Chapter 5

Projective and birational geometry

The first part of §5 aims to generalise the content of §§3–4 to projective varieties; this is fairly mechanical, with just a few essential points. The remainder of the section is concerned with birational geometry, taking up the function field $k(V)$ from the end of §4; this is material which fits equally well into the projective or affine context.

5.0 Why projective varieties?

The cubic curve

$$C : (Y^2Z = X^3 + aXZ^2 + bZ^3) \subset \mathbb{P}^2$$

is the union of two affine curves

$$\begin{aligned} &C_0 : (y^2 = x^3 + ax + b) \subset \mathbb{A}^2 \quad \text{(the piece } (Z=1) \text{ of } C) \quad \text{and} \\ &C_1 : (z_1 = x_1^3 + axz_1^2 + bz_1^3) \subset \mathbb{A}^2 \quad \text{(the piece } (Y=1)), \end{aligned}$$

glued together by the isomorphism

$$\text{by} \quad \begin{array}{ccc} C_0 \setminus (y=0) & \longrightarrow & C_1 \setminus (z_1 = 0) \\ (x,y) & \longmapsto & (x/y, 1/y). \end{array}$$

As a much simpler example, $\mathbb{P}^1$ with homogeneous coordinates (X,Y) is the union of 2 copies of $\mathbb{A}^1$ with coordinates x_0, y_1 respectively, glued together by the isomorphism

$$\text{by} \quad \begin{array}{ccc} \mathbb{A}^1 \setminus (x_0 = 0) & \longrightarrow & \mathbb{A}^1 \setminus (y_1 = 0) \\ x_0 & \longmapsto & 1/x_0. \end{array}$$