3.14 Remarks

(I) In fact, the proof of (3.13) shows that $y_1, \dots, y_m$ can be chosen to be m general linear forms in $a_1, \dots, a_n$. To understand the significance of (3.13), write $I = \ker\{k[X_1, \dots, X_n] \to k[a_1, \dots, a_n] = A\}$, and assume for simplicity that I is prime. Consider $V = V(I) \subset \mathbb{A}_k^n$; let $\pi\colon \mathbb{A}_k^n \to \mathbb{A}_k^m$ be the linear projection defined by $y_1, \dots, y_m$, and $p = \pi|V\colon V \to \mathbb{A}_k^m$. It can be seen that the conclusions (i) and (ii) of (3.13) imply that above every $P \in \mathbb{A}_k^m$, $p^{-1}(P)$ is a finite nonempty set (see Ex. 3.16).

(II) The proof of (3.13) has also a simple geometric interpretation: choosing $n-1$ linear forms in the n variables $X_1, \dots, X_n$ corresponds to making a linear projection $\pi\colon \mathbb{A}_k^n \to \mathbb{A}_k^{n-1}$; the fibres of π then form an $(n-1)$-dimensional family of parallel lines. Having chosen the polynomial $f \in I$, it is not hard to see that f gives rise to a monic relation in the final X_n if and only if none of the parallel lines are asymptotes of the variety $(f = 0)$; in terms of projective geometry, this means that the point at infinity $(0, \alpha_1, \dots, \alpha_{n-1}, 1) \in \mathbb{P}_k^{n-1}$ specifying the parallel projection does not belong to the projective closure of $(f = 0)$.

(III) The above proof of (3.13) does not work for a finite field (see Ex. 3.14). However, the theorem itself is true without any condition on k (see [Mumford, Introduction, p. 4] or [Atiyah and Macdonald, (7.9)]).

3.15 Proof of (3.8)

Let $A = k[a_1, \dots, a_n]$ be a finitely generated k-algebra and suppose that $y_1, \dots, y_m \in A$ are as in (3.13). Write $B = k[y_1, \dots, y_m]$. Then A is a finite B-algebra, and it is given that A is a field. If I knew that B is a field, it would follow at once that $m = 0$, so that A is a finite k-algebra, that is, a finite field extension of k, and (3.8) would be proved. Therefore it remains only to prove the following statement:

Lemma *If A is a field, and $B \subset A$ a subring such that A is a finite B-algebra, then B is a field.*

Proof For any $0 \neq b \in B$, the inverse $b^{-1} \in A$ exists in A. Now by (3.12, ii), the finiteness implies that b^{-1} satisfies a monic equation over B, that is, there exists a relation

$$b^{-n} + a_{n-1}b^{-(n-1)} + \cdots + a_1 b^{-1} + a_0 = 0, \quad \text{with } a_i \in B;$$

then multiplying through by b^{n-1},

$$b^{-1} = -(a_{n-1} + a_{n-2}b + \cdots + a_0 b^{n-1}) \in B.$$

Therefore B is a field. This proves (3.8) and completes the proof of NSS.

3.16 Separable addendum

For the purposes of arranging that everything goes through in characteristic p, it is useful to add a tiny precision. I'm only going to use this in one place in the sequel, so if you can't remember too much about separability from Galois theory, don't lose too much sleep over it (GOTO 3.17).