Proof Let m_∞ be the multiplicity of the zero of F at $(1,0)$; then by definition, $d - m_\infty$ is the degree of the inhomogeneous polynomial f, and the proposition reduces to the well known fact that a polynomial in one variable has at most $\deg f$ roots. Q.E.D.

Note that over an algebraically closed field, F will factorise as a product $F = \prod \lambda_i^{m_i}$ of linear forms $\lambda_i = (a_i U + b_i V)$, and treated in this way, the point $(1,0)$ corresponds to the form $\lambda_\infty = V$, and is on the same footing as all other points.

1.9 Easy cases of Bézout's Theorem

Bézout's theorem says that if C and D are plane curves of degrees $\deg C = m$, $\deg D = n$, then the number of points of intersection of C and D is mn, provided that (i) the field is algebraically closed; (ii) points of intersection are counted with the right multiplicities; (iii) we work in $\mathbb{P}^2$ to take right account of intersections 'at infinity'. See for example [Fulton, p. 112] for a self-contained proof. In this section I am going to treat the case when one of the curves is a line or conic.

Theorem *Let $L \subset \mathbb{P}^2_k$ be a line (respectively $C \subset \mathbb{P}^2_k$ a nondegenerate conic), and let $D \subset \mathbb{P}^2_k$ be a curve defined by $D : (G_d(X,Y,Z) = 0)$, where G is a form of degree d in X, Y, Z. Assume that $L \not\subset D$ (respectively, $C \not\subset D$); then*

$$\#\{L \cap D\} \le d \quad \textit{(respectively } \#\{C \cap D\} \le 2d).$$

In fact there is a natural definition of multiplicity of intersection such that the inequality still holds for 'number of points counted with multiplicities', and if k is algebraically closed then equality holds.

Proof A line $L \subset \mathbb{P}^2_k$ is given by an equation $\lambda = 0$, with λ a linear form; for my purpose, it is convenient to give it parametrically as

$$X = a(U,V), \quad Y = b(U,V), \quad Z = c(U,V),$$

where a, b, c are linear forms in U, V. So for example, if $\lambda = \alpha X + \beta Y + \gamma Z$, and $\gamma \ne 0$, then L can be given as

$$X = U, \quad Y = V, \quad Z = -\frac{\alpha}{\gamma}U - \frac{\beta}{\gamma}V.$$

Similarly, as explained in (1.7), a nondegenerate conic can be given parametrically as

$$X = a(U,V), \quad Y = b(U,V), \quad Z = c(U,V),$$

where a, b, c are quadratic forms in U, V. This is because C is a projective transformation of $(XZ = Y^2)$, which is parametrically $(X,Y,Z) = (U^2, UV, V^2)$, so C is given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M \begin{pmatrix} U^2 \\ UV \\ V^2 \end{pmatrix}$$

where M is a nonsingular 3×3 matrix.

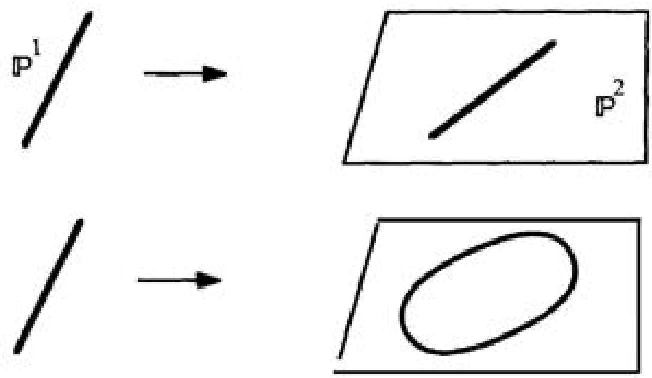

Figure 1.4: (a) Parametrised line; (b) parametrised conic

Then the intersection of L (respectively C) with D is given by finding the values of the ratios $(U : V)$ such that

$$F(U,V) = G_d(a(U,V), b(U,V), c(U,V)) = 0.$$

But F is a form of degree d (respectively $2d$) in U, V, so the result follows by (1.8). Q.E.D.

Corollary 1.10 *If $P_1, \dots, P_5 \in \mathbb{P}^2_{\mathbb{R}}$ are distinct points and no 4 are collinear, there exists at most one conic through $P_1, \dots, P_5$.*

Proof Suppose by contradiction that C_1 and C_2 are conics with $C_1 \neq C_2$ such that

$$C_1 \cap C_2 \supset \{P_1, \dots, P_5\}.$$

C_1 is nonempty, so that if it's nondegenerate, then by (1.7), it's projectively equivalent to the parametrised curve

$$C_1 = \left\{ (U^2, UV, V^2) \bigm| (U,V) \in \mathbb{P}^1 \right\};$$

by (1.9), $C_1 \subset C_2$. Now if Q_2 is the equation of C_2, it follows that $Q_2(U^2, UV, V^2) \equiv 0$ for all $(U,V) \in \mathbb{P}^1$, and an easy calculation (see Ex. 1.6) shows that Q_2 is a multiple of $(XZ - Y^2)$; this contradicts $C_1 \neq C_2$.

Now suppose C_1 is degenerate; by (1.6) again, it's either a line pair or a line, and one sees easily that

$$C_1 = L_0 \cup L_1, \quad C_2 = L_0 \cup L_2,$$

with L_1, L_2 distinct lines. Then $C_1 \cap C_2 = L_0 \cup (L_1 \cap L_2)$: