8.6 Choice of topics

The topics and examples treated in this book have been chosen partly pragmatically on the basis of small degree and ease of computation. However, they also hint at the 'classification of varieties': the material on conics applies in a sense to every rational curve, and cubic surfaces are the most essential examples of del Pezzo rational surfaces. Cubic curves with their group law are examples of Abelian varieties; the fact (2.2) that a nonsingular cubic is not rational is the very first step in classification. The intersection of two plane conics in (1.12–14) and the intersection of two quadrics of $\mathbb{P}^3_k$ referred to in Ex. 5.6 could also be fitted into a similar pattern, with the intersection of two quadrics in $\mathbb{P}^4_k$ providing another class of del Pezzo surfaces, and the family of lines on the intersection of two quadrics in $\mathbb{P}^5_k$ providing 2-dimensional Abelian varieties.

The genus of a curve, and the division into 3 cases tabulated on p. 46 is classification in a nutshell. I would have liked to include more material on the genus of a curve, in particular how to calculate it in terms of topological Euler characteristic or of intersection numbers in algebraic geometry, essential five finger exercises for young geometers. However, this would comfortably occupy a separate undergraduate lecture course, as would the complex analytic theory of elliptic curves.

8.7 Computation versus theory

Another point to make concerning the approach in these notes is that quite a lot of emphasis is given to cases that can be handled by explicit calculations. When general theory proves the existence of some construction, then doing it in terms of explicit coordinate expressions is a useful exercise that helps one to keep a grip on reality, and this is appropriate for an undergraduate textbook. This should not however be allowed to obscure the fact that the theory is really designed to handle the complicated cases, when explicit computations will often not tell us anything.

8.8 $\mathbb{R}$ versus $\mathbb{C}$

The reader with real interests may be disappointed that the treatment over $\mathbb{R}$ in §§1–2 gave way in §3 to considerations over an arbitrary field k, promptly assumed to be algebraically closed. I advise this class of reader to persevere; there are plenty of relations between real and complex geometry, including some that will come as a surprise. Asking about the real points of a real variety is a very hard question, and something of a minority interest in algebraic geometry; in any case, knowing all about its complex points will usually be an essential prerequisite. Another direct relation between geometry over $\mathbb{R}$ and $\mathbb{C}$ is that an n-dimensional nonsingular complex variety is a $2n$-dimensional real manifold – for example, algebraic surfaces are a principal source of constructions of smooth 4-manifolds.

As well as these fairly obvious relations, there are more subtle ones, for example: (a) singularities of plane curves $C \subset \mathbb{C}^2$ give rise to knots in S^3 by intersecting with the boundary of a small ball; and (b) the Penrose twistor construction views a 4-manifold (with a special kind of Riemannian metric) as the set of real valued points of a 4-dimensional complex variety that parametrises rational curves on a complex 3-dimensional variety (thus the real 4-sphere S^4 we live in can be identified as the real locus in the complex Grassmannian $\mathrm{Gr}(2,4)$ of lines in $\mathbb{P}^3_{\mathbb{C}}$).

8.9 Regular functions and sheaves

The reader who has properly grasped the notion of rational function $f \in k(X)$ on a variety X and of regularity of f at $P \in X$ ((4.7) and (5.4)) already has a pretty good intuitive idea of the structure sheaf $\mathcal{O}_X$. For an open set $U \subset X$, the set of regular functions $U \to k$

$$\mathcal{O}_X(U) = \big\{ f \in k(X) \bigm| f \text{ is regular } \forall P \in U \big\} = \bigcap_{P \in U} \mathcal{O}_{X,P}$$

is a subring of the field $k(X)$. The sheaf $\mathcal{O}_X$ is just the family of rings $\mathcal{O}_X(U)$ as U runs through the opens of X. Clearly, any element of the local ring $\mathcal{O}_{X,P}$ (see (4.7) and (5.4) for the definition) is regular in some neighbourhood U of P, so that $\mathcal{O}_{X,P} = \bigcup_{U \ni P} \mathcal{O}_X(U)$. There's no more to it than that; there's a fixed pool of rational sections $k(X)$, and sections of the sheaf over an open U are just rational sections with a regularity condition at every $P \in U$.

This language is adequate to describe any torsion free sheaf on an irreducible variety with the Zariski topology. Of course, you need the full definition of sheaves if X is reducible, or if you want to handle more complicated sheaves, or to use the complex topology.

8.10 Globally defined regular functions

If X is a projective variety then the only rational functions $f \in k(X)$ that are regular at every $P \in X$ are the constants. This is a general property of projective varieties, analogous to Liouville's theorem in functions of one complex variable; for a variety over $\mathbb{C}$ it comes from compactness and the maximum modulus principle ($X \subset \mathbb{P}^n_{\mathbb{C}}$ is compact in the complex topology, so the modulus of a global holomorphic function on X must take a maximum), but in algebraic geometry it is surprisingly hard to prove from scratch (see for example [Hartshorne, I.3.4]; it is essentially a finiteness result, related to the finite dimensionality of coherent cohomology groups).

8.11 The surprising sufficiency of projective algebraic geometry

Weil's abstract definition of a variety (affine algebraic sets glued together along isomorphic open sets) was referred to briefly in (0.4), and is quite easy to handle in terms of sheaves. Given this, the idea of working only with varieties embedded in a fixed ambient space $\mathbb{P}^N_k$ seems at first sight unduly restrictive. I want to describe briefly the modern point of view on this question.

(a) Polarisation and positivity

Firstly, varieties are usually considered up to isomorphism, so saying a variety X is *projective* means that X can be embedded in some $\mathbb{P}^N$, that is, is isomorphic to a closed subvariety $X \subset \mathbb{P}^N$ as in (5.1–7). *Quasiprojective* means isomorphic to a locally closed subvariety of $\mathbb{P}^N$, so an open dense subset of a projective variety; projectivity includes the property of *completeness*, that X cannot be embedded as a dense open set of any bigger variety.

The choice of an actual embedding $X \hookrightarrow \mathbb{P}^N$ (or of a very ample line bundle $\mathcal{O}_X(1)$ whose sections will be the homogeneous coordinates of $\mathbb{P}^N$) is often called a *polarisation*, and we write