Chapter 0

Woffle

This section is intended as a cultural introduction, and is not *logically* part of the course, so just skip through it.

0.1 What it's about

A variety is (roughly) a locus defined by polynomial equations:

$$V = \left\{ P \in k^n \;\middle|\; f_i(P) = 0 \right\} \subset k^n,$$

where k is a field and $f_i \in k[X_1, \ldots, X_n]$ are polynomials; so for example, the plane curves $C : (f(x, y) = 0) \subset \mathbb{R}^2$ or $\mathbb{C}^2$.

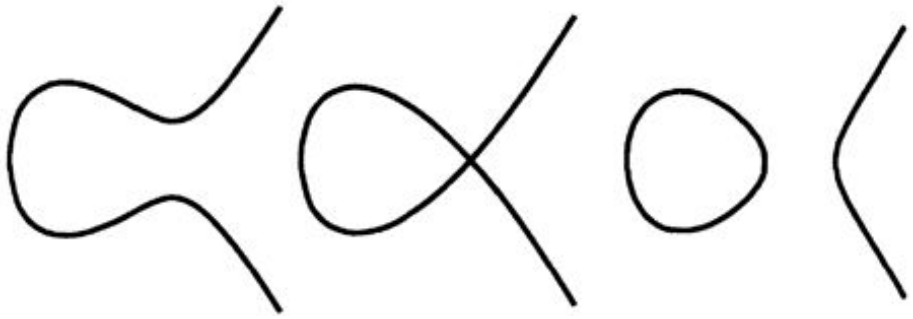

Figure 1: The cubic curves (a) $y^2 = (x+1)(x^2+\varepsilon)$, (b) $y^2 = (x+1)x^2$, and (c) $y^2 = (x+1)(x^2-\varepsilon)$.

I want to study V; several questions present themselves:

Number Theory For example, if $k = \mathbb{Q}$ and $V \subset \mathbb{Q}^n$, how can we tell if V is nonempty, or find all its points if it is? A specific case is historically of some significance: how many solutions are there to

$$x^n + y^n = 1, \quad \text{with} \quad x, y \in \mathbb{Q} \quad \text{and} \quad n \geq 3?$$

Questions of this kind are generally known as *Diophantine problems*.