Prerequisites for this course:

Algebra: Quadratic forms, easy properties of commutative rings and their ideals, principal ideal domains and unique factorisation.

Galois Theory: Fields, polynomial rings, finite extensions, algebraic versus transcendental extensions, separability.

Topology and geometry: Definition of topological space, projective space $\mathbb{P}^n$ (but I'll go through it again in detail).

Calculus in $\mathbb{R}^n$: Partial derivatives, implicit function theorem (but I'll remind you of what I need when we get there).

Commutative algebra: Other experience with commutative rings is desirable, but not essential.

Course relates to:

Complex Function Theory An algebraic curve over $\mathbb{C}$ is a 1-dimensional complex manifold, and regular functions on it are holomorphic, so that this course is closely related to complex function theory, even if the relation is not immediately apparent.

Algebraic Number Theory For example the relation with Fermat's Last Theorem.

Catastrophe Theory Catastrophes are singularities, and are essentially always given by polynomial functions, so that the analysis of the geometry of the singularities is pure algebraic geometry.

Commutative Algebra Algebraic geometry provides motivation for commutative algebra, and commutative algebra provides technical support for algebraic geometry, so that the two subjects enrich one another.

Exercises to Chapter 0

0.1 (a) Show that for fixed values of (y, z), x is a repeated root of $x^3 + xy + z = 0$ if and only if $x = -3z/2y$ and $4y^3 + 27z^2 = 0$;

(b) there are 3 distinct roots if and only if $4y^3 + 27z^2 < 0$;

(c) sketch the surface $S : (x^3 + xy + z = 0) \subset \mathbb{R}^3$ and its projection onto the (y, z)-plane;

(d) now open up any book or article on catastrophe theory and compare.

0.2 Let $f \in \mathbb{R}[X, Y]$ and let $C : (f = 0) \subset \mathbb{R}^2$; say that $P \in C$ is *isolated* if there is an $\varepsilon > 0$ such that $C \cap B(P, \varepsilon) = P$. Show by example that C can have isolated points. Prove that if $P \in C$ is an isolated point then $f\colon \mathbb{R}^2 \to \mathbb{R}$ must have a max or min at P, and deduce that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at P. This proves that an isolated point of a real curve is singular.

0.3 *Cubic curves:*

(i) Draw the graph of $y = 4x^3 + 6x^2$ and its intersection with the horizontal lines $y = t$ for integer values of $t \in [-1, 3]$;

(ii) draw the cubic curves $y^2 = 4x^3 + 6x^2 - t$ for the same values of t.

Books

Most of the following are textbooks at a graduate level, and some are referred to in the text:

W. Fulton, Algebraic curves, Springer. (This is the most down-to-earth and self-contained of the graduate texts; Ch. 1–6 are quite well suited to an undergraduate course, although the material is somewhat dry.)

I.R. Shafarevich, Basic algebraic geometry, Springer. (A graduate text, but Ch. I, and SII.1 are quite suitable material.)

P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley. (Gives the complex analytic point of view.)

David Mumford, Algebraic geometry I, Complex projective varieties, Springer.

D. Mumford, Introduction to algebraic geometry, Harvard notes. (Not immediately very readable, but goes directly to the main points; many algebraic geometers of my generation learned their trade from these notes. Recently reissued as Springer LNM 1358, and therefore no longer a little red book.)

K. Kendig, Elementary algebraic geometry, Springer. (Treats the relation between algebraic geometry and complex analytic geometry.)

R. Hartshorne, Algebraic geometry, Springer. (This is the professional's handbook, and covers much more advanced material; Ch. I is an undergraduate course in bare outline.)

M. Berger, Geometry I and II, Springer. (Some of the material of the sections on quadratic forms and quadric hypersurfaces in II is especially relevant.)

M.F. Atiyah and I.G. Macdonald, Commutative algebra, Addison-Wesley. (An invaluable textbook.)

E. Kunz, Introduction to commutative algebra and algebraic geometry, Birkhäuser.

H. Matsumura, Commutative ring theory, Cambridge. (A more detailed text on commutative algebra.)

D. Mumford, Curves and their Jacobians, Univ. of Michigan Press. (Colloquial lectures, going quite deep quite fast.)

C.H. Clemens, A scrapbook of complex curves, Plenum. (Lots of fun.).

E. Brieskorn and H. Knörrer, Plane algebraic curves, Birkhäuser.

A. Beauville, Complex algebraic surfaces, LMS Lecture Notes, Cambridge.

J. Kollár, The structure of algebraic threefolds: An introduction to Mori's program, Bull. Amer. Math. Soc. 17 (1987), 211–273. (A nicely presented travel brochure to one active area of research. Mostly harmless.)

J.G. Semple and L. Roth, Introduction to algebraic geometry, Oxford. (A marvellous old book, full of information, but almost entirely lacking in rigour.)

J.L. Coolidge, Treatise on algebraic plane curves, Oxford and Dover.