

3.6 The correspondence I

As a kind of inverse to V there is a correspondence

$$\{\text{ideals } J \subset A\} \xleftarrow{I} \{\text{subsets } X \subset \mathbb{A}_k^n\}$$

by

$$I(X) = \{f \in k[V] \mid f(P) = 0 \ \forall P \in X\} \longleftrightarrow X.$$

That is, I takes a subset X to the ideal of functions vanishing on it.

Proposition (a) $X \subset Y \implies I(X) \supset I(Y)$;

(b) for any subset $X \subset \mathbb{A}_k^n$, I have $X \subset V(I(X))$, with equality if and only if X is an algebraic set;

(c) for $J \subset A$, I have $J \subset I(V(J))$; the inclusion may well be strict.

Proof (a) is trivial. The two inclusion signs in (b) and (c) are tautologous: if $I(X)$ is defined as the set of functions vanishing at all points of X , then for any point of X , all the functions of $I(X)$ vanish at it. And indeed conversely, if not more so, just as I was about to say myself, Piglet.

The remaining part of (b) is easy: if $X = V(I(X))$ then X is certainly an algebraic set, since it's of the form $V(\text{ideal})$. Conversely, if $X = V(I_0)$ is an algebraic set, then $I(X)$ contains at least I_0 , so $V(I(X)) \subset V(I_0) = X$.

There are two different ways in which the inclusion $J \subset I(V(J))$ in (c) may be strict. It's most important to understand these, since they lead directly to the correct statement of the Nullstellensatz.

Example 1 Suppose that the field k is not algebraically closed, and let $f \in k[X]$ be a nonconstant polynomial not having a root in k . Consider the ideal $J = (f) \subset k[X]$. Then $J \neq k[X]$, since $1 \notin J$. But

$$V(J) = \{P \in \mathbb{A}_k^1 \mid f(P) = 0\} = \emptyset.$$

Therefore $I(V(J)) = k[X]$ (since any function vanishes at all points of the empty set).

So if your field is not algebraically closed, you may not get enough zeros. A rather similar example: in \mathbb{R}^2 , the polynomial $X^2 + Y^2$ defines the single point $P = (0, 0)$, so $V(X^2 + Y^2) = \{P\}$. But then many more polynomials vanish on $\{P\}$ than just the multiples of $X^2 + Y^2$, and in fact $I(P) = (X, Y)$.

Example 2 For any $f \in k[X_1, \dots, X_n]$ and $a \geq 2$, f^a defines the same locus as f , that is $f^a(P) = 0 \iff f(P) = 0$. So $V(f^a) = V(f)$, and $f \in I(V(f^a))$, but usually $f \notin (f^a)$. The trouble here is already present in \mathbb{R}^2 : in §1, mention was made of the 'double line' defined by $X^2 = 0$. The only meaning that can be attached to this is the line $(X = 0)$ deemed to have multiplicity 2; but the point set itself doesn't understand that it's being deemed.

3.7 Irreducible algebraic set

An algebraic set $X \subset \mathbb{A}_k^n$ is *irreducible* if there does not exist a decomposition

$$X = X_1 \cup X_2 \quad \text{with} \quad X_1, X_2 \subsetneq X$$

of X as a union of two strict algebraic subsets. For example, the algebraic subset $V(xy) \subset \mathbb{A}_k^2$ is the locus consisting of the two coordinate axes, and is obviously the union of $V(x)$ and $V(y)$, hence reducible.

Proposition (a) Let $X \subset \mathbb{A}_k^n$ be an algebraic set and $I(X)$ the corresponding ideal; then

$$X \text{ is irreducible} \iff I(X) \text{ is prime.}$$

(b) Any algebraic set X has a (unique) expression

$$X = X_1 \cup \cdots \cup X_r \tag{*}$$

with X_i irreducible and $X_i \not\subset X_j$ for $i \neq j$.

The X_i in (*) are the irreducible components of X .

Proof (a) In fact I prove that X is reducible $\iff I(X)$ is not prime.

(\implies) Suppose $X = X_1 \cup X_2$ with $X_1, X_2 \subsetneq X$ algebraic subsets. Then $X_1 \subsetneq X$ means that there exists $f_1 \in I(X_1) \setminus I(X)$, and similarly $X_2 \subsetneq X$ gives $f_2 \in I(X_2) \setminus I(X)$. Now the product $f_1 f_2$ vanishes at all points of X , and so $f_1 f_2 \in I(X)$. Therefore $I(X)$ is not prime.

(\impliedby) Suppose that $I(X)$ is not prime; then there exist $f_1, f_2 \notin I(X)$ such that $f_1 f_2 \in I(X)$. Let $I_1 = (I(X), f_1)$ and $V(I_1) = X_1$; then $X_1 \subsetneq X$ is an algebraic subset; similarly, setting $I_2 = (I(X), f_2)$ and $V(I_2) = X_2$ gives $X_2 \subsetneq X$. But $X \subset X_1 \cup X_2$, since for all $P \in X$, $f_1 f_2(P) = 0$ implies that either $f_1(P) = 0$ or $f_2(P) = 0$.

(b) First of all, I establish the following proposition: the algebraic subsets of \mathbb{A}_k^n satisfy the descending chain condition, that is, every chain

$$X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$$

eventually stops with $X_N = X_{N+1} = \cdots$. This is because

$$I(X_1) \subset I(X_2) \subset \cdots \subset I(X_n) \subset \cdots$$

is an ascending chain of ideals of A , and this stops, giving $X_N = X_{N+1} = \cdots$. Thus just as in (3.1),

$$\begin{aligned} &\text{any nonempty set } \Sigma \text{ of algebraic} \\ &\text{subsets of } \mathbb{A}_k^n \text{ has a minimal element.} \end{aligned} \tag{!}$$

Now to prove (b), let Σ be the set of algebraic subsets of \mathbb{A}_k^n which do not have a decomposition (*). If $\Sigma = \emptyset$ then (b) is proved. On the other hand, if $\Sigma \neq \emptyset$ then by (!), there must be a minimal element $X \in \Sigma$, and this leads speedily to one of two contradictions: if X is irreducible, then $X \notin \Sigma$, a contradiction; if X is reducible, then $X = X_1 \cup X_2$, with $X_1, X_2 \subsetneq X$, so that by minimality of $X \in \Sigma$, I get $X_1, X_2 \notin \Sigma$. So each of X_1, X_2 has a decomposition (*) as a union of irreducibles, and