Appendix to Part I: Curves and their genus

2.14 Topology of a nonsingular cubic

It is easy to see that a nonsingular plane cubic $C : (y^2 = x^3 + ax + b) \subset \mathbb{P}^2_{\mathbb{R}}$ has one of the two shapes

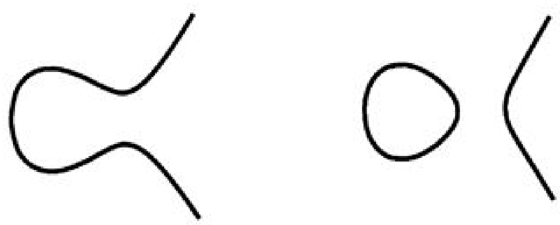

Figure 2.7: Real cubics

That is, topologically, C is either one or two circles (including the single point at infinity, of course). To look at the same question over $\mathbb{C}$, take the alternative normal form

$$C : (y^2 = x(x-1)(x-\lambda)) \cup \{\infty\};$$

what is the topology of $C \subset \mathbb{P}^2_{\mathbb{C}}$? The answer is a torus:

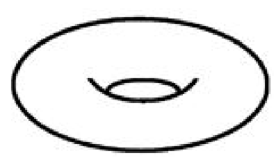

Figure 2.8: Torus

The idea of the proof is to consider the map

$$\pi\colon C \to \mathbb{P}^1_{\mathbb{C}} \text{ by } (X,Y,Z) \mapsto (X,Z) \quad \text{and } \infty \mapsto (1,0);$$

in affine coordinates this is $(x,y) \mapsto x$, so it's the 2-to-1 map corresponding to the graph of $y = \pm\sqrt{x(x-1)(x-\lambda)}$. Everyone knows that $\mathbb{P}^1_{\mathbb{C}}$ is homeomorphic to S^2, the Riemann sphere ('stereographic projection'); consider the 'function' $y(x) = \pm\sqrt{x(x-1)(x-\lambda)}$ on $\mathbb{P}^1_{\mathbb{C}}$. This is 2-valued outside $\{0, 1, \lambda, \infty\}$:

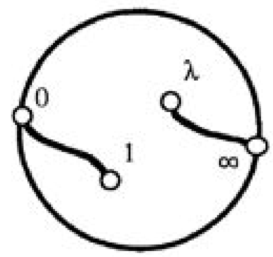

Figure 2.9: Two paths 01 and $\lambda\infty$

Now cut $\mathbb{P}^1_{\mathbb{C}}$ along two paths 01 and $\lambda\infty$; the double cover falls apart as 2 pieces, so that the function y is single valued on each sheet. So (the shading indicates how the two sheets match up

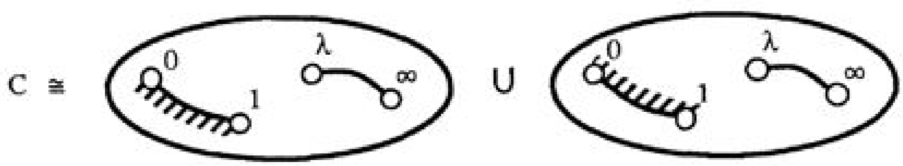

Figure 2.10: C as a union of two spheres with slits

under the glueing). To see what's going on, open up the slits:

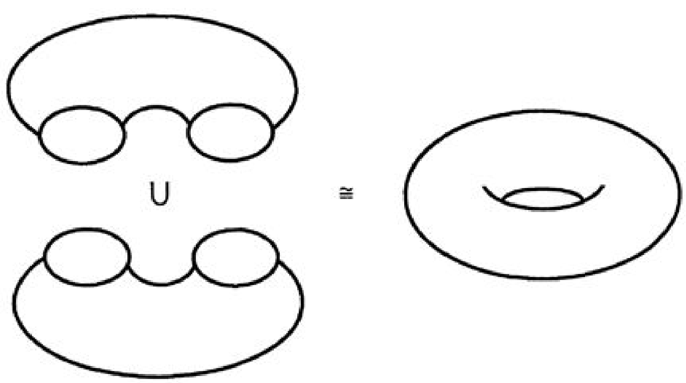

Figure 2.11: Union of two spheres with open slits

2.15 Discussion of genus

A nonsingular projective curve C over $\mathbb{C}$ has got just one topological invariant, its genus $g = g(C)$:

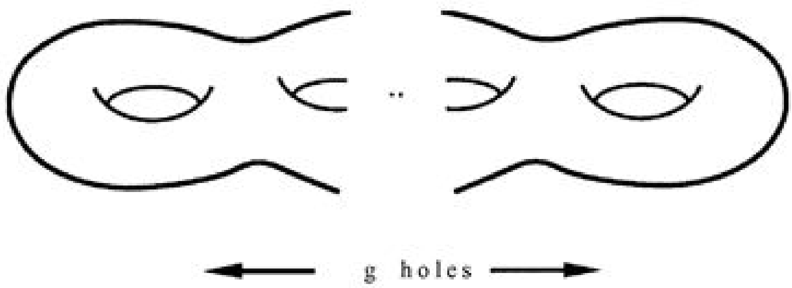

Figure 2.12: Surface of genus g

For example, the affine curve $C : (y^2 = f_{2g+1}(x) = \prod_i (x-a_i)) \subset \mathbb{C}^2$, where f_{2g+1} is a polynomial of degree $2g+1$ in x with distinct roots a_i, can be related to the Riemann surface of $\sqrt{f}$ exactly as in (2.13), and be viewed as a double cover of the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$ branched in the $2g+1$ points a_i and in ∞, and by the same argument, can be seen to have genus g. As another example, the genus of a nonsingular plane curve $C_d \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d is given by $g = g(C_d) = \binom{d-1}{2}$.

2.16 Commercial break

Complex curves (= compact Riemann surfaces) appear across a whole spectrum of math problems, from Diophantine arithmetic through complex function theory and low dimensional topology to differential equations of math physics. So go out and buy a complex curve today.

To a quite extraordinary degree, the properties of a curve are determined by its genus, and more particularly by the trichotomy $g = 0$, $g = 1$ or $g \geq 2$. Some of the more striking aspects of this are described in the table on the following page, and I give a brief discussion; this ought to be in the background culture of every mathematician.

To give a partial answer to the Diophantine question mentioned in (1.1–2) and again in (2.1), it is known that a curve can be parametrised by rational functions if and only if $g = 0$; if I'm working over a fixed field, a curve of genus 0 may have no k-valued points at all (for example, the conic in (1.2)), but if it has one point, it can be parametrised over k, so that its k-valued points are in bijection with $\mathbb{P}^1_k$. Any curve of genus 1 is isomorphic to a cubic as in this section, and a group law is defined on the k-valued points (provided of course that there exists at least one – there's no such thing as the empty group); if k is a number field (for example, $k = \mathbb{Q}$), the k-valued points form an Abelian group which is finitely generated (the Mordell–Weil Theorem). Whereas a curve of genus $g \geq 2$ is now known to have only a finite set of k-valued points; this is a famous theorem proved by Faltings in 1983, and for which he received the Fields medal in 1986. Thus for example, for any $n \geq 4$, the Fermat curve $x^n + y^n = 1$ has at most a finite number of rational points.