5.7 Examples

(I) Rational normal curve. This is a very easy example of an isomorphic embedding $f\colon \mathbb{P}^1 \xrightarrow{\cong} C \subset \mathbb{P}^m$ which generalises the parametrised conic of (1.7), and which occurs throughout projective and algebraic geometry. Define

$$f\colon \mathbb{P}^1 \to \mathbb{P}^m \quad \text{by} \quad (U:V) \mapsto (U^m : U^{m-1}V : \cdots : V^m)$$

(writing down all monomials of degree m in U, V). Arguing step by step:

(i) f is a rational map, since it's given by

$$((U/V)^m, (U/V)^{m-1}, \dots, 1);$$

(ii) f is a morphism wherever $V \neq 0$ by the formula just written, and if $V = 0$ then $U \neq 0$, so a similar trick with V/U works;

(iii) the image of f is the set of points $(X_0 : \cdots : X_m) \in \mathbb{P}^m$ such that

$$(X_0 : X_1) = (X_1 : X_2) = \cdots = (X_{m-1} : X_m),$$

that is,

$$X_0X_2 = X_1^2, \quad X_0X_3 = X_1X_2, \quad X_0X_4 = X_1X_3, \quad \text{etc.}$$

The equations can be written all together in the extremely convenient determinantal form

$$\operatorname{rank}\begin{pmatrix} X_0 & X_1 & X_2 & \dots & X_{m-1} \\ X_1 & X_2 & X_3 & \dots & X_m \end{pmatrix} \leq 1$$

(the rank condition means exactly that all 2×2 minors vanish). These are homogeneous equations defining an algebraic set $C \subset \mathbb{P}^m$;

(iv) the inverse morphism $g : C \to \mathbb{P}^1$ is not hard to find: just take a point of C into the common ratio $(X_0 : X_1) = \cdots = (X_{m-1} : X_m) \in \mathbb{P}^1$. As an exercise, find out for yourself what has to be checked, then check it all.

(II) Linear projection, parametrising a quadric. The map $\pi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ given by $(X_0, X_1, X_2, X_3) \mapsto (X_1, X_2, X_3)$ is a rational map, and a morphism outside the point $P_0 = (1, 0, 0, 0)$. Let $Q \subset \mathbb{P}^3$ be a quadric hypersurface with $P \in Q$. Then every point P of $\mathbb{P}^2$ corresponds to a line L of $\mathbb{P}^3$ through P, and L should in general meet Q at P_0 and a second point $\varphi(P)$: for example, if $Q : (X_0X_3 = X_1X_2)$, then $\pi|Q\colon Q \dashrightarrow \mathbb{P}^2$ has the inverse map

$$\varphi\colon \mathbb{P}^2 \dashrightarrow Q \ \text{ given by } \ (X_1, X_2, X_3) \mapsto (X_1X_2/X_3, X_1, X_2, X_3).$$

This is essentially the same idea as the parametrisation of the circle in (1.1).

It is a rewarding exercise (see Ex. 5.2) to find $\operatorname{dom}\pi$ and $\operatorname{dom}\varphi$, and to give a geometric interpretation of the singularities of π and φ.

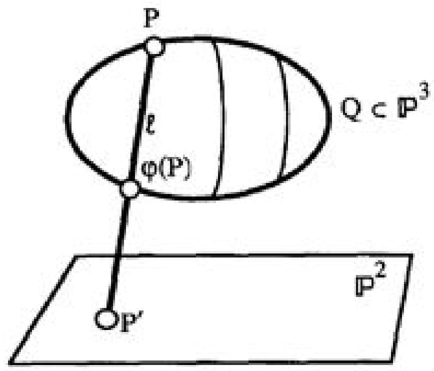

Figure 5.2: Projection of quadric surface

5.8 Birational maps

Definition Let V and W be (affine or projective) varieties; then a rational map $f\colon V \dashrightarrow W$ is *birational* (or is a *birational equivalence*) if it has a rational inverse, that is, if there exists a rational map $g\colon W \dashrightarrow V$ such that $f \circ g = \mathrm{id}_W$ and $g \circ f = \mathrm{id}_V$.

Proposition *The following three conditions on a rational map $f\colon V \dashrightarrow W$ are equivalent:*

(i) f is a birational equivalence;

(ii) f is dominant (see (4.10)), and $f^*\colon k(W) \to k(V)$ is an isomorphism;

(iii) there exist open sets $V_0 \subset V$ and $W_0 \subset W$ such that f restricted to V_0 is an isomorphism $f\colon V_0 \to W_0$.

Proof f^* is defined in the same way as for affine varieties, and (i) $\iff$ (ii) is as in (4.11). (iii) $\implies$ (i) is clear, since an isomorphism $f\colon V_0 \to W_0$ and its inverse $g = f^{-1}\colon W_0 \to V_0$ are by definition rational maps between V and W.

The essential implication (i) $\implies$ (iii) is tricky, although content-free (GOTO (5.9) if you want to avoid a headache): by assumption (i), there are inverse rational maps $f\colon V \dashrightarrow W$ and $g\colon W \dashrightarrow V$; now set $V' = \operatorname{dom} f \subset V$ and $\varphi = f|V'\colon V' \to W$, and similarly $W' = \operatorname{dom} g \subset W$ and $\psi = g|W'\colon W' \to V$. In the diagram

$$\begin{array}{ccccc} \psi^{-1}V' & \xrightarrow{\psi} & V' & \xrightarrow{\varphi} & W \\ \cap & & & & \\ W & & & & \end{array}$$

all the arrows are morphisms, and $\mathrm{id}_W|\psi^{-1}V' = \varphi \circ \psi$ (as morphisms) follows from $\mathrm{id}_W = f \circ g$ (as rational maps). Hence

$$\varphi(\psi(P)) = P \quad \text{for all } P \in \psi^{-1}V'.$$