Chapter 5

Projective and birational geometry

The first part of §5 aims to generalise the content of §§3–4 to projective varieties; this is fairly mechanical, with just a few essential points. The remainder of the section is concerned with birational geometry, taking up the function field $k(V)$ from the end of §4; this is material which fits equally well into the projective or affine context.

5.0 Why projective varieties?

The cubic curve

$$C : (Y^2Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2$$

is the union of two affine curves

$$\begin{aligned} &C_0 : (y^2 = x^3 + ax + b) \subset \mathbb{A}^2 \quad \text{(the piece } (Z = 1) \text{ of } C) \quad \text{and} \\ &C_1 : (z_1 = x_1^3 + axz_1^2 + bz_1^3) \subset \mathbb{A}^2 \quad \text{(the piece } (Y = 1)), \end{aligned}$$

glued together by the isomorphism

$$\text{by} \quad \begin{array}{rcl} C_0 \setminus (y = 0) & \longrightarrow & C_1 \setminus (z_1 = 0) \\ (x, y) & \longmapsto & (x/y, 1/y). \end{array}$$

As a much simpler example, $\mathbb{P}^1$ with homogeneous coordinates (X, Y) is the union of 2 copies of $\mathbb{A}^1$ with coordinates x_0, y_1 respectively, glued together by the isomorphism

$$\text{by} \quad \begin{array}{rcl} \mathbb{A}^1 \setminus (x_0 = 0) & \longrightarrow & \mathbb{A}^1 \setminus (y_1 = 0) \\ x_0 & \longmapsto & 1/x_0. \end{array}$$

The usual picture is

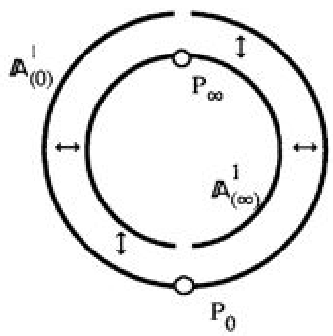

Figure 5.1: $\mathbb{P}^1$ glued from two $\mathbb{A}^1$s

(the arrows $\leftrightarrow$ denote glueing).

It's important to understand that *these varieties are strictly bigger than any affine variety*. In fact, with the natural notion of morphism (to be introduced shortly), it can be seen that there are no nonconstant morphisms $\mathbb{P}^1 \to \mathbb{A}^n$ or $C \to \mathbb{A}^n$ for any n (see Ex. 5.1 and Ex. 5.12, and the discussion in (8.10)).

One solution to this problem is to define the notion of 'abstract variety' V as a union $V = \bigcup V_i$ of affine varieties, modulo suitable glueing. By analogy with the definition of manifolds in topology, this is an attractive idea, but it leads to many more technical difficulties. Using projective varieties sidesteps these problems by working in the ready-made ambient space $\mathbb{P}^n$, so that (apart from a little messing about with homogeneous polynomials) they are not much harder to study than affine varieties. In fact, although this may not be clear at an elementary level, projective varieties to a quite remarkable extent provide a natural framework for studying varieties (this is briefly discussed from a more advanced point of view in (8.11)).

5.1 Graded rings and homogeneous ideals

Definition *A polynomial $f \in k[X_0, \dots, X_n]$ is* homogeneous of degree d *if*

$$f = \sum a_{i_0 \dots i_n} X_0^{i_0} \cdots X_n^{i_n} \text{ with } a_{i_0 \dots i_n} \neq 0 \text{ only if } i_0 + \cdots + i_n = d.$$

Any $f \in k[X_0, \dots, X_n]$ has a unique expression $f = f_0 + f_1 + \cdots + f_N$ in which f_d is homogeneous of degree d for each $d = 0, 1, \dots, N$.

Proposition *If f is homogeneous of degree d then*

$$f(\lambda X_0, \dots, \lambda X_n) = \lambda^d f(X_0, \dots, X_n) \quad \text{for all } \lambda \in k;$$

if k is an infinite field then the converse also holds.