**Proof** Let $r = \min\{\dim T_P V\}$, taken over all points $P \in V$. Then clearly

$$S(r-1) = \emptyset, \quad S(r) = V, \quad \text{and} \quad S(r+1) \subsetneq V;$$

therefore $S(r) \setminus S(r+1) = \{P \in V \mid \dim T_P V = r\}$ is open and nonempty. Q.E.D.

## 6.7 $\dim V = \operatorname{tr}\deg k(V)$ – the hypersurface case

It follows from Proposition 6.3 that if $V = V(f) \subset \mathbb{A}^n$ is a hypersurface defined by some nonconstant polynomial $f$, then $\dim V = n-1$. On the other hand, for a hypersurface, $k[V] = k[X_1, \dots, X_n]/(f)$, so that, assuming that $f$ involves $X_1$ in a nontrivial way, the function field of $V$ is of the form

$$k(V) = k(X_2, \dots, X_n)[X_1]/(f),$$

that is, it is built up from $k$ by adjoining $n-1$ algebraically independent elements, then making a primitive algebraic extension.

**Definition** If $k \subset K$ is a field extension, the *transcendence degree* of $K$ over $k$ is the maximum number of elements of $K$ algebraically independent over $k$. It is denoted $\operatorname{tr}\deg_k K$.

The elementary theory of transcendence degree of a field extension $K/k$ is formally quite similar to that of the dimension of a vector space: given $\alpha_1, \dots, \alpha_m \in K$, we know what it means for them to be *algebraically independent* over $k$ (see (3.13)); they *span* the transcendental part of the extension if $K/k(\alpha_1, \dots, \alpha_m)$ is algebraic; and they form a *transcendence basis* if they are algebraically independent and span. Then it is an easy theorem that a transcendence basis is a maximal algebraically independent set, and a minimal spanning set, and that any two transcendence bases of $K/k$ have the same number of elements (see Ex. 6.1).

Thus for a hypersurface $V \subset \mathbb{A}^n$, $\dim V = n - 1 = \operatorname{tr}\deg_k k(V)$. The rest of this section is concerned with proving that the equality $\dim V = \operatorname{tr}\deg_k k(V)$ holds for all varieties, by reducing to the case of a hypersurface. The first thing to show is that for a point $P \in V$ of a variety, the tangent space $T_P V$, which so far has been discussed in terms of a particular coordinate system in the ambient space $\mathbb{A}^n$, is in fact an intrinsic property of a neighbourhood of $P \in V$.

## 6.8 Intrinsic nature of $T_P V$

From now on, given $P = (a_1, \dots, a_n) \in V \subset \mathbb{A}^n$, I take new coordinates $X_i' = X_i - a_i$ to bring $P$ to the origin, and thus assume that $P = (0, \dots, 0)$. Then $T_P V \subset \mathbb{A}^n$ is a vector subspace of $k^n$.

**Notation** Write $m_P =$ ideal of $P$ in $k[V]$, and

$$M_P = \text{the ideal } (X_1, \dots, X_n) \subset k[X_1, \dots, X_n].$$

Then of course $m_P = M_P/I(V) \subset k[V]$.

**Theorem** *In the above notation,*

(a) *there is a natural isomorphism of vector spaces*

$$(T_PV)^* = m_P/m_P^2,$$

*where* $(\,)^*$ *denotes the dual of a vector space.*

(b) *If* $f \in k[V]$ *is such that* $f(P) \neq 0$*, and* $V_f \subset V$ *is the standard affine open as in (4.13), then the natural map*

$$T_P(V_f) \to T_PV$$

*is an isomorphism.*

**Proof of (a)** Write $(k^n)^*$ for the vector space of linear forms on $k^n$; this is the vector space with basis $X_1, \dots, X_n$. Since $P = (0, \dots, 0)$, for any $f \in k[X_1, \dots, X_n]$, the linear part $f_P^{(1)}$ is naturally an element of $(k^n)^*$; define a map $d\colon M_P \to (k^n)^*$ by taking $f \in M_P$ into $df = f_P^{(1)}$.

Now $d$ is surjective, since the $X_i \in M_P$ go into the natural basis of $(k^n)^*$; also $\ker d = M_P^2$, since

$$\begin{aligned} f_P^{(1)} = 0 &\iff f \text{ starts with quadratic terms in } X_1, \dots, X_n \\ &\iff f \in M_P^2. \end{aligned}$$

Hence $M_P/M_P^2 \cong (k^n)^*$. This is statement (a) for the special case $V = \mathbb{A}^n$. In the general case, dual to the inclusion $T_PV \subset k^n$, there is a restriction map $(k^n)^* \to (T_PV)^*$, taking a linear form $\lambda$ on $k^n$ into its restriction to $T_PV$; composing then defines a map

$$D : M_P \to (k^n)^* \to (T_PV)^*.$$

The composite $D$ is surjective since each factor is. I claim that the kernel of $D$ is just $M_P^2 + I(V)$, so that

$$m_P/m_P^2 = M_P/(M_P^2 + I(V)) \cong (T_PV)^*,$$

as required. To prove the claim,

$$\begin{aligned} f \in \ker D &\iff f_P^{(1)} | T_PV = 0 \\ &\iff f_P^{(1)} = \sum_i a_i g_{i,P}^{(1)} \text{ for some } g_i \in I(V) \end{aligned}$$

(since $T_PV \subset k^n$ is the vector subspace defined by $(g_P^{(1)} = 0)$ for $g \in I(V)$)

$$\iff f - \sum_i a_i g_i \in M_P^2 \text{ for some } g_i \in I(V) \iff f \in M_P^2 + I(V).$$

The proof of (b) of Theorem 6.8 is left to the reader (see Ex. 6.2). Q.E.D.

**Corollary 6.9** $T_PV$ *only depends on a neighbourhood of* $P \in V$ *up to isomorphism. More precisely, if* $P \in V_0 \subset V$ *and* $Q \in W_0 \subset W$ *are open subsets of affine varieties, and* $\varphi\colon V_0 \to W_0$ *an isomorphism taking* $P$ *into* $Q$*, there is a natural isomorphism* $T_PV_0 \to T_QW_0$*; hence* $\dim T_PV_0 = \dim T_QW_0$.

*In particular, if* $V$ *and* $W$ *are birationally equivalent varieties then* $\dim V = \dim W$.