(II) *Conversely, given any choice* $\{i,j,k\}$ *of* 3 *elements of the set* $\{1,2,3,4,5\}$, *there is a unique line* $\ell_{ijk} \subset S$ *meeting* ℓ_i, ℓ_j, ℓ_k.

Proof (I) Given four disjoint lines of S, it is clear that they do not all lie on a quadric Q, since otherwise $Q \subset S$, contradicting the irreducibility of S.

If n meets ≥ 4 of the ℓ_i then by Lemma 7.5, $n = \ell$ or m, which is a contradiction. If n meets ≤ 2 of the ℓ_i then it meets ≥ 3 of the ℓ'_i, and so meets say either $\ell'_2, \ell'_3, \ell'_4, \ell'_5$ or $\ell_1, \ell'_3, \ell'_4, \ell'_5$; but by what was said above, ℓ and ℓ''_1 are two common transversals of the 5 disjoint lines $\ell'_2, \ell'_3, \ell'_4, \ell'_5$ and ℓ_1, so that by Lemma 7.5 again, if n meets ≥ 4 of these then $n = \ell$ or ℓ''_1. This is the same contradiction.

(II) There are 10 lines meeting ℓ_1 by (7.3), of which so far only 4 have been accounted for (namely, ℓ, ℓ'_1, m and ℓ''_1). The six other lines must meet exactly 2 out of the 4 remaining lines $\ell_2, \ldots, \ell_5$, and there are exactly $6 = \binom{4}{2}$ possible choices; so they must all occur. Q.E.D.

This gives the lines of S as being

$$\{\ell, m, \ell_i, \ell'_i, \ell''_i, \ell_{ijk}\},$$

and the number of them is

$$1 + 1 + 5 + 5 + 5 + 10 = 27.$$

7.7 The configuration of lines

An alternative statement is that the lines of S are $\ell, \ell_1, \ldots, \ell_5, \ell'_1, \ldots, \ell'_5$, and 16 other lines which meet an odd number of $\ell_1, \ldots, \ell_5$:

$$\begin{aligned} \ell''_i &\text{ meets } \ell_i \text{ only} \\ \ell_{ijk} &\text{ meets } \ell_i, \ell_j, \ell_k \text{ only} \\ m &\text{ meets all of } \ell_1, \ldots, \ell_5. \end{aligned}$$

In the notation I have introduced, it is easy to see that the incidence relation between the 27 lines of S is as follows:

ℓ meets $\ell_1, \ldots, \ell_5, \ell'_1, \ldots, \ell'_5$;

ℓ_1 meets $\ell, m, \ell'_1, \ell''_1$, and ℓ_{1jk} for 6 choices of $\{j,k\} \subset \{2,3,4,5\}$;

ℓ'_1 meets ℓ, ℓ_1, ℓ''_j (for 4 choices of $j \neq 1$), and ℓ_{ijk} (for 4 choices of $\{i,j,k\} \subset \{2,3,4,5\}$);

ℓ''_1 meets m, ℓ_1, ℓ'_j (for 4 choices of $j \neq 1$), and ℓ_{ijk} (for 4 choices of $\{i,j,k\} \subset \{2,3,4,5\}$);

ℓ_{123} meets $\ell_1, \ell_2, \ell_3, \ell_{145}, \ell_{245}, \ell_{345}, \ell'_4, \ell'_5, \ell''_4, \ell''_5$.

This combinatorial configuration has many different representations, some of them much more symmetric than that given here; see for example [Semple and Roth, pp. 122–128 and 151–152].

Exercises to Chapter 7

7.1 Prove case (ii) of the claim in Proposition 7.3. [Hint: as in the given proof of case (i), $f = X(X-T)T + Zg$, so that $A = T + aZ$, $D = -T^2 + Z \cdot \ell$, where ℓ is linear, so that $Z \mid B, C, E, F$, and Z does not divide D; also, the nonsingularity of S at $(0,1,0,0)$ implies that $C = cZ$, with $c \neq 0$. Now calculate $\Delta(Z,T)$ modulo Z^2.]

7.2 Prove that given 3 disjoint lines $\ell_1, \dots, \ell_3 \subset \mathbb{P}^3$, there exists a nonsingular quadric $Q \supset \ell_1, \dots, \ell_3$. [Hint: on each line ℓ_i, take 3 points $P_i, P_i', P_i'' \in \ell_i$, and show as in (1.11) or (2.4) that there is at least one quadric Q through them; it follows that each $\ell_i \subset Q$. Now show that Q can't be singular: for example, what happens if Q is a pair of planes?]

7.3 *The Hessian.* Let $f = f_d(x_0, \dots, x_n)$ be a form of degree d in $x_0, \dots, x_n$, defining a hypersurface $V : (f = 0) \subset \mathbb{P}^n$; suppose for simplicity that the characteristic $\neq 2$ and does not divide $(d-1)$. Write $f_{x_i} = \partial f / \partial x_i$ and $f_{x_i x_j} = \partial^2 f / \partial x_i \partial x_j$ for the first and second derivatives of f. The Taylor expansion of f about a point $P \in \mathbb{P}^n$ is

$$f = f(P) + f^{(1)}(x) + f^{(2)}(x) + \cdots,$$

where $f^{(1)}$ and $f^{(2)}$ are linear and quadratic forms:

$$f^{(1)} = \sum f_{x_i}(P) \cdot x_i \quad \text{and} \quad f^{(2)} = (1/2) \sum f_{x_i x_j}(P) \cdot x_i x_j.$$

If $P \in V$ is singular then $f(P)$ and $f^{(1)}$ vanish at P, and the nature of V or of f near P is determined to second order by the quadratic form $f^{(2)}$. Similarly if $P \in V$ is nonsingular then the nature of f restricted to the hyperplane $T_P V$ (or of the singular hyperplane section $V \cap T_P V$) is determined by $f^{(2)}$. Define the *Hessian matrix* of f (w.r.t. coordinates $x_0, \dots, x_n$) by $H(f) = H(f,x) = \{f_{x_i x_j}\}_{i,j}$, and the *Hessian* $h(f) = h(f,x)$ to be the determinant $h(f) = \det H(f)$.

(i) Let $x_i' = \sum a_{ij} x_j$ be a projective coordinate change with $A = (a_{ij})$ a nonsingular $(n+1) \times (n+1)$ matrix. If $g(x') = f(Ax)$, prove that the Hessian matrix transforms as

$$H(g, x') = ({}^t A) H(f,x) A$$

where ${}^t A$ is the transpose matrix; deduce that $h(g,x') = (\det A)^2 h(f,x)$.

(ii) Consider an affine piece $V_{(i)} \subset \mathbb{A}^n_{(i)}$ of $V : (f = 0)$ as in (5.5). Let $P \in V_{(i)}$ be a nonsingular point, and $\Pi = T_P V_{(i)}$ the affine tangent plane; write φ for the restriction to Π of the defining equation $f / x_i d$ of $V_{(i)}$. Prove that the Taylor expansion of φ at P starts with a nondegenerate quadratic form $\varphi^{(2)}$ (in $n-1$ variables) if and only if $h(f)(P) \neq 0$.
[Hint: Reduce to $P = (1, 0, \dots, 0)$ and $T_P V : (x_1 = 0)$ using (i). Then $\varphi^{(2)}$ is the bottom right $(n-1) \times (n-1)$ block of the projective Hessian matrix $H(f)$. Use $f_{x_i}(P) = 0$ for $i \neq 1$ and Euler's formula $\sum_j f_{x_i x_j} \cdot x_j = (d-1) f x_i$ to show that the matrix $H(f)$ has exactly one nonzero entry in the zeroth row and column. Compare [Fulton, p. 116].]

(iii) Let $C : (f = 0) \subset \mathbb{P}^2$ be a nonsingular plane cubic curve; deduce from (ii) that $P \in C$ is an inflexion point if and only if $H(f)(P) = 0$. Bézout's theorem implies that $(f = H(f) = 0) \subset \mathbb{P}^2$ is nonempty (see (1.9) and [Fulton, p. 112]).