

Part III

Applications

Chapter 5

Projective and birational geometry

The first part of §5 aims to generalise the content of §§3–4 to projective varieties; this is fairly mechanical, with just a few essential points. The remainder of the section is concerned with birational geometry, taking up the function field $k(V)$ from the end of §4; this is material which fits equally well into the projective or affine context.

5.0 Why projective varieties?

The cubic curve

$$C : (Y^2Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2$$

is the union of two affine curves

$$\begin{aligned} C_0 : (y^2 = x^3 + ax + b) &\subset \mathbb{A}^2 \quad (\text{the piece } (Z = 1) \text{ of } C) \quad \text{and} \\ C_1 : (z_1 = x_1^3 + ax_1z_1^2 + bz_1^3) &\subset \mathbb{A}^2 \quad (\text{the piece } (Y = 1)), \end{aligned}$$

glued together by the isomorphism

$$\begin{aligned} C_0 \setminus (y = 0) &\longrightarrow C_1 \setminus (z_1 = 0) \\ \text{by} \quad (x, y) &\longmapsto (x/y, 1/y). \end{aligned}$$

As a much simpler example, \mathbb{P}^1 with homogeneous coordinates (X, Y) is the union of 2 copies of \mathbb{A}^1 with coordinates x_0, y_1 respectively, glued together by the isomorphism

$$\begin{aligned} \mathbb{A}^1 \setminus (x_0 = 0) &\longrightarrow \mathbb{A}^1 \setminus (y_1 = 0) \\ \text{by} \quad x_0 &\longmapsto 1/y_1. \end{aligned}$$