

Now set $V_0 = \varphi^{-1}\psi^{-1}V'$, and $W_0 = \psi^{-1}\varphi^{-1}W'$; then $\varphi: V_0 \rightarrow \psi^{-1}V'$ is a morphism by construction. However, $\psi^{-1}V' \subset W_0$, since $P \in \psi^{-1}V'$ implies that $\varphi(\psi(P)) = P$, so that $P \in \psi^{-1}\varphi^{-1}W' = W_0$. Therefore, $\varphi: V_0 \rightarrow W_0$ is a morphism, and similarly $\psi: W_0 \rightarrow V_0$. Q.E.D.

5.9 Rational varieties

The notion of birational equivalence discussed in (5.8) is of key importance in algebraic geometry. Condition (iii) in the proposition says that the ‘meat’ of the varieties V and W is the same, although they may differ a bit around the edges; an example of the use of birational transformations is blowing up a singular variety to obtain a nonsingular one, see (6.12) below. An important particular case of Proposition 5.8 is the following result.

Corollary *Given a variety V , the following two conditions are equivalent:*

- (a) *the function field $k(V)$ is a purely transcendental extension of k , that is $k(V) \cong k(t_1, \dots, t_n)$ for some n ;*
- (b) *there exists a dense open set $V_0 \subset V$ which is isomorphic to a dense open subset $U_0 \subset \mathbb{A}^n$.*

A variety satisfying these conditions is said to be *rational*. Condition (b) is a precise version of the statement that V can be parametrised by n independent variables. This notion has already appeared implicitly several times in these notes (for example, (1.1), (2.1), (3.11, b), (5.7, II)). A large proportion of the elementary applications of algebraic geometry to other branches of math are related one way or another to rational varieties.

5.10 Reduction to a hypersurface

An easy consequence of the discussion of Noether normalisation at the end of §3 is that every variety is birational to a hypersurface: firstly, since birational questions only depend on a dense open set, and any open set contains a dense open subset isomorphic to an affine variety (by (4.13)), I only need to consider an affine variety $V \subset \mathbb{A}^n$. It was proved in (3.18) that there exist elements $y_1, \dots, y_{m+1} \in k[V]$ which generate the field extension $k \subset k(V)$, and such that y_1, \dots, y_m are algebraically independent, and y_{m+1} is algebraic over $k(y_1, \dots, y_m)$. These elements thus define a morphism $V \rightarrow \mathbb{A}^{m+1}$ which is a birational equivalence of V with a hypersurface $V' \subset \mathbb{A}^{m+1}$.

5.11 Products

If V and W are two affine varieties then there is a natural sense in which $V \times W$ is again a variety: if $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ then $V \times W$ is the subset of \mathbb{A}^{n+m} given by

$$\left\{ ((\alpha_1, \dots, \alpha_n); (\beta_1, \dots, \beta_m)) \mid \begin{array}{l} f(\underline{\alpha}) = 0 \text{ for all } f \in I(V) \\ g(\underline{\beta}) = 0 \text{ for all } g \in I(W) \end{array} \right\}$$

It’s easy to check that $V \times W$ remains irreducible. Note however that the Zariski topology of the product is not the product of the Zariski topologies (see Ex. 5.10).

The case of projective varieties is not so obvious; to be able to define products, we need to know that $\mathbb{P}^n \times \mathbb{P}^m$ is itself a projective variety. Notice that it is definitely not isomorphic to \mathbb{P}^{n+m} (see Ex. 5.2, ii). To do this, I use a construction rather similar in spirit to that of (5.7, I): make an embedding (the ‘Segre embedding’)

$$\varphi: \mathbb{P}^n \times \mathbb{P}^m \rightarrow S_{n,m} \subset \mathbb{P}^N,$$

where $N = (n+1)(m+1) - 1$ as follows: \mathbb{P}^N is the projective space with homogeneous coordinates

$$(U_{ij})_{\substack{i=0,\dots,n \\ j=0,\dots,m}}.$$

It’s useful to think of the U_{ij} as being set out in a matrix

$$\begin{pmatrix} U_{00} & \dots & U_{0m} \\ U_{10} & \dots & \dots \\ \dots & \dots & U_{nm} \end{pmatrix}$$

Then define φ by $((X_0, \dots, X_n), (Y_0, \dots, Y_m)) \mapsto (X_i Y_j)_{\substack{i=0,\dots,n \\ j=0,\dots,m}}$. This is obviously a well defined morphism, and the image $S_{n,m}$ is easily seen to be the projective subvariety given by

$$\text{rank} \begin{pmatrix} U_{00} & \dots & U_{0m} \\ U_{10} & \dots & \dots \\ \dots & \dots & U_{nm} \end{pmatrix} \leq 1, \quad \text{that is, } \det \begin{vmatrix} U_{ik} & U_{i\ell} \\ U_{jk} & U_{j\ell} \end{vmatrix} = 0$$

for all $i, j = 0, \dots, n$ and $k, \ell = 0, \dots, m$.

We get an inverse map $S_{n,m} \rightarrow \mathbb{P}^n \times \mathbb{P}^m$ as follows. For $P \in S_{n,m}$ there exists at least one pair (i, j) such that $U_{ij}(P) \neq 0$; fixing this (i, j) , send

$$S_{n,m} \ni P \mapsto ((U_{0j}, \dots, U_{nj}), (U_{i0}, \dots, U_{im})) \in \mathbb{P}^n \times \mathbb{P}^m.$$

Note that the choice of (i, j) doesn’t matter, since the matrix $U_{ij}(P)$ has rank 1, and hence all its rows and all its columns are proportional.

From this it is not hard to see that if $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ are projective varieties, then $V \times W \subset \mathbb{P}^n \times \mathbb{P}^m \cong S_{n,m} \subset \mathbb{P}^N$ is again a projective variety (see Ex. 5.11).

Exercises to Chapter 5

- 5.1 Prove that a regular function on \mathbb{P}^1 is a constant. [Hint: use the notation of (5.0); suppose that $f \in k(\mathbb{P}^1)$ is regular at every point of \mathbb{P}^1 . Apply (4.8, II) to the affine piece $\mathbb{A}_{(0)}^1$, to show that $f = p(x_0) \in k[x_0]$; on the other affine piece $\mathbb{A}_{(\infty)}^1$, $f = p(1/y_1) \in k[y_1]$. Now, how can it happen that $p(1/y_1)$ is a polynomial?] Deduce that there are no nonconstant morphisms $\mathbb{P}^1 \rightarrow \mathbb{A}^m$ for any m .

- 5.2 *The quadric surface in \mathbb{P}^3 .*