Clearly $\sigma^{-1}(C) \subset B$ is the algebraic subset defined by $(f \circ \sigma)(u,v) = f(u,uv) = 0$; since $0 \in C$ by assumption, it follows that $\ell : (u = 0)$ is contained in $\sigma^{-1}(C)$, or equivalently, that $u \mid f(u,uv)$. It's easy to see that the highest power u^m of u dividing $f(u,uv)$ is equal to the smallest degree $m = a+b$ of the monomials $x^a y^b$ occurring in f, that is, the *multiplicity* of f at 0; so $\sigma^{-1}(C)$ decomposes as the union of the exceptional curve $\sigma^{-1}(0) = \ell$ (with multiplicity m), together with a new curve C_1 defined by $f_1(u,v) = f(u,uv)/u^m$. Consider the examples

(a) $f = \alpha x - y + \cdots$;

(b) $f = y^2 - x^2 + \cdots$;

(c) $f = y^2 - x^3$,

where ... denotes terms of higher degree. Clearly in (a) f has multiplicity 1, and $f_1 = \alpha - v + \cdots$ (where $\cdots$ consists of terms divisible by u), so C_1 is again nonsingular, and meets ℓ transversally at $(0,\alpha)$; thus σ replaces $0 \in \mathbb{A}^2$ with the line ℓ whose points correspond to tangent directions at 0 (excluding $(x = 0)$). In (b) $f_1 = v^2 - 1 + \cdots$, so C_1 has two nonsingular points $(0, \pm 1)$ above $0 \in C$; thus the blowup σ 'separates the two branches' of the singular curve C. In (c) $f_1 = v^2 - u$, so that C_1 is nonsingular, but above 0 it is tangent to the contracted curve ℓ.

In either case (b) or (c), σ replaces a singular curve C by a nonsingular one C_1 birational to C (by introducing 'new coordinates' $u = x, v = y/x$). This is what is meant by a *resolution of singularities*. In the case of plane curves, a resolution can always be obtained by a chain of blowups (see Ex. 6.6 for examples, and [Fulton, pp. 162–171] for more details), and the process of resolution gives detailed information about the singularities. A famous theorem of H. Hironaka guarantees the possibility of resolving singularities by blowups (in any dimension, over a field of characteristic zero). This is a crucial theoretical result that reduces the birational study of varieties to nonsingular ones; however, the actual process of resolution by blowups is in general extremely complicated, and does not necessarily contribute very much to the understanding of the singularities or varieties concerned.

Exercises to Chapter 6

6.1 Let $k \subset K$ be a field extension, and $(u_1, \dots, u_r)$, $(v_1, \dots, v_s)$ two sets of elements of K; suppose that $(u_1, \dots, u_r)$ are algebraically independent, and that $(v_1, \dots, v_s)$ span the extension $k \subset K$ algebraically. Prove that $r \leq s$. [Hint: the inductive step consists of assuming that $(u_1, \dots, u_i, v_{i+1}, \dots, v_s)$ span K/k algebraically, and considering u_{i+1}.] Deduce that any two transcendence bases of K/k have the same number of elements.

6.2 Prove Theorem 6.8, (b). [Hint:

$$I(V_f) = (I(V), Yf - 1) \subset k[X_1, \dots, X_n, Y],$$

so that if $Q = (a_1, \dots, a_n, b) \in V_f$, then $T_Q V_f \subset \mathbb{A}^{n+1}$ is defined by the equations for $T_P V \subset \mathbb{A}^n$, together with one equation involving Y.]

6.3 Determine all the singular points of the following curves in $\mathbb{A}^2$.

(a) $y^2 = x^3 - x$;

(b) $y^2 = x^3 - 6x^2 + 9x$;

(c) $x^2y^2 + x^2 + y^2 + 2xy(x+y+1) = 0$;

(d) $x^2 = x^4 + y^4$;

(e) $xy = x^6 + y^6$;

(f) $x^3 = y^2 + x^4 + y^4$;

(g) $x^2y + xy^2 = x^4 + y^4$.

6.4 Find all the singular points of the surfaces in $\mathbb{A}^3$ given by

(a) $xy^2 = z^2$;

(b) $x^2 + y^2 = z^2$;

(c) $xy + x^3 + y^3 = 0$.

(You will find it useful to sketch the real parts of the surfaces, to the limits of your ability; algebraic geometers usually can't draw.)

6.5 Show that the hypersurface $V_d \subset \mathbb{P}^n$ defined by

$$X_0^d + X_1^d + \cdots + X_n^d = 0$$

is nonsingular (if $\operatorname{char} k$ does not divide d).

6.6 (a) Let $C_n \subset \mathbb{A}^2$ be the curve given by $f_n : y^2 - x^{2n+1}$ and $\sigma\colon B \to \mathbb{A}^2$ be as in (6.12), with $\ell = \sigma^{-1}(0)$; show that $\sigma^{-1}(C_n)$ decomposes as the union of ℓ together with a curve isomorphic to C_{n-1}. Deduce that C_n can be resolved by a chain of n blowups.

(b) Show how to resolve the following curve singularities by making one or more blowups:

(i) $y^3 = x^4$;

(ii) $y^3 = x^5$;

(iii) $(y^2 - x^2)(y^2 - x^5) = x^8$.

6.7 Prove that the intersection of a hypersurface $V \subset \mathbb{A}^n$ (not a hyperplane) with the tangent hyperplane T_PV is singular at P.

Chapter 7

The 27 lines on a cubic surface

In this section $S \subset \mathbb{P}^3$ will be a nonsingular cubic surface, given by a homogeneous cubic $f = f(X,Y,Z,T)$. Consider the lines ℓ of $\mathbb{P}^3$ lying on S.

7.1 Consequences of nonsingularity

Proposition *(a) There exists at most* 3 *lines of* S *through any point* $P \in S$*; if there are* 2 *or* 3*, they must be coplanar. The picture is:*

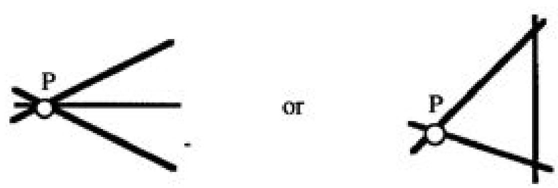

Figure 7.1: 3 concurrent lines or triangle

(b) Every plane $\Pi \subset \mathbb{P}^3$ *intersects* S *in one of the following:*

(i) an irreducible cubic; or

(ii) a conic plus a line; or

(iii) 3 distinct lines.

Proof (a) If $\ell \subset S$ then $\ell = T_P\ell \subset T_PS$, so that all lines of S through P are contained in the plane T_PS; there are at most 3 of them by (b).

(b) I have to prove that a multiple line is impossible: if $\Pi : (T = 0)$ and $\ell : (Z = 0) \subset \Pi$, then to say that ℓ is a multiple line of $S \cap \Pi$ means that f is of the form

$$f = Z^2 \cdot A(X,Y,Z,T) + T \cdot B(X,Y,Z,T),$$