(a) *there is a natural isomorphism of vector spaces*

$$(T_PV)^* = m_P/m_P^2,$$

where $(\ )^*$ *denotes the dual of a vector space.*

(b) *If* $f \in k[V]$ *is such that* $f(P) \neq 0$, *and* $V_f \subset V$ *is the standard affine open as in (4.13), then the natural map*

$$T_P(V_f) \to T_PV$$

is an isomorphism.

Proof of (a) Write $(k^n)^*$ for the vector space of linear forms on k^n; this is the vector space with basis $X_1, \dots, X_n$. Since $P = (0, \dots, 0)$, for any $f \in k[X_1, \dots, X_n]$, the linear part $f_P^{(1)}$ is naturally an element of $(k^n)^*$; define a map $d\colon M_P \to (k^n)^*$ by taking $f \in M_P$ into $df = f_P^{(1)}$.

Now d is surjective, since the $X_i \in M_P$ go into the natural basis of $(k^n)^*$; also $\ker d = M_P^2$, since

$$\begin{aligned} f_P^{(1)} = 0 &\iff f \text{ starts with quadratic terms in } X_1, \dots, X_n \\ &\iff f \in M_P^2. \end{aligned}$$

Hence $M_P/M_P^2 \cong (k^n)^*$. This is statement (a) for the special case $V = \mathbb{A}^n$. In the general case, dual to the inclusion $T_PV \subset k^n$, there is a restriction map $(k^n)^* \to (T_PV)^*$, taking a linear form λ on k^n into its restriction to T_PV; composing then defines a map

$$D : M_P \to (k^n)^* \to (T_PV)^*.$$

The composite D is surjective since each factor is. I claim that the kernel of D is just $M_P^2 + I(V)$, so that

$$m_P/m_P^2 = M_P/(M_P^2 + I(V)) \cong (T_PV)^*,$$

as required. To prove the claim,

$$\begin{aligned} f \in \ker D &\iff f_P^{(1)} | T_PV = 0 \\ &\iff f_P^{(1)} = \sum_i a_i g_{i,P}^{(1)} \text{ for some } g_i \in I(V) \end{aligned}$$

(since $T_PV \subset k^n$ is the vector subspace defined by $(g_P^{(1)} = 0)$ for $g \in I(V)$)

$$\iff f - \sum_i a_i g_i \in M_P^2 \text{ for some } g_i \in I(V) \iff f \in M_P^2 + I(V).$$

The proof of (b) of Theorem 6.8 is left to the reader (see Ex. 6.2). Q.E.D.

Corollary 6.9 T_PV *only depends on a neighbourhood of* $P \in V$ *up to isomorphism. More precisely, if* $P \in V_0 \subset V$ *and* $Q \in W_0 \subset W$ *are open subsets of affine varieties, and* $\varphi\colon V_0 \to W_0$ *an isomorphism taking* P *into* Q, *there is a natural isomorphism* $T_PV_0 \to T_QW_0$; *hence* $\dim T_PV_0 = \dim T_QW_0$.

In particular, if V *and* W *are birationally equivalent varieties then* $\dim V = \dim W$.

Proof By passing to a smaller neighbourhood of P in V, I can assume V_0 is isomorphic to an affine variety (Proposition 4.13). Then so is W_0, and φ induces an isomorphism $k[V_0] \cong k[W_0]$ taking m_P into m_Q. The final sentence holds because by (5.8), V and W contain dense open subsets which are isomorphic.

Theorem 6.10 *For any variety V, $\dim V = \operatorname{tr}\deg k(V)$.*

Proof This is known if V is a hypersurface. On the other hand, every variety is birational to a hypersurface (by (5.10)), and both sides of the required relation are the same for birationally equivalent varieties. Q.E.D.

6.11 Nonsingularity and projective varieties

Although the above results were discussed in terms of affine varieties, the idea of nonsingularity and of dimension applies directly to any variety V: a point $P \in V$ is nonsingular if it is a nonsingular point of an affine open $V_0 \subset V$ containing it; by Corollary 6.9, this notion does not depend on the choice of V_0. On the other hand, for a projective variety $V \subset \mathbb{P}^n$, it is sometimes useful to consider the tangent space to V at P as a projective subspace of $\mathbb{P}^n$. I give the definition for a hypersurface only: if $V = V(f)$ is a hypersurface defined by a form (= homogeneous polynomial) $f \in k[X_0, \dots, X_n]$ of degree d, and $V \ni P = (a_0, \dots, a_n)$, then $\sum \partial f/\partial X_i(P) \cdot X_i = 0$ is the equation of a hyperplane in $\mathbb{P}^n$ which plays the role of the tangent plane to V at P. If $P \in \mathbb{A}^n_{(0)}$, then this projective hyperplane is the projective closure of the affine tangent hyperplane to $V_{(0)}$ at P, as can be checked easily using Euler's formula:

$$\sum X_i \cdot \frac{\partial f}{\partial X_i} = df \quad \text{for } f \in k[X_0, \dots, X_n] \text{ homogeneous of degree } d.$$

Because of this formula, to find out whether a point $P \in \mathbb{P}^n$ is a singular point of V, we only have to check $(n+1)$ out of the $(n+2)$ conditions

$$f(P) = 0, \quad \frac{\partial f}{\partial X_i}(P) = 0 \text{ for } i = 0, \dots, n,$$

so that for example, if the degree of f is not divisible by $\operatorname{char} k$,

$$\frac{\partial f}{\partial X_i}(P) = 0 \text{ for } i = 0, \dots, n \implies f(P) = 0,$$

and $P \in V$ is a singularity.

6.12 Worked example: blowup

Let $B = \mathbb{A}^2$ with coordinates (u, v), and $\sigma\colon B \to \mathbb{A}^2$ the map $(u, v) \mapsto (x = u, y = uv)$; clearly, σ is a birational morphism: it contracts the v-axis $\ell : (u = 0)$ to the origin 0 and is an isomorphism outside this exceptional set. Let's find out what happens under σ to a curve $C : (f = 0) \subset \mathbb{A}^2$; the question will only be of interest if C passes through 0.