ISyE 6416: Computational Statistics
Homework 5
(100 points total.)

- This homework is due on Monday March 25.
- Please write your team member's name is you collaborate.

1. **Proof (20 points).** Prove the following results in the lecture slides for hidden Markov model:

$$\mathbb{P}(S_t = i|o_1, \cdots, o_T) \propto \underbrace{\mathbb{P}(S_t = i, o_1, \cdots, o_t)}_{\alpha_i(t)} \cdot \underbrace{\mathbb{P}(o_{t+1}, \cdots, o_T|S_t = i)}_{\beta_i(t)}$$

$$\mathbb{P}(S_t = i, S_{t+1} = j|o_1, \ldots, o_T) \propto \alpha_i(t)\beta_j(t+1)a_{i,j}b_{j,o_{t+1}}$$

2. **The occasionally dishonest casino (40 points).** Consider the occasionally dishonest casino problem we discussed in class. There are two states for the dice, the "fair" (F) and "loaded" (L) dices, and under each state, the emission probability matrix is given by

$$E = \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 1/10 & 1/10 & 1/10 & 1/10 & 1/10 & 1/2 \end{bmatrix}$$

   And the state transition matrix is

$$P(F|F) = 0.95, \quad P(L|F) = 0.05, \quad P(L|L) = 0.9, \quad P(H|L) = 0.1.$$

   Assume the observed sequence is 445436316566265666. Assume the initial distribution to be [1/2, 1/2].

   (a) Program your own code to answer this question: use the forward-backward algorithm to determine the probability of being in each state at time k given all three observed bits.

   (b) What is the most likely sequence of states given all the observations? Implement the computation.

3. **HMM and a two-bit register (40 points).** Consider a two-bit register. The register has four possible states: 00, 01, 10 and 11. Initially, at time 0, the contents of the register is chosen at random to be one of these four states, each with equal probability. At each time step, beginning at time 1, the register is randomly manipulated as follows: with probability 1/2, the register is left unchanged; with probability 1/4, the two bits of the register are exchanged (e.g., 01 becomes 10); and with probability 1/4, the right bit is flipped (e.g., 01 becomes 00). After the register has been manipulated in this fashion, the left bit is observed. Suppose that on the first three time steps, we observe 0, 0, 1. Solve the following question by hand (not using matlab code).

(a) Show how the register can be formulated as an HMM. What is the probability of transitioning from every state to every other state? What is the probability of observing each output (0 or 1) in each state?

(b) Use the forward-backward algorithm to determine the probability of being in each state at time k given all three observed bits, for $k = 0, 1, 2, 3$.

(c) What is the most likely sequence of states given all three observed bits? (Be sure to include the initial state at time 0 in your sequence.) Draw a trellis diagram to illustrate the path elimination process of the Viterbi algorithm.