The addition law is a map $\varphi\colon C \times C \to C$ given by $(A, B) \mapsto A + B$. By (i), φ is continuous, and hence so are the two maps (sorry!)

$$f = \varphi \circ (\varphi \times \mathrm{id}_C) \quad \text{and} \quad g = \varphi \circ (\mathrm{id}_C \times \varphi)\colon C \times C \times C \to C$$

given by $(A, B, C) \mapsto (A + B) + C$ and $A + (B + C)$. Also, by (ii), the subset $U \subset C \times C \times C$ consisting of triples (A, B, C) for which the 9 points of the construction are distinct is dense; by the above argument, f and g thus coincide on U, and since they are continuous, they coincide everywhere. Q.E.D.

Remark The continuity argument as it stands involves the topology of $\mathbb{C}$, and is thus not purely algebraic. In fact the addition map φ is a morphism of varieties $\varphi\colon C \times C \to C$, as will be proved later (see (4.14)), and the remainder of the argument can also be reformulated in this purely algebraic form: the subset of $C \times C \times C$ for which the 9 points are distinct is a dense open set for the Zariski topology, and two morphisms which coincide on a dense open set coincide everywhere. (I hope that this remark can provide useful motivation for the rest of the course; if you find it confusing, just ignore it for the moment.)

2.11 Pascal's Theorem (the mystic hexagon)

The diagram consists of a hexagon $ABCDEF$ in $\mathbb{P}^2_k$ with pairs of opposite sides extended until

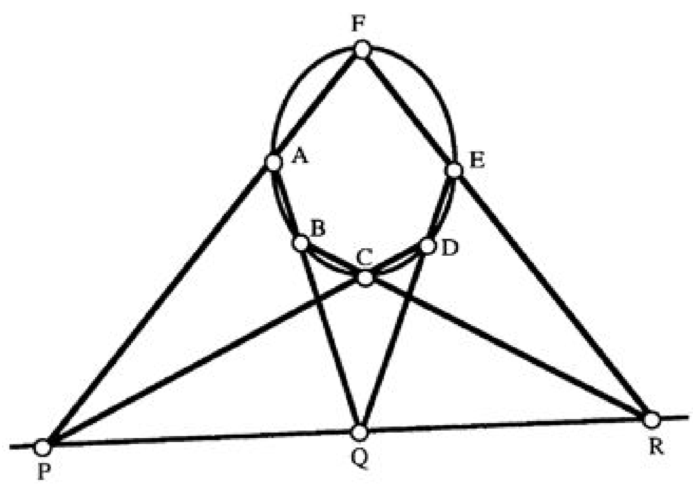

Figure 2.4: The mystic hexagon

they meet in points P, Q, R. Assume that the nine points and the six lines of the diagram are all distinct; then

$$ABCDEF \text{ are conconic} \iff PQR \text{ are collinear.}$$

This famous theorem is a rather similar application of (2.7), and is given just for fun; of course, other proofs are possible, see any text on geometry, for example [Berger, 16.2.10 and 16.8.3–5].

Proof In the diagram, consider the two triples of lines

$$L_1 : PAF, \quad L_2 : QDE, \quad L_3 : RBC,$$

and

$$M_1 : PCD, \quad M_2 : QAB, \quad M_3 : REF;$$

let $C_1 = L_1 + L_2 + L_3$ and $C_2 = M_1 + M_2 + M_3$. Now I'm all set to apply (2.7), since clearly C_1 and C_2 are two cubics such that

$$C_1 \cap C_2 = \{A, B, C, D, E, F, P, Q, R\}.$$

Suppose PQR are collinear, with $L = PQR$; let Γ be the conic through $ABCDE$ (the existence and unicity of which is provided by Proposition 1.11). Then by construction, $L+\Gamma$ is a cubic passing through the 8 points A, B, C, D, E, P, Q, R, and by (2.7), it must contain F; by assumption, $F \notin L$, so that necessarily $F \in \Gamma$, proving that the six points are conconic.

Now conversely, suppose that $ABCDEF$ are on a conic Γ, and let $L = PQ$; then $L + \Gamma$ is a cubic passing through A, B, C, D, E, F, P, Q, so by (2.7) it must pass through R. Now R can't be on the conic Γ (since otherwise Γ is a line pair, and some of the 6 lines of the diagram must coincide), so $R \in L$, that is, PQR are collinear. Q.E.D.

2.12 Inflexion, normal form

Every cubic in $\mathbb{P}^2_{\mathbb{R}}$ or $\mathbb{P}^2_{\mathbb{C}}$ can be put in the normal form

$$C : Y^2Z = X^3 + aXZ^2 + bZ^3, \qquad (**)$$

or in the affine form

$$y^2 = x^3 + ax + b.$$

Now consider the above curve C; where does it meet the line at infinity $L : (Z = 0)$? That's easy, just substitute $Z = 0$ in the defining polynomial $F = -Y^2Z + X^3 + aXZ^2 + bZ^3$ to get $F|L = X^3$; this means that $F|L$ has a triple zero at $P = (0, 1, 0)$. To see what this means geometrically, set $Y = 1$, to get the equation in affine coordinates (x, z) around $(0, 1, 0)$:

$$z = x^3 + axz^2 + bz^3.$$

This curve is approximated to a high degree of accuracy by $z = x^3$: the behaviour is described by

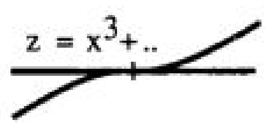

Figure 2.5: Inflexion point

saying that C has an *inflexion point* at $(0, 1, 0)$. More generally, an inflexion point P on a curve C is defined by the condition that there is a line $L \subset \mathbb{P}^2_k$ such that $F|L$ has a zero of multiplicity ≥ 3 at