(or each divided by 2 if ℓ, m are both odd). Note that the equation is homogeneous, so that if (X,Y,Z) is a solution, then so is $(\lambda X, \lambda Y, \lambda Z)$.

Maybe the parametrisation was already familiar from school geometry, and in any case, it's easy to verify that it works. However, if I didn't know it already, I could have obtained it by an easy geometric argument, namely linear projection from a given point: $P = (0,1) \in C$, and if $\lambda \in \mathbb{Q}$

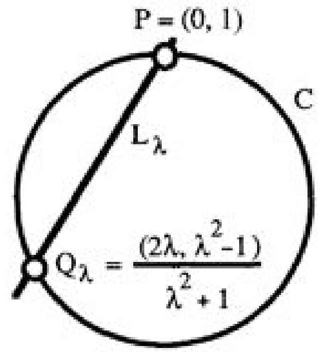

Figure 1.1: Linear projection of a conic to a line

is any value, then the line L_λ through P with slope $-\lambda$ meets C in a further point Q_λ. This construction of a map by means of linear projection will appear many times in what follows.

1.2 Similar example

$C : (2X^2 + Y^2 = 5Z^2)$. The same method leads to the parametrisation $\mathbb{R} \to C$ given by

$$x = \frac{2\sqrt{5}\lambda}{1+2\lambda^2}, \quad y = \frac{2\lambda^2 - 1}{1+2\lambda^2}.$$

This allows us to understand all about points of C with coefficients in $\mathbb{R}$, and there's no real difference from the previous example; what about $\mathbb{Q}$?

Proposition *If $(a,b,c) \in \mathbb{Q}$ satisfies $2a^2 + b^2 = 5c^2$ then $(a,b,c) = (0,0,0)$.*

Proof Multiplying through by a common denominator and taking out a common factor if necessary, I can assume that a, b, c are integers, not all of which are divisible by 5; also if $5 \mid a$ and $5 \mid b$ then $25 \mid 5c^2$, so that $5 \mid c$, which contradicts what I've just said. It is now easy to get a contradiction by considering the possible values of a and b mod 5: since any square is 0, 1 or 4 mod 5, clearly $2a^2 + b^2$ is one of $0+1$, $0+4$, $2+0$, $2+1$, $2+4$, $8+0$, $8+1$ or $8+4$ mod 5, none of which can be of the form $5c^2$. Q.E.D.

Note that this is a thoroughly arithmetic argument.

1.3 Conics in $\mathbb{R}^2$

A conic in $\mathbb{R}^2$ is a plane curve given by a quadratic equation

$$q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

Everyone has seen the classification of nondegenerate conics:

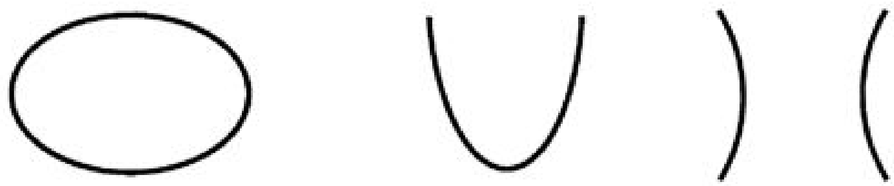

Figure 1.2: The nondegenerate conics: (a) ellipse; (b) parabola; (c) hyperbola.

in addition, there are a number of peculiar cases:
(d) single point given by $x^2 + y^2 = 0$;
(e, f, g) empty set given by any of the 3 equations: (e) $x^2 + y^2 = -1$, (f) $x^2 = -1$ or (g) $0 = 1$. These three equations are different, although they define the same locus of zeros in $\mathbb{R}^2$; consider for example their complex solutions.
(h) line $x = 0$;
(i) line pair $xy = 0$;
(j) parallel lines $x(x - 1) = 0$;
(k) 'double line' $x^2 = 0$; you can choose for yourself whether you'll allow the final case:
(l) whole plane given by $0 = 0$.

1.4 Projective plane

The definition 'out of the blue':

$$\begin{aligned} \mathbb{P}^2_{\mathbb{R}} &= \bigl\{\text{lines of } \mathbb{R}^3 \text{ through origin}\bigr\} \\ &= \bigl\{\text{ratios } X : Y : Z\bigr\} \\ &= \bigl(\mathbb{R}^3 \setminus \{0\}\bigr)/\sim, \quad \text{where } (X, Y, Z) \sim (\lambda X, \lambda Y, \lambda Z) \text{ if } \lambda \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

(The sophisticated reader will have no difficulty in generalising from $\mathbb{R}^3$ to an arbitrary vector space over a field, and in replacing work in a chosen coordinate system with intrinsic arguments.)

To represent a ratio $X : Y : Z$ for which $Z \neq 0$, I can set $x = X/Z$, $y = Y/Z$; this simplifies things, since the ratio corresponds to just two real numbers. In other words, the equivalence class of (X, Y, Z) under $\sim$ has a unique representative $(x, y, 1)$ with 3rd coordinate $= 1$. Unfortunately, sometimes Z might be $= 0$, so that this way of choosing a representative of the equivalence class is then no good. This discussion means that $\mathbb{P}^2_{\mathbb{R}}$ contains a copy of $\mathbb{R}^2$. A picture: