Robots and Wage Polarization: The Effects of Robot Capital across Occupations

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Abstract

What are the distributional and aggregate effects of the increase in the use of industrial robots across occupations? To answer this question, I construct a novel dataset that tracks quantities and values of robots in the US, which permits me to distinguish which occupations are more affected by robots. I then develop a general equilibrium model where industrial robots are an internationally traded durable good that have the possibility of substituting labor differently across occupations. The elasticities of substitution (EoS) between industrial robots and labor within an occupation crucially determine the occupation-specific real-wage effects of accumulated robots. To estimate the model parameters, I construct the model-implied optimal instrumental variable from the robot average prices in my dataset. I find that the EoS between robots and labor are heterogeneous across occupations, and higher than the EoS between general capital goods and labor in some occupations. These estimates imply that the increase of the robot use can explain 11.7% of the observed increase in wage inequality across occupations during 1990-2007 measured by the 90th-50th percentile ratio of the distribution. A counterfactual tax on the robot purchase increases real income in the short run, but leads to a decline in income in the long run.

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1. Introduction

In the last three decades, the global market size of industrial robots has grown by 12% annually, where robots are defined as multiple-axes manipulators and measured by the number of such manipulators, or robot arms.[1] International trade of robots is also sizable–41% of all robots are traded before adoption. Furthermore, robots have gradually replaced workers in some occupations, raising concerns about the distributional effects of such trends. Motivated by this concern, policymakers have proposed regulation on the penetration of automation technologies such as a robot tax.[2] On the academic front, emerging literature has found the relative effects of penetration of robots on employment and impacts of a counterfactual robot tax (e.g., Acemoglu and Restrepo, 2020; Humlum, 2019). However, due to the limited data measuring robot adoption across tasks and the lack of a model capturing the trade of robots and dynamic robot accumulation, our understanding of the distributional and aggregate impacts of industrial robots is still limited.

In this paper, I study how industrial robots affect the occupational wage inequality and aggregate income. In particular, I emphasize that robots are internationally traded, robots accumulate over the years, and particular robots compete with workers in particular occupations. I use a unique dataset of robots disaggregated by occupations that robots replace and a model where robots are an internationally traded durable good and endogenously accumulated, and substitution between robots and labor within an occupation. Based on these data and model, I estimate that the elasticities of substitution (EoS) are heterogeneous across occupation groups. For the group of production and material-

[1] More formally, I follow International Organization for Standardization (ISO), which defines robots in ISO 8373:2012 as "automatically controlled, reprogrammable, multipurpose manipulator, programmable in three or more axes, which can be either fixed in place or mobile for use in industrial automation applications." Many studies in the literature also use the same definition. This definition implies that any automation equipment that does not have multiple axes is out of the scope of the paper, even though some of them are sometimes called "robots" (e.g., autonomous home vacuum cleaners made by iRobot Corporation). Examples of the robots considered in this paper are shown in Section A1. I use the terms "industrial robots" and "robots" as those satisfying this definition and interchangeably throughout the paper.

[2] The European Parliament proposed a robot tax on robot owners in 2015, although it eventually rejected the proposal. (Delvaux et al., 2016) South Korea revised the corporate tax laws that downsize the "Tax Credit for Investment in Facilities for Productivity Enhancement" for enterprises investing in automation equipment. (MOEF, 2018)

moving occupations, the EoS are significantly higher than those between general capital and labor estimated in the literature, revealing the susceptibility of these workers to robots. I find that robots can explain a sizable part of the expansion of the US wage inequality across occupations during 1990-2007. Furthermore, a robot tax could increase the US's real income in the short run. However, it leads to a decline in income in the long run because of robot de-accumulation.

I construct the dataset that tracks the number of robot arms and average prices disaggregated by occupations that robots replace. To do so, I use the information from the Japan Robot Association (JARA) about task- and destination-country level shipment of Japanese robots. I then match the JARA data with match scores on occupation similarity from the Occupational Information Network (O*NET) to obtain the occupation-level measurement of robots. I further merge this dataset to the US Census/ACS to have occupational wages and employment. The dataset links robots imported from Japan, which comprise one-third of all robots in the US, to the US's occupational labor market outcomes. With the dataset, I show that over 1990-2007, robot prices fall heterogeneously across occupations. Furthermore, there is a positive correlation between the growth of robot unit values and US wage growth at the occupation level.

Guided by these facts, I develop a general equilibrium model with three key features. First, I incorporate the trade of robots prevalent in the data. Robot trade fits well with my dataset that measures the international shipment of robots from Japan to the US. Theoretically, the trade of robots in the large-open country setting implies that a robot tax affects the price of robots traded in the global market. Hence, a country may gain from the aggregate perspective if it can reduce the cost of adopting robots by imposing the robot tax. Second, I describe the endogenous accumulation of robots with an adjustment cost. The costly accumulation implies the sluggish adjustment of robot stock, making the aggregate income implication of the robot tax subtle and different over the time horizon. Finally, the model has within-occupation elasticities of substitution (EoS) between robots and labor that are heterogeneous across occupations. The production function with within-occupation EoS makes rich the prediction of the real-wage effect of robots at the occupation level because a unit of robots can substitute for workers differentially across occupations, as well as the accumulated stock of occupation-specific robots is different.

I estimate the model's structural parameters by applying the model-implied optimal instrumental variable based on the average prices of robots in my dataset. The structural parameters include the EoS between robots and labor. In my context, the identification challenge is that the cost shock inferred from the change in the average prices may be correlated with the automation shock, or equivalently quality upgrading of robots, affecting the labor market outcomes simultaneously. To overcome this challenge, I use the GE structure. Specifically, I take the wage change predicted by the GE that includes the effect of the automation shock on the wage. By removing the predicted wage change from the observed one, I obtain the structural residual that is free from the effect of the automation shock. I then assume such a structural residual is orthogonal to the robot cost shock measured from the robot price change in my dataset. Finally, I estimate the structural parameters with the optimal model-implied IV constructed from the GE solution, which increases the estimation precision.

I apply this estimation method to the US data on occupational labor market outcomes and robot adoption. The estimation result reveals EoS are heterogeneous across occupation groups. For routine occupations that perform production and material moving, the estimates are as high as around 4. These estimates are significantly higher than the values of the EoS between general capital and labor estimated in the literature, highlighting one of the main differences between robots and other capital goods. In contrast, the EoS in other occupations are close to 1, or robots and labor are neither substitutes nor complements in these occupations.

The estimated model and shocks backed out from the model predict occupational US wage changes in 1990-2007. The high EoS between robots and workers in production and material moving occupations imply that the robotization in this period significantly decreased the relative wage in these occupations. Since these occupations tend to be in the middle of the occupational wage distribution in 1990, this finding indicates that the automation shock compressed the wage growth of occupations in the middle deciles. Quantitatively, it explains 11.7% of the wage polarization measured by the change in the 90th-50th percentile wage ratio, a measure of wage inequality popularized by Goos and Manning (2007) and Autor et al. (2008).

Finally, I examine the counterfactual effect of introducing a tax on robot purchases. From a normative viewpoint, a robot tax could potentially increase the aggregate income of a country. Due to the trade of robots, a government can exert monopsony power in the global robot market by taxing robot purchases, leading to a decrease in the before-tax price of imported robots in each period. In contrast, the robot tax also disincentivizes the accumulation of robots in the long run, potentially reducing aggregate income. Quantitatively, the latter effect dominates the former in the long-run, and robot tax decreases the aggregate real income.

This paper contributes to the literature that studies the economic impacts of industrial robots. The closest papers are Acemoglu and Restrepo (2020) and Humlum (2019).[3] Acemoglu and Restrepo (2020) establishes the geographic relationship that the US commuting zones that experienced penetration of general robots over 1992-2007 also saw decreased wages and total employment. Humlum (2019) uses firm-level adoption measures of robots and firm-worker-level panel data, estimates a model that incorporates a small-open economy of robot importers, the binary decision of robot adoption, and the EoS between occupations.[4] He then studies the distributional effect of robots and a counterfactual robot tax. In contrast to these papers, my study features the following three elements. First, I consider the trade of robots in a large-open economy setting. This setting implies that the US real income effect of robots is positive in the short-run in my counterfactual exercise. Second, I model the endogenous and dynamic robot accumulation,

[3] Dauth et al. (2017); Graetz and Michaels (2018) also use the industry-level aggregate data of robot adoption and its impact on labor markets.

[4] There is also a growing body of studies that use the firm- and plant-level microdata to study the impact on workers in Canada (Dixon et al., 2019), France (Acemoglu et al., 2020; Bonfiglioli et al., 2020), Netherlands (Bessen et al., 2019), Spain (Koch et al., 2019), and the US (Dinlersoz et al., 2018).

which makes the efficiency loss due to regulating robots materialize not immediately but over the time horizon. Finally, I bring the robot data for each occupation robots replace and model the heterogeneous substitution between robots and labor across occupations. Combined with the high estimates of the EoS between robots and workers, this implies that the distributional effects of the increase in industrial robot use are larger.

Occupations are receiving attention in the literature of automation as they matter when considering the distributional effects. While Jäger et al. (2016) finds no association between industrial robot adoptions and total employment at the firm level, Dinlersoz et al. (2018) report the cost share of workers in the production occupation dropped after the adoption of robots within a firm. Jaimovich et al. (2020) construct a model to study the effect of automation on the labor market of routine and non-routine workers in the steady state. I contribute to this literature by estimating the within-occupation EoS between robots and labor with the occupation-level data of robot measures and labor market outcomes, as well as incorporating the endogenous trade of robots and characterizing the transition dynamics of the effect of robot tax.

Using the methodologies from the literature of the capital goods trade in the quantitative open-economy equilibrium model, I study the distributional and aggregate effects of robots. For example, extensive literature examines the international productivity difference through the trade of capital goods and capital accumulation (e.g., Eaton and Kortum, 2001; Mutreja et al., 2018; Anderson et al., 2019). Another strand of literature studies the role of international trade of capital goods in explaining the skill premia trend under the capital-skill complementarity framework (Burstein et al., 2013; Parro, 2013; Koren et al., 2020).

Since robots are one type of capital goods, my paper is also related to the vast literature of estimating the EoS between capital and labor (to name a few, Arrow et al., 1961; Chirinko, 2008; Oberfield and Raval, 2014). Although the literature yields a set of estimates with a wide range, the upper limit appears around 1.5 (Karabarbounis and Neiman, 2014; Hubmer, 2018). Therefore, the estimates as high as 4 in production and material-moving occupations among routine occupations are significantly higher than this upper limit. The intuition for the high estimates follows the literature. Namely, the strong positive correlation between robot price changes and wage changes is rationalized if the drop of robot price elastically induces a substantial increase (resp. decrease) of relative robot demand (resp. relative labor demand), which causes a large wage fall. The high EoS estimates for these routine occupations highlight one of the main differences between robots and other capital goods: these workers' vulnerability to robots.

2. Data and Stylized Facts

To obtain an under-explored variation of robots across occupation, I construct a novel dataset of occupational robots and labor. The main data sources are the Japan Robot Association (JARA), Occupational Information Network OnLine Code Connector (O*NET), and the US Census, American Community Survey (ACS), retrieved from the Integrated Public Use Microdata Series (IPUMS) USA (Ruggles et al., 2018). Using these sources, I develop a concordance between robot applications and labor occupations and the combined dataset of occupational robots from Japan and labor market outcomes across occupations. The dataset reveals the following two facts. First, there is a significant heterogeneity in the rate of decrease of robot prices across occupations. Second, US labor market outcomes are negatively correlated with robot adoption in non-US countries and positively correlated with robot prices in non-US countries. I relegate the discussion of supplementary data in Section A2.

2.1. Data Sources

JARA is a general incorporated association composed of robot producer companies in Japan. As of August 2020, the number of member companies is 381. JARA surveys the member and non-member companies annually, asking the sales units and monetary values of robots for each destination country and application. JARA releases several summary cross tables of the survey. Among them, I digitize and use the cross table of annual robot sales by destination countries and by applications. I call this data as JARA data below. Note that Japan has been a significant robot innovator, producer, and exporter. Using JARA data, I observe one-third of US robot purchases at the application level. Section A3. shows the international flow of robots, including Japan, the US, and the rest of the world.

I discuss the average price of robots, or the unit values, in detail since the previous literature does not focus on them. I measure the price of robots by average unit values throughout the paper. Note that this measure reflects the price of the set of robot joints and arms (hardware or unit), as opposed to the cost of the whole system installation. In particular, a modern industrial robot is typically not a stand-alone hardware, but an ecosystem that includes the hardware and control units operated by computers (software). Due to its complexity, installing robots in the production environment often requires hiring costly system integrators that offer specific engineering knowledge. A relevant cost of robots for adopters includes this hardware, software, and integration costs. However, as Leigh and Kraft (2018) pointed out, the current industry and occupation classifications do not allow separating system integrators, making it hard to estimate the cost from these classifications. They estimate that two-thirds of robot costs are the integration costs. In this sense, readers should interpret that my measure of robot price reflects the part of overall robot costs. I will address this issue in the model section by separately defining the overall robot costs and observed robot unit prices and an adjustment cost.

To match the JARA data with the labor market outcomes at the occupation level, I use O*NET. O*NET is an online database of occupational definitions sponsored by the US Department of Labor, Employment and Training Administration. O*NET Code Connector provides an occupational search service that helps workforce professionals determine the correct O*NET-SOC Occupation Codes for job orders. Along with the O*NET-SOC codes, the search algorithm provides (i) the textual description of each code and (ii) a match score that shows the relevance of the search target with the search query term. I use these textual descriptions and match scores for matching robot applications and labor occupations. The match score is further discussed in detail below.

2.2. Robot Applications and Labor Occupations

My dataset provides the employment of labor and robots at the occupation level, complementing data in the previous literature at the sector level or, more recently, firm level. This is made possible by observing robot application-level data, and converting robot applications to labor occupations. In this section, I describe robot applications in detail and propose a method to match the robot application codes in JARA to occupation codes in IPUMS USA. Robot applications and labor occupations are close concepts. On the one hand, robot application is a task where the robot is applied. On the other hand, labor occupation describes multiple types of tasks the person does on the job. Each task has different requirements for robotics automation. Therefore, a different mix of tasks in each occupation generates heterogeneous levels of automatability across occupations at a time, and heterogeneous penetration of robots within a period of time (Manyika et al., 2017).[5] Yet, there is no formal concordance between application and occupation codes.[6]

One might be concerned that the conceptual closeness actually indicates that robots complement workers as opposed to substitute them. To address this concern, I study the tasks performed by robots and labor as precise as possible. By doing so, robots and workers are more likely substitutes if they do exactly the same task. For this purpose, I use the *match score* from the O*NET Code Connector that contains detailed textual descriptions

[5] Concrete examples of pairs of an application and an occupation that are close are spot welding and material handling. Spot welding is a task of welding two or more metal sheets into one by applying heat and pressure to a small area called spot. It is one major application of robots. In contrast, O*NET-SOC Code 51-4121.06 has the title "Welders, Cutters, and Welder Fitters" ("Welders" below). Therefore, both spot welding robots and welders perform the welding task. Another example is material handling. Material handling is a short-distance movement of heavy materials. It is another major application of robots. In comparison, ONET-SOC Code 53-7062.00 has the title "Laborers and Freight, Stock, and Material Movers, Hand" ("Material Handler" below). Therefore, both material handling robots and material handlers perform the material handling task.

[6] In a related attempt, Webb (2019) provides a method to match technological advance embodied in each patent text to occupations. He applied the method to study the impact of new technology on occupations, with examples of (general) industrial robots, software, and artificial intelligence. One can interpret my application complements his study by matching between robots at a more detailed application level and occupations.

of occupations. The match score is an output of the *weighted search algorithm* used by the O*NET Code Connector, which is the internal search algorithm developed and employed by O*NET and since September 2005. Since then, the O*NET has continually updated the algorithm and improved the quality of the search results. Morris (2019) reports that the updated weighted search algorithm scored 95.9% based on the position and score of a target best occupation for a given query.

The concrete process of matching is the following. Denote a as robot application and o as labor occupation. JARA data measure robot sales quantity and total monetary transaction values for each application a. I write these as robot measures X_a^R, a generic notation that means both quantity and values. The goal is to convert an application-level robot measure X_a^R to an O*NET-SOC occupation-level one X_o^R. First, I search occupations in O*NET Code Connector by the title of robot application a. Second, I web-scrape the match score m_{oa} between a and o.[7] Finally, I allocate X_a^R to each occupation o according to m_{oa}-weight by

$$X_o^R = \sum_a \omega_{oa} X_a^R \text{ where } \omega_{oa} \equiv \frac{m_{oa}}{\sum_{o'} m_{o'a}}.$$

As a result, X_o^R measures the occupation-level robot measures such as quantity and monetary values. Several clarifications follow. First, note that $\sum_o \omega_{oa} X_a^R = X_a^R$ since $\sum_o \omega_{oa} = 1$. In other words, occupation-level robot measures sum back to the application level when summed across occupations, as a desired property of the allocation. Second, after the final step, I convert the O*NET-SOC-level occupation codes to OCC2010 codes for matching labor market measures available in IPUMS USA. Below I use notation o to denote OCC2010-level occupations.

[7] To obtain the consistent data across periods, I only focus on occupations that exist between the 1970 Census and the 2007 ACS, the sample period and the period of pre-trend analysis. By this choice, I may miss an extensive margin of the effect of automation that creates new labor intensive tasks and thus occupations (Acemoglu and Restrepo, 2018). Autor and Salomons (2019) discusses how to measure emerging occupations in data.

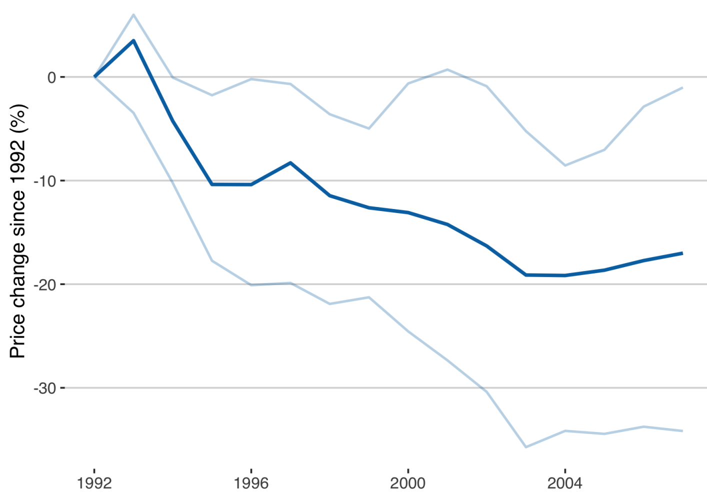
2.3. Facts

In the generated data, I have variables of robot sales in USD in real values deflated by the 1999 CPI, robot arm units, and stock for each occupation o. I define the initial year as 1992, in which the JARA data starts tracking destination-country level variable, and 1992-2007 as the sample period. I show the raw trends of occupational robot stocks and prices in Section A4.

Fact 1: Trends of Robot Prices I show the patterns of average prices of robots across occupations that is not intensively studied in the literature. Figure 1 shows the distribution (10-th 50-th, and 90-th percentile) of the growth rates of the price of robots in the US relative to the initial year. The figure represents two patterns. First, the robot prices show an overall decreasing trend, with the median growth rate of -17% from 1992 to 2007, or -1.1% annually. This pattern is consistent with the decreasing trend of prices of general investment goods since 1980, as Karabarbounis and Neiman (2014) reports 10% decrease per decade from their data sources. Second, there is significant heterogeneity in the rate of price falls across occupations. The 10-th percentile occupation experienced -34% growth (-2.8% per annum), while the 90-th percentile occupation almost did not change the price in the sample period.

There are several interpretations of the price trend, including cost reduction and quality changes. They are hard to separate in my data, and thus interpreted through the lens of the economic model in Section 3.. First, if the cost of producing robots decreases, the measured prices naturally drop. In the model, I will capture this pattern by positive Hicks-neutral productivity shock to robot producers. Second and importantly, if the quality of the robots increased over the period, the quality-adjusted prices may experience a larger decrease than what is observed. To take it into account, I also allow the quality changes in the model and examine its general equilibrium effects on robot prices and quantities. Taken together, the differences in the robot price (and quality) trends may affect the variation in robot adoptions and labor market impacts at the occupation level.

Figure 1: Variation in the Robot Price Trends across Occupations

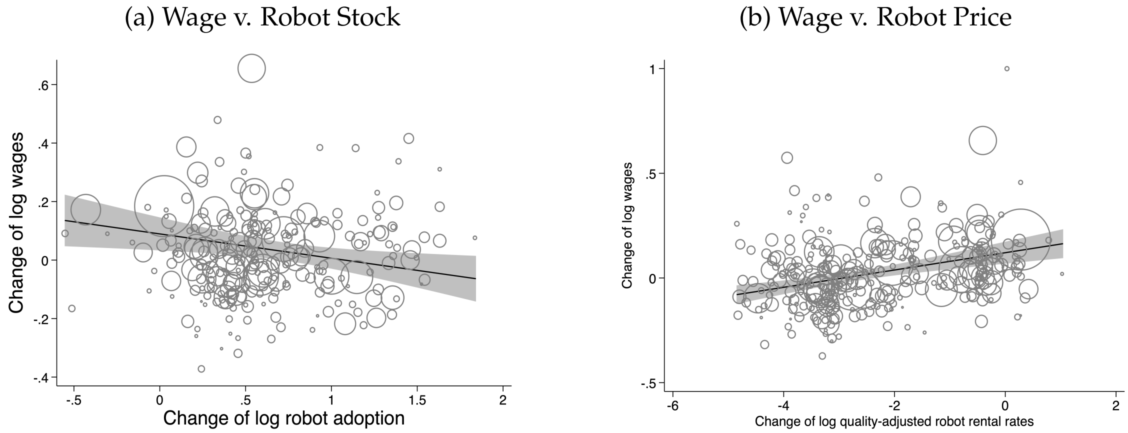


Notes: Authors' calculation based on JARA and O*NET. The figure shows the trend of prices of robots in the US for each occupation. The thick and dark line shows the median price in each year, and two thin and light lines are the 10-th and 90-th percentile. Three-year moving averages are taken to smooth out yearly noises.

Fact 2: Correlation between robot measures and wages I use changes in robot stock in non-US countries as a proxy for the robot technology changes in the US following Acemoglu and Restrepo (2020). Figure 2 plots the correlation between changes in robot measures in non-US countries and changes in log wages in the US at the occupation level, weighted by the size of occupation measured by initial the employment level. The two panels differ in the variables in the x-axis, changes in log monetary value of robot stock on the left panel, and changes in the log robot prices in the right panel.

The figure both corroborates and extends findings in the literature. In the left panel, I confirm that there is a negative correlation between robot adoption and labor market outcomes. Although this correlation is a reminiscent of the result in Acemoglu and Restrepo (2020) that finds the region-level negative correlation between wages and their measure of penetration of robots, the occupation-level negative correlation found in my dataset suggests more direct evidence of substitution of workers performing the task as the same occupation as robots. In the right panel, I find that there is a positive correlation between robot adoption and labor market outcomes at the occupation level. Although

Figure 2: Correlation of Wages and Robot Measures



Notes: Authors' calculation based on JARA, O*NET, and IPUMS USA. The figure shows the scatterplots and weighted fit line and 95 percent confidence interval of changes in log robot adoptions and changes in log wages. The left panel takes the change in log robot stock (measured in monetary value) in the x-axis, while the right panel takes the change in log robot average price in the x-axis. Each bubble represents the occupation and the size of bubbles indicate the baseline year (1990, which is the closest Census year to the initial year that we observe the robot adoption, 1992).

the change in my unit value measure is affected by concurrent quality upgrading as I already discussed, this finding suggests that the robots and workers are substitutable because when the price of robots falls in an occupation, the relative demand for robots (resp. labor) increases (resp. decreases) in the same occupation. I will formalize this idea in the model and estimation sections below. Finally, these findings are robust to including demographic control variables, taking the robot measures in third countries to remove the effect of occupational demand shocks in the US, quality adjustment following Khandelwal et al. (2013), using the quantity measure for robot stocks, and unweighted plots and regression, as shown in Section A6.

3. Model

Guided by the dataset, I develop an open-economy dynamic general equilibrium model. The model features three novelty: (i) a production function featuring the constant elasticity of substitution between robots and labor within an occupation, (ii) open-economy equilibrium with robot trades, and (iii) endogenous investment to robots with an adjustment cost. I show a sufficient statistic formula of real wages and clarify that it depends on the elasticity of substitution between robots and labor. I also show that robot tax may be used as a tool for terms-of-trade manipulation. I relegate discussions of assumptions and derivations to Section B.

3.1. Setup

I formalize the model settings, assumptions, and key characterizations. I relegate discussions and comparisons to the literature in Section B1. Other standard characterizations of equilibrium conditions are given in Section B4.

Environment Time is discrete and has infinite horizon $t = 0, 1, \ldots$. There is N number of countries and O number of occupations. There are two types of goods g, a non-robot good $g = G$ and robot $g = R$. Both goods are tradable. Countries are denoted by subscript i, j, and l, where l means a robot exporter, i means a robot importer and non-robot good exporter, and j means a non-robot good importer. The non-robot good G is differentiated by origin countries and can be consumed by households, used as intermediate goods, invested to produce robots, and used as an input for integration, which I will discuss in detail. Robot R is differentiated by origin countries and occupations. There are bilateral and good-specific iceberg trade costs $\tau^g_{ij,t}$ for each $g = G, R$. I use notation Y for the total production, while Q for the quantity arrived at the destination. For instance, non-robot good G shipped from i to j in period t satisfies $Y^G_{ij,t} = Q^G_{ij,t}\tau^G_{ij,t}$. There is no intra-country trade cost, thus $\tau^g_{ii,t} = 1$ for all i, g and t.

There are three factors for production of good G: labor by occupation L_o, robot capital by occupation K^R_o, and non-robot capital K. The stock of non-robot capital is exogenously given at any period for each country. There is no international movement of factors. Note that non-robot capital is not occupational. While producers rent non-robot capital from the rental market, they accumulate and own robot capital. All good and factor markets are perfectly competitive.

The government in each country exogenously sets the robot tax. Buyers of robot $Q^R_{li,o,t}$ have to pay ad-valorem robot tax $u_{li,t}$ on top of producer price $p^R_{li,o,t}$ to buy from l. The tax revenue is uniformly rebated to destination country i's workers.

Workers Workers' problem is characterized by a dynamic discrete choice problem of occupations (Traiberman, 2019; Humlum, 2019). The technique also follows the discrete sector choice problem in Dix-Carneiro (2014) and Caliendo et al. (2019) in that workers choose the occupations that maximize the lifetime utility based on the draw of the idiosyncratic shock, switching cost, and welfare gains from moving, which has a closed form solution when the idiosyncratic shock follows a suitable extreme value distribution (McFadden, 1973).[8] In Section B2., I formally define and show that the worker's problem can be characterized by, for each country i and period t, the transition probability $\mu_{i,oo',t}$ from occupation o in period t to occupation o' in period $t+1$, and the exponential expected value $V_{i,o,t}$ for occupation o that satisfy

$$\mu_{i,oo',t} = \frac{\left(\left(1-\chi_{i,oo',t}\right)\left(V_{i,o',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}}{\sum_{o''}\left(\left(1-\chi_{i,oo'',t}\right)\left(V_{i,o'',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}}, \tag{1}$$

$$V_{i,o,t} = \widetilde{\Gamma} C_{i,o,t}\left[\sum_{o'}\left(\left(1-\chi_{i,oo',t}\right)\left(V_{i,o',t+1}\right)^{\frac{1}{1+\iota}}\right)^{\phi}\right]^{\frac{1}{\phi}}, \tag{2}$$

respectively, where $C_{i,o,t+1}$ is the real consumption, $\chi_{i,oo',t}$ is an ad-valorem moving cost from occupation o to o', ϕ is the occupation-switch elasticity, $\widetilde{\Gamma} \equiv \Gamma\left(1-1/\phi\right)$ is a constant that depends on the Gamma function. For each i and t, due to these solutions, employment level satisfies the law of motion

$$L_{i,o,t+1} = \sum_{o'} \mu_{i,o'o,t} L_{i,o',t}, \tag{3}$$

[8] A difference from these past studies is that I characterize the moving cost by an ad-valorem term, which helps to perform log-linearization when solving the model.

with the total employment satisfying an adding-up constraint

$$\sum_o L_{i,o,t} = \overline{L}_{i,t}. \tag{4}$$

Production Function I describe a production function in country i in period t. For each good g, there is a given mass of producers. Non-robot good-G producers produce by aggregating occupation service $Q^{G,O}_{i,o,t}$, intermediate goods $M_{i,t}$, and non-robot capital $K_{i,t}$ by

$$Y^G_{i,t} = A^G_{i,t} \left[\sum_o \left(b_{i,o,t}\right)^{\frac{1}{\beta}} \left(Q^O_{i,o,t}\right)^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}\alpha_L} \left(M_{i,t}\right)^{\alpha_M} \left(K_{i,t}\right)^{1-\alpha_L-\alpha_M}, \tag{5}$$

where $Y^G_{i,t}$ is the production quantity, $A^G_{i,t}$ is a Hicks-neutral total factor productivity (TFP) shock, $b_{i,o,t}$ is the share parameter of occupation o, β is the elasticity of substitution between occupations, and $\alpha_{i,L}$, $\alpha_{i,M}$, and $1-\alpha_{i,L}-\alpha_{i,M}$ are Cobb-Douglas weights on occupations, intermediate goods, and non-robot capital, respectively. Parameters satisfy $b_{i,o,t} > 0$ for all i, o, and t, $\sum_o b_{i,o,t} = 1$, $\beta > 0$, and $\alpha_{i,L}, \alpha_{i,M}, 1-\alpha_L-\alpha_M > 0$. For simplification, I assume that robots R for occupation o are produced by investing non-robot goods $Q^V_{i,o,t}$ with productivity $A^R_{i,o,t}$:[9]

$$Y^R_{i,o,t} = A^R_{i,o,t} Q^V_{i,o,t}. \tag{6}$$

Note that the increase in the TFP term $A^R_{i,o,t}$ may drive a reduction in the robot prices. To perform each occupation o, producers hire labor $L_{i,o,t}$ and robot capital $K^R_{i,o,t}$

$$Q^O_{i,o,t} = \left[\left(1-a_{o,t}\right)^{\frac{1}{\theta_o}} \left(L_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}} + \left(a_{o,t}\right)^{\frac{1}{\theta_o}} \left(K^R_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}} \right]^{\frac{\theta_o}{\theta_o-1}}, \tag{7}$$

[9] The assumption greatly simplifies the solution of the model because I can simply assign all the demand for occupations, intermediate goods and non-robot capital to non-robot good producers as opposed to robot producers. Furthermore, I can simply use the estimates measured at the unit of output dollar values when taking the budget constraint of the model to the data in log-linearized solution. To conduct the estimation and counterfactual exercises without the simplification, one would need to observe the cost shares of intermediate goods and non-robot capital for robot producers. This task is left for future research.

where $\theta_o > 0$ is the elasticity of substitution between robots and labor within occupation o, and $a_{o,t}$ is the share of robot capital in the cost of occupation o. In the following sections, I use the shift of $a_{o,t}$ as a source of automation. I will discuss real-world examples and the relationship to the models in the literature in Section B1. The intermediate goods are aggregated by

$$M_{i,t} = \left[\sum_{l} (M_{li,t})^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{8}$$

where ε the elasticity of substitution. Since intermediate goods are traded across countries and aggregated by equation (8), elasticity parameter ε serves as the trade elasticity. Given the iceberg trade cost $\tau^G_{ij,t}$, the bilateral price of good G that country j pays to i is $p^G_{ij,t} = p^G_{i,t} \tau^G_{ij,t}$.

Discussion–Production Function and Automation It is worth mentioning the relationship between production functions (5) and (7) and the way automation is treated in the literature. A common approach to modeling robots in the literature, called the task-based approach, imply occupation production function (7). A large body of literature develops task-based approach to model industrial robots (e.g., Acemoglu and Restrepo, 2018) and more general automation (e.g., Autor et al., 2003; Acemoglu and Autor, 2011). In particular, task-based approach constructs the production function (task-based production function) based on the producers' allocation problem of production factors (e.g., robot capital, labor) to a set of production tasks (e.g., spot welding). In Lemma B.1 in Section B, I show that the solution to the allocation problem implies the resulting production function under the class of functions (5) and (7). In other words, production functions (5) and (7) are micro-founded by the task-based approach of industrial robots.

The cost share parameter $a_{o,t}$ of equation (7) has several interpretations worth mentioning. First, recall that the task-based approach implies allocation of factors to tasks. Therefore, given the above micro-foundation, the cost share parameter $a_{o,t}$ is the share of the space of tasks performed by robot capital. Given that the change in the task space performed by robots is a major source of robotics automation, I will perform the log-linearization analysis with respect to the change in $a_{o,t}$ and call the change as the expansion of the robot task space. Second, note that quality of good is a non-pecuniary "attribute whose valuation is agreed upon by all consumers" (Khandelwal, 2010). Therefore the increase in the cost share parameter $a_{o,t}$ can also be interpreted as a form of quality upgrading of robots, when combined with a suitable adjustment in the TFP term. In particular, equation (7) implies that in the long-run (hence dropping the time subscript) the demand for capital is

$$K_{i,o}^{R} = a_o \left(\frac{c_{i,o}^{R}}{P_{i,o}} \right)^{-\theta_o} Q_{i,o}^{O},$$

where $c_{i,o}^{R}$ is the long-run marginal cost of robot capital formally defined in Section B4., $P_{i,o}$ is the unit cost of performing occupation o. In this equation, a_o serves as the quality term defined above. For this reason, I use terms (positive) automation shocks and robot quality upgrading interchangeably to describe an exogenous increase in a_o.

The robot-labor substitution parameter θ_o is the key elasticity that affect the changes in real wages given the automation shocks. In Section 3.3., I show that θ_o is negatively related with the real wage changes conditional on the initial cost shares. Hence it is critical to know the value of the parameter to answer the welfare and policy questions. To the best of my knowledge, equation (7) is the most flexible formulation of substitution between robots and labor in the literature. For instance, I show that the unit cost function of Acemoglu and Restrepo (2020) can be obtained by $\theta_o \to 0$ for any o with suitable other parameter configurations in Lemma B.1 in Section B. I also show that my model can imply the production structure of Humlum (2019) in Lemma B.2.

Producers' Problem Producers' problem comprises two-tiers decisions–static optimization and dynamic optimization. The static optimization is to choose the employment and capital rental amount conditional on market prices and current stock of robot capital.

Namely, for each i and t, conditional on o-vector of stock of robot capital $\left\{K^R_{i,o,t}\right\}_o$,

$$\pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}_o\right) \equiv \max_{\left\{L_{i,o,t}\right\}_o, \left\{M_{li,t}\right\}_l, K_{i,t}} p^G_{i,t} Y^G_{i,t} - \sum_o w_{i,o,t} L_{i,o,t} - \sum_l p^G_{li,t} M_{li,t} - r_{i,t} K_{i,t}, \quad (9)$$

subject to production function (5).

The dynamic optimization is to choose the amount of purchase of new robots, or robot investment, given the current stock of robot capital. It requires the following three settings. First, for each i, o, and t, robot capital $K^R_{i,o,t}$ accumulates according to

$$K^R_{i,o,t+1} = (1-\delta) K^R_{i,o,t} + Q^R_{i,o,t}, \quad (10)$$

where $Q^R_{i,o,t}$ is the amount of new robot investment and δ is the depreciation rate of robots. Second, I assume that the new investment is given by CES aggregation of robot arms from country l, $Q^R_{li,o,t}$, and local non-robot-good input $Q^I_{i,o,t}$, called integration input,[10]

$$Q^R_{i,o,t} = \left[\sum_l \left(Q^R_{li,o,t}\right)^{\frac{\varepsilon^R-1}{\varepsilon^R}}\right]^{\frac{\varepsilon^R}{\varepsilon^R-1}\alpha^R} \left(Q^I_{i,o,t}\right)^{1-\alpha^R} \quad (11)$$

where l denotes the origin of the newly purchased robots, and α^R is the expenditure share of robot arms in the cost of investment. Given the iceberg trade cost $\tau^R_{ij,t}$, the bilateral price of robot R is $p^R_{ij,o,t} = p^R_{i,o,t}\tau^R_{ij,t}$. Write the unit investment price of robots as $P^R_{i,o,t}$. Third, installing robots is costly and requires a per-unit convex adjustment cost $\gamma Q^R_{i,o,t}/K^R_{i,o,t}$ in the robot unit, where γ governs the size of adjustment cost (Cooper and Haltiwanger, 2006).

Given these settings, a producer of non-robot good G in country i solves dynamic

[10]Note that equation (11) implies that the robots are traded because they are differentiated by origin country l. This follows the formulation of capital good trade in Anderson et al. (2019). Furthermore, combined with equation (10), equation (11) implies that the origin-differentiated investment good is aggregated at first, and then added to the stock of capital. This specification helps reduce the number of capital stock variables and is also used in Engel and Wang (2011).

optimization problem is

$$\max_{\left\{Q^R_{li,o,t}\right\},Q^I_{i,o,t}} \sum_{t=0}^{\infty}\left(\frac{1}{1+\iota}\right)^t\left[\pi_{i,t}\left(\left\{K^R_{i,o,t}\right\}_o\right)-\sum_{l,o}\left(p^R_{li,o,t}\left(1+u_{li,t}\right)Q^R_{li,o,t}+\gamma P^R_{i,o,t}Q^R_{i,o,t}\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\right], \tag{12}$$

subject to accumulation equation (10) and (11), and given $\left\{K^R_{i,o,0}\right\}_o$. Because producers are owned by households, the producer uses the household discount rate ι.

The first-order condition (FOC) of the Lagrangian function of problem (12) with respect to $Q^R_{li,o,t}$ implies the investment condition

$$p^R_{li,o,t}\left(1+u_{li,t}\right)+2\gamma\alpha^R\left(\frac{Q^R_{i,o,t}}{K^R_{i,o,t}}\right)\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}}=\lambda^R_{i,o,t}\frac{\partial Q^R_{i,o,t}}{\partial Q^R_{li,o,t}}, \tag{13}$$

where $\lambda^R_{li,o,t}$ is the marginal value of robot capital. In this equation, the left-hand side (LHS) represents the marginal cost of robot adoption (the cost of robot investment and adjustment cost), whereas the right-hand side (RHS) represents the marginal values of buying a robot from country l, $Q^R_{li,o,t}$. The FOC with respect to $K^R_{i,o,t}$ implies Euler equation

$$\lambda^R_{i,o,t}=\frac{1}{1+\iota}\left[\left(1-\delta\right)\lambda^R_{i,o,t+1}+\frac{\partial}{\partial K^R_{i,o,t+1}}\pi_{i,t+1}\left(\left\{K^R_{i,o,t+1}\right\}\right)+\gamma p^R_{i,o,t+1}\left(\frac{Q^R_{i,o,t+1}}{K^R_{i,o,t+1}}\right)^2\right]. \tag{14}$$

Equilibrium To close the model, the market clearing conditions equate demand and supply of non-robot goods, robot flows, and employment. The employment level must satisfy an adding-up constraint (4). I formalize the other conditions in Section B4. To define the equilibrium, I first define a temporary equilibrium in each period and then a sequential equilibrium. I finally discuss the steady state. Define the bold symbols as vectors of robot capital $\boldsymbol{K}^R_t \equiv \left\{K^R_{i,o,t}\right\}_{i,o}$, marginal values of robot capital $\boldsymbol{\lambda}^R_t \equiv \left\{\lambda^R_{i,o,t}\right\}_{i,o}$, employment $\boldsymbol{L}_t \equiv \left\{L_{i,o,t}\right\}_{i,o}$, workers' value functions $\boldsymbol{V}_t \equiv \left\{V_{i,o,t}\right\}_{i,o}$, non-robot good prices $\boldsymbol{p}^G_t \equiv \left\{p^G_{i,t}\right\}_i$ robot prices $\boldsymbol{p}^R_t \equiv \left\{p^R_{i,o,t}\right\}_{i,o}$, wages, $\boldsymbol{w}_t \equiv \left\{w_{i,o,t}\right\}_{i,o}$, bilateral non-robot good trade levels $\boldsymbol{Q}_t^G \equiv \left\{Q_{ij,t}^G\right\}_{i,j}$, bilateral non-robot good trade levels $\boldsymbol{Q}_t^R \equiv \left\{Q_{ij,o,t}^R\right\}_{i,j,o}$, and occupation transition shares $\boldsymbol{\mu}_t \equiv \left\{\mu_{i,oo',t}\right\}_{i,oo'}$. I write $\boldsymbol{S}_t \equiv \left\{\boldsymbol{K}_t^R, \boldsymbol{\lambda}_t^R, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$ as state variables.

Definition 1. In each period t, given state variables $\boldsymbol{S}_t$, a *temporary equilibrium* (TE) is prices $\boldsymbol{p}_t \equiv \left\{\boldsymbol{p}_t^G, \boldsymbol{p}_t^R, \boldsymbol{w}_t\right\}$ and flow quantities $\boldsymbol{Q}_t \equiv \left\{\boldsymbol{Q}_t^G, \boldsymbol{Q}_t^R, \boldsymbol{\mu}_t\right\}$ that satisfy:

1. Given $\boldsymbol{p}_t$, workers choose occupation optimally by equation (1).

2. Given $\boldsymbol{p}_t$, producers maximize flow profit (9) and optimize investment (13).

3. Factor markets clear and trade balances.

The temporary equilibrium inputs all state variables and outputs other endogenous variables that are determined contemporaneously. The following sequential equilibrium gives the determination of all state variables given initial conditions.

Definition 2. Given initial robot capital stocks and employment $\left\{\boldsymbol{K}_0^R, \boldsymbol{L}_0\right\}$, a *sequential equilibrium* (SE) is a sequence of vectors $\boldsymbol{y}_t \equiv \left\{\boldsymbol{x}_t, \boldsymbol{S}_t\right\}_t$ that satisfies the TE conditions and capital accumulation (10), the Euler equation (14), employment law of motion (3), value function (2), and the transversality condition: for any j and o,

$$\lim_{t\to\infty} e^{-\iota t}\lambda_{j,o,t}^R K_{j,o,t+1}^R = 0. \tag{15}$$

Finally, I define the steady state as a SE $\boldsymbol{y}$ that does not change over time.

3.2. Solution

I apply the log-linearization technique to solve the model. I study the effect of shocks on the sequential equilibrium $\boldsymbol{y}_t$. For this purpose, I discuss the nature of shocks in detail first. Then I preview the key conceptual steps to solve the model. The output of this subsection is a sequence of matrices $\left\{\overline{\boldsymbol{F}_t}\right\}_t$ and a matrix $\overline{\boldsymbol{E}}$ that summarizes the first-order effect on sequential equilibrium in transition dynamics, which is a key object in estimating the model in Section 4. Section C gives the details of the derivation.

In the economy described in Section 3.1., the shocks comprise changes in the economic environment and changes in policy. Consider an unexpected and permanent expansion of the robot task space $a_{o,t}$ and the change in TFP of robot producers $A^R_{i,o,t}$ as discussed in Section B1. For instance, consider the increase of robot task space $a_{o,t}$ in baseline period t_0 by Δ_o percent, or

$$a_{o,t} = \begin{cases} a_{o,t_0} & \text{if } t < t_0 \\ a_{o,t_0} \times (1 + \Delta_o) & \text{if } t \geq t_0 \end{cases}.$$

In this formulation, Δ_o is interpreted as the size of the expansion of the robot task space. I combine all these changes into a column vector $\boldsymbol{\Delta}$. I take the following three steps to solve the model. Write state variables $\boldsymbol{S}_t = \left\{\boldsymbol{K}^R_t, \boldsymbol{\lambda}^R_t, \boldsymbol{L}_t, \boldsymbol{V}_t\right\}$, and "hat" notation to denote changes from t_0: for any variable z_t, $\widehat{z_t} \equiv \ln(z_t) - \ln(z_{t_0})$.

Step 1. For a given period t, I combine the vector of shocks $\boldsymbol{\Delta}$ and (given) changes in state variables $\widehat{\boldsymbol{S}}_t$ into a (column) vector $\widehat{\boldsymbol{A}_t} = \left\{\boldsymbol{\Delta}, \widehat{\boldsymbol{S}}_t\right\}$. Log-linearizing TE conditions, I solve for matrices $\overline{\boldsymbol{D}^x}$ and $\overline{\boldsymbol{D}^A}$ such that

$$\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}}_t = \overline{\boldsymbol{D}^A}\widehat{\boldsymbol{A}_t}. \tag{16}$$

In this equation, $\overline{\boldsymbol{D}^x}$ is a substitution matrix and $\overline{\boldsymbol{D}^A}\widehat{\boldsymbol{A}_t}$ is a vector of partial equilibrium shifts in period t (Adao et al., 2019).[11] Note that there exists a block separation of matrix $\overline{\boldsymbol{D}^A} = \left[\overline{\boldsymbol{D}^{A,\Delta}}|\overline{\boldsymbol{D}^{A,S}}\right]$ such that equation (16) can be written as

$$\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}}_t - \overline{\boldsymbol{D}^{A,S}}\widehat{\boldsymbol{S}}_t = \overline{\boldsymbol{D}^{A,\Delta}}\boldsymbol{\Delta}. \tag{17}$$

[11] Since the temporary equilibrium vector $\widehat{\boldsymbol{x}}_t$ includes wages $\widehat{\boldsymbol{w}_t}$, equation (16) nests the general equilibrium comparative statics formulation in Adao et al. (2019).

Step 2. Log linearizing laws of motion and Euler equations around the old steady state, I solve for matrices $\overline{\boldsymbol{D}^{y,SS}}$ and $\overline{\boldsymbol{D}^{\Delta,SS}}$ such that $\overline{\boldsymbol{D}^{y,SS}}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{D}^{\Delta,SS}}\boldsymbol{\Delta}$, where the superscript SS denotes steady state. Combined with steady state version of equation (17), I have

$$\overline{\boldsymbol{E}^{y}}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}^{\Delta}}\boldsymbol{\Delta}, \tag{18}$$

where

$$\overline{\boldsymbol{E}^{y}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{x}} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,SS}}} \end{bmatrix}, \text{ and } \overline{\boldsymbol{E}^{\Delta}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \overline{\boldsymbol{D}^{\Delta,SS}} \end{bmatrix}.$$

Step 3. Log linearizing laws of motion and Euler equations around the new steady state, I solve for matrices $\overline{\boldsymbol{D}^{y,TD}_{t+1}}$ and $\overline{\boldsymbol{D}^{y,TD}_{t}}$ such that $\overline{\boldsymbol{D}^{y,TD}_{t+1}}\check{\boldsymbol{y}}_{t+1} = \overline{\boldsymbol{D}^{y,TD}_{t}}\check{\boldsymbol{y}}_{t}$, where the superscript TD stands for transition dynamics. Log-linearized sequential equilibrium satisfies the following first-order difference equation

$$\overline{\boldsymbol{F}^{y}_{t+1}}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^{y}_{t}}\widehat{\boldsymbol{y}}_{t} + \overline{\boldsymbol{F}^{\Delta}_{t+1}}\boldsymbol{\Delta}.$$

Combined with the transversality condition, there is a matrix $\overline{\boldsymbol{F}_t}$ such that

$$\widehat{\boldsymbol{y}}_t = \overline{\boldsymbol{F}_t}\boldsymbol{\Delta}. \tag{19}$$

3.3. Real-wage Effect of Automation

How does the flexible occupation production function (7) imply for the effect of automation? In this section, I show that the effect of automation on occupational real wages depends negatively on substitution elasticity parameters θ_o and β conditional on the changes in input and trade shares. The key insight is that the real wages are relative prices of labor to the bundle of factors, and the relative price changes are relevant input and trade shares inversely related to the elasticities of substitution. These elasticities are among the target parameters of the estimation in Section 4.

I modify notations of function (7) to express the result in a compact way. Define

$$A^K_{i,o,t} \equiv A^G_{i,t} a_{o,t},\ A^L_{i,o,t} \equiv A^G_{i,t} \left(1 - a_{o,t}\right). \tag{20}$$

Substituting these into production functions (5) and (7), I have

$$Q^G_{i,t} = \left[\sum_o \left(b_{i,o,t}\right)^{\frac{1}{\beta}} \left(Q^O_{i,o,t}\right)^{\frac{\beta-1}{\beta}}\right]^{\frac{\beta}{\beta-1}\alpha_{i,L}} \left(M_{i,t}\right)^{\alpha_{i,M}} \left(K_{i,t}\right)^{1-\alpha_{i,L}-\alpha_{i,M}},$$

and

$$Q^O_{i,o,t} = \left[\left(A^L_{i,o,t}\right)^{\frac{1}{\theta_o}} \left(L_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}} + \left(A^K_{i,o,t}\right)^{\frac{1}{\theta_o}} \left(K^R_{i,o,t}\right)^{\frac{\theta_o-1}{\theta_o}}\right]^{\frac{\theta_o}{\theta_o-1}}.$$

Therefore, one can interpret the newly defined terms $A^K_{i,o,t}$ and $A^L_{i,o,t}$ as the productivity shock on the robot and labor, respectively. The following proposition claims that the long-run real-wage implication of robot productivity change $\widehat{A^K_{i,o}}$ can be expressed by changes in input and trade shares and elasticities of substitutions.

Define the good G-producers' labor share within occupation $\widetilde{x}^L_{i,o,t}$, occupation cost share $\widetilde{x}_{i,o}$, and trade shares $\widetilde{x}_{ij,t}$ as

$$\widetilde{x}^L_{i,o,t} \equiv \frac{w_{i,o,t} L^G_{i,o,t}}{p^G_{i,o,t} Q^G_{i,o,t}},\ \widetilde{x}_{i,o,t} \equiv \frac{p^G_{i,o,t} Q^G_{i,o,t}}{p^G_{i,t} Q^G_{i,t}},\ \widetilde{x}_{ij,t} \equiv \frac{p^G_{i,t} Q^G_{ij,t}}{P^G_{i,t} Q^G_{i,t}}. \tag{21}$$

Given these, the following proposition characterizes the real-wage changes in the steady state.

Proposition 1. *Suppose robot productivity grows $\widehat{A^K_{i,o}} > 0$. For each country i and occupation o,*

$$\widehat{\left(\frac{w_{i,o}}{P^G_i}\right)} = \frac{1}{1-\alpha_{i,M}} \left(\frac{\widehat{\widetilde{x}^L_{i,o}}}{1-\theta_o} + \frac{\widehat{\widetilde{x}_{i,o}}}{1-\beta} + \frac{\widehat{\widetilde{x}_{ii}}}{1-\varepsilon}\right).$$

Proof. See Section B5. □

Proposition 1 clarifies how the elasticity parameters and change of shares of input and trade affect real wages at occupation level. Among them, one can observe that if $\theta_o > 1$, then (i) the larger the fall of the labor share within occupation $\widehat{\widetilde{x}_{i,o}^{L}}$, the larger the real wage gains, and (ii) pattern (i) is stronger if θ_o is small and close to 1. Therefore, conditional on other terms, the steady state changes of occupational real wages depend on the elasticity of substitution between robots and labor θ_o.

The intuition of Proposition 1 is the series of revealed cost reductions. The first term reveals the robot cost reduction relative to labor cost.[12] If $\theta_o > 1$, then the reduction of the price index or cost savings dominates the drop of nominal wage, increasing the real wage. Similar intuition holds for the second and third terms. The second term reveals the relative occupation cost reduction, whereas the last term reveals the relative sectoral cost reduction.

Proposition 1 also gives an extension to the welfare sufficient statistic in the trade literature. In particular, Arkolakis et al. (2012) showed that under a large class of trade models, the welfare effect of the reduction in trade costs can be summarized into a well-known ACR formula, or log-difference of the trade shares times the negative of trade elasticity. In fact, by dropping the robots and non-robot capital and aggregating occupations into one, the model reduces to:

$$\widehat{\left(\frac{w_i}{P_i^G}\right)} = \frac{1}{1-\alpha_{i,M}} \frac{1}{1-\varepsilon} \widehat{\widetilde{x}_{ii}},$$

which is a modified ACR term with intermediate goods as in Caliendo and Parro (2015) and Ossa (2015).

4. Estimation

Applying the log-linearized solution in Section 3., I show the estimation method based on the generalized method of moments (GMM), or in particular, the model-implied opti-

[12]Section B4. contains two discussions that facilitate the understandings of steady-state labor share $\chi_{i,o}^{L}$. First, I define the the steady state cost of robot capital $c_{i,o}^{g,R}$ by the steady state marginal cost of robot adoption in country i, good g, and occupation o. Second, as a special case, I show that if there is no adjustment cost $\gamma = 0$, $c_{i,o}^{g,R} = (\iota + \delta)\lambda_{i,o}^{g,R}$ and reduces to the user cost of Hall and Jorgenson (1967).

mal instrumental variable (MOIV, Adao et al., 2019). Using this approach, I address the identification challenge when estimating robot-labor EoS.

4.1. Target Parameters

To simplify the notation and tailor to my empirical application, I stick to country labels $i = 1$ as the US (USA), $i = 2$ as Japan (JPN), $i = 3$ as the Rest of the World (ROW). Following my data, I interpret country $i = 1$ as the country of interest in terms of labor market outcome variables, country $i = 2$ as the source country of automation shocks by robots, and country $i = 3$ as the (set of) countries in which the measurement of robots proxies the technological changes in country 2.

In the estimation, I allow heterogeneity across occupations of the within-occupation EoS between robots and labor. To do so, I define the occupation groups as follows. I first separate occupations into three broad occupation groups, Abstract, Manual, Routine following Acemoglu and Autor (2011). Routines occupations include production, transportation and material moving, sales, clerical, and administrative support.[13] Given the trend that production and transportation/material moving occupations introduced robots over the sample period, I further divide routine occupations into to three subcategories, Production, Transportation (which indicates transporation and material moving), and Others, where Others include sales, clerical, and administrative support. As a result, I obtain five occupation groups, which I denote as θ_{g_o}, across which I allow heterogeneity in the EoS between robot capital and labor.

I also write the vector of structural parameters as $\mathbf{\Theta}$ and its dimension as $d \equiv \dim(\mathbf{\Theta})$. To formally define $\mathbf{\Theta}$, I fix a subset of parameters of the model at conventional values. Table 1 gives such values. In particular, I assume that the annual discount rate is $\iota = 0.05$ and robot depreciation rate is 10 percent following Graetz and Michaels (2018).[14] I

[13]Abstract occupations are professional, managerial and technical occupations; manual occupations are protective service, food preparation, cleaning, personal care and personal services.

[14]For example, see King and Rebelo (1999) for the source of the conventional value of ι who matches the discount rate to the average real return on capital. For ε, see Simonovska and Waugh (2014) or Caliendo and Parro (2015).

Table 1: List of Fixed Parameters

Variable	Description	Value	Source
ι	Annual discount rate	0.05	Conventional
δ	Robot depreciation rate	0.1	Graetz and Michaels (2018)
ε	Trade elasticity	4	Conventional

take trade elasticity of $\varepsilon = 4$ from the large literature of trade elasticity estimation (e.g., Simonovska and Waugh, 2014). With these parametrizations, structural parameters to be estimated are $\mathbf{\Theta} \equiv \{\theta_{g_o}, \beta, \gamma, \phi\}$.

4.2. Estimation Method

I observe changes in endogenous variables, US occupational wages $\widehat{\boldsymbol{w}_1}$, US employment $\widehat{\boldsymbol{L}_1}$, robot shipment from Japan to the US $\widehat{\boldsymbol{Q}_{21}^R}$, and the corresponding unit values $\widehat{\boldsymbol{p}_{21}^R}$ between 1992 and 2007, as well as the initial equilibrium $\boldsymbol{y}_{t_0}$. I approximate the 15-year changes as the steady-state changes. To simplify, I focus on the expansion of robot task space $\widehat{a_o}$ and robot cost shocks in Japan $\widehat{A_{2,o}^R}$ as the source of the occupational shocks in this section. To back up the robot cost shock, I run the following fixed effect regression

$$\widehat{p_{2j,o,t}^R} = \psi_{j,t}^D + \psi_{o,t}^C + \widetilde{e}_{j,o,t}.$$

By the robot production function (6), $\psi_{j,t}^D$ captures the bilateral shocks between Japan and destination country j that may incorporate the destination country's robot demand shocks, while $\psi_{o,t}^C$ captures the cost shock to produce robots o in Japan. Since $\widehat{A_{2,o}^R}$ is the TFP shock to produce robots o that is negatively correlated with the cost shock, I measure it by $\widehat{A_{2,o}^R} = -\psi_{o,t}^C$.

To discuss the identification challenge and the countermeasure, I decompose the automation shock $\widehat{a_o}$ into observed component $\widehat{a_o^{\text{obs}}}$ and unobserved error component $\widehat{a_o^{\text{err}}}$ such that $\widehat{a_o} = \widehat{a_o^{\text{obs}}} + \widehat{a_o^{\text{err}}}$ for all o. The component $\widehat{a_o^{\text{obs}}}$ is observed conditional on parameter θ–namely, it satisfies the steady-state change of relative demand of robots and labor implied by Euler equations

$$\left(\widehat{\frac{p_{i,o}K_{i,o}}{w_{i,o}L_{i,o}}}\right) = (1-\theta)\left(\widehat{\frac{p_{i,o}}{w_{i,o}}}\right) + \frac{\widehat{a_o^{\text{obs}}}}{1-a_{o,t_0}}. \tag{22}$$

Equation (22) highlights the issues in identifying θ. First, the observed relative price change $(\widehat{p_{i,o}/w_{i,o}})$ does not identify θ because $(\widehat{p_{i,o}/w_{i,o}})$ is an endogenous and is correlated with the residual term $\widehat{a_o^{\text{obs}}}/\left(1-a_{o,t_0}\right)$ that represents the task-space expansion of robots.[15] Second, the cost shock to produce robots $\widehat{A_{2,o}^R}$ also does not also work as an instrumental variable (IV) in linear regression model of (22) because of a potential correlation between the cost shock and the observed task-space expansion shock $\widehat{a_o^{\text{obs}}}$. For example, an innovation in the robotics technology could entail both the expansion of the tasks that robots can perform ("product innovation"), and the cost saving to produce robots that performs the same task as before ("process innovation").

To overcome these identification issues, I employ a method based on the full GE model below. Conditional on $\widehat{a_o^{\text{obs}}}$, the error component $\widehat{a_o^{\text{err}}}$ can be inferred from each observed endogenous variable. Take the changes in occupational wages $\widehat{\boldsymbol{w}_1}$ for example. The steady-state solution matrix $\overline{\boldsymbol{E}}$ implies that there is a $O \times O$ submatrices $\overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}}$ and $\overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R}$ such that

$$\widehat{\boldsymbol{w}} = \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}}\widehat{\boldsymbol{a}} + \overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R}\widehat{\boldsymbol{A}_2^R}. \tag{23}$$

Since $\widehat{\boldsymbol{a}} = \widehat{\boldsymbol{a}^{\text{obs}}} + \widehat{\boldsymbol{a}^{\text{err}}}$, we have

$$\boldsymbol{\nu}_w = \widehat{\boldsymbol{w}} - \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}}\widehat{\boldsymbol{a}^{\text{obs}}} - \overline{\boldsymbol{E}}_{w_1,\boldsymbol{A}_2^R}\widehat{\boldsymbol{A}_2^R},$$

where $\boldsymbol{\nu}_w \equiv \overline{\boldsymbol{E}}_{w_1,\boldsymbol{a}}\widehat{\boldsymbol{a}^{\text{err}}}$ is the O-vector structural residual generated from the linear combination of the unobserved component of the automation shocks. Note that the structural residual depends on the structural parameters $\boldsymbol{\Theta}$. To clarify this, I occationally write the structural residual as $\boldsymbol{\nu}_w = \boldsymbol{\nu}_w(\boldsymbol{\Theta})$. For other endogenous variables $\left(\widehat{\boldsymbol{L}_1}, \widehat{\boldsymbol{p}_{21}^R}, \widehat{\boldsymbol{Q}_{21}^R}\right)$, I re-

[15] See Karabarbounis and Neiman (2014); Hubmer (2018) for this approach for a broader definition of capital

peat the same process and obtain corresponding structural errors $\left(\boldsymbol{\nu}_{\boldsymbol{L}}, \boldsymbol{\nu}_{\boldsymbol{p}^R}, \boldsymbol{\nu}_{\boldsymbol{Q}^R}\right)$. Then I stack these vectors into an $O \times 4$ matrix $\boldsymbol{\nu} \equiv \left[\boldsymbol{\nu}_{\boldsymbol{w}}, \boldsymbol{\nu}_{\boldsymbol{L}}, \boldsymbol{\nu}_{\boldsymbol{p}^R}, \boldsymbol{\nu}_{\boldsymbol{Q}^R}\right]$, and from its o-th row define 4×1 vector as $v_o = \left[\nu_{\boldsymbol{w},o}, \nu_{\boldsymbol{L},,o}, \nu_{\boldsymbol{p}^R,o}, \nu_{\boldsymbol{Q}^R,o}\right]^\top$. Given these structural residuals, I assume the following moment condition.

Assumption 1. *(Moment Condition)*

$$\mathbb{E}\left[\nu_o | \widehat{\boldsymbol{A}_2^R}\right] = \mathbf{0}.$$

Assumption 1 is a restriction on the structural residual $\boldsymbol{\nu}$ that it should not be predicted by the vector of cost shocks to produce robots. Note that the moment condition allows that the automation shock $\widehat{a_o}$ may correlate with the cost shock $\widehat{\boldsymbol{A}_2^R}$ due to, for example, the correlated process and product innovations of robots. Instead, the structural residual ν_o purges out all the predictions of the impacts of shocks $\widehat{\boldsymbol{a}}$ and $\widehat{\boldsymbol{A}_2^R}$ on endogenous variables, and I place the assumption that the remaining variation should not be predicted by cost shocks from the data.

Assumption 1 implies that, for any d-dimensional functions $\boldsymbol{H} \equiv \{H_o\}_o$, $\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\right] = 0$. The GMM estimator based on $\boldsymbol{H}$ is

$$\boldsymbol{\Theta}_{\boldsymbol{H}} \equiv \arg\min_{\boldsymbol{\Theta}} \sum_{o=1}^{O} \left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right]^\top \left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right] = 0, \tag{24}$$

which is consistent under the moment condition 1 if $\boldsymbol{H}$ satisfies the rank conditions in Newey and McFadden (1994). The exact specification of $\boldsymbol{H}$ determines the optimality, or the minimal variance, of estimator (24). To specify $\boldsymbol{H}$, I apply the approach that achieves the asymptotic optimality developed in Chamberlain (1987). Formally, define the instrumental variable Z_o as follows:

$$Z_o \equiv H_o^*\left(\widehat{\boldsymbol{A}_2^R}\right) \equiv \mathbb{E}\left[\nabla_{\boldsymbol{\Theta}}\nu_o\left(\boldsymbol{\Theta}\right)|\widehat{\boldsymbol{A}_2^R}\right]\mathbb{E}\left[\nu_o\left(\boldsymbol{\Theta}\right)\left(\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top|\widehat{\boldsymbol{A}_2^R}\right]^{-1}, \tag{25}$$

and assume the regularity conditions in Section B6.

Proposition 2. *Under Assumptions 1 and B.1, $\mathbf{\Theta}_{\boldsymbol{H}^*}$ is asymptotically normal with variance $\left(\boldsymbol{G}^\top \boldsymbol{\Omega}^{-1} \boldsymbol{G}\right)^{-1}$, which is the minimum among the asymptotic variances of the class of estimators in equation (24), where*

$$\boldsymbol{G} \equiv \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right) \nabla_{\mathbf{\Theta}} \nu_o\left(\mathbf{\Theta}\right)\right] \text{ and } \boldsymbol{\Omega} \equiv \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right) \nu_o\left(\mathbf{\Theta}\right)\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right) \nu_o\left(\mathbf{\Theta}\right)\right)^\top\right].$$

Proof. See Section B6. □

To understand the optimality of the IV in equation (25), note that it has two components. The first term is the conditional expected gradient vector $\mathbb{E}\left[\nabla_{\mathbf{\Theta}} \nu_o\left(\mathbf{\Theta}\right) | \widehat{\boldsymbol{A}_2^R}\right]$, which takes the gradient with respect to the structural parameter vector. Thus, it assigns large weight to occupation that changes the the predicted outcome variable sensitively to the parameters. The second term is the conditional inverse expected variance matrix $\mathbb{E}\left[\nu_o\left(\mathbf{\Theta}\right)\left(\nu_o\left(\mathbf{\Theta}\right)\right)^\top | \widehat{\boldsymbol{A}_2^R}\right]^{-1}$, which put large weight to occupation that has small variance of the structural residuals.

Substituting equation (25) to the general GMM estimator (24), we have an estimator $\mathbf{\Theta}_{\boldsymbol{H}^*} = \arg\min_{\mathbf{\Theta}} \left[\sum_o Z_o \nu_o\left(\mathbf{\Theta}\right)\right]^\top \left[\sum_o Z_o \nu_o\left(\mathbf{\Theta}\right)\right]$. Since Z_o depends on unknown parameters $\mathbf{\Theta}$, we implement the estimation by the following two-step feasible method: We first estimate the first-step estimate $\mathbf{\Theta}_1$ from arbitrary initial values $\mathbf{\Theta}_0$. Since the IV is a function of the cost shock $\widehat{\boldsymbol{A}_2^R}$, $\mathbf{\Theta}_1$ is consistent by Assumption 1. However, it is not optimal. To achieve the optimality, in the second step, I obtain the optimal IV using the consistent estimator $\mathbf{\Theta}_1$. To summarize the discussion so far, define IVs $Z_{o,n}$ where $n = 0, 1$ as follows:

$$Z_{o,n} \equiv H_{o,n}\left(\widehat{\boldsymbol{A}_2^R}\right) = \mathbb{E}\left[\nabla_{\mathbf{\Theta}} \nu_o\left(\mathbf{\Theta}_n\right) | \widehat{\boldsymbol{A}_2^R}\right] \mathbb{E}\left[\nu_o\left(\mathbf{\Theta}_n\right)\left(\nu_o\left(\mathbf{\Theta}_n\right)\right)^\top | \widehat{\boldsymbol{A}_2^R}\right]^{-1}. \tag{26}$$

Then I have the following result.

Proposition 3. *Under Assumptions 1 and B.1, the estimator $\mathbf{\Theta}_2$ obtained in the following procedure is consistent, asymptotically normal, and optimal:*

Step 1: With a guess $\mathbf{\Theta}_0$, estimate $\mathbf{\Theta}_1 = \mathbf{\Theta}_{\boldsymbol{H}_0}$ using $Z_{o,0}$ defined in equation (26).

Step 2: With $\mathbf{\Theta}_1$, estimate $\mathbf{\Theta}_2$ by $\mathbf{\Theta}_2 = \mathbf{\Theta}_{\boldsymbol{H}_1}$ using $Z_{o,1}$ defined in equation (26).

Proof. See Section B7. □

4.3. Result

To apply Proposition (3), I need measurement of the initial equilibrium $\boldsymbol{y}_{t_0}$ that is an input to the solution matrix $\overline{\boldsymbol{E}}$. I take these data from JARA, IFR, IPUMS USA and CPS, BACI, and World Input-Output Data (WIOD). The measurement of labor market outcomes is standard and relegated to Section A8. I set the initial period robot tax to be zero in all countries.

Table 2 gives the estimates of the structural parameters. The third column shows the estimation result when I restrict the EoS between robots and labor to be constant across occupation groups. The estimate of the within-occupation EoS between robots and labor, θ_{g_o}, implies that robots and labor are substitutes within an occupation, and rejects the Cobb-Douglas case $\theta_{g_o} = 1$ at a conventional significance level. The point estimate of the EoS between occupations, β, is 0.71, or occupation groups are complementary. The one-standard error bracket covers Humlum (2019)'s central estimate 0.49. The adjustment cost parameter γ is close to the estimate of Cooper and Haltiwanger (2006) when they restrict the model with only quadratic adjustment costs, like in my model. The one-standard error range of occupational dynamic labor supply elasticity ϕ is estimated to be $[0.55, 1.07]$, which contains an estimate of 0.6 in the dynamic occupation choice model in Traiberman (2019) in the case without the specific human capital accumulation.

The fourth column of Table 2 shows the estimation result when I allow the heterogeneity across occupation groups. The other structural estimates, (β, γ, ϕ), do not change qualitatively. Figure 3 shows the estimates of the within-occupation EoS between robots and labor, θ_{g_o}. I find that the EoS for routine production occupations and routine transportation occupations are significantly higher than other occupation groups.

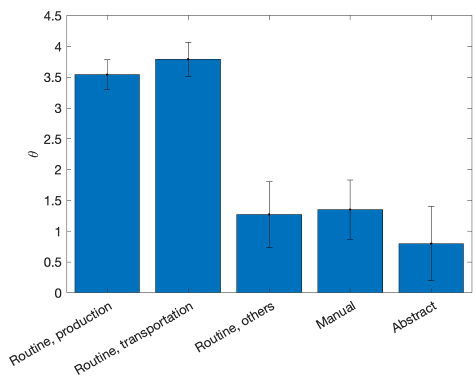
To examine the fit of the model, XXX should bring 'fig:robot-adoption-data-theory' and internal distribution of robot types.

Table 2: Parameter Estimates

Notation	Description	Homogeneous θ	Heterogeneous θ
θ	EoS between robots and labor	2.96 (0.17)	(see Figure 3)
β	EoS between occupations	0.71 (0.23)	0.73 (0.31)
γ	Capital adjustment cost	0.30 (0.11)	0.30 (0.14)
ϕ	Elasticity of occupation moving	0.81 (0.26)	0.81 (0.30)

Notes: Point estimates and standard error estimates of the structural parameters based on the two-step estimator in 3. The column "Homogeneous θ" is the estimation result that restricts θ to be constant across occupation groups. The column "Heterogeneous θ" allows heterogeneous θ for five occupation groups. See the main text for detail.

Figure 3: Heterogeneous Elasticity of Substitution between Robots and Labor

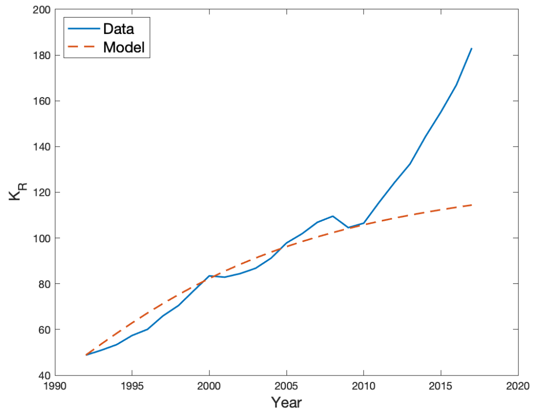


Notes: Point estimates and standard error estimates of the structural parameters based on the two-step estimator in Proposition 3, where the EoS between robots and labor θ are allowed to differ across occupation groups. The bars indicate the point estimates, and the errorbars indicate the standard error of the estimator.

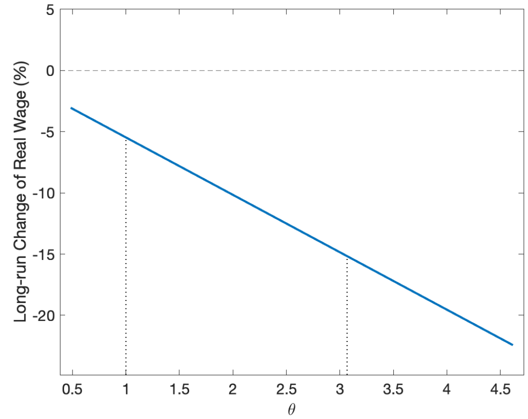
5. Counterfactual Exercises

In this section, I perform several counterfactual exercises using the estimated model. I use the estimated parameters with heterogeneous EoS between robots and labor and back out the observed shocks $\widehat{a_{o,t}^{\text{obs}}}$ from equation (22). Then I hit the economy with the backed-out $\widehat{a_{o,t}^{\text{obs}}}$ and measured robot cost shock $\widehat{A_{2,o,t}^{R}}$ and study the impact on endogenous variables through the model solution $\overline{\boldsymbol{F}}_t$. Figure 4a shows the trends of robot stock in the US in the data and the model. Although I do not match the overall robot capital stocks, the estimated model tracks the observed pattern well between 1992 and the late 2010s, consistent with the fact that I target the changes between 1992 and 2007. The model significantly underpredicts the observed accumulation of robots starting 2010, suggesting a structural change that the model does not take into account, such as the accerelation of adopting

Figure 4: Effects of Automation



(a) Trends of Overall Robot Adoptions



(b) Routine-production Workers' real wage

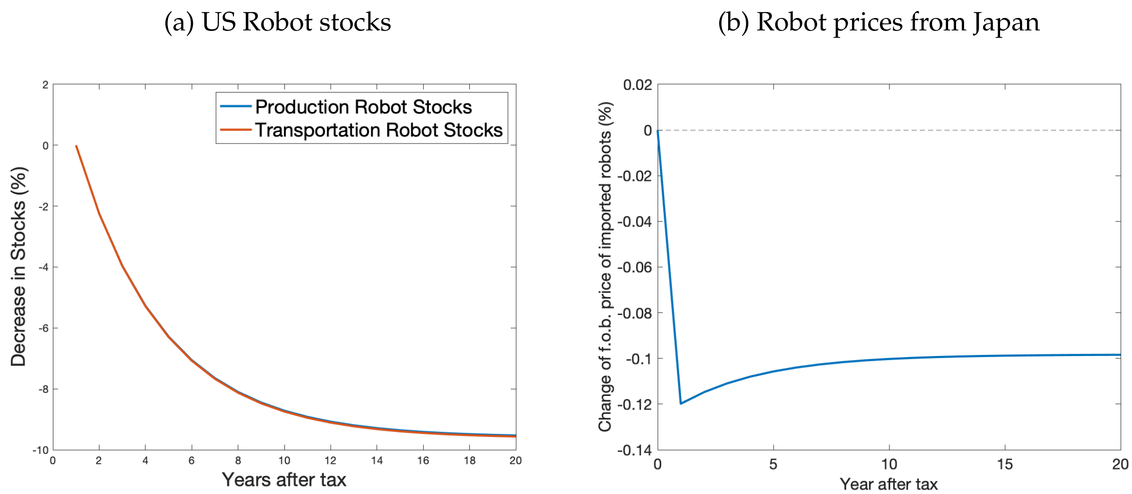
automation equipment triggered by the great recession.[16]

How does the estimated model predict the effect of automation shocks on occupational wages? To answer the question, I hit the economy with the backed-out automation shock $\widehat{a_{o,t}^{\text{obs}}}$ and derive the steady-state real-wage implications using the steady-state solution matrix $\overline{\boldsymbol{E}}$. I repeat the same process for different values of the EoS between robots and labor for the production workers, and plot the effects on the production workers' real wage against each parameter value. The result is shown in Figure 4b. In the plot, I compare the two values of the EoS, 1 and my point estimate, 3.07. The figure clarifies that under my estimates, the effect negative effect of observed robots on the real wage of production workers is roughly three-times larger.

To further study the implications of the estimated model, I consider a counterfactual rise of robot tax as well as the automation shock. In the baseline economy, all countries levied zero robot tax. Consider an unexpected, unilateral, and permanent increase in the robot tax by 30% in the US. Figure 5a shows the dynamic effect of the tax on robot stock accumulation. The tax significantly slows the accumulation of robot stocks, and decreases the steady-state stock of robots by 9.7 percent. Since the tax is uniform across

[16] Hershbein and Kahn (2018) discusses the accerelated adoption of routine-biased technologies since the great recession.

Figure 5: Effects of the Robot Tax



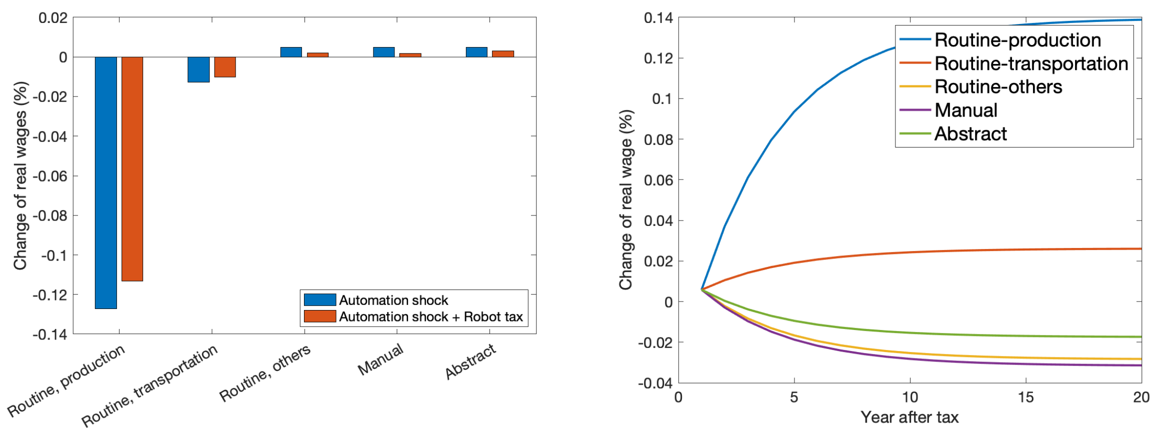
occupations that they replace, the percentage decrease of the robot stocks are very similar across occupations. In Figure 5b, I show the dynamic effect of the tax on the price of robots imported from Japan in the US. In the short-run, the price decreases due to the decreased demand from the US. As the sequential equilibrium reaches the new steady state where the US stock of robots is decreased, the marginal value of the robots is higher. The increased marginal value partially offsets the decreased price of robots in the short-run.

How does the robot tax affect real wages in each occupation? In Figure 6a, I show two scenarios of the steady-state changes in occupational real wages. On the one hand, I shock the economy only with the automation shocks. On the other hand, I shock the economy with both the automation shocks and the robot tax. The result shows heterogenous effects on occupational real wages of the robot tax. The tax mitigates the negative effect of automation on routine production workers and routine transportation workers, while the tax marginally decreases the small gains that workers in the other occupations would have enjoyed. Overall, the robot tax mitigates the large heterogeneous effects of the automation shocks, that could go negative and positive directions depending on occupation groups, and compresses the effects towards zero. Figure 6b shows the dynamics of

Figure 6: Effects of the Robot Tax on Occupational Real Wages

(a) Steady-state Comparison

(b) Transitional Effect of Tax

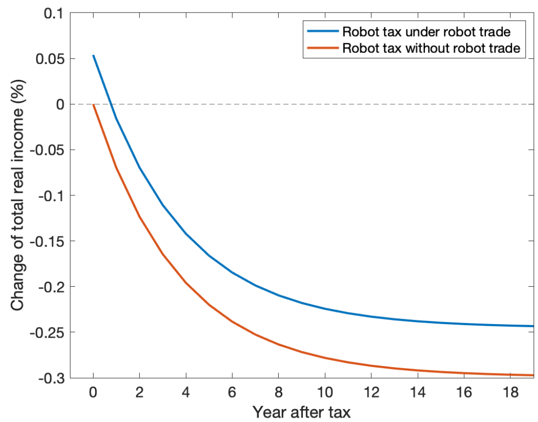


the effects of robot tax, net of the effects of automation shocks. Although the steady-state effects of robot tax were heterogeneous as shown in Figure 6a, the effect is not immediate but materializes after around 10 years, due to the sluggish adjustment in the accumulation of the robot capital stock.

How does the robot tax affect the real income at the country level? A country's total income comprises of the sum of workers' wages, the non-robot good producer's profit, and tax revenue. As I discussed in Section 3., there are two effects of the robot tax, the terms-of-trade effect in each period and the long-run capital decumulation effect. As an extreme case, I also consider an alternative model with no trade of robots due to prohibitively high robot cost, and give the robot tax counterfactual exercise.

Figure 7 shows the dynamic effect of the robot tax on the US real income. If the robot trade is not allowed, the robot tax does not increases the real income in any period since the terms-of-trade effect does not show up, but only the long-run capital decumulation effect does. On the other hand, once I allow the robot trade as we observe in the data, the robot tax may increase the real income because it decreases the price of imported robots. The effect is concentrated in the short-run before the capital decumulation process matures. In the long run, the negative decumulation effect dominates the positive terms-

Figure 7: Effects of the Robot Tax on the US Real Income



of-trade effect.

6. Conclusion

In this paper, I study the distributional and aggregate effects industrial robots, emphasizing the aspect that the robots perform specified tasks and robots are internationally traded. I make three contributions. First, I construct a dataset that tracks the number of robot arms and unit values disaggregated by occupations that robots replace. Second, I develop a general equilibrium model that features trade of robots in a large-open economy and endogenous robot accumulation with an adjustment cost. When estimating the model, to identify the within-occupation EoS between robots and labor, I construct a model-implied optimal instrumental variable from the average price of robots in my dataset.

I find that the within-occupation EoS between robots and labor are around 3 on average. These estimates are significantly larger than estimates of the EoS of capital goods and labor, with a maximum of about 1.5, revealing the susceptibility of labor to robot adaptation. I also show that the EoS are heterogeneous across occupation groups—higher in the routine production occupations than in other routine occupations such as clerical occupations. These estimates imply that robots contributed to the wage polarization across occupations in the US in 1990-2007. A commonly advertised robot tax could increase the US real income in the short run, but leads to a decline in income in the long run due to robot de-accumulation. These findings indicate that the robots can have larger distributional impacts than is considered in the previous literature, and regulating robots could potentially have a positive impact from the aggregate perspective.

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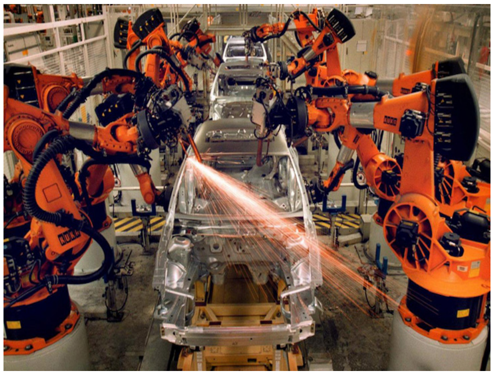
Figure A.1: Examples of Industrial Robots

(a) Spot Welding



(b) Material Handling

Sources: Autobot Systems and Automation (https://www.autobotsystems.com) and PaR Systems (https://www.par.com)

Appendix

A Data Appendix

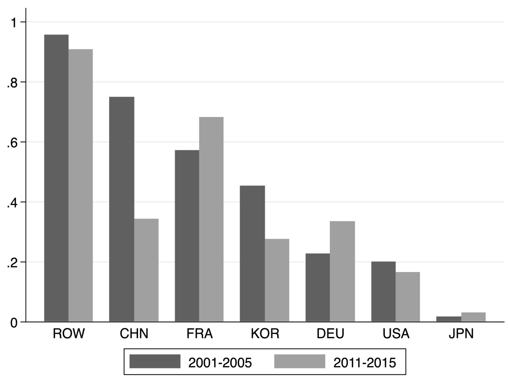
A1. Examples of Industrial Robots

Figure A.1 shows the pictures of examples of industrial robots considered in this paper. The left panel shows spot-welding robots, while the right panel shows the material-handling robots. The spot welding robots are an example of robots in routine-production occupations, while the material-handling robots are that in routine-transportation (material-moving) occupations.

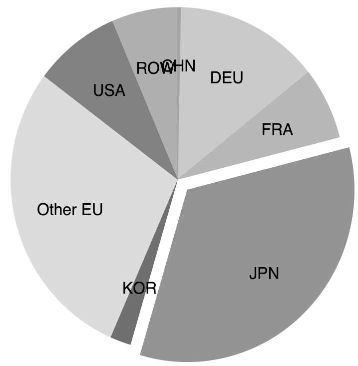
A2. Other Data Sources

I supplement data from JARA data and O*NET data by the ones from IFR, BACI, IPUMS USA. IFR is a standard data source of industrial robot adoption in several countries (Graetz and Michaels, 2018; Acemoglu and Restrepo, 2020, AR thereafter; among others), to which JARA provides the robot data of Japan. I use IFR data to show the total robot adoption in each destination country as opposed to the import from Japan. BACI provides disaggregated data on trade flows for more than 5000 products and 200 countries and is a standard data source of international trade (Gaulier and Zignago, 2010). I use BACI data to obtain the measure of international trade of industrial robots and baseline trade shares. IPUMS USA collects and harmonizes US census microdata (Ruggles et

Figure A.2: Trade of Industrial Robots



(a) Robot Import-Absorption Ratio



(b) World Robot Export Share, 2001-2005

al., 2018). I use Population Censuses (1970, 1980, 1990, and 2000) and American Community Surveys (ACS, 2006-2008 3-year sample and 2012-2016 5-year sample). I obtain occupational wages, employment, and labor cost shares from these data sources. To obtain the intermediate inputs shares, I take data from the World Input-Output Data (WIOD) in the closest year to my initial period, 1992.

A3. Trade of Industrial Robots

Figure A.2 the pattern of international trade of international robots.

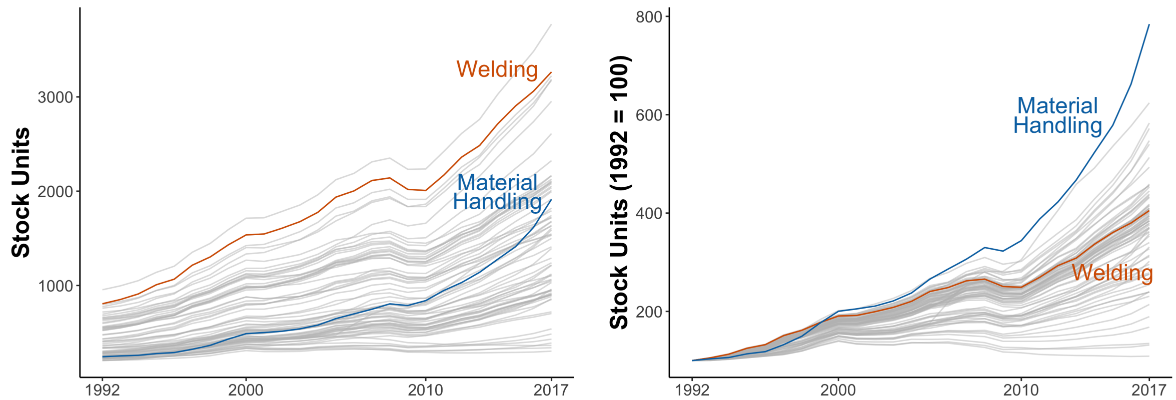
A4. Trends of Robot Stocks and Prices

I will show that different occupations experienced different trends in robot adoption. Figure A.3 shows the trend of US robot stocks at the occupation level. In the left panel, I show the trend of raw stock. First, the overall robot stocks increased rapidly in the period, as found in the previous literature. The panel also shows that the increase occurred in many occupations, but at differential rates. To highlight such a difference, in the right panel, I plot the normalized trend at 100 in the initial year. There is significant heterogeneity in the growth rates, ranging from a factor of one to eight.

For example, I color in the figure two occupations, robots that correspond to “Welding, Soldering, and Brazing Workers” (or “Welding”) and “Laborers and Freight, Stock, and Material Movers,

Figure A.3: US Robot Stocks at the Occupation Level

(a) Level (b) Normalized



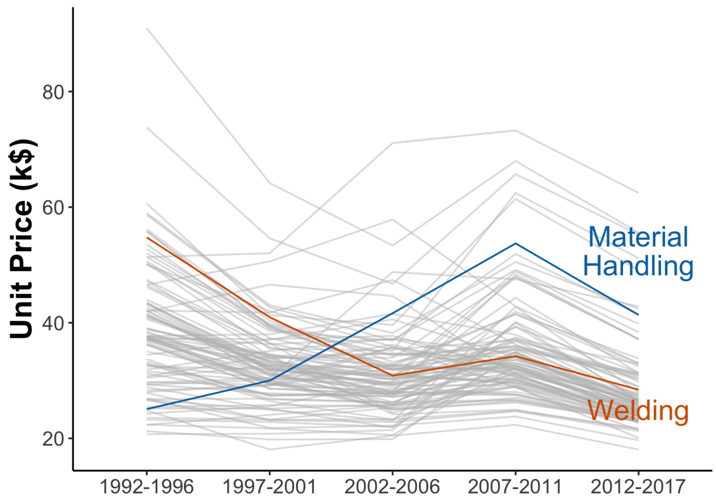
Notes: Authors' calculation based on JARA and O*NET. The figure shows the trend of stocks of robots in the US for each occupation. The left panel shows the level, whereas the right panel shows the normalized trend at 100 in 1992. In both panels, I highlight two occupations. "Welding" corresponds IPUMS USA OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds IPUMS USA OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises.

Hand" (or "Material handling"). On the one hand, welding is an occupation where the majority of robots were applied continuously throughout the period, as can be confirmed in the left panel. However, the growth rate of the stocks is not outstanding, but within the range of growth rates of other occupations. On the other hand, material handling was not a majority occupation as of the initial year, but it grew at the most rapid pace in the period.

These findings indicate the difference between the automation shocks each occupation received. Some occupations were already somewhat automated by robotics as of the initial year, and the automation process continued afterward (e.g., welding). There are a few occupations where robotics automation was not achieved initially, but the innovation and adoption occurred rapidly in the period (e.g., material handling). I propose a model that incorporates this heterogeneity and discuss how to exploit it in estimation in the following sections.

Figure A.4 shows the trend of prices of robots in the US for each occupation. In addition to the overall decreasing trend, there is significant heterogeneity in the pattern of price falls across occupations. For instance, although the welding robots saw a large drop in the price during the 1990s, the material handling robots did not but increased the price over the sample period.

Figure A.4: Robot Prices at the Occupation Level



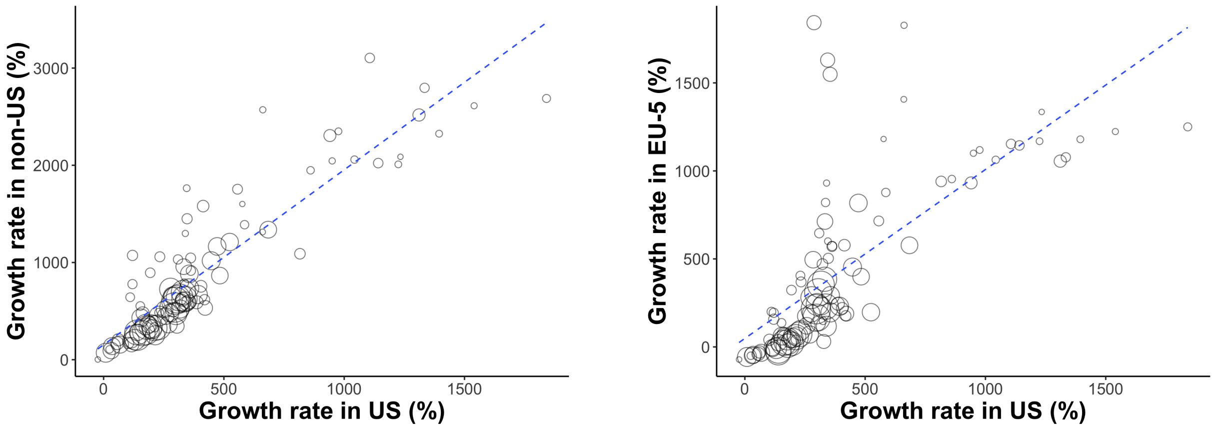
Notes: Authors' calculation based on JARA and O*NET. The figure shows the trend of prices of robots in the US for each occupation. I highlight two occupations. "Welding" corresponds IPUMS USA OCC2010 = 8140 "Welding, Soldering, and Brazing Workers." "Material Handling" corresponds IPUMS USA OCC2010 = 9620 "Laborers and Freight, Stock, and Material Movers, Hand." Years are aggregated into five-year bins (with the last bin 2012-2017 being six-year one) to smooth out yearly noises. The dollars are converted to 2000 US dollar using CPI.

A5. Robots from Japan in the US, Europe, and the Rest of the World

I review the international comparison of the pattern of robot adoption. I generate the growth rates of stock of robots between 1992 and 2017 at the occupation level for each group of destination countries. The groups are the US, the non-US countries, (namely, the world excluding the US and Japan), and five European countries (or "EU-5"), Denmark, Finland, France, Italy, and Sweden used in AR. To calculate the stock of robots, I employ the perpetual inventory method with depreciation rate of $\delta = 0.1$, following Graetz and Michaels (2018).

Figure A.5 shows scatterplots of the growth rates at the occupation level. The left panel shows the growth rates in the US on the horizontal axis and the ones in non-US countries on the vertical axis. The right panel shows the same measures on the horizontal axis, but the growth rates in the set of EU-5 countries on the vertical axis. These panels show that the stocks of robots at the occupation level grow (1992-2017) between the US and non-US proportionately relative to those between the US and EU-5. This finding is in contrast to AR, who find that the US aggregate robot stocks grew at a roughly similar rate as those did in EU-5. It also indicates that non-US growth patterns reflect growths of robotics technology at the occupation level available in the US. I will

Figure A.5: Growth Rates of Robots at the Occupation Level



(a) Comparison between the US and non-US

(b) Comparison between the US and EU-5

Notes: Authors' calculation based on JARA, and O*NET. The left panel shows the correlation between occupation-level growth rates of robot stock quantities from Japan to the US and the growth rates of the quantities to the non-US countries. The right one shows the correlation between growth rates of the quantities to the US and EU-5 countries. Non-US are the aggregate of all countries excluding the US and Japan. EU-5 are the aggregate of Denmark, France, Finland, Italy, and Sweden used in Acemoglu and Restrepo (2020). Each bubble shows each occupation, and the size of the bubble reflects the stock of robot in the US in the baseline year, 1992. See the main text for the detail of the method to create the variables.

use these patterns as the proxy for robotics technology available in the US. In Section 3. and on, I take a further step and solve for the robot adoption quantity and values in non-US countries in general equilibrium including the US and non-US countries.

It is worth mentioning that a potential cause of the difference between my finding and AR's is the difference in data sources. In contrast to the JARA data I use, AR use IFR data that include all robot seller countries. In particular, recall that EU-5 is closer to major robot producer countries other than Japan, including Germany, the robot adoption pattern across occupations may be influenced by their presence. If these close producers have a comparative advantage in producing robots for a particular occupation, then EU-5 may adopt the robots for such occupations intensively from close producers. In contrast, countries out of EU-5, including the US, may not benefit the closeness to these producers. Thus they are more likely to purchase robots from far producers from EU-5, such as Japan.

A6. Robustness Check of Fact 2.3.

To be added.

A7. Examples of Robotics Innovation

I use robot task space $a_{o,t}$ as the automation shock, and robot producer's TFP $A^R_{l,o,t}$ as the cost shock to produce robots. In this section, I show some examples of changes of robot technology and new patents to facilitate understandings of these interpretation. An example of task space expansion is adopting *Programmed Article Transfer* (PAT, Devol, 1961). PAT was machine that moves objects by a method called "teaching and playback". Teaching and playback method needs one-time teaching of how to move, after which the machine playbacks the movement repeatedly and automatically. This feature frees workers of performing repetitive tasks. PAT was intensively introduced in spot welding tasks. KHI (2018) reports that among 4,000 spot welding points, 30% were done be human previously, which PAT took over. Therefore, I interpret the adoption of PAT as the example of the expansion of the robot task space, or increase in $a_{o,t}$. Note that AR also analyze this type of technological change.

An example of cost reduction is adopting *Programmable Universal Manipulator for Assembly (PUMA).* PUMA was designed to quickly and accurately transport, handle and assemble automobile accessories. A new computer language, *Variable Assembly Language (VAL)*, made it possible because it made the teaching process less work and more sophisticated. In other words, PUMA made tasks previously done by other robots but at cheaper unit cost per unit of task.

It is also worth mentioning that introduction of a new robot brand typically contains both components of innovation (task space expansion and cost reduction). For example, PUMA also expanded task space of robots. Since VAL allowed the use of sensors and "expanded the range of applications to include assembly, inspection, palletizing, resin casting, arc welding, sealing and research" (KHI, 2018).

A8. Initial Share Data

Since the log-linearized sequential equilibrium solution depends on several initial share data generated from the initial equilibrium, I show the data sources and methods for measuring these shares. I set $t_0 = 1992$ and the time frequency is annual. I consider the world that consists of three countries $\{USA, JPN, ROW\}$.

Table A.1 summarizes the variable notations, descriptions, and data sources. First, I take matri-

Table A.1: List of Data Sources

Variable	Description	Source
$\widetilde{y}^G_{ij,t_0}, \widetilde{x}^G_{ij,t_0}, \widetilde{y}^R_{ij,t_0}, \widetilde{x}^R_{ij,t_0}$	Trade shares of goods and robots	BACI, IFR
$\widetilde{x}^O_{i,o,t_0}$	Occupation cost shares	IPUMS
l_{i,o,t_0}	Labor shares within occupation	JARA, IFR, IPUMS
$s^G_{i,t_0}, s^V_{i,t_0}, s^R_{i,t_0}$	Robot expenditure shares	BACI, IFR, WIOT
$\alpha_{i,M}$	Intermediate input share	WIOT

ces of trade of goods and robots by BACI data. As in Humlum (2019), I measure robots by HS code 847950 ("Industrial Robots For Multiple Uses") and approximate the initial year value by year of 1998, in which the robot HS code is first available. To obtain the domestic robot absorption data, I take from IFR data the flow quantity variable and the aggregate price variable for a selected set of countries. I then multiply these to obtain USA and JPN robot adoption value. For robot prices in ROW, I take the simple average of the prices among the set of countries (France, Germany, Italy, South Korea, and the UK, as well as Japan and the US) for which the price is available in 1999, the oldest year in which the price data are available. Graetz and Michaels (2018) discuss prices of robots with the same data source.

Second, I construct occupation cost shares $\widetilde{x}^O_{i,o,t_0}$ and labor shares within occupation l_{i,o,t_0} as follows. To measure $\widetilde{x}^O_{i,o,t_0}$, I aggregate the total wage income of workers that primarily works in each occupation o in year 1990, the Census year closest to t_0. I then take the share of this total compensation measure for each occupation. To measure l_{i,o,t_0}, I take the total compensation as the total labor cost and a measure of the user cost of robots for each occupation. The user cost of robots is calculated with the occupation-level robot price data available in JARA and the set of calibrated parameters in Table **??**. Due to the lack of data, I proxy the robot prices for non-Japanese robots with the JARA data.

Third, I take intermediate input share $\alpha_{i,M}$ from World Input-Output Tables (WIOT Timmer et al., 2015). Finally, I combine the trade matrix generated above and WIOT to construct the good and robot expenditure shares $s^G_{i,t_0}, s^V_{i,t_0}, s^R_{i,t_0}$. In particular, with the robot trade matrix, I take the total sales value by summing across importers for each exporter, and total absorption value by summing across exporters for each importers. I also obtain the total good absorption by WIOT. From these total values, I take expenditure shares.

B Theory Appendix

B1. Further Discussion of Model Assumptions

I highlight key features of the model, their comparisons to the literature, and possible interpretations.

Capital-Skill Complementarity Occupation production function (7) also nest the one in the literature of capital-skill complementarity (Krusell et al., 2000 among others). To simplify, I focus on individual producer's production function in the steady state. Thus I drop subscripts and superscripts of country i, good g, and time period t. Suppose the set of occupations is $O \equiv \{R, U\}$ and $a_U = 0$. R stands for the robotized occupation (e.g., spot welding) and U stands for "unrobotized" (e.g., programming). Note that since U is unrobotized $a_U = 0$. Then the unit cost of occupation aggregate (7), p^O, is

$$p^O = \left[(b_R)^{\frac{1}{\beta}} \left((1 - a_R)(w_R)^{1-\theta_R} + a_R (c_R)^{1-\theta_R} \right)^{\frac{1-\beta}{1-\theta_R}} + (b_U)^{\frac{1}{\beta}} (w_U)^{1-\beta} \right]^{\frac{1}{1-\beta}}.$$

Thus different skills R and U are substituted by robots with different substitution parameters θ_R and β, respectively. Since the literature of capital-skill complementarity studies the rising skill premium, the current model also has an ability to discuss the occupation (or skill) premium given the different level of automation across occupations.

Adjustment Cost of Robot Capital I give an interpretation of another key feature of the model, the convex adjustment cost of robot adoption. The interpretation is twofold–the cost of adopting new technology and of integration. With these convex adjustments costs, the model predict the staggered adoption of robots over years, which we observe in the data.

First, when adopting new technology including robots, it is necessary to re-optimize the overall production process since the production process is typically optimized to employ workers. More generally, the literature of organizational dynamics studies the difficulty, not to say the impossibility, of changing strategies of a company due to complementarities (see Brynjolfsson and Milgrom, 2013 for a review). Reoptimization incurs an additional cost of adoption in addition to the purchase of robot arms. Moreover, even within a production unit, there is a variation of this difficulty of adopting robots across production processes. In this case, the part where the adjustment is easy adopts the robots first, and vice versa. This allocation of robot adoptions over years may aggregate to make the robot stock increase slowly (Baldwin and Lin, 2002).

The second component of the adjustment cost may come from the cost of integration as I discussed in Section 2.1. The marginal integration cost may increase as the massive upgrading of robotics system may require large-scale overhaul of production process, which increases the complexity and so is costly. The adjustment cost may capture the increasing marginal cost component of the integration cost. Note that it explains a different component of the integration cost implied by constant returns-to-scale (CRS) robot aggregation (11) as it adds the increasing marginal cost component to the constant one based on the CRS structure.

Another potential choice of modeling a staggered growth of robot stocks is to assume a fixed cost of robot adoption and lumpy investment. Humlum (2019) finds that many plants buy robots only once during the sample period. Since JARA data does not observe plant-level adoptions, I do not separately identify lumpy investment from the staggered growth of robot stocks in the data. To the extent that fixed cost of investment may make the policy intervention less effective (e.g., Koby and Wolf, 2019), the counterfactual analysis in this paper may overestimate the effect of robot taxes since it does not take into account the fixed cost and lumpiness of investment.

B2. Derivation of Worker's Optimality Conditions

In this section, I formalize the assumptions behind the derivation and show equations (1) and (2).

Fix country i and period t. There is a mass $\overline{L}_{i,t}$ of workers. In the beginning of each period, worker $\omega \in \left[0, \overline{L}_{i,t}\right]$ draws a multiplicative idiosyncratic preference shock $\left\{Z_{i,o,t}\left(\omega\right)\right\}_{o}$ that follows an independent Fréchet distribution with scale parameter $A^{V}_{i,o,t}$ and shape parameter $1/\phi$. Note that one can simply extend that the idiosyncratic preference follows a correlated Fréchet distribution to allow correlated preference across occupations, as in Lind and Ramondo (2018). To keep the expression simple, I only consider the case of independent distribution. A worker ω then works in the current occupation, earns income, consumes and derives logarithmic utility, and then chooses the next period's occupation with discount rate ι. When choosing the next period occupation o', she pays an ad-valorem moving cost $\chi_{i,oo',t}$ in terms of consumption unit that depends on current occupation o. She consumes her income in each period. Thus, worker ω who currently works in occupation o_t maximizes the following objective function over the future stream of utilities by choosing occupations $\{o_s\}_{s=t+1}^{\infty}$:

$$E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota} \right)^{s-t} \left[\ln \left(C_{i,o_s,s} \right) + \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right]$$

where $C_{i,o,s}$ is a consumption bundle when working in occupation o in period $s \geq t$, and E_t is the expectation conditional on the value of $Z_{i,o_t,t}\left(\omega\right)$. Each worker owns occupation-specific labor endowment $l_{i,o,t}$. I assume that her income is comprised of labor income $w_{i,o,t}$ and occupation-specific ad-valorem government transfer with rate $T_{i,o,t}$. Given the consumption price $P_{i,t}$, the budget constraint is

$$P_{i,t} C_{i,o,t} = w_{i,o,t} l_{i,o,t} \left(1 + T_{i,o,t} \right)$$

for any worker.

By linearity of expectation,

$$\begin{aligned} & E_t \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota} \right)^{s-t} \left[\ln \left(C_{i,o_s,s} \right) + \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right] \\ = & \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota} \right)^{s-t} \left[E_t \ln \left(C_{i,o_s,s} \right) + E_t \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + E_t \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right]. \end{aligned}$$

By monotone transformation with exponential function,

$$\begin{aligned} & \exp \left\{ \sum_{s=t}^{\infty} \left(\frac{1}{1+\iota} \right)^{s-t} \left[E_t \ln \left(C_{i,o_s,s} \right) + E_t \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + E_t \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right]. \right\} \\ = & \prod_{s=t}^{\infty} \exp \left\{ \left(\frac{1}{1+\iota} \right)^{s-t} \left[E_t \ln \left(C_{i,o_s,s} \right) + E_t \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + E_t \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right] \right\}. \end{aligned}$$

Write the value function conditional on the realization of shocks at period t as follows:

$$V_{i,o_t,t}\left(\omega\right) \equiv \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp \left\{ \left(\frac{1}{1+\iota} \right)^{s-t} \left[E_t \ln \left(C_{i,o_s,s} \right) + E_t \ln \left(1 - \chi_{i,o_s o_{s+1},s} \right) + E_t \ln \left(Z_{i,o_{s+1},s} \left(\omega \right) \right) \right] \right\}.$$

I apply Bellman's principle of optimality as follows:

$$
\begin{aligned}
& V_{i,o_t,t}(\omega) \\
&= \max_{\{o_s\}_{s=t+1}^{\infty}} \prod_{s=t}^{\infty} \exp\left\{ \left(\frac{1}{1+\iota}\right)^{s-t} \left[E_t \ln\left(C_{i,o_s,s}\right) + E_t \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_t \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \right\} \\
&= \max_{o_{t+1}} \exp\left\{\ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right) + \ln\left(Z_{i,o_{t+1},t}(\omega)\right)\right\} \times \\
& \max_{\{o_s\}_{s=t+2}^{\infty}} \prod_{s=t+1}^{\infty} \exp\left\{ \left(\frac{1}{1+\iota}\right)^{s-(t+1)} \left[E_{t+1} \ln\left(C_{i,o_s,s}\right) + E_{t+1} \ln\left(1-\chi_{i,o_s o_{s+1},s}\right) + E_{t+1} \ln\left(Z_{i,o_{s+1},s}(\omega)\right)\right] \right\} \\
&= \max_{o_{t+1}} \exp\left\{\ln\left(Z_{i,o_t,t}(\omega)\right) + \ln\left(C_{i,o_t,t}\right) + \ln\left(1-\chi_{i,o_t o_{t+1},t}\right)\right\} V_{i,o_{t+1},t+1},
\end{aligned}
$$

where $V_{i,o_t,t}$ is the unconditional expected value function $V_{i,o_t,t} \equiv E_{t-1} V_{i,o_t,t}(\omega)$. Changing the notation from (o_t, o_{t+1}) into (o, o'), I have

$$
V_{i,o,t}(\omega) = \max_{o'} C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1}.
$$

Solving the worker's maximization problem is equivalent to

$$
\begin{aligned}
& \Pr\left(\text{worker } \omega \text{ in } o \text{ chooses occupation } o'\right) \\
&= \Pr\left(\max_{o''} C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) Z_{i,o'',t}(\omega) V_{i,o'',t+1} \leq C_{i,o,t}\left(1-\chi_{i,oo',t}\right) Z_{i,o',t}(\omega) V_{i,o',t+1}\right).
\end{aligned}
$$

By independent Fréchet assumption,

$$
\mu_{i,oo',t} = \frac{\left(C_{i,o,t}\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(C_{i,o,t}\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}} = \frac{\left(\left(1-\chi_{i,oo',t}\right) V_{i,o',t+1}\right)^{\phi}}{\sum_{o''}\left(\left(1-\chi_{i,oo'',t}\right) V_{i,o'',t+1}\right)^{\phi}},
$$

$$
V_{i,o,t+1} = \widetilde{\Gamma} C_{i,o,t} \left(\sum_{o'}\left(\left(1-\chi_{i,oo',t+1}\right) V_{i,o',t+2}\right)^{\phi}\right)^{\frac{1}{\phi}}.
$$

B3. Relationship with Other Models of Automation

The model in Section 3. is general enough to nest models of automation in the previous literature. In particular, I show how the production functions (5) and (7) imply to specifications in AR and Humlum (2019). Since the discussion is about individual producer's production function, I fix country i and focus on steady states and thus drop subscripts i and t throughout this Section B3..

B3.1. Relationship with the model in Acemoglu and Restrepo (2020, AR)

Following AR that abstract from occupations, I drop occupations by setting $O = 1$ in this paragraph. Therefore, the EoS between occupations β plays no role, and $\theta_o = \theta$ is a unique value. AR show that the unit cost (hence the price given perfect competition) function is written as

$$p^{AR} \equiv \frac{1}{\widetilde{A}} \left[(1 - \widetilde{a}) \frac{w}{A^L} + \widetilde{a} \frac{c^R}{A^R} \right]^{\alpha_L} r^{1-\alpha_L},$$

for each sector and location (See AR, Appendix A1, equation A5). In this equation, c^R is the steady state marginal cost of robot capital defined in equation (B.23) and A^L and A^R represent per-unit efficiency of labor and robots, respectively. In Lemma B.1 below, I prove that my model implies a unit cost function that is strict generalization of p^{AR} with proper modification to the shock terms and parameter configuration. I begin with the modification that allows per-unit efficiency terms in my model.

Definition B.1. For labor and robot per-unit efficiency terms $A^L > 0$ and $A^R > 0$ respectively, modified robot task space $\widetilde{a}$ and TFP term $\widetilde{A}$ are

$$\widetilde{a} \equiv \frac{a \left(A^L\right)^{\theta-1}}{a \left(A^L\right)^{\theta-1} + (1-a) \left(A^R\right)^{\theta-1}}, \tag{B.1}$$

$$\widetilde{A} \equiv \frac{A}{\left[(1-\widetilde{a}) \left(A^L\right)^{\theta-1} + \widetilde{a} \left(A^R\right)^{\theta-1} \right]}. \tag{B.2}$$

Lemma B.1. *Set the number of occupations $O = 1$. In the steady state,*

$$p = \frac{1}{\widetilde{A}} \left[(1-\widetilde{a}) \left(\frac{w}{A^L}\right)^{1-\theta} + \widetilde{a} \left(\frac{c^R}{A^R}\right)^{1-\theta} \right]^{\frac{\alpha_L}{1-\theta}} p^{\alpha_M} r^{1-\alpha_M-\alpha_L}. \tag{B.3}$$

Proof. Note that modified robot task space (B.1) and modified TFP (B.2) can be inverted to have

$$a \equiv \frac{\widetilde{a} \left(A^R\right)^{\theta-1}}{(1-\widetilde{a}) \left(A^L\right)^{\theta-1} + \widetilde{a} \left(A^R\right)^{\theta-1}}, \tag{B.4}$$

$$A \equiv \left[(1-\widetilde{a}) \left(A^L \right)^{\theta-1} + \widetilde{a} \left(A^R \right)^{\theta-1} \right] \widetilde{A}. \tag{B.5}$$

Cost minimization problem with the production functions (5) and (7) and perfect competition imply

$$p = \frac{1}{A} \left(p^O \right)^{\alpha_L} p^{\alpha_M} r^{1-\alpha_L-\alpha_M},$$

and

$$p^O = \left[(1-a) w^{1-\theta} + a^{1-\theta} \right]^{\frac{1}{1-\theta}},$$

where p^O is the unit cost of aggregated occupation $Q^O \equiv \left[\sum_o (Q_o)^{(\beta-1)/\beta} \right]^{\beta/(\beta-1)}$. Substituting equations (B.4) and (B.5) and rearranging, we have

$$p = \frac{1}{\widetilde{A}} \left(\widetilde{p^O} \right)^{\alpha_L} p^{\alpha_M} r^{1-\alpha_L-\alpha_M},$$

where

$$\widetilde{p^O} = \left[(1-\widetilde{a}) \left(\frac{w}{A^L} \right)^{1-\theta} + a \left(\frac{c^R}{A^R} \right)^{1-\theta} \right]^{\frac{1}{1-\theta}}.$$

□

Lemma B.1 immediately implies the following corollary that shows that the steady state modified unit cost (B.3) strictly nests the unit cost formulation of AR as a special case of Leontief occupation aggregation.

Corollary B.1. *Suppose $\alpha^M = 0$. Then as $\theta \to 0$, $p \to p^{AR}$.*

B3.2. Relationship with the model in Humlum (2019)

I show that production functions (5) and (7) nest the production function used by Humlum (2019). Namely, for each firm in each period, Humlum (2019) specifies

$$Q^D = \exp\left[\varphi_H^D + \gamma_H^D K \right] \left[\sum_o \left(\exp\left[\varphi_o^D + \gamma_o^D K \right] \right)^{\frac{1}{\beta}} (L_o)^{\frac{\beta-1}{\beta}} \right]^{\frac{\beta}{\beta-1}}, \tag{B.6}$$

where $K = \{0, 1\}$ is a binary choice, $\varphi_H^D, \gamma_H^D, \varphi_o^D$ and γ_o^D are parameters, and superscript D represents the discrete adoption problem of Humlum (2019). As normalization, suppose that

$$\sum_o \exp\left(\varphi_o^D + \gamma_o^D K\right) = 1.$$

I will start from production function (5) and (7), place restrictions, and arrive at equation (B.6). As a key observation, relative to the discrete choice of robot adoption in Humlum (2019), the continuous choice of robot *quantity* in production function (7) allows significant flexibility. In this paragraph, I assume away with intermediate inputs. This is because Humlum (2019) assumes that intermediate inputs enter in an element of CES, while production function (5) implies that intermediate inputs enter as an element of the Cobb-Douglas function.

Now, given our production functions (5) and (7), suppose producers follow the binary decision rule defined below.

Definition B.2. A binary decision rule by a producer is that producers can choose between two choices: adopting robots $K = 1$ or not $K = 0$. If they choose $K = 1$, they adopt robots at the same unit as labor $K_o^R = L_o \geq 0$ for all occupation o. If they choose $K = 0$, $K_o^R = 0$ for all o.

Note that the binary decision rule is nested in the original choice problem from $K_o^R \geq 0$ for each o. Set

$$A_o^D\left(K^R\right) \equiv \begin{cases} A_o\left((1-a_o)^{\frac{1}{\theta}} + (a_o)^{\frac{1}{\theta}}\right)^{\frac{\theta}{\theta-1}(\beta-1)} & \text{if } K^R = L_o \\ A_o\left(1-a_o\right)^{\frac{1}{\theta-1}(\beta-1)} & \text{if } K^R = 0 \end{cases}.$$

Then I have

$$Q = \left[\sum_o \left(A_o^D\left(K_o^R\right)\right)^{\frac{1}{\beta}} \left(L_o\right)^{\frac{\beta-1}{\beta}}\right]^{\frac{\beta}{\beta-1}}.$$

To normalize, define

$$\widetilde{A_o^D} \equiv \left(\sum_o A_o^D\left(K_o^R\right)\right)^{\frac{1}{\beta-1}}$$

and

$$a_o^D\left(K_o^R\right) \equiv \frac{A_o^D\left(K_o^R\right)}{\sum_{o'} A_{o'}^D\left(K_{o'}^R\right)}.$$

Then I have

$$Q=\widetilde{A_o^D}\left[\sum_o\left(a_o^D\left(K_o^R\right)\right)^{\frac{1}{\beta}}\left(L_o\right)^{\frac{\beta-1}{\beta}}\right]^{\frac{\beta}{\beta-1}}. \tag{B.7}$$

Finally, let

$$A_{o,0}\equiv\left[\exp\left(\varphi_H^D+\varphi_o^D\right)\right]^{\frac{\theta_o-1}{\beta-1}}$$

and

$$A_{o,1}\equiv\left[\left(\exp\left(\varphi_H^D+\varphi_o^D+\gamma_H^D+\gamma_o^D\right)\right)^{\frac{1}{\theta_o}\frac{\theta_o-1}{\beta-1}}-\left(\exp\left(\varphi_H^D+\varphi_o^D\right)\right)^{\frac{1}{\theta_o}\frac{\theta_o-1}{\beta-1}}\right]^{\theta_o}.$$

and also let A_o and a_o satisfy

$$A_o=\left(A_{o,0}+A_{o,1}\right)^{\frac{\beta-1}{\theta_o-1}} \tag{B.8}$$

and

$$a_o=\frac{A_{o,1}}{A_{o,0}+A_{o,1}}. \tag{B.9}$$

Then one can substitute equations (B.8) and (B.9) to equation (B.7) and confirm that $Q=Q^{Humlum}$. Summarizing the discussion above, I have the result that my model can be restricted to produce the production side of the model of Humlum (2019) as follows.

Lemma B.2. *Suppose that (i) producers follow the binary decision rule in Definition B.2 and that (ii) occupation productivity A_o and robot task space a_o satisfy equations (B.8) and (B.9) for each o. Then $Q=Q^D$.*

B4. Equilibrium Characterization

To characterize the producer problem, I show the static optimization conditions and then the dynamic ones. To solve for the static problem, consider the FOCs of equation (9)

$$p_{i,t}^G\alpha_{i,L}\frac{Y_{i,t}^G}{Q_{i,t}^O}\left(b_{i,o,t}\frac{Q_{i,t}^O}{Q_{i,o,t}^O}\right)^{\frac{1}{\beta}}\left(\left(1-a_{o,t}\right)\frac{Q_{i,o,t}^O}{L_{i,o,t}}\right)^{\frac{1}{\theta_o}}=w_{i,o,t}, \tag{B.10}$$

$$p_{i,t}^G\alpha_{i,M}\frac{Y_{i,t}^G}{M_{i,t}}\left(\frac{M_{i,t}}{M_{li,t}}\right)^{\frac{1}{\varepsilon}}=p_{li,t}^G, \tag{B.11}$$

and

$$p_{i,t}^G\alpha_{i,K}\frac{Y_{i,t}^G}{K_{i,t}}=r_{i,t}, \tag{B.12}$$

where $\alpha_{i,K} \equiv 1 - \alpha_{i,L} - \alpha_{i,M}$. Note also that by the envelope theorem,

$$\frac{\partial \pi_{i,t}\left(\left\{K_{i,o,t}^{R}\right\}\right)}{\partial K_{i,o,t}^{R}} = p_{i,t}^{G}\frac{\partial Y_{i,t}}{\partial K_{i,o,t}^{R}} = p_{i,t}^{G}\left(\alpha_{L}\frac{Y_{i,t}^{G}}{Q_{i,t}^{O}}\left(b_{i,o,t}\frac{Q_{i,t}^{O}}{Q_{i,o,t}^{O}}\right)^{\frac{1}{\beta}}\left(a_{o,t}\frac{Q_{i,o,t}^{O}}{K_{i,o,t}^{R}}\right)^{\frac{1}{\theta}}\right). \tag{B.13}$$

To solve the dynamic problem, set up the (current-value) Lagrangian function of good producers

$$\mathcal{L}_{i,t} = \sum_{t=0}^{\infty}\left\{\left(\frac{1}{1+\iota}\right)^{t}\left[\pi_{i,t}\left(\left\{K_{i,o,t}^{R}\right\}_{o}\right) - \sum_{l,o}\left(p_{li,o,t}^{R}\left(1+u_{li,t}\right)Q_{li,o,t}^{R} + P_{i,t}^{G}Q_{i,o,t}^{I} + \gamma P_{i,o,t}^{R}Q_{i,o,t}^{R}\frac{Q_{i,o,t}^{R}}{K_{i,o,t}^{R}}\right)\right]\right\}.$$
$$- \lambda_{i,o,t}^{R}\left\{K_{i,o,t+1}^{R} - (1-\delta)K_{i,o,t}^{R} - Q_{i,o,t}^{R}\right\}$$

Taking the FOC with respect to $Q_{li,o,t}^{R}$, I have

$$p_{li,o,t}^{R}\left(1+u_{li,t}\right) + 2\gamma P_{i,o,t}^{R}\left(\frac{Q_{i,o,t}^{R}}{K_{i,o,t}^{R}}\right)\frac{\partial Q_{i,o,t}^{R}}{\partial Q_{li,o,t}^{R}} = \lambda_{i,o,t}^{R}\frac{\partial Q_{i,o,t}^{R}}{\partial Q_{li,o,t}^{R}}. \tag{B.14}$$

Taking the FOC with respect to $Q_{i,o,t}^{I}$, I have

$$P_{i,t}^{G} + 2\gamma P_{i,o,t}^{R}\left(\frac{Q_{i,o,t}^{R}}{K_{i,o,t}^{R}}\right)\frac{\partial Q_{i,o,t}^{R}}{\partial Q_{i,o,t}^{I}} = \lambda_{i,o,t}^{R}\frac{\partial Q_{i,o,t}^{R}}{\partial Q_{i,o,t}^{I}}, \tag{B.15}$$

Taking the FOC with respect to $K_{i,o,t+1}^{R}$, I have

$$\left(\frac{1}{1+\iota}\right)^{t+1}\left[\frac{\partial \pi_{i,t+1}\left(\left\{K_{i,o,t+1}^{R}\right\}_{o}\right)}{\partial K_{i,o,t+1}^{R}} + \gamma P_{i,o,t+1}^{R}\left(\frac{Q_{i,o,t+1}^{R}}{K_{i,o,t+1}^{R}}\right)^{2} + (1-\delta)\lambda_{i,o,t+1}^{R}\right] - \left(\frac{1}{1+\iota}\right)^{t}\lambda_{i,o,t}^{R} = 0.$$

Rearranging, I have Euler equation (14).

The demand for non-robot good depends on bilateral intermediate good trade demand and total expenditure. Write $X_{j,t}^{G}$ the total purchase quantity (but not value) of good G in country j in period t. By equation (8), the bilateral trade demand is given by

$$p_{ij,t}^{G}Q_{ij,t}^{G} = \left(\frac{p_{ij,t}^{G}}{P_{j,t}^{G}}\right)^{1-\varepsilon}P_{j,t}^{G}X_{j,t}^{G}, \tag{B.16}$$

for any i, j, and t. In this equation, $P^G_{j,t}X^G_{j,t}$ is the total expenditures on non-robot goods. The total expenditure is the sum of final consumption $I_{j,t}$, payment to intermediate goods $\alpha_M p^G_{j,t}Y^G_{j,t}$, input to robot productions $\sum_o P^G_{j,t}Q^V_{j,o,t} = \sum_{o,k} p^R_{jk,o,t}Q^R_{jk,o,t}$, and payment to robot integration $\sum_o P^G_{j,t}Q^I_{j,o,t} = \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}$. Hence

$$P^G_{j,t}X^G_{j,t} = I_{j,t} + \alpha_M p^G_{j,t}Y^G_{j,t} + \sum_{o,k} p^R_{jk,o,t}Q^R_{jk,o,t} + \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}.$$

For country j and period t, by substituting into income $I_{j,t}$ the period cash flow of non-robot good producer that satisfies

$$\Pi_{j,t} \equiv \pi_{j,t}\left(\left\{K^R_{j,o,t}\right\}_o\right) - \sum_{i,o}\left(p^R_{ij,o,t}\left(1+u_{ij,t}\right)Q^R_{ij,o,t} + \sum_o P^G_{j,t}Q^I_{j,o,t} + \gamma P^R_{j,o,t}Q^R_{j,o,t}\left(\frac{Q^R_{j,o,t}}{K^R_{j,o,t}}\right)\right)$$

and robot tax revenue $T_{j,t} = \sum_{i,o} u_{ij,t}p^R_{ij,o,t}Q^R_{ij,o,t}$, I have

$$I_{j,t} = \left(1-\alpha_M\right)\sum_k p^G_{jk,t}Q^G_{jk,t} - \left(\sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t} + \left(1-\alpha^R\right)\sum_o P^R_{j,o,t}Q^R_{j,o,t}\right), \tag{B.17}$$

or in terms of variables in the definition of equilibrium,

$$I_{j,t} = \left(1-\alpha_M\right)\sum_k p^G_{jk,t}Q^G_{jk,t} - \frac{1}{\alpha^R}\sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}.$$

Hence, the total expenditure measured in terms of the production side as opposed to income side is

$$P^G_{j,t}X^G_{j,t} = \sum_k p^G_{jk,t}Q^G_{jk,t} - \sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}\left(1+\gamma\frac{Q^R_{ij,o,t}}{K^R_{j,o,t}}\right). \tag{B.18}$$

Note that this equation embeds the balanced-trade condition. By substituting equation (B.18) into equation (B.16), I have

$$p^G_{ij,t}Q^G_{ij,t} = \left(\frac{p^G_{ij,t}}{P^G_{j,t}}\right)^{1-\varepsilon^G}\left(\sum_k p^G_{jk,t}Q^G_{jk,t} + \sum_{k,o} p^R_{jk,o,t}Q^R_{jk,o,t} - \sum_{i,o} p^R_{ij,o,t}Q^R_{ij,o,t}\right). \tag{B.19}$$

The good and robot-o market-clearing conditions are given by,

$$Y_{i,t}^{R}=\sum_{j}Q_{ij,t}^{G}\tau_{ij,t}^{G}, \tag{B.20}$$

for all i and t, and

$$p_{i,o,t}^{R}=\frac{P_{i,t}^{G}}{A_{i,o,t}^{R}} \tag{B.21}$$

for all i,o, and t, respectively.

Conditional on state variables $\boldsymbol{S}_t=\left\{\boldsymbol{K}_t^R,\boldsymbol{\lambda}_t^R,\boldsymbol{L}_t,\boldsymbol{V}_t\right\}$, equations (1), (B.10), (B.14), (B.19), (B.20), and (B.21) characterize the temporary equilibrium $\left\{\boldsymbol{p}_t^G,\boldsymbol{p}_t^R,\boldsymbol{w}_t,\boldsymbol{Q}_t^G,\boldsymbol{Q}_t^R,\boldsymbol{L}_t\right\}$. In addition, conditional on initial conditions $\left\{\boldsymbol{K}_0^R,\boldsymbol{L}_0\right\}$, (10), (14), and transversality condition (15) characterize the sequential equilibrium.

Finally, the steady state conditions are given by imposing the time-invariance condition to equations (10) and (14):

$$Q_{i,o}^{R}=\delta K_{i,o}^{R}, \tag{B.22}$$

$$\frac{\partial}{\partial K_{i,o}^{R}}\pi_i\left(\left\{K_{i,o}^{R}\right\}\right)=\left(\iota+\delta\right)\lambda_{i,o}^{R}-\sum_{l}\gamma p_{li,o}^{R}\left(\frac{Q_{li,o}^{R}}{K_{i,o}^{R}}\right)^{2}\equiv c_{i,o}^{R}. \tag{B.23}$$

Note that equation (B.23) can be interpreted as the flow marginal profit of capital must be equalized to the marginal cost term. Thus I define the steady state marginal cost of robot capital $c_{i,o}^R$ from the right-hand side of equation (B.23). Note that if there is no adjustment cost $\gamma=0$, the steady state Euler equation (B.23) implies

$$\frac{\partial}{\partial K_{i,o}^{R}}\pi_i\left(\left\{K_{i,o}^{R}\right\}\right)=c_{i,o}^{R}=\left(\iota+\delta\right)\lambda_{i,o}^{R},$$

which states that the marginal profit of capital is the user cost of robots in the steady state (Hall and Jorgenson, 1967).

B5. Proof of Proposition 1

Proposition 1 states that under robot productivity growth $\widehat{A^K_{i,o}}$ defined in equation (20),

$$\widehat{\left(\frac{w_{i,o}}{P_i}\right)} = \frac{1}{1-\alpha_{i,M}} \left(\frac{\widehat{\widetilde{x}^L_{i,o}}}{1-\theta_o} + \frac{\widehat{\widetilde{x}_{i,o}}}{1-\beta} + \frac{\widehat{\widetilde{x}_{ij}}}{1-\varepsilon} \right). \tag{B.24}$$

To prove this equation, I take the following four conceptual steps. First, I will write the real wage change $(\widehat{w_{i,o}/P_i})$ in terms of the weighted average of relative price changes, making use of the fact that the sum of shares equals one. Second, I rewrite relative price change into layers of relative price changes with the technique of addition and subtraction. Third, I show that each layer of relative price changes is a change of relevant input or trade shares controlled by elasticity substitution. In other words, an input or trade shares reveals a layer of relative price changes. Finally, I make use of the fact that the sum of shares do not change after the shock to arrive at equation (B.24).

Cost minimization given production functions (5), (7), and (8) imply

$$\widehat{\left(\frac{w_{i,o}}{P_i}\right)} = \frac{1}{1-\alpha_{i,M}} \sum_l \widetilde{x}_{li,t_0} \sum_{o'} \widetilde{x}_{l,o',t_0} \left[\widetilde{x}^L_{l,o',t_0} \left(\widehat{w_{i,o}} - \widehat{w_{l,o'}}\right) + \left(1 - \widetilde{x}^L_{l,o',t_0}\right) \left(\widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta_o} + \widehat{c^R_{l,o'}} \right) \right) \right]. \tag{B.25}$$

Note that by additions and subtractions, I can rewrite

$$\begin{aligned} \widehat{w_{i,o}} - \widehat{w_{l,o'}} = {} & \left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right) - \left(\widehat{w_{l,o'}} - \widehat{p_{l,o'}}\right) + \left(\widehat{p_{i,o}} - \widehat{p^O_i}\right) - \left(\widehat{p_{l,o'}} - \widehat{p^O_l}\right) \\ & + \left(\widehat{p^O_i} - \widehat{p_i}\right) - \left(\widehat{p^O_l} - \widehat{p_l}\right) + \left(\widehat{p_i} - \widehat{P_i}\right) - \left(\widehat{p_l} - \widehat{P_i}\right) \end{aligned} \tag{B.26}$$

and

$$\begin{aligned} \widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta} + \widehat{c^R_{l,o'}} \right) = {} & \left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right) - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta} + \widehat{c^R_{l,o'}} - \widehat{p_{l,o'}} \right) + \left(\widehat{p_{i,o}} - \widehat{p^O_i}\right) - \left(\widehat{p_{l,o'}} - \widehat{p^O_l}\right) \\ & + \left(\widehat{p^O_i} - \widehat{p_i}\right) - \left(\widehat{p^O_l} - \widehat{p_l}\right) + \left(\widehat{p_i} - \widehat{P_i}\right) - \left(\widehat{p_l} - \widehat{P_i}\right). \end{aligned} \tag{B.27}$$

Note that the cost minimizing input and trade shares satisfy

$$\begin{cases} \widehat{\tilde{x}^L_{i,o}} = (1-\theta_o)\left(\widehat{w_{i,o}} - \widehat{p_{i,o}}\right), \; \widehat{1-\tilde{x}^L_{i,o}} = \widehat{A^K_{i,o}} + (1-\theta_o)\left(\widehat{c^R_{i,o}} - \widehat{p_{i,o}}\right) \\ \widehat{\tilde{x}_{i,o}} = (1-\beta)\left(\widehat{p_{i,o}} - \widehat{p^O_i}\right), \; \widehat{\tilde{x}_{li}} = (1-\varepsilon)\left(\widehat{p_l} - \widehat{P_i}\right) \end{cases} \tag{B.28}$$

Combined with the Cobb-Douglas assumption of production function (5), equations (B.26), (B.27), and (B.28) imply

$$\widehat{w_{i,o}} - \widehat{w_{l,o'}} = \frac{\widehat{\tilde{x}^L_{i,o}}}{1-\theta_o} - \frac{\widehat{\tilde{x}^L_{l,o'}}}{1-\theta_o} + \frac{\widehat{\tilde{x}_{i,o}}}{1-\beta} - \frac{\widehat{\tilde{x}_{l,o'}}}{1-\beta} + \frac{\widehat{\tilde{x}_{ii}}}{1-\varepsilon} - \frac{\widehat{\tilde{x}_{li}}}{1-\varepsilon}$$

$$\widehat{w_{i,o}} - \left(\frac{\widehat{A^K_{l,o'}}}{1-\theta_o} + \widehat{c^R_{l,o'}}\right) = \frac{\widehat{\tilde{x}^L_{i,o}}}{1-\theta_o} - \frac{\widehat{\left(1-\tilde{x}^L_{l,o'}\right)}}{1-\theta_o} + \frac{\widehat{\tilde{x}_{i,o}}}{1-\beta} - \frac{\widehat{\tilde{x}_{l,o'}}}{1-\beta} + \frac{\widehat{\tilde{x}_{ii}}}{1-\varepsilon} - \frac{\widehat{\tilde{x}_{li}}}{1-\varepsilon}.$$

Substituting these in equation (B.25) and using the facts that $\tilde{x}^L_{i,o,t_0}\widehat{\tilde{x}^L_{i,o}} + \left(1-\tilde{x}^L_{i,o,t_0}\right)\left(\widehat{1-\tilde{x}^L_{i,o}}\right) = 0$ for all i and o, $\sum_o \tilde{x}_{i,o,t_0}\widehat{\tilde{x}_{i,o}} = 0$, and $\sum_l \tilde{x}_{li,t_0}\widehat{\tilde{x}_{li}} = 0$ for all i, I have equation (B.24).

B6. Proof of Proposition 2

I follow the arguments made in Sections 2 and 3 of Newey and McFadden (1994). The proof consists of four sub results in the following Lemma. Proposition 2 can be obtained as a combination of the four results. The formal statement requires the following additional assumptions.

Assumption B.1. *(i) A function of $\tilde{\mathbf{\Theta}}$, $\mathbb{E}\left[H_o\left(\widehat{\mathbf{A}^R_2}\right)\nu_o\left(\tilde{\mathbf{\Theta}}\right)\right] \neq 0$ for any $\tilde{\mathbf{\Theta}} \neq \mathbf{\Theta}$. (ii) $\underline{\theta} \leq \theta_o \leq \overline{\theta}$ for any o, $\underline{\beta} \leq \beta \leq \overline{\beta}$, $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$, and $\underline{\phi} \leq \phi \leq \overline{\phi}$ for some positive values $\underline{\theta}, \underline{\beta}, \underline{\gamma}, \underline{\phi}, \overline{\theta}, \overline{\beta}, \overline{\gamma}, \overline{\phi}$. (iii) $\mathbb{E}\left[\sup_{\mathbf{\Theta}} \| H_o\left(\widehat{\mathbf{A}^R_2}\right)\nu_o\left(\tilde{\mathbf{\Theta}}\right)\|\right] < \infty$. (iv) $\mathbb{E}\left[\| H_o\left(\widehat{\mathbf{A}^R_2}\right)\nu_o\left(\tilde{\mathbf{\Theta}}\right)\|^2\right] < \infty$ (v) $\mathbb{E}\left[\sup_{\mathbf{\Theta}} \| H_o\left(\widehat{\mathbf{A}^R_2}\right)\nabla_{\tilde{\mathbf{\Theta}}}\nu_o\left(\tilde{\mathbf{\Theta}}\right)\|\right] < \infty$.*

Lemma B.3. *Assume Assumptions 1 and B.1(i)-(iii). (a) The estimator of the form (24) is consistent.*

Additionally, assume Assumtions B.1(iv)-(vi). (b) The estimator of the form (24) is asymptotically normal.

(c) $\sqrt{O}\left(\mathbf{\Theta}_{\mathbf{H}^*} - \mathbf{\Theta}\right) \to_d \mathcal{N}\left(0, \left(\mathbf{G}^\top \mathbf{\Omega}^{-1}\mathbf{G}\right)^{-1}\right)$, and the asymptotic variance is the minimum of that of the estimator of the form (24) for any function $\mathbf{H}$.

Proof. (a) I follow Theorems 2.6 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(iv) of this theorem are satisfied. Assumption B.1(i) guarantees condition (i). Condition (ii) is implied by Assumption B.1(ii). Condition (iii) follows because all supply and demand functions in the model is continuous. Condition (iv) is implied by Assumption B.1(iii).

(b) I follow Theorem 3.4 of Newey and McFadden (1994), which implies that it suffices to show conditions (i)-(v) of this theorem are satisfied. Condition (i) is satisfied by Assumption B.1(i). Condition (ii) follows because all supply and demand functions in the model is continuously differentiable. Condition (iii) is implied by Assumption 1 and Assumption B.1(iv). Assumption B.1(v) implies condition (iv). Finally, the gradient vectors of the structural residual is linear independent, guaranteeing the non-singularity of the variance matrix and condition (v).

(c) By Theorem 3.4 of Newey and McFadden (1994) and IV-generating function $\boldsymbol{H}$, we know that the asymptotic variance of $\boldsymbol{\Theta}_{\boldsymbol{H}^*}$ is

$$\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1},$$

where $H_o\left(\widehat{\boldsymbol{A}_2^R}\right)=Z_o$, hence $\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$. To show that this is minimal, we will show that

$$\begin{aligned}\boldsymbol{\Delta}\equiv&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\\&-\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\end{aligned}$$

is positive semi-definite. In fact, note that

$$\begin{aligned}\boldsymbol{\Delta}=&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}\times\\&\left\{\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right]-\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right\}\times\\&\left(\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\right)^{-1}.\end{aligned}$$

Define

$$\widetilde{\nu}_o=H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o-\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\left(\left(G_o\right)^\top\Omega_o^{-1}\nu_o\right)^{-1}\right]\mathbb{E}\left(\left(G_o\right)^\top\Omega_o^{-1}\nu_o\right)^{-1}\left(G_o\right)^\top\Omega_o^{-1}\nu_o,$$

where $G_o \equiv \mathbb{E}\left[\nabla_{\mathbf{\Theta}} \nu_o(\mathbf{\Theta}) \,|\widehat{\boldsymbol{A}_2^R}\right]$ and $\Omega_o \equiv \mathbb{E}\left[\nu_o(\mathbf{\Theta})(\nu_o(\mathbf{\Theta}))^\top |\widehat{\boldsymbol{A}_2^R}\right]$. Since any function of $\widehat{\boldsymbol{A}_2^R}$ gives the orthogonality with ν_o by Assumption 1, I have

$$\mathbb{E}\left[\widetilde{\nu}_o(\widetilde{\nu}_o)^\top\right] = \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\nu_o\nu_o^\top\left(H_o\left(\widehat{\boldsymbol{A}_2^R}\right)\right)^\top\right] - \mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right]\left(\boldsymbol{G}^\top\mathbf{\Omega}^{-1}\boldsymbol{G}\right)^{-1}\mathbb{E}\left[H_o\left(\widehat{\boldsymbol{A}_2^R}\right)G_o\right].$$

Since $\mathbb{E}\left[\widetilde{\nu}_o(\widetilde{\nu}_o)^\top\right]$ is positive semi-definite, $\mathbf{\Delta}$ is also positive semi-definite. □

B7. Proof of Proposition 3

I apply arguments in Section 6.1 of Newey and McFadden (1994). Namely, I define the joint estimator of the first-step and second-step estimator in Proposition 3 that falls into the class of general GMM estimation, and discuss the asymptotic property using the general result about GMM estimation. In the proof, I modify the notation of the set of functions that yield optimal IV, $\boldsymbol{H}^*$, to clarify that it depends on parameters $\mathbf{\Theta}$ as follows:

$$H_o^*\left(\widehat{\boldsymbol{A}_2^R};\mathbf{\Theta}\right) = \mathbb{E}\left[\nabla_{\mathbf{\Theta}}\nu_o(\mathbf{\Theta})\,|\widehat{\boldsymbol{A}_2^R}\right]\mathbb{E}\left[\nu_o(\mathbf{\Theta})(\nu_o(\mathbf{\Theta}))^\top|\widehat{\boldsymbol{A}_2^R}\right]^{-1}.$$

Define the joint estimator as follows:

$$\begin{pmatrix}\mathbf{\Theta}_2\\ \mathbf{\Theta}_1\end{pmatrix} \equiv \arg\min_{\mathbf{\Theta}_2,\mathbf{\Theta}_1}\left[\sum_o e_o(\mathbf{\Theta}_2,\mathbf{\Theta}_1)\right]^\top\left[\sum_o e_o(\mathbf{\Theta}_2,\mathbf{\Theta}_1)\right],$$

where

$$e_o(\mathbf{\Theta}_2,\mathbf{\Theta}_1) \equiv \begin{pmatrix} H_o^*\left(\widehat{\boldsymbol{A}_2^R};\mathbf{\Theta}_1\right)\nu_o(\mathbf{\Theta}_2)\\ H_o^*\left(\widehat{\boldsymbol{A}_2^R};\mathbf{\Theta}_0\right)\nu_o(\mathbf{\Theta}_1)\end{pmatrix}.$$

Since for any $\mathbf{\Theta}$, IV-generating function $H_o^*\left(\widehat{\boldsymbol{A}_2^R};\mathbf{\Theta}_0\right)$ gives the consistent estimator for $\mathbf{\Theta}$, we have $\mathbf{\Theta}_1 \to \mathbf{\Theta}$ and $\mathbf{\Theta}_2 \to \mathbf{\Theta}$. We also have the asymptotic variance

$$\text{Var}\begin{pmatrix}\mathbf{\Theta}_2\\ \mathbf{\Theta}_1\end{pmatrix} = \left[\left(\widetilde{\boldsymbol{G}}\right)^\top\widetilde{\mathbf{\Omega}}\widetilde{\boldsymbol{G}}\right]^{-1},$$

where

$$\widetilde{\boldsymbol{G}} \equiv \mathbb{E}\left[\nabla_{\begin{pmatrix}\boldsymbol{\Theta}_2\\ \boldsymbol{\Theta}_1\end{pmatrix}} e_o\left(\boldsymbol{\Theta}_2, \boldsymbol{\Theta}_1\right)\right]$$
$$= \mathbb{E}\begin{bmatrix} H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nabla\nu_o\left(\boldsymbol{\Theta}_2\right) & \nabla H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right) \\ \boldsymbol{0} & H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_0\right)\nabla\nu_o\left(\boldsymbol{\Theta}_1\right)\end{bmatrix}$$

and

$$\widetilde{\boldsymbol{\Omega}} \equiv \mathbb{E}\left[e_o\left(\boldsymbol{\Theta}_2, \boldsymbol{\Theta}_1\right)\left[e_o\left(\boldsymbol{\Theta}_2, \boldsymbol{\Theta}_1\right)\right]^\top\right]$$
$$= \mathbb{E}\begin{bmatrix} H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right]^\top & H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\right]^\top \\ H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right]^\top & H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_0\right)\nu_o\left(\boldsymbol{\Theta}_1\right)\right]^\top \end{bmatrix}.$$

Note that Assumption 1 implies that any function of $\widehat{\boldsymbol{A}_2^R}$ is orthogonal to ν_o, implying $\mathbb{E}\left[\nabla H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}_1\right)\nu_o\left(\boldsymbol{\Theta}_2\right)\right] = 0$. Therefore, $\widetilde{\boldsymbol{G}}$ is a block-diagonal matrix and thus the marginal asymptotic distribution of $\boldsymbol{\Theta}_2$ is normal with variance $\mathrm{Var}\left(\boldsymbol{\Theta}_2\right) = \left(\boldsymbol{G}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{G}\right)^{-1}$, noting that $\boldsymbol{G} = \mathbb{E}\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}\right)\nabla\nu_o\left(\boldsymbol{\Theta}\right)\right]$ and $\boldsymbol{\Omega} \equiv \mathbb{E}\left[H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}\right)\nu_o\left(\boldsymbol{\Theta}\right)\left(H_o^*\left(\widehat{\boldsymbol{A}_2^R}; \boldsymbol{\Theta}\right)\nu_o\left(\boldsymbol{\Theta}\right)\right)^\top\right]$.

C Detail of the GE Solution

In this section, I discuss the derivation log-linearization in equations (16), (18), and (19), so that I can bring the theory with computation. First, note that the CES production structure implies the share-weighted log-change expression for both prices and quantities. Namely, I have a formula for the change in destination price index $\widehat{P_{j,t}} = \sum_i x_{ij,t_0}\widehat{p_{ij,t}}$ and one for the change in destination expenditure $\widehat{P_{j,t}} + \widehat{Q_{j,t}} = \sum_i x_{ij,t_0}\left(\widehat{p_{ij,t}} + \widehat{Q_{ij,t}}\right)$. These imply that

$$\widehat{Q_{j,t}} = \sum_i x_{ij,t_0}\widehat{Q_{ij,t}},$$

or the changes of quantity aggregate $\widehat{Q_{j,t}}$ are also share-weighted average of changes of origin quantity $\widehat{Q_{ij,t}}$.

By log-linearizing equation (B.20) for any i ,

$$
\begin{aligned}
& -\alpha_M \widehat{p^G_{i,t}} + \alpha_M \sum_l \widetilde{x}^G_{li,t_0} \widehat{p^G_{l,t}} + (1-\alpha_M) \sum_j \widetilde{y}^G_{ij,t_0} \widehat{Q^G_{ij,t}} - \alpha_L \sum_o \widetilde{x}^O_{i,o,t_0} l^O_{i,o,t_0} \widehat{L_{i,o,t}} \\
&= \frac{\alpha_L}{\theta-1} \sum_o \frac{\widetilde{x}^O_{i,o,t_0}}{1-a_{o,t_0}} \left(-a_{o,t_0} l^O_{i,o,t_0} + (1-a_{o,t_0}) \left(1 - l^O_{i,o,t_0}\right)\right) \widehat{a_{o,t}} + \alpha_L \sum_o \widetilde{x}^O_{i,o,t_0} \frac{1}{\beta-1} \widehat{b_{i,o,t}} \\
&\quad + \widehat{A^G_{i,t}} + (1-\alpha_L-\alpha_M) \widehat{K_{i,t}} - \alpha_M \sum_l \widetilde{x}^G_{li,t_0} \widehat{\tau^G_{li,t}} - (1-\alpha_M) \sum_j \widetilde{y}^G_{ij,t_0} \widehat{\tau^G_{ij,t}} + \alpha_L \sum_o \widetilde{x}^O_{i,o,t_0} \left(1 - l^O_{i,o,t_0}\right) \widehat{K^R_{i,o,t}},
\end{aligned}
$$

To write a matrix notation, write

$$
\overline{\boldsymbol{M}^{yG}} \equiv \begin{bmatrix} \left[\widetilde{y}^G_{11,t_0}, \dots, \widetilde{y}^G_{1N,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{y}^G_{N1,t_0}, \dots, \widetilde{y}^G_{NN,t_0}\right] \end{bmatrix}
$$

a $N \times N^2$ matrix,

$$
\overline{\boldsymbol{M}^{xOl}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \widetilde{\boldsymbol{l}}_{N,\cdot,t_0}\right)^\top \end{bmatrix}
$$

a $N \times NO$ matrix where

$$
\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \equiv \left(\widetilde{x}^O_{1,o,t_0}\right)_o \text{ and } \widetilde{\boldsymbol{l}}_{1,\cdot,t_0} \equiv \left(l^O_{1,o,t_0}\right)_o \tag{C.1}
$$

are $O \times 1$ vectors, $\overline{\boldsymbol{M}^{al}}$ as a matrix with its element

$$
M^{al}_{i,o} = \frac{-a_{o,t_0} l^O_{i,o,t_0} + (1-a_{o,t_0}) \left(1 - l^O_{i,o,t_0}\right)}{1-a_{o,t_0}},
$$

and a $N \times O$ matrix,

$$
\overline{\boldsymbol{M}^{xO}} \equiv \begin{bmatrix} \left[\widetilde{x}^O_{1,1,t_0}, \dots, \widetilde{x}^O_{1,O,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{x}^O_{N,1,t_0}, \dots, \widetilde{x}^O_{N,O,t_0}\right] \end{bmatrix},
$$

a $N \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xG}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}^{G}_{1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\widetilde{x}^{G}_{N\cdot,t_0}\right) \end{bmatrix},$$

a $N \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xOl,2}} \equiv \begin{bmatrix} \left(\widetilde{\boldsymbol{x}}_{1,\cdot,t_0} \circ \left(\mathbf{1}_O - \widetilde{\boldsymbol{l}}_{1,\cdot,t_0}\right)\right)^{\top} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\widetilde{\boldsymbol{x}}_{N,\cdot,t_0} \circ \left(\mathbf{1}_O - \widetilde{\boldsymbol{l}}_{N,\cdot,t_0}\right)\right)^{\top} \end{bmatrix},$$

a $N \times NO$ matrix where $\widetilde{\boldsymbol{x}}_{1,\cdot,t_0}$ and $\widetilde{\boldsymbol{l}}_{1,\cdot,t_0}$ are defined in equation (C.1). Then I have

$$\begin{aligned} & -\alpha_M \left(\overline{\boldsymbol{I}} - \left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top}\right) \widehat{\boldsymbol{p}^{G}_t} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\widehat{\boldsymbol{Q}^{G}_t} - \alpha_L \overline{\boldsymbol{M}^{xOl}}\widehat{\boldsymbol{L}}_t \\ &= \frac{\alpha_L}{\theta-1}\left(\widetilde{\boldsymbol{x}}^{O}_{t_0} \circ \overline{\boldsymbol{M}^{al}}\right)\widehat{\boldsymbol{a}}_t + \frac{\alpha_L}{\beta-1}\overline{\boldsymbol{M}^{xO}}\widehat{\boldsymbol{b}}_t + \widehat{\boldsymbol{A}^{G}_t} + (1-\alpha_L-\alpha_M)\,\widehat{\boldsymbol{K}_t} \\ &\quad - \left[\alpha_M \overline{\boldsymbol{M}^{xG}} + (1-\alpha_M)\,\overline{\boldsymbol{M}^{yG}}\right]\widehat{\boldsymbol{\tau}^{G}_t} + \alpha_L \overline{\boldsymbol{M}^{xOl,2}}\widehat{\boldsymbol{K}^{R}_t}, \end{aligned}$$

By log-linearizing equation (B.21) for any i and o,

$$\widehat{p^{R}_{i,o,t}} = \widehat{P^{G}_{i,t}} - \widehat{A^{R}_{i,o,t}}$$

$$-\sum_l \widetilde{x}^{G}_{li,t_0}\widehat{p^{G}_{l,t}} + \widehat{p^{R}_{i,o,t}} = -\widehat{A^{R}_{i,o,t}} + \sum_l \widetilde{x}^{G}_{li,t_0}\widehat{\tau^{G}_{li,t}}.$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{xG,2}} \equiv \begin{bmatrix} \mathbf{1}_O\left[\widetilde{x}^{G}_{11,t_0},\dots,\widetilde{x}^{G}_{N1,t_0}\right] \\ \vdots \\ \mathbf{1}_O\left[\widetilde{x}^{G}_{1N,t_0},\dots,\widetilde{x}^{G}_{NN,t_0}\right] \end{bmatrix}$$

a $NO \times N$ matrix, and

$$\overline{\boldsymbol{M}^{xG,3}} \equiv \begin{bmatrix} \widetilde{x}^{G}_{11,t_0} & \dots & \widetilde{x}^{G}_{N1,t_0} & & & \mathbf{0} \\ & & & \ddots & & \\ \mathbf{0} & & & \widetilde{x}^{G}_{1N,t_0} & \dots & \widetilde{x}^{G}_{NN,t_0} \end{bmatrix} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix. Then I have

$$-\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}_t^G} + \widehat{\boldsymbol{p}_t^R} = -\widehat{\boldsymbol{A}_t^R} + \overline{\boldsymbol{M}^{xG,3}}\widehat{\boldsymbol{\tau}_t^G}.$$

By log-linearizing equations (1), (2), and (3) for any i, o, and o', I have

$$\widehat{\mu_{i,oo',t}} = \phi\left(-d\chi_{i,oo',t} + \frac{1}{1+\iota}\widehat{V_{i,o',t+1}}\right) - \sum_{o''}\mu_{i,oo'',t_0}\left(-d\chi_{i,oo'',t} + \frac{1}{1+\iota}\widehat{V_{i,o'',t+1}}\right), \tag{C.2}$$

$$\widehat{V_{i,o,t+1}} = \widehat{w_{i,o,t+1}} + dT_{i,o,t+1} - \widehat{P_{i,t+1}} + \sum_{o'}\mu_{i,oo',t_0}\left(-d\chi_{i,oo',t+1} + \frac{1}{1+\iota}\widehat{V_{i,o',t+2}}\right), \tag{C.3}$$

and

$$\widehat{L_{i,o,t+1}} = \sum_{o'}\frac{L_{i,o',t_0}}{L_{i,o,t_0}}\mu_{i,o'o,t_0}\left(\widehat{\mu_{i,o'o,t}} + \widehat{L_{i,o',t}}\right). \tag{C.4}$$

In matrix notation, by equation (C.2),

$$\widehat{\boldsymbol{\mu}_t^{\text{vec}}} = -\phi\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)d\boldsymbol{\chi}_t^{\text{vec}} + \frac{\phi}{1+\iota}\left(\overline{\boldsymbol{I}_{NO^2}} - \overline{\boldsymbol{M}^{\mu}}\right)\left(\overline{\boldsymbol{I}_{NO}} \otimes \mathbf{1}_O\right)\widehat{\boldsymbol{V}_{t+1}}.$$

where

$$\overline{\boldsymbol{M}^{\mu}} \equiv \overline{\boldsymbol{M}^{\mu,3}} \otimes \mathbf{1}_O,$$

$$\overline{\boldsymbol{M}^{\mu,3}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{i,1\cdot,t_0}\right)^\top & & & & & & \\ & \ddots & & & & \mathbf{0} & \\ & & \left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right)^\top & & & & \\ & & & \ddots & & & \\ & & & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^\top & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & \left(\boldsymbol{\mu}_{i,O\cdot 1,t_0}\right)^\top \end{bmatrix},$$

$$d\chi_t^{\text{vec}} \equiv \begin{bmatrix} d\chi_{1,1\cdot,t} \\ \vdots \\ d\chi_{1,O\cdot,t} \\ \vdots \\ d\chi_{N,1\cdot,t} \\ \vdots \\ d\chi_{N,O\cdot,t} \end{bmatrix},$$

and

$$\boldsymbol{\mu}_{i,o\cdot,t_0} \equiv \left(\mu_{i,oo',t_0}\right)_{o'} \text{ and } d\boldsymbol{\chi}_{1,o\cdot,t} \equiv \left(d\chi_{1,oo',t}\right)_{o'} \tag{C.5}$$

are $O \times 1$ vectors. By equation (C.3),

$$\frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\check{\boldsymbol{V}}_{t+2} = \overline{\boldsymbol{M}^{xG,2}}\boldsymbol{p}_{t+1}^{\check{G}} - \check{\boldsymbol{w}}_{t+1} + \check{\boldsymbol{V}}_{t+1}.$$

where

$$\overline{\boldsymbol{M}^{\mu,2}} \equiv \begin{bmatrix} \left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right)^\top & & & \\ \vdots & & \mathbf{0} & \\ \left(\boldsymbol{\mu}_{1,O\cdot,t_0}\right)^\top & & & \\ & \ddots & & \\ & & & \left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right)^\top \\ & \mathbf{0} & & \vdots \\ & & & \left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right)^\top \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (C.5) for any i and o. By equation (C.3),

$$\check{\boldsymbol{L}}_{t+1} = \overline{\boldsymbol{M}^{\mu L,2}}\check{\boldsymbol{\mu}}_t^{\text{vec}} + \overline{\boldsymbol{M}^{\mu L}}\check{\boldsymbol{L}}_t$$

where $\overline{\boldsymbol{M}^{\mu L}}$ being the $NO \times NO$ matrix

$$\overline{\boldsymbol{M}^{\mu L}} = \overline{\boldsymbol{M}^{\mu,2}} \circ \left(\begin{bmatrix} \left(\boldsymbol{L}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\boldsymbol{L}_{N,\cdot,t_0}\right)^\top \end{bmatrix} \otimes \mathbf{1}_O \right) \oslash \left(\begin{bmatrix} \boldsymbol{L}_{1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \boldsymbol{L}_{N,\cdot,t_0} \end{bmatrix} \otimes \left(\mathbf{1}_O\right)^\top \right)$$

and $\overline{\boldsymbol{M}^{\mu L,2}}$ being the $NO \times NO^2$ matrix

$$\overline{\boldsymbol{M}^{\mu L,2}} = \overline{\boldsymbol{M}^{\mu,4}} \circ \left(\begin{bmatrix} \left(\boldsymbol{L}_{1,\cdot,t_0}\right)^\top & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\boldsymbol{L}_{N,\cdot,t_0}\right)^\top \end{bmatrix} \otimes \overline{\boldsymbol{I}_O} \right) \oslash$$
$$\begin{pmatrix} \left(\mathbf{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{1,o,t_0}\right) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left(\mathbf{1}_O\right)^\top \otimes \operatorname{diag}\left(L_{N,o,t_0}\right) \end{pmatrix},$$

where

$$\overline{\boldsymbol{M}^{\mu,4}} \equiv \begin{bmatrix} \operatorname{diag}\left(\boldsymbol{\mu}_{1,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{i,O\cdot,t_0}\right) & & & \mathbf{0} & \\ & & & \ddots & & & \\ & \mathbf{0} & & & \operatorname{diag}\left(\boldsymbol{\mu}_{N,1\cdot,t_0}\right) & \dots & \operatorname{diag}\left(\boldsymbol{\mu}_{N,O\cdot,t_0}\right) \end{bmatrix},$$

and $\boldsymbol{\mu}_{i,o\cdot,t_0}$ is given by equation (C.5) for any i and o.

By log-linearizing equation (B.19) for each i and j,

$$\widehat{Q^G_{ij,t}} = -\varepsilon^G \widehat{p^G_{ij,t}} - \left(1-\varepsilon^G\right) \widehat{P^G_{j,t}} + \left[s^G_{j,t_0} \sum_k \widehat{p^G_{jk,t} Q^G_{jk,t}} + s^V_{j,t_0} \sum_{i,o} \widehat{p^R_{ij,o,t} Q^R_{ij,o,t}} + s^R_{j,t_0} \sum_{o,k} \widehat{p^R_{jk,o,t} Q^R_{jk,o,t}} \right]$$

where

$$s^G_{j,t_0} \equiv \frac{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}$$

is the baseline share of non-robot good production in income,

$$s^R_{j,t_0} \equiv \frac{\sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the baseline share of robot production, and

$$s^V_{j,t_0} \equiv -\frac{\sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0}}{\sum_k p^G_{jk,t_0} Q^G_{jk,t_0} - \sum_{i,o} p^R_{ij,o,t_0} Q^R_{ij,o,t_0} + \sum_{o,k} p^R_{jk,o,t_0} Q^R_{jk,o,t_0}},$$

is the (negative) baseline absorption share of robots. Thus

$$\left[\varepsilon^G\widehat{p^G_{i,t}}+\left(1-\varepsilon^G\right)\sum_l\widetilde{x}^G_{lj,t_0}\widehat{p^G_{l,t}}-s^G_{j,t_0}\widehat{p^G_{j,t}}\right]-\left[s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,o,t_0}\widetilde{x}^R_{j,o,t_0}\widehat{p^R_{l,o,t}}+s^R_{t_0}\sum_o\widetilde{y}^R_{j,o,t_0}\widehat{p^R_{j,o,t}}\right]$$
$$+\left(\widehat{Q^G_{ij,t}}-s^G_{j,t_0}\sum_k\widetilde{y}^G_{jk,t_0}\widehat{Q^G_{jk,t}}\right)-\left(s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,o,t_0}\widetilde{x}^R_{j,o,t_0}\widehat{Q^R_{lj,o,t}}+s^R_{j,t_0}\sum_{k,o}\widetilde{y}^R_{jk,o,t_0}\widetilde{y}^R_{j,o,t_0}\widehat{Q^R_{jk,o,t}}\right)$$
$$=-\left[\varepsilon^G\widehat{\tau^G_{ij,t}}+\left(1-\varepsilon^G\right)\sum_l\widetilde{x}^G_{lj,t_0}\widehat{\tau^G_{lj,t}}-s^G_{j,t_0}\sum_k\widetilde{y}^G_{jk,t_0}\widehat{\tau^G_{jk,t}}\right]+\left[s^V_{j,t_0}\sum_{l,o}\widetilde{x}^R_{lj,t_0}\widehat{\tau^R_{lj,t}}+s^R_{j,t_0}\sum_{k,o}\widetilde{y}^R_{jk,t_0}\widehat{\tau^R_{jk,t}}\right]$$

where

$$\widetilde{x}^R_{ij,o,t_0}\equiv\frac{p^R_{ij,o,t_0}Q^R_{ij,o,t_0}}{P^R_{j,o,t_0}Q^R_{j,o,t_0}},\widetilde{x}^R_{j,o,t_0}\equiv\frac{P^R_{j,o,t_0}Q^R_{j,o,t_0}}{P^R_{j,t_0}Q^R_{j,t_0}},\widetilde{x}^R_{ij,t_0}\equiv\frac{\sum_o p^R_{ij,o,t_0}Q^R_{ij,o,t_0}}{P^R_{j,t_0}Q^R_{j,t_0}},$$

$$\widetilde{y}^R_{ij,o,t_0}\equiv\frac{p^R_{ij,o,t_0}Q^R_{ij,o,t_0}}{\sum_k p^R_{ik,o,t_0}Q^R_{ik,o,t_0}},\widetilde{y}^R_{i,o,t_0}\equiv\frac{\sum_k p^R_{ik,o,t_0}Q^R_{ik,o,t_0}}{\sum_{k,o'} p^R_{ik,o',t_0}Q^R_{ik,o',t_0}},\widetilde{y}^R_{ij,t_0}\equiv\frac{\sum_o p^R_{ij,o,t_0}Q^R_{ij,o,t_0}}{\sum_{k,o} p^R_{ik,o,t_0}Q^R_{ik,o,t_0}}.$$

In matrix notation, define

$$\overline{\boldsymbol{M}^{xR}}\equiv\mathbf{1}_N\otimes\begin{bmatrix}\widetilde{\boldsymbol{x}}^R_{t_0}\circ\widetilde{\boldsymbol{x}}^R_{\cdot 1,\cdot,t_0} & \ldots & \widetilde{\boldsymbol{x}}^R_{t_0}\circ\widetilde{\boldsymbol{x}}^R_{\cdot N,\cdot,t_0}\end{bmatrix},$$

a $N^2\times NO$ matrix,

$$\overline{\boldsymbol{M}^{yR}}\equiv\mathbf{1}_N\otimes\begin{bmatrix}\widetilde{y}^R_{1,1} & \ldots & \widetilde{y}^R_{1,O} & & & \mathbf{0}\\ & & & \ddots & & \\ & \mathbf{0} & & \widetilde{y}^R_{N,1} & \ldots & \widetilde{y}^R_{N,O}\end{bmatrix},$$

a $N^2\times NO$ matrix,

$$\overline{\boldsymbol{M}^{yG,2}}\equiv\mathbf{1}_N\otimes\overline{\boldsymbol{M}^{yG}}.$$

a $N^2\times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,2}}\equiv\mathbf{1}_N\otimes\begin{bmatrix}\left[\widetilde{x}^R_{1,o,t_0}\widetilde{x}^R_{11,o,t_0}\right]_o & & \mathbf{0} & & \left[\widetilde{x}^R_{1,o,t_0}\widetilde{x}^R_{N1,o,t_0}\right]_o & & \mathbf{0}\\ & \ddots & & \ldots & & \ddots & \\ \mathbf{0} & & \left[\widetilde{x}^R_{N,o,t_0}\widetilde{x}^R_{1N,o,t_0}\right]_o & & \mathbf{0} & & \left[\widetilde{x}^R_{N,o,t_0}\widetilde{x}^R_{NN,o,t_0}\right]_o\end{bmatrix}$$

a $N^2 \times N^2 O$ matrix ,

$$\overline{\boldsymbol{M}^{yR,2}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{11,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{1N,o,t_0}\right]_o & & & \mathbf{0} & \\ & & & \ddots & & & \\ & \mathbf{0} & & & \left[\widetilde{y}^R_{1,o,t_0}\widetilde{y}^R_{N1,o,t_0}\right]_o & \cdots & \left[\widetilde{y}^R_{N,o,t_0}\widetilde{y}^R_{NN,o,t_0}\right]_o \end{bmatrix}$$

a $N^2 \times N^2 O$ matrix,

$$\overline{\boldsymbol{M}^{xG,4}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xG}}$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,3}} \equiv \mathbf{1}_N \otimes \left[\begin{array}{ccc} \operatorname{diag}\left(\widetilde{x}^R_{1\cdot,t_0}\right) & \ldots & \operatorname{diag}\left(\widetilde{x}^R_{N\cdot,t_0}\right) \end{array} \right]$$

a $N^2 \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{yR,3}} \equiv \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{y}^R_{11,t_0}, \ldots, \widetilde{y}^R_{1N,t_0}\right] & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \left[\widetilde{y}^R_{N1,t_0}, \ldots, \widetilde{y}^R_{NN,t_0}\right] \end{bmatrix}$$

a $N^2 \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{xO,2}} \equiv \mathbf{1}_N \otimes \overline{\boldsymbol{M}^{xO}},$$

a $N^2 \times NO$ matrix. Then I have

$$\begin{aligned}
&\left(\varepsilon^G \left[\overline{\boldsymbol{I}_N} \otimes \mathbf{1}_N\right] + \left(1-\varepsilon^G\right)\left[\mathbf{1}_N \otimes \left(\widetilde{\boldsymbol{x}}^G_{t_0}\right)^\top\right] - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\left[\mathbf{1}_N \otimes \overline{\boldsymbol{I}_N}\right]\right)\widehat{\boldsymbol{p}^G_t} \\
&- \left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR}}\right)\widehat{\boldsymbol{p}^R_t} \\
&+ \left(\overline{\boldsymbol{I}_{N^2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{Q}^G_t} - \left[\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,2}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,2}}\right]\widehat{\boldsymbol{Q}^R_t} \\
&= -\left(\varepsilon^G + \left(1-\varepsilon^G\right)\overline{\boldsymbol{M}^{xG,4}} - \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^G_{t_0}\right)\overline{\boldsymbol{M}^{yG,2}}\right)\widehat{\boldsymbol{\tau}^G_t} \\
&\qquad + \left(\operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^V_{t_0}\right)\overline{\boldsymbol{M}^{xR,3}} + \operatorname{diag}\left(\mathbf{1}_N \otimes \boldsymbol{s}^R_{t_0}\right)\overline{\boldsymbol{M}^{yR,3}}\right)\widehat{\boldsymbol{\tau}^R_t}
\end{aligned}$$

By log-linearizing equation (B.14) for each i, j, and o,,

$$
\begin{aligned}
&\left(1-\alpha^{R}\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\sum_{l}\widetilde{x}^{G}_{lj,t_0}\widehat{p^{G}_{l,t}}+\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\widehat{p^{R}_{i,o,t}}\\
&+\left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}-\left(1-\alpha^{R}\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_{l}\widetilde{x}^{R}_{lj,o,t_0}\widehat{p^{R}_{l,o,t}}\\
&+\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\frac{1}{\varepsilon^{R}}\widehat{Q^{R}_{ij,o,t}}+\left[-\frac{1}{\varepsilon^{R}}\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}+\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_{l}\widetilde{x}^{R}_{lj,o,t_0}\widehat{Q^{R}_{lj,o,t}}\\
&=-\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}du_{ij,t}-\left(1-\alpha^{R}\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\sum_{l}\widetilde{x}^{G}_{lj,t_0}\widehat{\tau^{G}_{lj,t}}-\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\widehat{\tau^{R}_{ij,t}}\\
&-\left[\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}-\left(1-\alpha^{R}\right)\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}\right]\sum_{l}\widetilde{x}^{R}_{lj,o,t_0}\widehat{\tau^{R}_{lj,t}}+\widehat{\lambda^{R}_{j,o,t}}+\frac{2\gamma\delta}{1+u_{ij,t_0}+2\gamma\delta}\widehat{K^{R}_{j,o,t}}.
\end{aligned}
$$

In matrix notation, write a preliminary $N\times N$ matrix $\overline{\widetilde{\boldsymbol{u}_{t_0}}}$ as such that the (i,j)-element is

$$
\frac{1+u_{ij,t_0}}{1+u_{ij,t_0}+2\gamma\delta}.
$$

Then $\mathbf{1}_N\left(\mathbf{1}_N\right)^{\top}-\overline{\widetilde{\boldsymbol{u}_{t_0}}}$ is a matrix that is filled with $2\gamma\delta/\left(1+u_{ij,t_0}+2\gamma\delta\right)$ for its (i,j) element and

$$
\overline{\boldsymbol{M}^{u}}\equiv\operatorname{diag}\left(\left[\widetilde{u_{1\cdot,t_0}},\ldots,\widetilde{u_{N\cdot,t_0}}\right]^{\top}\right).
$$

Using these, write

$$
\overline{\boldsymbol{M}^{xG,5}}\equiv\left(\overline{\boldsymbol{M}^{u}}\otimes\overline{\boldsymbol{I}_O}\right)\left(\mathbf{1}_N\otimes\left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top}\otimes\mathbf{1}_O\right)
$$

a $N^2O\times N$ matrix,

$$
\overline{\boldsymbol{M}^{u,2}}\equiv\begin{bmatrix}\widetilde{\boldsymbol{u}_{1\cdot,t_0}} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{u}_{N\cdot,t_0}}\end{bmatrix}\otimes\overline{\boldsymbol{I}_O},
$$

a $N^2O \times NO$ matrix where $\widetilde{\boldsymbol{u}_{i\cdot,t_0}} \equiv (\widetilde{u_{i\cdot,t_0}})_j$ is a $N \times 1$ vector,

$$\overline{\boldsymbol{M}^{xR,4}} \equiv \left\{ \left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) - \left(1 - \alpha^R \right) \overline{\boldsymbol{M}^u} \right] \otimes \overline{\boldsymbol{I}_O} \right\} \times . \left(\mathbf{1}_N \otimes \begin{bmatrix} \operatorname{diag}\left(\left\{ \widetilde{x}^R_{11,o,t_0} \right\}_o \right) & \dots & \operatorname{diag}\left(\left\{ \widetilde{x}^R_{N1,o,t_0} \right\}_o \right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\left\{ \widetilde{x}^R_{1N,o,t_0} \right\}_o \right) & \dots & \operatorname{diag}\left(\left\{ \widetilde{x}^R_{NN,o,t_0} \right\}_o \right) \end{bmatrix} \right)$$

a $N^2O \times NO$ matrix,

$$\overline{\boldsymbol{M}^{xR,5}} \equiv \left\{ \left[-\frac{1}{\varepsilon^R} \overline{\boldsymbol{M}^u} + \left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) \right] \otimes \overline{\boldsymbol{I}_O} \right\} \times \left\{ \mathbf{1}_N \otimes \left[\operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^R_{11,1,t_0} \\ \vdots \\ \widetilde{x}^R_{11,O,t_0} \\ \vdots \\ \widetilde{x}^R_{1N,O,t_0} \end{bmatrix} \right) \quad \dots \quad \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^R_{N1,1,t_0} \\ \vdots \\ \widetilde{x}^R_{N1,O,t_0} \\ \vdots \\ \widetilde{x}^R_{NN,O,t_0} \end{bmatrix} \right) \right] \right\}$$

a $N^2O \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xG,6}} \equiv \left(\overline{\boldsymbol{M}^u} \otimes \overline{\boldsymbol{I}_O} \right) \left\{ \mathbf{1}_N \otimes \left[\operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^G_{11,t_0} \\ \vdots \\ \widetilde{x}^G_{1N,t_0} \end{bmatrix} \right) \quad \dots \quad \operatorname{diag}\left(\begin{bmatrix} \widetilde{x}^G_{N1,t_0} \\ \vdots \\ \widetilde{x}^G_{NN,t_0} \end{bmatrix} \right) \right] \otimes \mathbf{1}_O \right\}$$

a $N^2O \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xR,6}} \equiv \left\{ \left[\left(\overline{\boldsymbol{I}_{N^2}} - \overline{\boldsymbol{M}^u} \right) - \left(1 - \alpha^R \right) \overline{\boldsymbol{M}^u} \right] \otimes \overline{\boldsymbol{I}_O} \right\} \times \left\{ \mathbf{1}_N \otimes \begin{bmatrix} \left[\widetilde{x}^R_{11,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} & \dots & \left[\widetilde{x}^R_{N1,o,t_0} \right]_o & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} & & \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{1N,o,t_0} \right]_o & & \mathbf{0} & \mathbf{0} & \left[\widetilde{x}^R_{N3,o,t_0} \right]_o \end{bmatrix} \right\}$$

a $N^2O \times N^2$ matrix, and

$$\overline{\boldsymbol{M}^{u,3}} \equiv \begin{bmatrix} 1-\widetilde{\widehat{u_{11,t_0}}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1-\widetilde{\widehat{u_{1N,t_0}}} \\ & \vdots & \\ 1-\widetilde{\widehat{u_{N1,t_0}}} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & 1-\widetilde{\widehat{u_{NN,t_0}}} \end{bmatrix} \otimes \overline{\boldsymbol{I}_O}$$

a $N^2O \times NO$ matrix. Finally, I have

$$\begin{aligned} &\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,5}}\widehat{\boldsymbol{p}_t^G} + \left[\overline{\boldsymbol{M}^{u,2}} + \overline{\boldsymbol{M}^{xR,4}}\right]\widehat{\boldsymbol{p}_t^R} + \left\{\frac{1}{\varepsilon^R}\left(\overline{\boldsymbol{M}^u} \otimes \overline{\boldsymbol{I}_O}\right) + \overline{\boldsymbol{M}^{xR,5}}\right\}\widehat{\boldsymbol{Q}_t^R} \\ &= -\left(\overline{\boldsymbol{M}^u} \otimes \mathbf{1}_O\right) d\boldsymbol{u}_t - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,6}}\widehat{\boldsymbol{\tau}_t^G} - \left[\left(\overline{\boldsymbol{M}^u} \otimes \mathbf{1}_O\right) + \overline{\boldsymbol{M}^{xR,6}}\right]\widehat{\boldsymbol{\tau}_t^R} + \left(\mathbf{1}_N \otimes \overline{\boldsymbol{I}_{NO}}\right)\widehat{\boldsymbol{\lambda}_t^R} + \overline{\boldsymbol{M}^{u,3}}\widehat{\boldsymbol{K}_t^R}. \end{aligned}$$

By log-linearizing equation and (B.10) for each i and o,

$$\begin{aligned} &\widehat{p_{i,t}^G} + \sum_j \widetilde{y}_{ij,t_0}^G \widehat{Q_{ij,t}^G} - \widehat{w_{i,o,t}} + \left[-\frac{1}{\theta} + \left(-\frac{1}{\beta} + \frac{1}{\theta}\right) l_{i,o,t_0}^O\right]\widehat{L_{i,o,t}} + \left(-1+\frac{1}{\beta}\right)\sum_{o'} \widetilde{x}_{i,o',t_0}^O l_{i,o',t_0}^O \widehat{L_{i,o',t}} \\ &= -\frac{1}{\beta}\widehat{b_{i,o,t}} + \frac{1}{\theta}\frac{a_{o,t_0}}{1-a_{o,t_0}}\widehat{a_{o,t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\left[-\left(1-l_{i,o,t_0}^O\right) + l_{i,o,t_0}^O \frac{a_{o,t_0}}{1-a_{o,t_0}}\right]\widehat{a_{o,t}} \\ &\quad + \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'} \widetilde{x}_{i,o',t_0}^O \left[-\left(1-l_{i,o',t_0}^O\right) + l_{i,o',t_0}^O \frac{a_{o',t_0}}{1-a_{o',t_0}}\right]\widehat{a_{o',t}} \\ &\quad - \sum_j y_{ij,t_0}^G \widehat{\tau_{ij,t}^G} - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l_{i,o,t_0}^O\right)\widehat{K_{i,o,t}^R} - \left(-1+\frac{1}{\beta}\right)\sum_{o'} \widetilde{x}_{i,o',t_0}^O \left(1-l_{i,o',t_0}^O\right)\widehat{K_{i,o',t}^R}, \end{aligned}$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{yG,3}} \equiv \overline{\boldsymbol{M}^{yG}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix,

$$\overline{\boldsymbol{M}^{xOl,3}} \equiv \overline{\boldsymbol{M}^{xOl}} \otimes \mathbf{1}_O$$

a $NO \times NO$ matrix,

$$\overline{\boldsymbol{M}^a} \equiv \mathbf{1}_N \otimes \operatorname{diag}\left(\frac{a_{o,t_0}}{1-a_{o,t_0}}\right)$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,2}} \equiv \begin{bmatrix} \operatorname{diag}\left(-\left(1-l^{O}_{1,o,t_0}\right)+l^{O}_{1,o,t_0}\frac{a_{o,t_0}}{1-a_{o,t_0}}\right) \\ \vdots \\ \operatorname{diag}\left(-\left(1-l^{O}_{N,o,t_0}\right)+l^{O}_{N,o,t_0}\frac{a_{o,t_0}}{1-a_{o,t_0}}\right) \end{bmatrix}$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{al,3}} \equiv \left(\widetilde{\boldsymbol{x}}^{O}_{t_0} \circ \overline{\boldsymbol{M}^{al}}\right) \otimes \mathbf{1}_O$$

a $NO \times O$ matrix,

$$\overline{\boldsymbol{M}^{xOl,4}} \equiv \overline{\boldsymbol{M}^{xOl,2}} \otimes \mathbf{1}_O,$$

a $NO \times NO$ matrix. I have

$$\begin{aligned}
&\left(\boldsymbol{I}_N \otimes \mathbf{1}_O\right)\widehat{\boldsymbol{p}^{G}_t} - \widehat{\boldsymbol{w}}_t + \overline{\boldsymbol{M}^{yG,3}}\widehat{\boldsymbol{Q}^{G}_t} + \left(-\frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(\boldsymbol{l}^{O}_{t_0}\right) + \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}}\right)\widehat{\boldsymbol{L}}_t \\
&= -\frac{1}{\beta}\widehat{\boldsymbol{b}}_t + \left[\frac{1}{\theta}\overline{\boldsymbol{M}^{a}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,2}} + \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,3}}\right]\widehat{\boldsymbol{a}}_t - \overline{\boldsymbol{M}^{yG,3}}\widehat{\boldsymbol{\tau}^{G}_t} \\
&\quad + \left[-\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(1-l^{O}_{i,o,t_0}\right) - \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}}\right]\widehat{\boldsymbol{K}^{R}_t}.
\end{aligned}$$

Hence the log-linearized temporary equilibrium system is

$$\overline{\boldsymbol{D}^{x}}\widehat{\boldsymbol{x}} = \overline{\boldsymbol{D}^{A}}\widehat{\boldsymbol{A}}$$

where matrices $\overline{\boldsymbol{D}^{x}}$ and $\overline{\boldsymbol{D}^{A}}$ are defined as

$$\overline{\boldsymbol{D}^{x}} \equiv \begin{bmatrix}
\overline{\boldsymbol{D}^{x}_{11}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{14}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{16}} \\
-\overline{\boldsymbol{M}^{xG,2}} & \overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\phi\overline{\boldsymbol{M}^{xG,2}} & \mathbf{0} & -\phi\overline{\boldsymbol{I}_{NO}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{M}^{l}} \\
\overline{\boldsymbol{D}^{x}_{41}} & \overline{\boldsymbol{D}^{x}_{42}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{44}} & \overline{\boldsymbol{D}^{x}_{45}} & \mathbf{0} \\
\overline{\boldsymbol{D}^{x}_{51}} & \overline{\boldsymbol{D}^{x}_{52}} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{55}} & \mathbf{0} \\
\overline{\boldsymbol{D}^{x}_{61}} & \mathbf{0} & -\overline{\boldsymbol{I}_{NO}} & \overline{\boldsymbol{M}^{yG,3}} & \mathbf{0} & \overline{\boldsymbol{D}^{x}_{66}}
\end{bmatrix},$$

where

$$\overline{\boldsymbol{D}^{x}_{11}} \equiv -\alpha_M\left(\overline{\boldsymbol{I}_N} - \left(\widetilde{\boldsymbol{x}}^{G}_{t_0}\right)^{\top}\right),\ \overline{\boldsymbol{D}^{x}_{14}} \equiv \left(1-\alpha_M\right)\overline{\boldsymbol{M}^{yG}},\ \overline{\boldsymbol{D}^{x}_{16}} \equiv -\alpha_L\overline{\boldsymbol{M}^{xOl}},$$

$$\overline{D^x_{41}} \equiv \varepsilon^G \left[\overline{I_N} \otimes \mathbf{1}_N\right] + \left(1-\varepsilon^G\right)\left[\mathbf{1}_N \otimes \left(\widetilde{x}^G_{t_0}\right)^\top\right] - \operatorname{diag}\left(\mathbf{1}_N \otimes s^G_{t_0}\right)\left[\mathbf{1}_N \otimes \overline{I_N}\right],$$

$$\overline{D^x_{42}} \equiv \operatorname{diag}\left(\mathbf{1}_N \otimes s^V_{t_0}\right)\overline{M^{xR}} + \operatorname{diag}\left(\mathbf{1}_N \otimes s^R_{t_0}\right)\overline{M^{yR}},$$

$$\overline{D^x_{44}} \equiv \overline{I_{N^2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes s^G_{t_0}\right)\overline{M^{yG,2}},$$

$$\overline{D^x_{45}} \equiv -\operatorname{diag}\left(\mathbf{1}_N \otimes s^V_{t_0}\right)\overline{M^{xR,2}} - \operatorname{diag}\left(\mathbf{1}_N \otimes s^R_{t_0}\right)\overline{M^{yR,2}},$$

$$\overline{D^x_{51}} \equiv \left(1-\alpha^R\right)\overline{M^{xG,5}},\ \overline{D^x_{52}} \equiv \overline{M^{u,2}} + \overline{M^{xR,4}},\ \overline{D^x_{55}} \equiv \frac{1}{\varepsilon^R}\left(\overline{M^u} \otimes \overline{I_O}\right) + \overline{M^{xR,5}},$$

$$\overline{D^x_{61}} \equiv I_N \otimes \mathbf{1}_N,\ \overline{D^x_{66}} \equiv -\frac{1}{\theta} + \left(-\frac{1}{\beta} + \frac{1}{\theta}\right)\operatorname{diag}\left(l^O_{t_0}\right) + \left(-1 + \frac{1}{\beta}\right)\overline{M^{xOl,3}},$$

and

$$\overline{D^A} \equiv \begin{bmatrix} \mathbf{0} & \overline{D^A_{12}} & \overline{D^A_{13}} & \overline{I_N} & \mathbf{0} & \overline{D^A_{16}} & \overline{D^A_{17}} & \mathbf{0} & \alpha_L\overline{M^{xOl,2}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\overline{I_{NO}} & \mathbf{0} & \overline{M^{xG}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\phi\overline{M^{xG,3}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{D^A_{47}} & \overline{D^A_{48}} & \mathbf{0} & \mathbf{0} \\ \overline{D^A_{51}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{D^A_{57}} & \overline{D^A_{58}} & \overline{M^{u,3}} & \overline{D^A_{5,10}} \\ \mathbf{0} & \overline{D^A_{62}} & -\frac{1}{\beta}\overline{I_{NO}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\overline{M^{yG,3}} & \mathbf{0} & \overline{D^A_{69}} & \mathbf{0} \end{bmatrix},$$

where

$$\overline{D^A_{12}} \equiv \frac{\alpha_L}{\theta-1}\left(\widetilde{x}^O_{t_0} \otimes \overline{M^{al}}\right),\ \overline{D^A_{13}} \equiv \frac{\alpha_L}{\beta-1}\overline{M^{xO}},$$

$$\overline{D^A_{16}} \equiv \left(1-\alpha_L-\alpha_M\right)\overline{I_N},\ \overline{D^A_{17}} \equiv -\left[\alpha_M\overline{M^{xG}} + \left(1-\alpha_M\right)\overline{M^{yG}}\right],$$

$$\overline{D^A_{47}} \equiv -\varepsilon^G + \left(1-\varepsilon^G\right)\overline{M^{xG,4}} + \operatorname{diag}\left(\mathbf{1}_N \otimes s^G_{t_0}\right)\overline{M^{yG,2}},$$

$$\overline{D^A_{48}} \equiv \operatorname{diag}\left(\mathbf{1}_N \otimes s^V_{t_0}\right)\overline{M^{xR,3}} + \operatorname{diag}\left(\mathbf{1}_N \otimes s^R_{t_0}\right)\overline{M^{yR,3}},$$

$$\overline{D^A_{51}} \equiv -\left(\overline{M^u} \otimes \mathbf{1}_O\right),\ \overline{D^A_{57}} \equiv -\left(1-\alpha^R\right)\overline{M^{xG,6}},$$

$$\overline{D^A_{58}} \equiv -\left[\left(\overline{M^u} \otimes \mathbf{1}_O\right) + \overline{M^{xR,6}}\right],\ \overline{D^A_{5,10}} \equiv \mathbf{1}_N \otimes \overline{I_{NO}},$$

$$\overline{D^A_{62}} \equiv \frac{1}{\theta}\overline{M^a} + \left(-\frac{1}{\beta} + \frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{M^{al,2}} + \left(-1 + \frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{M^{al,3}},$$

and

$$\overline{\boldsymbol{D}_{69}^{A}} \equiv -\left(-\frac{1}{\beta}+\frac{1}{\theta}\right) \operatorname{diag}\left(1-l_{i,o,t_0}^{O}\right)-\left(-1+\frac{1}{\beta}\right) \overline{\boldsymbol{M}^{xOl,4}}.$$

Note that to normalize the price, one of the good-demand equation must be replaced with log-linearized nureraire condition $\widehat{P_{1,t}^{G}}=\sum_i x_{i1,t_0}^{G}\left(\widehat{p_{i,t}^{G}}+\widehat{\tau_{i1,t}^{G}}\right)=0$, or

$$\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{p}_t^{G}}=-\overline{\boldsymbol{M}^{xG,num}}\widehat{\boldsymbol{\tau}_t^{G}},$$

where $\overline{\boldsymbol{M}^{xG,num}} \equiv \left[x_{11,t_0}^{G}, x_{21,t_0}^{G}, x_{31,t_0}^{G}\right]$.

To analyze the steady state conditions, first note that the steady state accumulation condition (B.22) implies $\widehat{Q_{i,o}^{R}}=\widehat{K_{i,o}^{R}}$. Using robot integration function, integration demand and unit cost formula, I have

$$\widehat{Q_{i,o}^{R}}=\sum_l x_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}}+\left(1-\alpha^{R}\right)\left(\sum_l \widetilde{x}_{ij,o,t_0}^{R}\widehat{p_{li,o}^{R}}-\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{p_{li,t}^{G}}\right) \tag{C.6}$$

Thus the condition is

$$\begin{aligned}
&\sum_l \widetilde{x}_{li,o,t_0}^{R}\widehat{Q_{li,o}^{R}}+\left(1-\alpha^{R}\right)\sum_l \widetilde{x}_{li,o,t_0}^{R}\widehat{p_{l,o}^{R}}-\left(1-\alpha^{R}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{p_l^{G}}-\widehat{K_{i,o}^{R}}\\
&=\left(1-\alpha^{R}\right)\sum_l \widetilde{x}_{li,t_0}^{G}\widehat{\tau_{li}^{G}}-\left(1-\alpha^{R}\right)\sum_l \widetilde{x}_{li,o,t_0}^{R}\widehat{\tau_{li}^{R}}.
\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{xR,7}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}_{1\cdot,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{N\cdot,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times N^2O$ matrix,

$$\overline{\boldsymbol{M}^{xR,8}} \equiv \begin{bmatrix} \operatorname{diag}\left(\widetilde{x}_{11,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{N1,\cdot,t_0}^{R}\right) \\ \vdots & & \vdots \\ \operatorname{diag}\left(\widetilde{x}_{1N,\cdot,t_0}^{R}\right) & \dots & \operatorname{diag}\left(\widetilde{x}_{NN,\cdot,t_0}^{R}\right) \end{bmatrix}$$

a $NO \times NO$ matrix, and

$$\overline{\boldsymbol{M}^{xG,7}} \equiv \begin{bmatrix} \widetilde{x}^G_{11,t_0} & & & \cdots & \widetilde{x}^G_{N1,t_0} & & \mathbf{0} \\ & \ddots & & & & \ddots & \\ \mathbf{0} & & \widetilde{x}^G_{1N,t_0} & & & & \widetilde{x}^G_{NN,t_0} \end{bmatrix} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix.

$$\overline{\boldsymbol{M}^{xR,9}} \equiv \begin{bmatrix} \widetilde{\boldsymbol{x}}^R_{11,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{N1,\cdot,t_0} & & \mathbf{0} \\ & \ddots & & \cdots & & \ddots & \\ \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{1N,\cdot,t_0} & & \mathbf{0} & & \widetilde{\boldsymbol{x}}^R_{NN,\cdot,t_0} \end{bmatrix},$$

a $NO \times N^2$ matrix, where $\widetilde{\boldsymbol{x}}^R_{ij,\cdot,t_0} \equiv \left(\widetilde{x}^R_{ij,o,t_0}\right)_o$ is an $O \times 1$ vector for any i and j. Then I have

$$-\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} + \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\widehat{\boldsymbol{p}^R} + \overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} - \widehat{\boldsymbol{K}^R} = \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} - \left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}$$

Next, to study the steady state Euler equation (B.23) , note that by equation (B.13),

$$\begin{aligned}
\frac{\partial \pi_{i,t}\left(\widehat{\left\{K^R_{i,o,t}\right\}}\right)}{\partial K^R_{i,o,t}} &= \sum_j \widetilde{y}^G_{ij,t}\left(\widehat{p^G_{ij,t}} + \widehat{Q^G_{ij,t}}\right) + \left[-\frac{1}{\beta}\sum_{o'} x^O_{i,o',t_0}\widehat{b_{i,o',t}} + \frac{1}{\beta}\widehat{b_{i,o,t}}\right] \\
&+ \left\{\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'}\frac{\widetilde{x}^O_{i,o',t_0}}{1-a_{o,t_0}}\left[-l^O_{i,o',t_0}a_{o,t_0} + \left(1-l^O_{i,o',t_0}\right)\left(1-a_{o,t_0}\right)\right]\widehat{a_{o',t}}\right. \\
&+ \left.\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\frac{-l^O_{i,o,t_0}a_{o,t_0} + \left(1-l^O_{i,o,t_0}\right)\left(1-a_{o,t_0}\right)}{1-a_{o,t_0}} + \frac{1}{\theta}\right\}\widehat{a_{o,t}}\right\} \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'} x^O_{i,o',t_0} l^O_{i,o',t_0}\widehat{L_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l^O_{i,o,t_0}\widehat{L_{i,o,t}}\right] \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}^O_{i,o',t_0}\left(1-l^O_{i,o',t_0}\right)\widehat{K^R_{i,o',t}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l^O_{i,o,t_0}\right)\widehat{K^R_{i,o,t}} + \left(-\frac{1}{\theta}\right)\widehat{K^R_{i,o,t}}\right].
\end{aligned} \tag{C.7}$$

Note that by the steady state accumulation condition (B.22), $Q^R_{i,o,t_0}/K^R_{i,o,t_0} = \delta$. Note also that investment function implies that, in the steady state,

$$\frac{\lambda^R_{j,o}}{P^R_{j,o}} = \left(\sum_i \frac{x^R_{ij,o}}{\left(1+u_{ij}\right)^{1-\varepsilon^R}}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta. \tag{C.8}$$

To simplify the notation, set

$$\widetilde{u}^{SS}_{j,o,t_0} \equiv \frac{\left(\iota+\delta\right)\left[\left(\sum_i x^R_{ij,o,t_0}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right]}{\left(\iota+\delta\right)\left[\left(\sum_i x^R_{ij,o,t_0}\left(1+u_{ij,t_0}\right)^{-\left(1-\varepsilon^R\right)}\right)^{\frac{1}{1-\varepsilon^R}\alpha^R} + 2\gamma\delta\right] - \gamma\delta^2},$$

Then by log-linearizing equation (B.23) implies, after rearranging,

$$\begin{aligned}
&\left[\widehat{p^G_i} + 2\left(1-\alpha^R\right)\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\sum_l \widetilde{x}^G_{li,t_0}\widehat{p^G_{l,t}}\right] - \left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\widehat{p^R_{i,o}} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\sum_l \widetilde{x}^R_{ij,o,t_0}\widehat{p^R_{l,o}} \\
&+ \sum_j \widetilde{y}^G_{ij,t_0}\widehat{Q^G_{ij}} - 2\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\sum_l \widetilde{x}^R_{li,o,t_0}\widehat{Q^R_{li,o}} + \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}^O_{i,o',t_0} l^O_{i,o',t_0}\widehat{L_{i,o'}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right) l^O_{i,o,t_0}\widehat{L_{i,o}}\right] \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}\widetilde{x}^O_{i,o',t_0}\left(1-l^O_{i,o',t_0}\right)\widehat{K^R_{i,o'}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l^O_{i,o,t_0}\right)\widehat{K^R_{i,o}} + \left(-\frac{1}{\theta}\right)\widehat{K^R_{i,o}} + 2\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\widehat{K^R_{i,o}}\right] \\
&- \widetilde{u}^{SS}_{i,o,t_0}\widehat{\lambda^R_{i,o}} \\
&= -\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\sum_{o'}\frac{\widetilde{x}^O_{i,o',t_0}}{1-a_{o,t_0}}\left[\left(1-l^O_{i,o',t_0}\right)\left(1-a_{o',t_0}\right) - l^O_{i,o',t_0}a_{o',t_0}\right]\widehat{a_{o'}} \\
&- \left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\frac{1}{1-a_{o,t_0}}\left[\left(1-l^O_{i,o,t_0}\right)\left(1-a_{o,t_0}\right) - l^O_{i,o,t_0}a_{o,t_0}\right] + \frac{1}{\theta}\right\}\widehat{a_o} \\
&- \left[-\frac{1}{\beta}\sum_{o'}\widetilde{x}^O_{i,o',t_0}\widehat{b_{i,o'}} + \frac{1}{\beta}\widehat{b_{i,o}}\right] + \left[-\sum_j \widetilde{y}^G_{ij,t_0}\widehat{\tau^G_{ij}} - 2\left(1-\alpha^R\right)\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\sum_l \widetilde{x}^G_{li,t_0}\widehat{\tau^G_{li,t}}\right] \\
&+ 2\left(1-\alpha^R\right)\left(1-\widetilde{u}^{SS}_{i,o,t_0}\right)\sum_l \widetilde{x}^R_{ij,o,t_0}\widehat{\tau^R_{li}}
\end{aligned}$$

In matrix notation, write

$$\overline{\boldsymbol{M}^{xO,3}} \equiv \overline{\boldsymbol{M}^{xO}} \otimes \mathbf{1}_O$$

a $NO \times N^2$ matrix. Then

$$\begin{aligned}
&\left[\left(\overline{\boldsymbol{I}_N} \otimes \boldsymbol{1}_O\right) + 2\left(1-\alpha^R\right) \operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right) \overline{\boldsymbol{M}^{xG,2}}\right] \widehat{\boldsymbol{p}^G} - \operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\left(\overline{\boldsymbol{I}_{NO}} - 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\widehat{\boldsymbol{p}^R} \\
&+ \overline{\boldsymbol{M}^{yG,3}}\widehat{\boldsymbol{Q}^G} - 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,7}}\widehat{\boldsymbol{Q}^R} + \left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(l^O_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{L}} \\
&+ \left[\left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(1-l^O_{\cdot,\cdot,t_0}\right) - \frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + 2\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\right]\widehat{\boldsymbol{K}^R} - \operatorname{diag}\left(\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\widehat{\boldsymbol{\lambda}^R} \\
&= -\left[\left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,3}} - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,2}}\right]\widehat{\boldsymbol{a}} - \frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right)\widehat{\boldsymbol{b}} \\
&+ \left[-\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xG,7}}\right]\widehat{\boldsymbol{\tau}^G} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(1-\widetilde{u}^{SS}_{\cdot,\cdot,t_0}\right)\overline{\boldsymbol{M}^{xR,9}}\widehat{\boldsymbol{\tau}^R}
\end{aligned}$$

In the steady state, I write equations (C.3) and (C.4) as

$$\overline{\boldsymbol{M}^{xG,2}}\widehat{\boldsymbol{p}^G} - \widehat{\boldsymbol{w}} + \left[\overline{\boldsymbol{I}_{NO}} - \frac{1}{1+\iota}\overline{\boldsymbol{M}^{\mu,2}}\right]\widehat{\boldsymbol{V}} = -\overline{\boldsymbol{M}^{xG,7}}\widehat{\boldsymbol{\tau}^G} + d\boldsymbol{T} - \overline{\boldsymbol{M}^{\mu,3}}d\boldsymbol{\chi}^{\text{vec}}$$

and

$$\left[\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{\mu L}}\right]\widehat{\boldsymbol{L}} - \overline{\boldsymbol{M}^{\mu L,2}}\widehat{\boldsymbol{\mu}^{\text{vec}}} = \boldsymbol{0}.$$

respectively.

Hence the log-linearized steady state system is

$$\overline{\boldsymbol{E}^y}\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}^\Delta}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{E}^y} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ \multicolumn{2}{c}{\overline{\boldsymbol{D}^{y,SS}}} \end{bmatrix}, \text{ and } \overline{\boldsymbol{E}^\Delta} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,\Delta}} \\ \overline{\boldsymbol{D}^{\Delta,SS}} \end{bmatrix},$$

$\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,T}} & \overline{\boldsymbol{D}^{A,\Delta}} \end{bmatrix}$, and matrices $\overline{\boldsymbol{D}^{y,SS}}$ and $\overline{\boldsymbol{D}^{\Delta,SS}}$ are defined as

$$\overline{\boldsymbol{D}^{y,SS}} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{y,SS}_{11}} & \overline{\boldsymbol{D}^{y,SS}_{12}} & \boldsymbol{0} & \boldsymbol{0} & \overline{\boldsymbol{M}^{xR,7}} & \boldsymbol{0} & -\overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} \\ \overline{\boldsymbol{D}^{y,SS}_{21}} & \overline{\boldsymbol{D}^{y,SS}_{22}} & \boldsymbol{0} & \overline{\boldsymbol{M}^{yG,3}} & \overline{\boldsymbol{D}^{y,SS}_{25}} & \overline{\boldsymbol{D}^{y,SS}_{26}} & \overline{\boldsymbol{D}^{y,SS}_{27}} & \overline{\boldsymbol{D}^{y,SS}_{28}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}^{y,SS}_{11}} \equiv -\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{12}^{y,SS}} \equiv \left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xR,8}},$$

$$\overline{\boldsymbol{D}_{21}^{y,SS}} \equiv \left(\overline{\boldsymbol{I}_{N}} \otimes \mathbf{1}_{O}\right) + 2\left(1-\alpha^{R}\right)\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xG,2}},$$

$$\overline{\boldsymbol{D}_{22}^{y,SS}} \equiv -\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xR,8}}\right),$$

$$\overline{\boldsymbol{D}_{25}^{y,SS}} \equiv -2\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xR,7}},$$

$$\overline{\boldsymbol{D}_{26}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,3}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(l_{\cdot,\cdot,t_0}^{O}\right),$$

$$\overline{\boldsymbol{D}_{27}^{y,SS}} \equiv \left(-1+\frac{1}{\beta}\right)\overline{\boldsymbol{M}^{xOl,4}} + \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(1-l_{\cdot,\cdot,t_0}^{O}\right) - \frac{1}{\theta}\overline{\boldsymbol{I}_{NO}} + 2\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right),$$

$$\overline{\boldsymbol{D}_{28}^{y,SS}} \equiv -\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right),$$

and

$$\overline{\boldsymbol{D}^{\Delta,SS}} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{17}^{\Delta,SS}} & \overline{\boldsymbol{D}_{18}^{\Delta,SS}} \\ \mathbf{0} & \overline{\boldsymbol{D}_{22}^{\Delta,SS}} & \overline{\boldsymbol{D}_{23}^{\Delta,SS}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \overline{\boldsymbol{D}_{27}^{\Delta,SS}} & \overline{\boldsymbol{D}_{28}^{\Delta,SS}} \end{bmatrix},$$

where

$$\overline{\boldsymbol{D}_{17}^{\Delta,SS}} \equiv \left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xG,7}},$$

$$\overline{\boldsymbol{D}_{18}^{\Delta,SS}} \equiv -\left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xR,9}},$$

$$\overline{\boldsymbol{D}_{22}^{\Delta,SS}} \equiv \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,2}} - \left(-1+\frac{1}{\beta}\right)\frac{1}{\theta-1}\overline{\boldsymbol{M}^{al,3}},$$

$$\overline{\boldsymbol{D}_{23}^{\Delta,SS}} \equiv -\frac{1}{\beta}\left(\overline{\boldsymbol{I}_{NO}} - \overline{\boldsymbol{M}^{xO,3}}\right),$$

$$\overline{\boldsymbol{D}_{27}^{\Delta,SS}} \equiv -\overline{\boldsymbol{M}^{yG,3}} - 2\left(1-\alpha^{R}\right)\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xG,7}},$$

and

$$\overline{\boldsymbol{D}_{28}^{\Delta,SS}} \equiv 2\left(1-\alpha^{R}\right)\operatorname{diag}\left(1-\widetilde{u}_{\cdot,\cdot,t_0}^{SS}\right)\overline{\boldsymbol{M}^{xR,9}}.$$

If $\overline{\boldsymbol{E}^{y}}$ is invertible, I have $\overline{\boldsymbol{E}} \equiv \left(\overline{\boldsymbol{E}^{y}}\right)^{-1}\overline{\boldsymbol{E}^{\Delta}}$ such that $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$. Write dimensions of $\boldsymbol{y}$ and $\boldsymbol{\Delta}$ as $n_y \equiv N + 3NO + N^2 + N^2O$ and $n_\Delta \equiv 3N^2 + O + 2NO + 2N$, respectively.

Finally, to study the transitional dynamics, the capital accumulation dynamics (10) implies

$$K_{i,o,t+1}^{\check{R}}=-\delta\left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,t_0}^{G}p_{l,t}^{\check{G}}+\delta\left(1-\alpha^{R}\right)\sum_{l}\widetilde{x}_{li,o}^{R}p_{l,o,t}^{\check{R}}+\delta\sum_{l}\widetilde{x}_{li,o}^{R}Q_{li,o,t}^{\check{R}}+(1-\delta)K_{i,o,t}^{\check{R}}.$$

In a matrix form, write

$$\boldsymbol{K}_{t+1}^{\check{R}}=-\delta\left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xG,2}}\boldsymbol{p}_{t}^{\check{G}}+\delta\left(1-\alpha^{R}\right)\overline{\boldsymbol{M}^{xR,8}}\boldsymbol{p}_{t}^{\check{R}}+\delta\overline{\boldsymbol{M}^{xR,7}}\boldsymbol{Q}_{t}^{\check{R}}+(1-\delta)\overline{\boldsymbol{I}_{NO}}\boldsymbol{K}_{t}^{\check{R}}.$$

Next, to study the Euler equation, define

$$\widetilde{u}_{i,o}^{TD,1}\equiv\frac{-(\iota+\delta)\left[\left(\sum_{l}x_{li,o}^{R}\left(1+u_{li}\right)^{-\left(1-\varepsilon^{R}\right)}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}}+2\gamma\delta\right]+\gamma\delta^{2}}{(1-\delta)\left[\left(\sum_{l}x_{li,o}^{R}\left(1+u_{li}\right)^{-\left(1-\varepsilon^{R}\right)}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}}+2\gamma\delta\right]}$$

and

$$\widetilde{u}_{i,o}^{TD,2}\equiv\frac{-\gamma\delta^{2}}{(1-\delta)\left[\left(\sum_{l}x_{li,o}^{R}\left(1+u_{li}\right)^{-\left(1-\varepsilon^{R}\right)}\right)^{\frac{1}{1-\varepsilon^{R}}\alpha^{R}}+2\gamma\delta\right]}.$$

Then I have

$$\begin{aligned}&\left[-\widetilde{u}_{i,o}^{TD,1}p_{i,t+1}^{\check{G}}+2\left(1-\alpha^{R}\right)\widetilde{u}_{i,o}^{TD,2}\sum_{l}\widetilde{x}_{li}^{G}p_{l,t+1}^{\check{G}}\right]+\left[-\widetilde{u}_{i,o}^{TD,2}p_{i,o,t+1}^{\check{R}}-2\left(1-\alpha^{R}\right)\widetilde{u}_{i,o}^{TD,2}\sum_{l}\widetilde{x}_{li,o}^{R}p_{l,o,t+1}^{\check{R}}\right]\\&-\widetilde{u}_{i,o}^{TD,1}\sum_{j}\widetilde{y}_{ij}^{G}Q_{ij,t+1}^{\check{G}}-2\widetilde{u}_{i,o}^{TD,2}\sum_{l}\widetilde{x}_{li,o}^{R}Q_{li,o,t+1}^{\check{R}}-\widetilde{u}_{i,o}^{TD,1}\left(-1+\frac{1}{\beta}\right)\sum_{o'}x_{i,o'}^{O}\left(1-l_{i,o'}^{O}\right)K_{i,o',t+1}^{\check{R}}\\&-\widetilde{u}_{i,o}^{TD,1}\left[\left(-1+\frac{1}{\beta}\right)\sum_{o'}x_{i,o'}^{O}l_{i,o'}^{O}L_{i,o',t+1}^{\check{}}+\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)l_{i,o}^{O}L_{i,o,t+1}^{\check{}}\right]\\&-\left[\widetilde{u}_{i,o}^{TD,1}\left\{\left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\left(1-l_{i,o}^{O}\right)+\left(-\frac{1}{\theta}\right)\right\}-2\widetilde{u}_{i,o}^{TD,2}\right]K_{i,o,t+1}^{\check{R}}+\lambda_{i,o,t+1}^{\check{R}}=\frac{1+\iota}{1-\delta}\lambda_{i,o,t}^{\check{R}}\end{aligned}$$

In a matrix form, write

$$\overline{\boldsymbol{M}^{u,4}}=\begin{bmatrix}\widetilde{\boldsymbol{u}}_{1,\cdot}^{TD,1}&&\boldsymbol{0}\\&\ddots&\\\boldsymbol{0}&&\widetilde{\boldsymbol{u}}_{N,\cdot}^{TD,1}\end{bmatrix},$$

a $NO \times N$ matrix where $\widetilde{\boldsymbol{u}}_{i,\cdot}^{TD,1} \equiv \left(\widetilde{u}_{i,o}^{TD,1}\right)_o$ is an $O \times 1$ vector for any i. Then

$$
\begin{aligned}
&\left(-\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\overline{\boldsymbol{M}^{xG,2}}\right)\boldsymbol{p}_{t+1}^{\check{G}} - \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right)\boldsymbol{p}_{t+1}^{\check{R}} \\
&\quad - \left[\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_N\right)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}}\right]\boldsymbol{Q}_{t+1}^{\check{G}} - 2\left(\left(\boldsymbol{1}_N\right)^\top \otimes \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}}\boldsymbol{Q}_{t+1}^{\check{R}} \\
&\quad + \left[-\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_O\right)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1} l_{\cdot,\cdot}^{O}\right)\right]\check{\boldsymbol{L}}_{t+1} \\
&\quad + \left\{\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_O\right)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\left(1-l_{\cdot,\cdot}^{O}\right)\right)\right. \\
&\quad \left. + \frac{1}{\theta}\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\right) + 2\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right\}\boldsymbol{K}_{t+1}^{\check{R}} + \overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}_{t+1}^{\check{R}} = \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}}\boldsymbol{\lambda}_{t}^{\check{R}}.
\end{aligned}
$$

Hence the log-linearized transitional dynamic system is $\overline{\boldsymbol{D}_{t+1}^{y,TD}}\check{\boldsymbol{y}}_{t+1} = \overline{\boldsymbol{D}_{t}^{y,TD}}\check{\boldsymbol{y}}_{t}$, where matrices $\overline{\boldsymbol{D}_{t+1}^{y,TD}}$ and $\overline{\boldsymbol{D}_{t}^{y,TD}}$ are defined as

$$
\overline{\boldsymbol{D}_{t+1}^{y,TD}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \overline{\boldsymbol{I}_{NO}} & \boldsymbol{0} \\ \overline{\boldsymbol{D}_{21,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{22,t+1}^{y,TD}} & \boldsymbol{0} & \overline{\boldsymbol{D}_{24,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{25,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{26,t+1}^{y,TD}} & \overline{\boldsymbol{D}_{27,t+1}^{y,TD}} & \overline{\boldsymbol{I}_{NO}} \end{bmatrix},
$$

where

$$
\overline{\boldsymbol{D}_{21,t+1}^{y,TD}} \equiv -\overline{\boldsymbol{M}^{u,4}} + 2\left(1-\alpha^R\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\overline{\boldsymbol{M}^{xG,2}},
$$

$$
\overline{\boldsymbol{D}_{22,t+1}^{y,TD}} \equiv -\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\left(\overline{\boldsymbol{I}_{NO}} + 2\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}}\right),
$$

$$
\overline{\boldsymbol{D}_{24,t+1}^{y,TD}} \equiv -\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_N\right)^\top\right) \circ \overline{\boldsymbol{M}^{yG,3}},
$$

$$
\overline{\boldsymbol{D}_{25,t+1}^{y,TD}} \equiv -2\left(\left(\boldsymbol{1}_N\right)^\top \otimes \operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right)\right) \circ \overline{\boldsymbol{M}^{xR,7}},
$$

$$
\overline{\boldsymbol{D}_{26,t+1}^{y,TD}} \equiv -\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_O\right)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,3}}\right) - \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1} l_{\cdot,\cdot}^{O}\right),
$$

$$
\begin{aligned}
\overline{\boldsymbol{D}_{27,t+1}^{y,TD}} \equiv &\left(-1+\frac{1}{\beta}\right)\left(\left(\overline{\boldsymbol{M}^{u,4}} \otimes \left(\boldsymbol{1}_O\right)^\top\right) \circ \overline{\boldsymbol{M}^{xOl,4}}\right) \\
&- \left(-\frac{1}{\beta}+\frac{1}{\theta}\right)\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\left(1-l_{\cdot,\cdot}^{O}\right)\right) + \frac{1}{\theta}\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,1}\right) + 2\operatorname{diag}\left(\widetilde{u}_{\cdot,\cdot}^{TD,2}\right),
\end{aligned}
$$

and

$$\overline{\boldsymbol{D}_t^{y,TD}} = \left[\begin{array}{cccccccc} -\delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xG,2}} & \delta\left(1-\alpha^R\right)\overline{\boldsymbol{M}^{xR,8}} & \mathbf{0} & \mathbf{0} & \delta\overline{\boldsymbol{M}^{xR,7}} & \mathbf{0} & (1-\delta)\,\overline{\boldsymbol{I}_{NO}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1+\iota}{1-\delta}\overline{\boldsymbol{I}_{NO}} \end{array} \right]. \tag{C.9}$$

Since $\check{\boldsymbol{y}}_t = \widehat{\boldsymbol{y}}_t - \widehat{\boldsymbol{y}}$ for any $t \geq t_0$ and $\widehat{\boldsymbol{y}} = \overline{\boldsymbol{E}}\boldsymbol{\Delta}$, I have

$$\overline{\boldsymbol{D}_{t+1}^{y,TD}}\left(\widehat{\boldsymbol{y}_{t+1}} - \widehat{\boldsymbol{y}}\right) = \overline{\boldsymbol{D}_t^{y,TD}}\left(\widehat{\boldsymbol{y}}_t - \widehat{\boldsymbol{y}}\right)$$

$$\iff \overline{\boldsymbol{D}_{t+1}^{y,TD}}\,\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{D}_t^{y,TD}}\widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{D}_{t+1}^{y,TD}} - \overline{\boldsymbol{D}_t^{y,TD}}\right)\overline{\boldsymbol{E}}\boldsymbol{\Delta}.$$

Recall the temporary equilibrium condition $\overline{\boldsymbol{D}^x}\widehat{\boldsymbol{x}}_t - \overline{\boldsymbol{D}^{A,S}}\widehat{\boldsymbol{S}}_t = \overline{\boldsymbol{D}^{A,\Delta}}\widehat{\boldsymbol{\Delta}}$ for any t. Thus

$$\overline{\boldsymbol{F}_{t+1}^y}\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}_t^y}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}_{t+1}^\Delta}\boldsymbol{\Delta},$$

where

$$\overline{\boldsymbol{F}_{t+1}^y} \equiv \left[\begin{array}{cc} \overline{\boldsymbol{D}^x} & -\overline{\boldsymbol{D}^{A,T}} \\ & \overline{\boldsymbol{D}_{t+1}^{y,TD}} \end{array} \right], \overline{\boldsymbol{F}_t^y} \equiv \left[\begin{array}{c} \mathbf{0} \\ \overline{\boldsymbol{D}_t^{y,TD}} \end{array} \right], \overline{\boldsymbol{F}_{t+1}^\Delta} \equiv \left[\begin{array}{c} \overline{\boldsymbol{D}^{A,\Delta}} \\ \left(\overline{\boldsymbol{D}_{t+1}^{y,TD}} - \overline{\boldsymbol{D}_t^{y,TD}}\right)\overline{\boldsymbol{E}} \end{array} \right],$$

or with $\overline{\boldsymbol{F}^y} \equiv \left(\overline{\boldsymbol{F}_{t+1}^y}\right)^{-1}\overline{\boldsymbol{F}_t^y}$ and $\overline{\boldsymbol{F}^\Delta} \equiv \left(\overline{\boldsymbol{F}_{t+1}^y}\right)^{-1}\overline{\boldsymbol{F}_{t+1}^\Delta}$, one can write

$$\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t + \overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{C.10}$$

It remains to find the initial values of the system (C.10) that satisfies the transversality condition. To this end, I apply Theorem 6.4 of Stokey and Lucas (1989). In particular, I first homogenize the system: Note that equation (C.10) can be rewritten as $\widehat{\boldsymbol{y}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{y}}_t + \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)\left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$ and thus

$$\widehat{\boldsymbol{z}_{t+1}} = \overline{\boldsymbol{F}^y}\widehat{\boldsymbol{z}}_t \tag{C.11}$$

where

$$\widehat{\boldsymbol{z}}_t \equiv \widehat{\boldsymbol{y}}_t - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}. \tag{C.12}$$

Next, for the transversality condition to be satisfied, the system (C.11) must not explode. Thus it must be that $\widehat{\boldsymbol{z}_t} \to \boldsymbol{0}$ or $\widehat{\boldsymbol{y}_t} \to \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1} \overline{\boldsymbol{F}^\Delta}\boldsymbol{\Delta}$. To find the condition, write Jordan decomposition of $\overline{\boldsymbol{F}^y}$ as $\overline{\boldsymbol{F}^y} = \overline{\boldsymbol{B}}^{-1}\overline{\boldsymbol{\Lambda}\boldsymbol{B}}$. Then Theorem 6.4 of Stokey and Lucas (1989) implies that it must be that out of n_y vector of $\overline{\boldsymbol{B}}\widehat{\boldsymbol{z}_{t_0}}$, n-th element must be zero if $|\lambda_n| > 1$. Since $\widehat{\boldsymbol{K}^R_{t_0}} = \boldsymbol{0}$, we can write

$$\widehat{\boldsymbol{z}_{t_0}} = \overline{\boldsymbol{F}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{F}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

where

$$\overline{\boldsymbol{F}^\Delta_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\Delta}} \\ \boldsymbol{0}_{2NO\times n_\Delta} \end{bmatrix} - \left(\overline{\boldsymbol{I}} - \overline{\boldsymbol{F}^y}\right)^{-1}\overline{\boldsymbol{F}^\Delta} \text{ and } \overline{\boldsymbol{F}^\lambda_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\overline{\boldsymbol{D}^{A,\lambda}} \\ \boldsymbol{0}_{NO\times NO} \\ \overline{\boldsymbol{I}_{NO}} \end{bmatrix}$$

and $\overline{\boldsymbol{D}^{A,\lambda}}$ is the right block matrix of $\overline{\boldsymbol{D}^A} \equiv \begin{bmatrix} \overline{\boldsymbol{D}^{A,K}} & \overline{\boldsymbol{D}^{A,\lambda}} \end{bmatrix}$ that corresponds to vector $\widehat{\boldsymbol{\lambda}^R}$. Extracting n-th row from $\overline{\boldsymbol{F}^\Delta_{t_0}}$ and $\overline{\boldsymbol{F}^\lambda_{t_0}}$ where $|\lambda_n| > 1$ and writing them as a $NO \times n_\Delta$ matrix $\overline{\boldsymbol{G}^\Delta_{t_0}}$ and $NO \times NO$ matrix $\overline{\boldsymbol{G}^\lambda_{t_0}}$, the condition of the Theorem is

$$\boldsymbol{0} = \overline{\boldsymbol{G}^\Delta_{t_0}}\boldsymbol{\Delta} + \overline{\boldsymbol{G}^\lambda_{t_0}}\widehat{\boldsymbol{\lambda}^R_{t_0}},$$

or $\widehat{\boldsymbol{\lambda}^R_{t_0}} = \overline{\boldsymbol{G}_{t_0}}\boldsymbol{\Delta}$ where $\overline{\boldsymbol{G}_{t_0}} \equiv -\left(\overline{\boldsymbol{G}^\lambda_{t_0}}\right)^{-1}\overline{\boldsymbol{G}^\Delta_{t_0}}$. Finally, tracing back to obtain the initial conditions for $\widehat{\boldsymbol{y}_t}$, it must be $\widehat{\boldsymbol{y}_{t_0}} = \overline{\boldsymbol{F}^y_{t_0}}\boldsymbol{\Delta}$,where

$$\overline{\boldsymbol{F}^y_{t_0}} \equiv \begin{bmatrix} \left(\overline{\boldsymbol{D}^x}\right)^{-1}\left(\overline{\boldsymbol{D}^{A,\Delta}} + \overline{\boldsymbol{D}^{A,\lambda}}\overline{\boldsymbol{G}_{t_0}}\right) \\ \boldsymbol{0}_{NO\times n_\Delta} \\ \overline{\boldsymbol{G}_{t_0}} \end{bmatrix}.$$