where $\sim$ is the equivalence relation

$$\frac{g}{h}\sim\frac{g'}{h'} \iff h'g - g'h \in I(V).$$

$k(V)$ is the (rational) *function field* of V.

The following definitions are just as in the affine case. For $f \in k(V)$ and $P \in V$, say that f is *regular* at P if there exists an expression $f = g/h$, with g, h homogeneous polynomials of the same degree, such that $h(P) \neq 0$. Write

$$\operatorname{dom} f = \{P \in V \mid f \text{ is regular at } P\}$$

and

$$\mathcal{O}_{V,P} = \{f \in k(V) \mid f \text{ is regular at } P\}.$$

Clearly, $\operatorname{dom} f \subset V$ is a dense Zariski open set in V (the proof is as in (4.8, I), and $\mathcal{O}_{V,P} \subset k(V)$ is a subring.

5.5 Affine covering of a projective variety

Let $V \subset \mathbb{P}^n$ be an irreducible algebraic set, and suppose for simplicity that $V \not\subset (X_i = 0)$ for any i. We know that $\mathbb{P}^n$ is covered by $n+1$ affine pieces $\mathbb{A}^n_{(i)}$, with affine (inhomogeneous) coordinates

$$X_0^{(i)}, \dots, X_{i-1}^{(i)}, X_{i+1}^{(i)}, \dots, X_n^{(i)}, \quad \text{where} \quad X_j^{(i)} = X_j/X_i \quad \text{for } j \neq i.$$

Write $V_{(i)} = V \cap \mathbb{A}^n_{(i)}$. Then $V_{(i)} \subset \mathbb{A}^n_{(i)}$ is clearly an affine algebraic set, because

$$\begin{aligned} V_{(0)} \ni P = (1, x_1^{(0)}, \dots, x_n^{(0)}) & \\ & \iff f(1, x_1^{(0)}, \dots, x_n^{(0)}) = 0 \quad \text{for all homogeneous } f \in I(V), \end{aligned}$$

which is a set of polynomial relations in the coordinates $(x_1^{(0)}, \dots, x_n^{(0)})$ of P. For clarity, I have taken $i = 0$ in the argument, and will continue to do so whenever convenient. The reader should remember that the same result applies to any of the other affine pieces $V_{(i)}$. The $V_{(i)}$ are called *standard affine pieces* of V.

Proposition *(i) The correspondence $V \mapsto V_{(0)} = V \cap \mathbb{A}^n_{(0)}$ gives a bijection*

$$\left\{ \begin{matrix} \textit{irreducible alg.} \\ \textit{subsets } V \subset \mathbb{P}^n \end{matrix} \;\middle|\; V \not\subset (X_0 = 0) \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{irreducible alg.} \\ \textit{subsets } V_0 \subset \mathbb{A}^n_{(0)} \end{matrix} \right\};$$

the inverse correspondence is given by taking the closure in the Zariski topology.

(ii) Write $I^h(V) \subset k[X_0, \dots, X_n]$ for the homogeneous ideal of $V \subset \mathbb{P}^n$ introduced in this section and $I^a(V_{(0)}) \subset k[X_1, \dots, X_n]$ for the usual (as in §3) inhomogeneous ideal of $V_{(0)} \subset \mathbb{A}^n_{(0)}$; then $I^h(V)$ and $I^a(V_{(0)})$ are related as follows:

$$I^a = \{f(1, X_1, \dots, X_n) \mid f \in I^h(V)\},$$

and

$$I^h(V)_d = \left\{X_0^d f\left(\tfrac{X_1}{X_0}, \dots, \tfrac{X_n}{X_0}\right) \;\middle|\; f \in I^a(V_{(0)}), \textit{ with } \deg f \leq d\right\},$$

where the subscript in $I^h(V)_d$ denotes the piece of degree d.

(iii) $k(V) \cong k(V_{(0)})$*, and for* $f \in k(V)$*, the domain of* f *as a function on* $V_{(0)}$ *is* $V_{(0)} \cap \operatorname{dom} f$*.*

Proof (i) and (ii) are easy. (iii) If $g, h \in k[X_0, \dots, X_n]$ are homogeneous of degree d, and $h \notin I(V)$, then $g/h \in k(V)$ restricted to $V_{(0)}$ is the function

$$\frac{g(1, X_1/X_0, \dots, X_n/X_0)}{h(1, X_1/X_0, \dots, X_n/X_0)};$$

this defines a map $k(V) \to k(V_{(0)})$, and it's easy to see what its inverse is.

5.6 Rational maps and morphisms

Rational maps between projective (or affine) varieties are defined using $k(V)$: if $V \subset \mathbb{P}^n$ is an irreducible algebraic set, a rational map $V \dashrightarrow \mathbb{A}^m$ is a (partially defined) map given by $P \mapsto (f_1(P), \dots, f_m(P))$, where $f_1, \dots, f_m \in k(V)$. A rational map $V \dashrightarrow \mathbb{P}^m$ is defined by $P \mapsto (f_0(P) : f_1(P) : \cdots : f_m(P))$ where $f_0, f_1, \dots, f_m \in k(V)$. Notice that if $g \in k(V)$ is a nonzero element, then $gf_0, gf_1, \dots, gf_m$ defines the same rational map. Therefore (assuming that V does not map into the smaller projective space $(X_0 = 0)$), it would be possible to assume throughout that $f_0 = 1$.

Clearly then, there is a bijection between the two sets

$$\left\{\text{rational maps } f\colon V \dashrightarrow \mathbb{A}^m \subset \mathbb{P}^m\right\}$$

and

$$\left\{\text{rational maps } f\colon V \dashrightarrow \mathbb{P}^m \;\middle|\; f(V) \not\subset (X_0 = 0)\right\},$$

since either kind of maps is given by m elements $f_i \in k(V)$.

Definition A rational map $f\colon V \dashrightarrow \mathbb{P}^m$ is *regular* at $P \in V$ if there exists an expression $f = (f_0, f_1, \dots, f_m)$ such that

(i) each of $f_0, \dots, f_m$ is regular at P; and

(ii) at least one $f_i(P) \neq 0$.

The second condition is required here in order that the ratio between the f_i is defined at P. If f is regular at P (as before, this is also expressed $P \in \operatorname{dom} f$) then $f\colon U \to \mathbb{A}^m_{(i)} \subset \mathbb{P}^m$ is a morphism for a suitable open neighbourhood $P \in U \subset V$: just take $U = \bigcap_j \operatorname{dom}(f_j/f_i)$ where $f_i(P) \neq 0$; then f is the morphism given by $\left\{f_j/f_i\right\}_{j=0,1,\dots,m}$.

If $U \subset V$ is an open subset of a projective variety V then a *morphism* $f\colon U \to W$ is a rational map $f\colon V \dashrightarrow W$ such that $\operatorname{dom} f \supset U$. So a morphism is just a rational map that is everywhere regular on U.