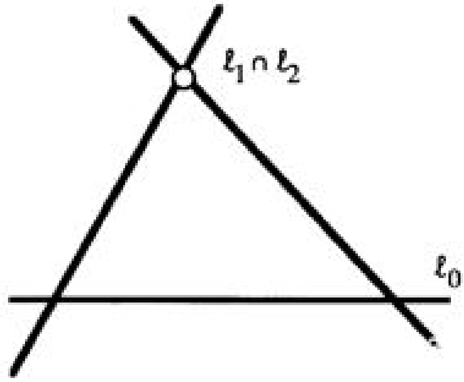



Figure 1.5: Lines meeting

thus 4 points out of $P_1, \dots, P_5$ lie on $L_0$, a contradiction. Q.E.D.

## 1.11 Space of all conics

Let

$$S_2 = \{\text{quadratic forms on } \mathbb{R}^3\} = \{3 \times 3 \text{ symmetric matrixes}\} \cong \mathbb{R}^6.$$

If $Q \in S_2$, write $Q = aX^2 + 2bXY + \cdots + fZ^2$; for $P_0 = (X_0, Y_0, Z_0) \in \mathbb{P}^2_{\mathbb{R}}$, consider the relation $P_0 \in C : (Q = 0)$. This is of the form

$$Q(X_0, Y_0, Z_0) = aX_0^2 + 2bX_0Y_0 + \cdots + fZ_0^2 = 0,$$

and for fixed $P_0$, this is a linear equation in $(a, b, \dots, f)$. So

$$S_2(P_0) = \bigl\{Q \in S_2 \bigm| Q(P_0) = 0\bigr\} \cong \mathbb{R}^5 \subset S_2 = \mathbb{R}^6$$

is a 5-dimensional hyperplane. For $P_1, \dots, P_n \in \mathbb{P}^2_{\mathbb{R}}$, define similarly

$$S_2(P_1, \dots, P_n) = \bigl\{Q \in S_2 \bigm| Q(P_i) = 0 \text{ for } i = 1, \dots, n\bigr\};$$

then there are $n$ linear equations in the 6 coefficients $(a, b, \dots, f)$ of $Q$. This gives the result:

**Proposition** $\dim S_2(P_1, \dots, P_n) \geq 6 - n$.

We can also expect that 'equality holds if $P_1, \dots, P_n$ are general enough'. More precisely:

**Corollary** *If $n \leq 5$ and no 4 of $P_1, \dots, P_n$ are collinear, then*

$$\dim S_2(P_1, \dots, P_n) = 6 - n.$$

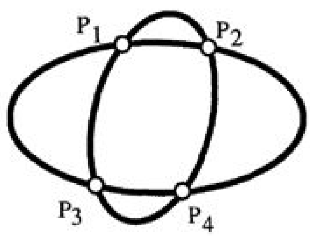
**Proof** Corollary 1.10 implies that if $n = 5$, $\dim S_2(P_1, \ldots, P_5) \le 1$, which gives the corollary in this case. If $n \le 4$, then I can add in points $P_{n+1}, \ldots, P_5$ while preserving the condition that no 4 points are collinear, and since each point imposes at most one linear condition, this gives

$$1 = \dim S_2(P_1, \ldots, P_5) \ge \dim S_2(P_1, \ldots, P_n) - (5 - n). \quad \text{Q.E.D.}$$

Note that if 6 points $P_1, \ldots, P_6 \in \mathbb{P}^2_{\mathbb{R}}$ are given, they may or may not lie on a conic.

## 1.12 Intersection of two conics

As we have seen above, it often happens that two conics meet in 4 points:




Conversely according to Corollary 1.11, given 4 points $P_1, \ldots, P_4 \in \mathbb{P}^2$, under suitable conditions $S_2(P_1, \ldots, P_4)$ is a 2-dimensional vector space, so choosing a basis $Q_1, Q_2$ for $S_2(P_1, \ldots, P_4)$ gives 2 conics $C_1, C_2$ such that $C_1 \cap C_2 = \{P_1, \ldots, P_4\}$. There are lots of possibilities for multiple intersections of nonsingular conics:

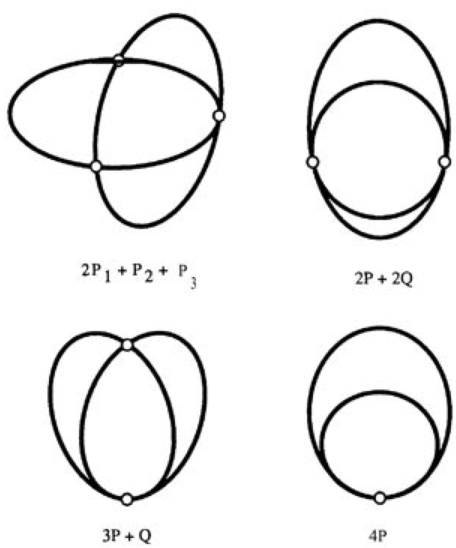



Figure 1.6: (a) $2P_1 + P_2 + P_3$; (b) $2P + 2Q$; (c) $3P + Q$; (d) $4P$

see Ex. 1.9 for suitable equations.