1.7 Parametrisation of a conic

Let C be a nondegenerate, nonempty conic of $\mathbb{P}^2_{\mathbb{R}}$. Then by Corollary 1.6, taking new coordinates $(X+Z, Y, Z-X)$, C is projectively equivalent to the curve $(XZ = Y^2)$; this is the curve parametrised by

$$\begin{aligned} \Phi\colon \mathbb{P}^1_{\mathbb{R}} &\longrightarrow C \subset \mathbb{P}^2_{\mathbb{R}}, \\ (U:V) &\mapsto (U^2 : UV : V^2). \end{aligned}$$

Remarks 1 The inverse map $\Psi\colon C \to \mathbb{P}^1_{\mathbb{R}}$ is given by

$$(X:Y:Z) \mapsto (X:Y) = (Y:Z);$$

here the left-hand ratio is defined if $X \neq 0$, and the right-hand ratio if $Z \neq 0$. In terminology to be introduced later, Φ and Ψ are inverse isomorphisms of varieties.

2 Throughout §§1–2, nonempty nondegenerate conics are tacitly assumed to be projectively equivalent to $(XZ - Y^2)$; over a field of characteristic $\neq 2$, this is justified in Ex. 1.5. (The reader interested in characteristic 2 should take this as the definition of a nondegenerate conic.)

1.8 Homogeneous form in 2 variables

Let $F(U, V)$ be a nonzero homogeneous polynomial of degree d in U, V, with coefficients in a fixed field k; (I will follow tradition, and use the word *form* for 'homogeneous polynomial'):

$$F(U,V) = a_d U^d + a_{d-1}U^{d-1}V + \cdots + a_i U^i V^{d-i} + \cdots + a_0 V^d.$$

F has an associated inhomogeneous polynomial in 1 variable,

$$f(u) = a_d u^d + a_{d-1}u^{d-1} + \cdots + a_i u^i + \cdots + a_0.$$

Clearly for $\alpha \in k$,

$$\begin{aligned} f(\alpha) = 0 &\iff (u-\alpha) \mid f(u) \\ &\iff (U - \alpha V) \mid F(U,V) \iff F(\alpha, 1) = 0; \end{aligned}$$

so zeros of f correspond to zeros of F on $\mathbb{P}^1$ away from the point $(1, 0)$, the 'point $\alpha = \infty$.' What does it mean for F to have a zero at infinity?

$$F(1,0) = 0 \iff a_d = 0 \iff \deg f < d.$$

Now define the *multiplicity* of a zero of F on $\mathbb{P}^1$ to be

(i) the multiplicity of f at the corresponding $\alpha \in k$; or

(ii) $d - \deg f$ if $(1, 0)$ is the zero.

So the multiplicity of zero of F at a point $(\alpha, 1)$ is the greatest power of $(U - \alpha V)$ dividing F, and at $(1, 0)$ it is the greatest power of V dividing F.

Proposition *Let $F(U, V)$ be a nonzero form of degree d in U, V. Then F has at most d zeros on $\mathbb{P}^1$; furthermore, if k is algebraically closed, then F has exactly d zeros on $\mathbb{P}^1$ provided these are counted with multiplicities as defined above.*