Proof Let $r = \min\{\dim T_P V\}$, taken over all points $P \in V$. Then clearly

$$S(r-1) = \emptyset, \quad S(r) = V, \quad \text{and} \quad S(r+1) \subsetneq V;$$

therefore $S(r) \setminus S(r+1) = \{P \in V \mid \dim T_P V = r\}$ is open and nonempty. Q.E.D.

6.7 $\dim V = \operatorname{tr}\deg k(V)$ – the hypersurface case

It follows from Proposition 6.3 that if $V = V(f) \subset \mathbb{A}^n$ is a hypersurface defined by some nonconstant polynomial f, then $\dim V = n-1$. On the other hand, for a hypersurface, $k[V] = k[X_1, \dots, X_n]/(f)$, so that, assuming that f involves X_1 in a nontrivial way, the function field of V is of the form

$$k(V) = k(X_2, \dots, X_n)[X_1]/(f),$$

that is, it is built up from k by adjoining $n-1$ algebraically independent elements, then making a primitive algebraic extension.

Definition If $k \subset K$ is a field extension, the *transcendence degree* of K over k is the maximum number of elements of K algebraically independent over k. It is denoted $\operatorname{tr}\deg_k K$.

The elementary theory of transcendence degree of a field extension K/k is formally quite similar to that of the dimension of a vector space: given $\alpha_1, \dots, \alpha_m \in K$, we know what it means for them to be *algebraically independent* over k (see (3.13)); they *span* the transcendental part of the extension if $K/k(\alpha_1, \dots, \alpha_m)$ is algebraic; and they form a *transcendence basis* if they are algebraically independent and span. Then it is an easy theorem that a transcendence basis is a maximal algebraically independent set, and a minimal spanning set, and that any two transcendence bases of K/k have the same number of elements (see Ex. 6.1).

Thus for a hypersurface $V \subset \mathbb{A}^n$, $\dim V = n-1 = \operatorname{tr}\deg_k k(V)$. The rest of this section is concerned with proving that the equality $\dim V = \operatorname{tr}\deg_k k(V)$ holds for all varieties, by reducing to the case of a hypersurface. The first thing to show is that for a point $P \in V$ of a variety, the tangent space $T_P V$, which so far has been discussed in terms of a particular coordinate system in the ambient space $\mathbb{A}^n$, is in fact an intrinsic property of a neighbourhood of $P \in V$.

6.8 Intrinsic nature of $T_P V$

From now on, given $P = (a_1, \dots, a_n) \in V \subset \mathbb{A}^n$, I take new coordinates $X_i' = X_i - a_i$ to bring P to the origin, and thus assume that $P = (0, \dots, 0)$. Then $T_P V \subset \mathbb{A}^n$ is a vector subspace of k^n.

Notation Write $m_P =$ ideal of P in $k[V]$, and

$$M_P = \text{the ideal } (X_1, \dots, X_n) \subset k[X_1, \dots, X_n].$$

Then of course $m_P = M_P/I(V) \subset k[V]$.

Theorem *In the above notation,*