Proposition 3.2 *(i) Suppose that A is Noetherian, and $I \subset A$ an ideal; then the quotient ring $B = A/I$ is Noetherian.*

(ii) Let A be a Noetherian integral domain, and $A \subset K$ its field of fractions; let $0 \notin S \subset A$ be a subset, and set

$$B = A[S^{-1}] = \left\{ \frac{a}{b} \in K \;\middle|\; \begin{array}{l} a \in A, \text{ and } b = 1 \text{ or a} \\ \text{product of elements of } S \end{array} \right\}.$$

Then B is again Noetherian.

Proof Exercise: in either case the ideals of B can be described in terms of certain ideals of A; see Ex. 3.4 for hints.

Theorem 3.3 (Hilbert Basis Theorem) *For a ring A,*

$$A \text{ Noetherian} \implies A[X] \text{ Noetherian}.$$

Proof Let $J \subset A[X]$ be any ideal; I prove that J is finitely generated. Define the ideal of leading terms of degree n in J to be

$$J_n = \left\{ a \in A \;\middle|\; \exists f = aX^n + b_{n-1}X^{n-1} + \cdots + b_0 \in J \right\}.$$

Then J_n is an ideal of A and $J_n \subset J_{n+1}$ (please provide your own proofs). Hence, using the a.c.c., there exists N such that

$$J_N = J_{N+1} = \cdots.$$

Now build a set of generators of J as follows: for $i \le N$, let $a_{i_1}, \dots, a_{i_{m_i}}$ be generators of J_i and, as in the definition of J_i, for each of the a_{ik}, let $f_{ik} = a_{ik}X^i + \cdots \in J$ be an element of degree i and leading term a_{ik}.

I claim that the set

$$\left\{ f_{ik} \;\middle|\; i = 0, \dots, N, k = 1, \dots, m_i \right\}$$

just constructed generates J: for given $g \in J$, suppose $\deg g = m$. Then the leading term of g is bX^m with $b \in J_m$, so that by what I know about J_m, I can write $b = \sum c_{m'k}a_{m'k}$ (here $m' = m$ if $m \le N$, otherwise $m' = N$). Then consider $g_1 = g - X^{m-m'} \cdot \sum c_{m'k}f_{m'k}$: by construction the term of degree m is zero, so that $\deg g_1 \le \deg g - 1$; by induction, I can therefore write out g as a combination of f_{ik}, so that these generate J. Q.E.D.

Corollary *For k a field, a finitely generated k-algebra is Noetherian.*

A finitely generated k-algebra is a ring of the form $A = k[a_1, \dots, a_n]$, so that A is generated as a ring by k and $a_1, \dots, a_n$; clearly, every such ring is isomorphic to a quotient of the polynomial ring, $A \cong k[X_1, \dots, X_n]/I$. A field is Noetherian, and by induction on (3.3), $k[X_1, \dots, X_n]$ is Noetherian; finally, passing to the quotient is OK by Proposition 3.2, (i). Q.E.D.