where $P = (a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ is the slope or direction vector of ℓ. Then $f|\ell = f(, \dots, a_i + b_iT, \dots,) = g(T)$ is a polynomial in T, and we know that $(T = 0)$ is one root of g. Hence

$$0 \text{ is a multiple root of } g \iff \frac{\partial g}{\partial T}(0) = 0,$$

that is,

$$\iff \sum_i b_i \frac{\partial f}{\partial X_i}(P) = 0 \iff \ell \subset T_P V.$$

Definition $P \in V \subset \mathbb{A}^n$ is a *nonsingular point* of V if $\partial f/\partial X_i(P) \neq 0$ for some i; otherwise P is a *singular point*, or a *singularity* of V.

Obviously $T_P V$ is an $(n-1)$-dimensional affine subspace of $\mathbb{A}^n$ if P is nonsingular, and $T_P V = \mathbb{A}^n$ if $P \in V$ is singular.

6.2 Remarks

(a) The derivatives $\partial f/\partial X_i(P)$ appearing above are formal algebraic operations (that is, $\partial/\partial X_i$ takes X_i^n into nX_i^{n-1}); no calculus is involved.

(b) Suppose $k = \mathbb{R}$ or $\mathbb{C}$, and that $\partial f/\partial X_i(P) \neq 0$; for clarity let me take $i = 1$. Then the map $p \colon \mathbb{A}^n \to \mathbb{A}^n$ defined by $(X_1, \dots, X_n) \mapsto (f, X_2, \dots, X_n)$ has nonvanishing Jacobian determinant at P, so that by the inverse function theorem, there exists a neighbourhood $P \in U \subset \mathbb{A}^n$ such that $p|U \colon U \to p(U) \subset \mathbb{A}^n$ is a diffeomorphism of the neighbourhood U with an open set $p(U)$ of $\mathbb{A}^n$ (in the usual topology of $\mathbb{R}^n$ or $\mathbb{C}^n$); that is, $p|U$ is bijective, and both p and p^{-1} are differentiable functions of real or complex variables. In other words, $(f, X_2, \dots, X_n)$ form a new differentiable coordinate system on $\mathbb{A}^n$ near P; this implies that a neighbourhood of P in $V : (f = 0)$ is diffeomorphic to an open set in $\mathbb{A}^{n-1}$ with coordinates $(X_2, \dots, X_n)$. Thus near a nonsingular point P, V is a *manifold* with $(X_2, \dots, X_n)$ as local parameters.

Proposition 6.3 $V_{\text{nonsing}} = \{P \in V \mid P \textit{ is nonsingular}\}$ *is a dense open set of* V *for the Zariski topology.*

Proof The complement of V_{nonsing} is the set V_{sing} of singular points, which is defined by $\partial f/\partial X_i(P) = 0$ for all i, that is

$$V_{\text{sing}} = V\Big(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\Big) \subset \mathbb{A}^n,$$

which is closed by definition of the Zariski topology. Since V is irreducible (by (3.11, a), to show that the open V_{nonsing} is dense, I only have to show it's nonempty (by Proposition 4.2); arguing by contradiction, suppose that it's empty, that is, suppose $V = V(f) = V_{\text{sing}}$. Then each of the polynomials $\partial f/\partial X_i$ must vanish on V, therefore (by (3.11) once more) they must be divisible by f in $k[X_1, \dots, X_n]$; but viewed as a polynomial in X_i, $\partial f/\partial X_i$ has degree strictly smaller than f, so that f divides $\partial f/\partial X_i$ implies that in fact $\partial f/\partial X_i = 0$ as a polynomial. Over $\mathbb{C}$, this is obviously only possible if X_i does not appear in f, and if this happens for all i then $f = \text{const.} \in \mathbb{C}$, which is excluded. Over a general field k, $\partial f/\partial X_i = 0$ is only possible if f is an inseparable polynomial in

X_i, that is, $\operatorname{char} k = p$, and X_i only appears in f as the pth power X_i^p. If this happens for each i, then by the argument given in (3.16), f is a pth power in $k[X_1, \dots, X_n]$; this contradicts the fact that f is irreducible. Q.E.D.

6.4 Tangent space

Definition Let $V \subset \mathbb{A}^n$ be a subvariety, with $V \ni P = (a_1, \dots, a_n)$. For any $f \in k[X_1, \dots, X_n]$, write

$$f_P^{(1)} = \sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i).$$

This is an affine linear polynomial (that is, linear plus constant), the 'first order part' of f at P. Now define the *tangent space* to V at P by

$$T_P V = \bigcap \left(f_P^{(1)} = 0 \right) \subset \mathbb{A}^n,$$

where the intersection takes place over all $f \in I(V)$.

Proposition 6.5 *The function $V \to \mathbb{N}$ defined by $P \mapsto \dim T_P V$ is an upper semicontinuous function (in the Zariski topology of V). In other words, for any integer r, the subset*

$$S(r) = \left\{ P \in V \,\middle|\, \dim T_P V \geq r \right\} \subset V$$

is closed.

Proof Let $(f_1, \dots, f_m)$ be a set of generators of $I(V)$; it is easy to see that for any $g \in I(V)$, the linear part $g_P^{(1)}$ of g is a linear combination of those of the f_i, so that the definition of $T_P V$ simplifies to

$$T_P V = \bigcap_{i=1}^{m} \left(f_{i,P}^{(1)} = 0 \right) \subset \mathbb{A}^n.$$

Then by elementary linear algebra,

$$\begin{aligned} P \in S(r) &\iff \text{the matrix } \left(\frac{\partial f}{\partial X_i}(P) \right)_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \text{ has rank } \leq n - r \\ &\iff \text{every } (n - r + 1) \times (n - r + 1) \text{ minor vanishes.} \end{aligned}$$

Now each entry $\partial f_i / \partial X_j (P)$ of the matrix is a polynomial function of P; thus each minor is a determinant of a matrix of polynomials, and so is itself a polynomial. Hence $S(r) \subset V \subset \mathbb{A}^n$ is an algebraic subset. Q.E.D.

Corollary-Definition 6.6 *There exists an integer r and a dense open subset $V_0 \subset V$ such that*

$$\dim T_P V = r \text{ for } P \in V_0, \text{ and } \dim T_P V \geq r \text{ for all } P \in V.$$

Define r to be the dimension *of V, and write* $\dim V = r$*; and say that $P \in V$ is* nonsingular *if* $\dim T_P V = r$*, and* singular *if* $\dim T_P V > r$*. A variety V is* nonsingular *if it is nonsingular at each point $P \in V$.*