# Part I

# Playing with plane curves

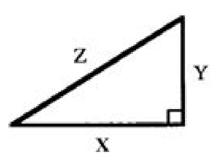
# Chapter 1

# Plane conics

I start by studying the geometry of conics as motivation for the projective plane $\mathbb{P}^2$. Projective geometry is usually mentioned in 2nd year undergraduate geometry courses, and I recall some of the salient features, with some emphasis on homogeneous coordinates, although I completely ignore the geometry of linear subspaces and the 'cross-ratio'. The most important aim for the student should be to grasp the way in which geometric ideas (for example, the idea that 'points at infinity' correspond to asymptotic directions of curves) are expressed in terms of coordinates. The interplay between the intuitive geometric picture (which tells you what you should be expecting), and the precise formulation in terms of coordinates (which allows you to cash in on your intuition) is a fascinating aspect of algebraic geometry.

## 1.1 Example of a parametrised curve

Pythagoras' Theorem says that, in the diagram




$$X^2 + Y^2 = Z^2,$$

so $(3,4,5)$ and $(5,12,13)$, as every ancient Egyptian knew. How do you find all integer solutions? The equation is homogeneous, so that $x = X/Z$, $y = Y/Z$ gives the circle $C : (x^2 + y^2 = 1) \subset \mathbb{R}^2$, which can easily be seen to be parametrised as

$$x = \frac{2\lambda}{\lambda^2+1}, \quad y = \frac{\lambda^2-1}{\lambda^2+1}, \quad \text{where} \quad \lambda = \frac{x}{1-y};$$

so this gives all solutions:

$$X = 2\ell m, \quad Y = \ell^2 - m^2, \quad Z = \ell^2 + m^2 \quad \text{with } \ell, m \in \mathbb{Z} \text{ coprime}$$