**Topology** If $k$ is $\mathbb{R}$ or $\mathbb{C}$ (which it quite often is), what kind of topological space is $V$? For example, the connected components of the above cubics are obvious topological invariants.

**Singularity theory** What kind of topological space is $V$ near $P \in V$; if $f\colon V_1 \to V_2$ is a regular map between two varieties (for example, a polynomial map $\mathbb{R}^2 \to \mathbb{R}$), what kind of topology and geometry does $f$ have near $P \in V_1$?

## 0.2 Specific calculations versus general theory

There are two possible approaches to studying varieties:

**Particular** Given specific polynomials $f_i$, we can often understand the variety $V$ by explicit tricks with the $f_i$; this is fun if the dimension $n$ and the degrees of the $f_i$ are small, or the $f_i$ are specially nice, but things get progressively more complicated, and there rapidly comes a time when mere ingenuity with calculations doesn't tell you much about the problem.

**General** The study of properties of $V$ leads at once to basic notions such as regular functions on $V$, nonsingularity and tangent planes, the dimension of a variety: the idea that curves such as the above cubics are 1-dimensional is familiar from elementary Cartesian geometry, and the pictures suggest at once what singularity should mean.

Now a basic problem in giving an undergraduate algebraic geometry course is that an adequate treatment of the 'general' approach involves so many definitions that they fill the entire course and squeeze out all substance. Therefore one has to compromise, and my solution is to cover a small subset of the general theory, with constant reference to specific examples. These notes therefore contain only a fraction of the 'standard bookwork' which would form the compulsory core of a 3–year undergraduate math course devoted entirely to algebraic geometry. On the other hand, I hope that each section contains some exercises and worked examples of substance.

## 0.3 Rings of functions and categories of geometry

The specific flavour of algebraic geometry comes from the use of only polynomial functions (together with rational functions); to explain this, if $U \subset \mathbb{R}^2$ is an open interval, one can reasonably consider the following rings of functions on $U$:

- $C^0(U)$ = all continuous functions $f\colon U \to \mathbb{R}$;
- $C^\infty(U)$ = all smooth functions (that is, differentiable to any order);
- $C^\omega(U)$ = all analytic functions (that is, convergent power series);
- $\mathbb{R}[X]$ = the polynomial ring, viewed as polynomial functions on $U$.

There are of course inclusions $\mathbb{R}[X] \subset C^\omega(U) \subset C^\infty(U) \subset C^0(U)$.

These rings of functions correspond to some of the important categories of geometry: $C^0(U)$ to the topological category, $C^\infty(U)$ to the differentiable category (differentiable manifolds), $C^\omega$ to real analytic geometry, and $\mathbb{R}[X]$ to algebraic geometry. The point I want to make here is that each of these inclusion signs represents an absolutely *huge* gap, and that this leads to the main characteristics of geometry in the different categories. Although it's not stressed very much in school and first year university calculus, any reasonable way of measuring $C^0(U)$ will reveal that the differentiable functions have measure 0 in the continuous functions (so if you pick a continuous function at random then with probability 1 it will be nowhere differentiable, like Brownian motion). The gap between $C^\omega(U)$ and $C^\infty(U)$ is exemplified by the behaviour of $\exp(-1/x^2)$, the standard function which is differentiable infinitely often, but for which the Taylor series (at 0) does not converge to $f$; using this, you can easily build a $C^\infty$ 'bump function' $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ if $|x| \leq 0.9$, and $f(x) = 0$ if $|x| \geq 1$: In contrast, an analytic function on $U$ extends (as a convergent

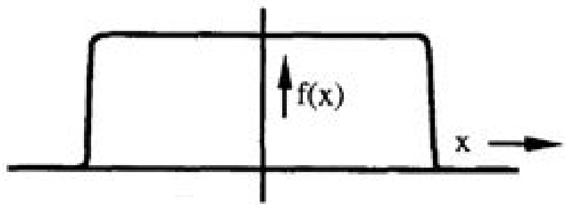


Figure 2: A $C^\infty$ bump function.

power series) to an analytic function of a complex variable on a suitable domain in $\mathbb{C}$, so that (using results from complex analysis), if $f \in C^\omega(U)$ vanishes on a real interval, it must vanish identically. This is a kind of 'rigidity' property which characterises analytic geometry as opposed to differential topology.

## 0.4 Geometry from polynomials

There are very few polynomial functions: the polynomial ring $\mathbb{R}[X]$ is just a countable dimensional $\mathbb{R}$-vector space, whereas $C^\omega(U)$ is already uncountable. Even allowing rational functions – that is, extending $\mathbb{R}[X]$ to its field of fractions $\mathbb{R}(X)$ – doesn't help much. (2.2) will provide an example of the characteristic rigidity of the algebraic category. The fact that it is possible to construct a geometry using only this set of functions is itself quite remarkable. Not surprisingly, there are difficulties involved in setting up this theory:

**Foundations via commutative algebra** Topology and differential topology can rely on the whole corpus of $\varepsilon$-$\delta$ analysis taught in a series of 1st and 2nd year undergraduate courses; to do algebraic geometry working only with polynomial rings, we need instead to study rings such as the polynomial ring $k[X_1, \ldots, X_n]$ and their ideals. In other words, we have to develop commutative algebra in place of calculus. The Nullstellensatz (§3 below) is a typical example of a statement having direct intuitive geometric content (essentially, "different ideals of functions in $k[X_1, \ldots, X_n]$ define different varieties $V \subset k^n$") whose proof involves quite a lengthy digression through finiteness conditions in commutative algebra.

**Rational maps and functions** Another difficulty arising from the decision to work with polynomials is the necessity of introducing 'partially defined functions'; because of the 'rigidity' hinted