Chapter 7

The 27 lines on a cubic surface

In this section $S \subset \mathbb{P}^3$ will be a nonsingular cubic surface, given by a homogeneous cubic $f = f(X,Y,Z,T)$. Consider the lines ℓ of $\mathbb{P}^3$ lying on S.

7.1 Consequences of nonsingularity

Proposition *(a) There exists at most 3 lines of S through any point $P \in S$; if there are 2 or 3, they must be coplanar. The picture is:*

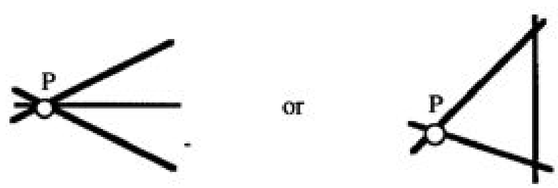

Figure 7.1: 3 concurrent lines or triangle

(b) Every plane $\Pi \subset \mathbb{P}^3$ intersects S in one of the following:

(i) an irreducible cubic; or

(ii) a conic plus a line; or

(iii) 3 distinct lines.

Proof (a) If $\ell \subset S$ then $\ell = T_P\ell \subset T_PS$, so that all lines of S through P are contained in the plane T_PS; there are at most 3 of them by (b).

(b) I have to prove that a multiple line is impossible: if $\Pi : (T = 0)$ and $\ell : (Z = 0) \subset \Pi$, then to say that ℓ is a multiple line of $S \cap \Pi$ means that f is of the form

$$f = Z^2 \cdot A(X,Y,Z,T) + T \cdot B(X,Y,Z,T),$$

with A a linear form, B a quadratic form. Then $S : (f = 0)$ is singular at a point where $Z = T = B = 0$; this is a nonempty set, since it is the set of roots of B on the line $\ell : (Z = T = 0)$.

Proposition 7.2 *There exists at least one line ℓ on S.*

There are several approaches to proving this. A standard argument is by a dimension count: lines of $\mathbb{P}^3$ are parametrised by a 4-dimensional variety, and for a line ℓ to lie on S imposes 4 conditions on ℓ (because the restriction of f to ℓ is a cubic form, the 4 coefficients of which must vanish). A little work is needed to turn this into a rigorous proof, since a priori it shows only that the set of lines has dimension ≥ 0, and not that it is nonempty (see the highbrow notes (8.15) for a discussion of the traditional proof and the difficulties involved in it).

It is also perfectly logical to assume the proposition (restrict attention only to cubic surfaces containing lines). I now explain how (7.2) can be proved by direct coordinate geometry and elimination. The proof occupies the next 3 pages, and divides up into 4 steps; you can skip it if you prefer (GOTO 7.3).

Step 1 (Preliminary construction) For any point $P \in S$, the intersection of S with the tangent plane T_PS is a plane cubic $C = S \cap T_PS$, which by Ex. 6.7 is singular at P. I assume that C is irreducible, since otherwise P is on a line of S, and I'm home; then C is a nodal or cuspidal cubic, and the coordinates (X, Y, Z, T) of $\mathbb{P}^3$ can be chosen such that $T_PS : (T = 0)$, $P = (0, 0, 1, 0)$, and

$$C : (XYZ = X^3 + Y^3) \text{ or } (X^2Z = Y^3).$$

Whether C is nodal or cuspidal for given $P \in S$ depends on the matrix of second derivatives (or *Hessian* matrix) of f at P; this is discussed in more detail in Ex. 7.3, which proves (in characteristic $\neq 2$) that the cuspidal case must occur for some point $P \in S$. For simplicity, I prove (7.2) in the cuspidal case; in principle, the proof goes through in exactly the same way in the nodal case, but the elimination calculation gets much nastier (see Ex. 7.10). Thus assume that

$$f = X^2Z - Y^3 + gT,$$

where $g = g_2(X, Y, Z, T)$ is a quadratic form; $g(0, 0, 1, 0) \neq 0$ by nonsingularity of S at P, so I can assume that $g(0, 0, 1, 0) = 1$.

Step 2 (Statement of main claim) Consider the variable point $P_\alpha = (1, \alpha, \alpha^3, 0)$ of $C \subset S$. Any line of $\mathbb{P}^3$ through P_α meets the complementary plane $\Pi : (X = 0)$ in a point $Q = (0, Y, Z, T)$. I write out the equations for the line $P_\alpha Q$ to be contained in S in terms of α and Q; expanding $f(\lambda P_\alpha + \mu Q)$ in powers of λ and μ gives

$$P_\alpha Q \subset S \iff A(Y, Z, T) = B(Y, Z, T) = C(Y, Z, T) = 0,$$

where A, B and C are forms of degree 1, 2 and 3 in (Y, Z, T), whose coefficients involve α.

Main Claim *There exists a 'resultant' polynomial $R_{27}(\alpha)$, which is monic of degree 27 in α, such that*

$$R(\alpha) = 0 \iff A = B = C = 0 \textit{ have a common zero } (\eta : \zeta : \tau) \textit{ in } \mathbb{P}^2.$$

This statement proves (7.2), since it implies that for every root α of R, there exists a point $Q = (0 : \eta : \zeta : \tau)$ in Π for which the line $P_\alpha Q$ is contained in S. The idea here is a standard elimination calculation based on Ex. 1.10; the rest of the proof is concerned with writing out A, B and C explicitly to prove the claim.