Proof Try it and see.

Definition *An ideal $I \subset k[X_0, \dots, X_n]$ is* homogeneous *if for all $f \in I$, the homogeneous decomposition $f = f_0 + f_1 + \dots + f_N$ of f satisfies $f_i \in I$ for all i.*

It is equivalent to say that I is generated by (finitely many) homogeneous polynomials.

5.2 The homogeneous V-I correspondences

Let $\mathbb{P}^n_k$ be n-dimensional projective space over a field k, with $X_0, \dots, X_n$ as homogeneous coordinates. Then $f \in k[X_0, \dots, X_n]$ is *not* a function on $\mathbb{P}^n_k$: by definition, $\mathbb{P}^n_k = k^{n+1} \setminus \{0\}/{\sim}$, where $\sim$ is the equivalence relation given by $(X_0, \dots, X_n) {\sim} (\lambda X_0, \dots, \lambda X_n)$ for $\lambda \in k \setminus \{0\}$; f is a function on k^{n+1}. Nevertheless, for $P \in \mathbb{P}^n$, the condition $f(P) = 0$ is well defined provided that f is homogeneous: suppose $P = (X_0 : \dots : X_n)$, so that $(X_0, \dots, X_n)$ is a representative in $k^{n+1} \setminus \{0\}$ of the equivalence class of P. Then since $f(\lambda X) = \lambda^d f(X)$, if $f(X_0, \dots, X_n) = 0$ then also $f(\lambda X_0, \dots, \lambda X_n) = 0$, so that the condition $f(P) = 0$ is independent of the choice of representative. With this in mind, define as before correspondences

$$\left\{\text{homog. ideals } J \subset k[X_0, \dots, X_n]\right\} \xleftrightarrow{V\text{-}I} \left\{\text{subsets } X \subset \mathbb{P}^n_k\right\}$$

by

$$V(J) = \left\{P \in \mathbb{P}^n_k \;\middle|\; f(P) = 0 \ \forall \text{ homogeneous } f \in J\right\}$$

and

$$I(X) = \left\{f \in k[X_0, \dots, X_n] \;\middle|\; f(P) = 0 \text{ for all } P \in X\right\}.$$

As an exercise, check that you understand why $I(X)$ is a homogeneous ideal.

The correspondences V and I satisfy the same formal properties as the affine V and I correspondences introduced in §3 (for example $V(J_1 + J_2) = V(J_1) \cap V(J_2)$). A subset of the form $V(I)$ is an *algebraic subset* of $\mathbb{P}^n_k$, and as in the affine case, $\mathbb{P}^n_k$ has a *Zariski topology* in which the closed sets are the algebraic subsets.

5.3 Projective Nullstellensatz

As with the affine correspondences, it is purely formal that $I(V(J)) \supset \operatorname{rad} J$ for any ideal J, and that for an algebraic set, $V(I(X)) = X$. There's just one point where care is needed: the trivial ideal $(1) = k[X_0, \dots, X_n]$ (the whole ring) defines the empty set in k^{n+1}, hence also in $\mathbb{P}^n_k$, which is as it should be; however, the ideal $(X_0, \dots, X_n)$ defines $\{0\}$ in k^{n+1}, which also corresponds to the empty set in $\mathbb{P}^n_k$. The ideal $(X_0, \dots, X_n)$ is an awkward (empty-set theoretical) exception to several statements in the theory, and is traditionally known as the 'irrelevant ideal'.

The homogeneous version of the Nullstellensatz thus becomes:

Theorem *Assume that k is an algebraically closed field. Then*

(i) $V(J) = \emptyset \iff \operatorname{rad} J \supset (X_0, \dots, X_n)$;

(ii) if $V(J) \neq \emptyset$ *then* $I(V(J)) = \operatorname{rad} J$.

Corollary *I and V determine inverse bijections*

$$\left\{\begin{array}{c}\textit{homogeneous radical}\\ \textit{ideals } J \subset k[x_0,\dots,x_n]\\ \textit{with } J \not\supset (x_0,\dots,x_n)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{algebraic subsets}\\ X \subset \mathbb{P}^n\end{array}\right\}$$

$$\bigcup \qquad\qquad\qquad\qquad \bigcup$$

$$\left\{\begin{array}{c}\textit{homogeneous prime}\\ \textit{ideals } J \subset k[x_0,\dots,x_n]\\ \textit{with } J \not\supset (x_0,\dots,x_n)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{irreducible algebraic}\\ \textit{subsets } X \subset \mathbb{P}^n\end{array}\right\}$$

Proof Let $\pi\colon \mathbb{A}^{n+1}\setminus\{0\} \to \mathbb{P}^n$ be the map defining $\mathbb{P}^n$. For a homogeneous ideal $J \subset k[X_0,\dots,X_n]$, write (in temporary notation) $V^a(J) \subset \mathbb{A}^{n+1}$ for the affine algebraic set defined by J. Then since J is homogeneous, $V^a(J)$ has the property

$$(\alpha_0,\dots,\alpha_n) \in V^a(J) \iff (\lambda\alpha_0,\dots,\lambda\alpha_n) \in V^a(J),$$

and $V(J) = V^a(J) \setminus \{0\}/{\sim} \subset \mathbb{P}^n$. Hence

$$V(J) = \emptyset \iff V^a(J) \subset \{0\} \iff \operatorname{rad} J \supset (X_0,\dots,X_n),$$

where the last implication uses the affine Nullstellensatz. Also, if $V(J) \neq \emptyset$ then

$$f \in I(V(J)) \iff f \in I(V^a(J)) \iff f \in \operatorname{rad} J. \quad \text{Q.E.D.}$$

The affine subset $V^a(J)$ occurring above is called the *affine cone* over the projective algebraic subset $V(J)$.

5.4 Rational functions on V

Let $V \subset \mathbb{P}^n_k$ be an irreducible algebraic set, and $I(V) \subset k[X_0,\dots,X_n]$ its ideal; there is no direct way of defining regular functions on V in terms of polynomials: an element $F \in k[X_0,\dots,X_n]$ gives a function on the affine cone over V, but (by case $d = 0$ of Proposition 5.1) this will be constant on equivalence classes only if F is homogeneous of degree 0, that is, a constant. So from the start, I work with rational functions only:

Definition A *rational function* on V is a (partially defined) function $f\colon V \dashrightarrow k$ given by $f(P) = g(P)/h(P)$, where $g, h \in k[X_0,\dots,X_n]$ are homogeneous polynomials of the same degree d.

Note here that provided $h(P) \neq 0$, the quotient $g(P)/h(P)$ is well defined, since

$$g(\lambda\underline{X})/h(\lambda\underline{X}) = \lambda^d g(\underline{X})/\lambda^d h(\underline{X}) = g(\underline{X})/h(\underline{X}) \quad \text{for } 0 \neq \lambda \in k.$$

Now obviously g/h and g'/h' define the same rational function on V if and only if $h'g - g'h \in I(V)$, so that the set of all rational functions is the field

$$k(V) = \left\{ \frac{g}{h} \;\middle|\; \begin{array}{l} g, h \in k[X_0,\dots,X_n] \text{ homogeneous}\\ \text{of the same degree, and } h \notin I(V)\end{array} \right\} /{\sim},$$