**Lemma 2.5** *Suppose that $k$ is an infinite field, and let $F \in S_d$.*

*(i) Let $L \subset \mathbb{P}^2_k$ be a line; if $F \equiv 0$ on $L$, then $F$ is divisible in $k[X,Y,Z]$ by the equation of $L$. That is, $F = H \cdot F'$ where $H$ is the equation of $L$ and $F' \in S_{d-1}$.*

*(ii) Let $C \subset \mathbb{P}^2_k$ be a nonempty nondegenerate conic; if $F \equiv 0$ on $C$, then $F$ is divisible in $k[X,Y,Z]$ by the equation of $C$. That is, $F = Q \cdot F'$ where $Q$ is the equation of $C$ and $F' \in S_{d-2}$.*

If you think this statement is obvious, congratulations on your intuition: you have just guessed a particular case of the Nullstellensatz. Now find your own proof (GOTO 2.6).

**Proof** (i) By a change of coordinates, I can assume $H = X$. Then for any $F \in S_d$, there exists a unique expression $F = X \cdot F'_{d-1} + G(Y,Z)$: just gather together all the monomials involving $X$ into the first summand, and what's left must be a polynomial in $Y, Z$ only. Now

$$F \equiv 0 \text{ on } L \iff G \equiv 0 \text{ on } L \iff G(Y,Z) = 0.$$

The last step holds because of (1.8): if $G(Y,Z) \neq 0$ then it has at most $d$ zeros on $\mathbb{P}^1_k$, whereas if $k$ is infinite, then so is $\mathbb{P}^1_k$.

(ii) By a change of coordinates, $Q = XZ - Y^2$. Now let me prove that for any $F \in S_d$, there exists a unique expression

$$F = Q \cdot F'_{d-2} + A(X,Z) + YB(X,Z):$$

if I just substitute $XZ - Q$ for $Y^2$ wherever it occurs in $F$, what's left has degree $\leq 1$ in $Y$, and is therefore of the form $A(X,Z) + YB(X,Z)$. Now as in (1.7), $C$ is the parametrised conic given by $X = U^2, Y = UV, Z = V^2$, so that

$$\begin{aligned} F \equiv 0 \text{ on } C &\iff A(U^2,V^2) + UVB(U^2,V^2) \equiv 0 \text{ on } C \\ &\iff A(U^2,V^2) + UVB(U^2,V^2) = 0 \in k[U,V] \\ &\iff A(X,Z) = B(X,Z) = 0. \end{aligned}$$

Here the last equality comes by considering separately the terms of even and odd degrees in the form $A(U^2,V^2) + UVB(U^2,V^2)$. Q.E.D.

Ex. 2.2 gives similar cases of 'explicit' Nullstellensatz.

**Corollary** *Let $L : (H = 0) \subset \mathbb{P}^2_k$ be a line (or $C : (Q = 0) \subset \mathbb{P}^2_k$ a nondegenerate conic); suppose that points $P_1, \dots, P_n \in \mathbb{P}^2_k$ are given, and consider $S_d(P_1, \dots, P_n)$ for some fixed $d$. Then*

*(i) If $P_1, \dots, P_a \in L, P_{a+1}, \dots, P_n \notin L$ and $a > d$, then*

$$S_d(P_1, \dots, P_n) = H \cdot S_{d-1}(P_{a+1}, \dots, P_n).$$

*(ii) If $P_1, \dots, P_a \in C, P_{a+1}, \dots, P_n \notin C$ and $a > 2d$, then*

$$S_d(P_1, \dots, P_n) = Q \cdot S_{d-2}(P_{a+1}, \dots, P_n).$$

**Proof** (i) If $F$ is homogeneous of degree $d$, and the curve $D : (F = 0)$ meets $L$ in points $P_1, \dots, P_a$ with $a > d$, then by (1.9), I must have $L \subset D$, so that by the lemma, $F = H \cdot F'$; now since $P_{a+1}, \dots, P_n \notin L$, obviously $F' \in S_{d-1}(P_{a+1}, \dots, P_n)$. (ii) is exactly the same. Q.E.D.

**Proposition 2.6** *Let $k$ be an infinite field, and $P_1, \dots, P_8 \in \mathbb{P}^2_k$ distinct points; suppose that no 4 of $P_1, \dots, P_8$ are collinear, and no 7 of them lie on a nondegenerate conic; then*

$$\dim S_3(P_1, \dots, P_8) = 2.$$

**Proof** For brevity, let me say that a set of points are *conconic* if they all lie on a nondegenerate conic. The proof of (2.6) breaks up into several cases.

**Main case** No 3 points are collinear, no 6 conconic. This is the 'general position' case.

Suppose for a contradiction that $\dim S_3(P_1, \dots, P_8) \geq 3$, and let $P_9, P_{10}$ be distinct points on the line $L = P_1P_2$. Then

$$\dim S_3(P_1, \dots, P_{10}) \geq \dim S_3(P_1, \dots, P_8) - 2 \geq 1,$$

so that there exists $0 \neq F \in S_3(P_1, \dots, P_{10})$. By Corollary 2.5, $F = H \cdot Q$, with $Q \in S_2(P_3, \dots, P_8)$. Now I have a contradiction to the case assumption: if $Q$ is nondegenerate then the 6 points $P_3, \dots, P_8$ are conconic, whereas if $Q$ is a line pair or a double line, then at least 3 of them are collinear.

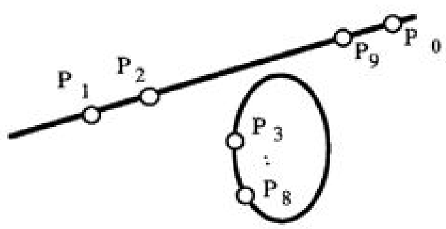


Figure 2.2: 10 points on a reducible cubic

**First degenerate case** Suppose $P_1, P_2, P_3 \in L$ are collinear, and let $L : (H = 0)$. Let $P_9$ be a 4th point on the line $L$. Then by Corollary 2.5,

$$S_3(P_1, \dots, P_9) = H \cdot S_2(P_4, \dots, P_8).$$

Also, since no 4 of $P_4, \dots, P_8$ are collinear, by Corollary 1.11,

$$\dim S_2(P_4, \dots, P_8) = 1, \quad \text{and then} \quad \dim S_3(P_1, \dots, P_9) = 1,$$

which implies $\dim S_3(P_1, \dots, P_8) \leq 2$.