The case of projective varieties is not so obvious; to be able to define products, we need to know that $\mathbb{P}^n \times \mathbb{P}^m$ is itself a projective variety. Notice that it is definitely not isomorphic to $\mathbb{P}^{n+m}$ (see Ex. 5.2, ii). To do this, I use a construction rather similar in spirit to that of (5.7, I): make an embedding (the 'Segre embedding')

$$\varphi \colon \mathbb{P}^n \times \mathbb{P}^m \to S_{n,m} \subset \mathbb{P}^N,$$

where $N = (n+1)(m+1) - 1$ as follows: $\mathbb{P}^N$ is the projective space with homogeneous coordinates

$$(U_{ij})_{\substack{i=0,\dots,n \\ j=0,\dots,m}}.$$

It's useful to think of the U_{ij} as being set out in a matrix

$$\begin{pmatrix} U_{00} & \dots & U_{0m} \\ U_{10} & \dots & \dots \\ \dots & \dots & U_{nm} \end{pmatrix}$$

Then define φ by $((X_0,\dots,X_n),(Y_0,\dots,Y_m)) \mapsto (X_iY_j)_{\substack{i=0,\dots,n \\ j=0,\dots,m}}$. This is obviously a well defined morphism, and the image $S_{n,m}$ is easily seen to be the projective subvariety given by

$$\operatorname{rank} \begin{pmatrix} U_{00} & \dots & U_{0m} \\ U_{10} & \dots & \dots \\ \dots & \dots & U_{nm} \end{pmatrix} \le 1, \quad \text{that is, } \det \begin{vmatrix} U_{ik} & U_{i\ell} \\ U_{jk} & U_{j\ell} \end{vmatrix} = 0$$

for all $i,j = 0,\dots,n$ and $k,\ell = 0,\dots,m$.

We get an inverse map $S_{n,m} \to \mathbb{P}^n \times \mathbb{P}^m$ as follows. For $P \in S_{n,m}$ there exists at least one pair (i,j) such that $U_{ij}(P) \neq 0$; fixing this (i,j), send

$$S_{n,m} \ni P \mapsto ((U_{0j},\dots,U_{nj}),(U_{i0},\dots,U_{im})) \in \mathbb{P}^n \times \mathbb{P}^m.$$

Note that the choice of (i,j) doesn't matter, since the matrix $U_{ij}(P)$ has rank 1, and hence all its rows and all its columns are proportional.

From this it is not hard to see that if $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ are projective varieties, then $V \times W \subset \mathbb{P}^n \times \mathbb{P}^m \cong S_{n,m} \subset \mathbb{P}^N$ is again a projective variety (see Ex. 5.11).

Exercises to Chapter 5

5.1 Prove that a regular function on $\mathbb{P}^1$ is a constant. [Hint: use the notation of (5.0); suppose that $f \in k(\mathbb{P}^1)$ is regular at every point of $\mathbb{P}^1$. Apply (4.8, II) to the affine piece $\mathbb{A}^1_{(0)}$, to show that $f = p(x_0) \in k[x_0]$; on the other affine piece $\mathbb{A}^1_{(\infty)}$, $f = p(1/y_1) \in k[y_1]$. Now, how can it happen that $p(1/y_1)$ is a polynomial?] Deduce that there are no nonconstant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$ for any m.

5.2 *The quadric surface in* $\mathbb{P}^3$.

(i) Show that the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ (as in (5.10)) gives an isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric
$$S_{1,1} = Q : (X_0X_3 = X_1X_2) \subset \mathbb{P}^3.$$

(ii) What are the images in Q of the two families of lines $\{p\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{p\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$? Use this to find some disjoint lines in $\mathbb{P}^1 \times \mathbb{P}^1$, and conclude from this that $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$.
(The fact that a quadric surface has two rulings by straight lines has applications in civil engineering: if you're trying to build a curved surface out of concrete, it's an obvious advantage to be able to determine the shape of the surface by imposing linear constraints. See [M. Berger, 14.4.6–7 and 15.3.3] for a discussion and pictures.)

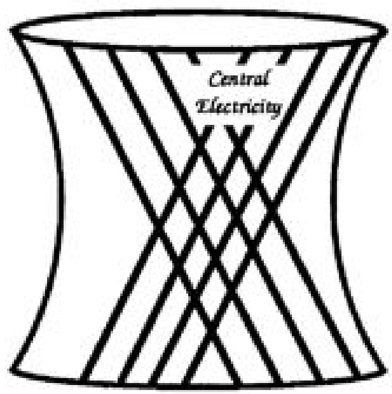

Figure 5.3: Quadrics surface as cooling tower

(iii) Show that there are two lines of Q passing through the point $P = (1,0,0,0)$, and that the complement U of these two lines is the image of $\mathbb{A}^1 \times \mathbb{A}^1$ under the Segre embedding.

(iv) Show that under the projection $\pi|Q \colon Q \dashrightarrow \mathbb{P}^2$ (in the notation of (5.7, II)), U maps isomorphically to a copy of $\mathbb{A}^2$, and the two lines through P are mapped to two points of $\mathbb{P}^2$.

(v) In the notation of (5.7, II), find $\operatorname{dom}\pi$ and $\operatorname{dom}\varphi$, and give a geometric interpretation of the singularities of π and φ.

5.3 Which of the following expressions define rational maps $\varphi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ (with $n, m = 1$ or 2) between projective spaces of the appropriate dimensions? In each case, determine $\operatorname{dom}\varphi$, say if φ is birational, and if so describe the inverse map.

(a) $(x,y,z) \mapsto (x,y)$;
(b) $(x,y) \mapsto (x,y,1)$;
(c) $(x,y) \mapsto (x,y,0)$;
(d) $(x,y,z) \mapsto (1/x, 1/y, 1/z)$;
(e) $(x,y,z) \mapsto ((x^3+y^3)/z^3, y^2/z^2, 1)$;
(f) $(x,y,z) \mapsto (x^2+y^2, y^2, y^2)$.

5.4 The rational normal curve (see (5.7, I)) of degree 3 is the curve $C \subset \mathbb{P}^3$ defined by the 3 quadrics $C = Q_1 \cap Q_2 \cap Q_3$, where

$$Q_1 : (XZ = Y^2), \quad Q_2 : (XT = YZ), \quad Q_3 : (YT = Z^2);$$

this curve is also well known as the *twisted cubic*, where 'twisted' refers to the fact that it is not a plane curve. Check that for any two of the quadrics Q_i, Q_j, the intersection $Q_i \cap Q_j = C \cup \ell_{ij}$, where ℓ_{ij} is a certain line. So this curve in 3-space is not the intersection of any 2 of the quadrics.

5.5 Let $Q_1 : (XZ = Y^2)$ and $F : (XT^2 - 2YZT + Z^3 = 0)$; prove that $C = Q_1 \cap F$ is the twisted cubic curve of Ex. 5.4. [Hint: start by multiplying F by X; subtracting a suitable multiple of Q_1, this becomes a perfect square)]

5.6 Let $C \subset \mathbb{P}^3$ be an irreducible curve defined by $C = Q_1 \cap Q_2$, where $Q_1 : (TX = q_1)$, $Q_2 : (TY = q_2)$, with q_1, q_2 quadratic forms in X, Y, Z. Show that the projection $\pi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ defined by $(X, Y, Z, T) \mapsto (X, Y, Z)$ restricts to an isomorphism of C with the plane curve $D \subset \mathbb{P}^2$ given by $Xq_2 = Yq_1$.

5.7 Let $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ be an isomorphism; identify the graph of φ as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^1 \cong Q \subset \mathbb{P}^3$. Now do the same if $\varphi \colon \mathbb{P}^1 \to \mathbb{P}^1$ is the two-to-one map given by $(X, Y) \mapsto (X^2, Y^2)$.

5.8 Prove that any irreducible quadric $Q \subset \mathbb{P}^{n+1}$ is rational; that is, as in the picture of (5.7, II), show that if $P \in Q$ is a nonsingular point, then the linear projection of $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$ induces a birational map $Q \dashrightarrow \mathbb{P}^n$.

5.9 For each of the following plane curves, write down the 3 standard affine pieces, and determine the intersection of the curve with the 3 coordinate axes:

(a) $y^2z = x^3 + axz^2 + bz^3$;

(b) $x^2y^2 + x^2z^2 + y^2z^2 = 2xyz(x + y + z)$;

(c) $xz^3 = (x^2 + z^2)y^2$.

5.10 (i) Prove that the product of two irreducible algebraic sets is again irreducible [Hint: the subsets $V \times \{w\}$ are irreducible for $w \in W$; given an expression $V \times W = U_1 \cup U_2$, consider the subsets

$$W_i = \{w \in W \mid V \times \{w\} \subset U_i\}$$

for $i = 1, 2$].

(ii) Describe the closed sets of the topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ which is the product of the Zariski topologies on the two factors; now find a closed subset of the Zariski topology of $\mathbb{A}^2$ not of this form.

5.11 (a) If $\mathbb{A}^n_{(0)}$ and $\mathbb{A}^m_{(0)}$ are standard affine pieces of $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, verify that the Segre embedding of (5.11) maps $\mathbb{A}^n_{(0)} \times \mathbb{A}^m_{(0)}$ isomorphically to an affine piece of the variety $S_{n,m} \subset \mathbb{P}^N$, say $S_{(0)} \subset \mathbb{A}^N$, and that the N coordinates of $\mathbb{A}^N$ restrict to $X_1, \dots, X_n, Y_1, \dots, Y_m$ and the nm terms X_iY_j.

(b) If $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$, prove that the product $V \times W$ is a projective subvariety of $\mathbb{P}^n \times \mathbb{P}^m = S_{n,m} \subset \mathbb{P}^N$. [Hint: the product of the affine pieces $V_{(0)} \times W_{(0)} \subset \mathbb{A}^{n+m}$ is a subvariety defined by polynomials as explained in (5.11); show that each of these is the restriction to $\mathbb{A}^{n+m} \cong S_{(0)}$ of a homogeneous polynomial in the U_{ij}.]

5.12 Let C be the cubic curve of (5.0); prove that any regular function f on C is constant. Proceed in the following steps:

Step 1 Applying (4.8, II) to the affine piece $C_{(0)}$, write $f = p(x,y) \in k[x,y]$.

Step 2 Subtracting a suitable multiple of the relation $y^2 - x^3 - ax - b$, assume that $p(x,y) = q(x) + yr(x)$, with $q, r \in k[x]$.

Step 3 Applying (4.8, II) to the affine piece $C_{(\infty)}$ gives

$$f = q(x_1/z_1) + (1/z_1)r(x_1/z_1) \in k[C_{(\infty)}],$$

and hence there exists a polynomial $S(x_1, z_1)$ such that

$$q(x_1/z_1) + (1/z_1)r(x_1/z_1) = S(x_1, z_1);$$

Step 4 Clear the denominator, and use the fact that $k[C_{(\infty)}] = k[x_1, z_1]/g$, where $g = z_1 - x_1^3 - ax_1z_1^2 - bz_1^3$, to deduce a polynomial identity

$$Q_m(x_1, z_1) + R_{m-1}(x_1, z_1) \equiv S(x_1, z_1)z_1^m + A(x_1, z_1)g$$

in $k[x_1, z_1]$, with Q_m and R_{m-1} homogeneous of the indicated degrees.

Step 5 Now if we write $S = S^+ + S^-$ and $A = A^+ + A^-$ for the decomposition into terms of even and odd degree, and note that g has only terms of odd degree, this identity splits into two:

$$Q_m \equiv S^+z_1^m + A^-g \quad \text{and} \quad R_{m-1} \equiv S^-z_1^m + A^+g$$

if m is even, and an analogous expression if m is odd.

Step 6 Q_m is homogeneous of degree m, and hence A^-g has degree $\geq m$; by considering the term of least degree in A^-g, prove that Q_m is divisible by z_1. Similarly for R_{m-1}. By taking the minimum value of m in the identity of Step 4, deduce that $q(x)$ has degree 0 and $r(x) = 0$.

5.13 *Veronese surface* Study the embedding $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^5$ given by $(X,Y,Z) \mapsto (X^2, XY, XZ, Y^2, YZ, Z^2)$; write down the equations defining the image $S = \varphi(\mathbb{P}^2)$, and prove that φ is an isomorphism (by writing down the equations of the inverse morphism). Prove that the lines of $\mathbb{P}^2$ go over into conics of $\mathbb{P}^5$, and that conics of $\mathbb{P}^2$ go over into twisted quartics of $\mathbb{P}^5$ (see (5.7)).

For any line $\ell \subset \mathbb{P}^2$, write $\pi(\ell) \subset \mathbb{P}^5$ for the projective plane spanned by the conic $\varphi(\ell)$. Prove that the union of $\pi(\ell)$ taken over all $\ell \subset \mathbb{P}^2$ is a cubic hypersurface $\Sigma \subset \mathbb{P}^5$. [Hint: as in (5.7) and (5.11), you can write the equations defining S in the form $\operatorname{rank} M \leq 1$, where M is a symmetric 3×3 matrix with entries the 6 coordinates of $\mathbb{P}^5$; then show that $\Sigma : (\det M = 0)$. See [Semple and Roth, p. 128] for more details.]

Chapter 6

Tangent space and nonsingularity, dimension

6.1 Nonsingular points of a hypersurface

Suppose $f \in k[X_1, \dots, X_n]$ is irreducible, $f \notin k$, and set $V = V(f) \subset \mathbb{A}^n$; let $P = (a_1, \dots, a_n) \in V$, and ℓ be a line through P. Since $P \in V$, obviously P is a root of $f|\ell$.

Question: When is P a multiple root of $f|\ell$?

Answer: If and only if ℓ is contained in the affine linear subspace

$$T_P V : \Big(\sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i) = 0\Big) \subset \mathbb{A}^n,$$

called the *tangent space* to V at P.

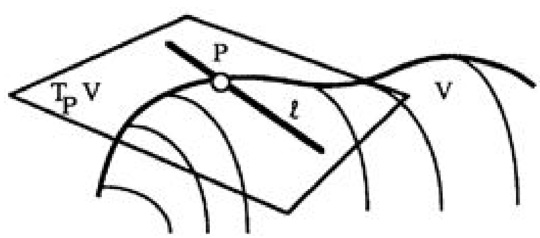

Figure 6.1: Tangent space

To prove this, parametrise ℓ as

$$\ell : X_i = a_i + b_i T,$$