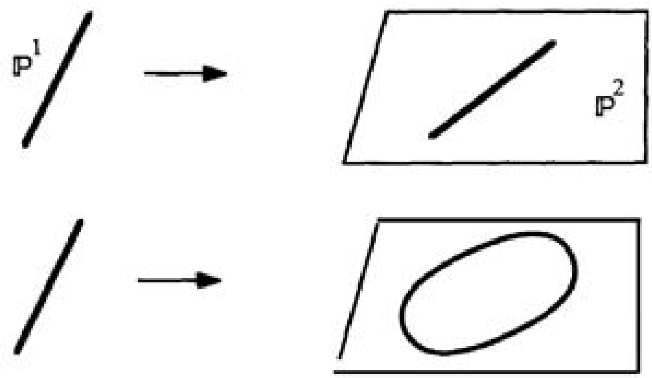

Figure 1.4: (a) Parametrised line; (b) parametrised conic

Then the intersection of L (respectively C) with D is given by finding the values of the ratios $(U:V)$ such that

$$F(U,V) = G_d(a(U,V), b(U,V), c(U,V)) = 0.$$

But F is a form of degree d (respectively $2d$) in U, V, so the result follows by (1.8). Q.E.D.

Corollary 1.10 *If $P_1, \dots, P_5 \in \mathbb{P}^2_{\mathbb{R}}$ are distinct points and no 4 are collinear, there exists at most one conic through $P_1, \dots, P_5$.*

Proof Suppose by contradiction that C_1 and C_2 are conics with $C_1 \neq C_2$ such that

$$C_1 \cap C_2 \supset \{P_1, \dots, P_5\}.$$

C_1 is nonempty, so that if it's nondegenerate, then by (1.7), it's projectively equivalent to the parametrised curve

$$C_1 = \left\{(U^2, UV, V^2) \mid (U,V) \in \mathbb{P}^1\right\};$$

by (1.9), $C_1 \subset C_2$. Now if Q_2 is the equation of C_2, it follows that $Q_2(U^2, UV, V^2) \equiv 0$ for all $(U,V) \in \mathbb{P}^1$, and an easy calculation (see Ex. 1.6) shows that Q_2 is a multiple of $(XZ - Y^2)$; this contradicts $C_1 \neq C_2$.

Now suppose C_1 is degenerate; by (1.6) again, it's either a line pair or a line, and one sees easily that

$$C_1 = L_0 \cup L_1, \quad C_2 = L_0 \cup L_2,$$

with L_1, L_2 distinct lines. Then $C_1 \cap C_2 = L_0 \cup (L_1 \cap L_2)$:

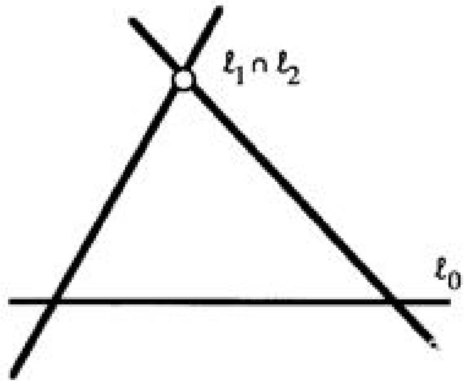

Figure 1.5: Lines meeting

thus 4 points out of $P_1, \dots, P_5$ lie on L_0, a contradiction. Q.E.D.

1.11 Space of all conics

Let

$$S_2 = \{\text{quadratic forms on } \mathbb{R}^3\} = \{3 \times 3 \text{ symmetric matrixes}\} \cong \mathbb{R}^6.$$

If $Q \in S_2$, write $Q = aX^2 + 2bXY + \cdots + fZ^2$; for $P_0 = (X_0, Y_0, Z_0) \in \mathbb{P}^2_{\mathbb{R}}$, consider the relation $P_0 \in C : (Q = 0)$. This is of the form

$$Q(X_0, Y_0, Z_0) = aX_0^2 + 2bX_0Y_0 + \cdots + fZ_0^2 = 0,$$

and for fixed P_0, this is a linear equation in $(a, b, \dots, f)$. So

$$S_2(P_0) = \{Q \in S_2 \bigm| Q(P_0) = 0\} \cong \mathbb{R}^5 \subset S_2 = \mathbb{R}^6$$

is a 5-dimensional hyperplane. For $P_1, \dots, P_n \in \mathbb{P}^2_{\mathbb{R}}$, define similarly

$$S_2(P_1, \dots, P_n) = \{Q \in S_2 \bigm| Q(P_i) = 0 \text{ for } i = 1, \dots, n\};$$

then there are n linear equations in the 6 coefficients $(a, b, \dots, f)$ of Q. This gives the result:

Proposition $\dim S_2(P_1, \dots, P_n) \geq 6 - n$.

We can also expect that 'equality holds if $P_1, \dots, P_n$ are general enough'. More precisely:

Corollary *If $n \leq 5$ and no 4 of $P_1, \dots, P_n$ are collinear, then*

$$\dim S_2(P_1, \dots, P_n) = 6 - n.$$