3.4 The correspondence V

k is any field, and $A = k[X_1, \dots, X_n]$. Following an almost universal idiosyncracy of algebraic geometers[1], I write $\mathbb{A}_k^n = k^n$ for the n-dimensional affine space over k; given a polynomial $f(X_1, \dots, X_n) \in A$ and a point $P = (a_1, \dots, a_n) \in \mathbb{A}_k^n$, the element $f(a_1, \dots, a_n) \in k$ is thought of as 'evaluating the function f at P'. Define a correspondence

$$\big\{\text{ideals } J \subset A\big\} \longrightarrow \big\{\text{subsets } X \subset \mathbb{A}_k^n\big\}$$

by

$$J \longmapsto V(J) = \big\{P \in \mathbb{A}_k^n \bigm| f(P) = 0 \ \forall f \in J\big\}.$$

Definition A subset $X \subset \mathbb{A}_k^n$ is an *algebraic set* if $X = V(I)$ for some I. (This is the same thing as a variety, but I want to reserve the word.) Notice that by Corollary 3.3, I is finitely generated. If $I = (f_1, \dots, f_r)$ then clearly

$$V(I) = \big\{P \in \mathbb{A}_k^n \bigm| f_i(P) = 0 \quad \text{for } i = 1, \dots, r\big\},$$

so that an algebraic set is just a locus of points satisfying a finite number of polynomial equations.

If $I = (f)$ is a principal ideal, then I usually write $V(f)$ for $V(I)$; this is of course the same thing as $V : (f = 0)$ in the notation of §§1–2.

3.5 Definition: the Zariski topology

Proposition-Definition *The correspondence V satisfies the following formal properties:*

(i) $V(0) = \mathbb{A}_k^n$; $V(A) = \emptyset$;

(ii) $I \subset J \implies V(I) \supset V(J)$;

(iii) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$;

(iv) $V\big(\sum_{\lambda \in \Lambda} I_\lambda\big) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Hence the algebraic subsets of $\mathbb{A}_k^n$ form the closed sets of a topology on $\mathbb{A}_k^n$, the Zariski topology.

The above properties are quite trivial, with the exception of the inclusion $\subset$ in (iii). For this, suppose $P \notin V(I_1) \cup V(I_2)$; then there exist $f \in I_1, g \in I_2$ such that $f(P) \neq 0, g(P) \neq 0$. So $fg \in I_1 \cap I_2$, but $fg(P) \neq 0$, and therefore $P \notin V(I_1 \cap I_2)$. Q.E.D.

The Zariski topology on $\mathbb{A}_k^n$ induces a topology on any algebraic set $X \subset \mathbb{A}_k^n$: the closed subsets of X are the algebraic subsets.

It's important to notice that the Zariski topology on a variety is very weak, and is quite different from the familiar topology of metric spaces like $\mathbb{R}^n$. As an example, a Zariski closed subset of $\mathbb{A}_k^1$ is either the whole of $\mathbb{A}_k^1$ or is finite; see Ex. 3.12 for a description of the Zariski topology on $\mathbb{A}_k^2$. If $k = \mathbb{R}$ or $\mathbb{C}$ then Zariski closed sets are also closed for the ordinary topology, since polynomial functions are continuous. In fact they're very special open or closed subsets: a nonempty Zariski open subset of $\mathbb{R}^n$ is the complement of a subvariety, so automatically dense in $\mathbb{R}^n$.

The Zariski topology may cause trouble to some students; since it is only being used as a language, and has almost no content, the difficulty is likely to be psychological rather than technical.

[1] $\mathbb{A}^n$ is thought of as a variety, whereas k^n is just a point set. Think of this as pure pedantry if you like; compare (4.6) below, as well as (8.3).