Proof Try it and see.

Definition *An ideal $I \subset k[X_0, \ldots, X_n]$ is* homogeneous *if for all $f \in I$, the homogeneous decomposition $f = f_0 + f_1 + \cdots + f_N$ of f satisfies $f_i \in I$ for all i.*

It is equivalent to say that I is generated by (finitely many) homogeneous polynomials.

5.2   The homogeneous V-I correspondences

Let $\mathbb{P}_k^n$ be n-dimensional projective space over a field k, with $X_0, \ldots, X_n$ as homogeneous coordinates. Then $f \in k[X_0, \ldots, X_n]$ is *not* a function on $\mathbb{P}_k^n$: by definition, $\mathbb{P}_k^n = k^{n+1} \setminus \{0\}/\sim$, where $\sim$ is the equivalence relation given by $(X_0, \ldots, X_n) \sim (\lambda X_0, \ldots, \lambda X_n)$ for $\lambda \in k \setminus \{0\}$; f is a function on k^{n+1}. Nevertheless, for $P \in \mathbb{P}^n$, the condition $f(P) = 0$ is well defined provided that f is homogeneous: suppose $P = (X_0 : \cdots : X_n)$, so that $(X_0, \ldots, X_n)$ is a representative in $k^{n+1} \setminus \{0\}$ of the equivalence class of P. Then since $f(\lambda X) = \lambda^d f(X)$, if $f(X_0, \ldots, X_n) = 0$ then also $f(\lambda X_0, \ldots, \lambda X_n) = 0$, so that the condition $f(P) = 0$ is independent of the choice of representative. With this in mind, define as before correspondences

$$\big\{\text{homog. ideals } J \subset k[X_0, \ldots, X_n]\big\} \xleftrightarrow{\;V\text{-}I\;} \big\{\text{subsets } X \subset \mathbb{P}_k^n\big\}$$

by

$$V(J) = \big\{P \in \mathbb{P}_k^n \bigm| f(P) = 0 \ \forall \text{ homogeneous } f \in J\big\}$$

and

$$I(X) = \big\{f \in k[X_0, \ldots, X_n] \bigm| f(P) = 0 \text{ for all } P \in X\big\}.$$

As an exercise, check that you understand why $I(X)$ is a homogeneous ideal.

The correspondences V and I satisfy the same formal properties as the affine V and I correspondences introduced in §3 (for example $V(J_1 + J_2) = V(J_1) \cap V(J_2)$). A subset of the form $V(I)$ is an *algebraic subset* of $\mathbb{P}_k^n$, and as in the affine case, $\mathbb{P}_k^n$ has a *Zariski topology* in which the closed sets are the algebraic subsets.

5.3   Projective Nullstellensatz

As with the affine correspondences, it is purely formal that $I(V(J)) \supset \operatorname{rad} J$ for any ideal J, and that for an algebraic set, $V(I(X)) = X$. There's just one point where care is needed: the trivial ideal $(1) = k[X_0, \ldots, X_n]$ (the whole ring) defines the empty set in k^{n+1}, hence also in $\mathbb{P}_k^n$, which is as it should be; however, the ideal $(X_0, \ldots, X_n)$ defines $\{0\}$ in k^{n+1}, which also corresponds to the empty set in $\mathbb{P}_k^n$. The ideal $(X_0, \ldots, X_n)$ is an awkward (empty-set theoretical) exception to several statements in the theory, and is traditionally known as the 'irrelevant ideal'.

The homogeneous version of the Nullstellensatz thus becomes:

Theorem *Assume that k is an algebraically closed field. Then*

(i) $V(J) = \emptyset \iff \operatorname{rad} J \supset (X_0, \ldots, X_n)$;

(ii) if $V(J) \neq \emptyset$ then $I(V(J)) = \operatorname{rad} J$.