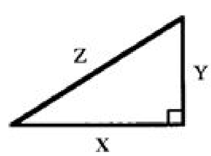
Chapter 1

Plane conics

I start by studying the geometry of conics as motivation for the projective plane $\mathbb{P}^2$. Projective geometry is usually mentioned in 2nd year undergraduate geometry courses, and I recall some of the salient features, with some emphasis on homogeneous coordinates, although I completely ignore the geometry of linear subspaces and the 'cross-ratio'. The most important aim for the student should be to grasp the way in which geometric ideas (for example, the idea that 'points at infinity' correspond to asymptotic directions of curves) are expressed in terms of coordinates. The interplay between the intuitive geometric picture (which tells you what you should be expecting), and the precise formulation in terms of coordinates (which allows you to cash in on your intuition) is a fascinating aspect of algebraic geometry.

1.1 Example of a parametrised curve

Pythagoras' Theorem says that, in the diagram



$$X^2 + Y^2 = Z^2,$$

so $(3,4,5)$ and $(5,12,13)$, as every ancient Egyptian knew. How do you find all integer solutions? The equation is homogeneous, so that $x = X/Z$, $y = Y/Z$ gives the circle $C : (x^2 + y^2 = 1) \subset \mathbb{R}^2$, which can easily be seen to be parametrised as

$$x = \frac{2\lambda}{\lambda^2+1}, \quad y = \frac{\lambda^2-1}{\lambda^2+1}, \quad \text{where} \quad \lambda = \frac{x}{1-y};$$

so this gives all solutions:

$$X = 2\ell m, \quad Y = \ell^2 - m^2, \quad Z = \ell^2 + m^2 \quad \text{with } \ell, m \in \mathbb{Z} \text{ coprime}$$

(or each divided by 2 if ℓ, m are both odd). Note that the equation is homogeneous, so that if (X, Y, Z) is a solution, then so is $(\lambda X, \lambda Y, \lambda Z)$.

Maybe the parametrisation was already familiar from school geometry, and in any case, it's easy to verify that it works. However, if I didn't know it already, I could have obtained it by an easy geometric argument, namely linear projection from a given point: $P = (0,1) \in C$, and if $\lambda \in \mathbb{Q}$

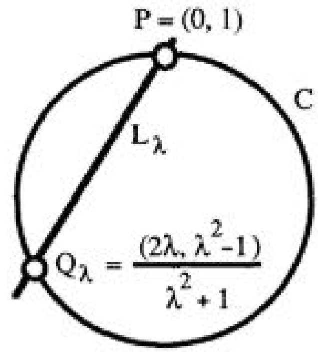


Figure 1.1: Linear projection of a conic to a line

is any value, then the line L_λ through P with slope $-\lambda$ meets C in a further point Q_λ. This construction of a map by means of linear projection will appear many times in what follows.

1.2 Similar example

$C : (2X^2 + Y^2 = 5Z^2)$. The same method leads to the parametrisation $\mathbb{R} \to C$ given by

$$x = \frac{2\sqrt{5}\lambda}{1+2\lambda^2}, \quad y = \frac{2\lambda^2 - 1}{1+2\lambda^2}.$$

This allows us to understand all about points of C with coefficients in $\mathbb{R}$, and there's no real difference from the previous example; what about $\mathbb{Q}$?

Proposition *If $(a,b,c) \in \mathbb{Q}$ satisfies $2a^2 + b^2 = 5c^2$ then $(a,b,c) = (0,0,0)$.*

Proof Multiplying through by a common denominator and taking out a common factor if necessary, I can assume that a, b, c are integers, not all of which are divisible by 5; also if $5 \mid a$ and $5 \mid b$ then $25 \mid 5c^2$, so that $5 \mid c$, which contradicts what I've just said. It is now easy to get a contradiction by considering the possible values of a and b mod 5: since any square is 0, 1 or 4 mod 5, clearly $2a^2 + b^2$ is one of $0+1$, $0+4$, $2+0$, $2+1$, $2+4$, $8+0$, $8+1$ or $8+4$ mod 5, none of which can be of the form $5c^2$. Q.E.D.

Note that this is a thoroughly arithmetic argument.