$ is an $O \times 1$ vector for any i. Then

$$\begin{aligned}
&\left(-\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\overline{\boldsymbol{M}^{xG,2}}\right)\boldsymbol{p}_{t+1}^{\check{G}} - \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\boldsymbol{p}_{t+1}^{\check{R}} \\
&\quad - \left[\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_N)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}}\right]\boldsymbol{Q}_{t+1}^{\check{G}} - 2\left((\mathbf{1}_N)^\top \otimes \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}}\boldsymbol{Q}_{t+1}^{\check{R}} \\
&\quad + \left[-\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1} l_{\cdot,\cdot}^{O}\right)\right]\boldsymbol{L}_{t+1}^{\check{}} \\
&\quad + \left\{\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\left(1-l_{\cdot,\cdot}^{O}\right)\right)\right. \\
&\quad \left. + \frac{1}{\theta}\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\right) + 2\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right\}\boldsymbol{K}_{t+1}^{\check{R}} + \overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}_{t+1}^{\check{R}} = \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}_t^{\check{R}}.
\end{aligned}$$

Hence the log-linearized transitional dynamic system is $\overline{\boldsymbol{D}_{t+1}^{y,TD}}\check{\boldsymbol{y}}_{t+1} = \overline{\boldsymbol{D}_t^{y,TD}}\check{\boldsymbol{y}}_t$, where matrices $\overline{\boldsymbol{D}_{t+1}^{y,TD}}$ and $\overline{\boldsymbol{D}_t^{y,TD}}$ are defined as

$$\overline{\boldsymbol{D}_{t+1}^{y,TD}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{I}_{NO}} & \mathbf{0} \\ \overline{\boldsymbol{D}_{21,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{22,t+1}^{y,TD}} & \mathbf{0} & \overline{\boldsymbol{D}_{24,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{25,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{26,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{27,t+1}^{y,TD}} & \overline{\boldsymbol{I}_{NO}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{21,t+1}^{y,TD}} \equiv -\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{22,t+1}^{y,TD}} \equiv -\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right),$$

$$\overline{\boldsymbol{D}_{24,t+1}^{y,TD}} \equiv -\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_N)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}},$$

$$\overline{\boldsymbol{D}_{25,t+1}^{y,TD}} \equiv -2\left((\mathbf{1}_N)^\top \otimes \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}},$$

$$\overline{\boldsymbol{D}_{26,t+1}^{y,TD}} \equiv -\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1} l_{\cdot,\cdot}^{O}\right),$$

$$\overline{\boldsymbol{D}^{y,TD}_{27,t+1}} \equiv \left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\mathbf{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right)$$
$$-\left(-\frac{1}{\beta}+\frac{1}{\theta_o}\right)\operatorname{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\left(1-l^O_{\cdot,\cdot}\right)\right)+\frac{1}{\theta_o}\operatorname{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\right)+2\operatorname{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right),$$

and

$$\overline{\boldsymbol{D}^{y,TD}_t} = \begin{bmatrix} -\delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}} & \delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}} & \mathbf{0} & \mathbf{0} & \delta\overline{\boldsymbol{M}^{xR,7}} & \mathbf{0} & (1-\delta)\overline{\boldsymbol{I}_{NO}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}} \end{bmatrix}. \tag{K.28}$$

Since $\check{\boldsymbol{y}}_t = \widehat{\boldsymbol{y}}_t - \widehat{\boldsymbol{y}}$ for any $t \geq t_0$ and $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$, I have

$$\overline{\boldsymbol{D}^{y,TD}_{t+1}}\left(\widehat{\boldsymbol{y}_{t+1}}-\widehat{\boldsymbol{y}}\right) = \overline{\boldsymbol{D}^{y,TD}_t}\left(\widehat{\boldsymbol{y}}_t-\widehat{\boldsymbol{y}}\right)$$

$$\iff \overline{\boldsymbol{D}^{y,TD}_{t+1}}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{D}^{y,TD}_t}\widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{D}^{y,TD}_{t+1}}-\overline{\boldsymbol{D}^{y,TD}_t}\right)\overline{\boldsymbol{E}}\boldsymbol{\Delta}.$$

Recall the temporary equilibrium condition $\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}}_t - \overline{\boldsymbol{D}^{A,S}}\widehat{\boldsymbol{S}}_t = \overline{\boldsymbol{D}^{A,\Delta}}\widehat{\boldsymbol{\Delta}}$ for any t. Thus

$$\overline{\boldsymbol{F}^y_{t+1}}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y_t}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}^\Delta_{t+1}}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{F}^y_{t+1}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,TD}_{t+1}}} \end{bmatrix}, \quad \overline{\boldsymbol{F}^y_t} \equiv \begin{bmatrix} \mathbf{0} \\ \overline{\boldsymbol{D}^{y,TD}_t} \end{bmatrix}, \quad \overline{\boldsymbol{F}^\Delta_{t+1}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \left(\overline{\boldsymbol{D}^{y,TD}_{t+1}}-\overline{\boldsymbol{D}^{y,TD}_t}\right)\overline{\boldsymbol{E}} \end{bmatrix},$$

or with $\overline{\boldsymbol{F}^y} \equiv \left(\overline{\boldsymbol{F}^y_{t+1}}\right)^{-1}\overline{\boldsymbol{F}^y_t}$ and $\overline{\boldsymbol{F}^\Delta} \equiv \left(\overline{\boldsymbol{F}^y_{t+1}}\right)^{-1}\overline{\boldsymbol{F}^\Delta_{t+1}}$, one can write

$$\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{K.29}$$

It remains to find the initial values of the system (K.29) that converges to the steady state. To this end, I apply a standard method in Stokey and Lucas (1989). In particular, I first homogenize the system: Note that equation (K.29) can be rewritten as $\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t +$

$\left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)\left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$ and thus

$$\widehat{\boldsymbol{z}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{z}}_t \tag{K.30}$$

where

$$\widehat{\boldsymbol{z}}_t \equiv \widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{K.31}$$

The system (K.30) must not explode, or it must be that $\widehat{\boldsymbol{z}}_t \to \boldsymbol{0} \iff \widehat{\boldsymbol{y}}_t \to \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$. I follow Blanchard and Kahn (1980) to find such a condition. Write Jordan decomposition of $\overline{\boldsymbol{F}^y}$ as $\overline{\boldsymbol{F}^y} = \overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{\Lambda}\boldsymbol{B}}$. Then Theorem 6.4 of Stokey and Lucas (1989) implies that it must be that out of n_y vector of $\overline{\boldsymbol{B}}\widehat{\boldsymbol{z}_{t_0}}$, n-th element must be zero if $|\lambda_n| > 1$. Since $\widehat{\boldsymbol{K}^R_{t_0}} = \boldsymbol{0}$, I can write

$$\widehat{\boldsymbol{z}_{t_0}} = \overline{\boldsymbol{F}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{F}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

where

$$\overline{\boldsymbol{F}^\Delta_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\Delta}} \\ \boldsymbol{0}_{2NO\times n_\Delta} \end{bmatrix} - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta} \text{ and } \overline{\boldsymbol{F}^\lambda_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\lambda}} \\ \boldsymbol{0}_{NO\times NO} \\ \overline{\boldsymbol{I}_{NO}} \end{bmatrix}$$

and $\overline{\boldsymbol{D}^{A,\lambda}}$ is the right block matrix of $\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,K}} & \overline{\boldsymbol{D}^{A,\lambda}} \end{bmatrix}$ that corresponds to vector $\widehat{\boldsymbol{\lambda}^R}$. Extracting n-th row from $\overline{\boldsymbol{F}^\Delta_{t_0}}$ and $\overline{\boldsymbol{F}^\lambda_{t_0}}$ where $|\lambda_n| > 1$ and writing them as a $NO \times n_\Delta$ matrix $\overline{\boldsymbol{G}^\Delta_{t_0}}$ and $NO \times NO$ matrix $\overline{\boldsymbol{G}^\lambda_{t_0}}$, the condition of the Theorem is

$$\boldsymbol{0} = \overline{\boldsymbol{G}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{G}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

or $\widehat{\boldsymbol{\lambda}^R_{t_0}} = \overline{\boldsymbol{G}_{t_0}}\boldsymbol{\Delta}$ where $\overline{\boldsymbol{G}_{t_0}} \equiv -\left(\overline{\boldsymbol{G}^\lambda_{t_0}}\right)^{-1}\overline{\boldsymbol{G}^\Delta_{t_0}}$. Finally, tracing back to obtain the initial conditions for $\widehat{\boldsymbol{y}}_t$, it must be $\widehat{\boldsymbol{y}_{t_0}} = \overline{\boldsymbol{F}_{t_0}^{y}}\boldsymbol{\Delta}$,where

$$\overline{\boldsymbol{F}_{t_0}^{y}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^{x}}\right)^{-1}\left(\overline{\boldsymbol{D}^{A,\Delta}} + \overline{\boldsymbol{D}^{A,\lambda}}\,\overline{\boldsymbol{G}_{t_0}}\right) \\ \boldsymbol{0}_{NO\times n_{\Delta}} \\ \overline{\boldsymbol{G}_{t_0}} \end{bmatrix}.$$

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