1.5 Equation of a conic

The inhomogeneous quadratic polynomial

$$q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

corresponds to the homogeneous quadratic

$$Q(X,Y,Z) = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2;$$

the correspondence is easy to understand as a recipe, or you can think of it as the bijection $q \leftrightarrow Q$ given by

$$q(x,y) = Q(X/Z, Y/Z, 1) \quad \text{with} \quad x = X/Z,\ \ y = Y/Z$$

and inversely,

$$Q = Z^2 q(X/Z, Y/Z).$$

A *conic* $C \subset \mathbb{P}^2$ is the curve given by $C : (Q(X,Y,Z) = 0)$, where Q is a homogeneous quadratic expression; note that the condition $Q(X,Y,Z) = 0$ is well defined on the equivalence class, since $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ for any $\lambda \in \mathbb{R}$. As an exercise, check that the projective curve C meets the affine piece $\mathbb{R}^2$ in the affine conic given by $(q = 0)$.

'Line at infinity' and asymptotic directions

Points of $\mathbb{P}^2$ with $Z = 0$ correspond to ratios $(X : Y : 0)$. These points form the *line at infinity*, a copy of $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (since $(X : Y) \mapsto X/Y$ defines a bijection $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$).

A line in $\mathbb{P}^2$ is by definition given by $L : (aX + bY + cZ = 0)$, and

$$L \text{ passes through } (X, Y, 0) \iff aX + bY = 0.$$

In affine coordinates the same line is given by $ax + by + c = 0$, so that all lines with the same ratio $a : b$ pass through the same point at infinity. This is called 'parallel lines meet at infinity'.

Example (a) The hyperbola $(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1)$ in $\mathbb{R}^2$ corresponds in $\mathbb{P}^2_{\mathbb{R}}$ to $C : (\frac{X^2}{a^2} - \frac{Y^2}{b^2} = Z^2)$; clearly this meets $(Z = 0)$ in the two points $(a, \pm b, 0) \in \mathbb{P}^2_{\mathbb{R}}$, corresponding in the obvious way to the asymptotic lines of the hyperbola.

Note that in the affine piece $(X \neq 0)$ of $\mathbb{P}^2_{\mathbb{R}}$, the affine coordinates are $u = Y/X, v = Z/X$, so that C becomes the ellipse $(\frac{u^2}{b^2} + v^2 = \frac{1}{a^2})$. See Ex. 1.7 for an artistic interpretation.

(b) The parabola $(y = mx^2)$ in $\mathbb{R}^2$ corresponds to $C : (YZ = mX^2)$ in $\mathbb{P}^2_{\mathbb{R}}$; this now meets $(Z = 0)$ at the single point $(0, 1, 0)$. So in $\mathbb{P}^2$, the 'two branches of the parabola meet at infinity'; note that this is a statement with intuitive content (maybe you feel it's pretty implausible?), but is not a result you could arrive at just by contemplating within $\mathbb{R}^2$ – maybe it's not even meaningful.

1.6 Classification of conics in $\mathbb{P}^2$

Let k be any field of characteristic $\neq 2$; recall two results from the linear algebra of quadratic forms:

Proposition (A) *There are natural bijections*

$$\left\{\begin{matrix}\text{homogeneous}\\ \text{quadratic polys.}\end{matrix}\right\} = \left\{\begin{matrix}\text{quad. forms}\\ k^3 \to k\end{matrix}\right\} \overset{\text{bij}}{\longleftrightarrow} \left\{\begin{matrix}\text{symmetric bilinear}\\ \text{forms on } k^3\end{matrix}\right\}$$

given in formulas by

$$aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2 \longleftrightarrow \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

A quadratic form is *nondegenerate* if the corresponding bilinear form is nondegenerate, that is, its matrix is nonsingular.

Theorem (B) *Let V be a vector space over k and $Q\colon V \to k$ a quadratic form; then there exists a basis of V such that*

$$Q = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2, \quad \textit{with } \varepsilon_i \in k.$$

(This is proved by *Gram-Schmidt orthogonalisation*, if that rings a bell.) Obviously, for $\lambda \in k \setminus \{0\}$ the substitution $x_i \mapsto \lambda x_i$ takes ε_i into $\lambda^{-2}\varepsilon_i$.

Corollary *In a suitable system of coordinates, any conic in $\mathbb{P}^2_{\mathbb{R}}$ is one of the following:*

(α) nondegenerate conic, $C : (X^2 + Y^2 - Z^2 = 0)$;

(β) empty set, given by $(X^2 + Y^2 + Z^2 = 0)$;

(γ) line pair, given by $(X^2 - Y^2 = 0)$;

(δ) one point $(0, 0, 1)$, given by $(X^2 + Y^2 = 0)$;

(ε) double line, given by $(X^2 = 0)$.

(Optionally you have the whole of $\mathbb{P}^2_{\mathbb{R}}$ given by $(0 = 0)$.)

Proof Any real number ε is either 0, or $\pm$ a square, so that I only have to consider Q as in the theorem with $\varepsilon_i = 0$ or ± 1. In addition, since I'm only interested in the locus $(Q = 0)$, I'm allowed to multiply Q through by -1. This leads at once to the given list. Q.E.D.

There are two points to make about this corollary: firstly, the list is quite a lot shorter than that in (1.3); for example, the 3 nondegenerate cases (ellipse, parabola, hyperbola) of (1.3) all correspond to case (α), and the 2 cases of intersecting and parallel line pairs are not distinguished in the projective case. Secondly, the derivation of the list from general algebraic principles is much simpler.