is allowed as a $\mathbb{C}$-valued point of C. Since π is transcendental over $\mathbb{Q}$, any polynomial $f \in \mathbb{Q}[x, y]$ vanishing at P_π is a multiple of $x^2 + y^2 - 1$; so P_π is a $\mathbb{Q}$-*generic* point of C – it's not in any smaller subvariety of C defined over $\mathbb{Q}$. In other words, the conjugates of P_π under $\operatorname{Aut}\mathbb{C}$ ("$= \operatorname{Gal}(\mathbb{C}/\mathbb{Q})$") are dense in C. Since P_π is $\mathbb{Q}$-generic, if you prove a statement only involving polynomials over $\mathbb{Q}$ about P_π, the same statement will be true for every point of C.

In fact this idea is already covered by the notion of an L-valued point described in (b), and the geometric content of generic points can be seen most clearly in this language. For example, the field $\mathbb{Q}(\pi)$ is just the purely transcendental extension, so $\mathbb{Q}(\pi) \cong \mathbb{Q}(\lambda)$ and the morphism $\operatorname{Spec}\mathbb{Q}(\lambda) \to C$ is the rational parametrisation of C discussed in (1.1): roughly, you're allowed to substitute any 'sufficiently general' value for the transcendental or unknown π. More generally, a finitely generated extension $k \subset L$ is the function field of a variety W over k; suppose that $\varphi\colon \operatorname{Spec} L \to V = \operatorname{Spec} A$ is a point corresponding to a k-algebra homomorphism $A \to L$, having kernel P. Then φ extends to a rational map $f\colon W \dashrightarrow V$ whose image is dense in the subvariety $Y = V(P) \subset V$, so φ or $\varphi(\operatorname{Spec} L)$ is a field-valued generic point of Y.

(d) Points as morphisms in scheme theory

The discussion in (c) shows that an L-valued point of a variety V contains implicitly a rational map $W \dashrightarrow V$, where W is a variety birational to $\operatorname{Spec} L$ (that is, $L = k(W)$); a geometer could think of this as a family of points parametrised by W.

More generally, for X a variety (or scheme) we are interested in, an S-valued point of X (where S is any scheme) can just be defined as a morphism $S \to X$. If $X = V(I) \subset \mathbb{A}^n_k$ is affine with coordinate ring $k[X]$ and $S = \operatorname{Spec} A$, then an S-valued point corresponds under (4.4) to a k-algebra homomorphism $k[X] \to A$, that is, to an n-tuple $(a_1, \dots, a_n)$ of elements of A satisfying $f(a) = 0$ for all $f \in I$.

In a highbrow sense, this is the final apotheosis of the notion of a variety: if a point of a variety X is just a morphism, then X itself is just the functor

$$S \mapsto X(S) = \bigl\{\text{morphisms } S \to X\bigr\}$$

on the category of schemes. (The fuss I made about the notation $\mathbb{A}^n_k$ in the footnote on p. 59 already reflect this.) Unlikely as it may seem, these metaphysical incantations are technically very useful, and varieties defined as functors are basic in the modern view of moduli spaces. Given a geometric construction that can 'depend algebraically on parameters' (such as space curves of fixed degree and genus), you can ask to endow the set of all possible constructions with the structure of an algebraic variety. Even better, you could ask for a family of constructions over a parameter space that is 'universal', or 'contains all possible constructions'; the parameter variety of this universal family can usually be defined most directly as a functor (you still have to prove that the variety exists). For example the Chow variety referred to in (8.2) represents the functor

$$S \mapsto \bigl\{\text{families of curves parametrised by } S\bigr\}.$$

8.14 How schemes are more general than varieties

I now discuss in isolation 3 ways in which affine schemes are more general than affine varieties; in cases of severe affliction, these complications may occur in combination with each other, with the global problems discussed in (8.11), or even in combination with new phenomena such as p-adic convergence or Arakelov Hermitian metrics. Considerations of space fortunately save me from having to say more on these fascinating topics.

(i) Not restricted to finitely generated algebras

Suppose $C \subset S$ is a curve on a nonsingular affine surface (over $\mathbb{C}$, if you must). The ring

$$\mathcal{O}_{S,C} = \left\{ f \in k(S) \;\middle|\; f = g/h \text{ with } h \notin IC \right\} \subset k(S)$$

is the *local ring* of S at C; elements $f \in \mathcal{O}_{S,C}$ are regular on an open set of S containing a dense open subset of C. Divisibility theory in this ring is very splendid, and relates to the geometric idea of zeros and poles of a meromorphic function: C is locally defined by a single equation ($y = 0$) with $y \in I_C$ a local generator, and every nonzero element $f \in \mathcal{O}_S, C$ is of the form $f = y^n \cdot f_0$, where $n \in \mathbb{Z}$ and f_0 is an invertible element of $\mathcal{O}_{S,C}$. A ring with this property is called a *discrete valuation ring* (d.v.r.), in honour of the discrete valuation $f \mapsto n$, which counts the order of zero of f along C ($n < 0$ corresponds to poles); the element y is called a *local parameter* of $\mathcal{O}_{S,C}$.

Now scheme theory allows us boldly to consider $\operatorname{Spec} \mathcal{O}_{S,C}$ as a geometric object, the topological space ($\cdot -$) with only two points: a closed point, the maximal ideal (y) (= the generic point of C) and a nonclosed point, the zero ideal 0 (= the generic point of S). The advantage here is not so much technical: the easy commutative algebra of discrete valuation rings was of course used to prove results in algebraic geometry and complex function theory (for example, about ideals of functions, or about the local behaviour above C of a branched cover $T \to S$ in terms of the field extension $k(S) \subset k(T)$) long before schemes were invented. More important, it gives us a precise geometric language, and a simple picture of the local algebra.

The above is just one example, related to localisation, or the idea of 'neighbourhood of a generic point of a subvariety', of benefits to ordinary geometry from taking Spec of a ring more general than a finitely generated algebra over a field; a similar example is thinking of the generic point $\operatorname{Spec} k(W)$ of a variety W as the variety obtained as the intersection of all nonempty open sets of W (compare (8.13, c)), like the grin remaining after the Cheshire cat's face has disappeared.

(ii) Nilpotents

The ring A can have nilpotent elements; for example $A = k[x,y]/(y^2 = 0)$ corresponds to the 'double line' $2\ell \subset \mathbb{A}^2_k$, to be thought of as an infinitesimal strip neighbourhood of the line. An element of A is of the form $f(x) + \varepsilon f_1(x)$ (with $\varepsilon^2 = 0$), so it looks like a Taylor series expansion of a polynomial about ℓ truncated to first order. If you practise hard several times a day, you should be able to visualise this as a function on the double line 2ℓ.

Nilpotents allow scheme theory to deal in Taylor series truncated to any order, so for example to deal with points of a variety by power series methods. They are crucial in the context of the moduli problems discussed at the end of (8.14, d): for example, they provide a precise language for handling first order infinitesimal deformations of a geometric construction (as a construction over the parameter space $\operatorname{Spec}(k[\varepsilon]/(\varepsilon^2 = 0))$), and viewing these as tangent vectors to the universal parameter variety. They also open up a whole range of phenomena for which there was no classical analogue, for example relations between inseparable field extensions and Lie algebras of vector fields on varieties in characteristic p.

(iii) No base field

Let p be a prime number, and $\mathbb{Z}_{(p)} \subset \mathbb{Q}$ the subring of rationals with no p in the denominator; $\mathbb{Z}_{(p)}$ is another discrete valuation ring, with parameter p. It has a unique maximal ideal $0 \neq p\mathbb{Z}_{(p)}$, with residue field $\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)} \cong \mathbb{F}_p = \mathbb{Z}_{(p)}/(p)$. If $F \in \mathbb{Z}_{(p)}[X,Y]$, then it makes sense to consider the curve $C_{\mathbb{C}} : (F = 0) \subset \mathbb{A}^2_{\mathbb{C}}$, or alternatively to take the reduction f of F mod p, and to consider the curve $C_p : (f = 0) \subset \mathbb{A}^2\mathbb{F}_p$. What kind of geometric object is it that contains both a curve over the complexes and a curve over a finite field? Whether you consider it to be truly geometric is a matter of opinion, but the scheme $\operatorname{Spec} \mathbb{Z}_{(p)}[X,Y]/(F)$ does exactly this.

Again, this is technically not a new idea: reducing a curve mod p has been practised since the 18th century, and Weil foundations contained a whole theory of 'specialisation' to deal with it. The advantage is a better conceptual picture of the curve $\operatorname{Spec} \mathbb{Z}_{(p)}[X,Y]/(F)$ over the d.v.r. $\mathbb{Z}_{(p)}$ as a geometric object fibred over $\operatorname{Spec}(\mathbb{Z}_{(p)})$ ('$=$ ($\cdot$-)'), with the two curve $C_{\mathbb{C}}$ and C_p as generic and special fibres.

In the same way, for $F \in \mathbb{Z}[X,Y]$, the scheme $\operatorname{Spec} \mathbb{Z}[X,Y]/(F)$ is a geometric object containing for every prime p the curve $C_p : (f_p = 0) \subset \mathbb{A}^2_{\mathbb{F}_p}$ over $\mathbb{F}_p$, where f_p is the reduction of F mod p, and at the same time the curve $C_{\mathbb{C}} : (F = 0) \subset \mathbb{A}^2_{\mathbb{C}}$, and is called an *arithmetic surface*; it contains quite a lot besides: in particular, for every point $c \in C_{\mathbb{C}}$ with algebraic numbers as coordinates, it contains a copy of $\operatorname{Spec} \mathbb{Q}[c]$, hence essentially all the information about the ring of integers of the number field $\mathbb{Q}(c)$ of definition of c.

However grotesquely implausible this object may seem at first sight (you can again get used to it if you practise), it is a key ingredient in modern number theory, and is the basic foundation on which the work of Arakelov and Faltings rests.

8.15 Proof of the existence of lines on a cubic surface

Every adult algebraic geometer knows the traditional proof of (7.2) by dimension counting (see for example [Beauville, Complex algebraic surfaces, p. 50], or [Mumford, Algebraic geometry I, Complex projective varieties, p. 174]). I run through this before commenting on the difficulties.

The set of lines of $\mathbb{P}^3$ is parametrised by the 4-dimensional Grassmannian $\mathrm{Gr} = \mathrm{Gr}(2,4)$, and cubic surfaces by the projective space $S = \mathbb{P}^N$ of cubic forms in (X,Y,Z,T) (in fact $N = 19$). Write $Z \subset \mathrm{Gr} \times S$ for the incidence subvariety

$$Z = \bigl\{(\ell, X) \bigm| \ell \in \mathrm{Gr}, X \in S \text{ s.t. } \ell \subset X\bigr\}.$$

Since cubic forms vanishing on a given line ℓ form a $\mathbb{P}^{N-4}$, it is easy to deduce from the first projection $Z \to \mathrm{Gr}$ that Z is a rational N-dimensional variety. So the second projection $p\colon Z \to S$ is a morphism between two N-dimensional varieties, and therefore

(i) *either* the image $p(Z)$ is an N-dimensional variety in S, and so contains a dense open of S,
or every fibre of p has dimension ≥ 1.

(ii) Z is a projective variety, so that the image $p(Z)$ is closed in S.

Since cubic surfaces containing only finitely many lines do exist, the second possibility in (i) doesn't occur, so every sufficiently general cubic surface contains lines. Then (ii) ensures that $p(Z) = S$, and every cubic surface contains lines.