Consider first $J = (uw - v^2, u^3 - vw)$. Then J is not prime, since

$$J \ni w(uw - v^2) - v(u^3 - vw) = u(w^2 - u^2v),$$

but $u, w^2 - u^2v \notin J$. Therefore

$$V(J) = V(J, u) \cup V(J, w^2 - u^2v);$$

obviously, $V(J, u)$ is the w-axis ($u = v = 0$). I claim that the other component $C = V(J, w^2 - u^2v)$ is an irreducible curve; indeed, C is given by

$$uw = v^2, \quad u^3 = vw, \quad w^2 = u^2v.$$

I claim that $C \subset \mathbb{A}^3$ is the image of the map $\varphi\colon \mathbb{A}^1 \to C \subset \mathbb{A}^3$ given by $t \mapsto t^3, t^4, t^5$: to see this, if $u \neq 0$ then $v, w \neq 0$. Set $t = v/u$, then $t = w/v$ and $t^2 = (v/u)(w/v) = w/u$. Hence $v = w^2/u^2 = t^4$, $u = v/(v/u) = t^4/t = t^3$, and $w = tv = t^5$. Now C is irreducible, since if $C = X_1 \cup X_2$ with $X_i \subset C$, and $f_i(u, v, w) \in I(X_i)$, then for all t, one of $f_i(t^3, t^4, t^5)$ must vanish. Since a nonzero polynomial has at most a finite number of zeros, one of f_1, f_2 must vanish identically, so $f_i \in I(C)$.

This example is of a nice 'monomial' kind; in general it might be quite tricky to guess the irreducible components of a variety, and even more so to prove that they are irreducible. A similar example is given in Ex. 3.11.

3.12 Finite algebras

I now start on the proof of (3.8). Let $A \subset B$ be rings. As usual, B is said to be *finitely generated* over A (or f.g. as A-algebra) if there exist finitely many elements $b_1, \dots, b_n$ such that $B = A[b_1, \dots, b_n]$, so that B is generated as a ring by A and $b_1, \dots, b_n$.

Contrast with the following definition: B is a *finite A-algebra* if there exist finitely many elements $b_1, \dots, b_n$ such that $B = Ab_1 + \dots + Ab_n$, that is, B is finitely generated as A-module. The crucial distinction here is between generation as ring (when you're allowed any polynomial expressions in the b_i), and as module (the b_i can only occur linearly). For example, $k[X]$ is a finitely generated k-algebra (it's generated by one element X), but is not a finite k-algebra (since it has infinite dimension as k-vector space).

Proposition *(i) Let $A \subset B \subset C$ be rings; then*

B a finite A-algebra and C a finite B-algebra

$$\implies \text{ C a finite A-algebra.}$$

(ii) If $A \subset B$ is a finite A-algebra and $x \in B$ then x satisfies a monic equation over A, that is, there exists a relation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0 \quad \text{with } a_i \in A$$

(note that the leading coefficient is 1).

(iii) Conversely, if x satisfies a monic equation over A, then $B = A[x]$ is a finite A-algebra.

Proof (i) and (iii) are easy exercises (compare similar results for field extensions). For (ii), I use a rather nonobvious 'determinant trick' (I didn't think of it for myself): suppose $B = \sum Ab_i$; for each i, $xb_i \in B$, so there exist constants $a_{ij} \in A$ such that

$$xb_i = \sum_j a_{ij} b_j.$$

This can be written

$$\sum_j (x\delta_{ij} - a_{ij}) b_j = 0,$$

where δ_{ij} is the identity matrix. Now let M be the matrix with

$$M_{ij} = x\delta_{ij} - a_{ij},$$

and set $\Delta = \det M$. Then by standard linear algebra, (writing $\mathbf{b}$ for the column vector with entries $(b_1, \dots, b_n)$ and M^{adj} for the adjoint matrix of M),

$$M\mathbf{b} = 0, \quad \text{hence} \quad 0 = (M^{\mathrm{adj}})M\mathbf{b} = \Delta\mathbf{b},$$

and therefore $\Delta b_i = 0$ for all i. However, $1_B \in B$ is a linear combination of the b_i, so that $\Delta = \Delta \cdot 1_B = 0$, and I've won my relation:

$$\det(x\delta_{ij} - a_{ij}) = 0.$$

This is obviously a monic relation for x with coefficients in A. Q.E.D.

3.13 Noether normalisation

Theorem (Noether normalisation lemma) *Let k be an infinite field, and $A = k[a_1, \dots, a_n]$ a finitely generated k-algebra. Then there exist $m \le n$ and $y_1, \dots, y_m \in A$ such that*

(i) $y_1, \dots, y_m$ are algebraically independent over k; and

(ii) A is a finite $k[y_1, \dots, y_m]$-algebra.

((i) means as usual that there are no nonzero polynomial relations holding between the y_i; an algebraist's way of saying this is that the natural (surjective) map $k[Y_1, \dots, Y_m] \to k[y_1, \dots, y_m] \subset A$ is injective.)

It is being asserted that, as you might expect, the extension of rings can be built up by first throwing in algebraically independent elements, then 'making an algebraic extension'; however, the statement (ii) is far more precise than this, since it says that every element of A is not just algebraic over $k[y_1, \dots, y_m]$, but satisfies a *monic* equation over it.