Exercises to Chapter 7

7.1 Prove case (ii) of the claim in Proposition 7.3. [Hint: as in the given proof of case (i), $f = X(X-T)T + Zg$, so that $A = T + aZ$, $D = -T^2 + Z \cdot \ell$, where ℓ is linear, so that $Z \mid B, C, E, F$, and Z does not divide D; also, the nonsingularity of S at $(0,1,0,0)$ implies that $C = cZ$, with $c \neq 0$. Now calculate $\Delta(Z,T)$ modulo Z^2.]

7.2 Prove that given 3 disjoint lines $\ell_1, \dots, \ell_3 \subset \mathbb{P}^3$, there exists a nonsingular quadric $Q \supset \ell_1, \dots, \ell_3$. [Hint: on each line ℓ_i, take 3 points $P_i, P_i', P_i'' \in \ell_i$, and show as in (1.11) or (2.4) that there is at least one quadric Q through them; it follows that each $\ell_i \subset Q$. Now show that Q can't be singular: for example, what happens if Q is a pair of planes?]

7.3 *The Hessian.* Let $f = f_d(x_0, \dots, x_n)$ be a form of degree d in $x_0, \dots, x_n$, defining a hypersurface $V : (f = 0) \subset \mathbb{P}^n$; suppose for simplicity that the characteristic $\neq 2$ and does not divide $(d-1)$. Write $f_{x_i} = \partial f / \partial x_i$ and $f_{x_i x_j} = \partial^2 f / \partial x_i \partial x_j$ for the first and second derivatives of f. The Taylor expansion of f about a point $P \in \mathbb{P}^n$ is

$$f = f(P) + f^{(1)}(x) + f^{(2)}(x) + \cdots,$$

where $f^{(1)}$ and $f^{(2)}$ are linear and quadratic forms:

$$f^{(1)} = \sum f_{x_i}(P) \cdot x_i \quad \text{and} \quad f^{(2)} = (1/2) \sum f_{x_i x_j}(P) \cdot x_i x_j.$$

If $P \in V$ is singular then $f(P)$ and $f^{(1)}$ vanish at P, and the nature of V or of f near P is determined to second order by the quadratic form $f^{(2)}$. Similarly if $P \in V$ is nonsingular then the nature of f restricted to the hyperplane $T_P V$ (or of the singular hyperplane section $V \cap T_P V$) is determined by $f^{(2)}$. Define the *Hessian matrix* of f (w.r.t. coordinates $x_0, \dots, x_n$) by $H(f) = H(f,x) = \{f_{x_i x_j}\}_{i,j}$, and the *Hessian* $h(f) = h(f,x)$ to be the determinant $h(f) = \det H(f)$.

(i) Let $x_i' = \sum a_{ij} x_j$ be a projective coordinate change with $A = (a_{ij})$ a nonsingular $(n+1) \times (n+1)$ matrix. If $g(x') = f(Ax)$, prove that the Hessian matrix transforms as

$$H(g, x') = ({}^t A) H(f,x) A$$

where ${}^t A$ is the transpose matrix; deduce that $h(g,x') = (\det A)^2 h(f,x)$.

(ii) Consider an affine piece $V_{(i)} \subset \mathbb{A}^n_{(i)}$ of $V : (f = 0)$ as in (5.5). Let $P \in V_{(i)}$ be a nonsingular point, and $\Pi = T_P V_{(i)}$ the affine tangent plane; write φ for the restriction to Π of the defining equation $f / x_i d$ of $V_{(i)}$. Prove that the Taylor expansion of φ at P starts with a nondegenerate quadratic form $\varphi^{(2)}$ (in $n-1$ variables) if and only if $h(f)(P) \neq 0$.
[Hint: Reduce to $P = (1, 0, \dots, 0)$ and $T_P V : (x_1 = 0)$ using (i). Then $\varphi^{(2)}$ is the bottom right $(n-1) \times (n-1)$ block of the projective Hessian matrix $H(f)$. Use $f_{x_i}(P) = 0$ for $i \neq 1$ and Euler's formula $\sum_j f_{x_i x_j} \cdot x_j = (d-1) f x_i$ to show that the matrix $H(f)$ has exactly one nonzero entry in the zeroth row and column. Compare [Fulton, p. 116].]

(iii) Let $C : (f = 0) \subset \mathbb{P}^2$ be a nonsingular plane cubic curve; deduce from (ii) that $P \in C$ is an inflexion point if and only if $H(f)(P) = 0$. Bézout's theorem implies that $(f = H(f) = 0) \subset \mathbb{P}^2$ is nonempty (see (1.9) and [Fulton, p. 112]).

(iv) Let $S:(f=0)\subset\mathbb{P}^3$ be a nonsingular cubic surface; for $P\in S$ prove that if P is not on a line of S then the intersection $S\cap T_PS$ is a cuspidal cubic if and only if $H(f)(P)=0$. Deduce that cuspidal cubic sections exist, as required in Step 1 of the proof of (7.2).

7.4 (i) Prove that if $P\in S$ is a singular point of a cubic surface then there is at least one line $\ell\subset S$ through P (and 'in general' 6).

(ii) If $X\subset\mathbb{P}^4$ is a nonsingular cubic hypersurface (a cubic 3-fold) and $P\in X$ then there is at least one line $\ell\subset X$ through P (and 'in general' 6). [Hint: write down the equation of X in coordinates with $P=(1,0,\dots,0)$.]

7.5 Prove that the rational map $\varphi\colon S\dashrightarrow \ell\times m$ of Corollary 7.4, (b) is in fact a morphism; prove that it contracts 5 lines of S to points.

7.6 Find all 27 lines of the diagonal (or 'Fermat') cubic surface

$$S:(X^3+Y^3+Z^3+T^3=0)\subset\mathbb{P}^3$$

in terms of planes such as $(X=\rho Y)$, where $\rho^3=1$.

7.7 Let $S\subset\mathbb{P}^3$ be the cubic surface given by $S:(f=0)$, where

$$f(X,Y,Z,T)=ZX^2+TY^2+(Z-d^2T)(Z-e^2T)(Z-f^2T),$$

with d,e,f distinct nonzero elements of k, and $\ell\subset S$ the line given by $Z=T=0$. By considering as in (7.3) the variable plane through ℓ, write down the equations of the 10 lines of S meeting ℓ.

7.8 *suggested by R. Casdagli.* Consider the cubic surface $S_{(0)}\subset\mathbb{R}^3$ given in affine coordinates by

$$x^2+y^2+z^2-2xyz=1+\lambda^2, \qquad (*)$$

where $\lambda\in\mathbb{R}$, $\lambda>0$ is a constant. (i) By rewriting $(*)$ as

$$(x-yz)^2=(y^2-1)(z^2-1)+\lambda^2,$$

show that $S_{(0)}$ has 4 tubes going off to infinity. On the other hand, the corresponding projective surface $S\subset\mathbb{P}^3_{\mathbb{R}}$ meets infinity in 3 lines $XYZ=0$. Use this to describe the topology of S.

(ii) By considering $(*)$ as the equation of a variable conic in the (x,y)-plane with parameter z, show that the four pairs of lines of $S_{(0)}$ which meet $(Z=0)$ asymptotically are given by

$$\begin{aligned} & z=\mu, \quad x=(\mu\pm\lambda)y;\\ & z=-\mu, \quad x=(-\mu\pm\lambda)y;\\ & z=1, \quad x-y=\pm\lambda;\\ \text{and}\quad & z=-1, \quad x+y=\pm\lambda, \end{aligned}$$

where $\mu^2=1+\lambda^2$.

Represent the surface $S_{(0)}$ in $\mathbb{R}^3$ and its 24 lines by computer graphics, or by making a plaster model.

7.9 *A case when all the lines are rational.* Suppose char $k \neq 2$ and let $S : (f = 0)$ be a nonsingular cubic surface, with

$$f = A(X,Y) \cdot T - B(X,Y) \cdot Z + (\text{terms of degree} \geq 2 \text{ in } Z \text{ and } T).$$

Then $S : (f = 0)$ contains $\ell : (Z = T = 0)$, and the tangent plane at $P = (1, \lambda, 0, 0)$ is $T_P S : A(1,\lambda)T = B(1,\lambda)Z$.

(i) Use linear coordinate changes in (X,Y) and (Z,T) to reduce A, B to $A = X^2 + \Delta Y^2$, $B = XY$ (with $\Delta \in k$), and if Δ is a perfect square to $A = X^2$, $B = Y^2$.

(ii) Suppose that S also contains the line $m : (X = Y = 0)$, and for ease of notation that $A = X^2$, $B = Y^2$. Let ℓ_i for $i = 1, \dots, 5$ be the 5 common transversals of ℓ and n, and write $P_i = (1, \lambda i, 0, 0) = \ell_i \cap \ell$ for the points of intersection of ℓ and ℓ_i. Prove that

$$\ell_i : (Y = \lambda_i X, T = \lambda_i^2 Z) \quad \text{for } i = 1, \dots, 5,$$

and that

$$f = X^2T - Y^2Z + X(\sigma_5 Z^2 + \sigma_3 ZT + \sigma_1 T^2) \\ - Y(\sigma_4 Z^2 + \sigma_2 ZT + T^2)$$

where $\sigma_1, \dots, \sigma_5$ are the elementary symmetric functions in $\lambda_1, \dots, \lambda_5$.

(iii) Find the remaining lines on S. [Hint: ℓ_i' and ℓ_i'' are contained in planes you already know. Arguing as in (7.6), it's not hard to show that every line meeting all 3 of ℓ_1, ℓ_2, ℓ_3 is given by $(\tau_2 Z + T) : X = (\tau_3 Z + \tau_1 T) : Y = \alpha : \beta$ for some $\alpha : \beta \in \mathbb{P}^1$, where $\tau_1, \dots, \tau_3$ are the elementary symmetric functions in $\lambda_1, \dots, \lambda_3$.]

7.10 This exercise is for the reader who likes big calculations, or has access to a computer algebra system. If a nonsingular cubic surface S has a nodal cubic curve C as a section, its equation can be written as

$$f = XYZ - X^3 - Y^3 + Tg.$$

Let $P_\alpha = (\alpha, \alpha^2, 1 + \alpha^3, 0)$ with $\alpha \neq 0, \infty$ be a variable point of C, and $Q = (0, Y, Z, T)$. Then expanding out $f(\lambda P_\alpha + \mu Q)$ in terms of the polar of f as in (7.2), Step 3, show that the line $P_\alpha Q \subset S$ if and only if $A = B = C = 0$, where

$$\begin{aligned} A &= (-2\alpha^4 + \alpha)Y + \alpha^3 Z + g(\alpha, \alpha^2, 1 + \alpha^3, 0)T; \\ B &= \alpha YZ - 3\alpha^2 Y^2 + g_1(\alpha, \alpha^2, 1 + \alpha^3, 0; 0, Y, Z, T)T; \\ C &= -Y^3 + g(0, Y, Z, T)T. \end{aligned}$$

Prove that there is a 'resultant' polynomial $R_{27}(\alpha)$, which is monic in α of degree 27 and with constant term 1, such that for $\alpha \neq 0$,

$$R(\alpha) = 0 \iff A = B = C = 0 \text{ have a} \\ \text{common zero } (\eta : \zeta : \tau) \in \mathbb{P}^2.$$

[Hint: solve $A = 0$ for Z (this introduces a term α^3 in the denominator), substitute for Z in B and C to get a binary quadratic and cubic in Y, T, then use the Sylvester determinant to eliminate Y and T. What makes this case hard is that the determinant formed by the leading term in each entry vanishes. The reason for this is that A, B, C do have the trivial common solution $Q = P_\alpha = (0,0,1,0)$ when $\alpha = 0$ or ∞. A priori, the determinant has terms in $\alpha^{18}, \ldots, \alpha^{-15}$, and you have to calculate the first and last 4 coefficients to prove that in fact it is $-1 \cdot \alpha^{15} + \cdots - 1 \cdot \alpha^{-12}$.]

Chapter 8

Final comments

This final section is not for examination, but some of the topics may nevertheless be of interest to the student.

History and sociology of the modern subject

8.1 Introduction

Algebraic geometry has over the last thirty years or so enjoyed a position in math similar to that of math in the world at large, being respected and feared much more than understood. At the same time, the 'service' questions I am regularly asked by British colleagues or by Warwick graduate students are generally of an elementary kind: as a rule, they are either covered in this book or in [Atiyah and Macdonald]. What follows is a view of the recent development of the subject, attempting to explain this paradox. I make no pretence at objectivity.

8.2 Prehistory

Algebraic geometry developed in the 19th century from several different sources. Firstly, the geometric tradition itself: projective geometry (and descriptive geometry, of great interest to the military at the time of Napoleon), the study of curves and surfaces for their own sake, configuration geometry; then complex function theory, the view of a compact Riemann surface as an algebraic curve, and the purely algebraic reconstruction of it from its function field. On top of this, the deep analogy between algebraic curves and the ring of integers of a number field, and the need for a language in algebra and geometry for invariant theory, which played an important role in the development of abstract algebra at the start of the 20th century.

The first decades of the 20th century saw a deep division. On the one hand, the geometric tradition of studying curves and surfaces, as pursued notably by the brilliant Italian school; alongside its own quite considerable achievements, this played a substantial motivating role in the development of topology and differential geometry, but became increasingly dependent on arguments 'by geometric intuition' that even the *Maestri* were unable to sustain rigorously. On the other hand, the newly emerging forces of commutative algebra were laying foundations and providing techniques of