Theorem 4.8 *(I)* $\operatorname{dom} f$ *is open and dense in the Zariski topology.*

Suppose that the field k is algebraically closed; then

(II)

$$\operatorname{dom} f = V \iff f \in k[V];$$

(that is polynomial function = regular rational function*). Furthermore, for any $h \in k[V]$, let*

$$V_h = V \setminus V(h) = \{P \in V \mid h(P) \neq 0\};$$

then

(III)

$$\operatorname{dom} f \supset V_h \iff f \in k[V][h^{-1}].$$

Proof Define the *ideal of denominators* of $f \in k(V)$ by

$$\begin{aligned} D_f &= \{h \in k[V] \mid hf \in k[V]\} \subset k[V] \\ &= \{h \in k[V] \mid \exists \text{ an expression } f = g/h \text{ with } g \in k[V]\} \cup \{0\}. \end{aligned}$$

From the first line, D_f is obviously an ideal of $k[V]$. Then formally,

$$V \setminus \operatorname{dom} f = \{P \in V \mid h(P) = 0 \text{ for all } h \in D_f\} = V(D_f),$$

so that $V \setminus \operatorname{dom} f$ is an algebraic set of V; hence $\operatorname{dom} f = V \setminus V(D_f)$ is the complement of a closed set, so open in the Zariski topology. It is obvious that $\operatorname{dom} f$ is nonempty, hence dense by Proposition 4.2.

Now using (b) of the Nullstellensatz,

$$\operatorname{dom} f = V \iff V(D_f) = \emptyset \iff 1 \in D_f, \quad \text{that is, } f \in k[V].$$

Finally,

$$\operatorname{dom} f \supset V_h \iff h \text{ vanishes on } V(D_f),$$

and using (c) of the Nullstellensatz,

$$\iff h^n \in D_f \text{ for some } n, \text{ that is, } f = g/h^n \in k[V][h^{-1}]. \quad \text{Q.E.D.}$$

4.9 Rational maps

Let V be an affine variety.

Definition A *rational map* $f\colon V \dashrightarrow \mathbb{A}^n_k$ is a partially defined map given by rational functions $f_1, \dots, f_n$, that is,

$$f(P) = f_1(P), \dots, f_n(P) \quad \text{for all } P \in \bigcap \operatorname{dom} f_i.$$

By definition, $\operatorname{dom} f = \bigcap \operatorname{dom} f_i$; as before, F is said to be *regular* at $P \in V$ if and only if $P \in \operatorname{dom} f$. A rational map $V \dashrightarrow W$ between two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is defined to be a rational map $f\colon V \dashrightarrow \mathbb{A}^m$ such that $f(\operatorname{dom} f) \subset W$.

Two examples of rational maps were described at the end of (4.3).

4.10 Composition of rational maps

The composite $g \circ f$ of rational maps $f\colon V \dashrightarrow W$ and $g\colon W \dashrightarrow U$ may not be defined. This is a difficulty caused by the fact that a rational map is not a map: in a natural and obvious sense, the composite is a map defined on $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$; however, it can perfectly well happen that this is empty (see Ex. 4.10).

Expressed algebraically, the same problem also occurs: suppose that f is given by $f_1, \dots, f_m \in k(V)$, so that

$$\text{by} \quad \begin{array}{ccc} f\colon V & \dashrightarrow & W \subset \mathbb{A}^m \\ P & \mapsto & f_1(P), \dots, f_m(P) \end{array}$$

for $P \in \bigcap \operatorname{dom} f_i$; any $g \in k[W]$ is of the form $g = G \bmod I(W)$ for some $G \in k[Y_1, \dots, Y_m]$, and $g \circ f = G(f_1, \dots, f_m)$ is well defined in $k(V)$. So exactly as in (4.4), there is a k-algebra homomorphism

$$f^*\colon k[W] \to k(V)$$

corresponding to f. However, if $h \in k[W]$ is in the kernel of f^*, then no meaning can be attached to $f^*(g/h)$, so that f^* cannot be extended to a field homomorphism $k(W) \to k(V)$.

Definition $f\colon V \dashrightarrow W$ is *dominant* if $f(\operatorname{dom} f)$ is dense in W for the Zariski topology.

Geometrically, this means that $f^{-1}(\operatorname{dom} g) \subset \operatorname{dom} f$ is a dense open set for any rational map $g\colon W \dashrightarrow U$, so that $g \circ f$ is defined on a dense open set of V, so is a partially defined map $V \dashrightarrow U$.

Algebraically,

$$f \text{ is dominant} \iff f^*\colon k[W] \to k(V) \text{ is injective.}$$

For given $g \in k[W]$,

$$g \in \ker f^* \iff f(\operatorname{dom} f) \subset V(g),$$

that is, f^* is not injective if and only if $f(\operatorname{dom} f)$ is contained in a strict algebraic subset of W.

Clearly, the composite $g \circ f$ of rational maps f and g is defined provided that f is dominant: $g \circ f$ is the rational map whose components are $f^*(g_i)$. Notice that the domain of $g \circ f$ certainly contains $f^{-1}(\operatorname{dom} g) \cap \operatorname{dom} f$, but may very well be larger (see Ex. 4.6).

Theorem 4.11 *(I) A dominant rational map $f\colon V \dashrightarrow W$ defines a field homomorphism $f^*\colon k(W) \to k(V)$.*

(II) Conversely, a k-homomorphism $\Phi\colon k(W) \to k(V)$ comes from a uniquely defined dominant rational map $f\colon V \dashrightarrow W$.

(III) If f and g are dominant then $(g \circ f)^* = f^* \circ g^*$.

The proof requires only minor modifications to that of (4.4).

4.12 Morphisms from an open subset of an affine variety

Let V, W be affine varieties, and $U \subset V$ an open subset.