Chapter 3

Affine varieties and the Nullstellensatz

Much of the first half of this section is pure commutative algebra; note that throughout these notes, *ring* means a commutative ring with a 1. Since this is not primarily a course in commutative algebra, I will hurry over several points.

3.1 Definition of Noetherian ring

Proposition-Definition *The following conditions on a ring A are equivalent.*

(i) Every ideal $I \subset A$ is finitely generated; that is, for every ideal $I \subset A$, there exist $f_1, \dots, f_k \in I$ such that $I = (f_1, \dots, f_k)$.

(ii) Every ascending chain

$$I_1 \subset \cdots \subset I_m \subset \cdots$$

of ideals of A terminates, that is the chain is eventually stationary, with $I_N = I_{N+1} = \cdots$ (the ascending chain condition*, or* a.c.c.*).*

(iii) Every nonempty set of ideals of A has a maximal element.

If they hold, A is a Noetherian *ring.*

Proof (i) $\Longrightarrow$ (ii) Given $I_1 \subset \cdots \subset I_m \subset \cdots$, set $I = \bigcup I_m$. Then clearly I is still an ideal. If $I = (f_1, \dots, f_k)$, then each f_i is an element of some I_{m_i} for some m_i, so that taking $m = \max\{m_i\}$ gives $I = I_m$, and the chain stops at I_m.

(ii) $\Longrightarrow$ (iii) is clear. (Actually, it uses the axiom of choice.)

(iii) $\Longrightarrow$ (i) Let I be any ideal; write $\Sigma = \{J \subset I \mid J \text{ is a f.g. ideal}\}$. Then by (iii), Σ has a maximal element, say J_0. But then $J_0 = I$, because otherwise any $f \in I \; J_0$ gives an ideal $J_0 + Af$ which is still finitely generated, but strictly bigger than J_0. Q.E.D.

As a thought experiment, prove that $\mathbb{Z}$ and $k[X]$ are Noetherian.