MATH 239 Lecture Notes

Daisy Ye

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Contents

Sep 7 Lecture

Definition of a Cardinality of a Set

Principle of Inclusion and Exclusion

Theorem of Permutation

Sep 9 Lecture

set A AND set B is the cartesian product $A * B$
set A OR set B is the disjoint union $A \cup B$

Partial List
We are interested in length-k partial lists of n-element sets.
Tuples of the form: $(S_1, S_2, ..., S_k)$.
There are n choices for S_1, $(n-1)$ choices for S_2, $(n+k-1)$ choices for S_k.
Total number of options $= n(n-1)...(n-k+1)$.

Theorem 1.4
For $n, k \geq 0$, the number of partial list of length k of an n-element set is $n(n-1)...(n-k+1)$.
For $0 \leq k \leq n$, we can write the above as follows:

$$\frac{n!}{(n-k)!}$$

For $n, k \geq 0$, let $\binom{n}{k}$ denote the number of k-element subsets of an n-element set.
We can see easily:
$\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{n} = 1$, $\binom{n}{k}$ where $k > n = 0$.

Theorem 1.5
For $0 \leq k \leq n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Pascal's Triangle
See examples from the textbook.

Multi-sets (1.1.4)
Suppose I have a bag of 8 marbles, each of which is either red, blue, or green. How many different bag contents are possible?
We want to count the number of three-tuples of non-negative integers (n_1, n_2, n_3) such that $n_1 + n_2 + n_3 = 8$.

Definition 1.8
Let $n \geq 0$, $t \geq 1$ be integers. A multi-set of size n with elements of t types us a sequence of non-negative integers $(m_1, m_2, ..., m_t)$ such that $m_1 + m_2 + ... + m_t = n$.

Theorem 1.9
For any $n \geq 0$, $t \geq 1$, the number of n-element multi-sets with elements of t types is $\binom{n+t-1}{t-1}$.

Sep 12 Lecture

Last Lecture Recap
- number of length-k partial list of an n-element set is $\frac{n!}{(n-k)!}$.
- proved Theorem 1.5: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ utilizing a combinatorial proof.
- introduced the definition of multi-sets of size n with elements of t types.

In a combinatorial proof, the fundamental concept is to count the same set in two different ways (some algebraic manipulations are acceptable) (definition of double counting).

Theorem 1.4
For any $n \geq 0$, $t \geq 1$, the number of n-element multisets with t types is $\binom{n+t-1}{t-1}$.

Combinatorial proof of Theorem 1.4:

We know that $\binom{n+t-1}{t-1}$ is the number of $(t-1)$-element subsets of an $(n+t-1)$-element set. We will show how to translate between $(t-1)$-element subsets of an $(n+t-1)$-element set and n-element multisets with t types.
Let us write down a row of $(n+t-1)$ circles:
o o o o o o o o o o o
Choose a $(t-1)$-element subset and cross them out:
o o x o o o x o o o
Our row now in general has n circles grouped into t segments, each containing 0 or more circles.
Let m_i be the length of the ith segment of consecutive circles.
Then $m_1 + m_2 + ... + m_t = n$, so $(m_1, m_2, ..., m_t)$ is a multiset of size n with t types.

Now let us check the other direction.
Let $(m_1, ..., m_t)$ be an n-element multiset with t types.
Write down a sequence of m_1 circles, then an x, then m_2 circles, then an x, and so on, up to a sequence of m_t circles.
We will have written down $(n+t-1)$ symbols with the $(t-1)$ x's indicating a $(t-1)$-element subset of an $(n+t-1)$-element set.

Bijections (Section 1.1.5)
Definition 1.10
Let A and B be sets.
Let $f : A- > B$.
f is surjective (onto) if for every b in B there exists an a in A such that $f(a) = b$.
f is injective (one-to-one) if for every a and a' in A, if $f(a) = f(a')$, then $a = a'$.
f is bijective if it is both surjective and injective.

Proposition 1.11
A function is bijective if and only if there exists an inverse function that inverts the mapping.

Corollary
If A and B are in bijection, and at least one of A or B is finite, then in fact both are finite and have the same number of elements in them. We write $|A| = |B|$.

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Last Lecture Recap
Bijection means both injective and surjective and have an inverse.
A in bijection of B implies that $|A| = |B|$.

Generating Series (Section 2)
Definition: A formal power series is an expression of the form

$$G(x) = \sum_{n=0}^{\infty} g_n x^n$$

in which the coefficients $(g_0, g_1, ...)$ are a sequence of integers.
x is called the indeterminate (meaning that we are not going to substitute some particular values for x, it is just used here to serve as a placeholder).
The coefficients are all finite.

Proposition
The inverse of $F(x) = \sum_{n=0}^{\infty} f_n x^n$ exists if and only if $f_0 ! = 0$.

Theorem 2.2 (Binomial Theorem)
For any natural number $n \in \mathcal{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Sep 16 Lecture

$\boxed{\text{Last time}}$

☆ formal power series

$G(x) = \sum_{n\geq 0} g_n x^n$ — call x indeterminate

s.t. $(g_0, g_1, \dots)$ is an integer sequence

can add and multiply formal power series.

inverse exists iff $g_0 \neq 0$.

☆ Binomial theorem $(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$, $n \geq 0$

☆ Theorem (Negative Binomial Theorem) 2.4

If $t \geq 1$, then

$$(1-x)^{-t} = \sum_{n\geq 0} \binom{n+t-1}{t-1} x^n$$

proof: (another combinatorial proof)

Let $m(n,t)$ denote the set of n-element multisets with t types.

Or $m(t)$ denote the set of multisets of t-types.

so $m(t) = \bigcup_{n\geq 0} m(n,t)$.

we can see there is a bijection between $m(t)$ and $\mathbb{N}^t$:

every multiset $\mu \in m(t)$ corresponds to a non-negative integer sequence $(m_1, m_2, \dots, m_t) \in \mathbb{N}^t$, and vice versa.

moreover, the # of elements in μ is $|\mu| = m_1 + m_2 + \dots + m_t$.

$$\begin{aligned}
\sum_{n\geq 0} \binom{n+t-1}{t-1} x^n &= \sum_{n\geq 0} |m(n,t)|\, x^n && \text{(Theorem 1.9)} \\
&= \sum_{\mu \in m(t)} x^{|\mu|} && \text{(another way of counting multisets)} \\
&= \sum_{(m_1, m_2 \dots m_t) \in \mathbb{N}^t} x^{m_1+m_2+\dots+m_t} && \text{(By bijection)} \\
&= \sum_{m_1\geq 0} \sum_{m_2 \geq 0} \dots \sum_{m_t\geq 0} x^{m_1+m_2+\dots+m_t} \\
&= \Big(\sum_{m_1\geq 0} x^{m_1}\Big)\Big(\sum_{m_2\geq 0} x^{m_2}\Big) \dots \Big(\sum_{m_t \geq 0} x^{m_t}\Big) && \text{(factoring out)} \\
&= \Big(\frac{1}{1-x}\Big) \dots \Big(\frac{1}{1-x}\Big) && \text{(By geometric series)} \\
&= \Big(\frac{1}{1-x}\Big)^t = (1-x)^{-t} \qquad \square
\end{aligned}$$

☆ Generating Series (2.1.1)

ex. write $\left(\frac{x^2}{1+3x}\right)^4$ as formal power series.

$$\left(\frac{x^2}{1+3x}\right)^4 = x^8\left(\frac{1}{1-(-3x)}\right)^4$$
$$= x^8\sum_{n\geq 0}\binom{n+4-1}{4-1}(-3x)^n \quad \text{By negative binomial theorem}$$
$$= x^8\sum_{n\geq 0}\binom{n+3}{3}(-3)^n x^n$$
$$= \sum_{n\geq 0}\binom{n+3}{3}(-3)^n x^{n+8}$$
$$= \sum_{n\geq 8}\binom{n-5}{3}(-3)^{n-8} x^n \quad \text{Re-index the sum}$$

☆ Definition (coefficient extraction) 2.8

Let $G(x)=\sum_{n\geq 0} g_n x^n$ be a formal power series. For $k \geq N$, the coefficient extraction operator denoted $[x^k]$ is defined by $[x^k]G(x) = g_k$.

ex (cont'd) $[x^k]\left(\frac{x^2}{1+3x}\right)^4$

$$= [x^k]\sum_{n\geq 0}\binom{n-5}{3}(-3)^{n-8}x^n$$
$$= \begin{cases} 0, & \text{if } k<8 \\ \binom{k-5}{3}(-3)^{k-8}, & \text{if } k\geq 8 \end{cases}$$

☆ Rules about coefficient extraction

- $[x^k](aF(x)+bG(x))$
  $= a[x^k]F(x)+b[x^k]G(x)$ for $a,b\in\mathbb{N}$.
- $[x^k]x^i F(x) = [x^{k-i}]F(x)$
- $[x^k](F(x)G(x)) = \sum_{i=0}^{k}([x^i]F(x))([x^{k-i}]G(x))$

We will use formal power series to encode counting information about a set, in which we call it a generating series.

ex. Let $M = \{\text{Jan, Feb}, \ldots, \text{Dec}\}$

Let $M_n = \{\alpha \in M: \alpha \text{ has } n \text{ days, ignoring leap years}\}$

$$\sum_{n\geq 0}|M_n|x^n = 1x^{28}+4x^{30}+7x^{31}$$
$$= \sum_{\alpha\in M} x^{(\#\text{ of days in }\alpha)}$$
$$= x^{31}+x^{28}+x^{31}+x^{30}+\cdots+x^{31}$$

☆ Definition 2.5. Let S be a set. A weight function is a function $w: S\to\mathbb{N}$ if, for every $n\in\mathbb{N}$, the number of elements of S of n is finite.

☆ Definition 2.6. Let S be a set and w be a weight function on S. The generating series of S w.r.t weight function w is $\Phi_S^w(x) = \sum_{\alpha\in S} x^{w(\alpha)} = \Phi_S(x)$ (can omit w)

For our previous ex, $w(\alpha)$ counts # days in the month α, and generating series of M w.r.t to w is $\Phi_M(x) = \sum_{\alpha\in M} x^{w(\alpha)} = x^{28}+4x^{30}+7x^{31}$.

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Last Lecture Recap
Negative Binomial Theorem

$$(1-x)^{-t} = \Sigma_{n\geq 0}\binom{n+t-1}{t-1}x^n$$

Coefficient Extraction

$$\Sigma_{k\geq 0} g_k x^t = g_n$$

Rules for Coefficient Extraction

Weight Function
Generating Series
The generating series for a set S with respect to a weight function:

$$\Phi_s^\omega(x) = \Phi_s(x) = \Sigma_{\alpha\in s} x^{w(\alpha)}$$

S = {months of year}
w(α) = number of days in month α (no leap years)

$$\Phi_s^\omega(x) = \Sigma_{\alpha\in months} x^{\text{number of days } \in\alpha}$$
$$= x^{31} + x^{28} + x^{31} + ... + x^{31}$$
$$= x^{28} + 4x^{30} + 7x^{31}$$

Proposition 2.7
Let ω be a weight function on a set S. Then:

$$\Phi_s^\omega(x) = \Sigma_{n\geq 0}|\{\alpha \in S : \omega(\alpha) = n\}|x^n$$

where the things in the absolute value represents the number of elements of weight n.

Example: Write the generating series for the set S of binary strings with respect to weight function ω that records the length of the string.

$$\Phi_x(x) = \Sigma_{\alpha\in S} x^{length of \alpha}$$

By Proposition 2.7:

$$= \Sigma_{n\geq 0}(\text{number of binary strings of length n})x^n$$

$$= \Sigma_{n\geq 0} 2^n x^n$$

$$= \Sigma_{n\geq 0}(2x)^m$$

By Geometric Series:

$$= \frac{1}{1-2x}$$

Example: Write the generating series for the set
$S = \{\text{subsets of}\{1, ..., t\}\}$
$\omega(\alpha) = |\alpha|$

$$\Phi_s^\alpha = \Sigma_{\alpha\subseteq\{1,\ldots,t\}} x^{|\alpha|}$$

$$= \Sigma_{n\geq 0}(\text{number of n-element subsets of1, ..., t by Prop 2.7})x^n$$

$$= \Sigma_{n\geq 0}\binom{t}{n}x^n$$

By Binomial Theorem:

$$= (1+x)^t$$

Sum and Product Lemmas (2.2.2)

Example: Let $S_1 = \{$omelette(10), waffles(10), pancakes(8), eggs(8), cereal(5)$\}$, where the numbers represent ω_1.

$$\Phi_{S_1}^{\omega_1} = 2x^{10} + 2x^8 + 1x^5$$

Let $S_2 = \{$bacon(5), hash-browns(4), toast(3)$\}$, where the numbers represent ω_2.

$$\Phi_{S_2}^{\omega_2} = x^3 + x^4 + x^5 5$$

Let $S_1 \cup S_2 = \{$omelette(10), waffles(10), pancakes(8), eggs(8), cereal(5), bacon(5), hash-browns(4), toast(3)$\}$.

$$\Phi_{S_1\cup S_2}^{\omega} = x^3 + x^4 + 2x^5 + 2x^8 + 2x^{10}$$

$$= \Phi_{S_1}^{\omega_1} + \Phi_{S_2}^{\omega_2}$$

$\Phi_{S_1\cup S_2}^{\omega}$ is the generating series for the number of breakfast items (mains or sides) of each price.
Let $S_1 \times S_2 = \{$(omelette, bacon), (omelette, hash-browns), ...$\}$.
Let $\omega((a, b)) = \omega_1(a) + \omega_2(b)$.

$$\Phi_{S1\times S1}^{\omega} = x^{10} + 2x^{11} + 5x^{12} + 4x^{13} + 2x^{14} + 2x^{15}$$

$$= (2x^{10} + 2x^8 + x^5)(x^3 + x^4 + x^5)$$

Sum Lemma, 2.10
Let S_1, S_2 be two disjoint sets, and let ω be a weight function on $S_1 \cup S_2$. Then:

$$\Phi_{S_1}(x) + \Phi_{S_2}(x) = \Phi_{S_1 \cup S_2}(x)$$

Prove the main idea: Sums correspond to disjoint union.
Proof:
By Definition:

$$\Phi_{S_1}(x) + \Phi_{S_2}(x) = \Sigma_{\alpha \in S_1} x^{\omega(\alpha)} + \Sigma_{\alpha \in S_2} x^{\omega(\alpha)}$$

Since the sets are disjoint:

$$= \Sigma_{\alpha \in S_1 \cup S_2} x^{\omega(\alpha)}$$

By Definition:

$$\Phi_{S_1 \cup S_2}(x)$$

We have completed the proof.

Lemma (Infinite Sum Lemma, 2.11)
Let S_0, S_1, S_2, ... be disjoint sets with union S. Let ω be a weight function on S. Then:

$$\Phi_S(x) = \Sigma_{n \geq 0} \Phi_{S_n}(x)$$

Lemma (Product Lemma, 2.12)
Let S_1 and S_2 be sets, and let ω_1 and ω_2 be weight functions on S_1 and S_2 respectively. Then:

$$\Phi_{S_1}^{\omega_1}(x) \times \Phi_{S_2}^{\omega_2}(x) = \Phi_{S_1 \times S_2}^{\omega}(x)$$

where ω is the weight function on $S_1 \times S_2$ defined by $\omega((a, b)) = \omega_1(a) + \omega_2(b)$.
Prove the main idea: Products correspond to cartesian product of sets.
Proof:

$$\begin{aligned}
\Phi_{S_1}^{\omega_1}(x) \times \Phi_{S_2}^{\omega_2}(x) &= \Sigma_{\alpha_1 \in S_1} x^{\omega_1(\alpha_1)} \times \Sigma_{\alpha_2 \in S_2} x^{\omega_2(\alpha_2)} \\
&= \Sigma_{\alpha_1 \in S_1} \Sigma_{\alpha_2 \in S_2} x^{\omega_1(\alpha_1) + \omega_2(\alpha_2)} \\
&= \Sigma_{(\alpha_1, \alpha_2) \in S_1 \times S_2} x^{\omega_1(\alpha_1) + \omega_2(\alpha_2)} \\
&= \Phi_{S_1 \times S_2}^{\omega}(x)
\end{aligned}$$

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Product Lemma

$$\Phi_\alpha(x)\Phi_\beta(x) = \Phi_{\alpha\times\beta}(x)$$

Example: Let $S = \{2, 4, 6, 8\}$. How many pairs (a, b) with $a, b \in S$ such that $a + b = 50$?
Let $\omega(\alpha) = \alpha, \forall\alpha \in S$, then count the number of pairs such that $a + b = 50$ is:

$$[x^{50}]\Phi_{S\times S}(x)$$

$$= [x^{50}](\Phi_S(x))(\Phi_S(x))$$

$$= [x^{50}](x^2 + x^4 + x^6 + ...)^2$$

$$= [x^{50}](x^2(1 + x^2 + x^4 + ...))^2$$

By Geometric Series:

$$= [x^{50}](x^2\frac{1}{1-x^2})^2$$

$$= [x^{50}]x^4\frac{1}{(1-x^2)^2}$$

$$= [x^{46}]\frac{1}{(1-x^2)^2}$$

By Negative Binomial Theorem:

$$= [x^{46}]\sum_{n=0}^{\infty}\binom{n+2-1}{2-1}(x^2)^n$$

$$= [x^{46}]\sum_{n=0}^{\infty}\binom{n+1}{1}x^{2n}$$

$$= [x^{46}]\sum_{n=0}^{\infty}(n+1)x^{2n}$$

$$= [x^{23}]\sum_{n=0}^{\infty}(n+1)x^{n}$$

Plug in $n = 23$, the coefficient is:

$$23 + 1 = 24$$

String Lemma
Let A be a set.

$$\bigcup_{k=0}^{\infty} = A^k = A^* = \text{all tuples of all lengths of elements of A}$$

Think of A^* as strings using alphabet A.

Example: $\{0,1\}^2 = \{\textit{binary strings}\} = \{(0),(1),(0,0,1,1,0),(),...\}$.
The () is called the empty string aka string of nothing, we use ϵ to denote empty.

Given a weight function ω on A, then we define $\omega^* : A^* -> \mathbf{N}$ by:

$$\omega^*((\alpha_1,...,\alpha_k)) = \Sigma_{i=1}^k \omega(\alpha_i)$$

In particular, $\omega^*(\epsilon) = 0$.
Warning: ω^* is only a weight function if $A_0 = \emptyset$. (A has no elements with weight 0).

Example: $A = \{0,1\}$, $\omega(0) = \omega(1) = 1$.
Then $\omega^*(01101) = 5, \omega^*(01) = 2, \omega^*(\epsilon) = 0$, meaning ω^* counts length.

$$\Phi_A(x) = \Sigma_{\alpha \in A} x^{\omega(\alpha)} = x^1 + x^1 = 2x = 2x$$

Then by application of Product Lemma:

$$\Phi_{A^*}(x) = 1 + 2x + 4x^2 + ... + 2^n x^n + ... = \frac{1}{1-2x} = \frac{1}{1-\Phi_A(x)}$$

Definition:
Let A be a set with weight function $\omega : A -> \mathbf{N}$ such that no elements have weight 0. Then:

$$\Phi_{A^*}(x) = \frac{1}{1-\Phi_{A(x)}}$$

Proof of String Lemma:

$$\Phi_{A^*}(x) = \Sigma_{k=0}^{\infty} \Sigma_{(\alpha_1,...,\alpha_k) \in A^k} x^{\omega(\alpha_1)+\omega(\alpha_2)+...+\omega(\alpha_k)}$$
$$= \Sigma_{k=0}^{\infty} \Phi_{A^k}(x)$$

By Product Lemma:

$$= \Sigma_{k=0}^{\infty} (\Phi_A(x))^k$$

By Geometric Series:

$$= \Sigma_{k=0}^{\infty} \frac{1}{1 - \Phi_A(x)}$$

Composition

Definition: A composition is a finite sequence of positive integers.

$$r = (c_1, .., c_k)$$

where $c_k \in \mathbf{I}^+$

Sep 23 Lecture

Theorem Review
Let $P = \{1, 2, ...\}$ be the set of positive integers.
(a) The set of all compositions is $C = P^*$.
(b) The generating series for C with respect to size is $\Phi_C(x) = 1 + \frac{x}{1-2x}$.
(c) For $n \in \mathbf{N}$, the number of compositions of size n is:

$$|C_n| = \begin{cases} 1, & n = 0 \\ 2^{n-1}, & n > 0 \end{cases}$$

(c) Proof:
By Geometric Series:

$$\Phi_C(x) = 1 + \frac{x`}{1-2x} = 1 + x(\frac{1}{1-2x})$$

$$= 1 + x\sum_{k=0}^{\infty}(2x)^k$$

$$= 1 + \sum_{k=0}^{\infty} 2^k x^{k+1}$$

$$= 1 + \sum_{n=1}^{\infty} 2^{n-1} x^n$$

Now if we want to extract the coefficient of any power, then:

$$[x^n]\Phi_C(x) = \begin{cases} 1, & n = 0 \\ 2^{n-1}, & n > 0 \end{cases}$$

Bijective Proof
Number of compositions of size n = number of subsets of $\{1, ..., n-1\}$, that $|C_n| = 2^{n-1} for\, n > 0$.
Let $\gamma = (c_1, c_2, ..., c_k)$ be a composition of size n.
Definition:

$$f(\gamma) = \{c_1, c_1 + c_2, c_1 + c_2 + c_3, ..., c_1 + c_2 + ... + c_{k-1}\}$$

Is this well-defined? Yes, because $k \geq 1$.
Is $f(\gamma)$ a subset of $\{1, ..., n-1\}$?
Every element of $f(\gamma)$ is at least $c_1 \geq 1$ and at most $c_1 + ... + c_{k-1} = n - c_k \leq n-1$.

Let B_n be the set of subsets of $\{1, ..., n-1\}$.

I claim that $f : C_n -> B_n$ is a bijection.
The inverse map $g : B_n -> C_n$ is as follows:
For $S \in B_n$, write the elements of S in increasing order:

$$1 \leq s_1 < s_2 < ... < s_j \leq n-1$$

Define $g(S)$ by:

$$g(S) = (s_1, s_2 - s_1, s_3 - s_2, ..., s_j - s_{j-1}, n - s_j)$$

Check that $f(g(S)) = S$ and $g(f(\gamma)) = \gamma$, which would prove the validity of the bijective proof.

Example: Composition of size 3.
$C_3, 3, 21, 12, 111$
$B_3, \emptyset, \{1\}, \{2\}, \{1,2\}$
Note that 3 is sending to $\emptyset$, 21 is sending to $\{2\}$, 12 is sending to $\{1\}$, and lastly, 111 is sending to $\{1,2\}$.

Corollary
number of compositions of size n and length k
= number of subsets of $\{1, ..., n-1\}$ of size k-1

$$= \binom{n-1}{k-1}$$

Compositions with Restrictions
Example: Let F be the set of compositions where each part is 1 or 2.
The alphabet is $A = \{1, 2\}$. For example, $F = A^*$.
We have $\Phi_A(x) = x + x^2$, which is a very small generating function. By String Lemma:

$$\Phi_F(x) = \Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)} = \frac{1}{1 - (x + x^2)} = \frac{1}{1 - x - x^2}$$

which represents a fibonacci sequence, number of S.

Example: Let H be the set of all compositions where each part of at least 2.
The alphabet is $A = \{2, 3, ...\}$. We have:

$$\Phi_A(x) = x^2 + x^3 + x^4 + ...$$

$$= x^2(1 + x + x^2 + ...)$$

$$= \frac{x^2}{1-x}$$

So by String Lemma,

$$\Phi_H(x) = \Phi_{A^*}(x)$$
$$= \frac{1}{1 - \frac{x^2}{1-x}}$$
$$= 1 + \frac{x^2}{1-x-x^2}$$

which is also Fibonacci.

Example: Let J be the set of compositions with all parts odd. The alphabet is $A = \{1, 3, 5, 7, ...\}$.
Then:

$$\Phi_A(x) = x + x^3 + x^5 + x^7 + ...$$
$$= x(1 + x^2 + x^4 + ...)$$
$$= \frac{x}{1-x^2}$$
$$= \frac{1}{1 - \frac{x}{1-x^2}}$$
$$= 1 + \frac{x}{1-x-x^2}$$

also Fibonacci numbers Therefore:

$$\Phi_J(x) = \Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}$$

Sep 26 Lecture

Last Time
composition $\gamma = (c_1, c_2, ..., c_k)$, t = length, and size $|\gamma| = c_1 + c_2 + ... + c_k$.
Theorem:
The generating series of all integer compositions with respect to size is $\frac{1-x}{1-2x}$.
Theorem:
The number of compositions of size n is $\begin{cases} 1 & if \text{ n=0} \\ 2^{n-1} & if \text{ n} \geq 1 \end{cases}$

Binary Strings (Chapter 3)
Definition:
A binary string of length $n \geq 0$ is a finite sequence $\sigma = b_1 b_2 ... b_n$ where each bit $b_i \in \{0, 1\}$.
Examples:
ϵ (empty string, length 0), 0, 1, 00, 01, 10, 11, ...

Binary strings have a bijective correspondence with $\{0, 1^*\} = \{(), (0), (1), (0, 0), (0, 1), ...\}$.

We already know that the number of binary strings of length n is 2^n, but we can prove this in different ways using the String Lemma:

$$[x^n]\Phi_{\{0,1^*\}}(x) = [x^n]\frac{1}{1 - \Phi_{\{0,1\}}(x)}$$

By generating series for $\{0, 1\}$ with respect to length is $x + x = 2x$:

$$= [x^n]\frac{1}{1 - 2x}$$

By geometric series:

$$= [x^n]\sum_{k \geq 0}(2x)^k$$
$$= 2^n$$

Concatenation
We can concatenate binary strings if: $\sigma = a_1 a_2 ....a_m$ and $\tau = b_1 b_2 ...b_n$ are binary strings. Then the concatenation $\sigma\tau$ is the binary string $\sigma\tau = a_1 a_2 ...a_m b_1 b_2 ...b_n$. And σ^k denotes the k-fold concatenation of σ with itself.

Substrings
We say that σ is a substring of τ if there exist binary strings γ_1 and γ_2 such

that $\tau = \gamma_1\sigma\gamma_2$.

Concatenation Products, Definition 3.3
If S and T are sets of binary strings, then:

$$ST = \{\sigma\tau : \sigma \in S, \tau \in T\}$$

.

$$S^k = SS....S \text{ (k times)}$$

Example:

$$\tau = \gamma\sigma$$

$$\tau = 001110$$

$$\sigma = 111$$

Then we can re-write τ as $\tau = \gamma_1\sigma\gamma_2$ where $\gamma_1 = 00, \gamma_2 = 0$, and $\sigma = 111$ as defined above.
Both γ_1 and γ_2 can be empty as long as τ can be divided into three chunks. In other words, the empty string ϵ is a valid substring of every string.

Example:
$\{0, 1\}\{\epsilon, 00, 11\} = \{0, 1, 000, 011, 100, 111\}$.
It is important to note that the concatenation should be ORDERED.

$\{0, 01\}\{10, 0\} = \{010, 00, 0110\}$.
In the case above, there are duplicates of items (only counted once). In other words, the size of not preserved by the products.

$\{00, 01\}^2 = \{00, 01\}\{00, 01\} = \{0000, 0001, 0100, 0101\}$.

Warning: We cannot think of concatenation with itself as Cartesian products since Cartesians are a set of ordered pairs rather than a set of strings. These two are mathematically different things.

Regular Expressions (3.1)
Definition 3.2:
A regular expression is defined recursively as any of the followings:
- $\epsilon, 0, 1$
- the expression $R \smile S$ where R and S are regular expressions
- the expression RS where R and S are regular expressions (with $R^k = RR...R$ (k times))
- the expression R^* where R is a regular expression

Examples: The followings are all regular expressions.
- 010
- $010 \smile 01$
- $(010 \smile 01)^*$
- $(11)(010 \smile 01)^*(\epsilon \smile 0^*)$

$\{\epsilon, 01, 0011, 000111, 00001111, ...\}$ is NOT a regular expression.

A regular expression produces a set of binary strings.
Examples:
$\epsilon \smile 0 \smile 1$ produces $\{\epsilon, 0, 1\}$.
$(01)(0 \smile 11)$ produces $\{010, 0111\}$.
$(00 \smile 11)^*$ produces $\{\epsilon, 00, 11, 0000, 0011, 1100, 1111, 000000, ...\}$.

Production
We define production recursively as follows:
- the regular expression $\epsilon, 0, 1$ produces the sets of binary strings $\{\epsilon\}$, $\{0\}$, and $\{1\}$ respectively.
- if R and S are regular expressions that produce sets of binary strings $\mathcal{R}$ and $\mathcal{S}$ respectively, then:

1. $R \smile S$ produces $\mathcal{R} \smile \mathcal{S}$ (set union)
2. RS produces $\mathcal{RS}$ (concatenation product)
3. R^* produces $\mathcal{R}^* = \bigcup_{k \geq 0} \mathcal{R}^k$ (concatenation power)

Sep 28 Lecture

Last time
Binary String
Concatenation
Substring
Concatenation Product of sets of binary strings $\mathcal{R}$ and $\mathcal{S}$:

$$\mathcal{R}\mathcal{S} = \{\sigma\tau : \sigma \in \mathcal{R}, \tau \in \mathcal{S}\}$$

Regular expressions:
- ϵ, 0, 1
- If R and S are regular expressions, then so is $R \smile S$, RS, and R^*.

Regular expressions produce a set of binary strings. (review notes from last lecture).

Examples:
A list of regular expressions and the sets they produce:
- 0^* and $\{\epsilon, 0, 00, 000, ...\}$
- $(0 \smile 1)^*$ and $\{$all binary strings$\}$
- $(11)0^*$ and $\{11, 110, 1100, 11000, ...\}$
- $(0 \smile 00)*$ and $\{\epsilon, 0, 00, 000, ...\}$ (same as the first one)
- $(00 \smile 000)^*$ and $\{\epsilon, 00, 000, 0000, ...\}$ (missing the single zero element)

Rational Language
Definition: if $\mathcal{R}$ is a set of strings that can be produced by a regular expression, then we say that $\mathcal{R}$ is a rational language.

Unambiguous Expressions (3.2)
Example:
$(0 \smile 01)(0 \smile 10)$ produces $\{0, 01\}\{0, 10\} = \{00, 010, 010, 0110\}$.
We will notice that 010 is produced twice by this regular expression.
$(0 \smile 1)^*$ produces all binary strings exactly once.
$(0 \smile 1 \smile 01)^*$ produces all binary strings, but some are produced more than once. An example of this would be 0101, which can be produced by both 0.1.0.1 or 01.01.

Definition 3.8
A regular expression R is called unambiguous if every string in the language $\mathcal{R}$ produced by R is produced in exactly one way by R. Otherwise, it is called ambiguous.

Lemma 3.9
- The regular expressions $\epsilon, 0, 1$ are unambiguous.
- If R and S are unambiguous regular expressions that produce sets $\mathcal{R}$ and $\mathcal{S}$, then:
1. $R \smile S$ is unambiguous if and only if $\mathcal{R} \cap \mathcal{S} = \emptyset$
2. RS is unambiguous if and only if there is a bijection between $\mathcal{RS}$ (concatenation product) and $\mathcal{R} \times \mathcal{S}$ (Cartesian product). (In other words, for every $\alpha \in \mathcal{RS}$, there is a unique way to write $\alpha = \sigma\tau$ for $\sigma \in \mathcal{R}$ and $\tau \in \mathcal{S}$).

Example: Which is the followings are unambiguous?
- 010 is unambiguous.
- $(0 \smile 1)^*$ is unambiguous.
- $(0 \smile 1)^*(0 \smile 1)^*$ is ambiguous. For example, strings with only zeros can be produced in multiple ways.

Example:
$0^*(10^*)^*$ produces the set of all binary strings unambiguously.
How to justify this?
Notice that a string produced by $0^*(10^*)^*$ has the form 0...0(10...)(10...)...(10...). We label the groups of 0s as $\{m_0, m_1, ..., m_k\}$ for $k \geq 0$ and $m_0, m_1, ..., m_k \geq 0$. Every binary string $\sigma \in \{0,1\}^*$ can be written in this way: k is the number of 1's, m_i is the number of 0s in the block immediately before the $(m+1)$st 1.

Example:
$0^*(10000^*)^*$ does not produce the set of all binary strings.
$1^*(01^*)^*$ does produce the set of binary strings unambiguously.

Example:
$0^*(10000^*)^*$ produces the set of all binary strings where every 1 is followed by at least three 0s.

Sep 30 Lecture

Last time
rational language - set of binary strings that can be produced by a regular expression.
unambiguous regular expression

Translating Regular Expressions into Generating Series
Example: How many strings of length 10 are there where every block of 1's is of even length?
$(11 \smile 0)^*$ unambiguously produces the set of $\mathcal{R}$ of all binary strings where every block of 1's has even length.
Equivalently, the set $\mathcal{R}$ is in bijection to $\{0,11\}^*$.
The number of binary strings of length 10 in $\mathcal{R}$ is equal to the number of strings in $\{0,11\}^*$ of length 10:

$$= [x^{10}]\Phi_{\{0,11\}^*}(x)$$

By String Lemma, we have:

$$= [x^{10}]\frac{1}{1-\Phi_{\{0,11\}}(x)}$$

$$= [x^{10}]\frac{1}{1-x-x^2}$$

Regular expression $(11 \smile 0)^*$ leads to the rational function $\frac{1}{1-x-x^2}$.

Definition 3.11
A regular expression leads to a rational function is defined recursively as follows:
- Regular expression $\epsilon, 0, 1$ lead to the rational functions $1, x, x$.
- If regular expressions R and S lead to $f(x)$ and $g(x)$, then $R \smile S$ leads to $f(x)+g(x)$, RS leads to $f(x)g(x)$, and R^* leads to $\frac{1}{1-f(x)}$.

Example:
1100 leads to $x^1 * x^1 * x^1 * x^1 = x^4$.
$11 \smile 000$ leads to $x^2 + x^3$.
$0^*11(11 \smile 000)^*$ leads to $\frac{1}{1-x}x^2\frac{1}{1-x^2-x^3}$.

Theorem 3.13
Let R be a regular expression that unambiguously produces the set of binary strings $\mathcal{R}$. Also suppose the regular expression R leads to rational function $f(x)$, then $f(x)$ is the generating series for $\mathcal{R}$ with weight function length. In other words, $\Phi_{\mathcal{R}}(x) = f(x)$.

Definition 3.15
A block of a binary string is a non-empty maximal substring of equal bits.

Example: 00011010000 has five blocks.

Proposition 3.17 (Block Decomposition)
The regular expression $0^*(11^*00^*)^*1^*$ is unambiguous and produces the set of all binary strings, same for $1^*(00^*11^*)^*0^*$.

Example: 00011010000 can be produces using the regular expressions in the proposition, thus it is unambiguous.
The regular expressions in the block decomposition work by splitting the strings into blocks. The "forced" 1 and 0 inside the parenthesis act as block delimiters.

Example: Give a regular expression for the set of binary strings where each block of 1's has even length.
First solution: $(0 \smile 11)^*$
Or we can start from the block decomposition:

$$R = 0^*(11(11)^*00^*)^*(11)^*$$

In order to prove that the expression is unambiguous, we can just give a heuristic proof.

Example: How many binary strings of length 15 do not contain 0000?
Start from block decomposition $0^*(11^*00^*)1^*$, and restrict blocks of 0's to be length three of less:

$$R = (\epsilon \smile 0 \smile 00 \smile 000)(11^*(0 \smile 00 \smile 000))^*1^*$$

This unambiguous regular expression leads to generating series

$$\Phi(x) = (1 + x + x^2 + x^3)\frac{1}{1 - x\frac{1}{1-x}(x + x^2 + x^3)}\frac{1}{1-x}$$

We would want $[x^{15}]\Phi(x)$.

$$\begin{aligned}&[x^{15}]\Phi(x)\\ &= [x^{15}](1 + x + x^2 + x^3)\frac{1}{1 - x\frac{1}{1-x}(x + x^2 + x^3)}\frac{1}{1-x}\end{aligned}$$

Oct 3 Lecture

Last time
regular expression leads to a rational function
Theorem: If a regular expression is unambiguous, then it leads to the generating series for the corresponding set.

Blocks
Block Decomposition
Both $0^*(11^*00^*)^*1^*$ and $1^*(00^*11^*)^*0^*$ are unambiguous.

Example: How many strings of length 15 are there where every block has length 1 or 2?
Start from block decomposition $1^*(00^*11^*)^*0^*$ and restrict to blocks of length 1 or 2:

$$R = (\epsilon \smile 1 \smile 11)((0 \smile 00)(1 \smile 11))^*(\epsilon \smile 0 \smile 00)$$

This unambiguous regular expression leads to the generating series:

$$f(x) = (1 + x + x^2)\frac{1}{1 - (x + x^2)(x + x^2)}(1 + x + x^2)$$

$$= \frac{1 + x + x^2}{1 - x - x^2}$$

So the number of strings of length 15 with blocks of length 1 or 2 is the coefficient of:

$$[x^{15}]\frac{1 + x + x^2}{1 - x - x^2}$$

Length: $0, 1, 2, 3, 4$
Strings:
ϵ
$0, 1,\ 00, 11, 10, 01$
$001, 010, 011, 100, 101, 110$, etc.
Turned out that the numbers of strings follow the pattern of two times of the Fibonacci sequence.

Example: Find the generating series for the set of all strings not containing 011 as a substring.
Start from block decomposition $1^*(00^*11^*)^*0^*$ and restrict it.
Idea: after a zero, we can only have at most a single 1; therefore, the regular expression becomes:

$$1^*(00^*1)^*0^*$$

Note that within the inner bracket we should not write things like $\epsilon \smile 1$. This set is still valid, but ambiguous.

Prefix Decomposition (3.2.3)
Another unambiguous regular expressions for the set of all binary strings are

$$(0^*1)^*0^*$$

$$(1^*0)^*1^*$$

The prefix decomposition is something of the form A^*B.

Postfix Decomposition
Similar to prefix decomposition, except that it is in the form of B^*A.

$$0^*(10^*)^*$$

$$1^*(01^*)^*$$

Recursive Decomposition (3.3)
A recursive expression can reference itself and this can lead to something that is more powerful than the regular expressions.

Example:
A recursive expression that generates the set of all binary strings is as follows:

$$S = \epsilon \smile S(0 \smile 1)$$

This is unambiguous because every non-empty binary string can be uniquely expressed as a binary string with a 0 or 1 appended.
If S leads to a generating series $f(x)$, then the expression $S = \epsilon \smile S(0 \smile 1)$ leads to the equation $f(x) = x^0 + f(x)(x^1 + x^1)$.
Solve for $f(x)$:

$$f(x)(1 - 2x) = 1$$

$$f(x) = \frac{1}{1 - 2x} = \Phi_{\{0,1\}^*}(x)$$

which we have seen before is the generating series for the set of all binary strings.

Example:
$\{\epsilon, 01, 0011, 000111, ...\}$ cannot be produced by a regular expression, but can be produced using a recursive expression.

$$S = \epsilon \smile 0S1$$

Definition of Recursive Decomposition

A recursive decomposition of a set S describes S in terms of itself using the language of regular expressions together with the symbol S which produces the set S.

A recursive decomposition for S is unambiguous if each side of the equation produces each string exactly once.

Example: $S = 1S1 \smile 0$ describes the set of strings $\mathcal{S}$:

$$\mathcal{S} = \{0, 101, 11011, 1110111, ...\}$$

Example: $S = \epsilon \smile 0 \smile 1S1$ describes the set of strings $\mathcal{S}$:

$$\mathcal{S} = \{\epsilon, 0, 11, 101, 1111, 11011, 111111, 1110111, ...\}$$

Example: $S = 0 \smile 00 \smile 0S$ describes the set of strings $\mathcal{S}$:

$$\mathcal{S} = \{0, 00, 000, 0000, ...\}$$

the 00 in the middle makes the set ambiguous.

$S = \epsilon \smile 0 \smile 1S1$ leads to the generating series for the corresponding set:

$$\Phi_S(x) = \Phi_{\{\epsilon\}}(x) + \Phi_{\{0\}}(x) + \Phi_{\{1\}}(x)\Phi_{\{S\}}(x)\Phi_{\{1\}}(x)$$

Solving for $\Phi_S(x)$

$$\Phi_S(x) = \frac{1+x}{1-x^2} = \frac{1}{1-x}$$

Oct 5 Lecture

Last Time
Block Decomposition

$$0^*(11^*00^*)^*1^*$$

Predix Decomposition

$$(0^*1)^*0^*$$

Recursive Decomposition

$$S = \epsilon \smile S(0 \smile 1)$$

Example:

$$f(x) = 1 + f(x)(x + x)$$

$$f(x) = \frac{1}{1 - 2x}$$

Example: $\{\epsilon, 01, 0011, 000111, 00001111, ...\}$ can be generated by a recursive expression:

$$S = \epsilon \smile 0S1$$

Example: Find the generating series for the set S which is the set of all strings without having 101 as a substring.
Define A to be the set of all strings with exactly one occurrence of 101, appearing in the last three bits.
Define B to be the set of all strings with exactly two occurrences of 101, appearing in the last five bits.
We can see that $\epsilon \smile S(0 \smile 1) = A \smile S$ leads to the generating series:

$$1 + \Phi_S x \times 2x = \Phi_A(x) + \Phi_S(x)$$

We now solve the system for linear equations for $\Phi_S(x)$, and we get:

$$\Phi_S(x) = \frac{1 + x^2}{1 - 2x + x^2 - x^3}$$

Partial Fractions and Recurrence Relations (4)
Example:

$$[x^7]\frac{1 + 7x}{1 - x - 6x^2}$$

Factor the denominator:

$$= [x^7]\frac{1 + 7x}{(1 - 3x)(1 + 2x)}$$

Due to Partial Fractions:

$$= [x^7]\left(\frac{2}{1 - 3x} + \frac{-1}{1 + 2x}\right)$$

Apply the Negative Binomial Theorem

$$= [x^7]\left(2\sum_{n\geq 0}(3x)^n + (-1)\sum_{n\geq 0}(-2x)^n\right)$$

Simplify the expression:

$$= [x^7]\left(\sum_{n\geq 0}(2\times 2^n - (-2)^n)\right)x^n$$
$$= 2\times 3^7 - (-2)^7$$

Partial Fraction Simple Version (Theorem 4.12)
Let

$$G(x) = \frac{P(x)}{(1-\lambda_1 x)(1-\lambda_2 x)...(1-\lambda_s x)}$$

where P is a polynomial of degree less than s and $\lambda_i \in \mathcal{C}$ are distinct.
Then there exists values $c_1, c_2, ..., c_s \in \mathcal{C}$ such that

$$G(x) = \frac{c_1}{1-\lambda_1 x} + \frac{c_2}{1-\lambda_2 x} + ... + \frac{c_s}{1-\lambda_s x}$$

To find the c_i, we simply cross multiply and equate the coefficients:

Example:

$$\frac{1+7x}{(1-3x)(1+2x)} = \frac{c_1}{1-3x} + \frac{c_2}{1+2x}$$
$$1+7x = (1+2x)c_1 + (1-3x)c_2$$
$$= (c_1+c_2) + (2c_1 - 3c_2)x$$

Therefore,

$$c_1 + c_2 = 1, 2c_1 - 3c_2 = 7$$

We can now solve for the values of c_1 and c_2, which are 2 and -1.

Partial Fraction Full Version (Theorem 4.12)
Let

$$G(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(1-\lambda_1 x)^{d_1}...(1-\lambda_1 s)^{d_s}}$$

where $deg(P) < deg(Q)$, $\lambda_i \in \mathcal{C}$, $d_i \geq 1$.
Then,

$$G(x) = \sum_{i=1}^{s}\sum_{j=1}^{d_i}\frac{c_i^j}{(1-\lambda_i x)^j}$$

Example:

$$G(x) = \frac{2+5x}{1-3x^2-2x^3}$$

By factoring:

$$= \frac{2+5x}{(1+x)^2(1-2x)}$$

By Partial Fraction (full version):

$$= \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-2x}$$

Clearing the denominator gives:

$$2+5x = A(1+x)(1-2x) + B(1-2x) + C(1-x)^2$$

$$= (A+B+C) + (-A-2B+2C)x + (-2A+0B+C)x^2$$

Equate the coefficients gives: $\begin{cases} 2 = \text{A + B + C} \\ 5 = \text{-A - 2B + 2C} \\ 0 = \text{-2A + 0B + C} \end{cases}$

Solve for the values and we get $A = 1, B = -1, C = 2$.

Oct 7 Lecture

Last Time
Partial Fractions
If $G(x) = \frac{P(x)}{Q(x)} = \frac{P(x)}{(1-\lambda_1 x_1)^{d_1} \ldots (1-\lambda_s x_s)^{d_s}}$ where $deg(P) < deg(Q)$ and λ_i are all distinct.
Then $\exists c_{i,j} \in \mathcal{C}$ such that

$$G(x) = \sum_{i=1}^{s} \sum_{j=1}^{d_i} \frac{c_{i,j}}{(1-\lambda_i x)^j}$$

To analyze coefficients of a geometric series

$$G(x) = \frac{2+5x}{1-3x^2-2x^3}$$

1) Factor the denominator

$$G(x) = \frac{2+5x}{(1+x)^2(1-2x)}$$

2) Write $G(x)$ as a sum of negative powers using partial fractions:

$$G(x) = \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-2x}$$

3) Solve for A, B, C by clearing the denominator
4) Apply Geometric Series and Negative Binomial Theorem to write $G(x)$ as a single sum

Example continued:

$$G(x) = \frac{2+5x}{1-3x^2-2x^3} = \frac{1}{1+x} + \frac{1}{(1+x)^2} + \frac{2}{1-2x}$$

Apply Geometric Series and Negative Binomial Theorem:

$$G(x) = \sum_{n\geq 0}(-x)^n - \sum_{n\geq 0}\binom{n+2-1}{2-1}(-x)^n + 2\sum_{n\geq 0}(2x)^n$$

$$= \sum_{n\geq 0}((-1)^n - (n+1)(-1)^n + 22^n)x^n$$

$$= \sum_{n\geq 0}(2^{n+1} - n(-1)^n)x^n$$

Recurrence Relations
Example: We can also find a recurrence relation for coefficients.

$$G(x) = \frac{1-x^2}{1-x-x^2}$$
$$= \sum_{n\geq 0} g_n x^n$$

Multiply both sides by the denominator:

$$1 + 0x - x^2 = (1-x-x^2)(g_0 + g_1 x + g_2 x^2 + ...)$$

$$= (g_0) + (g_1 - g_0)x + (g_2 - g_1 - g_0)x^2 + (g_3 - g_2 - g_1)x^3 + ...$$

Equating coefficients on the LHS and RHS, we get:

$$1 = g_0, 0 = g_1 - g_0, -1 = g_3 - g_2 - g_1, 0 = g_n - g_{n-1} - g_{n-2} \forall n \geq 3$$

Simplifying this yields:

$$g_0 = 1, g_1 = 1, g_2 = 1, g_n = g_{n-1} + g_{n-2} \forall n \geq 3$$

Thinking of g_n as $[x^n]G(x)$, then for $n \geq 2$, we have:

$$0 = g_n - g_{n-1} - g_{n-2}$$
$$= [x^n]G(x) - [x^{n-1}]G(x) - [x^{n-2}]G(x)$$
$$= [x^n]G(x) - [x^n]xG(x) - [x^n]x^2G(x)$$
$$= [x^n](1-x-x^2)G(x)$$

So:

$$(1-x-x^2)G(x) = c_1 + c_2 x$$

We can now apply Partial Fractions:

$$G(x) = \frac{c_1 + c_2 x}{1-x-x^2}$$

$$= \frac{c_1 + c_2 x}{(1-\alpha x)(1-\beta x)}$$

where $\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$.

So, by Partial Fractions:

$$G(x) = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

By Geometric Series:

$$= \sum_{n \geq 0} (A\alpha^n + B\beta^n)x^n$$

Big Picture

1. Start from recurrence, that is

$$g_n - g_{n-1} - g_{n-2} = 0, n \geq 2$$

2. Write a rational expression

$$G(x) = \frac{c_1 + c_2 x}{1 - x - x^2}$$

$$= \frac{c_1 + c_2 x}{(1 - \alpha x)(1 - \beta x)}$$

3. Apply Partial Fractions to write as a sum

$$G(x) = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

4. Apply Geometric Series and Negative Binomial Theorem as necessary

$$g_n = [x^n]G(x) = A\alpha^n + B\beta^n$$

Continuing Example:

To solve for A, B, use the initial conditions $g_0 = 1, g_1 = 1$.

$$g_0 = 1 = A\alpha^0 + B\beta^0 = A + B$$

$$g_1 = 1 = a\alpha^1 + B\beta^1 = A\frac{1+\sqrt{5}}{2} + B\frac{5-\sqrt{5}}{10}$$

which implies

$$A = \frac{5+\sqrt{5}}{10}, B = \frac{5-\sqrt{5}}{10}$$

so

$$g_n = [x^n]G(x) = [x^n]\frac{1}{1 - x - x^2}$$

$$= A\alpha^n + B\beta^n$$

$$= \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{5-\sqrt{5}}{10}\right)^n$$

Theorem
Let $c_1, ..., c_k, \lambda_1, ..., \lambda_k \in \mathcal{C}$ with distinct λ_i such that

$$1 + c_1 x + c_2 x^n + ... + c_k x^k = (1 - \lambda_1 x)^{d_1} ... (1 - \lambda_s x)^{d_s}$$

in which the LHS is called the characteristic polynomial.
If $a_0, a_1, a_2, ..$ is a sequence satisfying the recurrence relation, then

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} = 0$$

for all $n \geq k$, then

$$a_n = p_1(n)\lambda_1^n + p_2(n)\lambda_2^n + ... + p_s(n)\lambda_s^n$$

where each p_i is a polynomial of degree less than d_i.

Oct 17 Lecture

Last Time
Big picture for solving recurrence relations:
1. Start from recurrence

$$f_n - f_{n-1} - f_{n-2} = 0$$

2. Write a rational expression and factor the denominator

$$F(x) = \frac{c_1 + c_2 x}{1 - x - x^2} = \frac{c_1 + c_2 x}{(1 - \alpha x)(1 - \beta x)}$$

in which the denominator is called the characteristic polynomial
3. Apply partial fractions

$$F(x) = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x}$$

4. Apply the geometric series and the negative binomial theorem to obtain a formula for the coefficients

$$f_n = [x^n]F(x) = A\alpha^n + B\beta^n$$

5. Substitute the initial conditions of the recurrence to solve for A and B

Theorem
Let $c_1, ..., c_k, \lambda_1, ..., \lambda_s \in \mathcal{C}$ with λ_i distinct.
Let

$$1 + c_1 x + c_2 x^2 + ... + c_k x^k = (1 - \lambda_1 x)_1^d ... (1 - \lambda_s x)_s^d$$

in which the LHS is called the characteristic polynomial.
If $a_0, a_1, ...$ satisfies the recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} = 0$$

then

$$a_n = p_1(n)\lambda_1^n + ... + p_s(n)\lambda_s^m$$

where $deg(p_i) < d_s$.

Example: Suppose a_n is a sequence given by $a_0 = 0, a_1 = 5, a_2 = -1, a_n = 3a_{n-2} - 2a_{n-3}$.
Give a formula for a_n as a function of n.

Write the recurrence as

$$a_n - 0a_{n-1} - 3a_{n-2} - 2a_{n-3} = 0$$

The characteristic polynomial is

$$1 - 0x - 3x^2 - 2x^3 = 0$$

which factors as

$$(1-x)^2(1+2x) = 0$$

So by Theorem, we have

$$a_n = p_1(n)(1)^n + p_2(n)(-2)^n$$

$$a_n = (An + B) + C(-2)^n$$

Sub in initial conditions

$$a_0 = 0 = A(0) + B + C(-2)^0 = B + C$$

$$a_1 = 5 = A(1) + B + C(-2)^1 = A + B - 2C$$

$$a_2 = -1 = A(2) + B + C(-2)^2 = 2A + B + 4C$$

Solve the system of linear equations and we get $(A, B, C) = (-2, -1, 1)$.
So, $a_n = -2n - 1 + (-2)^n$

Quadratic Recurrence Relations (4.4)
Motivation: We previously looked at generating series that satisfy linear recurrence relations, and could be written as rational functions:

$$G(x) = \frac{P(x)}{Q(x)}$$

or equivalently as

$$Q(x)G(x) - P(x) = 0$$

which is a linear function of $G(x)$.

Definition
A sequence $g_0, g_1, ...$ satisfies a quadratic recurrence if its generating series satisfies a quadratic equation

$$A(x)G(x)^2 + B(x)G(x) + C(x) = 0$$

where $A(x), B(x), C(x)$ are formal power series in x.

Given a quadratic equation that our generating series satisfies, we can solve for the generating series using the quadratic formula:

$$G_1(x), G_2(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}$$

One of these (namely the solution with non-negative coefficients and non-negative powers of x) is our generating series.

Definition
Let $\alpha \in \mathcal{C}$, $k \in \mathcal{N}$. The kth binomial coefficient of α is

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!}$$

Theorem (Complex Binomial Series)
For any $\alpha \in \mathcal{C}$,

$$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k$$

Proposition

$$\begin{aligned}\sqrt{1-4x} &= 1 - 2\sum_{k\geq 1} \frac{1}{k}\binom{2k-2}{k-1} x^k \\ &= 1 - 2\sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} x^{k+1}\end{aligned}$$

which can be proven using complex binomial series theorem. See course notes for full proof.

Catalan Numbers (4.4.2)
The nth catalan number is

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

or equivalently

$$c_n = \binom{2n}{n} - \binom{2n}{n+1}$$

The first few are

$$c_0 = 1, c_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, c_5 = 42, c_6 = 132$$

The catalan numbers pop up in a bunch of places:
- c_n is the number of full binary trees with $n+1$ leaves.
- c_n is the number of ways a convex polygon with $n+2$ sides can be cut in triangles by connecting the vertices with non-crossing line segments.
- c_n is the number of well-formed parenthesizations of size n.
()(()) is WFP of size 3.

Oct 19 lecture

Quadratic Recurrences and Catalan Numbers
Recall that if

$$W(x) = \frac{P(x)}{Q(x)}$$

is a rational function, then the coefficients w_n satisfy a linear recurrence relation

$$w_h = \sum_{i=1}^{k+1} c_i v_{h-i}$$

Question: Why is this called linear?
Answer: Each w_i term is only linear, and there is no quadratics.
Note: Quadratic recurrences would allow terms like w_i^2 or $w_i w_j$.

w_i is a rational function if and only if

$$Q(x)W(x) - P(x) = 0$$

Note that this is a linear equaton for $W(x)$ where the coefficeints are polynomials.
So instead, we can study quadratic recurrences from the function perspective, meaning

$$A(x)W(x)^2 + B(x)W(x) + C(x) = 0$$

for polynomials A, B, C.

If we have such an equation for $W(x)$ we can solve it for $W(x)$ by quadractic equation

$$W(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}$$

Question: How to calculate square root of polynomial as a formal power series?

Definition
Let α be a complex number, and k be a non-negative integer. We define the kth binomial coefficient of α as:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!}$$

General Binomial Theorem
Let α be a non-negative real number, then

$$(1+x)^\alpha = \sum_{k\geq 0} \binom{\alpha}{k} x^k$$

Corollary

$$\begin{aligned}\sqrt{1-4x} &= 1 - 2\sum_{k\geq 1} \frac{1}{k}\binom{2k-2}{k-1} x^k \\ &= 1 - 2\sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k} x^{k+1}\end{aligned}$$

Proof:
Use the Binomial Theorem with $\alpha = \frac{1}{2}$, and we obtain

$$(1-4x)^{\frac{1}{2}} = \sum_{k\geq 0} \binom{\frac{1}{2}}{k} (-4)^k x^k$$

and we do the calculations to obtain the correct result.

Catalan Numbers
The nth Catalan number is defined as

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

Review last lecture's notes.
There are over 75 definitions of these.

Question: What is the generating series for Catalan numbers?
Answer:

$$\begin{aligned} &\frac{1-\sqrt{1-4x}}{2x} \\ &= \frac{2\sum_{k\geq 0} c_k x^{k+1}}{2x} \\ &= \sum_{k\geq 0} c_k x^k \end{aligned}$$

Side note: This means the Catalan numbers do not form a rational language, meaning they form a "irrational" generating series.

Well-formed Parathesizations
Definition: A is a sequence of n opening and n closing parentheses that "match". Match means that for any starting sub-sequence, the number of opens is at least the number closes and they have equal numbers.

Example: Here are the WFPs when $n = 3$. (Five possibilities=c_3)

$$()()(), (())(), ()(()), ((())), (()())$$

We call n the size.

Let $W(x) = \sum_{n\geq 0} w_n x^n$ be the generating series where w_n is the number of WFPs of size n.

$$\forall n \geq 0, w_n = c_n$$

or equivalently

$$W(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$$

Question: How to prove this?
Answer: We will find a recursive unambiguous expression for W.

$$W = \epsilon$$

which is the empty WFP

$$W = \epsilon \smile (W)W$$

which is a recursive expression for W.

Question: Is this expression unambiguous?
Answer: Yes because we decomposed uniquely every nonempty WFP via its first open parenthesis and matching closing parenthesis.

Convert our expression to an equation for generating series:

$$W = \epsilon \smile (W)W$$

$$W(x) = 1 + xW(x)W(x)$$

these products come from concatenation.
We get

$$W(x) = 1 + xW(x)^2$$

$$xW(x)^2 - W(x) + 1 = 0$$

By Quadratic Equation

$$W(x) = \frac{1 \pm \sqrt{(-1)^2 - 4x(1)}}{2x}$$

$$W(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

which is the generating series for Catalan number.
If we take the plus solution, then

$$W(x) = \frac{1}{2x} + \frac{\sqrt{1-4x}}{2x}$$

$$= \frac{1}{2x} + \frac{1}{2x}(1 + 2\sum_{k\geq 0} c_k x^{k+1}$$

$$= \frac{1}{x} - \sum_{k\geq 0} c_k x^k$$

in which the first term is not well-defined because $a_0 = 0$.
Therefore, we only take the negative solution.

Oct 21 Lecture

Last time
A graph G consists of a finite non-empty $V(G)$ of objects called vertices and a set $E(G)$ of edges each of which is an ordered paris of vertices.
1. Two vertices u, v are adjacent if edge $\{u, v\}$ exists.
2. If $e = \{u, v\}$ is an edge, then e is incident to vertices u, v.
3. Vertices adjacent to a vertex v are called its neighbours.
4. Set of neighbours of v: $N(v)$

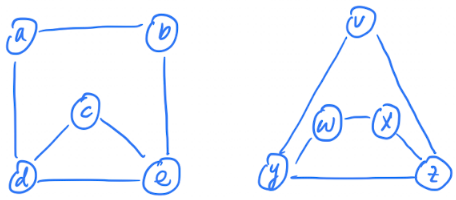
Graph: Vertex set and edge set in the form of $e = \{u, v\}$.

Definition (Planar)
A graph is planar if it can be drawn in the 2-D plane without edges crossing.

The keyword of the above sentence is "can be". A graph with edges crossing is still considered planar if it "can be" drawn with no edges crossing.

Isomorphism

- Are these 2 graphs the same?



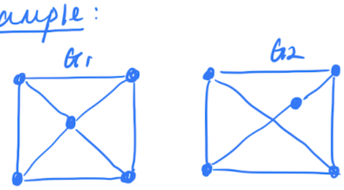
<u>Defn</u>: 2 graphs G_1 and G_2 are <u>isomorphic</u> if there exists a bijection $f: V(G_1) \to V(G_2)$. such that u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. Such a bijection is called an <u>isomorphism</u>.

To show 2 graphs are isomorphic, write down the isomorphism (bijection).

To show 2 graphs are not isomorphic, find some property that holds in one graph but not the other.

- different number of vertices
- different number of edges.
- patterns in the number of neighbours.
- many other options as well.

<u>example</u>:



These are not isomorphic since G_1 has a vertex with four neighbours but all vertices in G_2 have at most three neighbours.

Degree (Section 4.3)

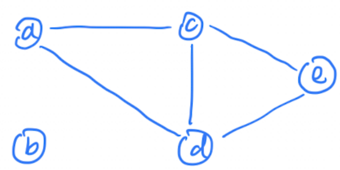
Defn: the degree of a vertex v in a graph G is the number of edges in G incident with v. It's denoted by $\deg(v)$ or $\deg_G(v)$.

Equivalently, for simple graphs

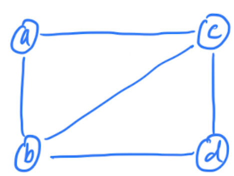
$$\deg_G(v) = |N(v)|$$

size of neighbourhood of v.

example:



v	a	b	c	d	e
deg	2	0	3	3	2



v	a	b	c	d
deg	2	3	2	3

Theorem (Handshaking Lemma 4.3.1)

for every graph G, we have that $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$.

Oct 24 Lecture

Last time
- Graphs
- Isomorphism

Definition
A graph G consists of a finite non-empty set $V(G)$ of objects, called vertices, and a set $E(G)$ of edges, which are unordered pairs of distinct vertices.

Handshaking Lemma
For every graph G, we have that

$$\sum_{v \in V(G)} deg(v) = 2|E(G)|$$

Informally, handshakes use two hands, which edges equal handshakes in this case.
2 $\times$ total number of handshakes = total number of hands
sum of all vertices = number of hands of a vertex

Formal proof: (double counting)
We count the sets in two different ways.
Let $S = \{(e, v)\}$ in which v is incident with e, $v \in V(G)$, $e \in E(G)$. Therefore,

$$|S| = \sum_{e \in E(G)} |\{(e, v)\} : v \text{ is incident with e}\}|$$

$$= \sum_{e \in E(G)} 2$$

$$= 2 \times |E(G)|$$

We now count by vertex:

$$|S| = \sum_{v \in V(G)} |\{(e, v) : \text{e is incident with v}\}|$$

$$= \sum_{v \in V(G)} deg_G(v)$$

Corollary 4.3.2
The number of odd degree vertices in a graph is even.

Proof: Let O be the number of odd degree vertices of G, and let E be the number of even degree vertices of G.
By handshaking lemma, we have

$$2 \times |E(G)| = \sum_{v \in O} deg_G(v) + \sum_{v \in E} deg_G(v)$$

Take the equation by mod 2:

$$O = \sum_{v \in O} 1 + \sum_{v \in E} 0$$

$$O = 1 \times \text{number of odd degree vertices} + 0$$

$$\text{number of add degree vertices} = O(mod2)$$

We have now completed this proof.

Corollary 4.3.3
The average degree of a graph G is $\frac{2E(G)}{V(G)}$.

Definition
The degree sequence of a graph G is $d_1 \geq d_2 \geq ... \geq d_n$ is the degree listed in decreasing order.
So handshaking tells you that certain degree sequences do not occur in graphs.

Example: $3, 3, 3, 2, 2$ is not feasible since there is an odd number of odd degrees.

Definition of k-regular
A graph G is k-regular if every vertex has a degree of k.

More Terminologies
We call $N(v)$ the neighbour pool of v.
If $e = u \times v$, then e joins u and v, and we call u and v the ends (or endpoints) of e.

Examples of Graphs
Definition of a Complete Graph
The complete graph on n vertices, denoted as K_n, is

$$V(K_n) = \{v_i : i \in [n]\}$$

$$E(K_n) = \{v_i v_j : i < j \in [n]\}$$

The degree of K_n is

$$\forall v_i \in V(K_n) : deg(v_i) = n - 1$$

Therefore, K_n is $(n-1)$-regular.

$$|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$$

Also according to handshaking lemma, we have

$$2 \times |E(K_n)| = n(n-1) = \sum_{v \in V(K_n)} deg(v)$$

More generally, if G is k-regular, then $|E(G)| = \frac{KV(G)}{2}$ by our handshaking lemma.

Path
The path on n vertices, denoted P_n is

$$V(P_n) = \{v_i : i \in [n]\}$$

$$E(P_n) = \{v_i v_{i+1} : i \in \{1, ..., n-1\}\}$$

Facts about path:
- all paths are planar
- What about the degrees?

$$|E(P_n)| = n - 1$$

Cycle
For all $n \geq 3$, a cycle on n vertices, denoted C_n, is

$$V(C_n) = \{v_i : i \in [n]\}$$

$$E(C_n) = \{v_i v_{i+1} : i \in \{1, ..., n-1\}\} \cup v_1 v_n$$

Facts about cycles:

$$|E(C_n)| = n$$

C_n is 2-regular.

Bipartite Graph
A graph is bipartite is there exists a bi-partition (A, B) of $V(G)$ such that every edge has exactly one end in A and one end in B.

Bipartition just means partition with 2 parts that can be possibly empty.

General question: When is a graph bipartite?
Specific question: Which of K_n, C_n, P_n are bipartite?
K_n is bipartite only for $n = 1, 2$, not bipartite for all $n \geq 3$.

P_n is bipartite for all n:

$$A = \{v_i : i \text{ odd}\}$$

$$A = \{v_i : i \text{ even}\}$$

C_n is bipartite if n is even:

$$A = \{v_i : i \text{ odd}\}$$

$$A = \{v_i : i \text{ even}\}$$

C_n is NOT bipartite if n is odd.

Oct 26 Lecture

Last Time
Handshaking Lemma

$$\sum_{v} deg(v) = 2|E(G)|$$

Corollary: Number of vertices of odd degree is even.
K-regular graph: All the vertices have degree equal to k.
Complete graph K_n.
Bipartite graph: Vertex set can be partitioned into two sets such that each edge has one endpoint in each set.

Complete Bipartite Graph
For positive integers m, n, the complete bipartite graph $K_{m,n}$ is the graph with bipartition A, B where $|A| = m$, $|B| = n$, containing all possible edges joining vertices in A to B.

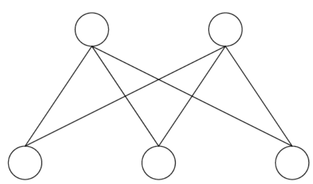


Figure 4.11: The complete bipartite graph $K_{2,3}$

Note: $K_{m,n}$ has $m + n$ vertices and mn edges.

Definition
For $n \geq 1$, the n-cube (hypercube) is a graph whose vertices are all binary string of length n and two vertices are adjacent if the strings differ in exactly one position. 1-cube is a line segment, and 2-cube is a square.

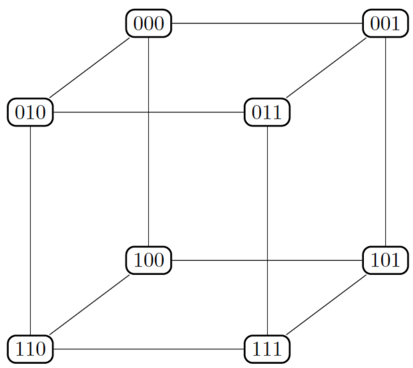


Figure 4.12: The 3-cube

Number of vertices of n-cube is 2^n.

n-cube is n-regular.

Number of edges of n-cube is $n \times 2^{n-1}$ by the handshaking lemma.

n-cube is bipartite with bipartition $A = \{\text{set of vertices with an odd number of 1s}\}$ and $B = \{\text{set of vertices with an even number of 1s}\}$.
Then if $e = uv$ is an edge, then u and v differ in exactly one position. WLOG, if u has an odd number of 1s, then v will have an even number of 1s, and vice versa.

How to specify a graph? (4.5)
Adjacency Matrix
Definition: The adjacency matrix of a graph G with vertices $v_1, ..., v_n$ is the $n \times n$ matrix $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ otherwise} \end{cases}$

Incidence Matrix
Definition: The incidence matrix of a graph G with vertices $v_1, ..., v_n$ and edges $e_1, ..., e_m$ is the $n \times m$ matrix $B = [b_{ij}]$ where $b_{ij} = \begin{cases} 1, \text{ if } v_i \text{ is incident with } e_j \\ 0, \text{ otherwise} \end{cases}$

Example:

$$BB^t = A + diag(deg(v_1), ..., deg(v_n))$$

Adjacency List
An adjacency list is a data structure where for each vertex, we have a list of the

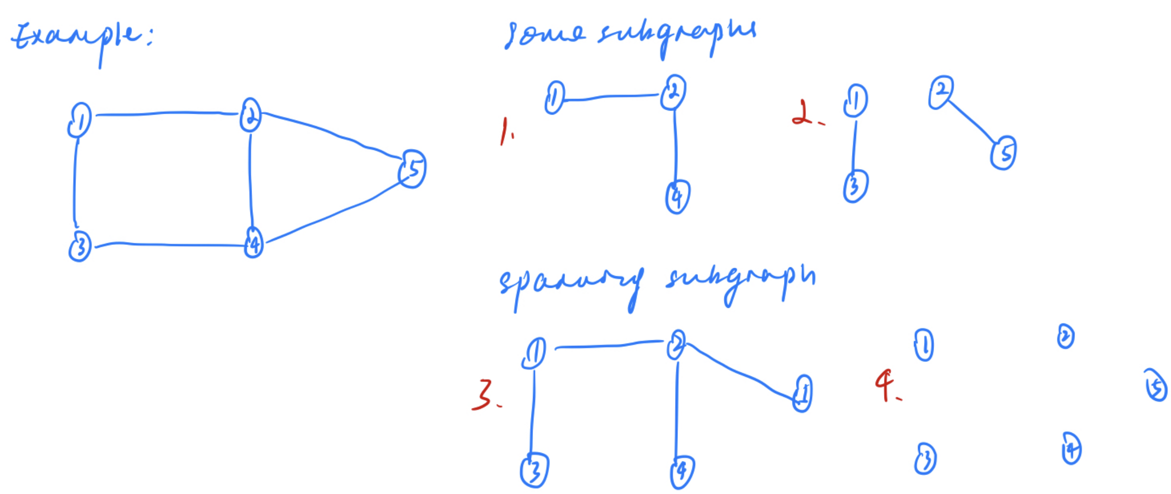
vertices it's adjacent to.

vertex	adjacent vertices
1	3 4 5 7
2	3 5 7
3	1 2 4 5 6
4	1 3 5 6
5	1 2 3 4
6	3 4 7
7	1 2 6

Paths and Cycles (4.6)

Definition: Let G be a graph. We say that H is a subgraph of G (sometimes denoted as $H \leq G$) if $V(G) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and every edge in $E(H)$ has its endpoints in $V(H)$.

Additionally, if $V(H) = V(G)$, then we call H a spanning subgraph of G. If $H \neq G$, then we say that H is a proper subgraph of G.



Notation

$H - e$ is the subgraph of H with vertex set $V(H)$ and edge set $E(H) \backslash \{e\}$.
$H + e$ is the subgraph of H with vertex set $V(H)$ and edge set $E(H) \cup \{e\}$.

Oct 28 Lecture

Last Time
Complete Bipartite Graph $K_{\{}m, n\}$
n-cube
Adjacency Matrix
Incidence Matrix
Adjacency List
Subgraph

Definition of a Walk
A u,v-walk is a sequence of alternating vertices and edges of the form $v_0, v_0v_1, v_1, v_1v_2, ..., v_{k-1}, v_{k-1}v_k, v_k$ where $v_0 = u, v_k = v$, and v_iv_j is an edge in the graph.
This is a walk of length k.
The walk is called closed if $v_0 = v_k$.

Example: A walk of length 5.

$$1, 12, 2, 24, 4, 45, 5, 52, 2, 23, 3$$

In a walk we are allowed to repeat vertices.

Definition of a Path
A u,v-path is a u,v-walk without repeated vertices.

Example: A path of length 3.

$$1, 12, 2, 25, 5, 54, 4$$

Theorem
If there is a u,v-walk in G, then there is a u,v-path in G.

Proof:
Induction on k in the following statement:
"If there is a u,v-walk in G with k repeated vertices, then there is a u,v-path in G."

Base case $k = 0$: Follows by definition of a u,v-path as a u,v-walk with 0 repeated vertices.

Inductive Hypothesis: If there is a u,v-walk with less than k repeated vertices, then there is a u,v-path.

Inductive Step: Suppose there is a u,v-walk with k repeated vertices $v_0v_1v_2...v_t$ with $v_0 = u$ and $v_t = v$. Let w be a repeated vertex such that $v_i = v_j = w$ for $i \neq j$.
Then $v_0v_1...v_iv_{j+1}...v_t$ is a u,v-walk with less than k repeated vertices.

Apply Inductive Hypothesis, and we complete the proof.

Corollary
If there is a u,v-path in G and a v,w-path in G, then there is a u,w-path in G.
For vertices in a graph G, "u and v are in a path" is an equivalence relation.

Definition of a Cycle
A cycle in a graph G is a subgraph with k distinct vertices and k distinct edges.
$v_1v_2, v_2v_3, ..., v_{k-1}v_k, v_kv_1$ where $k \geq 1$ is a cycle of length k.

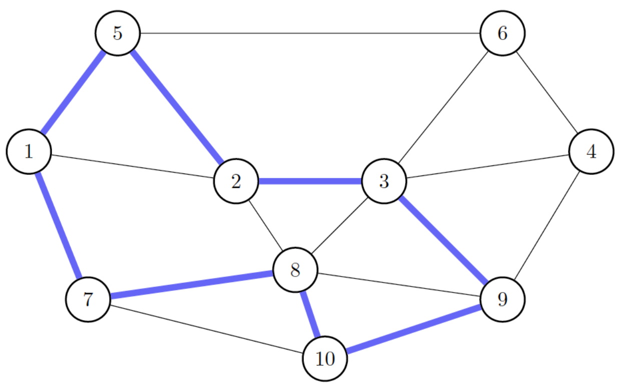


Figure 4.20: A cycle

The smallest cycle is a 3-cycle.
Every cycle is a closed-walk, but not every closed walk is a cycle.

If C is a cycle, and uv is an edge in C, then $C - uv$ is a u,v-path.
Consequently, if P is a u,v-path that does not contain the edge uv, then $P + uv$ is a cycle.

Question: Let u and v be vertices. If there is a closed walk containing u and v, is there a cycle containing u and v?
Answer: No. Consider a line segment with endpoints a and b. Then a, ab, b, ba, a is a closed walk. But since there are only two vertices, no cycles could be formed.

Oct 31 Lecture

Last Time
walk, closed-walk
path (differ from walk, meaning we have no repeated vertices or edges)
Theorem: If there is a u,v-walk, then there is a u,v-path.
Cycle: Sequence of alternating vertices and edges (no repeats except for beginning and end).

Theorem 4.6.4
If every vertex in a graph G has degree at least 2, then G contains a cycle.

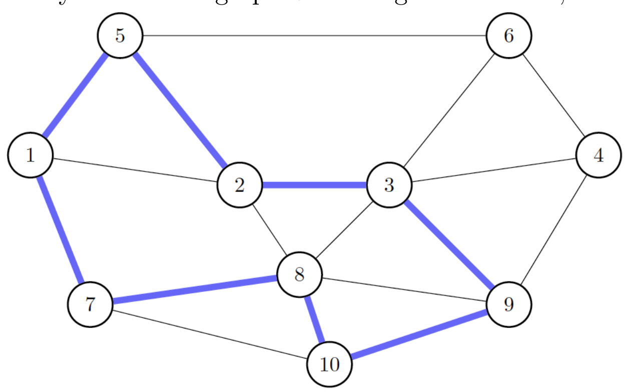


Figure 4.20: A cycle

Proof: Let $v_0 v_1 v_2 ... v_k$ be a longest path in G. Vertex v_0 has at least two neighbours. v_1 and some other vertex w. If w is not among $v_2 ... v_k$, then $w v_0 v_1 ... v_k$ is a longer path, which is a contradiction. Therefore, w must be v_i for some $i \in \{2...k\}$.
Thus, $w v_0 v_1 ... v_i$ is a cycle in G.

Question: Is the converse true? If G contains a cycle, then does every vertex in G has degree at least 2?
Answer: No. Consider a graph which contains a diamond and an additional line segment.

Definition of Girth
The girth of a graph is the length of the shortest cycle in graph. If G has no cycles, then we say that its girth is infinite.
The girth sometimes is also being referred to as the density of the graph.

Definition of Hamiltonian Cycle

A spanning cycle (a cycle that visits all of the vertices of the graph) is called a Hamiltonian Cycle.

Definition of Connected
A graph G is connected if there exists a u,v-path for every pair of vertices u and v.

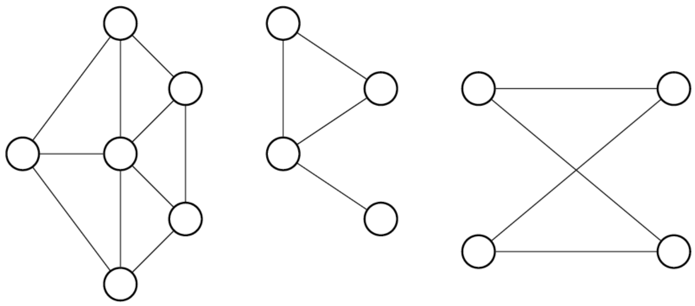


Figure 4.21: A disconnected graph

Theorem
Let G be a graph and u be a vertex of this graph. If a u,v-path exists for all $v \in V(G) \backslash \{u\}$, then G is connected.

Proof: Let $s, t \in V(G)$. By assumption, there is a u,s-path and a u,t-path. By Corollary 4.8.3, there exists an s,t-path. Hence, G is connected.

Proof: The n-cube is connected.
Fix a string u, say u is the all-zero string. How can we find a path to an arbitrary vertex $v = b_1 b_2 ... b_n$?
Suppose v has k 1s in position $i_1 i_2 ... i_k$. We construct a path through k vertices as follows: v_j is the string of length n with j 1s at positions $i_1 ... i_j$. So v_i and v_{i+1} differ in exactly one position. So $u = v_0 v_1 v_2 ... v_k$ is a u,v-path in the n-cube.

Definition of Component
A component of a graph G is the maximally connected subgraph. In other words, it is a connected subgraph that is not a proper subgraph of another connected subgraph of G.

Example: Figure 4.21 above has three components.

A connected graph has a single component which is just the whole graph.
The maximum number of components a graph can have is the number of vertices.

Definition of Disconnected
A graph is disconnected if there is a pair of vertices u,v with no u,v-path and disconnected graphs have at least two components.

Definition of Cut
Let $X \subseteq V(G)$. The cut induced by X is the set of all edges in G with exactly one end in X.