Chapter 4

Functions on varieties

In this section I work over a fixed field k; from (4.8, II) onwards, k will be assumed to be algebraically closed. The reader who assumes throughout that $k = \mathbb{C}$ will not lose much, and may gain a psychological crutch. I sometimes omit mention of the field k to simplify notation.

4.1 Polynomial functions

Let $V \subset \mathbb{A}^n_k$ be an algebraic set, and $I(V)$ its ideal. Then the quotient ring $k[V] = k[X_1, \dots, X_n]/I(V)$ is in a natural way a ring of functions on V. In more detail, define a *polynomial function* on V to be a map $f\colon V \to k$ of the form $P \mapsto F(P)$, with $F \in k[X_1, \dots, X_n]$; this just means that f is the restriction of a map $F\colon \mathbb{A}^n \to k$ defined by a polynomial. By definition of $I(V)$, two elements $F, G \in k[X_1, \dots, X_n]$ define the same function on V if and only if

$$F(P) - G(P) = 0 \text{ for all } P \in V,$$

that is, if and only if $F - G \in I(V)$. Thus I define the *coordinate ring* $k[V]$ by

$$\begin{aligned} k[V] &= \bigl\{f\colon V \to k \bigm| f \text{ is a polynomial function}\bigr\} \\ &\cong k[X_1, \dots, X_n]/I(V). \end{aligned}$$

This is the smallest ring of functions on V containing the coordinate functions X_i (together with k), so for once the traditional terminology is not too obscure.

4.2 $k[V]$ and algebraic subsets of V

An algebraic set $X \subset \mathbb{A}^n$ is contained in V if and only if $I(X) \supset I(V)$. On the other hand, ideals of $k[X_1, \dots, X_n]$ containing $I(V)$ are in obvious bijection with ideals of $k[X_1, \dots, X_n]/I(V)$. (Think about this if it's not obvious to you: the ideal J with $I(V) \subset J \subset k[X_1, \dots, X_n]$ corresponds to $J/I(V)$; and conversely, an ideal J_0 of $k[X_1, \dots, X_n]/I(V)$ corresponds to its inverse image in $k[X_1, \dots, X_n]$.)

Hence the I and V correspondences

$$\{\text{ideals } I \subset k[V]\} \xrightarrow{V} \{\text{subsets } X \subset V\}$$

by

$$I \longmapsto V(I) = \big\{P \in V \bigm| f(P) = 0\ \forall f \in I\big\}$$

and

$$\{\text{ideals } J \subset k[V]\} \xleftarrow{I} \{\text{subsets} X \subset V\}$$

by

$$I(X) = \big\{f \in k[V] \bigm| f(P) = 0\ \forall P \in X\big\} \longleftarrow X$$

are defined as in §3, and have similar properties. In particular V has a Zariski topology, in which the closed sets are the algebraic subsets (this is of course the subspace topology of the Zariski topology of $\mathbb{A}^n$).

Proposition *Let $V \subset \mathbb{A}^n$ be an algebraic subset. The following conditions are equivalent:*

(i) V is irreducible;

(ii) any two open subsets $\emptyset \neq U_1, U_2 \subset V$ have $U_1 \cap U_2 \neq \emptyset$;

(iii) any nonempty open subset $U \subset V$ is dense.

This is all quite trivial: V is irreducible means that V is not a union of two proper closed subsets; (ii) is just a restatement in terms of complements, since

$$U_1 \cap U_2 = \emptyset \iff V = (V - U_1) \cup (V - U_2).$$

A subset of a topological space is dense if and only if it meets every open, so that (iii) is just a restatement of (ii).

4.3 Polynomial maps

Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic sets; write $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ for the coordinates on $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively.

Definition A map $f\colon V \to W$ is a *polynomial map* if there exist m polynomials $F_1, \dots, F_m \in k[X_1, \dots, X_n]$ such that

$$f(P) = \big(F_1(P), \dots, F_m(P)\big) \in \mathbb{A}^m_k \quad \text{for all } P \in V.$$

This is an obvious generalisation of the above notion of a polynomial function.

Claim *A map $f\colon V \to W$ is a polynomial map if and only if for all j, the composite map $f_j = Y_j \circ f \in k[V]$:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "f_j"'] & W \subset \mathbb{A}^m_k \arrow[d, "Y_j"] \\ & k \end{tikzcd} \qquad \textit{(jth coordinate function).}$$