Proof There exists a smooth quadric $Q \supset \ell_1, \dots, \ell_3$: several proofs of this are possible (see Ex. 7.2).

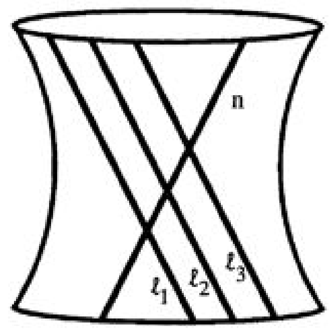

Figure 7.3: Quadric surface through 3 lines

Then in some choice of coordinates, $Q : (XT - YZ)$, and Q has two families of lines, or generators: any transversal of $\ell_1, \dots, \ell_3$ must lie in Q, since it has 3 points in Q. Now if $\ell_4 \not\subset Q$, then $\ell_4 \cap Q = \{1 \text{ or } 2 \text{ points}\}$, and the generators of the other family through these points are all the common transversals of $\ell_1, \dots, \ell_4$. Q.E.D.

7.6 The 27 lines

Let ℓ and m be two disjoint lines of S; as already observed, m meets exactly one out of each of the 5 pairs (ℓ_i, ℓ_i') of lines meeting ℓ. By renumbering the pairs, I assume that m meets ℓ_i for $i = 1, \dots, 5$. Introduce the following notation for the lines meeting ℓ or m:

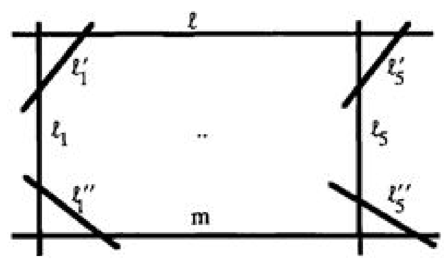

Figure 7.4: Configuration of lines on $S_3 \subset \mathbb{P}^3$

thus the 5 pairs of lines meeting m are (ℓ_i, ℓ_i'') for $i = 1, \dots, 5$. By (7.3, ii) applied to m, for $i \neq j$, the line ℓ_i'' does not meet ℓ_j. On the other hand, every line of S must meet one of ℓ, ℓ_j or ℓ_j', hence ℓ_i'' meets ℓ_j' for $i \neq j$.

Claim *(I) If $n \subset S$ is any line other than these 17, then n meets exactly 3 out of the 5 lines $\ell_1, \dots, \ell_5$.*

(II) Conversely, given any choice $\{i,j,k\}$ *of* 3 *elements of the set* $\{1,2,3,4,5\}$*, there is a unique line* $\ell_{ijk} \subset S$ *meeting* ℓ_i, ℓ_j, ℓ_k.

Proof (I) Given four disjoint lines of S, it is clear that they do not all lie on a quadric Q, since otherwise $Q \subset S$, contradicting the irreducibility of S.

If n meets ≥ 4 of the ℓ_i then by Lemma 7.5, $n = \ell$ or m, which is a contradiction. If n meets ≤ 2 of the ℓ_i then it meets ≥ 3 of the ℓ'_i, and so meets say either $\ell'_2, \ell'_3, \ell'_4, \ell'_5$ or $\ell_1, \ell'_3, \ell'_4, \ell'_5$; but by what was said above, ℓ and ℓ''_1 are two common transversals of the 5 disjoint lines $\ell'_2, \ell'_3, \ell'_4, \ell'_5$ and ℓ_1, so that by Lemma 7.5 again, if n meets ≥ 4 of these then $n = \ell$ or ℓ''_1. This is the same contradiction.

(II) There are 10 lines meeting ℓ_1 by (7.3), of which so far only 4 have been accounted for (namely, ℓ, ℓ'_1, m and ℓ''_1). The six other lines must meet exactly 2 out of the 4 remaining lines $\ell_2, \ldots, \ell_5$, and there are exactly $6 = \binom{4}{2}$ possible choices; so they must all occur. Q.E.D.

This gives the lines of S as being

$$\{\ell, m, \ell_i, \ell'_i, \ell''_i, \ell_{ijk}\},$$

and the number of them is

$$1 + 1 + 5 + 5 + 5 + 10 = 27.$$

7.7 The configuration of lines

An alternative statement is that the lines of S are $\ell, \ell_1, \ldots, \ell_5, \ell'_1, \ldots, \ell'_5$, and 16 other lines which meet an odd number of $\ell_1, \ldots, \ell_5$:

$$\begin{aligned} \ell''_i &\text{ meets } \ell_i \text{ only} \\ \ell_{ijk} &\text{ meets } \ell_i, \ell_j, \ell_k \text{ only} \\ m &\text{ meets all of } \ell_1, \ldots, \ell_5. \end{aligned}$$

In the notation I have introduced, it is easy to see that the incidence relation between the 27 lines of S is as follows:

ℓ meets $\ell_1, \ldots, \ell_5, \ell'_1, \ldots, \ell'_5$;

ℓ_1 meets $\ell, m, \ell'_1, \ell''_1$, and ℓ_{1jk} for 6 choices of $\{j,k\} \subset \{2,3,4,5\}$;

ℓ'_1 meets ℓ, ℓ_1, ℓ''_j (for 4 choices of $j \neq 1$), and ℓ_{ijk} (for 4 choices of $\{i,j,k\} \subset \{2,3,4,5\}$);

ℓ''_1 meets m, ℓ_1, ℓ'_j (for 4 choices of $j \neq 1$), and ℓ_{ijk} (for 4 choices of $\{i,j,k\} \subset \{2,3,4,5\}$);

ℓ_{123} meets $\ell_1, \ell_2, \ell_3, \ell_{145}, \ell_{245}, \ell_{345}, \ell'_4, \ell'_5, \ell''_4, \ell''_5$.

This combinatorial configuration has many different representations, some of them much more symmetric than that given here; see for example [Semple and Roth, pp. 122–128 and 151–152].