(iii) $k(V) \cong k(V_{(0)})$*, and for* $f \in k(V)$*, the domain of* f *as a function on* $V_{(0)}$ *is* $V_{(0)} \cap \operatorname{dom} f$*.*

Proof (i) and (ii) are easy. (iii) If $g, h \in k[X_0, \dots, X_n]$ are homogeneous of degree d, and $h \notin I(V)$, then $g/h \in k(V)$ restricted to $V_{(0)}$ is the function

$$\frac{g(1, X_1/X_0, \dots, X_n/X_0)}{h(1, X_1/X_0, \dots, X_n/X_0)};$$

this defines a map $k(V) \to k(V_{(0)})$, and it's easy to see what its inverse is.

5.6 Rational maps and morphisms

Rational maps between projective (or affine) varieties are defined using $k(V)$: if $V \subset \mathbb{P}^n$ is an irreducible algebraic set, a rational map $V \dashrightarrow \mathbb{A}^m$ is a (partially defined) map given by $P \mapsto (f_1(P), \dots, f_m(P))$, where $f_1, \dots, f_m \in k(V)$. A rational map $V \dashrightarrow \mathbb{P}^m$ is defined by $P \mapsto (f_0(P) : f_1(P) : \cdots : f_m(P))$ where $f_0, f_1, \dots, f_m \in k(V)$. Notice that if $g \in k(V)$ is a nonzero element, then $gf_0, gf_1, \dots, gf_m$ defines the same rational map. Therefore (assuming that V does not map into the smaller projective space $(X_0 = 0)$), it would be possible to assume throughout that $f_0 = 1$.

Clearly then, there is a bijection between the two sets

$$\bigl\{\text{rational maps } f\colon V \dashrightarrow \mathbb{A}^m \subset \mathbb{P}^m\bigr\}$$

and

$$\bigl\{\text{rational maps } f\colon V \dashrightarrow \mathbb{P}^m \bigm| f(V) \not\subset (X_0 = 0)\bigr\},$$

since either kind of maps is given by m elements $f_i \in k(V)$.

Definition A rational map $f\colon V \dashrightarrow \mathbb{P}^m$ is *regular* at $P \in V$ if there exists an expression $f = (f_0, f_1, \dots, f_m)$ such that

(i) each of $f_0, \dots, f_m$ is regular at P; and

(ii) at least one $f_i(P) \neq 0$.

The second condition is required here in order that the ratio between the f_i is defined at P. If f is regular at P (as before, this is also expressed $P \in \operatorname{dom} f$) then $f\colon U \to \mathbb{A}^m_{(i)} \subset \mathbb{P}^m$ is a morphism for a suitable open neighbourhood $P \in U \subset V$: just take $U = \bigcap_j \operatorname{dom}(f_j/f_i)$ where $f_i(P) \neq 0$; then f is the morphism given by $\bigl\{f_j/f_i\bigr\}_{j=0,1,\dots,m}$.

If $U \subset V$ is an open subset of a projective variety V then a *morphism* $f\colon U \to W$ is a rational map $f\colon V \dashrightarrow W$ such that $\operatorname{dom} f \supset U$. So a morphism is just a rational map that is everywhere regular on U.

5.7 Examples

(I) Rational normal curve. This is a very easy example of an isomorphic embedding $f\colon \mathbb{P}^1 \xrightarrow{\cong} C \subset \mathbb{P}^m$ which generalises the parametrised conic of (1.7), and which occurs throughout projective and algebraic geometry. Define

$$f\colon \mathbb{P}^1 \to \mathbb{P}^m \quad \text{by} \quad (U:V) \mapsto (U^m : U^{m-1}V : \cdots : V^m)$$

(writing down all monomials of degree m in U, V). Arguing step by step:

(i) f is a rational map, since it's given by

$$((U/V)^m, (U/V)^{m-1}, \dots, 1);$$

(ii) f is a morphism wherever $V \neq 0$ by the formula just written, and if $V = 0$ then $U \neq 0$, so a similar trick with V/U works;

(iii) the image of f is the set of points $(X_0 : \cdots : X_m) \in \mathbb{P}^m$ such that

$$(X_0 : X_1) = (X_1 : X_2) = \cdots = (X_{m-1} : X_m),$$

that is,

$$X_0X_2 = X_1^2, \quad X_0X_3 = X_1X_2, \quad X_0X_4 = X_1X_3, \quad \text{etc.}$$

The equations can be written all together in the extremely convenient determinantal form

$$\operatorname{rank}\begin{pmatrix} X_0 & X_1 & X_2 & \dots & X_{m-1} \\ X_1 & X_2 & X_3 & \dots & X_m \end{pmatrix} \leq 1$$

(the rank condition means exactly that all 2×2 minors vanish). These are homogeneous equations defining an algebraic set $C \subset \mathbb{P}^m$;

(iv) the inverse morphism $g : C \to \mathbb{P}^1$ is not hard to find: just take a point of C into the common ratio $(X_0 : X_1) = \cdots = (X_{m-1} : X_m) \in \mathbb{P}^1$. As an exercise, find out for yourself what has to be checked, then check it all.

(II) Linear projection, parametrising a quadric. The map $\pi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by $(X_0, X_1, X_2, X_3) \mapsto (X_1, X_2, X_3)$ is a rational map, and a morphism outside the point $P_0 = (1, 0, 0, 0)$. Let $Q \subset \mathbb{P}^3$ be a quadric hypersurface with $P \in Q$. Then every point P of $\mathbb{P}^2$ corresponds to a line L of $\mathbb{P}^3$ through P, and L should in general meet Q at P_0 and a second point $\varphi(P)$: for example, if $Q : (X_0X_3 = X_1X_2)$, then $\pi|Q\colon Q \dashrightarrow \mathbb{P}^2$ has the inverse map

$$\varphi\colon \mathbb{P}^2 \dashrightarrow Q \text{ given by } (X_1, X_2, X_3) \mapsto (X_1X_2/X_3, X_1, X_2, X_3).$$

This is essentially the same idea as the parametrisation of the circle in (1.1).

It is a rewarding exercise (see Ex. 5.2) to find $\operatorname{dom}\pi$ and $\operatorname{dom}\varphi$, and to give a geometric interpretation of the singularities of π and φ.