1.3 Conics in $\mathbb{R}^2$

A conic in $\mathbb{R}^2$ is a plane curve given by a quadratic equation

$$q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Everyone has seen the classification of nondegenerate conics:

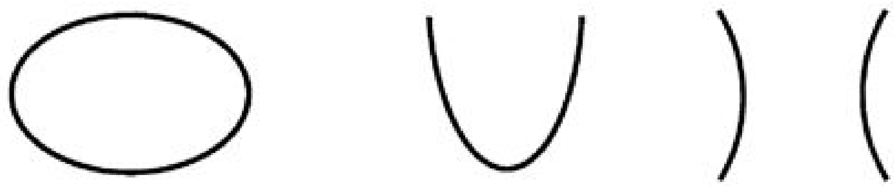

Figure 1.2: The nondegenerate conics: (a) ellipse; (b) parabola; (c) hyperbola.

in addition, there are a number of peculiar cases:
(d) single point given by $x^2 + y^2 = 0$;
(e, f, g) empty set given by any of the 3 equations: (e) $x^2 + y^2 = -1$, (f) $x^2 = -1$ or (g) $0 = 1$. These three equations are different, although they define the same locus of zeros in $\mathbb{R}^2$; consider for example their complex solutions.
(h) line $x = 0$;
(i) line pair $xy = 0$;
(j) parallel lines $x(x - 1) = 0$;
(k) 'double line' $x^2 = 0$; you can choose for yourself whether you'll allow the final case:
(l) whole plane given by $0 = 0$.

1.4 Projective plane

The definition 'out of the blue':

$$\begin{aligned}\mathbb{P}^2_{\mathbb{R}} &= \big\{\text{lines of } \mathbb{R}^3 \text{ through origin}\big\}\\ &= \big\{\text{ratios } X : Y : Z\big\}\\ &= \big(\mathbb{R}^3 \setminus \{0\}\big)/\!\sim, \quad \text{where } (X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z) \text{ if } \lambda \in \mathbb{R} \setminus \{0\}.\end{aligned}$$

(The sophisticated reader will have no difficulty in generalising from $\mathbb{R}^3$ to an arbitrary vector space over a field, and in replacing work in a chosen coordinate system with intrinsic arguments.)

To represent a ratio $X : Y : Z$ for which $Z \neq 0$, I can set $x = X/Z$, $y = Y/Z$; this simplifies things, since the ratio corresponds to just two real numbers. In other words, the equivalence class of (X, Y, Z) under $\sim$ has a unique representative $(x, y, 1)$ with 3rd coordinate $= 1$. Unfortunately, sometimes Z might be $= 0$, so that this way of choosing a representative of the equivalence class is then no good. This discussion means that $\mathbb{P}^2_{\mathbb{R}}$ contains a copy of $\mathbb{R}^2$. A picture: