4.6 Affine variety

Let k be a field; I want an *affine variety* to be an irreducible algebraic subset $V \subset \mathbb{A}^n_k$, defined up to isomorphism.

Theorem 4.4 tells us that the coordinate ring $k[V]$ is an invariant of the isomorphism class of V. This allows me to give a definition of a variety making less use of the ambient space $\mathbb{A}^n_k$; the reason for wanting to do this is rather obscure, and for practical purposes you will not miss much if you ignore it: subsequent references to an affine variety will always be taken in the sense given above (GOTO 4.7).

Definition An affine variety over a field k is a set V, together with a ring $k[V]$ of k-valued functions $f\colon V \to k$ such that

(i) $k[V]$ is a finitely generated k-algebra, and

(ii) for some choice $x_1, \dots, x_n$ of generators of $k[V]$ over k, the map

$$\text{by} \qquad \begin{array}{ccc} V & \to & \mathbb{A}^n_k \\ P & \mapsto & x_1(P), \dots, x_n(P) \end{array}$$

embeds V as an irreducible algebraic set.

4.7 Function field

Let V be an affine variety; then the coordinate ring $k[V]$ of V is an integral domain whose elements are k-valued functions of V.

Definition The *function field* $k(V)$ of V is the field of fractions $k(V) = \mathrm{Quot}(k[V])$ of $k[V]$. An element $f \in k(V)$ is a *rational function* on V; note that $f \in k(V)$ is by definition a quotient $f = g/h$ with $g, h \in k[V]$ and $h \neq 0$.

A priori f is not a function on V, because of the zeros of h; however, f is well defined at $P \in V$ whenever $h(P) \neq 0$, so is at least a 'partially defined function'. I now introduce terminology to shore up this notion.

Definition Let $f \in k(V)$ and $P \in V$; I say that f is *regular* at P, or that P is in the *domain of definition* of f if there exists an expression $f = g/h$ with $g, h \in k[V]$ and $h(P) \neq 0$.

An important point to bear in mind is that usually $k[V]$ will not be a UFD, so that $f \in k(V)$ may well have essentially different representations as $f = g/h$; see Ex. 4.9 for an example.

Write

$$\operatorname{dom} f = \bigl\{P \in V \bigm| f \text{ is regular at } P\bigr\}$$

for the *domain of definition* of f, and

$$\mathcal{O}_{V,P} = \bigl\{f \in k(V) \bigm| f \text{ is regular at } P\bigr\} = k[V][\{h^{-1} | h(P) \neq 0\}].$$

Then $\mathcal{O}_{V,P} \subset k(V)$ is a subring, the *local ring* of V at P.