where

$$b_0 = -3\alpha, \quad b_1 = g_1(1, \alpha, \alpha^3, 0; 0, 1, 3\alpha^2, 0) = 6\alpha^5 + \cdots,$$
$$b_2 = g_1(1, \alpha, \alpha^3, 0; 0, 0, -a(6), 1) = -2\alpha^9 + \cdots.$$

Similarly, substituting for Z in C, and expanding the quadratic form g gives

$$C = -Y^3 + g(0, Y, 3\alpha^2 Y - a^{(6)}T, T)T = c_0Y^3 + c_1Y^2T + c_2YT^2 + c_3T^3,$$

where

$$c_0 = -1, \quad c_1 = g(0, 1, 3\alpha^2, 0) = 9\alpha^4 + \cdots,$$
$$c_2 = g_1(0, 1, 3\alpha^2, 0; 0, 0, -a(6), 1) = -6\alpha^8 + \cdots,$$
$$c_3 = g(0, 0, -a(6), 1) = \alpha^{12} + \cdots.$$

Now by the result of Ex. 1.10, B' and C' have a common zero $(\eta : \tau)$ if and only if

$$\det \begin{vmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{vmatrix} = 0.$$

The determinant is a polynomial in α, and it's not hard to see that its leading term comes from taking the leading term in each entry of the determinant:

$$\det \begin{vmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{vmatrix} = \alpha^{27} \cdot \det \begin{vmatrix} -3 & 6 & 2 & & \\ & -3 & 6 & 2 & \\ & & -3 & 6 & 2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{vmatrix}$$
$$= \alpha^{27}.$$

This completes the proof of the main claim. Q.E.D.

Proposition 7.3 *Given a line $\ell \subset S$, there exist exactly 5 pairs (ℓ_i, ℓ_i') of lines of S meeting ℓ, in such a way that*

(i) for $i = 1, \ldots, 5$, $\ell \cup \ell_i \cup \ell_i'$ is coplanar, and

(ii) for $i \neq j$, $(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \emptyset$.

Proof *(taken from [Beauville, p. 51])* If Π is a plane of $\mathbb{P}^3$ containing ℓ then $\Pi \cap S = \ell +$ conic (since $f|\Pi$ is divisible by the equation of ℓ). This conic can either be singular or nonsingular:

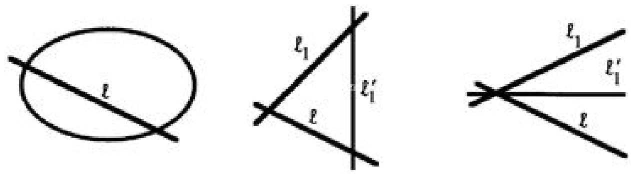

Figure 7.2: Line plus conic

I have to prove that there are exactly 5 distinct planes $\Pi_i \supset \ell$ for which the singular case occurs. The fact stated as property (ii) that lines in different planes are disjoint will then follow from (7.1, a).

Suppose that $\ell : (Z = T = 0)$; then I can expand f out as

$$f = AX^2 + BXY + CY^2 + DX + EY + F, \tag{$*$}$$

where $A, B, C, D, E, F \in k[Z, T]$, with A, B and C linear forms, D and E quadratic forms, and F a cubic form. If I consider this equation as a variable conic in X and Y, it is singular if and only if

$$\Delta(Z,T) = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} = 4ACF + BDE - AE^2 - B^2F - CD^2 = 0.$$

(Here Δ is 4 times the usual determinant if char $\neq 2$; in characteristic 2 the statement is an easy exercise.)

To be more precise, any plane through ℓ is given by $\Pi : (\mu Z = \lambda T)$; if $\mu \neq 0$, I can assume $\mu = 1$, so that $Z = \lambda T$. Then in terms of the homogeneous coordinates (X, Y, T) on Π, $f|\Pi = T \cdot Q(X, Y, T)$, where

$$\begin{aligned} Q = A(\lambda, 1)X^2 + B(\lambda, 1)XY + C(\lambda, 1)Y^2 \\ + D(\lambda, 1)TX + E(\lambda, 1)TY + F(\lambda, 1)T^2. \end{aligned}$$

Now $\Delta(Z, T)$ is a homogeneous quintic, so by (1.8), it has 5 roots counted with multiplicities. To prove the proposition, I have to show that it doesn't have multiple roots; this also is a consequence of the nonsingularity of S.

Claim $\Delta(Z, T)$ *has only simple roots.*

Suppose $Z = 0$ is a root of Δ, and let $\Pi : (Z = 0)$ be the corresponding plane; I have to prove that Δ is not divisible by Z^2. By the above picture, $\Pi \cap S$ is a set of 3 lines, and according to whether they are concurrent, I can arrange the coordinates so that

either (i) $\ell : (T = 0)$, $\ell_1 : (X = 0)$, $\ell_1' : (Y = 0)$,

or (ii) $\ell : (T = 0)$, $\ell_1 : (X = 0)$, $\ell_1' : (X = T)$.

Hence, in case (i), $f = XYT + Zg$, with g quadratic, and in terms of the expression $(*)$, this means that $B = T + aZ$, and $Z \mid A, C, D, E, F$. Therefore, modulo terms divisible by Z^2,

$$\Delta \equiv -T^2 F \mod Z^2.$$

In addition, the point $P = (0,0,0,1) \in S$, and nonsingularity at P means that F must contain the term ZT^2 with nonzero coefficient. In particular, Z^2 does not divide F. Therefore $(Z = 0)$ is a simple root of Δ.

Case (ii) is a similar calculation (see Ex. 7.1).

Corollary 7.4 *1. There exist two disjoint lines $\ell, m \subset S$.*

2. S is rational (that is, birational to $\mathbb{P}^2$, see (5.9)).

Proof (a) By (7.3, ii), just take ℓ_1 and ℓ_2.

(b) Consider two disjoint lines $\ell, m \subset S$, and define rational maps

$$\varphi\colon S \dashrightarrow \ell \times m \quad \text{and} \quad \psi\colon \ell \times m \dashrightarrow S$$

as follows. If $P \in \mathbb{P}^3 \setminus (\ell \cup m)$ then there exists a unique line n through P which meets both ℓ and m:

$$P \in n, \quad \text{and} \quad \ell \cap n \neq \emptyset,\ m \cap n \neq \emptyset.$$

Set $\Phi(P) = (\ell \cap n, m \cap n) \in \ell \times m$. This defines a morphism

$$\Phi\colon \mathbb{P}^3 \setminus (\ell \cup m) \to \ell \times m,$$

whose fibre above $(Q, R) \in \ell \times m$ is the line QR of $\mathbb{P}^3$. Define $\varphi\colon S \dashrightarrow \ell \times m$ as the restriction to S of Φ.

Conversely, for $(Q, R) \in \ell \times m$, let n be the line $n = QR$ in $\mathbb{P}^3$. By (7.3), there are only finitely many lines of S meeting ℓ, so that for almost all values of (Q, R), n intersects S in 3 points $\{P, Q, R\}$, of which Q and R are the given points on ℓ and m. Thus define $\psi\colon \ell \times m \dashrightarrow S$ by $(Q, R) \mapsto P$; then ψ is a rational map, since the ratios of coordinates of P are rational functions of those of Q, R.

Obviously φ and ψ are mutual inverses. Q.E.D.

7.5 Finding all the lines of S

I want to find all the lines of S in terms of the configuration given by Proposition 7.3 of a line ℓ and 5 disjoint pairs (ℓ_i, ℓ_i'). Any other line $n \subset S$ must meet exactly one of ℓ_i and ℓ_i' for $i = 1, \dots, 5$: this is because in $\mathbb{P}^3$, n meets the plane Π_i, and $\Pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$; also, n cannot meet both ℓ_i and ℓ_i', since this would contradict (7.1, a). The key to sorting out the remaining lines is the following lemma, which tells us that n is uniquely determined by which of the ℓ_i and ℓ_i' it meets. Let me say that a line n is a *transversal* of a line ℓ if $\ell \cap n \neq \emptyset$.

Lemma *If $\ell_1, \dots, \ell_4 \subset \mathbb{P}^3$ are disjoint lines then*

either *all 4 lines ℓ_i lie on a smooth quadric $\ell_1, \dots, \ell_4 \subset Q \subset \mathbb{P}^3$; and then they have an infinite number of common transversals;*

or *the 4 lines ℓ_i do not lie on any quadric $\ell_1, \dots, \ell_4 \not\subset Q$; and then they have either 1 or 2 common transversals.*