4.10 Composition of rational maps

The composite $g \circ f$ of rational maps $f\colon V \dashrightarrow W$ and $g\colon W \dashrightarrow U$ may not be defined. This is a difficulty caused by the fact that a rational map is not a map: in a natural and obvious sense, the composite is a map defined on $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$; however, it can perfectly well happen that this is empty (see Ex. 4.10).

Expressed algebraically, the same problem also occurs: suppose that f is given by $f_1, \dots, f_m \in k(V)$, so that

$$f\colon V \dashrightarrow W \subset \mathbb{A}^m \quad \text{by} \quad P \mapsto f_1(P), \dots, f_m(P)$$

for $P \in \bigcap \operatorname{dom} f_i$; any $g \in k[W]$ is of the form $g = G \bmod I(W)$ for some $G \in k[Y_1, \dots, Y_m]$, and $g \circ f = G(f_1, \dots, f_m)$ is well defined in $k(V)$. So exactly as in (4.4), there is a k-algebra homomorphism

$$f^*\colon k[W] \to k(V)$$

corresponding to f. However, if $h \in k[W]$ is in the kernel of f^*, then no meaning can be attached to $f^*(g/h)$, so that f^* cannot be extended to a field homomorphism $k(W) \to k(V)$.

Definition $f\colon V \dashrightarrow W$ is *dominant* if $f(\operatorname{dom} f)$ is dense in W for the Zariski topology.

Geometrically, this means that $f^{-1}(\operatorname{dom} g) \subset \operatorname{dom} f$ is a dense open set for any rational map $g\colon W \dashrightarrow U$, so that $g \circ f$ is defined on a dense open set of V, so is a partially defined map $V \dashrightarrow U$.

Algebraically,

$$f \text{ is dominant} \iff f^*\colon k[W] \to k(V) \text{ is injective.}$$

For given $g \in k[W]$,

$$g \in \ker f^* \iff f(\operatorname{dom} f) \subset V(g),$$

that is, f^* is not injective if and only if $f(\operatorname{dom} f)$ is contained in a strict algebraic subset of W.

Clearly, the composite $g \circ f$ of rational maps f and g is defined provided that f is dominant: $g \circ f$ is the rational map whose components are $f^*(g_i)$. Notice that the domain of $g \circ f$ certainly contains $f^{-1}(\operatorname{dom} g) \cap \operatorname{dom} f$, but may very well be larger (see Ex. 4.6).

Theorem 4.11 *(I) A dominant rational map $f\colon V \dashrightarrow W$ defines a field homomorphism $f^*\colon k(W) \to k(V)$.*

(II) Conversely, a k-homomorphism $\Phi\colon k(W) \to k(V)$ comes from a uniquely defined dominant rational map $f\colon V \dashrightarrow W$.

(III) If f and g are dominant then $(g \circ f)^* = f^* \circ g^*$.

The proof requires only minor modifications to that of (4.4).

4.12 Morphisms from an open subset of an affine variety

Let V, W be affine varieties, and $U \subset V$ an open subset.