Definition A *morphism* $f\colon U \to W$ is a rational map $f\colon V \dashrightarrow W$ such that $U \subset \operatorname{dom} f$, so that f is regular at every $P \in U$.

If $U_1 \subset V$ and $U_2 \subset W$ are opens, then a morphism $f\colon U_1 \to U_2$ is just a morphism $f\colon U_1 \to W$ such that $f(U_1) \subset U_2$. An *isomorphism* is a morphism which has a two-sided inverse morphism.

Note that if V, W are affine varieties, then by Theorem 4.8, (II),

$$\big\{\text{morphisms } f\colon V \to W\big\} = \big\{\text{polynomial maps } f\colon V \to W\big\};$$

the left-hand side of the equation consists of rational objects subject to regularity conditions, whereas the right-hand side is more directly in terms of polynomials.

Example The parametrisation of the cuspidal cubic $\mathbb{A}^1 \to C : (Y^2 = X^3)$ of (2.1) induces an isomorphism $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0,0)\}$; see Ex. 4.5 for details.

4.13 Standard open subsets

Let V be an affine variety. For $f \in k[V]$, write V_f for the open set $V_f = V \setminus V(f) = \big\{P \in V \mid f(P) \neq 0\big\}$. The V_f are called *standard open sets* of V.

Proposition *V_f is isomorphic to an affine variety, and*

$$k[V_f] = k[V][f^{-1}].$$

Proof The idea is to consider the graph of the function f^{-1}; a similar trick was used for (b) $\Longrightarrow$ (c) in the proof of NSS (3.10).

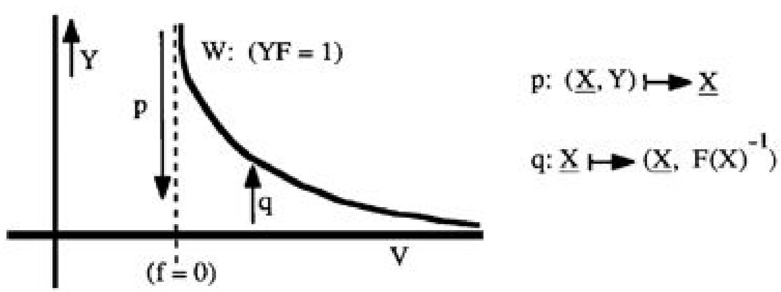

Figure 4.1: Graph of $1/f$

Let $J = I(V) \subset k[X_1, \dots, X_n]$, and choose $F \in k[X_1, \dots, X_n]$ such that $f = F \bmod I(V)$. Now define $I = (J, YF - 1) \subset k[X_1, \dots, X_n, Y]$, and let

$$V(I) = W \subset \mathbb{A}^{n+1}.$$

It is easy to check that the maps indicated in the diagram are inverse morphisms between W and V_f. The statement about the coordinate ring is contained in (4.8, III). Q.E.D.

The standard open sets V_f are important because they form a basis for the Zariski topology of V: every open set $U \subset V$ is a union of V_f (since every closed subset is of the form $V(I) = \bigcap_{f \in I} V(f)$ for some ideal). Thus the point of the result just proved is that every open set $U \subset V$ is a union of open sets V_f which are affine varieties.

4.14 Worked example

In §2 I discussed the addition law $(A, B) \mapsto A+B$ on a plane nonsingular (projective) cubic $C \subset \mathbb{P}^2$. Let $C_0 : (y^2 = x^3 + ax + b)$ be a nonsingular affine cubic:

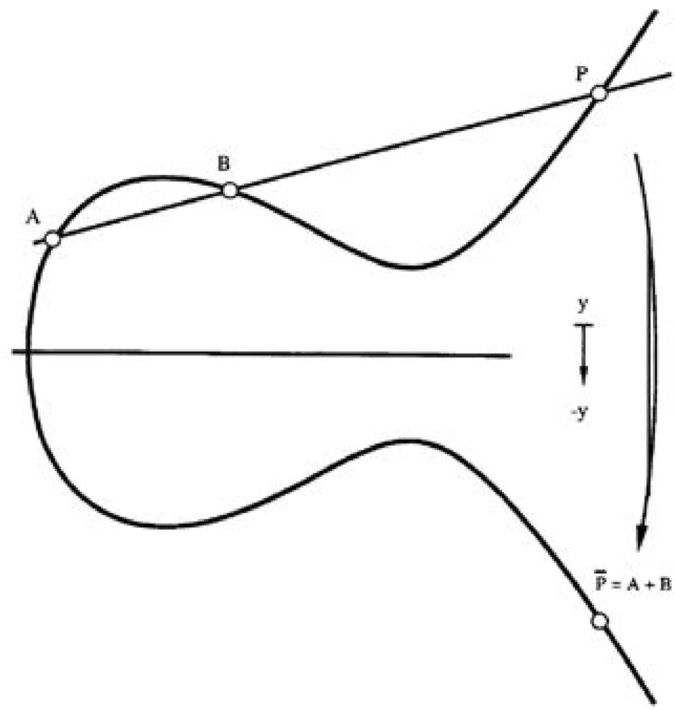

Figure 4.2: Group law on cubic as a morphism

I show here that the addition law defines a rational map $\varphi \colon C_0 \times C_0 \dashrightarrow C_0$, and that φ is a morphism wherever it should be. Although I will not labour the point, this argument can be used to give another proof 'by continuity' of the associativity of the group law valid for any field (see the discussion in (2.10)).

It is not difficult to see (compare Ex. 2.7) that if $A = (x, y)$, $B = (x', y')$, and $x \neq x'$ then setting $u = (y - y')/(x - x')$, the third point of intersection is $P = (x'', y'')$, where

$$x'' = f(x, y, x', y') = u^2 - (x + x'),$$
$$y'' = g(x, y, x', y') = u^3 + xu + y'.$$