Chapter 3

Affine varieties and the Nullstellensatz

Much of the first half of this section is pure commutative algebra; note that throughout these notes, *ring* means a commutative ring with a 1. Since this is not primarily a course in commutative algebra, I will hurry over several points.

3.1 Definition of Noetherian ring

Proposition-Definition *The following conditions on a ring A are equivalent.*

(i) *Every ideal $I \subset A$ is finitely generated; that is, for every ideal $I \subset A$, there exist $f_1, \dots, f_k \in I$ such that $I = (f_1, \dots, f_k)$.*

(ii) *Every ascending chain*

$$I_1 \subset \cdots \subset I_m \subset \cdots$$

of ideals of A terminates, that is the chain is eventually stationary, with $I_N = I_{N+1} = \cdots$ (the ascending chain condition, *or* a.c.c.*).*

(iii) *Every nonempty set of ideals of A has a maximal element.*

If they hold, A is a Noetherian *ring.*

Proof (i) $\implies$ (ii) Given $I_1 \subset \cdots \subset I_m \subset \cdots$, set $I = \bigcup I_m$. Then clearly I is still an ideal. If $I = (f_1, \dots, f_k)$, then each f_i is an element of some I_{m_i} for some m_i, so that taking $m = \max\{m_i\}$ gives $I = I_m$, and the chain stops at I_m.

(ii) $\implies$ (iii) is clear. (Actually, it uses the axiom of choice.)

(iii) $\implies$ (i) Let I be any ideal; write $\Sigma = \{J \subset I \mid J \text{ is a f.g. ideal}\}$. Then by (iii), Σ has a maximal element, say J_0. But then $J_0 = I$, because otherwise any $f \in I \setminus J_0$ gives an ideal $J_0 + Af$ which is still finitely generated, but strictly bigger than J_0. Q.E.D.

As a thought experiment, prove that $\mathbb{Z}$ and $k[X]$ are Noetherian.

Proposition 3.2   *(i) Suppose that A is Noetherian, and $I \subset A$ an ideal; then the quotient ring $B = A/I$ is Noetherian.*

(ii) Let A be a Noetherian integral domain, and $A \subset K$ its field of fractions; let $0 \notin S \subset A$ be a subset, and set

$$B = A[S^{-1}] = \left\{ \frac{a}{b} \in K \, \middle| \, \begin{array}{l} a \in A, \ and \ b = 1 \ or \ a \\ product \ of \ elements \ of \ S \end{array} \right\}.$$

Then B is again Noetherian.

Proof   Exercise: in either case the ideals of B can be described in terms of certain ideals of A; see Ex. 3.4 for hints.

Theorem 3.3 (Hilbert Basis Theorem) *For a ring A,*

$$A \ Noetherian \implies A[X] \ Noetherian.$$

Proof   Let $J \subset A[X]$ be any ideal; I prove that J is finitely generated. Define the ideal of leading terms of degree n in J to be

$$J_n = \left\{ a \in A \mid \exists f = aX^n + b_{n-1}X^{n-1} + \cdots + b_0 \in J \right\}.$$

Then J_n is an ideal of A and $J_n \subset J_{n+1}$ (please provide your own proofs). Hence, using the a.c.c., there exists N such that

$$J_N = J_{N+1} = \cdots .$$

Now build a set of generators of J as follows: for $i \leq N$, let $a_{i_1}, \ldots, a_{i_{m_i}}$ be generators of J_i and, as in the definition of J_i, for each of the a_{ik}, let $f_{ik} = a_{ik}X^i + \cdots \in J$ be an element of degree i and leading term a_{ik}.

I claim that the set

$$\left\{ f_{ik} \mid i = 0, \ldots, N, k = 1, \ldots, m_i \right\}$$

just constructed generates J: for given $g \in J$, suppose $\deg g = m$. Then the leading term of g is bX^m with $b \in J_m$, so that by what I know about J_m, I can write $b = \sum c_{m'k} a_{m'k}$ (here $m' = m$ if $m \leq N$, otherwise $m' = N$). Then consider $g_1 = g - X^{m-m'} \cdot \sum c_{m'k} f_{m'k}$: by construction the term of degree m is zero, so that $\deg g_1 \leq \deg g - 1$; by induction, I can therefore write out g as a combination of f_{ik}, so that these generate J.   Q.E.D.

Corollary *For k a field, a finitely generated k-algebra is Noetherian.*

A finitely generated k-algebra is a ring of the form $A = k[a_1, \ldots, a_n]$, so that A is generated as a ring by k and $a_1, \ldots, a_n$; clearly, every such ring is isomorphic to a quotient of the polynomial ring, $A \cong k[X_1, \ldots, X_n]/I$. A field is Noetherian, and by induction on (3.3), $k[X_1, \ldots, X_n]$ is Noetherian; finally, passing to the quotient is OK by Proposition 3.2, (i).   Q.E.D.