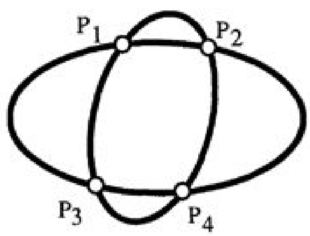
Proof Corollary 1.10 implies that if $n = 5$, $\dim S_2(P_1, \dots, P_5) \le 1$, which gives the corollary in this case. If $n \le 4$, then I can add in points $P_{n+1}, \dots, P_5$ while preserving the condition that no 4 points are collinear, and since each point imposes at most one linear condition, this gives

$$1 = \dim S_2(P_1, \dots, P_5) \ge \dim S_2(P_1, \dots, P_n) - (5 - n). \quad \text{Q.E.D.}$$

Note that if 6 points $P_1, \dots, P_6 \in \mathbb{P}^2_{\mathbb{R}}$ are given, they may or may not lie on a conic.

1.12 Intersection of two conics

As we have seen above, it often happens that two conics meet in 4 points:



Conversely according to Corollary 1.11, given 4 points $P_1, \dots, P_4 \in \mathbb{P}^2$, under suitable conditions $S_2(P_1, \dots, P_4)$ is a 2-dimensional vector space, so choosing a basis Q_1, Q_2 for $S_2(P_1, \dots, P_4)$ gives 2 conics C_1, C_2 such that $C_1 \cap C_2 = \{P_1, \dots, P_4\}$. There are lots of possibilities for multiple intersections of nonsingular conics:

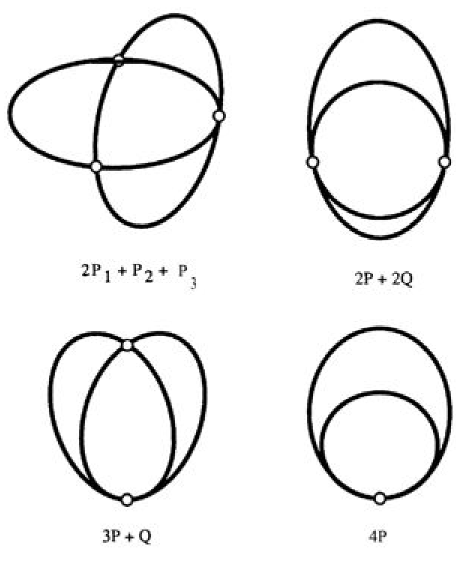


Figure 1.6: (a) $2P_1 + P_2 + P_3$; (b) $2P + 2Q$; (c) $3P + Q$; (d) $4P$

see Ex. 1.9 for suitable equations.

1.13 Degenerate conics in a pencil

Definition A *pencil of conics* is a family of the form

$$C_{(\lambda,\mu)} : (\lambda Q_1 + \mu Q_2 = 0);$$

each element is a plane curve, depending in a linear way on the parameters (λ, μ); think of the ratio $(\lambda : \mu)$ as a point of $\mathbb{P}^1$.

Looking at the examples, one expects that for special values of $(\lambda : \mu)$ the conic $C_{(\lambda,\mu)}$ is degenerate. In fact, writing $\det(Q)$ for the determinant of the symmetric 3×3 matrix corresponding to the quadratic form Q, it is clear that

$$C_{(\lambda,\mu)} \text{ is degenerate} \iff \det(\lambda Q_1 + \mu Q_2) = 0.$$

Writing out Q_1 and Q_2 as symmetric matrixes expresses this condition as

$$F(\lambda,\mu) = \det \left| \lambda \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} + \mu \begin{pmatrix} a' & b' & d' \\ b' & c' & e' \\ d' & e' & f' \end{pmatrix} \right| = 0.$$

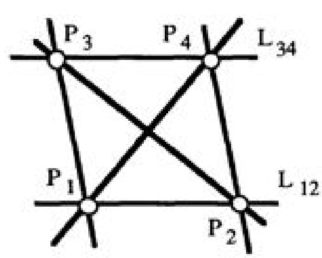
Now notice that $F(\lambda,\mu)$ is a homogeneous cubic form in λ, μ. In turn I can apply (1.8) to F to deduce:

Proposition *Suppose $C_{(\lambda,\mu)}$ is a pencil of conics of $\mathbb{P}^2_k$, with at least one nondegenerate conic (so that $F(\lambda,\mu)$ is not identically zero). Then the pencil has at most 3 degenerate conics. If $k = \mathbb{R}$ then the pencil has at least one degenerate conic.*

Proof A cubic form has ≤ 3 zeros. Also over $\mathbb{R}$, it must have at least one zero.

1.14 Worked example

Let $P_1, \dots, P_4$ be 4 points of $\mathbb{P}^2_{\mathbb{R}}$ such that no 3 are collinear; then the pencil of conics $C_{(\lambda,\mu)}$ through $P_1, \dots, P_4$ has 3 degenerate elements, namely the line pairs $L_{12} + L_{34}, L_{13} + L_{24}, L_{14} + L_{23}$, where L_{ij} is the line through P_i, P_j:



Next, suppose that I start from the pencil of conics generated by $Q_1 = Y^2 + rY + sX + t$ and $Q_2 = Y - X^2$, and try to find the points $P_1, \dots, P_4$ of intersection.