2.15 Discussion of genus

A nonsingular projective curve C over $\mathbb{C}$ has got just one topological invariant, its genus $g = g(C)$:

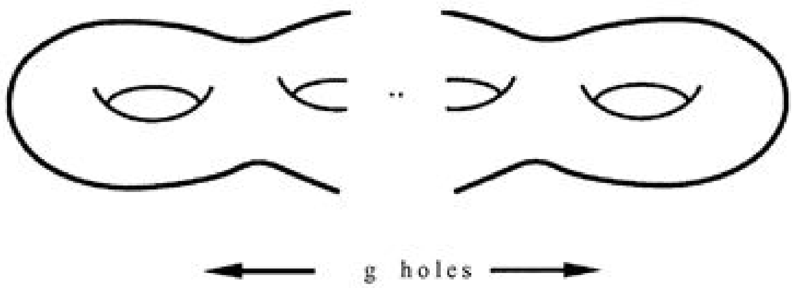

Figure 2.12: Surface of genus g

For example, the affine curve $C : (y^2 = f_{2g+1}(x) = \prod_i(x-a_i)) \subset \mathbb{C}^2$, where f_{2g+1} is a polynomial of degree $2g+1$ in x with distinct roots a_i, can be related to the Riemann surface of $\sqrt{f}$ exactly as in (2.13), and be viewed as a double cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ branched in the $2g+1$ points a_i and in ∞, and by the same argument, can be seen to have genus g. As another example, the genus of a nonsingular plane curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d is given by $g = g(C_d) = \binom{d-1}{2}$.

2.16 Commercial break

Complex curves (= compact Riemann surfaces) appear across a whole spectrum of math problems, from Diophantine arithmetic through complex function theory and low dimensional topology to differential equations of math physics. So go out and buy a complex curve today.

To a quite extraordinary degree, the properties of a curve are determined by its genus, and more particularly by the trichotomy $g = 0$, $g = 1$ or $g \geq 2$. Some of the more striking aspects of this are described in the table on the following page, and I give a brief discussion; this ought to be in the background culture of every mathematician.

To give a partial answer to the Diophantine question mentioned in (1.1–2) and again in (2.1), it is known that a curve can be parametrised by rational functions if and only if $g = 0$; if I'm working over a fixed field, a curve of genus 0 may have no k-valued points at all (for example, the conic in (1.2)), but if it has one point, it can be parametrised over k, so that its k-valued points are in bijection with $\mathbb{P}^1_k$. Any curve of genus 1 is isomorphic to a cubic as in this section, and a group law is defined on the k-valued points (provided of course that there exists at least one – there's no such thing as the empty group); if k is a number field (for example, $k = \mathbb{Q}$), the k-valued points form an Abelian group which is finitely generated (the Mordell–Weil Theorem). Whereas a curve of genus $g \geq 2$ is now known to have only a finite set of k-valued points; this is a famous theorem proved by Faltings in 1983, and for which he received the Fields medal in 1986. Thus for example, for any $n \geq 4$, the Fermat curve $x^n + y^n = 1$ has at most a finite number of rational points.