1.3 Conics in $\mathbb{R}^2$

A conic in $\mathbb{R}^2$ is a plane curve given by a quadratic equation

$$q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Everyone has seen the classification of nondegenerate conics:

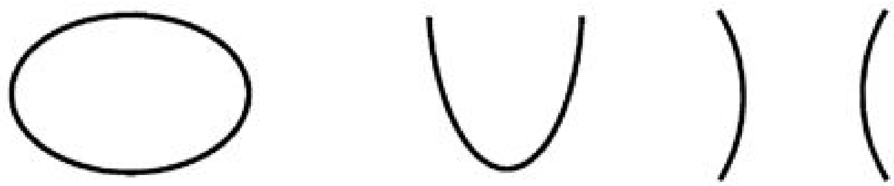

Figure 1.2: The nondegenerate conics: (a) ellipse; (b) parabola; (c) hyperbola.

in addition, there are a number of peculiar cases:
(d) single point given by $x^2 + y^2 = 0$;
(e, f, g) empty set given by any of the 3 equations: (e) $x^2 + y^2 = -1$, (f) $x^2 = -1$ or (g) $0 = 1$. These three equations are different, although they define the same locus of zeros in $\mathbb{R}^2$; consider for example their complex solutions.
(h) line $x = 0$;
(i) line pair $xy = 0$;
(j) parallel lines $x(x - 1) = 0$;
(k) 'double line' $x^2 = 0$; you can choose for yourself whether you'll allow the final case:
(l) whole plane given by $0 = 0$.

1.4 Projective plane

The definition 'out of the blue':

$$\begin{aligned}\mathbb{P}^2_{\mathbb{R}} &= \bigl\{\text{lines of } \mathbb{R}^3 \text{ through origin}\bigr\} \\ &= \bigl\{\text{ratios } X : Y : Z\bigr\} \\ &= \bigl(\mathbb{R}^3 \setminus \{0\}\bigr)/\sim, \quad \text{where } (X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z) \text{ if } \lambda \in \mathbb{R} \setminus \{0\}.\end{aligned}$$

(The sophisticated reader will have no difficulty in generalising from $\mathbb{R}^3$ to an arbitrary vector space over a field, and in replacing work in a chosen coordinate system with intrinsic arguments.)

To represent a ratio $X : Y : Z$ for which $Z \neq 0$, I can set $x = X/Z$, $y = Y/Z$; this simplifies things, since the ratio corresponds to just two real numbers. In other words, the equivalence class of (X, Y, Z) under $\sim$ has a unique representative $(x, y, 1)$ with 3rd coordinate $= 1$. Unfortunately, sometimes Z might be $= 0$, so that this way of choosing a representative of the equivalence class is then no good. This discussion means that $\mathbb{P}^2_{\mathbb{R}}$ contains a copy of $\mathbb{R}^2$. A picture:

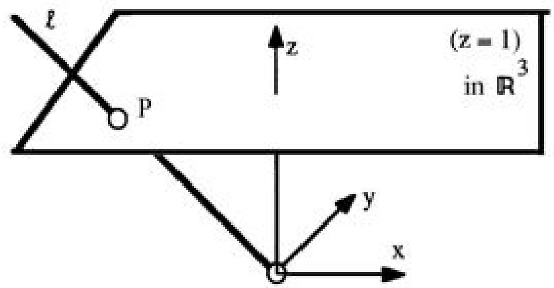

Figure 1.3: $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\} \to \mathbb{P}^2_{\mathbb{R}}$ by $(x, y) \mapsto (x, y, 1)$

The general line in $\mathbb{R}^3$ through 0 is not contained in the plane $(Z = 0)$, so that it meets $(Z = 1)$ in exactly one point, which is a representative for that equivalence class. The lines in $(Z = 0)$ never meet $(Z = 1)$, so they correspond not to points of $\mathbb{R}^2$, but to *asymptotic directions*, or to pencils of parallel lines of $\mathbb{R}^2$; so you can think of $\mathbb{P}^2_{\mathbb{R}}$ as consisting of $\mathbb{R}^2$ together with one 'point at infinity' for every pencil of parallel lines. From this point of view, you calculate in $\mathbb{R}^2$, try to guess what's going on at infinity by some kind of 'asymptotic' argument, then (if necessary), prove it in terms of homogeneous coordinates. The definition in terms of lines in $\mathbb{R}^3$ makes this respectable, since it treats all points of $\mathbb{P}^2_{\mathbb{R}}$ on an equal footing.

Groups of transformations are of central importance throughout geometry; properties of a geometric figure must be invariant under the appropriate kind of transformations before they are significant. An *affine* change of coordinates in $\mathbb{R}^2$ is of the form $T(\mathbf{x}) = A\mathbf{x} + B$, where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and A is a 2×2 invertible matrix, B a translation vector; if A is orthogonal then the transformation T is *Euclidean*. As everyone knows, every nondegenerate conic can be reduced to one of the standard forms (a–c) above by a Euclidean transformation. It is an exercise to the reader to show that every conic can be reduced to one of the forms (a–l) by an affine transformation.

A *projectivity*, or projective transformation of $\mathbb{P}^2_{\mathbb{R}}$ is a map of the form $T(\mathbf{X}) = M\mathbf{X}$, where M is an invertible 3×3 matrix. It's easy to understand the effect of this transformation on the affine piece $\mathbb{R}^2 \subset \mathbb{P}^2_{\mathbb{R}}$: as a partially defined map $\mathbb{R}^2 \dashrightarrow \mathbb{R}^2$, it is the fractional-linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(A \begin{pmatrix} x \\ y \end{pmatrix} + B \right) \Big/ (cx + dy + e), \quad \text{where} \quad M = \left(\begin{array}{c|c} A & B \\ \hline c\ d & e \end{array} \right).$$

T is of course not defined when $cx + dy + e = 0$. Perhaps this looks rather unintuitive, but it really occurs in nature: two different photographs of the same (plane) object are obviously related by a projectivity; see for example [Berger, 4.7.4] for pictures. So a math graduate getting a job interpreting satellite photography (whether for the peaceful purposes of the Forestry Commission, or as part of the vast career prospects opened up by President Reagan's defence policy) will spend a good part of his or her time computing projectivities.

Projective transformations are used implicitly throughout these notes, usually in the form 'by a suitable choice of coordinates, I can assume …'.

1.5 Equation of a conic

The inhomogeneous quadratic polynomial

$$q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

corresponds to the homogeneous quadratic

$$Q(X,Y,Z) = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2;$$

the correspondence is easy to understand as a recipe, or you can think of it as the bijection $q \leftrightarrow Q$ given by

$$q(x,y) = Q(X/Z, Y/Z, 1) \quad \text{with} \quad x = X/Z,\ \ y = Y/Z$$

and inversely,

$$Q = Z^2 q(X/Z, Y/Z).$$

A *conic* $C \subset \mathbb{P}^2$ is the curve given by $C : (Q(X,Y,Z) = 0)$, where Q is a homogeneous quadratic expression; note that the condition $Q(X,Y,Z) = 0$ is well defined on the equivalence class, since $Q(\lambda \mathbf{X}) = \lambda^2 Q(\mathbf{X})$ for any $\lambda \in \mathbb{R}$. As an exercise, check that the projective curve C meets the affine piece $\mathbb{R}^2$ in the affine conic given by $(q = 0)$.

'Line at infinity' and asymptotic directions

Points of $\mathbb{P}^2$ with $Z = 0$ correspond to ratios $(X : Y : 0)$. These points form the *line at infinity*, a copy of $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (since $(X : Y) \mapsto X/Y$ defines a bijection $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$).

A line in $\mathbb{P}^2$ is by definition given by $L : (aX + bY + cZ = 0)$, and

$$L \text{ passes through } (X,Y,0) \iff aX + bY = 0.$$

In affine coordinates the same line is given by $ax + by + c = 0$, so that all lines with the same ratio $a : b$ pass through the same point at infinity. This is called 'parallel lines meet at infinity'.

Example (a) The hyperbola $(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1)$ in $\mathbb{R}^2$ corresponds in $\mathbb{P}^2_{\mathbb{R}}$ to $C : (\frac{X^2}{a^2} - \frac{Y^2}{b^2} = Z^2)$; clearly this meets $(Z = 0)$ in the two points $(a, \pm b, 0) \in \mathbb{P}^2_{\mathbb{R}}$, corresponding in the obvious way to the asymptotic lines of the hyperbola.

Note that in the affine piece $(X \neq 0)$ of $\mathbb{P}^2_{\mathbb{R}}$, the affine coordinates are $u = Y/X, v = Z/X$, so that C becomes the ellipse $(\frac{u^2}{b^2} + v^2 = \frac{1}{a^2})$. See Ex. 1.7 for an artistic interpretation.

(b) The parabola $(y = mx^2)$ in $\mathbb{R}^2$ corresponds to $C : (YZ = mX^2)$ in $\mathbb{P}^2_{\mathbb{R}}$; this now meets $(Z = 0)$ at the single point $(0, 1, 0)$. So in $\mathbb{P}^2$, the 'two branches of the parabola meet at infinity'; note that this is a statement with intuitive content (maybe you feel it's pretty implausible?), but is not a result you could arrive at just by contemplating within $\mathbb{R}^2$ – maybe it's not even meaningful.