8.9 Regular functions and sheaves

The reader who has properly grasped the notion of rational function $f \in k(X)$ on a variety X and of regularity of f at $P \in X$ ((4.7) and (5.4)) already has a pretty good intuitive idea of the structure sheaf $\mathcal{O}_X$. For an open set $U \subset X$, the set of regular functions $U \to k$

$$\mathcal{O}_X(U) = \bigl\{ f \in k(X) \bigm| f \text{ is regular } \forall P \in U \bigr\} = \bigcap_{P \in U} \mathcal{O}_{X,P}$$

is a subring of the field $k(X)$. The sheaf $\mathcal{O}_X$ is just the family of rings $\mathcal{O}_X(U)$ as U runs through the opens of X. Clearly, any element of the local ring $\mathcal{O}_{X,P}$ (see (4.7) and (5.4) for the definition) is regular in some neighbourhood U of P, so that $\mathcal{O}_{X,P} = \bigcup_{U \ni P} \mathcal{O}_X(U)$. There's no more to it than that; there's a fixed pool of rational sections $k(X)$, and sections of the sheaf over an open U are just rational sections with a regularity condition at every $P \in U$.

This language is adequate to describe any torsion free sheaf on an irreducible variety with the Zariski topology. Of course, you need the full definition of sheaves if X is reducible, or if you want to handle more complicated sheaves, or to use the complex topology.

8.10 Globally defined regular functions

If X is a projective variety then the only rational functions $f \in k(X)$ that are regular at every $P \in X$ are the constants. This is a general property of projective varieties, analogous to Liouville's theorem in functions of one complex variable; for a variety over $\mathbb{C}$ it comes from compactness and the maximum modulus principle ($X \subset \mathbb{P}^n_{\mathbb{C}}$ is compact in the complex topology, so the modulus of a global holomorphic function on X must take a maximum), but in algebraic geometry it is surprisingly hard to prove from scratch (see for example [Hartshorne, I.3.4]; it is essentially a finiteness result, related to the finite dimensionality of coherent cohomology groups).

8.11 The surprising sufficiency of projective algebraic geometry

Weil's abstract definition of a variety (affine algebraic sets glued together along isomorphic open sets) was referred to briefly in (0.4), and is quite easy to handle in terms of sheaves. Given this, the idea of working only with varieties embedded in a fixed ambient space $\mathbb{P}^N_k$ seems at first sight unduly restrictive. I want to describe briefly the modern point of view on this question.

(a) Polarisation and positivity

Firstly, varieties are usually considered up to isomorphism, so saying a variety X is *projective* means that X can be embedded in some $\mathbb{P}^N$, that is, is isomorphic to a closed subvariety $X \subset \mathbb{P}^N$ as in (5.1–7). *Quasiprojective* means isomorphic to a locally closed subvariety of $\mathbb{P}^N$, so an open dense subset of a projective variety; projectivity includes the property of *completeness*, that X cannot be embedded as a dense open set of any bigger variety.

The choice of an actual embedding $X \hookrightarrow \mathbb{P}^N$ (or of a very ample line bundle $\mathcal{O}_X(1)$ whose sections will be the homogeneous coordinates of $\mathbb{P}^N$) is often called a *polarisation*, and we write