Proof (I) By what I said in (4.3), $f^*(g)$ is a polynomial map $V \to k$, hence $f^*(g) \in k[V]$. Obviously $f^*(a) = a$ for all $a \in k$ (since k is being considered as the constant functions on V, W). Finally the fact that f^* is a ring homomorphism is formal, since both $k[W]$ and $k[V]$ are rings of functions. (The ring structure is defined pointwise, so for example, for $g_1, g_2 \in k[W]$, the sum $g_1 + g_2$ is defined as the function on W such that $(g_1 + g_2)(P) = g_1(P) + g_2(P)$ for all $P \in W$; therefore $f^*(g_1 + g_2)(Q) = (g_1 + g_2)(f(Q)) = g_1(f(Q)) + g_2(f(Q)) = f^*g_1(Q) + f^*g_2(Q)$. No-one's going to read this rubbish, are they?)

(III) is just the fact that composition of maps is associative.

(II) is a little more tricky to get right, although it's still content-free. For $i = 1, \dots, m$, let $y_i \in k[W]$ be the ith coordinate function on W, so that

$$k[W] = k[y_1, \dots, y_m] = k[Y_1, \dots, Y_m]/I(W).$$

Now $\Phi\colon k[W] \to k[V]$ is given, so I can define $f_i \in k[V]$ by $f_i = \Phi(y_i)$.

Consider the map $f\colon V \to \mathbb{A}^m_k$ defined by $f(P) = f_1(P), \dots, f_m(P)$. This is a polynomial map since $f_i \in k[V]$. Furthermore, I claim that f takes V into W, that is, $f(V) \subset W$. Indeed, suppose that $G \in I(W) \subset k[Y_1, \dots, Y_m]$; then

$$G(y_1, \dots, y_m) = 0 \in k[W],$$

where the left-hand side means that I substitute the ring elements y_i into the polynomial expression G. Therefore, $\Phi(G(y_1, \dots, y_m)) = 0 \in k[V]$; but Φ is a k-algebra homomorphism, so that

$$k[V] \ni 0 = \Phi(G(y_1, \dots, y_m)) = G(\Phi(y_1), \dots, \Phi(y_m)) = G(f_1, \dots, f_m).$$

The f_i are functions on V, and $G(f_1, \dots, f_m) \in k[V]$ is by definition the function $P \mapsto G(f_1(P), \dots, f_m(P))$. This proves that for $P \in V$, and for every $G \in I(W)$, the coordinates $(f_1(P), \dots, f_m(P))$ of $f(P)$ satisfy $G(f_1(P), \dots, f_m(P)) = 0$. Since W is the subset of $\mathbb{A}^m_k$ defined by the vanishing of $G \in I(W)$, it follows that $f(P) \in W$. This proves that f given above is a polynomial map $f\colon V \to W$. To check that the two k-algebra homomorphisms f^*, $\Phi\colon k[W] \to k[V]$ coincide, it's enough to check that they agree on the generators, that is $f^*(y_i) = \Phi(y_i)$; a minute inspection of the construction of f (at the start of the proof of (II) above) will reveal that this is in fact the case. An exactly similar argument shows that the map f is uniquely determined by the condition $f^*(y_i) = \Phi(y_i)$. Q.E.D.

Corollary 4.5 *A polynomial map $f\colon V \to W$ is an isomorphism if and only if $f^*\colon k[W] \to k[V]$ is an isomorphism.*

Example Over an infinite field k, the polynomial map

$$\varphi\colon \mathbb{A}^1_k \to C : (Y^2 = X^3) \subset \mathbb{A}^2_k \text{ given by } T \mapsto (T^2, T^3)$$

is not an isomorphism. For in this case, the homomorphism

$$\varphi^*\colon k[C] = k[X,Y]/(Y^2 - X^3) \to k[T]$$

is given by $X \mapsto T^2, Y \mapsto T^3$. The image of φ^* is the k-algebra generated by T^2, T^3, that is $k[T^2, T^3] \subsetneq k[T]$. (Please make sure you understand why T^2, T^3 don't generate $k[T]$; I can't help you on this.)

Notice that φ is bijective, and so has a perfectly good inverse map $\psi\colon C \to \mathbb{A}^1_k$ given by $(X, Y) \mapsto 0$ if $X = Y = 0$ and Y/X otherwise. So why isn't φ an isomorphism? The point is that C has fewer polynomial functions on it than $\mathbb{A}^1$; in a sense you can see that for yourself, since $k[\mathbb{A}^1] = k[T]$ has a polynomial function with nonzero derivative at 0. The gut feeling is that φ 'squashes up the tangent vector at 0'.

4.6 Affine variety

Let k be a field; I want an *affine variety* to be an irreducible algebraic subset $V \subset \mathbb{A}^n_k$, defined up to isomorphism.

Theorem 4.4 tells us that the coordinate ring $k[V]$ is an invariant of the isomorphism class of V. This allows me to give a definition of a variety making less use of the ambient space $\mathbb{A}^n_k$; the reason for wanting to do this is rather obscure, and for practical purposes you will not miss much if you ignore it: subsequent references to an affine variety will always be taken in the sense given above (GOTO 4.7).

Definition An affine variety over a field k is a set V, together with a ring $k[V]$ of k-valued functions $f\colon V \to k$ such that

(i) $k[V]$ is a finitely generated k-algebra, and

(ii) for some choice $x_1, \dots, x_n$ of generators of $k[V]$ over k, the map

$$\begin{array}{rrcl} & V & \to & \mathbb{A}^n_k \\ \text{by} & & & \\ & P & \mapsto & x_1(P), \dots, x_n(P) \end{array}$$

embeds V as an irreducible algebraic set.

4.7 Function field

Let V be an affine variety; then the coordinate ring $k[V]$ of V is an integral domain whose elements are k-valued functions of V.

Definition The *function field* $k(V)$ of V is the field of fractions $k(V) = \operatorname{Quot}(k[V])$ of $k[V]$. An element $f \in k(V)$ is a *rational function* on V; note that $f \in k(V)$ is by definition a quotient $f = g/h$ with $g, h \in k[V]$ and $h \neq 0$.

A priori f is not a function on V, because of the zeros of h; however, f is well defined at $P \in V$ whenever $h(P) \neq 0$, so is at least a 'partially defined function'. I now introduce terminology to shore up this notion.

Definition Let $f \in k(V)$ and $P \in V$; I say that f is *regular* at P, or that P is in the *domain of definition* of f if there exists an expression $f = g/h$ with $g, h \in k[V]$ and $h(P) \neq 0$.

An important point to bear in mind is that usually $k[V]$ will not be a UFD, so that $f \in k(V)$ may well have essentially different representations as $f = g/h$; see Ex. 4.9 for an example.

Write

$$\operatorname{dom} f = \bigl\{P \in V \bigm| f \text{ is regular at } P\bigr\}$$

for the *domain of definition* of f, and

$$\mathcal{O}_{V,P} = \bigl\{f \in k(V) \bigm| f \text{ is regular at } P\bigr\} = k[V][\{h^{-1} | h(P) \neq 0\}].$$

Then $\mathcal{O}_{V,P} \subset k(V)$ is a subring, the *local ring* of V at P.