(iii) No base field

Let p be a prime number, and $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ the subring of rationals with no p in the denominator; $\mathbb{Z}_{(p)}$ is another discrete valuation ring, with parameter p. It has a unique maximal ideal $0 \neq p\mathbb{Z}_{(p)}$, with residue field $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{F}_p = \mathbb{Z}_{(p)}/(p)$. If $F \in \mathbb{Z}_{(p)}[X,Y]$, then it makes sense to consider the curve $C_{\mathbb{C}} : (F=0) \subset \mathbb{A}^2_{\mathbb{C}}$, or alternatively to take the reduction f of F mod p, and to consider the curve $C_p : (f=0) \subset \mathbb{A}^2\mathbb{F}_p$. What kind of geometric object is it that contains both a curve over the complexes and a curve over a finite field? Whether you consider it to be truly geometric is a matter of opinion, but the scheme $\operatorname{Spec}\mathbb{Z}_{(p)}[X,Y]/(F)$ does exactly this.

Again, this is technically not a new idea: reducing a curve mod p has been practised since the 18th century, and Weil foundations contained a whole theory of 'specialisation' to deal with it. The advantage is a better conceptual picture of the curve $\operatorname{Spec}\mathbb{Z}_{(p)}[X,Y]/(F)$ over the d.v.r. $\mathbb{Z}_{(p)}$ as a geometric object fibred over $\operatorname{Spec}(\mathbb{Z}_{(p)})$ ('$=$ (·-)'), with the two curve $C_{\mathbb{C}}$ and C_p as generic and special fibres.

In the same way, for $F \in \mathbb{Z}[X,Y]$, the scheme $\operatorname{Spec}\mathbb{Z}[X,Y]/(F)$ is a geometric object containing for every prime p the curve $C_p : (f_p = 0) \subset \mathbb{A}^2_{\mathbb{F}_p}$ over $\mathbb{F}_p$, where f_p is the reduction of F mod p, and at the same time the curve $C_{\mathbb{C}} : (F=0) \subset \mathbb{A}^2_{\mathbb{C}}$, and is called an *arithmetic surface*; it contains quite a lot besides: in particular, for every point $c \in C_{\mathbb{C}}$ with algebraic numbers as coordinates, it contains a copy of $\operatorname{Spec}\mathbb{Q}[c]$, hence essentially all the information about the ring of integers of the number field $\mathbb{Q}(c)$ of definition of c.

However grotesquely implausible this object may seem at first sight (you can again get used to it if you practise), it is a key ingredient in modern number theory, and is the basic foundation on which the work of Arakelov and Faltings rests.

8.15 Proof of the existence of lines on a cubic surface

Every adult algebraic geometer knows the traditional proof of (7.2) by dimension counting (see for example [Beauville, Complex algebraic surfaces, p. 50], or [Mumford, Algebraic geometry I, Complex projective varieties, p. 174]). I run through this before commenting on the difficulties.

The set of lines of $\mathbb{P}^3$ is parametrised by the 4-dimensional Grassmannian $\mathrm{Gr} = \mathrm{Gr}(2,4)$, and cubic surfaces by the projective space $S = \mathbb{P}^N$ of cubic forms in (X,Y,Z,T) (in fact $N = 19$). Write $Z \subset \mathrm{Gr} \times S$ for the incidence subvariety

$$Z = \big\{(\ell, X) \bigm| \ell \in \mathrm{Gr}, X \in S \text{ s.t. } \ell \subset X\big\}.$$

Since cubic forms vanishing on a given line ℓ form a $\mathbb{P}^{N-4}$, it is easy to deduce from the first projection $Z \to \mathrm{Gr}$ that Z is a rational N-dimensional variety. So the second projection $p\colon Z \to S$ is a morphism between two N-dimensional varieties, and therefore

(i) *either* the image $p(Z)$ is an N-dimensional variety in S, and so contains a dense open of S, *or* every fibre of p has dimension ≥ 1.

(ii) Z is a projective variety, so that the image $p(Z)$ is closed in S.

Since cubic surfaces containing only finitely many lines do exist, the second possibility in (i) doesn't occur, so every sufficiently general cubic surface contains lines. Then (ii) ensures that $p(Z) = S$, and every cubic surface contains lines.

This argument seems to me to be unsuitable for an undergraduate course for two reasons: statement (i) assumes results about the dimension of fibres, which however intuitively acceptable (especially to students in the last week of a course) are hard to do rigorously; whereas (ii) is the theorem that a projective variety is complete, that again requires proof (by elimination theory, compactness, or a full-scale treatment of the valuative criterion for properness).

To the best of my knowledge, my proof in (7.2) is new; the knowledgeable reader will of course see its relation to the other traditional argument by vector bundles: the Grassmannian $\mathrm{Gr}(2,4)$ has a tautological rank 2 vector bundle E (consisting of linear forms on the lines of $\mathbb{P}^3$); restricting the equation f of a cubic surface to every line $\ell \subset \mathbb{P}^3$ defines a section $s(f) \in S^3E$ of the 3rd symmetric power of E. Finally, every section of S^3E must have a zero, either by ampleness of E or by a Chern class argument (that also gives the magic number 27).

Substitute for preface

8.16 Acknowledgements and name dropping

It would be futile to try to list all the mathematicians who have contributed to my education. I owe a great debt to both my formal supervisors Pierre Deligne and Peter Swinnerton-Dyer (before he became a successful politician and media personality); I probably learned most from the books of David Mumford, and my understanding (such as it is) of the Grothendieck legacy derives largely from Mumford and Deligne. My view of the world, both as a mathematician and as a human being, has been strongly influenced by Andrei Tyurin.

My approach to what an undergraduate algebraic geometry course should be is partly based on a course designed around 1970 by Peter Swinnerton-Dyer for the Cambridge tripos, and taught in subsequent years by him and Barry Tennison; my book is in some ways a direct descendant of this, and some of the exercises have been taken over verbatim from Tennison's example sheets. However, I have benefitted enormously from the freedom allowed under the Warwick course structure, especially the philosophy of teaching (explicitly stated by Christopher Zeeman) that research experience must serve as one's main guideline in deciding how and what to teach.

The 'winking torus' appearing in (2.14–) comes to me from Jim Eells, who informs me he learnt it from H. Hopf (and that it probably goes back to an older German tradition of mathematical art work). I must thank Caroline Series, Frans Oort, Paul Cohn, John Jones, Ulf Persson, David Fowler, an anonymous referee and David Tranah from C.U.P. for helpful comments on the preprint version of this book, and apologise if on occasions I have either not been fully able to accommodate their suggestions, or preferred my own counsel.

I am grateful to Martina Jaeger for a number of corrections to the first printing, and to Isao Wakabayashi for a detailed reading, which uncovered many inaccuracies. I thank especially R.J. Chapman and Bill Bruce for pointing out the most serious error of the first printing (I avoided mention of the Hessian at the start of (7.2) by appealing to a false statement left as an exercise).