

1.13 Degenerate conics in a pencil

Definition A *pencil of conics* is a family of the form

$$C_{(\lambda, \mu)} : (\lambda Q_1 + \mu Q_2 = 0);$$

each element is a plane curve, depending in a linear way on the parameters (λ, μ) ; think of the ratio $(\lambda : \mu)$ as a point of \mathbb{P}^1 .

Looking at the examples, one expects that for special values of $(\lambda : \mu)$ the conic $C_{(\lambda, \mu)}$ is degenerate. In fact, writing $\det(Q)$ for the determinant of the symmetric 3×3 matrix corresponding to the quadratic form Q , it is clear that

$$C_{(\lambda, \mu)} \text{ is degenerate} \iff \det(\lambda Q_1 + \mu Q_2) = 0.$$

Writing out Q_1 and Q_2 as symmetric matrixes expresses this condition as

$$F(\lambda, \mu) = \det \left| \lambda \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} + \mu \begin{pmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{pmatrix} \right| = 0.$$

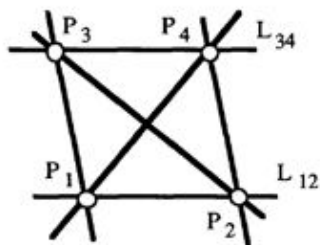
Now notice that $F(\lambda, \mu)$ is a homogeneous cubic form in λ, μ . In turn I can apply (1.8) to F to deduce:

Proposition Suppose $C_{(\lambda, \mu)}$ is a pencil of conics of \mathbb{P}_k^2 , with at least one nondegenerate conic (so that $F(\lambda, \mu)$ is not identically zero). Then the pencil has at most 3 degenerate conics. If $k = \mathbb{R}$ then the pencil has at least one degenerate conic.

Proof A cubic form has ≤ 3 zeros. Also over \mathbb{R} , it must have at least one zero.

1.14 Worked example

Let P_1, \dots, P_4 be 4 points of $\mathbb{P}_{\mathbb{R}}^2$ such that no 3 are collinear; then the pencil of conics $C_{(\lambda, \mu)}$ through P_1, \dots, P_4 has 3 degenerate elements, namely the line pairs $L_{12} + L_{34}, L_{13} + L_{24}, L_{14} + L_{23}$, where L_{ij} is the line through P_i, P_j :



Next, suppose that I start from the pencil of conics generated by $Q_1 = Y^2 + rY + sX + t$ and $Q_2 = Y - X^2$, and try to find the points P_1, \dots, P_4 of intersection.