

Chapter 6

Tangent space and nonsingularity, dimension

6.1 Nonsingular points of a hypersurface

Suppose $f \in k[X_1, \dots, X_n]$ is irreducible, $f \notin k$, and set $V = V(f) \subset \mathbb{A}^n$; let $P = (a_1, \dots, a_n) \in V$, and ℓ be a line through P . Since $P \in V$, obviously P is a root of $f|\ell$.

Question: When is P a multiple root of $f|\ell$?

Answer: If and only if ℓ is contained in the affine linear subspace

$$T_P V : \left(\sum_i \frac{\partial f}{\partial X_i}(P) \cdot (X_i - a_i) = 0 \right) \subset \mathbb{A}^n,$$

called the *tangent space* to V at P .

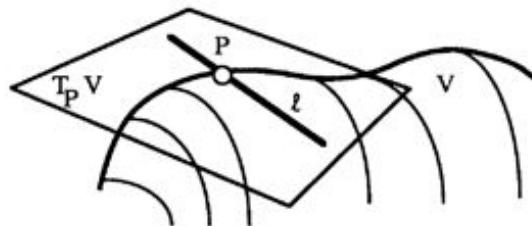


Figure 6.1: Tangent space

To prove this, parametrise ℓ as

$$\ell : X_i = a_i + b_i T,$$

where $P = (a_1, \dots, a_n)$ and (b_1, \dots, b_n) is the slope or direction vector of ℓ . Then $f|\ell = f(\dots, a_i + b_i T, \dots) = g(T)$ is a polynomial in T , and we know that $(T = 0)$ is one root of g . Hence

$$0 \text{ is a multiple root of } g \iff \frac{\partial g}{\partial T}(0) = 0,$$

that is,

$$\iff \sum_i b_i \frac{\partial f}{\partial X_i}(P) = 0 \iff \ell \subset T_P V.$$

Definition $P \in V \subset \mathbb{A}^n$ is a *nonsingular point* of V if $\partial f / \partial X_i(P) \neq 0$ for some i ; otherwise P is a *singular point*, or a *singularity* of V .

Obviously $T_P V$ is an $(n-1)$ -dimensional affine subspace of \mathbb{A}^n if P is nonsingular, and $T_P V = \mathbb{A}^n$ if $P \in V$ is singular.

6.2 Remarks

- (a) The derivatives $\partial f / \partial X_i(P)$ appearing above are formal algebraic operations (that is, $\partial / \partial X_i$ takes X_i^n into nX_i^{n-1}); no calculus is involved.
- (b) Suppose $k = \mathbb{R}$ or \mathbb{C} , and that $\partial f / \partial X_i(P) \neq 0$; for clarity let me take $i = 1$. Then the map $p: \mathbb{A}^n \rightarrow \mathbb{A}^n$ defined by $(X_1, \dots, X_n) \mapsto (f, X_2, \dots, X_n)$ has nonvanishing Jacobian determinant at P , so that by the inverse function theorem, there exists a neighbourhood $P \in U \subset \mathbb{A}^n$ such that $p|U: U \rightarrow p(U) \subset \mathbb{A}^n$ is a diffeomorphism of the neighbourhood U with an open set $p(U)$ of \mathbb{A}^n (in the usual topology of \mathbb{R}^n or \mathbb{C}^n); that is, $p|U$ is bijective, and both p and p^{-1} are differentiable functions of real or complex variables. In other words, (f, X_2, \dots, X_n) form a new differentiable coordinate system on \mathbb{A}^n near P ; this implies that a neighbourhood of P in $V : (f = 0)$ is diffeomorphic to an open set in \mathbb{A}^{n-1} with coordinates (X_2, \dots, X_n) . Thus near a nonsingular point P , V is a *manifold* with (X_2, \dots, X_n) as local parameters.

Proposition 6.3 $V_{\text{nonsing}} = \{P \in V \mid P \text{ is nonsingular}\}$ is a dense open set of V for the Zariski topology.

Proof The complement of V_{nonsing} is the set V_{sing} of singular points, which is defined by $\partial f / \partial X_i(P) = 0$ for all i , that is

$$V_{\text{sing}} = V\left(f, \frac{\partial f}{\partial X_1}, \dots, \frac{\partial f}{\partial X_n}\right) \subset \mathbb{A}^n,$$

which is closed by definition of the Zariski topology. Since V is irreducible (by (3.11, a)), to show that the open V_{nonsing} is dense, I only have to show it's nonempty (by Proposition 4.2); arguing by contradiction, suppose that it's empty, that is, suppose $V = V(f) = V_{\text{sing}}$. Then each of the polynomials $\partial f / \partial X_i$ must vanish on V , therefore (by (3.11) once more) they must be divisible by f in $k[X_1, \dots, X_n]$; but viewed as a polynomial in X_i , $\partial f / \partial X_i$ has degree strictly smaller than f , so that f divides $\partial f / \partial X_i$ implies that in fact $\partial f / \partial X_i = 0$ as a polynomial. Over \mathbb{C} , this is obviously only possible if X_i does not appear in f , and if this happens for all i then $f = \text{const.} \in \mathbb{C}$, which is excluded. Over a general field k , $\partial f / \partial X_i = 0$ is only possible if f is an inseparable polynomial in