X_i, that is, $\operatorname{char} k = p$, and X_i only appears in f as the pth power X_i^p. If this happens for each i, then by the argument given in (3.16), f is a pth power in $k[X_1, \dots, X_n]$; this contradicts the fact that f is irreducible. Q.E.D.

6.4 Tangent space

Definition Let $V \subset \mathbb{A}^n$ be a subvariety, with $V \ni P = (a_1, \dots, a_n)$. For any $f \in k[X_1, \dots, X_n]$, write

$$f_P^{(1)} = \sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i).$$

This is an affine linear polynomial (that is, linear plus constant), the 'first order part' of f at P. Now define the *tangent space* to V at P by

$$T_P V = \bigcap \left(f_P^{(1)} = 0 \right) \subset \mathbb{A}^n,$$

where the intersection takes place over all $f \in I(V)$.

Proposition 6.5 *The function $V \to \mathbb{N}$ defined by $P \mapsto \dim T_P V$ is an upper semicontinuous function (in the Zariski topology of V). In other words, for any integer r, the subset*

$$S(r) = \left\{ P \in V \;\middle|\; \dim T_P V \ge r \right\} \subset V$$

is closed.

Proof Let $(f_1, \dots, f_m)$ be a set of generators of $I(V)$; it is easy to see that for any $g \in I(V)$, the linear part $g_P^{(1)}$ of g is a linear combination of those of the f_i, so that the definition of $T_P V$ simplifies to

$$T_P V = \bigcap_{i=1}^{m} \left(f_{i,P}^{(1)} = 0 \right) \subset \mathbb{A}^n.$$

Then by elementary linear algebra,

$$\begin{aligned} P \in S(r) &\iff \text{the matrix } \left(\frac{\partial f}{\partial X_i}(P) \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \text{ has rank } \le n - r \\ &\iff \text{every } (n-r+1) \times (n-r+1) \text{ minor vanishes.} \end{aligned}$$

Now each entry $\partial f_i / \partial X_j (P)$ of the matrix is a polynomial function of P; thus each minor is a determinant of a matrix of polynomials, and so is itself a polynomial. Hence $S(r) \subset V \subset \mathbb{A}^n$ is an algebraic subset. Q.E.D.

Corollary-Definition 6.6 *There exists an integer r and a dense open subset $V_0 \subset V$ such that*

$$\dim T_P V = r \textit{ for } P \in V_0, \textit{ and } \dim T_P V \ge r \textit{ for all } P \in V.$$

Define r to be the dimension *of V, and write* $\dim V = r$*; and say that $P \in V$ is* nonsingular *if* $\dim T_P V = r$*, and* singular *if* $\dim T_P V > r$*. A variety V is* nonsingular *if it is nonsingular at each point $P \in V$.*