with A a linear form, B a quadratic form. Then $S : (f = 0)$ is singular at a point where $Z = T = B = 0$; this is a nonempty set, since it is the set of roots of B on the line $\ell : (Z = T = 0)$.

Proposition 7.2 *There exists at least one line ℓ on S.*

There are several approaches to proving this. A standard argument is by a dimension count: lines of $\mathbb{P}^3$ are parametrised by a 4-dimensional variety, and for a line ℓ to lie on S imposes 4 conditions on ℓ (because the restriction of f to ℓ is a cubic form, the 4 coefficients of which must vanish). A little work is needed to turn this into a rigorous proof, since a priori it shows only that the set of lines has dimension ≥ 0, and not that it is nonempty (see the highbrow notes (8.15) for a discussion of the traditional proof and the difficulties involved in it).

It is also perfectly logical to assume the proposition (restrict attention only to cubic surfaces containing lines). I now explain how (7.2) can be proved by direct coordinate geometry and elimination. The proof occupies the next 3 pages, and divides up into 4 steps; you can skip it if you prefer (GOTO 7.3).

Step 1 (Preliminary construction) For any point $P \in S$, the intersection of S with the tangent plane $T_P S$ is a plane cubic $C = S \cap T_P S$, which by Ex. 6.7 is singular at P. I assume that C is irreducible, since otherwise P is on a line of S, and I'm home; then C is a nodal or cuspidal cubic, and the coordinates (X, Y, Z, T) of $\mathbb{P}^3$ can be chosen such that $T_P S : (T = 0)$, $P = (0, 0, 1, 0)$, and

$$C : (XYZ = X^3 + Y^3) \text{ or } (X^2 Z = Y^3).$$

Whether C is nodal or cuspidal for given $P \in S$ depends on the matrix of second derivatives (or *Hessian* matrix) of f at P; this is discussed in more detail in Ex. 7.3, which proves (in characteristic $\neq 2$) that the cuspidal case must occur for some point $P \in S$. For simplicity, I prove (7.2) in the cuspidal case; in principle, the proof goes through in exactly the same way in the nodal case, but the elimination calculation gets much nastier (see Ex. 7.10). Thus assume that

$$f = X^2 Z - Y^3 + gT,$$

where $g = g_2(X, Y, Z, T)$ is a quadratic form; $g(0, 0, 1, 0) \neq 0$ by nonsingularity of S at P, so I can assume that $g(0, 0, 1, 0) = 1$.

Step 2 (Statement of main claim) Consider the variable point $P_\alpha = (1, \alpha, \alpha^3, 0)$ of $C \subset S$. Any line of $\mathbb{P}^3$ through P_α meets the complementary plane $\Pi : (X = 0)$ in a point $Q = (0, Y, Z, T)$. I write out the equations for the line $P_\alpha Q$ to be contained in S in terms of α and Q; expanding $f(\lambda P_\alpha + \mu Q)$ in powers of λ and μ gives

$$P_\alpha Q \subset S \iff A(Y, Z, T) = B(Y, Z, T) = C(Y, Z, T) = 0,$$

where A, B and C are forms of degree 1, 2 and 3 in (Y, Z, T), whose coefficients involve α.

Main Claim *There exists a 'resultant' polynomial $R_{27}(\alpha)$, which is monic of degree 27 in α, such that*

$$R(\alpha) = 0 \iff A = B = C = 0 \textit{ have a common zero } (\eta : \zeta : \tau) \textit{ in } \mathbb{P}^2.$$

This statement proves (7.2), since it implies that for every root α of R, there exists a point $Q = (0 : \eta : \zeta : \tau)$ in Π for which the line $P_\alpha Q$ is contained in S. The idea here is a standard elimination calculation based on Ex. 1.10; the rest of the proof is concerned with writing out A, B and C explicitly to prove the claim.

Step 3 (Polar form) Define the *polar* of f to be the form in two sets of variables (X, Y, Z, T) and (X', Y', Z', T') given by

$$f_1(X, Y, Z, T; X', Y', Z', T') = \frac{\partial f}{\partial X} \cdot X' + \frac{\partial f}{\partial Y} \cdot Y' + \frac{\partial f}{\partial Z} \cdot Z' + \frac{\partial f}{\partial T} \cdot T'.$$

It's clear from the definition of tangent space (see (6.4) and (6.10)) that for $P = (X, Y, Z, T) \in S$ and $P \neq Q = (X', Y', Z', T') \in \mathbb{P}^3$,

$$f_1(P; Q) = 0 \iff \text{the line } PQ \text{ is tangent to } S \text{ at } P.$$

Clearly

$$f(\lambda P + \mu Q) = \lambda^3 f(P) + \lambda^2 \mu f_1(P; Q) + \lambda \mu^2 f_1(Q; P) + \mu^3 f(Q),$$

so that for $P \neq Q \in \mathbb{P}^3$, the 4 conditions

$$f(P) = f_1(P; Q) = f_1(Q; P) = f(Q)$$

are the equations for the line $\ell = PQ$ to be contained in $S : (f = 0)$. More geometrically, these say that ℓ is tangent to S at both P and Q, so that $f|\ell$ has double roots at both points, and then $\ell \subset S$ follows from Proposition 1.8.

The polar of $f = X^2 Z - Y^3 + gT$ is

$$f_1 = 2XZ \cdot X' - 3Y^2 \cdot Y' + X^2 \cdot Z' + g(X, Y, Z, T) \cdot T' + T g_1.$$

Here $g_1 = g_1(X, Y, Z, T; X', Y', Z', T')$ is the polar form of g defined in the same way as above; since g is quadratic, g_1 is a symmetric bilinear form such that $g_1(P, P) = 2g(P)$.

Substituting $P_\alpha = (1, \alpha, \alpha^3, 0)$ and $Q = (0, Y, Z, T)$ gives the equations for $P_\alpha Q \subset S$ as $A = B = C = 0$, where

$$\begin{aligned} A &= Z - 3\alpha^2 Y + g(1, \alpha, \alpha^3, 0)T, \\ B &= -3\alpha Y^2 + g_1(1, \alpha, \alpha^3, 0; 0, Y, Z, T)T, \\ C &= -Y^3 + g(0, Y, Z, T)T. \end{aligned}$$

Step 4 (Elimination calculation) I now eliminate Y, Z, T from the above 3 equations, paying attention to the highest powers of α occurring. Note that since $g(0, 0, 1, 0) = 1$, it follows that

$$g(1, \alpha, \alpha^3, 0) = \alpha^6 + \cdots = a^{(6)},$$

where $\ldots$ denotes terms of lower degree in α; thus $a^{(6)}$ is monic of degree 6. Then $A = 0$ gives Z as a linear form in Y and T,

$$Z = 3\alpha^2 Y - a^{(6)} T.$$

Substituting in B, and using the bilinearity of g_1 gives

$$\begin{aligned} B &= -3\alpha Y^2 + g_1(1, \alpha, \alpha^3, 0; 0, Y, 3\alpha^2 Y - a^{(6)} T, T)T \\ &= b_0 Y^2 + b_1 YT + b_2 T^2, \end{aligned}$$

where

$$b_0 = -3\alpha, \quad b_1 = g_1(1, \alpha, \alpha^3, 0; 0, 1, 3\alpha^2, 0) = 6\alpha^5 + \cdots,$$
$$b_2 = g_1(1, \alpha, \alpha^3, 0; 0, 0, -a(6), 1) = -2\alpha^9 + \cdots.$$

Similarly, substituting for Z in C, and expanding the quadratic form g gives

$$C = -Y^3 + g(0, Y, 3\alpha^2 Y - a^{(6)}T, T)T = c_0 Y^3 + c_1 Y^2 T + c_2 Y T^2 + c_3 T^3,$$

where

$$c_0 = -1, \quad c_1 = g(0, 1, 3\alpha^2, 0) = 9\alpha^4 + \cdots,$$
$$c_2 = g_1(0, 1, 3\alpha^2, 0; 0, 0, -a(6), 1) = -6\alpha^8 + \cdots,$$
$$c_3 = g(0, 0, -a(6), 1) = \alpha^{12} + \cdots.$$

Now by the result of Ex. 1.10, B' and C' have a common zero $(\eta : \tau)$ if and only if

$$\det \begin{vmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{vmatrix} = 0.$$

The determinant is a polynomial in α, and it's not hard to see that its leading term comes from taking the leading term in each entry of the determinant:

$$\det \begin{vmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{vmatrix} = \alpha^{27} \cdot \det \begin{vmatrix} -3 & 6 & 2 & & \\ & -3 & 6 & 2 & \\ & & -3 & 6 & 2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{vmatrix}$$
$$= \alpha^{27}.$$

This completes the proof of the main claim. Q.E.D.

Proposition 7.3 *Given a line $\ell \subset S$, there exist exactly 5 pairs (ℓ_i, ℓ_i') of lines of S meeting ℓ, in such a way that*

(i) for $i = 1, \ldots, 5$, $\ell \cup \ell_i \cup \ell_i'$ is coplanar, and

(ii) for $i \neq j$, $(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \emptyset$.