1 Parametric Amplifier

1.1 Non-degenerate JPC

$$H = \frac{{Q_a}^2}{2C_a} + \frac{{Q_b}^2}{2C_b} + \frac{{Q_c}^2}{2C_c} + \frac{{\Phi_a}^2}{2\Phi_a} + \frac{{\Phi_b}^2}{2\Phi_b} + \frac{{\Phi_c}^2}{2\Phi_c} + K\Phi_a\Phi_b\Phi_c \tag{1}$$

Where

$$\omega_i = \frac{1}{\sqrt{L_i C_i}}, \quad Z_i = \sqrt{\frac{L_i}{C_i}}, \quad \Phi_i = \sqrt{\frac{\hbar Z_i}{2}}\left(i + i^\dagger\right), \quad Q_i = \mathrm{i}\sqrt{\frac{\hbar}{2Z_i}}\left(i - i^\dagger\right), \quad i \mathrm{in} \{a, b, c\}$$

1.2 4-wave mixing degenerate paramp

The Hamiltonian of a non-linear harmonic oscillator with 4th order non-linearity (Kerr) reads:

$$H/\hbar = \omega_a a^\dagger a + K\left(a^\dagger + a\right)^4 \tag{2}$$

And we can get the QLE:

$$\dot{a} = -\mathrm{i}\omega_a a - 4\mathrm{i}K\left(a^\dagger + a\right)^3 - \frac{\kappa}{2}a + \sqrt{\kappa}a_{\mathrm{in}} \tag{3}$$

1.3 Two-mode description

It is sometimes convenient to describe an one-mode degenerate paramp with an effective two-mode model, the signal and idler as two seperate Bosonic modes (although they are actually slightly different frequency components of the same physical cavity mode).

2 SPA

2.1 SNAIL

A Josephson junction of critical current I_c can be described equivalently with a Josephson inductence:

$$L_j = \frac{\Phi_0}{2\pi I_c} = \frac{\phi_0}{I_c} \tag{4}$$

where $\Phi_0 = \frac{\hbar}{2e}$ being the reduced flux quantum, or with a Josephson energy:

$$E_j = \phi_0 I_c = L_j I_c^2 = \frac{\phi_0^2}{L_j} \tag{5}$$

Under such definition, we can write the Josephson potential as:

$$U_j(\Phi) = -E_j \cos\frac{2\pi\Phi}{\Phi_0} \tag{6}$$

where we usually denote RF flux across the junction Φ as $\varphi \equiv \frac{2\pi\Phi}{\Phi_0}$ for simplicity.

A SNAIL, consisting of three junctions with E_j on one branch, and one junction with αE_j on the other branch, has the potential of:

$$U_S(\varphi_s) = -\alpha E_j \cos\varphi_s - 3E_j \cos\frac{\varphi_{\text{ext}} - \varphi_s}{3} \tag{7}$$

where φ_s is the reduced RF flux across the SNAIL, and there is an external magnetic flux φ_{ext} tunnelling through the loop formed by the SNAIL.

Expanding near the minimum $\Phi_s = \Phi_{\text{min}}$:

$$U_S(\tilde{\varphi}_s)/E_j = c_0 + \frac{c_2}{2!}\tilde{\varphi}_s^2 + \frac{c_3}{3!}\tilde{\varphi}_s^3 + \frac{c_4}{4!}\tilde{\varphi}_s^4 + \cdots \tag{8}$$

where

$$\tilde{\varphi}_s \equiv 2\pi\frac{\Phi_s - \Phi_{\text{min}}}{\Phi_0} \tag{9}$$

2.2 Lumped SPA model

$$p = \frac{NL_S}{L_{\text{ext}} + NL_S} \tag{10}$$

$$\omega_a = \frac{1}{\sqrt{C(L + ML_S)}} \tag{11}$$

$$H/\hbar = \Delta a^\dagger a + g\left(a^2 + a^{\dagger 2}\right) + \frac{K}{2}a^{\dagger 2}a^2 \tag{12}$$

$$U(\varphi) = MU_S(\varphi_s[\varphi]) + \frac{1}{2}E_L(\varphi - M\varphi_s[\varphi])^2 \tag{13}$$

$$U(\tilde{\varphi})/E_j = \tilde{c}_0 + \frac{\tilde{c}_2}{2!}\tilde{\varphi}^2 + \frac{\tilde{c}_3}{3!}\tilde{\varphi}^3 + \frac{\tilde{c}_4}{4!}\tilde{\varphi}^4 + \cdots \tag{14}$$

2.3 Distributed model

$$\mathcal{L} = \int_{-d_1}^{0-}\left[\frac{c}{2}(\partial_t\phi)^2 - \frac{1}{2l}(\partial_x\phi)^2\right]\mathrm{d}x + \int_{0+}^{d_2}\left[\frac{c}{2}(\partial_t\phi)^2 - \frac{1}{2l}(\partial_x\phi)^2\right]\mathrm{d}x - MU_S\left(\frac{\phi_+ - \phi_-}{M\phi_0}\right) \tag{15}$$

at $x = -d_1$ and $x = d_2$, we have the boundary condition:

$$\partial_x\phi = 0 \tag{16}$$

at $x = 0+$ and $x = 0-$, we have the current consevation relation:

$$-U_S'\left(\frac{\phi_+ - \phi_-}{M\phi_0}\right) + \frac{1}{l}\partial_x\phi_- = 0 \tag{17}$$

$$U_S'\left(\frac{\phi_+ - \phi_-}{M\phi_0}\right) - \frac{1}{l}\partial_x\phi_+ = 0 \tag{18}$$

2.4 Halved SPA model

For a perfectly halved SPA:

$$\mathcal{L} = \int_{-d}^{0-} \left[\frac{c}{2}(\partial_t \phi)^2 - \frac{1}{2l}(\partial_x \phi)^2 \right] \mathrm{d}x - MU_S(\frac{\phi_+ - \phi_-}{M\phi_0}) \tag{19}$$

at $x = -d$:

$$\partial_x \phi|_{x=-d} = 0 \tag{20}$$

at $x = 0-$, the current consevation relation is the same:

$$-U'_S\left(\frac{\phi_+ - \phi_-}{M\phi_0}\right) + \frac{1}{l}\partial_x \phi_- = 0 \tag{21}$$

and the boundary condition at $x = 0+$:

$$\phi_+ = 0 \tag{22}$$

Expanding U_S according to Eq.(8) and recall Josephson relations:

$$E_j = \frac{\phi_0^2}{L_j} \tag{23}$$

Eq.(21) gives rise to:

$$-E_j\left(c_2(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min}) + \frac{c_3}{2!}(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min})^2 + \frac{c_4}{3!}(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min})^3\right) + \frac{1}{l}\partial_x \phi_- = 0 \tag{24}$$

Expansion:

$$\phi(x,t) = (A_0 + B_0 x) + \sum_{n=1}^{\infty}(A_n \cos k_n x + B_n \sin k_n x)\mathrm{e}^{j\omega_n t} \tag{25}$$

Therefore, at $x = 0-$:

$$\phi_-(t) = A_0 + \sum_{n=1}^{\infty} A_n \mathrm{e}^{j\omega_n t} \tag{26}$$

and

$$\partial_x \phi = B_0 + \sum_{n=1}^{\infty} k_n(-A_n \sin k_n x + B_n \cos k_n x)\mathrm{e}^{j\omega_n t} \tag{27}$$

which, combining with Eq.(24), gives:

$$\begin{aligned} E_j\left(c_2(\sum_{n=0}^{\infty}\frac{-A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min}) + \frac{c_3}{2!}(\sum_{n=0}^{\infty}\frac{-A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min})^2 + \frac{c_4}{3!}(\sum_{n=0}^{\infty}\frac{-A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min})^3\right) \\ = \frac{1}{l}\left(B_0 + \sum_{n=1}^{\infty} k_n B_n \mathrm{e}^{j\omega_n t}\right) \end{aligned} \tag{28}$$

In addition, from Eq.(20), we have:

$$0 = \partial_x \phi|_{x=-d} = B_0 + \sum_{n=1}^{\infty} k_n(A_n \sin k_n d + B_n \cos k_n d)\mathrm{e}^{j\omega_n t} \tag{29}$$

such that $B_0 = 0$ and:

$$A_n \sin k_n d + B_n \cos k_n d = 0 \tag{30}$$

Starting from the linear part of Eq.(28), we get:

$$\frac{c_2}{L_j}\left(\frac{A_0}{M\phi_0} + \varphi_{\min}\right) + \frac{1}{l}B_0 = 0 \tag{31}$$

$$\frac{c_2}{L_j}\frac{A_n}{M} + \frac{k_n}{l}B_n = 0 \tag{32}$$

Since $B_0 = 0$, we get the phase offset:

$$\varphi_{\min} = -\frac{A_0}{M} \tag{33}$$

For any n, Eq.(30) and Eq.(32) should be solved for all A_n and B_n to get the eigenmode, so:

$$\begin{pmatrix} \sin k_n d & \cos k_n d \\ \frac{c_2}{ML_j} & \frac{k_n}{l} \end{pmatrix}\begin{pmatrix} A_n \\ B_n \end{pmatrix} = 0 \tag{34}$$

where this determinant should be zero. Write $k_n = \omega_n/v$, $l = Z_c/v$, it gives us the eigenmode equation:

$$\frac{ML_j}{c_2 Z_C}\omega_n \tan\frac{\omega_n d}{v} = 1 \tag{35}$$

For fitting, denote $\gamma = \frac{2Z_C}{L_j}$, and ω_0 as the frequency if there were not the SNAIL array (as we did for usual SPA). Notice that:

$$\frac{\omega_0}{v}d = \frac{\pi}{2} \tag{36}$$

Then Eq.(35) can be equivalently written into:

$$\frac{\gamma c_2}{2M} = \omega_n \tan\frac{\pi}{2}\frac{\omega_n}{\omega_0} \tag{37}$$

which is almost the same as a perfectly-centered SPA (i.e. $\mu = 0.5$, see Vlad's note equation 336), except for the difference between M and $2M$. This also makes sense: M for a halved-SPA should also be half of that for a perfectly-centered SPA in order to give the same eigenmodes.

2.5 Non-perfectly halved-SPA

Under the case of a non-perfect halved SPA, there would be a small imaginary impedance jX at 0+ instead of a perfect short. (For now I'm taking X as a constant value, although in reality this would be a frequency dependent phase shift.) Then instead of Eq.(22), boundary condition at $x = 0+$ should become:

$$U_S'\left(\frac{\phi_+ - \phi_-}{M\phi_0}\right) = \frac{1}{l}\partial_x\phi_+ = -\frac{\partial_t\phi_+}{jX} \tag{38}$$

which leads to:

$$E_j\left(c_2(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min}) + \frac{c_3}{2!}(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min})^2 + \frac{c_4}{3!}(\frac{\phi_+ - \phi_-}{M\phi_0} - \varphi_{\min})^3\right) = -\frac{\partial_t\phi_+}{jX} \tag{39}$$

The expansion for $x < 0$ section transmission line Eq.(53), still holds. Therefore, now combining Eq.(24) and Eq.(39), we have a new equation that replaces Eq.(28):

$$
\begin{aligned}
&E_j\left(c_2(\sum_{n=0}^{\infty}\frac{\phi_+ - A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min}) + \frac{c_3}{2!}(\sum_{n=0}^{\infty}\frac{\phi_+ - A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min})^2 + \frac{c_4}{3!}(\sum_{n=0}^{\infty}\frac{\phi_+ - A_n \mathrm{e}^{j\omega_n t}}{M\phi_0} - \varphi_{\min})^3\right)\\
&\quad = \frac{1}{l}\left(B_0 + \sum_{n=1}^{\infty} k_n B_n \mathrm{e}^{j\omega_n t}\right)\\
&\quad = -\frac{1}{jX}\partial_t \phi_+
\end{aligned}
\tag{40}
$$

such that

$$
\begin{aligned}
\phi_+(t) &= -\frac{jXB_0}{l}t - \sum_{n=1}^{\infty}\frac{Xk_nB_n}{\omega_n}\mathrm{e}^{j\omega_n t}\\
&= \sum_{n=1}^{\infty}\frac{X\tan k_n d}{Z_C}A_n \mathrm{e}^{j\omega_n t}
\end{aligned}
\tag{41}
$$

where the second equality has used Eq.(20): the boundary condition at $x = -d$.

Therefore, when Eq.(32) becomes:

$$
\frac{c_2}{ML_j}\left(1 - \frac{X\tan k_n d}{Z_C}\right)A_n + \frac{k_n}{l}B_n = 0 \tag{42}
$$

Using the same determinant to solve for eigenmodes, except that compared to now there's one more term in the bottom-left component in the determinant. And the final eigenmode equation is:

$$
\frac{ML_j}{c_2 Z_C}\omega_n \tan\frac{\omega_n d}{v} = 1 - \frac{X}{Z_C}\tan\frac{\omega_n d}{v} \tag{43}
$$

Using the same defined γ and ω_0 as before, just pay attention that now "bare mode" frequency ω_0 would no longer satisfy Eq.(36), instead it'll be X-dependent:

$$
\frac{\omega_0}{v}(d + d_X(\omega_0)) = \frac{\pi}{2} \tag{44}
$$

with $d_X(\omega) \equiv \frac{v}{\omega}\arctan\frac{X(\omega)}{Z_C}$ the effective electrical length of $jX(\omega)$.

While X can be (and usually is) a function of frequency ω, here we treat two simplest cases, i.e. $X(\omega) being constant (displaced)$, and $d_X(\omega)$ being constant (a small inductance).

In the first case:

$$
\frac{\omega_0}{v}d = \frac{\pi}{2} - \arctan\frac{X}{Z_C} \tag{45}
$$

In the second case:

$$
\frac{\omega_0}{v}(d + d_X) = \frac{\pi}{2} \tag{46}
$$

here we can define $\mu \equiv \frac{d}{d+d_X}$, using the same symbol as the μ that characterize the asymmetry in SPA.

So Eq.(43) can be written as:

$$
\frac{\gamma c_2}{2M} = \omega_n \frac{\tan\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}}{1 - \tan\frac{\pi}{2}\frac{(1-\mu)\omega_n}{\omega_0}\tan\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}} \tag{47}
$$

$$\tan\frac{\pi}{2}\frac{(1-\mu)\omega_n}{\omega_0}+\tan\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}=\frac{2M}{\gamma c_2}\omega_n\tan\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}\tan\frac{\pi}{2}\frac{\omega_n}{\omega_0} \tag{48}$$

$$\frac{\gamma c_2}{2M}=\omega_n\frac{\sin\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}\cos\frac{\pi}{2}\frac{(1-\mu)\omega_n}{\omega_0}}{\cos\frac{\pi}{2}\frac{\omega_n}{\omega_0}} \tag{49}$$

$$\frac{\gamma c_2}{M}=\omega_n\left(\tan\frac{\pi}{2}\frac{\mu\omega_n}{\omega_0}+\frac{\sin\frac{\pi}{2}\frac{(2\mu-1)\omega_n}{\omega_0}}{\cos\frac{\pi}{2}\frac{\omega_n}{\omega_0}}\right) \tag{50}$$

2.6 Capacitively coupled SPA

The old way to study the coupling is to calculate an effective mode L and C, and fitting the coupling C to the mearsured mode. While it should be a better fitting if we can directly take that coupling into the theory.

At $x=-d$, the boundary condition is no a perfect open, but instead:

$$\frac{1}{l}\partial_x\phi_{-d}=-\frac{\partial_t\phi_{-d}}{Z_{\text{couple}}} \tag{51}$$

with $Z_{\text{couple}}=50\Omega+\frac{1}{j\omega C_c}$, is the impedance seen by the device at $x=-d$.

Again, putting in the expansion:

$$\phi(x,t)=(A_0+B_0x)+\sum_{n=1}^{\infty}(A_n\cos k_nx+B_n\sin k_nx)\mathrm{e}^{j\omega_nt} \tag{52}$$

$$\partial_x\phi=B_0+\sum_{n=1}^{\infty}k_n(-A_n\sin k_nx+B_n\cos k_nx)\mathrm{e}^{j\omega_nt} \tag{53}$$

Now instead of Eq.(30), we arrive at:

$$\left(k_nA_n-\frac{j\omega_nl}{Z_{\text{couple}}}B_n\right)\sin k_nd+\left(k_nB_n+\frac{j\omega_nl}{Z_{\text{couple}}}A_n\right)\cos k_nd=0 \tag{54}$$

This together with Eq.(42) can give rise to the new derterminant and resulting in new eigenmode functions.

For an arbitrarily coupled, halved SPA:

$$\left(\sin k_nd+\frac{jZ_C}{Z_{\text{couple}}}\cos k_nd\right)A_n+\left(\cos k_nd-\frac{jZ_C}{Z_{\text{couple}}}\sin k_nd\right)B_n=0$$
$$\frac{c_2}{ML_j}\left(1-\frac{X\tan k_nd}{Z_C}\right)A_n+\frac{k_n}{l}B_n=0$$

And this finally gives rise to:

$$\frac{ML_j}{c_2Z_C}\omega_n\tan\frac{\omega_nd}{v}-\left(1-\frac{X}{Z_C}\tan\frac{\omega_nd}{v}\right)=\frac{jZ_C}{Z_C+1/j\omega C_c}\left(\frac{X}{Z_C}\tan^2\frac{\omega_nd}{v}-\tan\frac{\omega_nd}{v}-\frac{ML_j}{c_2Z_C}\omega_n\right) \tag{55}$$

which requires the RHS (introduced due to coupling) to have a zero imaginary part, and real part equal to LHS (eigenmode condition without coupling).

BTW, here I'm using the assumption that the resonator has the same Z_C as the TL session beyond coupling capacitor (that is supposed to be near 50Ω), which doesn't always be the case, but I'm doing this to all my PPFSPA devices.

$$\frac{ML_j}{c_2Z_C}\omega_n\tan\frac{\omega_nd}{v}=1-\frac{jZ_C}{Z_C+1/j\omega C_c}\left(\tan\frac{\omega_nd}{v}+\frac{ML_j}{c_2Z_C}\omega_n\right) \tag{56}$$

2.7 Old way

As Vlad did in distributed-model SPA fitting, we rewrite the Lagrangian with respect to a canonical coordinate $\phi(-d,t)$, i.e. the flux at $x=-d$ point, where the device is capacitively coupled to the environment.

Having the Lagrangian, we can effectively express the system as a LC seen at this particular point, and then fit kappa with a coupling capacitor C_c. And update the mode frequency fitting according to the loaded-LC model, with L and C represented from distributed-model parameters. And update C_c from kappa fitting... This iterative fitting is rather slow (runs more than 10min on PC), and sometimes doesn't give a good-looking fitting (fitted frequency being around 50MHz different from measured data). But let's first see how this works for PPFSPA:

We can take the time dependence $e^{j\omega t}$ into $A_n(t)$, making it a dynamical coordinate. Since $B_n(t) = -\tan k_n d A_n(t)$, we have:

$$\phi(x,t) = \sum_{n=1}^{\infty} A_n(t)(\cos k_n x - \tan k_n d \sin k_n x) \tag{57}$$

When we consider only the first mode (keeping only $n=1$), this system actually has only one degree of freedom. And we should be able to represent the Lagrangian with respect to this coordinate $A_1(t)$:

$$\begin{aligned}
\mathcal{L} &= \int_{-d}^{0-} \left[\frac{c}{2}(\partial_t\phi)^2 - \frac{1}{2l}(\partial_x\phi)^2\right] \mathrm{d}x - MU_S(\frac{\phi_+ - \phi_-}{M\phi_0}) \\
&= \int_{-d}^{0-} \left[\frac{c}{2}\dot{A_1}^2(\cos k_1 x - \tan k_1 d \sin k_1 x)^2 - \frac{1}{2l}A_1^2 k_1^2(-\sin k_1 x - \tan k_1 d \cos k_1 x)^2\right] \mathrm{d}x - ME_j\frac{c_2}{2}\left(\frac{0-A_1}{M\phi_0}\right)^2 \\
&= \frac{c}{2}\dot{A_1}^2\left[\frac{d}{2\cos^2 k_1 x} + \frac{1}{2k_1}\left(\sin k_1 d \cos k_1 d + \frac{\sin^3 k_1 d}{\cos k_1 d}\right)\right] - \\
&\quad \frac{1}{2l}A_1^2 k_1^2\left[\frac{d}{2\cos^2 k_1 x} - \frac{1}{2k_1}\left(\sin k_1 d \cos k_1 d + \frac{\sin^3 k_1 d}{\cos k_1 d}\right)\right] - \frac{c_2}{2ML_j}A_1^2 \\
&= \frac{c}{2}\dot{A_1}^2\left[\frac{d}{2\cos^2 k_1 x} + \frac{1}{2k_1}\left(\sin k_1 d \cos k_1 d + \frac{\sin^3 k_1 d}{\cos k_1 d}\right)\right] - \\
&= \frac{\dot{A_1}^2}{4}\left(\frac{dc}{\cos^2 k_1 x} + \frac{\tan k_1 d}{\omega_1 Z_C}\right) - \frac{{A_1}^2}{4}{\omega_1}^2\left(\frac{dc}{\cos^2 k_1 x} - \frac{\tan k_1 d}{\omega_1 Z_C} + \frac{2c_2}{ML_j}\right)
\end{aligned} \tag{58}$$

It makes more sense to use $\phi(-d,t)$ instead of $A_1(t)$ as the dynamical coordinate:

$$\phi_\omega(t) \equiv \phi(-d,t) = A_1(t)(\cos k_1 d + \tan k_1 d \sin k_1 d) = \frac{A_1(t)}{\cos k_1 d} \tag{59}$$

And Lagrangian in terms of this coordinate is:

$$\mathcal{L} \equiv \frac{C_\omega \dot{\phi_\omega}^2}{2} - \frac{{\phi_\omega}^2}{2L_\omega} \tag{60}$$

Therefore, we get:

$$C_\omega = \frac{dc}{2} + \frac{\sin 2k_1 d}{4\omega_1 Z_C} \tag{61}$$

$$\begin{aligned}
\frac{1}{L_\omega} &= {\omega_1}^2\left(\frac{dc}{2} - \frac{\sin 2k_1 d}{4\omega_1 Z_C} + \frac{c_2\cos^2 k_1 d}{{\omega_1}^2 ML_j}\right) \\
&= (\omega_1)^2 C_\omega
\end{aligned} \tag{62}$$

(for non-perfectly halved case, there's and extra term in L, but the expression for C doesn't change, and the last equality always holds.)

2.8 New way

3 Flip - Chip

$$W_{\text{eff}} = W + \frac{T}{\pi} \ln \left(1 + \frac{4}{\sqrt{\left(\frac{T}{H}\right)^2 + \left(\frac{T}{W\pi + 1.1T\pi}\right)^2}} \right) \frac{\epsilon_1 + \epsilon_2}{2\pi\epsilon_1} \tag{63}$$

$$X_1 = \frac{4H}{W_{\text{eff}}} \frac{14\epsilon_1 + 8\epsilon_2}{11\epsilon_1} \tag{64}$$

$$X_2 = \sqrt{X_1^2 + \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1}\pi^2} \tag{65}$$

$$Z = \frac{376.73\Omega}{2\pi\sqrt{2(\epsilon_1 + \epsilon_2)}} \ln \left(1 + \frac{4H(X_1 + X_2)}{W_{\text{eff}}} \right) \tag{66}$$