Taken together, these results identify affine varieties V with the affine schemes corresponding to geometric rings (compare also Definition 4.6).

The *prime spectrum* $\operatorname{Spec} A$ is defined for an arbitrary ring (commutative with a 1) as the set of prime ideals of A. It has a Zariski topology and a structure sheaf; this is the *affine scheme* corresponding to A (for details see [Mumford, Introduction, or Hartshorne, Ch. II]). There are several quite distinct ways in which affine schemes are more general than affine varieties; each of these is important, and I run through them briefly in (8.14).

It's important to understand that for a geometric ring $A = k[V]$, the prime spectrum $\operatorname{Spec} A$ contains exactly the same information as the variety V, and no more. The NSS tells us there's a plentiful supply of maximal ideals (m_v for points $v \in V$), and every other prime P of A is the intersection of maximal ideals over the points of an irreducible subvariety $Y \subset V$:

$$P = I(Y) = \bigcap_{v \in Y} m_v;$$

It's useful and (roughly speaking, at least) permissible to ignore the distinction between varieties and schemes, writing $V = \operatorname{Spec} A$, v for m_v, and imagining the prime $P = I(Y)$ ('generic point') as a kind of laundry mark stitched everywhere dense into the fabric of the subvariety Y.

8.13 What's the point?

A majority of students will never need to know any more about scheme theory than what is contained in (8.9) and (8.12), beyond the warning that the expression *generic point* is used in several technical senses, often meaning something quite different from *sufficiently general* point.

This section is intended for the reader who faces the task of working with the modern literature, and offers some comments on the various notions of point in scheme theory, potentially a major stumbling block for beginners.

(a) Scheme theoretic points of a variety

Suppose that k is a field (possibly not algebraically closed), and $A = k[X_1, \dots, X_n]/I$ with $I \subset k[X_1, \dots, X_n]$ an ideal; write $V = V(I) \subset K^n$ where $k \subset K$ is a chosen algebraic closure. The points of $\operatorname{Spec} A$ are only a bit more complicated than for a geometric ring in (8.12). By an obvious extension of the NSS, a maximal ideal of A is determined by a point $v = (a_1, \dots, a_n) \in V \subset K^n$, that is, it's of the form

$$m_v = \{f \in A \mid f(P) = 0\} = (x_1 - a_1, \dots, x_n - a_n) \cap A.$$

It's easy to see that different points $v \in V \subset K^n$ give rise to the same maximal ideal m_v of A if and only if they are conjugate over k in the sense of Galois theory (since A consists of polynomials with coefficients in k). So the maximal spectrum $\operatorname{Specm} A$ is just V 'up to conjugacy' (the orbit space of $\operatorname{Gal} K/k$ on V). Every other prime P of A corresponds as in (8.12) to an irreducible subvariety $Y = V(P) \subset V$ (up to conjugacy over k); $P \in \operatorname{Spec} A$ is the scheme theoretic *generic point* of Y, and is again to be thought of as a laundry mark on Y. The Zariski topology of $\operatorname{Spec} A$ is fixed up so that P is everywhere dense in Y. The maximal ideals of A are called *closed points* to distinguish them. If $C : (f = 0) \subset \mathbb{A}^2_{\mathbb{C}}$ is an irreducible curve, it has just one scheme theoretic generic point, corresponding to the ideal (0) of $\mathbb{C}[X,Y]/(f)$, whereas a surface S will have one generic point in each irreducible curve $C \subset S$ as well as its own generic point dense in S.

Scheme theoretic points are crucial in writing down the definition of $\operatorname{Spec} A$ as a set with a topology and a sheaf of rings (and are also important in commutative algebra, and in the treatment in algebraic geometry of notions like the neighbourhood of a generic point of an irreducible subvariety, see (8.14, i)); however, points of $V \subset K^n$ with values in the algebraic closure $k \subset K$ correspond more to the geometric idea of a point, and are called *geometric points*. This is similar to the way that the Zariski topology of a variety V serves more as a vehicle for the structure sheaf $\mathcal{O}_V$ than as a geometric object in its own right.

(b) Field-valued points in scheme theory

If P is a prime ideal of A (so $P \in \operatorname{Spec} A$ a point) the residue field at P is the field of fractions of the integral domain A/P, written $k(P)$; it is an algebraic extension of the ground field k if and only if P is maximal. A point of V with coefficients in a field extension $k \subset L$ (a point $(a_1, \dots, a_n) \in V(I) \subset L^n$) clearly corresponds to a homomorphism $A \to L$ (given by $X_i \mapsto a_i$), with kernel a prime ideal P of A, or equivalently, to an embedding $k(P) \hookrightarrow L$. If $P = m_v$ is a maximal ideal, and $L = K$ is the algebraic closure of k, it is the choice of the embedding $A/m_v = k(v) \hookrightarrow K$ that determines the coordinates of the corresponding point of $V \subset K^n$, or in other words, distinguishes this point from its Galois conjugates. These are the geometric points of V.

For any extension $k \subset L$, the k-algebra homomorphism $A \to L$ corresponding to an L-valued point of V can be dressed up to seem more reasonable. Recall first that a variety is more than a point set; even if it's only a single point, you have to say what field it's defined over. So

$$\operatorname{Spec} L = \frac{L}{\cdot} = \mathrm{pt}_L$$

is the variety consisting of a single point defined over L. By the equivalence of categories (4.4), a morphism $\operatorname{Spec} L \to V$ (the inclusion of a point defined over L) should be the same thing as a k-algebra homomorphism $A = k[V] \to L = k[\mathrm{pt}_L]$.

To summarise the relation between scheme theoretic points and field-valued points: a point $P \in \operatorname{Spec} A = V$ is a prime ideal of A, so corresponds to the quotient homomorphism $A \to A/P \subset \operatorname{Quot}(A/P) = k(P)$ to a field. If L is any field, an L-valued point of V is a homomorphism $A \to L$; a scheme theoretic point P corresponds in a tautological way to a field-valued point, but with the field $k(P)$ varying with P. If K is the algebraic closure of k then K-valued points of $V \subset K^n$ are just geometric points; a K-valued point v sits at a closed scheme theoretic point m_v, with a specified inclusion $A/m_v = k(v) \hookrightarrow K$.

(c) Generic points in Weil foundations

I mentioned in (8.3) the peculiarity of points in Weil foundations: a variety V defined over a field k is allowed to have L-valued points for any field extension $k \subset L$. This clearly derives from number theory, but it also has consequences in geometry. For example, if C is the circle $x^2 + y^2 = 1$ defined over $k = \mathbb{Q}$, then

$$P_\pi = \left(\frac{2\pi}{\pi^2 + 1}, \frac{\pi^2 - 1}{\pi^2 + 1)} \right)$$

is allowed as a $\mathbb{C}$-valued point of C. Since π is transcendental over $\mathbb{Q}$, any polynomial $f \in \mathbb{Q}[x,y]$ vanishing at P_π is a multiple of x^2+y^2-1; so P_π is a $\mathbb{Q}$-*generic* point of C – it's not in any smaller subvariety of C defined over $\mathbb{Q}$. In other words, the conjugates of P_π under $\operatorname{Aut}\mathbb{C}$ ("$=\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$") are dense in C. Since P_π is $\mathbb{Q}$-generic, if you prove a statement only involving polynomials over $\mathbb{Q}$ about P_π, the same statement will be true for every point of C.

In fact this idea is already covered by the notion of an L-valued point described in (b), and the geometric content of generic points can be seen most clearly in this language. For example, the field $\mathbb{Q}(\pi)$ is just the purely transcendental extension, so $\mathbb{Q}(\pi) \cong \mathbb{Q}(\lambda)$ and the morphism $\operatorname{Spec}\mathbb{Q}(\lambda) \to C$ is the rational parametrisation of C discussed in (1.1): roughly, you're allowed to substitute any 'sufficiently general' value for the transcendental or unknown π. More generally, a finitely generated extension $k \subset L$ is the function field of a variety W over k; suppose that $\varphi\colon \operatorname{Spec} L \to V = \operatorname{Spec} A$ is a point corresponding to a k-algebra homomorphism $A \to L$, having kernel P. Then φ extends to a rational map $f\colon W \dashrightarrow V$ whose image is dense in the subvariety $Y = V(P) \subset V$, so φ or $\varphi(\operatorname{Spec} L)$ is a field-valued generic point of Y.

(d) Points as morphisms in scheme theory

The discussion in (c) shows that an L-valued point of a variety V contains implicitly a rational map $W \dashrightarrow V$, where W is a variety birational to $\operatorname{Spec} L$ (that is, $L = k(W)$); a geometer could think of this as a family of points parametrised by W.

More generally, for X a variety (or scheme) we are interested in, an S-valued point of X (where S is any scheme) can just be defined as a morphism $S \to X$. If $X = V(I) \subset \mathbb{A}^n_k$ is affine with coordinate ring $k[X]$ and $S = \operatorname{Spec} A$, then an S-valued point corresponds under (4.4) to a k-algebra homomorphism $k[X] \to A$, that is, to an n-tuple $(a_1, \dots, a_n)$ of elements of A satisfying $f(a) = 0$ for all $f \in I$.

In a highbrow sense, this is the final apotheosis of the notion of a variety: if a point of a variety X is just a morphism, then X itself is just the functor

$$S \mapsto X(S) = \bigl\{\text{morphisms } S \to X\bigr\}$$

on the category of schemes. (The fuss I made about the notation $\mathbb{A}^n_k$ in the footnote on p. 59 already reflect this.) Unlikely as it may seem, these metaphysical incantations are technically very useful, and varieties defined as functors are basic in the modern view of moduli spaces. Given a geometric construction that can 'depend algebraically on parameters' (such as space curves of fixed degree and genus), you can ask to endow the set of all possible constructions with the structure of an algebraic variety. Even better, you could ask for a family of constructions over a parameter space that is 'universal', or 'contains all possible constructions'; the parameter variety of this universal family can usually be defined most directly as a functor (you still have to prove that the variety exists). For example the Chow variety referred to in (8.2) represents the functor

$$S \mapsto \bigl\{\text{families of curves parametrised by } S\bigr\}.$$

8.14 How schemes are more general than varieties

I now discuss in isolation 3 ways in which affine schemes are more general than affine varieties; in cases of severe affliction, these complications may occur in combination with each other, with