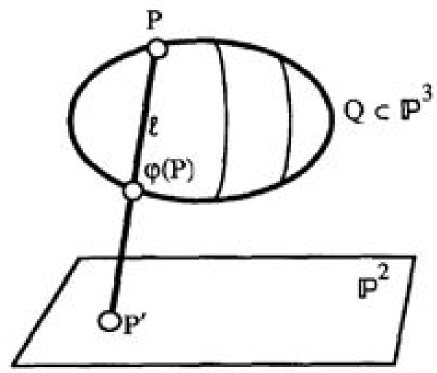

Figure 5.2: Projection of quadric surface

5.8 Birational maps

Definition Let V and W be (affine or projective) varieties; then a rational map $f\colon V \dashrightarrow W$ is *birational* (or is a *birational equivalence*) if it has a rational inverse, that is, if there exists a rational map $g\colon W \dashrightarrow V$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$.

Proposition *The following three conditions on a rational map $f\colon V \dashrightarrow W$ are equivalent:*

(i) f is a birational equivalence;

(ii) f is dominant (see (4.10)), and $f^*\colon k(W) \to k(V)$ is an isomorphism;

(iii) there exist open sets $V_0 \subset V$ and $W_0 \subset W$ such that f restricted to V_0 is an isomorphism $f\colon V_0 \to W_0$.

Proof f^* is defined in the same way as for affine varieties, and (i) $\iff$ (ii) is as in (4.11). (iii) $\implies$ (i) is clear, since an isomorphism $f\colon V_0 \to W_0$ and its inverse $g = f^{-1}\colon W_0 \to V_0$ are by definition rational maps between V and W.

The essential implication (i) $\implies$ (iii) is tricky, although content-free (GOTO (5.9) if you want to avoid a headache): by assumption (i), there are inverse rational maps $f\colon V \dashrightarrow W$ and $g\colon W \dashrightarrow V$; now set $V' = \operatorname{dom} f \subset V$ and $\varphi = f|V'\colon V' \to W$, and similarly $W' = \operatorname{dom} g \subset W$ and $\psi = g|W'\colon W' \to V$. In the diagram

$$\begin{array}{ccccc} \psi^{-1}V' & \xrightarrow{\psi} & V' & \xrightarrow{\varphi} & W \\ \cap & & & & \\ W & & & & \end{array}$$

all the arrows are morphisms, and $\mathrm{id}_W|\psi^{-1}V' = \varphi \circ \psi$ (as morphisms) follows from $\mathrm{id}_W = f \circ g$ (as rational maps). Hence

$$\varphi(\psi(P)) = P \quad \text{for all } P \in \psi^{-1}V'.$$

Now set $V_0 = \varphi^{-1}\psi^{-1}V'$, and $W_0 = \psi^{-1}\varphi^{-1}W'$; then $\varphi\colon V_0 \to \psi^{-1}V'$ is a morphism by construction. However, $\psi^{-1}V' \subset W_0$, since $P \in \psi^{-1}V'$ implies that $\varphi(\psi(P)) = P$, so that $P \in \psi^{-1}\varphi^{-1}W' = W_0$. Therefore, $\varphi\colon V_0 \to W_0$ is a morphism, and similarly $\psi\colon W_0 \to V_0$. Q.E.D.

5.9 Rational varieties

The notion of birational equivalence discussed in (5.8) is of key importance in algebraic geometry. Condition (iii) in the proposition says that the 'meat' of the varieties V and W is the same, although they may differ a bit around the edges; an example of the use of birational transformations is blowing up a singular variety to obtain a nonsingular one, see (6.12) below. An important particular case of Proposition 5.8 is the following result.

Corollary *Given a variety V, the following two conditions are equivalent:*

(a) the function field $k(V)$ is a purely transcendental extension of k, that is $k(V) \cong k(t_1, \dots, t_n)$ for some n;

(b) there exists a dense open set $V_0 \subset V$ which is isomorphic to a dense open subset $U_0 \subset \mathbb{A}^n$.

A variety satisfying these conditions is said to be *rational*. Condition (b) is a precise version of the statement that V can be parametrised by n independent variables. This notion has already appeared implicitly several times in these notes (for example, (1.1), (2.1), (3.11, b), (5.7, II)). A large proportion of the elementary applications of algebraic geometry to other branches of math are related one way or another to rational varieties.

5.10 Reduction to a hypersurface

An easy consequence of the discussion of Noether normalisation at the end of §3 is that every variety is birational to a hypersurface: firstly, since birational questions only depend on a dense open set, and any open set contains a dense open subset isomorphic to an affine variety (by (4.13)), I only need to consider an affine variety $V \subset \mathbb{A}^n$. It was proved in (3.18) that there exist elements $y_1, \dots, y_{m+1} \in k[V]$ which generate the field extension $k \subset k(V)$, and such that $y_1, \dots, y_m$ are algebraically independent, and y_{m+1} is algebraic over $k(y_1, \dots, y_m)$. These elements thus define a morphism $V \to \mathbb{A}^{m+1}$ which is a birational equivalence of V with a hypersurface $V' \subset \mathbb{A}^{m+1}$.

5.11 Products

If V and W are two affine varieties then there is a natural sense in which $V \times W$ is again a variety: if $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ then $V \times W$ is the subset of $\mathbb{A}^{n+m}$ given by

$$\left\{ ((\alpha_1, \dots, \alpha_n); (\beta_1, \dots, \beta_m)) \;\middle|\; \begin{array}{l} f(\underline{\alpha}) = 0 \text{ for all } f \in I(V) \\ g(\underline{\beta}) = 0 \text{ for all } g \in I(W) \end{array} \right\}$$

It's easy to check that $V \times W$ remains irreducible. Note however that the Zariski topology of the product is not the product of the Zariski topologies (see Ex. 5.10).