Second degenerate case Suppose $P_1, \dots, P_6 \in C$ are conconic, with $C : (Q = 0)$ a nondegenerate conic. Then choose $P_9 \in Q$ distinct from $P_1, \dots, P_6$. By Corollary 2.5 again,

$$S_3(P_1, \dots, P_9) = Q \cdot S_1(P_7, P_8);$$

the line $L = P_7P_8$ is unique, so that $S_3(P_1, \dots, P_9)$ is the 1-dimensional space spanned by QL, and hence $\dim S_3(P_1, \dots, P_8) \leq 2$. Q.E.D.

Corollary 2.7 *Let C_1, C_2 be two cubic curves whose intersection consists of 9 distinct points, $C_1 \cap C_2 = \{P_1, \dots, P_9\}$. Then a cubic D through $P_1, \dots, P_8$ also passes through P_9.*

Proof If 4 of the points $P_1, \dots, P_8$ were on a line L, then each of C_1 and C_2 would meet L in ≥ 4 points, and thus contain L, which contradicts the assumption on $C_1 \cap C_2$. For exactly the same reason, no 7 of the points can be conconic. Therefore the assumptions of (2.6) are satisfied, so I can conclude that

$$\dim S_3(P_1, \dots, P_8) = 2;$$

this means that the equations F_1, F_2 of the two cubics C_1, C_2 form a basis of $S_3(P_1, \dots, P_8)$, and hence $D : (G = 0)$, where $G = \lambda F_1 + \mu F_2$. Now F_1, F_2 vanish at P_9, hence so does G. Q.E.D.

2.8 Group law on a plane cubic

Suppose $k \subset \mathbb{C}$ is a subfield of $\mathbb{C}$, and $F \in k[X, Y, Z]$ a cubic form defining a (nonempty) plane curve $C : (F = 0) \subset \mathbb{P}^2_k$. Assume that F satisfies the following two conditions:

(a) F is irreducible (so that C does not contain a line or conic);

(b) for every point $P \in C$, there exists a unique line $L \subset \mathbb{P}^2_k$ such that P is a repeated zero of $F|L$.

Note that geometrically, the condition in (b) is that C should be nonsingular, and the line L referred to is the tangent line $L = T_PC$ (see Ex. 2.3). This will be motivation for the general definition of nonsingularity and tangent spaces to a variety in §6.

Fix any point $O \in C$, and make the following construction:

Construction *(i) For $A \in C$, let $\overline{A}$ = 3rd point of intersection of C with the line OA;*

(ii) for $A, B \in C$, write R = 3rd point of intersection of AB with C, and define $A + B$ by $A + B = \overline{R}$ (see diagram below).

Theorem *The above construction defines an Abelian group law on C, with O as zero (= neutral element).*