each of these inclusion signs represents an absolutely *huge* gap, and that this leads to the main characteristics of geometry in the different categories. Although it's not stressed very much in school and first year university calculus, any reasonable way of measuring $C^0(U)$ will reveal that the differentiable functions have measure 0 in the continuous functions (so if you pick a continuous function at random then with probability 1 it will be nowhere differentiable, like Brownian motion). The gap between $C^\omega(U)$ and $C^\infty(U)$ is exemplified by the behaviour of $\exp(-1/x^2)$, the standard function which is differentiable infinitely often, but for which the Taylor series (at 0) does not converge to f; using this, you can easily build a C^∞ 'bump function' $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $|x| \leq 0.9$, and $f(x) = 0$ if $|x| \geq 1$: In contrast, an analytic function on U extends (as a convergent

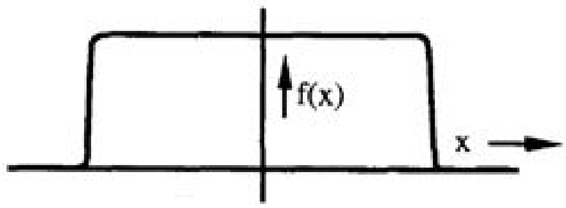

Figure 2: A C^∞ bump function.

power series) to an analytic function of a complex variable on a suitable domain in $\mathbb{C}$, so that (using results from complex analysis), if $f \in C^\omega(U)$ vanishes on a real interval, it must vanish identically. This is a kind of 'rigidity' property which characterises analytic geometry as opposed to differential topology.

0.4 Geometry from polynomials

There are very few polynomial functions: the polynomial ring $\mathbb{R}[X]$ is just a countable dimensional $\mathbb{R}$-vector space, whereas $C^\omega(U)$ is already uncountable. Even allowing rational functions – that is, extending $\mathbb{R}[X]$ to its field of fractions $\mathbb{R}(X)$ – doesn't help much. (2.2) will provide an example of the characteristic rigidity of the algebraic category. The fact that it is possible to construct a geometry using only this set of functions is itself quite remarkable. Not surprisingly, there are difficulties involved in setting up this theory:

Foundations via commutative algebra Topology and differential topology can rely on the whole corpus of ε-δ analysis taught in a series of 1st and 2nd year undergraduate courses; to do algebraic geometry working only with polynomial rings, we need instead to study rings such as the polynomial ring $k[X_1, \ldots, X_n]$ and their ideals. In other words, we have to develop commutative algebra in place of calculus. The Nullstellensatz (§3 below) is a typical example of a statement having direct intuitive geometric content (essentially, "different ideals of functions in $k[X_1, \ldots, X_n]$ define different varieties $V \subset k^n$") whose proof involves quite a lengthy digression through finiteness conditions in commutative algebra.

Rational maps and functions Another difficulty arising from the decision to work with polynomials is the necessity of introducing 'partially defined functions'; because of the 'rigidity' hinted at above, we'll see that for some varieties (in fact for all projective varieties), there do not exist any nonconstant regular functions (see Ex. 5.1, Ex. 5.12 and the discussion in (8.10)). Rational functions (that is, 'functions' of the form $f = g/h$, where g, h are polynomial functions) are not defined at points where the denominator vanishes. Although reprehensible, it is a firmly entrenched tradition among algebraic geometers to use 'rational function' and 'rational map' to mean 'only partially defined function (or map)'. So a rational map $f\colon V_1 \dashrightarrow V_2$ is not a map at all; the broken arrow here is also becoming traditional. Students who disapprove are recommended to give up at once and take a reading course in Category Theory instead.

This is not at all a frivolous difficulty. Even regular maps (= morphisms, these are genuine maps) have to be defined as rational maps which are regular at all points $P \in V$ (that is, well defined, the denominator can be chosen not to vanish at P). Closely related to this is the difficulty of giving a proper intrinsic definition of a variety: in this course (and in others like it, in my experience), affine varieties $V \subset \mathbb{A}^n$ and quasiprojective varieties $V \subset \mathbb{P}^n$ will be defined, but there will be no proper definition of 'variety' without reference to an ambient space. Roughly speaking, a variety should be what you get if you glue together a number of affine varieties along isomorphic open subsets. But our present language, in which isomorphisms are themselves defined more or less explicitly in terms of rational functions, is just too cumbersome; the proper language for this glueing is sheaves, which are well treated in graduate textbooks.

0.5 "Purely algebraically defined"

So much for the drawbacks of the algebraic approach to geometry. Having said this, almost all the algebraic varieties of importance in the world today are quasiprojective, and we can have quite a lot of fun with varieties without worrying overmuch about the finer points of definition.

The main advantages of algebraic geometry are that it is purely algebraically defined, and that it applies to any field, not just $\mathbb{R}$ or $\mathbb{C}$; we can do geometry over fields of characteristic p. Don't say 'characteristic p – big deal, that's just the finite fields'; to start with, very substantial parts of group theory are based on geometry over finite fields, as are large parts of combinatorics used in computer science. Next, there are lots of interesting fields of characteristic p other than finite ones. Moreover, at a deep level, the finite fields are present and working inside $\mathbb{Q}$ and $\mathbb{C}$. Most of the deep results on arithmetic of varieties over $\mathbb{Q}$ use a considerable amount of geometry over $\mathbb{C}$ or over the finite fields and their algebraic closures.

This concludes the introduction; see the informal discussion in (2.15) and the final §8 for more general culture.

0.6 Plan of the book

As to the structure of the book, Part I and Part III aim to indicate some worthwhile problems which can be studied by means of algebraic geometry. Part II is an introduction to the commutative algebra referred to in (0.4) and to the categorical framework of algebraic geometry; the student who is prone to headaches could perhaps take some of the proofs for granted here, since the material is standard, and the author is a professional algebraic geometer of the highest moral fibre.

§8 contains odds and ends that may be of interest or of use to the student, but that don't fit in the main text: a little of the history and sociology of the modern subject, hints as to relations of the subject matter with more advanced topics, technical footnotes, etc.