Proposition 3.2 *(i) Suppose that A is Noetherian, and $I \subset A$ an ideal; then the quotient ring $B = A/I$ is Noetherian.*

(ii) Let A be a Noetherian integral domain, and $A \subset K$ its field of fractions; let $0 \notin S \subset A$ be a subset, and set

$$B = A[S^{-1}] = \left\{ \frac{a}{b} \in K \;\middle|\; \begin{array}{l} a \in A, \text{ and } b = 1 \text{ or a} \\ \text{product of elements of } S \end{array} \right\}.$$

Then B is again Noetherian.

Proof Exercise: in either case the ideals of B can be described in terms of certain ideals of A; see Ex. 3.4 for hints.

Theorem 3.3 (Hilbert Basis Theorem) *For a ring A,*

$$A \text{ Noetherian} \implies A[X] \text{ Noetherian.}$$

Proof Let $J \subset A[X]$ be any ideal; I prove that J is finitely generated. Define the ideal of leading terms of degree n in J to be

$$J_n = \left\{ a \in A \mid \exists f = aX^n + b_{n-1}X^{n-1} + \cdots + b_0 \in J \right\}.$$

Then J_n is an ideal of A and $J_n \subset J_{n+1}$ (please provide your own proofs). Hence, using the a.c.c., there exists N such that

$$J_N = J_{N+1} = \cdots.$$

Now build a set of generators of J as follows: for $i \le N$, let $a_{i_1}, \dots, a_{i_{m_i}}$ be generators of J_i and, as in the definition of J_i, for each of the a_{ik}, let $f_{ik} = a_{ik}X^i + \cdots \in J$ be an element of degree i and leading term a_{ik}.

I claim that the set

$$\left\{ f_{ik} \mid i = 0, \dots, N, k = 1, \dots, m_i \right\}$$

just constructed generates J: for given $g \in J$, suppose $\deg g = m$. Then the leading term of g is bX^m with $b \in J_m$, so that by what I know about J_m, I can write $b = \sum c_{m'k} a_{m'k}$ (here $m' = m$ if $m \le N$, otherwise $m' = N$). Then consider $g_1 = g - X^{m-m'} \cdot \sum c_{m'k} f_{m'k}$: by construction the term of degree m is zero, so that $\deg g_1 \le \deg g - 1$; by induction, I can therefore write out g as a combination of f_{ik}, so that these generate J. Q.E.D.

Corollary *For k a field, a finitely generated k-algebra is Noetherian.*

A finitely generated k-algebra is a ring of the form $A = k[a_1, \dots, a_n]$, so that A is generated as a ring by k and $a_1, \dots, a_n$; clearly, every such ring is isomorphic to a quotient of the polynomial ring, $A \cong k[X_1, \dots, X_n]/I$. A field is Noetherian, and by induction on (3.3), $k[X_1, \dots, X_n]$ is Noetherian; finally, passing to the quotient is OK by Proposition 3.2, (i). Q.E.D.

3.4 The correspondence V

k is any field, and $A = k[X_1, \dots, X_n]$. Following an almost universal idiosyncracy of algebraic geometers[1], I write $\mathbb{A}^n_k = k^n$ for the n-dimensional affine space over k; given a polynomial $f(X_1, \dots, X_n) \in A$ and a point $P = (a_1, \dots, a_n) \in \mathbb{A}^n_k$, the element $f(a_1, \dots, a_n) \in k$ is thought of as 'evaluating the function f at P'. Define a correspondence

$$\{\text{ideals } J \subset A\} \longrightarrow \{\text{subsets } X \subset \mathbb{A}^n_k\}$$

by

$$J \longmapsto V(J) = \{P \in \mathbb{A}^n_k \mid f(P) = 0 \ \forall f \in J\}.$$

Definition A subset $X \subset \mathbb{A}^n_k$ is an *algebraic set* if $X = V(I)$ for some I. (This is the same thing as a variety, but I want to reserve the word.) Notice that by Corollary 3.3, I is finitely generated. If $I = (f_1, \dots, f_r)$ then clearly

$$V(I) = \{P \in \mathbb{A}^n_k \mid f_i(P) = 0 \quad \text{for } i = 1, \dots, r\},$$

so that an algebraic set is just a locus of points satisfying a finite number of polynomial equations.

If $I = (f)$ is a principal ideal, then I usually write $V(f)$ for $V(I)$; this is of course the same thing as $V : (f = 0)$ in the notation of §§1–2.

3.5 Definition: the Zariski topology

Proposition-Definition *The correspondence V satisfies the following formal properties:*

(i) $V(0) = \mathbb{A}^n_k$; $V(A) = \emptyset$;

(ii) $I \subset J \implies V(I) \supset V(J)$;

(iii) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$;

(iv) $V\left(\sum_{\lambda \in \Lambda} I_\lambda\right) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Hence the algebraic subsets of $\mathbb{A}^n_k$ form the closed sets of a topology on $\mathbb{A}^n_k$, the Zariski topology.

The above properties are quite trivial, with the exception of the inclusion $\subset$ in (iii). For this, suppose $P \notin V(I_1) \cup V(I_2)$; then there exist $f \in I_1, g \in I_2$ such that $f(P) \neq 0, g(P) \neq 0$. So $fg \in I_1 \cap I_2$, but $fg(P) \neq 0$, and therefore $P \notin V(I_1 \cap I_2)$. Q.E.D.

The Zariski topology on $\mathbb{A}^n_k$ induces a topology on any algebraic set $X \subset \mathbb{A}^n_k$: the closed subsets of X are the algebraic subsets.

It's important to notice that the Zariski topology on a variety is very weak, and is quite different from the familiar topology of metric spaces like $\mathbb{R}^n$. As an example, a Zariski closed subset of $\mathbb{A}^1_k$ is either the whole of $\mathbb{A}^1_k$ or is finite; see Ex. 3.12 for a description of the Zariski topology on $\mathbb{A}^2_k$. If $k = \mathbb{R}$ or $\mathbb{C}$ then Zariski closed sets are also closed for the ordinary topology, since polynomial functions are continuous. In fact they're very special open or closed subsets: a nonempty Zariski open subset of $\mathbb{R}^n$ is the complement of a subvariety, so automatically dense in $\mathbb{R}^n$.

The Zariski topology may cause trouble to some students; since it is only being used as a language, and has almost no content, the difficulty is likely to be psychological rather than technical.

[1] $\mathbb{A}^n$ is thought of as a variety, whereas k^n is just a point set. Think of this as pure pedantry if you like; compare (4.6) below, as well as (8.3).