1.5 Equation of a conic

The inhomogeneous quadratic polynomial

$$q(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$$

corresponds to the homogeneous quadratic

$$Q(X,Y,Z) = aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2;$$

the correspondence is easy to understand as a recipe, or you can think of it as the bijection $q \leftrightarrow Q$ given by

$$q(x,y) = Q(X/Z, Y/Z, 1) \quad \text{with} \quad x = X/Z,\ \ y = Y/Z$$

and inversely,

$$Q = Z^2 q(X/Z, Y/Z).$$

A *conic* $C \subset \mathbb{P}^2$ is the curve given by $C : (Q(X,Y,Z) = 0)$, where Q is a homogeneous quadratic expression; note that the condition $Q(X,Y,Z) = 0$ is well defined on the equivalence class, since $Q(\lambda\mathbf{X}) = \lambda^2 Q(\mathbf{X})$ for any $\lambda \in \mathbb{R}$. As an exercise, check that the projective curve C meets the affine piece $\mathbb{R}^2$ in the affine conic given by $(q = 0)$.

'Line at infinity' and asymptotic directions

Points of $\mathbb{P}^2$ with $Z = 0$ correspond to ratios $(X : Y : 0)$. These points form the *line at infinity*, a copy of $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (since $(X : Y) \mapsto X/Y$ defines a bijection $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$).

A line in $\mathbb{P}^2$ is by definition given by $L : (aX + bY + cZ = 0)$, and

$$L \text{ passes through } (X,Y,0) \iff aX + bY = 0.$$

In affine coordinates the same line is given by $ax + by + c = 0$, so that all lines with the same ratio $a : b$ pass through the same point at infinity. This is called 'parallel lines meet at infinity'.

Example (a) The hyperbola $(\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1)$ in $\mathbb{R}^2$ corresponds in $\mathbb{P}^2_{\mathbb{R}}$ to $C : (\frac{X^2}{a^2} - \frac{Y^2}{b^2} = Z^2)$; clearly this meets $(Z = 0)$ in the two points $(a, \pm b, 0) \in \mathbb{P}^2_{\mathbb{R}}$, corresponding in the obvious way to the asymptotic lines of the hyperbola.

Note that in the affine piece $(X \neq 0)$ of $\mathbb{P}^2_{\mathbb{R}}$, the affine coordinates are $u = Y/X, v = Z/X$, so that C becomes the ellipse $(\frac{u^2}{b^2} + v^2 = \frac{1}{a^2})$. See Ex. 1.7 for an artistic interpretation.

(b) The parabola $(y = mx^2)$ in $\mathbb{R}^2$ corresponds to $C : (YZ = mX^2)$ in $\mathbb{P}^2_{\mathbb{R}}$; this now meets $(Z = 0)$ at the single point $(0, 1, 0)$. So in $\mathbb{P}^2$, the 'two branches of the parabola meet at infinity'; note that this is a statement with intuitive content (maybe you feel it's pretty implausible?), but is not a result you could arrive at just by contemplating within $\mathbb{R}^2$ – maybe it's not even meaningful.