Chapter 0

Woffle

This section is intended as a cultural introduction, and is not *logically* part of the course, so just skip through it.

0.1 What it's about

A variety is (roughly) a locus defined by polynomial equations:

$$V = \{P \in k^n \mid f_i(P) = 0\} \subset k^n,$$

where k is a field and $f_i \in k[X_1, \ldots, X_n]$ are polynomials; so for example, the plane curves $C : (f(x, y) = 0) \subset \mathbb{R}^2$ or $\mathbb{C}^2$.

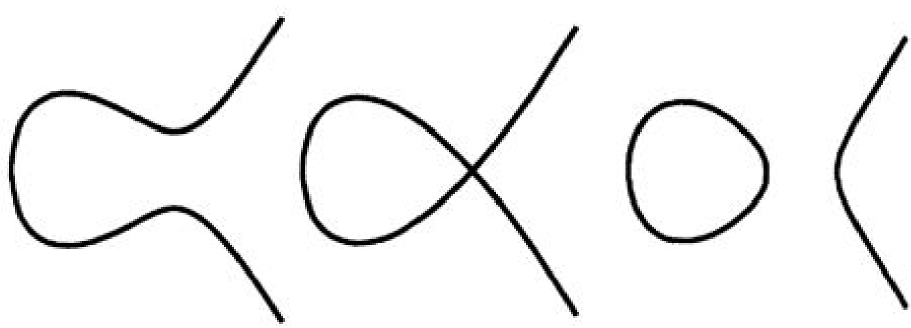

Figure 1: The cubic curves (a) $y^2 = (x+1)(x^2+\varepsilon)$, (b) $y^2 = (x+1)x^2$, and (c) $y^2 = (x+1)(x^2-\varepsilon)$.

I want to study V; several questions present themselves:

Number Theory For example, if $k = \mathbb{Q}$ and $V \subset \mathbb{Q}^n$, how can we tell if V is nonempty, or find all its points if it is? A specific case is historically of some significance: how many solutions are there to

$$x^n + y^n = 1, \quad \text{with} \quad x, y \in \mathbb{Q} \quad \text{and} \quad n \geq 3?$$

Questions of this kind are generally known as *Diophantine problems*.

Topology If k is $\mathbb{R}$ or $\mathbb{C}$ (which it quite often is), what kind of topological space is V? For example, the connected components of the above cubics are obvious topological invariants.

Singularity theory What kind of topological space is V near $P \in V$; if $f\colon V_1 \to V_2$ is a regular map between two varieties (for example, a polynomial map $\mathbb{R}^2 \to \mathbb{R}$), what kind of topology and geometry does f have near $P \in V_1$?

0.2 Specific calculations versus general theory

There are two possible approaches to studying varieties:

Particular Given specific polynomials f_i, we can often understand the variety V by explicit tricks with the f_i; this is fun if the dimension n and the degrees of the f_i are small, or the f_i are specially nice, but things get progressively more complicated, and there rapidly comes a time when mere ingenuity with calculations doesn't tell you much about the problem.

General The study of properties of V leads at once to basic notions such as regular functions on V, nonsingularity and tangent planes, the dimension of a variety: the idea that curves such as the above cubics are 1-dimensional is familiar from elementary Cartesian geometry, and the pictures suggest at once what singularity should mean.

Now a basic problem in giving an undergraduate algebraic geometry course is that an adequate treatment of the 'general' approach involves so many definitions that they fill the entire course and squeeze out all substance. Therefore one has to compromise, and my solution is to cover a small subset of the general theory, with constant reference to specific examples. These notes therefore contain only a fraction of the 'standard bookwork' which would form the compulsory core of a 3–year undergraduate math course devoted entirely to algebraic geometry. On the other hand, I hope that each section contains some exercises and worked examples of substance.

0.3 Rings of functions and categories of geometry

The specific flavour of algebraic geometry comes from the use of only polynomial functions (together with rational functions); to explain this, if $U \subset \mathbb{R}^2$ is an open interval, one can reasonably consider the following rings of functions on U:

- $C^0(U)$ = all continuous functions $f\colon U \to \mathbb{R}$;
- $C^\infty(U)$ = all smooth functions (that is, differentiable to any order);
- $C^\omega(U)$ = all analytic functions (that is, convergent power series);
- $\mathbb{R}[X]$ = the polynomial ring, viewed as polynomial functions on U.

There are of course inclusions $\mathbb{R}[X] \subset C^\omega(U) \subset C^\infty(U) \subset C^0(U)$.

These rings of functions correspond to some of the important categories of geometry: $C^0(U)$ to the topological category, $C^\infty(U)$ to the differentiable category (differentiable manifolds), C^ω to real analytic geometry, and $\mathbb{R}[X]$ to algebraic geometry. The point I want to make here is that