1.7 Parametrisation of a conic

Let C be a nondegenerate, nonempty conic of $\mathbb{P}^2_{\mathbb{R}}$. Then by Corollary 1.6, taking new coordinates $(X+Z, Y, Z-X)$, C is projectively equivalent to the curve $(XZ = Y^2)$; this is the curve parametrised by

$$\begin{aligned} \Phi \colon \mathbb{P}^1_{\mathbb{R}} &\longrightarrow C \subset \mathbb{P}^2_{\mathbb{R}}, \\ (U : V) &\mapsto (U^2 : UV : V^2). \end{aligned}$$

Remarks 1 The inverse map $\Psi \colon C \to \mathbb{P}^1_{\mathbb{R}}$ is given by

$$(X : Y : Z) \mapsto (X : Y) = (Y : Z);$$

here the left-hand ratio is defined if $X \neq 0$, and the right-hand ratio if $Z \neq 0$. In terminology to be introduced later, Φ and Ψ are inverse isomorphisms of varieties.

2 Throughout §§1–2, nonempty nondegenerate conics are tacitly assumed to be projectively equivalent to $(XZ - Y^2)$; over a field of characteristic $\neq 2$, this is justified in Ex. 1.5. (The reader interested in characteristic 2 should take this as the definition of a nondegenerate conic.)

1.8 Homogeneous form in 2 variables

Let $F(U, V)$ be a nonzero homogeneous polynomial of degree d in U, V, with coefficients in a fixed field k; (I will follow tradition, and use the word *form* for 'homogeneous polynomial'):

$$F(U, V) = a_d U^d + a_{d-1} U^{d-1} V + \cdots + a_i U^i V^{d-i} + \cdots + a_0 V^d.$$

F has an associated inhomogeneous polynomial in 1 variable,

$$f(u) = a_d u^d + a_{d-1} u^{d-1} + \cdots + a_i u^i + \cdots + a_0.$$

Clearly for $\alpha \in k$,

$$\begin{aligned} f(\alpha) = 0 &\iff (u - \alpha) \mid f(u) \\ &\iff (U - \alpha V) \mid F(U, V) \iff F(\alpha, 1) = 0; \end{aligned}$$

so zeros of f correspond to zeros of F on $\mathbb{P}^1$ away from the point $(1, 0)$, the 'point $\alpha = \infty$.' What does it mean for F to have a zero at infinity?

$$F(1, 0) = 0 \iff a_d = 0 \iff \deg f < d.$$

Now define the *multiplicity* of a zero of F on $\mathbb{P}^1$ to be

(i) the multiplicity of f at the corresponding $\alpha \in k$; or

(ii) $d - \deg f$ if $(1, 0)$ is the zero.

So the multiplicity of zero of F at a point $(\alpha, 1)$ is the greatest power of $(U - \alpha V)$ dividing F, and at $(1, 0)$ it is the greatest power of V dividing F.

Proposition *Let $F(U, V)$ be a nonzero form of degree d in U, V. Then F has at most d zeros on $\mathbb{P}^1$; furthermore, if k is algebraically closed, then F has exactly d zeros on $\mathbb{P}^1$ provided these are counted with multiplicities as defined above.*

Proof Let m_∞ be the multiplicity of the zero of F at $(1,0)$; then by definition, $d - m_\infty$ is the degree of the inhomogeneous polynomial f, and the proposition reduces to the well known fact that a polynomial in one variable has at most $\deg f$ roots. Q.E.D.

Note that over an algebraically closed field, F will factorise as a product $F = \prod \lambda_i^{m_i}$ of linear forms $\lambda_i = (a_i U + b_i V)$, and treated in this way, the point $(1,0)$ corresponds to the form $\lambda_\infty = V$, and is on the same footing as all other points.

1.9 Easy cases of Bézout's Theorem

Bézout's theorem says that if C and D are plane curves of degrees $\deg C = m$, $\deg D = n$, then the number of points of intersection of C and D is mn, provided that (i) the field is algebraically closed; (ii) points of intersection are counted with the right multiplicities; (iii) we work in $\mathbb{P}^2$ to take right account of intersections 'at infinity'. See for example [Fulton, p. 112] for a self-contained proof. In this section I am going to treat the case when one of the curves is a line or conic.

Theorem *Let $L \subset \mathbb{P}^2_k$ be a line (respectively $C \subset \mathbb{P}^2_k$ a nondegenerate conic), and let $D \subset \mathbb{P}^2_k$ be a curve defined by $D : (G_d(X,Y,Z) = 0)$, where G is a form of degree d in X, Y, Z. Assume that $L \not\subset D$ (respectively, $C \not\subset D$); then*

$$\#\{L \cap D\} \le d \quad (\textit{respectively } \#\{C \cap D\} \le 2d).$$

In fact there is a natural definition of multiplicity of intersection such that the inequality still holds for 'number of points counted with multiplicities', and if k is algebraically closed then equality holds.

Proof A line $L \subset \mathbb{P}^2_k$ is given by an equation $\lambda = 0$, with λ a linear form; for my purpose, it is convenient to give it parametrically as

$$X = a(U,V), \quad Y = b(U,V), \quad Z = c(U,V),$$

where a, b, c are linear forms in U, V. So for example, if $\lambda = \alpha X + \beta Y + \gamma Z$, and $\gamma \ne 0$, then L can be given as

$$X = U, \quad Y = V, \quad Z = -\frac{\alpha}{\gamma}U - \frac{\beta}{\gamma}V.$$

Similarly, as explained in (1.7), a nondegenerate conic can be given parametrically as

$$X = a(U,V), \quad Y = b(U,V), \quad Z = c(U,V),$$

where a, b, c are quadratic forms in U, V. This is because C is a projective transformation of $(XZ = Y^2)$, which is parametrically $(X,Y,Z) = (U^2, UV, V^2)$, so C is given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M \begin{pmatrix} U^2 \\ UV \\ V^2 \end{pmatrix}$$

where M is a nonsingular 3×3 matrix.