Chapter 7

The 27 lines on a cubic surface

In this section $S \subset \mathbb{P}^3$ will be a nonsingular cubic surface, given by a homogeneous cubic $f = f(X, Y, Z, T)$. Consider the lines ℓ of $\mathbb{P}^3$ lying on S.

7.1 Consequences of nonsingularity

Proposition *(a) There exists at most* 3 *lines of* S *through any point* $P \in S$*; if there are* 2 *or* 3*, they must be coplanar. The picture is:*

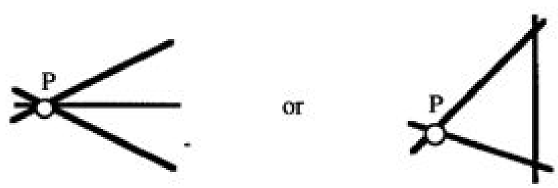

Figure 7.1: 3 concurrent lines or triangle

(b) Every plane $\Pi \subset \mathbb{P}^3$ *intersects* S *in one of the following:*

(i) an irreducible cubic; or

(ii) a conic plus a line; or

(iii) 3 distinct lines.

Proof (a) If $\ell \subset S$ then $\ell = T_P\ell \subset T_PS$, so that all lines of S through P are contained in the plane T_PS; there are at most 3 of them by (b).

(b) I have to prove that a multiple line is impossible: if $\Pi : (T = 0)$ and $\ell : (Z = 0) \subset \Pi$, then to say that ℓ is a multiple line of $S \cap \Pi$ means that f is of the form

$$f = Z^2 \cdot A(X, Y, Z, T) + T \cdot B(X, Y, Z, T),$$