Chapter 8

Final comments

This final section is not for examination, but some of the topics may nevertheless be of interest to the student.

History and sociology of the modern subject

8.1 Introduction

Algebraic geometry has over the last thirty years or so enjoyed a position in math similar to that of math in the world at large, being respected and feared much more than understood. At the same time, the 'service' questions I am regularly asked by British colleagues or by Warwick graduate students are generally of an elementary kind: as a rule, they are either covered in this book or in [Atiyah and Macdonald]. What follows is a view of the recent development of the subject, attempting to explain this paradox. I make no pretence at objectivity.

8.2 Prehistory

Algebraic geometry developed in the 19th century from several different sources. Firstly, the geometric tradition itself: projective geometry (and descriptive geometry, of great interest to the military at the time of Napoleon), the study of curves and surfaces for their own sake, configuration geometry; then complex function theory, the view of a compact Riemann surface as an algebraic curve, and the purely algebraic reconstruction of it from its function field. On top of this, the deep analogy between algebraic curves and the ring of integers of a number field, and the need for a language in algebra and geometry for invariant theory, which played an important role in the development of abstract algebra at the start of the 20th century.

The first decades of the 20th century saw a deep division. On the one hand, the geometric tradition of studying curves and surfaces, as pursued notably by the brilliant Italian school; alongside its own quite considerable achievements, this played a substantial motivating role in the development of topology and differential geometry, but became increasingly dependent on arguments 'by geometric intuition' that even the *Maestri* were unable to sustain rigorously. On the other hand, the newly emerging forces of commutative algebra were laying foundations and providing techniques of proof. An example of the difference between the two approaches was the argument between Chow and van der Waerden, who established rigorously the existence of an algebraic variety parametrising space curves of given degree and genus, and Severi, who had been making creative use of such parameter spaces all his working life, and who in his declining years bitterly resented the intrusion of algebraists (nonItalians at that!) into his field, and most especially the implicit suggestion that the work of his own school lacked rigour.

8.3 Rigour, the first wave

Following the introduction of abstract algebra by Hilbert and Emmy Noether, rigorous foundations for algebraic geometry were laid in the 1920s and 1930s by van der Waerden, Zariski and Weil (van der Waerden's contribution is often suppressed, apparently because a number of mathematicians of the immediate postwar period, including some of the leading algebraic geometers, considered him a Nazi collaborator).

A central plank of their program was to make algebraic geometry work over an arbitrary field. In this connection, a key foundational difficulty is that you can't just define a variety to be a point set: if you start life with a variety $V \subset \mathbb{A}^n_k$ over a given field k then V is not just a subset of k^n; you must also allow K-valued points of V for field extensions $k \subset K$ (see (8.13, c) for a discussion). This is one reason for the notation $\mathbb{A}^n_k$, to mean the k-valued points of a variety $\mathbb{A}^n$ that one would like to think of as existing independently of the specified field k.

The necessity of allowing the ground field to change throughout the argument added enormously to the technical and conceptual difficulties (to say nothing of the notation). However, by around 1950, Weil's system of foundations was accepted as the norm, to the extent that traditional geometers (such as Hodge and Pedoe) felt compelled to base their books on it, much to the detriment, I believe, of their readability.

8.4 The Grothendieck era

From around 1955 to 1970, algebraic geometry was dominated by Paris mathematicians, first Serre then more especially Grothendieck and his school. It is important not to underestimate the influence of Grothendieck's approach, especially now that it has to some extent gone out of fashion. This was a period in which tremendous conceptual and technical advances were made, and thanks to the systematic use of the notion of scheme (more general than a variety, see (8.12–14) below), algebraic geometry was able to absorb practically all the advances made in topology, homological algebra, number theory, etc., and even to play a dominant role in their development. Grothendieck himself retired from the scene around 1970 in his early forties, which must be counted a tragic waste (he initially left the IHES in a protest over military funding of science). As a practising algebraic geometer, one is keenly aware of the large blocks of powerful machinery developed during this period, many of which still remain to be written up in an approachable way.

On the other hand, the Grothendieck personality cult had serious side effects: many people who had devoted a large part of their lives to mastering Weil foundations suffered rejection and humiliation, and to my knowledge only one or two have adapted to the new language; a whole generation of students (mainly French) got themselves brainwashed into the foolish belief that a problem that can't be dressed up in high powered abstract formalism is unworthy of study, and were thus excluded from the mathematician's natural development of starting with a small problem he