Theorem 4.8 *(I)* $\operatorname{dom} f$ *is open and dense in the Zariski topology.*

Suppose that the field k is algebraically closed; then

(II)

$$\operatorname{dom} f = V \iff f \in k[V];$$

(that is polynomial function = regular rational function*). Furthermore, for any $h \in k[V]$, let*

$$V_h = V \setminus V(h) = \{P \in V \mid h(P) \neq 0\};$$

then

(III)

$$\operatorname{dom} f \supset V_h \iff f \in k[V][h^{-1}].$$

Proof Define the *ideal of denominators* of $f \in k(V)$ by

$$\begin{aligned} D_f &= \{h \in k[V] \mid hf \in k[V]\} \subset k[V] \\ &= \{h \in k[V] \mid \exists \text{ an expression } f = g/h \text{ with } g \in k[V]\} \cup \{0\}. \end{aligned}$$

From the first line, D_f is obviously an ideal of $k[V]$. Then formally,

$$V \setminus \operatorname{dom} f = \{P \in V \mid h(P) = 0 \text{ for all } h \in D_f\} = V(D_f),$$

so that $V \setminus \operatorname{dom} f$ is an algebraic set of V; hence $\operatorname{dom} f = V \setminus V(D_f)$ is the complement of a closed set, so open in the Zariski topology. It is obvious that $\operatorname{dom} f$ is nonempty, hence dense by Proposition 4.2.

Now using (b) of the Nullstellensatz,

$$\operatorname{dom} f = V \iff V(D_f) = \emptyset \iff 1 \in D_f, \quad \text{that is, } f \in k[V].$$

Finally,

$$\operatorname{dom} f \supset V_h \iff h \text{ vanishes on } V(D_f),$$

and using (c) of the Nullstellensatz,

$$\iff h^n \in D_f \text{ for some } n, \text{ that is, } f = g/h^n \in k[V][h^{-1}]. \quad \text{Q.E.D.}$$

4.9 Rational maps

Let V be an affine variety.

Definition A *rational map* $f\colon V \dashrightarrow \mathbb{A}^n_k$ is a partially defined map given by rational functions $f_1, \dots, f_n$, that is,

$$f(P) = f_1(P), \dots, f_n(P) \quad \text{for all } P \in \bigcap \operatorname{dom} f_i.$$

By definition, $\operatorname{dom} f = \bigcap \operatorname{dom} f_i$; as before, F is said to be *regular* at $P \in V$ if and only if $P \in \operatorname{dom} f$. A rational map $V \dashrightarrow W$ between two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is defined to be a rational map $f\colon V \dashrightarrow \mathbb{A}^m$ such that $f(\operatorname{dom} f) \subset W$.

Two examples of rational maps were described at the end of (4.3).