1.6 Classification of conics in $\mathbb{P}^2$

Let k be any field of characteristic $\neq 2$; recall two results from the linear algebra of quadratic forms:

Proposition (A) *There are natural bijections*

$$\left\{\begin{matrix}\text{homogeneous}\\ \text{quadratic polys.}\end{matrix}\right\} = \left\{\begin{matrix}\text{quad. forms}\\ k^3 \to k\end{matrix}\right\} \xleftrightarrow{\text{bij}} \left\{\begin{matrix}\text{symmetric bilinear}\\ \text{forms on } k^3\end{matrix}\right\}$$

given in formulas by

$$aX^2 + 2bXY + cY^2 + 2dXZ + 2eYZ + fZ^2 \longleftrightarrow \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}$$

A quadratic form is *nondegenerate* if the corresponding bilinear form is nondegenerate, that is, its matrix is nonsingular.

Theorem (B) *Let V be a vector space over k and $Q\colon V \to k$ a quadratic form; then there exists a basis of V such that*

$$Q = \varepsilon_1 x_1^2 + \varepsilon_2 x_2^2 + \cdots + \varepsilon_n x_n^2, \text{ with } \varepsilon_i \in k.$$

(This is proved by *Gram-Schmidt orthogonalisation*, if that rings a bell.) Obviously, for $\lambda \in k \setminus \{0\}$ the substitution $x_i \mapsto \lambda x_i$ takes ε_i into $\lambda^{-2}\varepsilon_i$.

Corollary *In a suitable system of coordinates, any conic in $\mathbb{P}^2_{\mathbb{R}}$ is one of the following:*

(α) *nondegenerate conic, $C : (X^2 + Y^2 - Z^2 = 0)$;*

(β) *empty set, given by $(X^2 + Y^2 + Z^2 = 0)$;*

(γ) *line pair, given by $(X^2 - Y^2 = 0)$;*

(δ) *one point $(0, 0, 1)$, given by $(X^2 + Y^2 = 0)$;*

(ε) *double line, given by $(X^2 = 0)$.*

(Optionally you have the whole of $\mathbb{P}^2_{\mathbb{R}}$ given by $(0 = 0)$.)

Proof Any real number ε is either 0, or $\pm$ a square, so that I only have to consider Q as in the theorem with $\varepsilon_i = 0$ or ± 1. In addition, since I'm only interested in the locus $(Q = 0)$, I'm allowed to multiply Q through by -1. This leads at once to the given list. Q.E.D.

There are two points to make about this corollary: firstly, the list is quite a lot shorter than that in (1.3); for example, the 3 nondegenerate cases (ellipse, parabola, hyperbola) of (1.3) all correspond to case (α), and the 2 cases of intersecting and parallel line pairs are not distinguished in the projective case. Secondly, the derivation of the list from general algebraic principles is much simpler.

1.7 Parametrisation of a conic

Let C be a nondegenerate, nonempty conic of $\mathbb{P}^2_{\mathbb{R}}$. Then by Corollary 1.6, taking new coordinates $(X+Z, Y, Z-X)$, C is projectively equivalent to the curve $(XZ = Y^2)$; this is the curve parametrised by

$$\begin{aligned} \Phi\colon \mathbb{P}^1_{\mathbb{R}} &\longrightarrow C \subset \mathbb{P}^2_{\mathbb{R}}, \\ (U : V) &\mapsto (U^2 : UV : V^2). \end{aligned}$$

Remarks 1 The inverse map $\Psi\colon C \to \mathbb{P}^1_{\mathbb{R}}$ is given by

$$(X : Y : Z) \mapsto (X : Y) = (Y : Z);$$

here the left-hand ratio is defined if $X \neq 0$, and the right-hand ratio if $Z \neq 0$. In terminology to be introduced later, Φ and Ψ are inverse isomorphisms of varieties.

2 Throughout §§1–2, nonempty nondegenerate conics are tacitly assumed to be projectively equivalent to $(XZ - Y^2)$; over a field of characteristic $\neq 2$, this is justified in Ex. 1.5. (The reader interested in characteristic 2 should take this as the definition of a nondegenerate conic.)

1.8 Homogeneous form in 2 variables

Let $F(U, V)$ be a nonzero homogeneous polynomial of degree d in U, V, with coefficients in a fixed field k; (I will follow tradition, and use the word *form* for 'homogeneous polynomial'):

$$F(U, V) = a_d U^d + a_{d-1} U^{d-1} V + \cdots + a_i U^i V^{d-i} + \cdots + a_0 V^d.$$

F has an associated inhomogeneous polynomial in 1 variable,

$$f(u) = a_d u^d + a_{d-1} u^{d-1} + \cdots + a_i u^i + \cdots + a_0.$$

Clearly for $\alpha \in k$,

$$\begin{aligned} f(\alpha) = 0 &\iff (u - \alpha) \mid f(u) \\ &\iff (U - \alpha V) \mid F(U, V) \iff F(\alpha, 1) = 0; \end{aligned}$$

so zeros of f correspond to zeros of F on $\mathbb{P}^1$ away from the point $(1, 0)$, the 'point $\alpha = \infty$.' What does it mean for F to have a zero at infinity?

$$F(1, 0) = 0 \iff a_d = 0 \iff \deg f < d.$$

Now define the *multiplicity* of a zero of F on $\mathbb{P}^1$ to be

(i) the multiplicity of f at the corresponding $\alpha \in k$; or

(ii) $d - \deg f$ if $(1, 0)$ is the zero.

So the multiplicity of zero of F at a point $(\alpha, 1)$ is the greatest power of $(U - \alpha V)$ dividing F, and at $(1, 0)$ it is the greatest power of V dividing F.

Proposition *Let $F(U, V)$ be a nonzero form of degree d in U, V. Then F has at most d zeros on $\mathbb{P}^1$; furthermore, if k is algebraically closed, then F has exactly d zeros on $\mathbb{P}^1$ provided these are counted with multiplicities as defined above.*