Wei's notes on Quantum Amplifiers

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Contents

1 Transmission Line Theory

1.1 Classical TL

Wave amplitude $A^{\rightarrow}(x,t)$
Fourier components of Φ

$$V(x=0,t) = V_{\text{in}}(t) + V_{\text{out}}(t) \tag{1}$$
$$I(x=0,t) = \frac{1}{Z_C}(V_{\text{out}}(t) - V_{\text{in}}(t)) \tag{2}$$

Power flow

1.2 Canonical Quantization for infinite TL

Vlad's note p11
Power flow
Why the relation can't be transformed directly into operators

1.3 Quantum Langevin Equation

For arbitrary system coupled to a bath with constant density of states, the time evolution of any operator $\hat{Y}$ is given by:

$$\dot{Y} = \frac{\mathrm{i}}{\hbar}[H_{\text{sys}}, Y] + \frac{\mathrm{i}}{2\hbar Z_C}\left\{\dot{\Phi} - V_{\text{in}}, [\Phi, Y]\right\} \tag{3}$$

For the system of an LC circuit, $H_{\text{sys}} = \frac{Q^2}{2C_a} + \frac{\Phi^2}{2L_a}$, and we can get the EOMs:

$$\dot{\Phi} = \frac{Q}{C_a} \tag{4}$$
$$\dot{Q} = -\frac{\Phi}{L_a} - \frac{1}{Z_C}(\dot{\Phi} - V_{\text{in}}) \tag{5}$$

Introducing annihilation/creation operator according to $\Phi = \Phi_{\text{ZPF}}(a + a^\dagger)$ and $Q = \mathrm{i}Q_{\text{ZPF}}(a^\dagger - a)$, where

$$\Phi_{\text{ZPF}} = \sqrt{\frac{\hbar}{2}Z_{\text{ZPF}}}, \qquad Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2Z_{\text{ZPF}}}}$$

and $Z_{\text{ZPF}} = \sqrt{L_a/C_a}$. Then Eq.(4) is equivalent to:

$$\dot{a} + \dot{a}^\dagger = \frac{\mathrm{i}Q_{\text{ZPF}}}{\Phi_{\text{ZPF}}C_a}(a^\dagger - a) = \mathrm{i}\omega_a(a^\dagger - a) \tag{6}$$

We can decompose the Heisenberg picture operator $a(t)$ in Fourier space:

$$a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a[\omega]\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \tag{7}$$

And Eq.(6) in Fourier domain can be written as:

$$a^\dagger[-\omega] = -\frac{\omega - \omega_a}{\omega + \omega_a}a[\omega] \tag{8}$$

And RWA, cavity theory......

1.4 Arbitrary coupling

The total admittance seen by the system is:

$$Y_{\text{tot}}[\omega] = \frac{1}{Z[\omega] + Z_C} \tag{9}$$

And the Quantum Langevin Equations under arbitrary coupling, which are the generalized case of Eq.(4) and Eq.(5), should be written as:

$$\dot{\Phi} = \frac{Q}{C_a} \tag{10}$$

$$\dot{Q} = -\frac{\Phi}{L_a} - I_{\text{out}}(t) \tag{11}$$

$$I_{\text{out}}[\omega] = \left(j\omega\Phi[\omega] - V_{\text{in}}[\omega]\right) Y_{\text{tot}}[\omega] \tag{12}$$

where we notice that the EOM of Φ doesn't change, so Eq.(6) still stands. And Eq.(11) can be written into:

$$-\mathrm{i}(\dot{a}^\dagger - \dot{a}) = \omega_a(a^\dagger + a) + \frac{1}{Q_{\text{ZPF}}} I_{\text{out}} \tag{13}$$

Using Eq.(7) and Eq.(8) we can obtain:

$$\begin{aligned}
-\mathrm{i}(\dot{a}^\dagger - \dot{a}) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\omega a[\omega]\mathrm{e}^{-\mathrm{i}\omega t} + \omega a^\dagger[\omega]\mathrm{e}^{\mathrm{i}\omega t}\right) \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega(a[\omega] - a^\dagger[-\omega])\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\omega^2}{\omega+\omega_a} a[\omega]\mathrm{e}^{j\omega t}\,\mathrm{d}\omega
\end{aligned} \tag{14}$$

$$\begin{aligned}
\dot{a}^\dagger + \dot{a} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(a[\omega]\mathrm{e}^{-\mathrm{i}\omega t} + a^\dagger[\omega]\mathrm{e}^{\mathrm{i}\omega t}\right) \mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (a[\omega] + a^\dagger[-\omega])\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\omega_a}{\omega+\omega_a} a[\omega]\mathrm{e}^{j\omega t}\,\mathrm{d}\omega
\end{aligned} \tag{15}$$

And using Eq.(12), we can write down the last term in Eq.(13) from anti-Fourier transform:

$$\begin{aligned}
I_{\text{out}}(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} I_{\text{out}}[\omega]\mathrm{e}^{j\omega t}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(j\omega\Phi_{\text{ZPF}}(a[\omega] + a^\dagger[-\omega]) - V_{\text{in}}[\omega]\right) Y_{\text{tot}}[\omega]\mathrm{e}^{j\omega t}\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(j\frac{2\omega\omega_a}{\omega+\omega_a}\Phi_{\text{ZPF}} a[\omega] - V_{\text{in}}[\omega]\right) Y_{\text{tot}}[\omega]\mathrm{e}^{j\omega t}\,\mathrm{d}\omega
\end{aligned} \tag{16}$$

Now we can finally rewrite the Quantum Langevin Equation Eq.(13) in Fourier domain:

$$\frac{2\omega^2}{\omega+\omega_a} a[\omega] = \frac{2\omega_a^2}{\omega+\omega_a} a[\omega] + \left(j\frac{2\omega\omega_a}{\omega+\omega_a}\Phi_{\text{ZPF}} a[\omega] - V_{\text{in}}[\omega]\right)\frac{Y_{\text{tot}}[\omega]}{Q_{\text{ZPF}}} \tag{17}$$

$$\left(\omega-\omega_a-j\frac{\omega\omega_a}{\omega+\omega_a}Z_{\mathrm{ZPF}}Y_{\mathrm{tot}}[\omega]\right)a[\omega]=-\frac{Y_{\mathrm{tot}}[\omega]}{2Q_{\mathrm{ZPF}}}V_{\mathrm{in}}[\omega] \tag{18}$$

Coupling kappa for arbitrary coupling

$$\kappa[\omega]=\frac{2\omega_a\omega}{\omega_a+\omega}Z_{\mathrm{ZPF}}\mathrm{Re}Y_{\mathrm{tot}}[\omega] \tag{19}$$

where for an resonance mode $\omega_a=1/\sqrt{L_aC_a}$, $Z_{\mathrm{ZPF}}=1/\omega_aC_a$

And for direct coupling:$Y=1/Z_C$, $\kappa_d=1/C_aZ_C$.

For capacitive coupling:

$$Y_{\mathrm{tot}}[\omega]=\frac{1}{Z_C+\frac{1}{j\omega C_c}}\approx j\omega C_c(1-j\omega C_cZ_C) \tag{20}$$

Therefore:

$$\kappa_c=\frac{Z_C}{L_a}\frac{C_c^2}{C_a^2} \tag{21}$$

When we describe the system dynamics, we usually include the driving in the Hamiltonian with a semiclassical term:

$$H_{\mathrm{drive}}=\left(u_\omega\mathrm{e}^{j\omega t}+u_\omega^*\mathrm{e}^{-j\omega t}\right)\left(a+a^\dagger\right) \tag{22}$$

where usually the drive is a single-frequency tone sent by the generator. u_ω is a classical amplitude that describes "how hard we drive the system", which intuitively should be proportional to the RF voltage we are sending in, and also related to the characteristic impedance of the transmission-line. Actually when we look at Eq.(18), it's natural to define the right part of the equation as our classical drive strength:

$$u_\omega=-\frac{Y_{\mathrm{tot}}[\omega]}{2Q_{\mathrm{ZPF}}}V_{\mathrm{in}}[\omega] \tag{23}$$

The input power:

$$P_{\mathrm{in}}=\frac{|V_{\mathrm{in}}|^2}{2Z_C}=\frac{\hbar\,|u_\omega|^2}{|Y_{\mathrm{tot}}[\omega]|^2}\frac{1}{Z_aZ_C} \tag{24}$$

$$|u_p|^2=\frac{\kappa_c}{\omega_a}P_p\frac{\omega_p^2}{\omega_a^2} \tag{25}$$

$$Y_{\mathrm{tot}}[\omega]=\frac{1}{Z[\omega]+Z_C} \tag{26}$$

$$\left(\omega_p-\omega_a-j\frac{\omega_p\omega_a}{\omega_p+\omega_a}\frac{Z_{\mathrm{ZPF}}}{Z_C+1/j\omega C_c}\right)\alpha_p=u_p \tag{27}$$

$$\alpha_p=\frac{1}{\omega_p-\omega_a}\sqrt{\frac{\kappa_c}{\omega_a}\frac{\omega_p^2}{\omega_a^2}P_p}\propto\sqrt{\frac{P_p}{Q_c}}\equiv\sqrt{P^{\mathrm{enter}}} \tag{28}$$

2 Parametric Amplifier

2.1 Non-degenerate JPC

$$H = \frac{{Q_a}^2}{2C_a} + \frac{{Q_b}^2}{2C_b} + \frac{{Q_c}^2}{2C_c} + \frac{{\Phi_a}^2}{2\Phi_a} + \frac{{\Phi_b}^2}{2\Phi_b} + \frac{{\Phi_c}^2}{2\Phi_c} + K\Phi_a\Phi_b\Phi_c \tag{29}$$

Where

$$\omega_i = \frac{1}{\sqrt{L_i C_i}}, \quad Z_i = \sqrt{\frac{L_i}{C_i}}, \quad \Phi_i = \sqrt{\frac{\hbar Z_i}{2}}\left(i + i^\dagger\right), \quad Q_i = \mathrm{i}\sqrt{\frac{\hbar}{2Z_i}}\left(i - i^\dagger\right), \quad i \mathrm{in}\{a, b, c\}$$

2.2 4-wave mixing degenerate paramp

The Hamiltonian of a non-linear harmonic oscillator with 4th order non-linearity (Kerr) reads:

$$H = \omega_a a^\dagger a + K\left(a^\dagger + a\right)^4 \tag{30}$$

And we can get the QLE:

$$\dot{a} = \frac{\mathrm{i}}{\hbar}[H, a] - \frac{\kappa}{2}a + \sqrt{\kappa}a_{\mathrm{in}} \tag{31}$$

Where

$$\omega_i = \frac{1}{\sqrt{L_i C_i}}, \quad Z_i = \sqrt{\frac{L_i}{C_i}}, \quad \Phi_i = \sqrt{\frac{\hbar Z_i}{2}}\left(i + i^\dagger\right), \quad Q_i = \mathrm{i}\sqrt{\frac{\hbar}{2Z_i}}\left(i - i^\dagger\right), \quad i \mathrm{in}\{a, b, c\}$$

3 SPA

SPA

3.1 Two-mode description

It is sometimes convenient to describe an one-mode degenerate paramp with an effective two-mode model, the signal and idler as two seperate Bosonic modes (although they are actually slightly different frequency components of the same physical cavity mode).

3.2 Lumped SPA model

$$p = \frac{NL_S}{L_{\text{ext}} + NL_S} \tag{32}$$

$$\omega_a = \frac{1}{\sqrt{C(L + ML_S)}} \tag{33}$$

$$H/\hbar = \Delta a^\dagger a + g\left(a^2 + a^{\dagger 2}\right) + \frac{K}{2} a^{\dagger 2} a^2 \tag{34}$$

$$U_S(\varphi_s) = -\alpha E_J \cos\varphi_s - 3E_J \cos\frac{\varphi_{\text{ext}} - \varphi_s}{3} \tag{35}$$

$$U_S(\tilde{\varphi}_s)/E_J = c_0 + c_2\tilde{\varphi}_s^2 + c_3\tilde{\varphi}_s^3 + c_4\tilde{\varphi}_s^4 + \cdots \tag{36}$$

$$U(\varphi) = MU_S(\varphi_s[\varphi]) + \frac{1}{2}E_L(\varphi - M\varphi_s[\varphi])^2 \tag{37}$$

$$U(\tilde{\varphi})/E_J = \tilde{c}_0 + \tilde{c}_2\tilde{\varphi}^2 + \tilde{c}_3\tilde{\varphi}^3 + \tilde{c}_4\tilde{\varphi}^4 + \cdots \tag{38}$$

3.3 Distributed SPA model

$$L = \int_{-d_1}^{0-} \left[\frac{c}{2}(\partial_t\phi)^2 - \frac{1}{2l}(\partial_x\phi)^2\right] \mathrm{d}x \tag{39}$$

$$y_c = \frac{N}{\pi}\sqrt{\frac{C_0}{C_S}} \tag{40}$$

$$-c_0\partial_t^2\Phi + c_S\partial_x^2\partial_t^2\Phi + \frac{1}{\ell_S}\partial_x^2\Phi = 0 \tag{41}$$

$$\omega^2 = \frac{1}{\ell_S c_S}\frac{k^2}{k^2 + \frac{c_0}{c_S}} \tag{42}$$

$$Z_S = \sqrt{L_S/C_0} \tag{43}$$

3.4 JAMPA

$$\omega_p = \frac{1}{\sqrt{L_S C_S}} \tag{44}$$

$$y_c = \frac{N}{\pi}\sqrt{\frac{C_0}{C_S}} \tag{45}$$

$$-c_0 \partial_t^2 \Phi + c_S \partial_x^2 \partial_t^2 \Phi + \frac{1}{\ell_S} \partial_x^2 \Phi = 0 \tag{46}$$

$$\omega^2 = \frac{1}{\ell_S c_S} \frac{k^2}{k^2 + \frac{c_0}{c_S}} \tag{47}$$

$$Z_S = \sqrt{L_S / C_0} \tag{48}$$