$(X, \mathcal{O}_X(1))$ to indicate that the choice has been made. In addition to completeness, a projective variety $X \subset \mathbb{P}^N$ satisfies a key condition of 'positive degree': if $V \subset X$ is a k-dimensional subvariety then V intersects a general linear subspace $\mathbb{P}^{N-k}$ in a positive finite number of points. Conversely, the Kleiman criterion says that some multiple of a line bundle on a complete variety X can be used to provide a projective embedding of X if its degree on every curve $C \subset X$ is consistently greater than zero (that is, $\geq \varepsilon \cdot$ (any reasonable measure of C)). This kind of positivity relates closely to the choice of a Kähler metric on a complex manifold (a Riemannian metric with the right kind of compatibility with the complex structure). So we understand projectivity as a kind of 'positive definiteness'.

(b) Sufficiency

The surprising thing is the many problems of algebraic geometry having answers within the framework of projective varieties. The construction of Chow varieties mentioned in (8.2) is one such example; another is Mumford's work of the 1960s, in which he constructed Picard varieties and many moduli spaces as quasiprojective varieties (schemes). Mori theory (responsible for important conceptual advances in classification of varieties related to rationality, see [Kollár]) is the most recent example; here the ideas and techniques are inescapably projective in nature.

(c) Insufficiency of abstract varieties

Curves and nonsingular surfaces are automatically quasiprojective; but abstract varieties that are not quasiprojective do exist (singular surfaces, or nonsingular varieties of dimension ≥ 3). However, if you feel the need for these constructions, you will almost certainly also want Moishezon varieties (M. Artin's algebraic spaces), objects of algebraic geometry more general than abstract varieties, obtained by a somewhat more liberal interpretation of 'glueing local pieces'.

Theorems on abstract varieties are often proved by a reduction to the quasiprojective case, so whether the quasiprojective proof or the detail of the reduction process is more useful, interesting, essential or likely-to-lead-to-cheap-publishable-work will depend on the particular problem and the individual student's interests and employment situation. It has recently been proved that a nonsingular abstract variety or Moishezon variety that is not quasiprojective necessarily contains a rational curve; however, the proof (due to J. Kollár) is Mori theoretic, so hardcore projective algebraic geometry.

8.12 Affine varieties and schemes

The coordinate ring $k[V]$ of an affine algebraic variety V over an algebraically closed field k (Definition 4.1) satisfies two conditions: (i) it is a finitely generated k-algebra; and (ii) it is an integral domain. A ring satisfying these two conditions is obviously of the form $k[V]$ for some variety V, and is called a *geometric ring* (or *geometric k-algebra*).

There are two key theoretical results in Part II; one of these is Theorem 4.4, which states precisely that $V \mapsto k[V] = A$ is an equivalence of categories between affine algebraic varieties and the opposite of the category of geometric k-algebras (although I censored out all mention of categories as unsuitable for younger readers). The other is the Nullstellensatz (3.10), that prime ideals of $k[V]$ are in bijection with irreducible subvarieties of V; the points of V are in bijection with maximal ideals.

Taken together, these results identify affine varieties V with the affine schemes corresponding to geometric rings (compare also Definition 4.6).

The *prime spectrum* $\operatorname{Spec} A$ is defined for an arbitrary ring (commutative with a 1) as the set of prime ideals of A. It has a Zariski topology and a structure sheaf; this is the *affine scheme* corresponding to A (for details see [Mumford, Introduction, or Hartshorne, Ch. II]). There are several quite distinct ways in which affine schemes are more general than affine varieties; each of these is important, and I run through them briefly in (8.14).

It's important to understand that for a geometric ring $A = k[V]$, the prime spectrum $\operatorname{Spec} A$ contains exactly the same information as the variety V, and no more. The NSS tells us there's a plentiful supply of maximal ideals (m_v for points $v \in V$), and every other prime P of A is the intersection of maximal ideals over the points of an irreducible subvariety $Y \subset V$:

$$P = I(Y) = \bigcap_{v \in Y} m_v;$$

It's useful and (roughly speaking, at least) permissible to ignore the distinction between varieties and schemes, writing $V = \operatorname{Spec} A$, v for m_v, and imagining the prime $P = I(Y)$ ('generic point') as a kind of laundry mark stitched everywhere dense into the fabric of the subvariety Y.

8.13 What's the point?

A majority of students will never need to know any more about scheme theory than what is contained in (8.9) and (8.12), beyond the warning that the expression *generic point* is used in several technical senses, often meaning something quite different from *sufficiently general* point.

This section is intended for the reader who faces the task of working with the modern literature, and offers some comments on the various notions of point in scheme theory, potentially a major stumbling block for beginners.

(a) Scheme theoretic points of a variety

Suppose that k is a field (possibly not algebraically closed), and $A = k[X_1, \dots, X_n]/I$ with $I \subset k[X_1, \dots, X_n]$ an ideal; write $V = V(I) \subset K^n$ where $k \subset K$ is a chosen algebraic closure. The points of $\operatorname{Spec} A$ are only a bit more complicated than for a geometric ring in (8.12). By an obvious extension of the NSS, a maximal ideal of A is determined by a point $v = (a_1, \dots, a_n) \in V \subset K^n$, that is, it's of the form

$$m_v = \{f \in A \mid f(P) = 0\} = (x_1 - a_1, \dots, x_n - a_n) \cap A.$$

It's easy to see that different points $v \in V \subset K^n$ give rise to the same maximal ideal m_v of A if and only if they are conjugate over k in the sense of Galois theory (since A consists of polynomials with coefficients in k). So the maximal spectrum $\operatorname{Specm} A$ is just V 'up to conjugacy' (the orbit space of $\operatorname{Gal} K/k$ on V). Every other prime P of A corresponds as in (8.12) to an irreducible subvariety $Y = V(P) \subset V$ (up to conjugacy over k); $P \in \operatorname{Spec} A$ is the scheme theoretic *generic point* of Y, and is again to be thought of as a laundry mark on Y. The Zariski topology of $\operatorname{Spec} A$ is fixed up so that P is everywhere dense in Y. The maximal ideals of A are called *closed points* to distinguish them. If $C : (f = 0) \subset \mathbb{A}^2_{\mathbb{C}}$ is an irreducible curve, it has just one scheme theoretic generic point,