3.4 The correspondence V

k is any field, and $A = k[X_1, \dots, X_n]$. Following an almost universal idiosyncracy of algebraic geometers[1], I write $\mathbb{A}^n_k = k^n$ for the n-dimensional affine space over k; given a polynomial $f(X_1, \dots, X_n) \in A$ and a point $P = (a_1, \dots, a_n) \in \mathbb{A}^n_k$, the element $f(a_1, \dots, a_n) \in k$ is thought of as 'evaluating the function f at P'. Define a correspondence

$$\{\text{ideals } J \subset A\} \longrightarrow \{\text{subsets } X \subset \mathbb{A}^n_k\}$$

by

$$J \longmapsto V(J) = \{P \in \mathbb{A}^n_k \mid f(P) = 0 \ \forall f \in J\}.$$

Definition A subset $X \subset \mathbb{A}^n_k$ is an *algebraic set* if $X = V(I)$ for some I. (This is the same thing as a variety, but I want to reserve the word.) Notice that by Corollary 3.3, I is finitely generated. If $I = (f_1, \dots, f_r)$ then clearly

$$V(I) = \{P \in \mathbb{A}^n_k \mid f_i(P) = 0 \quad \text{for } i = 1, \dots, r\},$$

so that an algebraic set is just a locus of points satisfying a finite number of polynomial equations.

If $I = (f)$ is a principal ideal, then I usually write $V(f)$ for $V(I)$; this is of course the same thing as $V : (f = 0)$ in the notation of §§1–2.

3.5 Definition: the Zariski topology

Proposition-Definition *The correspondence V satisfies the following formal properties:*

(i) $V(0) = \mathbb{A}^n_k$; $V(A) = \emptyset$;

(ii) $I \subset J \implies V(I) \supset V(J)$;

(iii) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$;

(iv) $V\big(\sum_{\lambda \in \Lambda} I_\lambda\big) = \bigcap_{\lambda \in \Lambda} V(I_\lambda)$.

Hence the algebraic subsets of $\mathbb{A}^n_k$ form the closed sets of a topology on $\mathbb{A}^n_k$, the Zariski topology.

The above properties are quite trivial, with the exception of the inclusion $\subset$ in (iii). For this, suppose $P \notin V(I_1) \cup V(I_2)$; then there exist $f \in I_1, g \in I_2$ such that $f(P) \neq 0, g(P) \neq 0$. So $fg \in I_1 \cap I_2$, but $fg(P) \neq 0$, and therefore $P \notin V(I_1 \cap I_2)$. Q.E.D.

The Zariski topology on $\mathbb{A}^n_k$ induces a topology on any algebraic set $X \subset \mathbb{A}^n_k$: the closed subsets of X are the algebraic subsets.

It's important to notice that the Zariski topology on a variety is very weak, and is quite different from the familiar topology of metric spaces like $\mathbb{R}^n$. As an example, a Zariski closed subset of $\mathbb{A}^1_k$ is either the whole of $\mathbb{A}^1_k$ or is finite; see Ex. 3.12 for a description of the Zariski topology on $\mathbb{A}^2_k$. If $k = \mathbb{R}$ or $\mathbb{C}$ then Zariski closed sets are also closed for the ordinary topology, since polynomial functions are continuous. In fact they're very special open or closed subsets: a nonempty Zariski open subset of $\mathbb{R}^n$ is the complement of a subvariety, so automatically dense in $\mathbb{R}^n$.

The Zariski topology may cause trouble to some students; since it is only being used as a language, and has almost no content, the difficulty is likely to be psychological rather than technical.

[1] $\mathbb{A}^n$ is thought of as a variety, whereas k^n is just a point set. Think of this as pure pedantry if you like; compare (4.6) below, as well as (8.3).

3.6 The correspondence I

As a kind of inverse to V there is a correspondence

$$\big\{\text{ideals } J \subset A\big\} \xleftarrow{\;I\;} \big\{\text{subsets } X \subset \mathbb{A}^n_k\big\}$$

by

$$I(X) = \big\{f \in k[V] \bigm| f(P) = 0 \ \forall P \in X\big\} \longleftarrow\!\shortmid X.$$

That is, I takes a subset X to the ideal of functions vanishing on it.

Proposition *(a)* $X \subset Y \implies I(X) \supset I(Y)$*;*

(b) for any subset $X \subset \mathbb{A}^n_k$*, I have* $X \subset V(I(X))$*, with equality if and only if* X *is an algebraic set;*

(c) for $J \subset A$*, I have* $J \subset I(V(J))$*; the inclusion may well be strict.*

Proof (a) is trivial. The two inclusion signs in (b) and (c) are tautologous: if $I(X)$ is defined as the set of functions vanishing at all points of X, then for any point of X, all the functions of $I(X)$ vanish at it. And indeed conversely, if not more so, just as I was about to say myself, Piglet.

The remaining part of (b) is easy: if $X = V(I(X))$ then X is certainly an algebraic set, since it's of the form $V(\text{ideal})$. Conversely, if $X = V(I_0)$ is an algebraic set, then $I(X)$ contains at least I_0, so $V(I(X)) \subset V(I_0) = X$.

There are two different ways in which the inclusion $J \subset I(V(J))$ in (c) may be strict. It's most important to understand these, since they lead directly to the correct statement of the Nullstellensatz.

Example 1 Suppose that the field k is not algebraically closed, and let $f \in k[X]$ be a nonconstant polynomial not having a root in k. Consider the ideal $J = (f) \subset k[X]$. Then $J \neq k[X]$, since $1 \notin J$. But

$$V(J) = \big\{P \in \mathbb{A}^1_k \bigm| f(P) = 0\big\} = \emptyset.$$

Therefore $I(V(J)) = k[X]$ (since any function vanishes at all points of the empty set).

So if your field is not algebraically closed, you may not get enough zeros. A rather similar example: in $\mathbb{R}^2$, the polynomial X^2+Y^2 defines the single point $P = (0,0)$, so $V(X^2+Y^2) = \{P\}$. But then many more polynomials vanish on $\{P\}$ than just the multiples of X^2+Y^2, and in fact $I(P) = (X, Y)$.

Example 2 For any $f \in k[X_1, \dots, X_n]$ and $a \geq 2$, f^a defines the same locus as f, that is $f^a(P) = 0 \iff f(P) = 0$. So $V(f^a) = V(f)$, and $f \in I(V(f^a))$, but usually $f \notin (f^a)$. The trouble here is already present in $\mathbb{R}^2$: in §1, mention was made of the 'double line' defined by $X^2 = 0$. The only meaning that can be attached to this is the line $(X = 0)$ deemed to have multiplicity 2; but the point set itself doesn't understand that it's being deemed.