

Proof (i) If F is homogeneous of degree d , and the curve $D : (F = 0)$ meets L in points P_1, \dots, P_a with $a > d$, then by (1.9), I must have $L \subset D$, so that by the lemma, $F = H \cdot F'$; now since $P_{a+1}, \dots, P_n \notin L$, obviously $F' \in S_{d-1}(P_{a+1}, \dots, P_n)$. (ii) is exactly the same. Q.E.D.

Proposition 2.6 Let k be an infinite field, and $P_1, \dots, P_8 \in \mathbb{P}_k^2$ distinct points; suppose that no 4 of P_1, \dots, P_8 are collinear, and no 7 of them lie on a nondegenerate conic; then

$$\dim S_3(P_1, \dots, P_8) = 2.$$

Proof For brevity, let me say that a set of points are *conconic* if they all lie on a nondegenerate conic. The proof of (2.6) breaks up into several cases.

Main case No 3 points are collinear, no 6 conconic. This is the ‘general position’ case.

Suppose for a contradiction that $\dim S_3(P_1, \dots, P_8) \geq 3$, and let P_9, P_{10} be distinct points on the line $L = P_1 P_2$. Then

$$\dim S_3(P_1, \dots, P_{10}) \geq \dim S_3(P_1, \dots, P_8) - 2 \geq 1,$$

so that there exists $0 \neq F \in S_3(P_1, \dots, P_{10})$. By Corollary 2.5, $F = H \cdot Q$, with $Q \in S_2(P_3, \dots, P_8)$. Now I have a contradiction to the case assumption: if Q is nondegenerate then the 6 points P_3, \dots, P_8 are conconic, whereas if Q is a line pair or a double line, then at least 3 of them are collinear.

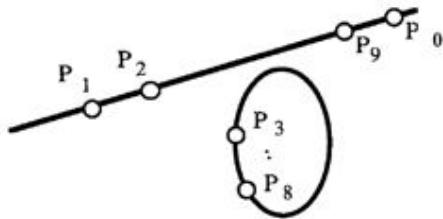


Figure 2.2: 10 points on a reducible cubic

First degenerate case Suppose $P_1, P_2, P_3 \in L$ are collinear, and let $L : (H = 0)$. Let P_9 be a 4th point on the line L . Then by Corollary 2.5,

$$S_3(P_1, \dots, P_9) = H \cdot S_2(P_4, \dots, P_8).$$

Also, since no 4 of P_4, \dots, P_8 are collinear, by Corollary 1.11,

$$\dim S_2(P_4, \dots, P_8) = 1, \quad \text{and then} \quad \dim S_3(P_1, \dots, P_9) = 1,$$

which implies $\dim S_3(P_1, \dots, P_8) \leq 2$.

Second degenerate case Suppose $P_1, \dots, P_6 \in C$ are conconic, with $C : (Q = 0)$ a nondegenerate conic. Then choose $P_9 \in Q$ distinct from P_1, \dots, P_6 . By Corollary 2.5 again,

$$S_3(P_1, \dots, P_9) = Q \cdot S_1(P_7, P_8);$$

the line $L = P_7P_8$ is unique, so that $S_3(P_1, \dots, P_9)$ is the 1-dimensional space spanned by QL , and hence $\dim S_3(P_1, \dots, P_8) \leq 2$. Q.E.D.

Corollary 2.7 Let C_1, C_2 be two cubic curves whose intersection consists of 9 distinct points, $C_1 \cap C_2 = \{P_1, \dots, P_9\}$. Then a cubic D through P_1, \dots, P_8 also passes through P_9 .

Proof If 4 of the points P_1, \dots, P_8 were on a line L , then each of C_1 and C_2 would meet L in ≥ 4 points, and thus contain L , which contradicts the assumption on $C_1 \cap C_2$. For exactly the same reason, no 7 of the points can be conconic. Therefore the assumptions of (2.6) are satisfied, so I can conclude that

$$\dim S_3(P_1, \dots, P_8) = 2;$$

this means that the equations F_1, F_2 of the two cubics C_1, C_2 form a basis of $S_3(P_1, \dots, P_8)$, and hence $D : (G = 0)$, where $G = \lambda F_1 + \mu F_2$. Now F_1, F_2 vanish at P_9 , hence so does G . Q.E.D.

2.8 Group law on a plane cubic

Suppose $k \subset \mathbb{C}$ is a subfield of \mathbb{C} , and $F \in k[X, Y, Z]$ a cubic form defining a (nonempty) plane curve $C : (F = 0) \subset \mathbb{P}_k^2$. Assume that F satisfies the following two conditions:

- (a) F is irreducible (so that C does not contain a line or conic);
- (b) for every point $P \in C$, there exists a unique line $L \subset \mathbb{P}_k^2$ such that P is a repeated zero of $F|L$.

Note that geometrically, the condition in (b) is that C should be nonsingular, and the line L referred to is the tangent line $L = T_P C$ (see Ex. 2.3). This will be motivation for the general definition of nonsingularity and tangent spaces to a variety in §6.

Fix any point $O \in C$, and make the following construction:

Construction

- (i) For $A \in C$, let $\bar{A} = 3rd$ point of intersection of C with the line OA ;
- (ii) for $A, B \in C$, write $R = 3rd$ point of intersection of AB with C , and define $A + B$ by $A + B = \bar{R}$ (see diagram below).

Theorem The above construction defines an Abelian group law on C , with O as zero (= neutral element).