

Corollary I and V determine inverse bijections

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{homogeneous radical} \\ \text{ideals } J \subset k[x_0, \dots, x_n] \\ \text{with } J \not\ni (x_0, \dots, x_n) \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{algebraic subsets} \\ X \subset \mathbb{P}^n \end{array} \right\} \\ \cup & & \cup \\ \left\{ \begin{array}{l} \text{homogeneous prime} \\ \text{ideals } J \subset k[x_0, \dots, x_n] \\ \text{with } J \not\ni (x_0, \dots, x_n) \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{irreducible algebraic} \\ \text{subsets } X \subset \mathbb{P}^n \end{array} \right\} \end{array}$$

Proof Let $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$ be the map defining \mathbb{P}^n . For a homogeneous ideal $J \subset k[X_0, \dots, X_n]$, write (in temporary notation) $V^a(J) \subset \mathbb{A}^{n+1}$ for the affine algebraic set defined by J . Then since J is homogeneous, $V^a(J)$ has the property

$$(\alpha_0, \dots, \alpha_n) \in V^a(J) \iff (\lambda\alpha_0, \dots, \lambda\alpha_n) \in V^a(J),$$

and $V(J) = V^a(J) \setminus \{0\}/\sim \subset \mathbb{P}^n$. Hence

$$V(J) = \emptyset \iff V^a(J) \subset \{0\} \iff \text{rad } J \supset (X_0, \dots, X_n),$$

where the last implication uses the affine Nullstellensatz. Also, if $V(J) \neq \emptyset$ then

$$f \in I(V(J)) \iff f \in I(V^a(J)) \iff f \in \text{rad } J. \quad \text{Q.E.D.}$$

The affine subset $V^a(J)$ occurring above is called the *affine cone* over the projective algebraic subset $V(J)$.

5.4 Rational functions on V

Let $V \subset \mathbb{P}_k^n$ be an irreducible algebraic set, and $I(V) \subset k[X_0, \dots, X_n]$ its ideal; there is no direct way of defining regular functions on V in terms of polynomials: an element $F \in k[X_0, \dots, X_n]$ gives a function on the affine cone over V , but (by case $d = 0$ of Proposition 5.1) this will be constant on equivalence classes only if F is homogeneous of degree 0, that is, a constant. So from the start, I work with rational functions only:

Definition A *rational function* on V is a (partially defined) function $f: V \dashrightarrow k$ given by $f(P) = g(P)/h(P)$, where $g, h \in k[X_0, \dots, X_n]$ are homogeneous polynomials of the same degree d .

Note here that provided $h(P) \neq 0$, the quotient $g(P)/h(P)$ is well defined, since

$$g(\lambda\underline{X})/h(\lambda\underline{X}) = \lambda^d g(\underline{X})/\lambda^d h(\underline{X}) = g(\underline{X})/h(\underline{X}) \quad \text{for } 0 \neq \lambda \in k.$$

Now obviously g/h and g'/h' define the same rational function on V if and only if $h'g - g'h \in I(V)$, so that the set of all rational functions is the field

$$k(V) = \left\{ \frac{g}{h} \mid \begin{array}{l} g, h \in k[X_0, \dots, X_n] \text{ homogeneous} \\ \text{of the same degree, and } h \notin I(V) \end{array} \right\} / \sim,$$

where \sim is the equivalence relation

$$\frac{g}{h} \sim \frac{g'}{h'} \iff h'g - g'h \in I(V).$$

$k(V)$ is the (rational) *function field* of V .

The following definitions are just as in the affine case. For $f \in k(V)$ and $P \in V$, say that f is *regular* at P if there exists an expression $f = g/h$, with g, h homogeneous polynomials of the same degree, such that $h(P) \neq 0$. Write

$$\text{dom } f = \{P \in V \mid f \text{ is regular at } P\}$$

and

$$\mathcal{O}_{V,P} = \{f \in k(V) \mid f \text{ is regular at } P\}.$$

Clearly, $\text{dom } f \subset V$ is a dense Zariski open set in V (the proof is as in (4.8, I)), and $\mathcal{O}_{V,P} \subset k(V)$ is a subring.

5.5 Affine covering of a projective variety

Let $V \subset \mathbb{P}^n$ be an irreducible algebraic set, and suppose for simplicity that $V \not\subset (X_i = 0)$ for any i . We know that \mathbb{P}^n is covered by $n+1$ affine pieces $\mathbb{A}_{(i)}^n$, with affine (inhomogeneous) coordinates

$$X_0^{(i)}, \dots, X_{i-1}^{(i)}, X_{i+1}^{(i)}, \dots, X_n^{(i)}, \quad \text{where } X_j^{(i)} = X_j/X_i \quad \text{for } j \neq i.$$

Write $V_{(i)} = V \cap \mathbb{A}_{(i)}^n$. Then $V_{(i)} \subset \mathbb{A}_{(i)}^n$ is clearly an affine algebraic set, because

$$\begin{aligned} V_{(0)} \ni P = (1, x_1^{(0)}, \dots, x_n^{(0)}) \\ \iff f(1, x_1^{(0)}, \dots, x_n^{(0)}) = 0 \quad \text{for all homogeneous } f \in I(V), \end{aligned}$$

which is a set of polynomial relations in the coordinates $(x_1^{(0)}, \dots, x_n^{(0)})$ of P . For clarity, I have taken $i = 0$ in the argument, and will continue to do so whenever convenient. The reader should remember that the same result applies to any of the other affine pieces $V_{(i)}$. The $V_{(i)}$ are called *standard affine pieces* of V .

Proposition (i) *The correspondence $V \mapsto V_{(0)} = V \cap \mathbb{A}_{(0)}^n$ gives a bijection*

$$\left\{ \begin{array}{l} \text{irreducible alg.} \\ \text{subsets } V \subset \mathbb{P}^n \end{array} \middle| V \not\subset (X_0 = 0) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irreducible alg.} \\ \text{subsets } V_0 \subset \mathbb{A}_{(0)}^n \end{array} \right\};$$

the inverse correspondence is given by taking the closure in the Zariski topology.

(ii) *Write $I^h(V) \subset k[X_0, \dots, X_n]$ for the homogeneous ideal of $V \subset \mathbb{P}^n$ introduced in this section and $I^a(V_{(0)}) \subset k[X_1, \dots, X_n]$ for the usual (as in §3) inhomogeneous ideal of $V_{(0)} \subset \mathbb{A}_{(0)}^n$; then $I^h(V)$ and $I^a(V_{(0)})$ are related as follows:*

$$I^a = \{f(1, X_1, \dots, X_n) \mid f \in I^h(V)\},$$

and

$$I^h(V)_d = \left\{ X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) \mid f \in I^a(V_{(0)}), \text{ with } \deg f \leq d \right\},$$

where the subscript in $I^h(V)_d$ denotes the piece of degree d .