1.13 Degenerate conics in a pencil

Definition A *pencil of conics* is a family of the form

$$C_{(\lambda,\mu)} : (\lambda Q_1 + \mu Q_2 = 0);$$

each element is a plane curve, depending in a linear way on the parameters (λ, μ); think of the ratio $(\lambda : \mu)$ as a point of $\mathbb{P}^1$.

Looking at the examples, one expects that for special values of $(\lambda : \mu)$ the conic $C_{(\lambda,\mu)}$ is degenerate. In fact, writing $\det(Q)$ for the determinant of the symmetric 3×3 matrix corresponding to the quadratic form Q, it is clear that

$$C_{(\lambda,\mu)} \text{ is degenerate} \iff \det(\lambda Q_1 + \mu Q_2) = 0.$$

Writing out Q_1 and Q_2 as symmetric matrixes expresses this condition as

$$F(\lambda,\mu) = \det \left| \lambda \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} + \mu \begin{pmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{pmatrix} \right| = 0.$$

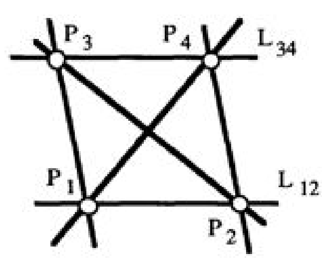
Now notice that $F(\lambda,\mu)$ is a homogeneous cubic form in λ, μ. In turn I can apply (1.8) to F to deduce:

Proposition *Suppose $C_{(\lambda,\mu)}$ is a pencil of conics of $\mathbb{P}^2_k$, with at least one nondegenerate conic (so that $F(\lambda,\mu)$ is not identically zero). Then the pencil has at most 3 degenerate conics. If $k = \mathbb{R}$ then the pencil has at least one degenerate conic.*

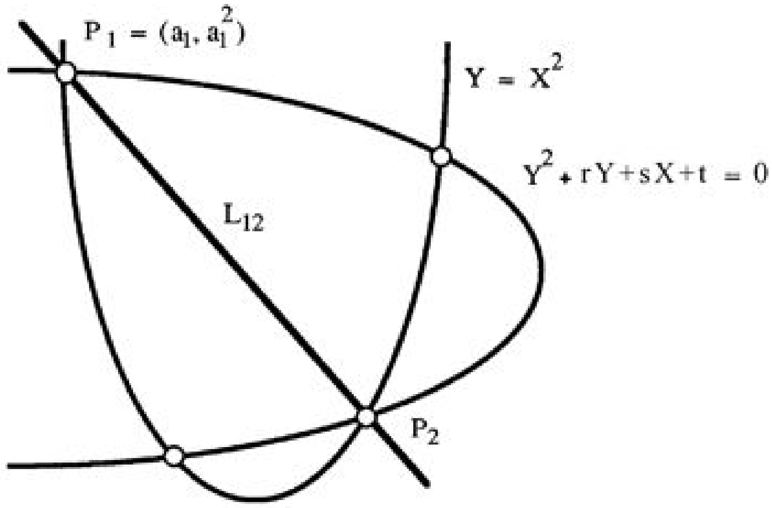
Proof A cubic form has ≤ 3 zeros. Also over $\mathbb{R}$, it must have at least one zero.

1.14 Worked example

Let $P_1, \dots, P_4$ be 4 points of $\mathbb{P}^2_{\mathbb{R}}$ such that no 3 are collinear; then the pencil of conics $C_{(\lambda,\mu)}$ through $P_1, \dots, P_4$ has 3 degenerate elements, namely the line pairs $L_{12} + L_{34}, L_{13} + L_{24}, L_{14} + L_{23}$, where L_{ij} is the line through P_i, P_j:



Next, suppose that I start from the pencil of conics generated by $Q_1 = Y^2 + rY + sX + t$ and $Q_2 = Y - X^2$, and try to find the points $P_1, \dots, P_4$ of intersection.



This can be done as follows: (1) find the 3 ratios $(\lambda : \mu)$ for which $C_{(\lambda,\mu)}$ are degenerate conics. Using what has been said above, this just means that I have to find the 3 roots of the cubic

$$F(\lambda, \mu) = \det \left| \lambda \begin{pmatrix} 0 & 0 & s/2 \\ 0 & 1 & r/2 \\ s/2 & r/2 & t \end{pmatrix} + \mu \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \right|$$

$$= -\frac{1}{4}(s^2\lambda^3 + (4t - r^2)\lambda^2\mu - 2r\lambda\mu^2 - \mu^3).$$

(2) Separate out 2 of the degenerate conics into pairs of lines (this involves solving 2 quadratic equations). (3) The 4 points P_i are the points of intersection of the lines.

This procedure gives a geometric interpretation of the reduction of the general quartic in Galois theory (see for example [van der Waerden, Algebra, Ch. 8, §64]): let k be a field, and $f(X) = X^4 + rX^2 + sX + t \in k[X]$ a quartic polynomial. Then the two parabolas C_1 and C_2 meet in the 4 points $P_i = (a_i, a_i^2)$ for $i = 1, \dots, 4$, where the a_i are the 4 roots of f.

Then the line $L_{ij} = P_iP_j$ is given by

$$L_{ij} : (Y = (a_i + a_j)X - a_ia_j),$$

and the reducible conic $L_{12} + L_{34}$ is given by

$$Y^2 + (a_1a_2 + a_3a_4)Y + (a_1 + a_2)(a_3 + a_4)X^2 + sX + t = 0,$$

that is, by $Q_1 - (a_1 + a_2)(a_3 + a_4)Q_2 = 0$. Hence the 3 values of μ/λ for which the conic $\lambda Q_1 + \mu Q_2$ breaks up as a line pair are

$$-(a_1 + a_2)(a_3 + a_4), \quad -(a_1 + a_3)(a_2 + a_4), \quad -(a_1 + a_4)(a_2 + a_3).$$

The cubic equation whose roots are these 3 quantities is called the *auxiliary cubic* associated with the quartic; it can be calculated using the theory of elementary symmetric functions; this is a fairly laborious procedure. On the other hand, the geometric method sketched above gives an elegant derivation of the auxiliary cubic which only involves evaluating a 3×3 determinant.

The above treatment is taken from [M.Berger, 16.4.10 and 16.4.11.1].

Exercises to Chapter 1

1.1 Parametrise the conic $C : (x^2+y^2 = 5)$ by considering a variable line through $(2,1)$ and hence find all rational solutions of $x^2 + y^2 = 5$.

1.2 Let p be a prime; by experimenting with various p, guess a necessary and sufficient condition for $x^2+y^2 = p$ to have rational solutions; prove your guess (a hint is given after Ex. 1.9 below – bet you can't do it for yourself!).

1.3 Prove the statement in (1.3), that an affine transformation can be used to put any conic of $\mathbb{R}^2$ into one of the standard forms (a–l). [Hint: use a linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ to take the leading term $ax^2 + bxy + cy^2$ into one of $\pm x^2 \pm y^2$ or $\pm x^2$ or 0; then complete the square in x and y to get rid of as much of the linear part as possible.]

1.4 Make a detailed comparison of the affine conics in (1.3) with the projective conics in (1.6).

1.5 Let k be any field of characteristic $\neq 2$, and V a 3-dimensional k-vector space; let $Q\colon V \to k$ be a nondegenerate quadratic form on V. Show that if $0 \neq e_1 \in V$ satisfies $Q(e_1) = 0$ then V has a basis e_1, e_2, e_3 such that $Q(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_3 + ax_2^2$. [Hint: work with the symmetric bilinear form φ associated to Q; since φ is nondegenerate, there is a vector e_3 such that $\varphi(e_1, e_3) = 1$. Now find a suitable e_2.]

Deduce that a nonempty, nondegenerate conic $C \subset \mathbb{P}^2_k$ is projectively equivalent to $(XZ = Y^2)$.

1.6 Let k be a field with at least 4 elements, and $C : (XZ = Y^2) \subset \mathbb{P}^2_k$; prove that if $Q(X,Y,Z)$ is a quadratic form which vanishes on C then $Q = \lambda(XZ - Y^2)$. [Hint: if you really can't do this for yourself, compare with the argument in the proof of Lemma 2.5.]

1.7 In $\mathbb{R}^3$, consider the two planes $A : (Z = 1)$ and $B : (X = 1)$; a line through 0 meeting A in $(x, y, 1)$ meets B in $(1, y/x, 1/x)$. Consider the map $\varphi\colon A \dashrightarrow B$ defined by $(x,y) \mapsto (y' = y/x, z' = 1/x)$; what is the image under φ of

(i) the line $ax = y + b$; the pencil of parallel lines $ax = y + b$ (fixed a and variable b);

(ii) circles $(x-1)^2 + y^2 = c$ for variable c (distinguish the 3 cases $c > 1$, $c = 1$ and $c < 1$).

Try to imagine the above as a perspective drawing by an artist sitting at $0 \in \mathbb{R}^3$, on a plane $(X = 1)$, of figures from the plane $(Z = 1)$. Explain what happens to the points of the two planes where φ and φ^{-1} are undefined.

1.8 Let $P_1, \dots, P_4$ be distinct points of $\mathbb{P}^2$ with no 3 collinear. Prove that there is a unique coordinate system in which the 4 points are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(1,1,1)$. Find all conics passing through $P_1, \dots, P_5$, where $P_5 = (a,b,c)$ is some other point, and use this to give another proof of Corollary 1.10 and Proposition 1.11.