8.9 Regular functions and sheaves

The reader who has properly grasped the notion of rational function $f \in k(X)$ on a variety X and of regularity of f at $P \in X$ ((4.7) and (5.4)) already has a pretty good intuitive idea of the structure sheaf $\mathcal{O}_X$. For an open set $U \subset X$, the set of regular functions $U \to k$

$$\mathcal{O}_X(U) = \left\{ f \in k(X) \bigm| f \text{ is regular } \forall P \in U \right\} = \bigcap_{P \in U} \mathcal{O}_{X,P}$$

is a subring of the field $k(X)$. The sheaf $\mathcal{O}_X$ is just the family of rings $\mathcal{O}_X(U)$ as U runs through the opens of X. Clearly, any element of the local ring $\mathcal{O}_{X,P}$ (see (4.7) and (5.4) for the definition) is regular in some neighbourhood U of P, so that $\mathcal{O}_{X,P} = \bigcup_{U \ni P} \mathcal{O}_X(U)$. There's no more to it than that; there's a fixed pool of rational sections $k(X)$, and sections of the sheaf over an open U are just rational sections with a regularity condition at every $P \in U$.

This language is adequate to describe any torsion free sheaf on an irreducible variety with the Zariski topology. Of course, you need the full definition of sheaves if X is reducible, or if you want to handle more complicated sheaves, or to use the complex topology.

8.10 Globally defined regular functions

If X is a projective variety then the only rational functions $f \in k(X)$ that are regular at every $P \in X$ are the constants. This is a general property of projective varieties, analogous to Liouville's theorem in functions of one complex variable; for a variety over $\mathbb{C}$ it comes from compactness and the maximum modulus principle ($X \subset \mathbb{P}^n_{\mathbb{C}}$ is compact in the complex topology, so the modulus of a global holomorphic function on X must take a maximum), but in algebraic geometry it is surprisingly hard to prove from scratch (see for example [Hartshorne, I.3.4]; it is essentially a finiteness result, related to the finite dimensionality of coherent cohomology groups).

8.11 The surprising sufficiency of projective algebraic geometry

Weil's abstract definition of a variety (affine algebraic sets glued together along isomorphic open sets) was referred to briefly in (0.4), and is quite easy to handle in terms of sheaves. Given this, the idea of working only with varieties embedded in a fixed ambient space $\mathbb{P}^N_k$ seems at first sight unduly restrictive. I want to describe briefly the modern point of view on this question.

(a) Polarisation and positivity

Firstly, varieties are usually considered up to isomorphism, so saying a variety X is *projective* means that X can be embedded in some $\mathbb{P}^N$, that is, is isomorphic to a closed subvariety $X \subset \mathbb{P}^N$ as in (5.1–7). *Quasiprojective* means isomorphic to a locally closed subvariety of $\mathbb{P}^N$, so an open dense subset of a projective variety; projectivity includes the property of *completeness*, that X cannot be embedded as a dense open set of any bigger variety.

The choice of an actual embedding $X \hookrightarrow \mathbb{P}^N$ (or of a very ample line bundle $\mathcal{O}_X(1)$ whose sections will be the homogeneous coordinates of $\mathbb{P}^N$) is often called a *polarisation*, and we write

$(X, \mathcal{O}_X(1))$ to indicate that the choice has been made. In addition to completeness, a projective variety $X \subset \mathbb{P}^N$ satisfies a key condition of 'positive degree': if $V \subset X$ is a k-dimensional subvariety then V intersects a general linear subspace $\mathbb{P}^{N-k}$ in a positive finite number of points. Conversely, the Kleiman criterion says that some multiple of a line bundle on a complete variety X can be used to provide a projective embedding of X if its degree on every curve $C \subset X$ is consistently greater than zero (that is, $\geq \varepsilon \cdot$ (any reasonable measure of C)). This kind of positivity relates closely to the choice of a Kähler metric on a complex manifold (a Riemannian metric with the right kind of compatibility with the complex structure). So we understand projectivity as a kind of 'positive definiteness'.

(b) Sufficiency

The surprising thing is the many problems of algebraic geometry having answers within the framework of projective varieties. The construction of Chow varieties mentioned in (8.2) is one such example; another is Mumford's work of the 1960s, in which he constructed Picard varieties and many moduli spaces as quasiprojective varieties (schemes). Mori theory (responsible for important conceptual advances in classification of varieties related to rationality, see [Kollár]) is the most recent example; here the ideas and techniques are inescapably projective in nature.

(c) Insufficiency of abstract varieties

Curves and nonsingular surfaces are automatically quasiprojective; but abstract varieties that are not quasiprojective do exist (singular surfaces, or nonsingular varieties of dimension ≥ 3). However, if you feel the need for these constructions, you will almost certainly also want Moishezon varieties (M. Artin's algebraic spaces), objects of algebraic geometry more general than abstract varieties, obtained by a somewhat more liberal interpretation of 'glueing local pieces'.

Theorems on abstract varieties are often proved by a reduction to the quasiprojective case, so whether the quasiprojective proof or the detail of the reduction process is more useful, interesting, essential or likely-to-lead-to-cheap-publishable-work will depend on the particular problem and the individual student's interests and employment situation. It has recently been proved that a nonsingular abstract variety or Moishezon variety that is not quasiprojective necessarily contains a rational curve; however, the proof (due to J. Kollár) is Mori theoretic, so hardcore projective algebraic geometry.

8.12 Affine varieties and schemes

The coordinate ring $k[V]$ of an affine algebraic variety V over an algebraically closed field k (Definition 4.1) satisfies two conditions: (i) it is a finitely generated k-algebra; and (ii) it is an integral domain. A ring satisfying these two conditions is obviously of the form $k[V]$ for some variety V, and is called a *geometric ring* (or *geometric k-algebra*).

There are two key theoretical results in Part II; one of these is Theorem 4.4, which states precisely that $V \mapsto k[V] = A$ is an equivalence of categories between affine algebraic varieties and the opposite of the category of geometric k-algebras (although I censored out all mention of categories as unsuitable for younger readers). The other is the Nullstellensatz (3.10), that prime ideals of $k[V]$ are in bijection with irreducible subvarieties of V; the points of V are in bijection with maximal ideals.