Hence, in case (i), $f = XYT + Zg$, with g quadratic, and in terms of the expression $(*)$, this means that $B = T + aZ$, and $Z \mid A, C, D, E, F$. Therefore, modulo terms divisible by Z^2,

$$\Delta \equiv -T^2 F \mod Z^2.$$

In addition, the point $P = (0,0,0,1) \in S$, and nonsingularity at P means that F must contain the term ZT^2 with nonzero coefficient. In particular, Z^2 does not divide F. Therefore $(Z = 0)$ is a simple root of Δ.

Case (ii) is a similar calculation (see Ex. 7.1).

Corollary 7.4 *1. There exist two disjoint lines $\ell, m \subset S$.*

2. S is rational (that is, birational to $\mathbb{P}^2$, see (5.9)).

Proof (a) By (7.3, ii), just take ℓ_1 and ℓ_2.

(b) Consider two disjoint lines $\ell, m \subset S$, and define rational maps

$$\varphi\colon S \dashrightarrow \ell \times m \quad \text{and} \quad \psi\colon \ell \times m \dashrightarrow S$$

as follows. If $P \in \mathbb{P}^3 \setminus (\ell \cup m)$ then there exists a unique line n through P which meets both ℓ and m:

$$P \in n, \quad \text{and} \quad \ell \cap n \neq \emptyset,\ m \cap n \neq \emptyset.$$

Set $\Phi(P) = (\ell \cap n, m \cap n) \in \ell \times m$. This defines a morphism

$$\Phi\colon \mathbb{P}^3 \setminus (\ell \cup m) \to \ell \times m,$$

whose fibre above $(Q, R) \in \ell \times m$ is the line QR of $\mathbb{P}^3$. Define $\varphi\colon S \dashrightarrow \ell \times m$ as the restriction to S of Φ.

Conversely, for $(Q, R) \in \ell \times m$, let n be the line $n = QR$ in $\mathbb{P}^3$. By (7.3), there are only finitely many lines of S meeting ℓ, so that for almost all values of (Q, R), n intersects S in 3 points $\{P, Q, R\}$, of which Q and R are the given points on ℓ and m. Thus define $\psi\colon \ell \times m \dashrightarrow S$ by $(Q, R) \mapsto P$; then ψ is a rational map, since the ratios of coordinates of P are rational functions of those of Q, R.

Obviously φ and ψ are mutual inverses. Q.E.D.

7.5 Finding all the lines of S

I want to find all the lines of S in terms of the configuration given by Proposition 7.3 of a line ℓ and 5 disjoint pairs (ℓ_i, ℓ_i'). Any other line $n \subset S$ must meet exactly one of ℓ_i and ℓ_i' for $i = 1, \dots, 5$: this is because in $\mathbb{P}^3$, n meets the plane Π_i, and $\Pi_i \cap S = \ell \cup \ell_i \cup \ell_i'$; also, n cannot meet both ℓ_i and ℓ_i', since this would contradict (7.1, a). The key to sorting out the remaining lines is the following lemma, which tells us that n is uniquely determined by which of the ℓ_i and ℓ_i' it meets. Let me say that a line n is a *transversal* of a line ℓ if $\ell \cap n \neq \emptyset$.

Lemma *If $\ell_1, \dots, \ell_4 \subset \mathbb{P}^3$ are disjoint lines then*

either *all 4 lines ℓ_i lie on a smooth quadric $\ell_1, \dots, \ell_4 \subset Q \subset \mathbb{P}^3$; and then they have an infinite number of common transversals;*

or *the 4 lines ℓ_i do not lie on any quadric $\ell_1, \dots, \ell_4 \not\subset Q$; and then they have either 1 or 2 common transversals.*