1 General picture

In typical amplifier terminology, α refers to a dimensionless "intra-cavity light strength", which can be understood as a semiclassical treatment of of bosonic annihilation operator a. i.e. $|\alpha|^2$ can be viewed as "intra-cavity photon number".

We kind of know where the cavity boundary is say in a 3D cavity, or Fabry-Perot like structure (e.g. TL with capacitors on both sides). While generally speaking, a mode doesn't have to be confined in particular spatial region with clear geometric boundaries. A *mode* stands for the eigenvalue (frequency) and eigenvector (field distribution function) for Maxwell Eqs with certain boundary condition, which I interprete as the "undriven" EM field allowed in the system. My point being: the term "intra-cavity" strictly speaking refers to photon occupation of the EM field structure corresponding to a particular mode, which has no physical meaning at off-resonance (non-eigenvalue) frequencies.

Instead of photon number existing within certain spatial region, a better quantity we can always count on is photon flux (photon number per unit time) that flows through a certain cross section. The photon scattering picture stands for light at whatever frequency. All we need to do then is to determine a particular cross section in evaluating photon flux and be consistant with the choice.

In our superconducting circuit system, Josephson elements, which provides nonlinearity for the system, are of key importance. While Josephson relation has an easy expression in terms of electric current and superconducting phase, describing a Josephson junction in photon scattering language is much less familiar to us (at least to me).

Therefore, in the following I'm going to simply describe the whole system in electrical language instead of microwave photon scattering language.

Josephson relation links current $I(t)$ to the superconducting phase $\varphi(t) = \Phi(t)/\phi_0$ across a JJ. Put it more general to other Josephson elements such as a SNAIL, which can always be expressed in a nonlinear potential such as:

$$U = E_j \left(c_0 + \frac{c_2}{2!}\tilde{\varphi}^2 + \frac{c_3}{3!}\tilde{\varphi}^3 + \frac{c_4}{4!}\tilde{\varphi}^4 + \cdots \right) \tag{1}$$

and therefore a current-flux relation:

$$I = \frac{E_j}{\phi_0} \left(c_2\tilde{\varphi} + \frac{c_3}{2}\tilde{\varphi}^2 + \frac{c_4}{6}\tilde{\varphi}^3 + \cdots \right) \tag{2}$$

The size of Josephson elements in a circuit are usually much smaller compared to microwave wavelength, allowing us to treat them as lumped elements (except for the case of Josephson transmission-line). For simplicity, let's assume there's only one Josephson element in the circuit and it's a dipole-like element (two terminals). The EM environment seen by the Josephson element will include the linear response of the circuit excluding the JJ, all the loadings, and all possible external drives. According to Thevenin theorem, such a linear one-port (two terminals) network can always be represented by an equivalent impedance $Z^{\mathrm{Th}}[\omega]$ and an equivalent voltage source $V^{\mathrm{Th}}(t)$ in series.

In the cases of interest, our external drives always consist of discrete frequency harmonic components, say ω_s, ω_i, ω_p etc. So linear response theory (applied on the Thevenin circuit) guarantees that $V^{\mathrm{Th}}(t)$ also have only these harmonic components. i.e.

$$V^{\mathrm{Th}}(t) = \mathrm{Re}\left(V^{\mathrm{Th}}_{\omega_s}\mathrm{e}^{j\omega_s t} + V^{\mathrm{Th}}_{\omega_p}\mathrm{e}^{j\omega_p t} + V^{\mathrm{Th}}_{\omega_i}\mathrm{e}^{j\omega_i t}\right) \tag{3}$$

where we have used the phasor representation. And similarly we can write the resulting current as:

$$\begin{aligned} I(t) &= \mathrm{Re}\left(I_{\omega_s}\mathrm{e}^{j\omega_s t} + I_{\omega_p}\mathrm{e}^{j\omega_p t} + I^*_{\omega_i}\mathrm{e}^{-j\omega_i t}\right) \\ &= \frac{1}{2}\left(I_{\omega_s}\mathrm{e}^{j\omega_s t} + I_{\omega_p}\mathrm{e}^{j\omega_p t} + I^*_{\omega_i}\mathrm{e}^{-j\omega_i t} + I^*_{\omega_s}\mathrm{e}^{-j\omega_s t} + I^*_{\omega_p}\mathrm{e}^{-j\omega_p t} + I_{\omega_i}\mathrm{e}^{j\omega_i t}\right) \end{aligned}$$

As mentioned before, another dynamical variable we want to look at is phase across the Josephson element:

$$\begin{aligned} \tilde{\varphi}(t) &= \mathrm{Re}\left(\varphi_{\omega_s}\mathrm{e}^{j\omega_s t} + \varphi_{\omega_p}\mathrm{e}^{j\omega_p t} + \varphi^*_{\omega_i}\mathrm{e}^{-j\omega_i t}\right) \\ &= \frac{1}{2}\left(\varphi_{\omega_s}\mathrm{e}^{j\omega_s t} + \varphi_{\omega_p}\mathrm{e}^{j\omega_p t} + \varphi^*_{\omega_i}\mathrm{e}^{-j\omega_i t} + \varphi^*_{\omega_s}\mathrm{e}^{-j\omega_s t} + \varphi^*_{\omega_p}\mathrm{e}^{-j\omega_p t} + \varphi_{\omega_i}\mathrm{e}^{j\omega_i t}\right) \end{aligned}$$

Or equivalently, voltage drop on the Josephson element, whose harmonic components are simply:

$$V_\omega = j\omega\phi_0\varphi_\omega$$

The whole system will satisfy:

$$I_\omega Z^{\mathrm{Th}}[\omega] + V_\omega = V^{\mathrm{Th}}_\omega \tag{4}$$

The only tricky part is that the Josephson response between I_ω and V_ω is nonlinear. For example, considering the c_3 term in SNAIL potential, different harmonic components are linked by:

$$I_{\omega_s} = \frac{E_j}{\phi_0}\left(c_2\varphi_{\omega_s} + \frac{c_3}{2}\varphi_{\omega_p}\varphi^*_{\omega_i}\right)$$

($\omega_p = \omega_s + \omega_i$ in typical amplifier terminology)

A general treatment is to put the above equations at all relavent harmonic components together, and solve them all self-consistently. This method is called "harmonic balance" calculation.

However, in some easy but common cases, we can assume one or several of the harmonic components are "stiff", i.e. a drive being strong enough that the responding strength at that frequency is almost indepent of the other harmonic components. Strictly speaking $\varphi_{\omega_p}, \varphi_{\omega_s}, \varphi_{\omega_i}$ values are linked to each other in the nonlinear dynamics, but the fluctuation in φ_{ω_p} is much less than its steady value induced by the external pump, we say the pump is stiff and simply take φ_{ω_p} as fixed (corresponding to pump strength).

In this case, we can evaluate the steady linear response φ_{ω_p} independent of what happens at ω_s and ω_i. Then, using this φ_{ω_p} as a parameter, we joyfully realize that the Josephson response at the other frequencies can also be linearized. This trick was introduced as "pumpistor model".

E.g. we can write

$$\begin{aligned} Y_J[\omega_s] &= \frac{I_{\omega_s}}{V_{\omega_s}} \\ &= \frac{1}{j\omega_s L_J} + \frac{\frac{c_3\varphi_{\omega_p}}{2c_2}}{j\omega_s L_J}\frac{\varphi^*_{\omega_i}}{\varphi_{\omega_s}} \end{aligned} \tag{5}$$

($L_J = \frac{\phi_0^2}{E_j c_2}$ is the total Josephson inductance)

Combined with similar relation at $-\omega_i$, we arrive at:

$$\begin{pmatrix} I_{\omega_s} \\ I^*_{\omega_i} \end{pmatrix} = \frac{1}{jL_J}\begin{pmatrix} \frac{1}{\omega_s} & \frac{\epsilon}{-\omega_i} \\ \frac{\epsilon^*}{\omega_s} & \frac{1}{-\omega_i} \end{pmatrix}\begin{pmatrix} V_{\omega_s} \\ V^*_{\omega_i} \end{pmatrix} \tag{6}$$

where we named $\epsilon = \frac{c_3\varphi_{\omega_p}}{2c_2}$

This is the linearized response of a 3-wave-mixing Josephson element under stiff pump.

We might also want to include 4th order nonlinearity, i.e. Kerr effect due to φ_{ω_p} (while unfortunately, in pumpistor model we can not include Kerr effect due to φ_{ω_s} itself in the dynamics at ω_s).

And strictly speaking, we should always include all "relavent" frequency components linked by the stiff φ_{ω_p}. E.g. the frequency $\omega_h = \omega_s + \omega_p$ enters the nonlinear dynamics for ω_s in lowest order, so it should be included, even if we're not driving nor measuring at that frequency.

What we get is:

$$I_{\omega_s} = I_c\left(c_2\varphi_{\omega_s} + \frac{c_3}{2}\left(\varphi_{\omega_p}\varphi^*_{\omega_i} + \varphi^*_{\omega_p}\varphi_{\omega_h}\right) + \frac{c_4}{4}|\varphi_{\omega_p}|^2\varphi_{\omega_s}\right) \tag{7}$$

And resulting linearized response:

$$\begin{pmatrix} I_{\omega_s} \\ I^*_{\omega_i} \\ I_{\omega_h} \end{pmatrix} = \frac{1}{jL_J}\begin{pmatrix} \frac{1+\delta}{\omega_s} & \frac{-\epsilon}{\omega_i} & \frac{\epsilon^*}{\omega_h} \\ \frac{-\epsilon^*}{\omega_s} & \frac{1+\delta}{\omega_i} & 0 \\ \frac{\epsilon}{\omega_s} & 0 & \frac{1+\delta}{\omega_h} \end{pmatrix}\begin{pmatrix} V_{\omega_s} \\ V^*_{\omega_i} \\ V_{\omega_h} \end{pmatrix} \tag{8}$$

which we should put together with:

$$\begin{aligned} I_{\omega_s} Z^{\mathrm{Th}}[\omega_s] + V_{\omega_s} &= V^{\mathrm{Th}}_{\omega_s} \\ I^*_{\omega_i} Z^{\mathrm{Th}}[-\omega_i] + V^*_{\omega_i} &= V^{\mathrm{Th}^*}_{\omega_i} \\ I_{\omega_h} Z^{\mathrm{Th}}[\omega_h] + V_{\omega_h} &= 0 \end{aligned}$$

and solve self-consistently (now as a linear system, this calculation is much easier than harmonic balance and we can expect some closed form expressions for the results).

Next:

- Engineering efficient pump coupling is equivalent to engineering a linear network in figure 1 that enables sufficient φ_{ω_p} with less input power from the physical port.

  This converts to two questions for the Thevenin circuit:

  1. What properties do we want of $Z^{\mathrm{Th}}[\omega]$?

  2. How do the pumps from physical ports go into V^{Th}?

  (My work on PPFSPA was trying to answer these questions for a specific application, i.e. a JPA with two external ports.)

- Engineering the gain profile of an amplifier. Amplifers should have $|S_{11}[\omega]| > 1$ satisfied for a decent range of ω. Usual resonance-based parametric amplifiers typically shows a Lorentzian lineshape gain profile versus frequency, and have a gain-bandwidth tradeoff. While this can be improved by engineering the linear coupling network.

- Figure out how noise (unexpected voltage sources at all possible frequencies) affects the SNR we care about. On one hand, we'd like to know the added noise performance for our parametric processes, e.g. guarantee it's quantum limited. On the other hand, we can intentionally make our system more susceptible or less susceptible to certain frequency noise from certain input channels (physical ports). Noise spectrometry, and offer instructions on line attenuation and filering.

- A more systematic approach of describing the system is always favored, especially as we have multiple requirements on a single module (?) and as we afford having more complex systems our single module.

  Instead of starting the design from minimal "atoms", perhaps we'll start with some overall requirements on the system response function, controlability, observability etc.

  Knowing what macroscopic properties we want, how to achieve that with basic circuit building blocks (such as L, C, TL) can become an optimization problem.

2 Evaluate flux induced by external drive

Let's first treat the linear response of the circuit by assuming the SNAIL array to behave like a linear impedance $Z_J[\omega]$.

The 2-port scattering parameters (evaluated assuming port impedance Z_c) is given by:

$$S = \begin{pmatrix} R & T \\ T & R \end{pmatrix}$$

with voltage reflection and transmission coefficient respectively:

$$R = \frac{Z_J}{Z_J + 2Z_c} \tag{9}$$

$$T = \frac{2Z_c}{Z_J + 2Z_c} \tag{10}$$

A two-port SPA consists of the concatenation of: signal port coupling network - SNAIL array - pump port coupling network. For whatever linear coupling networks, voltage drop across the SNAIL array can be treated as linear response to the external voltage drive applied from signal and pump ports.

In SPA application, the external drive we apply is typically a CW tone at fixed frequency, injecting from either one of the two ports. For the case of interest in next section, let's take the pump (ω_p frequency component, incoming from pump port) as an example:

From a signal flow chart analysis:

When we inject CW tone pump voltage $V_p \cos(\omega_p t) = \mathrm{Re}\left(V_p \mathrm{e}^{j\omega_p t}\right) = \frac{V_p}{2}\mathrm{e}^{j\omega_p t} + \frac{V_p^*}{2}\mathrm{e}^{-j\omega_p t}$ into pump port:

$$V_0 = \frac{S_{12}^{(p)} V_p}{1 - \left(R + \frac{S_{22}^{(s)} T^2}{1 - S_{22}^{(s)} R}\right) S_{11}^{(p)}} \tag{11}$$

$$V_L = \frac{T}{1 - S_{22}^{(s)} R}\left(1 + S_{22}^{(s)}\right) V_0 \tag{12}$$

$$V_R = \left(1 + R + \frac{S_{22}^{(s)} T^2}{1 - S_{22}^{(s)} R}\right) V_0 \tag{13}$$

So voltage across the SNAIL is simply $V_L - V_R$, which can be written into incoming wave voltage amplitude V_p.

We can convert voltage into phase difference $\varphi_{\omega_p} = \frac{V_L - V_R}{j\omega_p}$.

In next section, we'll view φ_{ω_p} as already-computed, and use it in the pumpistor model.

3 Pumpistor model for current pumped SNAIL array

The whole point of pumpistor model is to linearize a nonlinear parametric process, specifically parametric amplification. When using an amplifier in its linear amplification regime (which is typically the case in cQED application), we can always introduce a linear circuit including negative resistance to effectively describe the amplification behavior.

Such a linear circuit model shows its advantage when we want to synthesize an amplifier along with some complex linear networks, e.g. filters, impedance transformers, or hybrids.

A SNAIL has a nonlinear potential in the form:

$$U = E_j\left(c_0 + \frac{c_2}{2!}\tilde{\varphi}^2 + \frac{c_3}{3!}\tilde{\varphi}^3 + \frac{c_4}{4!}\tilde{\varphi}^4 + \cdots\right) \tag{14}$$

Similarly, current flowing through a SNAIL is given by:

$$I(t) = \frac{E_j}{\phi_0}\left(c_2\tilde{\varphi} + \frac{c_3}{2}\tilde{\varphi}^2 + \frac{c_4}{6}\tilde{\varphi}^3 + \cdots\right) \tag{15}$$

In phase preserving amplification process, $\tilde{\varphi}(t)$ composes of only three frequency components, namely signal ω_s, pump ω_p and idler $\omega_i = \omega_p - \omega_s$ (to be exact, it is the counter-rotating idle component, i.e. $-\omega_i$ that is of interest). Using phasor representation, we can write:

$$\begin{aligned}\tilde{\varphi}(t) &= \mathrm{Re}\left(\varphi_{\omega_s}\mathrm{e}^{j\omega_s t} + \varphi_{\omega_p}\mathrm{e}^{j\omega_p t} + \varphi^*_{\omega_i}\mathrm{e}^{-j\omega_i t}\right)\\ &= \frac{1}{2}\left(\varphi_{\omega_s}\mathrm{e}^{j\omega_s t} + \varphi_{\omega_p}\mathrm{e}^{j\omega_p t} + \varphi^*_{\omega_i}\mathrm{e}^{-j\omega_i t} + \varphi^*_{\omega_s}\mathrm{e}^{-j\omega_s t} + \varphi^*_{\omega_p}\mathrm{e}^{-j\omega_p t} + \varphi_{\omega_i}\mathrm{e}^{j\omega_i t}\right)\end{aligned} \tag{16}$$

and similarly

$$\begin{aligned}I(t) &= \mathrm{Re}\left(I_{\omega_s}\mathrm{e}^{j\omega_s t} + I_{\omega_p}\mathrm{e}^{j\omega_p t} + I^*_{\omega_i}\mathrm{e}^{-j\omega_i t}\right)\\ &= \frac{1}{2}\left(I_{\omega_s}\mathrm{e}^{j\omega_s t} + I_{\omega_p}\mathrm{e}^{j\omega_p t} + I^*_{\omega_i}\mathrm{e}^{-j\omega_i t} + I^*_{\omega_s}\mathrm{e}^{-j\omega_s t} + I^*_{\omega_p}\mathrm{e}^{-j\omega_p t} + I_{\omega_i}\mathrm{e}^{j\omega_i t}\right)\end{aligned} \tag{17}$$

Consider only the first two terms in Eq.(15), the $\mathrm{e}^{j\omega_s t}$ harmonic component of $I(t)$ will yield:

$$I_{\omega_s} = I_c\left(c_2\varphi_{\omega_s} + \frac{c_3}{2}\varphi_{\omega_p}\varphi^*_{\omega_i}\right)$$

($I_c = E_j/\phi_0$, is critical current of the large junction in a SNAIL.)

In the following we'll treat an array of M SNAILs in series as one component. Voltage across the SNAIL array is given by $V(t) = M\phi_0\dot{\varphi}(t)$, which has $\mathrm{e}^{j\omega_s t}$ harmonic component:

$$V_{\omega_s} = j\omega_s M\phi_0\varphi_{\omega_s} \tag{18}$$

In order to linearize the SNAIL array, we'll model its response at ω_s as a linear admittance:

$$\begin{aligned}Y_J[\omega_s] &= \frac{I_{\omega_s}}{V_{\omega_s}}\\ &= \frac{c_2}{j\omega_s M L_j} + \frac{\frac{c_3}{2}\frac{\varphi_{\omega_p}\varphi^*_{\omega_i}}{\varphi_{\omega_s}}}{j\omega_s M L_j}\end{aligned} \tag{19}$$

($L_j = \phi_0/I_c$ is Josephson inductance of the large junction in SNAIL.)

In the case of linear amplification, we want the total admitance to have a negative real part which, under stiff pump approximation, is determined by the system linear response at ω_p frequency.

In order to verify that, we'll need to apply the same treatments to $\mathrm{e}^{-j\omega_i t}$ harmonic component:

$$V^*_{\omega_i} = -j\omega_i M\phi_0\varphi^*_{\omega_i} \tag{20}$$

$$
\begin{aligned}
Y_J[-\omega_i] &= \frac{I^*_{\omega_i}}{V^*_{\omega_i}} \\
&= \frac{c_2}{-j\omega_i M L_j} + \frac{\frac{c_3}{2}\frac{\varphi^*_{\omega_p}\varphi_{\omega_s}}{\varphi^*_{\omega_i}}}{-j\omega_s M L_j}
\end{aligned}
\tag{21}
$$

Taking a second look into Eq.(19), let's rewrite:

$$
\begin{aligned}
\frac{\frac{c_3}{2}\frac{\varphi_{\omega_p}\varphi^*_{\omega_i}}{\varphi_{\omega_s}}}{j\omega_s M L_j} V_{\omega_s} &= \frac{c_2}{M L_j}\frac{c_3\varphi_{\omega_p}}{2c_2}\frac{V_{\omega_s}}{j\omega_s\varphi_{\omega_s}}\varphi^*_{\omega_i} \\
&= \frac{c_2}{M L_j}\frac{c_3\varphi_{\omega_p}}{2c_2}\frac{V^*_{\omega_i}}{-j\omega_i}
\end{aligned}
$$

and name $\epsilon = \frac{c_3\varphi_{\omega_p}}{2c_2}$.

Combining with a similar current-voltage relation at $-\omega_i$, we get the linearized response of SNAIL array to signal and idler voltage:

$$
\begin{pmatrix} I_{\omega_s} \\ I^*_{\omega_i} \end{pmatrix} = \frac{1}{jL_J}\begin{pmatrix} \frac{1}{\omega_s} & \frac{\epsilon}{-\omega_i} \\ \frac{\epsilon^*}{\omega_s} & \frac{1}{-\omega_i} \end{pmatrix}\begin{pmatrix} V_{\omega_s} \\ V^*_{\omega_i} \end{pmatrix}
\tag{22}
$$

where $L_J = ML_j/c_2$ is total inductacne of the SNAIL array.

In application, there's no external probe at ω_i. Therefore, the response at idler frequency should satisfy:

$$
I^*_{\omega_i} Z_E[-\omega_i] + V^*_{\omega_i} = 0
\tag{23}
$$

where Z_E is the total external impedance seen by the SNAIL array, which is fully determined by the linear coupling networks and environment.

Consequently:

$$
\frac{V^*_{\omega_i}}{V_{\omega_s}} = \frac{\epsilon^*}{\frac{\omega_s}{\omega_i} - \frac{jL_J\omega_s}{Z_E[-\omega_i]}}
\tag{24}
$$

Finally, Eq.(22) can be converted to:

$$
Y_J[\omega_s] = \frac{1}{jL_J\omega_s} - \frac{|\epsilon|^2}{jL_J\omega_s + \frac{L_J^2\omega_s\omega_i}{Z^*_E[\omega_i]}}
\tag{25}
$$

Therefore, the pumped SNAIL array acts like two elements in parallel. The first term corresponds to the linear inductance. The second term, which is called a "pumpistor", is determined by pump amplitude φ_{ω_p} (that goes into ϵ) and pump frequency ω_p (which determines idler frequency $\omega_i = \omega_p - \omega_s$).

The pumpistor has both an inductive part and a resistive part, while in the case of linear amplification, we're working in the regime of a negative resistive part.

Equivalently, we can express this linearized response as an impedance:

$$Z_J[\omega_s] = \frac{jL_J\omega_s}{1 - \frac{|\epsilon|^2}{1 - jL_J\omega_i/Z_E^*[\omega_i]}} \tag{26}$$

As a quick reminder:

The response at ω_s is linear as long as the pump is stiff, i.e. $\epsilon = \frac{c_3\varphi_{\omega_p}}{2c_2}$ stays constant.

Eq.(26) gives the linearized impedance Z_J of a pumped SNAIL array at signal frequency ω_s. It depends on magnetic flux bias (that determines c_2,c_3 etc.), pump frequency ω_p and pump amplitude φ_{ω_p}, for given junction parameters and external linear network .

In next section, we'll see what's the general guidelines for external linear network design. Or more straightforward, what $Z_E[\omega]$ do we want for an SPA.

4 Input impedance required for resonance

Outgoing wave solution: (argue that a discrete set)

There's a discrete set of ω_k that satisfies $j\mathrm{Im}Z_E[\omega_k] + jL_J\omega_k = 0$

We assume that the near-resonance response can be well-characterized by a LCR circuit.

We should have:

$$m := \frac{\mathrm{Im}Z'_E[\omega_a]}{L_J} = \frac{2L - L_J}{L_J} \tag{27}$$

participation ratio:

$$p := \frac{L_J}{L} = \frac{2}{m+1} \tag{28}$$

Pump frequency: $\omega_p = 2(\omega_a + \Delta)$

Signal frequency: $\omega_s = \omega_p/2 + \omega = \omega_a + \Delta + \omega$

Idler frequency: $\omega_i = \omega_p - \omega_s = \omega_a + \Delta - \omega$

$$Z_E[\omega_s] = \frac{\kappa_a}{2}L_J - j\omega_a L_J + jmL_J(\omega_s - \omega_a) \tag{29}$$

Specifically, we want the SPA mode linewidth κ_a to be dominated by signal port coupling. i.e. $\mathrm{Re}Z^{(s)}[\omega_a] = \frac{\kappa_a}{2}L_J$, $\mathrm{Re}Z^{(p)}[\omega_a] \approx 0$

In this case, assuming both signal and idler are near resonance, Eq.(26) can be written as:

$$\begin{aligned} Z_J[\omega_s] &= \frac{jL_J\omega_s}{1 - \frac{|\epsilon|^2}{1 - jL_J\omega_i/Z_E^*[\omega_i]}} \\ &= \frac{jL_J(\omega_a + \Delta + \omega)\left[\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta - \omega)\right]}{(1 - |\epsilon|^2)\left[\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta - \omega)\right] - j\,|\epsilon|^2\,(\omega_a + \Delta - \omega)} \end{aligned} \tag{30}$$

To do:

......

At the same time, the signal port network can be characterized by input impedance $Z^{(s)}$ (with the other port 50 Ω loaded), resulting in linear response between current I and node voltage $V_L = \dot{\Phi}_L$:

$$V_L(t) = -\int Z^{(s)}(t-\tau)I(\tau)\,\mathrm{d}\tau \tag{31}$$

and similarly for the right hand side:

$$V_R(t) = \int Z^{(p)}(t-\tau)I(\tau)\,\mathrm{d}\tau \tag{32}$$

Laplace transform on Eq.(31) and 32, as well as Eq.(**??**), we get:

$$\begin{aligned} I[s] &= \frac{\Phi_L[s]-\Phi_R[s]}{L_J} \\ -s\Phi_L[s] &= Z^{(s)}[s]I[s] \\ s\Phi_R[s] &= Z^{(p)}[s]I[s] \end{aligned} \tag{33}$$

or written equivalently:

$$\begin{pmatrix} 1 & \frac{s}{Z^{(s)}} & 0 \\ 1 & 0 & -\frac{s}{Z^{(p)}} \\ 1 & -1/L_J & 1/L_J \end{pmatrix} \begin{pmatrix} I \\ \Phi_L \\ \Phi_R \end{pmatrix} = 0 \tag{34}$$

Existance of non-trivial solutions requires:

$$Z^{(s)}[s] + Z^{(p)}[s] = -sL_J \tag{35}$$

If we write $s = -\kappa_a/2 + j\omega$:

$$\mathrm{Im}Z^{(s)}[-\kappa_a/2 + j\omega] + \mathrm{Im}Z^{(p)}[-\kappa_a/2 + j\omega] = -\omega L_J \tag{36}$$

$$\mathrm{Re}Z^{(s)}[-\kappa_a/2 + j\omega] + \mathrm{Re}Z^{(p)}[-\kappa_a/2 + j\omega] = \frac{\kappa_a}{2}L_J \tag{37}$$

Assuming a set of $\{\omega_k\}$ and $\{\kappa_k\}$ that satisfy these conditions, solution to the system will be in the form of linear combination of eigenmodes:

$$I(t) = \sum_k I_{\omega_k}\mathrm{e}^{-\frac{\kappa_k}{2}t}\mathrm{e}^{j\omega_k t} \tag{38}$$

with ω_k being frequency of the k-th eigenmode, and $\kappa_k/2$ being amplitude damping rate of that mode.

To design an SPA mode with arbitrary coupling networks, imaginary (reactive) parts of input impedance $Z^{(s)}$ and $Z^{(p)}$ has to satisfy requirement 36 at desired mode frequency ω_a. And reasonable κ_a for this mode has to be achieved by real part of $Z^{(s)}$ (since $\mathrm{Re}Z^{(p)}$ should be close to 0 to prevent signal leakage from pump port). Note that SNAIL expansion parameter c_2 is dependent on external flux bias. As L_J varies with flux, so does the ω that satisfies Eq.(36), thus tunes the mode frequency. For degenerate amplification, no other modes should exist within the frequency range of interest.

(Extract SNAIL inductance participation from slope of imaginary part of input impedance)

We can always get **energy participation** of a component from field simulation. And what we usually do is assuming the SNAIL is a lumped inductive element (no E field energy stored in it), then **inductance participation** is always twice the energy perticipation.

Here I propose an easier way to do so by extracting SNAIL **inductance participation** in the mode, which lay a requirement for the input impedances "apriori" before a full field simulation. This would be easier than "run HFSS and change paramters" especially when we want to optimize multiple things iteratively.

We can generally define total admittance of the loop $Y_{\text{sys}}[\omega] = \left(Z^{(s)}[\omega] + Z^{(p)}[\omega] + j\omega L_J\right)^{-1}$.

An assumption I make is that behaviour of $Y_{\text{sys}}[\omega]$ near ω_a (for whatever frequency my signal could lie) can be modeled for by a simple RLC. (This assumption is much weaker than Foster theorem: I'm only talking about one mode.)

The signal Thevenin voltage source $V^{(s)}$ sees the whole system as $Y_{\text{sys}}[\omega_s]$, that responds exactly the same (for near resonance ω_s) as a series RLC.

The RLC model should account for the mode frequency ω_a, residual of $\text{Im}Y_{\text{sys}}$ at ω_a, and energy damping rate κ_a of the mode. (To account for amplification, we should put a pumpistor in parallel with L_J.)

**How this model helps:

Using this model, we can calculate equivalent L_a and C_a from the values and slopes of imaginary part of the two input impedances (from linear simulation). All later nonlinearity calculation can be easily done analytically, and all existing results from mode amplitude a and $a^\dagger$ language applies directly. Doesn't involve the simulated $Z[\omega]$ function in calculation, and doesn't need harmonic balance simulation.

Note that 4-th and higher orders nonlinearity calculated from this circuit are different from Foster circuit BBQ. Following the current conservation argument in Appendix A of [**?**], we have to treat the nonlinear inductance in series with linear inductance, unlike in Foster circuit BBQ where nonlinearity is treated in parallel with linear part. Another way to interprate the reason why Foster BBQ break down for SPA is that SNAIL participation in higher modes are non-negligible, therefore renormalization from higher modes has to be accounted for.

5 Evaluate SPA reflection

Recall that the reflection and transmission coefficients (in S-matrix) for a lumped impedance Z_J is:

$$R = \frac{Z_J}{Z_J + 2Z_c} \qquad T = \frac{2Z_c}{Z_J + 2Z_c}$$

As the effective impedance for a pumped SNAIL array is derived in Eq.(26), it applies directly into the concatenated SPA:

$$V_0 = \frac{S_{21}^{(s)} V_s}{1 - \left(R + \frac{S_{11}^{(p)} T^2}{1 - S_{11}^{(p)} R}\right) S_{22}^{(s)}} \tag{39}$$

Overall reflection coefficient of the SPA is given by:

$$\frac{V_{\text{out}}}{V_{\text{in}}} = S_{11}^{(s)} + \frac{S_{21}^{(s)} S_{12}^{(s)} \Gamma}{1 - \Gamma S_{22}^{(s)}} \tag{40}$$

where we name:

$$\begin{aligned} \Gamma &= R + \frac{S_{11}^{(p)} T^2}{1 - S_{11}^{(p)} R} \\ &= \frac{Z_J}{Z_J + 2Z_c} + \frac{\frac{Z^{(p)} - Z_c}{Z^{(p)} + Z_c}\left(\frac{2Z_c}{Z_J + 2Z_c}\right)^2}{1 - \frac{Z^{(p)} - Z_c}{Z^{(p)} + Z_c}\frac{Z_J}{Z_J + 2Z_c}} \\ &= \frac{1}{Z_J + 2Z_c}\left(Z_J + \frac{2Z_c}{\frac{Z_J + 2Z_c}{Z^{(p)} - Z_c} + 1}\right) \\ &= \frac{Z_J + Z^{(p)} - Z_c}{Z_J + Z^{(p)} + Z_c} \end{aligned} \tag{41}$$

in which we have introduced pump port output impedance (seen by the SNAIL array):

$$Z^{(p)} = Z_c \frac{1 + S_{11}^{(p)}}{1 - S_{11}^{(p)}}$$

Similarly, signal port output impedance (seen by the SNAIL array) should be:

$$Z^{(s)} = Z_c \frac{1 + S_{22}^{(s)}}{1 - S_{22}^{(s)}}$$

In that case, the total external impedance $Z_E[\omega]$ seen by the SNAIL array which appears in Eq.(**??**) would compose of:

$$Z_E = Z^{(s)} + Z^{(p)} \tag{42}$$

And Eq.(40) is equivalent to:

$$\begin{aligned} \frac{V_{\text{out}}}{V_{\text{in}}} &= S_{11}^{(s)} + S_{21}^{(s)} S_{12}^{(s)} \frac{(Z^{(s)} + Z_c)(Z_J + Z^{(p)} - Z_c)}{2Z_c(Z_J + Z^{(s)} + Z^{(p)})} \\ &= S_{11}^{(s)} + S_{21}^{(s)} S_{12}^{(s)} \frac{Z^{(s)} + Z_c}{2Z_c}\left(1 - \frac{Z^{(s)} + Z_c}{Z_J + Z_E}\right) \end{aligned} \tag{43}$$

Such a signal port filter network should be reciprocal: $S_{12}^{(s)} = S_{21}^{(s)}$ and unitary: $S^{(s)\dagger}S^{(s)} = I$ (assuming lossless).

After some math, we have $S_{11}^{(s)}S_{22}^{(s)} - S_{21}^{(s)2} = \mathrm{e}^{j\theta}$ where $S_{11}^{(s)} = \mathrm{e}^{j\theta}S_{22}^{(s)*}$

Plug into Eq.(43):

$$
\begin{aligned}
\frac{V_\text{out}}{V_\text{in}} &= \mathrm{e}^{j\theta}S_{22}^{(s)*} + \mathrm{e}^{j\theta}\left(\left|S_{22}^{(s)}\right|^2 - 1\right)\frac{Z^{(s)}+Z_c}{2Z_c}\left(1 - \frac{Z^{(s)}+Z_c}{Z_J+Z_E}\right) \\
&= \mathrm{e}^{j\theta}\left[\frac{Z^{(s)*}-Z_c}{Z^{(s)*}+Z_c} + \left(\left|\frac{Z^{(s)}-Z_c}{Z^{(s)}+Z_c}\right|^2 - 1\right)\frac{Z^{(s)}+Z_c}{2Z_c}\left(1 - \frac{Z^{(s)}+Z_c}{Z_J+Z_E}\right)\right] \\
&= \mathrm{e}^{j\theta}\left[\frac{Z^{(s)*}-Z_c}{Z^{(s)*}+Z_c} + \left(\frac{\left|Z^{(s)}-Z_c\right|^2 - \left|Z^{(s)}+Z_c\right|^2}{(Z^{(s)}+Z_c)(Z^{(s)*}+Z_c)}\right)\frac{Z^{(s)}+Z_c}{2Z_c}\left(1 - \frac{Z^{(s)}+Z_c}{Z_J+Z_E}\right)\right] \\
&= \frac{\mathrm{e}^{j\theta}}{Z^{(s)*}+Z_c}\left[Z^{(s)*} - Z_c + \frac{-4\mathrm{Re}Z^{(s)}Z_c}{2Z_c}\left(1 - \frac{Z^{(s)}+Z_c}{Z_J+Z_E}\right)\right] \\
&= \frac{\mathrm{e}^{j\theta}}{Z^{(s)*}+Z_c}\left[Z^{(s)*} - 2\mathrm{Re}Z^{(s)} - Z_c + 2\mathrm{Re}Z^{(s)}\frac{Z^{(s)}+Z_c}{Z_J+Z_E}\right] \\
&= \mathrm{e}^{j\theta}\frac{Z^{(s)}+Z_c}{Z^{(s)*}+Z_c}\left(\frac{2\mathrm{Re}Z^{(s)}}{Z_J+Z_E} - 1\right)
\end{aligned}
$$

So if we look at power gain $G[\omega] = \left|\frac{V_\text{out}[\omega]}{V_\text{in}[\omega]}\right|^2$:

$$
\begin{aligned}
G[\omega_s] &= \left|\frac{2\mathrm{Re}Z^{(s)}[\omega_s]}{Z_J[\omega_s]+Z_E[\omega_s]} - 1\right|^2 \\
&= \left|\frac{\kappa_a}{\frac{Z_J[\omega_s]}{L_J} + \frac{\kappa_a}{2} - j(\omega_a - m(\Delta+\omega))} - 1\right|^2
\end{aligned}
\tag{44}
$$

Recall that:

$$
Z_J[\omega_s] = \frac{jL_J(\omega_a+\Delta+\omega)\left[\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta-\omega)\right]}{(1-|\epsilon|^2)\left[\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta-\omega)\right] - j|\epsilon|^2(\omega_a+\Delta-\omega)}
$$

Specifically, $Z_J[\omega_s] = jL_J(\omega_a+\Delta+\omega)$ without pump (i.e. when $\epsilon = 0$). In this case:

$$
\begin{aligned}
\frac{V_\text{out}[\omega_s]}{V_\text{in}[\omega_s]} &= \mathrm{e}^{j\theta}\frac{Z^{(s)}[\omega_s]+Z_c}{Z^{(s)*}[\omega_s]+Z_c}\left(\frac{\kappa_a}{\frac{\kappa_a}{2} + j\frac{2}{p}(\omega_s-\omega_a)} - 1\right) \\
&= \mathrm{e}^{j(\theta+2\theta_{\omega_s})}\frac{\frac{p}{2}\frac{\kappa_a}{2} + j(\omega_s-\omega_a)}{\frac{p}{2}\frac{\kappa_a}{2} - j(\omega_s-\omega_a)}
\end{aligned}
\tag{45}
$$

Here θ_{ω_s} is phase factor of $\mathrm{Re}Z^{(s)}[\omega_s] + Z_c + j\mathrm{Im}Z^{(s)}[\omega_s]$, which should be smooth over frequency. So we can take it as adding an overall slope on the phase roll.

$$G = 1 + \frac{\kappa_a^2 \left|\epsilon\right|^2 [(\omega_a + \Delta)^2 - \omega^2]}{(1 + m - m\left|\epsilon\right|^2)^2 \kappa_a^2 \omega^2 + \left(\frac{\kappa_a^2}{4}(1 - \left|\epsilon\right|^2) + \Delta^2(1 + 2m + m^2(1 - \left|\epsilon\right|^2)) - \omega^2(1 + 2m + m^2(1 - \left|\epsilon\right|^2)) + 2m} \tag{46}$$

$$G = 1 + \frac{\kappa_a^2 \left|\epsilon\right|^2 [(\omega_a + \Delta)^2 - \omega^2]}{(1 + m)^2 \kappa_a^2 \omega^2 + \left(\frac{\kappa_a^2}{4} + \Delta^2(1 + m)^2 - \omega^2(1 + m)^2 + 2m\left|\epsilon\right|^2 \Delta\omega_a - \left|\epsilon\right|^2 \omega_a^2\right)^2} \tag{47}$$

$$G = 1 + \frac{\kappa^2 \left|\epsilon' \omega_p / 2\right|^2}{\kappa^2 \omega^2 + \left(\frac{\kappa^2}{4} + \Delta^2 - \omega^2 - \left|\epsilon' \omega_p / 2\right|^2\right)^2} \tag{48}$$

When signal detuning $\omega \approx 0$:

$$G = 1 + \frac{\kappa^2 \left|\epsilon' \omega_p / 2\right|^2}{\left(\frac{\kappa^2}{4} + \Delta^2 - \left|\epsilon' \omega_p / 2\right|^2\right)^2} \tag{49}$$

where $\epsilon' = \frac{\epsilon}{1+m} = \frac{p c_3 \varphi_{\omega_p}}{4 c_2}$, $\kappa = \frac{p}{2}\kappa_a$

6 Verification: A lumped SPA

Impedance seen by the SNAIL:

$$Z_E[\omega] = \frac{R}{1 + j\omega RC} + j\omega L$$

We can evaluate its real and imaginary part respectively:

$$\begin{aligned}
\mathrm{Re} Z_E[\omega] &= \frac{R}{1 + \omega^2 R^2 C^2} \\
&\approx \frac{1}{\omega^2 R C^2}\left(1 - \frac{1}{\omega^2 R^2 C^2}\right) \\
&\approx \frac{1}{\omega_a^2 R C^2}\left(1 - \frac{2\Delta}{\omega_a}\right)\left(1 - \frac{1}{\omega_a^2 R^2 C^2}\right) \\
&= \frac{L_J + L}{RC}\left(1 - \frac{2\Delta}{\omega_a}\right)\left(1 - \frac{Z_a^2}{R^2}\right)
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Im} Z_E[\omega] &= \omega L - \frac{\omega R^2 C}{1+\omega^2 R^2 C^2} \\
&\approx \omega \left(L - \frac{1}{\omega^2 C} \left(1 - \frac{Z_a^2}{R^2} \right) \right) \\
&\approx (\omega_a + \Delta) \left(L - (L_J + L) \left(1 - \frac{2\Delta}{\omega_a} \right) \right) \\
&= -\omega_a L_J + L_J \left(\frac{2L}{L_J} + 1 \right) \Delta
\end{aligned}
$$

According to the relations in Section 4

$$
\mathrm{Im} Z_E[\omega_a] = -\omega_a L_J
$$

$$
m = \frac{\mathrm{Im} Z_E[\omega_a]}{L_J} = \frac{2L}{L_J} + 1
$$

And we see that SNAIL participation ratio is:

$$
p = \frac{2}{m+1} = \frac{L_J}{L + L_J}
$$

$$
\kappa = \frac{p}{2} \kappa_a = \frac{L_J}{2(L+L_J)} \frac{2 \mathrm{Re} Z_E[\omega_a]}{L_J} = \frac{1}{RC}
$$

7 Include Kerr into pumpistor model

$$
I(t) = \frac{E_j}{\phi_0} \left(c_2 \tilde{\varphi} + \frac{c_3}{2} \tilde{\varphi}^2 + \frac{c_4}{6} \tilde{\varphi}^3 + \cdots \right)
$$

$$
\begin{aligned}
\tilde{\varphi}(t) &= \mathrm{Re} \left(\varphi_{\omega_s} \mathrm{e}^{j\omega_s t} + \varphi_{\omega_p} \mathrm{e}^{j\omega_p t} + \varphi_{\omega_i}^* \mathrm{e}^{-j\omega_i t} \right) \\
&= \frac{1}{2} \left(\varphi_{\omega_s} \mathrm{e}^{j\omega_s t} + \varphi_{\omega_p} \mathrm{e}^{j\omega_p t} + \varphi_{\omega_i}^* \mathrm{e}^{-j\omega_i t} + \varphi_{\omega_s}^* \mathrm{e}^{-j\omega_s t} + \varphi_{\omega_p}^* \mathrm{e}^{-j\omega_p t} + \varphi_{\omega_i} \mathrm{e}^{j\omega_i t} \right)
\end{aligned}
$$

$$
\begin{aligned}
I(t) &= \mathrm{Re} \left(I_{\omega_s} \mathrm{e}^{j\omega_s t} + I_{\omega_p} \mathrm{e}^{j\omega_p t} + I_{\omega_i}^* \mathrm{e}^{-j\omega_i t} \right) \\
&= \frac{1}{2} \left(I_{\omega_s} \mathrm{e}^{j\omega_s t} + I_{\omega_p} \mathrm{e}^{j\omega_p t} + I_{\omega_i}^* \mathrm{e}^{-j\omega_i t} + I_{\omega_s}^* \mathrm{e}^{-j\omega_s t} + I_{\omega_p}^* \mathrm{e}^{-j\omega_p t} + I_{\omega_i} \mathrm{e}^{j\omega_i t} \right)
\end{aligned}
$$

$$
I_{\omega_s} = I_c \left(c_2 \varphi_{\omega_s} + \frac{c_3}{2} \varphi_{\omega_p} \varphi_{\omega_i}^* + \frac{c_4}{8} \left(|\varphi_{\omega_s}|^2 + 2|\varphi_{\omega_i}|^2 + 2|\varphi_{\omega_p}|^2 \right) \varphi_{\omega_s} \right) \tag{50}
$$

$$
Y_S[\omega_s] = \frac{c_2}{j\omega_s M L_J} \left(1 + \frac{c_4}{8c_2} \left(|\varphi_{\omega_s}|^2 + 2|\varphi_{\omega_i}|^2 + 2|\varphi_{\omega_p}|^2 \right) \right) + \frac{\frac{c_3}{2} \frac{\varphi_{\omega_p} \varphi_{\omega_i}^*}{\varphi_{\omega_s}}}{j\omega_s M L_J} \tag{51}
$$

$$\begin{pmatrix} I_{\omega_s} \\ I_{\omega_i}^* \end{pmatrix} = \frac{1}{jL_J} \begin{pmatrix} \frac{1+\delta_s}{\omega_s} & \frac{-\epsilon}{\omega_i} \\ \frac{-\epsilon^*}{\omega_s} & \frac{1+\delta_i}{\omega_i} \end{pmatrix} \begin{pmatrix} V_{\omega_s} \\ V_{\omega_i}^* \end{pmatrix} \tag{52}$$

To get the 2nd order perturbation correction (current conservation analysis): need also to include the higher idler frequency $\omega_h = \omega_s + \omega_p = 2\omega_p - \omega_i$

$$I_{\omega_s} = I_c \left(c_2 \varphi_{\omega_s} + \frac{c_3}{2} \left(\varphi_{\omega_p} \varphi_{\omega_i}^* + \varphi_{\omega_p}^* \varphi_{\omega_h} \right) + \frac{c_4}{4} |\varphi_{\omega_p}|^2 \varphi_{\omega_s} \right) \tag{53}$$

Let's consider the Kerr effect due to pump only, and denote:

$$\delta = \frac{c_4}{4c_2} |\varphi_{\omega_p}|^2 \tag{54}$$

We get:

$$\begin{pmatrix} I_{\omega_s} \\ I_{\omega_i}^* \\ I_{\omega_h} \end{pmatrix} = \frac{1}{jL_J} \begin{pmatrix} \frac{1+\delta}{\omega_s} & \frac{-\epsilon}{\omega_i} & \frac{\epsilon^*}{\omega_h} \\ \frac{-\epsilon^*}{\omega_s} & \frac{1+\delta}{\omega_i} & 0 \\ \frac{\epsilon}{\omega_s} & 0 & \frac{1+\delta}{\omega_h} \end{pmatrix} \begin{pmatrix} V_{\omega_s} \\ V_{\omega_i}^* \\ V_{\omega_h} \end{pmatrix} \tag{55}$$

which should be put together with other linear elements in the circuit and solved self-consistently, i.e.

$$\begin{aligned} I_{\omega_i}^* Z_E[-\omega_i] + V_{\omega_i}^* &= 0 \\ I_{\omega_h} Z_E[\omega_h] + V_{\omega_h} &= 0 \end{aligned}$$

Consequently:

$$\begin{aligned} \frac{V_{\omega_i}^*}{V_{\omega_s}} &= \frac{\epsilon^*}{(1+\delta)\frac{\omega_s}{\omega_i} - \frac{jL_J\omega_s}{Z_E[-\omega_i]}} \\ \frac{V_{\omega_h}}{V_{\omega_s}} &= \frac{-\epsilon}{(1+\delta)\frac{\omega_s}{\omega_h} + \frac{jL_J\omega_s}{Z_E[\omega_h]}} \end{aligned}$$

So the full expression should be:

$$Z_J[\omega_s] = \frac{jL_J\omega_s}{1 + \delta - \frac{|\epsilon|^2}{1+\delta - jL_J\omega_i/Z_E^*[\omega_i]} - \frac{|\epsilon|^2}{1+\delta+jL_J\omega_i/Z_E[\omega_h]}} \tag{56}$$

While let us neglect $Z_E[\omega_h]$ for now (assuming it's a short). Keeping up to 1st order in δ and $|\epsilon|^2$:

$$Z_J[\omega_s] = \frac{jL_J(\omega_a + \Delta + \omega)\left[(1+\delta)\left(\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta - \omega)\right) + j\delta(\omega_a + \Delta - \omega)\right]}{(1 + 2\delta - |\epsilon|^2)\left(\frac{\kappa_a}{2} - j\frac{2}{p}(\Delta - \omega)\right) + j(\delta - |\epsilon|^2)(\omega_a + \Delta - \omega)} \tag{57}$$

8 Impedance engineered broadband JPA

We have already shown that the gain profile has a frequency (signal detuning from $\omega_p/2$) dependence:

$$G[\omega] \approx \frac{\kappa^2 \left|\epsilon' \omega_p/2\right|^2}{\kappa^2\omega^2 + \left(\frac{\kappa^2}{4} + \Delta^2 - \omega^2 - \left|\epsilon'\omega_p/2\right|^2\right)^2}$$

When using a JPA, we usually talk about its gain at the peak, i.e. at $\omega \approx 0$:

$$G_0 \approx \frac{\kappa^2 \left|\epsilon' \omega_p/2\right|^2}{\left(\frac{\kappa^2}{4} + \Delta^2 - \left|\epsilon'\omega_p/2\right|^2\right)^2}$$

After some maths, we get:

$$G[\omega] = \frac{G_0}{1 + (\omega/\Gamma)^2} \tag{58}$$

where

$$\Gamma = \frac{1}{\sqrt{G_0}} \frac{\kappa \left|\epsilon'\omega_p/2\right|}{\sqrt{2\left(\frac{\kappa^2}{4} - \Delta^2 + \left|\epsilon'\omega_p/2\right|^2\right)}} \tag{59}$$

Worth noting, in the case of zero pump detuning ($\Delta \approx 0$) and high gain ($\frac{\kappa^2}{4} + \Delta^2 - \left|\epsilon'\omega_p/2\right|^2 \approx 0$), Eq.(60) have a prettier looking:

$$2\Gamma = \frac{\kappa}{\sqrt{G_0}} \tag{60}$$

i.e. FWHM in gain profile is approximately resonance linewidth κ over root gain. This is the famous gain-bandwidth tradeoff.

According to literature, engineering a slope in the real part of output impedance could made a broadband JPA beyond the gain-bandwidth tradeoff.

As a reminder, previously in Eq.(29) we're assuming the real part of output impedance being constant around ω_a (i.e. the slope being negligible).

Now we want to include the

$$Z_E[\omega_s] = \frac{\kappa_a}{2} L_J + rL_J(\omega_s - \omega_a) - j\omega_a L_J + jmL_J(\omega_s - \omega_a) \tag{61}$$

where we're assuming $\mathrm{Re}Z^{(p)}[\omega_a] = \mathrm{Re}Z^{(p)'}[\omega_a] = 0$ and:

$$r := \frac{\mathrm{Re}Z^{(s)'}[\omega_a]}{L_J} \tag{62}$$

In this case:

$$\begin{aligned} G[\omega_s] &= \left|\frac{2\mathrm{Re}Z^{(s)}[\omega_s]}{Z_J[\omega_s] + Z_E[\omega_s]} - 1\right|^2 \\ &= \left|\frac{\kappa_a + 2r(\Delta + \omega)}{\frac{Z_J[\omega_s]}{L_J} + \frac{\kappa_a}{2} + r(\Delta + \omega) - j(\omega_a - m(\Delta + \omega))} - 1\right|^2 \end{aligned} \tag{63}$$

And

$$Z_J[\omega_s] = \frac{jL_J(\omega_a+\Delta+\omega)\left[\frac{\kappa_a}{2}+r(\Delta-\omega)-j\frac{2}{p}(\Delta-\omega)\right]}{(1-|\epsilon|^2)\left[\frac{\kappa_a}{2}+r(\Delta-\omega)-j\frac{2}{p}(\Delta-\omega)\right]-j\,|\epsilon|^2(\omega_a+\Delta-\omega)} \tag{64}$$

Like what we did in section 5, let's start with the case of no pump: $Z_J[\omega_s] = jL_J\omega_s$

$$\begin{aligned}\frac{V_{\text{out}}[\omega_s]}{V_{\text{in}}[\omega_s]} &= \mathrm{e}^{j\theta}\,\frac{Z^{(s)}[\omega_s]+Z_c}{Z^{(s)*}[\omega_s]+Z_c}\left(\frac{\kappa_a+2r(\omega_s-\omega_a)}{\frac{\kappa_a}{2}+r(\omega_s-\omega_a)+j(m+1)(\omega_s-\omega_a)}-1\right)\\ &= \mathrm{e}^{j(\theta+2\theta_{\omega_s})}\,\frac{\frac{p}{2}\left(\frac{\kappa_a}{2}+r(\omega_s-\omega_a)\right)+j(\omega_s-\omega_a)}{\frac{p}{2}\left(\frac{\kappa_a}{2}+r(\omega_s-\omega_a)\right)-j(\omega_s-\omega_a)}\end{aligned} \tag{65}$$

$$G = 1+\frac{(\kappa_a+2r(\Delta+\omega))(\kappa_a+2r(\Delta-\omega))\,|\epsilon|^2\left[(\omega_a+\Delta)^2-\omega^2\right]}{\omega^2\left((1+m)\kappa_a-2r\,|\epsilon|^2\,\omega_a\right)^2+\left(\frac{\kappa_a^2}{4}+\Delta^2(1+m)^2-\omega^2(1+m)^2-|\epsilon|^2\,\omega_a^2+r^2(\Delta^2-\omega^2)+r\Delta\kappa_a\right)^2} \tag{66}$$

$$G = 1+\frac{\left(\kappa'^2-(p\omega/2)^2\right)|\epsilon'\omega_p/2|^2}{\omega^2(\kappa^2-4r\kappa\,|\epsilon'|^2\,\omega_a)+\left(\frac{\kappa'^2}{4}+\Delta^2-(1+r^2)\omega^2-|\epsilon'\omega_p/2|^2\right)\Big)^2} \tag{67}$$

where $\kappa' = \kappa + pr\Delta = \frac{\kappa_a+2r\Delta}{m+1}$

9 Noise

$$\begin{aligned}\text{NVR} &= P_{\text{on}}/P_{\text{off}}\\ &= \frac{G_{\text{sys}}(T_{\text{sys}}+G(T_Q+T_{\text{add}}))}{G_{\text{sys}}(T_{\text{sys}}+T_Q)}\\ &= \frac{G}{T_{\text{sys}}+T_Q}T_{\text{add}}+\frac{T_{\text{sys}}+GT_Q}{T_{\text{sys}}+T_Q}\end{aligned}$$