

**Corollary**  $I$  and  $V$  determine inverse bijections

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{homogeneous radical} \\ \text{ideals } J \subset k[x_0, \dots, x_n] \\ \text{with } J \not\supset (x_0, \dots, x_n) \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{algebraic subsets} \\ X \subset \mathbb{P}^n \end{array} \right\} \\ \cup & & \cup \\ \left\{ \begin{array}{l} \text{homogeneous prime} \\ \text{ideals } J \subset k[x_0, \dots, x_n] \\ \text{with } J \not\supset (x_0, \dots, x_n) \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{irreducible algebraic} \\ \text{subsets } X \subset \mathbb{P}^n \end{array} \right\} \end{array}$$

**Proof** Let  $\pi: \mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$  be the map defining  $\mathbb{P}^n$ . For a homogeneous ideal  $J \subset k[X_0, \dots, X_n]$ , write (in temporary notation)  $V^a(J) \subset \mathbb{A}^{n+1}$  for the affine algebraic set defined by  $J$ . Then since  $J$  is homogeneous,  $V^a(J)$  has the property

$$(\alpha_0, \dots, \alpha_n) \in V^a(J) \iff (\lambda\alpha_0, \dots, \lambda\alpha_n) \in V^a(J),$$

and  $V(J) = V^a(J) \setminus \{0\} / \sim \subset \mathbb{P}^n$ . Hence

$$V(J) = \emptyset \iff V^a(J) \subset \{0\} \iff \text{rad } J \supset (X_0, \dots, X_n),$$

where the last implication uses the affine Nullstellensatz. Also, if  $V(J) \neq \emptyset$  then

$$f \in I(V(J)) \iff f \in I(V^a(J)) \iff f \in \text{rad } J. \quad \text{Q.E.D.}$$

The affine subset  $V^a(J)$  occurring above is called the *affine cone* over the projective algebraic subset  $V(J)$ .

## 5.4 Rational functions on $V$

Let  $V \subset \mathbb{P}_k^n$  be an irreducible algebraic set, and  $I(V) \subset k[X_0, \dots, X_n]$  its ideal; there is no direct way of defining regular functions on  $V$  in terms of polynomials: an element  $F \in k[X_0, \dots, X_n]$  gives a function on the affine cone over  $V$ , but (by case  $d = 0$  of Proposition 5.1) this will be constant on equivalence classes only if  $F$  is homogeneous of degree 0, that is, a constant. So from the start, I work with rational functions only:

**Definition** A *rational function* on  $V$  is a (partially defined) function  $f: V \dashrightarrow k$  given by  $f(P) = g(P)/h(P)$ , where  $g, h \in k[X_0, \dots, X_n]$  are homogeneous polynomials of the same degree  $d$ .

Note here that provided  $h(P) \neq 0$ , the quotient  $g(P)/h(P)$  is well defined, since

$$g(\lambda \underline{X})/h(\lambda \underline{X}) = \lambda^d g(\underline{X})/\lambda^d h(\underline{X}) = g(\underline{X})/h(\underline{X}) \quad \text{for } 0 \neq \lambda \in k.$$

Now obviously  $g/h$  and  $g'/h'$  define the same rational function on  $V$  if and only if  $h'g - g'h \in I(V)$ , so that the set of all rational functions is the field

$$k(V) = \left\{ \frac{g}{h} \left| \begin{array}{l} g, h \in k[X_0, \dots, X_n] \text{ homogeneous} \\ \text{of the same degree, and } h \notin I(V) \end{array} \right. \right\} / \sim,$$

where  $\sim$  is the equivalence relation

$$\frac{g}{h} \sim \frac{g'}{h'} \iff h'g - g'h \in I(V).$$

$k(V)$  is the (rational) *function field* of  $V$ .

The following definitions are just as in the affine case. For  $f \in k(V)$  and  $P \in V$ , say that  $f$  is *regular* at  $P$  if there exists an expression  $f = g/h$ , with  $g, h$  homogeneous polynomials of the same degree, such that  $h(P) \neq 0$ . Write

$$\text{dom } f = \{P \in V \mid f \text{ is regular at } P\}$$

and

$$\mathcal{O}_{V,P} = \{f \in k(V) \mid f \text{ is regular at } P\}.$$

Clearly,  $\text{dom } f \subset V$  is a dense Zariski open set in  $V$  (the proof is as in (4.8, I), and  $\mathcal{O}_{V,P} \subset k(V)$  is a subring.

## 5.5 Affine covering of a projective variety

Let  $V \subset \mathbb{P}^n$  be an irreducible algebraic set, and suppose for simplicity that  $V \not\subset (X_i = 0)$  for any  $i$ . We know that  $\mathbb{P}^n$  is covered by  $n+1$  affine pieces  $\mathbb{A}_{(i)}^n$ , with affine (inhomogeneous) coordinates

$$X_0^{(i)}, \dots, X_{i-1}^{(i)}, X_{i+1}^{(i)}, \dots, X_n^{(i)}, \quad \text{where } X_j^{(i)} = X_j/X_i \text{ for } j \neq i.$$

Write  $V_{(i)} = V \cap \mathbb{A}_{(i)}^n$ . Then  $V_{(i)} \subset \mathbb{A}_{(i)}^n$  is clearly an affine algebraic set, because

$$\begin{aligned} V_{(0)} \ni P = (1, x_1^{(0)}, \dots, x_n^{(0)}) \\ \iff f(1, x_1^{(0)}, \dots, x_n^{(0)}) = 0 \quad \text{for all homogeneous } f \in I(V), \end{aligned}$$

which is a set of polynomial relations in the coordinates  $(x_1^{(0)}, \dots, x_n^{(0)})$  of  $P$ . For clarity, I have taken  $i = 0$  in the argument, and will continue to do so whenever convenient. The reader should remember that the same result applies to any of the other affine pieces  $V_{(i)}$ . The  $V_{(i)}$  are called *standard affine pieces* of  $V$ .

**Proposition** (i) *The correspondence  $V \mapsto V_{(0)} = V \cap \mathbb{A}_{(0)}^n$  gives a bijection*

$$\left\{ \begin{array}{l} \text{irreducible alg.} \\ \text{subsets } V \subset \mathbb{P}^n \end{array} \mid V \not\subset (X_0 = 0) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{irreducible alg.} \\ \text{subsets } V_0 \subset \mathbb{A}_{(0)}^n \end{array} \right\};$$

*the inverse correspondence is given by taking the closure in the Zariski topology.*

(ii) *Write  $I^h(V) \subset k[X_0, \dots, X_n]$  for the homogeneous ideal of  $V \subset \mathbb{P}^n$  introduced in this section and  $I^a(V_{(0)}) \subset k[X_1, \dots, X_n]$  for the usual (as in §3) inhomogeneous ideal of  $V_{(0)} \subset \mathbb{A}_{(0)}^n$ ; then  $I^h(V)$  and  $I^a(V_{(0)})$  are related as follows:*

$$I^a = \{f(1, X_1, \dots, X_n) \mid f \in I^h(V)\},$$

and

$$I^h(V)_d = \{X_0^d f(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}) \mid f \in I^a(V_{(0)}), \text{ with } \deg f \leq d\},$$

*where the subscript in  $I^h(V)_d$  denotes the piece of degree  $d$ .*