4.10 Composition of rational maps

The composite $g \circ f$ of rational maps $f\colon V \dashrightarrow W$ and $g\colon W \dashrightarrow U$ may not be defined. This is a difficulty caused by the fact that a rational map is not a map: in a natural and obvious sense, the composite is a map defined on $\operatorname{dom} f \cap f^{-1}(\operatorname{dom} g)$; however, it can perfectly well happen that this is empty (see Ex. 4.10).

Expressed algebraically, the same problem also occurs: suppose that f is given by $f_1, \dots, f_m \in k(V)$, so that

$$f\colon V \dashrightarrow W \subset \mathbb{A}^m \quad \text{by} \quad P \mapsto f_1(P), \dots, f_m(P)$$

for $P \in \bigcap \operatorname{dom} f_i$; any $g \in k[W]$ is of the form $g = G \bmod I(W)$ for some $G \in k[Y_1, \dots, Y_m]$, and $g \circ f = G(f_1, \dots, f_m)$ is well defined in $k(V)$. So exactly as in (4.4), there is a k-algebra homomorphism

$$f^*\colon k[W] \to k(V)$$

corresponding to f. However, if $h \in k[W]$ is in the kernel of f^*, then no meaning can be attached to $f^*(g/h)$, so that f^* cannot be extended to a field homomorphism $k(W) \to k(V)$.

Definition $f\colon V \dashrightarrow W$ is *dominant* if $f(\operatorname{dom} f)$ is dense in W for the Zariski topology.

Geometrically, this means that $f^{-1}(\operatorname{dom} g) \subset \operatorname{dom} f$ is a dense open set for any rational map $g\colon W \dashrightarrow U$, so that $g \circ f$ is defined on a dense open set of V, so is a partially defined map $V \dashrightarrow U$.

Algebraically,

$$f \text{ is dominant} \iff f^*\colon k[W] \to k(V) \text{ is injective.}$$

For given $g \in k[W]$,

$$g \in \ker f^* \iff f(\operatorname{dom} f) \subset V(g),$$

that is, f^* is not injective if and only if $f(\operatorname{dom} f)$ is contained in a strict algebraic subset of W.

Clearly, the composite $g \circ f$ of rational maps f and g is defined provided that f is dominant: $g \circ f$ is the rational map whose components are $f^*(g_i)$. Notice that the domain of $g \circ f$ certainly contains $f^{-1}(\operatorname{dom} g) \cap \operatorname{dom} f$, but may very well be larger (see Ex. 4.6).

Theorem 4.11 *(I) A dominant rational map $f\colon V \dashrightarrow W$ defines a field homomorphism $f^*\colon k(W) \to k(V)$.*

(II) Conversely, a k-homomorphism $\Phi\colon k(W) \to k(V)$ comes from a uniquely defined dominant rational map $f\colon V \dashrightarrow W$.

(III) If f and g are dominant then $(g \circ f)^* = f^* \circ g^*$.

The proof requires only minor modifications to that of (4.4).

4.12 Morphisms from an open subset of an affine variety

Let V, W be affine varieties, and $U \subset V$ an open subset.

Definition A *morphism* $f: U \to W$ is a rational map $f: V \dashrightarrow W$ such that $U \subset \operatorname{dom} f$, so that f is regular at every $P \in U$.

If $U_1 \subset V$ and $U_2 \subset W$ are opens, then a morphism $f: U_1 \to U_2$ is just a morphism $f: U_1 \to W$ such that $f(U_1) \subset U_2$. An *isomorphism* is a morphism which has a two-sided inverse morphism.

Note that if V, W are affine varieties, then by Theorem 4.8, (II),

$$\big\{\text{morphisms } f: V \to W\big\} = \big\{\text{polynomial maps } f: V \to W\big\};$$

the left-hand side of the equation consists of rational objects subject to regularity conditions, whereas the right-hand side is more directly in terms of polynomials.

Example The parametrisation of the cuspidal cubic $\mathbb{A}^1 \to C : (Y^2 = X^3)$ of (2.1) induces an isomorphism $\mathbb{A}^1 \setminus \{0\} \cong C \setminus \{(0,0)\}$; see Ex. 4.5 for details.

4.13 Standard open subsets

Let V be an affine variety. For $f \in k[V]$, write V_f for the open set $V_f = V \setminus V(f) = \big\{P \in V \mid f(P) \neq 0\big\}$. The V_f are called *standard open sets* of V.

Proposition *V_f is isomorphic to an affine variety, and*

$$k[V_f] = k[V][f^{-1}].$$

Proof The idea is to consider the graph of the function f^{-1}; a similar trick was used for (b) $\Longrightarrow$ (c) in the proof of NSS (3.10).

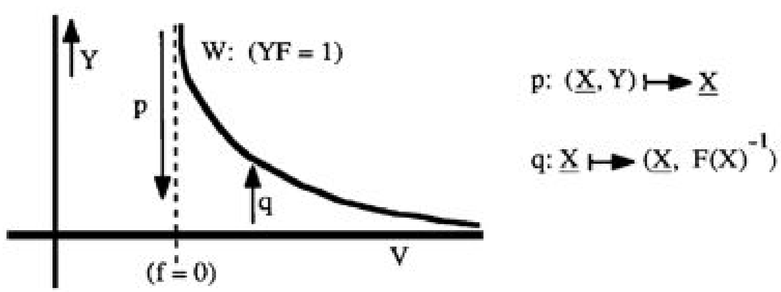

Figure 4.1: Graph of $1/f$

Let $J = I(V) \subset k[X_1, \ldots, X_n]$, and choose $F \in k[X_1, \ldots, X_n]$ such that $f = F \bmod I(V)$. Now define $I = (J, YF - 1) \subset k[X_1, \ldots, X_n, Y]$, and let

$$V(I) = W \subset \mathbb{A}^{n+1}.$$

It is easy to check that the maps indicated in the diagram are inverse morphisms between W and V_f. The statement about the coordinate ring is contained in (4.8, III). Q.E.D.