

X_i , that is, $\text{char } k = p$, and X_i only appears in f as the p th power X_i^p . If this happens for each i , then by the argument given in (3.16), f is a p th power in $k[X_1, \dots, X_n]$; this contradicts the fact that f is irreducible. Q.E.D.

6.4 Tangent space

Definition Let $V \subset \mathbb{A}^n$ be a subvariety, with $V \ni P = (a_1, \dots, a_n)$. For any $f \in k[X_1, \dots, X_n]$, write

$$f_P^{(1)} = \sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i).$$

This is an affine linear polynomial (that is, linear plus constant), the ‘first order part’ of f at P . Now define the *tangent space* to V at P by

$$T_P V = \bigcap \left(f_P^{(1)} = 0 \right) \subset \mathbb{A}^n,$$

where the intersection takes place over all $f \in I(V)$.

Proposition 6.5 *The function $V \rightarrow \mathbb{N}$ defined by $P \mapsto \dim T_P V$ is an upper semicontinuous function (in the Zariski topology of V). In other words, for any integer r , the subset*

$$S(r) = \{P \in V \mid \dim T_P V \geq r\} \subset V$$

is closed.

Proof Let (f_1, \dots, f_m) be a set of generators of $I(V)$; it is easy to see that for any $g \in I(V)$, the linear part $g_P^{(1)}$ of g is a linear combination of those of the f_i , so that the definition of $T_P V$ simplifies to

$$T_P V = \bigcap_{i=1}^m \left(f_{i,P}^{(1)} = 0 \right) \subset \mathbb{A}^n.$$

Then by elementary linear algebra,

$$\begin{aligned} P \in S(r) &\iff \text{the matrix } \left(\frac{\partial f}{\partial X_i}(P) \right)_{\substack{i=1, \dots, m \\ j=1, \dots, n}} \text{ has rank } \leq n - r \\ &\iff \text{every } (n - r + 1) \times (n - r + 1) \text{ minor vanishes.} \end{aligned}$$

Now each entry $\partial f_i / \partial X_j(P)$ of the matrix is a polynomial function of P ; thus each minor is a determinant of a matrix of polynomials, and so is itself a polynomial. Hence $S(r) \subset V \subset \mathbb{A}^n$ is an algebraic subset. Q.E.D.

Corollary-Definition 6.6 *There exists an integer r and a dense open subset $V_0 \subset V$ such that*

$$\dim T_P V = r \text{ for } P \in V_0, \text{ and } \dim T_P V \geq r \text{ for all } P \in V.$$

Define r to be the dimension of V , and write $\dim V = r$; and say that $P \in V$ is nonsingular if $\dim T_P V = r$, and singular if $\dim T_P V < r$. A variety V is nonsingular if it is nonsingular at each point $P \in V$.

Proof Let $r = \min\{\dim T_P V\}$, taken over all points $P \in V$. Then clearly

$$S(r-1) = \emptyset, \quad S(r) = V, \quad \text{and} \quad S(r+1) \subsetneq V;$$

therefore $S(r) \setminus S(r+1) = \{P \in V \mid \dim T_P V = r\}$ is open and nonempty. Q.E.D.

6.7 $\dim V = \text{tr deg } k(V)$ – the hypersurface case

It follows from Proposition 6.3 that if $V = V(f) \subset \mathbb{A}^n$ is a hypersurface defined by some nonconstant polynomial f , then $\dim V = n-1$. On the other hand, for a hypersurface, $k[V] = k[X_1, \dots, X_n]/(f)$, so that, assuming that f involves X_1 in a nontrivial way, the function field of V is of the form

$$k(V) = k(X_2, \dots, X_n)[X_1]/(f),$$

that is, it is built up from k by adjoining $n-1$ algebraically independent elements, then making a primitive algebraic extension.

Definition If $k \subset K$ is a field extension, the *transcendence degree* of K over k is the maximum number of elements of K algebraically independent over k . It is denoted $\text{tr deg}_k K$.

The elementary theory of transcendence degree of a field extension K/k is formally quite similar to that of the dimension of a vector space: given $\alpha_1, \dots, \alpha_m \in K$, we know what it means for them to be *algebraically independent* over k (see (3.13)); they *span* the transcendental part of the extension if $K/k(\alpha_1, \dots, \alpha_m)$ is algebraic; and they form a *transcendence basis* if they are algebraically independent and span. Then it is an easy theorem that a transcendence basis is a maximal algebraically independent set, and a minimal spanning set, and that any two transcendence bases of K/k have the same number of elements (see Ex. 6.1).

Thus for a hypersurface $V \subset \mathbb{A}^n$, $\dim V = n-1 = \text{tr deg}_k k(V)$. The rest of this section is concerned with proving that the equality $\dim V = \text{tr deg}_k k(V)$ holds for all varieties, by reducing to the case of a hypersurface. The first thing to show is that for a point $P \in V$ of a variety, the tangent space $T_P V$, which so far has been discussed in terms of a particular coordinate system in the ambient space \mathbb{A}^n , is in fact an intrinsic property of a neighbourhood of $P \in V$.

6.8 Intrinsic nature of $T_P V$

From now on, given $P = (a_1, \dots, a_n) \in V \subset \mathbb{A}^n$, I take new coordinates $X'_i = X_i - a_i$ to bring P to the origin, and thus assume that $P = (0, \dots, 0)$. Then $T_P V \subset \mathbb{A}^n$ is a vector subspace of k^n .

Notation Write $m_P = \text{ideal of } P \text{ in } k[V]$, and

$$M_P = \text{the ideal } (X_1, \dots, X_n) \subset k[X_1, \dots, X_n].$$

Then of course $m_P = M_P/I(V) \subset k[V]$.

Theorem *In the above notation,*