2.15 Discussion of genus

A nonsingular projective curve C over $\mathbb{C}$ has got just one topological invariant, its genus $g = g(C)$:

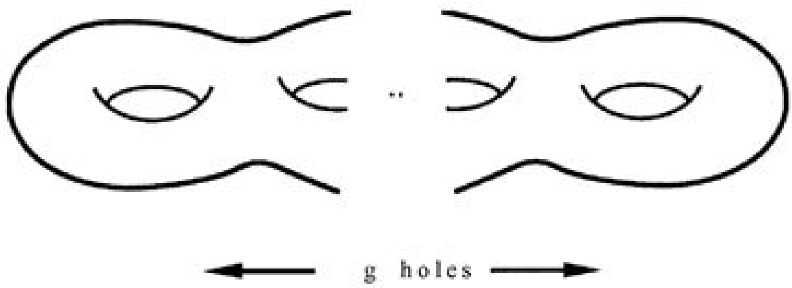


Figure 2.12: Surface of genus g

For example, the affine curve $C : (y^2 = f_{2g+1}(x) = \prod_i (x - a_i)) \subset \mathbb{C}^2$, where f_{2g+1} is a polynomial of degree $2g+1$ in x with distinct roots a_i, can be related to the Riemann surface of $\sqrt{f}$ exactly as in (2.13), and be viewed as a double cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ branched in the $2g+1$ points a_i and in ∞, and by the same argument, can be seen to have genus g. As another example, the genus of a nonsingular plane curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d is given by $g = g(C_d) = \binom{d-1}{2}$.

2.16 Commercial break

Complex curves (= compact Riemann surfaces) appear across a whole spectrum of math problems, from Diophantine arithmetic through complex function theory and low dimensional topology to differential equations of math physics. So go out and buy a complex curve today.

To a quite extraordinary degree, the properties of a curve are determined by its genus, and more particularly by the trichotomy $g = 0$, $g = 1$ or $g \geq 2$. Some of the more striking aspects of this are described in the table on the following page, and I give a brief discussion; this ought to be in the background culture of every mathematician.

To give a partial answer to the Diophantine question mentioned in (1.1–2) and again in (2.1), it is known that a curve can be parametrised by rational functions if and only if $g = 0$; if I'm working over a fixed field, a curve of genus 0 may have no k-valued points at all (for example, the conic in (1.2)), but if it has one point, it can be parametrised over k, so that its k-valued points are in bijection with $\mathbb{P}^1_k$. Any curve of genus 1 is isomorphic to a cubic as in this section, and a group law is defined on the k-valued points (provided of course that there exists at least one – there's no such thing as the empty group); if k is a number field (for example, $k = \mathbb{Q}$), the k-valued points form an Abelian group which is finitely generated (the Mordell–Weil Theorem). Whereas a curve of genus $g \geq 2$ is now known to have only a finite set of k-valued points; this is a famous theorem proved by Faltings in 1983, and for which he received the Fields medal in 1986. Thus for example, for any $n \geq 4$, the Fermat curve $x^n + y^n = 1$ has at most a finite number of rational points.

Over $\mathbb{C}$, a curve C of genus 1 is topologically a torus, and has a group law, so that it is analytically of the form $C \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \tau)$:

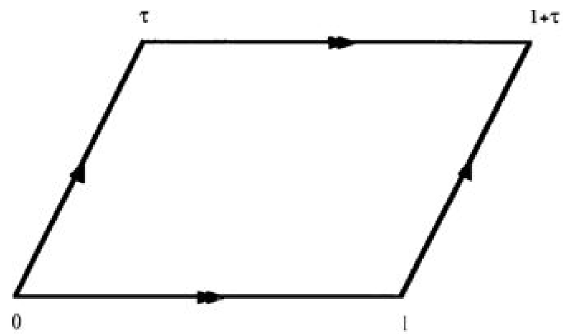
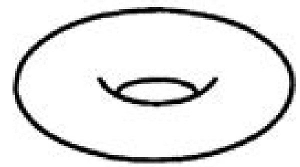


Figure 2.13: Genus 1 curve as $C \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \cdot \tau)$

The isomorphism between this quotient and a plane curve $C_3 \subset \mathbb{P}^2_{\mathbb{C}}$ is given by a holomorphic map $\varphi\colon \mathbb{C} \to C_3$, that is, a kind of 'parametrisation' of C_3; but φ cannot be in terms of rational functions (by (2.2)), and is infinity-to-one; this is the theory of doubly periodic functions of a complex variable, which was one of the mainstays of 19th century analysis (Weierstrass $\wp$-function, Riemann theta function).

Another important thing to notice is that different periods τ will usually lead to different curves; they're all homeomorphic to the standard torus $S^1 \times S^1$, but as algebraic curves, or complex analytic curves, they're not isomorphic. The period τ is a *modulus*, that is, a complex parameter which governs variation of the complex structure C on the fixed topological object $S^1 \times S^1$.

The student interested in more on curves should look at [D. Mumford, Curves and their Jacobians], the first part of which is fairly colloquial, or [Clemens].

	$g = 0$	$g = 1$	$g \geq 2$
Topology C is homeomorphic to:			
fundamental group:	simply-connected	$\pi_1 = \mathbb{Z} \oplus \mathbb{Z}$	like free group on 2g generators
Algebraic/complex analytic geometry embeddings, concrete descriptions:	$C \cong \mathbb{P}^1_{\mathbb{C}}$ $\cong C_2 \subset \mathbb{P}^2_{\mathbb{C}}$	$C \cong C_3 \subset \mathbb{P}^2_{\mathbb{C}}$ $\cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\cdot\tau)$	no simple description, but e.g. most curves of genus 3 are $\cong$ non-sing. $C_4 \subset \mathbb{P}^2_{\mathbb{C}}$
automorphisms:	3-dimensional group of projective transformations	translations in group law x finite group	finite group
moduli:	none	1 modulus (cross-ratio or j-invariant)	3g-3 moduli
Differential geometry there exists a natural class of Riemannian metrics with constant curvature:	constant positive curvature	zero curvature (that is, flat)	constant negative curvature
Diophantine problems if k= $\mathbb{Q}$ oranumberfield (that is, $[k : \mathbb{Q}] < \infty$) then:	$C_k = \emptyset$ or $\mathbb{P}^1_k$	C_k is a finitely generated Abelian group (Mordell-Weil theorem)	C_k is a finite set (Faltings Theorem, Mordell conjecture)

Part II

The category of affine varieties