This is clear: if $f$ is given by $F_1,\dots,F_m$, then the composite is just $P \mapsto F_j(P)$, which is a polynomial function. Conversely, if $f_j \in k[V]$ for each $j$, then for any choice of $F_j \in k[X_1,\dots,X_n]$ such that $f_j = F_j \bmod I(V)$, I get a description of $f$ as the polynomial map given by $(F_1,\dots,F_m)$.

In view of this claim, the map $f$ can be written $f = (f_1,\dots,f_m)$.

The composite of polynomial maps is defined in the obvious way: if $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ and $U \subset \mathbb{A}^\ell$ are algebraic sets, and $f\colon V \to W$, $g\colon W \to U$ are polynomial maps, then $g \circ f\colon V \to U$ is again a polynomial map; for if $f$ is given by $F_1,\dots,F_m \in k[X_1,\dots,X_n]$, and $g$ by $G_1,\dots,G_\ell \in k[Y_1,\dots,Y_m]$, then $g \circ f$ is given by

$$G_1(F_1,\dots,F_m),\dots,G_\ell(F_1,\dots,F_m) \in k[X_1,\dots,X_n].$$

**Definition** A polynomial map $f\colon V \to W$ between algebraic sets is an *isomorphism* if there exists a polynomial map $g\colon W \to V$ such that $f \circ g = g \circ f = \mathrm{id}$.

Several examples of polynomial maps have already been given: the parametrisations $\mathbb{R}^1 \to C \subset \mathbb{R}^2$ by $t \mapsto (t^2,t^3)$ or $(t^2-1,t^3-t)$ given in (2.1), and the map $k \to C \subset \mathbb{A}^3_k$ by $t \mapsto (t^3,t^4,t^5)$ discussed in (3.11, b) are clearly of this kind. Also, while discussing Noether normalisation, I had an algebraic set $V \subset \mathbb{A}^n_k$, and considered the general projection $p\colon V \to \mathbb{A}^m_k$ defined by $m$ 'fairly general' linear forms $Y_1,\dots,Y_m$; since the $Y_i$ are linear forms in the coordinates $X_i$ of $\mathbb{A}^n_k$, this projection is a polynomial map.

On the other hand the parametrisation of the circle given in (1.1) is given by rational functions (there's a term $\lambda^2+1$ in the denominator); and the inverse map $(X,Y) \dashrightarrow t = Y/X$ from either of the singular cubics $C \subset \mathbb{R}^2$ back to $\mathbb{R}^1$ is also disqualified (or at least, doesn't qualify *as written*) for the same reason.

## 4.4 Polynomial maps and $k[V]$

**Theorem** *Let $V \subset \mathbb{A}^n_k$ and $W \subset \mathbb{A}^m_k$ be algebraic sets as above.*

(1) *A polynomial map $f\colon V \to W$ induces a ring homomorphism $f^*\colon k[W] \to k[V]$, defined by composition of functions; that is, if $g \in k[W]$ is a polynomial function then so is $f^*(g) = g \circ f$, and $g \mapsto g \circ f$ defines a ring homomorphism, in fact a $k$-algebra homomorphism $f^*\colon k[W] \to k[V]$. (Note that it goes backwards.)*

(2) *Conversely, any $k$-algebra homomorphism $\Phi\colon k[W] \to k[V]$ is of the form $\Phi = f^*$ for a uniquely defined polynomial map $f\colon V \to W$.*

*Thus (I) and (II) show that*

$$\left\{\begin{matrix}\textit{polynomial}\\ \textit{maps } f\colon V \to W\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}k\textit{-algebra homs.}\\ \Phi\colon k[W] \to k[V]\end{matrix}\right\}$$

*by*

$$f \longmapsto f^*$$

*is a bijection.*

(3) *If $f\colon V \to W$ and $g\colon W \to U$ are polynomial maps then the two ring homomorphisms $(g \circ f)^* = f^* \circ g^*\colon k[U] \to k[V]$ coincide.*

**Proof** (I) By what I said in (4.3), $f^*(g)$ is a polynomial map $V \to k$, hence $f^*(g) \in k[V]$. Obviously $f^*(a) = a$ for all $a \in k$ (since $k$ is being considered as the constant functions on $V, W$). Finally the fact that $f^*$ is a ring homomorphism is formal, since both $k[W]$ and $k[V]$ are rings of functions. (The ring structure is defined pointwise, so for example, for $g_1, g_2 \in k[W]$, the sum $g_1 + g_2$ is defined as the function on $W$ such that $(g_1 + g_2)(P) = g_1(P) + g_2(P)$ for all $P \in W$; therefore $f^*(g_1 + g_2)(Q) = (g_1 + g_2)(f(Q)) = g_1(f(Q)) + g_2(f(Q)) = f^*g_1(Q) + f^*g_2(Q)$. No-one's going to read this rubbish, are they?)

(III) is just the fact that composition of maps is associative.

(II) is a little more tricky to get right, although it's still content-free. For $i = 1, \dots, m$, let $y_i \in k[W]$ be the $i$th coordinate function on $W$, so that

$$k[W] = k[y_1, \dots, y_m] = k[Y_1, \dots, Y_m]/I(W).$$

Now $\Phi\colon k[W] \to k[V]$ is given, so I can define $f_i \in k[V]$ by $f_i = \Phi(y_i)$.

Consider the map $f\colon V \to \mathbb{A}^m_k$ defined by $f(P) = f_1(P), \dots, f_m(P)$. This is a polynomial map since $f_i \in k[V]$. Furthermore, I claim that $f$ takes $V$ into $W$, that is, $f(V) \subset W$. Indeed, suppose that $G \in I(W) \subset k[Y_1, \dots, Y_m]$; then

$$G(y_1, \dots, y_m) = 0 \in k[W],$$

where the left-hand side means that I substitute the ring elements $y_i$ into the polynomial expression $G$. Therefore, $\Phi(G(y_1, \dots, y_m)) = 0 \in k[V]$; but $\Phi$ is a $k$-algebra homomorphism, so that

$$k[V] \ni 0 = \Phi(G(y_1, \dots, y_m)) = G(\Phi(y_1), \dots, \Phi(y_m)) = G(f_1, \dots, f_m).$$

The $f_i$ are functions on $V$, and $G(f_1, \dots, f_m) \in k[V]$ is by definition the function $P \mapsto G(f_1(P), \dots, f_m(P))$. This proves that for $P \in V$, and for every $G \in I(W)$, the coordinates $(f_1(P), \dots, f_m(P))$ of $f(P)$ satisfy $G(f_1(P), \dots, f_m(P)) = 0$. Since $W$ is the subset of $\mathbb{A}^m_k$ defined by the vanishing of $G \in I(W)$, it follows that $f(P) \in W$. This proves that $f$ given above is a polynomial map $f\colon V \to W$. To check that the two $k$-algebra homomorphisms $f^*$, $\Phi\colon k[W] \to k[V]$ coincide, it's enough to check that they agree on the generators, that is $f^*(y_i) = \Phi(y_i)$; a minute inspection of the construction of $f$ (at the start of the proof of (II) above) will reveal that this is in fact the case. An exactly similar argument shows that the map $f$ is uniquely determined by the condition $f^*(y_i) = \Phi(y_i)$. Q.E.D.

**Corollary 4.5** *A polynomial map $f\colon V \to W$ is an isomorphism if and only if $f^*\colon k[W] \to k[V]$ is an isomorphism.*

**Example** Over an infinite field $k$, the polynomial map

$$\varphi\colon \mathbb{A}^1_k \to C : (Y^2 = X^3) \subset \mathbb{A}^2_k \text{ given by } T \mapsto (T^2, T^3)$$

is not an isomorphism. For in this case, the homomorphism

$$\varphi^*\colon k[C] = k[X, Y]/(Y^2 - X^3) \to k[T]$$

is given by $X \mapsto T^2, Y \mapsto T^3$. The image of $\varphi^*$ is the $k$-algebra generated by $T^2, T^3$, that is $k[T^2, T^3] \subsetneq k[T]$. (Please make sure you understand why $T^2, T^3$ don't generate $k[T]$; I can't help you on this.)

Notice that $\varphi$ is bijective, and so has a perfectly good inverse map $\psi\colon C \to \mathbb{A}^1_k$ given by $(X, Y) \mapsto 0$ if $X = Y = 0$ and $Y/X$ otherwise. So why isn't $\varphi$ an isomorphism? The point is that $C$ has fewer polynomial functions on it than $\mathbb{A}^1$; in a sense you can see that for yourself, since $k[\mathbb{A}^1] = k[T]$ has a polynomial function with nonzero derivative at 0. The gut feeling is that $\varphi$ 'squashes up the tangent vector at 0'.