Proof By passing to a smaller neighbourhood of P in V, I can assume V_0 is isomorphic to an affine variety (Proposition 4.13). Then so is W_0, and φ induces an isomorphism $k[V_0] \cong k[W_0]$ taking m_P into m_Q. The final sentence holds because by (5.8), V and W contain dense open subsets which are isomorphic.

Theorem 6.10 *For any variety V, $\dim V = \operatorname{tr}\deg k(V)$.*

Proof This is known if V is a hypersurface. On the other hand, every variety is birational to a hypersurface (by (5.10)), and both sides of the required relation are the same for birationally equivalent varieties. Q.E.D.

6.11 Nonsingularity and projective varieties

Although the above results were discussed in terms of affine varieties, the idea of nonsingularity and of dimension applies directly to any variety V: a point $P \in V$ is nonsingular if it is a nonsingular point of an affine open $V_0 \subset V$ containing it; by Corollary 6.9, this notion does not depend on the choice of V_0. On the other hand, for a projective variety $V \subset \mathbb{P}^n$, it is sometimes useful to consider the tangent space to V at P as a projective subspace of $\mathbb{P}^n$. I give the definition for a hypersurface only: if $V = V(f)$ is a hypersurface defined by a form (= homogeneous polynomial) $f \in k[X_0, \dots, X_n]$ of degree d, and $V \ni P = (a_0, \dots, a_n)$, then $\sum \partial f/\partial X_i(P) \cdot X_i = 0$ is the equation of a hyperplane in $\mathbb{P}^n$ which plays the role of the tangent plane to V at P. If $P \in \mathbb{A}^n_{(0)}$, then this projective hyperplane is the projective closure of the affine tangent hyperplane to $V_{(0)}$ at P, as can be checked easily using Euler's formula:

$$\sum X_i \cdot \frac{\partial f}{\partial X_i} = df \quad \text{for } f \in k[X_0, \dots, X_n] \text{ homogeneous of degree } d.$$

Because of this formula, to find out whether a point $P \in \mathbb{P}^n$ is a singular point of V, we only have to check $(n+1)$ out of the $(n+2)$ conditions

$$f(P) = 0, \quad \frac{\partial f}{\partial X_i}(P) = 0 \text{ for } i = 0, \dots, n,$$

so that for example, if the degree of f is not divisible by $\operatorname{char} k$,

$$\frac{\partial f}{\partial X_i}(P) = 0 \text{ for } i = 0, \dots, n \implies f(P) = 0,$$

and $P \in V$ is a singularity.

6.12 Worked example: blowup

Let $B = \mathbb{A}^2$ with coordinates (u, v), and $\sigma\colon B \to \mathbb{A}^2$ the map $(u, v) \mapsto (x = u, y = uv)$; clearly, σ is a birational morphism: it contracts the v-axis $\ell : (u = 0)$ to the origin 0 and is an isomorphism outside this exceptional set. Let's find out what happens under σ to a curve $C : (f = 0) \subset \mathbb{A}^2$; the question will only be of interest if C passes through 0.

Clearly $\sigma^{-1}(C) \subset B$ is the algebraic subset defined by $(f \circ \sigma)(u, v) = f(u, uv) = 0$; since $0 \in C$ by assumption, it follows that $\ell : (u = 0)$ is contained in $\sigma^{-1}(C)$, or equivalently, that $u \mid f(u, uv)$. It's easy to see that the highest power u^m of u dividing $f(u, uv)$ is equal to the smallest degree $m = a + b$ of the monomials $x^a y^b$ occurring in f, that is, the *multiplicity* of f at 0; so $\sigma^{-1}(C)$ decomposes as the union of the exceptional curve $\sigma^{-1}(0) = \ell$ (with multiplicity m), together with a new curve C_1 defined by $f_1(u, v) = f(u, uv)/u^m$. Consider the examples

(a) $f = \alpha x - y + \cdots$;

(b) $f = y^2 - x^2 + \cdots$;

(c) $f = y^2 - x^3$,

where $\ldots$ denotes terms of higher degree. Clearly in (a) f has multiplicity 1, and $f_1 = \alpha - v + \cdots$ (where $\cdots$ consists of terms divisible by u), so C_1 is again nonsingular, and meets ℓ transversally at $(0, \alpha)$; thus σ replaces $0 \in \mathbb{A}^2$ with the line ℓ whose points correspond to tangent directions at 0 (excluding $(x = 0)$). In (b) $f_1 = v^2 - 1 + \cdots$, so C_1 has two nonsingular points $(0, \pm 1)$ above $0 \in C$; thus the blowup σ 'separates the two branches' of the singular curve C. In (c) $f_1 = v^2 - u$, so that C_1 is nonsingular, but above 0 it is tangent to the contracted curve ℓ.

In either case (b) or (c), σ replaces a singular curve C by a nonsingular one C_1 birational to C (by introducing 'new coordinates' $u = x, v = y/x$). This is what is meant by a *resolution of singularities*. In the case of plane curves, a resolution can always be obtained by a chain of blowups (see Ex. 6.6 for examples, and [Fulton, pp. 162–171] for more details), and the process of resolution gives detailed information about the singularities. A famous theorem of H. Hironaka guarantees the possibility of resolving singularities by blowups (in any dimension, over a field of characteristic zero). This is a crucial theoretical result that reduces the birational study of varieties to nonsingular ones; however, the actual process of resolution by blowups is in general extremely complicated, and does not necessarily contribute very much to the understanding of the singularities or varieties concerned.

Exercises to Chapter 6

6.1 Let $k \subset K$ be a field extension, and $(u_1, \ldots, u_r)$, $(v_1, \ldots, v_s)$ two sets of elements of K; suppose that $(u_1, \ldots, u_r)$ are algebraically independent, and that $(v_1, \ldots, v_s)$ span the extension $k \subset K$ algebraically. Prove that $r \leq s$. [Hint: the inductive step consists of assuming that $(u_1, \ldots, u_i, v_{i+1}, \ldots, v_s)$ span K/k algebraically, and considering u_{i+1}.] Deduce that any two transcendence bases of K/k have the same number of elements.

6.2 Prove Theorem 6.8, (b). [Hint:

$$I(V_f) = (I(V), Yf - 1) \subset k[X_1, \ldots, X_n, Y],$$

so that if $Q = (a_1, \ldots, a_n, b) \in V_f$, then $T_Q V_f \subset \mathbb{A}^{n+1}$ is defined by the equations for $T_P V \subset \mathbb{A}^n$, together with one equation involving Y.]

6.3 Determine all the singular points of the following curves in $\mathbb{A}^2$.

(a) $y^2 = x^3 - x$;