Chapter 4

Functions on varieties

In this section I work over a fixed field k; from (4.8, II) onwards, k will be assumed to be algebraically closed. The reader who assumes throughout that $k = \mathbb{C}$ will not lose much, and may gain a psychological crutch. I sometimes omit mention of the field k to simplify notation.

4.1 Polynomial functions

Let $V \subset \mathbb{A}^n_k$ be an algebraic set, and $I(V)$ its ideal. Then the quotient ring $k[V] = k[X_1, \dots, X_n]/I(V)$ is in a natural way a ring of functions on V. In more detail, define a *polynomial function* on V to be a map $f\colon V \to k$ of the form $P \mapsto F(P)$, with $F \in k[X_1, \dots, X_n]$; this just means that f is the restriction of a map $F\colon \mathbb{A}^n \to k$ defined by a polynomial. By definition of $I(V)$, two elements $F, G \in k[X_1, \dots, X_n]$ define the same function on V if and only if

$$F(P) - G(P) = 0 \text{ for all } P \in V,$$

that is, if and only if $F - G \in I(V)$. Thus I define the *coordinate ring* $k[V]$ by

$$\begin{aligned} k[V] &= \bigl\{ f\colon V \to k \bigm| f \text{ is a polynomial function} \bigr\} \\ &\cong k[X_1, \dots, X_n]/I(V). \end{aligned}$$

This is the smallest ring of functions on V containing the coordinate functions X_i (together with k), so for once the traditional terminology is not too obscure.

4.2 $k[V]$ and algebraic subsets of V

An algebraic set $X \subset \mathbb{A}^n$ is contained in V if and only if $I(X) \supset I(V)$. On the other hand, ideals of $k[X_1, \dots, X_n]$ containing $I(V)$ are in obvious bijection with ideals of $k[X_1, \dots, X_n]/I(V)$. (Think about this if it's not obvious to you: the ideal J with $I(V) \subset J \subset k[X_1, \dots, X_n]$ corresponds to $J/I(V)$; and conversely, an ideal J_0 of $k[X_1, \dots, X_n]/I(V)$ corresponds to its inverse image in $k[X_1, \dots, X_n]$.)