Hence the I and V correspondences

$$\begin{array}{rcl} \{\text{ideals } I \subset k[V]\} & \xrightarrow{V} & \{\text{subsets } X \subset V\} \\ I & \longmapsto & V(I) = \big\{P \in V \bigm| f(P) = 0 \ \forall f \in I\big\} \end{array}$$

by

and

$$\begin{array}{rcl} \{\text{ideals } J \subset k[V]\} & \xleftarrow{I} & \{\text{subsets} X \subset V\} \\ I(X) = \big\{f \in k[V] \bigm| f(P) = 0 \ \forall P \in X\big\} & \longleftarrow & X \end{array}$$

by

are defined as in §3, and have similar properties. In particular V has a Zariski topology, in which the closed sets are the algebraic subsets (this is of course the subspace topology of the Zariski topology of $\mathbb{A}^n$).

Proposition *Let $V \subset \mathbb{A}^n$ be an algebraic subset. The following conditions are equivalent:*

(i) V is irreducible;

(ii) any two open subsets $\emptyset \neq U_1, U_2 \subset V$ have $U_1 \cap U_2 \neq \emptyset$;

(iii) any nonempty open subset $U \subset V$ is dense.

This is all quite trivial: V is irreducible means that V is not a union of two proper closed subsets; (ii) is just a restatement in terms of complements, since

$$U_1 \cap U_2 = \emptyset \iff V = (V - U_1) \cup (V - U_2).$$

A subset of a topological space is dense if and only if it meets every open, so that (iii) is just a restatement of (ii).

4.3 Polynomial maps

Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be algebraic sets; write $X_1, \dots, X_n$ and $Y_1, \dots, Y_m$ for the coordinates on $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively.

Definition A map $f\colon V \to W$ is a *polynomial map* if there exist m polynomials $F_1, \dots, F_m \in k[X_1, \dots, X_n]$ such that

$$f(P) = \big(F_1(P), \dots, F_m(P)\big) \in \mathbb{A}^m_k \quad \text{for all } P \in V.$$

This is an obvious generalisation of the above notion of a polynomial function.

Claim *A map $f\colon V \to W$ is a polynomial map if and only if for all j, the composite map $f_j = Y_j \circ f \in k[V]$:*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "f_j"'] & W \subset \mathbb{A}^m_k \arrow[d, "Y_j"] \\ & k \end{tikzcd} \qquad \textit{(jth coordinate function).}$$

This is clear: if f is given by $F_1, \dots, F_m$, then the composite is just $P \mapsto F_j(P)$, which is a polynomial function. Conversely, if $f_j \in k[V]$ for each j, then for any choice of $F_j \in k[X_1, \dots, X_n]$ such that $f_j = F_j \bmod I(V)$, I get a description of f as the polynomial map given by $(F_1, \dots, F_m)$.

In view of this claim, the map f can be written $f = (f_1, \dots, f_m)$.

The composite of polynomial maps is defined in the obvious way: if $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ and $U \subset \mathbb{A}^\ell$ are algebraic sets, and $f\colon V \to W$, $g\colon W \to U$ are polynomial maps, then $g \circ f\colon V \to U$ is again a polynomial map; for if f is given by $F_1, \dots, F_m \in k[X_1, \dots, X_n]$, and g by $G_1, \dots, G_\ell \in k[Y_1, \dots, Y_m]$, then $g \circ f$ is given by

$$G_1(F_1, \dots, F_m), \dots, G_\ell(F_1, \dots, F_m) \in k[X_1, \dots, X_n].$$

Definition A polynomial map $f\colon V \to W$ between algebraic sets is an *isomorphism* if there exists a polynomial map $g\colon W \to V$ such that $f \circ g = g \circ f = \mathrm{id}$.

Several examples of polynomial maps have already been given: the parametrisations $\mathbb{R}^1 \to C \subset \mathbb{R}^2$ by $t \mapsto (t^2, t^3)$ or $(t^2 - 1, t^3 - t)$ given in (2.1), and the map $k \to C \subset \mathbb{A}^3_k$ by $t \mapsto (t^3, t^4, t^5)$ discussed in (3.11, b) are clearly of this kind. Also, while discussing Noether normalisation, I had an algebraic set $V \subset \mathbb{A}^n_k$, and considered the general projection $p\colon V \to \mathbb{A}^m_k$ defined by m 'fairly general' linear forms $Y_1, \dots, Y_m$; since the Y_i are linear forms in the coordinates X_i of $\mathbb{A}^n_k$, this projection is a polynomial map.

On the other hand the parametrisation of the circle given in (1.1) is given by rational functions (there's a term $\lambda^2 + 1$ in the denominator); and the inverse map $(X, Y) \dashrightarrow t = Y/X$ from either of the singular cubics $C \subset \mathbb{R}^2$ back to $\mathbb{R}^1$ is also disqualified (or at least, doesn't qualify *as written*) for the same reason.

4.4 Polynomial maps and $k[V]$

Theorem *Let $V \subset \mathbb{A}^n_k$ and $W \subset \mathbb{A}^m_k$ be algebraic sets as above.*

(1) A polynomial map $f\colon V \to W$ induces a ring homomorphism $f^*\colon k[W] \to k[V]$, defined by composition of functions; that is, if $g \in k[W]$ is a polynomial function then so is $f^*(g) = g \circ f$, and $g \mapsto g \circ f$ defines a ring homomorphism, in fact a k-algebra homomorphism $f^*\colon k[W] \to k[V]$. (Note that it goes backwards.)

(2) Conversely, any k-algebra homomorphism $\Phi\colon k[W] \to k[V]$ is of the form $\Phi = f^*$ for a uniquely defined polynomial map $f\colon V \to W$.

Thus (I) and (II) show that

$$\left\{\begin{array}{l}\text{polynomial}\\ \text{maps } f\colon V \to W\end{array}\right\} \longrightarrow \left\{\begin{array}{l}k\text{-algebra homs.}\\ \Phi\colon k[W] \to k[V]\end{array}\right\}$$

by

$$f \longmapsto f^*$$

is a bijection.

(3) If $f\colon V \to W$ and $g\colon W \to U$ are polynomial maps then the two ring homomorphisms $(g \circ f)^* = f^* \circ g^*\colon k[U] \to k[V]$ coincide.