4.6 Affine variety

Let k be a field; I want an *affine variety* to be an irreducible algebraic subset $V \subset \mathbb{A}^n_k$, defined up to isomorphism.

Theorem 4.4 tells us that the coordinate ring $k[V]$ is an invariant of the isomorphism class of V. This allows me to give a definition of a variety making less use of the ambient space $\mathbb{A}^n_k$; the reason for wanting to do this is rather obscure, and for practical purposes you will not miss much if you ignore it: subsequent references to an affine variety will always be taken in the sense given above (GOTO 4.7).

Definition An affine variety over a field k is a set V, together with a ring $k[V]$ of k-valued functions $f\colon V \to k$ such that

(i) $k[V]$ is a finitely generated k-algebra, and

(ii) for some choice $x_1, \dots, x_n$ of generators of $k[V]$ over k, the map

$$\text{by} \qquad \begin{array}{ccc} V & \to & \mathbb{A}^n_k \\ P & \mapsto & x_1(P), \dots, x_n(P) \end{array}$$

embeds V as an irreducible algebraic set.

4.7 Function field

Let V be an affine variety; then the coordinate ring $k[V]$ of V is an integral domain whose elements are k-valued functions of V.

Definition The *function field* $k(V)$ of V is the field of fractions $k(V) = \operatorname{Quot}(k[V])$ of $k[V]$. An element $f \in k(V)$ is a *rational function* on V; note that $f \in k(V)$ is by definition a quotient $f = g/h$ with $g, h \in k[V]$ and $h \neq 0$.

A priori f is not a function on V, because of the zeros of h; however, f is well defined at $P \in V$ whenever $h(P) \neq 0$, so is at least a 'partially defined function'. I now introduce terminology to shore up this notion.

Definition Let $f \in k(V)$ and $P \in V$; I say that f is *regular* at P, or that P is in the *domain of definition* of f if there exists an expression $f = g/h$ with $g, h \in k[V]$ and $h(P) \neq 0$.

An important point to bear in mind is that usually $k[V]$ will not be a UFD, so that $f \in k(V)$ may well have essentially different representations as $f = g/h$; see Ex. 4.9 for an example.

Write

$$\operatorname{dom} f = \left\{ P \in V \mid f \text{ is regular at } P \right\}$$

for the *domain of definition* of f, and

$$\mathcal{O}_{V,P} = \left\{ f \in k(V) \mid f \text{ is regular at } P \right\} = k[V][\{h^{-1} | h(P) \neq 0\}].$$

Then $\mathcal{O}_{V,P} \subset k(V)$ is a subring, the *local ring* of V at P.

Theorem 4.8 *(I)* $\operatorname{dom} f$ *is open and dense in the Zariski topology.*

Suppose that the field k is algebraically closed; then

(II)

$$\operatorname{dom} f = V \iff f \in k[V];$$

(that is polynomial function = regular rational function*). Furthermore, for any $h \in k[V]$, let*

$$V_h = V \setminus V(h) = \{P \in V \mid h(P) \neq 0\};$$

then

(III)

$$\operatorname{dom} f \supset V_h \iff f \in k[V][h^{-1}].$$

Proof Define the *ideal of denominators* of $f \in k(V)$ by

$$\begin{aligned} D_f &= \{h \in k[V] \mid hf \in k[V]\} \subset k[V] \\ &= \{h \in k[V] \mid \exists \text{ an expression } f = g/h \text{ with } g \in k[V]\} \cup \{0\}. \end{aligned}$$

From the first line, D_f is obviously an ideal of $k[V]$. Then formally,

$$V \setminus \operatorname{dom} f = \{P \in V \mid h(P) = 0 \text{ for all } h \in D_f\} = V(D_f),$$

so that $V \setminus \operatorname{dom} f$ is an algebraic set of V; hence $\operatorname{dom} f = V \setminus V(D_f)$ is the complement of a closed set, so open in the Zariski topology. It is obvious that $\operatorname{dom} f$ is nonempty, hence dense by Proposition 4.2.

Now using (b) of the Nullstellensatz,

$$\operatorname{dom} f = V \iff V(D_f) = \emptyset \iff 1 \in D_f, \quad \text{that is, } f \in k[V].$$

Finally,

$$\operatorname{dom} f \supset V_h \iff h \text{ vanishes on } V(D_f),$$

and using (c) of the Nullstellensatz,

$$\iff h^n \in D_f \text{ for some } n, \text{ that is, } f = g/h^n \in k[V][h^{-1}]. \quad \text{Q.E.D.}$$

4.9 Rational maps

Let V be an affine variety.

Definition A *rational map* $f\colon V \dashrightarrow \mathbb{A}^n_k$ is a partially defined map given by rational functions $f_1, \dots, f_n$, that is,

$$f(P) = f_1(P), \dots, f_n(P) \quad \text{for all } P \in \bigcap \operatorname{dom} f_i.$$

By definition, $\operatorname{dom} f = \bigcap \operatorname{dom} f_i$; as before, F is said to be *regular* at $P \in V$ if and only if $P \in \operatorname{dom} f$. A rational map $V \dashrightarrow W$ between two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is defined to be a rational map $f\colon V \dashrightarrow \mathbb{A}^m$ such that $f(\operatorname{dom} f) \subset W$.

Two examples of rational maps were described at the end of (4.3).