

## Chapter 6

# Tangent space and nonsingularity, dimension

### 6.1 Nonsingular points of a hypersurface

Suppose  $f \in k[X_1, \dots, X_n]$  is irreducible,  $f \notin k$ , and set  $V = V(f) \subset \mathbb{A}^n$ ; let  $P = (a_1, \dots, a_n) \in V$ , and  $\ell$  be a line through  $P$ . Since  $P \in V$ , obviously  $P$  is a root of  $f|_{\ell}$ .

**Question:** When is  $P$  a multiple root of  $f|_{\ell}$ ?

**Answer:** If and only if  $\ell$  is contained in the affine linear subspace

$$T_P V : \left( \sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i) = 0 \right) \subset \mathbb{A}^n,$$

called the *tangent space* to  $V$  at  $P$ .

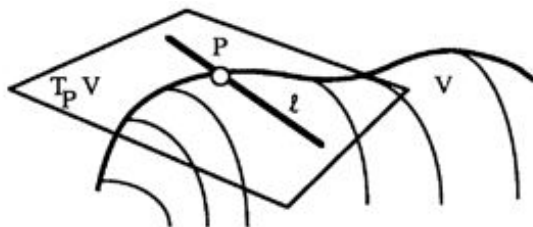


Figure 6.1: Tangent space

To prove this, parametrise  $\ell$  as

$$\ell : X_i = a_i + b_i T,$$

where  $P = (a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  is the slope or direction vector of  $\ell$ . Then  $f|_\ell = f(\dots, a_i + b_i T, \dots) = g(T)$  is a polynomial in  $T$ , and we know that  $(T = 0)$  is one root of  $g$ . Hence

$$0 \text{ is a multiple root of } g \iff \frac{\partial g}{\partial T}(0) = 0,$$

that is,

$$\iff \sum_i b_i \frac{\partial f}{\partial X_i}(P) = 0 \iff \ell \subset T_P V.$$

**Definition**  $P \in V \subset \mathbb{A}^n$  is a *nonsingular point* of  $V$  if  $\partial f / \partial X_i(P) \neq 0$  for some  $i$ ; otherwise  $P$  is a *singular point*, or a *singularity* of  $V$ .

Obviously  $T_P V$  is an  $(n-1)$ -dimensional affine subspace of  $\mathbb{A}^n$  if  $P$  is nonsingular, and  $T_P V = \mathbb{A}^n$  if  $P \in V$  is singular.

## 6.2 Remarks

- (a) The derivatives  $\partial f / \partial X_i(P)$  appearing above are formal algebraic operations (that is,  $\partial / \partial X_i$  takes  $X_i^n$  into  $nX_i^{n-1}$ ); no calculus is involved.
- (b) Suppose  $k = \mathbb{R}$  or  $\mathbb{C}$ , and that  $\partial f / \partial X_i(P) \neq 0$ ; for clarity let me take  $i = 1$ . Then the map  $p: \mathbb{A}^n \rightarrow \mathbb{A}^n$  defined by  $(X_1, \dots, X_n) \mapsto (f, X_2, \dots, X_n)$  has nonvanishing Jacobian determinant at  $P$ , so that by the inverse function theorem, there exists a neighbourhood  $P \in U \subset \mathbb{A}^n$  such that  $p|_U: U \rightarrow p(U) \subset \mathbb{A}^n$  is a diffeomorphism of the neighbourhood  $U$  with an open set  $p(U)$  of  $\mathbb{A}^n$  (in the usual topology of  $\mathbb{R}^n$  or  $\mathbb{C}^n$ ); that is,  $p|_U$  is bijective, and both  $p$  and  $p^{-1}$  are differentiable functions of real or complex variables. In other words,  $(f, X_2, \dots, X_n)$  form a new differentiable coordinate system on  $\mathbb{A}^n$  near  $P$ ; this implies that a neighbourhood of  $P$  in  $V: (f = 0)$  is diffeomorphic to an open set in  $\mathbb{A}^{n-1}$  with coordinates  $(X_2, \dots, X_n)$ . Thus near a nonsingular point  $P$ ,  $V$  is a *manifold* with  $(X_2, \dots, X_n)$  as local parameters.

**Proposition 6.3**  $V_{\text{nonsing}} = \{P \in V \mid P \text{ is nonsingular}\}$  is a dense open set of  $V$  for the Zariski topology.

**Proof** The complement of  $V_{\text{nonsing}}$  is the set  $V_{\text{sing}}$  of singular points, which is defined by  $\partial f / \partial X_i(P) = 0$  for all  $i$ , that is

$$V_{\text{sing}} = V\left(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right) \subset \mathbb{A}^n,$$

which is closed by definition of the Zariski topology. Since  $V$  is irreducible (by (3.11, a)), to show that the open  $V_{\text{nonsing}}$  is dense, I only have to show it's nonempty (by Proposition 4.2); arguing by contradiction, suppose that it's empty, that is, suppose  $V = V(f) = V_{\text{sing}}$ . Then each of the polynomials  $\partial f / \partial X_i$  must vanish on  $V$ , therefore (by (3.11) once more) they must be divisible by  $f$  in  $k[X_1, \dots, X_n]$ ; but viewed as a polynomial in  $X_i$ ,  $\partial f / \partial X_i$  has degree strictly smaller than  $f$ , so that  $f$  divides  $\partial f / \partial X_i$  implies that in fact  $\partial f / \partial X_i = 0$  as a polynomial. Over  $\mathbb{C}$ , this is obviously only possible if  $X_i$  does not appear in  $f$ , and if this happens for all  $i$  then  $f = \text{const.} \in \mathbb{C}$ , which is excluded. Over a general field  $k$ ,  $\partial f / \partial X_i = 0$  is only possible if  $f$  is an inseparable polynomial in