at above, we'll see that for some varieties (in fact for all projective varieties), there do not exist any nonconstant regular functions (see Ex. 5.1, Ex. 5.12 and the discussion in (8.10)). Rational functions (that is, 'functions' of the form $f = g/h$, where g, h are polynomial functions) are not defined at points where the denominator vanishes. Although reprehensible, it is a firmly entrenched tradition among algebraic geometers to use 'rational function' and 'rational map' to mean 'only partially defined function (or map)'. So a rational map $f\colon V_1 \dashrightarrow V_2$ is not a map at all; the broken arrow here is also becoming traditional. Students who disapprove are recommended to give up at once and take a reading course in Category Theory instead.

This is not at all a frivolous difficulty. Even regular maps (= morphisms, these are genuine maps) have to be defined as rational maps which are regular at all points $P \in V$ (that is, well defined, the denominator can be chosen not to vanish at P). Closely related to this is the difficulty of giving a proper intrinsic definition of a variety: in this course (and in others like it, in my experience), affine varieties $V \subset \mathbb{A}^n$ and quasiprojective varieties $V \subset \mathbb{P}^n$ will be defined, but there will be no proper definition of 'variety' without reference to an ambient space. Roughly speaking, a variety should be what you get if you glue together a number of affine varieties along isomorphic open subsets. But our present language, in which isomorphisms are themselves defined more or less explicitly in terms of rational functions, is just too cumbersome; the proper language for this glueing is sheaves, which are well treated in graduate textbooks.

0.5 "Purely algebraically defined"

So much for the drawbacks of the algebraic approach to geometry. Having said this, almost all the algebraic varieties of importance in the world today are quasiprojective, and we can have quite a lot of fun with varieties without worrying overmuch about the finer points of definition.

The main advantages of algebraic geometry are that it is purely algebraically defined, and that it applies to any field, not just $\mathbb{R}$ or $\mathbb{C}$; we can do geometry over fields of characteristic p. Don't say 'characteristic p – big deal, that's just the finite fields'; to start with, very substantial parts of group theory are based on geometry over finite fields, as are large parts of combinatorics used in computer science. Next, there are lots of interesting fields of characteristic p other than finite ones. Moreover, at a deep level, the finite fields are present and working inside $\mathbb{Q}$ and $\mathbb{C}$. Most of the deep results on arithmetic of varieties over $\mathbb{Q}$ use a considerable amount of geometry over $\mathbb{C}$ or over the finite fields and their algebraic closures.

This concludes the introduction; see the informal discussion in (2.15) and the final §8 for more general culture.

0.6 Plan of the book

As to the structure of the book, Part I and Part III aim to indicate some worthwhile problems which can be studied by means of algebraic geometry. Part II is an introduction to the commutative algebra referred to in (0.4) and to the categorical framework of algebraic geometry; the student who is prone to headaches could perhaps take some of the proofs for granted here, since the material is standard, and the author is a professional algebraic geometer of the highest moral fibre.

§8 contains odds and ends that may be of interest or of use to the student, but that don't fit in the main text: a little of the history and sociology of the modern subject, hints as to relations of the subject matter with more advanced topics, technical footnotes, etc.

Prerequisites for this course:

Algebra: Quadratic forms, easy properties of commutative rings and their ideals, principal ideal domains and unique factorisation.

Galois Theory: Fields, polynomial rings, finite extensions, algebraic versus transcendental extensions, separability.

Topology and geometry: Definition of topological space, projective space $\mathbb{P}^n$ (but I'll go through it again in detail).

Calculus in $\mathbb{R}^n$: Partial derivatives, implicit function theorem (but I'll remind you of what I need when we get there).

Commutative algebra: Other experience with commutative rings is desirable, but not essential.

Course relates to:

Complex Function Theory An algebraic curve over $\mathbb{C}$ is a 1-dimensional complex manifold, and regular functions on it are holomorphic, so that this course is closely related to complex function theory, even if the relation is not immediately apparent.

Algebraic Number Theory For example the relation with Fermat's Last Theorem.

Catastrophe Theory Catastrophes are singularities, and are essentially always given by polynomial functions, so that the analysis of the geometry of the singularities is pure algebraic geometry.

Commutative Algebra Algebraic geometry provides motivation for commutative algebra, and commutative algebra provides technical support for algebraic geometry, so that the two subjects enrich one another.

Exercises to Chapter 0

0.1 (a) Show that for fixed values of (y, z), x is a repeated root of $x^3 + xy + z = 0$ if and only if $x = -3z/2y$ and $4y^3 + 27z^2 = 0$;

(b) there are 3 distinct roots if and only if $4y^3 + 27z^2 < 0$;

(c) sketch the surface $S : (x^3 + xy + z = 0) \subset \mathbb{R}^3$ and its projection onto the (y, z)-plane;

(d) now open up any book or article on catastrophe theory and compare.

0.2 Let $f \in \mathbb{R}[X, Y]$ and let $C : (f = 0) \subset \mathbb{R}^2$; say that $P \in C$ is *isolated* if there is an $\varepsilon > 0$ such that $C \cap B(P, \varepsilon) = P$. Show by example that C can have isolated points. Prove that if $P \in C$ is an isolated point then $f \colon \mathbb{R}^2 \to \mathbb{R}$ must have a max or min at P, and deduce that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at P. This proves that an isolated point of a real curve is singular.

0.3 *Cubic curves:*

(i) Draw the graph of $y = 4x^3 + 6x^2$ and its intersection with the horizontal lines $y = t$ for integer values of $t \in [-1, 3]$;

(ii) draw the cubic curves $y^2 = 4x^3 + 6x^2 - t$ for the same values of t.