Robots and Wage Polarization: The Effects of Robot Capital by Occupations*

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Abstract

What are the distributional and aggregate effects of the rising use of industrial robots across occupations? To answer this question, I construct a novel dataset that tracks the shipment of the total number and values of robots by occupations. I develop a general equilibrium model where robots are internationally traded durable goods that may substitute for labor differently across occupations. The elasticities of substitution between robots and labor within an occupation determine the occupation-specific real-wage effects of accumulated robots. I estimate the model using the robot cost shock derived from my dataset and the optimal instrumental variable implied by the model. I find that the elasticities of substitution between robots and labor are heterogeneous across occupations, and higher than those between general capital goods and labor in production and material-moving occupations. These estimates imply that the increased use of robots can explain over 10% of the observed rise in the 90th-50th wage percentile ratio in the US during 1990-2007. A counterfactual tax on the robot purchase increases the US real income in the short run, but leads to a long-run decline.

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1. Introduction

In the last three decades, the global market size of industrial robots has grown by 12% annually.[1] International trade of robots is also sizable, with 41% of all robots imported. Workers in different occupations are differentially susceptible to robots, raising concerns about the distributional effects of such trends. Motivated by this concern, policymakers have proposed various restrictions on automation, such as a robot tax.[2] An emerging literature has estimated the relative effects of robot penetration on employment and the potential impact of such taxes (e.g., Acemoglu and Restrepo, 2020; Humlum, 2019). However, due to the limited data measuring the cost of robots across occupations and the lack of a model capturing the trade of robots and their dynamic accumulation, our understanding of the distributional and aggregate impacts of industrial robots is still limited.

In this paper, I study how industrial robots affect wage inequality between occupations and aggregate income. I assemble a new dataset of the cost of robots by 4-digit occupations and show that robot cost reduction drives wage drop. I develop a model where robots are internationally traded durable goods, are endogenously accumulated, and substitutes for labor within occupations. Based on these data and model, I provide the first estimate of the elasticity of substitution (EoS) between robots and workers heterogeneous across occupation groups. For the group of production (e.g., welders) and material-moving (e.g., hand laborers) occupations, the EoS estimates are around 4. I find that robots can explain a sizable part of the expansion of US wage inequality between occupations from 1990 to 2007. Furthermore, a robot tax could increase US real income in

[1] In this paper, industrial robots (or robots) are defined as multiple-axes manipulators and measured by the number of such manipulators, or robot arms, a standard in the literature. More formally, following International Organization for Standardization (ISO), I define robots as "automatically controlled, reprogrammable, multipurpose manipulator, programmable in three or more axes, which can be either fixed in place or mobile for use in industrial automation applications" (ISO 8373:2012). This definition implies that any automation equipment that does not have multiple axes is out of the scope of the paper, even though some of them are often called "robots" (e.g., autonomous home vacuum cleaners made by iRobot Corporation). Example images of the robots considered in this paper are shown in Section A8.

[2] The European Parliament proposed a robot tax on robot owners in 2015, although it eventually rejected the proposal. (Delvaux et al., 2016) South Korea revised the corporate tax laws that downsize the "Tax Credit for Investment in Facilities for Productivity Enhancement" for enterprises investing in automation equipment. (MOEF, 2018)

the short run because the tax reduces the price of imported robots. However, it leads to the long-run decline because the steady-state robot capital is small.

I construct a dataset unique in two ways. First, the dataset tracks the robots' monetary value as well as their number. Second, the observation is disaggregated by the adopting country and the 4-digit occupation that robots replace. To obtain such a dataset, I first use the information from the Japan Robot Association (JARA) about the shipment of Japanese robots to each country and by tasks. I then combine the JARA data with the O*NET Code Connector's match score and the US Census/ACS data. Finally, I derive the robot cost shock by occupations from the average price variable for each occupation and destination country. As a result, I obtain a dataset that links the US occupational labor market outcomes to the cost shock of robots imported from Japan, which comprise one-third of all robots in the US.

The dataset reveals two stylized facts. First, over 1990-2007, the cost of robots falls heterogeneously across occupations. Second, the fall of robot cost drives the drop in wages and employment by occupations in the US. A relative decrease in one standard deviation of robots' cost is associated with an 8 percent decrease in occupational wages. This finding suggests high responsiveness of relative robot demand to the cost reduction due to the strong substitutability of robots for labor. However, my measure of the robot cost is subject to a concern that it may reflect robot quality upgrading during the sample period. To deal with this issue, I employ an equilibrium model of automation.

I develop a general equilibrium (GE) model with robotics automation and the following three key features. First, I incorporate the trade of robots, which is prevalently observed and fits well with my dataset that measures the international shipment of robots from Japan. Theoretically, trade of robots in the large-open economy setting implies that a robot tax affects the price of robots traded in the global market. Hence, a country may gain from the aggregate perspective if it can reduce the cost of adopting robots by imposing the robot tax. Second, the model describes the endogenous investment of robots with an adjustment cost, which implies sluggish accumulation of robot capital. Therefore, the aggregate income implication of the robot tax is nuanced and different over the time horizon. Finally, the model has EoS between robots and labor, which varies across occupations. The production function with occupation-specific EoS yields rich predictions regarding the real-wage effect of robot capital for the following two reasons. Firstly, the accumulated stock of robots is different across occupations. Secondly, a unit of robots can substitute for workers differentially in each occupation.

To identify the EoS between robots and labor, I confront a challenge that the cost shock is correlated with the quality upgrading of robots, affecting the labor market outcomes simultaneously. To overcome this challenge, I use the GE structure and obtain the structural residual of labor market outcomes, which is free from the effect of quality upgrading. I construct a moment condition in which this structural residual is orthogonal to the robot cost shock. Using this moment condition, I generate an optimal model-implied instrumental variable, which increases the estimation precision.

I apply this estimation method to the US data on occupational labor market outcomes and robot adoption. The estimation result reveals that the EoS between robots and workers is heterogeneous across occupation groups. For routine occupations that perform production and material moving, the estimates are as high as around 4. These estimates are significantly higher than the values of the EoS between general capital and labor estimated in the literature, highlighting one of the main differences between robots and other capital goods. In contrast, the EoS in other occupations is close to 1, or robots and labor are neither substitutes nor complements in these other occupations.

The estimated model and shocks backed out from the model predict occupational US wage changes from 1990-2007. The high EoS between robots and workers in production and material moving occupations implies that the robotization in this period significantly decreased relative wage in these occupations. Since these occupations tend to be in the middle of the occupational wage distribution in 1990, this finding indicates that the automation shock compressed the wage growth of occupations in the middle deciles. Quantitatively, it explains 11.7% of the wage polarization measured by the change in the

90th-50th percentile wage ratio, a measure of wage inequality popularized by Goos and Manning (2007) and Autor et al. (2008).

Finally, I examine the counterfactual effect of introducing a tax on robot purchases. Such a robot tax could potentially increase the aggregate income of a country. Due to the trade of robots, a government can exert monopsony power in the global robot market by taxing robot purchases, leading to a decrease in the before-tax price of imported robots in each period. In contrast, the robot tax also disincentivizes the accumulation of robots in the long run, potentially reducing aggregate income. Quantitatively, the latter effect dominates the former in the long-run, and the robot tax decreases aggregate real income.

This paper contributes to the literature that studies the economic impacts of industrial robots by finding a sizable impact of robots on US wage inequality and a short-run positive aggregate effect of a robot tax. The closest papers are Acemoglu and Restrepo (2020) and Humlum (2019). Acemoglu and Restrepo (2020) establishes that the US commuting zones experiencing penetration of general robots over 1992-2007 also saw decreased wages and total employment.[3] Humlum (2019) uses firm-level adoption measures of robots and firm-worker-level panel data and estimates a model that incorporates a small-open economy of robot importers, a binary decision of robot adoption, and an EoS between occupations.[4] He then studies the distributional effect of robots and a counterfactual robot tax. In contrast to these papers, my study features the following three elements. First, I consider the trade of robots in a large-open economy setting. This setting implies that the US real income effect of robots is positive in the short-run in my counterfactual exercise. Second, I model the endogenous and dynamic robot accumulation, which delays the efficiency loss due to regulating robots. Finally, I use the robot measures for each occupation that robots replace and estimate occupation-specific substitution between robots and labor across occupations. Combined with the high estimates

[3] Dauth et al. (2017) and Graetz and Michaels (2018) also use the industry-level aggregate data of robot adoption and its impact on labor markets.

[4] There is also a growing body of studies that use the firm- and plant-level microdata to study the impact on workers in Canada (Dixon et al., 2019), France (Acemoglu et al., 2020; Bonfiglioli et al., 2020), Netherlands (Bessen et al., 2019), Spain (Koch et al., 2019), and the US (Dinlersoz et al., 2018).

of the EoS between robots and workers, this implies that the distributional effects of the increase in industrial robot use are larger.

Occupations are receiving attention in the literature of automation as they matter when considering the distributional effects. While Jäger et al. (2016) finds no association between industrial robot adoptions and total employment at the firm level, Dinlersoz et al. (2018) report the cost share of workers in the production occupation dropped after the adoption of robots within a firm. Jaimovich et al. (2020) construct a model to study the effect of automation on the labor market of routine and non-routine workers in the steady state. I contribute to this literature by estimating the within-occupation EoS between robots and labor with the occupation-level data of robot measures and labor market outcomes, as well as incorporating the endogenous trade of robots and characterizing the transition dynamics of the effect of robot tax.

Since robots are one type of capital goods, my paper is also related to the vast literature of estimating the EoS between capital and labor (to name a few, Arrow et al., 1961; Chirinko, 2008; Oberfield and Raval, 2014). Although the literature yields a set of estimates with a wide range, the upper limit appears around 1.5 (Karabarbounis and Neiman, 2014; Hubmer, 2018). Therefore, the estimates as high as 4 in production and material-moving occupations among routine occupations are significantly higher than this upper limit. They highlight one of the main differences between robots and other capital goods: these workers' vulnerability to robots.

2. Data and Stylized Facts

I construct a dataset containing the unique robot cost measure with an under-explored variation by occupation. I discuss the main data sources in Section 2.1., define the matching method to build robot measures by occupations in Section 2.2., and show stylized facts that motivates the general equilibrium and structural estimation in Section 2.3. The detailed discussion of the dataset is relegated to Section A1.

2.1. Data Sources

Japan Robot Association (JARA) is a general incorporated association composed of robot producer companies in Japan. As of August 2020, the number of member companies is 381. JARA surveys the member and non-member companies annually, asking the sales units and monetary values of robots for each destination country and application. Among summary cross-tables of the survey, I digitize and use the one about annual robot sales by destination country and by applications.

Japan has been a significant robot innovator, producer, and exporter. Using the JARA data, I observe one-third of world supply of robots disaggregated by applications. For example, US imports 5 billion-dollar worth of Japanese robots. Section A2. shows the international flow of robots, including Japan, the US, and the rest of the world.

It is worthwhile to clarify the concept and measurement of the average price of robots, or the unit values as the previous literature has not focused on them. I measure the price of robots by average unit values throughout the paper. Note that this measure reflects the price of the set of robot joints and arms (hardware or unit), as opposed to the cost of the whole system installation. In particular, a modern industrial robot is typically not standalone hardware but an ecosystem that includes the hardware and control units operated by computers (software). Due to its complexity, installing robots in the production environment often requires hiring costly system integrators that offer specific engineering knowledge. A relevant cost of robots for adopters includes this hardware, software, and integration costs. However, as Leigh and Kraft (2018) pointed out, the current industry and occupation classifications do not allow separating system integrators, making it hard to estimate the cost from these classifications. They estimate that two-thirds of robot costs are the integration costs. In this sense, readers should interpret that my measure of robot price reflects a portion of overall robot costs. I will address this issue in the model section by separately defining the overall robot costs and observed robot unit prices and an adjustment cost.

To match the JARA data with the labor market outcomes at the occupation level, I use Occupational Information Network OnLine (O*NET) Code Connector. O*NET is an online database of occupational definitions sponsored by the US Department of Labor, Employment, and Training Administration. O*NET Code Connector provides an occupational search service that helps workforce professionals determine relevant 4-digit level O*NET-SOC Occupation Codes for job orders. Along with the O*NET-SOC codes, the search algorithm provides (i) the textual description of each code and (ii) a match score that shows the relevance of the search target with the search query term. I use these textual descriptions and match scores for matching robot applications and labor occupations. The match score is further discussed in detail below.

2.2. Robot Applications and Labor Occupations

My dataset provides the employment of labor and robots at the occupation level, complementing data in the previous literature at the sector level or, more recently, firm level. This is made possible by observing robot application-level data, and converting robot applications to labor occupations. In this section, I describe robot applications in detail and propose a method to match the robot application codes in JARA to occupation codes in IPUMS USA. Robot applications and labor occupations are close concepts. On the one hand, robot application is a task where the robot is applied. On the other hand, labor occupation describes multiple types of tasks the person does on the job. Each task has different requirements for robotics automation. Therefore, a heterogeneous mix of tasks in each occupation generates a difference in the ease of automation across occupations and, thus, heterogeneous penetration of robots (Manyika et al., 2017).[5] Yet, there is no

[5]Concrete examples of pairs of an application and an occupation that are close are spot welding and material handling. Spot welding is a task of welding two or more metal sheets into one by applying heat and pressure to a small area called spot. It is one major application of robots. In contrast, O*NET-SOC Code 51-4121.06 has the title "Welders, Cutters, and Welder Fitters" ("Welders" below). Therefore, both spot welding robots and welders perform the welding task. Another example is material handling. Material handling is a short-distance movement of heavy materials. It is another major application of robots. In comparison, ONET-SOC Code 53-7062.00 has the title "Laborers and Freight, Stock, and Material Movers, Hand" ("Material Handler" below). Therefore, both material handling robots and material handlers perform the material handling task.

formal concordance between application and occupation codes.[6]

One might be concerned that the conceptual closeness actually indicates that robots complement workers as opposed to substitute them. To address this concern, I study the tasks performed by robots and labor as precisely as possible. By doing so, robots and workers are more likely substitutes if they do exactly the same task. For this purpose, I use the *match score* from the O*NET Code Connector that contains detailed textual descriptions of 4-digit occupations. The match score is an output of the *weighted search algorithm* used by the O*NET Code Connector, which is the internal search algorithm developed and employed by O*NET and since September 2005. Since then, the O*NET has continually updated the algorithm and improved the quality of the search results. Morris (2019) reports that the updated weighted search algorithm scored 95.9% based on the position and score of a target best 4-digit occupation for a given query.

The concrete process of matching is the following. Let a denote robot application and o labor occupation. JARA data measure robot sales quantity and total monetary transaction values for each application a. I write these as robot measures X_a^R, a generic notation that means both quantity and values. The goal is to convert an application-level robot measure X_a^R to an O*NET-SOC occupation-level one X_o^R. First, I search occupations in O*NET Code Connector by the title of robot application a. Second, I web-scrape the match score m_{oa} between a and o.[7] Finally, I allocate X_a^R to each occupation o according to m_{oa}-weight by

$$X_o^R = \sum_a \omega_{oa} X_a^R \text{ where } \omega_{oa} \equiv \frac{m_{oa}}{\sum_{o'} m_{o'a}}.$$

As a result, X_o^R measures the occupation-level robot measures such as quantity and mon-

[6] Webb (2019) provides a method to match technological advance embodied in each patent text to occupations. He applied the method to study the impact of new technology on occupations, with examples of (general) industrial robots, software, and artificial intelligence. One can interpret my application complements his study by matching between robots at a more detailed application level and occupations.

[7] To obtain the consistent data across periods, I only focus on occupations that exist between the 1970 Census and the 2007 ACS, covering the sample period and the period of pre-trend analysis. By this choice, I may miss an extensive margin of the effect of automation that creates new labor intensive tasks and thus occupations (Acemoglu and Restrepo, 2018). Autor and Salomons (2019) discusses how to measure emerging occupations in data.

etary values. Note $\sum_o \omega_{oa} X_a^R = X_a^R$ since $\sum_o \omega_{oa} = 1$. In other words, occupation-level robot measures sum back to the application level when summed across occupations, as a desired property of the allocation.

I then convert the O*NET-SOC-level occupation codes to OCC2010 occupation codes to match the labor market measures from the US Census, American Community Survey (ACS), retrieved from the Integrated Public Use Microdata Series (IPUMS) USA (Ruggles et al., 2018). The resulting dataset permits me to study the robot cost variation by occupations and the corresponding occupation's labor market outcomes with demographic controls, which I explore in the next subsection.

2.3. Stylized Facts

First, I obtain the robot cost shocks from my dataset's rich variation across dimensions. Write $p_{i,o,t}^R$ the average price measure of robots of occupation o in destination country i in year t. I fit the fixed-effect regression

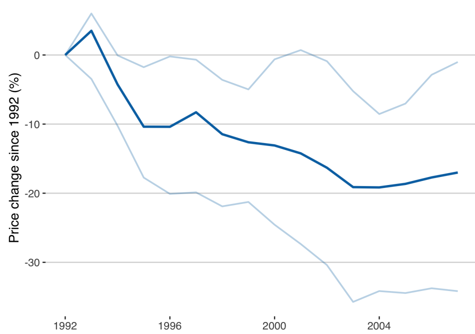
$$\ln\left(p_{i,o,t}^R\right) - \ln\left(p_{i,o,t_0}^R\right) = \psi_{i,t}^D + \psi_{o,t}^C + \epsilon_{i,o,t} \tag{1}$$

where t_0 is the initial year, $\psi_{i,t}^D$ is destination-year fixed effect, $\psi_{o,t}^C$ is occupation-year fixed effect, and $\epsilon_{i,o,t}$ is the residual. This regression controls any country-year specific effect $\psi_{i,t}^D$, which includes country i's demand shock or trade shock between Japan and i that are independent of occupations. I use the remaining variation across occupations $\psi_{o,t}^C$ as a cost shock of robots by occupations and let it explain the US occupational labor market outcomes. Throughout the paper, I define the initial year $t_0 = 1992$, in which the JARA data starts tracking the destination-country level variable, and 1992-2007 as the sample period, with notation $t_1 = 2007$. I show the raw trends of occupational robot stocks and prices in Section A3.

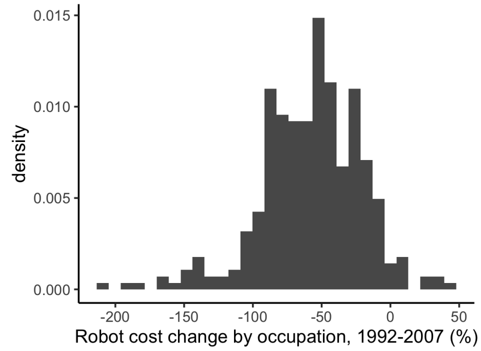
Fact 1: Trends of Robot Costs I show the patterns of average prices of robots across occupations that are not intensively studied in the literature. Figure 1a plots the distri-

Figure 1: Distribution of the Cost of Robots

(a) US Robot Price by Occupations



(b) Histogram of Long-run Cost Shocks



Note: The author's calculation based on JARA and O*NET. The lett panel shows the trend of prices of robots in the US by occupations, $p^R_{USA,o,t}$. The thick and dark line shows the median price in each year, and two thin and light lines are the 10-th and 90-th percentile. Three-year moving averages are taken to smooth out yearly noises. The right panel shows the histogram of long-run (1992-2007) cost shock of robots measured by the fixed effect ψ^C_{o,t_1} in equation (1).

bution (10-th 50-th, and 90-th percentile) of the growth rates of the price of robots in the US relative to the initial year. The figure shows two patterns: (i) the robot prices show an overall decreasing trend, with the median growth rate of -17% from 1992 to 2007, or -1.1% annually, and (ii) a significant heterogeneity in the rate of price falls across occupations, with the 10-th percentile occupation experienced -34% growth (-2.8% per annum), while the 90-th percentile occupation almost did not change the price in the sample period.[8] In contrast, Figure 1b shows the distribution of the long-run cost shock of robots in Japan from 1992 to 2007, or ψ^C_{i,t_1} in equation (1). It reveals that the large variation by occupations persists even after controlling for the destination-year fixed effect $\psi^D_{i,t}$.

There are several interpretations of the price trend, including the reduction in the cost to produce robots and quality changes. First, if the cost of producing robots decreases, the measured prices naturally drop. In the model, I will capture this pattern by positive Hicks-neutral productivity shock to robot producers. Second, if the quality of the robots increased over the period, the quality-adjusted prices may experience a larger decrease than what is observed in the average price measure. They are hard to separate in my data

[8] The price drop is consistent with the decreasing trend of prices of general investment goods since 1980, as Karabarbounis and Neiman (2014) report a 10% decrease per decade from their data sources.

and thus interpreted through the lens of the general equilibrium model in Section 3. by incorporating the quality change and examining its effects on robot prices and quantities. As a result, the differences in the robot cost shock and the quality change may affect the robot adoption and the labor market impacts by occupations.

Fact 2: Correlation between robot measures and wages Figure 2 plots the correlation between the changes in robot measures and the changes in log labor market outcomes in the US at the occupation level, weighted by the size of occupation measured by initial the employment level. The top two panels take the occupational wage as a labor market outcome on the y-axis, while the bottom two take the occupational employment. The left two panels take log monetary value of robot stock in non-US countries as a robot measure on the left panel, and the right two take the log robot cost shock ψ^{C}_{o,t_1}.

First, I offer a piece of evidence that robots have replaced workers at the occupation level. To control the demand factor in the US, Acemoglu and Restrepo (2020) used the robot stock changes in the other countries that show a similar trend of robot stocks as a proxy for the robot technological change and find the negative impact on the US regional labor market. Following this approach, using the changes in robot stocks in non-US countries (all countries except for the US and Japan), I find that the robot penetration measure negatively affects and labor market outcomes of wages and employment by occupation.[9] This result provides direct evidence of the substitution of robots for workers who perform the same task as robots, as well as corroborating the finding of Acemoglu and Restrepo (2020).

The right two panels, I find that the robot cost decline drives the drop of the labor market outcomes (wages and employment) by occupation. Quantitatively, one standard-deviation decrease of the robot cost implies the fall of occupational wage by 8 percent. This finding reveals the first evidence of the substitutability between robots and workers

[9] In Section A4., I show that the robot stock growths are similar between the US and the non-US countries by occupations. In contrast, the occupationl-level trend in the five countries Acemoglu and Restrepo (2020) used as comparison (Denmark, Finland, France, Italy, and Sweden) is less similar to the US trend than the non-US countries.

Figure 2: Correlation between Wages and Robot Measures

Note: The author's calculation based on JARA, O*NET, and the US Census/ACS. The figures show the scatterplot, weighted fit line, and the 95 percent confidence interval of the changes in log robot measures and changes in log labor market outcomes. On the y-axis, the top figures take occupational wages, while the bottom figures take occupational employment. The left panels take the change in log robot stocks (measured in monetary value) on the x-axis, while the right panels take the change in log robot average prices on the x-axis. Each bubble represents a 4-digit occupation. The bubble size reflects the employment in the baseline year (1990, which is the closest Census year to the initial year that I observe the robot adoption, 1992).

because when the price of robots falls in an occupation, the relative demand for robots (resp. labor) increases (resp. decreases) in the same occupation. The findings in Figure 2 are robust to a number of sensitivity analyses: to list a few, including demographic control variables, taking the robot measures in third countries to remove the effect of occupational demand shocks in the US, quality adjustment following Khandelwal et al. (2013), and measuring robot stocks by quantity, as shown in Section A5. This finding motivates me to develop a theory that addresses the quality change of robots and models the substitutability of robots with workers.

3. Model

I develop an open-economy dynamic general equilibrium model characterized by three features, (i) occupation-specific substitution of robots for workers, (ii) robot trade in a large-open economy, and (iii) endogenous investment in robots with an adjustment cost. Section 3.1. states the assumptions, agents' optimization, and the equilibrium definition. After showing the solution method in Section 3.2., I discuss an analytical result that underscores the relevance of occupation-specific substitution in Section 3.3. I relegate discussions of assumptions and derivations to Section B.

3.1. Setup

I formalize the model settings, assumptions, and key characterizations. I relegate discussions and comparisons to the literature in Section B1. Other standard characterizations of equilibrium conditions are given in Section B4.

Environment Time is discrete and has infinite horizon $t = 0, 1, \ldots$. There is N number of countries and O number of occupations. There are two types of goods g, a non-robot good $g = G$ and robot $g = R$. Both goods are tradable. Countries are denoted by subscript i, j, and l, where l means a robot exporter, i means a robot importer and non-robot good exporter, and j means a non-robot good importer. The non-robot good G is differentiated by origin countries and can be consumed by households, used as intermediate goods, invested to produce robots, and used as an input for integration, which I will discuss in detail. Robot R is differentiated by origin countries and occupations. There are bilateral and good-specific iceberg trade costs $\tau^{g}_{ij,t}$ for each $g = G, R$. I use notation Y for the total production, while Q for the quantity arrived at the destination. For instance, non-robot good G shipped from i to j in period t satisfies $Y^{G}_{ij,t} = Q^{G}_{ij,t}\tau^{G}_{ij,t}$. There is no intra-country trade cost, thus $\tau^g_{ii,t} = 1$ for all i, g and t.

There are three factors for production of good G: labor by occupation L_o, robot capital by occupation K^R_o, and non-robot capital K. The stock of non-robot capital is exogenously given at any period for each country. There is no international movement of factors. Note that non-robot capital is not occupational. While producers rent non-robot capital from the rental market, they accumulate and own robot capital. All good and factor markets are perfectly competitive.

The government in each country exogenously sets the robot tax. Buyers of robot $Q^R_{li,o,t}$ have to pay ad-valorem robot tax $u_{li,t}$ on top of producer price $p^R_{li,o,t}$ to buy from l. The tax revenue is uniformly rebated to destination country i's workers.

Workers Workers' problem is characterized by a dynamic discrete choice problem of occupations (Traiberman, 2019; Humlum, 2019). The technique also follows the discrete sector choice problem in Dix-Carneiro (2014) and Caliendo et al. (2019) in that workers choose the occupations that maximize the lifetime utility based on the draw of the idiosyncratic shock, switching cost, and welfare gains from moving, which has a closed form solution when the idiosyncratic shock follows a suitable extreme value distribution (McFadden, 1973).[10] In Section B2., I formally define the problem and show that the worker's problem can be characterized by, for each country i and period t, the transition probability $\mu_{i,oo',t}$ from occupation o in period t to occupation o' in period $t+1$, and the exponential expected value $V_{i,o,t}$ for occupation o that satisfy

$$\mu_{i,oo',t} = \frac{\left(\left(1-\chi_{i,oo',t}\right)\left(V_{i,o',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}}{\sum_{o''}\left(\left(1-\chi_{i,oo'',t}\right)\left(V_{i,o'',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}}, \tag{2}$$

$$V_{i,o,t} = \widetilde{\Gamma} C_{i,o,t}\left[\sum_{o'}\left(\left(1-\chi_{i,oo',t}\right)\left(V_{i,o',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}\right]^{\frac{1}{\phi}}, \tag{3}$$

[10]One of the differences from these past studies is that I characterize the moving cost by an ad-valorem term, which helps to perform log-linearization when solving the model.

respectively, where $C_{i,o,t+1}$ is the real consumption, $\chi_{i,oo',t}$ is an ad-valorem moving cost from occupation o to o', ϕ is the occupation-switch elasticity, $\widetilde{\Gamma} \equiv \Gamma\left(1 - 1/\phi\right)$ is a constant that depends on the Gamma function. For each i and t, due to these solutions, employment level satisfies the law of motion

$$L_{i,o,t+1} = \sum_{o'} \mu_{i,o'o,t} L_{i,o',t}, \tag{4}$$

with the total employment satisfying an adding-up constraint

$$\sum_{o} L_{i,o,t} = \overline{L}_{i,t}. \tag{5}$$

Production Function I describe a production function in country i in period t. For each good g, there is a given mass of producers. Non-robot good-G producers produce by aggregating occupation service $Q^{O}_{i,o,t}$, intermediate goods $M_{i,t}$, and non-robot capital $K_{i,t}$ by

$$Y^{G}_{i,t} = A^{G}_{i,t} \left[\sum_{o} \left(b_{i,o,t}\right)^{\frac{1}{\beta}} \left(Q^{O}_{i,o,t}\right)^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}\alpha_L} \left(M_{i,t}\right)^{\alpha_M} \left(K_{i,t}\right)^{1-\alpha_L-\alpha_M}, \tag{6}$$

where $Y^{G}_{i,t}$ is the production quantity, $A^{G}_{i,t}$ is a Hicks-neutral total factor productivity (TFP) shock, $b_{i,o,t}$ is the share parameter of occupation o, β is the elasticity of substitution between occupations, and $\alpha_{i,L}$, $\alpha_{i,M}$, and $1 - \alpha_{i,L} - \alpha_{i,M}$ are Cobb-Douglas weights on occupations, intermediate goods, and non-robot capital, respectively. Parameters satisfy $b_{i,o,t} > 0$ for all i, o, and t, $\sum_o b_{i,o,t} = 1$, $\beta > 0$, and $\alpha_{i,L}, \alpha_{i,M}, 1 - \alpha_L - \alpha_M > 0$. For simplification, I assume that robots R for occupation o are produced by investing non-robot goods $Q^V_{i,o,t}$ with productivity $A^R_{i,o,t}$:[11]

$$Y^R_{i,o,t} = A^R_{i,o,t} Q^V_{i,o,t}. \tag{7}$$

Note that the increase in the TFP term $A^R_{i,o,t}$ may drive a reduction in the robot prices. To perform each occupation o, producers hire labor $L_{i,o,t}$ and robot capital $K^R_{i,o,t}$

$$Q^O_{i,o,t} = \left[(1 - a_{o,t})^{\frac{1}{\theta_o}} (L_{i,o,t})^{\frac{\theta_o - 1}{\theta_o}} + (a_{o,t})^{\frac{1}{\theta_o}} \left(K^R_{i,o,t} \right)^{\frac{\theta_o - 1}{\theta_o}} \right]^{\frac{\theta_o}{\theta_o - 1}}, \tag{8}$$

where $\theta_o > 0$ is the elasticity of substitution between robots and labor within occupation o, and $a_{o,t}$ is the share of robot capital in the cost of occupation o. In the following sections, I use the shift of $a_{o,t}$ as a source of automation. I will discuss real-world examples and the relationship to the models in the literature in Section B1. The intermediate goods are aggregated by

$$M_{i,t} = \left[\sum_l (M_{li,t})^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon - 1}}, \tag{9}$$

where ε the elasticity of substitution. Since intermediate goods are traded across countries and aggregated by equation (9), elasticity parameter ε serves as the trade elasticity. Given the iceberg trade cost $\tau^G_{ij,t}$, the bilateral price of good G that country j pays to i is $p^G_{ij,t} = p^G_{i,t} \tau^G_{ij,t}$.

Discussion–Production Function and Automation It is worth mentioning the relationship between production functions (6) and (8) and the way automation is treated in the literature. A common approach to modeling robots in the literature, called the task-based approach, implies occupation production function (8). A large body of literature develops

[11] The assumption greatly simplifies the solution of the model because I can simply assign all the demand for occupations, intermediate goods and non-robot capital to non-robot good producers as opposed to robot producers. Furthermore, I can simply use the estimates measured at the unit of output dollar values when taking the budget constraint of the model to the data in log-linearized solution. To conduct the estimation and counterfactual exercises without the simplification, one would need to observe the cost shares of intermediate goods and non-robot capital for robot producers. This work is left for future research.

the task-based approach to model industrial robots (e.g., Acemoglu and Restrepo, 2018) and more general automation (e.g., Autor et al., 2003; Acemoglu and Autor, 2011). In particular, the task-based approach constructs the production function (task-based production function) based on the producers' allocation problem of production factors (e.g., robot capital, labor) to a set of production tasks (e.g., spot welding). In Lemma B.1 in Section B, I show that the solution to the allocation problem implies the resulting production function under the class of functions (6) and (8). In other words, production functions (6) and (8) are micro-founded by the task-based approach of industrial robots.

The cost-share parameter $a_{o,t}$ of equation (8) has several interpretations worth mentioning. First, recall that the task-based approach implies the allocation of factors to tasks. Therefore, given the above micro-foundation, the cost-share parameter $a_{o,t}$ is the share of the space of tasks performed by robot capital. Given that the change in the task space performed by robots is a major source of robotics automation, I will perform the log-linearization analysis with respect to the change in $a_{o,t}$ and call the change as the expansion of the robot task space. Second, note that quality of good is a non-pecuniary "attribute whose valuation is agreed upon by all consumers" (Khandelwal, 2010). Therefore the increase in the cost-share parameter $a_{o,t}$ can also be interpreted as a form of quality upgrading of robots, when combined with a suitable adjustment in the TFP term. In particular, equation (8) implies that in the long-run (hence dropping the time subscript) the demand for capital is

$$K_{i,o}^{R} = a_o \left(\frac{c_{i,o}^{R}}{P_{i,o}} \right)^{-\theta_o} Q_{i,o}^{O},$$

where $c_{i,o}^{R}$ is the long-run marginal cost of robot capital formally defined in Section B4., $P_{i,o}$ is the unit cost of performing occupation o. In this equation, a_o serves as the quality term defined above. For this reason, I use terms (positive) automation shocks and robot quality upgrading interchangeably to describe an exogenous increase in a_o.

The robot-labor substitution parameter θ_o is the key elasticity that affects the changes in real wages given the automation shocks. In Section 3.3., I show that θ_o is negatively related to the real wage changes conditional on the initial cost shares. Hence it is critical to know the value of the parameter to answer the welfare and policy questions. To the best of my knowledge, equation (8) is the most flexible formulation of substitution between robots and labor in the literature. For instance, I show that the unit cost function of Acemoglu and Restrepo (2020) can be obtained by $\theta_o \to 0$ for any o with suitable other parameter configurations in Lemma B.1 in Section B. I also show that my model can imply the production structure of Humlum (2019) in Lemma B.2.

Producers' Problem Producers' problem comprises two-tiers decisions–static optimization and dynamic optimization. The static optimization is to choose the employment and capital rental amount conditional on market prices and current stock of robot capital. Namely, for each i and t, conditional on o-vector of stock of robot capital $\left\{K_{i,o,t}^R\right\}_o$,

$$\pi_{i,t}\left(\left\{K_{i,o,t}^R\right\}_o\right) \equiv \max_{\left\{L_{i,o,t}\right\}_o, \left\{M_{li,t}\right\}_l, K_{i,t}} p_{i,t}^G Y_{i,t}^G - \sum_o w_{i,o,t} L_{i,o,t} - \sum_l p_{li,t}^G M_{li,t} - r_{i,t} K_{i,t}, \tag{10}$$

subject to production function (6).

The dynamic optimization is to choose the amount of purchase of new robots, or robot investment, given the current stock of robot capital. It requires the following three settings. First, for each i, o, and t, robot capital $K_{i,o,t}^R$ accumulates according to

$$K_{i,o,t+1}^R = (1-\delta) K_{i,o,t}^R + Q_{i,o,t}^R, \tag{11}$$

where $Q_{i,o,t}^R$ is the amount of new robot investment and δ is the depreciation rate of robots. Second, I assume that the new investment is given by CES aggregation of robot arms from country l, $Q^R_{li,o,t}$, and local non-robot-good input $Q^I_{i,o,t}$, called integration input,[12]

$$Q^R_{i,o,t} = \left[\sum_l \left(Q^R_{li,o,t}\right)^{\frac{\varepsilon^R-1}{\varepsilon^R}}\right]^{\frac{\varepsilon^R}{\varepsilon^R-1}\alpha^R} \left(Q^I_{i,o,t}\right)^{1-\alpha^R} \tag{12}$$

where l denotes the origin of the newly purchased robots, and α^R is the expenditure share of robot arms in the cost of investment. Given the iceberg trade cost $\tau^R_{ij,t}$, the bilateral price of robot R is $p^R_{ij,o,t} = p^R_{i,o,t}\tau^R_{ij,t}$. Write the unit investment price of robots as $P^R_{i,o,t}$. Third, installing robots is costly and requires a per-unit convex adjustment cost $\gamma Q^R_{i,o,t}/K^R_{i,o,t}$ in the robot unit, where γ governs the size of adjustment cost (Cooper and Haltiwanger, 2006).

Given these settings, a producer of non-robot good G in country i solves dynamic optimization problem is

$$\max_{\left\{Q^R_{li,o,t}\right\},Q^I_{i,o,t}} \sum_{t=0}^{\infty} \left(\frac{1}{1+\iota}\right)^t \left[\pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}_o\right) - \sum_{l,o}\left(p^R_{li,o,t}\left(1+u_{li,t}\right)Q^R_{li,o,t} + \gamma P^R_{i,o,t}Q^R_{i,o,t}\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\right], \tag{13}$$

subject to accumulation equation (11) and (12), and given $\left\{K^R_{i,o,0}\right\}_o$. Because producers are owned by households, the producer uses the household discount rate ι.

The first-order condition (FOC) of the Lagrangian function of problem (13) with respect to $Q^R_{li,o,t}$ implies the investment condition

$$p^R_{li,o,t}\left(1+u_{li,t}\right) + 2\gamma\alpha^R\left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}} = \lambda^R_{i,o,t}\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}}, \tag{14}$$

where $\lambda^R_{li,o,t}$ is the marginal value of robot capital. In this equation, the left-hand side (LHS) represents the marginal cost of robot adoption (the cost of robot investment and

[12]Note that equation (12) implies that the robots are traded because they are differentiated by origin country l. This follows the formulation of capital good trade in Anderson et al. (2019). Furthermore, combined with equation (11), equation (12) implies that the origin-differentiated investment good is aggregated at first, and then added to the stock of capital. This specification helps reduce the number of capital stock variables and is also used in Engel and Wang (2011).

adjustment cost), whereas the right-hand side (RHS) represents the marginal values of buying a robot from country l, $Q^R_{li,o,t}$. The FOC with respect to $K^R_{i,o,t}$ implies Euler equation

$$\lambda^R_{i,o,t} = \frac{1}{1+\iota}\left[(1-\delta)\,\lambda^R_{i,o,t+1} + \frac{\partial}{\partial K^R_{i,o,t+1}}\pi_{i,t+1}\left(\left\{K^R_{i,o,t+1}\right\}\right) + \gamma p^R_{i,o,t+1}\left(\frac{Q^R_{i,o,t+1}}{K^R_{i,o,t+1}}\right)^2\right]. \tag{15}$$

Equilibrium To close the model, the market clearing conditions equate demand and supply of non-robot goods, robot flows, and employment. The employment level must satisfy an adding-up constraint (5). I formalize the other conditions in Section B4. To define the equilibrium, I first define a temporary equilibrium in each period and then a sequential equilibrium. I finally discuss the steady state. Define the bold symbols as vectors of robot capital $\boldsymbol{K}^R_t \equiv \left\{K^R_{i,o,t}\right\}_{i,o}$, marginal values of robot capital $\boldsymbol{\lambda}^R_t \equiv \left\{\lambda^R_{i,o,t}\right\}_{i,o}$, employment $\boldsymbol{L}_t \equiv \{L_{i,o,t}\}_{i,o}$, workers' value functions $\boldsymbol{V}_t \equiv \{V_{i,o,t}\}_{i,o}$, non-robot good prices $\boldsymbol{p}^G_t \equiv \left\{p^G_{i,t}\right\}_i$ robot prices $\boldsymbol{p}^R_t \equiv \left\{p^R_{i,o,t}\right\}_{i,o}$, wages, $\boldsymbol{w}_t \equiv \{w_{i,o,t}\}_{i,o}$, bilateral non-robot good trade levels $\boldsymbol{Q}^G_t \equiv \left\{Q^G_{ij,t}\right\}_{i,j}$, bilateral non-robot good trade levels $\boldsymbol{Q}^R_t \equiv \left\{Q^R_{ij,o,t}\right\}_{i,j,o}$, and occupation transition shares $\boldsymbol{\mu}_t \equiv \{\mu_{i,oo',t}\}_{i,oo'}$. I write $\boldsymbol{S}_t \equiv \left\{\boldsymbol{K}^R_t, \boldsymbol{\lambda}^R_t, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$ as state variables.

Definition 1. In each period t, given state variables $\boldsymbol{S}_t$, a *temporary equilibrium* (TE) is prices $\boldsymbol{p}_t \equiv \left\{\boldsymbol{p}^G_t, \boldsymbol{p}^R_t, \boldsymbol{w}_t\right\}$ and flow quantities $\boldsymbol{Q}_t \equiv \left\{\boldsymbol{Q}^G_t, \boldsymbol{Q}^R_t, \boldsymbol{\mu}_t\right\}$ that satisfy:

1. Given $\boldsymbol{p}_t$, workers choose occupation optimally by equation (2).

2. Given $\boldsymbol{p}_t$, producers maximize flow profit (10) and optimize investment (14).

3. Factor markets clear and trade balances.

The temporary equilibrium inputs all state variables and outputs other endogenous variables that are determined contemporaneously. The following sequential equilibrium gives the determination of all state variables given initial conditions.

Definition 2. Given initial robot capital stocks and employment $\left\{\boldsymbol{K}_0^R, \boldsymbol{L}_0\right\}$, a *sequential equilibrium* (SE) is a sequence of vectors $\boldsymbol{y}_t \equiv \left\{\boldsymbol{x}_t, \boldsymbol{S}_t\right\}_t$ that satisfies the TE conditions and capital accumulation (11), the Euler equation (15), employment law of motion (4), value function (3), and the transversality condition: for any j and o,

$$\lim_{t\to\infty} e^{-\iota t} \lambda_{j,o,t}^R K_{j,o,t+1}^R = 0. \tag{16}$$

Finally, I define the steady state as a SE $\boldsymbol{y}$ that does not change over time.

3.2. Solution

I log-linearize around the initial equilibrium in order to solve the model. In particular, I study the effect of shocks on the sequential equilibrium $\boldsymbol{y}_t$. The output of this subsection is a sequence of matrices $\left\{\overline{\boldsymbol{F}_t}\right\}_t$ and a matrix $\overline{\boldsymbol{E}}$ that summarizes the first-order effect on sequential equilibrium in transition dynamics, which is a key object in estimating the model in Section 4. Section D gives the details of the derivation.

In the economy described in Section 3.1., the shocks comprise changes in the economic environment and changes in policy. For instance, consider the increase of robot task space $a_{o,t}$ in baseline period t_0 by Δ_o percent, or

$$a_{o,t} = \begin{cases} a_{o,t_0} & \text{if } t < t_0 \\ a_{o,t_0} \times (1+\Delta_o) & \text{if } t \geq t_0 \end{cases}.$$

In this formulation, Δ_o is interpreted as the size of the expansion of the robot task space. I combine all these changes into a column vector $\boldsymbol{\Delta}$. I take the following three steps to solve the model. Write state variables $\boldsymbol{S}_t = \left\{\boldsymbol{K}_t^R, \boldsymbol{\lambda}_t^R, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$, and use "hat" notation to denote changes from t_0: for any variable z_t, $\widehat{z_t} \equiv \ln(z_t) - \ln(z_{t_0})$.

Step 1. For a given period t, I combine the vector of shocks $\boldsymbol{\Delta}$ and (given) changes in state variables $\widehat{\boldsymbol{S}_t}$ into a (column) vector $\widehat{\boldsymbol{A}_t} = \left\{\boldsymbol{\Delta}, \widehat{\boldsymbol{S}_t}\right\}$. Log-linearizing TE conditions, I solve for matrices $\overline{\boldsymbol{D}^x}$ and $\overline{\boldsymbol{D}^A}$ such that

$$\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}_t} = \overline{\boldsymbol{D}^A}\widehat{\boldsymbol{A}_t}. \tag{17}$$

In this equation, $\overline{\boldsymbol{D}^x}$ is a substitution matrix and $\overline{\boldsymbol{D}^A}\widehat{\boldsymbol{A}_t}$ is a vector of partial equilibrium shifts in period t (Adao et al., 2019).[13] Note that there exists a block separation of matrix $\overline{\boldsymbol{D}^A} = \left[\overline{\boldsymbol{D}^{A,\Delta}}|\overline{\boldsymbol{D}^{A,S}}\right]$ such that equation (17) can be written as

$$\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}_t} - \overline{\boldsymbol{D}^{A,S}}\widehat{\boldsymbol{S}_t} = \overline{\boldsymbol{D}^{A,\Delta}}\boldsymbol{\Delta}. \tag{18}$$

Step 2. Log linearizing laws of motion and Euler equations around the old steady state, I solve for matrices $\overline{\boldsymbol{D}^{y,SS}}$ and $\overline{\boldsymbol{D}^{\Delta,SS}}$ such that $\overline{\boldsymbol{D}^{y,SS}}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{D}^{\Delta,SS}}\boldsymbol{\Delta}$, where superscript SS denotes steady state. Combined with steady state version of equation (18), I have

$$\overline{\boldsymbol{E}^y}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}^\Delta}\boldsymbol{\Delta}, \tag{19}$$

where

$$\overline{\boldsymbol{E}^y} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,SS}}} \end{bmatrix}, \text{ and } \overline{\boldsymbol{E}^\Delta} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \overline{\boldsymbol{D}^{\Delta,SS}} \end{bmatrix},$$

which implies the first-order steady state matrix $\overline{\boldsymbol{E}}$ that satisfies $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$.

Step 3. Log linearizing laws of motion and Euler equations around the new steady state, I solve for matrices $\overline{\boldsymbol{D}^{y,TD}_{t+1}}$ and $\overline{\boldsymbol{D}^{y,TD}_{t}}$ such that $\overline{\boldsymbol{D}^{y,TD}_{t+1}}\check{\boldsymbol{y}}_{t+1} = \overline{\boldsymbol{D}^{y,TD}_{t}}\check{\boldsymbol{y}}_t$, where the superscript TD stands for transition dynamics. Log-linearized sequential equilibrium satisfies the following first-order difference equation

$$\overline{\boldsymbol{F}^y_{t+1}}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y_t}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}^\Delta_{t+1}}\boldsymbol{\Delta}.$$

[13] Since the temporary equilibrium vector $\widehat{x}_t$ includes wages $\widehat{w_t}$, equation (17) nests the general equilibrium comparative statics formulation in Adao et al. (2019).

Combined with the transversality condition, there is a matrix representing the first-order transitional dynamics $\overline{\boldsymbol{F}_t}$ such that

$$\widehat{\boldsymbol{y}_t} = \overline{\boldsymbol{F}_t}\boldsymbol{\Delta}. \tag{20}$$

3.3. Real-wage Effect of Automation

How does the flexible occupation production function (8) imply the effect of automation? In this section, I show that the effect of automation on occupational real wages depends negatively on substitution elasticity parameters θ_o and β conditional on the changes in input and trade shares. The key insight is that the real wages are relative prices of labor to the bundle of factors, and the relative price changes are relevant input and trade shares inversely related to the elasticities of substitution. These elasticities are among the target parameters of the estimation in Section 4.

I modify notations in equation (8) to express the result in a concise way. Define

$$A^K_{i,o,t} \equiv A^G_{i,t} a_{o,t},\ A^L_{i,o,t} \equiv A^G_{i,t}\left(1 - a_{o,t}\right). \tag{21}$$

Substituting these into production functions (6) and (8), I have

$$Q^G_{i,t} = \left[\sum_o \left(b_{i,o,t}\right)^{\frac{1}{\beta}} \left(Q^O_{i,o,t}\right)^{\frac{\beta-1}{\beta}}\right]^{\frac{\beta}{\beta-1}\alpha_{i,L}} \left(M_{i,t}\right)^{\alpha_{i,M}} \left(K_{i,t}\right)^{1-\alpha_{i,L}-\alpha_{i,M}},$$

and

$$Q^O_{i,o,t} = \left[\left(A^L_{i,o,t}\right)^{\frac{1}{\theta_o}} \left(L_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}} + \left(A^K_{i,o,t}\right)^{\frac{1}{\theta_o}} \left(K^R_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}}\right]^{\frac{\theta_o}{\theta_o-1}}.$$

Therefore, one can interpret the newly defined terms $A^K_{i,o,t}$ and $A^L_{i,o,t}$ as the productivity shock on the robot and labor, respectively. The following proposition claims that the long-run real-wage implication of robot productivity change $\widehat{A^K_{i,o}}$ can be expressed by changes in input and trade shares and elasticities of substitutions.

Define the good G-producers' labor share within occupation $\widetilde{x}^L_{i,o,t}$, occupation cost share $\widetilde{x}_{i,o}$, and trade shares $\widetilde{x}_{ij,t}$ as

$$\widetilde{x}^L_{i,o,t} \equiv \frac{w_{i,o,t} L^G_{i,o,t}}{p^G_{i,o,t} Q^G_{i,o,t}}, \ \widetilde{x}_{i,o,t} \equiv \frac{p^G_{i,o,t} Q^G_{i,o,t}}{p^G_{i,t} Q^G_{i,t}}, \ \widetilde{x}_{ij,t} \equiv \frac{p^G_{i,t} Q^G_{ij,t}}{P^G_{i,t} Q^G_{i,t}}. \tag{22}$$

Given these, the following proposition characterizes the real-wage changes in the steady state.

Proposition 1. *Suppose robot productivity grows $\widehat{A^K_{i,o}} > 0$. For each country i and occupation o,*

$$\widehat{\left(\frac{w_{i,o}}{P^G_i}\right)} = \frac{1}{1-\alpha_{i,M}} \left(\frac{\widehat{\widetilde{x}^L_{i,o}}}{1-\theta_o} + \frac{\widehat{\widetilde{x}_{i,o}}}{1-\beta} + \frac{\widehat{\widetilde{x}_{ii}}}{1-\varepsilon} \right). \tag{23}$$

Proof. See Section B5. □

Proposition 1 clarifies how the elasticity parameters and change of shares of input and trade affect real wages at occupation level. Among them, one can observe that if $\theta_o > 1$, then (i) the larger the fall of the labor share within occupation $\widehat{\widetilde{x}^L_{i,o}}$, the larger the real wage gains, and (ii) pattern (i) is stronger if θ_o is small and close to 1. Therefore, conditional on other terms, the steady state changes of occupational real wages depend on the elasticity of substitution between robots and labor θ_o.

The intuition of Proposition 1 is the series of revealed cost reductions. The first term reveals the robot cost reduction relative to labor cost.[14] If $\theta_o > 1$, then the reduction of the price index or cost savings dominates the drop of nominal wage, increasing the real wage. Similar intuition holds for the second and third terms. The second term reveals the relative occupation cost reduction, whereas the last term reveals the relative sectoral cost reduction.

Proposition 1 also gives an extension to the welfare sufficient statistic in the trade literature. In particular, Arkolakis et al. (2012, ACR) showed that under a large class of

[14]Section B4. contains two discussions that facilitate the understandings of steady-state labor share $\chi^L_{i,o}$. First, I define the the steady state cost of robot capital $c^R_{i,o}$ by the steady state marginal cost of robot adoption in country i, good g, and occupation o. Second, as a special case, I show that if there is no adjustment cost $\gamma = 0$, $c^R_{i,o} = (\iota + \delta)\,\lambda^R_{i,o}$ and reduces to the steady-state user cost formula of Hall and Jorgenson (1967).

trade models, the welfare effect of the reduction in trade costs can be summarized into the well-known ACR formula, or log-difference of the trade shares times the negative of trade elasticity. In fact, by dropping the robots and non-robot capital and aggregating occupations into one, the model reduces to:

$$\widehat{\left(\frac{w_i}{P_i^G}\right)} = \frac{1}{1-\alpha_{i,M}}\frac{1}{1-\varepsilon}\widehat{\widetilde{x}_{ii}},$$

which is a modified ACR formula with intermediate goods as in Caliendo and Parro (2015) and Ossa (2015).

4. Estimation

Using the occupation-level robot cost shocks described in Section 2. and the solution to the general equilibrium model in Section 3., I show an estimation method based on the generalized method of moments (GMM), in particular, the model-implied optimal instrumental variable (MOIV, Adao et al., 2019). To do so, Section 4.1. sets the stage for the structural estimation by giving the implementation detail. I formalize the MOIV estimator in Section 4.2., which gives the structural estimates in Section **??**

4.1. Target Parameters

To simplify the notation and tailor to my empirical application, I stick to country labels $i = 1$ as the US (USA), $i = 2$ as Japan (JPN), $i = 3$ as the Rest of the World (ROW). Following my data, I interpret country $i = 1$ as the country of interest in terms of labor market outcome variables, country $i = 2$ as the source country of automation shocks by robots, and country $i = 3$ as the (set of) countries in which the measurement of robots proxies the technological changes in country 2.

In the estimation, I allow heterogeneity across occupations of the within-occupation EoS between robots and labor. To do so, I define the occupation groups as follows. I first separate occupations into three broad occupation groups, Abstract, Manual, Routine following Acemoglu and Autor (2011). Routines occupations include production, transportation and material moving, sales, clerical, and administrative support.[15] Given the trend that production and transportation/material moving occupations introduced robots over the sample period, I further divide routine occupations into three sub-categories, Production (e.g., welders), Transportation (indicating transportation and material-moving, e.g., hand laborer), and Others (e.g., repairer), where Others include sales, clerical, and administrative support. As a result, I obtain five occupation groups, for each of which I assume a constant EoS between robots and labor. With each occupation group (or mapping from 4-digit occupation o to the group) represented by g, notation θ_g denotes the robot-labor EoS for occupation group.

The vector of structural parameters are denoted as $\mathbf{\Theta}$ and its dimension is $d \equiv \dim(\mathbf{\Theta})$. To formally define $\mathbf{\Theta}$, I fix a subset of parameters of the model at conventional values. In particular, I assume that the annual discount rate is $\iota = 0.05$ and the robot depreciation rate is 10 percent following Graetz and Michaels (2018).[16] I take trade elasticity of $\varepsilon = 4$ from the large literature of trade elasticity estimation (e.g., Simonovska and Waugh, 2014). With this parametrization, structural parameters to be estimated are $\mathbf{\Theta} \equiv \{\theta_g, \beta, \gamma, \phi\}$.

4.2. Estimation Method

I observe changes in endogenous variables, US occupational wages $\widehat{\boldsymbol{w}_1}$, US employment $\widehat{\boldsymbol{L}_1}$, robot shipment from Japan to the US $\widehat{\boldsymbol{Q}_{21}^R}$, and the corresponding unit values $\widehat{\boldsymbol{p}_{21}^R}$ between 1992 and 2007, as well as the initial equilibrium $\boldsymbol{y}_{t_0}$. I approximate the 15-year changes as the steady-state changes. To simplify, I focus on the expansion of robot task space $\widehat{a_o}$ and robot cost shocks in Japan $\widehat{A_{2,o}^R}$ as the source of the occupational shocks in

[15] Abstract occupations are professional, managerial and technical occupations; manual occupations are protective service, food preparation, cleaning, personal care and personal services.

[16] For example, see King and Rebelo (1999) for the source of the conventional value of ι who matches the discount rate to the average real return on capital. For ε, see Simonovska and Waugh (2014) or Caliendo and Parro (2015).

this section. To back up the robot cost shock, I run the following fixed effect regression

$$\widehat{p_{2j,o,t}^{R}} = \psi_{j,t}^{D} + \psi_{o,t}^{C} + \tilde{e}_{j,o,t}.$$

By the robot production function (7), $\psi_{j,t}^{D}$ captures the bilateral shocks between Japan and destination country j that may incorporate the destination country's robot demand shocks, while $\psi_{o,t}^{C}$ captures the cost shock to produce robots o in Japan. Since $\widehat{A_{2,o}^{R}}$ is the TFP shock to produce robots o that is negatively correlated with the cost shock, I measure it by

$$\widehat{A_{2,o}^{R}} = -\left(\psi_{o,t_1}^{C} - \psi_{o,t_0}^{C}\right). \tag{24}$$

To discuss the identification challenge and the countermeasure, I decompose the automation shock $\widehat{a_o}$ into observed component $\widehat{a_o^{\text{obs}}}$ and unobserved error component $\widehat{a_o^{\text{err}}}$ such that $\widehat{a_o} = \widehat{a_o^{\text{obs}}} + \widehat{a_o^{\text{err}}}$ for all o. The component $\widehat{a_o^{\text{obs}}}$ is observed conditional on parameter θ_o–namely, it satisfies the steady-state change of relative demand of robots and labor implied by the Euler equation

$$\left(\widehat{\frac{p_{i,o}K_{i,o}}{w_{i,o}L_{i,o}}}\right) = (1-\theta_o)\left(\widehat{\frac{p_{i,o}}{w_{i,o}}}\right) + \frac{\widehat{a_o^{\text{obs}}}}{1-a_{o,t_0}}. \tag{25}$$

Equation (25) highlights the issues in identifying θ. First, the observed relative price change $(\widehat{p_{i,o}/w_{i,o}})$ does not identify θ_g because $(\widehat{p_{i,o}/w_{i,o}})$ is endogenous and is correlated with the residual term $\widehat{a_o^{\text{obs}}}/\left(1-a_{o,t_0}\right)$ that represents the task-space expansion of robots.[17] Second, the cost shock to produce robots $\widehat{A_{2,o}^{R}}$ also does not also work as an instrumental variable (IV) in the linear regression model of (25) because of a potential correlation between the cost shock and the observed task-space expansion shock $\widehat{a_o^{\text{obs}}}$. For example, innovation in robotics technology could entail both the expansion of the tasks that robots can perform ("product innovation"), and the cost-saving to produce robots that perform the same task as before ("process innovation").

[17]See Karabarbounis and Neiman (2014); Hubmer (2018) for this approach for a broader definition of capital

To overcome these identification issues, I employ a method based on the full GE model below. Conditional on $\widehat{a_o^{\text{obs}}}$, the error component $\widehat{a_o^{\text{err}}}$ can be inferred from each observed endogenous variable. Take the changes in occupational wages $\widehat{\boldsymbol{w}_1}$ for example. The steady-state solution matrix $\overline{\boldsymbol{E}}$ implies that there is a $O \times O$ sub-matrices $\overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}}$ and $\overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R}$ such that

$$\widehat{\boldsymbol{w}} = \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}} \widehat{\boldsymbol{a}} + \overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R} \widehat{\boldsymbol{A}_2^R}. \tag{26}$$

Since $\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{a}^{\text{obs}}} + \widehat{\boldsymbol{a}^{\text{err}}}$, I have

$$\boldsymbol{\nu}_{\boldsymbol{w}} = \widehat{\boldsymbol{w}} - \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}} \widehat{\boldsymbol{a}^{\text{obs}}} - \overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R} \widehat{\boldsymbol{A}_2^R},$$

where $\boldsymbol{\nu}_{\boldsymbol{w}} \equiv \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}} \widehat{\boldsymbol{a}^{\text{err}}}$ is the O-vector structural residual generated from the linear combination of the unobserved component of the automation shocks. Note that the structural residual depends on the structural parameters $\boldsymbol{\Theta}$. To clarify this, I occasionally write the structural residual as $\boldsymbol{\nu}_{\boldsymbol{w}} = \boldsymbol{\nu}_{\boldsymbol{w}}(\boldsymbol{\Theta})$. For other endogenous variables $\left(\widehat{\boldsymbol{L}_1}, \widehat{\boldsymbol{p}_{21}^R}, \widehat{\boldsymbol{Q}_{21}^R}\right)$, I repeat the same process and obtain corresponding structural errors $\left(\boldsymbol{\nu}_{\boldsymbol{L}}, \boldsymbol{\nu}_{\boldsymbol{p}^R}, \boldsymbol{\nu}_{\boldsymbol{Q}^R}\right)$. Then I stack these vectors into an $O \times 4$ matrix $\boldsymbol{\nu} \equiv \left[\boldsymbol{\nu}_{\boldsymbol{w}}, \boldsymbol{\nu}_{\boldsymbol{L}}, \boldsymbol{\nu}_{\boldsymbol{p}^R}, \boldsymbol{\nu}_{\boldsymbol{Q}^R}\right]$, and from its o-th row define 4×1 vector as $v_o = \left[\nu_{\boldsymbol{w},o}, \nu_{\boldsymbol{L},,o}, \nu_{\boldsymbol{p}^R,o}, \nu_{\boldsymbol{Q}^R,o}\right]^\top$. Given these structural residuals, I assume the following moment condition.

Assumption 1. *(Moment Condition)*

$$\mathbb{E}\left[\nu_o | \widehat{\boldsymbol{A}_2^R}\right] = \boldsymbol{0}.$$

Assumption 1 is a restriction on the structural residual $\boldsymbol{\nu}$ that it should not be predicted by the vector of cost shocks to produce robots. Note that the moment condition allows that the automation shock $\widehat{a_o}$ may correlate with the cost shock $\widehat{\boldsymbol{A}_2^R}$ due to, for example, the correlated process and product innovations of robots. Instead, the structural residual ν_o purges out all the predictions of the impacts of shocks $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{A}_2^R}$ on endogenous variables, and I place the assumption that the remaining variation should not be predicted by cost shocks from the data.

Assumption 1 implies that, for any d-dimensional functions $\boldsymbol{H} \equiv \{H_o\}_o$, $\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\right] = 0$. The GMM estimator based on $\boldsymbol{H}$ is

$$\boldsymbol{\Theta}_{\boldsymbol{H}} \equiv \arg\min_{\boldsymbol{\Theta}} \sum_{o=1}^{O} \left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right]^\top \left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right] = 0, \tag{27}$$

which is consistent under the moment condition 1 if $\boldsymbol{H}$ satisfies the rank conditions in Newey and McFadden (1994). The exact specification of $\boldsymbol{H}$ determines the optimality, or the minimal variance, of estimator (27). To specify $\boldsymbol{H}$, I apply the approach that achieves the asymptotic optimality developed in Chamberlain (1987). Formally, define the instrumental variable Z_o as follows:

$$Z_o \equiv H_o^*\left(\widehat{\boldsymbol{A}_2^R}\right) \equiv \mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)|\widehat{\boldsymbol{A}_2^R}\right]\mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\widehat{\boldsymbol{A}_2^R}\right]^{-1}, \tag{28}$$

and assume the regularity conditions B.1 in Section B6.

Proposition 2. *Under Assumptions 1 and B.1, $\boldsymbol{\Theta}_{\boldsymbol{H}^*}$ is asymptotically normal with variance $\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$, which is the minimum among the asymptotic variances of the class of estimators in equation (27), where*

$$\boldsymbol{G} \equiv \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)\right] \text{ and } \boldsymbol{\Omega} \equiv \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top\right].$$

Proof. See Section B6. □

To understand the optimality of the IV in equation (28), note that it has two components. The first term is the conditional expected gradient vector $\mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)|\widehat{\boldsymbol{A}_2^R}\right]$, which takes the gradient with respect to the structural parameter vector. Thus, it assigns large weight to occupation that changes the the predicted outcome variable sensitively to the parameters. The second term is the conditional inverse expected variance matrix $\mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\widehat{\boldsymbol{A}_2^R}\right]^{-1}$, which put large weight to occupation that has small variance of the structural residuals.

Substituting equation (28) to the general GMM estimator (27), I have an estimator $\Theta_{H^*} = \arg\min_{\Theta} \left[\sum_o Z_o \nu_o(\Theta)\right]^\top \left[\sum_o Z_o \nu_o(\Theta)\right]$. Since Z_o depends on unknown parameters Θ, I implement the estimation by the two-step feasible method, or the model-implied optimal IV (Adao et al., 2019). I first estimate the first-step estimate Θ_1 from arbitrary initial values Θ_0. Since the IV is a function of the cost shock $\widehat{A_2^R}$, Θ_1 is consistent by Assumption 1. However, it is not optimal. To achieve the optimality, in the second step, I obtain the optimal IV using the consistent estimator Θ_1. To summarize the discussion so far, define IVs $Z_{o,n}$ where $n = 0, 1$ as follows:

$$Z_{o,n} \equiv H_{o,n}\left(\widehat{A_2^R}\right) = \mathbb{E}\left[\nabla_{\Theta}\nu_o(\Theta_n) \,|\widehat{A_2^R}\right] \mathbb{E}\left[\nu_o(\Theta_n)(\nu_o(\Theta_n))^\top |\widehat{A_2^R}\right]^{-1}. \tag{29}$$

Then I have the following result.

Proposition 3. *Under Assumptions 1 and B.1, the estimator Θ_2 obtained in the following procedure is consistent, asymptotically normal, and optimal:*

Step 1: With a guess Θ_0, estimate $\Theta_1 = \Theta_{H_0}$ using $Z_{o,0}$ defined in equation (29).

Step 2: With Θ_1, estimate Θ_2 by $\Theta_2 = \Theta_{H_1}$ using $Z_{o,1}$ defined in equation (29).

Proof. See Section B7. □

4.3. Estimation Result

To apply Proposition 3, I need to measure the initial equilibrium y_{t_0}, which is an input to the solution matrix $\overline{E}$ in equation (19). I take these data from JARA, IFR, IPUMS USA and CPS, BACI, and World Input-Output Data (WIOD). The measurement of labor market outcomes is standard and relegated to Section A7. I set the initial period robot tax to be zero in all countries.

Table 1a gives the estimates of the structural parameters. Panel 1a shows the estimation result when I restrict the EoS between robots and labor to be constant across occupation groups. The estimate of the within-occupation EoS between robots and labor, θ_g, implies that robots and labor are substitutes within an occupation, and rejects

Table 1: Parameter Estimates

(a) All Parameters

Parameter	$\theta_g = \theta$	Free θ_g
θ	2.96 (0.17)	[Table 1b]
β	0.71 (0.23)	0.73 (0.31)
γ	0.30 (0.11)	0.30 (0.14)
ϕ	0.81 (0.26)	0.81 (0.30)

(b) Heterogeneous EoS θ_g

Routine	Production	4.04 (0.24)
	Transportation	4.29 (0.28)
	Others	1.27 (0.53)
Service		1.35 (0.48)
Abstract		0.80 (0.60)

Note: The estimates of the structural parameters based on the estimator in Proposition 3. Standard errors are in parentheses. In the left panel, parameter θ is the within-occupation elasticity of substitution between robots and labor. Parameter β is the elasticity of substitution between occupations. Parameter γ is the capital adjustment cost. Parameter ϕ is the occupation switch elasticity. The column "$\theta_g = \theta$" shows the result with the restriction that θ_o is constant across occupation groups. The column "Free θ_g" shows the result with θ_g allowed to be heterogeneous across five occupation groups. In the right panel, estimates for parameters θ_g with heterogeneity are shown. Transportation indicates "Transportation and Material Moving" occupations in the Census 4-digit occupation codes (OCC2010 from 9000 to 9750). See the main text for other details.

the Cobb-Douglas case $\theta_g = 1$ at a conventional significance level. The point estimate of the EoS between occupations, β, is 0.71, or occupation groups are complementary. The one-standard error bracket covers Humlum's (2019) central estimate of 0.49. The adjustment cost parameter γ is close to the estimate of Cooper and Haltiwanger (2006) when they restrict the model with only quadratic adjustment costs, like in my model. The one-standard error range of occupational dynamic labor supply elasticity ϕ is estimated to be $[0.55, 1.07]$, which contains an estimate of 0.6 in the dynamic occupation choice model in Traiberman (2019) in the case without the specific human capital accumulation.

Panel 1b shows the estimation result when I allow the heterogeneity across occupation groups. The other structural estimates, (β, γ, ϕ), do not change qualitatively. Table 1b shows the estimates of the within-occupation EoS between robots and labor, θ_g. I find that the EoS for routine production occupations and routine transportation occupations is around 4, while those for other occupation groups (other occupations in routine group, service, and abstract occupations) are not significantly different from 1, the case of Cobb-Douglas. The estimates for routine production and transportation indicate the susceptibility of workers in these occupations to accumulated robot capital.

What is the source of identification of these large and heterogeneous EoS between robots and labor identified? As in the literature of estimating the capital-labor substitution elasticity, the positive correlation between the robot price and the wage (labor market outcome) suggests robots and labor are substitutes, or large θ_g. Intuitively, if θ_g is large, then given a percentage decrease in the cost of robots, the steady-state relative robot (resp. labor) demand responds strongly in the positive (resp. negative) direction. Reducing the occupation wage through the labor demand equation, the large robot-labor EoS yields a positive correlation between the robot price trend and the wage trend, as found in Figure 2. Section A6. further discusses this source of identification of the EoS for each occupation group.

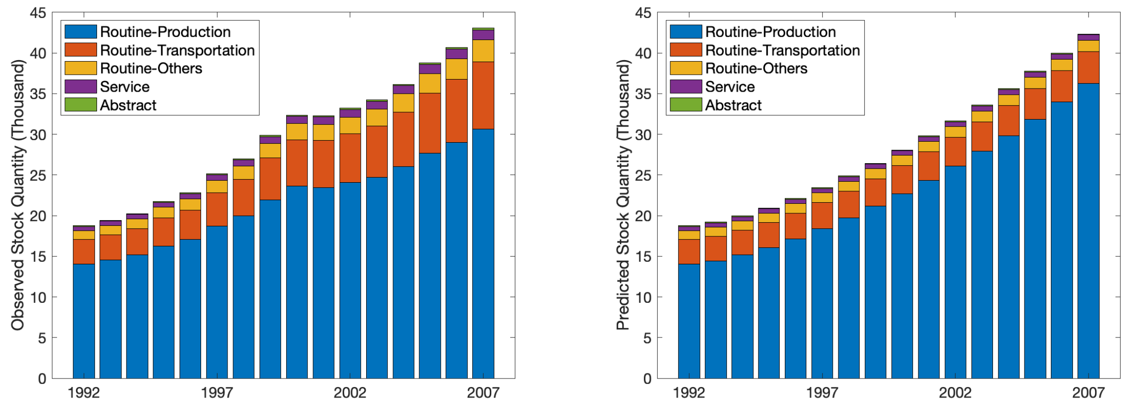
4.3.1. Measuring Shocks and Model Fit

To examine the plausibility of these parameter estimates, I check the model's fit. The process comprises two steps. First, I back out the observed shocks from the estimated model for each year between 1992 and 2007. Namely, with the point estimates in Table 1b, I obtain the Japanese cost shock $\widehat{A_{2,o,t}^{R}}$ using (24), equation the observed automation shock $\widehat{a_{o,t}^{\text{obs}}}$ using (25), and the US occupation demand shock $\widehat{b_{1,o,t}}$. To back out the cost shock in the other countries, I assume that $\widehat{A_{i,o,t}^{R}} = \widehat{A_{i,t}^{R}}$ for $i = 1, 3$. Then by the robot trade prices $p_{ij,t}^{R}$ from BACI, I fit fixed effect regression $\widehat{p_{ij,t}^{R}} = \widetilde{\psi}_{j,t}^{D} + \widetilde{\psi}_{i,t}^{C} + \widetilde{e}_{ij,t}$, and use $\widehat{A_{i,t}^{R}} = -\left(\widetilde{\psi}_{i,t_1}^{C} - \widetilde{\psi}_{i,t_0}^{C}\right)$. Second, applying the backed-out shocks $\widehat{A_{i,o,t}^{R}}$, $\widehat{a_{o,t}^{\text{obs}}}$, and $\widehat{b_{1,o,t}}$ to the first-order solution of the GE in equation (20), I obtain the prediction of changes of endogenous variables to these shocks to the first-order. Finally, applying the predicted changes to the initial data in $t_0 = 1992$, I obtain the predicted level of endogenous variables.

Figure 3 shows the trends of robot stock in the US in the data and the model. Although I do not match the overall robot capital stocks, the estimated model tracks the observed pattern well between 1992 and the late 2010s, consistent with the fact that I target the changes between 1992 and 2007. Overall, the model's prediction follows the observed

Figure 3: Trends of Robot Stocks

(a) Data (b) Model



Notes: Figures show the trend of the observed (left) and predicted (right) stock of robots for each occupation group measured by quantities. The predicted robot stocks are computed by shocks backed out from the estimated model and applying the first-order solution to the general equilibrium described in equation (20).

growth of robot stocks over 1992 and 2007, even though the robot stocks are not the directly targeted moments. There is a slight over-prediction of the growth of production robots and under-prediction of the growth of transportation (material moving) robots between occupation groups.

5. Simulation Exercises

This section tackles the following questions using my estimated model and backed-out shocks in the previous section. The first question is the distributional effects of robots. Autor et al. (2008) argue that the wage inequality measured by the ratio of the wages between the 90-th percentile and the 50-th percentile (90-50 ratio) steadily increased since 1980.[18] Although my data and model predicts the changes from 1990, can the increased use of industrial robots explain the 90-50 ratio? If so, how much? The second question concerns the policy implication of robot regulation. Due to the fear of automation, poli-

[18] Heathcote et al. (2010) argue that a sizable part of the US economic inequality roots in the wage inequality. Furthermore, the polarization is not a unique phenomenon in the US, but found in the other context such as the UK (Goos and Manning, 2007).

cymakers have proposed regulating industrial robots using robot taxes. What would be the effect of taxing on robot purchases?

5.1. The Distributional Effects of Robot Adoption

To study the contribution of robots to wage polarization, I begin by showing the pattern of robot accumulations over the occupational wage distribution.[19] Figure 4a shows the average annual growth rates of observed robot stock between 1992 and 2007 for every ten deciles of the occupational wage distribution in 1990. The figure clarifies that the occupations in the middle deciles of the distribution received relatively many robots. Conditional on robot prices, this pattern implies there are relatively large automation shocks on these occupations. The right panel shows the steady-state annualized predicted wage growths due to the shocks backed out in Section 4.3.1. and the estimated model with the first-order solution given in equation (20). Consistent with the high growth rate of robot stocks in the middle of the wage distribution and the estimation results that indicate the strong substitutability between robots and labor, I find that the wage effect in the middle deciles of the initial wage distribution is strongly negative. Quantitatively, the 90-50 ratio observed in 1990 and 2007 is, respectively, 1.588 and 1.668. On the other hand, the 90-50 ratio predicted by the initial 1990 data and the first-order solution (20) is 1.597. These findings indicate that 11.7 percent of the observed change in the 90-50 ratio between 1990 and 2007.

5.2. Robot Tax and Aggregate Income

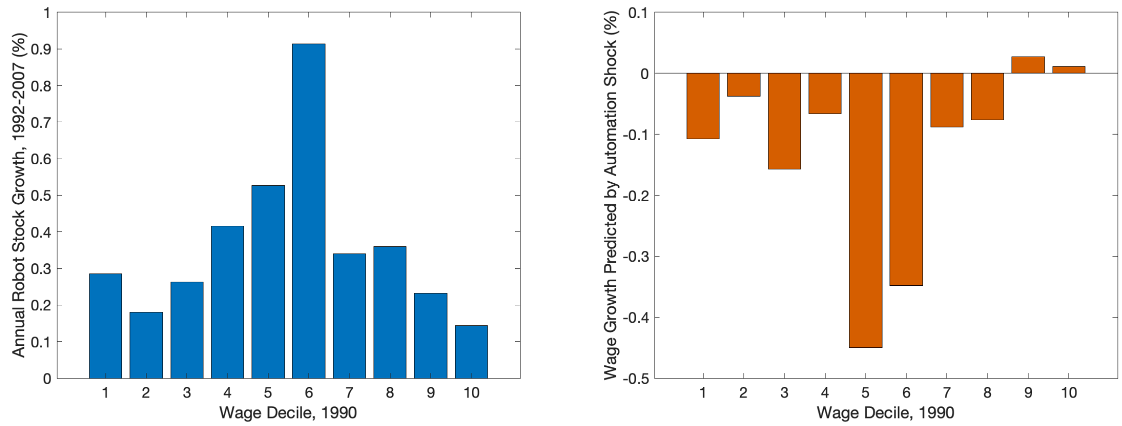
Next, I consider a counterfactual rise of robot tax as well as the automation shock. In the baseline economy, all countries levied zero robot tax. On the one hand, consider an unexpected, unilateral, and permanent increase in the robot tax by 30% in the US

[19] Across occupation groups, there are large drops of relative occupational wages in production and transportation (material-moving) occupations. This is a natural consequence of (i) the high estimates of EoS between robots and labor in these occupations (Table 1b), and the large quantities robots adopted (Figure 3). In Section C1., I confirm this pattern.

Figure 4: Robots, Wage Inequality, and Polarization

(a) Robot growths by Wage Decile

(b) Robots' Effect on Occupational Wages



Notes: The left panel shows the average annual growth rates of the observed robot stock between 1992 and 2007 for every ten deciles of the occupational wage distribution in 1990. The right panel shows the steady-state annualized wage growth rates predicted by the backed-out shocks and the estimated model's first-order steady-state solution given in equation (19).

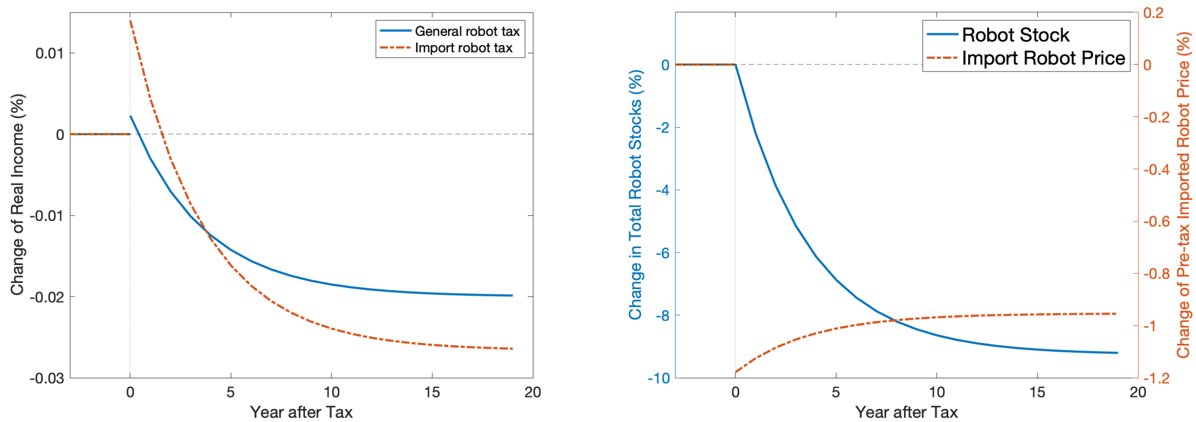
following Humlum (2019), or the general tax scenario. I also consider the tax on only imported robots by 168%, or the import tax scenario, which implies the same amount of tax revenue as in the general tax scenario. How does these robot tax affect the US real income? In Figure 5a, the solid line tracks the real-income effect of the general robot tax over a 20-year time horizon after the imposition. First, the magnitude of the effect is small because the cost of buying robots and the contribution of robot capital in the aggregate production are small. Second, in the short-run, there is a positive effect while the effect turns negative quickly and continues so in the long-run.

Why is there a short-run positive effect on real income? A country's total income comprises the sum of workers' wages, the non-robot good producer's profit, and tax revenue. Since robots are traded, and the US is a large economy that can affect the robot price produced in other countries, there is a terms-of-trade effect of robot tax in the US. Namely, the robot tax reduces the demand for robots produced in the other country, let the equilibrium robot price go down along the supply curve.[20] This reduction in the robot price

[20]The terms-of-trade manipulation is well-studied in the trade policy literature. This paper offers the upward sloping export supply curve from the general equilibrium, as opposed to the supply curve that is assumed upward sloping (e.g., Broda et al., 2008). Namely, when the demand for robots in a robot

Figure 5: Effects of the Robot Tax

(a) US Real Income (b) US Robot Stock and Import Robot Price



Notes: The left panel shows the counterfactual effect on the US real income of the two robot tax scenarios described in the main text over a 20-year time horizon. The right panel shows that of the import robot tax on the US total robot stocks (solid line) and the pre-tax robot price from Japan (dash-dot line) over the same time horizon.

contributes to the increase in the firm's profit, raising the real income in the short-run. The short-run positive effect is stronger in the import robot tax scenario because the higher tax rate induces a more substantial drop in the import robot price.

Why do I have the subtle effects on real income depending on the time horizon? The solid line in Figure 5b shows the dynamic impact of the import robot tax on robot stock accumulation. The tax significantly slows the accumulation of robot stocks, and decreases the steady-state stock of robots by 9.7 percent compared to the no-tax case. The smaller quantity of robot stocks reduces the firm profit, which contributes to smaller real income.[21] These results highlight the role of costly robot capital (de-)accumulation in the effect of the robot tax on aggregate income.

In Figure 5b, The dash-dot line shows a distinct dynamic effect: the effect of the robot

exporter country decreases, the resource to produce robots in the exporter country is freed and reallocated to produce the non-robot goods. In my case, the resource is simply the non-robot goods that are input to robot production in equation (7). This increases the supply of non-robot goods in the robot-exporting country, depressing the price of non-robot goods. Again due to robot production function (7), this decrease in the non-robot goods price means the decrease of the cost of producing robots, which in turn reduces the price of robots produced in the exporter country.

[21] For each occupation, the counterfactual evolution of robot stocks is similar to each other in percentage and thus, similar to the aggregate trend in percentage. This is not surprising since the robot tax is ad-valorem and uniform across occupations.

tax on the price of robots imported from Japan in the US. In the short-run, the price decreases due to the decreased demand from the US. As the sequential equilibrium reaches the new steady state where the US stock of robots is decreased, the marginal value of the robots is higher. This increased marginal value partially offsets the reduced price of robots in the short-run, pushing back the cost of robots from Japan. This figure shows the effect of the international trade of robots in a large country as well as the accumulation of robots. As an extreme case, I also consider an alternative model with no trade of robots due to prohibitively high robot cost and give the robot tax counterfactual exercise in Section C2.

5.3. Other Exercises

What does the same robot tax do to each occupation? The robot tax rolls back the long-run real wage effect of automation. Workers in occupations that experienced significant automation shocks (e.g., production and transportation in the routine occupation groups) who would have been substituted by accumulated robots benefit from the tax, while the others lose. Section C3. discusses the effect of the robot tax on occupational wages in detail. My model also allows the counterfactual exercise regarding robots trade liberalization. As mentioned in Section C4., I find that trade liberalization would benefit all countries in the long-run. The benefit in the US appears immediately after the reduction in the trade cost.

6. Conclusion

In this paper, I study the distributional and aggregate effects of industrial robots, emphasizing that robots perform specified tasks and are internationally traded. I make three contributions. First, I construct a first dataset that tracks the number of robot arms and unit values disaggregated by occupations that robots replace. Second, I develop a general equilibrium model that features the trade of robots in a large-open economy and endogenous robot accumulation with an adjustment cost. When estimating the model, to identify the occupation-specific EoS between robots and labor, I construct a model-implied optimal instrumental variable from the average price of robots in my dataset.

The estimates of within-occupation EoS between robots and labor is heterogeneous and as high as 4 in production and material moving occupations. These estimates are significantly larger than estimates of the EoS of capital goods and workers, with a maximum of about 1.5, revealing the susceptibility of workers in the occupations to robot adaptation. These estimates imply that robots contributed to the wage polarization across occupations in the US from 1990-2007. A commonly advertised robot tax could increase the US real income in the short-run but leads to a decline in the income in the long run due to the small steady-state robot stock. These findings indicate that the accumulated robots may have more massive distributional impacts than is considered in the previous literature, and regulating robots could have a positive effect from the aggregate perspective due to the trade of robots.

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Appendix

A Data Appendix

A1. Full Data Sources

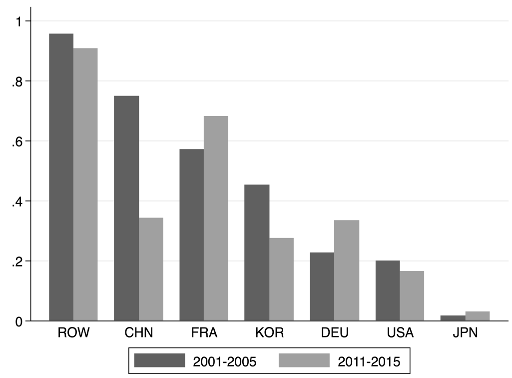
I supplement data from JARA data and O*NET data by the ones from IFR, BACI, IPUMS USA. IFR is a standard data source of industrial robot adoption in several countries (e.g., Graetz and Michaels, 2018; Acemoglu and Restrepo, 2020, AR thereafter), to which JARA provides the robot data of Japan. I use IFR data to show the total robot adoption in each destination country as opposed to the import from Japan. BACI provides disaggregated data on trade flows for more than 5000 products and 200 countries and is a standard data source of international trade (Gaulier and Zignago, 2010). I use BACI data to obtain the measure of international trade of industrial robots and baseline trade shares. IPUMS USA collects and harmonizes US census microdata (Ruggles et al., 2018). I use Population Censuses (1970, 1980, 1990, and 2000) and American Community Surveys (ACS, 2006-2008 3-year sample and 2012-2016 5-year sample). I obtain occupational wages, employment, and labor cost shares from these data sources. To obtain the intermediate inputs shares, I take data from the World Input-Output Data (WIOD) in the closest year to the initial year, 1992.

A2. Trade of Industrial Robots

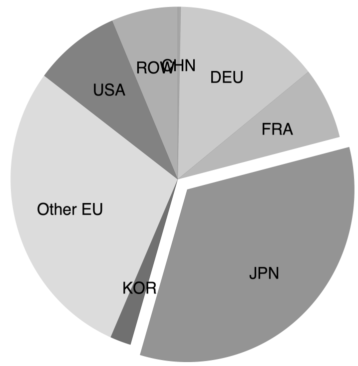
To compute the trade shares of industrial robots, I combine BACI and IFR data. In particular, I use the HS code 847950 ("Industrial Robots For Multiple Uses") to measure the robots, following Humlum (2019). I approximate the initial year value by year of 1998, when the this HS code of robots is first available. To calculate the total absorption value of robots in each country, I use the IFR data's robot units (quantities), combined with the price indices of robots occasionally released by IFR's annual reports for selected countries. These price indices do not give disaggregation by robot tasks or occupations, highlighting the value added of the JARA data. Figure A.1 the pattern of international trade of international robots. In the left panel, I compute the import-absorption ratio. To remove the noise of yearly data, I aggregate by five-year bins 2001-2005 and 2011-2015.

Figure A.1: Trade of Industrial Robots

(a) Robot Import-Absorption Ratio



(b) World Robot Export Share, 2001-2005



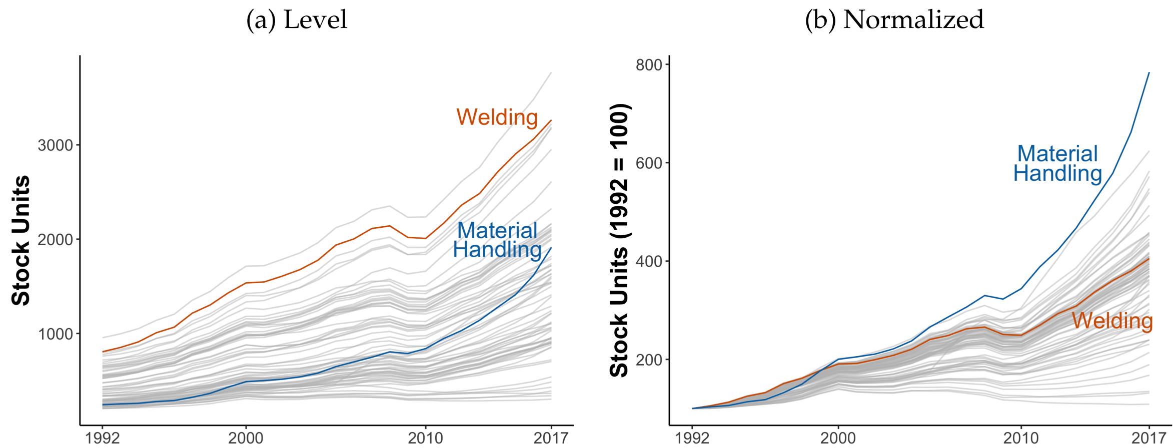
The result indicates that many countries import robots as opposed to produce in their countries. Japan's low import ratio is outstanding, revealing that its comparative advantage in this area. It is noteworthy that China largely domesticated the production of robots over the sample period. Another way to show grasp the comparative advantage of the robot industry is to examine the share of exports as in the right panel of Figure A.1. Roughly speaking, the half of the world robot market was dominated by EU and one-third by Japan in 2001-2005. The rest 20% is shared by the rest of the world, mostly by the US and South Korea.

A3. Trends of Robot Stocks and Prices

I will show that different occupations experienced different trends in robot adoption. Figure A.2 shows the trend of US robot stocks at the occupation level. In the left panel, I show the trend of raw stock. First, the overall robot stocks increased rapidly in the period, as found in the previous literature. The panel also shows that the increase occurred in many occupations, but at differential rates. To highlight such a difference, in the right panel, I plot the normalized trend at 100 in the initial year. There is significant heterogeneity in the growth rates, ranging from a factor of one to eight.

For example, I color in the figure two occupations, robots that correspond to "Welding, Soldering, and Brazing Workers" (or "Welding") and "Laborers and Freight, Stock, and Material Movers, Hand" (or "Material handling"). On the one hand, welding is an occupation where the majority

Figure A.2: US Robot Stocks at the Occupation Level



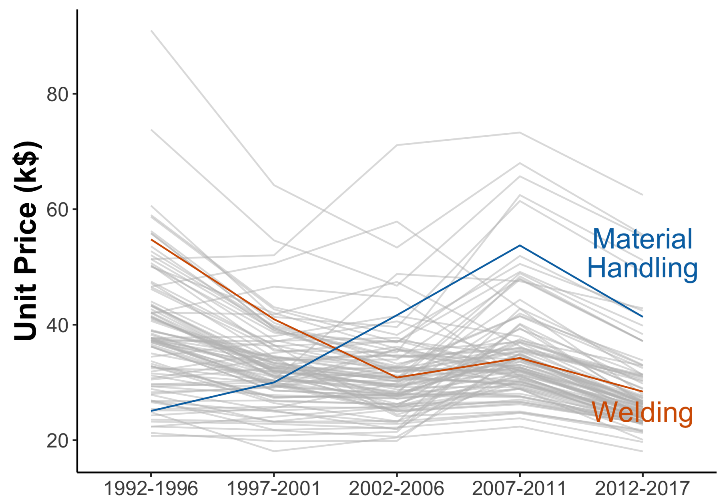
Notes: The author's calculation based on JARA and O*NET. The figure shows the trend of stocks of robots in the US for each occupation. The left panel shows the level, whereas the right panel shows the normalized trend at 100 in 1992. In both panels, I highlight two occupations. "Welding" corresponds the occupation code in IPUMS USA, OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds the occupation code OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises.

of robots were applied continuously throughout the period, as can be confirmed in the left panel. However, the growth rate of the stocks is not outstanding, but within the range of growth rates of other occupations. On the other hand, material handling was not a majority occupation as of the initial year, but it grew at the most rapid pace in the period.

These findings indicate the difference between the automation shocks each occupation received. Some occupations were already somewhat automated by robotics as of the initial year, and the automation process continued afterward (e.g., welding). There are a few occupations where robotics automation was not achieved initially, but the innovation and adoption occurred rapidly in the period (e.g., material handling). I propose a model that incorporates this heterogeneity and discuss how to exploit it in estimation in the following sections.

Figure A.3 shows the trend of prices of robots in the US for each occupation. In addition to the overall decreasing trend, there is significant heterogeneity in the pattern of price falls across occupations. For instance, although the welding robots saw a large drop in the price during the 1990s, the material handling robots did not but increased the price over the sample period.

Figure A.3: Robot Prices at the Occupation Level



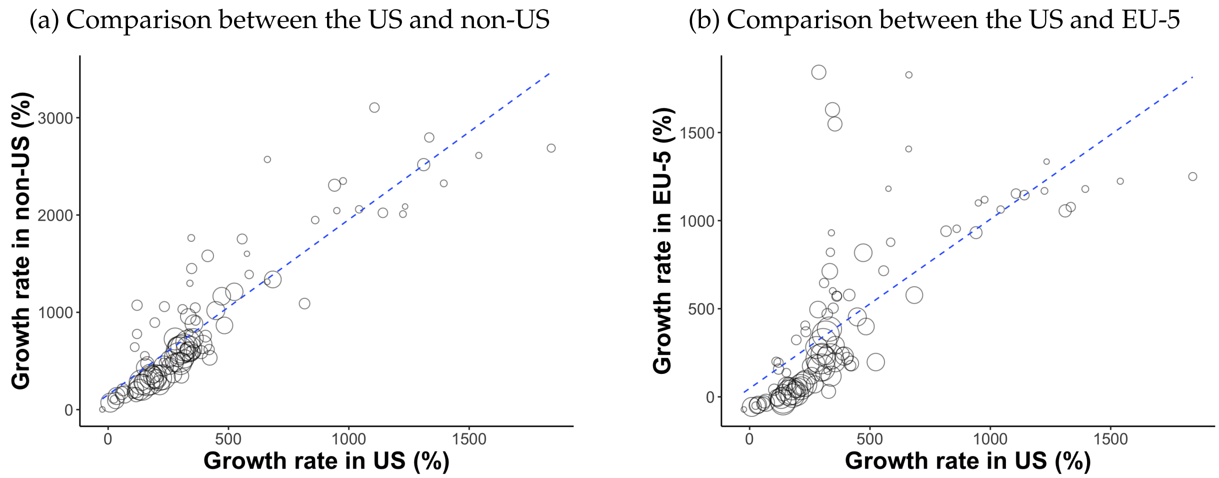
Notes: The author's calculation based on JARA and O*NET. The figure shows the trend of prices of robots in the US for each occupation. I highlight two occupations. "Welding" corresponds the occupation code in IPUMS USA, OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds the occupation code OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises. The dollars are converted to 2000 real US dollar using CPI.

A4. Robots from Japan in the US, Europe, and the Rest of the World

I review the international comparison of the pattern of robot adoption. I generate the growth rates of stock of robots between 1992 and 2017 at the occupation level for each group of destination countries. The groups are the US, the non-US countries, (namely, the world excluding the US and Japan), and five European countries (or "EU-5"), Denmark, Finland, France, Italy, and Sweden used in AR. To calculate the stock of robots, I employ the perpetual inventory method with depreciation rate of $\delta = 0.1$, following Graetz and Michaels (2018).

Figure A.4 shows scatterplots of the growth rates at the occupation level. The left panel shows the growth rates in the US on the horizontal axis and the ones in non-US countries on the vertical axis. The right panel shows the same measures on the horizontal axis, but the growth rates in the set of EU-5 countries on the vertical axis. These panels show that the stocks of robots at the occupation level grow (1992-2017) between the US and non-US proportionately relative to those between the US and EU-5. This finding is in contrast to AR, who find that the US aggregate robot stocks grew at a roughly similar rate as those did in EU-5. It also indicates that non-US growth patterns reflect growths of robotics technology at the occupation level available in the US. I will

Figure A.4: Growth Rates of Robots at the Occupation Level



Notes: The author's calculation based on JARA, and O*NET. The left panel shows the correlation between occupation-level growth rates of robot stock quantities from Japan to the US and the growth rates of the quantities to the non-US countries. The right one shows the correlation between growth rates of the quantities to the US and EU-5 countries. Non-US are the aggregate of all countries excluding the US and Japan. EU-5 are the aggregate of Denmark, France, Finland, Italy, and Sweden used in Acemoglu and Restrepo (2020). Each bubble shows an occupation. The bubble size reflects the stock of robot in the US in the baseline year, 1992. See the main text for the detail of the method to create the variables.

use these patterns as the proxy for robotics technology available in the US. In Section 3. and on, I take a further step and solve for the robot adoption quantity and values in non-US countries in general equilibrium including the US and non-US countries.

It is worth mentioning that a potential cause of the difference between my finding and AR's is the difference in data sources. In contrast to the JARA data I use, AR use IFR data that include all robot seller countries. In particular, recall that EU-5 is closer to major robot producer countries other than Japan, including Germany, the robot adoption pattern across occupations may be influenced by their presence. If these close producers have a comparative advantage in producing robots for a particular occupation, then EU-5 may adopt the robots for such occupations intensively from close producers. In contrast, countries out of EU-5, including the US, may not benefit the closeness to these producers. Thus they are more likely to purchase robots from far producers from EU-5, such as Japan.

A5. Robustness Check of Fact 2

Figure A.5 shows the results of a set of robustness checks for Fact 2 in Section 2.3. with an emphasis on the correlation between the wage changes and the changes in robot measures. Figures A.5a and A.5b show the wage correlation after residualizing the demographic control variables (initial-year female share, college graduates share, age 35-49 share, age 50-64 share, and foreign-born share in each occupation) for the robot stock and robot prices, respectively. Figures A.5c and A.5d show the correlation with the robot measures in the rest of the world (namely, world excluding the US and Japan) after residualizing the demographic control variables. The motivation of this exercise follows the intuition of AR–using the technological change proxied by the rest-of-the-world change in robot measures to move away from the occupational demand shocks since the US occupational robot adoption may be affected by occupational demand shocks such as occupational productivity changes. Figure A.5e shows the result with the measurement of the robot stock by quantities of machines as opposed to monetary value, which follows the approach in the past literature such as AR. Figure A.5f shows the result of correlation using quality-adjusted robot prices, where the method of quality adjustment follows the spirit of Khandelwal et al. (2013). Namely, I fit the following equation with the fixed-effect regression:

$$\ln\left(X^R_{JPNi,o,t}\right) = -\varsigma \ln\left(p^R_{JPNi,o,t}\right) + a^R_{o,t} + e^R_{i,o,t},$$

from which I obtain the fixed effect $a^R_{o,t}$, which absorbs the occupation-o specific log sales component that is not explained by the prices. I then proxy the quality change by the change in such fixed effects, $\Delta a^R_{o,t} \equiv a^R_{o,t} - a^R_{o,t_0}$. The (log) quality-adjusted price is then obtained by $\ln\left(p^R_{JPNi,o,t}\right) - \Delta a^R_{o,t}$. All the results are robust to these considerations–wage growths are negatively correlated with stock growths, and positively correlated with price growths, both across occupations.

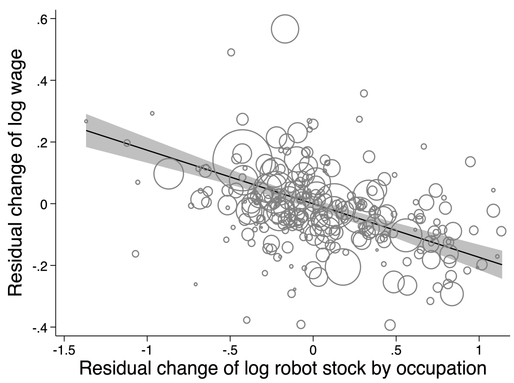
To further check the correlation systematically, I run the following regressions and report the results in Table A.1:

$$\Delta \ln\left(y_o\right) = a_R \Delta \ln\left(R_o\right) + \left(X_o\right)^\top \boldsymbol{a} + e_o,$$

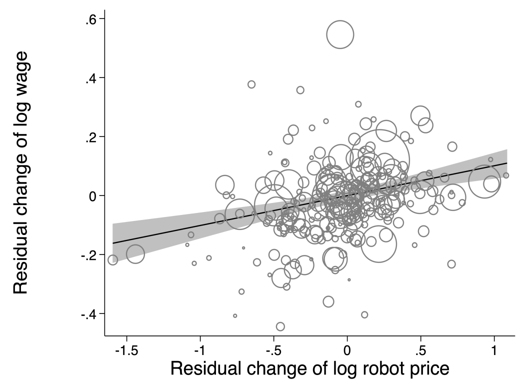
where y_o is labor-market outcome of occupation o (wage and employment), R_o is the measure of robots (stocks and prices), X_o are the demographic control variables, e_o is the regression residual, and Δ indicates the long-difference between 1990 (1992 for $\Delta \ln\left(R_o\right)$) and 2007. The coefficient of interest is a_R–I expect negative a_R if I take robot stocks as the explanatory variable, while I expect positive a_R when I take robot price as the right-hand side variable.

Figure A.5: Correlation between Occupational Wage and Occupational Robot Measures

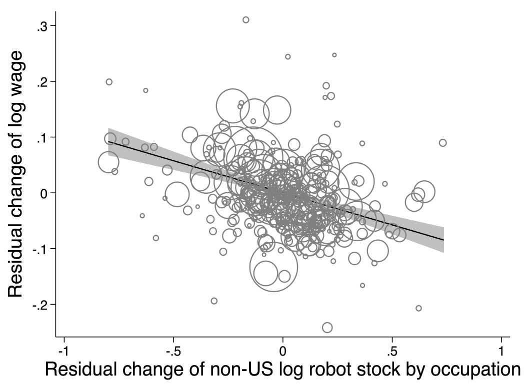
(a) Residualizing Control Variables, Stocks



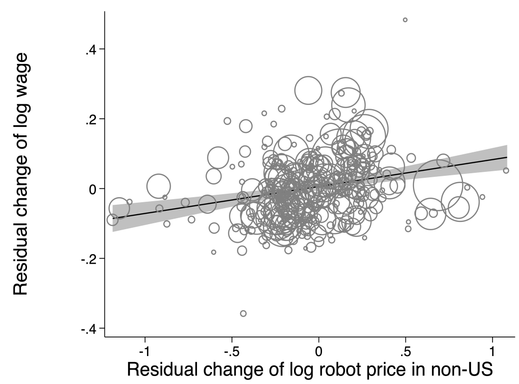
(b) Residualizing Control Variables, Prices



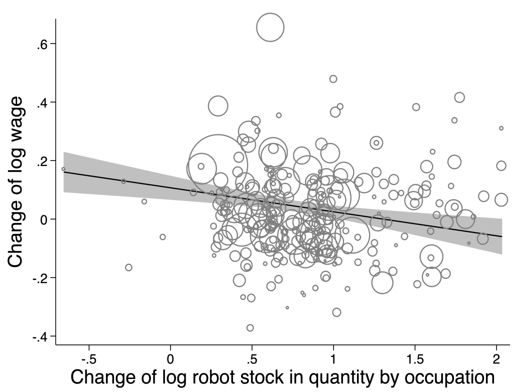
(c) ROW Stocks



(d) ROW, Prices



(e) Stock Quantity



(f) Quality-adjusted prices

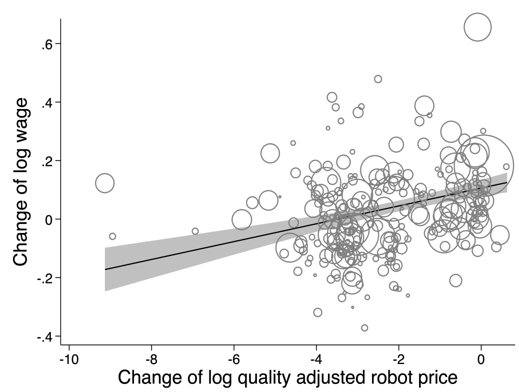


Table A.1: Regression Result of Labor Market Outcome on Robot Measures

VARIABLES	(1) $\Delta \ln(w)$	(2) $\Delta \ln(w)$	(3) $\Delta \ln(w)$	(4) $\Delta \ln(w)$	(5) $\Delta \ln(L)$	(6) $\Delta \ln(L)$	(7) $\Delta \ln(L)$	(8) $\Delta \ln(L)$
$\Delta \ln(K^R_{USA})$	-0.174***				-0.532***			
	(0.0251)				(0.203)			
$\Delta \ln(p^R_{USA})$		0.0969***				0.507***		
		(0.0263)				(0.141)		
$\Delta \ln(K^R_{ROW})$			-0.116***				-0.575***	
			(0.0162)				(0.0953)	
$\Delta \ln(p^R_{ROW})$				0.0999***				0.458***
				(0.0257)				(0.148)
Female share	0.0366	0.0391	0.0383	0.0361	-0.0658	-0.0663	-0.0563	-0.0616
	(0.0320)	(0.0340)	(0.0328)	(0.0335)	(0.175)	(0.178)	(0.175)	(0.181)
Col. grad. share	0.399***	0.379***	0.401***	0.399***	0.114	0.113	0.119	0.107
	(0.0684)	(0.0673)	(0.0707)	(0.0691)	(0.284)	(0.285)	(0.281)	(0.287)
Age 35-49 share	-0.768*	-0.594	-0.697*	-0.672*	0.399	0.449	0.325	0.427
	(0.395)	(0.405)	(0.405)	(0.404)	(1.281)	(1.331)	(1.308)	(1.320)
Age 50-64 share	0.778**	0.797**	0.787**	0.765**	-1.636	-1.650	-1.541	-1.576
	(0.345)	(0.345)	(0.365)	(0.376)	(1.166)	(1.134)	(1.208)	(1.170)
Foreign-born share	-0.0905	-0.0250	-0.0241	-0.00227	-0.255	-0.221	-0.322	-0.197
	(0.225)	(0.213)	(0.230)	(0.221)	(1.142)	(1.073)	(1.074)	(1.044)
Observations	324	324	324	324	324	324	324	324
R-squared	0.467	0.344	0.398	0.367	0.138	0.104	0.199	0.106

Notes: The author's calculation based on JARA, O*NET, and US Census/ACS. Observations are 4-digit level occupations, and the sample is all occupations that existed throughout 1970 and 2007. In each country $i \in \{USA, ROW\}$, K_i^R stands for the 2000-dollar value of the robot stock in the occupation and p_i^R stands for the average price of robot transacted in each year. All time differences (Δ) are taken as the long difference between 1990 and 2007. All demographic control variables are as of 1990. "Col. Grad. Share" stands for the college graduate share. Robust standard errors are reported in the parentheses. *** p<0.01, ** p<0.05, * p<0.1.

A6. Robot Price Trends by Occupation Groups

In this section, I examine the facts discussed in Section 2.3. for each occupation group described in Section 4.1. First, Figure A.6 shows the plot of the trend of the robot price distribution since 1992 for each occupation group, a version of Figure 1a, disaggregated by occupation groups. The top three panels show the trends for routine occupations, namely, from the left, routine-production, routine-transportion, and routine-others. The bottom two panels show the trends for service occupations and abstract occupations, from the left. All these panels show the overall decreasing trend of robot prices, and the dispersion of prices within each occupation group. Having such a dispersion is important because in Section 4. when I estimate heterogeneous EoS between robots and labor, I use the price variation within each occupation group. Next, Figure A.7 shows the correspondent of Figure 2 for each occupation group. The alignment of occupation groups is the same as Figure A.6. Interestingly, the positive correlation between occupational wage changes and occupational robot price changes, robustly found in Figure 2 and Section A5., is seen only in the group of production occupations and transportation occupations. Given that strong positive correlation yields a high elasticity of substitution, the finding in this figure is consistent with the heterogeneous elasticity of substitution between robots and labor found in Table 1b.

Figure A.6: Robot Price Trends by Occupation Groups

(a) Routine, Production

(b) Routine, Transportation

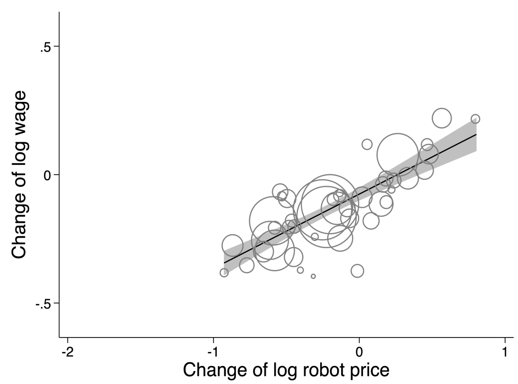
(c) Routine, Others

(d) Service

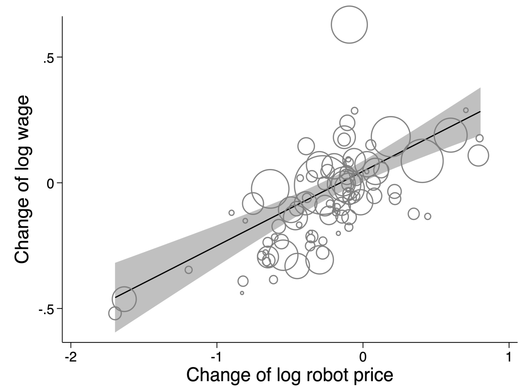
(e) Abstract

Figure A.7: Correlation between Wage and Robot Prices by Ocuupation Groups

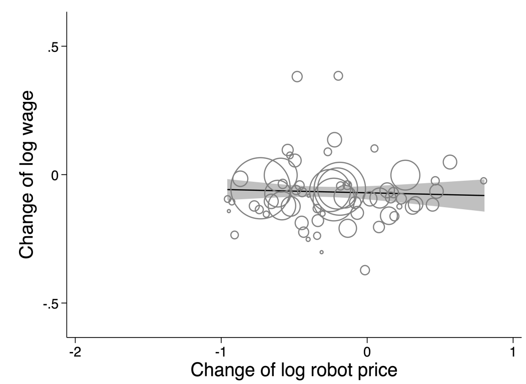
(a) Routine, Production



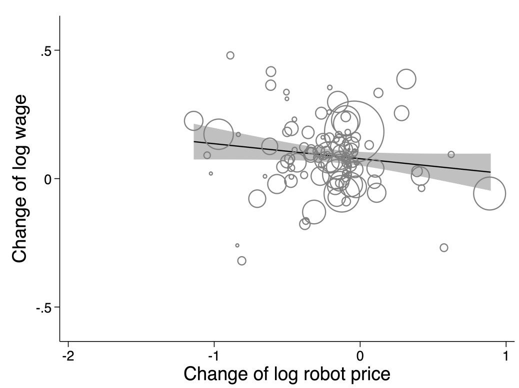
(b) Routine, Transportation



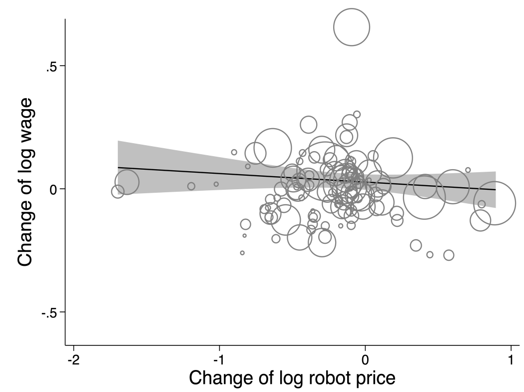
(c) Routine, Others



(d) Service



(e) Abstract



A7. Initial Share Data

Since the log-linearized sequential equilibrium solution depends on several initial share data generated from the initial equilibrium, I show the data sources and methods for measuring these shares. I define $t_0 = 1992$ and the time frequency is annual. I consider the world that consists of three countries $\{USA, JPN, ROW\}$.

Table A.2 summarizes the variable notations, descriptions, and data sources. First, I take matrices of trade of goods and robots by BACI data. As in Humlum (2019), I measure robots by HS code 847950 ("Industrial Robots For Multiple Uses") and approximate the initial year value by year of 1998, in which the robot HS code is first available. To obtain the domestic robot absorption data, I take from IFR data the flow quantity variable and the aggregate price variable for a selected set of countries. I then multiply these to obtain USA and JPN robot adoption value. For robot prices in ROW, I take the simple average of the prices among the set of countries (France, Germany, Italy, South Korea, and the UK, as well as Japan and the US) for which the price is available in 1999, the oldest year in which the price data are available. Graetz and Michaels (2018) discuss prices of robots with the same data source.

Second, I construct occupation cost shares $\tilde{x}^O_{i,o,t_0}$ and labor shares within occupation l_{i,o,t_0} as follows. To measure $\tilde{x}^O_{i,o,t_0}$, I aggregate the total wage income of workers that primarily works in each occupation o in year 1990, the Census year closest to t_0. I then take the share of this total compensation measure for each occupation. To measure l_{i,o,t_0}, I take the total compensation as the total labor cost and a measure of the user cost of robots for each occupation. The user cost of robots is calculated with the occupation-level robot price data available in JARA and the set of calibrated parameters in Section 4.1.. Due to the lack of data, I proxy the robot prices for non-Japanese robots with the JARA data.

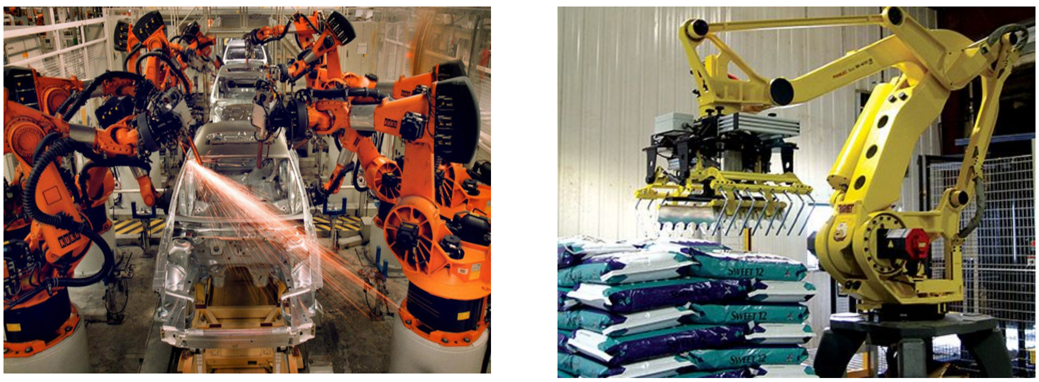
Third, I take intermediate input share $\alpha_{i,M}$ from World Input-Output Tables (WIOT Timmer et al., 2015). Finally, I combine the trade matrix generated above and WIOT to construct the good and robot expenditure shares $s^G_{i,t_0}, s^V_{i,t_0}, s^R_{i,t_0}$. In particular, with the robot trade matrix, I take the total sales value by summing across importers for each exporter, and total absorption value by summing across exporters for each importers. I also obtain the total good absorption by WIOT. From these total values, I take expenditure shares.

Table A.2: List of Data Sources

Variable	Description	Source
$\widetilde{y}^G_{ij,t_0}, \widetilde{x}^G_{ij,t_0}, \widetilde{y}^R_{ij,t_0}, \widetilde{x}^R_{ij,t_0}$	Trade shares of goods and robots	BACI, IFR
$\widetilde{x}^O_{i,o,t_0}$	Occupation cost shares	IPUMS
l_{i,o,t_0}	Labor shares within occupation	JARA, IFR, IPUMS
$s^G_{i,t_0}, s^V_{i,t_0}, s^R_{i,t_0}$	Robot expenditure shares	BACI, IFR, WIOT
$\alpha_{i,M}$	Intermediate input share	WIOT

Figure A.8: Examples of Industrial Robots

(a) Spot Welding (b) Material Handling



Sources: Autobot Systems and Automation (https://www.autobotsystems.com) and PaR Systems (https://www.par.com)

A8. Examples of Industrial Robots

Figure A.8 shows the pictures of examples of industrial robots considered in this paper. The left panel shows spot-welding robots, while the right panel shows the material-handling robots. The spot welding robots are an example of robots in routine-production occupations, while the material-handling robots are that in routine-transportation (material-moving) occupations.

The full list of robot applications avaliable in JARA data is Die casting; Forging; Resin molding; Pressing; Arc welding; Spot welding; Laser welding; Painting; Load / unload; Mechanical cutting; Polishing and deburring; Gas cutting; Laser cutting; Water jet cutting; General assembly; Inserting; Mounting; Bonding; Soldering; Sealing and gluing; Screw tightening; Picking alignment and packaging; Palletizing; Measurement / inspection / test; and Material handling.

A9. Examples of Robotics Innovation

I use robot task space $a_{o,t}$ as the automation shock, and robot producer's TFP $A^R_{l,o,t}$ as the cost shock to produce robots. In this section, I show some examples of changes of robot technology and new patents to facilitate understandings of these interpretation. An example of task space expansion is adopting *Programmed Article Transfer* (PAT, Devol, 1961). PAT was machine that moves objects by a method called "teaching and playback". Teaching and playback method needs one-time teaching of how to move, after which the machine playbacks the movement repeatedly and automatically. This feature frees workers of performing repetitive tasks. PAT was intensively introduced in spot welding tasks. KHI (2018) reports that among 4,000 spot welding points, 30% were done be human previously, which PAT took over. Therefore, I interpret the adoption of PAT as the example of the expansion of the robot task space, or increase in $a_{o,t}$. Note that AR also analyze this type of technological change.

An example of cost reduction is adopting *Programmable Universal Manipulator for Assembly (PUMA).* PUMA was designed to quickly and accurately transport, handle and assemble automobile accessories. A new computer language, *Variable Assembly Language (VAL)*, made it possible because it made the teaching process less work and more sophisticated. In other words, PUMA made tasks previously done by other robots but at cheaper unit cost per unit of task.

It is also worth mentioning that introduction of a new robot brand typically contains both components of innovation (task space expansion and cost reduction). For example, PUMA also expanded task space of robots. Since VAL allowed the use of sensors and "expanded the range of applications to include assembly, inspection, palletizing, resin casting, arc welding, sealing and research" (KHI, 2018).

B Theory Appendix

B1. Further Discussion of Model Assumptions

I highlight key features of the model, their comparisons to the literature, and possible interpretations.

Capital-Skill Complementarity Occupation production function (8) also nest the one in the literature of capital-skill complementarity (Krusell et al., 2000 among others). To simplify, I focus on individual producer's production function in the steady state. Thus I drop subscripts and superscripts of country i, good g, and time period t. Suppose the set of occupations is $O \equiv \{R, U\}$ and $a_U = 0$. R stands for the robotized occupation (e.g., spot welding) and U stands for "unrobotized" (e.g., programming). Note that since U is unrobotized $a_U = 0$. Then the unit cost of occupation aggregate (8), p^O, is

$$p^O = \left[(b_R)^{\frac{1}{\beta}} \left((1 - a_R) (w_R)^{1-\theta_R} + a_R (c_R)^{1-\theta_R} \right)^{\frac{1-\beta}{1-\theta_R}} + (b_U)^{\frac{1}{\beta}} (w_U)^{1-\beta} \right]^{\frac{1}{1-\beta}}.$$

Thus different skills R and U are substituted by robots with different substitution parameters θ_R and β, respectively. Since the literature of capital-skill complementarity studies the rising skill premium, the current model also has an ability to discuss the occupation (or skill) premium given the different level of automation across occupations.

Adjustment Cost of Robot Capital I give an interpretation of another key feature of the model, the convex adjustment cost of robot adoption. The interpretation is twofold–the cost of adopting new technology and of integration. With these convex adjustments costs, the model predict the staggered adoption of robots over years, which I observe in the data.

First, when adopting new technology including robots, it is necessary to re-optimize the overall production process since the production process is typically optimized to employ workers. More generally, the literature of organizational dynamics studies the difficulty, not to say the impossibility, of changing strategies of a company due to complementarities (see Brynjolfsson and Milgrom, 2013 for a review). Such a re-optimization incurs an additional cost of adoption in addition to the purchase of robot arms. Moreover, even within a production unit, there is a variation of this difficulty of adopting robots across production processes. In this case, the part where the adjustment is easy adopts the robots first, and vice versa. This allocation of robot adoptions over years may aggregate to make the robot stock increase slowly (Baldwin and Lin, 2002).

The second component of the adjustment cost may come from the cost of integration as I discussed in Section 2.1. The marginal integration cost may increase as the massive upgrading of robotics system may require large-scale overhaul of production process, which increases the complexity and so is costly. The adjustment cost may capture the increasing marginal cost component of the integration cost. Note that it explains a different component of the integration cost implied by constant returns-to-scale (CRS) robot aggregation (12) as it adds the increasing marginal cost component to the constant one based on the CRS structure.

Another potential choice of modeling a staggered growth of robot stocks is to assume a fixed cost of robot adoption and lumpy investment. Humlum (2019) finds that many plants buy robots only once during the sample period. Since JARA data does not observe plant-level adoptions, I do not separately identify lumpy investment from the staggered growth of robot stocks in the data. To the extent that fixed cost of investment may make the policy intervention less effective (e.g., Koby and Wolf, 2019), the counterfactual analysis in this paper may overestimate the effect of robot taxes since it does not take into account the fixed cost and lumpiness of investment.

B2. Derivation of Worker's Optimality Conditions

In this section, I formalize the assumptions behind the derivation and show equations (2) and (3).

Fix country i and period t. There is a mass $\overline{L}_{i,t}$ of workers. In the beginning of each period, worker $\omega \in \left[0, \overline{L}_{i,t}\right]$ draws a multiplicative idiosyncratic preference shock $\left\{Z_{i,o,t}\left(\omega\right)\right\}_{o}$ that follows an independent Fréchet distribution with scale parameter $A^{V}_{i,o,t}$ and shape parameter $1/\phi$. Note that one can simply extend that the idiosyncratic preference follows a correlated Fréchet distribution to allow correlated preference across occupations, as in Lind and Ramondo (2018). To keep the expression simple, I only consider the case of independent distribution. A worker ω then works in the current occupation, earns income, consumes and derives logarithmic utility, and then chooses the next period's occupation with discount rate ι. When choosing the next period occupation o', she pays an ad-valorem moving cost $\chi_{i,oo',t}$ in terms of consumption unit that depends on current occupation o. She consumes her income in each period. Thus, worker ω who currently works in occupation o_t maximizes the following objective function over the future stream of utilities by choosing occupations $\{o_s\}_{s=t+1}^{\infty}$:

$$E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[\ln\left(C_{i,o_s,s}\right) + \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + \ln\left(Z_{i,o_{s+1},s}\left(\omega\right)\right)\right]$$

where $C_{i,o,s}$ is a consumption bundle when working in occupation o in period $s \geq t$, and E_t is the expectation conditional on the value of $Z_{i,o_t,t}(\omega)$. Each worker owns occupation-specific labor endowment $l_{i,o,t}$. I assume that her income is comprised of labor income $w_{i,o,t}$ and occupation-specific ad-valorem government transfer with rate $T_{i,o,t}$. Given the consumption price $P_{i,t}$, the budget constraint is

$$P_{i,t}C_{i,o,t} = w_{i,o,t}l_{i,o,t}\left(1 + T_{i,o,t}\right)$$

for any worker.

By linearity of expectation,

$$\begin{aligned}
&E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[\ln\left(C_{i,o_s,s}\right) + \ln\left(1 - \chi_{i,o_s o_{s+1},s}\right) + \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \\
&= \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1 - \chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right].
\end{aligned}$$

By monotone transformation with exponential function,

$$\begin{aligned}
&\exp\left\{\sum_{s=t}^{\infty} \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1 - \chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right].\right\} \\
&= \prod_{s=t}^{\infty} \exp\left\{\left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1 - \chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right]\right\}.
\end{aligned}$$

Write the value function conditional on the realization of shocks at period t as follows:

$$V_{i,o_t,t}(\omega) \equiv \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp\left\{\left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1 - \chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right]\right\}.$$

I apply Bellman's principle of optimality as follows:

$$
\begin{aligned}
& V_{i,o_t,t}(\omega) \\
&= \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp\left\{\left(\frac{1}{1+\iota}\right)^{s-t}\left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right]\right\} \\
&= \max_{o_{t+1}} \exp\left\{\ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right) + \ln\left(Z_{i,o_{t+1},t}(\omega)\right)\right\} \times \\
& \max_{\{o_s\}_{s=t+2}^{\infty}} \prod_{s=t+1}^{\infty} \exp\left\{\left(\frac{1}{1+\iota}\right)^{s-(t+1)}\left[E_{t+1} \ln\left(C_{i,o_s,s}\right) + E_{t+1} \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_{t+1} \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right]\right\} \\
&= \max_{o_{t+1}} \exp\left\{\ln\left(Z_{i,o_t,t}(\omega)\right) + \ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right)\right\} V_{i,o_{t+1},t+1},
\end{aligned}
$$

where $V_{i,o_t,t}$ is the unconditional expected value function $V_{i,o_t,t} \equiv E_{t-1} V_{i,o_t,t}(\omega)$. Changing the notation from (o_t, o_{t+1}) into (o, o'), I have

$$
V_{i,o,t}(\omega) = \max_{o'} C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1}.
$$

Solving the worker's maximization problem is equivalent to

$$
\begin{aligned}
& \Pr\left(\text{worker } \omega \text{ in } o \text{ chooses occupation } o'\right) \\
&= \Pr\left(\max_{o''} C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) Z_{i,o'',t}(\omega) V_{i,o'',t+1} \leq C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1}\right).
\end{aligned}
$$

By independent Fréchet assumption,

$$
\mu_{i,oo',t} = \frac{\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}} = \frac{\left(\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}},
$$

$$
V_{i,o,t+1} = \widetilde{\Gamma} C_{i,o,t}\left(\sum_{o'}\left(\left(1-\chi_{i,oo',t+1}\right) V_{i,o',t+2}\right)^{\phi}\right)^{\frac{1}{\phi}}.
$$

B3. Relationship with Other Models of Automation

The model in Section 3. is general enough to nest models of automation in the previous literature. In particular, I show how the production functions (6) and (8) imply to specifications in AR and Humlum (2019). Throughout Section B3., I fix country i and focus on steady states and thus drop subscripts i and t since the discussion is about individual producer's production function.

B3.1. Relationship with the model in Acemoglu and Restrepo (2020, AR)

Following AR that abstract from occupations, I drop occupations by setting $O = 1$ in this paragraph. Therefore, the EoS between occupations β plays no role, and $\theta_o = \theta$ is a unique value. AR show that the unit cost (hence the price given perfect competition) function is written as

$$p^{AR} \equiv \frac{1}{\widetilde{A}} \left[(1 - \widetilde{a}) \frac{w}{A^L} + \widetilde{a} \frac{c^R}{A^R} \right]^{\alpha_L} r^{1-\alpha_L},$$

for each sector and location (See AR, Appendix A1, equation A5). In this equation, c^R is the steady state marginal cost of robot capital defined in equation (B.23) and A^L and A^R represent per-unit efficiency of labor and robots, respectively. In Lemma B.1 below, I prove that my model implies a unit cost function that is strict generalization of p^{AR} with proper modification to the shock terms and parameter configuration. I begin with the modification that allows per-unit efficiency terms in my model.

Definition B.1. For labor and robot per-unit efficiency terms $A^L > 0$ and $A^R > 0$ respectively, modified robot task space $\widetilde{a}$ and TFP term $\widetilde{A}$ are

$$\widetilde{a} \equiv \frac{a \left(A^L\right)^{\theta-1}}{a \left(A^L\right)^{\theta-1} + (1-a) \left(A^R\right)^{\theta-1}}, \tag{B.1}$$

$$\widetilde{A} \equiv \frac{A}{\left[(1-\widetilde{a}) \left(A^L\right)^{\theta-1} + \widetilde{a} \left(A^R\right)^{\theta-1} \right]}. \tag{B.2}$$

Lemma B.1. *Set the number of occupations $O = 1$. In the steady state,*

$$p = \frac{1}{\widetilde{A}} \left[(1-\widetilde{a}) \left(\frac{w}{A^L} \right)^{1-\theta} + \widetilde{a} \left(\frac{c^R}{A^R} \right)^{1-\theta} \right]^{\frac{\alpha_L}{1-\theta}} p^{\alpha_M} r^{1-\alpha_M-\alpha_L}. \tag{B.3}$$

Proof. Note that modified robot task space (B.1) and modified TFP (B.2) can be inverted to have

$$a \equiv \frac{\widetilde{a} \left(A^R\right)^{\theta-1}}{(1-\widetilde{a}) \left(A^L\right)^{\theta-1} + \widetilde{a} \left(A^R\right)^{\theta-1}}, \tag{B.4}$$

$$A \equiv \left[(1-\widetilde{a}) \left(A^L\right)^{\theta-1} + \widetilde{a} \left(A^R\right)^{\theta-1} \right] \widetilde{A}. \tag{B.5}$$

Cost minimization problem with the production functions (6) and (8) and perfect competition imply

$$p = \frac{1}{A} \left(p^O\right)^{\alpha_L} p^{\alpha_M} r^{1-\alpha_L-\alpha_M},$$

and

$$p^O = \left[(1-a) w^{1-\theta} + a^{1-\theta} \right]^{\frac{1}{1-\theta}},$$

where p^O is the unit cost of aggregated occupation $Q^O \equiv \left[\sum_o (Q_o)^{(\beta-1)/\beta}\right]^{\beta/(\beta-1)}$. Substituting equations (B.4) and (B.5) and rearranging, I have

$$p = \frac{1}{\widetilde{A}} \left(\widetilde{p^O}\right)^{\alpha_L} p^{\alpha_M} r^{1-\alpha_L-\alpha_M},$$

where

$$\widetilde{p^O} = \left[(1-\widetilde{a}) \left(\frac{w}{A^L}\right)^{1-\theta} + a \left(\frac{c^R}{A^R}\right)^{1-\theta} \right]^{\frac{1}{1-\theta}}.$$

□

Lemma B.1 immediately implies the following corollary that shows that the steady state modified unit cost (B.3) strictly nests the unit cost formulation of AR as a special case of Leontief occupation aggregation.

Corollary B.1. *Suppose $\alpha_M = 0$. Then as $\theta \to 0$, $p \to p^{AR}$.*

B3.2. Relationship with the model in Humlum (2019)

I show that production functions (6) and (8) nest the production function used by Humlum (2019). Namely, for each firm in each period, Humlum (2019) specifies

$$Q^D = \exp\left[\varphi_H^D + \gamma_H^D K\right] \left[\sum_o \left(\exp\left[\varphi_o^D + \gamma_o^D K\right] \right)^{\frac{1}{\beta}} (L_o)^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}}, \tag{B.6}$$

where $K = \{0, 1\}$ is a binary choice, $\varphi_H^D, \gamma_H^D, \varphi_o^D$ and γ_o^D are parameters, and superscript D represents the discrete adoption problem of Humlum (2019). As normalization, suppose that

$$\sum_o \exp\left(\varphi_o^D + \gamma_o^D K\right) = 1.$$

I will start from production function (6) and (8), place restrictions, and arrive at equation (B.6). As a key observation, relative to the discrete choice of robot adoption in Humlum (2019), the continuous choice of robot *quantity* in production function (8) allows significant flexibility. In this paragraph, I assume away with intermediate inputs. This is because Humlum (2019) assumes that intermediate inputs enter in an element of CES, while production function (6) implies that intermediate inputs enter as an element of the Cobb-Douglas function.

Now, given our production functions (6) and (8), suppose producers follow the binary decision rule defined below.

Definition B.2. A binary decision rule by a producer is that producers can choose between two choices: adopting robots $K = 1$ or not $K = 0$. If they choose $K = 1$, they adopt robots at the same unit as labor $K_o^R = L_o \geq 0$ for all occupation o. If they choose $K = 0$, $K_o^R = 0$ for all o.

Note that the binary decision rule is nested in the original choice problem from $K_o^R \geq 0$ for each o. Set

$$A_o^D\left(K^R\right) \equiv \begin{cases} A_o\left((1-a_o)^{\frac{1}{\theta}} + (a_o)^{\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}(\beta-1)} & \text{if } K^R = L_o \\ A_o\,(1-a_o)^{\frac{1}{\theta-1}(\beta-1)} & \text{if } K^R = 0 \end{cases}.$$

Then I have

$$Q = \left[\sum_o \left(A_o^D\left(K_o^R\right)\right)^{\frac{1}{\beta}} (L_o)^{\frac{\beta-1}{\beta}}\right]^{\frac{\beta}{\beta-1}}.$$

To normalize, define

$$\widetilde{A_o^D} \equiv \left(\sum_o A_o^D\left(K_o^R\right)\right)^{\frac{1}{\beta-1}}$$

and

$$a_o^D\left(K_o^R\right) \equiv \frac{A_o^D\left(K_o^R\right)}{\sum_{o'} A_{o'}^D\left(K_{o'}^R\right)}.$$

Then I have

$$Q = \widetilde{A_o^D} \left[\sum_o \left(a_o^D \left(K_o^R \right) \right)^{\frac{1}{\beta}} \left(L_o \right)^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}}. \tag{B.7}$$

Finally, let

$$A_{o,0} \equiv \left[\exp \left(\varphi_H^D + \varphi_o^D \right) \right]^{\frac{\theta_o - 1}{\beta - 1}}$$

and

$$A_{o,1} \equiv \left[\left(\exp \left(\varphi_H^D + \varphi_o^D + \gamma_H^D + \gamma_o^D \right) \right)^{\frac{1}{\theta_o} \frac{\theta_o - 1}{\beta - 1}} - \left(\exp \left(\varphi_H^D + \varphi_o^D \right) \right)^{\frac{1}{\theta_o} \frac{\theta_o - 1}{\beta - 1}} \right]^{\theta_o}.$$

and also let A_o and a_o satisfy

$$A_o = \left(A_{o,0} + A_{o,1} \right)^{\frac{\beta-1}{\theta_o - 1}} \tag{B.8}$$

and

$$a_o = \frac{A_{o,1}}{A_{o,0} + A_{o,1}}. \tag{B.9}$$

Then one can substitute equations (B.8) and (B.9) to equation (B.7) and confirm that $Q = Q^{Humlum}$. Summarizing the discussion above, I have the result that my model can be restricted to produce the production side of the model of Humlum (2019) as follows.

Lemma B.2. *Suppose that (i) producers follow the binary decision rule in Definition B.2 and that (ii) occupation productivity A_o and robot task space a_o satisfy equations* (B.8) *and* (B.9) *for each o. Then $Q = Q^D$.*

B4. Equilibrium Characterization

To characterize the producer problem, I show the static optimization conditions and then the dynamic ones. To solve for the static problem, consider the FOCs of equation (10)

$$p_{i,t}^G \alpha_{i,L} \frac{Y_{i,t}^G}{Q_{i,t}^O} \left(b_{i,o,t} \frac{Q_{i,t}^O}{Q_{i,o,t}^O} \right)^{\frac{1}{\beta}} \left(\left(1 - a_{o,t} \right) \frac{Q_{i,o,t}^O}{L_{i,o,t}} \right)^{\frac{1}{\theta_o}} = w_{i,o,t}, \tag{B.10}$$

$$p_{i,t}^G \alpha_{i,M} \frac{Y_{i,t}^G}{M_{i,t}} \left(\frac{M_{i,t}}{M_{li,t}} \right)^{\frac{1}{\varepsilon}} = p_{li,t}^G, \tag{B.11}$$

and

$$p_{i,t}^G \alpha_{i,K} \frac{Y_{i,t}^G}{K_{i,t}} = r_{i,t}, \tag{B.12}$$

where $\alpha_{i,K} \equiv 1 - \alpha_{i,L} - \alpha_{i,M}$. Note also that by the envelope theorem,

$$\frac{\partial \pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}\right)}{\partial K^R_{i,o,t}} = p^G_{i,t}\frac{\partial Y_{i,t}}{\partial K^R_{i,o,t}} = p^G_{i,t}\left(\alpha_L \frac{Y^G_{i,t}}{Q^O_{i,t}}\left(b_{i,o,t}\frac{Q^O_{i,t}}{Q^O_{i,o,t}}\right)^{\frac{1}{\beta}}\left(a_{o,t}\frac{Q^O_{i,o,t}}{K^R_{i,o,t}}\right)^{\frac{1}{\theta}}\right). \tag{B.13}$$

To solve the dynamic problem, set up the (current-value) Lagrangian function of good producers

$$\mathcal{L}_{i,t} = \sum_{t=0}^{\infty}\left\{\left(\frac{1}{1+\iota}\right)^t\left[\pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}_o\right) - \sum_{l,o}\left(p^R_{li,o,t}\left(1+u_{li,t}\right)Q^R_{li,o,t} + P^G_{i,t}Q^I_{i,o,t} + \gamma P^R_{i,o,t}Q^R_{i,o,t}\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\right]\right\}.$$
$$- \lambda^R_{i,o,t}\left\{K^R_{i,o,t+1} - (1-\delta)K^R_{i,o,t} - Q^R_{i,o,t}\right\}$$

Taking the FOC with respect to $Q^R_{li,o,t}$, I have

$$p^R_{li,o,t}\left(1+u_{li,t}\right) + 2\gamma P^R_{i,o,t}\left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}} = \lambda^R_{i,o,t}\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}}. \tag{B.14}$$

Taking the FOC with respect to $Q^I_{i,o,t}$, I have

$$P^G_{i,t} + 2\gamma P^R_{i,o,t}\left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\frac{\partial Q^R_{i,o,t}}{\partial Q^I_{i,o,t}} = \lambda^R_{i,o,t}\frac{\partial Q^R_{i,o,t}}{\partial Q^I_{i,o,t}}, \tag{B.15}$$

Taking the FOC with respect to $K^R_{i,o,t+1}$, I have

$$\left(\frac{1}{1+\iota}\right)^{t+1}\left[\frac{\partial \pi_{i,t+1}\left(\left\{K^R_{i,o,t+1}\right\}_o\right)}{\partial K^R_{i,o,t+1}} + \gamma P^R_{i,o,t+1}\left(\frac{Q^R_{i,o,t+1}}{K^R_{i,o,t+1}}\right)^2 + (1-\delta)\lambda^R_{i,o,t+1}\right] - \left(\frac{1}{1+\iota}\right)^t\lambda^R_{i,o,t} = 0.$$

Rearranging, I have Euler equation (15).

The demand for non-robot good depends on bilateral intermediate good trade demand and total expenditure. Write $X^G_{j,t}$ the total purchase quantity (but not value) of good G in country j in period t. By equation (9), the bilateral trade demand is given by

$$p^G_{ij,t}Q^G_{ij,t} = \left(\frac{p^G_{ij,t}}{P^G_{j,t}}\right)^{1-\varepsilon}P^G_{j,t}X^G_{j,t}, \tag{B.16}$$

for any i, j, and t. In this equation, $P^G_{j,t}X^G_{j,t}$ is the total expenditures on non-robot goods. The total expenditure is the sum of final consumption $I_{j,t}$, payment to intermediate goods $\alpha_M p^G_{j,t}Y^G_{j,t}$, input to robot productions $\sum_o P^G_{j,t}Q^V_{j,o,t} = \sum_{o,k} p^R_{jk,o,t}Q^R_{jk,o,t}$, and payment to robot integration $\sum_o P^G_{j,t}Q^I_{j,o,t} = \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}$. Hence

$$P^G_{j,t}X^G_{j,t} = I_{j,t} + \alpha_M p^G_{j,t}Y^G_{j,t} + \sum_{o,k} p^R_{jk,o,t}Q^R_{jk,o,t} + \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}.$$

For country j and period t, by substituting into income $I_{j,t}$ the period cash flow of non-robot good producer that satisfies

$$\Pi_{j,t} \equiv \pi_{j,t}\left(\left\{K^R_{j,o,t}\right\}_o\right) - \sum_{i,o}\left(p^R_{ij,o,t}\left(1+u_{ij,t}\right)Q^R_{ij,o,t} + \sum_o P^G_{j,t}Q^I_{j,o,t} + \gamma P^R_{j,o,t}Q^R_{j,o,t}\left(\frac{Q^R_{j,o,t}}{K^R_{j,o,t}}\right)\right)$$

and robot tax revenue $T_{j,t} = \sum_{i,o} u_{ij,t}p^R_{ij,o,t}Q^R_{ij,o,t}$, I have

$$I_{j,t} = (1-\alpha_M)\sum_k p^G_{jk,t}Q^G_{jk,t} - \left(\sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t} + \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}\right), \tag{B.17}$$

or in terms of variables in the definition of equilibrium,

$$I_{j,t} = (1-\alpha_M)\sum_k p^G_{jk,t}Q^G_{jk,t} - \frac{1}{\alpha^R}\sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}.$$

Hence, the total expenditure measured in terms of the production side as opposed to income side is

$$P^G_{j,t}X^G_{j,t} = \sum_k p^G_{jk,t}Q^G_{jk,t} - \sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}\left(1+\gamma\frac{Q^R_{ij,o,t}}{K^R_{j,o,t}}\right). \tag{B.18}$$

Note that this equation embeds the balanced-trade condition. By substituting equation (B.18) into equation (B.16), I have

$$p^G_{ij,t}Q^G_{ij,t} = \left(\frac{p^G_{ij,t}}{P^G_{j,t}}\right)^{1-\varepsilon^G}\left(\sum_k p^G_{jk,t}Q^G_{jk,t} + \sum_{k,o} p^R_{jk,o,t}Q^R_{jk,o,t} - \sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}\right). \tag{B.19}$$

The good and robot-o market-clearing conditions are given by,

$$Y_{i,t}^{R} = \sum_{j} Q_{ij,t}^{G} \tau_{ij,t}^{G}, \tag{B.20}$$

for all i and t, and

$$p_{i,o,t}^{R} = \frac{P_{i,t}^{G}}{A_{i,o,t}^{R}} \tag{B.21}$$

for all i, o, and t, respectively.

Conditional on state variables $\boldsymbol{S}_t = \left\{\boldsymbol{K}_t^R, \boldsymbol{\lambda}_t^R, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$, equations (2), (B.10), (B.14), (B.19), (B.20), and (B.21) characterize the temporary equilibrium $\left\{\boldsymbol{p}_t^G, \boldsymbol{p}_t^R, \boldsymbol{w}_t, \boldsymbol{Q}_t^G, \boldsymbol{Q}_t^R, \boldsymbol{L}_t\right\}$. In addition, conditional on initial conditions $\left\{\boldsymbol{K}_0^R, \boldsymbol{L}_0\right\}$, equations (11), (15), and transversality condition (16) characterize the sequential equilibrium.

Finally, the steady state conditions are given by imposing the time-invariance condition to equations (11) and (15):

$$Q_{i,o}^{R} = \delta K_{i,o}^{R}, \tag{B.22}$$

$$\frac{\partial}{\partial K_{i,o}^{R}} \pi_i \left(\left\{K_{i,o}^{R}\right\}\right) = (\iota + \delta)\, \lambda_{i,o}^{R} - \sum_{l} \gamma p_{li,o}^{R} \left(\frac{Q_{li,o}^{R}}{K_{i,o}^{R}}\right)^2 \equiv c_{i,o}^{R}. \tag{B.23}$$

Note that equation (B.23) can be interpreted as the flow marginal profit of capital must be equalized to the marginal cost term. Thus I define the steady state marginal cost of robot capital $c_{i,o}^{R}$ from the right-hand side of equation (B.23). Note that if there is no adjustment cost $\gamma = 0$, the steady state Euler equation (B.23) implies

$$\frac{\partial}{\partial K_{i,o}^{R}} \pi_i \left(\left\{K_{i,o}^{R}\right\}\right) = c_{i,o}^{R} = (\iota + \delta)\, \lambda_{i,o}^{R},$$

which states that the marginal profit of capital is the user cost of robots in the steady state (Hall and Jorgenson, 1967).

B5. Proof of Proposition 1

The proof takes the following four conceptual steps. First, I will write the real wage change $(\widehat{w_{i,o}/P_i})$ in terms of the weighted average of relative price changes, making use of the fact that the sum of shares equals one. Second, I rewrite relative price change into layers of relative price changes with the technique of addition and subtraction. Third, I show that each layer of relative price changes is a change of relevant input or trade shares controlled by elasticity substitution. In other words, an input or trade shares reveals a layer of relative price changes. Finally, I make use of the fact that the sum of shares do not change after the shock to arrive at equation (23).

Cost minimization given production functions (6), (8), and (9) imply

$$\widehat{\left(\frac{w_{i,o}}{P_i}\right)} = \frac{1}{1-\alpha_{i,M}} \sum_l \tilde{x}_{li,t_0} \sum_{o'} \tilde{x}_{l,o',t_0} \left[\tilde{x}^L_{l,o',t_0} \left(\widehat{w_{i,o}} - \widehat{w_{l,o'}}\right) + \left(1 - \tilde{x}^L_{l,o',t_0}\right) \left(\widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta_o} + \widehat{c^R_{l,o'}} \right) \right) \right]. \tag{B.24}$$

Note that by additions and subtractions, I can rewrite

$$\begin{aligned} \widehat{w_{i,o}} - \widehat{w_{l,o'}} = {} & \left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right) - \left(\widehat{w_{l,o'}} - \widehat{p_{l,o'}}\right) + \left(\widehat{p_{i,o}} - \widehat{p^O_i}\right) - \left(\widehat{p_{l,o'}} - \widehat{p^O_l}\right) \\ & + \left(\widehat{p^O_i} - \widehat{p}_i\right) - \left(\widehat{p^O_l} - \widehat{p}_l\right) + \left(\widehat{p}_i - \widehat{P}_i\right) - \left(\widehat{p}_l - \widehat{P}_i\right) \end{aligned} \tag{B.25}$$

and

$$\begin{aligned} \widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta} + \widehat{c^R_{l,o'}} \right) = {} & \left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right) - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta} + \widehat{c^R_{l,o'}} - \widehat{p_{l,o'}} \right) + \left(\widehat{p_{i,o}} - \widehat{p^O_i}\right) - \left(\widehat{p_{l,o'}} - \widehat{p^O_l}\right) \\ & + \left(\widehat{p^O_i} - \widehat{p}_i\right) - \left(\widehat{p^O_l} - \widehat{p}_l\right) + \left(\widehat{p}_i - \widehat{P}_i\right) - \left(\widehat{p}_l - \widehat{P}_i\right). \end{aligned} \tag{B.26}$$

Note that the cost minimizing input and trade shares satisfy

$$\begin{cases} \widehat{\tilde{x}^L_{i,o}} = (1-\theta_o)\left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right), \; \widehat{1 - \tilde{x}^L_{i,o}} = \widehat{A^K_{i,o}} + (1-\theta_o)\left(\widehat{c^R_{i,o}} - \widehat{p_{i,o}}\right) \\ \widehat{\tilde{x}_{i,o}} = (1-\beta)\left(\widehat{p_{i,o}} - \widehat{p^O_i}\right), \; \widehat{\tilde{x}_{li}} = (1-\varepsilon)\left(\widehat{p}_l - \widehat{P}_i\right) \end{cases} \tag{B.27}$$

Combined with the Cobb-Douglas assumption of production function (6), equations (B.25), (B.26), and (B.27) imply

$$\widehat{w_{i,o}} - \widehat{w_{l,o'}} = \frac{\widehat{\tilde{x}^L_{i,o}}}{1-\theta_o} - \frac{\widehat{\tilde{x}^L_{l,o'}}}{1-\theta_o} + \frac{\widehat{\tilde{x}_{i,o}}}{1-\beta} - \frac{\widehat{\tilde{x}_{l,o'}}}{1-\beta} + \frac{\widehat{\tilde{x}_{ii}}}{1-\varepsilon} - \frac{\widehat{\tilde{x}_{li}}}{1-\varepsilon}$$

$$\widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta_o} + \widehat{c^R_{l,o'}} \right) = \frac{\widehat{\tilde{x}^L_{i,o}}}{1-\theta_o} - \frac{\left(\widehat{1-\tilde{x}^L_{l,o'}}\right)}{1-\theta_o} + \frac{\widehat{\tilde{x}_{i,o}}}{1-\beta} - \frac{\widehat{\tilde{x}_{l,o'}}}{1-\beta} + \frac{\widehat{\tilde{x}_{ii}}}{1-\varepsilon} - \frac{\widehat{\tilde{x}_{li}}}{1-\varepsilon}.$$

Substituting these in equation (B.24) and using the facts that $\tilde{x}^L_{i,o,t_0}\widehat{\tilde{x}^L_{i,o}} + \left(1-\tilde{x}^L_{i,o,t_0}\right)\left(\widehat{1-\tilde{x}^L_{i,o}}\right) = 0$ for all i and o, $\sum_o \tilde{x}_{i,o,t_0}\widehat{\tilde{x}_{i,o}} = 0$, and $\sum_l \tilde{x}_{li,t_0}\widehat{\tilde{x}_{li}} = 0$ for all i, I have equation (23).

B6. Proof of Proposition 2

I follow the arguments made in Sections 2 and 3 of Newey and McFadden (1994). The proof consists of four sub results in the following Lemma. Proposition 2 can be obtained as a combination of the four results. The formal statement requires the following additional assumptions.

Assumption B.1. *(i) A function of* $\widetilde{\boldsymbol{\Theta}}$*,* $\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}^R_2}\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right)\right] \neq 0$ *for any* $\widetilde{\boldsymbol{\Theta}} \neq \boldsymbol{\Theta}$*. (ii)* $\underline{\theta} \leq \theta_o \leq \overline{\theta}$ *for any* o*,* $\underline{\beta} \leq \beta \leq \overline{\beta}$*,* $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$*, and* $\underline{\phi} \leq \phi \leq \overline{\phi}$ *for some positive values* $\underline{\theta}, \underline{\beta}, \underline{\gamma}, \underline{\phi}, \overline{\theta}, \overline{\beta}, \overline{\gamma}, \overline{\phi}$*. (iii)* $\mathbb{E}\left[\sup_{\boldsymbol{\Theta}} \| H_o\left(\widehat{\boldsymbol{A}^R_2}\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right) \|\right] < \infty$*. (iv)* $\mathbb{E}\left[\| H_o\left(\widehat{\boldsymbol{A}^R_2}\right)\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right) \|^2\right] < \infty$ *(v)* $\mathbb{E}\left[\sup_{\boldsymbol{\Theta}} \| H_o\left(\widehat{\boldsymbol{A}^R_2}\right)\nabla_{\widetilde{\boldsymbol{\Theta}}}\nu_o\left(\widetilde{\boldsymbol{\Theta}}\right) \|\right] < \infty$*.*

Lemma B.3. *Assume Assumptions 1 and B.1(i)-(iii). (a) The estimator of the form (27) is consistent.*

Additionally, assume Assumtions B.1(iv)-(vi). (b) The estimator of the form (27) is asymptotically normal.

(c) $\sqrt{O}\left(\boldsymbol{\Theta}_{\boldsymbol{H}^*} - \boldsymbol{\Theta}\right) \to_d \mathcal{N}\left(0, \left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\right)$*, and the asymptotic variance is the minimum of that of the estimator of the form (27) for any function* $\boldsymbol{H}$*.*

Proof. (a) I follow Theorems 2.6 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(iv) of this theorem are satisfied. Assumption B.1(i) guarantees condition (i). Condition (ii) is implied by Assumption B.1(ii). Condition (iii) follows because all supply and demand functions in the model is continuous. Condition (iv) is implied by Assumption B.1(iii).

(b) I follow Theorem 3.4 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(v) of this theorem are satisfied. Condition (i) is satisfied by Assumption B.1(i). Condition (ii) follows because all supply and demand functions in the model is continuously differentiable. Condition (iii) is implied by Assumption 1 and Assumption B.1(iv). Assumption B.1(v) implies condition (iv). Finally, the gradient vectors of the structural residual is linear independent, guaranteeing the non-singularity of the variance matrix and condition (v).

(c) By Theorem 3.4 of Newey and McFadden (1994) and IV-generating function $\boldsymbol{H}$, the asymptotic variance of $\boldsymbol{\Theta}_{\boldsymbol{H}^*}$ is

$$\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1},$$

where $H_o\left(\widehat{\boldsymbol{A}_2^R}\right)=Z_o$, hence $\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$. To show that this is minimal, I will show that

$$\begin{aligned}\boldsymbol{\Delta}\equiv&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\\&-\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\end{aligned}$$

is positive semi-definite. In fact, note that

$$\begin{aligned}\boldsymbol{\Delta}=&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\times\\&\left\{\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]-\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right\}\times\\&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}.\end{aligned}$$

Define

$$\widetilde{\nu}_o=H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o-\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left((G_o)^\top\Omega_o^{-1}\nu_o\right)^{-1}\right]\mathbb{E}\left((G_o)^\top\Omega_o^{-1}\nu_o\right)^{-1}(G_o)^\top\Omega_o^{-1}\nu_o,$$

where $G_o\equiv\mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o(\boldsymbol{\Theta})\,|\widehat{\boldsymbol{A}_2^R}\right]$ and $\Omega_o\equiv\mathbb{E}\left[\nu_o(\boldsymbol{\Theta})(\nu_o(\boldsymbol{\Theta}))^\top|\widehat{\boldsymbol{A}_2^R}\right]$. Since any function of $\widehat{\boldsymbol{A}_2^R}$ gives the orthogonality with ν_o by Assumption 1, I have

$$\mathbb{E}\left[\widetilde{\nu}_o(\widetilde{\nu}_o)^\top\right]=\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]-\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right].$$

Since $\mathbb{E}\left[\widetilde{\nu}_o(\widetilde{\nu}_o)^\top\right]$ is positive semi-definite, $\boldsymbol{\Delta}$ is also positive semi-definite. □

B7. Proof of Proposition 3

I apply arguments in Section 6.1 of Newey and McFadden (1994). Namely, I define the joint estimator of the first-step and second-step estimator in Proposition 3 that falls into the class of general

GMM estimation, and discuss the asymptotic property using the general result about GMM estimation. In the proof, I modify the notation of the set of functions that yield optimal IV, $\boldsymbol{H}^*$, to clarify that it depends on parameters $\boldsymbol{\Theta}$ as follows:

$$H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}\right) = \mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)|\widehat{\boldsymbol{A}_2^R}\right]\mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\widehat{\boldsymbol{A}_2^R}\right]^{-1}.$$

Define the joint estimator as follows:

$$\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix} \equiv \arg\min_{\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1}\left[\sum_o e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right]^\top\left[\sum_o e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right],$$

where

$$e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right) \equiv \begin{pmatrix} H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\\ H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\end{pmatrix}.$$

Since for any $\boldsymbol{\Theta}$, IV-generating function $H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)$ gives the consistent estimator for $\boldsymbol{\Theta}$, I have $\boldsymbol{\Theta}_1 \to \boldsymbol{\Theta}$ and $\boldsymbol{\Theta}_2 \to \boldsymbol{\Theta}$. I also have the asymptotic variance

$$\mathrm{Var}\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix} = \left[\left(\widetilde{\boldsymbol{G}}\right)^\top\widetilde{\boldsymbol{\Omega}}\widetilde{\boldsymbol{G}}\right]^{-1},$$

where

$$\begin{aligned}\widetilde{\boldsymbol{G}} &\equiv \mathbb{E}\left[\nabla_{\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix}} e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right]\\ &= \mathbb{E}\begin{bmatrix} H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nabla\nu_o\left(\boldsymbol{\Theta}_2\right) & \nabla H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\\ \mathbf{0} & H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nabla\nu_o\left(\boldsymbol{\Theta}_1\right)\end{bmatrix}\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\boldsymbol{\Omega}} &\equiv \mathbb{E}\left[e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\left[e_o\left(\boldsymbol{\Theta}_2,\boldsymbol{\Theta}_1\right)\right]^\top\right] \\
&= \mathbb{E}\left[\begin{array}{cc}
H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right]^\top & H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\right]^\top \\
H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right]^\top & H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\right]^\top
\end{array}\right].
\end{aligned}
$$

Note that Assumption 1 implies that any function of $\widehat{\boldsymbol{A}_2^R}$ is orthogonal to ν_o, implying $\mathbb{E}\left[\nabla H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right] = 0$. Therefore, $\widetilde{\boldsymbol{G}}$ is a block-diagonal matrix and thus the marginal asymptotic distribution of $\boldsymbol{\Theta}_2$ is normal with variance $\mathrm{Var}\left(\boldsymbol{\Theta}_2\right) = \left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$, noting that $\boldsymbol{G} = \mathbb{E}\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}\right)\nabla\nu_o\left(\boldsymbol{\Theta}\right)\right]$ and $\boldsymbol{\Omega} \equiv \mathbb{E}\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}\right)\nu_o\left(\boldsymbol{\Theta}\right)\left(H_o^*\left(\widehat{\boldsymbol{A}_2^R};\boldsymbol{\Theta}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top\right]$.

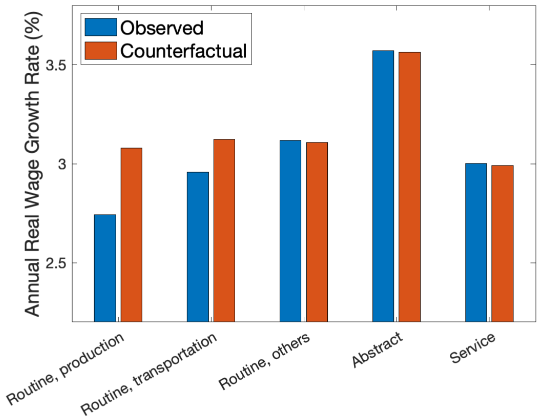
C Further Simulation Results

C1. Automation and Wages at Occupations

Figure C.1 shows the observed and counterfactual growth rate of real wages for each occupation, where the counterfactual change means the simulated change absent the automation shock. Figure C.1a shows the results aggregated at the 5 occupations groups defined in Section 4.1. I compute the counterfactual growth rate from the observed rate of the wage change, subtracted by the change predicted by the first-order steady-state solution $\overline{\boldsymbol{E}}$ and the observed automation shock $\widehat{\boldsymbol{a}^{\mathrm{obs}}}$. The result is based on the observed high growth rates of robots in routine production and transportation (material moving) occupations, and these occupations' high EoS estimates between robots and workers. In particular, at the 5-occupation aggregate level, most of the observed differences in the real wage growth rates in the three routine occupation groups are closed absent the automation shock. Applying the similar exercise for all occupations in my sample, Figure C.1b shows a more granular result, where occupations are sorted by the observed changes of wages from 1990-2007.

Figure C.1: The Steady-state Effect of Robots on Wages

(a) Occupation Groups



(b) All Occupations

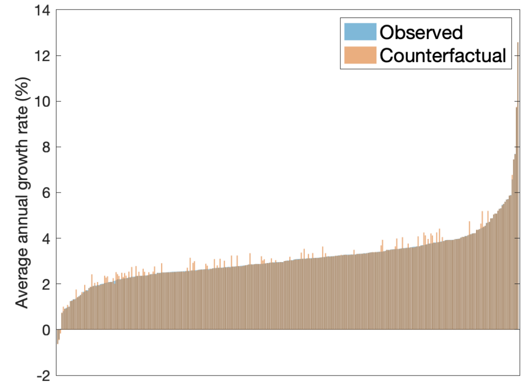
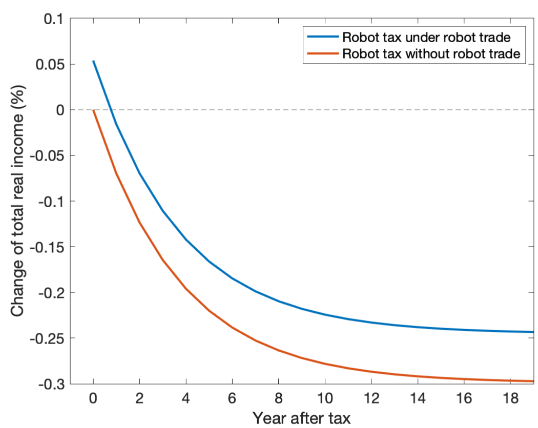


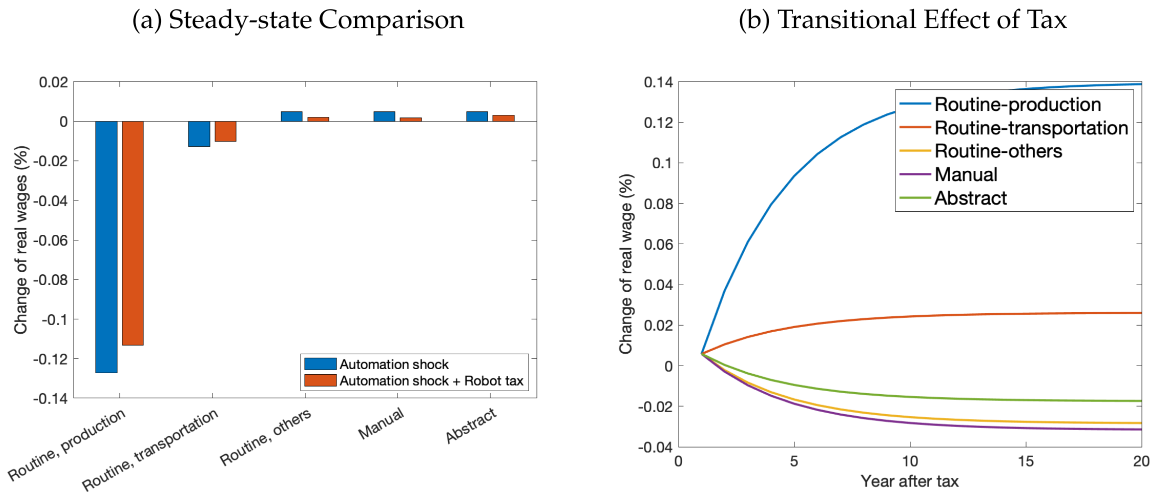
Figure C.2: Effects of the Robot Tax on the US Real Income



C2. The Role of Trade that Plays in the Robot Tax Effect

Figure C.2 shows the dynamic effect of the robot tax on the US real income. If the robot trade is not allowed, the robot tax does not increases the real income in any period since the terms-of-trade effect does not show up, but only the long-run capital decumulation effect does. On the other hand, once I allow the robot trade as observed in the data, the robot tax may increase the real income because it decreases the price of imported robots. The effect is concentrated in the short-run before the capital decumulation process matures. In the long run, the negative decumulation effect dominates the positive terms-of-trade effect.

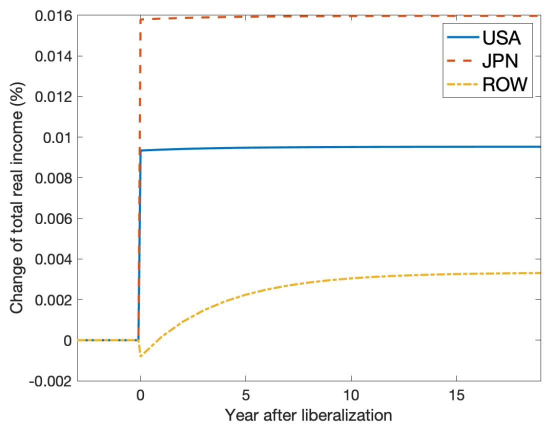
Figure C.3: Effects of the Robot Tax on Occupational Real Wages



C3. The Occupational Real Wage Effect of the Robot Tax

In Figure C.3a, I show two scenarios of the steady-state changes in occupational real wages. On the one hand, I shock the economy only with the automation shocks. On the other hand, I shock the economy with both the automation shocks and the robot tax. The result shows heterogenous effects on occupational real wages of the robot tax. The tax mitigates the negative effect of automation on routine production workers and routine transportation workers, while the tax marginally decreases the small gains that workers in the other occupations would have enjoyed. Overall, the robot tax mitigates the large heterogeneous effects of the automation shocks, that could go negative and positive directions depending on occupation groups, and compresses the effects towards zero. Figure C.3b shows the dynamics of the effects of robot tax, net of the effects of automation shocks. Although the steady-state effects of robot tax were heterogeneous as shown in Figure C.3a, the effect is not immediate but materializes after around 10 years, due to the sluggish adjustment in the accumulation of the robot capital stock. Overall, I find that since the robot tax slows down the adoption of robots, it rolls back the real wage effect of automation–workers in occupations that experienced significant automation shocks (e.g., production and transportation in the routine occupation groups) benefit from the tax, while the others lose.

Figure C.4: The Effect of Robot Trade Cost Reduction



C4. Trade Liberalization of Robots

Following Ravikumar et al. (2019), I consider unexpected and permanent 20% reduction in the bilateral trade costs to study the effect trade liberalization of capital good and dynamics gains from trade. Figure C.4 shows the result of such a simulation for a 20-years time horizon. All country groups in the model gain from the trade liberalization. The US gain materialize almost immediately after the trade cost change. A possible explanation is the combination of the following two observation. First, it takes time to accumulate robots after the trade liberalization, which makes the gains from trade liberalization sluggish. Second, by exporting robots to ROW, the US increases the revenue of robot sales immediately after the trade cost drop, improving the short-run real income gain. The real income gain is the largest for Japan, a large net robot exporter. It is noteworthy that ROW loses from the reduction in the robot trade cost, possibly due to the terms-of-trade deterioration in the short-run.

D Detail of the GE Solution

I discuss the derivation log-linearization in equations (17), (19), and (20), so that I can bring the theory with computation. Throughout the section, relational operator $\circ$ is Hadamard product, $\oslash$ indicates Hadamard division, and $\otimes$ means Kronecker product.

It is useful to show that the CES production structure implies the share-weighted log-change expression for both prices and quantities. Namely, I have a formula for the change in destination price index $\widehat{P_{j,t}} = \sum_i x_{ij,t_0} \widehat{p_{ij,t}}$ and one for the change in destination expenditure $\widehat{P_{j,t}} + \widehat{Q_{j,t}} = \sum_i x_{ij,t_0} \left(\widehat{p_{ij,t}} + \widehat{Q_{ij,t}} \right)$. These imply that

$$\widehat{Q_{j,t}} = \sum_i x_{ij,t_0} \widehat{Q_{ij,t}},$$

or the changes of quantity aggregate $\widehat{Q_{j,t}}$ are also share-weighted average of changes of origin quantity $\widehat{Q_{ij,t}}$.

By log-linearizing equation (B.20) for any i ,

$$\begin{aligned}
&-\alpha_M \widehat{p_{i,t}^G} + \alpha_M \sum_l \widetilde{x}_{li,t_0}^G \widehat{p_{l,t}^G} + (1-\alpha_M) \sum_j \widetilde{y}_{ij,t_0}^G \widehat{Q_{ij,t}^G} - \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O l_{i,o,t_0}^O \widehat{L_{i,o,t}} \\
&= \frac{\alpha_L}{\theta - 1} \sum_o \frac{\widetilde{x}_{i,o,t_0}^O}{1 - a_{o,t_0}} \left(-a_{o,t_0} l_{i,o,t_0}^O + (1 - a_{o,t_0}) \left(1 - l_{i,o,t_0}^O \right) \right) \widehat{a_{o,t}} + \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O \frac{1}{\beta - 1} \widehat{b_{i,o,t}} \\
&\quad + \widehat{A_{i,t}^G} + (1 - \alpha_L - \alpha_M) \widehat{K_{i,t}} - \alpha_M \sum_l \widetilde{x}_{li,t_0}^G \widehat{\tau_{li,t}^G} - (1-\alpha_M) \sum_j \widetilde{y}_{ij,t_0}^G \widehat{\tau_{ij,t}^G} + \alpha_L \sum_o \widetilde{x}_{i,o,t_0}^O \left(1 - l_{i,o,t_0}^O \right) \widehat{K_{i,o,t}^R},
\end{aligned}$$

To write a matrix notation, write

$$\overline{\boldsymbol{M}^{yG}} \equiv \begin{bmatrix} \left[\widetilde{y}_{11,t_0}^G, \dots, \widetilde{y}_{1N,t_0}^G \right] & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left[\widetilde{y}_{N1,t_0}^G, \dots, \widetilde{y}_{NN,t_0}^G \right] \end{bmatrix}$$

a $N \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xOl}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{1,\cdot,t_0} \right)^\top & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{N,\cdot,t_0} \right)^\top \end{bmatrix}$$

a $N \times NO$ matrix where

$$\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \equiv \left(\widetilde{x}_{1,o,t_0}^O \right)_o \text{ and } \widetilde{\boldsymbol{l}}_{1,\cdot,t_0} \equiv \left(l_{1,o,t_0}^O \right)_o \tag{D.1}$$

are $O \times 1$ vectors, $\overline{\boldsymbol{M}^{al}}$ as a matrix with its element

$$M_{i,o}^{al} = \frac{-a_{o,t_0} l_{i,o,t_0}^O + (1 - a_{o,t_0}) \left(1 - l_{i,o,t_0}^O \right)}{1 - a_{o,t_0}},$$

and a $N \times O$ matrix,

$$\overline{\boldsymbol{M}^{xO}} \equiv \begin{bmatrix} \left[\widetilde{x}^O_{1,1,t_0}, \ldots, \widetilde{x}^O_{1,O,t_0}\right] & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left[\widetilde{x}^O_{N,1,t_0}, \ldots, \widetilde{x}^O_{N,O,t_0}\right] \end{bmatrix},$$

a $N \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xG}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}^G_{1\cdot,t_0}\right) & \ldots & \operatorname{diag}\left(\widetilde{x}^G_{N\cdot,t_0}\right) \end{bmatrix},$$

a $N \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xOl,2}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \left(\boldsymbol{1}_O - \widetilde{\boldsymbol{l}}_{1,\cdot,t_0}\right)\right)^\top & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \left(\boldsymbol{1}_O - \widetilde{\boldsymbol{l}}_{N,\cdot,t_0}\right)\right)^\top \end{bmatrix},$$

a $N \times NO$ matrix where $\widetilde{\boldsymbol{x}}_{1,\cdot,t_0}$ and $\widetilde{\boldsymbol{l}}_{1,\cdot,t_0}$ are defined in equation (D.1). Then I have

$$\begin{aligned} & -\alpha_M \left(\overline{\boldsymbol{I}} - \left(\overline{\widetilde{\boldsymbol{x}}^G_{t_0}}\right)^\top\right) \widehat{\boldsymbol{p}^G_t} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\widehat{\boldsymbol{Q}^G_t} - \alpha_L \overline{\boldsymbol{M}^{xOl}}\widehat{\boldsymbol{L}}_t \\ &= \frac{\alpha_L}{\theta-1}\left(\widetilde{\boldsymbol{x}}^O_{t_0} \circ \overline{\boldsymbol{M}^{al}}\right)\widehat{\boldsymbol{a}}_t + \frac{\alpha_L}{\beta-1}\overline{\boldsymbol{M}^{xO}}\widehat{\boldsymbol{b}}_t + \widehat{\boldsymbol{A}^G_t} + (1-\alpha_L-\alpha_M)\,\widehat{\boldsymbol{K}_t} \\ &\quad - \left[\alpha_M \overline{\boldsymbol{M}^{xG}} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\right]\widehat{\boldsymbol{\tau}^G_t} + \alpha_L \overline{\boldsymbol{M}^{xOl,2}}\widehat{\boldsymbol{K}^R_t}, \end{aligned}$$

By log-linearizing equation (B.21) for any i and o,

$$\widehat{p^R_{i,o,t}} = \widehat{P^G_{i,t}} - \widehat{A^R_{i,o,t}}$$

$$-\sum_l \widetilde{x}^G_{li,t_0}\widehat{p^G_{l,t}} + \widehat{p^R_{i,o,t}} = -\widehat{A^R_{i,o,t}} + \sum_l \widetilde{x}^G_{li,t_0}\widehat{\tau^G_{li,t}}.$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{xG,2}} \equiv \begin{bmatrix} \boldsymbol{1}_O\left[\widetilde{x}^G_{11,t_0}, \ldots, \widetilde{x}^G_{N1,t_0}\right] \\ \vdots \\ \boldsymbol{1}_O\left[\widetilde{x}^G_{1N,t_0}, \ldots, \widetilde{x}^G_{NN,t_0}\right] \end{bmatrix}$$

a $NO \times N$ matrix, and

$$\overline{\boldsymbol{M}^{xG,3}} \equiv \begin{bmatrix} \widetilde{x}^G_{11,t_0} & \cdots & \widetilde{x}^G_{N1,t_0} & & & \mathbf{0} & \\ & & & \ddots & & & \\ & \mathbf{0} & & & \widetilde{x}^G_{1N,t_0} & \cdots & \widetilde{x}^G_{NN,t_0} \end{bmatrix} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix. Then I have

$$-\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G_t} + \widehat{\boldsymbol{p}^R_t} = -\widehat{\boldsymbol{A}^R_t} + \overline{\boldsymbol{M}^{xG,3}}\widehat{\boldsymbol{\tau}^G_t}.$$

By log-linearizing equations (2), (3), and (4) for any i, o, and o', I have

$$\widehat{\mu_{i,oo',t}} = \phi\left(-d\chi_{i,oo',t} + \frac{1}{1+\iota}\widehat{V_{i,o',t+1}}\right) - \sum_{o''}\mu_{i,oo'',t_0}\left(-d\chi_{i,oo'',t} + \frac{1}{1+\iota}\widehat{V_{i,o'',t+1}}\right), \tag{D.2}$$

$$\widehat{V_{i,o,t+1}} = \widehat{w_{i,o,t+1}} + dT_{i,o,t+1} - \widehat{P_{i,t+1}} + \sum_{o'}\mu_{i,oo',t_0}\left(-d\chi_{i,oo',t+1} + \frac{1}{1+\iota}\widehat{V_{i,o',t+2}}\right), \tag{D.3}$$

and

$$\widehat{L_{i,o,t+1}} = \sum_{o'}\frac{L_{i,o',t_0}}{L_{i,o,t_0}}\mu_{i,o'o,t_0}\left(\widehat{\mu_{i,o'o,t}} + \widehat{L_{i,o',t}}\right). \tag{D.4}$$

In matrix notation, by equation (D.2),

$$\widehat{\boldsymbol{\mu}^{\text{vec}}_t} = -\phi\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)d\boldsymbol{\chi}^{\text{vec}}_t + \frac{\phi}{1+\iota}\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)\left(\overline{\boldsymbol{I}_{NO}} \otimes \mathbf{1}_O\right)\widehat{\boldsymbol{V}_{t+1}}.$$

where

$$\overline{\boldsymbol{M}^{\mu}} \equiv \overline{\boldsymbol{M}^{\mu,3}} \otimes \mathbf{1}_O,$$

$$\overline{\boldsymbol{M}^{\mu,3}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{i,1\cdot,t_0}\right)^\top & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & \left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right)^\top & & & & \\ & & & \ddots & & & \\ & & & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^\top & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & \left(\boldsymbol{\mu}_{i,O\cdot 1,t_0}\right)^\top \end{bmatrix},$$

$$d\chi_t^{\text{vec}} \equiv \begin{bmatrix} d\chi_{1,1\cdot,t} & \dots & d\chi_{1,O\cdot,t} & \dots & d\chi_{N,1\cdot,t} & \dots & d\chi_{N,O\cdot,t} \end{bmatrix}^\top,$$

and

$$\boldsymbol{\mu}_{i,o\cdot,t_0} \equiv (\mu_{i,oo',t_0})_{o'} \text{ and } d\boldsymbol{\chi}_{1,o\cdot,t} \equiv (d\chi_{1,oo',t})_{o'} \tag{D.5}$$

are $O \times 1$ vectors. By equation (D.3),

$$\frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\check{\boldsymbol{V}}_{t+2} = \overline{\boldsymbol{M}^{xG,2}}\boldsymbol{p}^{\check{G}}_{t+1} - \check{\boldsymbol{w}}_{t+1} + \check{\boldsymbol{V}}_{t+1}.$$

where

$$\overline{\boldsymbol{M}^{\mu,2}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right)^\top & & & \\ \vdots & & \mathbf{0} & \\ \left(\boldsymbol{\mu}_{1,O\cdot,t_0}\right)^\top & & & \\ & \ddots & & \\ & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^\top & \\ \mathbf{0} & & \vdots & \\ & & \left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right)^\top & \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (D.5) for any i and o. By equation (D.3),

$$\check{\boldsymbol{L}}_{t+1} = \overline{\boldsymbol{M}^{\mu L,2}}\check{\boldsymbol{\mu}}_t^{\text{vec}} + \overline{\boldsymbol{M}^{\mu L}}\check{\boldsymbol{L}}_t$$

where $\overline{\boldsymbol{M}^{\mu L}}$ being the $NO \times NO$ matrix

$$\overline{\boldsymbol{M}^{\mu L}} = \overline{\boldsymbol{M}^{\mu,2}} \circ \left(\begin{bmatrix} (\boldsymbol{L}_{1,\cdot,t_0})^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & (\boldsymbol{L}_{N,\cdot,t_0})^\top \end{bmatrix} \otimes \mathbf{1}_O \right) \oslash \left(\begin{bmatrix} \boldsymbol{L}_{1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{L}_{N,\cdot,t_0} \end{bmatrix} \otimes (\mathbf{1}_O)^\top \right)$$

and $\overline{\boldsymbol{M}^{\mu L,2}}$ being the $NO \times NO^2$ matrix

$$\overline{\boldsymbol{M}^{\mu L,2}} = \overline{\boldsymbol{M}^{\mu,4}} \circ \left(\begin{bmatrix} \left(\boldsymbol{L}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\boldsymbol{L}_{N,\cdot,t_0}\right)^\top \end{bmatrix} \otimes \overline{\boldsymbol{I}_O} \right) \oslash$$
$$\begin{pmatrix} \left(\mathbf{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{1,o,t_0}\right) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\mathbf{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{N,o,t_0}\right) \end{pmatrix},$$

where

$$\overline{\boldsymbol{M}^{\mu,4}} \equiv \begin{bmatrix} \operatorname{diag}\left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right) & & & & \mathbf{0} \\ & & & \ddots & & & \\ & \mathbf{0} & & & \operatorname{diag}\left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right) \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (D.5) for any i and o.

By log-linearizing equation (B.19) for each i and j,

$$\widehat{Q^G_{ij,t}} = -\varepsilon^G \widehat{p^G_{ij,t}} - \left(1-\varepsilon^G\right) \widehat{P^G_{j,t}} + \left[s^G_{j,t_0} \sum_k \widehat{p^G_{jk,t} Q^G_{jk,t}} + s^V_{j,t_0} \sum_{i,o} \widehat{p^R_{ij,o,t} Q^R_{ij,o,t}} + s^R_{j,t_0} \sum_{o,k} \widehat{p^R_{jk,o,t} Q^R_{jk,o,t}} \right]$$

where

$$s^G_{j,t_0} \equiv \frac{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}$$

is the baseline share of non-robot good production in income,

$$s^R_{j,t_0} \equiv \frac{\sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the baseline share of robot production, and

$$s^V_{j,t_0} \equiv -\frac{\sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the (negative) baseline absorption share of robots. Thus

$$\left[\varepsilon^{G}\widehat{p_{i,t}^{G}}+\left(1-\varepsilon^{G}\right)\sum_{l}\widetilde{x}_{lj,t_0}^{G}\widehat{p_{l,t}^{G}}-s_{j,t_0}^{G}\widehat{p_{j,t}^{G}}\right]-\left[s_{j,t_0}^{V}\sum_{l,o}\widetilde{x}_{lj,o,t_0}^{R}\widetilde{x}_{j,o,t_0}^{R}\widehat{p_{l,o,t}^{R}}+s_{t_0}^{R}\sum_{o}\widetilde{y}_{j,o,t_0}^{R}\widehat{p_{j,o,t}^{R}}\right]$$
$$+\left(\widehat{Q_{ij,t}^{G}}-s_{j,t_0}^{G}\sum_{k}\widetilde{y}_{jk,t_0}^{G}\widehat{Q_{jk,t}^{G}}\right)-\left(s_{j,t_0}^{V}\sum_{l,o}\widetilde{x}_{lj,o,t_0}^{R}\widetilde{x}_{j,o,t_0}^{R}\widehat{Q_{lj,o,t}^{R}}+s_{j,t_0}^{R}\sum_{k,o}\widetilde{y}_{jk,o,t_0}^{R}\widetilde{y}_{j,o,t_0}^{R}\widehat{Q_{jk,o,t}^{R}}\right)$$
$$=-\left[\varepsilon^{G}\widehat{\tau_{ij,t}^{G}}+\left(1-\varepsilon^{G}\right)\sum_{l}\widetilde{x}_{lj,t_0}^{G}\widehat{\tau_{lj,t}^{G}}-s_{j,t_0}^{G}\sum_{k}\widetilde{y}_{jk,t_0}^{G}\widehat{\tau_{jk,t}^{G}}\right]+\left[s_{j,t_0}^{V}\sum_{l,o}\widetilde{x}_{lj,t_0}^{R}\widehat{\tau_{lj,t}^{R}}+s_{j,t_0}^{R}\sum_{k,o}\widetilde{y}_{jk,t_0}^{R}\widehat{\tau_{jk,t}^{R}}\right]$$

where

$$\widetilde{x}_{ij,o,t_0}^{R}\equiv\frac{p_{ij,o,t_0}^{R}Q_{ij,o,t_0}^{R}}{P_{j,o,t_0}^{R}Q_{j,o,t_0}^{R}},\widetilde{x}_{j,o,t_0}^{R}\equiv\frac{P_{j,o,t_0}^{R}Q_{j,o,t_0}^{R}}{P_{j,t_0}^{R}Q_{j,t_0}^{R}},\widetilde{x}_{ij,t_0}^{R}\equiv\frac{\sum_o p_{ij,o,t_0}^{R}Q_{ij,o,t_0}^{R}}{P_{j,t_0}^{R}Q_{j,t_0}^{R}},$$

$$\widetilde{y}_{ij,o,t_0}^{R}\equiv\frac{p_{ij,o,t_0}^{R}Q_{ij,o,t_0}^{R}}{\sum_k p_{ik,o,t_0}^{R}Q_{ik,o,t_0}^{R}},\widetilde{y}_{i,o,t_0}^{R}\equiv\frac{\sum_k p_{ik,o,t_0}^{R}Q_{ik,o,t_0}^{R}}{\sum_{k,o'} p_{ik,o',t_0}^{R}Q_{ik,o',t_0}^{R}},\widetilde{y}_{ij,t_0}^{R}\equiv\frac{\sum_o p_{ij,o,t_0}^{R}Q_{ij,o,t_0}^{R}}{\sum_{k,o} p_{ik,o,t_0}^{R}Q_{ik,o,t_0}^{R}}.$$

In matrix notation, define

$$\overline{\boldsymbol{M}^{xR}}\equiv\mathbf{1}_N\otimes\left[\begin{array}{ccc}\widetilde{\boldsymbol{x}}_{t_0}^{R}\circ\widetilde{\boldsymbol{x}}_{\cdot 1,\cdot,t_0}^{R} & \dots & \widetilde{\boldsymbol{x}}_{t_0}^{R}\circ\widetilde{\boldsymbol{x}}_{\cdot N,\cdot,t_0}^{R}\end{array}\right],$$

a $N^2\times NO$ matrix,

$$\overline{\boldsymbol{M}^{yR}}\equiv\mathbf{1}_N\otimes\left[\begin{array}{cccccc}\widetilde{y}_{1,1}^{R} & \dots & \widetilde{y}_{1,O}^{R} & & \mathbf{0} & \\ & & & \ddots & & \\ & \mathbf{0} & & \widetilde{y}_{N,1}^{R} & \dots & \widetilde{y}_{N,O}^{R}\end{array}\right],$$

a $N^2\times NO$ matrix,

$$\overline{\boldsymbol{M}^{yG,2}}\equiv\mathbf{1}_N\otimes\overline{\boldsymbol{M}^{yG}}.$$

a $N^2\times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,2}}\equiv\mathbf{1}_N\otimes\left[\begin{array}{ccccc}\left[\widetilde{x}_{1,o,t_0}^{R}\widetilde{x}_{11,o,t_0}^{R}\right]_o & \mathbf{0} & & \left[\widetilde{x}_{1,o,t_0}^{R}\widetilde{x}_{N1,o,t_0}^{R}\right]_o & \mathbf{0}\\ & \ddots & \dots & & \ddots\\ \mathbf{0} & \left[\widetilde{x}_{N,o,t_0}^{R}\widetilde{x}_{1N,o,t_0}^{R}\right]_o & & \mathbf{0} & \left[\widetilde{x}_{N,o,t_0}^{R}\widetilde{x}_{NN,o,t_0}^{R}\right]_o\end{array}\right]$$

a $N^2 \times N^2 O$ matrix ,

$$\overline{\boldsymbol{M}^{yR,2}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{11,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{1N,o,t_0}\right]_o & & & \mathbf{0} \\ & & & \ddots & & \\ & \mathbf{0} & & & \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{N1,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{NN,o,t_0}\right]_o \end{bmatrix}$$

a $N^2 \times N^2 O$ matrix,

$$\overline{\boldsymbol{M}^{xG,4}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xG}}$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,3}} \equiv \mathbf{1}_N \otimes \left[\begin{array}{ccc} \operatorname{diag}\left(\widetilde{x}^R_{1\cdot,t_0}\right) & \ldots & \operatorname{diag}\left(\widetilde{x}^R_{N\cdot,t_0}\right) \end{array} \right]$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{yR,3}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{11,t_0}, \ldots, \widetilde{y}^R_{1N,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{y}^R_{N1,t_0}, \ldots, \widetilde{y}^R_{NN,t_0}\right] \end{bmatrix}$$

a $N^2 \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xO,2}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xO}},$$

a $N^2 \times NO$ matrix. Then I have

$$\begin{aligned}
&\left(\varepsilon^G \left[\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_N\right] + \left(1-\varepsilon^G\right)\left[\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}^G_{t_0}\right)^\top\right] - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\left[\mathbf{1}_N \otimes \overline{\boldsymbol{I}_N}\right]\right)\widehat{\boldsymbol{p}^G_t} \\
&- \left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR}}\right)\widehat{\boldsymbol{p}^R_t} \\
&+ \left(\overline{\boldsymbol{I}_{N^2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{Q}^G_t} - \left[\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,2}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,2}}\right]\widehat{\boldsymbol{Q}^R_t} \\
&= -\left(\varepsilon^G + \left(1-\varepsilon^G\right)\overline{\boldsymbol{M}^{xG,4}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{\tau}^G_t} \\
&\qquad + \left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,3}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,3}}\right)\widehat{\boldsymbol{\tau}^R_t}
\end{aligned}$$

By log-linearizing equation (B.14) for each i, j, and o,,

$$
\begin{aligned}
&\left(1-\alpha^{R}\right) \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} \sum_{l} \widetilde{x}^{G}_{lj,t_0} \widehat{p^{G}_{l,t}} + \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} \widehat{p^{R}_{i,o,t}} \\
&+ \left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta} - \left(1-\alpha^{R}\right) \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right] \sum_{l} \widetilde{x}^{R}_{lj,o,t_0} \widehat{p^{R}_{l,o,t}} \\
&+ \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} \frac{1}{\varepsilon^{R}} \widehat{Q^{R}_{ij,o,t}} + \left[-\frac{1}{\varepsilon^{R}} \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} + \frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}\right] \sum_{l} \widetilde{x}^{R}_{lj,o,t_0} \widehat{Q^{R}_{lj,o,t}} \\
&= -\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} du_{ij,t} - \left(1-\alpha^{R}\right) \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} \sum_{l} \widetilde{x}^{G}_{lj,t_0} \widehat{\tau^{G}_{lj,t}} - \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta} \widehat{\tau^{R}_{ij,t}} \\
&- \left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta} - \left(1-\alpha^{R}\right) \frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right] \sum_{l} \widetilde{x}^{R}_{lj,o,t_0} \widehat{\tau^{R}_{lj,t}} + \widehat{\lambda^{R}_{j,o,t}} + \frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta} \widehat{K^{R}_{j,o,t}}.
\end{aligned}
$$

In matrix notation, write a preliminary $N \times N$ matrix $\overline{\widetilde{\boldsymbol{u}_{t_0}}}$ as such that the (i, j)-element is

$$
\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}.
$$

Then $\mathbf{1}_N \left(\mathbf{1}_N\right)^{\top} - \overline{\widetilde{\boldsymbol{u}_{t_0}}}$ is a matrix that is filled with $2\gamma\delta / \left(1+u_{ij,t_0}+2\gamma\delta\right)$ for its (i, j) element and

$$
\overline{\boldsymbol{M}^{u}} \equiv \operatorname{diag}\left(\left[\widetilde{u_{1\cdot,t_0}}, \ldots, \widetilde{u_{N\cdot,t_0}}\right]^{\top}\right).
$$

Using these, write

$$
\overline{\boldsymbol{M}^{xG,5}} \equiv \left(\overline{\boldsymbol{M}^{u}} \otimes \overline{\boldsymbol{I}_O}\right) \left(\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top} \otimes \mathbf{1}_O\right)
$$

a $N^2O \times N$ matrix,

$$
\overline{\boldsymbol{M}^{u,2}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{u}_{1\cdot,t_0}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{u}_{N\cdot,t_0}} \end{bmatrix} \otimes \overline{\boldsymbol{I}_O},
$$

a $N^2O \times NO$ matrix where $\widetilde{\boldsymbol{u}_{i\cdot,t_0}} \equiv \left(\widetilde{u_{i\cdot,t_0}}\right)_j$ is a $N \times 1$ vector,

$$\overline{\boldsymbol{M}^{xR,4}} \equiv \left\{ \left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) - \left(1-\alpha^R\right) \overline{\boldsymbol{M}^u} \right] \otimes \overline{\boldsymbol{I}_O} \right\} \times . \\ \left(\mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\left\{ \widetilde{x}^R_{11,o,t_0} \right\}_o \right) & \ldots & \operatorname{diag}\left(\left\{ \widetilde{x}^R_{N1,o,t_0} \right\}_o \right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\left\{ \widetilde{x}^R_{1N,o,t_0} \right\}_o \right) & \ldots & \operatorname{diag}\left(\left\{ \widetilde{x}^R_{NN,o,t_0} \right\}_o \right) \end{bmatrix} \right)$$

a $N^2O \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xR,5}} \equiv \left\{ \left[-\frac{1}{\varepsilon^R} \overline{\boldsymbol{M}^u} + \left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) \right] \otimes \overline{\boldsymbol{I}_O} \right\} \times \\ \left\{ \mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^R_{11,1,t_0} \\ \vdots \\ \widetilde{x}^R_{11,O,t_0} \\ \vdots \\ \widetilde{x}^R_{1N,O,t_0} \end{bmatrix} \right) & \ldots & \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^R_{N1,1,t_0} \\ \vdots \\ \widetilde{x}^R_{N1,O,t_0} \\ \vdots \\ \widetilde{x}^R_{NN,O,t_0} \end{bmatrix} \right) \end{bmatrix} \right\}$$

a $N^2O \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xG,6}} \equiv \left(\overline{\boldsymbol{M}^u} \otimes \overline{\boldsymbol{I}_O} \right) \left\{ \mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^G_{11,t_0} \\ \vdots \\ \widetilde{x}^G_{1N,t_0} \end{bmatrix} \right) & \ldots & \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^G_{N1,t_0} \\ \vdots \\ \widetilde{x}^G_{NN,t_0} \end{bmatrix} \right) \end{bmatrix} \otimes \mathbf{1}_O \right\}$$

a $N^2O \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,6}} \equiv \left\{ \left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) - \left(1-\alpha^R\right) \overline{\boldsymbol{M}^u} \right] \otimes \overline{\boldsymbol{I}_O} \right\} \\ \times \left\{ \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{x}^R_{11,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} & \ldots & \left[\widetilde{x}^R_{N1,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{1N,o,t_0} \right]_o & & \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{N3,o,t_0} \right]_o \end{bmatrix} \right\}$$

a $N^2O \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{u,3}} \equiv \begin{bmatrix} 1-\widetilde{\widehat{u_{11,t_0}}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1-\widetilde{\widehat{u_{1N,t_0}}} \\ & \vdots & \\ 1-\widetilde{\widehat{u_{N1,t_0}}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1-\widetilde{\widehat{u_{NN,t_0}}} \end{bmatrix} \otimes \overline{\boldsymbol{I}_O}$$

a $N^2O \times NO$ matrix. Finally, I have

$$\begin{aligned}
&\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,5}}\widehat{\boldsymbol{p}_t^G} + \left[\overline{\boldsymbol{M}^{u,2}} + \overline{\boldsymbol{M}^{xR,4}}\right]\widehat{\boldsymbol{p}_t^R} + \left\{\frac{1}{\varepsilon^R}\left(\overline{\boldsymbol{M}^u}\otimes\overline{\boldsymbol{I}_O}\right) + \overline{\boldsymbol{M}^{xR,5}}\right\}\widehat{\boldsymbol{Q}_t^R} \\
&= -\left(\overline{\boldsymbol{M}^u}\otimes\mathbf{1}_O\right)d\boldsymbol{u}_t - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,6}}\widehat{\boldsymbol{\tau}_t^G} - \left[\left(\overline{\boldsymbol{M}^u}\otimes\mathbf{1}_O\right) + \overline{\boldsymbol{M}^{xR,6}}\right]\widehat{\boldsymbol{\tau}_t^R} + \left(\mathbf{1}_N\otimes\overline{\boldsymbol{I}_{NO}}\right)\widehat{\boldsymbol{\lambda}_t^R} + \overline{\boldsymbol{M}^{u,3}}\widehat{\boldsymbol{K}_t^R}.
\end{aligned}$$

By log-linearizing equation and (B.10) for each i and o,

$$\begin{aligned}
&\widehat{p_{i,t}^G} + \sum_j \widetilde{y}_{ij,t_0}^G \widehat{Q_{ij,t}^G} - \widehat{w_{i,o,t}} + \left[-\frac{1}{\theta} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l_{i,o,t_0}^O\right]\widehat{L_{i,o,t}} + \left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}_{i,o',t_0}^O l_{i,o',t_0}^O \widehat{L_{i,o',t}} \\
&= -\frac{1}{\beta}\widehat{b_{i,o,t}} + \frac{1}{\theta}\frac{a_{o,t_0}}{1-a_{o,t_0}}\widehat{a_{o,t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\left[-\left(1-l_{i,o,t_0}^O\right) + l_{i,o,t_0}^O\frac{a_{o,t_0}}{1-a_{o,t_0}}\right]\widehat{a_{o,t}} \\
&\quad + \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'}\widetilde{x}_{i,o',t_0}^O\left[-\left(1-l_{i,o',t_0}^O\right) + l_{i,o',t_0}^O\frac{a_{o',t_0}}{1-a_{o',t_0}}\right]\widehat{a_{o',t}} \\
&\quad - \sum_j y_{ij,t_0}^G\widehat{\tau_{ij,t}^G} - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l_{i,o,t_0}^O\right)\widehat{K_{i,o,t}^R} - \left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}_{i,o',t_0}^O\left(1-l_{i,o',t_0}^O\right)\widehat{K_{i,o',t}^R},
\end{aligned}$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{yG,3}} \equiv \overline{\boldsymbol{M}^{yG}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xOl,3}} \equiv \overline{\boldsymbol{M}^{xOl}} \otimes \mathbf{1}_O$$

a $NO \times NO$ matrix,

$$\overline{\boldsymbol{M}^a} \equiv \mathbf{1}_N \otimes \operatorname{diag}\left(\frac{a_{o,t_0}}{1-a_{o,t_0}}\right)$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,2}} \equiv \begin{bmatrix} \operatorname{diag}\left(-\left(1 - l^{O}_{1,o,t_0}\right) + l^{O}_{1,o,t_0} \frac{a_{o,t_0}}{1-a_{o,t_0}}\right) \\ \vdots \\ \operatorname{diag}\left(-\left(1 - l^{O}_{N,o,t_0}\right) + l^{O}_{N,o,t_0} \frac{a_{o,t_0}}{1-a_{o,t_0}}\right) \end{bmatrix}$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,3}} \equiv \left(\widetilde{\boldsymbol{x}}^{O}_{t_0} \circ \overline{\boldsymbol{M}^{al}}\right) \otimes \mathbf{1}_O$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{xOl,4}} \equiv \overline{\boldsymbol{M}^{xOl,2}} \otimes \mathbf{1}_O,$$

a $NO \times NO$ matrix. I have

$$\begin{aligned} &\left(\boldsymbol{I}_N \otimes \mathbf{1}_O\right) \widehat{\boldsymbol{p}^{G}_t} - \widehat{\boldsymbol{w}}_t + \overline{\boldsymbol{M}^{yG,3}} \widehat{\boldsymbol{Q}^{G}_t} + \left(-\frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + \left(-\frac{1}{\beta} + \frac{1}{\theta}\right) \operatorname{diag}\left(\boldsymbol{l}^{O}_{t_0}\right) + \left(-1 + \frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,3}}\right) \widehat{\boldsymbol{L}}_t \\ &= -\frac{1}{\beta}\widehat{\boldsymbol{b}}_t + \left[\frac{1}{\theta}\overline{\boldsymbol{M}^{a}} + \left(-\frac{1}{\beta} + \frac{1}{\theta}\right) \frac{1}{\theta - 1}\overline{\boldsymbol{M}^{al,2}} + \left(-1 + \frac{1}{\beta}\right) \frac{1}{\theta - 1}\overline{\boldsymbol{M}^{al,3}}\right] \widehat{\boldsymbol{a}}_t - \overline{\boldsymbol{M}^{yG,3}} \widehat{\boldsymbol{\tau}^{G}_t} \\ &\quad + \left[-\left(-\frac{1}{\beta} + \frac{1}{\theta}\right) \operatorname{diag}\left(1 - l^{O}_{i,o,t_0}\right) - \left(-1 + \frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,4}}\right] \widehat{\boldsymbol{K}^{R}_t}. \end{aligned}$$

Hence the log-linearized temporary equilibrium system is

$$\overline{\boldsymbol{D}^{x}} \widehat{\boldsymbol{x}} = \overline{\boldsymbol{D}^{A}} \widehat{\boldsymbol{A}}$$

where matrices $\overline{\boldsymbol{D}^{x}}$ and $\overline{\boldsymbol{D}^{A}}$ are defined as

$$\overline{\boldsymbol{D}^{x}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{x}_{11}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{14}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{16}} \\ -\overline{\boldsymbol{M}^{xG,2}} & \overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \phi\overline{\boldsymbol{M}^{xG,2}} & \mathbf{0} & -\phi\overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{M}^{l}} \\ \overline{\boldsymbol{D}^{x}_{41}} & \overline{\boldsymbol{D}^{x}_{42}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{44}} & \overline{\boldsymbol{D}^{x}_{45}} & \mathbf{0} \\ \overline{\boldsymbol{D}^{x}_{51}} & \overline{\boldsymbol{D}^{x}_{52}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{55}} & \mathbf{0} \\ \overline{\boldsymbol{D}^{x}_{61}} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \overline{\boldsymbol{M}^{yG,3}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{66}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}^{x}_{11}} \equiv -\alpha_M \left(\overline{\boldsymbol{I}_N} - \left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top}\right), \overline{\boldsymbol{D}^{x}_{14}} \equiv \left(1 - \alpha_M\right) \overline{\boldsymbol{M}^{yG}}, \overline{\boldsymbol{D}^{x}_{16}} \equiv -\alpha_L \overline{\boldsymbol{M}^{xOl}},$$

$$\overline{\boldsymbol{D}_{41}^{x}} \equiv \varepsilon^{G}\left[\overline{\boldsymbol{I}_{N}} \otimes \mathbf{1}_{N}\right]+\left(1-\varepsilon^{G}\right)\left[\mathbf{1}_{N} \otimes\left(\widetilde{\boldsymbol{x}}_{t_0}^{G}\right)^{\top}\right]-\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{G}\right)\left[\mathbf{1}_{N} \otimes \overline{\boldsymbol{I}_{N}}\right],$$

$$\overline{\boldsymbol{D}_{42}^{x}} \equiv \operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{V}\right) \overline{\boldsymbol{M}^{xR}}+\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{R}\right) \overline{\boldsymbol{M}^{yR}},$$

$$\overline{\boldsymbol{D}_{44}^{x}} \equiv \overline{\boldsymbol{I}_{N^2}}-\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{G}\right) \overline{\boldsymbol{M}^{yG,2}},$$

$$\overline{\boldsymbol{D}_{45}^{x}} \equiv -\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{V}\right) \overline{\boldsymbol{M}^{xR,2}}-\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{R}\right) \overline{\boldsymbol{M}^{yR,2}},$$

$$\overline{\boldsymbol{D}_{51}^{x}} \equiv\left(1-\alpha^{R}\right) \overline{\boldsymbol{M}^{xG,5}},\ \overline{\boldsymbol{D}_{52}^{x}} \equiv \overline{\boldsymbol{M}^{u,2}}+\overline{\boldsymbol{M}^{xR,4}},\ \overline{\boldsymbol{D}_{55}^{x}} \equiv \frac{1}{\varepsilon^{R}}\left(\overline{\boldsymbol{M}^{u}} \otimes \overline{\boldsymbol{I}_{O}}\right)+\overline{\boldsymbol{M}^{xR,5}},$$

$$\overline{\boldsymbol{D}_{61}^{x}} \equiv \boldsymbol{I}_{N} \otimes \mathbf{1}_{N},\ \overline{\boldsymbol{D}_{66}^{x}} \equiv -\frac{1}{\theta}+\left(-\frac{1}{\beta}+\frac{1}{\theta}\right) \operatorname{diag}\left(\boldsymbol{l}_{t_0}^{O}\right)+\left(-1+\frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,3}},$$

and

$$\overline{\boldsymbol{D}^{A}} \equiv\left[\begin{array}{cccccccccc}
\mathbf{0} & \overline{\boldsymbol{D}_{12}^{A}} & \overline{\boldsymbol{D}_{13}^{A}} & \overline{\boldsymbol{I}_{N}} & \mathbf{0} & \overline{\boldsymbol{D}_{16}^{A}} & \overline{\boldsymbol{D}_{17}^{A}} & \mathbf{0} & \alpha_{L} \overline{\boldsymbol{M}^{xOl,2}} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \overline{\boldsymbol{M}^{xG}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\phi \overline{\boldsymbol{M}^{xG,3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{47}^{A}} & \overline{\boldsymbol{D}_{48}^{A}} & \mathbf{0} & \mathbf{0} \\
\overline{\boldsymbol{D}_{51}^{A}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{57}^{A}} & \overline{\boldsymbol{D}_{58}^{A}} & \overline{\boldsymbol{M}^{u,3}} & \overline{\boldsymbol{D}_{5,10}^{A}} \\
\mathbf{0} & \overline{\boldsymbol{D}_{62}^{A}} & -\frac{1}{\beta} \overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\overline{\boldsymbol{M}^{yG,3}} & \mathbf{0} & \overline{\boldsymbol{D}_{69}^{A}} & \mathbf{0}
\end{array}\right],$$

where

$$\overline{\boldsymbol{D}_{12}^{A}} \equiv \frac{\alpha_{L}}{\theta-1}\left(\widetilde{\boldsymbol{x}}_{t_0}^{O} \otimes \overline{\boldsymbol{M}^{al}}\right),\ \overline{\boldsymbol{D}_{13}^{A}} \equiv \frac{\alpha_{L}}{\beta-1} \overline{\boldsymbol{M}^{xO}},$$

$$\overline{\boldsymbol{D}_{16}^{A}} \equiv\left(1-\alpha_{L}-\alpha_{M}\right) \overline{\boldsymbol{I}_{N}},\ \overline{\boldsymbol{D}_{17}^{A}} \equiv-\left[\alpha_{M} \overline{\boldsymbol{M}^{xG}}+\left(1-\alpha_{M}\right) \overline{\boldsymbol{M}^{yG}}\right],$$

$$\overline{\boldsymbol{D}_{47}^{A}} \equiv-\varepsilon^{G}+\left(1-\varepsilon^{G}\right) \overline{\boldsymbol{M}^{xG,4}}+\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{G}\right) \overline{\boldsymbol{M}^{yG,2}},$$

$$\overline{\boldsymbol{D}_{48}^{A}} \equiv \operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{V}\right) \overline{\boldsymbol{M}^{xR,3}}+\operatorname{diag}\left(\mathbf{1}_{N} \otimes \boldsymbol{s}_{t_0}^{R}\right) \overline{\boldsymbol{M}^{yR,3}},$$

$$\overline{\boldsymbol{D}_{51}^{A}} \equiv-\left(\overline{\boldsymbol{M}^{u}} \otimes \mathbf{1}_{O}\right),\ \overline{\boldsymbol{D}_{57}^{A}} \equiv-\left(1-\alpha^{R}\right) \overline{\boldsymbol{M}^{xG,6}},$$

$$\overline{\boldsymbol{D}_{58}^{A}} \equiv-\left[\left(\overline{\boldsymbol{M}^{u}} \otimes \mathbf{1}_{O}\right)+\overline{\boldsymbol{M}^{xR,6}}\right],\ \overline{\boldsymbol{D}_{5,10}^{A}} \equiv \mathbf{1}_{N} \otimes \overline{\boldsymbol{I}_{NO}},$$

$$\overline{\boldsymbol{D}_{62}^{A}} \equiv \frac{1}{\theta} \overline{\boldsymbol{M}^{a}}+\left(-\frac{1}{\beta}+\frac{1}{\theta}\right) \frac{1}{\theta-1} \overline{\boldsymbol{M}^{al,2}}+\left(-1+\frac{1}{\beta}\right) \frac{1}{\theta-1} \overline{\boldsymbol{M}^{al,3}},$$

and

$$\overline{\boldsymbol{D}_{69}^{A}} \equiv -\left(-\frac{1}{\beta}+\frac{1}{\theta}\right) \operatorname{diag}\left(1-l_{i,o,t_0}^{O}\right) - \left(-1+\frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,4}}.$$

Note that to normalize the price, one of the good-demand equation must be replaced with log-linearized nureraire condition $\widehat{P_{1,t}^{G}} = \sum_i x_{i1,t_0}^{G}\left(\widehat{p_{i,t}^{G}} + \widehat{\tau_{i1,t}^{G}}\right) = 0$, or

$$\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{p}_{t}^{G}} = -\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{\tau}_{t}^{G}},$$

where $\overline{\boldsymbol{M}^{xG,num}} \equiv \left[x_{11,t_0}^{G}, x_{21,t_0}^{G}, x_{31,t_0}^{G}\right]$.

To analyze the steady state conditions, first note that the steady state accumulation condition (B.22) implies $\widehat{Q_{i,o}^{R}} = \widehat{K_{i,o}^{R}}$. Using robot integration function, integration demand and unit cost formula, I have

$$\widehat{Q_{i,o}^{R}} = \sum_{l} x_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}} + \left(1-\alpha^{R}\right)\left(\sum_{l}\widetilde{x}_{ij,o,t_0}^{R}\widehat{p_{li,o}^{R}} - \sum_{l}\widetilde{x}_{li,t_0}^{G}\widehat{p_{li,t}^{G}}\right) \tag{D.6}$$

Thus the condition is

$$\begin{aligned}
&\sum_{l}\widetilde{x}_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}} + \left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,o,t_0}^{R}\widehat{p_{l,o}^{R}} - \left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,t_0}^{G}\widehat{p_{l}^{G}} - \widehat{K_{i,o}^{R}}\\
&= \left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,t_0}^{G}\widehat{\tau_{li}^{G}} - \left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,o,t_0}^{R}\widehat{\tau_{li}^{R}}.
\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{xR,7}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}_{1\cdot,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{N\cdot,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xR,8}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}_{11,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{N1,\cdot,t_0}^{R}\right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\widetilde{x}_{1N,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{NN,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times NO$ matrix, and

$$\overline{\boldsymbol{M}^{xG,7}} \equiv \begin{bmatrix} \widetilde{x}^G_{11,t_0} & & & \cdots & \widetilde{x}^G_{N1,t_0} & & \mathbf{0} \\ & \ddots & & & & \ddots & \\ \mathbf{0} & & \widetilde{x}^G_{1N,t_0} & & & & \widetilde{x}^G_{NN,t_0} \end{bmatrix} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix.

$$\overline{\boldsymbol{M}^{xR,9}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{x}}^R_{11,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{N1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & & \cdots & & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{1N,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{NN,\cdot,t_0} \end{bmatrix},$$

a $NO \times N^2$ matrix, where $\widetilde{\boldsymbol{x}}^R_{ij,\cdot,t_0} \equiv \left(\widetilde{x}^R_{ij,o,t_0}\right)_o$ is an $O \times 1$ vector for any i and j. Then I have

$$-\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} + \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\widehat{\boldsymbol{p}^R} + \overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} - \widehat{\boldsymbol{K}^R} = \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}$$

Next, to study the steady state Euler equation (B.23) , note that by equation (B.13),

$$\begin{aligned} \frac{\partial \pi_{i,t}\left(\widehat{\left\{K^R_{i,o,t}\right\}}\right)}{\partial K^R_{i,o,t}} &= \sum_j \widetilde{y}^G_{ij,t}\left(\widehat{p^G_{ij,t}} + \widehat{Q^G_{ij,t}}\right) + \left[-\frac{1}{\beta}\sum_{o'} x^O_{i,o',t_0}\widehat{b_{i,o',t}} + \frac{1}{\beta}\widehat{b_{i,o,t}}\right] \\ &+ \left\{\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'}\frac{\widetilde{x}^O_{i,o',t_0}}{1-a_{o,t_0}}\left[-l^O_{i,o',t_0}a_{o,t_0} + \left(1-l^O_{i,o',t_0}\right)\left(1-a_{o,t_0}\right)\right]\widehat{a_{o',t}}\right. \\ &+ \left.\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\frac{-l^O_{i,o,t_0}a_{o,t_0} + \left(1-l^O_{i,o,t_0}\right)\left(1-a_{o,t_0}\right)}{1-a_{o,t_0}} + \frac{1}{\theta}\right\}\widehat{a_{o,t}}\right\} \\ &+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} x^O_{i,o',t_0} l^O_{i,o',t_0}\widehat{L_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l^O_{i,o,t_0}\widehat{L_{i,o,t}}\right] \\ &+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}^O_{i,o',t_0}\left(1-l^O_{i,o',t_0}\right)\widehat{K^R_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l^O_{i,o,t_0}\right)\widehat{K^R_{i,o,t}} + \left(-\frac{1}{\theta}\right)\widehat{K^R_{i,o,t}}\right]. \end{aligned} \tag{D.7}$$

Note that by the steady state accumulation condition (B.22), $Q^R_{i,o,t_0}/K^R_{i,o,t_0} = \delta$. Note also that investment function implies that, in the steady state,

$$\frac{\lambda_{j,o}^{R}}{P_{j,o}^{R}} = \left(\sum_i \frac{x_{ij,o}^{R}}{\left(1+u_{ij}\right)^{1-\varepsilon^R}}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta. \tag{D.8}$$

To simplify the notation, set

$$\widetilde{u}_{j,o,t_0}^{SS} \equiv \frac{(\iota+\delta)\left[\left(\sum_i x_{ij,o,t_0}^{R}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right]}{(\iota+\delta)\left[\left(\sum_i x_{ij,o,t_0}^{R}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right] - \gamma\delta^2},$$

Then by log-linearizing equation (B.23) implies, after rearranging,

$$\begin{aligned}
&\left[\widehat{p_i^G} + 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{p_{l,t}^G}\right] - \left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\widehat{p_{i,o}^R} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{ij,o,t_0}^{R}\widehat{p_{l,o}^R} \\
&+\sum_j \widetilde{y}_{ij,t_0}^{G}\widehat{Q_{ij}^G} - 2\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,o,t_0}^{R}\widehat{Q_{li,o}^R} + \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} \widetilde{x}_{i,o',t_0}^{O} l_{i,o',t_0}^{O}\widehat{L_{i,o'}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l_{i,o,t_0}^{O}\widehat{L_{i,o}}\right] \\
&+\left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} \widetilde{x}_{i,o',t_0}^{O}\left(1-l_{i,o',t_0}^{O}\right)\widehat{K_{i,o'}^R} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l_{i,o,t_0}^{O}\right)\widehat{K_{i,o}^R} + \left(-\frac{1}{\theta}\right)\widehat{K_{i,o}^R} + 2\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\widehat{K_{i,o}^R}\right] \\
&-\widetilde{u}_{i,o,t_0}^{SS}\widehat{\lambda_{i,o}^R} \\
&= -\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'}\frac{\widetilde{x}_{i,o',t_0}^{O}}{1-a_{o,t_0}}\left[\left(1-l_{i,o',t_0}^{O}\right)\left(1-a_{o',t_0}\right) - l_{i,o',t_0}^{O}a_{o',t_0}\right]\widehat{a_{o'}} \\
&-\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\frac{1}{1-a_{o,t_0}}\left[\left(1-l_{i,o,t_0}^{O}\right)\left(1-a_{o,t_0}\right) - l_{i,o,t_0}^{O}a_{o,t_0}\right] + \frac{1}{\theta}\right\}\widehat{a_o} \\
&-\left[-\frac{1}{\beta}\sum_{o'}\widetilde{x}_{i,o',t_0}^{O}\widehat{b_{i,o'}} + \frac{1}{\beta}\widehat{b_{i,o}}\right] + \left[-\sum_j \widetilde{y}_{ij,t_0}^{G}\widehat{\tau_{ij}^G} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{\tau_{li,t}^G}\right] \\
&+2\left(1-\alpha^R\right)\left(1-\widetilde{u}_{i,o,t_0}^{SS}\right)\sum_l \widetilde{x}_{ij,o,t_0}^{R}\widehat{\tau_{li}^R}
\end{aligned}$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{xO,3}} \equiv \overline{\boldsymbol{M}^{xO}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix. Then

$$\begin{aligned}
&\left[\left(\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_O\right) + 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,2}}\right]\widehat{\boldsymbol{p}^G} - \operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\left(\overline{\boldsymbol{I}_{NO}} - 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\widehat{\boldsymbol{p}^R} \\
&+ \overline{\boldsymbol{M}^{yG,3}}\widehat{\boldsymbol{Q}^G} - 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} + \left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(l^O_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{L}} \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(1-l^O_{\cdot,\cdot,t_0}\right) - \frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{K}^R} - \operatorname{diag}\left(\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\widehat{\boldsymbol{\lambda}^R} \\
&= -\left[\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,3}} - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,2}}\right]\widehat{\boldsymbol{a}} - \frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right)\widehat{\boldsymbol{b}} \\
&+ \left[-\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,7}}\right]\widehat{\boldsymbol{\tau}^G} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}
\end{aligned}$$

In the steady state, I write equations (D.3) and (D.4) as

$$\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} - \widehat{\boldsymbol{w}} + \left[\overline{\boldsymbol{I}_{NO}} - \frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\right]\widehat{\boldsymbol{V}} = -\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} + d\boldsymbol{T} - \overline{\boldsymbol{M}^{\mu,3}}d\boldsymbol{\chi}^{\text{vec}}$$

and

$$\left[\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{\mu L}}\right]\widehat{\boldsymbol{L}} - \overline{\boldsymbol{M}^{\mu L,2}}\widehat{\boldsymbol{\mu}^{\text{vec}}} = \mathbf{0}.$$

respectively.

Hence the log-linearized steady state system is

$$\overline{\boldsymbol{E}^y}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}^\Delta}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{E}^y} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,SS}}} \end{bmatrix}, \text{ and } \overline{\boldsymbol{E}^\Delta} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \overline{\boldsymbol{D}^{\Delta,SS}} \end{bmatrix},$$

$\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,T}} & \overline{\boldsymbol{D}^{A,\Delta}} \end{bmatrix}$, and matrices $\overline{\boldsymbol{D}^{y,SS}}$ and $\overline{\boldsymbol{D}^{\Delta,SS}}$ are defined as

$$\overline{\boldsymbol{D}^{y,SS}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{y,SS}_{11}} & \overline{\boldsymbol{D}^{y,SS}_{12}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{M}^{xR,7}} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \mathbf{0} \\ \overline{\boldsymbol{D}^{y,SS}_{21}} & \overline{\boldsymbol{D}^{y,SS}_{22}} & \mathbf{0} & \overline{\boldsymbol{M}^{yG,3}} & \overline{\boldsymbol{D}^{y,SS}_{25}} & \overline{\boldsymbol{D}^{y,SS}_{26}} & \overline{\boldsymbol{D}^{y,SS}_{27}} & \overline{\boldsymbol{D}^{y,SS}_{28}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}^{y,SS}_{11}} \equiv -\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{12}^{y,SS}} \equiv \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}},$$

$$\overline{\boldsymbol{D}_{21}^{y,SS}} \equiv \left(\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_O\right) + 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{22}^{y,SS}} \equiv -\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right),$$

$$\overline{\boldsymbol{D}_{25}^{y,SS}} \equiv -2\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xR,7}},$$

$$\overline{\boldsymbol{D}_{26}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\text{diag}\left(l_{\cdot,\cdot,t_0}^{O}\right),$$

$$\overline{\boldsymbol{D}_{27}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\text{diag}\left(1-l_{\cdot,\cdot,t_0}^{O}\right) - \frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + 2\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right),$$

$$\overline{\boldsymbol{D}_{28}^{y,SS}} \equiv -\text{diag}\left(\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right),$$

and

$$\overline{\boldsymbol{D}^{\Delta,SS}} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{17}^{\Delta,SS}} & \overline{\boldsymbol{D}_{18}^{\Delta,SS}} \\ \mathbf{0} & \overline{\boldsymbol{D}_{22}^{\Delta,SS}} & \overline{\boldsymbol{D}_{23}^{\Delta,SS}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{27}^{\Delta,SS}} & \overline{\boldsymbol{D}_{28}^{\Delta,SS}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{17}^{\Delta,SS}} \equiv \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,7}},$$

$$\overline{\boldsymbol{D}_{18}^{\Delta,SS}} \equiv -\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,9}},$$

$$\overline{\boldsymbol{D}_{22}^{\Delta,SS}} \equiv \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,2}} - \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,3}},$$

$$\overline{\boldsymbol{D}_{23}^{\Delta,SS}} \equiv -\frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right),$$

$$\overline{\boldsymbol{D}_{27}^{\Delta,SS}} \equiv -\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xG,7}},$$

and

$$\overline{\boldsymbol{D}_{28}^{\Delta,SS}} \equiv 2\left(1-\alpha^R\right)\text{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xR,9}}.$$

If $\overline{\boldsymbol{E}^y}$ is invertible, I have $\overline{\boldsymbol{E}} \equiv \left(\overline{\boldsymbol{E}^y}\right)^{-1}\overline{\boldsymbol{E}^\Delta}$ such that $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$. Write dimensions of $\boldsymbol{y}$ and $\boldsymbol{\Delta}$ as $n_y \equiv N + 3NO + N^2 + N^2O$ and $n_\Delta \equiv 3N^2 + O + 2NO + 2N$, respectively.

Finally, to study the transitional dynamics, the capital accumulation dynamics (11) implies

$$K^{\check{R}}_{i,o,t+1} = -\delta\left(1-\alpha^R\right)\sum_l \widetilde{x}^G_{li,t_0} p^{\check{G}}_{l,t} + \delta\left(1-\alpha^R\right)\sum_l \widetilde{x}^R_{li,o} p^{\check{R}}_{l,o,t} + \delta\sum_l \widetilde{x}^R_{li,o} Q^{\check{R}}_{li,o,t} + (1-\delta) K^{\check{R}}_{i,o,t}.$$

In a matrix form, write

$$\boldsymbol{K}^{\check{R}}_{t+1} = -\delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}}\boldsymbol{p}^{\check{G}}_t + \delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\boldsymbol{p}^{\check{R}}_t + \delta\overline{\boldsymbol{M}^{xR,7}}\boldsymbol{Q}^{\check{R}}_t + (1-\delta)\overline{\boldsymbol{I}_{NO}}\boldsymbol{K}^{\check{R}}_t.$$

Next, to study the Euler equation, define

$$\widetilde{u}^{TD,1}_{i,o} \equiv \frac{-(\iota+\delta)\left[\left(\sum_l x^R_{li,o}\left(1+u_{li}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right] + \gamma\delta^2}{(1-\delta)\left[\left(\sum_l x^R_{li,o}\left(1+u_{li}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right]}$$

and

$$\widetilde{u}^{TD,2}_{i,o} \equiv \frac{-\gamma\delta^2}{(1-\delta)\left[\left(\sum_l x^R_{li,o}\left(1+u_{li}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right]}.$$

Then I have

$$\begin{aligned}
&\left[-\widetilde{u}^{TD,1}_{i,o} p^{\check{G}}_{i,t+1} + 2\left(1-\alpha^R\right)\widetilde{u}^{TD,2}_{i,o}\sum_l \widetilde{x}^G_{li} p^{\check{G}}_{l,t+1}\right] + \left[-\widetilde{u}^{TD,2}_{i,o} p^{\check{R}}_{i,o,t+1} - 2\left(1-\alpha^R\right)\widetilde{u}^{TD,2}_{i,o}\sum_l \widetilde{x}^R_{li,o} p^{\check{R}}_{l,o,t+1}\right] \\
&- \widetilde{u}^{TD,1}_{i,o}\sum_j \widetilde{y}^G_{ij} Q^{\check{G}}_{ij,t+1} - 2\widetilde{u}^{TD,2}_{i,o}\sum_l \widetilde{x}^R_{li,o} Q^{\check{R}}_{li,o,t+1} - \widetilde{u}^{TD,1}_{i,o}\left(-1+\frac{1}{\beta}\right)\sum_{o'} x^O_{i,o'}\left(1-l^O_{i,o'}\right) K^{\check{R}}_{i,o',t+1} \\
&- \widetilde{u}^{TD,1}_{i,o}\left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} x^O_{i,o'} l^O_{i,o'} L_{i,\check{o'},t+1} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l^O_{i,o} L_{i,\check{o},t+1}\right] \\
&- \left[\widetilde{u}^{TD,1}_{i,o}\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l^O_{i,o}\right) + \left(-\frac{1}{\theta}\right)\right\} - 2\widetilde{u}^{TD,2}_{i,o}\right] K^{\check{R}}_{i,o,t+1} + \lambda^{\check{R}}_{i,o,t+1} = \frac{1+\iota}{1-\delta}\lambda^{\check{R}}_{i,o,t}
\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{u,4}} = \begin{bmatrix} \widetilde{\boldsymbol{u}}^{TD,1}_{1,\cdot} & & \boldsymbol{0} \\ & \ddots & \\ \boldsymbol{0} & & \widetilde{\boldsymbol{u}}^{TD,1}_{N,\cdot} \end{bmatrix},$$

a $NO \times N$ matrix where $\widetilde{\boldsymbol{u}}^{TD,1}_{i,\cdot} \equiv \left(\widetilde{u}^{TD,1}_{i,o}\right)_o$ is an $O \times 1$ vector for any i. Then

$$\begin{aligned}
&\left(-\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\overline{\boldsymbol{M}^{xG,2}}\right)\boldsymbol{p}^{\check{G}}_{t+1} - \mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\boldsymbol{p}^{\check{R}}_{t+1} \\
&\quad - \left[\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_N)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}}\right]\boldsymbol{Q}^{\check{G}}_{t+1} - 2\left((\boldsymbol{1}_N)^\top \otimes \mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}}\boldsymbol{Q}^{\check{R}}_{t+1} \\
&\quad + \left[-\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot} l^O_{\cdot,\cdot}\right)\right]\boldsymbol{L}^{\check{}}_{t+1} \\
&\quad + \left\{\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\left(1-l^O_{\cdot,\cdot}\right)\right)\right. \\
&\quad \left. + \frac{1}{\theta}\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\right) + 2\mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\right\}\boldsymbol{K}^{\check{R}}_{t+1} + \overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}^{\check{R}}_{t+1} = \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}^{\check{R}}_t.
\end{aligned}$$

Hence the log-linearized transitional dynamic system is $\overline{\boldsymbol{D}^{y,TD}_{t+1}}\check{\boldsymbol{y}}_{t+1} = \overline{\boldsymbol{D}^{y,TD}_t}\check{\boldsymbol{y}}_t$, where matrices $\overline{\boldsymbol{D}^{y,TD}_{t+1}}$ and $\overline{\boldsymbol{D}^{y,TD}_t}$ are defined as

$$\overline{\boldsymbol{D}^{y,TD}_{t+1}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} \\ \overline{\boldsymbol{D}^{y,TD}_{21,t+1}} & \overline{\boldsymbol{D}^{y,TD}_{22,t+1}} & \boldsymbol{0} & \overline{\boldsymbol{D}^{y,TD}_{24,t+1}} & \overline{\boldsymbol{D}^{y,TD}_{25,t+1}} & \overline{\boldsymbol{D}^{y,TD}_{26,t+1}} & \overline{\boldsymbol{D}^{y,TD}_{27,t+1}} & \overline{\boldsymbol{I}_{NO}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}^{y,TD}_{21,t+1}} \equiv -\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}^{y,TD}_{22,t+1}} \equiv -\mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right),$$

$$\overline{\boldsymbol{D}^{y,TD}_{24,t+1}} \equiv -\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_N)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}},$$

$$\overline{\boldsymbol{D}^{y,TD}_{25,t+1}} \equiv -2\left((\boldsymbol{1}_N)^\top \otimes \mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}},$$

$$\overline{\boldsymbol{D}^{y,TD}_{26,t+1}} \equiv -\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot} l^O_{\cdot,\cdot}\right),$$

$$\begin{aligned}
\overline{\boldsymbol{D}^{y,TD}_{27,t+1}} \equiv &\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes (\boldsymbol{1}_O)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right) \\
&- \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\left(1-l^O_{\cdot,\cdot}\right)\right) + \frac{1}{\theta}\mathrm{diag}\left(\widetilde{u}^{TD,1}_{\cdot,\cdot}\right) + 2\mathrm{diag}\left(\widetilde{u}^{TD,2}_{\cdot,\cdot}\right),
\end{aligned}$$

and

$$\overline{\boldsymbol{D}_t^{y,TD}} = \begin{bmatrix} -\delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}} & \delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}} & \boldsymbol{0} & \boldsymbol{0} & \delta\overline{\boldsymbol{M}^{xR,7}} & \boldsymbol{0} & (1-\delta)\,\overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}} \end{bmatrix}. \tag{D.9}$$

Since $\check{\boldsymbol{y}}_t = \widehat{\boldsymbol{y}}_t - \widehat{\boldsymbol{y}}$ for any $t \geq t_0$ and $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$, I have

$$\overline{\boldsymbol{D}_{t+1}^{y,TD}}\left(\widehat{\boldsymbol{y}_{t+1}} - \widehat{\boldsymbol{y}}\right) = \overline{\boldsymbol{D}_t^{y,TD}}\left(\widehat{\boldsymbol{y}}_t - \widehat{\boldsymbol{y}}\right)$$

$$\iff \overline{\boldsymbol{D}_{t+1}^{y,TD}}\,\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{D}_t^{y,TD}}\widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{D}_{t+1}^{y,TD}} - \overline{\boldsymbol{D}_t^{y,TD}}\right)\overline{\boldsymbol{E}}\boldsymbol{\Delta}.$$

Recall the temporary equilibrium condition $\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}}_t - \overline{\boldsymbol{D}^{A,S}}\widehat{\boldsymbol{S}}_t = \overline{\boldsymbol{D}^{A,\Delta}}\widehat{\boldsymbol{\Delta}}$ for any t. Thus

$$\overline{\boldsymbol{F}_{t+1}^y}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}_t^y}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}_{t+1}^\Delta}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{F}_{t+1}^y} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}_{t+1}^{y,TD}}} \end{bmatrix}, \overline{\boldsymbol{F}_t^y} \equiv \begin{bmatrix} \boldsymbol{0} \\ \overline{\boldsymbol{D}_t^{y,TD}} \end{bmatrix}, \overline{\boldsymbol{F}_{t+1}^\Delta} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \left(\overline{\boldsymbol{D}_{t+1}^{y,TD}} - \overline{\boldsymbol{D}_t^{y,TD}}\right)\overline{\boldsymbol{E}} \end{bmatrix},$$

or with $\overline{\boldsymbol{F}^y} \equiv \left(\overline{\boldsymbol{F}_{t+1}^y}\right)^{-1}\overline{\boldsymbol{F}_t^y}$ and $\overline{\boldsymbol{F}^\Delta} \equiv \left(\overline{\boldsymbol{F}_{t+1}^y}\right)^{-1}\overline{\boldsymbol{F}_{t+1}^\Delta}$, one can write

$$\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{D.10}$$

It remains to find the initial values of the system (D.10) that satisfies the transversality condition. To this end, I apply a standard method in Stokey and Lucas (1989). In particular, I first homogenize the system: Note that equation (D.10) can be rewritten as $\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t + \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)\left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$ and thus

$$\widehat{\boldsymbol{z}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{z}}_t \tag{D.11}$$

where

$$\widehat{\boldsymbol{z}}_t \equiv \widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{D.12}$$

Next, for the transversality condition to be satisfied, the system (D.11) must not explode. Thus it must be that $\widehat{z_t} \to \mathbf{0}$ or $\widehat{\boldsymbol{y}_t} \to \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1} \overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$. To find the condition, write Jordan decomposition of $\overline{\boldsymbol{F}^y}$ as $\overline{\boldsymbol{F}^y} = \overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{\Lambda B}}$. Then Theorem 6.4 of Stokey and Lucas (1989) implies that it must be that out of n_y vector of $\overline{\boldsymbol{B}}\widehat{\boldsymbol{z}_{t_0}}$, n-th element must be zero if $|\lambda_n| > 1$. Since $\widehat{\boldsymbol{K}^R_{t_0}} = \mathbf{0}$, I can write

$$\widehat{\boldsymbol{z}_{t_0}} = \overline{\boldsymbol{F}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{F}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

where

$$\overline{\boldsymbol{F}^\Delta_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\Delta}} \\ \mathbf{0}_{2NO\times n_\Delta} \end{bmatrix} - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta} \text{ and } \overline{\boldsymbol{F}^\lambda_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\lambda}} \\ \mathbf{0}_{NO\times NO} \\ \overline{\boldsymbol{I}_{NO}} \end{bmatrix}$$

and $\overline{\boldsymbol{D}^{A,\lambda}}$ is the right block matrix of $\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,K}} & \overline{\boldsymbol{D}^{A,\lambda}} \end{bmatrix}$ that corresponds to vector $\widehat{\boldsymbol{\lambda}^R}$. Extracting n-th row from $\overline{\boldsymbol{F}^\Delta_{t_0}}$ and $\overline{\boldsymbol{F}^\lambda_{t_0}}$ where $|\lambda_n| > 1$ and writing them as a $NO \times n_\Delta$ matrix $\overline{\boldsymbol{G}^\Delta_{t_0}}$ and $NO \times NO$ matrix $\overline{\boldsymbol{G}^\lambda_{t_0}}$, the condition of the Theorem is

$$\mathbf{0} = \overline{\boldsymbol{G}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{G}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

or $\widehat{\boldsymbol{\lambda}^R_{t_0}} = \overline{\boldsymbol{G}_{t_0}}\boldsymbol{\Delta}$ where $\overline{\boldsymbol{G}_{t_0}} \equiv -\left(\overline{\boldsymbol{G}^\lambda_{t_0}}\right)^{-1}\overline{\boldsymbol{G}^\Delta_{t_0}}$. Finally, tracing back to obtain the initial conditions for $\widehat{\boldsymbol{y}_t}$, it must be $\widehat{\boldsymbol{y}_{t_0}} = \overline{\boldsymbol{F}^y_{t_0}}\boldsymbol{\Delta}$,where

$$\overline{\boldsymbol{F}^y_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\left(\overline{\boldsymbol{D}^{A,\Delta}} + \overline{\boldsymbol{D}^{A,\lambda}}\overline{\boldsymbol{G}_{t_0}}\right) \\ \mathbf{0}_{NO\times n_\Delta} \\ \overline{\boldsymbol{G}_{t_0}} \end{bmatrix}.$$