The usual picture is

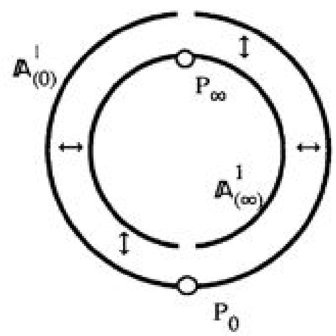

Figure 5.1: $\mathbb{P}^1$ glued from two $\mathbb{A}^1$s

(the arrows $\leftrightarrow$ denote glueing).

It's important to understand that *these varieties are strictly bigger than any affine variety.* In fact, with the natural notion of morphism (to be introduced shortly), it can be seen that there are no nonconstant morphisms $\mathbb{P}^1 \to \mathbb{A}^n$ or $C \to \mathbb{A}^n$ for any n (see Ex. 5.1 and Ex. 5.12, and the discussion in (8.10)).

One solution to this problem is to define the notion of 'abstract variety' V as a union $V = \bigcup V_i$ of affine varieties, modulo suitable glueing. By analogy with the definition of manifolds in topology, this is an attractive idea, but it leads to many more technical difficulties. Using projective varieties sidesteps these problems by working in the ready-made ambient space $\mathbb{P}^n$, so that (apart from a little messing about with homogeneous polynomials) they are not much harder to study than affine varieties. In fact, although this may not be clear at an elementary level, projective varieties to a quite remarkable extent provide a natural framework for studying varieties (this is briefly discussed from a more advanced point of view in (8.11)).

5.1 Graded rings and homogeneous ideals

Definition *A polynomial $f \in k[X_0, \dots, X_n]$ is* homogeneous of degree d *if*

$$f = \sum a_{i_0 \dots i_n} X_0^{i_0} \cdots X_n^{i_n} \text{ with } a_{i_0 \dots i_n} \neq 0 \text{ only if } i_0 + \cdots + i_n = d.$$

Any $f \in k[X_0, \dots, X_n]$ has a unique expression $f = f_0 + f_1 + \cdots + f_N$ in which f_d is homogeneous of degree d for each $d = 0, 1, \dots, N$.

Proposition *If f is homogeneous of degree d then*

$$f(\lambda X_0, \dots, \lambda X_n) = \lambda^d f(X_0, \dots, X_n) \quad \text{for all } \lambda \in k;$$

if k is an infinite field then the converse also holds.

Proof Try it and see.

Definition *An ideal $I \subset k[X_0, \dots, X_n]$ is* homogeneous *if for all $f \in I$, the homogeneous decomposition $f = f_0 + f_1 + \cdots + f_N$ of f satisfies $f_i \in I$ for all i.*

It is equivalent to say that I is generated by (finitely many) homogeneous polynomials.

5.2 The homogeneous V-I correspondences

Let $\mathbb{P}^n_k$ be n-dimensional projective space over a field k, with $X_0, \dots, X_n$ as homogeneous coordinates. Then $f \in k[X_0, \dots, X_n]$ is *not* a function on $\mathbb{P}^n_k$: by definition, $\mathbb{P}^n_k = k^{n+1} \setminus \{0\}/\sim$, where $\sim$ is the equivalence relation given by $(X_0, \dots, X_n) \sim (\lambda X_0, \dots, \lambda X_n)$ for $\lambda \in k \setminus \{0\}$; f is a function on k^{n+1}. Nevertheless, for $P \in \mathbb{P}^n$, the condition $f(P) = 0$ is well defined provided that f is homogeneous: suppose $P = (X_0 : \cdots : X_n)$, so that $(X_0, \dots, X_n)$ is a representative in $k^{n+1} \setminus \{0\}$ of the equivalence class of P. Then since $f(\lambda X) = \lambda^d f(X)$, if $f(X_0, \dots, X_n) = 0$ then also $f(\lambda X_0, \dots, \lambda X_n) = 0$, so that the condition $f(P) = 0$ is independent of the choice of representative. With this in mind, define as before correspondences

$$\big\{\text{homog. ideals } J \subset k[X_0, \dots, X_n]\big\} \xleftrightarrow{V\text{-}I} \big\{\text{subsets } X \subset \mathbb{P}^n_k\big\}$$

by

$$V(J) = \big\{P \in \mathbb{P}^n_k \bigm| f(P) = 0 \ \forall \text{ homogeneous } f \in J\big\}$$

and

$$I(X) = \big\{f \in k[X_0, \dots, X_n] \bigm| f(P) = 0 \text{ for all } P \in X\big\}.$$

As an exercise, check that you understand why $I(X)$ is a homogeneous ideal.

The correspondences V and I satisfy the same formal properties as the affine V and I correspondences introduced in §3 (for example $V(J_1 + J_2) = V(J_1) \cap V(J_2)$). A subset of the form $V(I)$ is an *algebraic subset* of $\mathbb{P}^n_k$, and as in the affine case, $\mathbb{P}^n_k$ has a *Zariski topology* in which the closed sets are the algebraic subsets.

5.3 Projective Nullstellensatz

As with the affine correspondences, it is purely formal that $I(V(J)) \supset \operatorname{rad} J$ for any ideal J, and that for an algebraic set, $V(I(X)) = X$. There's just one point where care is needed: the trivial ideal $(1) = k[X_0, \dots, X_n]$ (the whole ring) defines the empty set in k^{n+1}, hence also in $\mathbb{P}^n_k$, which is as it should be; however, the ideal $(X_0, \dots, X_n)$ defines $\{0\}$ in k^{n+1}, which also corresponds to the empty set in $\mathbb{P}^n_k$. The ideal $(X_0, \dots, X_n)$ is an awkward (empty-set theoretical) exception to several statements in the theory, and is traditionally known as the 'irrelevant ideal'.

The homogeneous version of the Nullstellensatz thus becomes:

Theorem *Assume that k is an algebraically closed field. Then*

(i) $V(J) = \emptyset \iff \operatorname{rad} J \supset (X_0, \dots, X_n)$;

(ii) if $V(J) \neq \emptyset$ *then* $I(V(J)) = \operatorname{rad} J$.